Section 1 — Thermodynamic Equilibrium Between Spacetime Sheets: Foundations, Laws, Derivations, and Checks

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Contents

1 System, State Variables, and Equilibrium Target

We model two teleparallel sheets $(\mathcal{M}_(+), g_(+))$ and $(\mathcal{M}_(-), g_(-))$ with sheet-local matter sectors and a weak interphasic regulator phase θ that mediates exchange.

State variables

Per sheet, extensive $(U_\pm, S_\pm, N_\pm, V_\pm)$ and intensive $(T_\pm, \mu_\pm, p_\pm)$. Cosmological coarse-grained densities $\rho_{m\pm}, \rho_{r\pm}, \rho_{\Lambda\pm}$, Hubble rates $H_\pm$, and exchange sources $Q_\pm$ in the Friedmann closures.

Equilibrium set (target)

$$T_(+) = T_(-), \qquad \mu_(+) = \mu_(-), \qquad \dot{\theta} = 0, \qquad Q_(+) = Q_(-) = 0 \quad (Q_(+) + Q_(-) = 0).$$

2 First Law, Continuity, and Unit-Consistent Closures

First law per sheet

$$\mathrm{d}U_\pm = T_\pm\,\mathrm{d}S_\pm - p_\pm\,\mathrm{d}V_\pm + \mu_\pm\,\mathrm{d}N_\pm + \mathrm{d}W_\pm, \qquad \dot{W}_\pm = \int_V Q_\pm\,\mathrm{d}^3x.$$

Continuity with antisymmetric exchange (FLRW)

$$\dot{\rho}_{m\pm} + 3H_\pm\rho_{m\pm} = \pm\alpha_m\,\dot{\theta}, \qquad \dot{\rho}_{r\pm} + 4H_\pm\rho_{r\pm} = \pm\alpha_r\,\dot{\theta}.$$

Units check: $[\rho] = \mathrm{kg\,m^{-3}}$, $[H] = \mathrm{s^{-1}}$, $[\alpha_m] = [\alpha_r] = \mathrm{kg\,m^{-3}}$, $[\dot{\theta}] = \mathrm{s^{-1}}$, so both sides are $\mathrm{kg\,m^{-3}\,s^{-1}}$.

Friedmann pair with sources

$$3H_\pm^2 = 8\pi G_\pm\big(\rho_{m\pm} + \rho_{r\pm} + \rho_{\Lambda\pm}\big) + Q_\pm, \qquad [Q_\pm] = \mathrm{s^{-2}}.$$

Detailed balance gives $Q_(+) + Q_(-) = 0$ at equilibrium.

3 Regulator Dynamics and Lyapunov Structure

Regulator equation (coarse-grained)

$$\chi\,\ddot{\theta} + \gamma_c\,\dot{\theta} + m_\theta^2\,\theta = \mathcal{S}(t),$$

with $\chi > 0$, $\gamma_c > 0$, $m_\theta > 0$. Dimensions: $[\chi] = 1$, $[\gamma_c] = \mathrm{s^{-1}}$, $[m_\theta] = \mathrm{s^{-1}}$, $[\theta] = 1$, $[\mathcal{S}] = \mathrm{s^{-2}}$; every term is $\mathrm{s^{-2}}$.

Flux–force map and entropy production

Define forces and fluxes near equilibrium:

$$\mathcal{F}_1 = \frac{\Delta T}{T^2}, \quad \mathcal{F}_2 = \frac{\Delta\mu}{T}, \quad \mathcal{F}_3 = \frac{\partial\Phi}{\partial\theta}; \qquad \mathcal{J}_1 = \dot{Q},\ \mathcal{J}_2 = \dot{N},\ \mathcal{J}_3 = \dot{\theta}.$$

Linear closure:

$$\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix} = \begin{pmatrix}L_{11} & L_{12} & L_{13}\\ L_{12} & L_{22} & L_{23}\\ L_{13} & L_{23} & L_{33}\end{pmatrix}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \mathcal{F}_3\end{pmatrix}, \qquad L = L^\top \succeq 0.$$

Entropy production density:

$$\sigma = \sum_{i,j} \mathcal{F}_i L_{ij} \mathcal{F}_j \geq 0.$$

Cauchy bounds: $|L_{ij}| \leq \sqrt{L_{ii}L_{jj}}$ for $i \neq j$.

Lyapunov functional and asymptotic stability

Define a coarse-grained free energy

$$\mathcal{F} = F_+(T_+,\mu_+) + F_-(T_-,\mu_-) - \Psi(\theta) + \tfrac{1}{2}\chi\dot{\theta}^{\,2},$$

with smooth Ψ. With no external drive ($\mathcal{S} = 0$) and $\gamma_c > 0$:

$$\dot{\mathcal{F}} = -\gamma_c\,\dot{\theta}^{\,2} \leq 0.$$

Hence $\mathcal{L} := \mathcal{F} - \mathcal{F}_{\mathrm{eq}}$ is a Lyapunov functional. The stationary set solves $\partial_\theta\big(m_\theta^2\theta - \Psi'(\theta)\big) = 0$, and with $\Delta T = \Delta\mu = 0$ one has $\dot{\theta} = 0$ and $Q_\pm = 0$.

4 Worked Calculations and Scaling Examples

Relaxation time (lab benchmark)

Choose $\gamma_c = 0.1\,\mathrm{s}^{-1}$, $m_\theta = 0.5\,\mathrm{s}^{-1}$ in the overdamped regime; then

$$\tau_\theta = \frac{\gamma_c}{m_\theta^2} = \frac{0.1}{(0.5)^2}\,\mathrm{s} = 0.4\,\mathrm{s},$$

and $t_{\mathrm{lock}} \approx 3\tau_\theta \approx 1.2\,\mathrm{s}$ for three e-folds of decay.

Dimensionless matter-fraction drift at $z \simeq 0$

Take $\alpha_m = 1.0 \cdot 10^{-27}\,\mathrm{kg\,m^{-3}}$, $\rho_{\mathrm{c0},+} = 8.5 \cdot 10^{-27}\,\mathrm{kg\,m^{-3}}$, and set $\dot{\theta} = \zeta H_+$ with $\zeta = 0.1$ at $a_+ = 1$:

$$\left.\frac{\mathrm{d}\Omega_{m+}}{\mathrm{d}\ln a_+}\right|_0 = \frac{\alpha_m\,\dot{\theta}}{H_+\,\rho_{\mathrm{c0},+}} = \zeta\,\frac{\alpha_m}{\rho_{\mathrm{c0},+}} = 0.1 \cdot \frac{1.0 \times 10^{-27}}{8.5 \times 10^{-27}} \approx 1.18 \times 10^{-2}.$$

Phase drift over a gigayear at constant fraction

Let $\dot{\theta} = \zeta H$ with $\zeta = 0.1$ and $H_0 = 2.27 \cdot 10^{-18}\,\mathrm{s}^{-1}$. For $\Delta t = 1.00 \cdot 10^9\,\mathrm{yr} \approx 3.156 \cdot 10^{16}\,\mathrm{s}$:

$$\Delta\theta \simeq \zeta H_0\,\Delta t \approx 0.1 \times 2.27 \times 10^{-18} \times 3.156 \times 10^{16} \approx 7.17 \times 10^{-3}\ \mathrm{rad}.$$

5 Operational Equilibrium Criteria and Failure Modes

Criteria

Thermal/chemical: $|\Delta T|/T \ll \varepsilon_T$, $|\Delta\mu|/T \ll \varepsilon_\mu$. Regulator: $|\dot\theta| < \varepsilon_\theta\, m_\theta$, $|\theta - \theta_*| < \varepsilon_\theta$. Budget: $|Q_(+) + Q_(-)|/H^2 < \varepsilon_Q$ and each $Q_\pm/H^2 \to 0$.

Failure modes

If $L \not\succeq 0$, then $\sigma < 0$ can occur; reject the point. If $\gamma_c \le 0$ or $m_\theta^2 \le 0$, Lyapunov decay fails; regulator unstable. If $|\mathrm{d}\Omega/\mathrm{d}\ln a|$ is large today from α_m, α_r, reduce ζ or re-fit cross-coefficients.

6 Geometric Chassis: Two Teleparallel Sheets and the Coupling Sector

Per-sheet teleparallel definitions

Each sheet $(\mathcal{M}_\pm, g_\pm)$ carries a tetrad $e^{a(\pm)}{}_\mu$ and a flat spin-connection $\omega^{a(\pm)}{}_{b\mu}$ (Weitzenböck). Define

$$T^{\rho(\pm)}{}_{\mu\nu} = e^{(\pm)}{}_a^{\ \rho}\Big(\partial_\mu e^{a(\pm)}{}_\nu - \partial_\nu e^{a(\pm)}{}_\mu + \omega^{a(\pm)}{}_{b\mu} e^{b(\pm)}{}_\nu - \omega^{a(\pm)}{}_{b\nu} e^{b(\pm)}{}_\mu\Big),$$

$$K^{\rho(\pm)}{}_{\mu\nu} = \tfrac{1}{2}(T_\mu{}^\rho{}_\nu - T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu})^{(\pm)}, \quad S_\rho^{(\pm)\,\mu\nu} = \tfrac{1}{2}\big(K^{\mu\nu}{}_\rho + \delta^\mu_\rho\, T^{\alpha\nu}{}_\alpha - \delta^\nu_\rho\, T^{\alpha\mu}{}_\alpha\big)^{(\pm)}.$$

Torsion scalar and TEGR identity:

$$T^{(\pm)} = \tfrac{1}{2}\, T^{\rho(\pm)}{}_{\mu\nu}\, S_\rho^{(\pm)\,\mu\nu}, \qquad R[g^{(\pm)}] = -\,T^{(\pm)} + B^{(\pm)}, \quad B^{(\pm)} = \frac{2}{e^{(\pm)}}\partial_\mu\Big(e^{(\pm)}\, T^{\mu(\pm)}\Big),$$

with $e^{(\pm)} = \det(e^{a(\pm)}{}_\mu)$ and $T^{\mu(\pm)} := T^{\alpha(\pm)}{}_\alpha{}^\mu$.

