Spin Connection and Vector Rotation in Teleparallel Bimetric Geometry

James Lockwood Christopher B. Cyrek Dustin Hansley Derek J. Burkeen

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Abstract

In the Teleparallel Equivalent of General Relativity (TEGR), gravity is encoded by torsion of a curvature-free (Weitzenböck) connection. As a result, parallel transport differs conceptually from the Levi–Civita picture: loops produce translation holonomy (closure failure) rather than rotational holonomy. In a *bimetric teleparallel* setting with two tetrads $e^a{}_\mu$ and $\tilde{e}^a{}_\mu$, a spacetime-dependent Lorentz map $\Lambda^a{}_b(x)$ and its Lie-algebra field $\Lambda^{-1}\mathrm{d}\Lambda \in \mathfrak{so}(1,3)$ govern how vector components "rotate" between frames as they move. This paper gives a compact, calculation-first account of: (i) vector transport under the Weitzenböck connection; (ii) the relative Lorentz rotation field and the square-root map $X = \sqrt{g^{-1}f}$; (iii) holonomy and autoparallels; and (iv) what changes in the bimetric teleparallel constraint structure when one tracks these rotation variables explicitly. All formulae are stated with clear conventions for reliable reuse.

1. Conventions and one-line summary

Indices. Greek $\mu, \nu, \ldots$ for spacetime; Latin $a, b, \ldots$ for Lorentz (frame). Signature $\eta_{ab} = \mathrm{diag}(-,+,+,+)$. Metrics: $g_{\mu\nu} = e^a{}_\mu\, e^b{}_\nu\, \eta_{ab}$, and similarly $f_{\mu\nu} = \tilde{e}^a{}_\mu \tilde{e}^b{}_\nu \eta_{ab}$. Weitzenböck connection:

$$\hat{\Gamma}^\lambda{}_{\mu\nu} = e_a{}^\mu \lambda\, \partial_\nu e^a{}_\mu \mu, \qquad T^\lambda{}_{\mu\nu} = \hat{\Gamma}^\lambda{}_{\nu\mu} - \hat{\Gamma}^\lambda{}_{\mu\nu}. \tag{1.1}$$

One-line summary. In TEGR, curvature $R(\hat{\Gamma}) = 0$. Hence rotational holonomy is pure gauge (a change of local Lorentz frame), while *torsion controls how far a transported vector fails to close in spacetime*. In bimetric teleparallel geometry, a relative Lorentz field $\Lambda(x)$ and the map $X = e_a{}^\mu \mu\, \Lambda^a{}_b\, \tilde{e}^b{}_\nu$ encode the *net* rotation/boost between the sheets.

2. Transport in TEGR: what "rotates" and what translates

Let v^a be frame components of a vector $V^\mu = e_a{}^\mu \mu v^a$. The *teleparallel* (Weitzenböck) covariant derivative acting on v^a reads

$$D_\mu v^a = \partial_\mu v^a + \omega^a{}_{b\mu}\, v^b, \qquad \omega^a{}_{b\mu} \in \mathfrak{so}(1,3), \tag{2.1}$$

with inertial spin connection $\omega^a{}_{b\mu}$ chosen so that $R^a{}_{b\mu\nu}(\omega) = 0$. Along a curve $\gamma : x^\mu(\lambda)$,

$$\frac{Dv^a}{d\lambda} = 0 \quad \Longrightarrow \quad v^a(\lambda) = U(\lambda, \lambda_0)^a{}_b\, v^b(\lambda_0), \quad U = \mathcal{P}\exp\int_{\lambda_0}^{\lambda} \omega_\mu\, \dot{x}^\mu\, d\lambda. \tag{2.2}$$

Because $R(\omega) = 0$, the holonomy is *pure gauge*:

$$U(\gamma) = \Lambda^{-1}(x(\lambda))\,\Lambda(x(\lambda_0)) \qquad \Rightarrow \qquad \text{no curvature-induced rotation.} \tag{2.3}$$

By contrast, spacetime closure Δx^μ around a small loop Σ is controlled by torsion:

$$\Delta x^\mu \;\simeq\; \frac{1}{2}\int_\Sigma T^\mu{}_{\alpha\beta}\, d\Sigma^{\alpha\beta}, \qquad d\Sigma^{\alpha\beta} = dx^\alpha \wedge dx^\beta. \tag{2.4}$$

Interpretation. In TEGR, vectors do not acquire curvature-driven *rotational* holonomy; instead, torsion produces *translational* holonomy (closure failure). Any actual "rotation" of components is a choice of Lorentz frame $\Lambda(x)$.

3. Bimetric teleparallel structure and the relative rotation field

Introduce a second tetrad $\tilde{e}^{\,a}{}_\mu$ with its own (inertial) spin connection $\tilde{\omega}^a{}_{b\mu}$ and torsion $\tilde{T}^\lambda{}_{\mu\nu}$. The two frames are related by a local Lorentz map $\Lambda^a{}_b(x)$:

$$\tilde{e}^{\,a}{}_\mu(x) = \Lambda^a{}_b(x)\, e^{\,b}{}_\mu(x), \qquad \Lambda(x) \in SO(1,3). \tag{3.1}$$

Define the *relative rotation one-form* (Maurer–Cartan form)

$$\mathcal{B}_\mu := \Lambda^{-1}\partial_\mu\Lambda \;\in\; \mathfrak{so}(1,3), \qquad \mathcal{B}_\mu = \frac{1}{2}\,\mathcal{B}^{ab}_\mu\, J_{ab}, \tag{3.2}$$

with J_{ab} Lorentz generators. The familiar bimetric square-root map follows in tetrad variables:

$$X^\mu{}_\nu \;=\; e_a{}^\mu\mu\,\Lambda^a{}_b\,\tilde{e}^{\,b}{}_\nu \;=\; \left(e^{-1}\Lambda\,\tilde{e}\right)^\mu{}_\nu, \qquad \text{so that} \quad X^\top g\, X = f. \tag{3.3}$$

Key point. $\mathcal{B}_\mu$ is the *differential* measure of how vectors "rotate" when moved from one sheet/frame to the other; X is the corresponding finite map on indices.

3.1 Component transport across sheets

Given V^μ, its frame components on the $(+)$ and $(-)$ sheets are $v^a = e^{\,a}{}_{\mu\mu}V^\mu$ and $\tilde{v}^a = \tilde{e}^{\,a}{}_\mu V^\mu$. Then

$$\tilde{v}^a = \Lambda^a{}_b\, v^b, \qquad \partial_\mu\tilde{v}^a = (\partial_\mu\Lambda^a{}_b)\, v^b + \Lambda^a{}_b\,\partial_\mu v^b. \tag{3.4}$$

Using D_μ and $\tilde{D}_\mu$, one gets the clean relation

$$\tilde{D}_\mu\tilde{v}^a = \Lambda^a{}_b\Big(D_\mu v^b + (\mathcal{B}_\mu)^b{}_c\, v^c\Big), \qquad \mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda + \Lambda^{-1}\omega_\mu\Lambda - \tilde{\omega}_\mu. \tag{3.5}$$

Thus $\mathcal{B}_\mu$ packages the *net* infinitesimal rotation/boost required to keep transport consistent between the two teleparallel frames.

4. Holonomy, autoparallels, and what a vector "does" along a path

4.1 Teleparallel autoparallels

Autoparallels of $\hat{\Gamma}$ satisfy

$$\ddot{x}^\lambda + \hat{\Gamma}^\lambda{}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0, \tag{4.1}$$

with $\dot{} \equiv d/d\lambda$. Because $R(\hat{\Gamma}) = 0$, parallel transport around a loop has *no rotational* holonomy. The loop effect is translational:

$$\Delta x^\mu \propto \int_\Sigma T^\mu{}_{\alpha\beta}\, d\Sigma^{\alpha\beta}. \tag{4.2}$$

4.2 Bimetric holonomy and relative rotation

Across sheets, the *relative* holonomy along γ is

$$U_{(-)\leftarrow(+)}(\gamma) = \mathcal{P}\exp\int_\gamma \mathcal{B}_\mu\, dx^\mu \ \in\ SO(1,3), \tag{4.3}$$

and the finite index map is X from (3.3). In particular, for a teleparallel gauge where $\omega = \tilde{\omega} = 0$,

$$\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda, \qquad U_{(-)\leftarrow(+)}(\gamma) = \Lambda^{-1}(x_{\text{end}})\,\Lambda(x_{\text{start}}). \tag{4.4}$$

Hence, the only "rotation" a vector experiences is the relative Lorentz twist between frames; torsion shows up as closure failure in the base manifold, not as curvature of the Lorentz bundle.

5. Square-root map and the dRGT/HR interaction (teleparallel reading)

In metric bimetric gravity one uses $X = \sqrt{g^{-1}f}$ to form the ghost-free potential $U(g,f) = \sum_{n=0}^{4}\beta_n e_n(X)$. In tetrad variables with Λ explicit, (3.3) makes the dependence on relative rotations manifest. Two practical consequences:

1. **Tetrad symmetry.** If the *symmetry condition* $e^a{}_{[\mu}X_{\nu]}{}^\mu = 0$ holds, then X can be taken symmetric, matching the metric HR construction.
2. **Constraint transparency.** Treating Λ and $\mathcal{B}_\mu$ as auxiliary (pure-gauge) fields clarifies that no new propagating spin-0 mode is introduced by relative rotations; the BD scalar remains absent on the healthy branch.

6. Worked micro-examples

6.1 Minkowski + sheet and a rotated $(-)$ sheet

Take $e^a{}_\mu = \delta^a{}_\mu$, $\omega = 0$. Let the other sheet be a rigid spatial rotation $R(\theta)$ about $\hat{\boldsymbol{z}}$: $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R(\theta) \end{pmatrix}$, $\theta = \theta(x)$. Then

$$\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \partial_\mu\theta\, J_z \end{pmatrix}, \qquad U_{(-)\leftarrow(+)}(\gamma) = \exp\Big(\Delta\theta(\gamma)\, J_z\Big). \tag{6.1}$$

Vectors "rotate" by the net angle difference $\Delta\theta$ accumulated along the path; all gravitational content here remains in torsion/translation, not in curvature.

6.2 Teleparallel loop: translational holonomy

On a small rectangular loop of sides $\epsilon^\alpha, \epsilon^\beta$,

$$\Delta x^\mu \ =\ \frac{1}{2}\, T^\mu{}_{\alpha\beta}\,\epsilon^\alpha\epsilon^\beta \ +\ \mathcal{O}(\epsilon^3), \tag{6.2}$$

while the Lorentz-frame holonomy (for $R(\omega) = 0$) is gauge and reduces to $\Lambda^{-1}(x_{\text{end}})\Lambda(x_{\text{start}})$.

7. Takeaways (for re-use in calculations)

- **TEGR transport.** Rotational holonomy is pure gauge; torsion controls closure failure.
- **Bimetric twist.** The relative Lorentz field $\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda + \Lambda^{-1}\omega_\mu\Lambda - \tilde{\omega}_\mu$ is the correct differential object for "how vectors rotate" between sheets.
- **Square-root clarity.** $X = e^{-1}\Lambda\,\tilde{e}$ cleanly separates index mapping from relative Lorentz rotation; this is the tetrad-side face of $X = \sqrt{g^{-1}f}$.
- **Constraints.** Treating Λ as auxiliary preserves the HR ghost-free count; no extra scalar propagates from relative rotations.

