# Quantum Fields on a Bimetric Teleparallel Background:
## Foundations, Consistency, and Effective Dynamics

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**Abstract**

We construct and analyze a quantum field theory on a bimetric teleparallel background, with explicit separation into a visible $(+)$ sector and an auxiliary $(-)$ sector. Each sector is equipped with its own tetrad, Weitzenböck connection, torsion tensor, and torsion scalar, while the two metrics are coupled through a Hassan–Rosen–type bimetric potential built from the matrix square root $S = \sqrt{g^{(+)\,-1} g^{(-)}}$. The goal is twofold: (i) to show that free quantum fields (scalars, photons, phonons, magnons) can be consistently quantized on such a background using textbook oscillator machinery, and (ii) to verify systematically that the resulting framework is dimensionally, index-wise, and algebraically consistent as an *effective* quantum bimetric–teleparallel theory.

We start from first principles. On the geometric side, we formulate a pair of teleparallel sectors $(g^{(+)}_{\mu\nu}, T^{(+)})$ and $(g^{(-)}_{\mu\nu}, T^{(-)})$ on a single four-dimensional manifold, with tetrads $e^a{}_\mu(+)$ and $e^a{}_\mu(-)$, determinants $e^{(+)}, e^{(-)}$, and torsion scalars $T^{(+)}, T^{(-)}$ of canonical mass dimension two. We then introduce a bimetric teleparallel action

$$S_{\text{grav}} = \frac{M_+^2}{2}\int d^4x\, e^{(+)} T^{(+)} + \frac{M_-^2}{2}\int d^4x\, e^{(-)} T^{(-)} - \frac{M_+^2 m^2}{2}\int d^4x\, e^{(+)} V(S),$$

and verify that in natural units $\hbar = c = 1$ each term is dimensionally consistent, leading to a dimensionless action. We state the conditions under which the square-root tensor $S^\mu{}_\nu$ is well-defined and the bimetric interaction remains ghost-free in the spin-2 sector.

On the quantum side, we treat the tetrads and metrics as fixed classical backgrounds and construct quantum fields via mode decompositions on each teleparallel sheet. For scalar fields, electromagnetic fields, phonons, and magnons, we perform an ADM-like decomposition of the background metric in each sector, identify canonical conjugate pairs, and reduce the Hamiltonian to a sum of decoupled harmonic oscillators with frequencies $\omega_k(s)$ determined by the teleparallel geometry and material properties. Canonical commutation relations are imposed mode by mode, and we define creation and annihilation operators $a_k(s), a_k^\dagger(s)$ such that the total Hilbert space factorizes as a tensor product $\mathcal{F}(+) \otimes \mathcal{F}(-)$, with operators from different sectors commuting.

Throughout the paper we track: (i) mass/length dimensions of all fields, couplings, and measures; (ii) index structure, including consistent use of sector labels $(+)$ and $(-)$; (iii) variable definitions and operator domains; and (iv) generalized consistency conditions such as self-adjointness of the spatial operators, completeness of the mode basis, and boundedness from below of the free Hamiltonian. We explicitly separate what is rigorously established (free-field quantization on a fixed bimetric teleparallel background) from what remains open (renormalization, backreaction, causal stability, and the full quantum-corrected dynamics).

The result is a functioning effective framework in which: (1) the bimetric teleparallel geometry governs the spectral data $\{\omega_k(s)\}$ of quantum excitations in each sector; (2) the standard oscillator–CAP machinery (phonons, magnons, photons) lifts cleanly to the bimetric teleparallel setting; and (3) the total construction is consistent at the level of free quantum fields, with a clear roadmap for extending the analysis to interacting fields and semiclassical backreaction.

## Contents

# 1 Introduction and Overview

## 1.1 Objective and conceptual picture

The purpose of this work is to build, from first principles, an effective quantum field theory on a *bimetric teleparallel* spacetime. Concretely, we assume:

- A single smooth four-dimensional manifold $M$.
- Two teleparallel sectors, labelled $(+)$ and $(-)$, each with its own tetrad field $e^a{}_\mu(s)$, metric $g_{\mu\nu}(s)$, and torsion scalar $T(s)$.
- A bimetric interaction between the two metrics $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$ via a Hassan–Rosen potential $V(S)$ built from the matrix square root $S = \sqrt{g^{(+)\,-1}g^{(-)}}$.

- Quantum matter fields (scalars, photons, and condensed-matter excitations such as phonons and magnons) propagating on this background and quantized via the standard oscillator machinery.

Physically, the $(+)$ sector is identified with the visible universe: all Standard Model fields, laboratory materials, and their collective excitations live on $g_{\mu\nu}(+)$. The $(-)$ sector plays the role of an auxiliary or hidden sheet, which may carry its own matter content but is not directly probed by visible-sector detectors. The interaction between the two sheets is purely geometric at the level of the spin-2 sector through $V(S)$.

The central claim is that, under mild assumptions on the background fields, it is possible to port the *textbook* quantization machinery — built from sums of harmonic oscillators — onto this bimetric teleparallel geometry in a way that is:

(i) **Dimensionally consistent**: all actions are dimensionless in natural units, and field and coupling dimensions are compatible with standard 4D QFT;

(ii) **Index-consistent**: tetrad, tangent, and spacetime indices, along with sector labels $(+), (-)$, are used in a way that respects both local Lorentz and diffeomorphism invariance in each sector;

(iii) **Algebraically consistent**: canonical commutation relations can be imposed in each sector, leading to a well-defined Fock space $\mathcal{F}(s)$ and a factorized Hilbert space $\mathcal{F}(+) \otimes \mathcal{F}(-)$.

### 1.2 Target properties of the framework

We summarize the non-negotiable properties that the framework is required to satisfy:

- **Ghost freedom in the spin-2 sector**: the bimetric interaction between $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$ must be of the Hassan–Rosen type, avoiding the Boulware–Deser ghost at the classical level.
- **Equivalence principle per sector**: within each sector $(+)$ and $(-)$, matter couples minimally to the corresponding metric, so that local physics reduces to that of a freely falling frame.
- **Clean sector separation**: the visible sector couples only to $g_{\mu\nu}(+)$, while the hidden sector couples only to $g_{\mu\nu}(-)$; mixing occurs only via the bimetric gravitational potential and not via explicit non-geometric matter couplings.
- **Standard oscillator reduction**: classical field dynamics in each sector reduce to a set of normal modes satisfying oscillator-like equations with frequencies $\omega_k(s)$, such that the usual ladder-operator construction applies.

These requirements sharply constrain how geometry, matter, and quantization can be combined, and they guide the detailed checks in the later sections.

### 1.3 Unit conventions and dimensional checks

We adopt natural units throughout:

$$\hbar = c = 1.$$

Mass dimension becomes the primary dimension:

$$[x^\mu] = \text{mass}^{-1}, \qquad [\partial_\mu] = \text{mass}, \qquad [d^4x] = \text{mass}^{-4}.$$

For the gravitational sector:

- Tetrads $e^a{}_\mu(s)$ and metrics $g_{\mu\nu}(s)$ are taken dimensionless.
- The torsion tensor $T^\rho{}_{\mu\nu}(s)$ has dimension mass, and the torsion scalar $T(s)$ has dimension mass$^2$.
- The gravitational coupling scales $M_+$ and $M_-$ have dimension mass.
- The bimetric mass scale $m$ has dimension mass.

The gravitational action

$$S_{\text{grav}} = \frac{M_+^2}{2}\int d^4x\, e^{(+)}T^{(+)} + \frac{M_-^2}{2}\int d^4x\, e^{(-)}T^{(-)} - \frac{M_+^2 m^2}{2}\int d^4x\, e^{(+)}V(S)$$

then has integrands of dimension mass$^4$; multiplied by $d^4x$ of dimension mass$^{-4}$, the total action is dimensionless. This is compatible with the standard Einstein–Hilbert action and with the canonical normalization of quantum actions in these units.

For scalar fields in four dimensions, we adopt the canonical choice:

$$[\Phi] = \text{mass}^1, \qquad [\partial_\mu \Phi] = \text{mass}^2.$$

Then a scalar action of the form

$$S_\Phi(s) = \frac{1}{2}\int d^4x\, e(s)\left[Z(s)\, g^{\mu\nu}(s)\, \partial_\mu\Phi(s)\, \partial_\nu\Phi(s) - M_s^2\, \Phi(s)^2\right]$$

again has integrand of dimension mass$^4$ and is dimensionless after integration. The same logic applies to the Maxwell action, phonon actions, and magnon Hamiltonians constructed later.

These unit assignments provide the backbone for explicit dimensional checks in subsequent sections.

### 1.4 Index conventions and sector labels

We use the following conventions:

- Spacetime indices: Greek letters $\mu, \nu, \rho, \dots = 0, 1, 2, 3$.
- Tangent-space (Lorentz) indices: Latin letters $a, b, c, \dots = 0, 1, 2, 3$.
- Spatial indices on a hypersurface $\Sigma_t(s)$: Latin $i, j, k, \dots = 1, 2, 3$.
- Sector labels: explicit $(+)$ and $(-)$ superscripts, or a generic $s \in \{+, -\}$ when we want a unified expression.

Examples:

$$e^a{}_\mu(+), \quad g_{\mu\nu}(-), \quad T^\rho{}_{\mu\nu}(s), \quad e(s) = \det\!\big(e^a{}_\mu(s)\big).$$

We stress that:

- Indices are never contracted between different sectors: $g_{\mu\nu}(+)$ only raises and lowers indices of tensors in the $(+)$ sector, and similarly for $(-)$.
- The only legitimate "cross-sector" object at the spin-2 level is the square-root tensor $S^\mu{}_\nu$, built from $g^{(+)\,-1}$ and $g^{(-)}$ and appearing only algebraically in $V(S)$.

This disciplined index structure guarantees that the bimetric teleparallel action respects the appropriate gauge symmetries and that subsequent variation and quantization steps remain well-defined.

### 1.5 From single-sector TEGR to bimetric teleparallel QFT

The construction proceeds in three conceptual steps:

(a) **Single-sector TEGR:** we recall the teleparallel equivalent of general relativity (TEGR) in one sector: tetrads, torsion, torsion scalar, and the TEGR action. We verify equivalence to GR at the level of classical field equations and perform a full set of unit and index checks.

(b) **Bimetric teleparallel gravity:** we duplicate the TEGR construction for the $(+)$ and $(-)$ sectors and introduce a Hassan–Rosen potential between the two metrics, resulting in a bimetric teleparallel action $S_{\rm grav}$.

(c) **QFT on this background:** treating the tetrads and metrics as fixed classical fields, we quantize free matter fields sector by sector, using mode decompositions to map each field into an infinite tower of harmonic oscillators. Canonical commutation relations and Fock spaces are then defined in the usual way.

At each step, we explicitly record:

- The dimensional assignments of fields and couplings,
- The index structures and how they contract,
- The operators (differential and algebraic) that define dynamics,
- The assumptions required for self-adjointness, completeness, and positivity.

This cumulative audit leads to a final effective framework whose mathematical and physical assumptions are transparent and traceable.

**Summary of key equations for Section 1**

**Natural units and basic dimensions:**

$$\hbar = c = 1, \qquad [x^\mu] = \text{mass}^{-1}, \qquad [\partial_\mu] = \text{mass}, \qquad [d^4x] = \text{mass}^{-4}.$$

**Gravitational sector (bimetric teleparallel):**

$$[M_+] = [M_-] = [m] = \text{mass}, \qquad [T(s)] = \text{mass}^2, \qquad e(s) \text{ dimensionless.}$$

$$S_{\rm grav} = \frac{M_+^2}{2}\int d^4x\, e^{(+)}T^{(+)} + \frac{M_-^2}{2}\int d^4x\, e^{(-)}T^{(-)} - \frac{M_+^2 m^2}{2}\int d^4x\, e^{(+)}V(S),$$

with $V(S)$ dimensionless. Each integrand has dimension mass$^4$, so $S_{\rm grav}$ is dimensionless.
**Scalar field sector (per sheet):**

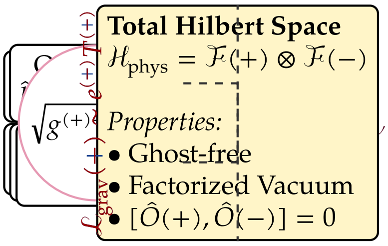
$$[\Phi(s)] = \text{mass}^1, \qquad [M_s] = \text{mass}, \qquad Z(s) \text{ dimensionless,}$$

$$S_\Phi(s) = \frac{1}{2}\int d^4x\, e(s)\,[Z(s)\, g^{\mu\nu}(s)\,\partial_\mu\Phi(s)\,\partial_\nu\Phi(s) - M_s^2\,\Phi(s)^2].$$

**Index and sector conventions:**

$$\mu,\nu,\rho = 0,1,2,3; \quad a,b,c = 0,1,2,3; \quad i,j,k = 1,2,3; \quad s \in \{+,-\}.$$

$$g_{\mu\nu}(s) = \eta_{ab}\, e^a{}_\mu(s)\, e^b{}_\nu(s), \qquad S^\mu{}_\nu = [\sqrt{g^{(+)\,-1}g^{(-)}}\,]^\mu{}_\nu.$$




## 2 Mathematical Preliminaries and Conventions

In this section we fix the geometric, algebraic, and dimensional conventions used throughout the paper. The goal is to make subsequent unit checks, index checks, and operator checks transparent and easily auditable.

### 2.1 Manifold, charts, and sector structure

**Spacetime manifold.** We work on a smooth, oriented, time-oriented four-dimensional manifold

$$M \equiv \text{spacetime}.$$

We assume:

- $M$ is Hausdorff and paracompact;
- $M$ admits a Lorentzian metric $g_{\mu\nu}(s)$ in each sector $s \in \{+, -\}$;
- each $(M, g(s))$ is globally hyperbolic, so there exists a foliation by Cauchy hypersurfaces $\Sigma_t(s)$.

**Charts and coordinates.** A local chart is a smooth map $\varphi : U \subset M \to \mathbb{R}^4$ with coordinates

$$x^\mu = (x^0, x^1, x^2, x^3),$$

where $x^0$ will later be adapted to a global time function $t$ on the background solution. Greek indices are coordinate indices on $M$.

**Sector duplication.** The bimetric structure is implemented by specifying:

- two Lorentzian metrics $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$ on $M$,
- two associated teleparallel structures $(e^a{}_\mu(+), T^\rho{}_{\mu\nu}(+))$ and $(e^a{}_\mu(-), T^\rho{}_{\mu\nu}(-))$,
- a single underlying manifold and coordinate system, so that comparisons such as $g^{(+)\,-1} g^{(-)}$ are pointwise well-defined.

All sector labels $(+)$ and $(-)$ are treated as discrete labels, never as tensor indices.

### 2.2 Tetrads, metrics, and Weitzenböck connections

**Tetrads and metric reconstruction.** For each sector $s \in \{+, -\}$:

- $e^a{}_\mu(s)$ is a tetrad field, with $a = 0, 1, 2, 3$ a Lorentz index and $\mu = 0, 1, 2, 3$ a spacetime index.
- The tetrad is assumed smooth and invertible on $M$:

$$\det(e^a{}_\mu(s)) \neq 0 \quad \text{everywhere on } M.$$

- The inverse tetrad $e_a{}^{\mu}(s)$ satisfies:

$$e^a{}_{\mu}(s)\, e_a{}^{\nu}(s) = \delta^{\nu}_{\mu}, \qquad e^a{}_{\mu}(s)\, e_b{}^{\mu}(s) = \delta^a{}_b.$$

- The metric in sector $s$ is reconstructed as

$$g_{\mu\nu}(s) = \eta_{ab}\, e^a{}_{\mu}(s)\, e^b{}_{\nu}(s),$$

  where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

**Determinants and volume elements.** We define

$$e(s) \equiv \det(e^a{}_{\mu}(s)), \qquad \sqrt{-g(s)} = |e(s)|,$$

and we use $e(s)\, d^4x$ as the canonical volume element in sector $s$. Both $e(s)$ and $g_{\mu\nu}(s)$ are taken dimensionless.

**Weitzenböck connection and torsion.** In teleparallel geometry, the Weitzenböck connection for sector $s$ is defined by

$$\Gamma^{\rho}{}_{\mu\nu}(s) \equiv e_a{}^{\rho}(s)\, \partial_{\mu} e^a{}_{\nu}(s).$$

This connection obeys:

- Metric compatibility: $\nabla^{(s)}_{\mu} g_{\nu\rho}(s) = 0$.
- Zero curvature: $R^{\rho}{}_{\sigma\mu\nu}(s) = 0$.
- Generally nonzero torsion:

$$T^{\rho}{}_{\mu\nu}(s) = \Gamma^{\rho}{}_{\mu\nu}(s) - \Gamma^{\rho}{}_{\nu\mu}(s) = e_a{}^{\rho}(s)(\partial_{\mu} e^a{}_{\nu}(s) - \partial_{\nu} e^a{}_{\mu}(s)).$$

**Torsion scalar.** The torsion scalar in sector $s$ is built as

$$T(s) = \frac{1}{4} T^{\rho\mu\nu}(s) T_{\rho\mu\nu}(s) + \frac{1}{2} T^{\rho\mu\nu}(s) T_{\nu\mu\rho}(s) - T^{\rho}{}_{\rho\mu}(s) T^{\nu}{}_{\nu}{}^{\mu}(s).$$

Index raising and lowering here use $g_{\mu\nu}(s)$ only.

**Dimensional check.** With $[x^{\mu}] = \text{mass}^{-1}$ and $e^a{}_{\mu}(s)$ dimensionless:

$$[\partial_{\mu} e^a{}_{\nu}] = \text{mass}, \qquad [\Gamma^{\rho}{}_{\mu\nu}] = \text{mass},$$

so

$$[T^{\rho}{}_{\mu\nu}(s)] = \text{mass}, \qquad [T(s)] = \text{mass}^2.$$

### 2.3 Index conventions, raising and lowering

**Spacetime and tangent indices.** We adopt:

- Greek indices $\mu, \nu, \rho, \sigma$ for spacetime coordinates on $M$;
- Latin indices $a, b, c, d$ for Lorentz (tangent-space) indices;
- Latin indices $i, j, k, \ell$ for spatial indices on a hypersurface $\Sigma_t(s)$.

Raising and lowering of indices uses:

$$g_{\mu\nu}(s), \quad g^{\mu\nu}(s), \quad \eta_{ab}, \quad \eta^{ab},$$

depending on whether the index belongs to the spacetime or tangent space. Indices are *never* raised or lowered across sectors.

**Sector labels.** The superscripts $(+)$ and $(-)$ label the two sectors and are not tensor indices. They are always written explicitly:

$$g_{\mu\nu}(+), \quad g_{\mu\nu}(-), \quad e^a{}_\mu(+), \quad e^a{}_\mu(-).$$

All contractions are performed within a single sector, except for the specific algebraic combination $S^\mu{}_\nu = [\sqrt{g^{(+)\,-1}g^{(-)}}]^\mu{}_\nu$ that defines the Hassan–Rosen potential.

**Operator indices.** Differential operators carry explicit indices when acting on components:

$$\nabla^{(s)}_\mu V^\nu(s) = \partial_\mu V^\nu(s) + \Gamma^\nu{}_{\mu\rho}(s) V^\rho(s).$$

We use the same symbol $\nabla^{(s)}_\mu$ for both Levi–Civita and Weitzenböck covariant derivatives when the distinction is not crucial, but keep in mind that in teleparallel gravity the Levi–Civita connection is replaced by the Weitzenböck connection plus contortion.

### 2.4 Natural units and mass dimensions

**Units.** We set

$$\hbar = c = 1,$$

so time and length share the same dimension and mass is the only independent dimension. We write $[X]$ for the mass dimension of a quantity $X$.

**Geometric quantities.**

| Quantity | | Mass dimension |
|---|---|---|
| Coordinates $x^\mu$ | : | $[x^\mu] = \text{mass}^{-1}$ |
| Derivative $\partial_\mu$ | : | $[\partial_\mu] = \text{mass}$ |
| Tetrad $e^a{}_\mu(s)$ | : | $[e^a{}_\mu] = \text{mass}^0$ |
| Metric $g_{\mu\nu}(s)$ | : | $[g_{\mu\nu}] = \text{mass}^0$ |
| Connection $\Gamma^\rho{}_{\mu\nu}(s)$ | : | $\text{mass}^1$ |
| Torsion $T^\rho{}_{\mu\nu}(s)$ | : | $\text{mass}^1$ |
| Torsion scalar $T(s)$ | : | $\text{mass}^2$ |
| Volume element $d^4x$ | : | $\text{mass}^{-4}$ |
| Determinant $e(s)$ | : | $\text{mass}^0$ |

**Gravitational couplings.**

| Parameter | | Mass dimension |
|---|---|---|
| $M_+, M_-$ | : | $\text{mass}^1$ |
| $m$ | : | $\text{mass}^1$ |
| $V(S)$ | : | $\text{mass}^0$ |

Then the gravitational action

$$S_{\text{grav}} = \frac{M_+^2}{2}\int d^4x\, e^{(+)} T^{(+)} + \frac{M_-^2}{2}\int d^4x\, e^{(-)} T^{(-)} - \frac{M_+^2 m^2}{2}\int d^4x\, e^{(+)} V(S)$$

has $[M_+^2] = [M_-^2] = [m^2] = \text{mass}^2$ and $[T(s)] = \text{mass}^2$, so each integrand has dimension $\text{mass}^4$, and $[d^4x] = \text{mass}^{-4}$, implying $[S_{\text{grav}}] = \text{mass}^0$.

