Sheet Coordinates σ^p, Spacetime Coordinates x^μ, and the Holographic Master Field $\Phi(\sigma; x)$

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Abstract

This work develops a first-principles mathematical foundation for the holographic Master Field $\Phi(\sigma; x)$, which depends simultaneously on sheet coordinates σ^p and spacetime coordinates x^μ. We model the physical spacetime as a four-dimensional Lorentzian manifold $(M_4, g_{\mu\nu})$ and introduce a two-dimensional spectral sheet (Σ_2, γ_{pq}) that encodes scale and chiral vortex structure in an auxiliary, non-spacetime sector. The combined kinematic domain is the product manifold $\mathcal{M} = \Sigma_2 \times M_4$, on which $\Phi(\sigma; x)$ is defined as a scalar field valued in an internal BT8g carrier space.

Section 1 constructs the geometric and dimensional infrastructure for this framework. We fix index sets and signatures for M_4 and Σ_2, assign consistent mass dimensions to x^μ, σ^p, the integration measures, and all basic geometric objects, and define the tangent and cotangent bases on $\mathcal{M}$. We emphasize a strict separation between sheet and spacetime indices, treating $\mathrm{Diff}(M_4)$ and $\mathrm{Diff}(\Sigma_2)$ as independent symmetry factors. On this background we introduce the Master Field $\Phi(\sigma; x)$ with an explicit amplitude–phase–internal decomposition and a canonical mass dimension chosen so that natural holographic actions on $\Sigma_2 \times M_4$ are dimensionally consistent.

The result is a minimal but complete kinematic basis for subsequent sections: the teleparallel description of M_4, the intrinsic geometry of Σ_2, and the holographic dynamics by which effective four-dimensional fields emerge from $\Phi(\sigma; x)$ via sheet projections. All later constructions in the Chiral Vortex and BT8g program are constrained to respect the index structure, units, and dimensional assignments established here.

1 Geometric and Dimensional Setup

The purpose of this section is to define, from first principles, the mathematical environment in which the holographic Master Field $\Phi(\sigma; x)$ is defined. The central objects are

- a four-dimensional Lorentzian spacetime $(M_4, g_{\mu\nu})$ with coordinates x^μ,
- a two-dimensional "sheet" manifold (Σ_2, γ_{pq}) with coordinates σ^p,
- their product $\mathcal{M} = \Sigma_2 \times M_4$ carrying the field $\Phi : \mathcal{M} \to \mathcal{V}$, where $\mathcal{V}$ is an internal BT8g carrier space.

We work in natural units $\hbar = c = 1$ and adopt a single fundamental dimension denoted $[M]$. All remaining quantities are expressed as powers of $[M]$.

1.1 Manifolds and coordinate charts

Spacetime. Let M_4 be a smooth, four-dimensional manifold equipped with a Lorentzian metric $g_{\mu\nu}$ of signature $(-+++)$. Local coordinates are denoted

$$x^\mu = (x^0, x^1, x^2, x^3), \qquad \mu, \nu, \rho, \sigma = 0, 1, 2, 3.$$

In natural units we assign

$$[x^\mu] = [M]^{-1}, \qquad [g_{\mu\nu}] = 1, \tag{1.1}$$

so that the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is dimensionless and the Christoffel symbols carry dimension $[\Gamma^\rho{}_{\mu\nu}] = [M]$.

Spectral sheet. Let Σ_2 be a smooth, oriented, two-dimensional Riemannian manifold with local coordinates

$$\sigma^p = (\sigma^1, \sigma^2), \qquad p, q, r, s = 1, 2.$$

We equip Σ_2 with a positive-definite metric $\gamma_{pq}(\sigma)$. Following the spectral interpretation, the sheet coordinates are taken to have the dimension of an energy/momentum scale,

$$[\sigma^p] = [M], \qquad [\gamma_{pq}] = 1. \tag{1.2}$$

This choice reflects the role of Σ_2 as a "spectral sheet" in momentum-space-like variables rather than an additional spacetime.

The associated volume element on Σ_2 is

$$d^2\sigma = \sqrt{\gamma}\, d\sigma^1 d\sigma^2, \qquad \gamma = \det(\gamma_{pq}), \tag{1.3}$$

and the Levi–Civita tensor is defined by

$$\epsilon_{12} = -\epsilon_{21} = \sqrt{\gamma}, \qquad \epsilon^{12} = -\epsilon^{21} = \frac{1}{\sqrt{\gamma}}. \tag{1.4}$$

Product manifold. The total kinematic domain of the Master Field is the direct product

$$\mathcal{M} = \Sigma_2 \times M_4, \tag{1.5}$$

with global coordinates

$$(y^A) = (\sigma^p, x^\mu), \qquad A, B, \dots \in \{p, \mu\}.$$

At this stage we treat Σ_2 and M_4 as independent factors; the holographic coupling is encoded in the dynamics of $\Phi(\sigma; x)$ rather than in a nontrivial fibration of $\mathcal{M}$.

1.2 Index sets and signatures

We use the following index conventions throughout:

Index type	Symbols	Role
Spacetime (coordinate)	μ, ν, ρ, σ	$0, \dots, 3$ on M_4
Lorentz (tetrad)	a, b, c, d	$0, \dots, 3$ in $T_x M_4$
Sheet (coordinate)	p, q, r, s	$1, 2$ on Σ_2
Sheet (orthonormal)	$\hat{p}, \hat{q}$	frame on Σ_2
Internal BT8g	A, B	$1, \dots, 8$

Spacetime carries Lorentzian signature

$$g_{\mu\nu} \sim \mathrm{diag}(-1,1,1,1),$$

while the sheet metric is positive definite:

$$\gamma_{pq} \text{ Riemannian}, \qquad \gamma_{pq} v^p v^q > 0 \text{ for } v^p \neq 0.$$

The combined index structure on $\mathcal{M}$ is block-diagonal: Greek indices are never raised or lowered with γ_{pq}, and sheet indices are never contracted with $g_{\mu\nu}$. This strict separation is essential for keeping the dimensional analysis and tensor types consistent.