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8. 3+1 Hamiltonian structure and constraint algebra

We perform a 3+1 split for each sheet ($\pm$) with coordinates $x^\mu = (t, x^i)$ and tetrads

$$e^a{}_\mu = \begin{pmatrix} e^a{}_0 & e^a{}_j \end{pmatrix}, \qquad \tilde{e}^a{}_\mu = \begin{pmatrix} \tilde{e}^a{}_0 & \tilde{e}^a{}_j \end{pmatrix}, \qquad i, j = 1, 2, 3. \tag{8.1}$$

Write the induced spatial triads $E^a{}_i := e^a{}_i$, $\tilde{E}^a{}_i := \tilde{e}^a{}_i$, the spatial metrics $\gamma_{ij} := E^a{}_i E^b{}_j \eta_{ab}$ and $\tilde{\gamma}_{ij} := \tilde{E}^a{}_i \tilde{E}^b{}_j \eta_{ab}$, and define lapses and shifts through $e^a{}_0 = N n^a + N^i E^a{}_i$ and $\tilde{e}^a{}_0 = \tilde{N}\,\tilde{n}^a + \tilde{N}^i \tilde{E}^a{}_i$, with $n^a \eta_{ab} E^b{}_i = 0 = \tilde{n}^a \eta_{ab} \tilde{E}^b{}_i$ and $n^a n^b \eta_{ab} = -1 = \tilde{n}^a \tilde{n}^b \eta_{ab}$.

8.1 Canonical variables and primary constraints

In TEGR the Lagrangian is $\mathcal{L}_{\text{TEGR}} = \frac{1}{2\kappa}\, e\, T(e^{\,a}{}_\mu \mu)$ (and analogously for $\tilde{e}$), where T is quadratic in torsion. The only time derivatives enter through $\dot{E}^a{}_i$ and $\dot{\tilde{E}}^a{}_i$; lapses, shifts, and inertial Lorentz variables carry no velocities. Define the canonical momenta

$$\Pi_a{}^i := \frac{\partial \mathcal{L}}{\partial \dot{E}^a{}_i}, \qquad \tilde{\Pi}_a{}^i := \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{E}}^a{}_i}. \tag{8.2}$$

The absence of $\dot{N}, \dot{N}^i, \dot{\tilde{N}}, \dot{\tilde{N}}^i$ yields the *primary* constraints

$$\pi_N \approx 0, \quad \pi_i \approx 0, \qquad \tilde{\pi}_{\tilde{N}} \approx 0, \quad \tilde{\pi}_i \approx 0. \tag{8.3}$$

Local Lorentz invariance on each sheet is carried by auxiliary inertial fields (or, in Weitzenböck gauge, by the purely algebraic boost/rotation components of $E^a{}_i$). The corresponding primary Lorentz constraints are

$$\mathcal{J}_{ab} \approx E_{[a}{}^i\, \Pi_{b]i} \approx 0, \qquad \tilde{\mathcal{J}}_{ab} \approx \tilde{E}_{[a}{}^i\, \tilde{\Pi}_{b]i} \approx 0, \tag{8.4}$$

which generate $SO(1, 3)$ rotations of the spatial frames.

8.2 Interaction potential and relative Lorentz sector

Let the interaction be the standard dRGT/HR ghost-free potential written in tetrad variables via the map

$$X^{\mu}{}_{\nu} = (e^{-1}\Lambda\,\tilde{e})^{\mu}{}_{\nu}, \qquad U = \sum_{n=0}^{4} \beta_n\, e_n(X), \tag{8.5}$$

with $\Lambda \in SO(1,3)$ the relative Lorentz field and e_n the elementary symmetric polynomials. We impose the *symmetric-vierbein* condition on the healthy branch,

$$e^{a}{}_{[\mu}\, X_{\nu]}{}^{\mu} = 0, \tag{8.6}$$

which removes boost-spin asymmetries and makes U depend only on the symmetric square-root $X = \sqrt{g^{-1}f}$. Then U is algebraic in $(N, N^i, \tilde{N}, \tilde{N}^i, E, \tilde{E}, \Lambda)$ and carries no time derivatives: no new canonical momenta are introduced for lapses, shifts, or Λ.

8.3 Total Hamiltonian and secondary constraints

The total Hamiltonian takes the form

$$\begin{aligned} H_{\text{tot}} = \int d^3x \Big[N\,\mathcal{C}_0 + N^i\,\mathcal{C}_i + \tilde{N}\,\tilde{\mathcal{C}}_0 + \tilde{N}^i\,\tilde{\mathcal{C}}_i + \lambda^{ab}\,\mathcal{J}_{ab} + \tilde{\lambda}^{ab}\,\tilde{\mathcal{J}}_{ab} + U_{\text{int}}(N, N^i, \tilde{N}, \tilde{N}^i; E, \tilde{E}; \Lambda)\Big] \\ + \int d^3x \big[u\,\pi_N + u^i \pi_i + \tilde{u}\,\tilde{\pi}_{\tilde{N}} + \tilde{u}^i \tilde{\pi}_i \big], \end{aligned} \tag{8.7}$$

where $(\mathcal{C}_0, \mathcal{C}_i)$ and $(\tilde{\mathcal{C}}_0, \tilde{\mathcal{C}}_i)$ are the TEGR scalar and vector densities on each sheet (they match the GR ones modulo boundary terms). On the symmetric branch (8.6), U_{int} is *linear* in one particular linear combination of lapses and shifts, typically the "diagonal" lapse N_{diag} and shift N^i_{diag} that generate the surviving diffeomorphisms. Linearity produces secondary constraints from the preservation in time of the primary lapse/shift constraints:

$$\dot{\pi}_N \approx -\frac{\delta H_{\text{tot}}}{\delta N} \;\Rightarrow\; \mathcal{C}_0^{\text{diag}} + \frac{\delta U_{\text{int}}}{\delta N} \approx 0, \qquad \dot{\pi}_i \approx -\frac{\delta H_{\text{tot}}}{\delta N^i} \;\Rightarrow\; \mathcal{C}_i^{\text{diag}} + \frac{\delta U_{\text{int}}}{\delta N^i} \approx 0, \tag{8.8}$$

and analogously for $(\tilde{N}, \tilde{N}^i)$. Crucially, one additional *scalar* secondary constraint $\mathcal{S} \approx 0$ emerges from the special structure of U; together with its partner it removes the would-be Boulware–Deser mode.

8.4 Constraint brackets and degree-of-freedom count

Let $\{\cdot,\cdot\}$ denote the equal-time Poisson bracket with the canonical pairs $(E^a{}_i, \Pi_a{}^i)$ and $(\tilde{E}^a{}_i, \tilde{\Pi}_a{}^i)$. The non-vanishing brackets close as

$$\{\mathcal{C}_i^{\text{diag}}(x), \mathcal{C}_j^{\text{diag}}(y)\} = \mathcal{C}_j^{\text{diag}}(x)\,\partial_i\delta - \mathcal{C}_i^{\text{diag}}(y)\,\partial_j\delta, \tag{8.9}$$

$$\{\mathcal{C}_0^{\text{diag}}(x), \mathcal{C}_i^{\text{diag}}(y)\} = \mathcal{C}_0^{\text{diag}}(y)\,\partial_i\delta, \tag{8.10}$$

$$\{\mathcal{J}_{ab}(x), \mathcal{J}_{cd}(y)\} = (\eta_{ad}\mathcal{J}_{bc} - \eta_{ac}\mathcal{J}_{bd} - \eta_{bd}\mathcal{J}_{ac} + \eta_{bc}\mathcal{J}_{ad})\,\delta, \tag{8.11}$$

up to standard TEGR boundary terms. On the symmetric branch and for the dRGT/HR form of U, one finds a pair of *second-class* scalar constraints $(\mathcal{S}, \mathcal{S}')$ whose Poisson matrix has non-zero determinant on the healthy sector; these eliminate the BD scalar. The remaining first-class set generates the single diagonal diffeomorphism group and the diagonal local Lorentz rotations.

Counting phase-space variables: two spatial triads $E^a{}_i$ and $\tilde{E}^a{}_i$ give 2×12 coordinates and momenta, i.e. 48 canonical dimensions. First-class constraints: 4 (diagonal diffeos) $+\,6$ (diagonal Lorentz) $= 10$; second-class: 2 (the BD pair). The physical phase-space dimension is

$$48 \;-\; 2\times 10 \;-\; 2 \;=\; 26 \quad\Rightarrow\quad \frac{26}{2} = 13 \text{ configuration dof} \;=\; 2 \;+\; 5 \;+\; 6_{\text{non-prop}}, \tag{8.12}$$

where 2+5 are the propagating massless and massive spin-2 polarisations, and the residual 6 are fixed by gauge and constraints at the level of initial data.[1] Thus, teleparallelising the kinetic terms preserves the standard HR 2+5 spectrum.

8.5 Role of the relative Lorentz field

The auxiliary Λ and its Maurer–Cartan one-form $\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda + \Lambda^{-1}\omega_\mu\Lambda - \tilde{\omega}_\mu$ enter only algebraically. Their equations of motion enforce (8.6) and align the two local frames; they do not generate independent canonical pairs. In particular,

$$\frac{\delta H_{\text{tot}}}{\delta\Lambda} \;=\; 0 \quad\Rightarrow\quad \text{algebraic conditions on boosts/rotations ensuring } X^\top g X = f, \tag{8.13}$$

which guarantees that no extra scalar or vector mode propagates from the relative-rotation sector.

8.6 Sufficient conditions for BD-ghost absence in teleparallel bimetric

1. **TEGR kinetic sector on each sheet:** $R(\hat{\Gamma}) = 0$, torsion-scalar kinetic term only, with inertial spin connection restoring local Lorentz symmetry.
2. **HR-type potential:** $U = \sum \beta_n e_n(X)$ with $X = e^{-1}\Lambda\,\tilde{e}$ and the symmetric-vierbein condition (8.6).
3. **Linearity in diagonal lapses/shifts:** U remains linear in the diagonal combination of $(N, N^i, \tilde{N}, \tilde{N}^i)$, yielding a secondary scalar constraint.
4. **Non-degenerate scalar pair:** the Poisson bracket $\{\mathcal{S}, \mathcal{S}'\}$ is non-vanishing on the healthy branch, making the pair second-class and removing the BD mode.

Under these conditions the 2+5 spectrum is retained, with no additional propagating scalar.

9. Autoparallel completion and motion of matter

We clarify which curves physical probes follow and how this ties to ghost freedom.

[1] Equivalently: 7 propagating metric/tetrad dof in four dimensions.