Two-sheet action and exchange Lagrangian

$$S = \int \mathrm{d}^4x \left[\frac{e^{()}\,T^{()}}{16\pi G_+} + \frac{e^{()}\,T^{()}}{16\pi G_-} + e^{()}\mathcal{L}_m^{()} + e^{()}\mathcal{L}_m^{()} + e^{()}e^{()}\,\mathcal{L}_{\text{couple}}(\theta, J_+, J_-)\right].$$

Take a unit-consistent representative:

$$\mathcal{L}_{\text{couple}} = -\lambda\,\sin\!\Big(\frac{\theta}{M}\Big)\,\Delta J - \frac{1}{2}\gamma_c\,\dot\theta^{\,2} - \frac{1}{2}m_\theta^2\,\theta^2, \qquad \Delta J := c_m(\rho_{m+} - \rho_{m-}) + c_r(\rho_{r+} - \rho_{r-}) + \cdots.$$

Units: $[\mathcal{L}_{\text{couple}}]$ = energy density. Choose $[\lambda] = \mathrm{kg}^{-1}\,\mathrm{m}^{-1}\,\mathrm{s}^2$, $[M] = 1$, $[\gamma_c] = \mathrm{s}^{-1}$, $[m_\theta] = \mathrm{s}^{-1}$, $[\theta] = 1$, $[c_{m,r}] = 1$.

Field equations and exchange currents

Variation w.r.t. $e^{a(\pm)}{}_\mu$ gives the TEGR equations sourced by matter and coupling. The coupling induces non-conservation:

$$\nabla_\nu^{(\pm)}\,\Theta^{(\pm)\,\mu\nu} = J_{\text{ex}}^{(\pm)\,\mu}, \qquad J_{\text{ex}}^{()\,\mu} + J_{\text{ex}}^{()\,\mu} = 0.$$

In homogeneous FLRW, only the temporal component is nonzero:

$$\dot\rho_\pm + 3H_\pm(\rho_\pm + p_\pm) = Q_\pm, \qquad Q_+ + Q_- = 0.$$

Deriving $Q_\pm$ near lock

Linearize at $\theta = \theta_* + \delta\theta$:

$$\mathcal{L}_{\rm couple} \approx -\lambda \sin\Big(\frac{\theta_*}{M}\Big)\Delta J - \frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)\Delta J\,\delta\theta - \tfrac{1}{2}\gamma_c\,\dot{\delta\theta}^{\,2} - \tfrac{1}{2}m_\theta^2\,\delta\theta^2.$$

Energy balance per comoving volume:

$$\dot\rho_{\rm couple} = -\frac{\partial \mathcal{L}_{\rm couple}}{\partial \delta\theta}\,\dot{\delta\theta} = \frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)\Delta J\,\dot{\delta\theta} + \mathcal{O}(\dot{\Delta J}).$$

Assigning antisymmetric exchange gives

$$Q_\pm = \pm\left[\frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)\Delta J\right]\dot\theta + \mathcal{O}(\dot{\Delta J}).$$

Species split (match to continuity form):

$$Q_{m\pm} = \pm\alpha_m\,\dot\theta, \qquad Q_{r\pm} = \pm\alpha_r\,\dot\theta,$$

with $\alpha_m = \big(\frac{\lambda}{M}\cos(\theta_*/M)\big)\,c_m\rho_{c0,\pm}$ and similarly for α_r.

FLRW reduction and units

For a proper diagonal tetrad with the flat connection,

$$T^{(\pm)} = -6\,H_\pm^2, \qquad 3H_\pm^2 = 8\pi G_\pm\,\rho_\pm + Q_\pm.$$

Units: $[T] = \mathrm{s}^{-2}$, $[H] = \mathrm{s}^{-1}$, $[Q] = \mathrm{s}^{-2}$.

Sanity checks. (i) Equilibrium $\dot\theta \to 0 \Rightarrow Q_\pm \to 0$. (ii) Decoupling $c_m = c_r = 0 \Rightarrow Q_\pm = 0$. (iii) Sign flip $\theta \to -\theta$ flips $Q_\pm$.

7 Interphasic Thermodynamics: Onsager Matrix, Reduced Dynamics, Stability

Flux–force structure and $\sigma \geq 0$

Forces

$$\mathcal{F}_1 = \frac{\Delta T}{T^2}, \quad \mathcal{F}_2 = \frac{\Delta\mu}{T}, \quad \mathcal{F}_3 = \frac{\partial\Phi}{\partial\theta},$$

fluxes

$$\mathcal{J}_1 = \dot Q, \quad \mathcal{J}_2 = \dot N, \quad \mathcal{J}_3 = \dot\theta.$$

Near equilibrium:

$$\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix} = \begin{pmatrix}L_{11} & L_{12} & L_{13}\\ L_{12} & L_{22} & L_{23}\\ L_{13} & L_{23} & L_{33}\end{pmatrix}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\\ \mathcal{F}_3\end{pmatrix}, \qquad L = L^\top \succeq 0.$$

Entropy production density:

$$\sigma = \sum_{i,j}\mathcal{F}_i L_{ij}\mathcal{F}_j \geq 0, \qquad |L_{ij}| \leq \sqrt{L_{ii}L_{jj}}\ (i \neq j).$$

Overdamped regulator elimination

With $\chi\ddot{\theta} \ll \gamma_c\dot{\theta}$,

$$\dot{\theta} \simeq -\frac{1}{\gamma_c}\left(m_\theta^2\,\theta - \Psi'(\theta) - \frac{\lambda}{M}\,\Delta J - \Delta\mu\right).$$

Insert into

$$\dot{\rho}_{m\pm} + 3H_\pm\rho_{m\pm} = \pm\alpha_m\,\dot{\theta}, \qquad \dot{\rho}_{r\pm} + 4H_\pm\rho_{r\pm} = \pm\alpha_r\,\dot{\theta},$$

to obtain a closed drift for $(\rho_{m\pm}, \rho_{r\pm}, \theta)$.

Linearization around equilibrium and eigenvalues

Expand about $\big(\Delta T, \Delta\mu, \theta - \theta_*\big) = (0, 0, 0)$ and define

$$\Gamma_\theta := \frac{m_\theta^2 - \Psi''(\theta_*)}{\gamma_c} > 0, \qquad \zeta := \frac{\dot{\theta}}{H}\Big|_{\text{eq pert}}.$$

The coupled $(\delta\theta, \delta\rho_{m\pm}, \delta\rho_{r\pm})$ system yields eigenvalues

$$\lambda_1 \simeq -\Gamma_\theta, \qquad \lambda_2 \simeq -3H + \mathcal{O}\!\left(\frac{\alpha_m\,\zeta}{\rho_c}\right), \qquad \lambda_3 \simeq -4H + \mathcal{O}\!\left(\frac{\alpha_r\,\zeta}{\rho_c}\right),$$

so stability holds if $\Gamma_\theta > 0$ and the exchange corrections are small compared to $3H$ and $4H$.

Free-energy decay

Define

$$\mathcal{F}_{\text{tot}} = F_+(T_+, \mu_+) + F_-(T_-, \mu_-) - \Psi(\theta) + \tfrac{1}{2}\chi\,\dot{\theta}^{\,2}.$$

With no external drive and $\gamma_c > 0$:

$$\dot{\mathcal{F}}_{\text{tot}} = -\gamma_c\,\dot{\theta}^{\,2} - \begin{pmatrix}\mathcal{F}_1 & \mathcal{F}_2\end{pmatrix}\begin{pmatrix}L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix}\begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{pmatrix} \le 0.$$

Quantitative bounds and example scales

Today, take $H_0 = 2.27\cdot 10^{-18}\,\mathrm{s}^{-1}$, $\alpha_m = 1.0\cdot 10^{-27}\,\mathrm{kg\,m^{-3}}$, $\rho_{c0} = 8.5\cdot 10^{-27}\,\mathrm{kg\,m^{-3}}$. With $\dot{\theta} = \zeta H_0$ and $\zeta = 0.1$,

$$\frac{\mathrm{d}\Omega_{m+}}{\mathrm{d}\ln a_+}\bigg|_0 = \zeta\,\frac{\alpha_m}{\rho_{c0}} \approx 1.18\times 10^{-2}, \qquad \Delta\theta(\Delta t) \simeq \zeta H_0\,\Delta t.$$

Lab-scale relaxation with $\gamma_c = 0.1\,\mathrm{s}^{-1},\ m_\theta = 0.5\,\mathrm{s}^{-1}$ gives

$$\tau_\theta = \frac{\gamma_c}{m_\theta^2} = 0.4\,\mathrm{s}, \quad t_{\text{lock}} \approx 3\tau_\theta \approx 1.2\,\mathrm{s}.$$

Acceptance criteria

$L \succeq 0$; $\Gamma_\theta > 0$; $|Q_\pm|/H^2 \ll 1$ at late times; $|\mathrm{d}\Omega/\mathrm{d}\ln a| \lesssim 10^{-2}$ today under chosen ζ.

8 Cosmological Background with Exchange: Friedmann Pair, Closed-Form Drifts, and Bounds

Continuity integration with $\dot{\theta} = \zeta H$ (constant fraction)

Assume a constant fraction benchmark $\dot{\theta} = \zeta H$ with small $|\zeta| \ll 1$. Continuity per sheet:

$$\frac{\mathrm{d}\rho_{m\pm}}{\mathrm{d}t} + 3H_\pm \rho_{m\pm} = \pm\alpha_m\,\dot{\theta}, \qquad \frac{\mathrm{d}\rho_{r\pm}}{\mathrm{d}t} + 4H_\pm \rho_{r\pm} = \pm\alpha_r\,\dot{\theta}.$$

Use $\mathrm{d}/\mathrm{d}t = H_\pm\,\mathrm{d}/\mathrm{d}\ln a_\pm$ and insert $\dot{\theta} = \zeta H_\pm$:

$$\frac{\mathrm{d}\rho_{m\pm}}{\mathrm{d}\ln a_\pm} + 3\rho_{m\pm} = \pm\alpha_m\,\zeta, \qquad \frac{\mathrm{d}\rho_{r\pm}}{\mathrm{d}\ln a_\pm} + 4\rho_{r\pm} = \pm\alpha_r\,\zeta.$$

Solve with integrating factors ($a_\pm$-power laws). For any reference a_i:

$$\rho_{m\pm}(a_\pm) = \rho_{m\pm}(a_\mathrm{i})\left(\frac{a_\pm}{a_\mathrm{i}}\right)^{-3} \;\pm\; \frac{\alpha_m\zeta}{3}\left[1 - \left(\frac{a_\pm}{a_\mathrm{i}}\right)^{-3}\right],$$

$$\rho_{r\pm}(a_\pm) = \rho_{r\pm}(a_\mathrm{i})\left(\frac{a_\pm}{a_\mathrm{i}}\right)^{-4} \;\pm\; \frac{\alpha_r\zeta}{4}\left[1 - \left(\frac{a_\pm}{a_\mathrm{i}}\right)^{-4}\right].$$

Checks: at $a_\pm = a_\mathrm{i}$ the source correction vanishes; as $a_\pm \to \infty$ the densities asymptote to $\pm$ constants set by $(\alpha_m\zeta/3,\ \alpha_r\zeta/4)$.

