Sheet Coordinates σ^p, Spacetime Coordinates x^μ, and the Holographic Master Field $\Phi(\sigma; x)$

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Abstract

This work develops a first-principles mathematical foundation for the holographic Master Field $\Phi(\sigma; x)$, which depends simultaneously on sheet coordinates σ^p and spacetime coordinates x^μ. We model the physical spacetime as a four-dimensional Lorentzian manifold $(M_4, g_{\mu\nu})$ and introduce a two-dimensional spectral sheet (Σ_2, γ_{pq}) that encodes scale and chiral vortex structure in an auxiliary, non-spacetime sector. The combined kinematic domain is the product manifold $\mathcal{M} = \Sigma_2 \times M_4$, on which $\Phi(\sigma; x)$ is defined as a scalar field valued in an internal BT8g carrier space.

Section 1 constructs the geometric and dimensional infrastructure for this framework. We fix index sets and signatures for M_4 and Σ_2, assign consistent mass dimensions to x^μ, σ^p, the integration measures, and all basic geometric objects, and define the tangent and cotangent bases on $\mathcal{M}$. We emphasize a strict separation between sheet and spacetime indices, treating $\mathrm{Diff}(M_4)$ and $\mathrm{Diff}(\Sigma_2)$ as independent symmetry factors. On this background we introduce the Master Field $\Phi(\sigma; x)$ with an explicit amplitude–phase–internal decomposition and a canonical mass dimension chosen so that natural holographic actions on $\Sigma_2 \times M_4$ are dimensionally consistent.

The result is a minimal but complete kinematic basis for subsequent sections: the teleparallel description of M_4, the intrinsic geometry of Σ_2, and the holographic dynamics by which effective four-dimensional fields emerge from $\Phi(\sigma; x)$ via sheet projections. All later constructions in the Chiral Vortex and BT8g program are constrained to respect the index structure, units, and dimensional assignments established here.

1 Geometric and Dimensional Setup

The purpose of this section is to define, from first principles, the mathematical environment in which the holographic Master Field $\Phi(\sigma; x)$ is defined. The central objects are

- a four-dimensional Lorentzian spacetime $(M_4, g_{\mu\nu})$ with coordinates x^μ,
- a two-dimensional "sheet" manifold (Σ_2, γ_{pq}) with coordinates σ^p,
- their product $\mathcal{M} = \Sigma_2 \times M_4$ carrying the field $\Phi : \mathcal{M} \to \mathcal{V}$, where $\mathcal{V}$ is an internal BT8g carrier space.

We work in natural units $\hbar = c = 1$ and adopt a single fundamental dimension denoted $[M]$. All remaining quantities are expressed as powers of $[M]$.

1.1 Manifolds and coordinate charts

Spacetime. Let M_4 be a smooth, four-dimensional manifold equipped with a Lorentzian metric $g_{\mu\nu}$ of signature $(-+++)$. Local coordinates are denoted

$$x^\mu = (x^0, x^1, x^2, x^3), \qquad \mu, \nu, \rho, \sigma = 0, 1, 2, 3.$$

In natural units we assign

$$[x^\mu] = [M]^{-1}, \qquad [g_{\mu\nu}] = 1, \tag{1.1}$$

so that the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is dimensionless and the Christoffel symbols carry dimension $[\Gamma^\rho{}_{\mu\nu}] = [M]$.

Spectral sheet. Let Σ_2 be a smooth, oriented, two-dimensional Riemannian manifold with local coordinates

$$\sigma^p = (\sigma^1, \sigma^2), \qquad p, q, r, s = 1, 2.$$

We equip Σ_2 with a positive-definite metric $\gamma_{pq}(\sigma)$. Following the spectral interpretation, the sheet coordinates are taken to have the dimension of an energy/momentum scale,

$$[\sigma^p] = [M], \qquad [\gamma_{pq}] = 1. \tag{1.2}$$

This choice reflects the role of Σ_2 as a "spectral sheet" in momentum-space-like variables rather than an additional spacetime.

The associated volume element on Σ_2 is

$$d^2\sigma = \sqrt{\gamma}\, d\sigma^1 d\sigma^2, \qquad \gamma = \det(\gamma_{pq}), \tag{1.3}$$

and the Levi–Civita tensor is defined by

$$\epsilon_{12} = -\epsilon_{21} = \sqrt{\gamma}, \qquad \epsilon^{12} = -\epsilon^{21} = \frac{1}{\sqrt{\gamma}}. \tag{1.4}$$

Product manifold. The total kinematic domain of the Master Field is the direct product

$$\mathcal{M} = \Sigma_2 \times M_4, \tag{1.5}$$

with global coordinates

$$(y^A) = (\sigma^p, x^\mu), \qquad A, B, \dots \in \{p, \mu\}.$$

At this stage we treat Σ_2 and M_4 as independent factors; the holographic coupling is encoded in the dynamics of $\Phi(\sigma; x)$ rather than in a nontrivial fibration of $\mathcal{M}$.

1.2 Index sets and signatures

We use the following index conventions throughout:

Index type	Symbols	Role
Spacetime (coordinate)	μ, ν, ρ, σ	$0, \dots, 3$ on M_4
Lorentz (tetrad)	a, b, c, d	$0, \dots, 3$ in $T_x M_4$
Sheet (coordinate)	p, q, r, s	$1, 2$ on Σ_2
Sheet (orthonormal)	$\hat{p}, \hat{q}$	frame on Σ_2
Internal BT8g	A, B	$1, \dots, 8$

Spacetime carries Lorentzian signature

$$g_{\mu\nu} \sim \mathrm{diag}(-1,1,1,1),$$

while the sheet metric is positive definite:

$$\gamma_{pq} \text{ Riemannian}, \qquad \gamma_{pq} v^p v^q > 0 \text{ for } v^p \neq 0.$$

The combined index structure on $\mathcal{M}$ is block-diagonal: Greek indices are never raised or lowered with γ_{pq}, and sheet indices are never contracted with $g_{\mu\nu}$. This strict separation is essential for keeping the dimensional analysis and tensor types consistent.