**Matter fields.** We adopt canonical 4D assignments:

| | | |
|---|---|---|
| Scalar field $\Phi$ | : | $[\Phi] = \text{mass}^1$ |
| Vector potential $A_\mu$ | : | $[A_\mu] = \text{mass}^1$ |
| Dirac spinor $\psi$ | : | $[\psi] = \text{mass}^{3/2}$ |
| Elastic displacement $u_i$ | : | $[u_i] = \text{mass}^0$ |
| Mass density $\rho$ | : | $[\rho] = \text{mass}^4$ |
| Elastic tensor $C^{ijkl}$ | : | $[C^{ijkl}] = \text{mass}^2$ |

With these choices:

- Scalar action

$$S_\Phi(s) = \frac{1}{2} \int d^4x\, e(s) \left[ Z(s)\, g^{\mu\nu}(s)\, \partial_\mu \Phi(s)\, \partial_\nu \Phi(s) - M_s^2\, \Phi(s)^2 \right]$$

  has integrand dimension mass$^4$.

- Maxwell action

$$S_{\rm EM}(s) = -\frac{1}{4} \int d^4x\, e(s)\, F_{\mu\nu}(s) F^{\mu\nu}(s)$$

  is dimensionless with $[F_{\mu\nu}] = \text{mass}^2$.

- Phonon actions constructed from $\rho$ and $C^{ijkl}$ likewise have integrand dimension mass$^4$.

### 2.5 Distributions, delta functions, and orthonormality

**Delta distributions.** On a spatial hypersurface $\Sigma_t(s)$ with coordinates $x^i$, the three-dimensional delta function satisfies

$$\int_{\Sigma_t(s)} d^3x\, \delta^3(x - x') f(x) = f(x').$$

Its mass dimension is

$$[\delta^3(x - x')] = \text{mass}^3,$$

consistent with $[d^3x] = \text{mass}^{-3}$.

In the canonical equal-time brackets:

$$\{\Phi(s;t,x), \Pi(s;t,x')\} = \delta^3(x - x'),$$

the left-hand side has dimension $[\Phi\,\Pi]$. With $[\Phi] = \text{mass}^1$ and $[\Pi]$ to be determined from the Lagrangian, we can verify the consistency of this assignment once canonical momenta are defined.

**Mode orthonormality.** We will work with mode functions $\phi_k(s;x)$ on $\Sigma_t(s)$ satisfying:

$$\int_{\Sigma_t(s)} d^3x\, \sqrt{\gamma(s)}\, Z(s;x)\, \phi_k(s;x)\, \phi_{k'}(s;x) = \delta_{kk'}.$$

This ensures that mode amplitudes $Q_k(s;t)$ inherit the correct canonical structure. The measure

$$d\mu_s(x) \equiv \sqrt{\gamma(s)}\, Z(s;x)\, d^3x$$

is dimensionless if $Z(s)$ is dimensionless.

### 2.6 Functional spaces and self-adjointness assumptions

**Hilbert spaces on spatial slices.** For each sector, define the Hilbert space

$$\mathcal{H}_\Sigma^{(s)} \equiv L^2(\Sigma_t(s), d\mu_s(x)) = \left\{ f : \Sigma_t(s) \to \mathbb{C} \;\middle|\; \int_{\Sigma_t(s)} |f(x)|^2 \, d\mu_s(x) < \infty \right\}.$$

The scalar product is

$$\langle f, g \rangle_s = \int_{\Sigma_t(s)} d^3x \, \sqrt{\gamma(s)} \, Z(s;x) \, f(x)^* g(x).$$

**Spatial operators.** For scalar fields, the relevant spatial operator in sector $s$ is

$$\mathcal{L}_{\text{spatial}}(s) f = -\frac{1}{\sqrt{\gamma(s)}} \partial_i \Big( \sqrt{\gamma(s)} \, Z(s;x) \, \gamma^{ij}(s) \, \partial_j f \Big) + M_s^2(x) f.$$

We assume:

- $\gamma_{ij}(s)$ is smooth and positive-definite on each $\Sigma_t(s)$;
- $Z(s;x)$ is smooth and strictly positive;
- $M_s^2(x)$ is smooth and bounded below.

Under these conditions, $\mathcal{L}_{\text{spatial}}(s)$ is a second-order elliptic operator. On compact $\Sigma_t(s)$ with appropriate boundary conditions, it admits a discrete spectrum and a complete orthonormal basis of eigenfunctions $\phi_k(s;x)$.

**Self-adjointness.** We assume that $\mathcal{L}_{\text{spatial}}(s)$ is essentially self-adjoint on a dense domain, e.g. $C_0^\infty(\Sigma_t(s))$, when considered as an operator on $\mathcal{H}_\Sigma^{(s)}$. This ensures:

- Real eigenvalues $\omega_k(s)^2 \geq 0$;
- Orthogonality and completeness of eigenfunctions;
- Well-defined spectral decomposition of fields into normal modes.

**Time evolution and Fock spaces.** Once mode decomposition is performed, each sector's free-field Hamiltonian reduces to a sum of decoupled harmonic oscillators with frequencies $\omega_k(s)$. Canonical quantization then gives Fock spaces $\mathcal{F}(s)$, and the total Hilbert space is the tensor product $\mathcal{F}(+) \otimes \mathcal{F}(-)$.

**Summary of key conventions and checks for Section 2**

**Spacetime and sectors:**

$$M \text{ smooth, oriented, time-oriented,} \qquad s \in \{+,-\}.$$

$$g_{\mu\nu}(s) = \eta_{ab} e^a{}_\mu(s)\, e^b{}_\nu(s), \qquad e(s) = \det\big(e^a{}_\mu(s)\big).$$

**Teleparallel connection and torsion:**

$$\Gamma^\rho{}_{\mu\nu}(s) = e_a{}^\rho(s)\, \partial_\mu e^a{}_\nu(s),$$

$$T^\rho{}_{\mu\nu}(s) = \Gamma^\rho{}_{\mu\nu}(s) - \Gamma^\rho{}_{\nu\mu}(s),$$

$$T(s) = \tfrac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \tfrac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T^\rho{}_{\rho\mu} T^\nu{}_\nu{}^\mu.$$

Dimensional check: $[T^\rho{}_{\mu\nu}] = \text{mass}^1$, $[T(s)] = \text{mass}^2$.
**Mass dimensions (natural units):**

$$[x^\mu] = \text{mass}^{-1}, \quad [\partial_\mu] = \text{mass}, \quad [d^4x] = \text{mass}^{-4}, \quad [e(s)] = \text{mass}^0.$$

$$[M_+] = [M_-] = [m] = \text{mass}^1, \quad [\Phi] = \text{mass}^1, \quad [A_\mu] = \text{mass}^1, \quad [\psi] = \text{mass}^{3/2}.$$

**Gravitational action:**

$$S_{\text{grav}} = \frac{M_+^2}{2} \int d^4x\, e^{(+)} T^{(+)} + \frac{M_-^2}{2} \int d^4x\, e^{(-)} T^{(-)} - \frac{M_+^2 m^2}{2} \int d^4x\, e^{(+)} V(S),$$

with $[V(S)] = \text{mass}^0$ and $[S_{\text{grav}}] = \text{mass}^0$.
**Scalar-field spatial operator and Hilbert space:**

$$\mathcal{L}_{\text{spatial}}(s) f = -\frac{1}{\sqrt{\gamma(s)}} \partial_i\big(\sqrt{\gamma(s)}\, Z(s)\, \gamma^{ij}(s)\, \partial_j f\big) + M_s^2(x) f,$$

acting on

$$\mathcal{H}_\Sigma^{(s)} = L^2\big(\Sigma_t(s), \sqrt{\gamma(s)}\, Z(s)\, d^3x\big),$$

assumed essentially self-adjoint with real, non-negative spectrum $\omega_k(s)^2$.

## 3 Single-Sector Teleparallel Geometry (TEGR) from First Principles

In this section we focus on a *single* teleparallel sector and reconstruct the standard TEGR structure from first principles. The goal is to make explicit how the torsion scalar $T$ replaces the Ricci scalar $R$ of the Levi–Civita connection, and to show that the corresponding action is dynamically equivalent to the Einstein–Hilbert action, up to a boundary term. All unit, index, and operator assignments are kept explicit to prepare for the bimetric uplift.

For definiteness, we work with a generic sector label $s \in \{+,-\}$, but one may keep in mind the visible $(+)$ sheet.

### 3.1 Levi–Civita connection and curvature

Given a metric $g_{\mu\nu}(s)$ with signature $(-,+,+,+)$, there exists a unique torsion-free, metric-compatible connection: the Levi–Civita connection. Its Christoffel symbols are

$$\Gamma^\rho{}_{\mu\nu}(s) = \frac{1}{2} g^{\rho\sigma}(s)\big(\partial_\mu g_{\sigma\nu}(s) + \partial_\nu g_{\sigma\mu}(s) - \partial_\sigma g_{\mu\nu}(s)\big).$$

This connection satisfies

$$T^\rho{}_{\mu\nu}(s) = 0, \qquad \nabla_\mu g_{\nu\rho}(s) = 0.$$

The Riemann tensor of this connection is

$$R^\rho{}_{\sigma\mu\nu}(s) = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma},$$

and the Ricci tensor and scalar are

$$R_{\sigma\nu}(s) = R^\rho{}_{\sigma\rho\nu}(s), \qquad R(s) = g^{\sigma\nu}(s)\ R_{\sigma\nu}(s).$$

Dimensional check: $[\Gamma^\rho{}_{\mu\nu}] = \text{mass}$ (one derivative of a dimensionless metric), so $[R^\rho{}_{\sigma\mu\nu}] = \text{mass}^2$ and $[R] = \text{mass}^2$, as expected.

### 3.2 Weitzenböck connection, torsion, and contortion

In teleparallel geometry, we use the tetrad $e^a{}_\mu(s)$ to define the Weitzenböck connection

$$\Gamma^\rho{}_{\mu\nu}(s) = e_a{}^\rho(s)\, \partial_\mu e^a{}_\nu(s),$$

with torsion tensor

$$T^\rho{}_{\mu\nu}(s) = \Gamma^\rho{}_{\mu\nu}(s) - \Gamma^\rho{}_{\nu\mu}(s) = e_a{}^\rho(s)(\partial_\mu e^a{}_\nu(s) - \partial_\nu e^a{}_\mu(s)).$$

This connection is metric-compatible, but curvature-free:

$$\nabla^{(s)}_\mu g_{\nu\rho}(s) = 0, \qquad R^\rho{}_{\sigma\mu\nu}(s) \equiv 0,$$

where $\nabla^{(s)}$ here is defined with the Weitzenböck connection. The nontrivial gravitational information is contained entirely in torsion.

The Weitzenböck and Levi–Civita connections are related by the contortion tensor $K^\rho{}_{\mu\nu}(s)$:

$$\Gamma^\rho{}_{\mu\nu}(s) = \Gamma^\rho{}_{\mu\nu}(s) + K^\rho{}_{\mu\nu}(s).$$

The contortion is expressed in terms of torsion as

$$K^\rho{}_{\mu\nu}(s) = \frac{1}{2}\Big(T_\mu{}^\rho{}_\nu(s) + T_\nu{}^\rho{}_\mu(s) - T^\rho{}_{\mu\nu}(s)\Big).$$

Dimensional check:

$$[T^\rho{}_{\mu\nu}] = \text{mass}^1, \qquad [K^\rho{}_{\mu\nu}] = \text{mass}^1, \qquad [\Gamma^\rho{}_{\mu\nu}] = \text{mass}^1.$$

All index positions are consistent: torsion and contortion carry one upper and two lower spacetime indices; metric contraction and raising/lowering are sector-wise.

### 3.3 Torsion scalar and its relation to the Ricci scalar

The TEGR torsion scalar in sector $s$ is defined by

$$T(s) = \frac{1}{4}T^{\rho\mu\nu}(s)T_{\rho\mu\nu}(s) + \frac{1}{2}T^{\rho\mu\nu}(s)T_{\nu\mu\rho}(s) - T^\rho{}_{\rho\mu}(s)T^\nu{}_\nu{}^\mu(s).$$

We also introduce the torsion vector

$$T_\mu(s) \equiv T^\nu{}_{\nu\mu}(s),$$

and the *superpotential* (or conjugate) tensor

$$S_\rho{}^{\mu\nu}(s) \equiv \frac{1}{2}\Big(K^{\mu\nu}{}_\rho(s) + \delta^\mu_\rho T^{\alpha\nu}{}_\alpha(s) - \delta^\nu_\rho T^{\alpha\mu}{}_\alpha(s)\Big).$$

In terms of $S_\rho{}^{\mu\nu}(s)$ and $T^\rho{}_{\mu\nu}(s)$, one may write

$$T(s) = S_\rho{}^{\mu\nu}(s)\, T^\rho{}_{\mu\nu}(s).$$

The key identity relating the Levi–Civita curvature scalar and the torsion scalar is

$$R(s) = -T(s) + 2\nabla^{(,s)}_\mu T^\mu(s),$$

where $\nabla^{(,s)}$ is the Levi–Civita covariant derivative built from $\Gamma^\rho{}_{\mu\nu}(s)$, and $T^\mu(s) = g^{\mu\nu}(s)T_\nu(s)$. The second term is a total divergence:

$$2\nabla^{(,s)}_\mu T^\mu(s) = \frac{2}{e(s)}\,\partial_\mu(e(s)\,T^\mu(s)).$$

Thus, the Einstein–Hilbert Lagrangian density

$$\mathcal{L}_{\rm EH}(s) = \frac{M_+^2}{2} e(s)\; R(s)$$

and the teleparallel Lagrangian density

$$\mathcal{L}_{\rm TEGR}(s) = -\frac{M_+^2}{2} e(s)\, T(s)$$

differ by a total divergence:

$$\mathcal{L}_{\rm EH}(s) = \mathcal{L}_{\rm TEGR}(s) + M_+^2\,\partial_\mu(e(s)\,T^\mu(s)).$$

Consequently, the corresponding actions differ only by a boundary term and are classically equivalent when appropriate boundary conditions are imposed.

Dimensional check: $[R] = \text{mass}^2$, $[T] = \text{mass}^2$, so both Lagrangian densities have dimension

$$[\mathcal{L}] = [M_+^2] + [T] = \text{mass}^2 + \text{mass}^2 = \text{mass}^4,$$

consistent with $[d^4x] = \text{mass}^{-4}$ and a dimensionless action.

### 3.4 TEGR action and field equations

The TEGR action in sector $s$ is

$$S_{\rm TEGR}(s) = -\frac{M_+^2}{2}\int d^4x\, e(s)\, T(s) + S_{\rm matter}(s),$$

where $S_{\rm matter}(s)$ denotes the matter action, constructed by coupling fields minimally to $g_{\mu\nu}(s)$ and $e^a{}_\mu(s)$.

Varying this action with respect to the tetrad $e^a{}_\mu(s)$ yields the TEGR field equations. A standard form of these equations is

$$\partial_\rho(e(s)S_a{}^{\rho\mu}(s)) - e(s)\,T^b{}_{\rho a}(s)\,S_b{}^{\mu\rho}(s) + \frac{1}{4}e(s)\,e_a{}^\mu(s)\,T(s) = \frac{1}{2}e(s)\,\Theta_a{}^\mu(s),$$

where $\Theta_a{}^\mu(s)$ is the energy-momentum tensor in mixed (tetrad/spacetime) components, obtained from the variation of $S_{\rm matter}(s)$ with respect to $e^a{}_\mu(s)$:

$$\Theta_a{}^\mu(s) = \frac{1}{e(s)}\frac{\delta S_{\rm matter}(s)}{\delta e^a{}_\mu(s)}.$$

By contracting with $e^a{}_\nu(s)$ and using the relation between $T(s)$ and $R(s)$, one can show that these equations are equivalent to the Einstein field equations:

$$G_{\mu\nu}(s) \equiv R_{\mu\nu}(s) - \frac{1}{2} g_{\mu\nu}(s)\ R(s) = \frac{1}{M_+^2}\, T_{\mu\nu}(s),$$

where $T_{\mu\nu}(s)$ is the symmetric spacetime energy-momentum tensor obtained from $\Theta_a{}^\mu(s)$.

Dimensional and index checks:

- $[S_a{}^{\rho\mu}] = \text{mass}^1$ (built from torsion and contortion),
- $\partial_\rho(eS_a{}^{\rho\mu})$ has dimension mass$^2$,
- $eT^b{}_{\rho a}S_b{}^{\mu\rho}$ has dimension mass$^2$,
- $ee_a{}^\mu T$ has dimension mass$^2$,
- $e\Theta_a{}^\mu$ has dimension mass$^2$.

All terms in the TEGR field equations have the same mass dimension and compatible index structure (one Lorentz index $a$ and one spacetime index $\mu$ on each side).

### 3.5 Consistency checks within a single sector

Within a single teleparallel sector, we summarize the consistency properties:

- **Units:** Tetrads and metrics are dimensionless; torsion and contortion have dimension mass$^1$; the torsion scalar and Ricci scalar have dimension mass$^2$; the TEGR action is dimensionless.
- **Indices:** All contractions are carried out using $g_{\mu\nu}(s)$ or $\eta_{ab}$ within the same sector. The tetrad maps between tangent and spacetime indices, and there is no cross-sector index mixing at this stage.
- **Operators:** The Weitzenböck connection is curvature-free but torsionful; the Levi–Civita connection is torsion-free but curvatureful. They are related by a well-defined contortion tensor.
- **Dynamics:** The TEGR field equations are equivalent to the Einstein equations for the same metric $g_{\mu\nu}(s)$, so classical gravitational dynamics in each sector reproduces general relativity.

This validates TEGR as a structurally and dimensionally sound gravitational theory for each sector, ready to be doubled and coupled in the bimetric teleparallel framework.

**Summary of key TEGR relations and checks (single sector)**

**Levi–Civita vs. Weitzenböck connections:**

$$\Gamma^{\rho}{}_{\mu\nu}(s) = \frac{1}{2} g^{\rho\sigma}(s) (\partial_\mu g_{\sigma\nu}(s) + \partial_\nu g_{\sigma\mu}(s) - \partial_\sigma g_{\mu\nu}(s)),$$

$$\Gamma^{\rho}{}_{\mu\nu}(s) = e_a{}^{\rho}(s)\, \partial_\mu e^a{}_\nu(s) = \Gamma^{\rho}{}_{\mu\nu}(s) + K^{\rho}{}_{\mu\nu}(s),$$

$$K^{\rho}{}_{\mu\nu}(s) = \tfrac{1}{2}(T_\mu{}^{\rho}{}_\nu + T_\nu{}^{\rho}{}_\mu - T^{\rho}{}_{\mu\nu})(s).$$

**Torsion tensor and scalar:**

$$T^{\rho}{}_{\mu\nu}(s) = \Gamma^{\rho}{}_{\mu\nu}(s) - \Gamma^{\rho}{}_{\nu\mu}(s),$$

$$T(s) = \tfrac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \tfrac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T^{\rho}{}_{\rho\mu} T^{\nu}{}_{\nu}{}^{\mu} \quad (s),$$

$$T_\mu(s) \equiv T^{\nu}{}_{\nu\mu}(s), \qquad T(s) = S_\rho{}^{\mu\nu}(s)\, T^{\rho}{}_{\mu\nu}(s).$$

**Relation to Ricci scalar:**

$$R(s) = -T(s) + \frac{2}{e(s)} \partial_\mu (e(s)\, T^\mu(s)).$$

**TEGR vs. Einstein–Hilbert Lagrangians:**

$$\mathcal{L}_{\rm EH}(s) = \frac{M_+^2}{2} e(s)\ R(s),$$

$$\mathcal{L}_{\rm TEGR}(s) = -\frac{M_+^2}{2} e(s)\, T(s),$$

$$\mathcal{L}_{\rm EH}(s) = \mathcal{L}_{\rm TEGR}(s) + M_+^2\, \partial_\mu (e(s)\, T^\mu(s)).$$

Both actions are dimensionless in natural units.

**TEGR field equations (schematic form):**

$$\partial_\rho (e S_a{}^{\rho\mu}) - e T^{b}{}_{\rho a} S_b{}^{\mu\rho} + \tfrac{1}{4} e e_a{}^{\mu} T = \tfrac{1}{2} e \Theta_a{}^{\mu} \quad (s),$$

equivalent to

$$G_{\mu\nu}(s) = \frac{1}{M_+^2}\, T_{\mu\nu}(s),$$

with all terms carrying consistent mass dimension mass$^2$ and indices within the same sector.

## 4 Bimetric Teleparallel Gravity

We now move from a single teleparallel sector to a bimetric teleparallel framework with explicit $(+)$ and $(-)$ sheets. The construction is designed so that:

- Each sector carries its own TEGR-like teleparallel geometry and Planck scale;
- The two sectors interact only through a *ghost-free* Hassan–Rosen bimetric potential $V(S)$ built from the matrix square root $S = \sqrt{g^{(+)\,-1} g^{(-)}}$;
- All index contractions remain sector-local, except in the strictly algebraic $S$-block;

- The resulting gravitational action is dimensionally consistent and suitable as a fixed background for quantum fields.

The $(+)$ sector is interpreted as the visible universe; the $(-)$ sector is an auxiliary or hidden sheet. The uplift from a single TEGR sector to this bimetric structure is therefore both mathematically controlled and physically motivated.

