

① Cosmology  
 FLRW metric cylinder-homo but not isotropic Both imply at any  
 Metric of homogeneous & isotropic space pt. there is isotropy

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

All symmetry apply on this  $= g_{00} dt^2 + 2g_{0i} dx^i dt - h_{ij} dx^i dx^j$

comoving coordinate instead

$$ds^2 = g_{00} dt^2 + 2g_{0i} dx^i dt - h_{ij} dx^i dx^j$$

e.g. sphere

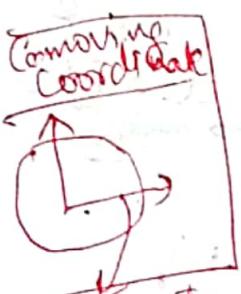
Maximally symmetric spatial geometry

### (i) Isotropy

Transformation of spatial coordinates

$$\text{as } x \rightarrow -x$$

$$ds^2 = g_{00} dt^2 - 2g_{0i} dx^i dt - h_{ij} dx^i dx^j \quad ②$$



Coordinate also expands

s.t. Adding ① & ②

coordinates B/w 2 objects remain at same

Distance

$$ds^2 = g_{00} dt^2 - h_{ij} dx^i dx^j$$

$$= 0$$

$\therefore g_{0i}$  (using this transformation)

$$(ii) d\tau = \sqrt{g_{00}} dt$$

$$ds^2 = d\tau^2 - h_{ij} dx^i dx^j$$

Writing  $t$  instead of  $\tau$

$$ds^2 = dt^2 - h_{ij} dx^i dx^j$$

### (iii) Isotropy

$$ds^2 = h_{ij} dx^i dx^j$$

Due to isotropy

Doubt

isotropy  
spherical symmetry?

spherical symmetry  
if there is one pt. in one direction is different from others

$$\text{i.e. } dx^1 dx^8 = dx + \alpha$$

due to ~~symmetry~~ symmetry

$$\boxed{dx dy = 0}$$

& due to isotropy  $h_{ij} = h(r) \alpha(t)$

$$ds^2 = a^2(t) r^2 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]$$

Taking  $\theta_2 = r$  & redefining  $\theta' = \frac{\theta}{r}$  (to make it appear only one)

$$\text{we get } ds^2 = a^2 [r^2 d\theta'^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \text{ just as } dt = f(r) d\tau \text{ was made}$$

This was all due to isotropy  
Now imposing Homogeneity  $\rightarrow$  why not 3D?

(iv)  $\therefore$  All pts should be same.

We now seek metric that describes S immersed in a spherical  $4+1$  Euclidean space.  $\rightarrow$  No time evolution

$$ds^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$x_1 = a \cos \chi \sin \theta \sin \phi$$

$$x_2 = a \cos \chi \cos \theta$$

$$x_3 = a \cos \chi \sin \theta \cos \phi$$

$$x_4 = a \sin \chi$$

- ③

$$\begin{cases} x = a \sin \theta \sin \phi \\ y = a \sin \theta \cos \phi \\ z = a \cos \theta \end{cases}$$



Diffr ③

$$x_i dx_i = -(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)$$

as

$$\cancel{dx_1^2 + dx_2^2 + dx_3^2}$$

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{x_4^2} \end{aligned}$$

$$dx_1 = -a \sin X \sin \theta \sin \phi dX + a c X \cos \theta \cos \phi d\theta \\ + a c X \cos \theta \cos \phi d\phi$$

$$dx_2 = -a s X c \theta dX - a c X \sin \theta d\theta$$

$$dx_3 = -a s X s \theta c \phi dX + a c X c \theta c \phi d\theta \\ - a c X s \theta s \phi d\phi$$

$$dx_4 = a c X dX$$

$$ds^2 = a^2 (dX^2 + \sin^2 X (d\theta^2 + \sin^2 \theta d\phi^2))$$

$$\rightarrow ds^2 = d^2 (\lambda^2 d\gamma^1)^2 + \gamma^1 d\gamma^1 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Both should be same

$$\text{Given: } \sin X = \gamma^1 \\ dX = \gamma^1 d\gamma^1$$

$$\cos X dX = d\gamma^1$$

$$\gamma^1 = \frac{1}{\cos X}$$

$$\lambda^2 = \frac{1}{1 - \gamma^1 d\gamma^1}$$

Why K should be discrete?

(V) Generalizing (IV)

$$a^2 = x_1^2 + x_2^2 + x_3^2 + k x_n^2$$

Going with same story we obtain

$$ds^2 = a^2 (dX^2 + F(X) (d\theta^2 + \sin^2 \theta d\phi^2))$$

$$\text{where } \left\{ \begin{array}{l} F(X) = \frac{\sin X}{X} \\ \frac{\sinh X}{X} \end{array} \right.$$

$$\left\{ \begin{array}{l} k=1 \\ k=0 \\ k=-1 \end{array} \right.$$

(14)

This metric is again  $\tilde{g}^{ab} ds^2$   
By multiplying  $\tilde{g}^{ab}$  with  $\lambda^{-2}$

### Conclusion

Finally, we obtain

$$ds^2 = dr^2 - \lambda^2(r) \left[ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]$$

### Finsler metric

We have used here  
in practice  $\lambda = \lambda(r)$  metric

### ② Cartesian form

$$ds^2 = dr^2 - \lambda^2(r) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]$$

$$\lambda' = \frac{1}{\sqrt{1-kr^2}}$$

$$\lambda' = r'$$

$$\lambda' = \frac{\lambda}{\frac{d\lambda}{dr} + r}$$

$$\frac{\lambda}{r \frac{d\lambda}{dr} + \lambda} = \frac{1}{\sqrt{1-kr^2}}$$

$$2(1-kr^2) = r^2 \left( \frac{d\lambda}{dr} \right)^2 + 2r \lambda \frac{d\lambda}{dr}$$

$$2-kr^2 r^2 = r^2 \left( \frac{d\lambda}{dr} \right)^2 + 2r \lambda \frac{d\lambda}{dr} + \lambda^2$$



$$\Rightarrow \lambda = \frac{1}{\sqrt{1-kr^2}}$$

$$ds^2 = dt^2 - \frac{a^2}{(1 + kr^2)^2} [dx^2 + dy^2 + dz^2] \quad (5)$$

③  $ds = 0$  (Null geodesic)

$$c^2 dt^2 = a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

Assuming  $d\theta = d\phi = 0$

$$cdt = \frac{a dr'}{\sqrt{1 - kr'^2}} = \frac{a dr}{\sqrt{1 - kr^2}}$$

Physical Distance  
a(t) Coord. Dist.

$\therefore cdt \approx a dr$

This is not real distance

for small propagation distances

r' small

$$\frac{kr^2}{2} \approx 1$$

$$kr^2 \ll 1$$

Hubble law

④  $c dt = D \approx a dr'$   $\downarrow$    
 Distance travelled by light (Null geodesic)  
 Not as coordinate distance is fixed (small propagation)

$D = \dot{a} dr'$  (as  $r, \theta, \phi, t$  are independent  
 $\frac{dr}{dt} \approx 0$ )

⑤  $D = H D$

$H = \frac{\dot{a}}{a}$  = Hubble Constant

= rate of exp. of Space

Hubble law apply to any system that expands in homog & isotropic way.

$$\text{as } \Delta r = \epsilon'$$

$r$

Hubble law applies only to small distances  
 or in any K-space or to only flat space at any distance

$$(x^1, \theta, \phi, t)$$

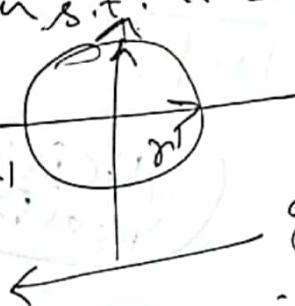
## ⑤ Physical Distance VS Comoving Distance

Coordinate system is chosen s.t. it is expanding at same rate as universe.

Assuming for small Distance or  $z \ll 1$ .

$$d = \dot{a}dr' = HD$$

$$D = a(t)dr' \quad \begin{matrix} \downarrow \\ \text{Physical} \end{matrix} \quad \begin{matrix} \downarrow \\ \text{Comoving distance} \end{matrix}$$



∴ Comoving distance b/w any two galaxies moving away from each other one is constant.

## ⑥ Redshift:

$$s. \lambda_{\text{em}} = c dt \quad \begin{matrix} \downarrow \\ \text{Time between crests} \end{matrix}$$



Assuming  $v \ll c$



$$\frac{d\lambda}{\lambda} = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{v}{c} = z \quad \leftarrow$$

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$$

If the source is following Hubble law  $D = HD \equiv v = HD \Rightarrow \frac{d\lambda}{\lambda} = \frac{v}{c} = \frac{Hdt}{a} - \frac{da}{a}$

$$dt = t_{\text{obs}} - t_{\text{em}} = \text{negative time}$$

$$\frac{d\lambda}{\lambda} = + \frac{da}{a} \Rightarrow \ln \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = + \ln \frac{a_0}{a_{\text{em}}} \Rightarrow \ln \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = + \ln \frac{a_0}{a_{\text{em}}}$$

Source moves away in  $dt$  time at  $v$  speed i.e.  $vdt \therefore \lambda_0 = cdt + vdt$



$$⑧ T^{00} = p \quad \text{Restframe } u^i = (t, 0, 0)$$

$$T_{ij}^{ij} = p - 3p$$

$$T^{ij} = (\rho + p) u^i u^j - p g^{ij} \quad i=1, 2, 3$$

$$= (\rho + p) 0 - \frac{p h^{ij}}{a^2} = - \frac{p h^{ij}}{a^2}$$

$$g^{ij} = \frac{h^{ij}}{a^2}$$

$$⑨ \nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma_{\alpha\gamma}^\alpha T^{\gamma\beta} + \Gamma_{\alpha\gamma}^\beta T^{\alpha\gamma}$$

$$\Gamma_{0j}^i = \Gamma_{j0}^i = \frac{\dot{a}}{a} \delta_j^i$$

$$\Gamma_{ij}^0 = a \dot{a} h_{ij}$$

~~$\Gamma_{jk}^i = 0$~~

$$\nabla_\alpha T^{\alpha 0} = \partial_\alpha T^{\alpha 0} + \Gamma_{\alpha\gamma}^0 T^{\gamma 0} + \Gamma_{\alpha\gamma}^0 T^{\gamma 0}$$

$$= \frac{\partial p}{\partial t} + \Gamma_{ij}^0 T^{ij} + \Gamma_{\alpha 0}^\alpha T^{00}$$

$$= \frac{\partial p}{\partial t} + a \dot{a} h_{ij} (-\rho g^{ij}) + \frac{\dot{a}}{a} p$$

$$\text{For Radiation} = \frac{\partial p}{\partial t} + \frac{3 \dot{a} p}{a} + p \frac{3 \ddot{a}}{a} \rightarrow$$

$$p = \frac{f}{3} \quad 0 = \frac{\partial f}{\partial t} + \frac{4 \dot{a}}{a} p \Rightarrow \frac{d}{dt} (p a^3) = 0$$

$$\varphi a^h = \epsilon = \text{const}$$

(9)

$$\rho = \frac{\epsilon}{a^h}$$

$$\frac{\partial \rho}{\partial t} + 3 \frac{\dot{a}}{a} (\rho + p) = 0$$

$$\frac{\partial \rho}{\partial t} + 3H(\rho + p) = 0$$

(10) Define Spatial Curvature Scalar?

