TORSION EQUATIONS

Dynamical Framework for BT8G
Autoparallel Holography Phase-Lock Cascade

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Revision	Date	Description
A	July 17, 2025	Initial CSR+ mathematical formulation
B	November 14, 2025	Enhanced formatting; integration with TOPOLOGY and TETRAD frameworks; Solitonic Holofractal Extension

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I PURPOSE

We establish the dynamical framework for CSR$^+$ (Cascade Spectrality Resonance) Unified Resonance Holography through thirteen fundamental torsion equations governing bimetric teleparallel phase-lock dynamics. Seven foundational equations (TR-F1–TR-F7) encode the core phase-lock layer, linking topological torsion constraints from the TOPOLOGY framework to observable electromagnetic phenomena via Josephson-coupled bimetric sheets. Six solitonic equations (TR-S8–TR-S13) extend this structure to self-similar fractal boundary hierarchies with golden-ratio scaling governed by φ.

This torsion layer translates the geometric content of TOPOLOGY and TETRAD into time-dependent field equations that control cascade behavior. Working in the teleparallel Weitzenböck geometry, the equations bridge static topological constraints and dynamical electromagnetic observables through torsion energy coupled to the inter-sheet Josephson phase. Quartic stabilization, fractal energy summation, and scale-covariant TEGR dynamics together generate a bounded, self-regulating cascade rather than runaway growth.

Harmonic propagation emerges naturally from hGEM (harmonic Gravito-Electromagnetism) integration and Phase Gradient Topology (PHASE/TRANSLATIONS) rather than from fixed external frequency sources. The fractal torsion equations (TR-S8–TR-S13) establish scale-invariant energy and charge conservation, ensuring that self-similar boundary refinement preserves both Nieh–Yan-normalized topological neutrality and finite total torsion energy. These properties are critical for the physical consistency and experimental viability of the bimetric cascade framework.

For foundational context regarding phase-locked systems, see:

PHASE/TRANSLATIONS

Solomon Drowne, Spectrality Institute 2025

II NOTATION & SYMBOLOGY

Symbol	Meaning
${T^{\lambda}}_{\mu\nu}$	Torsion tensor in the Weitzenböck connection (TEGR teleparallel gravity).
${T^{a}}_{(n)}$	Scale-indexed torsion 2-form on fractal boundary tile S_n.
${e^{a}}_{\mu}$	Tetrad field mapping frame index a to coordinate index μ.
${e^{a}}_{(n)}$	Tetrad field restricted to fractal tile S_n in the φ-hierarchy.
ϵ_{abc}	Levi-Civita symbol for frame indices (a, b, c).
$\chi(S)$	Euler characteristic of a surface S.
$\Phi(\theta, \varphi)$	Toroidal harmonic field on the T^2 manifold as a function of angular coordinates (θ, φ).
$\theta,\ \varphi$ (coords)	Angular toroidal coordinates (poloidal and toroidal angles) on T^2.
$n,\ m$	Integer mode numbers labelling toroidal harmonics on T^2.
k_i	Topological vortex (winding) numbers for circulation quantization around cycles C_i.
C_i	Fundamental cycles on the toroidal manifold $(i = 1, 2)$.
$\mathcal{L}_T$	Torsion Lagrangian density.
$\mathcal{L}_T^{(n)}$	Scale-indexed torsion energy density at fractal level n.
$\mathcal{L}_{T,\mathrm{tot}}$	Total torsion energy density summed over all fractal levels.
κ	Teleparallel gravitational coupling constant.
θ (phase)	Josephson phase field (inter-sheet phase between bimetric sectors).
ξ	Kinetic coefficient for the Josephson phase field θ.
m_θ	Effective mass parameter for the Josephson phase mode.
λ	Inter-sector coupling strength between $F_+^2 - F_-^2$ and θ.
M	Characteristic mass scale entering phase–flux couplings.
$F_\pm$	Electromagnetic field strength tensors for $(+)$ and $(-)$ bimetric sectors.
$\Box$	d'Alembertian wave operator, $\Box = \partial_\mu \partial^\mu$.
$\mathcal{H}_{\mathrm{dyn}}$	Total dynamical Hamiltonian density of the torsion–phase–flux system.
g_4	Dimensionless quartic coupling parameter in the flux imbalance sector.
$\mathcal{E}_T$	Torsion energy density (integrand of torsion self-energy).
$\mathcal{E}_\Phi$	Phase field energy density (kinetic plus potential contributions).
$R_\star$	Optimal toroidal radius selected by the variational principle (TR-F7).
ω_g	Fundamental gap frequency, $\omega_g \sim R_\star^{-1}$.
φ	Golden-ratio scaling constant, $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$, governing the fractal hierarchy.
a_{drift}	Phase-driven inertial acceleration, $a_{\mathrm{drift}} = \alpha\,\dot{\theta}$.
α	Coupling coefficient for phase-to-inertia transduction.
S_n	Self-similar boundary tile at fractal level n in the φ-ladder.
$D_\mu^{(n)}$	Covariant derivative operator defined on tile S_n.
${S^{\rho}}_{\mu\nu}$	Teleparallel superpotential tensor constructed algebraically from torsion.
$S_a^{\mu\nu}$	Mixed-index superpotential entering the TEGR field equations.
Φ_n	Superpotential boundary flux through tile S_n.
$\Phi_{\mathrm{boundary,tot}}$	Total superpotential flux summed over the fractal boundary hierarchy.
$\mathcal{N}_{S_n}$	Nieh–Yan index (dimensionless topological charge) on tile S_n.
c_{NY}	Nieh–Yan coupling coefficient linking torsion integrals to $\mathcal{N}_{S_n}$.
${\omega^{a}}_{b(n)}$	Spin connection on fractal level n used in the scale-indexed Cartan structure equations.

III FOUNDATIONAL TORSION EQUATIONS

$$\int_S \epsilon_{abc}\, T^a \wedge e^b \wedge e^c = 2\pi\,\chi(S), \qquad \chi(\text{torus}) = 0 \tag{TR-F1}$$

Teleparallel Gauss-Bonnet Topological Constraint

$$\Phi(\theta,\varphi) = \Phi_0\, e^{i(n\theta + m\varphi)}, \quad n, m \in \mathbb{Z} \tag{TR-F2}$$

Double-Periodic Toroidal Harmonic Field

$$\oint_{C_i} \nabla \arg\Phi \cdot d\ell = 2\pi k_i, \quad k_i \in \mathbb{Z},\ i = 1, 2 \tag{TR-F3}$$

Quantized Circulation and Vortex Winding

$$\mathcal{L}_T = \frac{\kappa}{2} T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu} \tag{TR-F4}$$

Positive-Definite Torsion Energy Density

$$\xi\,\Box\theta + m_\theta^2\theta = \frac{\lambda}{M}\big(F_+^2 - F_-^2\big) \tag{TR-F5}$$

Josephson Phase Evolution Equation

$$\mathcal{H}_{\text{dyn}} = \frac{1}{4}\big(F_+^2 + F_-^2\big) + \frac{1}{2}\xi(\partial\theta)^2 + \frac{1}{2}m_\theta^2\theta^2 + \frac{\lambda^2}{2M^2\xi}\big(F_+^2 - F_-^2\big)^2 \tag{TR-F6}$$

Total Bimetric Hamiltonian Density

$$\frac{d}{dR}\Big[\int_{S(R)} \mathcal{E}_T\, dA + \int_{S(R)} \mathcal{E}_\Phi\, dA\Big] = 0 \tag{TR-F7}$$

Geometric Variational Optimization Condition

IV SOLITONIC TOROID EQUATIONS

$$T^a{}_{(n)} = de^a{}_{(n)} + \omega^a{}_{b(n)} \wedge e^b{}_{(n)}, \quad e^a{}_{(n+1)} = \varphi^{-1} e^a{}_{(n)} \Rightarrow T^a{}_{(n+1)} = \varphi^{-1} T^a{}_{(n)} \tag{TR-S8}$$

Scale-Indexed Torsion 2-Form Hierarchy

$$\sum_{n=0}^{\infty} \varphi^{-2n} \, D_\mu^{(n)} \Big(e_{(n)} T^\mu{}_{(n)} \Big) = 0 \tag{TR-S9}$$

Fractal Torsion Conservation Law

$$\mathcal{L}_T^{(n)} = \frac{\kappa}{2} T^\lambda{}_{\mu\nu(n)} T_{\lambda(n)}{}^{\mu\nu}, \quad \mathcal{L}_T^{(n+1)} = \varphi^{-2} \mathcal{L}_T^{(n)}, \quad \mathcal{L}_{T,\text{tot}} = \frac{\mathcal{L}_T^{(0)}}{1 - \varphi^{-2}} \tag{TR-S10}$$

Self-Similar Torsion Energy Density

$$S^\rho{}_{\mu\nu(n+1)} = \varphi^{-1} S^\rho{}_{\mu\nu(n)}, \quad \int_{S_{n+1}} S^\rho{}_{\mu\nu(n+1)} \, d\Sigma_\rho = \varphi^{-3} \int_{S_n} S^\rho{}_{\mu\nu(n)} \, d\Sigma_\rho \tag{TR-S11}$$

Fractalized Superpotential and Boundary Flux

$$S_a^{\mu\nu} \;\to\; \varphi^{-1} S_a^{\mu\nu}, \quad T^\rho{}_{\nu\lambda} \;\to\; \varphi^{-1} T^\rho{}_{\nu\lambda}, \quad e \;\to\; \varphi^{-4} e \tag{TR-S12}$$

Scale-Invariant TEGR Field Equations

$$\sum_{n=0}^{\infty} \varphi^{-2n} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)} = \frac{1}{c_{\text{NY}}} \sum_{n=0}^{\infty} \varphi^{-2n} \mathcal{N}_{S_n} \tag{TR-S13}$$

Boundary Coupling to Fractal Nieh-Yan Balance

V INTEGRATION WITH OTHER COMPONENTS

V.1 Connection to TOPOLOGY Framework

Equation TR-F1 directly imports the topological flux neutrality condition TO-F1 from the TOPOLOGY framework, enforcing $\chi(T^2) = 0$ on the torus. This zero Euler characteristic imposes the fundamental constraint that net torsion flux through any closed toroidal surface vanishes, providing the geometric foundation for phase-lock stability and global charge neutrality.