1.3 Natural units and dimensional assignments

The fundamental dimension is mass $[M]$. We adopt the conventions

$$[d^4x] = [M]^{-4}, \qquad [d^2\sigma] = [M]^2, \qquad [g_{\mu\nu}] = [\gamma_{pq}] = 1. \tag{1.6}$$

The total integration measure on $\mathcal{M}$ then has

$$[d^4x\, d^2\sigma] = [M]^{-2}. \tag{1.7}$$

For the geometric objects on M_4 we take

$$[\partial_\mu] = [M], \quad [\Gamma^\rho{}_{\mu\nu}] = [M], \quad [R^\rho{}_{\sigma\mu\nu}] = [M]^2, \tag{1.8}$$

and analogously on Σ_2,

$$[\partial_p] = [M]^{-1}, \quad [\Gamma^r{}_{pq}(\gamma)] = [M]^{-1}, \quad [\mathcal{R}_{pq}(\gamma)] = [M]^{-2}, \tag{1.9}$$

where $\mathcal{R}_{pq}(\gamma)$ denotes the Ricci tensor of γ_{pq}. These assignments reflect the choice $[\sigma^p] = [M]$, so derivatives with respect to σ^p carry inverse mass dimension and the sheet curvature has the expected $[M]^{-2}$ scaling.

1.4 Tangent and cotangent bases on $\mathcal{M}$

The tangent space at a point $(\sigma, x) \in \mathcal{M}$ splits as a direct sum

$$T_{(\sigma,x)}\mathcal{M} \simeq T_\sigma\Sigma_2 \oplus T_x M_4, \tag{1.10}$$

with coordinate bases

$$\left\{ \frac{\partial}{\partial \sigma^p} \right\} \cup \left\{ \frac{\partial}{\partial x^\mu} \right\}. \tag{1.11}$$

The dual cotangent bases are

$$\{d\sigma^p\} \cup \{dx^\mu\}, \tag{1.12}$$

satisfying

$$d\sigma^p\left(\frac{\partial}{\partial\sigma^q}\right) = \delta^p{}_q, \qquad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu, \qquad d\sigma^p\left(\frac{\partial}{\partial x^\mu}\right) = 0, \qquad dx^\mu\left(\frac{\partial}{\partial\sigma^p}\right) = 0.$$

On spacetime we introduce a tetrad $e^a{}_\mu(x)$ and its inverse $e_a{}^\mu(x)$ satisfying

$$e^a{}_\mu e_b{}^\mu = \delta^a{}_b, \qquad e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu, \tag{1.13}$$

with

$$g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu e^b{}_\nu, \qquad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{1.14}$$

In teleparallel gauge the spin connection can be chosen flat, and the torsion tensor is constructed from first derivatives of the tetrad. The tetrad is taken dimensionless, $[e^a{}_\mu] = 1$, consistent with $[g_{\mu\nu}] = 1$.

If desired, an orthonormal frame on Σ_2 can be defined by a zweibein $f^{\hat{p}}{}_q(\sigma)$ such that

$$\gamma_{pq} = \delta_{\hat{p}\hat{q}} f^{\hat{p}}{}_p f^{\hat{q}}{}_q, \tag{1.15}$$

but for the present kinematic analysis we keep γ_{pq} in coordinate form.

1.5 Product structure and diffeomorphisms

Kinematically, the symmetry group of coordinate redefinitions is the direct product

$$\mathrm{Diff}(\mathcal{M}) \supset \mathrm{Diff}(M_4) \times \mathrm{Diff}(\Sigma_2), \tag{1.16}$$

where

$$x^\mu \mapsto x'^\mu(x), \tag{1.17}$$

$$\sigma^p \mapsto \sigma'^p(\sigma), \tag{1.18}$$

act independently on spacetime and sheet coordinates. The field $\Phi(\sigma; x)$ is taken to be a scalar under both sets of coordinate transformations,

$$\Phi'(\sigma'; x') = \Phi(\sigma; x), \tag{1.19}$$

while its internal indices (BT8g labels) transform under a fixed representation of the internal algebra.

This factorized diffeomorphism structure is the geometric starting point for holography: all dynamical couplings between Σ_2 and M_4 will be encoded in the action and field equations of $\Phi(\sigma; x)$, not in a mixing of the coordinate transformation groups.

1.6 Domain and basic normalization of the Master Field

The holographic Master Field is defined as a map

$$\Phi : \Sigma_2 \times M_4 \longrightarrow \mathcal{V}, \tag{1.20}$$

where $\mathcal{V}$ is an 8-dimensional internal space carrying a real BT8g Clifford representation. In components,

$$\Phi(\sigma; x) = \rho(\sigma; x)\, e^{i\theta(\sigma;x)}\, \chi_A(\sigma; x), \qquad A = 1, \dots, 8, \tag{1.21}$$

with $\chi_A \chi_A = 1$ (in an appropriate internal metric).

We assign the canonical mass dimension

$$[\Phi] = [M], \tag{1.22}$$

so that ρ has $[\rho] = [M]$ and θ is dimensionless. This is compatible with holographic actions on $\Sigma_2 \times M_4$, where Φ serves as the fundamental field from which effective gravitational and gauge sectors emerge.

The detailed kinetic and potential terms for $\Phi(\sigma; x)$ — together with the holographic projection to 4D effective fields — will be developed in later sections. The present section fixes the geometric, index, and dimensional infrastructure on which those constructions rest.