$$R = h^{ij} R_{ij}$$

(11) Taking spatial comp. of E-M Tensor

constant curvature

Taking  $\Lambda$  also

$$\frac{\partial (\rho + \frac{\Lambda}{8\pi G})}{\partial t} + a \dot{a} h^{ij} \left( \frac{\Lambda}{8\pi G} - p_{ij} \right) + \frac{3\ddot{a}}{a} \left( \rho + \frac{\Lambda}{8\pi G} \right)$$

$$\frac{\partial \rho}{\partial t} + \frac{a \dot{a}}{a} \left( -\frac{3\Lambda}{8\pi G} + 3p \right) + \frac{3\ddot{a}}{a} \left( \rho + \frac{\Lambda}{8\pi G} \right) = 0$$

$$\frac{\partial \rho}{\partial t} + 3H(\rho + p) = 0$$

Cy Eqn  
Gaussian

# Expanding Universe

(1)

$$\textcircled{1} \quad R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2} R = T_{\alpha\beta} \quad 8\pi G$$

$$\boxed{R_{00} - \frac{g_{00}}{2} R = T_{00} \quad 8\pi G} \quad \textcircled{1}$$

$$R^a_a - g_{RR} = -R = 8\pi T$$

$$\boxed{R = -8\pi T} \quad \textcircled{2}$$

Taking the FRW metric

$$ds^2 = dt^2 - a^2 \left[ \frac{dx^2}{1-kx^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right]$$

Calculating  $R$  & using (1) & (2) we obtain

Using  $R_{\alpha\beta} + \frac{g_{\alpha\beta}}{2} R = 8\pi T_{\alpha\beta}$

$$\textcircled{6} \quad H^2 = \frac{8\pi p}{3} - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(p+3p)$$

Friedmann Eqn

(2) In the previous section we have also obtained  
conservation Eqn  $\nabla_\alpha T^{\alpha 0} = 0$

$$\frac{dp}{dt} + 3H(p+p) = 0$$

Friedmann Eqn & conserv. Eqn are not find.  
Diff (1) & Friedm & using Conserv. Eqn, we obtain  
2) Fried

$$\textcircled{3} \text{ Critical Density } \rho_c = \frac{3H^2}{8\pi G}$$

$$\text{Density parameter } \Omega = \frac{\rho}{\rho_c}$$

using 1<sup>st</sup> Friedmann Eqn

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{k}{a^2 H^2} = \frac{\rho}{\rho_c} - \frac{k}{a^2 H^2}$$

$$\text{Let } G = 1$$

$$1 = \Omega - \frac{k}{a^2 H^2}$$

at any time

$$\Omega = \Omega(a) + \Omega_k(a)$$

$$\Omega_k(a) = -\frac{k}{a^2 H^2}$$

\textcircled{4} for  $k=0$  flat FLRW

$$\Omega_k = \frac{-k}{a^2 H^2} = \frac{\rho_k}{\rho_c}$$

$$\Omega = 1 \Rightarrow \rho = \rho_c \text{ Doubt}$$

$$\frac{-k^3}{a^2 8\pi G} = \rho_k$$

for  $k=+1$

for  $k=-1$

$$\Omega > 1$$

\textcircled{5} Non Relativistic fluid

$p=0$  [Dust matter (galaxy)]

[Non interacting particles with non relativistic vel. (CDM)]

Also Pressure of free particle with mean sq. vel.  $v$   $p = nmv^2 < \rho = nmc^2$  for  $v \ll c$ . By cont. Eq.  $\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}$

$$1 = \Omega(a) + \Omega_k(a) \text{ at every epoch}$$

This relation extends to models with diff comp.

$$\ln \rho = -3 \ln a$$

$$\rho \propto a^{-3} \Rightarrow \frac{\rho}{\rho_0} = \left(\frac{a_0}{a}\right)^3$$

$$\boxed{\rho = \rho_0 \left(\frac{a_0}{a}\right)^3}$$

Let  $a_0 = 1$

$$\rho = \frac{\rho_0}{a^3} = \rho_0 (1+z)^3 = \rho_c S_{NR} (1+z)^3$$

Present value  $\Rightarrow S_{NR} = \frac{\rho_0}{\rho_c}$

⑥ By F<sup>1st</sup> Eq.

$$k=0 \quad H_0^2 = \frac{8\pi G}{3} \rho_0$$

Flat space

also  $H^2 = \frac{8\pi G}{3} \rho$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{\rho_0} \rho = H_0^2 a_0^{-3} a^{-3} = H_0^2 a^{-3}$$

Integrate for time  
By conservation of mass

$$\dot{a}^2 = \frac{H_0^2}{a} \Rightarrow a^{1/2} da = H_0 dt$$

$$\int a^{3/2} = H_0 t$$

Assuming flat space  $\boxed{a = (2H_0)^{2/3} t^{2/3}}$

$H_0$  value we know

$\therefore$  By  $H_0^2 = \frac{8\pi G}{3} \rho_0$  we can find present matter density  $\Rightarrow$

$$\textcircled{1} H_0 = 100 h \frac{\text{km}}{\text{s.Mpc}}$$

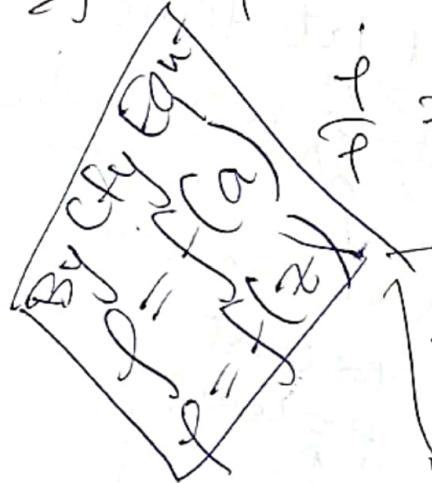
$$h = 0.70 \pm 0.04$$

(12)

## ② Relativistic Component

$$\phi = \frac{1}{3} \quad (\text{A photon gas distributed as black body has pressure } p = \frac{1}{3} \rho)$$

$$\text{Cyl Eq} \quad \ddot{p} + 3H\dot{p} = 0$$

$$\frac{\dot{p}}{p} = -3 \frac{\dot{a}}{a} \Rightarrow \ln p = -3 \ln a$$


$$p = p_0 \left(\frac{a_0}{a}\right)^3$$

current value

$$\text{let } a_0 = 1 \Rightarrow \phi_R = \frac{p_0}{a^3} = p_c \text{ s}_{\text{UR}} T^{(1+z)^3}$$

$$p_m \propto a^{-3} \quad \text{at Early time}$$

$$p_0 \propto a^{-n} \quad \text{Radiation Dominant}$$

$$p_{NR} = p_c \text{ s}_{NR} (1+z)^3$$

Radiation Density decreases as  $a^{-3}$  b.c. of vol. expn &  $a^{-1}$  b.c. of Energy Redshift.

With matter has energy

$$\textcircled{3} \text{ Photon gas in Eqn with matter has energy}$$

$$\text{Density } p_r = \frac{g\pi^2}{30} T^4 \quad \begin{cases} g=2 \text{ for photons} \\ g=3.36 \text{ massless neutrino} \end{cases}$$

$$p_r \propto a^{-4}, \quad T \propto \frac{1}{a} \Rightarrow \text{as } T_0 = 3K$$

$$T_0 = 2 \times 10^{-29} h^2 \text{ g cm}^{-3}$$

$$p_{r,m} > p_{r,0} \quad \begin{cases} \text{matter Dominating} \\ \text{at Epoch} \end{cases}$$

$$p_r = g 2.3 \times 10^{-34} \text{ g cm}^{-3}$$

$$\textcircled{10} \quad p_m = p_{m,0} \begin{cases} a^{-3} \\ a^{-4} \end{cases}$$

$p_r = p_{r,0}$

Equivalence Epoch where  $p_r = p_m$

$$a_e = \frac{p_{r,0}}{p_{m,0}} = \frac{p_{m,e} p_{r,0}}{p_{r,c} p_{m,0}} = \frac{\dot{S}_M}{\dot{S}_m}$$

assuming  $p_{r,c} = p_{m,c}$

Equivalence Redshift?

$$\textcircled{11} \quad \dot{\phi} = \omega \dot{p}$$

$$-3(\omega+1)$$

$$\dot{p} \propto a$$

By Ctry Eq<sup>V</sup>

at late time  
 $a > 1$   
 $\text{const} = p_r$  is Dominant

Vacuum  $\omega = -1$

$p_2 \text{ const}$   $\dot{p} = -\dot{p}$

$p = \text{const}$

$p = 0$	$\dot{p} \propto a^{-3}$
$p = \frac{q}{3}$	$\dot{p} \propto a^{-4}$
$\ddot{p} = \omega \dot{p}$	$\ddot{p} \propto a^{-3(\omega+1)}$

\textcircled{12} Assuming  $\lambda = 0$

$$\text{By } \textcircled{11} \text{ st F Eq} \rightarrow H^2 = \frac{8\pi G}{3} p$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{H_0^2}{a^{3(\omega+1)}}$$

$$\frac{\dot{a}^2}{a^2} = H_0^2 a^{-3\omega-1}$$

$$\frac{3\omega+1}{2} \dot{a}^2 = H_0^2 \Rightarrow \left(\frac{3\omega+1}{2}\right) a^{\frac{3\omega+1}{2}} = H_0^2$$

$$(13) H^2 = \frac{8\pi G}{3} (p_{m,0} a^{-3} + p_{r,0} a^{-4} + p_{k,0} a^{-2}) \quad (18)$$

$$f_c = \frac{3H_0^2}{8\pi G} \quad \Omega = \frac{f}{f_c}$$

$$p_{k,0} = -\frac{3k}{8\pi G}$$

$$(14) H^2 = H_0^2 (\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4} + \Omega_{k,0} a^{-2})$$

$$\text{s.t. } \Omega_{m,0} + \Omega_{r,0} + \Omega_{k,0} = 1$$

Every other component  $\propto$  can be added when its behaviour with  $a$  is known.