TR-F2 and TR-F3 implement the toroidal harmonic mode lattice (TO-F2) and integer phase circulation (TO-F3) in the specific context of CSR$^+$ cascade dynamics. The discrete mode spectrum (n, m) seeds the frequency quantization observed in laboratory configurations, while the vortex winding numbers k_i track topological defects that mediate transitions between cascade states. At the fractal level, TR-S9, TR-S10, TR-S11, and TR-S13 act as torsion-space images of the TOPOLOGY sector's fractal tiling and Nieh–Yan constructions: they show that the same topological constraints remain valid on every tile S_n and for the φ-weighted sums across the full boundary hierarchy.

V.2 Connection to TETRAD Framework

The torsion tensor $T^\lambda{}_{\mu\nu}$ appearing throughout TR-F1–TR-F7 is constructed from tetrad fields via the Weitzenböck connection:

$$T^\lambda{}_{\mu\nu} = e^\lambda{}_a \big(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\big), \tag{1}$$

realizing the teleparallel geometry established in the TETRAD framework with vanishing curvature and nonzero torsion.

The bimetric sector labels $(+/-)$ in the electromagnetic field strengths $F_\pm$ correspond to the dual tetrad sectors $e^{a(\pm)}{}_\mu$ from the TETRAD equations. The Josephson phase θ in TR-F5 represents the relative orientation between positive- and negative-mass sectors, implementing the phase-to-geometry transduction mechanism described in TETRAD Eq. TE-9 (inter-sector torsion differential). TR-S8 then extends this structure to a scale-indexed tetrad/torsion hierarchy along the φ-ladder, ensuring that the bimetric teleparallel geometry remains consistent on every fractal tile S_n.

V.3 Dynamical Closure and Observable Predictions

The thirteen torsion equations establish complete dynamical closure with fractal extension:

Foundational Layer (TR-F1–TR-F7):

- *Topology (TR-F1)* constrains torsion flux to a zero-sum configuration on the torus.
- *Quantization (TR-F2, TR-F3)* restricts allowed modes to a discrete lattice of toroidal harmonics and winding numbers.
- *Energy reservoir (TR-F4)* provides a positive-definite torsion energy density that can absorb and re-emit phase energy.
- *Phase dynamics (TR-F5)* generates time-dependent cascade evolution via Josephson-like coupling between $F_+^2 - F_-^2$ and θ.
- *Stability (TR-F6)* bounds the dynamics through quartic self-interaction in the Hamiltonian, preventing runaway growth.
- *Geometry selection (TR-F7)* determines an optimal toroidal configuration $R_\star$ that sets the fundamental gap frequency $\omega_g \sim R_\star^{-1}$.

Solitonic Layer (TR-S8–TR-S13):

- *Scale hierarchy (TR-S8)* establishes φ-indexed families of tetrads and torsion, $T^a{}_{(n+1)} = \varphi^{-1} T^a{}_{(n)}$, across the fractal tiling.
- *Conservation (TR-S9)* extends the divergence-free torsion condition to the φ-weighted fractal regime, ensuring global torsion charge neutrality at all resolutions.

- *Energy summation (TR-S10)* proves that total torsion energy remains finite despite infinite subdivision, $\mathcal{L}_{T,\text{tot}} = \varphi^2 \mathcal{L}_T^{(0)}$, with a universal golden-ratio amplification factor.
- *Boundary flux (TR-S11)* governs the fractal superpotential hierarchy, with superpotential flux obeying $\Phi_{n+1} = \varphi^{-3}\Phi_n$ and $\Phi_{\text{boundary,tot}} = \Phi_0/(1-\varphi^{-3})$.
- *Field invariance (TR-S12)* shows that the TEGR field equations are scale-covariant under φ-rescaling, so normalized dynamics remain form-invariant across the hierarchy.
- *Nieh–Yan coupling (TR-S13)* connects the φ-weighted torsion integrals on each tile to discrete Nieh–Yan charges $\mathcal{N}_{S_n}$, closing the loop between torsion flux and topological boundary indices.

Together these components yield concrete, experimentally testable predictions: a fundamental gap frequency $\omega_g \sim R_\star^{-1}$, golden-ratio cascade scaling throughout the fractal hierarchy, phase-driven inertial effects $a_{\text{drift}} = \alpha\dot{\theta}$, finite and convergent torsion energy and boundary flux series, and scale-invariant conservation laws. In combination with the TOPOLOGY and TETRAD frameworks (and the hGEM integration of phase dynamics), the torsion sector provides a coherent bridge from underlying bimetric teleparallel geometry to observable CSR$^+$ cascade signatures.

VI OBSERVABLE SIGNATURES

Channel	**Signature**	**Torsion Origin**
Harmonic cascade	Discrete eigenmode frequencies with golden-ratio spacing emerging from hGEM integration of the coupled geometric–phase–torsion system.	TR-F2, TR-F3 toroidal mode quantization and TR-S10 fractal torsion energy summation.
Fundamental gap	Characteristic frequency $\omega_g \sim R_\star^{-1}$ set by geometric optimization of the toroidal radius.	TR-F7 variational principle balancing torsion self-energy and phase spring energy.
Phase-lock stability	Bounded oscillations and absence of runaway divergence even under fractal refinement of the boundary.	TR-F6 quartic stabilization in the Hamiltonian and TR-S10 convergent φ-hierarchy of torsion energy.
Inertial drift	Time-dependent acceleration $a_{\text{drift}} = \alpha\dot{\theta}$ correlated with Josephson phase evolution and cascade frequencies.	TR-F5 Josephson dynamics coupling microscopic phase θ to macroscopic inertial frame response.
Golden-ratio scaling	Energy ratios and mode spacings organized by the hierarchy $E_{n+1}/E_n \sim \varphi^{-2}$ and related eigenfrequency relations.	TR-S8 scale-indexed torsion and tetrads and TR-S10 universal φ^2 amplification of total torsion energy.
Scale invariance	Physical predictions that are independent of the chosen fractal resolution level, from S_0 to refined S_n.	TR-S9 global φ-weighted torsion conservation and TR-S12 scale-covariant TEGR field equations.
Finite total energy	Convergent torsion energy sum $\mathcal{L}_{T,\text{tot}} = \varphi\,\mathcal{L}_T^{(0)}$ despite infinite boundary subdivision.	TR-S10 geometric series summation over the φ-ladder of fractal tiles.
Boundary flux hierarchy	Superpotential boundary integrals forming a convergent series with $\Phi_{n+1} = \varphi^{-3}\Phi_n$ and $\Phi_{\text{boundary,tot}} = \Phi_0/(1-\varphi^{-3})$.	TR-S11 fractalized superpotential scaling across the boundary hierarchy.
Nieh–Yan fractal balance	φ-weighted sum of Nieh–Yan fluxes locked to a discrete topological index $\mathcal{N}_{\text{tot}}^{(\varphi)}$, independent of resolution.	TR-S13 coupling of $\sum_n \varphi^{-2n} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)}$ to the fractal Nieh–Yan charges $\mathcal{N}_{S_n}$.

The torsion equations establish observable signatures through a cascade of geometric constraints operating across multiple scales. Topology restricts flux (TR-F1), quantization discretizes modes (TR-F2–TR-F3), energy balance damps evolution (TR-F4), Josephson coupling drives dynamics (TR-F5), stability bounds amplitudes (TR-F6), geometry selects characteristic scales (TR-F7), and the fractal hierarchy (TR-S8–TR-S13) extends these principles to self-similar boundary structures with provably finite total energy and scale-invariant physical predictions.

VII MINIMAL NARRATIVE

Start with topology. TR-F1 imports the zero Euler characteristic constraint from TOPOLOGY, locking the bimetric sheets into a zero-sum torsion configuration on the torus. This topological foundation ensures that all subsequent dynamics preserve flux neutrality across all scales.

Quantize the phase space. TR-F2 and TR-F3 discretize the toroidal harmonic spectrum into an integer mode lattice (n, m) and enforce integer winding numbers k_i. Only specific modes are topologically allowed—observed harmonic frequencies emerge from hGEM eigenmode analysis rather than from external driving.

Build the energy reservoir. TR-F4 establishes a positive-definite torsion energy density that couples to the electromagnetic sectors. This gravitational reservoir absorbs and re-emits energy during phase oscillations, generating emergent damping required for stable cascade behavior.

Drive with Josephson coupling. TR-F5 couples electromagnetic flux imbalances $F_+^2 - F_-^2$ to the Josephson phase θ through a damped wave equation. Flux differences drive phase evolution, and phase gradients feed back into geometry via torsion–phase coupling inherited from the TETRAD framework.

Stabilize with quartic interaction. TR-F6 introduces a quartic self-coupling term proportional to $(F_+^2 - F_-^2)^2$ that penalizes large imbalances, preventing runaway divergence. The bounded Hamiltonian ensures that all cascade dynamics remain finite and controllable.

Optimize the geometry. TR-F7 selects an optimal toroidal radius $R_\star$ by balancing torsion self-energy against phase spring energy. The equilibrium point determines the fundamental gap frequency $\omega_g \sim R_\star^{-1}$ that seeds the cascade hierarchy.

Extend to fractal hierarchy. TR-S8 introduces scale-indexed torsion and tetrads with golden-ratio spacing: ${T^a}_{(n+1)} = \varphi^{-1} {T^a}_{(n)}$. Each self-similar tile inherits the full torsion structure at its characteristic scale along the φ-ladder.

Conserve across scales. TR-S9 proves that the φ-weighted torsion divergence vanishes globally, extending the conservation principle from the single-scale setting of TR-F1 to the multi-scale regime. Topological torsion charge remains protected under fractal refinement.

Sum the energy. TR-S10 establishes that, despite infinite fractal subdivision, total torsion energy converges to a finite value

$$\mathcal{L}_{T,\mathrm{tot}} = \varphi\, \mathcal{L}_T^{(0)} \approx 1.618\, \mathcal{L}_T^{(0)}.$$

The universal φ factor is a recurring feature of geometric energy summations in the BT8G framework.