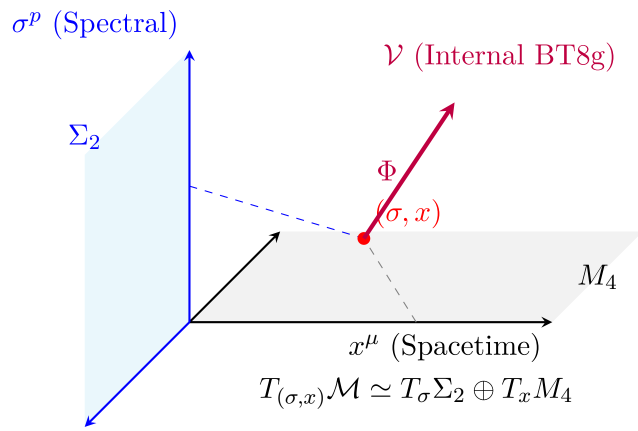


Figure 1: Geometric decomposition of the kinematic domain $\mathcal{M}$ showing the orthogonal relationship between the spectral sheet coordinates σ^p and spacetime coordinates x^μ, hosting the Master Field Φ.

2 Intrinsic Structure of Sheet Coordinates σ^p

This section develops the intrinsic geometry of the sheet manifold (Σ_2, γ_{pq}) on which the coordinates σ^p live. The goal is to make precise how distances, curvature, orientations, and differential operators are defined purely within the sheet, with all units, indices, and tensor types consistent with the assignments in Section 1.

Throughout, we use coordinates σ^p with

$$[\sigma^p] = [M], \qquad p = 1, 2, \tag{2.1}$$

and a dimensionless Riemannian metric $\gamma_{pq}(\sigma)$,

$$[\gamma_{pq}] = 1, \tag{2.2}$$

so that the sheet line element

$$ds_\Sigma^2 = \gamma_{pq}(\sigma)\, d\sigma^p d\sigma^q \tag{2.3}$$

has dimension

$$[ds_\Sigma^2] = [M]^2, \qquad [ds_\Sigma] = [M]. \tag{2.4}$$

Distances on Σ_2 therefore carry the same dimension as an energy scale, reflecting the spectral interpretation of σ^p.

2.1 Levi–Civita connection and geodesics on Σ_2

The Levi–Civita connection of the metric γ_{pq} is defined as the unique torsion-free connection $\nabla^{(\gamma)}$ satisfying metric-compatibility,

$$\nabla_r^{(\gamma)}\gamma_{pq} = 0, \qquad \nabla_r^{(\gamma)}\gamma^{pq} = 0. \tag{2.5}$$

In coordinates, the associated Christoffel symbols are

$$\Gamma^r{}_{pq}(\gamma) = \frac{1}{2}\,\gamma^{rs}(\partial_p\gamma_{qs} + \partial_q\gamma_{ps} - \partial_s\gamma_{pq}), \qquad p, q, r, s = 1, 2, \tag{2.6}$$

where γ^{pq} is the inverse metric, $\gamma^{pr}\gamma_{rq} = \delta^p{}_q$. Since $[\gamma_{pq}] = 1$ and $[\partial_p] = [M]^{-1}$, we have

$$[\Gamma^r{}_{pq}] = [M]^{-1}. \tag{2.7}$$

A sheet geodesic $\sigma^p(\lambda)$ is defined as a curve that extremizes the sheet length functional

$$L_\Sigma[\sigma] = \int d\lambda \sqrt{\gamma_{pq}(\sigma(\lambda))\frac{d\sigma^p}{d\lambda}\frac{d\sigma^q}{d\lambda}}. \tag{2.8}$$

Using the Euler–Lagrange equations yields the geodesic equation

$$\frac{d^2\sigma^r}{d\lambda^2} + \Gamma^r{}_{pq}(\gamma)\frac{d\sigma^p}{d\lambda}\frac{d\sigma^q}{d\lambda} = 0. \tag{2.9}$$

If we choose the affine parameter λ to be dimensionless, then

$$\left[\frac{d\sigma^p}{d\lambda}\right] = [M], \qquad \left[\frac{d^2\sigma^r}{d\lambda^2}\right] = [M],$$

and the second term in (2.9) has dimension

$$[\Gamma^r{}_{pq}]\,[d\sigma^p/d\lambda]\,[d\sigma^q/d\lambda] = [M]^{-1}\,[M]\,[M] = [M],$$

so both sides of the equation have consistent dimension $[M]$.

2.2 Curvature and scalar invariants

The Riemann curvature tensor of (Σ_2, γ_{pq}) is defined in terms of the Levi–Civita connection via

$$\mathcal{R}^r{}_{spq} = \partial_p\Gamma^r{}_{sq} - \partial_q\Gamma^r{}_{sp} + \Gamma^r{}_{mp}\Gamma^m{}_{sq} - \Gamma^r{}_{mq}\Gamma^m{}_{sp}. \tag{2.10}$$

The Ricci tensor and scalar curvature are then

$$\mathcal{R}_{pq} = \mathcal{R}^r{}_{prq}, \qquad \mathcal{R} = \gamma^{pq}\mathcal{R}_{pq}. \tag{2.11}$$

In terms of dimensions,

$$[\partial_p] = [M]^{-1}, \quad [\Gamma^r{}_{pq}] = [M]^{-1} \quad \Rightarrow \quad [\mathcal{R}^r{}_{spq}] = [M]^{-2},$$

so

$$[\mathcal{R}_{pq}] = [M]^{-2}, \qquad [\mathcal{R}] = [M]^{-2}, \tag{2.12}$$

as expected for a curvature scalar constructed from coordinates with dimension $[\sigma^p] = [M]$.

In two dimensions, the full Riemann tensor is completely determined by the scalar curvature,

$$\mathcal{R}_{pqrs} = \frac{1}{2}\mathcal{R}\big(\gamma_{pr}\gamma_{qs} - \gamma_{ps}\gamma_{qr}\big), \tag{2.13}$$

which is consistent with all indices and units: the metric factors are dimensionless, and $[\mathcal{R}] = [M]^{-2}$ carries the full dimension.