### (15) Qualitative Trends

$\rightarrow p+3p > 0$  till now  
 By 2<sup>nd</sup> F Egn  $\ddot{a} < 0$  Decelerating.  
 $\therefore \Omega a=0$  at some time (if initially  $a>0$ )  
 $\ddot{a}(t-t_0) = a_0 + \dot{a}(t-t_0)$   
 let  $\ddot{a} = \text{const} \Rightarrow a(t) = a_0 + K(t-t_0) = a_0 + \dot{a}(t-t_0)$

→ Proves Big Bang

$$\Rightarrow a \propto t^{\frac{2}{3w-1}}$$

(14) Raychaudhuri Eqn- VS (Observe  
Cosmological Constant)

Eqn

$$\frac{d\theta}{dt} = -\frac{\theta^2}{3} + \frac{\alpha\beta}{\alpha\beta + \omega\omega} - R_{\mu\nu}U^\mu U^\nu$$

$$SEC : R_{\mu\nu} U^\mu U^\nu > 0$$

$$\frac{d\theta}{dt} < 0$$

(Rate of Expansion  
Universe will decrease)



$$SEC: r^2(\rho + \alpha^2 p_1 + \beta^2 p_2 + \gamma^2 k_B) \geq \frac{(\rho - p_1 - p_2 - k_B)}{2}$$

$$-\alpha\beta = \begin{pmatrix} \rho \\ -p_1 - p_2 - p_3 \end{pmatrix}$$

$$\text{if } i) a=b=c=0 \quad v=0 \Rightarrow \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 \quad 17$$

$$t + p_1 + p_2 + p_3 \geq 0,$$

$$ii). \text{ if } b=c=0 \quad \gamma = \frac{1}{\sqrt{1-a^2}}$$

$$T + p_i \geq 0$$

Cosmological Const

$$\text{for } p+\beta a \quad p+3p > 0$$

Violate

$$p+3p > 0$$

SEC

$$q) \quad \dot{a} > 0 \quad \text{Exp.}$$

$$\dot{H} = \frac{\dot{a}^2 - \ddot{a}a}{a^2}$$

$$\ddot{H} = H^2 - \frac{\ddot{a}}{a}$$

$$\ddot{H}^2 - \dot{H}^2 = \frac{\ddot{a}^2}{a^2} > 0$$

$$\frac{\dot{H}}{H^2} < 0$$

age of Universe  $H_0 = T$

To obtain age of universe

$$- H_0^{-1} = T \text{ greater}$$

$$T^{\alpha\beta} = (p+\rho) g^{\alpha\beta} + \rho g^{\alpha\beta}$$

↓  
for observer which are comoving with the  
expn. For all others  $T^{\alpha\beta}$  would be  
different i.e. different content of Energy/  
pressure.

But if a case where  $\alpha^\beta$  same for every observer  
regardless of  $U^\alpha \Rightarrow$  This occurs when  $p = -\rho$

$$T^{\alpha\beta} = -\rho g^{\alpha\beta} = \rho g^{\alpha\beta}$$

$$\nabla_\alpha T^{\alpha\beta} = \rho \nabla_\alpha g^{\alpha\beta} + (\lambda \rho) g^{\alpha\beta} = 0$$

$$\Rightarrow \lambda \rho = 0 \Rightarrow \rho = \underline{\underline{\text{const}}}$$

~~Energy Cons~~  
Energy Cons

$$\therefore T_{\alpha\beta} = \rho g_{\alpha\beta}$$

$$T_{\alpha\beta} = \frac{\Lambda}{8\pi G} g_{\alpha\beta}$$

$T_{\alpha\beta}(\Lambda)$  = Vacuum Energy

$$T_{\alpha\beta} = (\rho, -p, -p, -p)$$

$$T_{\alpha\beta} = (\rho, p, p, p)$$

$$T_{\alpha\beta} = \left( \frac{\Lambda}{8\pi G}, \frac{-\Lambda}{8\pi G}, \frac{-\Lambda}{8\pi G}, \frac{-\Lambda}{8\pi G} \right)$$

$$\rho = \frac{\Lambda}{8\pi G}$$

$$p = -\rho$$

$$p = -\frac{\Lambda}{8\pi G}$$

$$w = \frac{p}{\rho} = -1$$

Violates  
SEC

$$\rho + 3p = \rho - 3\rho = -2\rho < 0$$

Age of Universe

$$\therefore EFE \Rightarrow \Box G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

$$G_{\alpha\beta} - g_{\alpha\beta} = 8\pi G T_{\alpha\beta} = 8\pi G T_{\alpha\beta} + \frac{8\pi G \Lambda}{8\pi G} g_{\alpha\beta}$$

## (B) Cosmological Observations

$$M = -2.5 \log_{10} L + \text{const.}$$

Absolute Magnitude: const. & can be chosen arbitrarily depending on observed wavelength.

(e.g.  $M_{\text{sun}, B} = 5.48$  (B is Blue band at 4400 Å)).

## (C) In Non-expanding Euclidean Geometry

$$f = \frac{L}{4\pi d^2}$$

$$m = -2.5 \log f + \text{const}$$

Apparent magnitude

Friedmann Eqn

$$H_0 - H_{100} = 8\pi G P_{100}$$

$$\frac{\dot{a}^2}{a^2} - \frac{1}{3} = 8\pi G \rho$$

$$\ddot{a} = -\frac{4\pi G}{3}(1+3\rho) + \frac{D}{3}$$

Cly Egn

const. chosen s.t. const = 0  
for  $f = 2.5 \times 10^{-5}$

$$m = -2.5 \log 2.5 \times 10^{-5}$$

$$M = -2.5 \log_{10} f 4\pi d^2$$

$$M = m - 2.5 \log_{10} 4\pi d^2$$

$$M = m - 5 \log_{10} d - 2.5 \log_{10} 4\pi$$

$$\rightarrow H^2 = H_0^2 \left( \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda a^{-4} + \Omega_{0,n} \right)$$

$$m = M + 2S + 5 \log d$$

$d : \text{Mpc}$

$M = m - m = \text{dist. Modulus} \& \text{a measure of distance.}$

# A CDM model

① 3 components  $\rightarrow$

- 1  $\Lambda = \text{Dark energy}$
- 2 CDM  $\rightarrow$  Non Rel. ( $p=0$ )
- 3 Ordinary matter Relativist. ( $p=\frac{1}{3} \rho a^3$ )

19

$$\rho \propto a^{-3}$$

Quintessence vs cosmological constant.

② Dark energy vs Ordinary Matter

Baryons

$$\rho \propto a^{-3}$$

$$p=0$$

Radiation

$$\rho \propto a^{-4}$$

$$p = \frac{1}{3}$$

CDM

$$\rho \propto a^{-3}$$

$$p=0$$

$$H(a) = \frac{\dot{a}}{a} = H_0 \sqrt{(S_c + S_b)a^{-3} + S_\gamma a^{-4} + S_k a^{-2} + S_{DE} a^{-3(1+w)}}$$

A

③ General

$$p = w\rho \xrightarrow{-3(w+1)} w=-1$$

$$S_{DE} = S_V$$

$\rho \propto a$   
 Now put in 1st F Eq to get  
 $a(t)$

⑨ FRW metric  
(homogeneity & isotropy)

(Spatial component  
is time dependent)

$$c^2 dx^2 = c^2 dt^2 - a(t)^2 d\Sigma^2$$

Scale factor

3 dimensional space of  
uniform curvature.  
i.e. Elliptical space,  
Euclidean or Hyperbolic

$$\Omega_m = \Omega_b + \Omega_c$$

⑩ Assuming  $\Omega_k = 0$

&  $\Omega_b \ll 1$

$$a(t) = \left(\frac{\Omega_m}{\Omega_b}\right)^{1/3} \sinh\left(\frac{t}{t_n}\right)^{2/3}$$