9.1 Levi–Civita vs. Weitzenböck transport

The Levi–Civita and Weitzenböck connections are related by the contorsion

$$ {}^{\{\}}\Gamma^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}{}_{\mu\nu} - K^{\lambda}{}_{\mu\nu}, \qquad K^{\lambda}{}_{\mu\nu} = \tfrac{1}{2}\big(T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu} - T^{\lambda}{}_{\mu\nu}\big). \tag{9.1} $$

Geodesics of $g_{\mu\nu}$ satisfy

$$ \ddot{x}^{\lambda} + {}^{\{\}}\Gamma^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0. \tag{9.2} $$

Autoparallels of $\hat{\Gamma}$ satisfy

$$ \ddot{x}^{\lambda} + \hat{\Gamma}^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0. \tag{9.3} $$

Using ${}^{\{\}}\Gamma = \hat{\Gamma} - K$, (9.2) is equivalent to the *force form*

$$ \ddot{x}^{\lambda} + \hat{\Gamma}^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = K^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}. \tag{9.4} $$

Thus, in TEGR the physical free fall can be described either by metric geodesics (9.2) or, equivalently, by the torsion "force" equation (9.4). *Autoparallels* (9.3) *coincide with geodesics only if* $K^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$ along the worldline, which is not true in general.

9.2 Matter coupling and the physical path

To preserve the constraint structure, matter couples minimally to a *single* metric/tetrad, say $g_{\mu\nu}(e)$. Then:

$$ S_{\rm m}[x;\, g] = -m\int d\lambda\, \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \quad \Rightarrow \quad \text{Euler–Lagrange} \equiv \text{geodesic (9.2)}, \tag{9.5} $$

$$ \text{equivalently torsion “force” form (9.4) under the TEGR identity } R({}^{\{\}}\Gamma) = -T + B, \tag{9.6} $$

with B a boundary term. Couplings to a *composite* or two-sheet metric generically spoil linearity in the lapses and endanger the secondary scalar constraint. Hence we impose single-sheet minimal coupling for all matter to maintain BD-ghost absence.

9.3 Photons, spin, and transport of frames

Null rays follow g-null geodesics; in the teleparallel picture their tangent k^{μ} obeys (9.4) with $k^2 = 0$. A spin frame s^a carried by a massive particle undergoes Fermi–Walker transport in the Levi–Civita picture; equivalently,

$$ \frac{Ds^a}{d\tau} = \omega^{a}{}_{b\mu}\, u^{\mu} s^{b}, \quad u^{\mu} = \frac{dx^{\mu}}{d\tau}, \tag{9.7} $$

with the inertial spin connection $\omega^{a}{}_{b\mu}$ ensuring local Lorentz covariance and $R(\omega) = 0$.

9.4 Bimetric teleparallel case

With two tetrads $(e, \tilde{e})$ and relative Lorentz field Λ:

- **Physical coupling rule:** Matter couples to $g(e)$ only. Then physical paths are g-geodesics, or equivalently TEGR force curves (9.4) built from $(e, \hat{\Gamma}[e])$.
- **Across-sheet comparison:** The frame components relate by $\tilde{v}^a = \Lambda^{a}{}_{b}v^{b}$ and evolve via the relative Maurer–Cartan field $\mathcal{B}_{\mu}$; this does *not* introduce new propagating modes.
- **EP and cones:** The physical light cone is that of $g(e)$. The second metric $f(\tilde{e})$ enters only through the potential $U(g, f)$ and does not define separate matter characteristics.

9.5 Autoparallel completion: when (9.3) is admissible

Using (9.4), autoparallels match geodesics iff the contorsion projection vanishes on the trajectory:

$$\boxed{K^{\lambda}{}_{\mu\nu}\, u^{\mu} u^{\nu} = 0} \quad \Longleftrightarrow \quad \text{TEGR autoparallels} \;=\; \text{Levi–Civita geodesics.} \tag{9.8}$$

Sufficient conditions include:

1. *Purely vector torsion aligned with u^{μ}:* $T^{\lambda}{}_{\mu\nu} = 2\,\delta^{\lambda}{}_{[\mu} t_{\nu]}$ with $t_{\mu} \propto u_{\mu}$ along the worldline.
2. *Gauge-aligned frames:* Choose inertial ω and tetrad gauge so that the induced $K^{\lambda}{}_{\mu\nu} u^{\mu} u^{\nu} = 0$ for the congruence under study.

In generic TEGR spacetimes (9.8) need not hold; the correct equation is the geodesic or, equivalently, the force form (9.4).

9.6 Deviation and conservation

Let u^{μ} be the tangent and ξ^{μ} a connecting vector. The g-geodesic deviation reads

$$\frac{D^2 \xi^{\mu}}{d\tau^2} = R(^{\{\}}\Gamma)^{\mu}{}_{\nu\alpha\beta} u^{\nu} u^{\alpha} \xi^{\beta}. \tag{9.9}$$

In teleparallel variables this decomposes into torsion and contorsion pieces via $R(^{\{\}}\Gamma) = -(\nabla K + K * K)$. Energy–momentum conservation for minimally coupled matter is

$$\nabla_{\mu}^{(\{\}\Gamma)} T^{\mu}{}_{\nu} = 0 \quad \Longleftrightarrow \quad \nabla_{\mu}^{(\hat{\Gamma})} T^{\mu}{}_{\nu} = K^{\rho}{}_{\mu\nu}\, T^{\mu}{}_{\rho}. \tag{9.10}$$

9.7 Ghost-freedom compatibility conditions for motion

To keep the BD scalar removed while defining physical motion:

$$\boxed{\text{(i) Single-sheet matter coupling } S_{\mathrm{m}}[g(e), \Psi]; \quad \text{(ii) HR/dRGT potential in the symmetric-vierbein branch;}} \tag{9.11}$$

$$\boxed{\text{(iii) No derivatives of } \Lambda \text{ in } S_{\mathrm{m}}; \quad \text{(iv) Matter Hamiltonian linear in the } g\text{-lapse/shift.}} \tag{9.12}$$

Under (i)–(iv), physical curves are g-geodesics (or (9.4)), and the constraint algebra retains the 2+5 propagating spectrum.

10. Failure modes and exclusions: when the BD ghost reappears

We list concrete modifications that spoil the secondary scalar constraint or add extra canonical pairs. Each item includes a minimal diagnostic.

10.1 Breaking the symmetric-vierbein branch

The HR potential is implemented via $X = e^{-1} \Lambda\, \tilde{e}$ with the symmetry condition

$$e^{a}{}_{[\mu}\, X_{\nu]}{}^{\mu} = 0. \tag{10.1}$$

Failure mode: dropping (10.1) activates boost/rotation antisymmetries and makes U depend on components that are not purely algebraic, obstructing the linearity-in-lapse structure.
Diagnostic: in 3+1, compute $\partial^2 U / \partial N^2$ at fixed $(E, \tilde{E})$. If it is nonzero anywhere on phase space, the scalar secondary constraint is lost.

10.2 Nonlinear lapse/shift dependence in the interaction

The HR/dRGT potential must be *linear* in the diagonal lapse and shift after solving auxiliary algebraic conditions.
Failure mode: any cross-torsion or mixed-tetrad kinetic insertion into U,

$$\Delta U_{\text{bad}} = \alpha\, T[e]\cdot T[\tilde{e}] \quad \text{or} \quad \beta\, S[e]:\partial X \tag{10.2}$$

feeds velocities or quadratic lapse dependence back into the "potential".
Diagnostic: compute the Hessian $\mathcal{H} = \begin{pmatrix} \partial^2 L/\partial N^2 & \partial^2 L/\partial N\,\partial N^i \\ \partial^2 L/\partial N^j\,\partial N & \partial^2 L/\partial N^i \partial N^j \end{pmatrix}$. For a healthy theory the rank of $\mathcal{H}$ must be $4-1=3$ (one null direction generating the scalar secondary constraint). Rank 4 indicates ghost.

10.3 Derivatives of the relative Lorentz field

The relative Maurer–Cartan field is

$$\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda + \Lambda^{-1}\omega_\mu\Lambda - \tilde{\omega}_\mu. \tag{10.3}$$

Failure mode: introducing $\partial\Lambda$ in the action beyond algebraic elimination, e.g.

$$\Delta S_{\text{bad}} = \int d^4x\, e\, \gamma_1\, \mathcal{B}_\mu \mathcal{B}^\mu \quad \text{or} \quad \gamma_2\, T^\mu{}_{\nu\rho}\, (\mathcal{B}_\mu)^{\nu\rho}, \tag{10.4}$$

promotes Λ to a propagating sector and adds extra dof.
Diagnostic: check whether $\Pi^\mu_{(\Lambda)} = \partial L/\partial(\partial_0\Lambda)$ vanishes identically. If not, new canonical pairs form and the HR counting is lost.

10.4 Matter coupling to both sheets or to X

Failure mode A: bi-metric matter,

$$S_{\text{m}}[g,f,\Psi] = \int d^4x\, \big(\sqrt{-g}\,\mathcal{L}_{\text{m}}(\Psi,g) + \sqrt{-f}\,\tilde{\mathcal{L}}_{\text{m}}(\Psi,f)\big), \tag{10.5}$$

reintroduces the BD mode by making both lapses enter nonlinearly in H_{m}.
Failure mode B: composite coupling

$$S_{\text{m}}[X,\Psi] = \int d^4x \sqrt{-g}\,\mathcal{L}_{\text{m}}\big(\Psi,\, g_{\mu\nu},\, X^\mu{}_\nu\big), \tag{10.6}$$

breaks linearity after solving the symmetry condition.
Diagnostic: compute the primary constraints from $\dot{N}, \dot{\tilde{N}}$ absence, then test preservation in time. If no additional scalar secondary arises, the BD scalar propagates.

10.5 Incorrect inertial-connection treatment

Local Lorentz symmetry in TEGR requires an inertial spin connection with $R(\omega)=0$.
Failure mode: fixing $\omega = 0$ without ensuring inertiality of the tetrad gauge may mimic $f(T)$-like behavior, turning Lorentz generators second-class and changing dof.
Diagnostic: verify $R^a{}_{b\mu\nu}(\omega) = 0$ and that the six Lorentz constraints $\mathcal{J}_{ab} \approx 0$ remain first-class on the chosen gauge.

10.6 Parity-odd and boundary-sensitive additions

Failure mode: adding Nieh–Yan with non-constant coefficient,

$$\Delta S_{\text{bad}} = \int d^4x\, \vartheta(x)\, \partial_\mu\big(\epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}\big), \tag{10.7}$$

moves a boundary into the bulk and can introduce derivatives of lapses via $\partial_0\vartheta$.
Diagnostic: ensure all parity-odd terms are either total derivatives with constant coefficients or are arranged so their time parts are boundary-exact on the Cauchy slice.

10.7 Turning TEGR into $f(T)$ in either sheet

Failure mode: replacing $T \mapsto f(T)$ on one sheet changes the primary constraint algebra and removes the linear dependence on lapse typical of TEGR/GR.
Diagnostic: the scalar constraint ceases to generate time reparametrizations; Poisson closure with the vector constraints fails to reproduce the hypersurface-deformation algebra.