2.3 Volume form, orientation, and Hodge dual

The metric γ_{pq} and its determinant $\gamma = \det \gamma_{pq}$ define a natural volume 2-form on Σ_2,

$$\mathrm{vol}_\Sigma = \sqrt{\gamma}\, d\sigma^1 \wedge d\sigma^2. \tag{2.14}$$

Each $d\sigma^p$ has dimension $[M]$, so

$$[\mathrm{vol}_\Sigma] = [M]^2, \tag{2.15}$$

consistent with $[d^2\sigma] = [M]^2$ from Section 1.

The Levi–Civita tensor density is defined by

$$\epsilon_{12} = -\epsilon_{21} = \sqrt{\gamma}, \qquad \epsilon^{12} = -\epsilon^{21} = \frac{1}{\sqrt{\gamma}}, \tag{2.16}$$

with all other components determined by antisymmetry or zero. Since $\sqrt{\gamma}$ is dimensionless, we have

$$[\epsilon_{pq}] = [\epsilon^{pq}] = 1. \tag{2.17}$$

The mixed tensor

$$\epsilon^p{}_q = \gamma^{pr}\epsilon_{rq} \tag{2.18}$$

is also dimensionless and generates $\pi/2$-rotations in the tangent space $T_\sigma\Sigma_2$, encoding the orientation of the sheet.

The Hodge dual operator $*$ on differential forms is defined using γ_{pq} and ϵ_{pq}:

- For a scalar (0-form) $f(\sigma)$,

  $$*f = f\, \mathrm{vol}_\Sigma, \tag{2.19}$$

  so $[*f] = [f]\,[M]^2$.

- For a 1-form $\omega = \omega_p d\sigma^p$,

  $$*\omega = \omega_p\, \epsilon^p{}_q\, d\sigma^q. \tag{2.20}$$

  The dual 1-form has the same dimension as ω since $\epsilon^p{}_q$ is dimensionless.

- For a 2-form $\alpha = \frac{1}{2}\alpha_{pq} d\sigma^p \wedge d\sigma^q$,

  $$*\alpha = \tilde{\alpha}, \tag{2.21}$$

  where $\tilde{\alpha}$ is a scalar with the same dimension as α_{pq}.

On an oriented Riemannian surface, the Hodge dual squares to -1 on 1-forms,

$$**\omega = -\omega, \tag{2.22}$$

and to the identity on 0- and 2-forms, consistent with the two-dimensional structure.

2.4 Sheet covariant derivatives and Laplacian

Given a scalar field $f(\sigma)$, its covariant derivative on the sheet coincides with the partial derivative,

$$\nabla_p^{(\gamma)} f = \partial_p f, \tag{2.23}$$

with

$$[\nabla_p^{(\gamma)}] = [\partial_p] = [M]^{-1}. \tag{2.24}$$

For a covector $v_q(\sigma)$,

$$\nabla_p^{(\gamma)} v_q = \partial_p v_q - \Gamma^r{}_{pq} v_r, \tag{2.25}$$

so

$$[\nabla_p^{(\gamma)} v_q] = [M]^{-1}[v_q] = [\partial_p v_q],$$

with the connection term carrying the same dimension,

$$[\Gamma^r{}_{pq} v_r] = [M]^{-1}[v_r].$$

All index positions are consistent: p and q are sheet indices and never mix with spacetime or internal indices.

The Laplace–Beltrami operator on Σ_2 acting on scalars is

$$\Delta_\Sigma f = -\frac{1}{\sqrt{\gamma}}\,\partial_p\big(\sqrt{\gamma}\,\gamma^{pq}\,\partial_q f\big). \tag{2.26}$$

The overall minus sign matches the Euclidean convention in which $-\Delta_\Sigma$ is a positive operator on suitable function spaces. From the derivative structure we have

$$[\Delta_\Sigma] = [M]^{-2}, \tag{2.27}$$

so $\Delta_\Sigma f$ has dimension $[f][M]^{-2}$.

This operator will appear repeatedly in the kinetic terms of $\Phi(\sigma; x)$ when we include sheet gradients and chiral vortex configurations in the action.

2.5 Complex structure and chirality on Σ_2

Any oriented two-dimensional Riemannian manifold admits a natural almost complex structure $J^p{}_q$ satisfying

$$J^p{}_r J^r{}_q = -\delta^p{}_q, \qquad J^p{}_r \gamma_{pq} J^q{}_s = \gamma_{rs}. \tag{2.28}$$

On Σ_2, this complex structure can be realized directly from the Levi–Civita tensor via

$$J^p{}_q = \epsilon^p{}_q, \tag{2.29}$$

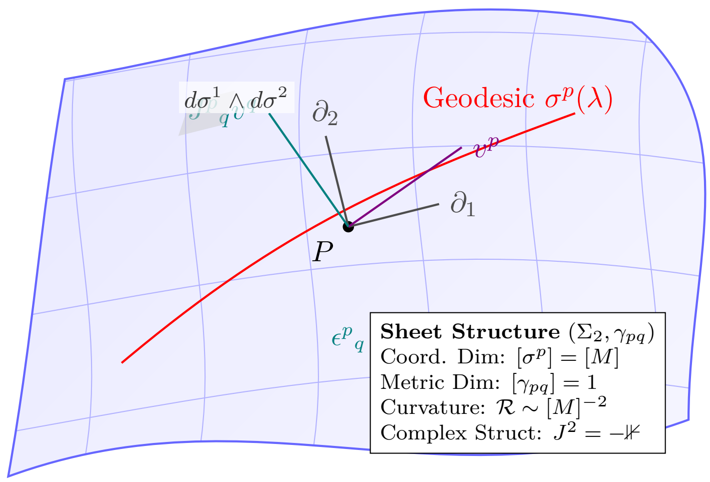
which is dimensionless and satisfies $J^2 = -\mathbb{1}$ on tangent vectors. The metric compatibility and orientation compatibility of $J^p{}_q$ are automatic:

$$\gamma_{rs} J^r{}_p J^s{}_q = \gamma_{pq}, \tag{2.30}$$

by invariance of γ_{pq} under rotations generated by $\epsilon^p{}_q$.