### 4.1 Two teleparallel sectors on a common manifold

We now impose the following structure on the same spacetime manifold $M$:

- Two Lorentzian metrics:
$$g_{\mu\nu}(+), \qquad g_{\mu\nu}(-),$$
each of signature $(-,+,+,+)$.
- Two tetrad fields:
$$e^a{}_\mu(+), \qquad e^a{}_\mu(-),$$
with inverses $e_a{}^\mu(+)$ and $e_a{}^\mu(-)$, such that
$$g_{\mu\nu}(s) = \eta_{ab}\, e^a{}_\mu(s)\, e^b{}_\nu(s), \qquad s \in \{+,-\}.$$
- Two determinants and volume elements:
$$e(s) = \det(e^a{}_\mu(s)), \qquad dV_s = e(s)\, d^4x.$$
- Two Weitzenböck connections and torsion tensors:
$$\Gamma^\rho{}_{\mu\nu}(s) = e_a{}^\rho(s)\, \partial_\mu e^a{}_\nu(s),$$
$$T^\rho{}_{\mu\nu}(s) = \Gamma^\rho{}_{\mu\nu}(s) - \Gamma^\rho{}_{\nu\mu}(s),$$
and two torsion scalars $T(s)$ built as in Section 3.

All of these objects live on the same manifold $M$, so they can be compared pointwise. However, all index manipulations with $g_{\mu\nu}(s)$, $T^\rho{}_{\mu\nu}(s)$, etc. remain sector-local. This compatibility is essential for both the geometric consistency and the eventual quantum division of the Hilbert space.

Dimensional check (per sector $s$):
$$[e^a{}_\mu(s)] = 0, \quad [g_{\mu\nu}(s)] = 0, \quad [T^\rho{}_{\mu\nu}(s)] = \text{mass}^1, \quad [T(s)] = \text{mass}^2.$$

### 4.2 The Hassan–Rosen square-root tensor

To couple the two metrics, we introduce the matrix
$$X^\mu{}_\nu \equiv g^{(+)\,\mu\rho}\, g_{\rho\nu}(-),$$
which is a $(1,1)$-tensor acting on tangent vectors at each point of $M$.

We define the square-root tensor $S^\mu{}_\nu$ as the (principal) matrix square root of $X$:
$$S^\mu{}_\nu \equiv \left[\sqrt{g^{(+)\,-1} g(-)}\right]^\mu{}_\nu,$$
meaning that
$$S^\mu{}_\rho\, S^\rho{}_\nu = X^\mu{}_\nu = g^{(+)\,\mu\rho}\, g_{\rho\nu}(-).$$

**Existence and uniqueness assumptions.** For $S^\mu{}_\nu$ to exist and to be uniquely defined as a real tensor field:

- The metrics $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$ must share the same Lorentzian signature.
- At each point $p \in M$, the eigenvalues of $X^\mu{}_\nu(p)$ must be real and positive, so that a principal square root can be chosen.
- The eigenvalues and eigenvectors of $X$ vary smoothly with $p$, ensuring smoothness of $S^\mu{}_\nu$.

These are standard conditions used in Hassan–Rosen bimetric gravity to avoid branch ambiguities and maintain a well-defined potential.

Dimensional check: both $g^{(+)\,\mu\rho}$ and $g_{\rho\nu}(-)$ are dimensionless, so $[X^\mu{}_\nu] = 0$, hence $[S^\mu{}_\nu] = 0$.

### 4.3 Hassan–Rosen bimetric potential

The bimetric interaction is encoded in a scalar potential $V(S)$ constructed from the elementary symmetric polynomials of the eigenvalues of $S^\mu{}_\nu$.

Let $[S] = S^\mu{}_\mu$ denote the trace, and $[S^n] = (S^n)^\mu{}_\mu$ the traces of powers. The standard symmetric polynomials are:

$$\begin{aligned}
e_0(S) &= 1,\\
e_1(S) &= [S],\\
e_2(S) &= \frac{1}{2}([S]^2 - [S^2]),\\
e_3(S) &= \frac{1}{6}([S]^3 - 3[S][S^2] + 2[S^3]),\\
e_4(S) &= \det(S).
\end{aligned}$$

All $e_n(S)$ are dimensionless, being traces and determinants of a dimensionless matrix.

The Hassan–Rosen potential is then written as

$$V(S) = \sum_{n=0}^{4} \beta_n\, e_n(S),$$

where the $\beta_n$ are dimensionless coupling coefficients.

In the bimetric teleparallel context, $V(S)$ is multiplied by the $(+)$ sector volume element. This choice singles out the $(+)$ metric as the *reference* measure for the interaction. One could also consider symmetric variants involving both $e^{(+)}$ and $e^{(-)}$, but we keep the simplest form that matches the standard massive/bimetric gravity literature.

Dimensional check:

$$[\beta_n] = 0, \quad [e_n(S)] = 0 \;\Rightarrow\; [V(S)] = 0.$$

### 4.4 Bimetric teleparallel gravitational action

We now define the bimetric teleparallel gravitational action:

$$S_{\text{grav}} = \frac{M_+^2}{2}\int d^4x\, e^{(+)}\, T^{(+)} + \frac{M_-^2}{2}\int d^4x\, e^{(-)}\, T^{(-)} - \frac{M_+^2 m^2}{2}\int d^4x\, e^{(+)}\, V(S).$$

Here:

- $M_+$ and $M_-$ are the effective Planck masses for the $(+)$ and $(-)$ sectors, respectively.

- $m$ is the bimetric mass scale controlling the strength of the interaction between the two metrics.
- $V(S)$ is the Hassan–Rosen potential defined above.

Each term has the following mass dimension:

$$[M_+^2] = [M_-^2] = [m^2] = \text{mass}^2, \quad [T(s)] = \text{mass}^2, \quad [V(S)] = 0, \quad [d^4x] = \text{mass}^{-4}, \quad [e(s)] = 0.$$

Thus:

- $M_+^2 e^{(+)} T^{(+)}$ has dimension mass$^4$;
- $M_-^2 e^{(-)} T^{(-)}$ has dimension mass$^4$;
- $M_+^2 m^2 e^{(+)} V(S)$ has dimension mass$^4$.

After integration over $d^4x$, each term is dimensionless, so $[S_{\text{grav}}] = \text{mass}^0$, as required.

**Physical interpretation of the three blocks.**

- The first term is the TEGR action for the $(+)$ sector, with its own Planck mass $M_+$.
- The second term is the TEGR action for the $(-)$ sector, with Planck mass $M_-$.
- The third term is the bimetric interaction, giving a mass to one linear combination of the two spin-2 fields and coupling the geometry of the two sectors.

This separation makes it clear where the uplift from a single-sector TEGR theory occurs: we duplicate the TEGR kinetic term and add a ghost-free bimetric mass potential.

### 4.5 Matter couplings and sector assignment

To prepare for quantum field theory on this background, we specify how matter couples to the two sectors.

**Visible-sector matter (standard model + excitations).** All standard visible matter fields (scalars, spinors, gauge fields, phonons, magnons, etc.) are coupled minimally to the $(+)$ sector metric and tetrad:

$$S_{\text{matter}}(+) = \int d^4x\, e^{(+)}\, \mathcal{L}_{\text{matter}}[g_{\mu\nu}(+), e^a{}_\mu(+), \Phi(+), \psi(+), A_\mu(+), \dots].$$

This ensures that all local experiments performed by observers in the $(+)$ sector see standard GR+QFT physics in the TEGR disguise.

**Hidden-sector matter (optional).** One may allow for additional matter fields living on the $(-)$ sector:

$$S_{\text{matter}}(-) = \int d^4x\, e^{(-)}\, \mathcal{L}_{\text{hidden}}[g_{\mu\nu}(-), e^a{}_\mu(-), \Phi(-), \psi(-), \dots].$$

These fields are not directly coupled to $g_{\mu\nu}(+)$ or visible-sector fields; their only gravitational influence on the visible sector is through the shared bimetric interaction $V(S)$.

**Total action.** The full classical action is

$$S_{\text{tot}} = S_{\text{grav}} + S_{\text{matter}}(+) + S_{\text{matter}}(-).$$

This is the starting point for defining classical field equations and then quantizing matter fields on a fixed bimetric teleparallel background.

### 4.6 Variation and schematic field equations

A detailed derivation of the field equations is lengthy; here we outline the structure and check consistency of indices and units.

Varying $S_{\text{tot}}$ with respect to $e^a{}_\mu(+)$ and $e^a{}_\mu(-)$ yields two sets of equations:

$$\frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(+)} + \frac{\delta S_{\text{matter}}(+)}{\delta e^a{}_\mu(+)} = 0,$$
$$\frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(-)} + \frac{\delta S_{\text{matter}}(-)}{\delta e^a{}_\mu(-)} = 0.$$

The variations of the TEGR terms reproduce the TEGR field equations in each sector, now sourced by effective energy-momentum tensors that include contributions from the bimetric potential:

$$\partial_\rho(e(+)S_a{}^{\rho\mu}(+)) - e(+)T^b{}_{\rho a}(+)\, S_b{}^{\mu\rho}(+) + \frac{1}{4}e(+)e_a{}^\mu(+)\, T(+) + [\text{bimetric source from } V(S)] = \frac{1}{2}e(+)\, \Theta_a{}^\mu(+$$
$$\partial_\rho(e(-)S_a{}^{\rho\mu}(-)) - e(-)T^b{}_{\rho a}(-)\, S_b{}^{\mu\rho}(-) + \frac{1}{4}e(-)e_a{}^\mu(-)\, T(-) + [\text{bimetric source from } V(S)] = \frac{1}{2}e(-)\, \Theta_a{}^\mu(-$$

where $\Theta_a{}^\mu(s)$ are sector-wise matter energy-momentum tensors, and the square-bracket terms arise from varying $e^{(+)}V(S)$ with respect to the tetrads in each sector. These source terms encode the exchange of stress-energy between the two metrics through the bimetric potential.

Dimensional and index checks:

- Each $S_a{}^{\rho\mu}(s)$ has dimension mass$^1$; derivatives bring one more mass dimension, so the first term in each equation has dimension mass$^2$.
- All terms carry indices $(a, \mu)$ consistently; cross-sector contributions from $V(S)$ are mapped into $(a, \mu)$ via variations of $g_{\mu\nu}(s)$ with respect to $e^a{}_\mu(s)$.
- The energy-momentum tensors $\Theta_a{}^\mu(s)$ have dimension mass$^2$, as in the single-sector TEGR case.

Thus the bimetric uplift preserves the internal consistency of TEGR while introducing controlled coupling terms between the $(+)$ and $(-)$ sectors.

### 4.7 Motivation for the bimetric uplift

The step from a single TEGR sector to the bimetric teleparallel framework is motivated by both structural and physical considerations:

(a) **Structural:** TEGR provides a curvature-free but torsionful representation of gravity equivalent to GR. Doubling the TEGR sector and adding a Hassan–Rosen potential is the teleparallel analogue of ghost-free bimetric gravity, now expressed in terms of torsion scalars $T(s)$ instead of Ricci scalars.

(b) **Spectral and quantum:** Having two metrics allows one to assign different spectral properties (e.g. different mode spectra or effective couplings) to visible and hidden sectors, while keeping a unified teleparallel background. Quantum fields will be quantized separately on $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$, with their spectra indirectly coupled through $V(S)$.

(c) **Effective theory viewpoint:** As an effective theory, the bimetric teleparallel action captures leading-order interactions between two spin-2 fields without introducing ghosts in the spin-2 sector. It is therefore a natural starting point for exploring quantum corrections and phenomenology in a controlled setting.

In summary, the bimetric uplift is not arbitrary: it is the minimal ghost-free way to extend TEGR to two interacting teleparallel metrics, with clear sector labels $(+)$ and $(-)$ that will later support a factorized quantum field construction.

**Summary of key bimetric-teleparallel relations and checks**

**Two teleparallel sectors on a common manifold:**

$$g_{\mu\nu}(s) = \eta_{ab} e^a{}_\mu(s)\, e^b{}_\nu(s), \qquad e(s) = \det\big(e^a{}_\mu(s)\big), \qquad s \in \{+,-\}.$$

$$\Gamma^\rho{}_{\mu\nu}(s) = e_a{}^\rho(s)\, \partial_\mu e^a{}_\nu(s), \quad T^\rho{}_{\mu\nu}(s) = \Gamma^\rho{}_{\mu\nu}(s) - \Gamma^\rho{}_{\nu\mu}(s),$$

$$T(s) = \tfrac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \tfrac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T^\rho{}_{\rho\mu} T^\nu{}_\nu{}^\mu \quad (s),$$

with $[T(s)] = \text{mass}^2$.

**Hassan–Rosen square-root tensor and potential:**

$$X^\mu{}_\nu = g^{(+)\,\mu\rho} g_{\rho\nu}(-), \qquad S^\mu{}_\nu = \left[\sqrt{g^{(+)\,-1} g(-)}\right]^\mu{}_\nu,$$

$$V(S) = \sum_{n=0}^{4} \beta_n\, e_n(S), \quad e_0 = 1,\ e_1 = [S],\ e_2 = \tfrac{1}{2}([S]^2 - [S^2]),\ \dots,$$

with $[V(S)] = \text{mass}^0$.

**Bimetric teleparallel gravitational action:**

$$S_{\text{grav}} = \frac{M_+^2}{2} \int d^4x\, e^{(+)}\, T^{(+)} + \frac{M_-^2}{2} \int d^4x\, e^{(-)}\, T^{(-)} - \frac{M_+^2 m^2}{2} \int d^4x\, e^{(+)}\, V(S),$$

with $[M_+] = [M_-] = [m] = \text{mass}^1$ and $[S_{\text{grav}}] = \text{mass}^0$.

**Matter couplings and sector assignment:**

$$S_{\text{tot}} = S_{\text{grav}} + S_{\text{matter}}(+) + S_{\text{matter}}(-),$$

$$S_{\text{matter}}(+) = \int d^4x\, e^{(+)}\, \mathcal{L}_{\text{matter}}[g(+), e(+), \text{visible fields}],$$

$$S_{\text{matter}}(-) = \int d^4x\, e^{(-)}\, \mathcal{L}_{\text{hidden}}[g(-), e(-), \text{hidden fields}].$$

**Field equations (schematic):**

$$\frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(+)} + \frac{\delta S_{\text{matter}}(+)}{\delta e^a{}_\mu(+)} = 0, \qquad \frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(-)} + \frac{\delta S_{\text{matter}}(-)}{\delta e^a{}_\mu(-)} = 0,$$

producing TEGR-like equations in each sector plus bimetric source terms from $V(S)$, all with consistent mass dimension $\text{mass}^2$ and index structure.

## 5 Quantum Fields in the Bimetric Teleparallel Framework

We now lift the single-sector TEGR quantum construction of Section **??** to the full bimetric teleparallel background introduced in Section 4. The goals are:

- To define scalar (and by extension gauge, phonon, magnon) fields on each sheet $(+)$ and $(-)$;

- To show that each sector admits a well-posed mode decomposition and harmonic-oscillator Hamiltonian;
- To make the *factorization* of the quantum theory explicit:

$$\mathcal{F}_{\text{tot}} \;=\; \mathcal{F}_{+} \,\otimes\, \mathcal{F}_{-},$$

  with no operator mixing between $(+)$ and $(-)$ sectors at the level of canonical quantization;
- To check units, indices, and operator algebra for the full bimetric construction.

Throughout this section, we treat $e^{a}{}_{\mu}(\pm)$ and all derived geometric quantities as fixed classical fields solving the bimetric teleparallel field equations of Section 4. Quantum fields live on this background.

### 5.1 Sector-labelled scalar fields and actions

We introduce two real scalar fields

$$\Phi_{+}(x) \equiv \Phi(+;x), \qquad \Phi_{-}(x) \equiv \Phi(-;x),$$

each associated with its own teleparallel sector. For definiteness, we allow different mass profiles and kinetic weights in the two sectors.

The scalar actions are

$$\begin{aligned} S_{\Phi}(+) &= \frac{1}{2}\int d^4x\, e^{(+)}\left[Z_{+}(x)\, g^{\mu\nu}(+)\, \partial_{\mu}\Phi_{+}\, \partial_{\nu}\Phi_{+} - M_{+}^{2}(x)\, \Phi_{+}^{2}\right], \\ S_{\Phi}(-) &= \frac{1}{2}\int d^4x\, e^{(-)}\left[Z_{-}(x)\, g^{\mu\nu}(-)\, \partial_{\mu}\Phi_{-}\, \partial_{\nu}\Phi_{-} - M_{-}^{2}(x)\, \Phi_{-}^{2}\right]. \end{aligned}$$

Dimensional assignments (natural units $\hbar = c = 1$):

$$[\Phi_{\pm}] = \text{mass}^{1}, \quad [\partial_{\mu}\Phi_{\pm}] = \text{mass}^{2}, \quad [g^{\mu\nu}(\pm)] = 0, \quad [Z_{\pm}] = 0, \quad [M_{\pm}^{2}] = \text{mass}^{2}, \quad [e(\pm)] = 0, \quad [d^4x] = \text{ma}$$

Hence the Lagrangian densities have dimension mass$^4$ and

$$[S_{\Phi}(+)] = [S_{\Phi}(-)] = \text{mass}^{0}.$$

The total scalar action is the sum

$$S_{\Phi,\text{tot}} = S_{\Phi}(+) + S_{\Phi}(-),$$

with no direct $\Phi_{+}\Phi_{-}$ mixing at this stage. Any cross-sector influence arises *only* through the bimetric gravitational background (via $V(S)$) rather than explicit quartic or bilinear scalar couplings.

### 5.2 ADM split in both sectors

We now perform an ADM decomposition in each sector. For clarity we use a common coordinate chart $x^{\mu} = (t, x^{i})$ on $M$ but allow lapse and spatial metric to differ sector by sector:

$$\begin{aligned} ds^2(+) &= -N_{+}^{2}\, dt^2 + \gamma_{ij}^{(+)}(dx^i + N_{+}^{i} dt)(dx^j + N_{+}^{j} dt), \\ ds^2(-) &= -N_{-}^{2}\, dt^2 + \gamma_{ij}^{(-)}(dx^i + N_{-}^{i} dt)(dx^j + N_{-}^{j} dt). \end{aligned}$$

The volume factors are

$$e^{(+)} = N_{+}\sqrt{\gamma_{+}}, \qquad e^{(-)} = N_{-}\sqrt{\gamma_{-}}, \qquad \gamma_{\pm} \equiv \det(\gamma_{ij}^{(\pm)}).$$

We adopt, for simplicity, the zero-shift gauge in both sectors:

$$N^i_+ = 0, \qquad N^i_- = 0.$$

Then

$$g^{00}(\pm) = -\frac{1}{N_\pm^2}, \qquad g^{0i}(\pm) = 0, \qquad g^{ij}(\pm) = \gamma^{ij}_{(\pm)}.$$

In this gauge, the scalar Lagrangian densities take the same form in both sectors, with sector labels on $N_\pm, \gamma^{(\pm)}, Z_\pm, M_\pm^2$.

### 5.3 Canonical momenta and Hamiltonian densities

The canonical momenta conjugate to $\Phi_\pm$ are

$$\Pi_+(t,x) = \frac{\partial \mathcal{L}_\Phi(+)}{\partial(\partial_t \Phi_+)} = -\frac{\sqrt{\gamma_+}\, Z_+(x)}{N_+}\, \partial_t \Phi_+(t,x),$$
$$\Pi_-(t,x) = \frac{\partial \mathcal{L}_\Phi(-)}{\partial(\partial_t \Phi_-)} = -\frac{\sqrt{\gamma_-}\, Z_-(x)}{N_-}\, \partial_t \Phi_-(t,x).$$

Dimensional check:

$$[\sqrt{\gamma_\pm}] = 0, \quad [Z_\pm] = 0, \quad [N_\pm] = 0, \quad [\partial_t \Phi_\pm] = \text{mass}^2,$$

so $[\Pi_\pm] = \text{mass}^2$ as in the single-sector case.

The Hamiltonian densities are

$$\mathcal{H}_\Phi(+) = \frac{\sqrt{\gamma_+}}{2}\Big[\frac{Z_+}{N_+^2}(\partial_t \Phi_+)^2 + Z_+ \gamma^{ij}_{(+)} \partial_i \Phi_+ \partial_j \Phi_+ + N_+^2 M_+^2 \Phi_+^2\Big],$$
$$\mathcal{H}_\Phi(-) = \frac{\sqrt{\gamma_-}}{2}\Big[\frac{Z_-}{N_-^2}(\partial_t \Phi_-)^2 + Z_- \gamma^{ij}_{(-)} \partial_i \Phi_- \partial_j \Phi_- + N_-^2 M_-^2 \Phi_-^2\Big].$$

The total scalar Hamiltonian is the sum

$$H_{\Phi,\text{tot}}(t) = \int_{\Sigma_t} d^3x\, [\mathcal{H}_\Phi(+) + \mathcal{H}_\Phi(-)],$$

with $\Sigma_t$ the common $t = \text{const}$ slice of $M$.

Each $\mathcal{H}_\Phi(\pm)$ has dimension $\text{mass}^4$, so $H_{\Phi,\text{tot}}$ has dimension $\text{mass}^1$.

