Visualizing Mobius strips and Knotted Tori Using Computer Graphics

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Abstract:

Through out the study of topology, mathematicians have attempted to visualize and model topological spaces so that further conclusions can be drawn, and a deeper understanding can be made. Through out this experiment, an attempt is made to model and study two famous topological spaces: The mobius band, and the torus knot. Using Mathematica, one can model the mobius strip in order to understand how to construct the topological space. Further, this allows us to visualize the space in order to fully understand the deformation retracts, homotopic equivalencies and applications of the Hairy Ball theorem. By modeling torus knots using Mathematica, conclusions can be drawn about the crossing number and, the Siefert surfaces, using the projection maps. Furthermore, when one makes these knots 3-D it allows us to understand the topology much better, giving us information about the genus and Euler Characteristic of the knot. These models allow applied topologists to model and visualize these surfaces to better understand fluid dynamics, electrodynamics, and quantum physics.

Limitations of these models:

While useful, there are limitations to the use for topological models. The biggest limitation of this experiment was the processing power of the machine used for modeling. The "Manipulate" function in Mathematica takes a large amount of processing power. To improve this in future models, it may be useful to switch to graphical modeling in C++. Additionally, the current models that are used as standard for modeling have large Big O running times. This limits what all mathematicians and computer scientists can do and significantly reduces some of the possibilities for future research, as the computer may take a long time to run the model. For example, in Mathematica, the original rendering of the (n, m) torus knot took two hours to fully load. Thus, to solve this, one of the modes uses a library to reduce the big-O running time of the model. Other models created were completely stationary and cannot be deformed or changed using the manipulate function in Mathematica. Mobius Strip:

Mobius Strips

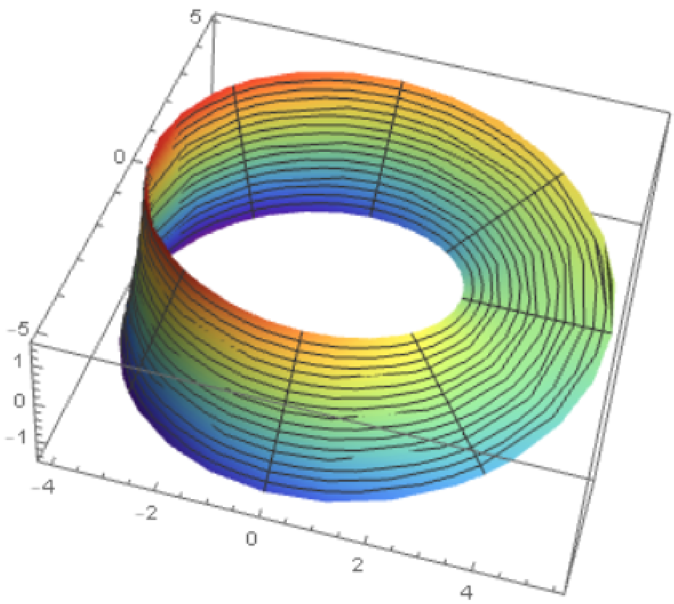
Construction:

The Mobius strip is one-sided, non-orientable surface that only has one boundary. To construct this model, the following parametric equation was considered:

$$\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} a*cos(u)+v*cos(\frac{u}{2})*cos(u) \\ a*sin(u)+v*cos(\frac{u}{2})*sin(u) \\ a*v*sin(\frac{u}{2}) \end{pmatrix}$$

Using the parametric plot 3D function in Mathematica, one can construct the model by changing the parameters to get the following parametric formula

Figure 1: Mobius Band or Strip



$$\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} 4 * cos(u) + rcos(\frac{u}{2}) * cos(u) \\ a * sin(u) + rcos(\frac{u}{2}) * sin(u) \\ rsin(\frac{u}{2}) \end{pmatrix}$$

giving us the following mathematica code:

```
rr = Sqrt[x^2 + y^2]

ParametricPlot3D[ {4 Cos[a] + r Cos[a] Cos[a/2], 4 Sin[a] + r Sin[a] Cos[a/2],
r Sin[a/2]}, {a, 0, 4 \[Pi]}, {r, -(3/2), 3/2}, ColorFunction -> "Rainbow"]
```

Euler Characteristic:

The Euler charactaristic is defined as either:

$X(g) = V - E + F$ where V is the number of verticies, E is the number of edges and F is the number of faces, or $X(g) = 2 - 2g$ where g is the genus, or the number of holes in the surface. So, by definition, since the genus of a

Mobius strip is 1, the Euler charactaristic as defined by $X(g) = 2 - 2g$ is 0. Furthermore, a Mobius strip is not orientable.