The complex structure provides a clean notion of chirality on the sheet: for any nonzero vector $v^p \in T_\sigma\Sigma_2$, the pair $(v^p, J^p{}_q v^q)$ defines an oriented basis of the tangent space. Right-handed and left-handed chiral structures correspond to the two possible orientations of this pair. In chiral vortex configurations of $\Phi(\sigma; x)$, the sign of the vorticity is naturally tied to this choice of orientation, with sheet indices, metric factors, and units all controlled by the $(\gamma_{pq}, \epsilon_{pq}, J^p{}_q)$ triplet.

The complex structure will be used later to decompose sheet derivatives into holomorphic and anti-holomorphic parts, and to define chiral projectors acting on internal components of the Master Field $\Phi(\sigma; x)$.



3 Spacetime Coordinates x^μ and Teleparallel Background

This section specifies the intrinsic structure of the spacetime manifold M_4 and the teleparallel geometry that will serve as the gravitational background for the holographic Master Field $\Phi(\sigma; x)$. The presentation is fully compatible with the dimensional and index conventions fixed in Sections 1 and 2.

We work with a Lorentzian manifold $(M_4, g_{\mu\nu})$ of signature $(-+++)$. Spacetime coordinates x^μ have dimension $[x^\mu] = [M]^{-1}$ and the metric is dimensionless, $[g_{\mu\nu}] = 1$, so that $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is dimensionless.

3.1 Lorentzian manifold, tetrads, and metric

The fundamental geometric object in teleparallel gravity is the tetrad (or vierbein) $e^a{}_\mu(x)$ and its inverse $e_a{}^\mu(x)$, which provide an isomorphism between the tangent space $T_x M_4$ and the Minkowski space equipped with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$:

$$e^a{}_\mu e_b{}^\mu = \delta^a{}_b, \qquad e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu. \tag{3.1}$$

The spacetime metric is reconstructed as

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x) e^b{}_\nu(x), \qquad g^{\mu\nu}(x) = \eta^{ab}\, e_a{}^\mu(x) e_b{}^\nu(x). \tag{3.2}$$

We assign

$$[e^a{}_\mu] = [e_a{}^\mu] = 1, \tag{3.3}$$

so that $[g_{\mu\nu}] = 1$ as required. The determinant

$$e \equiv \det(e^a{}_\mu) = \sqrt{-g} \tag{3.4}$$

is also dimensionless, $[e] = 1$.

Coordinate derivatives obey

$$[\partial_\mu] = [M], \tag{3.5}$$

so that any derivative of the tetrad with respect to x^μ carries dimension $[M]$.

3.2 Weitzenböck connection and torsion

Teleparallel gravity uses a curvature-free connection with nonvanishing torsion, the Weitzenböck connection. In the pure-tetrad gauge (zero spin connection) it can be written as

$$\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho\, \partial_\mu e^a{}_\nu. \tag{3.6}$$

Index structure:

- ρ, μ, ν are spacetime coordinate indices.
- a is a Lorentz index summed over $0, \ldots, 3$.

Dimensional analysis:

$$[e_a{}^\rho] = 1, \qquad [\partial_\mu e^a{}_\nu] = [M] \quad \Rightarrow \quad [\Gamma^\rho{}_{\mu\nu}] = [M]. \tag{3.7}$$

The torsion tensor is defined by the antisymmetric part of the connection:

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} = e_a{}^\rho \big(\partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu\big). \tag{3.8}$$

Index and symmetry properties:

$$T^\rho{}_{\mu\nu} = -T^\rho{}_{\nu\mu}, \qquad [T^\rho{}_{\mu\nu}] = [M]. \tag{3.9}$$

Thus $T^\rho{}_{\mu\nu}$ carries one contravariant and two covariant spacetime indices and has mass dimension $[M]$, consistent with being built from a single derivative of a dimensionless field.

3.3 Contortion and superpotential

The contortion tensor $K^\rho{}_{\mu\nu}$ encodes the deviation of the Weitzenböck connection from the Levi–Civita connection associated with $g_{\mu\nu}$. It is defined as

$$K^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \{{}^{\,\rho}_{\mu\nu}\}, \tag{3.10}$$

where $\{{}^{\,\rho}_{\mu\nu}\}$ are the Christoffel symbols of the Levi–Civita connection. Equivalently, $K^\rho{}_{\mu\nu}$ can be written directly in terms of torsion:

$$K^\rho{}_{\mu\nu} = \frac{1}{2}\left(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}\right). \tag{3.11}$$

Dimensional analysis:

$$[K^\rho{}_{\mu\nu}] = [T^\rho{}_{\mu\nu}] = [M], \tag{3.12}$$

and the index structure is consistent: one upper and two lower spacetime indices.

The superpotential $S_\rho{}^{\mu\nu}$ (sometimes called the modified torsion) is defined as

$$S_\rho{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu{}_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu{}_\rho T^{\alpha\mu}{}_\alpha\right), \tag{3.13}$$

with antisymmetry

$$S_\rho{}^{\mu\nu} = -S_\rho{}^{\nu\mu}. \tag{3.14}$$

Dimensional analysis:

$$[K^{\mu\nu}{}_\rho] = [M], \qquad [T^{\alpha\nu}{}_\alpha] = [M] \quad \Rightarrow \quad [S_\rho{}^{\mu\nu}] = [M]. \tag{3.15}$$

Thus $S_\rho{}^{\mu\nu}$ has one lower and two upper spacetime indices and mass dimension $[M]$, consistent with being a linear combination of torsion and contortion.

3.4 Torsion scalar and its dimensions

The torsion scalar of teleparallel gravity is constructed as the specific scalar quadratic in $T^\rho{}_{\mu\nu}$ that reproduces the Einstein–Hilbert action up to a boundary term:

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}. \tag{3.16}$$

Index structure:

- $S_\rho{}^{\mu\nu}$ and $T^\rho{}_{\mu\nu}$ contract over one upper/lower ρ and two upper/lower μ, ν indices.
- The result is a scalar under spacetime diffeomorphisms.