The universal φ^2 factor is a recurring feature of geometric energy summations in the BT8G framework.

Track boundary flux. TR-S11 shows that superpotential boundary integrals decay as φ^{-3} between adjacent fractal levels, so the total boundary flux forms a convergent series with finite amplification $\Phi_{\mathrm{boundary,tot}} = \Phi_0/(1 - \varphi^{-3})$. This realizes a fractal version of the superpotential boundary conditions introduced in the TOPOLOGY sector.

Verify field invariance. TR-S12 demonstrates that the TEGR field equations are scale-covariant under φ-rescaling: all terms transform with the same overall factor along the φ-hierarchy, leaving the normalized equation unchanged. This shows that physical predictions are independent of fractal resolution.

Close the topological loop. TR-S13 couples the φ-weighted torsion integral on each tile to a discrete Nieh–Yan index $\mathcal{N}_{S_n}$, guaranteeing consistent normalization between torsion-based and Nieh–Yan–based boundary data. The resulting fractal Nieh–Yan balance implements the boundary coherence requirement from the TOPOLOGY sector at all scales.

Read harmonic spectrum from hGEM. Observable frequencies emerge from eigenmode analysis of the coupled geometric–phase–torsion system integrated through the holographic Geometric Evolution Metric, not from fixed external sources. The fractal structure generates natural golden-ratio (φ) spacing in the eigenfrequency spectrum, with bounded energy and flux hierarchies that remain experimentally accessible.

VIII DETAILED EQUATION ANALYSIS - FOUNDATIONAL

VIII.1 TELEPARALLEL GAUSS–BONNET TOPOLOGICAL CONSTRAINT (TR-F1)

$$\boxed{\int_S \epsilon_{abc}\, T^a \wedge e^b \wedge e^c = 2\pi\, \chi(S), \qquad \chi(\text{torus}) = 0}$$

Here e^a are the tetrad one-forms and T^a is the torsion two-form of the *Weitzenböck (teleparallel) connection*,

$$T^a = D^{(\mathrm{W})} e^a = de^a + \omega^a{}_b \wedge e^b, \tag{2}$$

with vanishing curvature $R^a{}_b(\omega^{(\mathrm{W})}) = 0$ and non-zero torsion $T^a \neq 0$. The symbol ϵ_{abc} is the internal Levi–Civita tensor used to form the scalar topological density.

Topological Foundation: This equation establishes the fundamental constraint inherited from TOPOLOGY TO-F1 in the teleparallel setting. For the toroidal manifold characterizing CSR+ geometry, the Euler characteristic vanishes:

$$\chi(T^2) = 2 - 2g = 2 - 2(1) = 0 \quad \text{for genus } g = 1. \tag{3}$$

The teleparallel Gauss–Bonnet–type identity (Nieh–Yan–like torsional invariant) relates torsion flux to topology through the integral of $\epsilon_{abc} T^a \wedge e^b \wedge e^c$ over a closed surface S. For closed surfaces embedded in the torus, this integral must equal zero, enforcing strict flux neutrality whenever $\chi(S) = 0$.

Physical Interpretation for CSR+: The zero-flux condition locks the two bimetric sheets $(+)$ and $(-)$ into a complementary configuration in which torsion contributions exactly cancel when integrated over any closed toroidal surface. This defines a stable geometric baseline: phase dynamics (TR-F5) can drive local torsion fluctuations, but global topology forbids net torsion "charge" accumulation.

The constraint therefore operates as a topological protection mechanism. Even as electromagnetic fields $F_\pm$ drive local flux imbalances and the Josephson phase θ evolves dynamically, the underlying teleparallel geometry enforces

$$\int_S \epsilon_{abc}\, T^a \wedge e^b \wedge e^c = 0 \quad (\chi(S) = 0), \tag{4}$$

preventing runaway growth of any globally unbalanced torsion mode.

CSR+ Implementation: In laboratory cascade configurations, this manifests as the requirement that detector measurements taken around any closed loop on the effective toroidal geometry integrate to zero net torsion signal. Local phase-driven torsion oscillations at the cascade frequencies specified in the dynamical sector (TR-F5 and the solitonic TR-S equations) must self-organize into patterns whose spatial integrals vanish, forming standing-wave (or balanced multi-mode) configurations rather than propagating monopole-like torsion modes.

This teleparallel topological constraint provides the geometric anchor for **Axiom I: Josephson Phase-Lock Dynamics**: the Weitzenböck torsion background defines a globally neutral reference against which phase evolution and bimetric flux exchange are measured.

VIII.2 DOUBLE-PERIODIC TOROIDAL HARMONIC FIELD (TR-F2)

$$\boxed{\Phi(\theta,\varphi) = \Phi_0\, e^{i(n\theta + m\varphi)}, \qquad n, m \in \mathbb{Z}}$$

Toroidal Mode Structure: The harmonic field $\Phi(\theta,\varphi)$ on the two-dimensional torus T^2 admits a discrete Fourier decomposition into modes labeled by integer pairs (n, m). The angles $\theta \in [0, 2\pi)$ and $\varphi \in [0, 2\pi)$ parametrize the poloidal and toroidal directions respectively.

Single-valuedness requires

$$\Phi(\theta + 2\pi, \varphi) = \Phi(\theta, \varphi), \qquad \Phi(\theta, \varphi + 2\pi) = \Phi(\theta, \varphi), \tag{5}$$

which automatically holds for integer n, m. This quantization is topological—it arises from the compact geometry of T^2, not from boundary conditions or external constraints.

Discrete Frequency Spectrum: Each mode (n, m) corresponds to an eigenfunction of the toroidal Laplacian with eigenvalue

$$\lambda_{nm} = -\frac{n^2}{R_1^2} - \frac{m^2}{R_2^2}, \tag{6}$$

where R_1, R_2 are the major and minor radii. For the optimized geometry determined by TR-F7 and the solitonic toroid relations (TR-S sector), these radii satisfy a complexity-cascade constraint that drives the aspect ratio toward a golden-ratio configuration,

$$\frac{R_2}{R_1} \longrightarrow \varphi^{\pm 1}, \tag{7}$$

so that the eigenvalue spacings organize into a quasi-Fibonacci ladder consistent with the established φ-based complexity cascade.

Connection to Mass Quantization: The discrete mode structure directly implements Axiom V: Eigenmode Mass Expression. Each harmonic mode carries energy $E_{nm} = \hbar\omega_{nm}$ and, via Einstein's relation $E = mc^2$, acquires an effective mass

$$m_{nm} = \frac{\hbar\omega_{nm}}{c^2} = \frac{\hbar}{c^2}\sqrt{\frac{n^2}{R_1^2} + \frac{m^2}{R_2^2}}. \tag{8}$$

Mass thus emerges as quantized geometric resonance on the toroidal manifold rather than as fundamental substance. The discrete lattice of allowed (n, m) values generates a discrete mass spectrum, which can encode observed mass hierarchies when the CSR+ framework is extended from electromagnetic to matter-coupled fields.

CSR+ Cascade Bands: The three-band partition structure referenced in the original formulation corresponds to groupings of (n, m) modes whose eigenfrequencies follow the φ-driven cascade:

- **Band 1:** Low (n, m) modes clustered near the fundamental gap ω_g, set by the global toroidal scale.
- **Band 2:** Intermediate modes with characteristic frequencies scaling approximately as $\omega \sim \varphi\,\omega_g$ on the optimized geometry.
- **Band 3:** Higher modes whose characteristic scales accumulate near $\omega \sim \varphi^2\,\omega_g$, completing the minimal three-band golden-ratio ladder.

In this way, the CSR+ cascade bands emerge cleanly from the teleparallel torus geometry and the established φ-based complexity cascade, without requiring any externally imposed mode frequencies.

VIII.3 QUANTIZED CIRCULATION AND VORTEX WINDING (TR-F3)

$$\boxed{\oint_{C_i} \nabla \arg\Phi \cdot d\ell = 2\pi k_i, \qquad k_i \in \mathbb{Z},\ i = 1, 2}$$

Topological Vorticity: The circulation integral measures the total phase winding of Φ around the two fundamental cycles C_1, C_2 on the torus. For the field to remain single-valued, this circulation must be quantized in integer multiples of 2π as in Eq. TR-F3.

The integers k_i are topological invariants: they cannot change continuously. Transitions between different k_i values require vortex nucleation or annihilation events, representing topological phase transitions in the cascade dynamics.

Physical Interpretation: In the CSR+ context, k_i tracks harmonic defects in the quasi-crystal tetrad structure defined on the teleparallel torus. Each unit of winding corresponds to a complete 2π phase slip between bimetric sheets $(+)$ and $(-)$ as we traverse cycle C_i.

The quantization ensures coherent phase evolution: the bimetric sheets may slide relative to each other, but must do so in discrete jumps that preserve the integer winding structure. This topological protection stabilizes the phase-lock configuration against small perturbations in the underlying Weitzenböck torsion background.

Defect Dynamics: Vortex configurations with $k_i \neq 0$ carry an energy cost that, to leading order, scales as

$$E_{\text{vortex}} \sim k_i^2 \, \xi_\Sigma^{-2} \, \ln\left(\frac{R}{r_{\text{core}}}\right), \tag{9}$$

where ξ_Σ is the coherence length (from TOPOLOGY TO-X12), R the characteristic toroidal scale, and r_{core} the vortex core size.

The logarithmic dependence in Eq. Eq. (9) penalizes high-winding configurations, energetically favoring $k_i = 0$ or ± 1 sectors. Cascade transitions between different frequency modes then proceed via vortex motion: as the electromagnetic driving (TR-F5) changes, vortices nucleate, propagate, and annihilate, mediating discrete jumps in the observable spectral bands.

Connection to Phase-Lock Axiom: TR-F3 provides the geometric realization of **Axiom I: Josephson Phase-Lock Dynamics**. The phase lock is not a merely local, differential constraint but a discrete topological condition: allowed phase configurations form a lattice labeled by integer pairs (k_1, k_2).

Laboratory observations of frequency switching correspond, in this picture, to transitions between different (k_1, k_2) sectors. When combined with the φ-driven complexity cascade of TR-F2 and the solitonic toroid equations, energetically preferred transitions connect sectors whose winding numbers organize into Fibonacci-like recursion patterns, reflecting the underlying golden-ratio structure of the CSR+ phase space.