$Proof$: Let $M := \{(x, y)|x \in \mathbb{R}, -1 < y < 1\}$ be an infinite strip and let $L > 0$. The definition of a Mobius strip $\hat{M}$is defined by $(x + L, -y) \sim (x, y)$. Now let $\phi : M \to \hat{M}$ be the projection map. Assume $\hat{M}$is orientable. Thus $\exists$ some atlas (U_α, β_α) . For $x \in \mathbb{R}$, where $-1 \leq x \leq 1$, $\phi(x, 0) \in M$ and thus $\phi(x, 0) \in U_\alpha$. The map $f := \beta_\alpha^{-1} o\phi$ is a diffeomorphism in a neighborhood around V of (x,0). Now, take $sgn(J_f(x, 0))$ where J_f is the jacobian of f. Clearly this is localy constant on $\mathbb{R}$.However since $f(x + L, y) \sim f(x, -y)$, there is a contradiction and the Mobius strip is not orientable

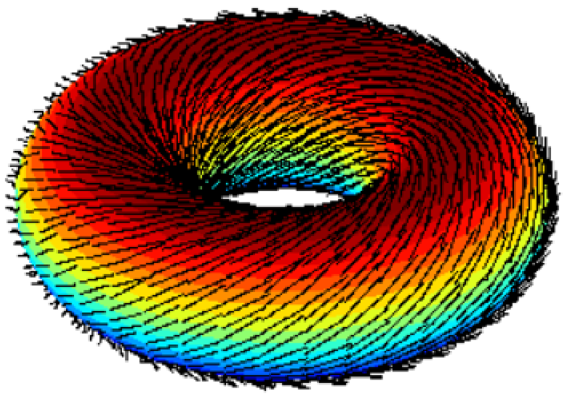
Because of this, one can use another definition of the Euler charactaristic $X(k) = 2 - k$ where k is "... the number of the number of real projective planes in a connected sum decomposition of the surface"(Hatcher) One can thus make the claim that the real projective planes in a connected sum decomposition of the Mobius strip is 2. or, the non-orientable genus is 2.

Extension of the Hairy Ball Theorem

The hairy ball theorem states that "... given a ball with hairs all over it, it is impossible to comb the hairs continuously and have all the hairs lay flat. Some hair must be sticking straight up". In other words, the hairy ball theorem of algebreic topology claims that $\exists$ no non-vanishing, continuous, tangent vector field. i.e. in $\mathbb{R}^3$, if $f : X \to Y$ is a continuous function that assigns a vector in $\mathbb{R}$ to every point p on a sphere such that f(p) is always tangent to the sphere at p, $\exists$ at least one p such that $f(p) = 0$.

$Proof$: Assume there exists a vector feild where $\exists$ no x such that $f(x) = 0$. Now, define v_x to be the vector at x. Now define the homotopy $H(x, t)$: $S^2 \times [0, 1] \to S^2$to be $t\pi$ away from x. Thus giving us the identity and antipodal

Figure 2: Hairy Ball Theorem on a Torus



map in S^2. Since the antipodal map has degree 1, there is a contradiction. $\therefore \exists$ no such vector field.

To extend this to the Mobius strips,one can see that since the Euler charactaristic is 0, the hairy ball theorem fails. This leads us to the Poincare-Hopf theorem. This theorem claims that the sum of the indices of the zeros of a vector field on a closed surface is equal to the Euler Charactaristic.This can be used to prove that the hairy ball theorem does not apply to a torus or Mobius strip.

Homotopy Equivilancies:

When trying to understand the structure of the Mobius strip we can do so by analyzing the homotopy equivilencies and the homotopy invariant properties of these. For example, the Mobius strip is homeomorphic to the surface that is swept in R^3 by a segment rotating in a half-plane around the midpoint, while the half-plane rotates around its boundary line.

