Linear Homogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear* ***homogeneous*** *recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c1, c2, \ldots, ck$ are real numbers, and $ck \neq 0$.

Example:

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

How to solve linear recurrence relation:
characteristic equation

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if *r* is a solution of this last equation. We call this the **characteristic equation** of the recurrence relation.

characteristic roots: The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Theorem1: Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0,$$

has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$$

is a solution of the recurrence relation.

Second, it must be shown that

if the sequence $\{a_n\}$ is a solution,

then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$

for some constants α_1 and α_2.

Now we will show that if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$$

then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0,$ it follows that

$$r_1^2 = c_1 r_1 + c_2,\ r_2^2 = c_1 r_2 + c_2.$$

From these equations, we see that

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$ is a solution of the recurrence relation.

Theorem2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0,$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.

Theorem3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots,$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Problem1: What is the solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$
with $a_0 = 2$ and $a_1 = 7$?

Solve: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0.$ Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2,$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Problem2: Find an explicit formula for the Fibonacci numbers.

Solve: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Therefore, from Theorem 1 it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants. We have

$$f_0 = \alpha_1 + \alpha_2 = 0,$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

The solution to these simultaneous equations for $\alpha 1$ and $\alpha 2$ is
$\alpha 1 = 1/\sqrt{5}, \alpha 2 = -1/\sqrt{5}$.
Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Problem3: What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solve: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$

$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that

$\alpha_1 = 1$ and $\alpha_2 =$. Consequently, the solution
to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

Problem4: Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solve: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$

$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$

$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Problem5: Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solve: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$a_0=1=\alpha_{1,0}$,

$a_1=-2=-\alpha_{1,0}-\alpha_{1,1}-\alpha_{1,2}$,

$a_2=-1=\alpha_{1,0} + 2\alpha_{1,1}+4\alpha_{1,2}$.

The simultaneous solution of these three equations is $\alpha_{1,0}=1$, $\alpha_{1,1}=3$, and $\alpha_{1,2} = -2$.

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: recurrence relation of the form

$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n)$,

is known as linear nonhomogeneous recurrence relation with constant coefficient.

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

Theorem4: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof: Because $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence,
say, $\{a_n^{(h)}\}$. Consequently,
$b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Theorem5: Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.When s is not a root of the characteristic equationof the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Problem6: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a1 = 3$?
Solve: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^r$ where α is a constant.
To determine whether there are any solutions of this form, suppose that

$p_n = cn + d$ is such a solution. Then the equation $a_n = 3a_{n-1} + 2n$ becomes
$cn + d = 3(c(n-1) + d) + 2n$.
Simplifying and combining like terms gives $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$
is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. This shows that $cn + d$ is a solution if
and only if $c = -1$ and $d = -3/2$.
Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n,$$

where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general
Solution.We find that $3 = -1 - 3/2 + 3\alpha$, which implies that $\alpha = 11/6$. The solution
is $a_n = -n - 3/2 + (11/6)3^n$.

Problem7: Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Solve: The solutions of its associated Homogeneous recurrence relation
$a_n = 5a_{n-1} - 6a_{n-2}$
are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n)$ =
$F(n) = 7^n$ a reasonable trial solution is
$a_n^{(p)} = C \cdot 7^n$,

where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$.

Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. All solutions are of the form $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n$.