$$t_n = \frac{2}{3\sqrt[3]{\Omega_b}}$$

solving for  $a(t) = 1$  we get  
at the age of the Universe

⑪ Success & failure

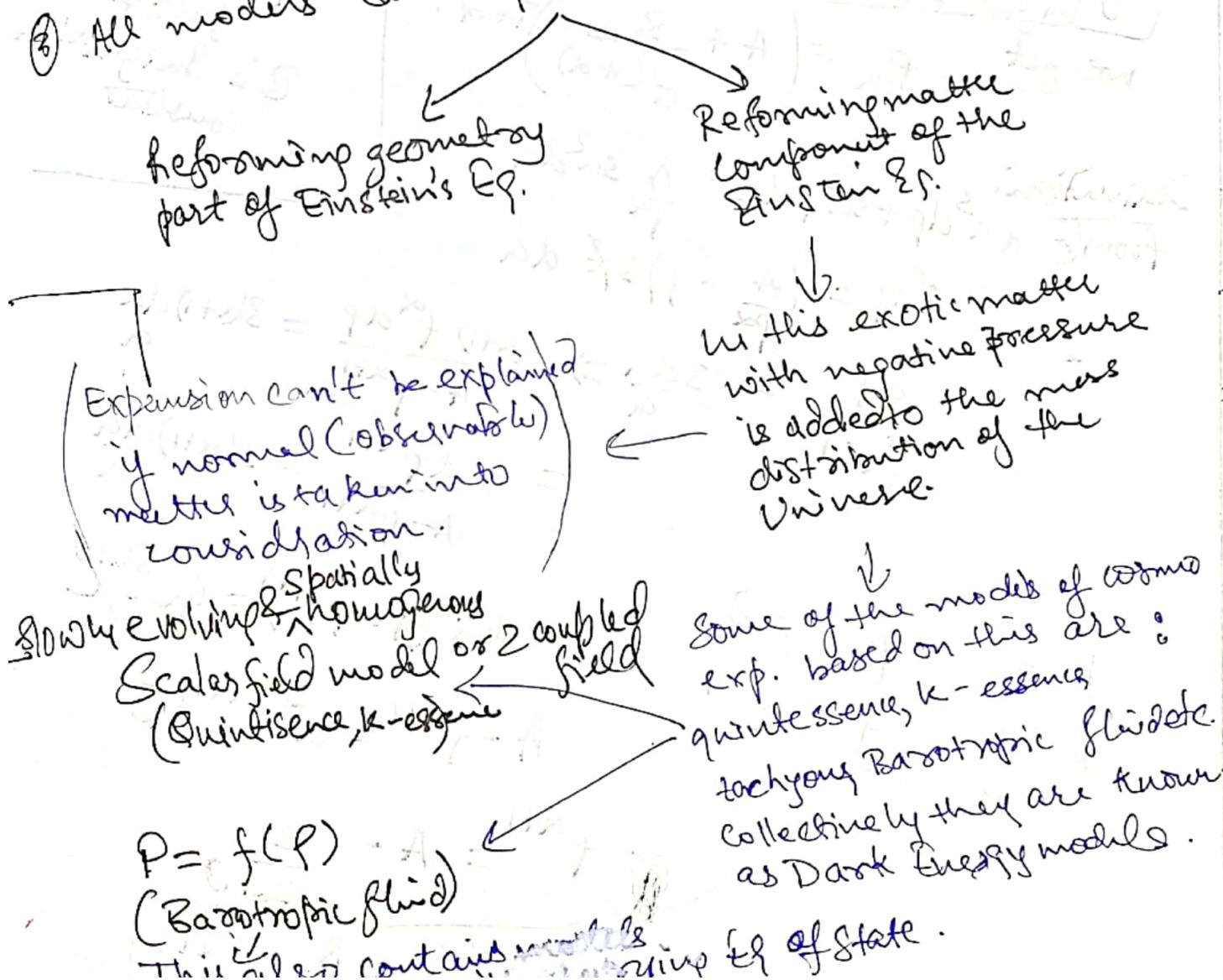
As these observations show that our universe is made up of 2 composite : Dark matter & Dark Energy.  
Dark Matter: contributes  $\frac{1}{3}$  of total energy density of universe.  
Dark Energy: Self interacting, negative pressure contributes  $\frac{2}{3}$  of total energy density of universe.  $10^{-10}$

The matter universe is made up of different components: matter, radiation & dark energy matter.

### Quintessence Model

This model also suffers from fine tuning problem. The problem is why does dark energy start dominating over matter content of universe recently.

- ① The Universe is expanding was found out by the observations. One such observation was of SNe - Ia. There are many # of observations after this discovery of SNe were done to confirm expansion of Universe. These observations are Baryon Acoustic Oscillations (BAO), CMB, growth of structures & GRBs.
- ② There are many different cosmological models which explain the expanding Universe. One such is  $\Lambda$ CDM model (or cold Dark Matter) where  $\Lambda$  is the cosmological constant. Accounts for the energy density of space or the vacuum energy. However this suffers from serious fine tuning problem. Hence, many models have been proposed.
- ③ All models can be put into 2 classes.



## ⑤ Barotropic fluid

$$P = f(\rho)$$

Relation b/w  $P$  &  $\rho$  (energy density) determines the dynamics of the fluid. For e.g. one such fluid is Chaplygin gas.

mimics dark energy & dark matter & is a possible substn of stand. model of cosm.

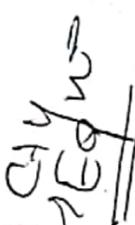
## ⑥ Chaplygin Gas Model

Eq of state

$$P = -\frac{A}{\rho}$$

( $A$  is positive constant)  
( $\rho$  &  $P$  both in comoving frame with  $P \geq 0$ )

## Generalized model of Chaplygin gas



$$\text{Eq of state } P_{ch} = -\frac{A}{\rho^\alpha}$$

$$A > 0$$

$$0 < \alpha \leq 1$$

By energy momentum eq.  
in flat FRW metric

$$d(\rho a^3) = -P d\rho a^3 \quad (*)$$

$$\text{we get } P_{ch} = \left( A + \frac{B}{\alpha(1+\alpha)} \right)^{\frac{1}{1+\alpha}}$$

$a$  is the scale factor  
 $B$  is Integration constant

$$\text{Derivation: } a^3 dp + 3a^2 P da = \frac{A}{\rho} a^2 da$$

$$dp = \left( \frac{A}{\rho} - P \right) 3 \rho^\alpha da$$

$$\begin{aligned} \frac{dp}{\frac{A}{\rho} - P} &= \frac{3da}{a} \Rightarrow \frac{(\alpha+1) \rho^\alpha df}{A - \rho^{\alpha+1}} = 3(\alpha+1) \frac{da}{a} \\ &= \frac{dP^{\alpha+1}}{A - P^{\alpha+1}} = 3(\alpha+1) \frac{da}{a} \end{aligned}$$

$$-\ln(A - P^{\alpha+1}) = \ln a^{\frac{3}{\alpha+1}} C$$

$$A - P^{\alpha+1} = a^{-\frac{3}{\alpha+1}} C$$

$$P = 0$$

$$P \propto a^{-3}$$

$$\rho = \frac{P}{3}$$

$$\rho \propto a^{-4}$$

$$P^{\alpha+1} = A - \frac{C}{a^{3(\alpha+1)}}$$

$$P = \left( A + \frac{B}{a^{3(\alpha+1)}} \right)^{\frac{1}{1+\alpha}}$$

Doubts

- ① How BAO, CMB, growth of structure & GRB explain the expansion of the Universe.
- ② How these evolving & spatially homogeneous scalar field ("quintessence" model) works?
- ③ What is CDM model?
- ④ What is Dark matter & Dark Energy?
- ⑤ Difference b/w Dark matter & Dark energy.
- ⑥ Difference b/w Energy momentum conservation statement & Energy conservation statement.
- ⑦ What is Energy momentum conservation statement in FLRW metric?
- ⑧ Why at early time  $a=1$

→ This is the fluid eqn which holds for all radiation, matter & dark matter

$$\dot{\rho}_i + \frac{3\dot{a}}{a} (\rho_i + p_i) = 0 \quad i \in \{r, b, ch\}$$

$$\frac{dp}{dt} + 3H(p + \rho) = 0 \quad \begin{matrix} \downarrow \\ \text{rad, Baryon} \end{matrix} \quad \begin{matrix} \downarrow \\ \text{matter} \end{matrix}$$

for Baryonic matter

$$p=0 \Rightarrow \dot{\rho}_b + \frac{3\dot{a}}{a} \rho_b = 0 \Rightarrow d\ln \rho = -3da/a$$

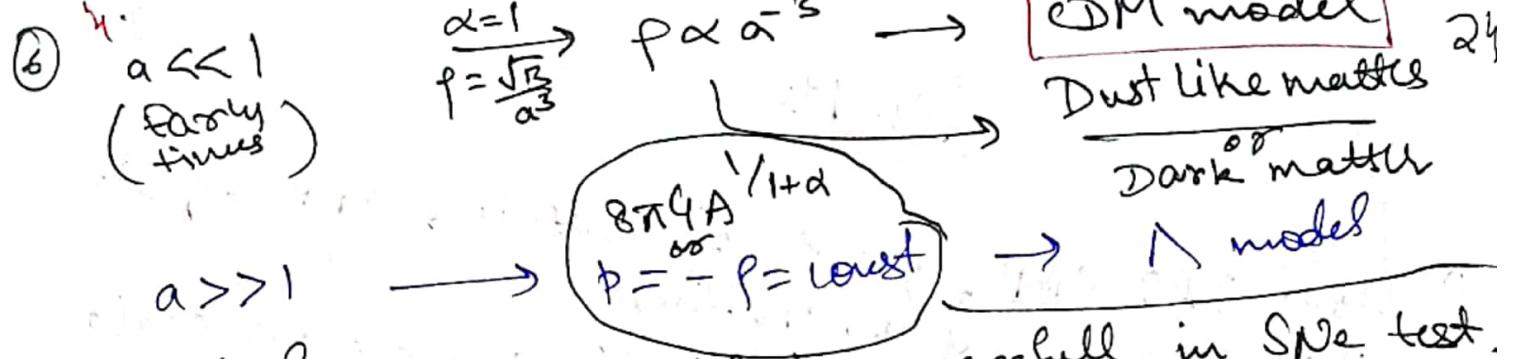
$$\Rightarrow \rho = \rho_{b0} a^{-3}$$

$\downarrow$   
Int. const.

for Radiation

$$p = \frac{f}{3} \Rightarrow \dot{\rho}_r + 4\rho_r \frac{\dot{a}}{a} = 0 \Rightarrow \rho = \rho_{r0} a^{-4}$$

$\downarrow$   
Int. const.



⑦ Generalized Chaplygin gas model is successful in SNe test.  
~~Chaplygin gas~~ model is successful in SNe test.  
 CMB peak locations, Gravitational lensing etc.

However it produces oscillations or exponential blowups of matter power spectrum which is inconsistent with observation.  $\therefore$  Modification of Gen. Chaplygin gas

Variable Chaplygin gas

### ⑧ Variable Chaplygin Gas

Eq of State  $P_{ch} = -\frac{A(a)}{f_{ch}}$

$$A(a) = A_0 a^{-n}$$

$A_0, n$  is constant  
 $A(a)$  is positive for  
 $a$  is scale factor.

Using energy momentum conservation in flat FRW metric we get

$$f_{ch} = \sqrt{\frac{6}{6-n} \frac{A_0}{a^n} + \frac{B}{a^6}}$$

Derivation:

$$f = A^{n+1} = \text{const} - \frac{1}{3\alpha}$$

$$P = -\frac{A}{A^{n+1}} \Rightarrow P = -A^{-n-1}$$

$$\therefore P = -\rho = \text{const.}$$

Transformation from  $z \rightarrow a$

$1+z = \frac{a_0}{a}$  where  $a_0$  is the scale factor at present time which we normalize to 1.