$Proof$: Consider the map: $f : [0, 2 * \pi]x[-1/2, 1/2] \rightarrow \mathbb{R}^3 : (x, y) \rightarrow$

$1 + y * sin(x/2))cos(x), (1 + y * sin(x/2))sin(x), ysin(x)$

Clearly, f maps the square onto the Mobius Strip and the mid of the unit square to the mid circle of the Mobius strip.Then one can map the vertical segments of the square onto a strip orthogonal to the mid-circle. The vertical sides of the square will map to one and the same segment and the opposite verticies of the square are mapped to each other. Thus, this is a homotopy equivalence.

One can also claim that there exists a homotopy equivalence between a Mobius Strip and S^1 x I

$Proof$:Construct a deformation retract $F : X \times I \to X$of X onto a subset of X. $F(-1) : X \to A$ to get the composition of equivalences $M \to S^1 \to S^1 \times I$.

Finally one can look at the deformation retracts of the Mobius strip. If C is the center circle, C is a deformation retract of M.

$Proof$: Suppose $h := Ix[-1, 1]xI \to Ix[-1, 1]$ where $h(x, y, t) = x, (1-t)y$. Since $h(x, y, t) \sim h(x, -y, t)$, there is a deformation retract from the Mobius band to the center circle.

Knotted Tori

Construction:

A Knotted Tori can be defined as a closed, non-self-intersecting curve that is embedded in three dimensions and cannot be untangled to produce a simple loop. The loops are constructed by the Reidemeister moves (see figure 3).

Using these one can construct knotted loops by using a composition of these moves. For the purposes of construction in Mathematica, one can begin to construct the Knotted Tori by begining with the simplest torus knot. To create this, the following parametric formula was input into the parametric plot 3D

Figure 3: Reidemeister Moves

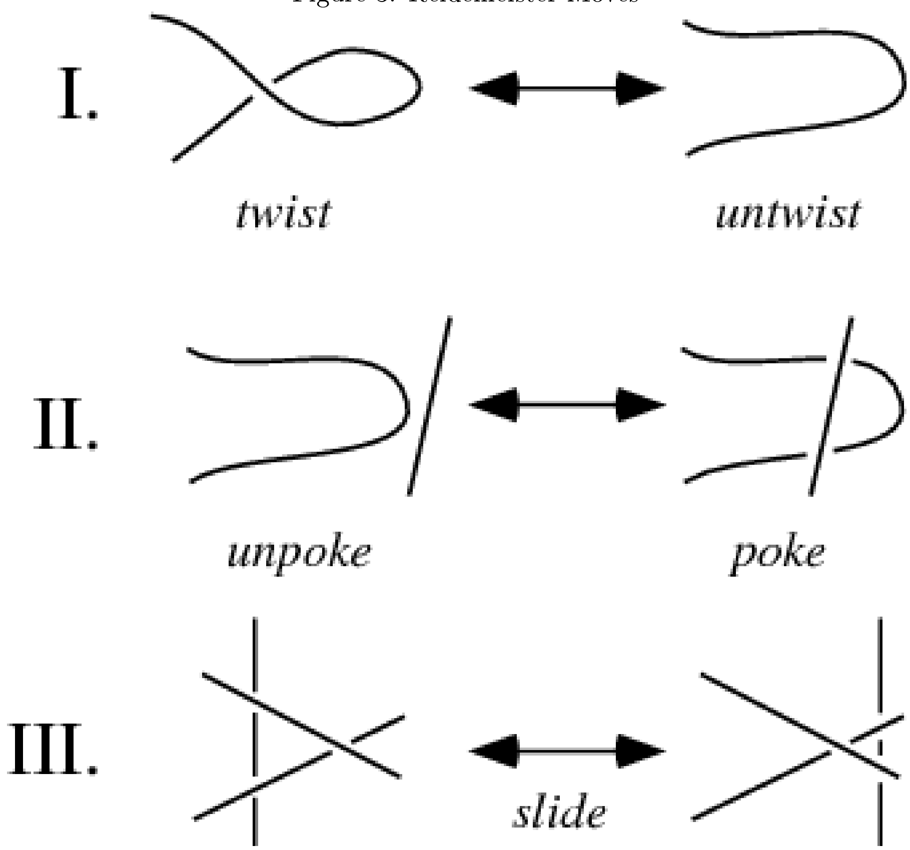
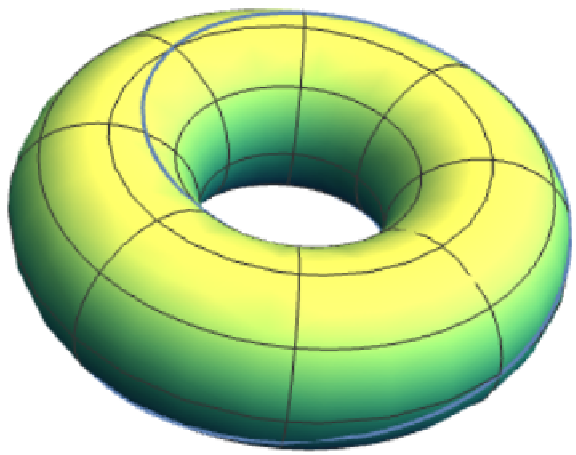