10.8 Summary: exclusion box

$$\boxed{\begin{aligned} &\text{Do } \textit{not}: \ \text{(i) break } e^a{}_{[\mu} X_{\nu]}{}^\mu = 0; \ \text{(ii) add velocities/cross-kinetics to } U; \\ &\quad \text{(iii) propagate } \Lambda; \ \text{(iv) couple matter to } f \text{ or } X; \ \text{(v) choose non-inertial } \omega; \\ &\quad \text{(vi) use non-constant Nieh–Yan couplings; (vii) replace } T \text{ by } f(T). \end{aligned}} \tag{10.8}$$

10.9 Quick rank test for implementations

Given a concrete discretization or symbolic implementation:

1. Build the lapse/shift Hessian $\mathcal{H}$ of the Lagrangian density.
2. Enforce the symmetric-vierbein condition algebraically.
3. Evaluate rank $\mathcal{H}$ on random phase-space points respecting primary constraints.
4. Accept only if rank $\mathcal{H} = 3$ and the scalar pair $(\mathcal{S}, \mathcal{S}')$ has $\det\{\mathcal{S}, \mathcal{S}'\} \neq 0$.

This suffices to exclude the BD scalar in numerical pipelines.

11. Gauge-fixed Poisson matrix and Dirac bracket (full calculation)

We present the constraint set, an explicit gauge fixing, and the resulting Dirac matrix in block form. This makes the BD-scalar removal transparent and ready for code.

11.1 Constraint set and gauge fixing

First-class generators on the symmetric-vierbein branch:

$$\mathcal{F} := \{\, \mathcal{C}_0^{\text{diag}},\ \mathcal{C}_i^{\text{diag}},\ \mathcal{J}_{ab}^{\text{diag}} \,\}, \qquad a < b,\ i = 1, 2, 3. \tag{11.1}$$

Intrinsic second-class pair from the HR potential:

$$\mathcal{S} \approx 0, \qquad \mathcal{S}' \approx 0, \qquad \mu(x) := \{\mathcal{S}(x), \mathcal{S}'(x)\} \neq 0 \text{ on the healthy branch.} \tag{11.2}$$

Choose gauge-fixing conditions that pair with $\mathcal{F}$:

$$\chi^0 \approx 0, \qquad \chi^i \approx 0, \qquad \chi_{\mathrm{L}}^{ab} \approx 0 \; (a < b), \tag{11.3}$$

with the following concrete, easy choices that keep algebra simple:

Time gauge: $e^0{}_i = 0,\ \tilde{e}^0{}_i = 0 \ \Rightarrow\ \chi_{\mathrm{L}}^{0i} := e^0{}_i \approx 0,\ \tilde{\chi}_{\mathrm{L}}^{0i} := \tilde{e}^0{}_i \approx 0,$ (11.4)

Diagonal-rotation alignment: $\chi_{\mathrm{L}}^{ij} := E^a{}_{[i}\, E_{aj]} \approx 0 \quad (i < j),$ (11.5)

Spatial diffeo gauge: $\chi^i := \partial_j \gamma^{ji} \approx 0$ (spatial de Donder, any elliptic choice with invertible FP operator work (11.6)

Time diffeo gauge: $\chi^0 := N - N_\star(E, \tilde{E}) \approx 0$ with a non-vanishing pairing $\{\mathcal{C}_0^{\mathrm{diag}}, \chi^0\}$. (11.7)

Any alternative set with the same non-degeneracy properties is acceptable.

11.2 Second-class set and Dirac matrix

Collect the full second-class vector

$$\Phi_A := \big(\underbrace{\mathcal{C}_0^{\mathrm{diag}},\, \chi^0}_{2}\,;\, \underbrace{\mathcal{C}_i^{\mathrm{diag}},\, \chi^i}_{3+3}\,;\, \underbrace{\mathcal{J}_{ab}^{\mathrm{diag}},\, \chi_{\mathrm{L}}^{ab}}_{6+6}\,;\, \underbrace{\mathcal{S},\, \mathcal{S}'}_{2}\big), \tag{11.8}$$

so $\dim \Phi = 2 + 6 + 12 + 2 = 22$. The Dirac (Poisson) matrix is

$$\Delta_{AB}(x, y) := \{\Phi_A(x), \Phi_B(y)\}. \tag{11.9}$$

By construction and up to standard boundary terms, Δ is block almost-lower-triangular:

$$\Delta = \begin{pmatrix} \boxed{A} & 0 & 0 & 0 \\ \star & \boxed{B} & 0 & 0 \\ \star & \star & \boxed{R} & 0 \\ \star & \star & \star & \boxed{M} \end{pmatrix}, \qquad \begin{array}{l} A \in \mathbb{R}^{2\times 2},\quad B \in \mathbb{R}^{6\times 6}, \\ R \in \mathbb{R}^{12\times 12},\quad M \in \mathbb{R}^{2\times 2}. \end{array} \tag{11.10}$$

The boxed diagonal blocks are:

$$A(x, y) = \begin{pmatrix} 0 & -\alpha(x) \\ +\alpha(x) & 0 \end{pmatrix} \delta(x - y), \quad \alpha(x) := \big\{\mathcal{C}_0^{\mathrm{diag}}(x), \chi^0(x)\big\}, \tag{11.11}$$

$$B^i{}_j(x, y) = \big\{\mathcal{C}_i^{\mathrm{diag}}(x), \chi^j(y)\big\} \;=\; -\mathcal{M}_i{}^j(x)\, \delta(x - y) \;+\; \text{derivative terms}, \tag{11.12}$$

$$R_{ab}{}^{cd}(x, y) = \big\{\mathcal{J}_{ab}^{\mathrm{diag}}(x), \chi_{\mathrm{L}}^{cd}(y)\big\} \;=\; \mathcal{R}_{ab}{}^{cd}(x)\, \delta(x - y), \tag{11.13}$$

$$M(x, y) = \begin{pmatrix} 0 & \mu(x) \\ -\mu(x) & 0 \end{pmatrix} \delta(x - y), \qquad \mu(x) := \{\mathcal{S}(x), \mathcal{S}'(x)\}. \tag{11.14}$$

Non-boxed entries (marked $\star$) either vanish weakly or contribute only lower-triangular structure that does not affect invertibility.

Non-degeneracy conditions. The block inverse exists provided:

$$\alpha(x) \neq 0, \qquad \det \mathcal{M} \neq 0, \qquad \det \mathcal{R} \neq 0, \qquad \mu(x) \neq 0. \tag{11.15}$$

These are satisfied for the choices above on generic data, and $\mu \neq 0$ holds on the HR healthy branch.

11.3 Explicit ingredients of the blocks

Time block A. With $\chi^0 = N - N_\star(E, \tilde{E})$,

$$\alpha(x) = \left\{ \mathcal{C}_0^{\text{diag}}(x), N(x) \right\} - \left\{ \mathcal{C}_0^{\text{diag}}(x), N_\star(x) \right\} = 1 - \frac{\delta N_\star}{\delta t}\Big|_{\text{PB}} \Rightarrow \alpha \neq 0. \tag{11.16}$$

Spatial block B. For de Donder $\chi^i = \partial_j \gamma^{ji}$ one has

$$\mathcal{M}_i{}^j = -\gamma^{jk}\partial_k\partial_i - \partial_i\partial^j + \cdots , \tag{11.17}$$

an elliptic FP operator on compact slices; invertible after boundary conditions are set.
Lorentz block R. With time gauge and antisymmetric-triad kill, the bracket reduces to the adjoint action of $\mathfrak{so}(1,3)$ on the six parameters:

$$\mathcal{R}_{ab}{}^{cd} = \eta_a{}^{[c}\delta_b{}^{d]} - \eta_b{}^{[c}\delta_a{}^{d]} \quad \Rightarrow \quad \det \mathcal{R} \neq 0. \tag{11.18}$$

BD-pair block M. On the symmetric-vierbein branch with $U = \sum_n \beta_n e_n(X)$ and linearity in the diagonal lapse,

$$\mu(x) = \{\mathcal{S}, \mathcal{S}'\} \ = \ \Xi^{ij}(E, \tilde{E}; \beta_n)\, \gamma_{ij} \ + \ \text{potential terms}, \tag{11.19}$$

where Ξ^{ij} is a non-vanishing tensor density determined by $e_n(X)$. For generic β_n on the healthy sector, $\mu(x) \neq 0$.

11.4 Inverse and Dirac bracket

The block inverse is immediate:

$$A^{-1} = \begin{pmatrix} 0 & +\alpha^{-1} \\ -\alpha^{-1} & 0 \end{pmatrix}, \quad B^{-1} = \mathcal{M}^{-1} + (\text{boundary}), \quad R^{-1} = (\mathcal{R})^{-1}, \quad M^{-1} = \begin{pmatrix} 0 & -\mu^{-1} \\ +\mu^{-1} & 0 \end{pmatrix}. \tag{11.20}$$

The Dirac bracket for any phase-space functionals F, G is

$$\{F, G\}_{\text{D}} = \{F, G\} - \iint d^3x\, d^3y\, \{F, \Phi_A(x)\}\, (\Delta^{-1})^{AB}(x, y)\, \{\Phi_B(y), G\}. \tag{11.21}$$

In particular, for the physical spatial triads on the diagonal sheet one gets

$$\{E^a{}_i(x), \Pi_b{}^j(y)\}_{\text{D}} = \delta^a{}_b\, \delta_i{}^j\, \delta(x - y) \ - \ \text{projectors along the fixed gauges}, \tag{11.22}$$

with all gauge and BD directions projected out.

11.5 Minimal implementation checklist

1. Build $\mathcal{C}_{0,i}^{\text{diag}}$, $\mathcal{J}_{ab}^{\text{diag}}$ for TEGR on each sheet, add $U = \sum \beta_n e_n(X)$.
2. Enforce symmetric-vierbein algebraically; solve inertial spin connections.
3. Choose gauges $(\chi^0, \chi^i, \chi_{\text{L}}^{ab})$; assemble Φ_A.
4. Form the four diagonal blocks A, B, R, M; verify non-degeneracy on random phase-space points respecting primaries.
5. Invert Δ blockwise and implement $\{\cdot,\cdot\}_{\text{D}}$. Confirm that only $2+5$ propagating tensor dof remain in the quadratic theory.

12. Stückelberg map and decoupling limit: coefficients $c_k(\beta_n)$

We expose the helicity decomposition and derive the scalar-sector coefficients that control the Λ_3 decoupling limit. TEGR kinetics do not modify the HR potential; the mapping equals the metric case.

12.1 Setup and fields

Work around a bi-flat background, $g_{\mu\nu} = \eta_{\mu\nu}$, $f_{\mu\nu} = \eta_{\mu\nu}$. Introduce fluctuations and Stückelberg fields:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}, \qquad f_{\mu\nu} = \partial_\mu \Phi^\alpha \, \partial_\nu \Phi^\beta \left(\eta_{\alpha\beta} + \frac{1}{M_f} \ell_{\alpha\beta} \right), \tag{12.1}$$

$$\Phi^\mu = x^\mu - \frac{A^\mu}{m \, M_{\text{eff}}} - \frac{\partial^\mu \pi}{m^2 M_{\text{eff}}}, \qquad M_{\text{eff}}^{-2} := M_g^{-2} + M_f^{-2}, \quad \Lambda_3 := (m^2 M_{\text{eff}})^{1/3}. \tag{12.2}$$

Define $\Pi_{\mu\nu} := \partial_\mu \partial_\nu \pi$ and use the symmetric-vierbein branch so that $X = \sqrt{g^{-1} f}$ is well-defined and symmetric.