(i)  $a < 1 \rightarrow a^{-3}$

Condition on  $n$ ?

$a > 1 \rightarrow P = -\rho = \text{const.}$

chaplygin gas evolves from dust dominated epoch to cosmological const. in present times

Derivation:

if  $n < 6$

$$\left( \frac{c}{6-n} \frac{a_0}{a^n} \right)^{1/2}$$

$$\rho \sim a^{-n/2}$$

(10) Defining  $\sigma_m = \frac{B}{\frac{6A_0}{6-n} + B}$

we get

$$f_{ch}(a) = f_{ch,0} \left( \frac{\sigma_m}{a^6} + \frac{1-\sigma_m}{a^n} \right)^{1/2}$$

$$f_{ch,0} = \sqrt{\frac{6}{6-n}} A_0 + B$$

$\uparrow$   
present value of  $f_{ch}$



(11) Fischman Eqn in terms of  $\sigma_m, \sigma_b, \sigma_g$

$$H^2 = \frac{8\pi G}{3} \left\{ f_{r0} (Hz)^4 + f_{bo} (Hz)^4 + f_{ch0} [(\sigma_m Hz)^6 + (-\sigma_m) (Hz)^6] \right\}$$

$$H^2 = \sigma_{ch,0} H_b^2 a^{-h} X^2(a)$$

$$X^2(a) = \frac{\sigma_m}{1 - \sigma_{bo} - \sigma_{go}} + \frac{\sigma_{bo} a}{1 - \sigma_b - \sigma_{bo}} + a^{\frac{n}{6}} \left( \frac{\sigma_m}{a^6} + \frac{1 - \sigma_m}{a^n} \right)^{1/2}$$

① Relation b/w  $a$  &  $z$ ?

② How to find condition on  $m$ ?

$$H^2 = H_0^2 \left[ S_g (1+z)^4 + S_b (1+z)^3 + S_{\text{cho}} \right]^{1/2}$$
$$= H_0^2 S_{\text{cho}} \left[ S_g + \frac{S_b}{(1+z)^4} + \frac{S_{\text{cho}}}{(1+z)^3} \right]^{1/2}$$

③ Luminosity Distance

$$d_L = \frac{c}{a H_0} \int_{a_{\text{es}}}^1 \frac{da}{S_{\text{cho}}^{1/2} x(a)}$$

or

$$d_L = c(1+z) \int_0^z \frac{dz'}{H(z', P)}$$

$$dz = -\frac{da}{a^2}$$

$$\mu = 5 \log \frac{H_0 d_L}{c h} + 42.38$$

$$h = \frac{H_0}{1000} \text{ km s}^{-1} \text{ Mpc}^{-1}$$

# Statistical Analysis

$$\textcircled{1} \quad S_{b0} = 0.02$$

$$S_{b0} = \frac{g_{b0}}{3H_0^2}$$

$$S_{r0} = 0.0000245$$

$$S_{r0} = \frac{p_{r0}}{3H_0^2}$$

$$\textcircled{2} \quad \frac{10^{M5+1}}{10^6} = \alpha_L \text{ in Mpc}$$

\textcircled{3} To determine the best fit parameters, we minimize

$$\chi^2 = \sum_i \left[ \frac{\mu_{\text{th}}^i - \mu_{\text{obs}}^i}{\sigma_i} \right]^2 - \frac{c_1}{c_2} \left( c_1 + \frac{2}{5} \ln 10 \right) - 2 \ln h$$

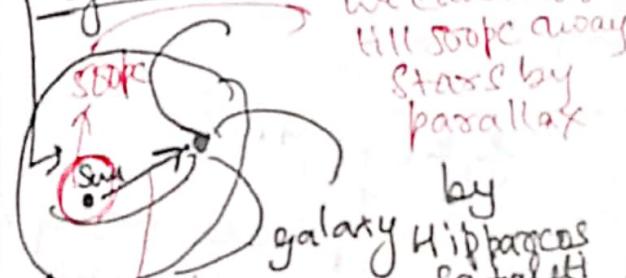
$$c_1 = \sum_i \frac{\mu_{\text{th}}^i - \mu_{\text{obs}}^i}{\sigma_i^2}$$

$$c_2 = \sum_i \frac{1}{\sigma_i^2}$$

## Key Properties of Gravitational Wave

also if the star is moving then we will get due to proper motion of star

we can learn about dist + rel. perpendicular of the star



Gaia Satellite measures till 30000 ly.  $F = \frac{L}{4\pi r^2}$

from this we can get luminosity. By putting filters through light from these stars we can get color of these stars.

Lum vs Color

HR diagram → Definite pattern which is used for finding

- ① Stellar evolution
- ② Distance

Calibration

## HR Diagram as Std Candle Method

### Main Sequence fitting

let say star is too far for parallax to find dist

\* Measure the color & luminosity

There would not be one star but cluster of them (globular cluster)

- ① measure color of each
- ② ~~estimate~~ guess dist. & then measure through  $F = \frac{L}{4\pi r^2}$

we can know the range of luminosity through HR diag. ∴ Range of distance

$$F = \frac{L}{4\pi r^2}$$

- ③ Match it with HR diagram
- ④ choose & which fit HR diagram

→ works till  $10^5$  pc

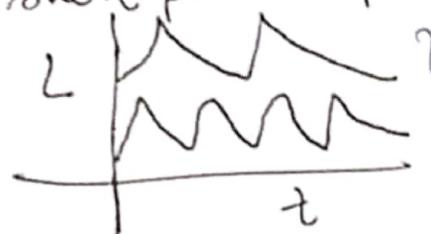
→ covers most of milky way

Can't cover whole universe because for longer range color of stars can't be distinguished

## Cosmic Distance Ladder: Cepheid Variables

- ① Stars usually evolve over long time scale i.e. they are ~~constant~~ form.  
But there is a type of star (Variable Star) which changes very quickly.
- ② One subtype of variable star is Cepheid Variables which can be used to measure distance.

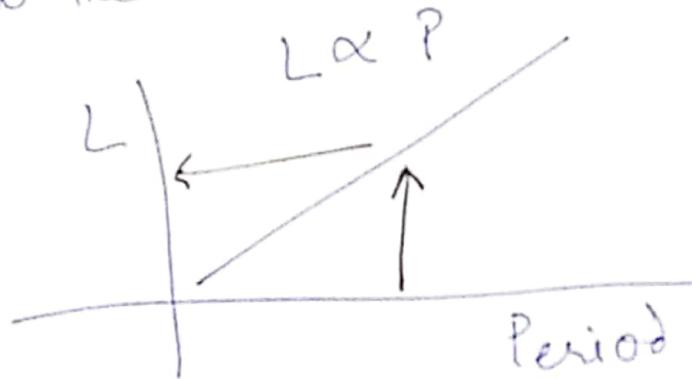
- ③ Period is related luminosity.  
Long period Cepheid Variable have higher luminosity than short period Cepheid Variables.



2 stars with same distance  $\therefore$  luminosity is not changing due to distance but it is intrinsic property

$L \propto P^2$  f period

There is a relationship b/w L & P  
If we know the period of Cepheid then we can know the lum. thus the distance.



But farther than few 10 Mly we can't measure  
Bec. star is not that bright.  
 $\therefore$  we need other std. candle wh. are bright

- ④ This method was applied to V1 in M31 galaxy which is closest galaxy to Milky Way by Hubble.  
~~This proved that V1 was outside our galaxy &~~  
Our Milky Way is not the only galaxy.  
 $\rightarrow$  was at 2.35 Mly away



34 ~~on easier to measure~~ ~~MMA technical~~  $\rightarrow$  errors carry through.

## Cosmic Distance Ladder: Std. Candles

(1)

$$F = \frac{L}{4\pi s^2}$$

[we know luminosity] Intrinsic flux is how many photons are hitting.

$\therefore$  we get  $s$ .

①  $L$  from HR diagram.

HR diagram fitting

from Observable property

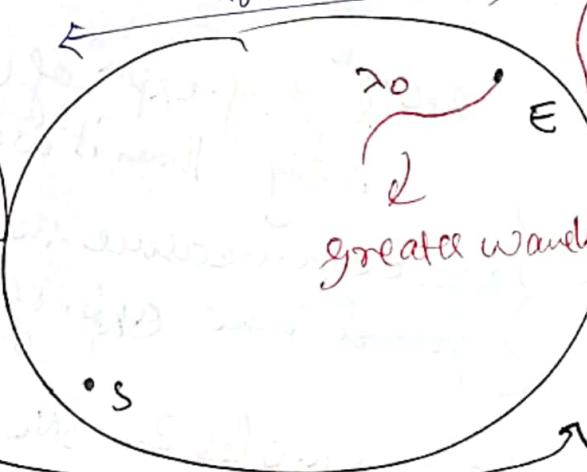
color

& then relate it to  $L$  & then find  $s$

Std. Candles  $a = \text{scale factor}$   $a_0 = 1$   $\therefore$  universe exp 10 times

② Redshift

$\Delta\lambda_e = 0.1$  expanding



$$\frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e}$$

greater wavelength

each element has unique Spectrograph

$\lambda_e$  can be found easily by Spectro techniques

$$\text{Redshift } z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{v}{c} \quad \text{recession of source (Assuming } v \ll c \text{) w.r.t. observer}$$

for nearby source  $v = H_0 d$  (ignoring vel. of galaxy through gravity)

$$\therefore z \approx \frac{H_0 d}{c}$$

$\therefore$  If I know redshift of galaxy, I know  $H_0$

const.  $\therefore$  I can get distance

$$H_0 = 70.4 \pm 1.4 \text{ km/s/Mpc}$$

if source is 1 Mpc ( $10^6 \times 3.26 \text{ ly}$ ) away

then that source is moving at 70.4 km/s away

### (3) How to measure redshift

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

(32)

By spectral lines

This redshift method can measure distance upto  $\geq 200 \text{ Mly}$

### Errors

- ① In lower redshift, galaxies have their own intrinsic motion in addition to the Universe being expanding.  
 $\therefore$  this adds error while finding distance  
 as  $v = H_0 d$  is no longer valid

- ② At higher redshifts i.e. early time of the Universe  $v = H_0 d$  is not valid  
 in fact  $v = H_0 d + \text{correction}$

Because rate of exp' of Universe is different today than it was billion yrs ago.