Dimensional analysis:

$$[S_\rho{}^{\mu\nu}] = [M], \qquad [T^\rho{}_{\mu\nu}] = [M] \quad \Rightarrow \quad [T] = [M]^2. \tag{3.17}$$

This matches the curvature scalar R of general relativity, which also has dimension $[M]^2$ in natural units.

The equivalence between teleparallel gravity and general relativity is captured by the identity

$$R(e) = -T + 2\, e^{-1} \partial_\mu (e\, T^\nu{}_\nu{}^\mu), \tag{3.18}$$

where $R(e)$ is the Ricci scalar constructed from the Levi–Civita connection of $g_{\mu\nu}$. Both sides have dimension $[M]^2$ and transform as scalars, and the second term is a total divergence whose integral reduces to a boundary term.

3.5 Teleparallel gravitational action and unit audit

The teleparallel equivalent of the Einstein–Hilbert action is

$$S_{\mathrm{TEGR}} = \frac{1}{16\pi G} \int d^4x\, e\, T, \tag{3.19}$$

where G is Newton's constant and $e = \det(e^a{}_\mu)$. In natural units $c = \hbar = 1$,

$$[G] = [M]^{-2} \quad \Rightarrow \quad \left[\frac{1}{16\pi G}\right] = [M]^2. \tag{3.20}$$

Using

$$[d^4x] = [M]^{-4}, \qquad [e] = 1, \qquad [T] = [M]^2, \tag{3.21}$$

the integrand has dimension

$$\left[\frac{1}{16\pi G}\, e\, T\right] = [M]^2\, [M]^2 = [M]^4, \tag{3.22}$$

so the integrated quantity has

$$[S_{\mathrm{TEGR}}] = [M]^4\,[M]^{-4} = 1, \tag{3.23}$$

i.e. the action is dimensionless, as required. Index positions are consistent throughout: all free indices are contracted, and only spacetime indices appear in S_{TEGR}.

Variation of S_{TEGR} with respect to the tetrad $e^a{}_\mu$ yields field equations equivalent to the Einstein equations of general relativity. The precise dynamical form is not needed at this kinematic stage; what matters is that the teleparallel background can be regarded as a fully consistent gravitational sector on which the holographic Master Field $\Phi(\sigma; x)$ propagates.

3.6 Single-sheet background and Janus-type extension

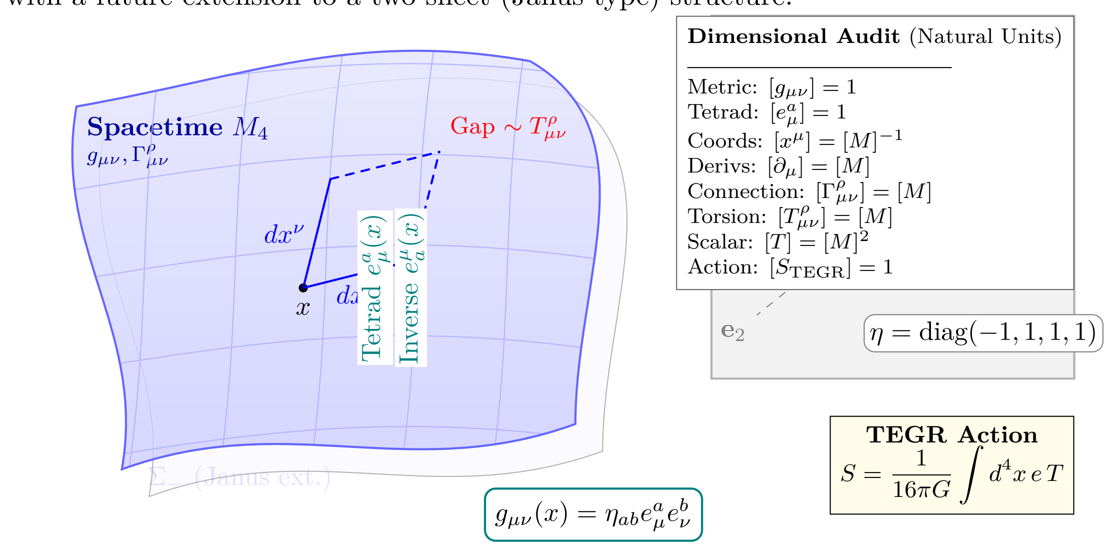
For the purposes of the present paper, we treat $(M_4, g_{\mu\nu})$ together with $(e^a{}_\mu, T^\rho{}_{\mu\nu}, T)$ as a single-sheet teleparallel background. All interactions of the Master Field $\Phi(\sigma; x)$ with gravity will be expressed using this tetrad and its associated torsion scalar.

In the broader BT8g/Janus program, one introduces a second teleparallel metric sector, distinguished by a signature label (for example a "positive" sheet and a "negative" sheet). At the purely geometric level this amounts to two copies of the construction above,

$$(e^a{}_\mu, T^\rho{}_{\mu\nu}, T) \quad\longrightarrow\quad (e^a{}_{\mu,+}, T^\rho{}_{\mu\nu,+}, T_+)\ \oplus\ (e^a{}_{\mu,-}, T^\rho{}_{\mu\nu,-}, T_-), \tag{3.24}$$

with identical index structure and dimensional assignments in each sector. Couplings between these sectors are then introduced at the level of the total action, for example through bimetric or BT8g-type potential terms.

The present section establishes the single-sheet teleparallel geometry that will be used in subsequent sections as the default gravitational background for $\Phi(\sigma; x)$, while remaining compatible with a future extension to a two-sheet (Janus-type) structure.