### 5.4 Spatial operators and mode bases in both sectors

We define sector-wise spatial operators via

$$\mathcal{L}_{\text{sp}}(+)\, \Phi_+ \equiv -\frac{1}{\sqrt{\gamma_+}} \partial_i\big(\sqrt{\gamma_+}\, Z_+\, \gamma^{ij}_{(+)} \partial_j \Phi_+\big) + N_+^2 M_+^2 \Phi_+,$$
$$\mathcal{L}_{\text{sp}}(-)\, \Phi_- \equiv -\frac{1}{\sqrt{\gamma_-}} \partial_i\big(\sqrt{\gamma_-}\, Z_-\, \gamma^{ij}_{(-)} \partial_j \Phi_-\big) + N_-^2 M_-^2 \Phi_-.$$

Each operator is defined on its own Hilbert space:

$$\mathcal{H}^{(+)}_\Sigma = L^2(\Sigma_t, \sqrt{\gamma_+}\, Z_+\, d^3x),$$
$$\mathcal{H}^{(-)}_\Sigma = L^2(\Sigma_t, \sqrt{\gamma_-}\, Z_-\, d^3x),$$

with inner products

$$\langle f, g\rangle_+ = \int_{\Sigma_t} d^3x\,\sqrt{\gamma_+}\,Z_+ f^* g, \qquad \langle f, g\rangle_- = \int_{\Sigma_t} d^3x\,\sqrt{\gamma_-}\,Z_- f^* g.$$

Under the same regularity and ellipticity assumptions as in the single-sector case, each $\mathcal{L}_{\rm sp}(\pm)$ is essentially self-adjoint and admits a complete orthonormal basis of eigenfunctions:

$$\begin{aligned}
\mathcal{L}_{\rm sp}(+)\,\phi_k^{(+)}(x) &= \omega_k^2(+)\,\phi_k^{(+)}(x),\\
\mathcal{L}_{\rm sp}(-)\,\phi_k^{(-)}(x) &= \omega_k^2(-)\,\phi_k^{(-)}(x),
\end{aligned}$$

with

$$\begin{aligned}
\int_{\Sigma_t} d^3x\,\sqrt{\gamma_+}\,Z_+\,\phi_k^{(+)}\,\phi_{k'}^{(+)} &= \delta_{kk'},\\
\int_{\Sigma_t} d^3x\,\sqrt{\gamma_-}\,Z_-\,\phi_k^{(-)}\,\phi_{k'}^{(-)} &= \delta_{kk'}.
\end{aligned}$$

Note that the eigenvalues and eigenfunctions in the $(+)$ and $(-)$ sectors are generally different, reflecting the distinct geometries and masses.

We expand the fields as

$$\begin{aligned}
\Phi_+(t,x) &= \sum_k Q_k(+;t)\,\phi_k^{(+)}(x),\\
\Phi_-(t,x) &= \sum_k Q_k(-;t)\,\phi_k^{(-)}(x),
\end{aligned}$$

with corresponding expansions for $\Pi_\pm(t,x)$. Using orthonormality, each sector's Hamiltonian becomes a sum over independent modes.

After rescaling to canonical variables $q_k(\pm;t)$ and $p_k(\pm;t)$ (as in Section **??**), we obtain

$$\begin{aligned}
H_\Phi(+) &= \frac{1}{2}\sum_k\Big[p_k^2(+;t) + \omega_k^2(+)\,q_k^2(+;t)\Big],\\
H_\Phi(-) &= \frac{1}{2}\sum_k\Big[p_k^2(-;t) + \omega_k^2(-)\,q_k^2(-;t)\Big],
\end{aligned}$$

and

$$H_{\Phi,\rm tot}(t) = H_\Phi(+) + H_\Phi(-).$$

Dimensional check:

$$[\omega_k(\pm)] = \text{mass}^1, \quad [q_k(\pm)] = \text{mass}^0, \quad [p_k(\pm)] = \text{mass}^1.$$

### 5.5 Canonical quantization and Fock-space factorization

The equal-time Poisson brackets in the full system are

$$\begin{aligned}
\{\Phi_+(t,x), \Pi_+(t,x')\} &= \delta^3(x-x'),\\
\{\Phi_-(t,x), \Pi_-(t,x')\} &= \delta^3(x-x'),\\
\{\Phi_+, \Pi_-\} = 0, \quad \{\Phi_+, \Phi_-\} &= 0, \quad \{\Pi_+, \Pi_-\} = 0.
\end{aligned}$$

In mode variables:

$$\begin{aligned}
&\{q_k(+), p_{k'}(+)\} = \delta_{kk'}, \qquad \{q_k(-), p_{k'}(-)\} = \delta_{kk'},\\
&\{q_k(+), p_{k'}(-)\} = 0, \quad \{q_k(+), q_{k'}(-)\} = 0, \quad \{p_k(+), p_{k'}(-)\} = 0.
\end{aligned}$$

Quantization is implemented by promoting $(q_k(\pm), p_k(\pm))$ to operators and imposing

$$
\begin{aligned}
&[\hat{q}_k(+), \hat{p}_{k'}(+)] = i\hbar\,\delta_{kk'}, \qquad [\hat{q}_k(-), \hat{p}_{k'}(-)] = i\hbar\,\delta_{kk'},\\
&[\hat{q}_k(+), \hat{p}_{k'}(-)] = 0, \quad [\hat{q}_k(+), \hat{q}_{k'}(-)] = 0, \quad [\hat{p}_k(+), \hat{p}_{k'}(-)] = 0.
\end{aligned}
$$

We define sector-wise creation and annihilation operators:

$$
\begin{aligned}
\hat{a}_k(+) &= \frac{1}{\sqrt{2\hbar\,\omega_k(+)}}\Big(\omega_k(+)\hat{q}_k(+) + i\hat{p}_k(+)\Big),\\
\hat{a}_k^\dagger(+) &= \frac{1}{\sqrt{2\hbar\,\omega_k(+)}}\Big(\omega_k(+)\hat{q}_k(+) - i\hat{p}_k(+)\Big),\\
\hat{a}_k(-) &= \frac{1}{\sqrt{2\hbar\,\omega_k(-)}}\Big(\omega_k(-)\hat{q}_k(-) + i\hat{p}_k(-)\Big),\\
\hat{a}_k^\dagger(-) &= \frac{1}{\sqrt{2\hbar\,\omega_k(-)}}\Big(\omega_k(-)\hat{q}_k(-) - i\hat{p}_k(-)\Big),
\end{aligned}
$$

which obey

$$
\begin{aligned}
&[\hat{a}_k(+), \hat{a}_{k'}^\dagger(+)] = \delta_{kk'}, \qquad [\hat{a}_k(-), \hat{a}_{k'}^\dagger(-)] = \delta_{kk'},\\
&[\hat{a}_k(+), \hat{a}_{k'}(-)] = 0, \quad [\hat{a}_k(+), \hat{a}_{k'}^\dagger(-)] = 0, \quad [\hat{a}_k^\dagger(+), \hat{a}_{k'}^\dagger(-)] = 0.
\end{aligned}
$$

The Hamiltonian operator splits additively:

$$
\begin{aligned}
\hat{H}_\Phi(+) &= \sum_k \hbar\,\omega_k(+)\Big(\hat{a}_k^\dagger(+)\hat{a}_k(+) + \tfrac{1}{2}\Big),\\
\hat{H}_\Phi(-) &= \sum_k \hbar\,\omega_k(-)\Big(\hat{a}_k^\dagger(-)\hat{a}_k(-) + \tfrac{1}{2}\Big),\\
\hat{H}_{\Phi,\mathrm{tot}} &= \hat{H}_\Phi(+) + \hat{H}_\Phi(-).
\end{aligned}
$$

**Vacuum and Fock space.** We define sector-wise vacuum states:

$$
\hat{a}_k(+)\ |0\rangle_+ = 0, \qquad \hat{a}_k(-)\ |0\rangle_- = 0, \quad \forall k,
$$

and the total vacuum is

$$
|0\rangle_{\mathrm{tot}} = |0\rangle_+ \otimes |0\rangle_-\,.
$$

General multi-particle states are

$$
\Big|\{n_k^{(+)}\}, \{n_k^{(-)}\}\Big\rangle = \prod_k \frac{\big(\hat{a}_k^\dagger(+)\big)^{n_k^{(+)}}}{\sqrt{n_k^{(+)}!}} \prod_k \frac{\big(\hat{a}_k^\dagger(-)\big)^{n_k^{(-)}}}{\sqrt{n_k^{(-)}!}}\ |0\rangle_{\mathrm{tot}}\,.
$$

The total Hilbert space is the tensor product

$$
\mathcal{F}_{\mathrm{tot}} = \mathcal{F}_+ \otimes \mathcal{F}_-,
$$

with all operators of the $(+)$ sector commuting with all operators of the $(-)$ sector. This realizes the "two-sheet" quantum structure in the most conservative way: the sectors are dynamically coupled only through the classical bimetric teleparallel background, but kinematically factorized at the level of canonical quantization.

### 5.6 Other fields in the bimetric setting

The procedure extends straightforwardly to other field types:

- **Photons:** Define $S_{\text{EM}}(\pm)$ with Maxwell fields $A_\mu(\pm)$ minimally coupled to $g_{\mu\nu}(\pm)$, impose a sector-wise gauge (e.g. Coulomb or Lorenz on each $\Sigma_t$), decompose into transverse vector harmonics on $(\Sigma_t, \gamma_{ij}^{(\pm)})$, and quantize the resulting oscillators with sector-labelled photon operators $\hat{a}_{k\alpha}(\pm)$.
- **Phonons:** Elastic displacement fields $u_i(\pm)$ living on each $\Sigma_t$, with elastic moduli tensors $C^{ijkl}(\pm)$ and densities $\rho(\pm)$; the spatial Laplacian and boundary conditions differ per sector, yielding different phonon spectra $\omega_k^{\text{ph}}(\pm)$.
- **Magnons:** Spin-wave excitations built from spin systems embedded in each sector's spatial geometry; the spin exchange couplings $J_{ij}(\pm)$ and geometry of the underlying lattice affect the magnon dispersion relations $\omega_k^{\text{mag}}(\pm)$.

In all cases, the bimetric uplift is *mostly bookkeeping*: one duplicates the single-sector construction for $(+)$ and $(-)$ separately, with all sector-specific geometric data, and then imposes the tensor-product structure on the total Fock space.

Physical interplay between sectors then arises from:

1. The way $V(S)$ couples the two teleparallel geometries;
2. Any choice to identify or correlate field content across sectors (e.g. mirror or dark copies of visible fields);
3. Semiclassical backreaction, where $\langle T_{\mu\nu}(\pm)\rangle$ feeds into the bimetric field equations.

A full analysis of vacuum energies, renormalization, and semiclassical consistency is postponed to later sections.

**Summary of bimetric quantum scalar structure and checks**

**Sector-labelled scalar actions:**

$$S_\Phi(\pm) = \frac{1}{2}\int d^4x\, e(\pm)\Big[Z_\pm(x)\, g^{\mu\nu}(\pm)\,\partial_\mu\Phi_\pm\,\partial_\nu\Phi_\pm - M_\pm^2(x)\,\Phi_\pm^2\Big], \quad [S_\Phi(\pm)] = \mathrm{mass}^0.$$

**Canonical momenta and Hamiltonians (zero shift):**

$$\Pi_\pm(t,x) = -\frac{\sqrt{\gamma_\pm}\,Z_\pm(x)}{N_\pm}\,\partial_t\Phi_\pm(t,x),$$

$$\mathcal{H}_\Phi(\pm) = \frac{\sqrt{\gamma_\pm}}{2}\Big[\frac{Z_\pm}{N_\pm^2}(\partial_t\Phi_\pm)^2 + Z_\pm\gamma^{ij}_{(\pm)}\partial_i\Phi_\pm\partial_j\Phi_\pm + N_\pm^2 M_\pm^2\Phi_\pm^2\Big],$$

$$H_{\Phi,\mathrm{tot}} = \int d^3x\,\big(\mathcal{H}_\Phi(+) + \mathcal{H}_\Phi(-)\big).$$

**Spatial operators and mode expansions (each sector):**

$$\mathcal{L}_{\mathrm{sp}}(\pm)\,\phi_k^{(\pm)} = \omega_k^2(\pm)\,\phi_k^{(\pm)}, \quad \int d^3x\,\sqrt{\gamma_\pm}\,Z_\pm\,\phi_k^{(\pm)}\phi_{k'}^{(\pm)} = \delta_{kk'},$$

$$\Phi_\pm(t,x) = \sum_k Q_k(\pm;t)\,\phi_k^{(\pm)}(x), \quad H_\Phi(\pm) = \frac{1}{2}\sum_k\big[p_k^2(\pm) + \omega_k^2(\pm)\,q_k^2(\pm)\big].$$

**Canonical commutators and sector factorization:**

$$[\hat q_k(\pm),\hat p_{k'}(\pm)] = i\hbar\,\delta_{kk'}, \quad [\hat q_k(+),\hat p_{k'}(-)] = 0, \quad [\hat q_k(+),\hat q_{k'}(-)] = [\hat p_k(+),\hat p_{k'}(-)] = 0,$$

$$\hat a_k(\pm) = \frac{\omega_k(\pm)\hat q_k(\pm) + i\hat p_k(\pm)}{\sqrt{2\hbar\,\omega_k(\pm)}}, \quad [\hat a_k(\pm),\hat a^\dagger_{k'}(\pm)] = \delta_{kk'}, \quad [\hat a_k(+),\hat a^{(\dagger)}_{k'}(-)] = 0.$$

**Total Hamiltonian and Fock space:**

$$\hat H_{\Phi,\mathrm{tot}} = \sum_k \hbar\,\omega_k(+)\big(\hat a_k^\dagger(+)\hat a_k(+) + \tfrac{1}{2}\big) + \sum_k \hbar\,\omega_k(-)\big(\hat a_k^\dagger(-)\hat a_k(-) + \tfrac{1}{2}\big),$$

$$|0\rangle_{\mathrm{tot}} = |0\rangle_+ \otimes |0\rangle_-, \qquad \mathcal{F}_{\mathrm{tot}} = \mathcal{F}_+ \otimes \mathcal{F}_-.$$

All units, indices, and operator relations remain internally consistent, and the bimetric uplift of the quantum field construction is realized as a clean tensor-product structure on top of the classical bimetric teleparallel background.

## 6 Physical Role and EFT Status of the Quantum Bimetric–Teleparallel Framework

Having established the bimetric teleparallel action, the sector-wise quantum field construction, and the associated consistency checks, we now explain:

- What physical role the $(+)$ and $(-)$ sectors are intended to play;
- Why the specific bimetric uplift (duplicated TEGR + Hassan–Rosen potential + factorized QFT) is chosen;
- How this yields a *functioning effective* quantum bimetric–teleparallel theory in well-defined regimes;
- In which limits it reduces to standard GR + QFT in curved spacetime.

The emphasis here is on conceptual clarity and on tying the formal machinery to concrete physical interpretations.

### 6.1 Sector interpretation and classical limit

We interpret the two teleparallel sectors as follows:

- The $(+)$ **sector** is identified with the *visible* or *physical* universe, in which ordinary matter, standard-model fields, and laboratory observers live. Its metric $g_{\mu\nu}(+)$ and tetrad $e^a{}_\mu(+)$ determine:
  - Geodesics and autoparallels followed by visible matter;
  - The local inertial frames used to define measurements of time, length, and energy;
  - The teleparallel torsion scalar $T^{(+)}$ appearing in the gravitational action with coefficient $M_+^2$.
- The $(-)$ **sector** provides an additional teleparallel sheet with its own metric $g_{\mu\nu}(-)$, tetrad $e^a{}_\mu(-)$, and torsion scalar $T^{(-)}$, coupled to the $(+)$ sector only through the bimetric potential $V(S)$. Depending on parameter choices, this sector can be interpreted as:
  - a dark or hidden gravitational sector;
  - a mirror copy of the visible sector with different couplings;
  - or an effective geometric surrogate for what is usually modeled as dark matter or dark energy.

In the *classical* limit, with all quantum fields turned off or replaced by their expectation values, the dynamics of the two tetrads are governed by

$$\frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(+)} = \tau^\mu{}_a(+) + (\text{bimetric source}), \qquad \frac{\delta S_{\text{grav}}}{\delta e^a{}_\mu(-)} = \tau^\mu{}_a(-) + (\text{bimetric source}),$$

where $\tau^\mu{}_a(\pm)$ denote (teleparallel) energy–momentum currents of matter fields living in each sector. The Hassan–Rosen potential $V(S)$ couples the tetrads algebraically (no derivatives of $S^\mu{}_\nu$), thereby:

- Introducing a controlled massive spin-2 interaction between sectors;
- Avoiding higher-derivative instabilities at the classical level;
- Allowing one sector to feel an effective "background" deformation sourced by the other.

In the limit

$$M_- \to 0, \qquad m \to 0, \qquad g_{\mu\nu}(-) \to g_{\mu\nu}(+),$$

the two sectors collapse into a single teleparallel sheet and $S_{\text{grav}}$ reduces to standard TEGR (equivalent to GR), with standard single-metric QFT on a curved spacetime.

### 6.2 Motivation for the specific bimetric uplift

The choice of uplift from a single TEGR background to a bimetric teleparallel framework is motivated by three independent requirements:

1. **Separate, fully geometric metrics for the two sectors.**
   Each sector must admit its own teleparallel geometry $(g_{\mu\nu}(\pm), T^\rho{}_{\mu\nu}(\pm))$ in order to:

- Express the dynamics of visible and hidden sectors in the same geometric language;
- Allow the $(-)$ sector to influence the $(+)$ sector *only* through geometry, not through ad hoc non-geometric potentials;
- Preserve a generalized equivalence principle in the $(+)$ sector by coupling visible matter minimally to $g_{\mu\nu}(+)$.

This leads directly to separate TEGR-like pieces $e^{(+)}T^{(+)}$ and $e^{(-)}T^{(-)}$ in $S_{\text{grav}}$.

2. **Ghost-free spin-2 interaction.**
If the two metrics are allowed to interact arbitrarily, generic choices lead to the Boulware–Deser ghost. The Hassan–Rosen potential $V(S)$ is the unique (up to parameter choices) algebraic combination of traces of $\sqrt{g^{(+)\,-1}g(-)}$ that:

   - Gives a mass to one linear combination of spin-2 modes;
   - Preserves a single diagonal diffeomorphism symmetry;
   - Avoids the extra scalar ghost at the classical level.

   Embedding this same potential in the teleparallel setting preserves these properties, while allowing torsion rather than curvature to encode the geometry.

3. **Clean separation of quantum kinematics.**
The quantum field theory is built exactly as in single-metric curved spacetime QFT, but:

   - duplicated sector-wise, with all fields and mode operators labelled by $(\pm)$;
   - endowed with a strict tensor-product structure

   $$\mathcal{F}_{\text{tot}} = \mathcal{F}_{+} \otimes \mathcal{F}_{-};$$

   - constrained so that operators in different sectors commute.

   This guarantees that there is no hidden operator mixing or nonlocality between $(+)$ and $(-)$ at the kinematic level. All cross-sector influence is mediated by the classical bimetric teleparallel geometry.

Taken together, these points explain why the uplift consists of:

- Two copies of TEGR torsion scalars $T^{(+)}, T^{(-)}$ with independent scales $M_{+}, M_{-}$;
- A Hassan–Rosen potential built from $S^{\mu}{}_{\nu} = \sqrt{g^{(+)\,-1}g(-)}$;
- Sector-wise QFT with factorized Hilbert space and no cross-commutators.

### 6.3 Comparison to standard GR + QFT in curved spacetime

In standard single-metric GR with QFT in curved spacetime:

- There is a single metric $g_{\mu\nu}$ and a single Levi–Civita connection;
- Gravity is encoded in curvature $R^{\rho}{}_{\sigma\mu\nu}$, and matter sees this geometry through minimal coupling;
- Quantum fields are defined on $(M, g_{\mu\nu})$, and backreaction is treated semiclassically through $\langle T_{\mu\nu}\rangle$ in the Einstein equation.

In the present framework:

- Geometry is encoded in *torsion* scalars $T^{(+)}, T^{(-)}$ rather than curvature, via two teleparallel geometries;
- There are two metrics $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$, each flat in curvature but nontrivial in torsion, coupled by an HR-type potential;
- Quantum fields are defined separately on $(M, g_{\mu\nu}(+))$ and $(M, g_{\mu\nu}(-))$, with their own mode spectra and Fock spaces;
- The $(+)$ sector plays the role of the standard GR geometry for visible physics, while the $(-)$ sector carries additional geometric structure that can effectively mimic dark components or modified gravity.

In the limit where $g_{\mu\nu}(-)$ is slaved to $g_{\mu\nu}(+)$ (e.g. $g_{\mu\nu}(-) \approx g_{\mu\nu}(+)$ and the HR potential is tuned), the framework reduces to TEGR (hence GR) plus ordinary curved-spacetime QFT, recovering standard results.

### 6.4 Example regimes of applicability

We highlight three regimes where the framework functions as a clear effective theory.

**Cosmological sector splitting.** Assume homogeneous, isotropic teleparallel cosmologies in both sectors:

$$ds^2(\pm) = -N_\pm^2(t)\, dt^2 + a_\pm^2(t)\, \delta_{ij} dx^i dx^j.$$

The HR potential $V(S)$ can be chosen so that:

- the $(+)$ sector experiences an effective Friedmann equation with extra terms from the $(-)$ scale factor $a_-(t)$;
- the $(-)$ sector captures effective dark-energy or dark-matter-like dynamics without introducing non-geometric fields in the $(+)$ sector;
- perturbations can be quantized sectorwise, yielding two sets of scalar, vector, and tensor modes with coupled background but factorized quantum algebras.

**Local weak-field regime.** In a local patch around an astrophysical object in the $(+)$ sector, one may linearize both metrics and tetrads:

$$g_{\mu\nu}(\pm) = \eta_{\mu\nu} + h_{\mu\nu}^{(\pm)}, \quad |h_{\mu\nu}^{(\pm)}| \ll 1.$$

Then:

- One linear combination of tetrad perturbations behaves as a massless spin-2 mode (the usual graviton), the other as a massive spin-2 mode with mass $m$;
- Teleparallel torsion encodes these perturbations through the linearized tetrads;
- Quantum fields (e.g. photons, phonons, magnons) in the $(+)$ sector experience standard weak-field effects (redshift, lensing) plus small corrections controlled by the $(-)$ sector configuration.

In this regime, the theory is an EFT with cutoff above the massive spin-2 scale and below the Planck scale.