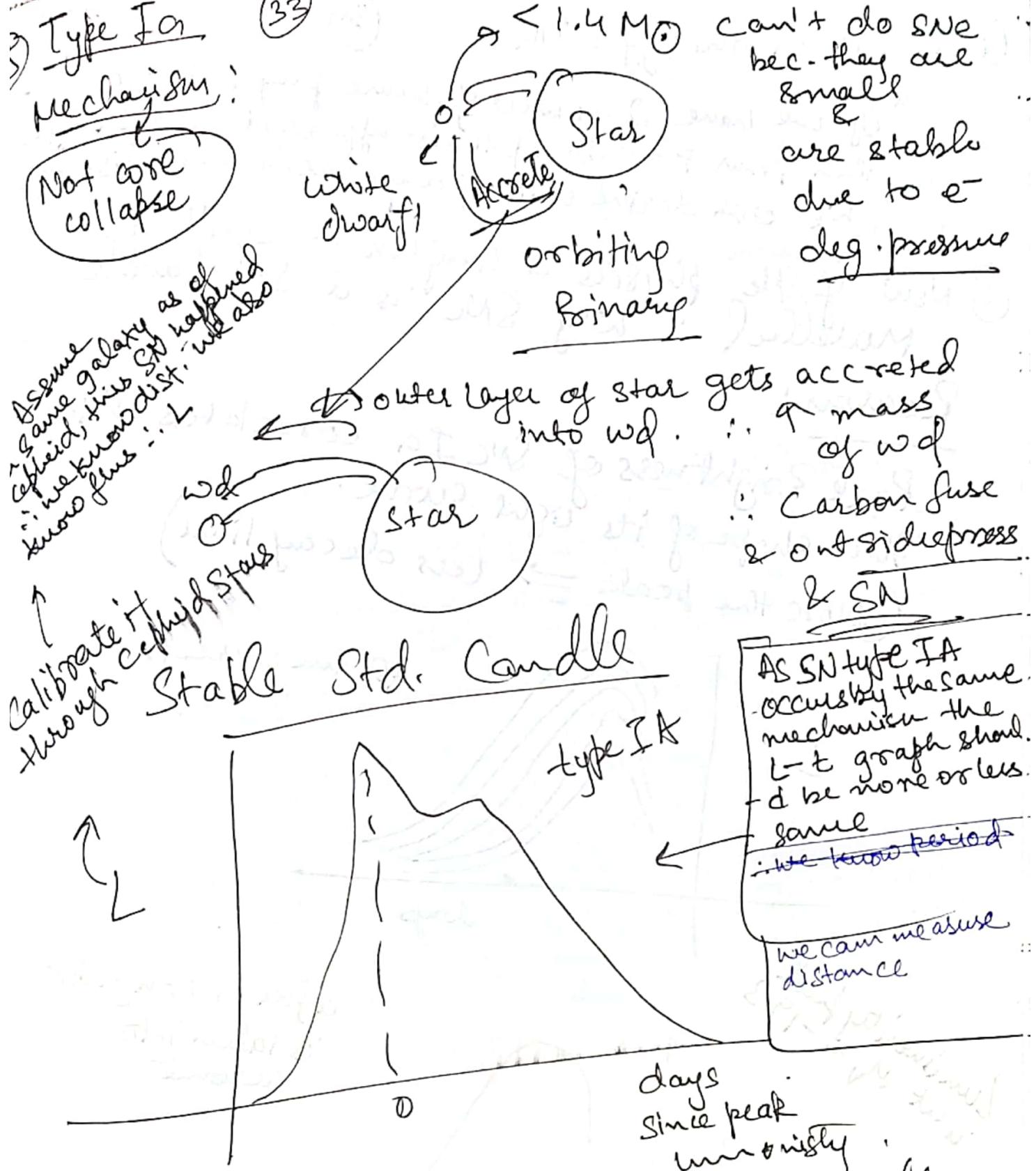
We can measure these corrections & we found that exp. of Universe is accelerating

Star explodes & SNe is brought

### SNe



There are different types of SNe and all types show variability i.e. the graph of luminosity is not constant every time except of type Ia SNe.



Now we know that SNe are standard candles  
 i.e. L is same  
 if 2 sources have same intrinsic lum ("Std Candles")  
 from the ratio of their apparent brightness we can derive the ratio of their ratio of unosity dist.

(7) This is in analogy with

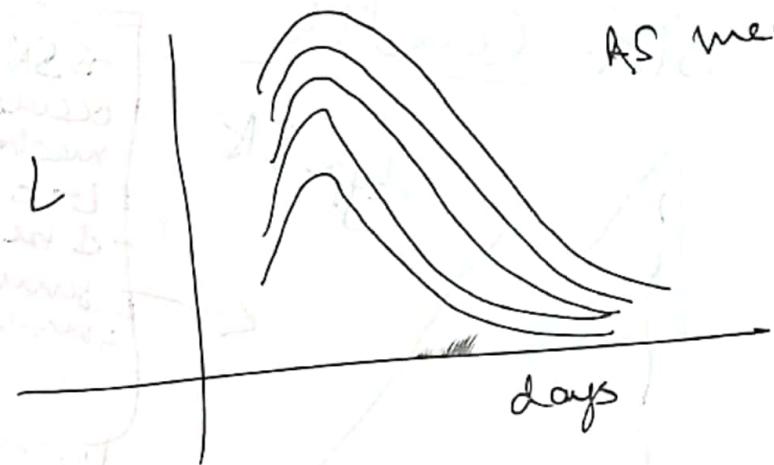
(34)

"If we have 2 sources of same physical size then from the ratio of their apparent angular size we can derive ratio of angular diameter distance."

(8) Now if the physics is complex & can't be modelled  $\therefore$  why SNe is a Std Candle.

Reason:

Peak Brightness of SNe Ia correlates with the shape of its light curve.  
(More the peak  $\Rightarrow$  less decay time)



Luminosity peak  
is at  $m \sim 19.3$



$\therefore$  They all follow same  
light curve shape

• Taking into stretch factor we can standardize ⑯  
the ~~standard~~ SNe. to  $10^{-1}$ . or better

However, the absolute zero-point of SNe a distance scale has to be calibrated externally eg - with cepheids. But cepheids are not many which has SNe also.

④ ~~We can~~

Cosmic distance ladder cons.

- ① Some object (earlier in the ladder) may not be present in the same galaxy of the other object which we are trying to use as std. Object which we are trying to use as cosmic distance ladder.
- ② Interaction of light with IS dust which makes light dim which can make us think source is farther

We can account for ② by

let say if we measure high  
~~short wavelength source~~  
~~longer wavelength~~  
we know Interstellar dust  
absorbs longer & but not  
short &  
∴ we can find that  
it would be due to  
IS dust

But if all  $\lambda$  are changed then due to distance.

We can study properties of IS dust.

- ③ errors in earlier ladder can keep on adding in later ladder  
Because we use later ladder the earlier ladder

'⑩ We can also standardize SNe not only by stretch factor but also by color information. 36

## Supernova Cosmology with Python

①

- SN cosmology
  - describe scientific problem
  - describe the data to set up the issues
  - describe the features
- Python Code SNCosmo & features
- Use of simulation catalogs being built for LSST survey.

"The Universe is an excellent lab for testing Physics"

→ Can probe physical effects at a very long time scales

→ Probes effects at large spatial scales

→ Observational evidence for gravity stronger than expected for observed particles

(

Dark Energy: Observational evidence for late time acc. of universe.

'③ SN cosmology was the first evidence that universe is expanding.

These people involved in this were given Nobel prize.

(4) How can we get information on cosmic expansion from data 37 19

- Combine knowledge of astrophysical systems, the impact of the expansion & observations of those system.
- for SN: This means comparing observed brightness which is affected by expansion of the universe to intrinsic brightness.

Data

(5)

10

David Holz  
Scott Hughes  
Bernard Schutz. Measuring Cosmic Distances with Standard Sirens

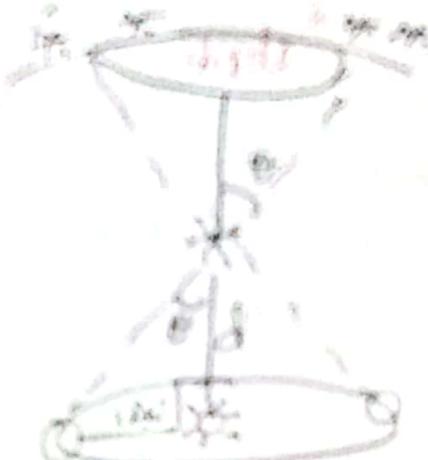
- ① Because  $g_w$  encodes the distance to its source, GW170817 provided astrophysical community with another advance: the first measurement of the local cosmic rate: Hubble constant - via  $g_w$ . This is the new ~~new~~ way for standard siren technique.
- ② If light emitted from source has wavelength  $\lambda$  & is at distance  $D$  from observer. Then observer will measure the light to have a wavelength of  $(1+z)\lambda_{em}$  where  $z$  is the redshift.
- ③ To leading order in  $z$ , (for nearby objects)  $cz = H_0 D \rightarrow$  Physical Distance
- ④  $\frac{1}{H_0} = [8] \equiv$  Hubble time  $\equiv$  Age of the Universe
- ⑤ (for faraway objects) higher order terms needs to be added
- ⑥ Redshifts can be determined from spectral measurements but determining astronomical distances is difficult

④ For nearby objects

## Distance through Parallax



How to measure it?  
(Through the eye)



If the star is at  
at  $90^\circ$  to Sun  
then it will be  
at  $0^\circ$  after  
one full revolution

Assuming Sun,  
ground stars  
are far away  
that parallax  
distance would  
be ignore.

$$d = \frac{1 \text{ AU}}{\tan \alpha}$$

$$d(\text{pc}) = \frac{1}{\alpha (\text{arcsec})}$$

$$1 \text{ pc} = 3.26 \text{ ly}$$

$$\alpha \text{ arcsec} = \frac{1}{3600}$$

You can measure stars closer to us by this method as  $\alpha$  considered too small to measure for large dist stars  
the technique doesn't work for large distances, at angular  
dist due to Earth's orbital motion becomes too small  
to measure.