1.3 Natural units and dimensional assignments

The fundamental dimension is mass $[M]$. We adopt the conventions

$$[d^4x] = [M]^{-4}, \qquad [d^2\sigma] = [M]^2, \qquad [g_{\mu\nu}] = [\gamma_{pq}] = 1. \tag{1.6}$$

The total integration measure on $\mathcal{M}$ then has

$$[d^4x\, d^2\sigma] = [M]^{-2}. \tag{1.7}$$

For the geometric objects on M_4 we take

$$[\partial_\mu] = [M], \quad [\Gamma^\rho{}_{\mu\nu}] = [M], \quad [R^\rho{}_{\sigma\mu\nu}] = [M]^2, \tag{1.8}$$

and analogously on Σ_2,

$$[\partial_p] = [M]^{-1}, \quad [\Gamma^r{}_{pq}(\gamma)] = [M]^{-1}, \quad [\mathcal{R}_{pq}(\gamma)] = [M]^{-2}, \tag{1.9}$$

where $\mathcal{R}_{pq}(\gamma)$ denotes the Ricci tensor of γ_{pq}. These assignments reflect the choice $[\sigma^p] = [M]$, so derivatives with respect to σ^p carry inverse mass dimension and the sheet curvature has the expected $[M]^{-2}$ scaling.

1.4 Tangent and cotangent bases on $\mathcal{M}$

The tangent space at a point $(\sigma, x) \in \mathcal{M}$ splits as a direct sum

$$T_{(\sigma,x)}\mathcal{M} \simeq T_\sigma\Sigma_2 \oplus T_x M_4, \tag{1.10}$$

with coordinate bases

$$\left\{\frac{\partial}{\partial\sigma^p}\right\} \cup \left\{\frac{\partial}{\partial x^\mu}\right\}. \tag{1.11}$$

The dual cotangent bases are

$$\{d\sigma^p\} \cup \{dx^\mu\}, \tag{1.12}$$

satisfying

$$d\sigma^p\left(\frac{\partial}{\partial\sigma^q}\right) = \delta^p{}_q, \qquad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu, \qquad d\sigma^p\left(\frac{\partial}{\partial x^\mu}\right) = 0, \qquad dx^\mu\left(\frac{\partial}{\partial\sigma^p}\right) = 0.$$

On spacetime we introduce a tetrad $e^a{}_\mu(x)$ and its inverse $e_a{}^\mu(x)$ satisfying

$$e^a{}_\mu e_b{}^\mu = \delta^a{}_b, \qquad e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu, \tag{1.13}$$

with

$$g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu e^b{}_\nu, \qquad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{1.14}$$

In teleparallel gauge the spin connection can be chosen flat, and the torsion tensor is constructed from first derivatives of the tetrad. The tetrad is taken dimensionless, $[e^a{}_\mu] = 1$, consistent with $[g_{\mu\nu}] = 1$.

If desired, an orthonormal frame on Σ_2 can be defined by a zweibein $f^{\hat{p}}{}_q(\sigma)$ such that

$$\gamma_{pq} = \delta_{\hat{p}\hat{q}} f^{\hat{p}}{}_p f^{\hat{q}}{}_q, \tag{1.15}$$

but for the present kinematic analysis we keep γ_{pq} in coordinate form.

1.5 Product structure and diffeomorphisms

Kinematically, the symmetry group of coordinate redefinitions is the direct product

$$\mathrm{Diff}(\mathcal{M}) \supset \mathrm{Diff}(M_4) \times \mathrm{Diff}(\Sigma_2), \tag{1.16}$$

where

$$x^\mu \mapsto x'^\mu(x), \tag{1.17}$$

$$\sigma^p \mapsto \sigma'^p(\sigma), \tag{1.18}$$

act independently on spacetime and sheet coordinates. The field $\Phi(\sigma; x)$ is taken to be a scalar under both sets of coordinate transformations,

$$\Phi'(\sigma'; x') = \Phi(\sigma; x), \tag{1.19}$$

while its internal indices (BT8g labels) transform under a fixed representation of the internal algebra.

This factorized diffeomorphism structure is the geometric starting point for holography: all dynamical couplings between Σ_2 and M_4 will be encoded in the action and field equations of $\Phi(\sigma; x)$, not in a mixing of the coordinate transformation groups.

1.6 Domain and basic normalization of the Master Field

The holographic Master Field is defined as a map

$$\Phi : \Sigma_2 \times M_4 \longrightarrow \mathcal{V}, \tag{1.20}$$

where $\mathcal{V}$ is an 8-dimensional internal space carrying a real BT8g Clifford representation. In components,

$$\Phi(\sigma; x) = \rho(\sigma; x)\, e^{i\theta(\sigma;x)}\, \chi_A(\sigma; x), \qquad A = 1, \ldots, 8, \tag{1.21}$$

with $\chi_A \chi_A = 1$ (in an appropriate internal metric).

We assign the canonical mass dimension

$$[\Phi] = [M], \tag{1.22}$$

so that ρ has $[\rho] = [M]$ and θ is dimensionless. This is compatible with holographic actions on $\Sigma_2 \times M_4$, where Φ serves as the fundamental field from which effective gravitational and gauge sectors emerge.

The detailed kinetic and potential terms for $\Phi(\sigma; x)$ — together with the holographic projection to 4D effective fields — will be developed in later sections. The present section fixes the geometric, index, and dimensional infrastructure on which those constructions rest.