Two equivalent closures for the background Friedmann law

Closure A (explicit source):

$$3H_\pm^2 = 8\pi G_\pm\big(\rho_{m\pm} + \rho_{r\pm} + \rho_{\Lambda\pm}\big) + Q_\pm, \qquad Q_+ + Q_- = 0.$$

Closure B (absorbed source): define $\rho'_{\mathrm{tot}\,\pm} := \rho_{m\pm} + \rho_{r\pm} + \rho_{\Lambda\pm} + \rho_{\mathrm{couple}\,\pm}$ so that

$$3H_\pm^2 = 8\pi G_\pm\,\rho'_{\mathrm{tot}\,\pm}, \qquad \dot{\rho}'_{\mathrm{tot}\,\pm} + 3H_\pm(\rho'_{\mathrm{tot}\,\pm} + p'_{\mathrm{tot}\,\pm}) = 0.$$

Both are consistent if $\rho_{\mathrm{couple}\,\pm}$ is chosen so that $\dot{\rho}_{\mathrm{couple}\,\pm} = -Q_\pm$. Use A when you want to track $Q_\pm$ explicitly; use B to fold it into the density budget.

Dimensionless form and small-ζ drift of fractions

Let today's critical densities be $\rho_{\mathrm{c}0,+},\ \rho_{\mathrm{c}0,-}$ and define $\Omega_{i\pm} := \rho_{i\pm}/\rho_{\mathrm{c}0,\pm}$ at $a_\pm = 1$. Set $a_\mathrm{i} = 1$. From the solutions above:

$$\Omega_{m\pm}(a_\pm) = \Omega_{m\pm}(1)\,a_\pm^{-3} \;\pm\; \frac{\alpha_m\zeta}{3\rho_{\mathrm{c}0,\pm}}\left(1 - a_\pm^{-3}\right),$$

$$\Omega_{r\pm}(a_\pm) = \Omega_{r\pm}(1)\,a_\pm^{-4} \;\pm\; \frac{\alpha_r\zeta}{4\rho_{\mathrm{c}0,\pm}}\left(1 - a_\pm^{-4}\right).$$

Local drift rates at $a_\pm = 1$:

$$\left.\frac{\mathrm{d}\Omega_{m\pm}}{\mathrm{d}\ln a_\pm}\right|_1 = \pm\frac{\alpha_m\,\zeta}{\rho_{\mathrm{c}0,\pm}}, \qquad \left.\frac{\mathrm{d}\Omega_{r\pm}}{\mathrm{d}\ln a_\pm}\right|_1 = \pm\frac{\alpha_r\,\zeta}{\rho_{\mathrm{c}0,\pm}}.$$

Units: $\alpha_m/\rho_{\mathrm{c}0,\pm}$ and $\alpha_r/\rho_{\mathrm{c}0,\pm}$ are dimensionless; signs are opposite across sheets.

Worked numbers (today, $a_\pm = 1$)

Example values (placeholders): $\alpha_m = 1.0 \times 10^{-27}\ \mathrm{kg\,m^{-3}}$, $\alpha_r = 1.0 \times 10^{-30}\ \mathrm{kg\,m^{-3}}$, $\rho_{c0,\pm} = 8.5 \times 10^{-27}\ \mathrm{kg\,m^{-3}}$, $\zeta = 0.1$. Then

$$\left.\frac{\mathrm{d}\Omega_{m+}}{\mathrm{d}\ln a_+}\right|_1 = +0.1 \times \frac{1.0}{8.5} \approx +1.18 \times 10^{-2}, \quad \left.\frac{\mathrm{d}\Omega_{m-}}{\mathrm{d}\ln a_-}\right|_1 = -1.18 \times 10^{-2},$$

$$\left.\frac{\mathrm{d}\Omega_{r+}}{\mathrm{d}\ln a_+}\right|_1 \approx +1.18 \times 10^{-3}, \quad \left.\frac{\mathrm{d}\Omega_{r-}}{\mathrm{d}\ln a_-}\right|_1 \approx -1.18 \times 10^{-3}.$$

These satisfy detailed balance to leading order: $\mathrm{d}(\Omega_{m+} + \Omega_{m-})/\mathrm{d}\ln a \simeq 0$ and same for radiation.

Impact on $H(a)$ to first order in ζ

Closure B with flat geometry gives

$$E_\pm^2(a) := \frac{H_\pm^2(a)}{H_{\pm 0}^2} = \Omega_{m\pm}(1)\,a^{-3} + \Omega_{r\pm}(1)\,a^{-4} + \Omega_{\Lambda\pm}(1) \ \pm\ \frac{\zeta}{\rho_{c0,\pm}}\left[\frac{\alpha_m}{3}\left(1 - a^{-3}\right) + \frac{\alpha_r}{4}\left(1 - a^{-4}\right)\right].$$

So the fractional H-shift at late times ($a \to 1$) is

$$\left.\frac{\Delta E_\pm^2}{E_\pm^2}\right|_{a=1} = \pm\frac{\zeta}{\rho_{c0,\pm}}\left(\frac{\alpha_m}{3} + \frac{\alpha_r}{4}\right) \Big/ \left[\Omega_{m\pm}(1) + \Omega_{r\pm}(1) + \Omega_{\Lambda\pm}(1)\right] = \pm\frac{\zeta}{\rho_{c0,\pm}}\left(\frac{\alpha_m}{3} + \frac{\alpha_r}{4}\right),$$

since the denominator equals 1 by definition at $a = 1$.

Bounds and acceptance window

Require small background deformation:

$$\frac{|\alpha_m|}{\rho_{c0,\pm}}\,|\zeta| \ll 1, \qquad \frac{|\alpha_r|}{\rho_{c0,\pm}}\,|\zeta| \ll 1.$$

Operational choice: cap the present drift by $|\mathrm{d}\Omega_{m\pm}/\mathrm{d}\ln a|_1 \le 10^{-2}$ and $|\mathrm{d}\Omega_{r\pm}/\mathrm{d}\ln a|_1 \le 10^{-3}$, which the numeric example satisfies on the $+$ sheet and is mirrored with opposite sign on the $-$ sheet.

Consistency checks

(i) Setting $\zeta \to 0$ reproduces standard scalings $\rho_m \propto a^{-3}$, $\rho_r \propto a^{-4}$. (ii) Exchanging sheet labels flips the signs of all source-induced drifts. (iii) At equilibrium ($\dot\theta = 0$) the source terms vanish and both closures A/B coincide.

9 Linear Response and Growth: Subhorizon Limit, Analytic Exponents, and Audits

Governing equation in $x = \ln a$ variables

Define prime $' = \mathrm{d}/\mathrm{d}x$ with $x = \ln a$. On subhorizon scales the density contrast obeys

$$\delta_\pm'' + \left(2 + \frac{H_\pm'}{H_\pm}\right)\delta_\pm' - \frac{3}{2}\,\frac{\pm}{G}\,\Omega_{m\pm}(a)\,\delta_\pm = \frac{S_\pm}{H_\pm^2},$$

with source

$$S_\pm = \pm\,\sigma_0\,\dot\theta\,\delta_\pm \quad \Rightarrow \quad \frac{S_\pm}{H_\pm^2} = \pm\,\sigma_0\,\frac{\dot\theta}{H_\pm}\,\delta_\pm.$$

Benchmark: constant fraction $\dot\theta = \zeta H_\pm$ gives a dimensionless drive

$$\frac{S_\pm}{H_\pm^2} = \pm\,(\sigma_0\zeta)\,\delta_\pm \equiv \pm\,\varepsilon\,\delta_\pm, \qquad \varepsilon := \sigma_0\zeta \ll 1.$$

Matter era analytic solution (EdS check)

In an Einstein–de Sitter limit ($\Omega_m = 1$, $= G$, $H'/H = -\frac{3}{2}$) the equation reduces to

$$\delta_\pm'' + \tfrac{1}{2}\,\delta_\pm' - \tfrac{3}{2}\,\delta_\pm = \pm\,\varepsilon\,\delta_\pm.$$

Seek power laws $\delta_\pm \propto a^{p_\pm}$, i.e. $\delta_\pm' = p_\pm\delta_\pm$, $\delta_\pm'' = p_\pm^2\delta_\pm$. The characteristic equation is

$$p_\pm^2 + \tfrac{1}{2}p_\pm - \tfrac{3}{2} \mp \varepsilon = 0,$$

with roots

$$p_{\pm,\{1,2\}} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6 \pm 4\varepsilon}}{2}.$$

Small-ε expansions about the standard $\{1, -\frac{3}{2}\}$ modes:

$$p_{\pm,1} \simeq 1 \ \pm\ 0.4\,\varepsilon, \qquad p_{\pm,2} \simeq -\tfrac{3}{2} \ \mp\ 0.4\,\varepsilon.$$

Hence the growing mode on the $+$ sheet is slightly enhanced for $\varepsilon > 0$, and suppressed on the $-$ sheet by the same amount; signs flip for $\varepsilon < 0$.

Growth rate $f = \mathrm{d}\ln\delta/\mathrm{d}\ln a$ and γ proxy

Define $f_\pm := \delta_\pm'/\delta_\pm$. For a pure power law $\delta_\pm \propto a^{p_{\pm,1}}$, $f_\pm = p_{\pm,1}$. In Λ-like backgrounds one uses $f_\pm \simeq \Omega_{m\pm}^{\gamma_\pm}$. A first-order shift at late times follows from linearizing the growth equation:

$$\Delta f_\pm \equiv f_\pm - f_{\Lambda\mathrm{CDM}} \ \approx\ \pm\,0.4\,\varepsilon \quad \Rightarrow \quad \Delta\gamma_\pm \approx \frac{\Delta f_\pm}{\ln\Omega_{m\pm}^{-1}}\bigg|_{z\approx 0}.$$

For $\Omega_{m\pm}(0) = 0.3$ one has $\ln\Omega_m^{-1} \approx 1.204$, so

$$\Delta\gamma_\pm \approx \pm\,0.33\,\varepsilon \quad \text{(order-of-magnitude)}.$$

Worked numbers

Choose $\sigma_0 = 0.02$, $\zeta = 0.1 \Rightarrow \varepsilon = 2.0\times10^{-3}$.

$$p_{+,1} \simeq 1 + 0.4\varepsilon = 1 + 8.0\times10^{-4}, \qquad p_{-,1} \simeq 1 - 8.0\times10^{-4}.$$

Late-time shift in the growth index proxy:

$$\Delta\gamma_+ \approx +0.33\times2.0\times10^{-3} \approx +6.6\times10^{-4}, \qquad \Delta\gamma_- \approx -6.6\times10^{-4}.$$

These are sub-permil changes for the chosen benchmark and satisfy the small-deformation bounds in the previous section.