Figure 4: Simple Torus Knot



function to produce a torus, otherwise known as the unknot.

$$\begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} (2+cos(2\pi u))cos(2\pi v) \\ (2+cos(2\pi u))sin(2\pi v) \\ sin(2\pi u) \end{pmatrix}$$

Then, one can overlay the knot by taking p = ParametricPlot3D[torus[u, 2 u]]. Thus, the following code is used to model a simple knot.

```
rr = 2
torus[u_, v_] := {(rr + Cos[2 Pi u]) Cos[2 Pi v], (rr + Cos[2 Pi u]) Sin[ 2 Pi v], Sin[2 Pi u]}
t = ParametricPlot3D[torus[u, v], {u, 0, 2}, {v, 0, 2}, Boxed -> False, Axes -> False, ColorFunction -> "BlueGreenYellow"]
p = ParametricPlot3D[torus[u, 2 u], {u, 0, 1}, Boxed -> False, Axes ->]
Show[t, p]
```

This produces the figure:

From here, to increase the complexity one can build a 3-D knotted torus by using the parametric formula:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} (1+0.3cos(11t/3))cos(t) \\ (1+0.3cos(11t/3))sin(t) \\ 0.3sin(11t/3) \end{pmatrix}$$

Then, we can take the first and second derivatives with respect to t of the parametric formula. From here we can create a function to normalize these by dividing x by the squeare root of x dotted with x. We normalize the first and second derivatie and then take the cross product of these. In doing so, the model is effectivley being blown up from 2 dimentions to 3. The remaining code is as follows:

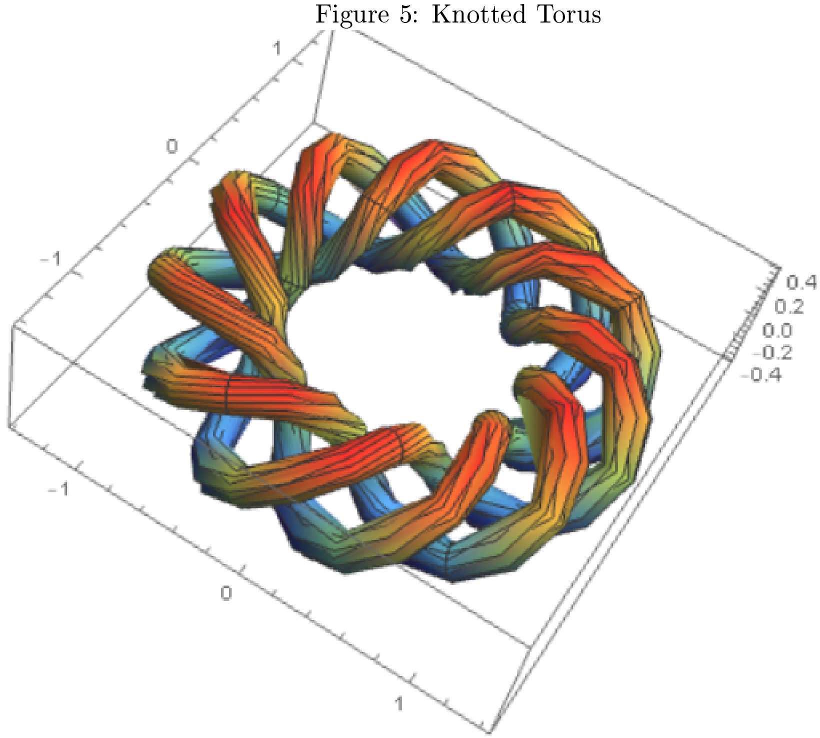
```
normalize[x_] := x/Sqrt[x.x];
k[t_] := {(1 + 0.3 Cos[11 t/3]) Cos[t], (1 + 0.3 Cos[11 t/3]) Sin[t], 0.3 Sin[11 t/3]};
f[t_, theta_] := Module[{dp = k'[t], ddp = k''[t], tangent, normal1, normal2},
tangent = Normalize[dp];
normal1 = Normalize[ddp (dp.dp) - dp (dp.ddp)];
normal2 = Cross[tangent, normal1];
k[t] + 0.1 (normal1 Cos[theta] + normal2 Sin[theta])];
ParametricPlot3D[f[t, theta], {t, 0, 6 Pi}, {theta, 0, 6 Pi}, ColorFunction -> "Rainbow"]
```