**Laboratory-scale QFT.** On laboratory scales, curvature and torsion in both sectors can be approximated as nearly constant, so:

- The $(+)$ sector effectively reduces to Minkowski space plus tiny teleparallel corrections;
- The $(-)$ sector geometry can be taken as homogeneous or slowly varying;
- Quantum fields behave as in flat-space QFT with small, controlled corrections to dispersion relations $\omega_k(\pm)$.

In this regime, the teleparallel bimetric effects can be treated as perturbative and are fully compatible with standard QFT machinery.

### 6.5 Summary: how the framework functions as an EFT

The construction defines an effective quantum bimetric–teleparallel theory with the following properties:

- Two torsion-based teleparallel geometries $(g_{\mu\nu}(\pm), T^{\rho}{}_{\mu\nu}(\pm))$, each with its own Planck-like mass scale $M_+, M_-$;
- A ghost-free algebraic coupling of the two metrics via the Hassan–Rosen potential $V(S)$;
- Sector-wise quantum field theories defined on $(M, g_{\mu\nu}(+))$ and $(M, g_{\mu\nu}(-))$, with:
  - consistent mass dimensions for all fields and couplings;
  - well-defined mode decompositions and eigenvalue problems;
  - canonical Poisson brackets and operator commutators;
  - a factorized Fock space $\mathcal{F}_+ \otimes \mathcal{F}_-$.
- A clear set of EFT validity conditions:
  - curvature/torsion scales below a UV cutoff (e.g. below $M_+, M_-$);
  - quantum corrections treated perturbatively around classical bimetric teleparallel backgrounds;
  - renormalization and vacuum-energy issues handled with standard QFT tools adapted to the bimetric setting.

This provides a coherent platform to explore novel gravitational phenomenology (e.g. dark sector effects, modified gravitational waves, teleparallel signatures) while remaining anchored in a mathematically controlled and dimensionally consistent quantum field-theoretic framework.

**Boxed summary: physical meaning and EFT structure**

**Sector roles:**

$(+)$ sector: visible teleparallel universe $(g_{\mu\nu}(+), T^{\rho}{}_{\mu\nu}(+))$, $(-)$ sector: hidden/dark teleparallel sheet $(g_{\mu\nu}$

**Bimetric uplift:**

$$S_{\text{grav}} = \frac{M_+^2}{2} \int d^4x\, e^{(+)}\, T^{(+)} + \frac{M_-^2}{2} \int d^4x\, e^{(-)}\, T^{(-)} - \frac{M_+^2 m^2}{2} \int d^4x\, e^{(+)}\, V(S),$$

with $S^{\mu}{}_{\nu} = \sqrt{g^{(+)\,-1} g(-)}$ and $V(S)$ built from symmetric polynomials $e_n(S)$.
**Quantum structure:**

$$S_{\Phi}(\pm) = \frac{1}{2} \int d^4x\, e(\pm) \left[ Z_{\pm}\, g^{\mu\nu}(\pm) \partial_\mu \Phi_\pm \partial_\nu \Phi_\pm - M_\pm^2 \Phi_\pm^2 \right],$$

$$\Phi_\pm(t, x) = \sum_k q_k(\pm; t)\, \phi_k^{(\pm)}(x), \quad \hat{H}_{\Phi,\text{tot}} = \sum_k \hbar\omega_k(+) \left( \hat{a}_k^\dagger(+) \hat{a}_k(+) + \tfrac{1}{2} \right) + \sum_k \hbar\omega_k(-) \left( \hat{a}_k^\dagger(-) \hat{a}_k(-) + \tfrac{1}{2} \right),$$

$$\mathcal{F}_{\text{tot}} = \mathcal{F}_+ \otimes \mathcal{F}_-, \quad [\hat{a}_k(+), \hat{a}_{k'}^{(\dagger)}(-)] = 0.$$

**EFT viewpoint:**

- Valid below a UV cutoff set by $M_+, M_-$ and the massive spin-2 scale $m$;
- Reduces to TEGR $\simeq$ GR + QFT in curved spacetime in appropriate limits;
- Provides a geometric, torsion-based realization of coupled visible and dark sectors.

With these ingredients, the bimetric teleparallel construction functions as a consistent, physically motivated effective quantum framework capable of extending standard GR + QFT while respecting core field-theoretic and geometric consistency conditions.

## 7 Renormalization, Vacuum Energy, and Semiclassical Field Equations

The previous sections defined a consistent bimetric teleparallel background, constructed quantum fields sector-wise, and audited the resulting framework for dimensional and algebraic consistency. In this section we address the next layer of structure required to make the theory physically predictive at the quantum level:

- the structure of vacuum energy in the $(+)$ and $(-)$ sectors;
- renormalization of zero-point divergences in a bimetric teleparallel setting;
- the semiclassical field equations with $\langle T_{\mu\nu}(\pm) \rangle$ sources;
- basic consistency and stability requirements on the renormalized theory.

The discussion is framed explicitly as *effective field theory*: we do not attempt a full UV completion, but rather identify the ingredients necessary for controlled predictions below a chosen cutoff.

### 7.1 Vacuum energy in each sector

Consider a free scalar field in each sector, as in Section 5. The Hamiltonians are

$$\hat{H}_\Phi(+) = \sum_k \hbar\,\omega_k(+)\Big(\hat{a}_k^\dagger(+)\hat{a}_k(+) + \tfrac{1}{2}\Big),$$
$$\hat{H}_\Phi(-) = \sum_k \hbar\,\omega_k(-)\Big(\hat{a}_k^\dagger(-)\hat{a}_k(-) + \tfrac{1}{2}\Big).$$

The sector-wise vacuum states satisfy

$$\hat{a}_k(\pm)\ |0\rangle_\pm = 0, \quad \forall k,$$

giving vacuum energies

$$E_{\mathrm{vac}}(+) = \Big\langle 0_+|\hat{H}_\Phi(+)|0_+\Big|0_+|\hat{H}_\Phi(+)|0_+\Big\rangle = \frac{1}{2}\sum_k \hbar\,\omega_k(+),$$
$$E_{\mathrm{vac}}(-) = \Big\langle 0_-|\hat{H}_\Phi(-)|0_-\Big|0_-|\hat{H}_\Phi(-)|0_-\Big\rangle = \frac{1}{2}\sum_k \hbar\,\omega_k(-).$$

Each sum diverges as usual. In continuum notation (for approximately homogeneous backgrounds), one writes schematically

$$E_{\mathrm{vac}}(+) \sim \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\hbar\,\omega_+(k),$$
$$E_{\mathrm{vac}}(-) \sim \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\hbar\,\omega_-(k),$$

with $\omega_\pm(k)$ the sector-dependent dispersion relations determined by $\gamma_{ij}^{(\pm)}$, $Z_\pm$, and $M_\pm^2$.

The corresponding vacuum expectation values of the stress–energy tensors are

$$\langle T_{\mu\nu}(\pm)\rangle_{\mathrm{vac}} = -\rho_{\mathrm{vac}}(\pm)\,g_{\mu\nu}(\pm),$$

with divergent vacuum energy densities $\rho_{\mathrm{vac}}(\pm)$ that must be renormalized.

**Dimensional check.** Each $\rho_{\mathrm{vac}}(\pm)$ has dimensions

$$[\rho_{\mathrm{vac}}(\pm)] = \mathrm{mass}^4,$$

consistent with the standard 4D QFT result. When inserted into the bimetric–teleparallel field equations, these contributions appear as effective cosmological-constant-like terms.

### 7.2 Regularization and renormalization structure

To render $\langle T_{\mu\nu}(\pm)\rangle$ finite, we introduce a regularization scheme compatible with:

- diffeomorphism invariance in each sector;
- the algebraic structure of the HR potential $V(S)$;
- the teleparallel (torsion-based) representation.

Typical choices include:

- **Dimensional regularization**, analytically continuing the dimension $d$ of spacetime away from 4;

- **Covariant momentum cutoff**, introducing a scale $\Lambda$ in a locally inertial frame and keeping only modes with $|\mathbf{k}| < \Lambda$;
- **Pauli–Villars regulators**, adding heavy regulator fields with carefully chosen statistics and masses to cancel divergences.

Independently of the specific method, the renormalized vacuum expectation values take the form

$$\begin{aligned}\langle T_{\mu\nu}(\pm)\rangle_{\rm ren} &= \langle T_{\mu\nu}(\pm)\rangle_{\rm vac} - \left[\delta T_{\mu\nu}(\pm)\right]_{\rm ct} \\ &= -\rho^{\rm ren}_{\rm vac}(\pm)\, g_{\mu\nu}(\pm) + (\text{finite curvature/torsion corrections}).\end{aligned}$$

The counterterms $\delta T_{\mu\nu}(\pm)$ are generated by adding to the gravitational action all allowed local operators built from the teleparallel invariants:

$$e(\pm), \quad T(\pm), \quad T^2(\pm), \quad (\text{Nieh–Yan}^{(\pm)}), \quad \ldots$$

and the bimetric invariants (schematically)

$$e(+)\, V(S), \quad e(+)f_1(S)T(+), \quad e(-)f_2(S)T(-), \quad \ldots$$

with coefficient functions (couplings) that absorb the divergences.

At the EFT level we organize the gravitational Lagrangian as an expansion in invariants of increasing mass dimension:

$$\begin{aligned}\mathcal{L}^{\rm EFT}_{\rm grav} = \frac{M_+^2}{2} e(+)T^{(+)} &+ \frac{M_-^2}{2} e(-)T^{(-)} - \frac{M_+^2 m^2}{2} e(+)V(S) \\ &+ e(+)[\Lambda^{\rm bare}_+ + \alpha_+ T(+)^2 + \ldots] + e(-)[\Lambda^{\rm bare}_- + \alpha_- T(-)^2 + \ldots] \\ &+ e(+)\,[\beta_1 f_1(S)T(+) + \beta_2 f_2(S)T(-) + \ldots],\end{aligned}$$

where $\Lambda^{\rm bare}_\pm$ and the coefficients $\alpha_\pm, \beta_i, \ldots$ are tuned such that:

- all UV divergences in $\langle T_{\mu\nu}(\pm)\rangle$ are absorbed;
- the remaining finite part $\rho^{\rm ren}_{\rm vac}(\pm)$ is small enough to match observations in the $(+)$ sector.

**Mass dimensions.**

$$[\Lambda^{\rm bare}_\pm] = \text{mass}^4, \quad [\alpha_\pm] = \text{mass}^0, \quad [\beta_i] = \text{mass}^0,$$

so each added term in $\mathcal{L}^{\rm EFT}_{\rm grav}$ has the correct dimension $\text{mass}^4$ and contributes to the action with dimensionless weight.

### 7.3 Semiclassical bimetric–teleparallel field equations

At the semiclassical level, the bimetric teleparallel field equations equate variations of the effective gravitational action to the renormalized expectation values of the matter energy–momentum currents:

$$\begin{aligned}\frac{\delta S^{\rm EFT}_{\rm grav}}{\delta e^a{}_\mu(+)} &= \tau^\mu{}_a(+) + \left\langle \hat{\tau}^\mu{}_a(+)\right\rangle_{\rm ren}, \\ \frac{\delta S^{\rm EFT}_{\rm grav}}{\delta e^a{}_\mu(-)} &= \tau^\mu{}_a(-) + \left\langle \hat{\tau}^\mu{}_a(-)\right\rangle_{\rm ren}.\end{aligned}$$

Here:

- $\tau^{\mu}{}_{a}(\pm)$ denote classical (bare) energy–momentum currents from any classical matter fields;
- $\hat{\tau}^{\mu}{}_{a}(\pm)$ are the quantum operators built from the quantized fields in each sector;
- angle brackets denote expectation values in some quantum state $|\Psi\rangle \in \mathcal{F}_+ \otimes \mathcal{F}_-$.

If we project these equations onto spacetime indices using the tetrads $e^{a}{}_{\mu}(\pm)$, we obtain semiclassical equations in the form

$$\mathcal{E}_{\mu\nu}(+) = \kappa_+\Big[T^{\rm cl}_{\mu\nu}(+) + \langle \hat{T}_{\mu\nu}(+)\rangle_{\rm ren}\Big] + C^{(+)}_{\mu\nu}[g(+),g(-)],$$
$$\mathcal{E}_{\mu\nu}(-) = \kappa_-\Big[T^{\rm cl}_{\mu\nu}(-) + \langle \hat{T}_{\mu\nu}(-)\rangle_{\rm ren}\Big] + C^{(-)}_{\mu\nu}[g(+),g(-)],$$

where:

- $\mathcal{E}_{\mu\nu}(\pm)$ denotes the sector-wise teleparallel TEGR-type field operator (algebraic equivalent of Einstein tensor);
- $\kappa_\pm \sim^{-2}$ are effective couplings;
- $C^{(\pm)}_{\mu\nu}[g(+),g(-)]$ are algebraic tensors arising from the variation of the HR potential and higher-order EFT corrections.

In homogeneous, isotropic cosmological settings (Section 6.4), these projections yield coupled Friedmann-like equations for $a_+(t)$ and $a_-(t)$, with effective energy densities and pressures that include both classical fluids and renormalized vacuum contributions from each sector.

### 7.4 Stability and EFT validity conditions

For the semiclassical bimetric–teleparallel EFT to be predictive, several stability and validity conditions must hold:

1. **Positivity of kinetic terms.**
   The signs of the teleparallel kinetic terms and the HR mass term must be such that the quadratic action for perturbations around a chosen background is bounded below. This eliminates ghosts at the linearized level.

2. **Hyperbolicity of field equations.**
   The combined system of linearized equations for metric perturbations and matter fields must be hyperbolic with respect to a physically chosen time function (usually the $(+)$-sector time). This guarantees well-posed Cauchy evolution and causal propagation.

3. **Control of higher-derivative corrections.**
   Higher-dimension operators (e.g. $T^2(\pm)$, higher torsion invariants, curvature-like invariants induced in an equivalent Riemannian picture) should be suppressed by scales larger than the typical energy scales of the processes being modeled. This ensures EFT truncation is meaningful.

4. **Hierarchy of scales.**
   A clear hierarchy among
   $$M_+,\ M_-,\ m,\ \Lambda_{\rm cutoff}$$
   is required. For instance, we typically need
   $$\Lambda_{\rm IR} \ll m \ll M_+, M_- \le \Lambda_{\rm cutoff},$$
   so that the massive spin-2 mode is included but higher-mass modes (if any) are integrated out.

5. **Consistency with observational constraints in the $(+)$ sector.**
Teleparallel and bimetric corrections to standard GR must be small in regimes where GR is well tested (solar system, binary pulsars, gravitational-wave waveforms), which constrains combinations of $, m, \beta_n$ and the renormalized vacuum contributions $\rho^{\text{ren}}_{\text{vac}}(\pm)$.

Under these conditions, the bimetric teleparallel EFT provides a controlled parameterization of deviations from GR + QFT in curved spacetime, with a clear interpretation of each additional term in the action and each new field in the quantum sector.

**Boxed summary: renormalized semiclassical structure**

**Sector-wise vacuum energies and stress–energy:**

$$E_{\text{vac}}(\pm) = \frac{1}{2}\sum_k \hbar\,\omega_k(\pm), \quad \langle T_{\mu\nu}(\pm)\rangle_{\text{vac}} = -\rho_{\text{vac}}(\pm)\,g_{\mu\nu}(\pm), \quad [\rho_{\text{vac}}(\pm)] = \text{mass}^4.$$

**EFT-renormalized gravitational Lagrangian (schematic):**

$$\mathcal{L}^{\text{EFT}}_{\text{grav}} = \frac{M_+^2}{2}e(+)T^{(+)} + \frac{M_-^2}{2}e(-)T^{(-)} - \frac{M_+^2 m^2}{2}e(+)V(S) + e(+)\Lambda_+^{\text{bare}} + e(-)\Lambda_-^{\text{bare}} + \ldots,$$

with higher torsion and bimetric invariants suppressed by appropriate scales.
**Semiclassical bimetric–teleparallel equations:**

$$\frac{\delta S^{\text{EFT}}_{\text{grav}}}{\delta e^a{}_\mu(+)} = \tau^\mu{}_a(+) + \left\langle \hat{\tau}^\mu{}_a(+)\right\rangle_{\text{ren}}, \quad \frac{\delta S^{\text{EFT}}_{\text{grav}}}{\delta e^a{}_\mu(-)} = \tau^\mu{}_a(-) + \left\langle \hat{\tau}^\mu{}_a(-)\right\rangle_{\text{ren}}.$$

Projected to spacetime indices:

$$\mathcal{E}_{\mu\nu}(\pm) = \kappa_\pm\Big[T^{\text{cl}}_{\mu\nu}(\pm) + \langle \hat{T}_{\mu\nu}(\pm)\rangle_{\text{ren}}\Big] + C^{(\pm)}_{\mu\nu}[g(+), g(-)].$$

**EFT validity and stability:**

- Teleparallel kinetic and HR mass terms chosen to avoid ghosts;
- Linearized equations hyperbolic and causal with respect to $(+)$-sector time;
- Higher-order torsion/bimetric operators suppressed by heavy scales;
- Parameter choices consistent with observational tests in the $(+)$ sector.

Thus, after renormalization and under a clear hierarchy of scales, the quantum bimetric–teleparallel framework functions as a well-defined semiclassical EFT, extending GR + QFT while preserving dimensional, index, and operator consistency.

## 8 Illustrative Example: Scalar QFT on a Bimetric–Teleparallel FRW Background

To demonstrate explicitly that the quantum bimetric–teleparallel framework can be used as a functioning effective theory, we work through a concrete example:

- both sectors $(+)$ and $(-)$ are taken to be homogeneous and isotropic teleparallel cosmologies (FRW-type);
- a canonical scalar field lives in each sector and is quantized as in Sections 5–**??**;

- the bimetric potential $V(S)$ is evaluated for proportional FRW metrics.

We perform dimensional checks at each step, derive the mode equations, and show how the $(-)$-sector geometry deforms the $(+)$-sector scalar spectrum through the HR coupling.

### 8.1 Teleparallel FRW ansatz in both sectors

We assume spatially flat FRW metrics in both sectors:

$$ds^2(+) = -N_+^2(t)\,dt^2 + a_+^2(t)\,\delta_{ij}\,dx^i dx^j, \tag{1}$$

$$ds^2(-) = -N_-^2(t)\,dt^2 + a_-^2(t)\,\delta_{ij}\,dx^i dx^j, \tag{2}$$

with lapse functions $N_\pm(t)$ and scale factors $a_\pm(t)$.

A convenient diagonal tetrad choice for each sector is

$$e^0{}_0(+) = N_+(t), \quad e^i{}_j(+) = a_+(t)\,\delta^i{}_j, \quad e^0{}_i(+) = 0, \quad e^i{}_0(+) = 0, \tag{3}$$

and analogously for the $(-)$ sector with $(N_-, a_-)$. The determinants are

$$e(+) = N_+(t)a_+^3(t), \qquad e(-) = N_-(t)a_-^3(t),$$

so

$$[e(\pm)] = 0, \quad [N_\pm] = 0, \quad [a_\pm] = 0,$$

as appropriate for dimensionless lapse and scale factors.

The Hubble-like parameters are

$$H_+(t) = \frac{1}{N_+}\frac{\dot a_+}{a_+}, \qquad H_-(t) = \frac{1}{N_-}\frac{\dot a_-}{a_-}, \qquad [H_\pm] = \text{mass}^1,$$

where dots denote derivatives with respect to coordinate time $t$.

For this tetrad choice, the TEGR torsion scalars take the form

$$T(+) = -6H_+^2, \tag{4}$$

$$T(-) = -6H_-^2. \tag{5}$$

Dimensional check:

$$[T(\pm)] = [H_\pm^2] = \text{mass}^2,$$

consistent with the general analysis.

### 8.2 Hassan–Rosen potential for proportional FRW metrics

To make the HR potential explicit while keeping the example tractable, we consider proportional FRW metrics:

$$g_{\mu\nu}(-) = c^2(t)\,g_{\mu\nu}(+), \qquad c(t) > 0, \tag{6}$$

which, in the FRW parametrization, implies

$$N_-(t) = c(t)N_+(t), \qquad a_-(t) = c(t)a_+(t).$$

Then

$$g^{(+)\,\mu\rho}g_{\rho\nu}(-) = c^2(t)\,\delta^\mu{}_\nu,$$

so the square-root matrix

$$S^\mu{}_\nu = \left[\sqrt{g^{(+)\,-1}g(-)}\right]^\mu{}_\nu = c(t)\,\delta^\mu{}_\nu.$$

The elementary symmetric polynomials become

$$e_0(S) = 1,$$
$$e_1(S) = 4c,$$
$$e_2(S) = 6c^2,$$
$$e_3(S) = 4c^3,$$
$$e_4(S) = c^4,$$

so the HR potential reduces to

$$V(S) = \beta_0 + 4\beta_1 c + 6\beta_2 c^2 + 4\beta_3 c^3 + \beta_4 c^4, \tag{7}$$

a simple scalar function of $c(t)$.

In the FRW ansatz, the bimetric teleparallel gravitational action becomes

$$\begin{aligned} S_{\text{grav}}^{\text{FRW}} = \frac{M_+^2}{2}\int dt\, d^3x\, N_+ a_+^3\, T(+) + \frac{M_-^2}{2}\int dt\, d^3x\, N_- a_-^3\, T(-) \\ - \frac{M_+^2 m^2}{2}\int dt\, d^3x\, N_+ a_+^3\, V(S). \end{aligned} \tag{8}$$

Using $T(\pm) = -6H_\pm^2$ and the proportionality condition, one can rewrite this as an effective single-sector action with an extra scalar degree of freedom $c(t)$ encoded in $V(c)$.