12.2 Square-root expansion

Write $X = \sqrt{\mathbf{1} + H}$ with

$$H^\mu{}_\nu := g^{\mu\alpha} \big(f_{\alpha\nu} - g_{\alpha\nu} \big) = \frac{1}{M_f} \, \ell^\mu{}_\nu - \frac{1}{M_g} \, h^\mu{}_\nu - \frac{2}{m^2 M_{\text{eff}}} \, \Pi^\mu{}_\nu + \mathcal{V}(A) + \mathcal{O}(h\ell, \Pi h, \Pi \ell), \tag{12.3}$$

and use $\sqrt{\mathbf{1} + H} = \mathbf{1} + \frac{1}{2} H - \frac{1}{8} H^2 + \frac{1}{16} H^3 + \cdots$. Insert into $U = \sum_{n=0}^{4} \beta_n e_n(X)$, where e_n are the elementary symmetric polynomials of X. Keep terms up to cubic order in fields and organize by helicity.

12.3 Helicity decomposition at Λ_3

At the Λ_3 scale the vector A_μ is subleading unless sourced; the scalar π yields Galileon invariants

$$\mathcal{L}_2 = (\partial \pi)^2, \tag{12.4}$$

$$\mathcal{L}_3 = (\partial \pi)^2 [\Pi], \tag{12.5}$$

$$\mathcal{L}_4 = (\partial \pi)^2 \big([\Pi]^2 - [\Pi^2] \big), \tag{12.6}$$

$$\mathcal{L}_5 = (\partial \pi)^2 \big([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3] \big), \tag{12.7}$$

with traces $[\Pi^k] := \Pi^{\mu}{}_{\nu_1}\Pi^{\nu_1}{}_{\nu_2}\cdots\Pi^{\nu_{k-1}}{}_{\mu}$. The scalar sector of the interaction reduces to

$$\mathcal{L}_\pi \;=\; -\frac{3}{2}c_2\,\mathcal{L}_2 + \frac{c_3}{\Lambda_3^3}\,\mathcal{L}_3 + \frac{c_4}{\Lambda_3^6}\,\mathcal{L}_4 + \frac{c_5}{\Lambda_3^9}\,\mathcal{L}_5 + \text{mixing with } h,\ell \;\; \text{(diagonalizable)}. \tag{12.8}$$

12.4 Coefficient matching

Expanding $e_n(X)$ in H and projecting onto the scalar sector yields linear combinations of β_n:

$$\boxed{\begin{aligned} c_2 &= a_{20}\beta_0 + a_{21}\beta_1 + a_{22}\beta_2 + a_{23}\beta_3 + a_{24}\beta_4, \\ c_3 &= a_{30}\beta_0 + a_{31}\beta_1 + a_{32}\beta_2 + a_{33}\beta_3 + a_{34}\beta_4, \\ c_4 &= a_{40}\beta_0 + a_{41}\beta_1 + a_{42}\beta_2 + a_{43}\beta_3 + a_{44}\beta_4, \\ c_5 &= a_{50}\beta_0 + a_{51}\beta_1 + a_{52}\beta_2 + a_{53}\beta_3 + a_{54}\beta_4, \end{aligned}} \tag{12.9}$$

where the a_{nk} depend only on the choice of normalization for U and the background convention. For the common normalization $U = m^2 M_g^2 \sum_{n=0}^{4} \beta_n e_n(X)$ about bi-flat backgrounds, one convenient basis gives the *relative* pattern

$$\boxed{c_2 \propto \beta_1 + 2\beta_2 + \beta_3, \quad c_3 \propto \beta_2 + 2\beta_3 + \beta_4, \quad c_4 \propto \beta_3 + 2\beta_4, \quad c_5 \propto \beta_4,} \tag{12.10}$$

up to an overall factor set by (m, M_g, M_f) and the chosen U prefactor. Exact prefactors follow from the H-expansion bookkeeping.

Relation to the (α_3, α_4) basis. If one reparametrizes the potential by two constants (α_3, α_4) via linear relations $\beta_n = \beta_n(\alpha_3, \alpha_4)$, then (12.10) induces

$$\boxed{c_3, c_4, c_5 \text{ are affine functions of } (\alpha_3, \alpha_4), \qquad c_2 \text{ fixed by canonically normalizing } \pi.} \tag{12.11}$$

This matches the standard dRGT decoupling-limit structure and ensures second-order π-equations (Galileon form).

12.5 Recipe for exact a_{nk} in any convention

1. Expand $X = \sqrt{\mathbf{1} + H}$ to third order using the series above.
2. Compute $e_n(X)$ to cubic order and insert into U. Drop pure tensors (h, ℓ) and vectors (A) to isolate the $(\partial\pi)^2[\Pi]^k$ sector.
3. Canonically normalize π by $\pi \to \hat{\pi}$ so that the kinetic term is $-\frac{3}{2}\mathcal{L}_2$.
4. Read off c_k as linear forms in β_n; the coefficients are the a_{nk} in (12.9).

12.6 Mixing removal and diagonal modes

The scalar mixes linearly with the trace of the massive tensor. Define the diagonal combinations

$$h_{\mu\nu}^{\text{diag}} = \frac{M_g h_{\mu\nu} + M_f \ell_{\mu\nu}}{\sqrt{M_g^2 + M_f^2}}, \qquad h_{\mu\nu}^{\text{mass}} = \frac{M_f h_{\mu\nu} - M_g \ell_{\mu\nu}}{\sqrt{M_g^2 + M_f^2}}, \tag{12.12}$$

and perform the standard field redefinition $h_{\mu\nu}^{\text{mass}} \to h_{\mu\nu}^{\text{mass}} + \zeta\,\eta_{\mu\nu}\pi$ with $\zeta = \mathcal{O}(1)$ fixed by the quadratic mixing. This isolates the π Galileon sector with coefficients $c_k(\beta_n)$.

12.7 Teleparallel note

The TEGR kinetic terms differ from GR by a boundary term. At quadratic order about Minkowski/FLRW they coincide, so the decoupling-limit map (12.9)–(12.10) is unchanged. No torsion insertion appears in U; thus the Galileon tuning that removes the BD scalar is the same as in metric bimetric.

12.8 Boxed summary for implementations

$$\boxed{\begin{aligned}&X=\sqrt{g^{-1}\partial\Phi\, f\,(\partial\Phi)^{\top}},\quad \Phi^{\mu}=x^{\mu}-\frac{A^{\mu}}{mM_{\text{eff}}}-\frac{\partial^{\mu}\pi}{m^2M_{\text{eff}}},\ \ \Lambda_3=(m^2M_{\text{eff}})^{1/3},\\ &\mathcal{L}_\pi=-\tfrac{3}{2}c_2\,(\partial\pi)^2+\frac{c_3}{\Lambda_3^3}\mathcal{L}_3+\frac{c_4}{\Lambda_3^6}\mathcal{L}_4+\frac{c_5}{\Lambda_3^9}\mathcal{L}_5,\quad c_k=\sum_{n=0}^{4}a_{kn}\beta_n,\\ &\text{TEGR}\ \Rightarrow\ \text{same } c_k(\beta_n) \text{ as metric bimetric on the symmetric branch.}\end{aligned}} \tag{12.13}$$

13. Quadratic actions on Minkowski and FLRW: mode matrices and stability

We expand the teleparallel bimetric action to second order about proportional backgrounds and extract the kinetic, gradient, and mass matrices in tensor, vector, and scalar sectors. TEGR kinetics coincide with GR up to a boundary term, so all mode conditions match the metric bimetric case on the symmetric-vierbein branch.

13.1 Backgrounds and proportional ansatz

Take proportional tetrads

$$\tilde{e}^{a}{}_{\mu}=c_0\,e^{a}{}_{\mu}\quad\Rightarrow\quad f_{\mu\nu}=c_0^2\,g_{\mu\nu}, \tag{13.1}$$

with constant $c_0>0$ fixed by the background equations from $U=\sum_{n=0}^{4}\beta_n e_n(X)$. We treat two cases:

1. **Minkowski:** $g_{\mu\nu}=\eta_{\mu\nu}$, $f_{\mu\nu}=c_0^2\eta_{\mu\nu}$.
2. **Spatially flat FLRW:** $g_{\mu\nu}=-dt^2+a^2(t)\delta_{ij}$, $f_{\mu\nu}=-\xi^2(t)dt^2+b^2(t)\delta_{ij}$ with proportional branch $b(t)=c_0a(t)$ and $\xi(t)=1$.

Write perturbations

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+\frac{1}{M_g}h_{\mu\nu},\qquad f_{\mu\nu}=\bar{f}_{\mu\nu}+\frac{1}{M_f}\ell_{\mu\nu}. \tag{13.2}$$

Define diagonal/massive combinations

$$h^{\text{diag}}_{\mu\nu}=\frac{M_gh_{\mu\nu}+M_f\ell_{\mu\nu}}{\sqrt{M_g^2+M_f^2}},\qquad h^{\text{mass}}_{\mu\nu}=\frac{M_fh_{\mu\nu}-M_g\ell_{\mu\nu}}{\sqrt{M_g^2+M_f^2}}. \tag{13.3}$$

13.2 Tensor sector

Decompose $h_{ij} = h^{\mathrm{T}}_{ij} + (\text{vector} + \text{scalar})$ with $\partial_i h^{\mathrm{T}}_{ij} = 0 = h^{\mathrm{T}}_{ii}$, and similarly for ℓ_{ij}. The quadratic action is

$$S^{\text{tens}}_{(2)} = \frac{1}{8} \int d^4x\, a^3 \Big[M_g^2 \big(\dot{h}^{\mathrm{T}}_{ij} \dot{h}^{\mathrm{T}}_{ij} - a^{-2} \partial_k h^{\mathrm{T}}_{ij} \partial_k h^{\mathrm{T}}_{ij} \big) + M_f^2 \big(\dot{\ell}^{\mathrm{T}}_{ij} \dot{\ell}^{\mathrm{T}}_{ij} - a^{-2} \partial_k \ell^{\mathrm{T}}_{ij} \partial_k \ell^{\mathrm{T}}_{ij} \big) \\ - \, a^3\, m_T^2\, \big(h^{\text{mass}}_{ij} \big)^2 \Big], \tag{13.4}$$

with m_T^2 the Fierz–Pauli tensor mass from U evaluated on the proportional background. On Minkowski,

$$\boxed{m_T^2 \;=\; m^2 \big(\beta_1 + 2\beta_2 + \beta_3 \big) \quad \text{(bi-Minkowski, common normalization)}} \tag{13.5}$$

and on the proportional FLRW branch

$$m_T^2(t) \;=\; m^2 \big(\beta_1 c_0 + 2\beta_2 c_0^2 + \beta_3 c_0^3 \big) \quad \text{(time-independent if } c_0 \text{ is constant)}. \tag{13.6}$$

Diagonalizing by $(h^{\text{diag}}, h^{\text{mass}})$ gives one massless $c_T^2 \;=\; 1$ mode and one massive mode with dispersion $\omega^2 = k^2/a^2 + m_T^2$.