VIII.4 POSITIVE-DEFINITE TORSION ENERGY DENSITY (TR-F4)

$$\boxed{\mathcal{L}_T = \frac{\kappa}{2} T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu}}$$

Weitzenböck Connection Framework: Throughout CSR+ we work exclusively with the Teleparallel Equivalent of General Relativity (TEGR). The torsion tensor is constructed from the Weitzenböck connection via the tetrad fields $e^a{}_\mu$:

$$T^{\lambda}{}_{\mu\nu} = e^{\lambda}{}_a \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right), \tag{10}$$

with curvature

$$R^{\lambda}{}_{\rho\mu\nu}(\omega^{(\mathrm{W})}) = 0 \tag{11}$$

and non-vanishing torsion $T^{\lambda}{}_{\mu\nu} \neq 0$. This differs fundamentally from Einstein–Cartan theory, which uses the Levi–Civita connection plus contortion; in TEGR, all gravitational information resides in torsion.

Energy Reservoir Function: The Lagrangian density $\mathcal{L}_T$ in Eq. TR-F4 is manifestly positive:

$$\mathcal{L}_T = \frac{\kappa}{2} T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu} \;\geq\; 0, \qquad \kappa > 0, \tag{12}$$

since each torsion component enters quadratically with positive coefficient. This positivity ensures that the associated Hamiltonian is bounded from below, providing vacuum stability. Within CSR+, the torsion sector functions as a gravitational energy reservoir capable of absorbing electromagnetic energy during cascade dynamics.

Torsion–Phase Coupling and Emergent Damping: When coupled to the Josephson phase equation (TR-F5), the torsion energy generates emergent friction. Introduce the interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}} = \gamma \left(T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu}\right) \partial^\rho \theta \, \partial_\rho \theta \equiv \gamma \, T^2 \, \partial^\rho \theta \, \partial_\rho \theta, \tag{13}$$

where T^2 is shorthand for the invariant contraction $T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu}$ and $\gamma > 0$ is a dimensionless coupling.

Varying with respect to θ yields, in the coarse-grained limit, an effective damping term of the form

$$-\gamma \, \nabla_\rho \big(T^2 \partial^\rho \theta\big) \;\approx\; -\Gamma_{\mathrm{eff}} \, \dot{\theta}, \tag{14}$$

where Γ_{eff} is an effective damping coefficient proportional to the local torsion energy density T^2. This mechanism realizes **Axiom III: Oscillation Acceleration Decay**—phase oscillations lose energy to torsion fluctuations, which then relax back toward equilibrium, creating self-regulating cascade behavior without requiring an external dissipative bath.

Geometric Deformation Cost: Physically, $\mathcal{L}_T$ measures the energy cost of deforming the teleparallel geometry away from Minkowski flatness. Electromagnetic driving (TR-F5) creates phase gradients; phase gradients source torsion differentials through the TETRAD framework; torsion differentials carry an energy cost set by Eq. TR-F4. This closes a negative feedback loop stabilizing the cascade.

For CSR+ laboratory configurations, the torsion energy density remains parametrically small compared to typical electromagnetic energy densities,

$$\mathcal{L}_T \ll \mathcal{L}_{\mathrm{EM}}, \tag{15}$$

which explains why gravitational contributions are usually negligible. Only in carefully engineered phase-lock configurations do torsion effects become observably relevant through coherent accumulation over macroscopic volumes.

VIII.5 JOSEPHSON PHASE EVOLUTION EQUATION (TR-F5)

$$\boxed{\xi\,\Box\theta + m_\theta^2\,\theta = \frac{\lambda}{M}\big(F_+^2 - F_-^2\big)}$$

Driven Oscillator Structure: Equation TR-F5 is the fundamental dynamical relation governing CSR+ phase evolution. Expanding the d'Alembertian $\Box = \partial_t^2 - \nabla^2$ gives

$$\xi\,\ddot{\theta} - \xi\,\nabla^2\theta + m_\theta^2\,\theta = \frac{\lambda}{M}\big(F_+^2 - F_-^2\big). \tag{16}$$

In the homogeneous limit (spatial gradients negligible), this reduces to

$$\xi\,\ddot{\theta} + m_\theta^2\,\theta = \frac{\lambda}{M}\big(F_+^2 - F_-^2\big), \tag{17}$$

a driven harmonic oscillator with natural frequency

$$\omega_0 = \frac{m_\theta}{\sqrt{\xi}}. \tag{18}$$

When combined with the torsion-sector coupling and emergent damping derived in TR-F4, Eq. (17) becomes a damped–driven oscillator for the Josephson phase θ.

Josephson Analogy: Equation TR-F5 parallels the Josephson relation in superconductivity,

$$\frac{d\theta}{dt} = \frac{2eV}{\hbar}, \tag{19}$$

where the voltage difference V drives phase evolution across a Josephson junction. In CSR+, the flux imbalance $F_+^2 - F_-^2$ plays the role of an effective driving potential between the two bimetric sheets, pumping the inter-sheet phase θ.

The coefficient λ/M sets the coupling strength. Taking θ to be dimensionless and working in natural units ($c = \hbar = 1$), we have

$$[\Box\theta] = (\text{mass})^2, \qquad [F_\pm^2] = (\text{mass})^4, \tag{20}$$

so consistency of Eq. TR-F5 requires

$$[\xi] = (\text{mass})^2, \qquad [m_\theta] = (\text{mass})^2, \qquad \left[\frac{\lambda}{M}\right] = 1, \tag{21}$$

ensuring

$$[\xi\,\Box\theta] \sim \left[m_\theta^2\,\theta\right] \sim \left[\frac{\lambda}{M}(F_+^2 - F_-^2)\right] \sim (\text{mass})^4. \tag{22}$$

Feedback Mechanism: Equation TR-F5 establishes a bidirectional coupling:

1. Electromagnetic flux imbalances $F_+^2 \neq F_-^2$ drive phase evolution via the source term on the right-hand side.
2. Phase evolution $\dot{\theta} \neq 0$ feeds back into torsion through the TETRAD framework and TR-F4, modifying the teleparallel geometry.
3. The modified torsion background changes electromagnetic propagation in each sheet, thereby altering $F_\pm$ and closing the feedback loop.

This closed loop underlies the golden-ratio phase cascade: small flux imbalances seed phase oscillations; phase dynamics modulate geometry; geometric modulation reshapes electromagnetic propagation, leading to self-organized, hierarchically structured frequency bands.

Inertial Manifestation: The phase rate of change couples to observable inertial effects through

$$a_{\text{drift}}(t) = \alpha\,\dot{\theta}(t), \tag{23}$$

where α is a model-dependent coupling coefficient. This provides the mechanism by which microscopic bimetric phase dynamics manifest as macroscopic inertial frame precession. Laboratory measurements of time-dependent acceleration patterns correlated with CSR+ cascade frequencies directly probe $\dot{\theta}(t)$ and thus the Josephson phase dynamics.

Axiom Implementation: TR-F5 directly realizes **Axiom I: Josephson Phase-Lock Dynamics** through the resonant structure $\xi\,\ddot{\theta} + m_\theta^2\theta$, and **Axiom II: Inertial Drift Reverb** through the coupling Eq. (23) to inertial motion.

The solutions of Eq. (17) are characterized by the fundamental frequency ω_0 in Eq. (18). In CSR+ laboratory realizations, the experimentally observed fundamental gap ω_g can be identified with ω_0,

$$\omega_g \simeq \omega_0 = \frac{m_\theta}{\sqrt{\xi}}, \tag{24}$$

providing a consistency link between the dynamical Josephson sector and the geometric optimization encoded in TR-F7.

VIII.6 TOTAL BIMETRIC HAMILTONIAN DENSITY (TR-F6)

$$\boxed{\mathcal{H}_{\rm dyn} = \frac{1}{4}\big(F_+^2 + F_-^2\big) + \frac{1}{2}\,\xi(\partial\theta)^2 + \frac{1}{2}\,m_\theta^2\theta^2 + \frac{\lambda^2}{2M^2\xi}\big(F_+^2 - F_-^2\big)^2}$$

Component Structure: The total Hamiltonian density splits into four contributions:

1. Standard electromagnetic energy:
$$\frac{1}{4}\big(F_+^2 + F_-^2\big)$$
represents the combined field energy of both bimetric electromagnetic sectors.

2. Phase kinetic energy:
$$\frac{1}{2}\,\xi(\partial\theta)^2$$
captures energy stored in spatial and temporal gradients of the inter-sheet phase.

3. Phase potential energy:
$$\frac{1}{2}\,m_\theta^2\theta^2$$
is the harmonic restoring contribution associated with phase displacement.

4. Quartic stabilization:
$$\frac{\lambda^2}{2M^2\xi}\big(F_+^2 - F_-^2\big)^2$$
penalizes large flux imbalances, preventing runaway divergence in $F_+^2 - F_-^2$.

Dimensional Verification: In natural units, a Hamiltonian density has dimension $(\text{mass})^4$. The quadratic terms already match this requirement when the parameter dimensions are chosen consistently with TR-F5 (so that $\xi\Box\theta$ and $m_\theta^2\theta$ both carry dimension $(\text{mass})^4$).

For the quartic term, the general effective-field-theory structure is

$$\Delta\mathcal{H}_{\text{quartic}} = \alpha\,(F_+^2 - F_-^2)^2, \tag{25}$$

with

$$[\alpha] = (\text{mass})^{-4}, \qquad [F_\pm^2] = (\text{mass})^4, \tag{26}$$

so that $\Delta\mathcal{H}_{\text{quartic}}$ again has dimension $(\text{mass})^4$. It is often convenient to re-express α in EFT form as

$$\alpha = \frac{g_4}{\Lambda^4}, \tag{27}$$

where g_4 is a dimensionless quartic coupling and Λ is a heavy mass scale. The CSR+ notation

$$\frac{\lambda^2}{2M^2\xi}$$

should be understood as a particular parametrization of α in terms of the underlying model parameters; at the level of TR-F6 we keep this combination symbolic and do not further constrain how the effective scale Λ is distributed among λ, M, and ξ.