Including a mild deformation

If $_\pm/G = 1 + \nu_\pm$ with $|\nu_\pm| \ll 1$ and slowly varying, the EdS characteristic equation becomes

$$p_\pm^2 + \tfrac{1}{2}p_\pm - \tfrac{3}{2}(1+\nu_\pm) \mp \varepsilon = 0,$$

so to first order

$$p_{\pm,1} \simeq 1 + 0.4\,(\pm\varepsilon + 1.5\,\nu_\pm), \qquad p_{\pm,2} \simeq -\tfrac{3}{2} - 0.4\,(\pm\varepsilon + 1.5\,\nu_\pm).$$

The sheet-asymmetric drive ($\pm\varepsilon$) and symmetric gravity shift ($\nu_\pm$ if common) separate cleanly.

Units and consistency

$S_\pm/H^2$ is dimensionless. The coefficients in the $x = \ln a$ form are dimensionless: $(2 + H'/H)$, $\frac{3}{2}(/G)\Omega_m$, and $\pm\varepsilon$. Power-law solutions are valid while these coefficients vary slowly across a decade in a.

Acceptance bounds

Adopt $|\varepsilon| \le 5 \times 10^{-3}$ to keep $|\Delta f| \lesssim 2 \times 10^{-3}$ and $|\Delta\gamma| \lesssim 10^{-3}$ at $z \approx 0$. Tighten if required by data or lab constraints.

10 Fluctuations, Noise, and FDT: Langevin, Spectra, and Entropy Rate

Overdamped Langevin for the regulator

Near the lock point, linearize the regulator about $\theta = \theta_* + \delta\theta$ and work in the overdamped regime ($\chi\,\ddot{\theta} \ll \gamma_c\dot{\theta}$):

$$\dot{\delta\theta} = -\Gamma_\theta\,\delta\theta + \xi(t) + s(t), \qquad \Gamma_\theta := \frac{m_\theta^2 - \Psi''(\theta_*)}{\gamma_c} > 0.$$

Here $\xi(t)$ is zero-mean *internal* noise and $s(t)$ is an optional small external drive (e.g. proportional to $\Delta\mu$ or ΔJ). We model ξ as white with

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\,\xi(t') \rangle = 2D\,\delta(t-t') \quad \text{with} \quad [D] = \mathrm{s}^{-2}.$$

Units: $\dot{\delta\theta}$ and $\Gamma_\theta\delta\theta$ are s^{-1}; $\xi(t)$ must be s^{-1}; therefore D carries s^{-2} since $\delta(\cdot)$ has s^{-1}.

Stationary solution, variance, and correlation time

For $s(t) \equiv 0$, the Ornstein–Uhlenbeck process has the unique stationary state

$$\langle \delta\theta \rangle = 0, \qquad \mathrm{Var}(\delta\theta) = \frac{D}{\Gamma_\theta}, \qquad C_\theta(\tau) := \langle \delta\theta(t)\delta\theta(t+\tau) \rangle = \frac{D}{\Gamma_\theta}\,e^{-\Gamma_\theta|\tau|}.$$

The correlation time is $\tau_\theta = \Gamma_\theta^{-1}$.

Power spectral density and response

Fourier transforming gives the one-sided power spectral density (angular frequency ω)

$$S_\theta(\omega) = \frac{2D}{\omega^2 + \Gamma_\theta^2} \quad [\text{units: s}],$$

which integrates to the variance: $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_\theta(\omega)\,\mathrm{d}\omega = D/\Gamma_\theta$. The linear susceptibility to a harmonic drive $s(t) = s_0 e^{i\omega t}$ is

$$\chi(\omega) = \frac{\delta\theta(\omega)}{s(\omega)} = \frac{1}{\Gamma_\theta + i\omega}, \qquad |\chi(\omega)|^2 = \frac{1}{\omega^2 + \Gamma_\theta^2}.$$

FDT (normalized form). The Einstein relation here is simply

$$D = \Gamma_\theta \operatorname{Var}(\delta\theta),$$

i.e. once the stationary variance is measured, the noise intensity is fixed by the relaxation rate. In a microphysical derivation with explicit energetic units one recovers $D \propto \gamma_c^{-1} k_B T_{\text{eff}}$; our normalized coarse-grained form absorbs those constants into D.

Worked numbers (consistent with earlier benchmarks)

Take $m_\theta = 0.5\,\mathrm{s}^{-1}$, $\gamma_c = 0.1\,\mathrm{s}^{-1}$, and $\Psi''(\theta_*) \approx 0$. Then

$$\Gamma_\theta = \frac{m_\theta^2}{\gamma_c} = \frac{0.25}{0.1} = 2.5\,\mathrm{s}^{-1}, \qquad \tau_\theta = \Gamma_\theta^{-1} = 0.4\,\mathrm{s}.$$

Choose a representative internal noise level $D = 1.0 \cdot 10^{-3}\,\mathrm{s}^{-2}$. Then

$$\operatorname{Var}(\delta\theta) = \frac{D}{\Gamma_\theta} = \frac{10^{-3}}{2.5} = 4.0 \times 10^{-4}, \quad \operatorname{Std}(\delta\theta) = \sqrt{4.0 \times 10^{-4}} \approx 2.0 \times 10^{-2} \text{ rad}.$$

The 3 dB cutoff (half-power) occurs at $\omega = \Gamma_\theta$:

$$f_c = \frac{\Gamma_\theta}{2\pi} \approx \frac{2.5}{2\pi} \approx 0.398\,\mathrm{Hz}.$$

Zero-frequency PSD:

$$S_\theta(0) = \frac{2D}{\Gamma_\theta^2} = \frac{2 \times 10^{-3}}{(2.5)^2} \approx 3.2 \times 10^{-4} \text{ s}.$$

Consistency check: $\frac{1}{2\pi}\int S_\theta(\omega)\mathrm{d}\omega = \frac{D}{\Gamma_\theta} = 4.0 \times 10^{-4}$ (matches the variance above).

Entropy production rate and detailed balance

Near equilibrium the generalized flux–force law (restricted to the regulator channel) reads

$$\mathcal{J}_3 = \dot{\theta} = L_{33}\,\mathcal{F}_3 + \text{noise}, \qquad L_{33} = \frac{1}{\gamma_c}.$$

The instantaneous entropy production density from this channel is

$$\sigma_\theta = \mathcal{J}_3\,\mathcal{F}_3 = L_{33}\,\mathcal{F}_3^2 + \text{fluctuations } \geq 0.$$

At strict equilibrium ($\mathcal{F}_3 = 0$) the *mean* entropy production from the θ-channel vanishes; any nonzero $\mathcal{F}_3$ (e.g. a bias from $\Delta\mu$ or ΔJ) produces $\langle\sigma_\theta\rangle > 0$ and relaxes the system back to the lock point at rate Γ_θ.

Laboratory inference recipe (OU fit)

1. Record $\theta(t)$ at sampling rate $\gg f_c$; detrend the mean.
2. Estimate Γ_θ by fitting $C_\theta(\tau) \sim e^{-\Gamma_\theta \tau}$ on $0 < \tau \lesssim 2\tau_\theta$.
3. Compute $\mathrm{Var}(\delta\theta)$ from the time series.
4. Infer $D = \Gamma_\theta \,\mathrm{Var}(\delta\theta)$; predict $S_\theta(\omega)$ and compare to the measured spectrum.

Unit and consistency audits

- Γ_θ has s^{-1}; D has s^{-2}; $S_\theta(\omega)$ has s; $\delta(\cdot)$ has s^{-1}.
- The Einstein relation $D = \Gamma_\theta \,\mathrm{Var}(\delta\theta)$ is dimensionally consistent: $\mathrm{s}^{-2} = \mathrm{s}^{-1} \times 1$.
- In the deterministic limit $D \to 0$ the PSD collapses and $\delta\theta \to 0$ as $t \to \infty$.

11 Observables and Inference: Background, Growth, Lensing, and Lab Probes

Background expansion $H(a)$

Using the absorbed-source closure (flat background),

$$E_\pm^2(a) := \frac{H_\pm^2(a)}{H_{\pm 0}^2} = \Omega_{m\pm}(1)\, a^{-3} + \Omega_{r\pm}(1)\, a^{-4} + \Omega_{\Lambda\pm}(1) \ \pm\ \frac{\zeta}{\rho_{c0,\pm}} \Big[\frac{\alpha_m}{3} \big(1 - a^{-3}\big) + \frac{\alpha_r}{4} \big(1 - a^{-4}\big) \Big].$$

Present-day fractional shift (at $a = 1$):

$$\frac{\Delta E_\pm^2}{E_\pm^2}\bigg|_{a=1} = \pm \frac{\zeta}{\rho_{c0,\pm}} \Big(\frac{\alpha_m}{3} + \frac{\alpha_r}{4} \Big), \qquad \frac{\Delta H_\pm}{H_\pm}\bigg|_{a=1} \approx \frac{1}{2} \frac{\Delta E_\pm^2}{E_\pm^2}\bigg|_{a=1}.$$

Worked numbers (benchmarks): $\zeta = 0.1$, $\alpha_m = 1.0 \times 10^{-27}$, $\alpha_r = 1.0 \times 10^{-30}$, $\rho_{c0,\pm} = 8.5 \times 10^{-27}$ (SI units omitted for brevity) give

$$\frac{\Delta E_\pm^2}{E_\pm^2}\bigg|_1 \simeq \pm \Big[0.1 \Big(\tfrac{1}{8.5} \cdot \tfrac{1}{3} + \tfrac{10^{-3}}{8.5} \cdot \tfrac{1}{4} \Big) \Big] \approx \pm 3.925 \times 10^{-3}, \quad \frac{\Delta H_\pm}{H_\pm}\bigg|_1 \approx \pm 1.96 \times 10^{-3}.$$

Growth and $f\sigma_8$ proxy

Subhorizon growth with drive $S_\pm = \pm \sigma_0 \dot\theta\, \delta_\pm$ (constant fraction $\dot\theta = \zeta H$) yields the EdS analytic exponents

$$p_{\pm,1} \simeq 1 \pm 0.4\, \varepsilon, \qquad \varepsilon := \sigma_0 \zeta \ll 1,$$

so in a late-time window $a \in [a_1, a_2]$ the fractional change of the linear growth factor D is

$$\Delta \ln D_\pm \simeq (\pm 0.4\, \varepsilon)\, \ln \frac{a_2}{a_1}.$$

Worked numbers: with $\sigma_0 = 0.02$, $\zeta = 0.1$ ($\varepsilon = 2.0 \times 10^{-3}$) and $a_1 = 0.5$, $a_2 = 1$,

$$\Delta \ln D_\pm \simeq (\pm 0.4)(2.0 \times 10^{-3}) \ln 2 \approx \pm 5.5 \times 10^{-4}.$$

Thus $\Delta\sigma_{8,\pm}/\sigma_8 \approx \pm 5.5 \times 10^{-4}$ over $z \in [1, 0]$ for the benchmark.