Tensor stability. Conditions:

$$M_g^2 > 0, \quad M_f^2 > 0, \quad m_T^2 > 0, \quad \text{and on de Sitter } m_T^2 >\, 2H^2 \text{ (Higuchi)}. \tag{13.7}$$

13.3 Vector sector

Use the standard transverse basis B_i and E_i with $\partial_i B_i = \partial_i E_i = 0$ on each sheet. After integrating out nondynamical components and forming (diag, mass) combinations, only the massive-vector pair propagates. The quadratic action is

$$S^{\text{vec}}_{(2)} = \frac{1}{2} \int d^4x\, a^3 \Big[K_V\, \dot{V}_i \dot{V}_i - G_V\, a^{-2} (\partial_j V_i)(\partial_j V_i) - M_V^2\, V_i V_i \Big], \tag{13.8}$$

with V_i the massive combination. TEGR yields $K_V \;>\; 0$ identical to GR. The coefficients are algebraic functions of (M_g, M_f, c_0, β_n).

Vector stability. Conditions:

$$K_V > 0, \quad G_V > 0, \quad M_V^2 \geq 0. \tag{13.9}$$

13.4 Scalar sector

Work in scalar SVT variables on each sheet and impose the symmetric-vierbein branch. Primary lapse/shift constraints remove nondynamical scalars, and the HR structure supplies the second-class pair that excises the BD ghost. One scalar helicity-0 mode from h^{mass} remains. After field redefinitions,

$$S^{\text{sc}}_{(2)} = \frac{1}{2} \int d^4x\, a^3 \Big[K_S\, \dot{\chi}^2 - G_S\, a^{-2} (\partial_i \chi)^2 - M_S^2\, \chi^2 \Big], \tag{13.10}$$

where χ is the canonically normalized scalar. For bi-Minkowski, $K_S > 0$ and $G_S > 0$ reduce to linear inequalities in (β_n); on FLRW they depend on (β_n, c_0, H).

Scalar stability. Conditions:

$$K_S > 0, \qquad G_S > 0, \qquad M_S^2 \text{ not tachyonic on the timescale of interest.} \tag{13.11}$$

13.5 Minkowski specialization (closed forms)

Setting $a = 1$ and $c_0 = \text{const}$, the kinetic matrices are diagonal in $(h^{\text{diag}}, h^{\text{mass}})$. With the common $U = m^2 M_g^2 \sum \beta_n e_n(X)$ normalization one finds

$$\text{Tensor:} \quad \mathcal{K}_{\text{tens}} = \frac{1}{4}\,\text{diag}(M_g^2,\, M_f^2), \quad \mathcal{M}_{\text{tens}} = \text{diag}(0,\, \tfrac{1}{4}(M_g^2 + M_f^2) m_T^2), \tag{13.12}$$

$$\text{Vector:} \quad K_V = \frac{M_g^2 M_f^2}{M_g^2 + M_f^2}, \quad G_V = K_V, \quad M_V^2 = \frac{M_g^2 + M_f^2}{4}\, m_T^2, \tag{13.13}$$

$$\text{Scalar:} \quad K_S = \frac{3 M_g^2 M_f^2}{M_g^2 + M_f^2}, \quad G_S = K_S, \quad M_S^2 = \frac{M_g^2 + M_f^2}{4}\, m_T^2. \tag{13.14}$$

These equal the metric bimetric results; TEGR adds only a boundary that does not affect quadratic dynamics.

13.6 FLRW specialization (propagation and bounds)

For $a(t)$ with $H = \dot{a}/a$ and constant c_0, tensors propagate luminally with time-independent m_T^2 given above. Vectors and the scalar have time-dependent K, G, M^2 but remain algebraic functions of (β_n, c_0, M_g, M_f) and $H(t)$.

$$\text{Tensor:} \quad \omega^2 = \frac{k^2}{a^2} + m_T^2, \quad c_T^2 = 1, \quad \text{Higuchi: } m_T^2 > 2H^2 \text{ if } \dot{H} \approx 0, \tag{13.15}$$

$$\text{Vector:} \quad c_V^2 = G_V/K_V > 0, \qquad \text{no-ghost: } K_V > 0, \tag{13.16}$$

$$\text{Scalar:} \quad c_S^2 = G_S/K_S > 0, \qquad \text{no-ghost: } K_S > 0. \tag{13.17}$$

On the proportional branch the background ratio is constant, so no extra mixing from $\dot{c}_0$ appears.

13.7 Algorithm to obtain coefficients from U

1. Fix a background and c_0 by the background equations from U.
2. Expand $X = \sqrt{g^{-1} f}$ to quadratic order in (h, ℓ) and insert into U.
3. Rotate to $(h^{\text{diag}}, h^{\text{mass}})$; integrate out nondynamical lapses/shifts using TEGR constraints.
4. Read off sector matrices $\mathcal{K}$, $\mathcal{G}$, and $\mathcal{M}$; enforce positivity of kinetic and gradient eigenvalues and $m_T^2 > 0$.

13.8 Boxed stability checklist

$$\boxed{\begin{array}{l} \text{Kinetics: } M_g^2 > 0,\ M_f^2 > 0,\ K_V > 0,\ K_S > 0; \\ \text{Gradients: } c_T^2 = 1,\ G_V > 0,\ G_S > 0; \\ \text{Masses: } m_T^2 > 0 \text{ (Minkowski)}, \quad m_T^2 > 2H^2 \text{ (de Sitter)}; \\ \text{Branch: proportional, symmetric-vierbein, TEGR on both sheets.} \end{array}} \tag{13.18}$$

14. Initial–boundary value problem (IBVP): hyperbolicity, characteristics, and well–posedness

We formulate the TEGR–bimetric field equations as a first–order system on a finite domain with boundaries, prove strong/symmetric hyperbolicity for the principal part, and state maximally dissipative boundary conditions that preserve constraints. The HR interaction is algebraic, so it does not affect the principal symbol.

14.1 First–order reduction and variables

Work on the proportional, symmetric–vierbein branch. Use the diagonal/massive metric combinations

$$h^{\rm diag}_{\mu\nu}, \quad h^{\rm mass}_{\mu\nu}$$

from the previous section. Adopt generalized harmonic (GH) gauge for the Levi–Civita connection of g, $H_\mu := g_{\mu\nu}\, g^{\rho\sigma}\, {}^{\{\}}\Gamma^\nu{}_{\rho\sigma} = 0$, which, by TEGR≡GR+boundary, yields identical principal parts for the g-sector. Introduce first–order variables for spatial derivatives,

$$P_{ij} := \partial_t h^{\rm diag}_{ij}, \quad Q_{kij} := \partial_k h^{\rm diag}_{ij}, \qquad \mathcal{P}_{ij} := \partial_t h^{\rm mass}_{ij}, \quad \mathcal{Q}_{kij} := \partial_k h^{\rm mass}_{ij}.$$

Collect the state vector

$$\mathbf{U} = \big(h^{\rm diag}_{ij}, P_{ij}, Q_{kij};\ h^{\rm mass}_{ij}, \mathcal{P}_{ij}, \mathcal{Q}_{kij}\big).$$

The evolution takes the first–order quasilinear form

$$\partial_t \mathbf{U} \ +\ A^k(\mathbf{U})\, \partial_k \mathbf{U} \ =\ S(\mathbf{U}), \quad S \text{ algebraic in } \mathbf{U} \text{ and includes } m_T^2 \text{ terms from } U. \tag{14.1}$$

Because $U = \sum \beta_n e_n(X)$ is algebraic, A^k does not depend on (β_n).

14.2 Principal symbol and characteristics

Fix a unit spatial covector s_k. The principal symbol is

$$\mathcal{P}(s) := A^k s_k \ = \begin{pmatrix} 0 & -\mathbf{1}\otimes s^0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\mathbf{1}\otimes s^0 & 0 & \frac{1}{2}\mathcal{S} & 0 & 0 & 0 \\ 0 & \mathcal{S} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{1}\otimes s^0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathbf{1}\otimes s^0 & 0 & \frac{1}{2}\mathcal{S} \\ 0 & 0 & 0 & 0 & \mathcal{S} & 0 \end{pmatrix}, \qquad (\mathcal{S}\cdot V)_{ij} := s^k V_{kij}, \tag{14.2}$$

written schematically in (h, P, Q) blocks for each sector. Its eigenvalues are

$$\lambda \in \{-1,\, 0,\, +1\} \quad \text{with multiplicities matching tensor SVT counts.} \tag{14.3}$$

Hence the system is *strongly hyperbolic*. The massive tensor mass m_T^2 appears only in $S(\mathbf{U})$ and does not change the characteristic speeds.

Characteristic fields. Along s_i, the gravitational tensorial characteristic combinations are

$$W^{\rm diag}_{\pm,ij} := P^{\rm T}_{ij} \ \pm\ \tfrac{1}{2}\, s^k Q^{\rm T}_{kij}, \qquad \text{speed } \pm 1, \tag{14.4}$$

$$W^{\rm diag}_{0,ij} := Q^{\rm T}_{\perp\, ij}, \qquad \text{speed } 0, \tag{14.5}$$

and analogously for the massive sector $(\mathcal{P}, \mathcal{Q})$. "T" denotes TT projection; "$\perp$" denotes components tangential to s_i.

14.3 Constraint system and damping

Primary constraints: GH gauge $H_\mu = 0$ and its first–order companions; TEGR Lorentz constraints $\mathcal{J}_{ab} \approx 0$; HR scalar pair $(\mathcal{S}, \mathcal{S}') \approx 0$. Introduce Z4–style damping for the GH sector,

$$\partial_t H_\mu + \cdots = -\kappa_1 H_\mu - \kappa_2 n_\mu n^\nu H_\nu, \quad \kappa_{1,2} \geq 0,$$

which leaves the principal part unchanged but yields exponential decay of gauge violations in energy estimates. The HR pair remains algebraic/elliptic and is enforced at each step (no propagation equation needed).

14.4 Boundary conditions: maximally dissipative and CPBC

On a timelike boundary with outward normal s_i, split characteristic fields into outgoing/incoming by sign of speed. Impose *maximally dissipative* linear conditions on the incoming set:

$$W^{\mathrm{diag}}_{-,ij} = \mathcal{R}_{ij}{}^{kl} W^{\mathrm{diag}}_{+,kl} + g^{\mathrm{diag}}_{ij}, \qquad W^{\mathrm{mass}}_{-,ij} = \mathcal{R}^{\mathrm{m}kl}_{ij} W^{\mathrm{mass}}_{+,kl} + g^{\mathrm{mass}}_{ij}, \quad \|\mathcal{R}\| \leq 1. \tag{14.6}$$

To preserve constraints (CPBC), set incoming gauge and constraint combinations to zero (or to prescribed data):

$$H^{\mathrm{in}}_\mu = 0, \qquad \mathcal{J}^{\mathrm{in}}_{ab} = 0, \qquad \mathcal{S} = \mathcal{S}' = 0 \text{ on the boundary.} \tag{14.7}$$

Physical data are supplied through the TT parts of g^{diag}_{ij} and, if desired, g^{mass}_{ij}.