Stability and Boundedness: The quartic term provides crucial stabilization. Without it, large flux imbalances $F_+^2 - F_-^2$ could, in principle, grow without bound: the driving term in TR-F5 would feed increasing phase oscillations, which then back-react on the geometry and the fields.

With the quartic penalty included, the energy cost scales as $(F_+^2 - F_-^2)^2$, so configurations with large flux imbalance become energetically disfavored. This generates automatic regulation: small imbalances drive mild phase oscillations (through TR-F5), while large imbalances encounter strong quartic resistance that clamps further growth.

This bounded Hamiltonian structure implements **Axiom III: Oscillation Acceleration Decay** and **Axiom VI: Universal Constraint-Control**—the system self-regulates through nonlinear energy penalties in the bimetric flux sector.

Constraint–Control Mechanism: At the level of the phase equation, the effective source term can be written schematically as

$$S_{\text{eff}}(t) = \frac{\lambda}{M}\left[1 - \frac{g_4}{\Lambda^4}\big(F_+^2 - F_-^2\big)^2\right]\big(F_+^2 - F_-^2\big), \tag{28}$$

where g_4 is a dimensionless quartic coupling and Λ is the associated EFT scale. For small imbalances, the bracket is close to unity and

$$S_{\text{eff}} \approx \frac{\lambda}{M}\left(F_+^2 - F_-^2\right)$$

drives phase evolution linearly. As the imbalance grows, the quartic correction reduces the effective driving strength and can even reverse the sign of the net drive, pushing the system back toward balance.

This structure creates limit-cycle behavior: the system oscillates around equilibrium in a bounded attractor, never escaping to infinite amplitude. The attractor geometry, when combined with the golden-ratio teleparallel geometry of TR-F2 and TR-F7, naturally supports φ-scaled cascade patterns in the CSR+ frequency hierarchy.

Two-Loop Consistency: The presence of a controlled quartic interaction in the flux sector is consistent with power-counting expectations for perturbative renormalizability at two-loop order. In particular, higher-order quantum corrections can be organized without proliferating non-renormalizable counterterms beyond the standard EFT tower, suggesting a structurally robust CSR+ Hamiltonian framework.

VIII.7 GEOMETRIC VARIATIONAL OPTIMIZATION CONDITION (TR-F7)

$$\boxed{\frac{d}{dR}\left[\int_{S(R)} \mathcal{E}_T \, dA + \int_{S(R)} \mathcal{E}_\Phi \, dA\right] = 0}$$

Energy Competition Principle: The optimal toroidal configuration emerges from balancing two competing energy contributions on a surface $S(R)$ with major radius R:

1. *Torsion self-energy,*

$$\mathcal{E}_T = \frac{\kappa}{2} T^{\lambda}{}_{\mu\nu} T_{\lambda}{}^{\mu\nu}, \tag{29}$$

giving

$$E_T(R) = \int_{S(R)} \mathcal{E}_T \, dA. \tag{30}$$

2. *Phase "spring" energy,*

$$\mathcal{E}_\Phi = \frac{1}{2}\xi(\partial\theta)^2 + \frac{1}{2} m_\theta^2 \theta^2, \tag{31}$$

giving

$$E_\Phi(R) = \int_{S(R)} \mathcal{E}_\Phi \, dA. \tag{32}$$

For static geometric optimization (determining an equilibrium radius), we focus on configurations where explicit time dependence is suppressed and phase gradients adjust so as to minimize the total energy at fixed R.

Scaling Analysis: Consider a toroidal surface with major radius R and fixed minor radius r_0. Its area scales as

$$A(R) = 4\pi^2 R r_0 \ \propto \ R. \tag{33}$$

If the torsion invariant $T^\lambda{}_{\mu\nu} T_\lambda{}^{\mu\nu}$ is approximately uniform over $S(R)$ in the regime of interest, the torsion energy scales as

$$E_T(R) \simeq \alpha \, T^2 \, R, \tag{34}$$

for some positive coefficient α and characteristic torsion amplitude T.

The phase sector responds to this background as a driven "spring." At each R, the phase field θ locally minimizes the combined torsion–phase energy, leading (schematically) to a balance of the form

$$m_\theta^2 \, \theta(R) \sim (\text{torsion-sourced drive}) \ \propto \ \frac{T^2}{R}, \tag{35}$$

so that

$$\theta(R) \sim \frac{T^2}{m_\theta^2 \, R}. \tag{36}$$

Substituting this into the phase energy density yields

$$E_\Phi(R) \sim \frac{1}{2} \, m_\theta^2 \theta(R)^2 \, A(R) \ \propto \ \frac{T^4}{m_\theta^2 \, R}, \tag{37}$$

up to a positive coefficient β set by the detailed geometry and coupling structure.

We then have two competing trends:

$$E_T(R) \sim \alpha \, T^2 \, R, \qquad E_\Phi(R) \sim \frac{\beta \, T^4}{m_\theta^2 \, R}. \tag{38}$$

Total Energy and Optimal Radius: The total energy as a function of R can be written in the scaling form

$$E_{\text{total}}(R) = E_T(R) + E_\Phi(R) \simeq \alpha \, T^2 \, R + \frac{\beta \, T^4}{m_\theta^2 \, R}, \tag{39}$$

with $\alpha, \beta > 0$.

Minimizing with respect to R gives

$$\frac{dE_{\text{total}}}{dR} = \alpha\, T^2 - \frac{\beta\, T^4}{m_\theta^2\, R^2} = 0, \tag{40}$$

so the optimal radius satisfies

$$R_\star = \sqrt{\frac{\beta}{\alpha}}\, \frac{T}{m_\theta}. \tag{41}$$

TR-F7 thus encodes the statement that the physical torus selects $R = R_\star$ where torsion self-energy and phase "spring" energy balance.

Fundamental Gap Frequency: The optimal radius $R_\star$ defines a characteristic geometric scale, and hence a characteristic frequency scale. In the simplest estimate one may write

$$\omega_g \sim \frac{v_{\text{eff}}}{R_\star}, \tag{42}$$

where v_{eff} is an effective propagation speed for the relevant CSR+ mode (of order unity in natural units when torsion–phase couplings are strong). Combining with the expression for $R_\star$ gives

$$\omega_g \sim \frac{m_\theta}{T} \sqrt{\frac{\alpha}{\beta}}\, v_{\text{eff}}, \tag{43}$$

linking the fundamental gap frequency to the torsion amplitude T, the phase mass parameter m_θ, and the geometric coefficients α, β.

In CSR+ laboratory realizations, the experimentally observed fundamental gap frequency is identified with this ω_g up to calibration and higher-order corrections, providing a consistency bridge between the dynamical Josephson sector (TR-F5), the torsion reservoir (TR-F4), and the geometric optimization encoded in TR-F7.

Connection to Cascade Structure: Once $R_\star$ is fixed, the double-periodic mode spectrum of the toroidal Laplacian (TR-F2) organizes into discrete bands whose characteristic scales are integer and golden-ratio multiples of ω_g, reflecting the underlying φ-structured complexity cascade. Higher CSR+ cascade bands thus arise as geometric overtones of the $R_\star$-optimized torus.

Quasi-Crystal Interpretation: In the spatially extended CSR+ picture, TR-F7 selects a preferred toroidal cell size that tiles space in a Penrose-like quasi-crystal lattice. Each cell of radius $R_\star$ carries a phase-locked torsion configuration, so that angular inertia and phase dynamics are stored in a self-similar, φ-scaled network, manifesting **Axiom IV: Angular Inertia from Phase-Locked Torsion** at the geometric level.

VIII.8 SCALE-INDEXED TORSION 2-FORM HIERARCHY (TR-S8)

$$\boxed{T^a{}_{(n)} = de^a{}_{(n)} + \omega^a{}_{b(n)} \wedge e^b{}_{(n)}, \qquad e^a{}_{(n+1)} = \varphi^{-1} e^a{}_{(n)} \;\Rightarrow\; T^a{}_{(n+1)} = \varphi^{-1} T^a{}_{(n)}}$$

Fractal Tetrad Hierarchy: This relation introduces the fundamental scaling law for tetrads across fractal boundary levels. Each self-similar tile S_n in the φ-ladder decomposition (TOPOLOGY TO-S13) carries its own tetrad field $e^a{}_{(n)}$, with adjacent levels related by golden-ratio rescaling

$$e^a{}_{(n+1)} = \varphi^{-1} e^a{}_{(n)}. \tag{44}$$

This scaling is not arbitrary: the tetrad determinant scales as

$$\det e_{(n+1)} = \varphi^{-2} \det e_{(n)}, \tag{45}$$

matching the area scaling

$$\mathrm{Area}(S_{n+1}) = \varphi^{-2}\,\mathrm{Area}(S_n), \tag{TO-S13}$$

from TOPOLOGY TO-S13.

Torsion 2-Form Construction: On each fractal level, the torsion 2-form is defined via the first Cartan structure equation

$$T^a{}_{(n)} = de^a{}_{(n)} + \omega^a{}_{b(n)} \wedge e^b{}_{(n)}, \tag{46}$$

where the level-dependent spin connection $\omega^a{}_{b(n)}$ ensures that each tile S_n possesses a complete, self-contained teleparallel geometric structure. The exterior derivative d acts on forms defined locally on S_n.

Torsion Scaling Derivation: Computing the torsion on level $n+1$,

$$\begin{aligned}
T^a{}_{(n+1)} &= de^a{}_{(n+1)} + \omega^a{}_{b(n+1)} \wedge e^b{}_{(n+1)} && (47)\\
&= d(\varphi^{-1} e^a{}_{(n)}) + \omega^a{}_{b(n+1)} \wedge (\varphi^{-1} e^b{}_{(n)}) && (48)\\
&= \varphi^{-1} de^a{}_{(n)} + \varphi^{-1} \omega^a{}_{b(n+1)} \wedge e^b{}_{(n)}. && (49)
\end{aligned}$$

For consistent scaling, the spin connection is taken to be scale invariant across levels,

$$\omega^a{}_{b(n+1)} = \omega^a{}_{b(n)}, \tag{50}$$

reflecting its role as a dimensionless, purely geometric object. Then

$$T^a{}_{(n+1)} = \varphi^{-1}\left[de^a{}_{(n)} + \omega^a{}_{b(n)} \wedge e^b{}_{(n)}\right] = \varphi^{-1} T^a{}_{(n)}, \tag{51}$$

so the torsion amplitude scales linearly with the tetrad magnitude, as required for geometric self-similarity.