⑤ Standard Candle: Astronomical source whose intrinsic luminosity is assumed to be known.

Cosmic Distance Ladder

There are multiple methods for measuring distances:  
Objects thought to be of std. candle is identified on each mng & calibrated  
in terms of measured to previous mng.

⑥ SNe are also std candles & they helped determine H<sub>0</sub>.  
They also implied non linear contribution to  
 $c_2 = 80$   
implying that universe is accelerating.

(i) Key Properties of GW

$\rightarrow h_{\mu\nu}$  is analogous to  $A_\mu$  of EM

(4)

$j$ : spatial

$$A_j = \frac{\mu_0}{4\pi} \frac{1}{D^2} \frac{dP_j}{dt}$$

(ii) for source moving at less than speed of light

$\vec{P}$  = Electric Dipole moment.

$D$  = Distance from source.

$\mu_0$  = Permeability of free space

$$P = \int p_e r dV$$

source

$p_e$ : Charge Density

volume of the source

$\rightarrow$  analogous result for GW

$$h_{ijk} = \frac{2q}{c^4} \frac{1}{D} \frac{d^2 I_{ijk}}{dt^2}$$

$i, k$ : spatial

as GW are orthogonal to propagation dir

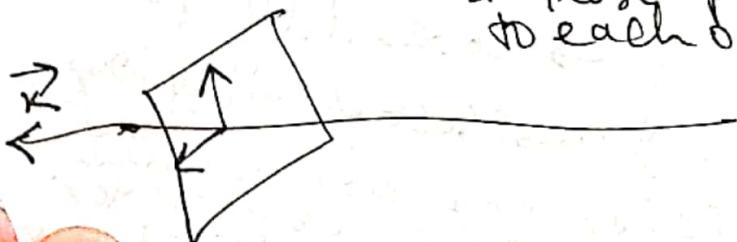
Amplitude falls off  $\propto \frac{1}{D}$

$I_{ijk}$ : Quadrupole moment

$$I_{ijk} = \int \rho_m \left[ r_j r_k - \frac{r^2}{3} \delta_{jk} \right] dV$$

(ii) Polarization of GW

In EM, polarizations are in plane  $\perp$  to propog.  
 & those polarizations are orthogonal to each other



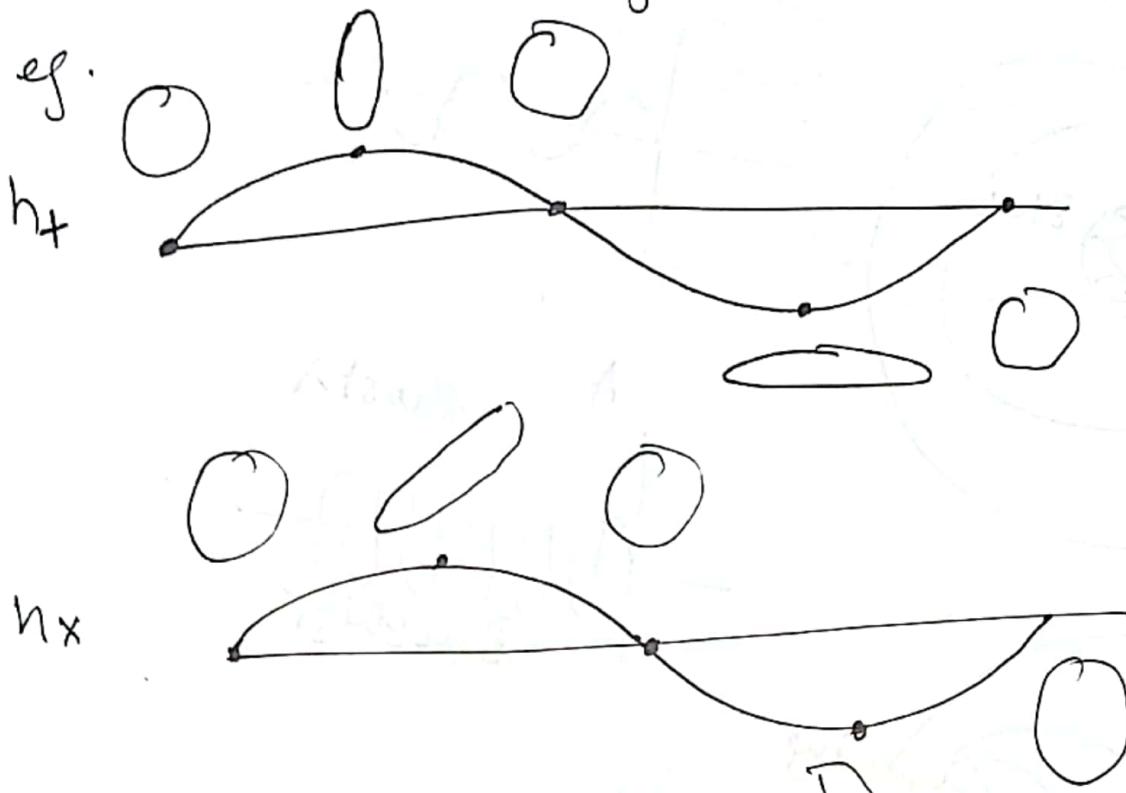
in GW, polarization basis are in plane for to propagate  
But they are rotated by 45° to each other.

(ii)



GW stretches & squeezes along polarization basis  
EM force exerts force on charge along polarization basis.

e.g.



② If somehow we know how source quad. mom. varies with time, then we can measure distance.

↓  
for case of binary inspiral quad. mom. ( $\chi$ ) can be known & hence distance. ∴ without any reference to cosmic distance ladder, distance can be measured of binary inspirals.

③ From Kepler's law & formula which relates  $\frac{d\ell}{dt}$  to IJK

$$\frac{d\omega}{dt} = \frac{96}{5} \left( \frac{GM}{c^3} \right)^{5/3} \omega^{11/3}; \quad \omega: \text{frequency of orbiting}$$

M: chirp mass

$$M = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

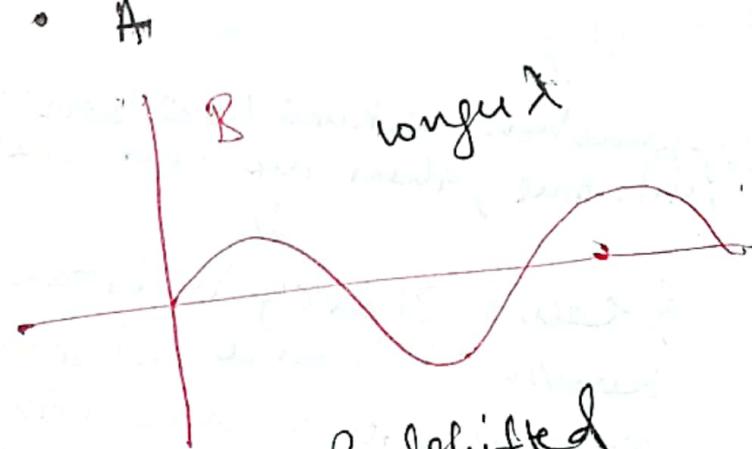
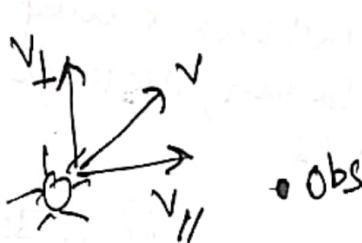
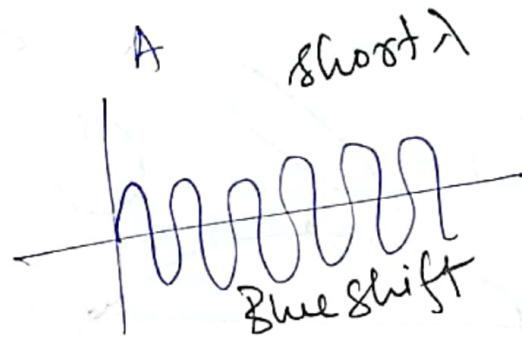
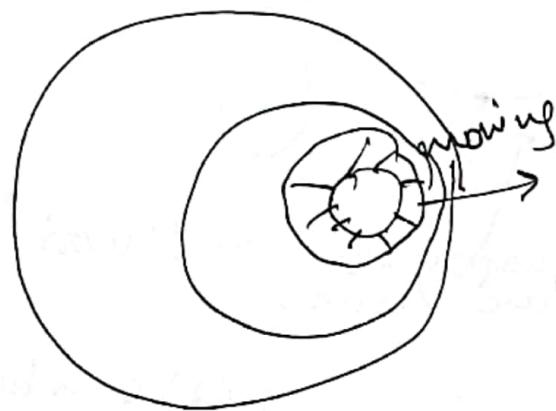
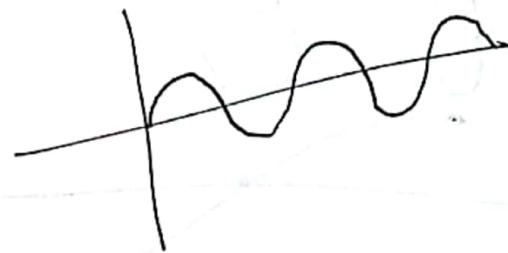
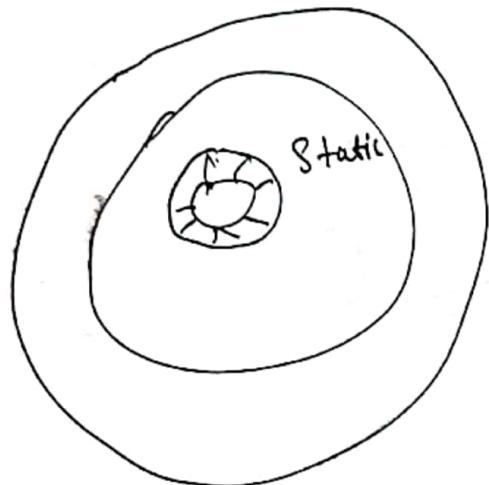
To the leading order

Once we know M, we know change in freq.  
Same M, same  $\omega(t)$  even though  $m_1$  &  $m_2$  of 2 system differ.

~~Redshift & Hubble's Law~~

~~Doppler effect~~

- ① frequency & wavelength depends on the source movement.