14.5 Energy estimate and well–posedness

Define the standard energy on a slice Σ_t,

$$E(t) := \frac{1}{2} \int_{\Sigma_t} d^3x \left(|P^{\mathrm{diag}}|^2 + \tfrac{1}{4} |Q^{\mathrm{diag}}|^2 + |\mathcal{P}|^2 + \tfrac{1}{4} |\mathcal{Q}|^2 + m_T^2 \, |h^{\mathrm{mass}}|^2 + \alpha_1 H_\mu H^\mu \right), \tag{14.8}$$

with $\alpha_1 > 0$. Using the symmetrizer induced by GH gauge, one obtains

$$\frac{dE}{dt} \; \leq \; - \int_{\partial\Sigma_t} dS \, \left(\langle W^{\mathrm{diag}}_-, W^{\mathrm{diag}}_- \rangle + \langle W^{\mathrm{mass}}_-, W^{\mathrm{mass}}_- \rangle \right) \; + \; C \, E(t), \tag{14.9}$$

for some $C \geq 0$ depending on background and lower–order terms. With maximally dissipative BC ($\|\mathcal{R}\| \leq 1$) and $\kappa_{1,2} \geq 0$, Grönwall implies

$$\boxed{E(t) \; \leq \; e^{Ct} \Big(E(0) + \int_0^t \|\text{boundary data}\|^2 \, ds \Big)}, \tag{14.10}$$

which yields *well–posedness* (existence, uniqueness, and continuous dependence) for the linearized IBVP. Nonlinear well–posedness follows by standard continuation for small data on smooth backgrounds.

14.6 Notes for numerics

- **Discretization:** SBP finite differences or spectral elements with SAT penalty terms to impose the above BC.
- **Constraint handling:** enforce $(\mathcal{S}, \mathcal{S}') = 0$ algebraically at each step; Z4–damp H_μ; monitor $\mathcal{J}_{ab}$.
- **Massive sector:** m_T^2 enters only in $S(\mathbf{U})$; no CFL impact since char speeds remain $\{0, \pm 1\}$.

14.7 Boxed IBVP checklist

$$\boxed{\begin{array}{l}\text{(i) GH gauge for } g;\ \text{TEGR kinetic } \Rightarrow \text{GR principal part;}\\ \text{(ii) First–order variables } (h, P, Q) \text{ for diag and mass sectors;}\\ \text{(iii) Strong hyperbolicity: char speeds } \{0, \pm 1\};\\ \text{(iv) Max. dissipative CPBC on incoming modes; Z4 damping for } H_\mu;\\ \text{(v) HR potential algebraic } \Rightarrow \text{ no change to } A^k;\ \text{well–posed linear IBVP.}\end{array}} \tag{14.11}$$

15. Matter sector variants and equivalence principle (EP) tests

We state admissible matter couplings, the observables that test them, and the minimal conditions that keep the BD mode excised.

15.1 Admissible couplings

Minimal single-sheet coupling (recommended).

$$S_{\mathrm{m}}[g, \Psi] = \int d^4x\, \sqrt{-g}\, \mathcal{L}_{\mathrm{m}}(\Psi; g, \nabla^{(\{\}\Gamma[g])}), \tag{15.1}$$

with all Standard Model fields using $g_{\mu\nu}(e)$ and its Levi–Civita connection. In TEGR this equals the usual GR coupling because $R(^{\{\}}\Gamma) = -T + B$.

Spinors.

$$\mathcal{L}_{\mathrm{D}} = e\, \bar{\psi}\, i\gamma^a e_a{}^\mu \Big(\partial_\mu + \tfrac{1}{4}\omega_{bc\mu}\sigma^{bc}\Big)\psi - e\, m\bar{\psi}\psi, \quad R(\omega) = 0 \;\Rightarrow\; \text{GR-equivalent dynamics.} \tag{15.2}$$

Electromagnetism.

$$\mathcal{L}_{\mathrm{EM}} = -\tfrac{1}{4}\, e\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \qquad \nabla_\mu^{(\{\}\Gamma)} F^{\mu\nu} = J^\nu. \tag{15.3}$$

Excluded (ghost/EP risk). (i) Direct matter coupling to $f_{\mu\nu}$, (ii) composite couplings $\mathcal{L}_{\mathrm{m}}(\Psi; g, X)$, (iii) kinetic terms for the relative Lorentz field Λ or $\mathcal{B}_\mu$. Each of these spoils lapse linearity or adds new dof.

15.2 Weak/strong EP and geodesic motion

With single-sheet coupling, test bodies follow g-geodesics or the TEGR force form,

$$\ddot{x}^\lambda + {}^{\{\}}\Gamma[g]^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \quad \Longleftrightarrow \quad \ddot{x}^\lambda + \hat{\Gamma}^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = K^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{15.4}$$

WEP holds exactly; SEP holds up to massive-mode backreaction, which is screened in the Vainshtein region.

15.3 Post-Newtonian and screening diagnostics

For a static, weak-field source of mass M, define the Vainshtein radius

$$r_V \sim \left(\frac{GM}{m^2}\right)^{1/3}, \tag{15.5}$$

with m the HR mass scale (tensor mass on proportional backgrounds). Inside r_V, non-linearities suppress helicity-0; outside, a Yukawa correction appears.

PPN near an isolated source. For $r \ll r_V$,

$$\gamma - 1 \approx 0, \qquad \beta - 1 \approx 0, \tag{15.6}$$

matching GR to leading order. For $r \gg r_V$,

$$V(r) \simeq -\frac{G_N M}{r}\left(1 + \alpha\, e^{-m_T r}\right), \tag{15.7}$$

with α an order-one function of (β_n, M_g, M_f, c_0). Solar-system tests require $|\alpha e^{-m_T r_{\rm ss}}| \ll 1$.

15.4 Light propagation and GW sector

Photons. Null rays obey g-null geodesics. Shapiro delay and lensing equal GR on screened scales.

Gravitational waves. Luminal principal speed ($c_T^2 = 1$) from TEGR kinetics; dispersion $\omega^2 = k^2 + m_T^2$ from U. Bounds apply to m_T; parameter fits must respect current GW limits.

15.5 Laboratory and astrophysical EP tests

- **Eötvös/WEP.** Composition independence holds exactly with single-sheet coupling. Any portal to f would induce $\eta \neq 0$ at $\mathcal{O}(\alpha)$.
- **Redshift and clock tests.** Redshift follows g; TEGR $\Rightarrow$ GR prediction.
- **Binary pulsars.** Dipole radiation absent for visible matter on g; constraints translate into limits on (β_n, c_0) via m_T and mixing angle.
- **Cosmology.** On the proportional branch, background expansion follows two-fluid effective stress from U; growth and lensing kernels match those in metric bimetric with identical parameter map.

15.6 Dark-sector variants (use with caution)

If a hidden sector $\Psi_{\rm d}$ couples to f,

$$S_{\rm d}[f, \Psi_{\rm d}] = \int d^4x\, \sqrt{-f}\, \mathcal{L}_{\rm d}(\Psi_{\rm d}; f), \tag{15.8}$$

the visible sector remains EP-safe, but the total Hamiltonian acquires nonlinear lapse dependence unless the hidden sector is pressureless on the backgrounds considered and is integrated as an external fluid. For fully dynamical $\Psi_{\rm d}$ this typically reintroduces the BD mode. We exclude it by default.

15.7 Observational fit knobs

$$\boxed{\text{Free parameters: } (m_T,\ c_0,\ \beta_n/M^2,\ M_g,\ M_f).\ \text{Screening: } r_V(M) \sim (GM/m_T^2)^{1/3}.\ \text{Linear regime: Yukawa tail}} \tag{15.9}$$

15.8 EP-safe matter sector: boxed conditions

$$\boxed{\begin{array}{l} \text{(i) All visible matter minimally coupled to } g(e);\ \text{no } X \text{ or } \Lambda \text{ in } S_{\mathrm{m}}. \\ \text{(ii) No direct coupling to } f;\ \text{no composite effective metrics in } S_{\mathrm{m}}. \\ \text{(iii) Inertial spin connection } \omega \text{ with } R(\omega) = 0;\ \text{TEGR } \Rightarrow \text{GR matter kinematics.} \\ \text{(iv) Choose } (\beta_n, c_0) \text{ so } r_V \text{ covers solar-system scales and } m_T \text{ obeys GW bounds.} \end{array}} \tag{15.10}$$

16. Definitive ghost-free conditions on the interaction potential

We collect necessary and sufficient conditions on $U(g, f)$ for BD-ghost absence in teleparallel bimetric on the symmetric-vierbein branch, and give a proof sketch combining Hamiltonian rank, decoupling-limit, and quadratic-mode analyses.

16.1 Setting and objects

Two TEGR sectors with inertial spin connections $R(\omega) = 0 = R(\tilde{\omega})$, tetrads $(e, \tilde{e})$, and relative Lorentz field $\Lambda \in SO(1, 3)$. Define

$$X^{\mu}{}_{\nu} = (e^{-1}\Lambda\,\tilde{e})^{\mu}{}_{\nu}, \qquad U[g, f] = \sum_{n=0}^{4} \beta_n\, e_n(X), \tag{16.1}$$

with e_n the elementary symmetric polynomials. Impose the *symmetric-vierbein* condition

$$e^{a}{}_{[\mu}\, X_{\nu]}{}^{\mu} = 0 \quad \Longleftrightarrow \quad X^{\top} g X = f. \tag{16.2}$$

16.2 Theorem (teleparallel HR ghost-freedom)

Assumptions.

1. **Kinetic:** Each sheet uses TEGR (torsion scalar) kinetics with inertial spin connection; no $f(T)$ deformations.
2. **Potential:** $U = \sum_{n=0}^{4} \beta_n e_n(X)$ with $X = e^{-1}\Lambda\tilde{e}$ and (16.2) enforced algebraically; no derivatives of X or Λ.
3. **Lapse/shift structure:** After solving algebraic auxiliaries (boost/rotation) the Lagrangian is *linear* in the diagonal lapse and shift.
4. **Matter:** All visible matter minimally couples to $g(e)$ only; S_{m} contains no f, X, or $\partial\Lambda$.

Claim. Under 1–4 the teleparallel bimetric theory propagates exactly 2+5 tensor dof (massless+massive spin-2) and no BD scalar.

Proof sketch. (i) *Hamiltonian rank*). In 3+1, TEGR shares the GR lapse/shift linearity. With (16.2), U depends algebraically on spatial triads and lapses/shifts; its Hessian in $(N, N^i)_{\text{diag}}$ has rank 3, yielding one primary-null direction. Time preservation produces the secondary scalar $\mathcal{S}$. Together with its partner $\mathcal{S}'$ (from the special e_n structure) they are second-class ($\{\mathcal{S}, \mathcal{S}'\} \neq 0$), removing the BD scalar. (ii) *Stückelberg/decoupling*). The Λ_3 limit reproduces the Galileon tuned scalar sector with coefficients $c_k(\beta_n)$ identical to metric bimetric; equations remain second order and no extra mode appears. (iii) *Quadratic spectrum*). About proportional Minkowski/FLRW, TEGR and GR coincide up to a boundary term, so the quadratic mode matrices yield one massless and one massive tensor, stable for standard HR ranges. (iv) *IBVP*). Principal part equals GR in generalized-harmonic reduction; well-posedness with constraint-preserving boundary conditions excludes hidden propagating scalars.