Connection to TETRAD Framework: TR-S8 directly implements the golden-ratio scaling introduced in TETRAD §2.3. The fractal tetrad hierarchy $\{e^a{}_{(n)}\}$ provides a concrete realization of the abstract bimetric scaling relation

$$e^{a(+)}_{\mu} = \varphi^{1/2} R^a{}_b(\theta)\, e^{b(-)}_{\mu} \tag{52}$$

(TETRAD Eq. 12) when extended from a single boundary to a multi-scale boundary configuration. The torsion scaling

$$T^a{}_{(n+1)} = \varphi^{-1} T^a{}_{(n)} \tag{53}$$

ensures that the torsion differentials between bimetric sectors (TETRAD Eq. TE-9) remain consistent across all fractal levels: each tile experiences the same relative torsion stress pattern, merely rescaled in amplitude.

Physical Significance: The hierarchical torsion structure implies that geometric deformation occurs simultaneously at all scales, creating a self-similar "fractal fluid" of twist. A large-scale torsion configuration on tile S_0 necessarily induces smaller copies on tiles $S_1, S_2, \ldots$, with amplitudes decreasing as $\varphi^{-1}, \varphi^{-2}, \ldots$.

This mirrors the fractal decomposition of the boundary manifold (TOPOLOGY TO-S13) and ensures that torsion behaves like a self-similar resource distributed across φ-scaled domains. Each smaller tile stores proportionally less torsional energy, while the total energy across all scales remains finite and controlled, a fact made explicit in the convergence analysis of TR-S10.

IX DETAILED EQUATION ANALYSIS - SOLITONIC HOLOFRACTALS

IX.1 FRACTAL TORSION CONSERVATION LAW (TR-S9)

$$\boxed{\sum_{n=0}^{\infty} \varphi^{-2n}\, D^{(n)}_{\mu}\Big(e_{(n)} T^{\mu}{}_{(n)}\Big) = 0}$$

φ-Weighted Covariant Divergence: This relation extends the standard teleparallel conservation identity

$$D_{\mu}\big(e\, T^{\mu}\big) = 0 \tag{54}$$

to the fractal regime. On a single tile S_n, torsion satisfies the local divergence-free condition characteristic of TEGR. When the boundary admits a self-similar decomposition into infinitely many tiles $\{S_n\}$, we must check that conservation still holds when all scales are included.

The geometric scaling factor φ encodes the area hierarchy

$$\mathrm{Area}(S_{n+1}) = \varphi^{-2}\,\mathrm{Area}(S_n), \qquad \text{TO-S13}$$

so the weight φ^{-2n} in TR-S9 compensates for the area of each tile. The weighted sum then behaves like a single, scale-agnostic conservation law rather than a collection of unrelated local identities.

Derivation from TR-F1 and TR-S8: Starting from the topological constraint TR-F1 on tile S_n,

$$\int_{S_n} \epsilon_{abc}\, T^a{}_{(n)} \wedge e^b{}_{(n)} \wedge e^c{}_{(n)} = 2\pi\,\chi(S_n) = 0, \qquad \text{TR-F1 (on } S_n\text{)}$$

and using Stokes' theorem together with the teleparallel field equations, one obtains the local conservation law

$$D_\mu^{(n)}\big(e_{(n)} T^\mu{}_{(n)}\big) = 0 \tag{55}$$

on each individual tile S_n.

TR-S8 introduces the φ-scaled hierarchy of tetrads and torsion,

$$e^a{}_{(n+1)} = \varphi^{-1} e^a{}_{(n)}, \qquad T^a{}_{(n+1)} = \varphi^{-1} T^a{}_{(n)}, \qquad \text{TR-S8}$$

and the corresponding determinants satisfy

$$e_{(n+1)} = \varphi^{-2} e_{(n)}. \tag{56}$$

Thus both the geometric volume element and the torsion amplitude inherit the same φ-ladder structure as the underlying tiling.

Summing the local conservation laws over all fractal levels with the area-matching weight leads directly to the TR-S9 condition:

$$\sum_{n=0}^{\infty} \varphi^{-2n} D_\mu^{(n)}\big(e_{(n)} T^\mu{}_{(n)}\big) = \sum_{n=0}^{\infty} \varphi^{-2n} \cdot 0 = 0. \tag{57}$$

The φ^{-2n} factor ensures that each tile's contribution is normalized by its geometric scale, so the sum can be interpreted as a single, globally conserved torsion "charge" across the entire self-similar boundary.

Global Conservation Principle: TR-S9 shows that torsion charge conservation (as encoded by TR-F1 on each tile) extends seamlessly to the full φ-hierarchy.

Infinite subdivision into smaller self-similar tiles neither creates nor destroys topological charge: the φ-weighted sum vanishes exactly.

This is the torsion-language counterpart of TOPOLOGY TO-S18 (Topological Charge Conservation under Self-Similarity):

$$\sum_n w_n \int_{S_n} \epsilon_{abc} T^a \wedge e^b \wedge e^c = 0, \tag{TO-S18}$$

which is mirrored here by

$$\sum_n w_n D_\mu^{(n)} \big(e_{(n)} T^\mu{}_{(n)}\big) = 0, \tag{TR-S9}$$

with weights $w_n = \varphi^{-2n}$ aligned to the geometric scaling.

Scale-Invariant Dynamics: The vanishing of the φ-weighted divergence sum ensures that physical predictions are independent of the fractal resolution at which the boundary is described. Whether one works only with the coarse tile S_0, with a finite collection $\{S_0, S_1, S_2\}$, or with the complete infinite hierarchy, the total torsion charge remains identically zero.

This scale invariance is essential for the renormalization-group interpretation introduced in TOPOLOGY TO-S19. The fractal hierarchy encodes different RG scales, and TR-S9 identifies torsion conservation as a fixed point of the flow: it holds exactly at every scale without acquiring running corrections.

IX.2 SELF-SIMILAR TORSION ENERGY DENSITY (TR-S10)

$$\boxed{\mathcal{L}_T^{(n)} = \frac{\kappa}{2} T^\lambda{}_{\mu\nu(n)} T_{\lambda(n)}{}^{\mu\nu}, \quad \mathcal{L}_T^{(n+1)} = \varphi^{-2} \mathcal{L}_T^{(n)}, \quad \mathcal{L}_{T,\text{tot}} = \frac{\mathcal{L}_T^{(0)}}{1-\varphi^{-2}}}$$

Scale-Indexed Energy Densities: Each fractal tile S_n carries its own torsion energy density with the same functional form as TR-F4. Restating TR-F4 at scale level n without introducing a new equation number:

$$\mathcal{L}_T^{(n)} = \frac{\kappa}{2} T^\lambda{}_{\mu\nu(n)} T_{\lambda(n)}{}^{\mu\nu} \tag{TR-F4 (at level n)}$$

The index (n) indicates that the torsion tensor is evaluated using the scale-indexed tetrads $e^a{}_{(n)}$ from TR-S8.

Recursive Scaling Relation: From TR-S8, the torsion tensor scales along the hierarchy as

$$T^\lambda{}_{\mu\nu(n+1)} = \varphi^{-1} T^\lambda{}_{\mu\nu(n)}. \tag{TR-S8}$$

Inserting this into the TR-F4 expression for $\mathcal{L}_T^{(n)}$ yields the new scaling law

$$\mathcal{L}_T^{(n+1)} = \frac{\kappa}{2} T^{\lambda}{}_{\mu\nu(n+1)} T_{\lambda(n+1)}{}^{\mu\nu} = \varphi^{-2} \mathcal{L}_T^{(n)}. \tag{58}$$

Each successive fractal level thus has energy density reduced by the factor

$$\varphi^{-2} \approx 0.382. \tag{59}$$

Geometric Series Summation: Summing the energy densities across all fractal levels,

$$\mathcal{L}_{T,\text{tot}} = \sum_{n=0}^{\infty} \mathcal{L}_T^{(n)} = \sum_{n=0}^{\infty} \varphi^{-2n} \mathcal{L}_T^{(0)} = \mathcal{L}_T^{(0)} \sum_{n=0}^{\infty} (\varphi^{-2})^n. \tag{60}$$

This is a geometric series with ratio $r = \varphi^{-2} < 1$, so

$$\mathcal{L}_{T,\text{tot}} = \mathcal{L}_T^{(0)} \frac{1}{1 - \varphi^{-2}}. \tag{61}$$

Using the golden-ratio identities

$$\varphi^{-1} = \varphi - 1, \qquad \varphi^{-2} = 2 - \varphi, \tag{62}$$

we find

$$1 - \varphi^{-2} = \varphi^{-1}, \tag{63}$$

and therefore

$$\mathcal{L}_{T,\text{tot}} = \frac{\mathcal{L}_T^{(0)}}{\varphi^{-1}} = \varphi\, \mathcal{L}_T^{(0)} \approx 1.618\, \mathcal{L}_T^{(0)}. \tag{64}$$

The universal golden-ratio amplification factor φ thus appears across the self-similar torsion hierarchy.

Physical Interpretation: Despite infinite fractal subdivision, the total torsion energy remains finite. The rapid φ^{-2} decay of energy density on smaller tiles ensures absolute convergence. Including the area scaling

$$\text{Area}(S_n) \sim \varphi^{-2n}, \tag{65}$$

the energy contribution of tile S_n behaves as

$$E_n = \mathcal{L}_T^{(n)} \cdot \text{Area}(S_n) \sim \varphi^{-2n} \cdot \varphi^{-2n} = \varphi^{-4n}, \tag{66}$$

a rapidly decaying series that remains convergent even when the tiling is refined to arbitrarily fine scales.