$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{emit}}} \approx \frac{v_{\parallel}}{c} \quad (\text{assuming source } v \ll c)$$

②  $\lambda_{em}$  can be found through spectrophotograph.  
if there are multiple lines at spectrophotograph & each shifted then we know which line corresponds to which & thus  $\lambda_{emit}$  can be known.

by same  
process

if only one line is there then we can figure if the shifted line we observe is of the same element

Doppler & SB

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## (b) Cosmological Observations

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$$M = -2.5 \log_{10} L + \text{const.}$$

Absolute Magnitude: const. & can be chosen arbitrarily depending on observed wavelength.

(e.g.  $M_{\text{Sun}, B} = 5.48$  (B is Blue band at 4400 Å).)

## (c) In Non-expanding Euclidean Geometry

$$f = \frac{L}{4\pi d^2}$$

$$m = -2.5 \log f + \text{const}$$

Apparent Magnitude

Friedmann Eqn

$$a_0 t - a_0 = 8\pi G P_0$$

$$\frac{H^2 + K}{a^2} - \frac{1}{3} = 8\pi G P$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(P + \frac{K}{a^2})$$

Cyber

[const. chosen s.t. const = 0]  
for  $f = 2.5 \times 10^{-5}$

$$m = -2.5 \log 2.5 \times 10^{-5}$$

$$M = -2.5 \log_{10} f 4\pi d^2$$

$$M = m - 2.5 \log_{10} 4\pi d^2$$

$$M = m - 5 \log_{10} d - 2.5 \log_{10} 4\pi$$

$$\rightarrow H^2 = H_0^2 (5r_m a^{-3} + 5r_K a^{-2} + 5r_\Lambda a^4 + r_{0,1})$$

$$\frac{1}{3} H^2 = r_\Lambda$$

$$m = M + 25 + 5 \log d$$

$d$ : Mpc

$\mu = m - m = \text{dist. Modulus} \& \text{a measure of distance.}$

(18) Absolute magnitude

Apparent Magnitude

Relativistic version

$$(9) f = \frac{L}{4\pi r^2 (1+z)^2}$$

$$d_L = \propto (1+z) \quad \therefore f = \frac{L}{4\pi d_L^2}$$

$$\therefore m - M = \mu = 5 \log d_L + 25$$

(ITE) appears Bcz. (1) Energy emitted is redshifted away (2) time interval of received =  $\frac{a_0}{a_1} dt$   $\Rightarrow$  time of emission.

(20) Coordinate distance  $\propto$  "along null geod. i.e.

$$ds^2 = c dt^2 - a^2 d\sigma^2 = 0$$

$$(i) R=0 \Rightarrow \sigma = \int_0^z d\sigma' = c \int_{t_1}^{t_0} \frac{dt}{a} = c \int_{a_1}^{a_0} \frac{da}{\dot{a}} = c \int_{a_1}^{a_0} \frac{da}{H(a)^2}$$

$$dz = -\frac{da}{a^2} \Rightarrow \sigma = c \int_0^z \frac{dz}{H(z)}$$

(ii) Non flat

$$H = H_0 E(z)$$

$$r = \frac{1}{H_0 \sqrt{S_{\text{curv}}}}$$

$$S \left[ \int_0^z \frac{dz'}{E(z')} \right]^2$$

$$S_{\text{curv}} = -\frac{k}{H_0^2}$$

$$S(x) = \begin{cases} \sin(x) & k=+1 \\ x & k=0 \\ \sin(kx) & k=-1 \end{cases}$$

(given any cosmological model  $(\Omega_m, \Omega_r, \dots)$  we can obtain  
 (1)  $\propto(z)$   
 (2)  $d_L(z)$   
 (3)  $\mu \xrightarrow{\text{knowing } M}$

# Cosmology

(46)

Daniel Holz

- ① Overview → (i) Cosmological Background  
 (ii) GW  
 (iii) Std. Sirens → (iv) SMBH  
 (v) GRBs  
 (vi) Grav. Lensing.

- ② Cosmology → We are trying to measure evolution history of the Universe.  
 i.e. we are trying to get Lumin. Dist. - Redshift Curve.

Lumin. Dist → tells how much time light took to reach us, tells abt time.  $t = \frac{d_L}{c}$

Redshift → tells abt size of Universe  
 $\therefore$  LD - Redshift tells about size as far of time of Universe.

- ③ Redshift → Scale Ratio of Universe at time of emission  
 $a(z) = \frac{a_0}{1+z}$

Measuring Redshift is straight forward.

- ④ Distance → Measuring distance luminosity is the hardest & we use std sirens for that.

⑤ Std. Sirens: SNe  $\rightarrow$  we know how bright they are & hence by luminosity we can tell how far they are.

SNe  $\rightarrow$  is good to about  $10 - 15\text{M}_\odot$ .

$\therefore$  The motivation is if we can do good more than  $10 - 15\text{M}_\odot$

- ⑥ To know the physics behind  $\Delta t - z$  of SNe is difficult  
 we have to know ① Element abundance of star  
 ② Convection  
 ③ Neutrinos.

$\therefore$  The best simulations are fairly poor

⑦ On the other hand, BH are described by just 3 no.

BH binary requires 15#.

$\therefore$  They are std. objects  $\therefore$  They can be modelled extremely well.

⑧ Strongest harmonic:

$$h(t) = \frac{M_2^{5/3} f(t)^{2/3}}{D_L} F(\text{angles}) \cos \phi(t)$$

Dimensionless strain  $h(t)$

luminosity distance  $D_L$

accumulated gw phase  $\cos(\phi(t))$

gw frequency  $f(t) = + \frac{d\phi}{2\pi dt}$

position & orientation dependence  $F(\text{angles})$  depends on the inclination of observer

Redshifted Chirp mass

$$M_2 = (1+z) \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

11. By GW templates we can measure  $M_2$  uniquely.  
Other things also can be taken care of except  $D_2$   
so we know  $h(t)$ ,  $M_2$   
we can infer  $D_2$

This is an absolute measure of  $D_2$   
in case of SNe  $D_2$  is relative measure of distance.  
(Schutz 1986)

### ⑨ Detecting GW

GW are weak.

Fractional strain due to strong GW  $h \sim 10^{-22}$

Further there is Noise

This much distance (m)  
we are trying to  
measure

10 Redshift is intrinsically built in chirp mass

$\therefore M_2$  has 3 parameters  $z$ ,  $m_1$ ,  $m_2$ .

$\therefore$  GW templates we can measure  $M_2$  but  
we can't tell redshift.  
So SMBH & stellar BH can have similar templates  
depending on the redshift of both.  $\therefore$  gravitational  
scale free

11 In Astronomy  $\rightarrow$  easier to detect  $z$   
difficult to detect  $D_2$

In GW  $\rightarrow$  easier to detect  $D_2$   
difficult to detect  $z$ .

12 GW provide direct measure of luminosity distance,  
but they give no independent information about  
redshift.

- (13) GW is scale free
- GW signal from local binary with masses  $(m_1, m_2)$  is indistinguishable from binary with masses  $\frac{m_1}{1+z} \quad \frac{m_2}{1+z}$  at redshift  $z$ .

$\therefore$  To measure cosmology, we need independent determination of redshift.  
 $\therefore$  EM counterparts do help in determining redshifts.

- (14) The other way to get redshift is by the Statistics.

i.e. let say we know the population of SMBH masses

We can know  $z$

or

if we know masses of NS

individually we can infer  $z$ .

- (15) We can measure very accurately  $dt^2$

But

Grav. lensing brings errors.

Motion due to  
expn of Univ + Motion due  
to other forces  
e.g. gravity

# ~~Doubts & S.R.~~

## Hypothesis Testing

### Basics

① Hypothesis: Something we can put to a test.

So, the question we are interested in is:  
let say, we took some sample, &  
then we put our hypothesis to test against  
this sample. & then ask ↓

How much does the sample  
data put doubt on our hypothesis?

Doubt | let say, sample isn't putting  
any doubt against hypothesis  
but is it still safe to say our  
hypothesis is correct by just  
looking one small sample size. → Type 2 errors

② Doubt against our hypothesis can be

made in 2 ways

- ① The result of the sample is far away from  $H_0$
- ② The result is far away but the sample size is small (i.e. less doubt on  $H_0$ )

③ Critical Values: Values beyond which we reject our  $H_0$

Critical values depend on the sample size:

for eg. for small sample  $\Rightarrow H_0: \mu = 20$  critical value.



for large sample size,  $H_0$  can be true.  
 $(\mu = 22)$   $\xrightarrow{\text{accept}} 20 \xrightarrow{\text{reject}} 23$

(F) Question: If  $H_0$  is true, how extreme is our sample?

- ① if sample mean is too far away from hyp. mean then prob.  $H_0$  is false
- ② if sample is too big & mean is still far it makes the case worse than ①
- formula:  $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

if  $z=0$  or  $\bar{x}=\mu$  then  $H_0$  True

if  $z \neq 0$  &  $n \gg 1$  then  $H_0$  False.

⑤ Type 1 errors: Occurs when you reject a null hypothesis that is in fact true.

Type 2 errors: Occurs when you accept a null hypothesis that is in fact false.

There is still a possibility that  $H_0$  True prob =  $\alpha$

We can choose level of significance.  
Larger the level of signif. less the critical values.

level of significance  
Convention  
5% level of sig

(More strict we were)

There is a possibility that  $H_0$  is false. Prob. =  $\beta$

$1 - \beta$  = Power of Hyp. Test

(i) 21/11/21

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Meeting

- ① What should be the  $\alpha$  value in case of GW as errors in it ~~are~~ have different up & down limit?
- ② How to make contours & what is the meaning of confidence intervals in your graphs?
- ③ Can we also use Chi-Square in case of DL?
- ④ Why Chi-Square for GW is so low??
- ⑤ Transformation
  - $y = \sqrt{2X^2 - \sqrt{2\text{dof} - 1}}$
  - should follow Gaussian
  - within  $1\sigma$  of mean  $68.3\%$ . prob of <sup>correct</sup> fit

$y < 1$	$\Rightarrow$	$0.6\%$ prob. of correctly fit
$y = 3$	$\Rightarrow$	

Chi-Square Sum  $\approx 1$  for good fit  
 No. of data sets

Chi-Square minimum for diff. parameter will give most prob. value of the parameter which fits well

- ⑥ How to take care of asymmetric errors in Curve-fit?