Lensing strength Σ and slip η

If the inter-sheet exchange does not alter the metric anisotropic stress (no new vector/tensor sources) and ${}_{\pm}/G = 1 + \nu_{\pm}$ with small $|\nu_{\pm}|$,

$$\eta := \frac{\Phi - \Psi}{\Psi} \approx 0, \qquad \Sigma := \frac{}{G}\,\frac{1+\eta}{2} \approx 1 + \nu_{\pm}.$$

In the minimal model ($\nu_{\pm} = 0$) lensing remains unchanged at leading order; any nonzero $\nu_{\pm}$ enters linearly and can be bounded independently of ε.

Cross-sheet diagnostics and null tests

- **Antisymmetry:** $\Delta H/H$ has opposite signs on the two sheets; likewise $\Delta \ln D$.
- **Budget closure:** $Q_{+} + Q_{-} = 0$ and $\mathrm{d}(\Omega_{m+} + \Omega_{m-})/\mathrm{d}\ln a \simeq 0$ at leading order.
- **Equilibrium limit:** setting $\zeta \to 0$ (or $\dot{\theta} \to 0$) restores standard ΛCDM background/growth on both sheets.

Laboratory resonator mapping (readout of θ)

For a cavity or oscillator whose eigenfrequency f depends on θ,

$$\frac{\delta f}{f} = \kappa_{\theta}\,\delta\theta, \qquad S_{\delta f/f}(\omega) = \kappa_{\theta}^{2}\,S_{\theta}(\omega), \qquad S_{\theta}(\omega) = \frac{2D}{\omega^{2} + \Gamma_{\theta}^{2}}.$$

Worked numbers: take $\kappa_{\theta} = 10^{-3}\ \mathrm{rad}^{-1}$, $\Gamma_{\theta} = 2.5\,\mathrm{s}^{-1}$, $D = 10^{-3}\,\mathrm{s}^{-2}$ (from the FDT section). Then

$$S_{\delta f/f}(0) = \kappa_{\theta}^{2}\,\frac{2D}{\Gamma_{\theta}^{2}} = 10^{-6} \cdot \frac{2 \times 10^{-3}}{(2.5)^{2}} \approx 3.2 \times 10^{-10}\ \mathrm{s}.$$

The stationary RMS for $\delta f/f$ is $\sqrt{\mathrm{Var}} = \kappa_{\theta}\sqrt{D/\Gamma_{\theta}} = 10^{-3}\sqrt{4.0 \times 10^{-4}} \approx 2.0 \times 10^{-5}$. For $f_0 = 10\,\mathrm{MHz}$ this corresponds to an RMS $\sigma_f \approx 200$ Hz.

Parameter combinations to report

The observables mainly constrain the combinations

$$\Xi_m := \frac{\alpha_m\,\zeta}{\rho_{c0,\pm}}, \qquad \Xi_r := \frac{\alpha_r\,\zeta}{\rho_{c0,\pm}}, \qquad \varepsilon := \sigma_0\zeta, \qquad \nu_{\pm} := \frac{\pm}{G} - 1, \qquad \Gamma_{\theta},\ D,\ \kappa_{\theta}\ \text{(lab)}.$$

Background probes are sensitive to Ξ_m, Ξ_r; growth to ε (and $\nu_{\pm}$ if present); lensing to $\nu_{\pm}$; lab readouts to $(\Gamma_{\theta}, D, \kappa_{\theta})$.

Acceptance bounds (operational)

Adopt conservative late-time bounds

$$|\Xi_m| \lesssim 10^{-2}, \qquad |\Xi_r| \lesssim 10^{-3}, \qquad |\varepsilon| \lesssim 5 \times 10^{-3}, \qquad |\nu_{\pm}| \ll 1,$$

which keep $|\Delta H/H| \lesssim 2\times 10^{-3}$, $|\Delta\sigma_8/\sigma_8| \lesssim 10^{-3}$, and lab RMS fractions $\lesssim 10^{-5}$ for the benchmark sensitivities above.

12 Consistency, Constraints, and Null Tests: Stability, Causality, and Bounds

Global conservation and Ward identities

Add the two continuity equations:

$$\big(\dot{\rho}_{m+} + 3H_+\rho_{m+}\big) + \big(\dot{\rho}_{m-} + 3H_-\rho_{m-}\big) = \alpha_m(\dot{\theta} - \dot{\theta}) = 0,$$

$$\big(\dot{\rho}_{r+} + 4H_+\rho_{r+}\big) + \big(\dot{\rho}_{r-} + 4H_-\rho_{r-}\big) = \alpha_r(\dot{\theta} - \dot{\theta}) = 0.$$

Hence the *total* comoving energy in $\{m, r\}$ sectors is conserved when summed over sheets. Covariantly (per sheet) one has $\nabla^{(\pm)}_\nu \Theta^{(\pm)\,\mu\nu} = J^{(\pm)\,\mu}_{\rm ex}$ with $J^{(+)\,\mu}_{\rm ex} + J^{(-)\,\mu}_{\rm ex} = 0$, so global diffeomorphism invariance is preserved by an antisymmetric exchange current. In FLRW this reduces to $Q_+ + Q_- = 0$.

Degrees of freedom and absence of pathologies

No BD ghost (teleparallel sheets). Each sheet is TEGR-equivalent to GR: 2 propagating tensor DOF per sheet. The coupling we use is

$$\mathcal{L}_{\rm couple} = -\lambda \,\sin(\theta/M)\,\Delta J - \tfrac{1}{2}\gamma_c\,\dot{\theta}^2 - \tfrac{1}{2}m_\theta^2\theta^2,$$

algebraic in the matter invariants (via ΔJ) and at most *first* time derivatives of θ (quadratic). There are no metric derivatives across sheets inside $\mathcal{L}_{\rm couple}$, hence no higher-derivative mixing and no Ostrogradski/BD mode is introduced. Total propagating DOF: $2 + 2$ tensors $+\,0$ extra metric modes $+\,$(θ optional scalar if its gradient terms are included).

Optional spatial stiffness for θ. If needed for microcausality, add a canonical gradient term

$$\mathcal{L}_{\nabla\theta} = -\frac{c_\theta^2}{2}(\nabla\theta)^2,$$

with $0 < c_\theta \le 1$ to ensure subluminal scalar characteristics. This does not alter the background results used above when spatial homogeneity is assumed.

Stability conditions (local and cosmological)

- **Regulator:** effective mass-squared positive near lock:

  $$m_{\rm eff}^2 := m_\theta^2 - \Psi''(\theta_*) > 0, \qquad \gamma_c > 0.$$

  Then the linearized rate $\Gamma_\theta = (m_{\rm eff}^2/\gamma_c) > 0$ and the OU spectrum is well-defined (no tachyon/negative diffusion).

- **Onsager matrix:** $L = L^\top \succeq 0$ so that $\sigma = \sum_{i,j}\mathcal{F}_i L_{ij}\mathcal{F}_j \ge 0$. Principal-minor tests:

  $$L_{ii} \ge 0, \quad L_{11}L_{22} - L_{12}^2 \ge 0, \quad \det L \ge 0.$$

- **Tensor sector:** since $\mathcal{L}_{\rm couple}$ has no tensor kinetic mixing, the tensor wave speed and kinetic term on each sheet equal those of TEGR/GR. Thus

  $$c_T^{(\pm)} = 1, \qquad \text{no gradient or ghost instabilities in the tensor sector.}$$

Causality and subluminal characteristics

With $\mathcal{L}_{\nabla\theta}$ the scalar dispersion is $\omega^2 = \Gamma_\theta^2$ (overdamped) at long times or, in the underdamped limit with explicit inertia χ, $\omega^2 = m_{\rm eff}^2 + c_\theta^2 k^2$. Choose $0 < c_\theta \le 1$ to keep the scalar light cone inside (or on) the metric light cone. No signal channel in $\mathcal{L}_{\rm couple}$ propagates superluminally because ΔJ is algebraic in local fluid variables.

Background deformation bounds (late time)

Using the derived expressions with $\dot\theta = \zeta H$,

$$\left.\frac{\Delta H_\pm}{H_\pm}\right|_{a=1} \approx \pm\frac{1}{2}\,\frac{\zeta}{\rho_{c0,\pm}}\left(\frac{\alpha_m}{3} + \frac{\alpha_r}{4}\right).$$

Impose a conservative bound $|\Delta H/H|_0 \le 2\times10^{-3}$ which yields

$$\left|\frac{\alpha_m}{\rho_{c0,\pm}}\right| |\zeta| \;\lesssim\; 1.2\times10^{-2}, \qquad \left|\frac{\alpha_r}{\rho_{c0,\pm}}\right| |\zeta| \;\lesssim\; 1.6\times10^{-2},$$

(the radiation coefficient is looser at $z \simeq 0$ but tighter at early times; see next).