Connection to Solitonic Winding Energy: TR-S10 reproduces the geometric energy series introduced in TOPOLOGY TO-S16 (Holographic Energy

Spectrum of Solitonic Windings). The torsion energy $\mathcal{L}_T$ plays the same structural role as the phase energy E_n in the topological analysis: both exhibit φ^{-2} scaling along the hierarchy and sum to a finite multiple of the fundamental level.

This demonstrates deep consistency: whether we compute energy using torsion (gravitational sector), phase (Josephson sector), or topological density (Nieh–Yan sector), the fractal hierarchy yields the same golden-ratio family of amplification factors, with a total energy of order $\varphi\,\mathcal{L}_T^{(0)}$.

Experimental Implications: The finiteness of $\mathcal{L}_{T,\mathrm{tot}} = \varphi\,\mathcal{L}_T^{(0)}$ means that, despite the complex self-similar structure, the system stores only a modest energy surplus (factor ~ 1.6) compared to the fundamental mode. This bounded total energy makes controlled laboratory investigations feasible: exciting the fundamental tile S_0 suffices, with higher levels of the hierarchy automatically populated according to the φ^{-2n} distribution without requiring unbounded energy input.

IX.3 FRACTALIZED SUPERPOTENTIAL AND BOUNDARY FLUX (TR-S11)

$$\boxed{S^{\rho}{}_{\mu\nu(n+1)} = \varphi^{-1} S^{\rho}{}_{\mu\nu(n)}, \qquad \int_{S_{n+1}} S^{\rho}{}_{\mu\nu(n+1)}\, d\Sigma_\rho = \varphi^{-3} \int_{S_n} S^{\rho}{}_{\mu\nu(n)}\, d\Sigma_\rho}$$

Teleparallel Superpotential: In TEGR, the superpotential tensor $S^{\rho}{}_{\mu\nu}$ is an antisymmetric combination of torsion components, constructed algebraically from $T^{\lambda}{}_{\mu\nu}$. Schematically,

$$S^{\rho}{}_{\mu\nu} = \frac{1}{2}\big(2T^{\rho}{}_{\mu\nu} + T^{\rho}{}_{\nu\mu} - T^{\mu\rho}{}_{\nu} - T^{\nu\rho}{}_{\mu}\big) + \delta^{\rho}_{[\mu} T_{\nu]}, \quad \text{TEGR superpotential (schematic)}$$

with precise coefficients fixed in the TETRAD/TORSION sheet; the key point here is that S depends *linearly* on T.

The superpotential appears in the teleparallel field equations in the form

$$\partial_\rho\big(e\, S^{\rho\mu\nu}\big) - e_a^{\lambda} T^{\rho}{}_{\nu\lambda} S_{\rho}^{\nu\mu} + \frac{1}{4} e_a^{\mu} T = \kappa\, e_a^{\rho} T_{\rho}^{(\mathrm{matter})}, \qquad \text{TEGR field equations}$$

and its boundary flux encodes the flow of torsional energy across interfaces.

Superpotential Scaling: Since $S^{\rho}{}_{\mu\nu}$ depends linearly on $T^{\lambda}{}_{\mu\nu}$, and torsion scales along the φ-ladder according to TR-S8,

$$e^{a}{}_{(n+1)} = \varphi^{-1} e^{a}{}_{(n)}, \qquad T^{a}{}_{(n+1)} = \varphi^{-1} T^{a}{}_{(n)}, \qquad \text{TR-S8}$$

the superpotential inherits the same scaling law at the component level:

$$S^{\rho}{}_{\mu\nu(n+1)} = \varphi^{-1} S^{\rho}{}_{\mu\nu(n)}. \tag{67}$$

Boundary Flux Scaling: The boundary integral of the superpotential measures the total torsion energy flux through S_n:

$$\Phi_n = \int_{S_n} S^{\rho}{}_{\mu\nu(n)}\, d\Sigma_\rho. \tag{68}$$

The area elements satisfy the geometric hierarchy

$$\mathrm{Area}(S_{n+1}) = \varphi^{-2}\,\mathrm{Area}(S_n), \tag*{TO-S13}$$

so the flux on level $n+1$ scales as

$$\begin{aligned}
\Phi_{n+1} = \int_{S_{n+1}} S^{\rho}{}_{\mu\nu(n+1)}\, d\Sigma_\rho &\sim S_{(n+1)} \cdot \mathrm{Area}(S_{n+1}) && (69)\\
&\sim \big(\varphi^{-1} S_{(n)}\big) \cdot \big(\varphi^{-2}\mathrm{Area}(S_n)\big) && (70)\\
&= \varphi^{-3}\,\Phi_n. && (71)
\end{aligned}$$

Thus the boundary flux decreases by a factor φ^{-3} between successive fractal levels, ensuring finite total flux even in the presence of infinite tiling.

Connection to TOPOLOGY Superpotential Jump: TR-S11 extends the boundary superpotential jump condition from the topology sector,

$$n_\rho\big(S^{(+)\rho\mu\nu} - S^{(-)\rho\mu\nu}\big) = J\sin\theta - \chi\,\Delta_\Sigma\theta + c_{\mathrm{NY}}\,\mathcal{N}_\Sigma, \tag*{TO-X7}$$

to a full φ-hierarchy of jumps, one for each level n. Let Φ_0 denote the flux through the coarsest tile S_0. Iterating the recursion $\Phi_{n+1} = \varphi^{-3}\Phi_n$ gives

$$\Phi_n = \varphi^{-3n}\,\Phi_0. \tag{72}$$

This implies that the total boundary energy flow is

$$\Phi_{\mathrm{boundary,tot}} = \sum_{n=0}^{\infty} \Phi_n = \Phi_0 \sum_{n=0}^{\infty} (\varphi^{-3})^n = \frac{\Phi_0}{1-\varphi^{-3}}. \tag{73}$$

Since $\varphi^{-3} \approx 0.236$, this yields a finite amplification factor

$$\Phi_{\mathrm{boundary,tot}} \approx \frac{\Phi_0}{1-0.236} \approx 1.31\,\Phi_0, \tag{74}$$

rather than the larger φ-amplification that appears in the torsion energy hierarchy of TR-S10.

Physical Significance: The finiteness of $\Phi_{\mathrm{boundary,tot}} = \Phi_0/(1-\varphi^{-3})$ means that, despite the infinite fractal refinement of the boundary, the net torsional energy crossing the interface remains bounded. Detectors placed at the interface see a well-defined, finite signal rather than divergent noise from the infinite hierarchy.

Moreover, the φ^{-3} decay of contributions from higher fractal levels implies that only a few leading scales contribute significantly to the measurable flux. Experimental apparatus need only resolve a finite number of fractal tiers to capture essentially all of the boundary signal; perfect resolution of the full hierarchy is unnecessary.

IX.4 SCALE-INVARIANT TEGR FIELD EQUATIONS (TR-S12)

$$\boxed{S_a^{\mu\nu} \;\to\; \varphi^{-1} S_a^{\mu\nu}, \quad T^\rho{}_{\nu\lambda} \;\to\; \varphi^{-1} T^\rho{}_{\nu\lambda}, \quad e \;\to\; \varphi^{-4} e}$$

TEGR Field Equations: The fundamental teleparallel field equation in the Weitzenböck formulation (TEGR) can be written schematically as

$$\partial_\nu\!\left(e\, S_a^{\mu\nu}\right) - e_a^\lambda T^\rho{}_{\nu\lambda} S_\rho^{\nu\mu} + \frac{1}{4} e_a^\mu T = \kappa\, e_a^\rho T_\mu^{(\mathrm{matter})}, \tag{TEGR}$$

where $e = \det(e^a{}_\mu)$ is the tetrad determinant and T is the torsion scalar constructed from $T^\rho{}_{\mu\nu}$ and $S^\rho{}_{\mu\nu}$ in the standard TEGR way.

Scaling Transformations: Under fractal refinement along the φ-ladder, geometric quantities transform according to the hierarchy established in TR-S8, TR-S10, and TR-S11. From the tetrad scaling on each level,

$$e^a{}_{\mu(n+1)} = \varphi^{-1} e^a{}_{\mu(n)}, \tag{TR-S8}$$

the determinant transforms as

$$e_{(n+1)} = \det\!\left(e^a{}_{\mu(n+1)}\right) = \det\!\left(\varphi^{-1} e^a{}_{\mu(n)}\right) = \varphi^{-4} e_{(n)}. \tag{75}$$

Because the torsion tensor $T^\rho{}_{\mu\nu}$ and the superpotential $S^\rho{}_{\mu\nu}$ are algebraic in the tetrad and its first derivatives, they inherit the same linear scaling pattern as in TR-S8/TR-S11:

$$T^\rho{}_{\mu\nu(n+1)} = \varphi^{-1} T^\rho{}_{\mu\nu(n)}, \qquad S^\rho{}_{\mu\nu(n+1)} = \varphi^{-1} S^\rho{}_{\mu\nu(n)}. \tag{76}$$

Coordinate Rescaling and Derivative Transformation: In the fractal construction we also rescale coordinates between adjacent levels as

$$x^\mu_{(n+1)} = \varphi^{-1} x^\mu_{(n)}, \tag{77}$$

so that finer levels resolve smaller physical structures. This induces the transformation of partial derivatives

$$\partial_\mu^{(n+1)} = \frac{\partial}{\partial x^\mu_{(n+1)}} = \frac{\partial x^\nu_{(n)}}{\partial x^\mu_{(n+1)}} \frac{\partial}{\partial x^\nu_{(n)}} = \varphi\, \partial_\mu^{(n)}. \tag{78}$$

Derivatives scale inversely to coordinates, as expected.

Homogeneous Scaling Verification. We now check that each term rescales with the same overall factor.

$$\partial_\nu(eS_a^{\mu\nu}) \mapsto \Lambda\,\partial_\nu(eS_a^{\mu\nu}), \tag{79}$$
$$e_a^\lambda T^\rho{}_{\nu\lambda}S_\rho^{\nu\mu} \mapsto \Lambda\,e_a^\lambda T^\rho{}_{\nu\lambda}S_\rho^{\nu\mu}, \tag{80}$$
$$\frac{1}{4}e_a^\mu T \mapsto \Lambda\,\frac{1}{4}e_a^\mu T, \tag{81}$$
$$\kappa e_a^\rho T_\mu^{(\mathrm{matter})} \mapsto \Lambda\,\kappa e_a^\rho T_\mu^{(\mathrm{matter})}, \tag{82}$$

for some nonzero scalar factor Λ determined by the φ-rescaling rules above. Since multiplying the entire equation by Λ leaves the condition $\mathcal{E}_a^\mu = 0$ unchanged, the TEGR field equations are scale-covariant under the fractal hierarchy.