This produced the figure 5. Similar methodology would work for different torus knots.

The (p,q) knot :

A torus knot can be rendered in multiple ways which are topologically equivalent but geometrically distinct. The (p,q) knot is defined by the parametric formula

Figure 5: Knotted Torus



$$\begin{pmatrix} x(\phi) \\ y(\phi) \\ z(\phi) \end{pmatrix} = \begin{pmatrix} (cos(q\phi)+2)*cos(p\phi) \\ (cos(q\phi)+2)*sin(p\phi) \\ -sin(q\phi) \end{pmatrix} \text{ where } 0 < \phi < 2\pi$$

The (p,q)-torus knot can be defined where the torus winds q times around a circle in the interior of the torus, and winds p times around its axis. To model this, one can use a KnotData library within Mathematica to create a p-q slider that will allow one to see how p and q affect the shape and topology of the torus knot. To impliment this library, one can use the manipulate function to create the sliders. Using the code:

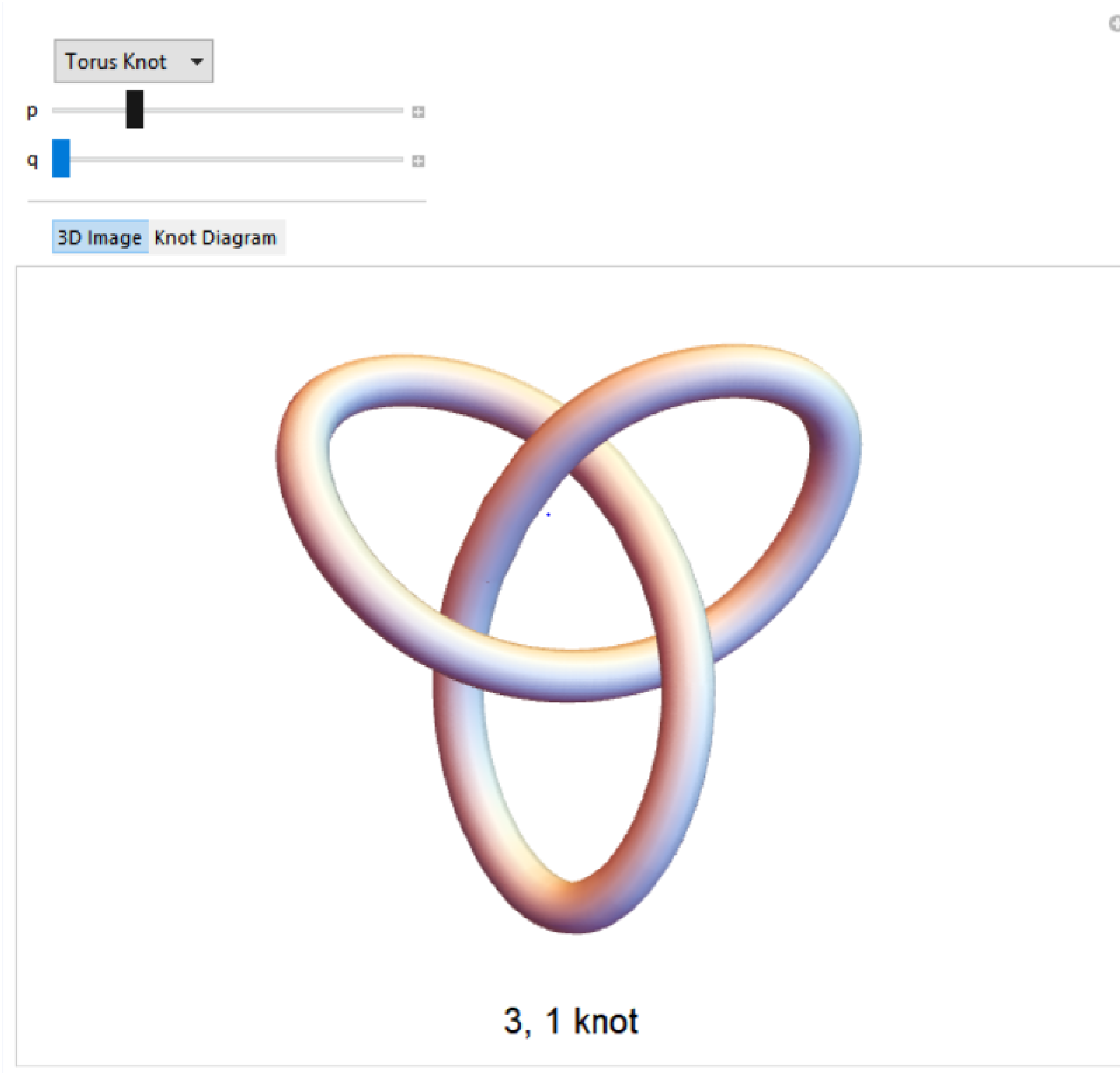
```
Manipulate[
With[{r = Quiet@KnotData[{# &, {"TorusKnot", #} &}[[type]][{i, j}], prop]},
Labeled[Pane[If[Head[r] === Missing || Head[r] === KnotData, Framed[
Text[Style["Error Type 1", Bold, 30]] Background -> Black], r], {600, 400}, Alignment -> Center],
Style[Text[ Row[{i, ", ", j, {"", " torus", " pretzel"}[[type]], " knot"}]], 20]]],
{{type, 1, ""}, {1 -> "Torus Knot"}},
{{i, 3, "p"}, 1, 10, 1}, {{j, 1, "q"}, 1, 10, 1}, Delimiter,
{{prop, "Image", ""}, {"Image" -> "3D Image", "KnotDiagram" -> "Knot Diagram"}}]
```