Radiation-era bounds (BBN and CMB)

The exchange shifts the radiation budget on each sheet. Interpreting the net late-time offset as an effective extra component on the + sheet,

$$\Delta\rho_{r+}(a) = +\frac{\alpha_r\zeta}{4}\left(1 - a^{-4}\right),$$

so near recombination ($a_{\rm rec} \simeq 1/1100$) the fractional shift is

$$\left.\frac{\Delta\rho_{r+}}{\rho_{r+}}\right|_{a_{\rm rec}} \simeq \frac{\alpha_r\zeta}{4\,\rho_{r+}(a_{\rm rec})}.$$

Expressed as $\Delta N_{\rm eff}$ using $\rho_{\rm rad} = \rho_\gamma\left[1 + \frac{7}{8}(\frac{4}{11})^{4/3} N_{\rm eff}\right]$,

$$\Delta N_{\rm eff} \;\simeq\; \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{\Delta\rho_{r+}}{\rho_\gamma} \;\propto\; \frac{\alpha_r\zeta}{\rho_\gamma(a_{\rm rec})}.$$

Impose $|\Delta N_{\rm eff}| \lesssim 0.2$ to obtain a bound on $\alpha_r\zeta$ (data choice dependent). Since $\rho_\gamma(a_{\rm rec}) \propto a^{-4}$ is large, this typically enforces $|\alpha_r\zeta|/\rho_{c0,\pm} \ll 10^{-3}$ if the drive is active that early. A safe operational choice is to suppress the drive before BBN/CMB or ensure α_r is sufficiently small.

Growth and lensing bounds

With $\varepsilon = \sigma_0\zeta$, EdS analytics give

$$\Delta f_\pm \approx \pm\,0.4\,\varepsilon, \qquad \Delta\gamma_\pm \approx \pm\,0.33\,\varepsilon \quad (z \approx 0).$$

Impose $|\Delta\gamma| \lesssim 10^{-3}$ to find $|\varepsilon| \lesssim 3\times10^{-3}$ (consistent with the acceptance window used above). In the minimal model, $c_T = 1$ and slip $\eta \simeq 0$, so weak-lensing deviations arise only through any explicit $\nu_\pm \equiv_\pm /G - 1$; require $|\nu_\pm| \ll 1$ (e.g. $|\nu_\pm| \lesssim 10^{-2}$ as a prior).

Laboratory bounds and thermodynamic positivity

From OU inference $D = \Gamma_\theta \operatorname{Var}(\delta\theta) \geq 0$; measured negative D signals model/systematics failure. Entropy production density in the θ-channel is

$$\sigma_\theta = L_{33}\,\mathcal{F}_3^2 = \frac{1}{\gamma_c}\,\mathcal{F}_3^2 \geq 0,$$

hence γ_c must be positive. If an external lock drive is used ($\mathcal{F}_3 \neq 0$), verify observed $\langle\sigma_\theta\rangle \propto \Gamma_\theta \operatorname{Var}(\delta\theta)$.

Null tests (diagnostics)

- **Antisymmetric budget:** verify $Q_+ + Q_- = 0$ and $d(\Omega_{m+} + \Omega_{m-})/d\ln a = 0$ to leading order.
- **Opposite-sign drifts:** $\Delta H/H$ and $\Delta \ln D$ must have opposite signs on the two sheets.
- **Equilibrium limit:** $\dot\theta \to 0$ restores standard ΛCDM on both sheets.
- **Tensor speed:** check $c_T^{(\pm)} = 1$ (no shift in GW arrival times).
- **Second law:** $\sigma \geq 0$; any negative estimate indicates $L \not\succeq 0$ or analysis error.

Unit audits (spot checks)

- $[\Delta H/H] = 1$ (dimensionless), $[\Delta N_{\text{eff}}] = 1$, $[\varepsilon] = 1$.
- $[m_{\text{eff}}] = \text{s}^{-1}$, $[\Gamma_\theta] = \text{s}^{-1}$, $[D] = \text{s}^{-2}$.
- $[\alpha_{m,r}/\rho_{c0,\pm}] = 1$; thus $|\alpha_{m,r}/\rho_{c0,\pm}| \cdot |\zeta|$ is a pure number (used in all bounds).

13 Canonical 3+1 Structure and Constraint Algebra: Closure, Counting, and Units

Setup: two ADM splits and a conservative regulator core

On each sheet $(\mathcal{M}_\pm, g_\pm)$ perform an ADM split with lapse $N_\pm$, shift $N_\pm^i$, and spatial metric $h_{ij}^{(\pm)}$. The teleparallel Hamiltonian equals the GR Hamiltonian up to a boundary term, so we use the standard ADM form for the gravitational sector per sheet:

$$\mathcal{H}_{\text{grav}}^{(\pm)} = N_\pm\,\mathcal{C}^{(\pm)} + N_\pm^i\,\mathcal{C}_i^{(\pm)} + \partial_i(\cdots),$$

with Hamiltonian (scalar) constraint density $\mathcal{C}^{(\pm)} \approx 0$ and momentum (vector) constraint density $\mathcal{C}_i^{(\pm)} \approx 0$. Matter on each sheet contributes $\mathcal{H}_{\text{m}}^{(\pm)}$ in the usual way. For the canonical analysis we use the *conservative* regulator (drop explicit damping here to avoid non-Hamiltonian friction):

$$\mathcal{L}_\theta = \frac{\chi}{2}\,\dot\theta^2 - \frac{m_\theta^2}{2}\,\theta^2 - \lambda\,\sin\!\Big(\frac{\theta}{M}\Big)\,\Delta J, \qquad \Delta J = c_m(\rho_{m+} - \rho_{m-}) + c_r(\rho_{r+} - \rho_{r-}) + \cdots.$$

Then $\pi_\theta := \partial\mathcal{L}_\theta/\partial\dot\theta = \chi\,\dot\theta$ and the regulator Hamiltonian density is

$$\mathcal{H}_\theta = \frac{\pi_\theta^2}{2\chi} + \frac{m_\theta^2}{2}\,\theta^2 + \lambda\,\sin\!\Big(\frac{\theta}{M}\Big)\,\Delta J.$$

Total Hamiltonian and primary constraints

Define for each sheet

$$\mathcal{H}_{\text{tot}}^{(\pm)} = N_\pm \, \mathcal{C}^{(\pm)} + N_\pm^i \, \mathcal{C}_i^{(\pm)} + \mathcal{H}_{\text{m}}^{(\pm)},$$

and the exchange/regulator part

$$\mathcal{H}_{\text{couple}} = \mathcal{H}_\theta \quad (\text{depends on } \theta, \pi_\theta \text{ and on densities inside } \Delta J).$$

The full Hamiltonian is

$$H_{\text{full}} = \int \mathrm{d}^3x \, \Big[\mathcal{H}_{\text{tot}}^{(+)} + \mathcal{H}_{\text{tot}}^{(-)} + \mathcal{H}_{\text{couple}}\Big].$$

Primary constraints: momenta conjugate to the lapses and shifts vanish on each sheet,

$$\pi_{N_\pm} \approx 0, \qquad \pi_{N_\pm^i} \approx 0,$$

as in GR/TEGR.

Secondary constraints and closure with the exchange potential

Preserving the primary constraints gives the standard secondary constraints on each sheet:

$$\mathcal{C}^{(\pm)} \approx 0, \qquad \mathcal{C}_i^{(\pm)} \approx 0.$$

Because $\mathcal{H}_{\text{couple}}$ is a *local scalar density* that depends only on sheet densities (configuration variables) and on (θ, π_θ) but not on gravitational momenta or their spatial derivatives across sheets, the Poisson brackets of constraints retain the Dirac (hypersurface-deformation) form up to the usual matter terms:

$$\{\, \mathcal{C}^{(\pm)}[N], \, \mathcal{C}^{(\pm)}[M] \,\} = \mathcal{C}_i^{(\pm)}\big[h_{(\pm)}^{ij}(N\partial_j M - M\partial_j N)\big],$$

$$\{\, \mathcal{C}_i^{(\pm)}[N^i], \, \mathcal{C}^{(\pm)}[N] \,\} = \mathcal{C}^{(\pm)}[\mathcal{L}_{\vec{N}} N], \qquad \{\, \mathcal{C}_i^{(\pm)}[N^i], \, \mathcal{C}_j^{(\pm)}[M^j] \,\} = \mathcal{C}_i^{(\pm)}[\mathcal{L}_{\vec{N}} M^i],$$

$$\{\, \mathcal{C}^{(+)}, \, \mathcal{C}^{(-)} \,\} = 0 \quad (\text{disjoint gravitational phase spaces}).$$

The exchange potential enters only additively in $\mathcal{C}^{(\pm)}$ as a matter-like piece and does not introduce momentum-dependent cross-terms; therefore the combined set

$$\mathfrak{C} := \{\mathcal{C}^{(+)}, \mathcal{C}^{(-)}, \mathcal{C}_i^{(+)}, \mathcal{C}_i^{(-)}\}$$

is first class, with the same algebra as the direct sum of two GR/TEGR algebras. The regulator adds the canonical pair (θ, π_θ) with Hamilton equations

$$\dot{\theta} = \frac{\pi_\theta}{\chi}, \qquad \dot{\pi}_\theta = -m_\theta^2 \, \theta - \frac{\lambda}{M} \cos\Big(\frac{\theta}{M}\Big) \Delta J,$$

and transforms as a scalar under spatial diffeomorphisms generated by $\mathcal{C}_i^{(\pm)}$.

Energy constraint in FLRW and recovery of the sourced Friedmann pair

In a homogeneous/isotropic reduction on each sheet, the Hamiltonian constraint gives

$$3H_\pm^2 = 8\pi G_\pm\,\rho_\pm + \underbrace{\left(\frac{\pi_\theta^2}{2\chi} + \frac{m_\theta^2}{2}\,\theta^2 + \lambda\,\sin\tfrac{\theta}{M}\,\Delta J\right)}_{:=\rho_{\mathrm{couple},\pm}},$$

where $\rho_{\mathrm{couple},+} = -\rho_{\mathrm{couple},-}$ under the antisymmetric assignment implicit in ΔJ. In the overdamped phenomenological limit used earlier (eliminate π_θ adiabatically and linearize near the lock), this reproduces

$$3H_\pm^2 = 8\pi G_\pm\,\rho_\pm + Q_\pm, \qquad Q_+ + Q_- = 0,$$

with $Q_\pm$ the source terms calibrated by $\alpha_{m,r}$.

Degrees of freedom (DOF) counting

Per sheet (TEGR/GR) there are 2 propagating tensor polarizations. The regulator provides one scalar DOF if $\chi > 0$ is kept; with the purely overdamped effective description the scalar becomes auxiliary (non-propagating).

$$\text{Total DOF} = \begin{cases} 2+2+1=5 & \text{(conservative regulator with } \chi > 0\text{)},\\ 2+2=4 & \text{(overdamped, auxiliary } \theta\text{)}. \end{cases}$$

No higher-derivative metric couplings are introduced by $\mathcal{H}_{\mathrm{couple}}$; hence no Ostrogradski/Boulware–Deser mode arises.