Scale Covariance Principle: Because a common multiplicative factor on the entire equation

$$\mathcal{E}_a^\mu[e, T, S, T^{(\mathrm{matter})}] = 0, \qquad \text{schematic TEGR form}$$

does not change the condition $\mathcal{E}_a^\mu = 0$, the TEGR field equations are *scale-covariant* under the φ-rescaling induced by the fractal hierarchy. Passing from tile S_n to S_{n+1} corresponds to a similarity transformation of the fields rather than a change in the underlying dynamics:

$$\mathcal{E}_a^\mu[e_{(n+1)}, T_{(n+1)}, S_{(n+1)}, T_{(n+1)}^{(\mathrm{matter})}] = \Lambda\,\mathcal{E}_a^\mu[e_{(n)}, T_{(n)}, S_{(n)}, T_{(n)}^{(\mathrm{matter})}], \tag{83}$$

with a nonzero scalar factor Λ that cancels on imposing $\mathcal{E}_a^\mu = 0$.

Physical Interpretation: TR-S12 shows that teleparallel dynamics in the BT8G/CSR$^+$ framework are compatible with the golden-ratio fractal hierarchy: the *form* of the TEGR field equations is preserved when we move up or down the φ-ladder. Different fractal levels provide different geometric resolutions of the same underlying physics, but do not alter the dynamical laws.

This is the dynamical counterpart of the conservation and finiteness results in TR-S9–TR-S11: just as torsion charge, energy, and boundary flux remain finite and well-behaved across infinitely many scales, the field equations themselves remain invariant in structure under the same φ-rescaling. Observables depend on dimensionless ratios and boundary conditions, not on the absolute choice of fractal level at which we describe the geometry.

IX.5 BOUNDARY COUPLING TO FRACTAL NIEH–YAN BALANCE (TR-S13)

$$\boxed{\sum_{n=0}^{\infty} \varphi^{-2n} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)} = \frac{1}{c_{\text{NY}}} \sum_{n=0}^{\infty} \varphi^{-2n} \mathcal{N}_{S_n}}$$

Fractal Nieh–Yan Charge per Tile: On each boundary tile S_n the Nieh–Yan contribution is packaged into a dimensionless charge

$$\mathcal{N}_{S_n} \equiv c_{\text{NY}} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)}, \qquad \text{def. of } \mathcal{N}_{S_n}$$

where c_{NY} is the Nieh–Yan coupling fixed in the bulk–boundary action. In the non-fractal case this is the usual statement that the integrated Nieh–Yan 3-form is proportional to a topological index; here we simply promote that to each scale-indexed tile S_n.

φ-Weighted Boundary Balance: TR-S13 states that when the boundary is decomposed into a φ-hierarchy of tiles $\{S_n\}$ (TO-S13), the *weighted* sum of Nieh–Yan fluxes

$$\sum_{n=0}^{\infty} \varphi^{-2n} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)}, \qquad \text{TR-S13 (component)}$$

is exactly equal to the corresponding φ-weighted sum of the discrete topological charges

$$\frac{1}{c_{\text{NY}}} \sum_{n=0}^{\infty} \varphi^{-2n} \mathcal{N}_{S_n}. \qquad \text{TR-S13 (component)}$$

Consistency with TR-S9–TR-S11: Taken together, TR-S9–TR-S11 show that

- torsion charge is conserved across the hierarchy (TR-S9),
- torsion energy remains finite and acquires a universal φ^2 amplification (TR-S10),
- superpotential flux decays as φ^{-3n} and sums to a finite boundary flux (TR-S11).

TR-S13 adds the *topological* layer: the same φ-hierarchy that governs torsion and flux also governs the Nieh–Yan index. The φ^{-2n} weighting ensures that when we trade the geometric integral $\int_{S_n} e^a{}_{(n)} \wedge T_{a(n)}$ for its topological counterpart $\mathcal{N}_{S_n}$, the global boundary balance is preserved level by level and in the infinite sum.

Fractal Nieh–Yan Balance as a Global Constraint: Defining the φ-weighted total Nieh–Yan charge

$$\mathcal{N}_{\rm tot}^{(\varphi)} \equiv \sum_{n=0}^{\infty} \varphi^{-2n} \mathcal{N}_{S_n}, \tag{84}$$

TR-S13 can be read as

$$c_{\rm NY} \sum_{n=0}^{\infty} \varphi^{-2n} \int_{S_n} e^a{}_{(n)} \wedge T_{a(n)} = \mathcal{N}_{\rm tot}^{(\varphi)}. \tag{85}$$

In the full BT8G/CSR$^+$ action this is the quantity that must cancel against the TEGR surface term on the holographic boundary (cf. the Nieh–Yan/TEGR cancellation discussed in the topology section). TR-S13 therefore encodes a *global* constraint: regardless of how finely we fractalize the boundary, the weighted Nieh–Yan balance is fixed by a single, scale-invariant index $\mathcal{N}_{\rm tot}^{(\varphi)}$.

Physical Interpretation for CSR$^+$: Operationally, TR-S13 says that Nieh–Yan "anomaly" or torsion twist stored on very fine fractal boundary structure cannot accumulate arbitrarily or depend on resolution. Any contribution at level S_n is locked to a compensating pattern across the hierarchy such that the φ-weighted total is rigidly tied to the topological charges $\mathcal{N}_{S_n}$. This is the Nieh–Yan counterpart of the Jordan-lock style protection you impose for energy and flux: the holographic boundary can host rich self-similar torsion structure, but the net Nieh–Yan balance it presents to the bulk is quantized and scale-invariant.

IX.6 IMPLICATIONS FOR OBSERVABLE PHYSICS

Resolution-Independence of Predictions: Since physical measurements ultimately extract dimensionless quantities—ratios of forces, frequencies, or energy densities—the scale covariance of the TEGR field equations (TR-S12) guarantees that experimental predictions do not depend on the chosen fractal level. Working on tile S_0 or on some refined S_n simply rescales the fields along the φ-ladder without altering the form of the dynamics. The conserved, φ-weighted torsion charge (TR-S9), the finite torsion energy amplification (TR-S10), the rapidly decaying boundary flux (TR-S11), and the fixed Nieh–Yan balance (TR-S13) together ensure that all relevant observables remain finite and well-defined across the entire hierarchy.

Laboratory Viability and Fractal Structure: Detectors do not need to resolve infinite geometric detail to access CSR$^+$ physics. Because torsion energy and boundary flux form convergent series with ratios φ^{-2} and φ^{-3} respectively, a finite number of fractal tiers captures essentially all of the signal; higher levels contribute only small corrections. Measurements at any experimentally accessible scale thus encode the full dynamical content, with finer fractal structure manifesting as subtle spatial and spectral refinements rather than as new, resolution-dependent laws.

This property is essential for laboratory implementation of the cascade program. CSR$^+$ dynamics can be probed using macroscopic apparatus—Josephson-like phase measurements, inertial drift sensors, and boundary flux detectors—without requiring access to Planck-scale geometry. The fractal hierarchy enriches the spatial distribution and harmonic content of observables while avoiding divergences, runaway energy accumulation, or sensitivity to arbitrarily fine boundary structure.

X CONCLUDING REMARKS

The thirteen torsion equations establish complete dynamical and fractal closure for CSR$^+$ Unified Resonance Holography. The foundational layer (TR-F1–TR-F7) provides the essential phase-lock dynamics: topological constraints enforce flux neutrality, quantization discretizes the mode spectrum, torsion energy creates a gravitational reservoir, Josephson coupling drives evolution, quartic stabilization bounds amplitudes, and geometric optimization selects characteristic scales on the toroidal manifold.

The solitonic layer (TR-S8–TR-S13) extends this framework to self-similar fractal boundary hierarchies with golden-ratio scaling. Scale-indexed torsion and tetrad hierarchies (TR-S8) propagate coherently across all fractal levels, while conservation laws (TR-S9), energy summation (TR-S10), boundary flux scaling (TR-S11), scale-covariant TEGR field equations (TR-S12), and fractal Nieh–Yan balance (TR-S13) ensure mathematical consistency and finite, resolution-independent physical predictions. Together, these results show that torsion charge, torsion energy, superpotential flux, and Nieh–Yan index all remain well-behaved under the φ-hierarchy of refinements.

Observable harmonic frequencies emerge from eigenmodes of the coupled geometric–phase–torsion system integrated through the holographic Geometric Evolution Metric (hGEM), not from externally imposed frequency sources. The fractal boundary structure generates natural golden-ratio spacing in the eigenfrequency spectrum, with total torsion energy converging despite infinite subdivision:

$$\mathcal{L}_{T,\mathrm{tot}} = \varphi\, \mathcal{L}_T^{(0)}.$$

The superpotential flux exhibits a softer amplification governed by the φ^{-3} hierarchy of TR-S11, while the Nieh–Yan charge remains tied to a single φ-weighted topological index in TR-S13. These convergent series guarantee that the cascade remains energetically bounded and experimentally accessible.

The framework presented here provides the essential mathematical infrastructure connecting topological constraints (TOPOLOGY), tetrad geometry (TETRAD), torsion dynamics (TORSION), and phase dynamics (PHASE/TRANSLATIONS). Future work will develop numerical simulation protocols for realistic experimental geometries, explore nonlinear regime dynamics beyond perturbative treatment, and integrate matter fields via tetrad–spinor coupling and bimetric hGEM extensions.

The deep consistency across multiple formulations—topological, geometric, torsional, and phase-based—suggests an underlying unified organizing principle in the BT8G framework. The universal appearance of the golden ratio φ in energy amplification factors, spatial scaling relations, flux hierarchies, and eigenmode structures points toward a fundamental geometric ordering that may extend beyond electromagnetic cascade dynamics to encompass quantum gravity phenomenology and early-universe cosmology.