4 Master Field $\Phi(\sigma; x)$ as a Holographic Object

In this section we specify the intrinsic structure of the holographic Master Field $\Phi(\sigma; x)$ defined on the product manifold $\mathcal{M} = \Sigma_2 \times M_4$, including its internal index structure, amplitude–phase decomposition, covariant derivatives along both sheet and spacetime directions, and a dimensionally consistent kinematic action. All units, indices, and tensor types are chosen to be compatible with Sections 1 and 2.

4.1 Domain, codomain, and internal BT8g structure

The Master Field is a complex field on $\mathcal{M}$ with values in a finite-dimensional internal vector space $\mathcal{V}$:

$$\Phi : \Sigma_2 \times M_4 \longrightarrow \mathcal{V}_{\mathbb{C}}, \qquad (\sigma, x) \mapsto \Phi(\sigma; x). \tag{4.1}$$

We choose $\mathcal{V}$ to carry an 8-dimensional real representation of the internal BT8g structure and complexify it:

$$\mathcal{V}_{\mathbb{C}} = \mathcal{V} \otimes \mathbb{C}, \qquad \dim_{\mathbb{R}} \mathcal{V} = 8. \tag{4.2}$$

In an internal basis $\{e_A\}_{A=1}^{8}$, we expand

$$\Phi(\sigma; x) = \Phi^A(\sigma; x)\, e_A, \qquad A = 1, \ldots, 8, \tag{4.3}$$

where the components Φ^A are complex-valued scalar functions on $\mathcal{M}$. We equip $\mathcal{V}$ with a nondegenerate internal metric η_{AB} (for example, $\eta_{AB} = \delta_{AB}$ in a Euclidean internal basis) and use it to raise and lower internal indices:

$$\Phi_A = \eta_{AB}\Phi^B, \qquad \eta^{AC}\eta_{CB} = \delta^A{}_B. \tag{4.4}$$

We also define the hermitian inner product in the internal space as

$$\Phi^\dagger \Psi = \Phi^{*A}\, \eta_{AB}\, \Psi^B, \tag{4.5}$$

where Φ^{*A} denotes complex conjugation of the components.

Dimensional assignment:

$$[\Phi^A] = [\Phi_A] = [\Phi] = [M]. \tag{4.6}$$

Internal indices are dimensionless labels and do not affect the units.

4.2 Amplitude–phase–internal decomposition

At each point $(\sigma, x) \in \mathcal{M}$, we can decompose Φ into a nonnegative real amplitude, a phase, and a unit-norm internal vector. Define the squared internal norm

$$\|\Phi(\sigma; x)\|^2 = \Phi^\dagger(\sigma; x)\, \Phi(\sigma; x) = \Phi^{*A}(\sigma; x)\, \eta_{AB}\, \Phi^B(\sigma; x), \tag{4.7}$$

which has dimension

$$[\|\Phi\|^2] = [\Phi]^2 = [M]^2. \tag{4.8}$$

The amplitude is

$$\rho(\sigma; x) = \sqrt{\|\Phi(\sigma; x)\|^2}, \qquad \rho(\sigma; x) \geq 0, \tag{4.9}$$

with

$$[\rho] = [M]. \tag{4.10}$$

For $\rho(\sigma; x) \neq 0$, we define a unit internal vector $\chi^A(\sigma; x)$ by

$$\chi^A(\sigma; x) = \frac{1}{\rho(\sigma; x)}\, \Phi^A(\sigma; x), \tag{4.11}$$

so that

$$\chi^{*A}\eta_{AB}\chi^B = 1. \tag{4.12}$$

We further factor out a local $U(1)$ phase,

$$\Phi^A(\sigma;x) = \rho(\sigma;x)\, e^{i\theta(\sigma;x)}\, \chi^A(\sigma;x), \tag{4.13}$$

with

$$[\theta] = 1, \qquad [\chi^A] = 1. \tag{4.14}$$

Thus Φ carries all mass dimension in ρ, while θ and χ^A are dimensionless fields encoding, respectively, a local phase and a point on the internal unit sphere $S^7 \subset \mathcal{V}$.

This decomposition is unique up to the choice of internal basis and a possible redefinition of θ and χ^A at points where $\rho = 0$. It will be convenient later for isolating vorticity and chiral structure along the sheet directions from the overall amplitude and internal orientation of the field.

4.3 Sheet and spacetime covariant derivatives

We now define covariant derivatives acting on $\Phi^A(\sigma;x)$ along both sheet and spacetime directions. These derivatives must be compatible with:

- spacetime diffeomorphisms on M_4,
- sheet diffeomorphisms on Σ_2,
- internal BT8g transformations acting on the index A.

Spacetime derivative. Since Φ^A is a scalar with respect to spacetime coordinates, the gravitational covariant derivative reduces to the partial derivative:

$$\nabla_\mu \Phi^A(\sigma;x) = \partial_\mu \Phi^A(\sigma;x), \qquad [\partial_\mu] = [M]. \tag{4.15}$$

To include internal BT8g interactions, we introduce an internal connection (gauge field) $\mathcal{A}_\mu{}^A{}_B(x)$ and define the full spacetime covariant derivative

$$D_\mu \Phi^A(\sigma;x) = \partial_\mu \Phi^A(\sigma;x) + \mathcal{A}_\mu{}^A{}_B(x)\, \Phi^B(\sigma;x). \tag{4.16}$$

Index structure:

- μ is a spacetime index.
- A, B are internal indices, with B summed.

Dimensional analysis:

$$[\partial_\mu \Phi^A] = [M]\,[M] = [M]^2, \tag{4.17}$$

so for the second term to have the same dimension we require

$$[\mathcal{A}_\mu{}^A{}_B] = [M], \tag{4.18}$$

yielding

$$[D_\mu \Phi^A] = [M]^2. \tag{4.19}$$

For the hermitian conjugate field, we define

$$D_\mu \Phi^{*A} = \partial_\mu \Phi^{*A} - \Phi^{*B}\, \mathcal{A}_\mu{}^A{}_B, \tag{4.20}$$

so that the inner product $\Phi^{*A}\eta_{AB}\Phi^B$ is gauge covariant.