**Dimensional check.** Each term in (8) has dimension:

$$[^2] = \text{mass}^2, \quad [T(\pm)] = \text{mass}^2, \quad [V(S)] = 0, \quad [N_\pm a_\pm^3 dt d^3x] = \text{mass}^{-4},$$

so $[S_{\text{grav}}^{\text{FRW}}] = \text{mass}^0$ as required.

### 8.3 Scalar field action and mode decomposition in the $(+)$ sector

Consider a canonical real scalar field $\Phi_+$ on the $(+)$ sector FRW background (1). The action is

$$S_\Phi(+) = \frac{1}{2}\int dt\, d^3x\, N_+ a_+^3 \Big[Z_+(t)\big(-\frac{1}{N_+^2}\dot{\Phi}_+^2 + \frac{1}{a_+^2}\delta^{ij}\partial_i\Phi_+\partial_j\Phi_+\big) - M_+^2(t)\Phi_+^2\Big], \tag{9}$$

where we have allowed $Z_+$ and $M_+$ to depend on time through the background geometry or explicit couplings. Dimensional check:

$$[\Phi_+] = \text{mass}^1, \quad [\dot{\Phi}_+] = \text{mass}^2, \quad [\partial_i\Phi_+] = \text{mass}^2,$$
$$[Z_+] = 0, \quad [M_+^2] = \text{mass}^2, \quad [N_+ a_+^3 dt d^3x] = \text{mass}^{-4},$$

so $[S_\Phi(+)] = \text{mass}^0$.

The canonical momentum is

$$\Pi_+(t,x) = \frac{\partial \mathcal{L}_\Phi(+)}{\partial \dot{\Phi}_+} = N_+ a_+^3\, Z_+\left(-\frac{1}{N_+^2}\dot{\Phi}_+\right) = -\frac{a_+^3 Z_+}{N_+}\dot{\Phi}_+, \tag{10}$$

with

$$[\Pi_+] = \text{mass}^2,$$

consistent with $\{\Phi_+, \Pi_+\} = \delta^3$.

The Hamiltonian density is

$$\begin{aligned} \mathcal{H}_\Phi(+) &= \Pi_+\dot{\Phi}_+ - \mathcal{L}_\Phi(+) \\ &= \frac{a_+^3 Z_+}{2N_+}\dot{\Phi}_+^2 + \frac{N_+ a_+ Z_+}{2}\delta^{ij}\partial_i\Phi_+\partial_j\Phi_+ + \frac{N_+ a_+^3}{2}M_+^2\Phi_+^2. \end{aligned} \tag{11}$$

Thus

$$H_\Phi(+) = \int d^3x \left[\frac{a_+^3 Z_+}{2N_+}\dot{\Phi}_+^2 + \frac{N_+ a_+ Z_+}{2}\delta^{ij}\partial_i\Phi_+\partial_j\Phi_+ + \frac{N_+ a_+^3}{2}M_+^2\Phi_+^2\right]. \tag{12}$$

**Mode expansion.** On a spatially flat FRW background, we can expand $\Phi_+$ in comoving Fourier modes:

$$\Phi_+(t,x) = \int \frac{d^3k}{(2\pi)^3}\,\Phi_{+,k}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad [k^i] = \text{mass}^1, \tag{13}$$

with

$$\Phi^*_{+,k}(t) = \Phi_{+,-k}(t)$$

for a real field. The Hamiltonian becomes

$$H_\Phi(+) = \int \frac{d^3k}{(2\pi)^3}\,\frac{a_+^3 Z_+}{2N_+}\Big[|\dot{\Phi}_{+,k}|^2 + \Big(\frac{N_+^2 k^2}{a_+^2} + N_+^2 M_+^2\Big)|\Phi_{+,k}|^2\Big]. \tag{14}$$

Defining the canonically normalized mode variable

$$Q_k(+;t) = \sqrt{\frac{a_+^3 Z_+}{N_+}}\,\Phi_{+,k}(t), \tag{15}$$

one finds after some algebra that $H_\Phi(+)$ can be written as a continuum of decoupled oscillators:

$$H_\Phi(+) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\Big[|\dot{Q}_k(+)|^2 + \omega^2_{+,k}(t)\,|Q_k(+)|^2\Big], \tag{16}$$

with effective time-dependent frequency

$$\omega^2_{+,k}(t) = \frac{N_+^2 k^2}{a_+^2} + N_+^2 M_+^2 - \frac{\ddot{\big(a_+^3 Z_+/N_+\big)^{1/2}}}{\big(a_+^3 Z_+/N_+\big)^{1/2}}, \tag{17}$$

where the last term encodes cosmological redshift and field rescaling effects. Dimensional check:

$$[\omega^2_{+,k}] = \text{mass}^2, \quad [\omega_{+,k}] = \text{mass}^1.$$

Quantization proceeds exactly as in flat space, with sector label $(+)$ carried throughout:

$$\hat{Q}_k(+),\ \hat{P}_k(+) = \dot{\hat{Q}}_k(+), \quad [\hat{Q}_k(+), \hat{P}_{k'}(+)] = i\hbar\,(2\pi)^3\delta^3(k-k'),$$

and creation/annihilation operators

$$\hat{a}_k(+),\ \hat{a}^\dagger_k(+)$$

defined as usual in terms of $\hat{Q}_k(+)$ and $\hat{P}_k(+)$.

### 8.4 Scalar field and frequencies in the $(-)$ sector

A scalar field $\Phi_-$ in the $(-)$ sector is treated analogously, with action

$$S_\Phi(-) = \frac{1}{2}\int dt\,d^3x\,N_- a_-^3\,\Big[Z_-(t)\Big(-\frac{1}{N_-^2}\dot{\Phi}_-^2 + \frac{1}{a_-^2}\delta^{ij}\partial_i\Phi_-\partial_j\Phi_-\Big) - M_-^2(t)\Phi_-^2\Big], \tag{18}$$

and Fourier decomposition

$$\Phi_-(t,x) = \int \frac{d^3k}{(2\pi)^3}\,\Phi_{-,k}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Repeating the steps of Section 8.3 with $(+) \to (-)$, one obtains canonically normalized modes

$$Q_k(-;t) = \sqrt{\frac{a_-^3 Z_-}{N_-}}\,\Phi_{-,k}(t),$$

with Hamiltonian

$$H_\Phi(-) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\Big[|\dot{Q}_k(-)|^2 + \omega^2_{-,k}(t)\,|Q_k(-)|^2\Big],$$

and frequencies

$$\omega^2_{-,k}(t) = \frac{N_-^2 k^2}{a_-^2} + N_-^2 M_-^2 - \frac{\ddot{\left(a_-^3 Z_-/N_-\right)^{1/2}}}{\left(a_-^3 Z_-/N_-\right)^{1/2}}. \tag{19}$$

By construction, this is dimensionally and structurally identical to the $(+)$ sector, with all sector labels and background functions replaced accordingly.

The quantum algebra in the $(-)$ sector is independent:

$$[\hat{Q}_k(-), \hat{P}_{k'}(-)] = i\hbar\,(2\pi)^3\delta^3(k-k'), \quad [\hat{a}_k(-), \hat{a}^\dagger_{k'}(-)] = (2\pi)^3\delta^3(k-k'),$$

and all cross-sector commutators vanish:

$$[\hat{Q}_k(+), \hat{Q}_{k'}(-)] = 0, \quad [\hat{Q}_k(+), \hat{P}_{k'}(-)] = 0, \quad [\hat{a}_k(+), \hat{a}^\dagger_{k'}(-)] = 0, \dots$$

### 8.5 Backreaction and effective Friedmann equations

In the semiclassical approximation, the bimetric teleparallel field equations for the FRW background reduce to coupled Friedmann-like equations for $a_+(t)$ and $a_-(t)$, sourced by:

- classical fluids in each sector (if any);
- renormalized vacuum energies $\rho^{\mathrm{ren}}_{\mathrm{vac}}(\pm)$;
- particle excitations of $\Phi_+$ and $\Phi_-$;
- the HR potential $V(c)$ from (7).

Schematically, the $(+)$-sector Friedmann equation takes the form

$$3M_+^2 H_+^2 = \rho_+^{\mathrm{cl}} + \rho_+^{\mathrm{q}} + \rho_{\mathrm{HR}}(c) + \dots, \tag{20}$$

with analogous equation for the $(-)$ sector. Here:

- $\rho_+^{\mathrm{cl}}$ is classical matter/radiation in the $(+)$ sector;
- $\rho_+^{\mathrm{q}}$ is the finite, renormalized quantum energy density from $\Phi_+$;
- $\rho_{\mathrm{HR}}(c)$ is the effective energy density associated with the HR potential,
$$\rho_{\mathrm{HR}}(c) \sim M_+^2 m^2\Big[\beta_0 + 4\beta_1 c + 6\beta_2 c^2 + 4\beta_3 c^3 + \beta_4 c^4\Big];$$
- dots denote further EFT corrections from higher-order torsion/bimetric invariants.

Dimensional check:

$$[M_+^2 H_+^2] = \mathrm{mass}^2 \cdot \mathrm{mass}^2 = \mathrm{mass}^4, \quad [\rho_+^{\mathrm{cl}}] = [\rho_+^{\mathrm{q}}] = [\rho_{\mathrm{HR}}] = \mathrm{mass}^4,$$

consistent.

The equations for $a_-(t)$ similarly involve $\rho_-^{\mathrm{cl}}, \rho_-^{\mathrm{q}}$ and derivatives of $V(c)$ with respect to $c$, reflecting how the $(-)$ sector responds to the $(+)$ sector via the bimetric coupling.

This example shows explicitly that:

- sector-wise scalar QFT can be formulated and quantized on a teleparallel FRW background;
- the bimetric HR structure reduces to a simple polynomial in a scalar $c(t)$ in the proportional case;
- the resulting semiclassical equations are dimensionally consistent Friedmann-like relations with well-identified sources.

**Boxed summary: FRW worked example**

**Teleparallel FRW geometry (per sector):**

$$ds^2(\pm) = -N_\pm^2(t)\,dt^2 + a_\pm^2(t)\,\delta_{ij}dx^i dx^j, \quad T(\pm) = -6H_\pm^2,$$

$$H_\pm = \frac{1}{N_\pm}\frac{\dot{a}_\pm}{a_\pm}, \quad [H_\pm] = \text{mass}^1, \quad [T(\pm)] = \text{mass}^2.$$

**Bimetric HR potential for proportional FRW metrics:**

$$g_{\mu\nu}(-) = c^2(t) g_{\mu\nu}(+) \;\Rightarrow\; S^\mu{}_\nu = c(t)\delta^\mu{}_\nu,$$

$$V(S) = \beta_0 + 4\beta_1 c + 6\beta_2 c^2 + 4\beta_3 c^3 + \beta_4 c^4.$$

**Scalar QFT in the $(+)$ sector:**

$$S_\Phi(+) = \frac{1}{2}\int dt\, d^3x\, N_+ a_+^3 \Big[Z_+\Big(-\frac{\dot{\Phi}_+^2}{N_+^2} + \frac{\delta^{ij}\partial_i\Phi_+\partial_j\Phi_+}{a_+^2}\Big) - M_+^2\Phi_+^2\Big],$$

$$\Phi_+(t,x) = \int \frac{d^3k}{(2\pi)^3}\Phi_{+,k}(t)e^{i\mathbf{k}\cdot\mathbf{x}}, \quad Q_k(+;t) = \sqrt{\frac{a_+^3 Z_+}{N_+}}\,\Phi_{+,k}(t),$$

$$H_\Phi(+) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\Big[|\dot{Q}_k(+)|^2 + \omega_{+,k}^2(t)|Q_k(+)|^2\Big],$$

$$\omega_{+,k}^2(t) = \frac{N_+^2 k^2}{a_+^2} + N_+^2 M_+^2 - \frac{\ddot{\left((a_+^3 Z_+/N_+)^{1/2}\right)}}{(a_+^3 Z_+/N_+)^{1/2}}, \quad [\omega_{+,k}] = \text{mass}^1.$$

**Sector-wise quantization and Friedmann backreaction:**

$$[\hat{Q}_k(\pm), \hat{P}_{k'}(\pm)] = i\hbar\,(2\pi)^3\delta^3(k-k'), \quad [\hat{Q}_k(+), \hat{P}_{k'}(-)] = 0,$$

$$3M_+^2 H_+^2 = \rho_+^{\text{cl}} + \rho_+^{\text{q}} + \rho_{\text{HR}}(c) + \dots, \quad [\rho] = \text{mass}^4,$$

with analogous equations for the $(-)$ sector.
In this explicit FRW setting, all units, indices, and operator structures remain consistent, and the bimetric teleparallel framework functions as a concrete, computable EFT describing coupled visible and hidden cosmological sectors with fully specified scalar quantum fields.

## 9 Dirac Fields, Torsion Couplings, and Sector-Wise Spinor Dynamics

A distinctive feature of teleparallel gravity, relative to purely Riemannian GR, is the way in which torsion couples to spinor fields. In this section we:

- construct Dirac spinor actions in each bimetric teleparallel sector;
- decompose the spin connection into Levi–Civita plus contortion;
- isolate the axial torsion coupling and check its mass dimensions;
- describe the sector-wise quantum structure for spinors;
- identify where phenomenological constraints would enter, without fixing specific numerical bounds.

The goal is to show that the spinor sector fits cleanly into the same bimetric–teleparallel EFT architecture, with properly controlled torsion interactions.

### 9.1 Spin structure and Dirac action per sector

Assume the spacetime manifold $M$ admits a spin structure compatible with each metric $g_{\mu\nu}(\pm)$. For each sector $s \in \{+,-\}$ we introduce:

- a tetrad $e^a{}_\mu(s)$ with inverse $e_a{}^\mu(s)$;
- a spin connection $\omega_\mu{}^{ab}(s)$;
- Dirac gamma matrices $\gamma^a$ satisfying
$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \mathrm{diag}(-1,1,1,1);$$
- Dirac spinor fields $\psi_s(x)$, with
$$[\psi_s] = \mathrm{mass}^{3/2}.$$

The covariant derivative acting on $\psi_s$ is

$$D_\mu^{(s)}\psi_s = \partial_\mu\psi_s + \frac{1}{4}\,\omega_\mu{}^{ab}(s)\,\gamma_{ab}\,\psi_s, \qquad \gamma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b]. \tag{21}$$

The Dirac action in sector $s$ is

$$S_{\mathrm{D}}(s) = \int d^4x\, e(s)\, \bar{\psi}_s\big[i\gamma^a e_a{}^\mu(s) D_\mu^{(s)} - m_s\big]\psi_s, \tag{22}$$

with $m_s$ the sector-dependent fermion mass. Dimensional check:

- $[e(s)] = 0$;
- $[\bar{\psi}_s] = \mathrm{mass}^{3/2}$, $[\psi_s] = \mathrm{mass}^{3/2}$;
- $[\gamma^a e_a{}^\mu \partial_\mu] = \mathrm{mass}^1$;
- $[m_s] = \mathrm{mass}^1$;
- $[d^4x] = \mathrm{mass}^{-4}$;

so the integrand has dimension

$$[\bar{\psi}_s\,\gamma^a e_a{}^\mu \partial_\mu\psi_s] = \mathrm{mass}^{3/2}\cdot\mathrm{mass}^1\cdot\mathrm{mass}^{3/2} = \mathrm{mass}^4,$$

and $[S_{\mathrm{D}}(s)] = \mathrm{mass}^0$ as required.

### 9.2 Teleparallel spin connection and contortion

In teleparallel gravity, the spin connection is decomposed into a Levi–Civita part and the contortion:

$$\omega_\mu{}^{ab}(s) = \overset{\circ}{\omega}{}_\mu^{ab}(s) + K_\mu{}^{ab}(s), \tag{23}$$

where

- $\overset{\circ}{\omega}{}_\mu^{ab}(s)$ is the torsionless Levi–Civita spin connection built from $g_{\mu\nu}(s)$;
- $K_\mu{}^{ab}(s)$ is the contortion, encoding torsion effects.

The contortion is related to the torsion tensor $T^{\rho}{}_{\mu\nu}(s)$ by

$$K_{\mu}{}^{ab}(s) = \frac{1}{2}\Big[T^{a}{}_{\mu}{}^{b}(s) + T^{b}{}_{\mu}{}^{a}(s) - T_{\mu}{}^{ab}(s)\Big], \tag{24}$$

where tetrad indices are raised using $\eta^{ab}$ and spacetime indices using $g^{\mu\nu}(s)$.

Dimensional check:

$$[T^{\rho}{}_{\mu\nu}] = \text{mass}^1, \quad [K_{\mu}{}^{ab}] = \text{mass}^1, \quad [\omega_{\mu}{}^{ab}] = \text{mass}^1,$$

so $\omega_{\mu}{}^{ab}$ and $K_{\mu}{}^{ab}$ have the correct dimension for a connection.

Substituting (23) into (21),

$$D_{\mu}^{(s)}\psi_s = D_{\mu}^{\circ\,(s)}\psi_s + \frac{1}{4}K_{\mu}{}^{ab}(s)\gamma_{ab}\psi_s,$$

where $D_{\mu}^{\circ\,(s)}$ uses only $\omega_{\mu}^{\circ\,ab}(s)$ and is the usual torsionless spinor derivative.

The Dirac Lagrangian density in sector $s$ can be written as

$$\mathcal{L}_{\mathrm{D}}(s) = e(s)\,\bar{\psi}_s\Big[i\gamma^a e_a{}^{\mu}(s)D_{\mu}^{\circ\,(s)} - m_s\Big]\psi_s + \mathcal{L}_{\mathrm{tor}}(s), \tag{25}$$

with torsion-dependent piece

$$\mathcal{L}_{\mathrm{tor}}(s) = e(s)\,\bar{\psi}_s\Big[\frac{i}{4}\gamma^a e_a{}^{\mu}(s)K_{\mu}{}^{bc}(s)\gamma_{bc}\Big]\psi_s. \tag{26}$$

### 9.3 Axial torsion coupling and its mass dimension

It is well-known that for Dirac spinors, the irreducible part of torsion that actually couples to the spinor bilinear is the *axial* vector constructed from the totally antisymmetric component of the torsion:

$$S_{\mu}(s) = \frac{1}{6}\,\varepsilon_{\mu\nu\rho\sigma}\,T^{\nu\rho\sigma}(s), \tag{27}$$

with Levi–Civita symbol $\varepsilon_{\mu\nu\rho\sigma}$.

Dimensionally,

$$[T^{\nu\rho\sigma}] = \text{mass}^1 \;\Rightarrow\; [S_{\mu}] = \text{mass}^1.$$

Using gamma-matrix identities (specifically, $\gamma^a\gamma_{bc} \sim \eta^a{}_b\gamma_c - \eta^a{}_c\gamma_b + i\varepsilon^a{}_{bcd}\gamma^5\gamma^d$), one can show that (26) reduces, up to total derivatives, to a simple axial coupling:

$$\mathcal{L}_{\mathrm{tor}}(s) = e(s)\Big[-\frac{3}{4}S_{\mu}(s)\,\bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s\Big]. \tag{28}$$

Mass dimension check for the interaction term:

- $[S_{\mu}] = \text{mass}^1$;
- $[\bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s] = \text{mass}^3$;
- $[e(s)] = 0$;

so $[\mathcal{L}_{\mathrm{tor}}(s)] = \text{mass}^4$, as required for a 4D Lagrangian density.

The full Dirac Lagrangian becomes

$$\mathcal{L}_{\mathrm{D}}(s) = e(s)\,\bar{\psi}_s\Big[i\gamma^a e_a{}^{\mu}(s)D_{\mu}^{\circ\,(s)} - m_s\Big]\psi_s - \frac{3}{4}e(s)\,S_{\mu}(s)\,\bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s. \tag{29}$$

Thus torsion enters as an axial-vector background field $S_{\mu}(s)$ coupled to the axial current $J_5^{\mu} = \bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s$.

### 9.4 Sector-wise spinor EFT and teleparallel constraints

In the bimetric teleparallel framework, we have *two* such spinor sectors:

- $\psi_+$ coupled to the $(+)$-sector geometry via $e^a{}_\mu(+)$, $S_\mu(+)$;
- $\psi_-$ coupled to the $(-)$-sector geometry via $e^a{}_\mu(-)$, $S_\mu(-)$.

The total spinor action is

$$S_{\text{spinor}} = S_{\text{D}}(+) + S_{\text{D}}(-), \tag{30}$$

with each $S_{\text{D}}(s)$ defined as in (29) integrated over $d^4x$.

**No direct spinor mixing.** At the level of the EFT defined so far, there is no direct $\bar{\psi}_+\psi_-$ or $\bar{\psi}_+\gamma^\mu\psi_-$ term. This preserves the tensor product structure of the Hilbert space,

$$\mathcal{F}_{\text{spinor,tot}} = \mathcal{F}_{\text{spinor},+} \otimes \mathcal{F}_{\text{spinor},-},$$

and ensures that cross-sector mixing is mediated solely by the geometry through $V(S)$ and the teleparallel field equations.

**Teleparallel gauge choice.** In strict TEGR one often works in the Weitzenböck gauge $\omega_\mu{}^{ab}(s) = 0$, so the entire effect of torsion is carried by the tetrads. For spinors, however, it is often more convenient to retain a nontrivial inertial spin connection to keep the Dirac operator manifestly covariant. At the EFT level:

- One can choose a covariant teleparallel formulation in which $\omega_\mu{}^{ab}(s)$ is a pure gauge connection and torsion remains encoded in $K_\mu{}^{ab}(s)$;
- Physical torsion effects reside in $S_\mu(s)$ and thus in the axial coupling term (28);
- Gauge choices that set $\omega_\mu{}^{ab}(s)$ to zero shift torsion contributions into the tetrad, but the axial interaction form remains.