Constraint preservation and exchange current

Constraint preservation requires $\dot{\mathcal{C}}^{(\pm)} \approx \{\mathcal{C}^{(\pm)}, H_{\mathrm{full}}\} \approx 0$. Because $\mathcal{H}_{\mathrm{couple}}$ is a scalar density and the regulator sector is canonical, the only effect is a shift of the matter energy density by $\pm\rho_{\mathrm{couple}}$, which is already included in $\mathcal{C}^{(\pm)}$. The antisymmetry implied by ΔJ ensures global energy conservation:

$$\dot\rho_{m+} + 3H_+(\rho_{m+} + p_{m+}) + \dot\rho_{m-} + 3H_-(\rho_{m-} + p_{m-}) \;=\; 0.$$

Units and sample numbers

Hamiltonian densities have units of energy per volume ($\mathrm{J\,m^{-3}}$). With χ measured in s^2 (since π_θ is dimensionless in this normalized coarse-grained choice), one has

$$\left[\frac{\pi_\theta^2}{2\chi}\right] = \mathrm{s}^{-2} \;\leftrightarrow\; \mathrm{J\,m^{-3}} \text{ via } c = \hbar = 1 \text{ or by inserting a conversion factor as needed.}$$

For a homogeneous estimate, set $\chi = 1$, $m_\theta = 0.5\,\mathrm{s}^{-1}$, $\theta = 10^{-2}\,\mathrm{rad}$, $\pi_\theta = 0$ (locked), $\lambda\,\Delta J/M = 10^{-3}\,\mathrm{s}^{-2}$. Then

$$\rho_{\mathrm{couple},\pm} \approx \tfrac{1}{2}m_\theta^2\theta^2 + \lambda\,\sin(\theta/M)\,\Delta J \approx \tfrac{1}{2}(0.25)\times10^{-4} + 10^{-3}\times10^{-2} \approx 1.25\times10^{-5} + 1.0\times10^{-5} \;\approx 2.25\times10^{-5}\,\mathrm{s}^{-2},$$

which corresponds to a fractional H-shift $\Delta H/H \sim \frac{1}{2}\,\rho_{\mathrm{couple}}/H^2$ at $a = 1$.

What to include in a full canonical appendix

For a full Claude build, include:

- Explicit expressions for $\mathcal{C}^{(\pm)}$ and $\mathcal{C}_i^{(\pm)}$ (ADM or TEGR form) and their Poisson brackets.
- Canonical variables and symplectic form on each sheet; explicit statement that cross-sheet brackets vanish.
- The regulator canonical pair (θ, π_θ), its Hamilton equations, and its transformation under spatial diffeomorphisms.
- Proof sketch that any scalar-density $\mathcal{H}_{\text{couple}} = \sqrt{h_+}\sqrt{h_-}\, V(\theta, \rho_{i\pm})$ preserves first-class nature of $\mathfrak{C}$ (no momentum dependence $\Rightarrow$ no new second-class constraints).
- Homogeneous reduction reproducing the sourced Friedmann pair and exchange continuity equations.

14 Microphysical Coupling Models: Calibration, Linearization, and Units

Representative microphysics and the continuity map

Adopt a Josephson-like exchange sector

$$\mathcal{L}_{\text{couple}}(\theta; \Delta J) = -\lambda \sin\Big(\frac{\theta}{M}\Big)\,\Delta J - \frac{1}{2}\gamma_c\,\dot{\theta}^2 - \frac{1}{2}m_\theta^2\theta^2, \qquad \Delta J := c_m(\rho_{m+} - \rho_{m-}) + c_r(\rho_{r+} - \rho_{r-}) + \cdots .$$

Energy balance per comoving volume ($\dot{\rho}_{\text{couple}} = -\partial_\theta \mathcal{L}_{\text{couple}}\,\dot{\theta}$) and antisymmetry of exchange imply

$$\dot{\rho}_{\text{couple}} = -(Q_+ + Q_-) = 0 \quad \Rightarrow \quad Q_\pm = \pm\Big[\frac{\lambda}{M}\cos\Big(\frac{\theta}{M}\Big)\Delta J\Big]\dot{\theta} + \mathcal{O}(\dot{\Delta J}).$$

Splitting by species and matching to the continuity equations,

$$\dot{\rho}_{m\pm} + 3H_\pm \rho_{m\pm} = \pm\alpha_m\,\dot{\theta}, \qquad \dot{\rho}_{r\pm} + 4H_\pm \rho_{r\pm} = \pm\alpha_r\,\dot{\theta},$$

fixes (near a lock point θ_*)

$$\alpha_m = \frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)c_m\,\rho_{c0,\pm}, \qquad \alpha_r = \frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)c_r\,\rho_{c0,\pm},$$

so that $\alpha_{m,r}/\rho_{c0,\pm}$ are dimensionless calibration numbers.

Linearization and small-oscillation scales

Expand $\theta = \theta_* + \delta\theta$ with $|\delta\theta| \ll 1$:

$$\mathcal{L}_{\text{couple}} \approx -\lambda \sin\Big(\frac{\theta_*}{M}\Big)\Delta J - \frac{\lambda}{M}\cos\Big(\frac{\theta_*}{M}\Big)\Delta J\,\delta\theta - \frac{1}{2}\gamma_c\,\dot{\delta\theta}^2 - \frac{1}{2}m_\theta^2\delta\theta^2.$$

In the overdamped regime ($\chi\,\ddot{\theta} \ll \gamma_c\dot{\theta}$),

$$\dot{\delta\theta} \simeq -\Gamma_\theta\,\delta\theta + \frac{1}{\gamma_c}\Big(\frac{\lambda}{M}\cos\tfrac{\theta_*}{M}\,\Delta J + \Delta\mu\Big), \qquad \Gamma_\theta := \frac{m_\theta^2 - \Psi''(\theta_*)}{\gamma_c} > 0.$$

Thus a static bias ΔJ produces an offset $\delta\theta_{\text{off}} = (\lambda/M)\cos(\theta_*/M)\,\Delta J/(\gamma_c\Gamma_\theta)$.

Dimensional calibration (SI)

Choose $\rho_{c0,\pm} = 8.5 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$ and a target $\alpha_m/\rho_{c0,\pm} = 10^{-2}$. With $c_m = 1$ and $\cos(\theta_*/M) \simeq 1$ this fixes

$$\frac{\lambda}{M} \simeq 10^{-2}.$$

If $M = 1$ (dimensionless normalization), then $\lambda \simeq 10^{-2}$. This choice keeps the present-day drifts $\mathrm{d}\Omega_{m\pm}/\mathrm{d}\ln a = \pm\alpha_m\zeta/\rho_{c0,\pm}$ at the $10^{-2}\zeta$ level (Section on background).

Worked numbers

With $\zeta = 0.1$ and $\alpha_m/\rho_{c0,\pm} = 10^{-2}$:

$$\left.\frac{\mathrm{d}\Omega_{m\pm}}{\mathrm{d}\ln a_\pm}\right|_1 = \pm 10^{-3}.$$

For $\alpha_r/\rho_{c0,\pm} = 10^{-3}$ one gets $\mathrm{d}\Omega_{r\pm}/\mathrm{d}\ln a = \pm 10^{-4}$.

15 Boundary Terms and Teleparallel Identities: $R = -T + B$, Nieh–Yan, and Audits

TEGR identity per sheet

Each sheet satisfies

$$R[g^{(\pm)}] = -\,T^{(\pm)} + B^{(\pm)}, \qquad B^{(\pm)} = \frac{2}{e^{(\pm)}}\partial_\mu\big(e^{(\pm)}T^{\mu(\pm)}\big),$$

with $T^{\mu(\pm)} := T^{\alpha(\pm)}{}_{\alpha}{}^{\mu}$ and $e^{(\pm)} = \det(e^{a(\pm)}{}_{\mu})$. Thus the action built from $T^{(\pm)}$ differs from the Einstein–Hilbert action by a total divergence $B^{(\pm)}$.

Nieh–Yan 4-form and teleparallel reduction

The Nieh–Yan density on each sheet is

$$\mathrm{NY}^{(\pm)} := \mathrm{d}\big(e^{a(\pm)} \wedge T_a^{(\pm)}\big) = T^{a(\pm)} \wedge T_a^{(\pm)} - e^{a(\pm)} \wedge e^{b(\pm)} \wedge R_{ab}^{(\pm)}.$$

For the flat spin connection ($R_{ab}^{(\pm)} = 0$) one has

$$\mathrm{NY}^{(\pm)} = \mathrm{d}\big(e^{a(\pm)} \wedge T_a^{(\pm)}\big),$$

a total derivative. Hence any NY-proportional term contributes only a boundary integral and can be fixed by boundary conditions.

Two-sheet boundary accounting

The total boundary contribution is

$$S_B = \int_{\partial\mathcal{M}_+}(\cdots) \;+\; \int_{\partial\mathcal{M}_-}(\cdots).$$

If the two sheets share the same physical boundary (e.g. asymptotically flat common boundary) and the regulator enforces equilibrium ($\dot\theta \to 0$), then the NY/TEGR boundary pieces can be chosen equal and opposite, or independently set to zero by standard falloff, so that S_B does not affect bulk equations. This preserves the first-class constraint algebra derived earlier.

Checks

(i) Adding a constant multiple of $\mathrm{NY}^{(\pm)}$ leaves bulk field equations unchanged. (ii) The exchange sector $\mathcal{L}_{\mathrm{couple}}(\theta, \Delta J)$ is algebraic in the fluid invariants and does not introduce boundary torsion terms. (iii) All extra terms are either total derivatives or enter as matter-like potentials and cannot generate higher-derivative metric couplings.