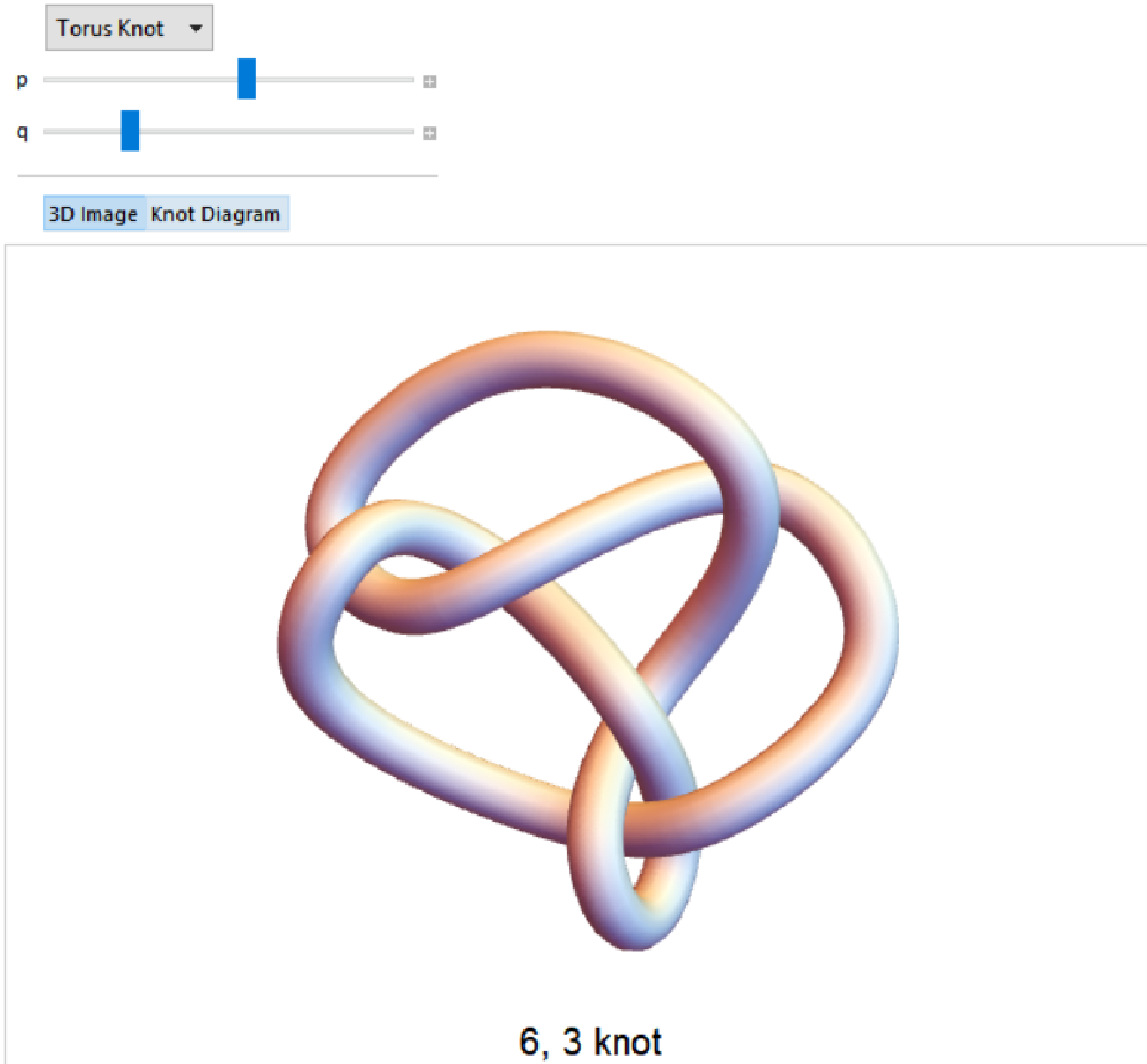
We can create the following knots where p>q,

And the figure:

(6,3) Torus Knot

Figure 6: Treefoil (3,1) Knot





Crossing number:

The crossing number of a knot can be defined by the smallest number of crossings in any diagram of a knot. This knot invariant property of knots is not well understood as there is no algorithm or function that will calculate the crossing number acurately. This is because there is no way to be sure that the knot has been unravled as much as possible. If one takes the projection map of a knot from the front, and a projection map of the knot from the side, some curves seem to become one and the crossing number seems to look very different given the orientation of the projection map.

Conclusions:

Practical Appications of Mobius Strips and Knotted Tori

Unsurprisingly, there are many applications of the Mobius strip in modern technology. The most previlent of which is the patent design for convayer belts. As described by Abram Teplitskiy ,

"There are existing technical applications of the Mobius strip. Mobius strips have been used as conveyor belts that last longer because the entire surface area of the belt gets the same amount of wear, and as continuous-loop recording tapes (to double the playing time). Mobius strips are common in the manufacturing of fabric computer printer and typewriter ribbons, allowing the ribbon to be twice as wide as the printhead while using both half-edges evenly." (Teplitskiy)

Torus knots have more interesting topologies and thus have been used to describe a wide range of phenomenon such as fluid dynamicsm electrodynamics, and quantum physics. The study of Mobius energy has brought knot theory to the forefront of mathematical thought in recent years. Mobius energy can be described as energy that blows up as the knot's strands get close to one another. According to the American Mathemtica Society, "Experiments with a discretized version of the Möbius energy, applicable to the study of arbitrary knots and links, are also described, and confirm the results of the analytic calculations."

Using these models the study of knots and the Mobius strip have become easier than before. We can use these models in the future to 3D print objects of topological interest and thus make knot theory and Mobius strips more tangable and comprehendable.

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