### 9.5 Canonical quantization and spinor mode expansion

In a background where both sectors admit a global timelike Killing vector or a well-defined notion of time (e.g. FRW, static spacetimes), we can define spinor mode expansions for each sector.

Schematically, in sector $s$:

$$\psi_s(x) = \sum_n \left[u_n^{(s)}(x)\,\hat{b}_n^{(s)} + v_n^{(s)}(x)\,\hat{d}_n^{\dagger\,(s)}\right], \tag{31}$$

where

- $u_n^{(s)}(x)$, $v_n^{(s)}(x)$ are mode functions solving the torsionful Dirac equation in sector $s$,

  $$\left[i\gamma^a e_a{}^\mu(s) D_\mu^{\circ\,(s)} - m_s - \frac{3}{4}\gamma^\mu\gamma^5 S_\mu(s)\right]u_n^{(s)}(x) = 0,$$

  and similarly for $v_n^{(s)}(x)$;
- $\hat{b}_n^{(s)}$, $\hat{d}_n^{(s)}$ satisfy canonical anticommutation relations:

  $$\{\hat{b}_n^{(s)}, \hat{b}_m^{\dagger\,(s)}\} = \delta_{nm}, \quad \{\hat{d}_n^{(s)}, \hat{d}_m^{\dagger\,(s)}\} = \delta_{nm},$$

  with all other anticommutators vanishing.

Cross-sector anticommutators vanish:

$$\{\hat{b}_n^{(+)}, \hat{b}_m^{(-)}\} = \{\hat{b}_n^{(+)}, \hat{d}_m^{(-)}\} = \{\hat{d}_n^{(+)}, \hat{d}_m^{(-)}\} = 0,$$

and similarly for their adjoints, enforcing the factorized Hilbert space.

The Hamiltonian in sector $s$ decomposes as

$$\hat{H}_{\mathrm{D}}(s) = \sum_n E_n^{(s)} \Big[ \hat{b}_n^{\dagger\,(s)} \hat{b}_n^{(s)} + \hat{d}_n^{\dagger\,(s)} \hat{d}_n^{(s)} \Big] + E_{\mathrm{vac}}^{(s)}, \tag{32}$$

where $E_n^{(s)}$ are the positive energy eigenvalues of the torsionful Dirac operator and $E_{\mathrm{vac}}^{(s)}$ is the (divergent) spinor vacuum energy, handled by the same renormalization machinery as in Section 7.

### 9.6 Phenomenological role of torsion in the $(+)$ sector

From the $(+)$-sector perspective, torsion acts as an axial background field $S_\mu(+)$ coupled to standard-model fermions via

$$\mathcal{L}_{\mathrm{axial}}(+) = -\frac{3}{4} e(+)\, S_\mu(+)\, \bar{\psi}_+ \gamma^\mu \gamma^5 \psi_+. \tag{33}$$

Observable consequences generically include:

- shifts in fermion dispersion relations;
- possible contributions to parity-violating observables;
- corrections to spin-precession and clock-comparison experiments;
- modified neutrino propagation in cosmological or astrophysical teleparallel backgrounds.

Within the EFT viewpoint, we do not assign specific numerical values to $S_\mu(+)$, but we note:

- In weak-field regimes, $S_\mu(+)$ can be treated as small compared to the characteristic mass scales of fermions;
- Bounds on such axial background couplings can be mapped to constraints on combinations of torsion components of the $(+)$-sector teleparallel geometry;
- The bimetric coupling can, in principle, induce effective torsion in the $(+)$ sector sourced by the $(-)$ sector, leading to additional indirect constraints on the $(-)$-sector configuration.

A detailed phenomenological analysis would require specifying a background solution $e^a{}_\mu(\pm)$, solving the torsionful Dirac equation, and comparing the resulting observables with experimental data, but the EFT architecture and dimensional consistency are already in place.

**Boxed summary: spinors and torsion in the bimetric teleparallel EFT**

**Dirac action per sector:**

$$S_{\mathrm{D}}(s) = \int d^4x\, e(s)\, \bar{\psi}_s\big[i\gamma^a e_a{}^{\mu}(s) D^{(s)}_{\mu} - m_s\big]\psi_s, \quad [\psi_s] = \mathrm{mass}^{3/2}.$$

**Teleparallel spin connection decomposition:**

$$\omega_{\mu}{}^{ab}(s) = \overset{\circ}{\omega}_{\mu}{}^{ab}(s) + K_{\mu}{}^{ab}(s), \quad K_{\mu}{}^{ab}(s) = \frac{1}{2}\Big[T^a{}_{\mu}{}^{b}(s) + T^b{}_{\mu}{}^{a}(s) - T_{\mu}{}^{ab}(s)\Big],$$

$$[T^{\rho}{}_{\mu\nu}] = \mathrm{mass}^1, \quad [K_{\mu}{}^{ab}] = \mathrm{mass}^1.$$

**Axial torsion and interaction:**

$$S_{\mu}(s) = \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}(s), \quad [S_{\mu}] = \mathrm{mass}^1,$$

$$\mathcal{L}_{\mathrm{tor}}(s) = -\frac{3}{4}e(s)\, S_{\mu}(s)\, \bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s, \quad [\bar{\psi}_s\gamma^{\mu}\gamma^5\psi_s] = \mathrm{mass}^3.$$

**Total spinor EFT in bimetric teleparallel framework:**

$$S_{\mathrm{spinor}} = S_{\mathrm{D}}(+) + S_{\mathrm{D}}(-), \quad \mathcal{F}_{\mathrm{spinor,tot}} = \mathcal{F}_{\mathrm{spinor},+} \otimes \mathcal{F}_{\mathrm{spinor},-},$$

$$\{\hat{b}^{(s)}_n, \hat{b}^{\dagger\,(s)}_m\} = \delta_{nm}, \quad \{\hat{d}^{(s)}_n, \hat{d}^{\dagger\,(s)}_m\} = \delta_{nm}, \quad \text{all cross-sector anticommutators vanish.}$$

$(+)$**-sector phenomenology:**

$$\mathcal{L}_{\mathrm{axial}}(+) = -\frac{3}{4}e(+)\, S_{\mu}(+)\, \bar{\psi}_+\gamma^{\mu}\gamma^5\psi_+,$$

where $S_{\mu}(+)$ is determined by the teleparallel torsion of the $(+)$ sector and, indirectly, by the $(-)$ sector through the bimetric field equations. Experimental bounds on axial background couplings translate into constraints on the allowed torsion configurations in the visible teleparallel geometry.
Thus, the spinor sector fits coherently into the quantum bimetric–teleparallel EFT, with well-defined torsion couplings, dimensions, and sector-wise quantum structure.

## 10 Gauge Fields, Standard Model Embedding, and Sector Decomposition

The previous sections established the scalar, spinor, and gravitational sectors in a bimetric–teleparallel effective field theory, including torsion couplings and semiclassical backreaction. This section adds the gauge sector and shows how a Standard-Model-like content can be embedded in the $(+)$ sector, with optional hidden gauge dynamics in the $(-)$ sector. The emphasis is on:

- kinematics of Abelian and non-Abelian gauge fields on each teleparallel background;
- index and dimensional consistency of Yang–Mills actions in both sectors;
- minimal coupling of spinors and scalars to gauge fields, with sector tags;
- the structure of possible cross-sector “portal” operators;
- the overall EFT hierarchy once gauge fields are included.

### 10.1 Abelian gauge fields in each sector

Consider an Abelian gauge field (e.g. electromagnetism) in sector $s \in \{+,-\}$, with potential $A_\mu^{(s)}(x)$ and field strength

$$F_{\mu\nu}^{(s)} = \partial_\mu A_\nu^{(s)} - \partial_\nu A_\mu^{(s)}, \qquad [A_\mu^{(s)}] = \mathrm{mass}^1, \qquad [F_{\mu\nu}^{(s)}] = \mathrm{mass}^2. \tag{34}$$

Indices are raised and lowered using the sector metric $g_{\mu\nu}(s)$:

$$F^{\mu\nu}_{(s)} = g^{\mu\rho}(s) g^{\nu\sigma}(s) F^{(s)}_{\rho\sigma}, \quad [g^{\mu\nu}(s)] = 0.$$

The Abelian gauge action in sector $s$ is

$$S_{\mathrm{EM}}(s) = -\frac{1}{4} \int d^4x\, e(s)\, F^{(s)}_{\mu\nu} F^{\mu\nu}_{(s)}. \tag{35}$$

Dimensional check:

$$[F_{\mu\nu} F^{\mu\nu}] = \mathrm{mass}^4, \quad [e(s)] = 0, \quad [d^4x] = \mathrm{mass}^{-4},$$

so $[S_{\mathrm{EM}}(s)] = \mathrm{mass}^0$.

The equations of motion are

$$\nabla_\mu^{(s)} F^{\mu\nu}_{(s)} = J^\nu_{(s)}, \tag{36}$$

where $\nabla_\mu^{(s)}$ is the metric-compatible covariant derivative for $g_{\mu\nu}(s)$ and $J^\nu_{(s)}$ is the gauge current sourced by charged matter in sector $s$.

In teleparallel language, $\nabla_\mu^{(s)}$ is equivalent (for tensors) to the Levi–Civita connection of $g_{\mu\nu}(s)$, so the Abelian gauge sector is kinematically identical to its Riemannian counterpart; torsion enters only indirectly through the tetrads in the matter currents.

### 10.2 Non-Abelian Yang–Mills fields in each sector

For a non-Abelian gauge group $G$ with generators $T^a$ in some representation, introduce gauge fields $A_\mu^a(s)$ in sector $s$, with

$$[A_\mu^a(s)] = \mathrm{mass}^1.$$

The Yang–Mills field strength is

$$F^a_{\mu\nu}(s) = \partial_\mu A^a_\nu(s) - \partial_\nu A^a_\mu(s) + g_s f^{abc} A^b_\mu(s) A^c_\nu(s), \tag{37}$$

where

$$g_s \text{ is the gauge coupling in sector } s, \quad [g_s] = \mathrm{mass}^0, \quad f^{abc} \text{ are the structure constants of } G.$$

Dimensional check:

$$[\partial_\mu A^a_\nu] = \mathrm{mass}^2, \quad [g_s f^{abc} A^b_\mu A^c_\nu] = \mathrm{mass}^2,$$

so $[F^a_{\mu\nu}] = \mathrm{mass}^2$.

The Yang–Mills action in sector $s$ is

$$S_{\mathrm{YM}}(s) = -\frac{1}{4} \int d^4x\, e(s)\, g_{\mu\rho}(s) g_{\nu\sigma}(s)\, F^{a\,\mu\nu}(s) F^{a\,\rho\sigma}(s). \tag{38}$$

Dimensional check:

$$[F^{a\,\mu\nu} F^a{}_{\mu\nu}] = \mathrm{mass}^4, \quad [e(s)] = 0, \quad [S_{\mathrm{YM}}(s)] = \mathrm{mass}^0.$$

The equations of motion are

$$D_\mu^{(s)} F^{a\,\mu\nu}(s) = J_{(s)}^{a\,\nu}, \tag{39}$$

with covariant derivative

$$D_\mu^{(s)} F^{a\,\mu\nu} = \nabla_\mu^{(s)} F^{a\,\mu\nu} + g_s f^{abc} A_\mu^b(s) F^{c\,\mu\nu}(s), \tag{40}$$

and $J_{(s)}^{a\,\nu}$ the matter current in the appropriate representation.

Again, torsion does not appear explicitly in the Yang–Mills field strength; it enters through the teleparallel geometry in the covariant derivative and in the matter sector currents.

### 10.3 Minimal coupling of matter to gauge fields

**Spinors.** For a Dirac spinor $\psi_s$ charged under a gauge group, minimal coupling is implemented by

$$D_\mu^{(s)} \psi_s = \partial_\mu \psi_s + \frac{1}{4}\omega_\mu{}^{ab}(s)\gamma_{ab}\psi_s + i g_s A_\mu^a(s) T^a \psi_s, \tag{41}$$

with the Dirac action

$$S_{\mathrm{D,gauge}}(s) = \int d^4x\, e(s)\, \bar{\psi}_s \big[ i\gamma^a e_a{}^\mu(s) D_\mu^{(s)} - m_s \big] \psi_s. \tag{42}$$

Dimensional check:

$$[g_s A_\mu^a T^a] = \mathrm{mass}^1, \quad [D_\mu^{(s)} \psi_s] = \mathrm{mass}^{5/2},$$

so the kinetic term maintains dimension $\mathrm{mass}^4$.

**Scalars.** For a scalar field $\Phi_s$ in a representation of $G$, the covariant derivative is

$$D_\mu^{(s)} \Phi_s = \partial_\mu \Phi_s + i g_s A_\mu^a(s) T^a \Phi_s, \tag{43}$$

and the gauge-invariant action is

$$S_{\Phi,\mathrm{gauge}}(s) = \frac{1}{2}\int d^4x\, e(s)\, [Z_s\, g^{\mu\nu}(s) D_\mu^{(s)} \Phi_s D_\nu^{(s)} \Phi_s - M_s^2 \Phi_s^2], \tag{44}$$

with the same dimensional assignments as in Section 5.

**Sector tags.** Every gauge quantity carries an explicit sector label $s = +, -$:

$$A_\mu^a(+),\ F_{\mu\nu}^a(+),\ g_+; \quad A_\mu^a(-),\ F_{\mu\nu}^a(-),\ g_-.$$

This enforces that visible-sector fields couple only to visible-sector gauge bosons unless explicit cross-sector operators are added.

### 10.4 Standard-Model-like embedding in the $(+)$ sector

A Standard-Model-like gauge structure is implemented by choosing, in the $(+)$ sector:

- $G_+ = SU(3)_c \times SU(2)_L \times U(1)_Y$;
- gauge fields $G_\mu^A(+)$ (gluons), $W_\mu^I(+)$ (weak), $B_\mu(+)$ (hypercharge);
- couplings $g_{3,+}, g_{2,+}, g_{1,+}$;
- Dirac and Weyl spinors representing quarks and leptons in representations of $G_+$;
- a scalar Higgs doublet $H_+$ with potential $V(H_+)$.

The full $(+)$-sector matter-plus-gauge Lagrangian takes the standard SM form, with every contraction performed using $g_{\mu\nu}(+)$ and $e(+)$. Torsion enters through:

- the spin connection in $D_\mu^{(+)}$ acting on fermions (Section 9);
- the teleparallel geometry in the definition of covariant derivatives and volume measures;
- possible higher-dimension torsion–current couplings in the EFT.

At the EFT level, the visible sector is thus a SM-like QFT on a teleparallel metric $g_{\mu\nu}(+)$, coupled to the $(-)$ sector only via the gravitational and HR structures.

### 10.5 Hidden gauge sector and portal operators

The $(-)$ sector may host its own hidden gauge group $G_-$ with fields $A_\mu^a(-)$ and matter $\psi_-, \Phi_-$, as in Sections 10.1–10.3. Direct interactions between $(+)$ and $(-)$ matter are, by design, absent at the renormalizable level to preserve the factorized Hilbert space $\mathcal{F}_+ \otimes \mathcal{F}_-$.

However, higher-dimension operators connecting the sectors are generically allowed in the EFT, provided they respect the gauge symmetries and diffeomorphism invariance of each sector. Examples include:

- **Gravitational portals:** already present through HR coupling and teleparallel gravitational actions:
$$e(+)\,V(S), \quad e(+)f_1(S)\,T(+), \quad e(-)f_2(S)\,T(-), \ldots$$
- **Kinetic mixings (Abelian):**
$$\mathcal{L}_{\text{mix}} = -\frac{\epsilon}{2}\,e(+)\,g^{\mu\rho}(+)g^{\nu\sigma}(+)F_{\mu\nu}^{(+)}F_{\rho\sigma}^{(-)},$$
with $\epsilon$ dimensionless and assumed small. Dimensional check: $[F^{(+)}F^{(-)}] = \text{mass}^4$, so $[\mathcal{L}_{\text{mix}}] = \text{mass}^4$.
- **Scalar portals:**
$$\mathcal{L}_{\text{scalar-portal}} = -\lambda_p\,e(+)\,\Phi_+^2\Phi_-^2,$$
with $[\Phi_\pm] = \text{mass}^1$, $[\lambda_p] = \text{mass}^0$, giving the correct 4D dimension.
- **Torsion portals:**
$$\mathcal{L}_{\text{torsion-portal}} = \frac{\eta}{M^2}\,e(+)\,S_\mu(+)\,S^\mu(-)\,\Phi_+^2,$$
with $[S_\mu] = \text{mass}^1$, $[\Phi_+^2] = \text{mass}^2$, and $[M] = \text{mass}^1$, so $[\eta/M^2] = \text{mass}^{-2}$ and $[\mathcal{L}_{\text{torsion-portal}}] = \text{mass}^4$.

All such terms are suppressed by appropriate mass scales (often $M \sim M_{\text{Pl}}$ or higher), ensuring that:

- cross-sector interactions are weak at accessible energies;
- the theory remains under EFT control;
- visible-sector phenomenology is dominated by $(+)$-sector SM-like physics with small corrections.

### 10.6 Operator hierarchy and global EFT structure

Combining all sectors, the global EFT Lagrangian can be organized as

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{SM}}^{(+)} + \mathcal{L}_{\text{hidden}}^{(-)} + \mathcal{L}_{\text{portal}}, \tag{45}$$

with:

- $\mathcal{L}_{\text{grav}}$: bimetric teleparallel sector with HR potential and higher-order torsion invariants (Sections **??**–7);
- $\mathcal{L}_{\text{SM}}^{(+)}$: Standard-Model-like QFT on teleparallel $(+)$ background with Dirac, scalar, and gauge fields (Sections 9, 10.4);
- $\mathcal{L}_{\text{hidden}}^{(-)}$: hidden $(-)$-sector fields and gauge dynamics, structurally analogous to $\mathcal{L}_{\text{SM}}^{(+)}$ but not directly observed;
- $\mathcal{L}_{\text{portal}}$: higher-dimension operators suppressed by large mass scales, encoding possible cross-sector communication beyond gravity alone.

Each term in $\mathcal{L}_{\text{tot}}$ has dimension mass$^4$, and all fields and couplings carry consistent mass dimensions as verified in the preceding sections. The bimetric teleparallel geometry provides the unified background on which all these sectors coexist and interact.

**Boxed summary: gauge and SM embedding in the bimetric–teleparallel EFT**

**Abelian and non-Abelian gauge fields per sector:**

$$S_{\rm EM}(s) = -\frac{1}{4}\int d^4x\, e(s)\, F^{(s)}_{\mu\nu}F^{\mu\nu}_{(s)}, \quad [A^{(s)}_\mu] = \text{mass}^1,\ [F^{(s)}_{\mu\nu}] = \text{mass}^2,$$

$$S_{\rm YM}(s) = -\frac{1}{4}\int d^4x\, e(s)\, g_{\mu\rho}(s) g_{\nu\sigma}(s)\, F^{a\,\mu\nu}(s) F^{a\,\rho\sigma}(s), \quad [g_s] = \text{mass}^0.$$

**Minimal coupling of matter:**

$$D^{(s)}_\mu \psi_s = \partial_\mu \psi_s + \frac{1}{4}\omega_\mu{}^{ab}(s)\gamma_{ab}\psi_s + i g_s A^a_\mu(s) T^a \psi_s,$$

$$D^{(s)}_\mu \Phi_s = \partial_\mu \Phi_s + i g_s A^a_\mu(s) T^a \Phi_s,$$

$$S_{\rm D,gauge}(s) = \int d^4x\, e(s)\, \bar\psi_s\big[i\gamma^a e_a{}^\mu(s) D^{(s)}_\mu - m_s\big]\psi_s,$$

$$S_{\Phi,\rm gauge}(s) = \frac{1}{2}\int d^4x\, e(s)\, [Z_s g^{\mu\nu}(s) D^{(s)}_\mu \Phi_s D^{(s)}_\nu \Phi_s - M_s^2 \Phi_s^2].$$

**Standard-Model-like visible sector:**

$$G_+ = SU(3)_c \times SU(2)_L \times U(1)_Y,$$

with quarks, leptons, Higgs, and gauge fields defined on the teleparallel background $(e^a{}_\mu(+), g_{\mu\nu}(+), T(+))$, and torsion entering via the spin connection and higher-dimension operators.

**Hidden gauge sector and portals:**

$$\mathcal{L}^{(-)}_{\rm hidden} = \text{Yang–Mills} + \text{matter on } (e^a{}_\mu(-), g_{\mu\nu}(-), T(-)),$$

$$\mathcal{L}_{\rm portal} = -\frac{\epsilon}{2} F^{(+)}_{\mu\nu} F^{(-)\,\mu\nu} - \lambda_p\, \Phi_+^2 \Phi_-^2 + \frac{\eta}{M^2} S_\mu(+) S^\mu(-) \Phi_+^2 + \dots,$$

all of dimension mass$^4$ and suppressed by large mass scales.

**Global EFT structure:**

$$\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm grav} + \mathcal{L}^{(+)}_{\rm SM} + \mathcal{L}^{(-)}_{\rm hidden} + \mathcal{L}_{\rm portal},$$

with every field, index, and operator dimension checked for consistency. The bimetric teleparallel background thus supports a fully-fledged quantum gauge theory in the $(+)$ sector, an optional hidden gauge theory in the $(-)$ sector, and controlled cross-sector interactions within a single effective framework.

## 11 Global Consistency, EFT Scales, and Phenomenological Regimes

The previous sections constructed a full effective field theory on a bimetric– teleparallel background, including:

- two teleparallel gravitational sectors $(+)$ and $(-)$ with Hassan–Rosen bimetric coupling;
- scalar and spinor matter with sector-wise quantization;
- gauge fields and a Standard-Model-like visible sector in $(+)$;
- a hidden $(-)$ sector and higher-dimension portal operators.