16 Laboratory Architecture: Power, Readout, and Minimum SNR

Dissipated power and drive power

From the regulator energy $E = \frac{1}{2}\chi\dot{\theta}^2 + \frac{1}{2}m_\theta^2\theta^2 - \Psi(\theta)$ and equation $\chi\ddot{\theta} + \gamma_c\dot{\theta} + m_\theta^2\theta = \mathcal{S}(t)$,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma_c\,\dot{\theta}^2 + \dot{\theta}\,\mathcal{S}(t).$$

Time-averaged (over $2\pi/\omega$) under a harmonic drive $\mathcal{S}(t) = S_0\cos\omega t$ and response $\theta(t) = \Re[\chi(\omega)S_0 e^{i\omega t}]$ with $\chi(\omega) = (\Gamma_\theta + i\omega)^{-1}$ (overdamped),

$$\langle P_{\mathrm{diss}}\rangle = \gamma_c\,\langle\dot{\theta}^2\rangle = \frac{\gamma_c}{2}\,\omega^2\,|\chi(\omega)|^2 S_0^2 = \frac{\gamma_c}{2}\,\frac{\omega^2}{\omega^2+\Gamma_\theta^2}\,\frac{S_0^2}{\Gamma_\theta^2+\omega^2}.$$

At $\omega = \Gamma_\theta$ this reduces to $\langle P_{\mathrm{diss}}\rangle = \frac{\gamma_c}{8}\,S_0^2/\Gamma_\theta^2$.

Readout via frequency pull

If a resonator has fractional pull $\delta f/f = \kappa_\theta\,\delta\theta$, the single-sided PSD of the readout is

$$S_{\delta f/f}(\omega) = \kappa_\theta^2\,S_\theta(\omega) = \kappa_\theta^2\,\frac{2D}{\omega^2+\Gamma_\theta^2}.$$

Integrated RMS in a bandwidth $[0,\Omega]$:

$$\mathrm{Var}\Big(\frac{\delta f}{f}\Big) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} S_{\delta f/f}(\omega)\,\mathrm{d}\omega = \kappa_\theta^2\,\frac{D}{\Gamma_\theta}\,\frac{2}{\pi}\arctan\Big(\frac{\Omega}{\Gamma_\theta}\Big).$$

Minimum integration time (white readout noise)

With detector white floor S_y for $y = \delta f/f$, the SNR for a DC offset $\delta\theta_{\mathrm{off}}$ after time τ is

$$\mathrm{SNR}^2 = \frac{(\kappa_\theta\delta\theta_{\mathrm{off}})^2\,\tau}{S_y}.$$

Setting $\mathrm{SNR} = 1$ and using $\delta\theta_{\mathrm{off}} = (\lambda/M)\cos(\theta_*/M)\,\Delta J/(\gamma_c\Gamma_\theta)$ gives

$$\tau_{\min} = \frac{S_y}{\kappa_\theta^2}\Big(\frac{\gamma_c\Gamma_\theta\,M}{\lambda\cos(\theta_*/M)\,\Delta J}\Big)^2.$$

Worked numbers

Take $\Gamma_\theta = 2.5\ \mathrm{s}^{-1}$, $\gamma_c = 0.1\ \mathrm{s}^{-1}$, $\kappa_\theta = 10^{-3}\ \mathrm{rad}^{-1}$, $M = 1$, $\lambda = 10^{-2}$, $\cos(\theta_*) \simeq 1$, $\Delta J = 10^{-2}$, and $S_y = 10^{-12}\ \mathrm{Hz}^{-1}$ (typical fractional frequency PSD for a quiet RF cavity). Then

$$\delta\theta_{\mathrm{off}} = \frac{10^{-2}\times10^{-2}}{0.1\times2.5} = 4.0\times10^{-3}\ \mathrm{rad}, \qquad \tau_{\min} = \frac{10^{-12}}{10^{-6}}\Big(\frac{0.25}{10^{-4}}\Big)^2 \approx 10^{-6}\times6.25\times10^{6}\ \mathrm{s} \approx 6.25\ \mathrm{s}.$$

Thus a few seconds of averaging would resolve the DC offset at $\mathrm{SNR} \approx 1$ under these benchmark conditions.

17 Inference Pipeline: Parameters, Likelihoods, and Fisher Pre-fit

Parameter vector and priors

Adopt

$$\mathbf{p} = \big(\Xi_m,\ \Xi_r,\ \varepsilon,\ \nu_+,\ \nu_-,\ \Gamma_\theta,\ D,\ \kappa_\theta\big), \qquad \Xi_{m,r} := \frac{\alpha_{m,r}\zeta}{\rho_{c0,\pm}}, \quad \varepsilon := \sigma_0\zeta.$$

Priors: $|\Xi_m| \lesssim 10^{-2}$, $|\Xi_r| \lesssim 10^{-3}$, $|\varepsilon| \lesssim 5\times10^{-3}$, $|\nu_\pm| \ll 1$, $\Gamma_\theta > 0$, $D \geq 0$, $\kappa_\theta > 0$.

Background and growth observable maps

Background:

$$E^2_\pm(a) = E^2_{\Lambda\mathrm{CDM},\pm}(a)\ \pm\ \Xi_m\Big(1-a^{-3}\Big)\,\frac{1}{3}\ \pm\ \Xi_r\Big(1-a^{-4}\Big)\,\frac{1}{4}.$$

Growth (late-time proxy over $a \in [a_1, a_2]$):

$$\Delta \ln D_\pm \simeq (\pm 0.4\,\varepsilon)\ \ln\frac{a_2}{a_1}, \qquad \Delta\gamma_\pm \approx \pm 0.33\,\varepsilon \quad (z \approx 0).$$

Lab PSD:

$$S_{\delta f/f}(\omega) = \kappa_\theta^2\,\frac{2D}{\omega^2+\Gamma_\theta^2}.$$

Gaussian Fisher pre-fit (schematic)

For a data vector $\mathbf{O}$ with covariance C,

$$F_{ij} = \Big(\frac{\partial \mathbf{O}}{\partial p_i}\Big)^{\top} C^{-1}\Big(\frac{\partial \mathbf{O}}{\partial p_j}\Big), \qquad \sigma(p_i) \geq \sqrt{(F^{-1})_{ii}}.$$

Derivatives needed at $a = 1$:

$$\frac{\partial E^2_\pm}{\partial \Xi_m}\Big|_1 = \pm\frac{1}{3}, \qquad \frac{\partial E^2_\pm}{\partial \Xi_r}\Big|_1 = \pm\frac{1}{4}, \qquad \frac{\partial f_\pm}{\partial \varepsilon} \approx \pm 0.4.$$

Lab:

$$\frac{\partial S_{\delta f/f}}{\partial D} = \kappa_\theta^2\,\frac{2}{\omega^2+\Gamma_\theta^2}, \qquad \frac{\partial S_{\delta f/f}}{\partial \Gamma_\theta} = \kappa_\theta^2\,2D\,\frac{-2\Gamma_\theta}{(\omega^2+\Gamma_\theta^2)^2}, \qquad \frac{\partial S_{\delta f/f}}{\partial \kappa_\theta} = 2\kappa_\theta\,\frac{2D}{\omega^2+\Gamma_\theta^2}.$$

Acceptance and identifiability checks

- Background-only data mainly constrain (Ξ_m, Ξ_r); growth constrains ε; lensing constrains $\nu_\pm$.
- Lab data constrain $(\Gamma_\theta, D, \kappa_\theta)$ independently of cosmology.
- Degeneracies: Ξ_m anti-correlates with H_0 and Ω_{m0} in background fits; include standard cosmological parameters in a joint analysis if needed.

18 Symbol Ledger and Units (Quick Reference)

Core symbols

Symbol	Meaning	Units
$H_\pm$	Hubble rate on sheet $\pm$	s^{-1}
$\rho_{m\pm},\ \rho_{r\pm}$	matter, radiation densities	$\mathrm{kg\,m^{-3}}$
$\rho_{c0,\pm}$	critical density today (sheet $\pm$)	$\mathrm{kg\,m^{-3}}$
θ	interphasic regulator phase	1 (radians)
γ_c	damping coefficient	s^{-1}
m_θ	regulator frequency scale	s^{-1}
Γ_θ	overdamped relaxation rate	s^{-1}
D	noise intensity (OU)	s^{-2}
λ/M	coupling calibration	1
$\alpha_{m,r}$	exchange coefficients (continuity)	$\mathrm{kg\,m^{-3}}$
$\Xi_{m,r}$	$\alpha_{m,r}\zeta/\rho_{c0,\pm}$	1
ζ	constant fraction $\dot\theta/H$	1
σ_0	source–contrast coupling	1
ε	$\sigma_0\zeta$	1
$\nu_\pm$	$\pm/G-1$	1
κ_θ	readout pull coeff.	rad^{-1}
$S_\theta(\omega)$	PSD of θ	s

Frequently used relations

$$\dot\rho_{m\pm} + 3H_\pm\rho_{m\pm} = \pm\alpha_m\dot\theta, \quad \dot\rho_{r\pm} + 4H_\pm\rho_{r\pm} = \pm\alpha_r\dot\theta, \quad Q_+ + Q_- = 0,$$

$$E_\pm^2(a) = E^2_{\Lambda\mathrm{CDM},\pm}(a) \ \pm\ \frac{\Xi_m}{3}\,(1-a^{-3}) \ \pm\ \frac{\Xi_r}{4}\,(1-a^{-4}),$$

$$S_\theta(\omega) = \frac{2D}{\omega^2+\Gamma_\theta^2}, \qquad \mathrm{Var}(\theta) = \frac{D}{\Gamma_\theta}, \qquad \Delta\ln D_\pm \simeq (\pm 0.4\,\varepsilon)\ln(a_2/a_1).$$

19 Conclusions and Roadmap

Summary

We constructed a two-sheet teleparallel thermodynamic framework with: (i) antisymmetric energy exchange conserving the global budget, (ii) a Josephson-like regulator obeying a Lyapunov decay, (iii) dimensionally consistent background/growth deformations controlled by $(\Xi_m, \Xi_r, \varepsilon)$, (iv) fluctuation–dissipation structure enabling laboratory inference of $(\Gamma_\theta, D, \kappa_\theta)$, (v) preserved constraint algebra and boundary-term consistency (TEGR identity, Nieh–Yan total derivative).

Next steps (theory and experiment)

- Extend beyond constant-fraction $\dot\theta = \zeta H$ to time-dependent $\zeta(a)$ and assess early-time bounds (ΔN_{eff}).
- Include a canonical gradient term for θ and analyze propagation/causality on inhomogeneous backgrounds.

- Build a cavity readout prototype meeting $\tau_{\min} \sim$ few seconds at SNR 1; validate OU statistics and FDT.
- Integrate with a CLASS/hi_class fork: add $(\Xi_m, \Xi_r, \varepsilon)$ hooks; produce mock constraints.

Validation checklist

Units pass, constraint algebra closes, entropy production $\sigma \geq 0$, $c_T = 1$, opposite-sign drifts across sheets, and equilibrium limit recovers ΛCDM.