16.3 Minimal exclusions (sharp form)

Any of the following invalidates the claim:

1. Break (16.2) or allow U to depend on antisymmetric vierbein parts.
2. Introduce ∂X, $\partial\Lambda$, or cross-kinetics such as $T[e]\cdot T[\tilde{e}]$ into U.
3. Couple matter to f or to a composite metric built from X.
4. Replace the TEGR kinetic with $f(T)$ or choose a non-inertial spin-connection gauge that breaks local Lorentz symmetry.

16.4 Equivalent working checklist

$$\boxed{\begin{array}{l} \text{Build } X = e^{-1}\Lambda\tilde{e},\;\; U = \displaystyle\sum_{n=0}^{4} \beta_n e_n(X);\; \text{solve boosts/rotations} \Rightarrow (16.2). \\ \text{Verify linearity in diagonal } (N, N^i);\; \text{rank}\,\mathcal{H} = 3;\; \exists\, \mathcal{S}, \mathcal{S}' :\; \det\{\mathcal{S}, \mathcal{S}'\} \neq 0. \\ \text{Check } \Lambda_3 \text{ map } c_k(\beta_n) \text{ Galileon};\; \text{quadratic modes } (2,5) \text{ healthy on background.} \\ \text{Matter: } S_{\mathrm{m}}[g(e), \Psi] \text{ only};\; R(\omega) = R(\tilde{\omega}) = 0. \end{array}} \tag{16.3}$$

16.5 Remarks on branches and orientations

The symmetry condition (16.2) admits disconnected branches distinguished by tetrad orientation and time gauge. Ghost-freedom holds on the *healthy* branch where X is symmetric and positive with respect to g. Branch changes that flip orientation can activate antisymmetric modes and break linearity; they are excluded.

16.6 Cosmology and parameters

On the proportional branch $f = c_0^2 g$, the tensor mass is

$$m_T^2 = m^2\big(\beta_1 c_0 + 2\beta_2 c_0^2 + \beta_3 c_0^3\big), \tag{16.4}$$

and stability requires $m_T^2 > 0$ (Minkowski) and $m_T^2 > 2H^2$ on quasi-de Sitter. These are constraints on $\{\beta_n, c_0\}$ independent of teleparallelization.

16.7 Boxed theorem (ready to cite)

$$\boxed{\text{\textbf{Teleparallel HR Ghost-Freedom.} Two TEGR sheets with inertial spin connections, interaction } U=\sum_{n=0}^{4}\beta_n e_n(X) \text{ built from } X=e^{-1}\Lambda\tilde{e} \text{ on the symmetric-vierbein branch, linear in the diagonal lapse/shift and with matter minimally coupled to } g(e)\text{, propagate exactly 2+5 tensor degrees of freedom. No BD scalar propagates. Any departure from symmetry, lapse-linearity, inertiality, or single-sheet matter coupling reintroduces the ghost.}} \tag{16.5}$$

17. Conclusion and outlook

We have isolated the kinematics of "rotation" in teleparallel bimetric gravity to the relative Lorentz field Λ and its Maurer–Cartan one–form $\mathcal{B}_\mu$, while showing that all gravitational holonomy in TEGR manifests as translational closure via torsion. On the symmetric–vierbein branch, the interaction $U = \sum_{n=0}^{4} \beta_n e_n(X)$ with $X = e^{-1}\Lambda\tilde{e}$ preserves lapse/shift linearity and yields the standard HR scalar pair that removes the BD mode. Autoparallel completion clarifies motion: with single–sheet matter coupling, physical trajectories are g–geodesics (or the equivalent TEGR force equation). The principal part of the field equations equals GR's, giving strong hyperbolicity and a well–posed linear IBVP with constraint–preserving boundary conditions. Quadratic spectra about Minkowski/FLRW match metric bimetric: one massless and one massive tensor, healthy vectors, and a single helicity–0 with Galileon decoupling at Λ_3 set by $c_k(\beta_n)$.

17.1 What is proved and what is required in practice

- **Kinetic sector:** Both sheets use TEGR with inertial spin connections ($R(\omega) = 0 = R(\tilde{\omega})$). No $f(T)$ deformations.
- **Potential sector:** HR/dRGT form in tetrads, $U = \sum \beta_n e_n(X)$, with the symmetric–vierbein condition solved algebraically so that $X^\top g X = f$.
- **Hamiltonian rank:** After eliminating auxiliaries, the Lagrangian is linear in the diagonal lapse/shift, the lapse–shift Hessian has rank 3, and the scalar pair $(\mathcal{S}, \mathcal{S}')$ is second–class with $\{\mathcal{S}, \mathcal{S}'\} \neq 0$.
- **Matter:** Minimal coupling to $g(e)$ only. No f or X in S_m. No derivatives of Λ in any sector.
- **Dynamics:** Strongly hyperbolic first–order reduction; maximally dissipative CPBC; GR–identical characteristic speeds $\{0, \pm 1\}$.

17.2 Open problems and extensions

1. **Non–proportional backgrounds.** Extend quadratic analysis and constraint closure to branches with $f \not\propto g$ while maintaining (16.2). Characterize the allowed time–dependent X with symmetric vierbeins.
2. **Cosmological perturbations.** Compute (K, G, M^2) for scalar–vector–tensor sectors on general FLRW with time–varying ratio $c(t)$; map observational windows for $(\beta_n, c, M_g/M_f)$ under screening.

3. **Boundary terms and topology.** Classify constant–coefficient parity–odd additions (Nieh–Yan, teleparallel Euler/Pontryagin) that remain boundary–exact on Cauchy slices and preserve the rank structure.
4. **Numerics.** Implement SBP–SAT discretizations with algebraic enforcement of $(\mathcal{S}, \mathcal{S}')$ and Z4–damped GH gauge; validate the rank test and Dirac–bracket projector on random phase–space samples.
5. **Matter portals.** Quantify how any dark–sector coupling to f can be approximated as an external fluid without spoiling lapse linearity; determine precise failure thresholds.

17.3 Minimal implementation recipe (one page)

1. Build $X = e^{-1}\Lambda\tilde{e}$ and enforce $e^a{}_{[\mu}X_{\nu]}{}^{\mu} = 0$ to solve boosts/rotations.
2. Verify linearity in diagonal (N, N^i) by evaluating the lapse/shift Hessian $\mathcal{H}$; accept only if $\operatorname{rank}\mathcal{H} = 3$.
3. Construct $(\mathcal{S}, \mathcal{S}')$ and check $\det\{\mathcal{S}, \mathcal{S}'\} \neq 0$ on representative data.
4. Linearize about the chosen background; read off mode matrices $(\mathcal{K}, \mathcal{G}, \mathcal{M})$ and ensure positivity of kinetic and gradient eigenvalues.
5. Reduce to first order in GH gauge; impose CPBC on incoming characteristics; confirm energy estimate closes.

17.4 Boxed synthesis

$$\boxed{\begin{aligned}&\textbf{Ghost-free teleparallel bimetric}: \ \big[\text{TEGR on both sheets}\big] + \big[U = \textstyle\sum_{n=0}^{4} \beta_n e_n(X),\ X = e^{-1}\Lambda\tilde{e},\ X^{\top} g X = f\big] \\ &\quad + \big[\text{linearity in diagonal lapse/shift}\big] + \big[\text{single-sheet matter}\big] \implies \text{propagating dof } = 2 \oplus 5,\ \text{no BD scalar.}\end{aligned}} \tag{17.1}$$

18. Boxed identities and implementation formulas

All at-a-glance, boxed for deployment.

$$\boxed{\hat{\Gamma}^{\lambda}{}_{\mu\nu} = e_a{}^{\mu}\lambda\, \partial_\nu e^a{}_{\mu}\mu} \tag{18.1}$$

$$\boxed{T^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}{}_{\nu\mu} - \hat{\Gamma}^{\lambda}{}_{\mu\nu}} \tag{18.2}$$

$$\boxed{K^{\lambda}{}_{\mu\nu} = \tfrac{1}{2}\Big(T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu} - T^{\lambda}{}_{\mu\nu}\Big)} \tag{18.3}$$

$$\boxed{{}^{\{\}}\Gamma^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}{}_{\mu\nu} - K^{\lambda}{}_{\mu\nu}} \tag{18.4}$$

$$\boxed{\ddot{x}^{\lambda} + \hat{\Gamma}^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = K^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \tag{18.5}$$

$$\boxed{\Delta x^{\mu} \simeq \tfrac{1}{2}\int_{\Sigma} T^{\mu}{}_{\alpha\beta}\, d\Sigma^{\alpha\beta}} \tag{18.6}$$

$$\boxed{\mathcal{B}_\mu = \Lambda^{-1}\partial_\mu\Lambda + \Lambda^{-1}\omega_\mu\Lambda - \tilde{\omega}_\mu} \tag{18.7}$$

$$\boxed{X^\mu{}_\nu = e_a{}^\mu \mu\, \Lambda^a{}_b\, \tilde{e}^b{}_\nu} \tag{18.8}$$

$$\boxed{X^\top g X = f \qquad \Longleftrightarrow \qquad e^a{}_{[\mu}\, X_{\nu]}{}^\mu = 0} \tag{18.9}$$

$$\boxed{U(g,f) = \sum_{n=0}^{4} \beta_n\, e_n(X)} \tag{18.10}$$

$$\boxed{\text{rank}\, \mathcal{H}_{(N,N^i)_{\text{diag}}} = 3 \quad \text{(lapse/shift Hessian)}} \tag{18.11}$$

$$\boxed{\mu(x) := \{\mathcal{S}(x), \mathcal{S}'(x)\} \neq 0 \quad \text{(BD pair second-class)}} \tag{18.12}$$

$$\boxed{c_2 \propto \beta_1 + 2\beta_2 + \beta_3,\;\; c_3 \propto \beta_2 + 2\beta_3 + \beta_4,\;\; c_4 \propto \beta_3 + 2\beta_4,\;\; c_5 \propto \beta_4} \tag{18.13}$$

$$\boxed{m_T^2 = m^2\big(\beta_1 c_0 + 2\beta_2 c_0^2 + \beta_3 c_0^3\big)} \tag{18.14}$$

$$\boxed{K^\lambda{}_{\mu\nu}\, u^\mu u^\nu = 0 \iff \text{autoparallels} = \text{geodesics}} \tag{18.15}$$

$$\boxed{E(t) \;\le\; e^{Ct}\left(E(0) + \int_0^t \|\text{boundary data}\|^2\, ds\right)} \tag{18.16}$$

$$\boxed{r_V \sim \left(\frac{GM}{m^2}\right)^{1/3}} \tag{18.17}$$

$$\boxed{c_T^2 = 1, \qquad \omega^2 = \frac{k^2}{a^2} + m_T^2} \tag{18.18}$$