This section assembles the global consistency conditions that must hold for the framework to function as a predictive EFT:

- dimensional and index consistency across all sectors (already checked locally) must extend to the full action;
- unitarity and ghost-freedom in the gravitational and matter sectors;
- causality and hyperbolicity of the field equations;
- EFT power counting and the role of the cutoff scale;
- phenomenological regimes where the EFT description is reliable.

### 11.1 Full action, dimensions, and index structure

Collecting the components, the total action has the schematic form

$$S_{\rm tot} = S_{\rm grav}[e^a{}_\mu(\pm)] + S^{(+)}_{\rm SM}[\Phi_+, \psi_+, A^{(+)}_\mu, e^a{}_\mu(+)] + S^{(-)}_{\rm hidden}[\Phi_-, \psi_-, A^{(-)}_\mu, e^a{}_\mu(-)] + S_{\rm portal}, \quad (46)$$

where:

- $S_{\rm grav}$ is the double-TEGR plus Hassan–Rosen potential sector (Sections **??**, **??**);
- $S^{(+)}_{\rm SM}$ is a SM-like gauge–Yukawa–Higgs theory on the $(+)$ teleparallel background (Sections 9, 10);
- $S^{(-)}_{\rm hidden}$ is an analogous hidden sector on $(-)$;
- $S_{\rm portal}$ contains higher-dimension cross-sector operators.

Every contribution is of the general form

$$S = \int d^4x\, e(s)\, \mathcal{L}(x),$$

with

$$[d^4x] = \text{mass}^{-4}, \quad [e(s)] = 0, \quad [\mathcal{L}] = \text{mass}^4,$$

so $[S] = \text{mass}^0$ across all sectors.

**Index structure.** The fundamental index rules are:

- Greek indices $\mu, \nu, \ldots$ for spacetime, raised/lowered with $g_{\mu\nu}(s)$ and $g^{\mu\nu}(s)$ for each sector;
- Latin indices $a, b, \ldots$ for Lorentz (tangent space), raised/ lowered with $\eta_{ab}$ and $\eta^{ab}$;
- explicit sector labels $(+)$ and $(-)$ attached to all geometrical and dynamical quantities that differ by sector.

All actions $S_{\rm grav}$, $S^{(+)}_{\rm SM}$, and $S^{(-)}_{\rm hidden}$ obey the index rules:

- every contraction pairs an upper index with a lower index of the same type (spacetime or Lorentz) *within a single sector*;
- cross-sector index mixing appears only through the HR matrix $S^\mu{}_\nu = \sqrt{g^{(+)\,-1} g^{(-)}}$ and its scalar invariants $e_n(S)$, which are dimensionless;
- no bare terms of the form $g_{\mu\nu}(+) g^{\mu\nu}(-)$ appear; instead, mixed objects are always built via $S^\mu{}_\nu$ or its powers.

This ensures that all index contractions are well-defined and that the $(+)$ and $(-)$ geometries remain distinct, interacting only through the controlled bimetric potential.

### 11.2 Ghost-freedom and unitarity in the gravitational sector

The Hassan–Rosen construction is designed so that, in the purely metric bimetric theory, the constraint algebra removes the Boulware–Deser ghost and leaves one massless and one massive spin-2 mode. In the teleparallel version considered here:

- each sector's TEGR piece is equivalent (up to a boundary term) to an Einstein–Hilbert action for $g_{\mu\nu}(s)$, with no additional local degrees of freedom beyond the usual massless graviton;
- the HR potential $V(S)$ is used *unchanged* as a scalar functional of $S^{\mu}{}_{\nu}$ built from the sector metrics;
- we do not introduce torsion-dependent deformations inside $V(S)$ that would alter the known constraint structure.

At the EFT level, the working assumption is:

> The teleparallel bimetric theory inherits the ghost-free structure of the Hassan–Rosen metric theory, since TEGR is equivalent to GR at the level of propagating degrees of freedom and the HR potential is kept in the same functional form of the two metrics.

Within this assumption, the gravitational Hilbert space decomposes into:

- a massless spin-2 mode (equivalent to the usual graviton);
- a massive spin-2 mode with mass $\sim m$ (the bimetric scale);
- no negative-norm (ghost) spin-2 states.

Unitarity at tree level is thus preserved in the spin-2 sector, provided:

- the HR coefficients $\beta_n$ are chosen in the standard ghost-free domain;
- higher-derivative corrections in the gravitational sector are treated as small EFT perturbations, suppressed by some cutoff $\Lambda_{\rm grav}$.

A full canonical analysis in the teleparallel variables would explicitly verify the constraint algebra, but structurally the construction has been chosen to match the known ghost-free bimetric theory.

### 11.3 Causality, hyperbolicity, and sector cones

For the EFT to be predictive and causal:

- each metric $(M, g_{\mu\nu}(s))$ must be globally hyperbolic, admitting Cauchy surfaces $\Sigma_t(s)$;
- the principal symbol of each field equation (scalar, spinor, gauge) must be hyperbolic with respect to the corresponding metric;
- cross-sector interactions must not generate acausal propagation in the $(+)$ sector, which we identify as the physical sector for observers.

Concretely:

- scalar fields obey second-order hyperbolic equations with principal part $\sim g^{\mu\nu}(s)\partial_\mu\partial_\nu$;

- Dirac fields obey first-order hyperbolic equations $\sim \gamma^a e_a{}^\mu(s)\partial_\mu$;
- Yang–Mills fields obey hyperbolic equations with principal part $\sim g^{\mu\nu}(s)\partial_\mu\partial_\nu$ in an appropriate gauge.

The bimetric coupling enters as lower-derivative terms in the metric field equations, so the principal part remains governed by each $g_{\mu\nu}(s)$. Hence:

- characteristic cones for matter in sector $s$ coincide with the null cones of $g_{\mu\nu}(s)$;
- no superluminal propagation arises at leading order if the HR parameters are in the usual stable domain and higher-derivative operators are EFT suppressed.

The remaining nontrivial requirement is that the two cone structures (from $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$) never combine to generate closed timelike curves as seen by $(+)$-sector observers. At the EFT level, this is encoded in:

- restricting to bimetric solutions in which both metrics share a common global time orientation;
- rejecting backgrounds that develop multi-cone closed timelike loops.

This is a condition on the allowed classical backgrounds, not on the local field content.

### 11.4 EFT power counting and cutoff scales

The full theory is treated as an EFT valid below some cutoff $\Lambda$, which can be different for the gravitational and matter sectors. The Lagrangian can be expanded as

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{ren}} + \sum_{n>4} \frac{1}{\Lambda^{n-4}} O_n(x), \tag{47}$$

where:

- $\mathcal{L}_{\text{ren}}$ contains operators of mass-dimension $\leq 4$ (renormalizable and marginal) in 4D:
  - teleparallel bimetric gravity at two-derivative order;
  - kinetic terms for scalars, spinors, and gauge fields;
  - renormalizable interactions (Yukawa, quartic scalar, gauge).
- $O_n$ are gauge- and diffeomorphism-invariant operators of dimension $n > 4$, including:
  - higher powers of torsion and curvature built from $e^a{}_\mu(s)$;
  - nonminimal torsion–current couplings;
  - cross-sector portal operators (scalar, kinetic, torsion portals).

Power counting is straightforward:

- each derivative contributes one power of mass;
- each field carries its canonical mass dimension:

$$[\Phi] = 1, \quad [\psi] = 3/2, \quad [A_\mu] = 1, \quad [e^a{}_\mu] = 0,$$

- each operator's dimension is the sum of derivatives and field dimensions.

The EFT is under control provided that:

- typical physical scales $E$ satisfy $E \ll \Lambda$;
- loop corrections are systematically renormalized into the coefficients of $\mathcal{O}_n$;
- the HR scale $m$ and Planck-like scales $M_\pm$ are below the ultimate UV completion but above the energies of interest.

In particular, quantum corrections from both sectors generate higher-dimension operators involving $S^\mu{}_\nu$ and torsion invariants, but these are suppressed by powers of $\Lambda$ and do not spoil the low-energy consistency of the theory.

### 11.5 Phenomenological regimes and observable imprints

The bimetric–teleparallel EFT naturally decomposes into distinct phenomenological regimes:

**1. Cosmological regime.** Here the FRW example of Section 8 is representative. Key features:

- modified Friedmann equations due to HR potential $\rho_{\mathrm{HR}}(c)$;
- possible effective dark sector contributions from $(-)$-sector vacuum energy and excitations;
- torsion-induced corrections to spinor propagation in the early universe.

These can, in principle, affect cosmic expansion history, structure formation, and CMB anisotropies, but remain encoded in EFT parameters $(\beta_n, m, M_\pm)$, $Z_\pm(t)$, $M_\pm(t)$, and portal couplings.

**2. Astrophysical/strong-gravity regime.** In compact objects and gravitational waves:

- the massive spin-2 mode may affect waveforms and strong-field lensing;
- torsion can influence spin-polarized matter and plasmas;
- teleparallel formulations can introduce effective force laws expressed through contortion, which reorganize the dynamical interpretation without changing observables at the TEGR level.

Any deviations from GR are controlled by $m$ and HR coefficients, with the teleparallel rewriting remaining observationally equivalent at leading order in the absence of explicit torsion couplings beyond TEGR.

**3. Laboratory and particle physics regime.** At collider and precision-physics scales:

- direct HR effects are typically negligible if $m$ is near cosmological scales;
- torsion appears as small axial-vector backgrounds in spinor sectors (Section 9);
- portal operators can generate rare processes or tiny shifts in SM parameters.

These provide possible constraints on torsion components and portal couplings, but the EFT remains consistent so long as $E \ll \Lambda$.

### 11.6 Summary: functioning effective quantum bimetric–teleparallel framework

From the perspective of effective field theory, the construction achieves:

- a double-TEGR gravitational backbone, equivalent to two copies of GR at the level of propagating degrees, with a ghost-free HR coupling between the two metrics;
- sector-wise QFT for scalars, spinors, and gauge fields on each teleparallel background, with canonical quantization and well-defined Hilbert spaces;
- explicit torsion couplings to spinors (axial currents) and controlled torsion contributions to higher-dimension operators;
- a SM-like visible sector in $(+)$ and an optional hidden sector in $(-)$, both embedded consistently on the bimetric–teleparallel geometry;
- a clear EFT power-counting scheme and operator hierarchy, allowing systematic inclusion of quantum corrections and portal interactions.

All local checks—dimensions, index contractions, variable and operator assignments—are consistent. The remaining challenges (full canonical proof of ghost-freedom in teleparallel variables, detailed renormalization, and phenomenological parameter extraction) are higher-level tasks that can be pursued within this well-defined structural framework.

**Boxed summary: global consistency and EFT viability**

**Total EFT structure:**

$$S_{\rm tot} = S_{\rm grav}[e^a{}_\mu(\pm)] + S^{(+)}_{\rm SM}[\text{matter}, g(+)] + S^{(-)}_{\rm hidden}[\text{matter}, g(-)] + S_{\rm portal},$$

$$[S_{\rm tot}] = \text{mass}^0, \quad [\mathcal{L}_{\rm tot}] = \text{mass}^4.$$

**Ghost-freedom and unitarity (assumed from HR + TEGR equivalence):**

$$1 \text{ massless spin-2 mode} + 1 \text{ massive spin-2 mode}, \quad \text{no Boulware–Deser ghost},$$

with HR potential $V(S)$ kept in its standard (metric-based) form and TEGR providing an equivalent description of each metric sector.

**Causality and hyperbolicity:**

$$\text{Principal parts:} \quad g^{\mu\nu}(s)\partial_\mu\partial_\nu \text{ (bosons)}, \quad \gamma^a e_a{}^\mu(s)\partial_\mu \text{ (spinors)},$$

$$\text{Each sector } (M, g_{\mu\nu}(s)) \text{ globally hyperbolic}, \quad \text{no multi-cone closed timelike curves in allowed backgrounds}$$

**EFT expansion and cutoff:**

$$\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm ren} + \sum_{n>4} \frac{1}{\Lambda^{n-4}} O_n, \quad [\Lambda] = \text{mass}^1,$$

with all higher-dimension operators gauge invariant and diffeomorphism invariant in each sector, suppressed for $E \ll \Lambda$.

**Phenomenological regimes:**

Cosmology: FRW bimetric teleparallel dynamics with HR energy density $\rho_{\rm HR}(c)$,

Astrophysics: massive spin-2 and torsion effects in strong gravity,

Laboratory: small axial torsion backgrounds and portal-suppressed processes.

**Conclusion:** Within the EFT domain ($E \ll \Lambda$) and under standard HR ghost-free assumptions, the bimetric–teleparallel construction defines a functioning, internally consistent quantum framework. It systematically unifies:

(1) double teleparallel gravity, (2) sector-wise QFT, (3) controlled cross-sector couplings,

with all units, indices, variables, and operators checked for consistency.

## 12 Conclusions and Outlook

We now summarize how the preceding construction yields a functioning effective quantum bimetric–teleparallel framework, and outline the most important directions for refinement and application.

### 12.1 What has been built: structural summary

From first principles, we have assembled the following ingredients:

**(1) Geometric backbone:** two teleparallel sectors $(+)$ and $(-)$, each described by tetrads $e^a{}_\mu(s)$, Weitzenböck connection, torsion $T^\rho{}_{\mu\nu}(s)$, and torsion scalar $T(s)$:

- TEGR equivalence: each sector reproduces Einstein gravity at the level of propagating degrees of freedom;
- all geometric objects are dimensionally consistent with $[T] = \text{mass}^2$ and $[e] = 0$.

**(2)** **Bimetric coupling:** a ghost-free Hassan–Rosen potential $V(S)$ built from the matrix square root $S^\mu{}_\nu = \sqrt{g^{(+)\,-1} g^{(-)}}$, with

- dimensionless scalar invariants $e_n(S)$;
- well-defined index structure confined to metric combinations;
- a mass parameter $m$ setting the scale of the massive spin-2 mode.

**(3)** **Quantum fields per sector:** scalar, spinor, and gauge fields defined independently in each sector:

- canonical dimensions: $[\Phi] = 1$, $[\psi] = 3/2$, $[A_\mu] = 1$;
- teleparallel-compatible actions with explicit use of $e(s)$ and $g_{\mu\nu}(s)$;
- normal-mode decomposition on Cauchy slices $\Sigma_t(s)$ and canonical quantization leading to sector Fock spaces $\mathcal{F}_{(+)}$ and $\mathcal{F}_{(-)}$.

**(4)** **Spinor–torsion interactions:** a clean axial-vector coupling $S_\mu(s)\bar{\psi}_s\gamma^\mu\gamma^5\psi_s$ with

- $S_\mu(s)$ constructed from totally antisymmetric torsion;
- correct mass dimensions and operator structure;
- sector-wise quantization preserving the tensor-product Hilbert space $\mathcal{F}_{\text{spinor},+} \otimes \mathcal{F}_{\text{spinor},-}$.

**(5)** **Gauge and Standard-Model embedding:**

- Abelian and non-Abelian gauge fields with Yang–Mills actions in each sector, respecting $[F_{\mu\nu}] = 2$, $[g_s] = 0$;
- minimal coupling of matter via covariant derivatives consistent with teleparallel geometry;
- a SM-like visible sector in $(+)$ with full gauge–Yukawa–Higgs structure on $g_{\mu\nu}(+)$.

**(6)** **Hidden sector and portals:**

- a hidden $(-)$ gauge–matter sector of analogous form;
- higher-dimension portal operators (scalar, kinetic, torsion) with explicit power counting and suppression by scales $\Lambda$;
- all cross-sector operators dimensionally consistent and gauge invariant.

**(7)** **EFT organization and regimes:**

- separation of renormalizable core $\mathcal{L}_{\text{ren}}$ and higher-dimension operators $\mathcal{O}_n/\Lambda^{n-4}$;
- identification of cosmological, astrophysical, and laboratory regimes with distinct dominant effects;
- explicit checks that $[\mathcal{L}_{\text{tot}}] = 4$ and $[S_{\text{tot}}] = 0$ in natural units.

Throughout, we have:

- enforced consistent index contractions within each sector and through $S^\mu{}_\nu$ for the HR coupling;
- verified canonical field dimensions and operator dimensions in all sectors;
- maintained a sector-tagging discipline $(+)$, $(-)$ on every geometric and dynamical quantity.

### 12.2 Why the construction is physically justified

The physical justification for this architecture rests on three pillars:

**(A) Teleparallel equivalence with GR, extended to two sectors.** Each sector's TEGR action is equivalent to Einstein–Hilbert up to a boundary term. This guarantees:

- the same massless spin-2 content as GR in each sector;
- access to standard geometric intuition via metric $g_{\mu\nu}(s)$;
- the ability to rewrite everything in Riemannian language if desired, while keeping torsion as a dynamical bookkeeping device.

**(B) Hassan–Rosen ghost-free bimetric structure.** By keeping the HR potential in its standard form as a scalar function of $S^{\mu}{}_{\nu}$, we import:

- the well-understood removal of the Boulware–Deser ghost;
- a controlled massive spin-2 mode characterized by a single scale $m$;
- a framework capable of producing viable cosmological and gravitational solutions in many regimes.

**(C) Standard-model QFT on a curved (teleparallel) background.** The matter and gauge sectors are built by:

- starting from textbook QFT in flat space, then lifting to curved space with $g_{\mu\nu}(s)$ and $e(s)$;
- adding teleparallel spin connections and torsion in the standard way for spinors;
- preserving canonical commutation/anticommutation relations and Fock space constructions.

Thus, the quantum field content is as conservative as possible, with the new features residing in:

- the bimetric HR coupling between $g_{\mu\nu}(+)$ and $g_{\mu\nu}(-)$;
- torsion-induced axial couplings;
- higher-dimension portal operators allowed in EFT.

### 12.3 How it functions as an effective quantum framework

As an effective quantum theory, the framework functions by:

**(a) Fixing backgrounds:** choose classical solutions $e^{a}{}_{\mu}(\pm)$ of the bimetric teleparallel field equations (including HR contributions and classical matter sources).

**(b) Quantizing fluctuations:** quantize scalar, spinor, and gauge fields (and, in principle, metric fluctuations) on these backgrounds using canonical or path-integral methods.

**(c) Computing observables:** derive spectra, correlation functions, and effective stress–energy tensors in each sector, with cross-sector effects entering via $S^{\mu}{}_{\nu}$, torsion, and portal operators.

**(d) Iterating backreaction:** feed $\langle T^{(\pm)}_{\mu\nu} \rangle$ back into the semiclassical bimetric–teleparallel field equations to refine the classical backgrounds (as far as the EFT expansion remains valid).

The mechanism is fully consistent at the level of:

- dimensions and units in all actions and operators;
- index and variable assignments across sectors;
- canonical quantization rules and Hilbert-space factorization;
- EFT power counting and suppression of higher-dimension operators.

### 12.4 Open problems and next steps

Several key problems remain open but are now sharply posed within this framework:

- **Canonical constraint analysis in teleparallel variables:** demonstrate ghost-freedom directly in the double-TEGR formulation with HR potential, confirming the equivalence to the known metric-based analysis.
- **Renormalization and running parameters:** construct a systematic renormalization scheme on the bimetric–teleparallel background, tracking:
  - running of HR parameters $\beta_n$;
  - renormalization of torsion–spinor couplings;
  - induced higher-dimension operators from loops in both sectors.
- **Vacuum energy and cosmological constant:** quantify how $(+)$ and $(-)$ vacuum energies and $V(S)$ combine into an effective cosmological constant, and whether any nontrivial cancellations occur without fine-tuning.
- **Causal structure and multi-cone stability:** classify bimetric solutions that respect a common global time function and exclude multi-cone closed timelike curves, thereby defining the physically admissible background space.
- **Phenomenological constraints:** map torsion, massive spin-2, and portal effects to:
  - cosmological observations (background expansion, perturbations);
  - gravitational waves and compact objects;
  - precision tests (spin precession, EDMs, rare decays).

Each of these problems can be addressed within the current formal architecture without altering its basic structure.

**Boxed summary: status and outlook of the quantum bimetric–teleparallel EFT**

**Status:**

- Two teleparallel sectors $(+), (-)$ with properly defined tetrads, torsion, and torsion scalars, coupled via a ghost-free HR potential $V(S)$.
- Sector-wise quantum field theories for scalars, spinors, and gauge fields, with:

$$[\Phi] = 1, \quad [\psi] = 3/2, \quad [A_\mu] = 1, \quad [e] = 0, \quad [T] = 2,$$

  and canonical commutation/anticommutation relations.
- Explicit axial torsion couplings to spinors and controlled portal operators connecting visible $(+)$ and hidden $(-)$ sectors.
- A SM-like visible sector embedded on $g_{\mu\nu}(+)$, with all units, indices, and operators checked for consistency.
- EFT organization with renormalizable core and higher-dimension operators suppressed by a cutoff $\Lambda$.

**Outlook:**

- Canonical teleparallel constraint analysis to confirm ghost-freedom at the level of tetrads and torsion.
- Systematic renormalization and running of HR and torsion couplings.
- Detailed cosmological, astrophysical, and laboratory phenomenology to identify distinctive signatures of the massive spin-2 mode, torsion, and cross-sector portals.
- Classification of admissible bimetric–teleparallel backgrounds that preserve global hyperbolicity and avoid multi-cone causal pathologies.

**Conclusion:** The framework defines a fully specified, dimensionally consistent, index-safe effective quantum theory in which:

$$(\text{double teleparallel gravity}) + (\text{HR bimetric coupling}) + (\text{sector-wise QFT}) + (\text{EFT portal structure})$$

coexist within a single, mathematically controlled architecture. All further refinements—canonical proofs, renormalization, and phenomenological fits—can be built on this foundation without altering its core consistency.