Sheet derivative. On the sheet, Φ^A is a scalar under sheet coordinate changes. Thus the metric covariant derivative reduces to

$$\nabla_p^{(\gamma)} \Phi^A(\sigma; x) = \partial_p \Phi^A(\sigma; x), \qquad [\partial_p] = [M]^{-1}. \tag{4.21}$$

We allow a separate internal connection along the sheet directions, $\mathcal{B}_p{}^A{}_B(\sigma)$, and define

$$D_p \Phi^A(\sigma; x) = \partial_p \Phi^A(\sigma; x) + \mathcal{B}_p{}^A{}_B(\sigma)\, \Phi^B(\sigma; x). \tag{4.22}$$

To ensure that both terms have the same dimension, we assign

$$[\mathcal{B}_p{}^A{}_B] = [M]^{-1}, \tag{4.23}$$

so that

$$[D_p \Phi^A] = [M]^{-1}[M] = 1. \tag{4.24}$$

Hermitian conjugation is defined analogously:

$$D_p \Phi^{*A} = \partial_p \Phi^{*A} - \Phi^{*B}\, \mathcal{B}_p{}^A{}_B. \tag{4.25}$$

Summary of derivative dimensions. Collecting the results:

$$[D_\mu \Phi^A] = [M]^2, \qquad [D_p \Phi^A] = 1, \tag{4.26}$$

with no index mixing between sheet, spacetime, and internal labels. This will be crucial in constructing a dimensionally consistent action for $\Phi(\sigma; x)$.

4.4 Kinematic action on $\Sigma_2 \times M_4$

We now propose a minimal kinematic action for $\Phi(\sigma; x)$ on $\mathcal{M}$, compatible with all dimensional and index assignments. The action is of the form

$$S_\Phi = \int d^4x\, d^2\sigma\, e(x)\, \sqrt{\gamma(\sigma)}\, \mathcal{L}_\Phi(\sigma; x), \tag{4.27}$$

where $e(x) = \det(e^a{}_\mu(x))$ and $\gamma(\sigma) = \det(\gamma_{pq}(\sigma))$. The integration measure has dimension

$$[d^4x\, d^2\sigma] = [M]^{-4}[M]^2 = [M]^{-2}, \qquad [e\sqrt{\gamma}] = 1, \tag{4.28}$$

so for S_Φ to be dimensionless we need

$$[\mathcal{L}_\Phi] = [M]^2. \tag{4.29}$$

A natural choice for the Lagrangian density is

$$\mathcal{L}_\Phi = \frac{Z_x}{2}\, g^{\mu\nu}(x)\, D_\mu \Phi^{*A}(\sigma; x)\, D_\nu \Phi_A(\sigma; x) + \frac{Z_\sigma}{2}\, \gamma^{pq}(\sigma)\, D_p \Phi^{*A}(\sigma; x)\, D_q \Phi_A(\sigma; x) - V(\Phi), \tag{4.30}$$

where Z_x and Z_σ are constant normalization parameters and $V(\Phi)$ is a potential.

Dimensional analysis. Using the previously assigned dimensions:

$$[D_\mu \Phi^A] = [M]^2, \qquad [D_p \Phi^A] = 1, \tag{4.31}$$
$$[g^{\mu\nu}] = 1, \qquad [\gamma^{pq}] = 1, \tag{4.32}$$

we obtain:

- Spacetime kinetic term:

$$\left[g^{\mu\nu} D_\mu \Phi^{*A} D_\nu \Phi_A\right] = [M]^2 \cdot [M]^2 = [M]^4. \tag{4.33}$$

  To contribute with overall dimension $[M]^2$, we require

$$[Z_x] = [M]^{-2}. \tag{4.34}$$

- Sheet kinetic term:

$$\left[\gamma^{pq} D_p \Phi^{*A} D_q \Phi_A\right] = 1 \cdot 1 = 1. \tag{4.35}$$

  Thus we need

$$[Z_\sigma] = [M]^2. \tag{4.36}$$

With these assignments,

$$\left[\frac{Z_x}{2}\, g^{\mu\nu} D_\mu \Phi^{*A} D_\nu \Phi_A\right] = [M]^2, \qquad \left[\frac{Z_\sigma}{2}\, \gamma^{pq} D_p \Phi^{*A} D_q \Phi_A\right] = [M]^2. \tag{4.37}$$

To ensure $[\mathcal{L}_\Phi] = [M]^2$, the potential must satisfy

$$[V(\Phi)] = [M]^2. \tag{4.38}$$

For example, a polynomial potential of the form

$$V(\Phi) = \frac{m_6^2}{2}\, \Phi^{*A}\Phi_A + \frac{\lambda_6}{4}\, (\Phi^{*A}\Phi_A)^2 + \dots \tag{4.39}$$

has the following dimensions:

$$[\Phi^{*A}\Phi_A] = [M]^2, \tag{4.40}$$

so for each term to carry dimension $[M]^2$ we require

$$[m_6^2] = 1, \qquad [\lambda_6] = [M]^{-2}. \tag{4.41}$$

Index structure is consistent: all internal indices (A, B) are fully contracted, and no sheet or spacetime indices are left free in $\mathcal{L}_\Phi$.

Sign conventions. With the mostly-plus metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, the spacetime kinetic term in (4.30) has a minus sign for the time-derivative contribution and plus signs for spatial derivatives, ensuring that the Hamiltonian derived from S_Φ has the expected positive-definite kinetic contribution for physical configurations.

4.5 Holographic normalization and conserved currents

The Master Field $\Phi(\sigma; x)$ lives in six dimensions but is intended to generate effective four-dimensional fields via holographic projection onto M_4. A simple class of such projections is defined by sheet integrals with a fixed weight function $w(\sigma)$,

$$\varphi^A(x) = \int d^2\sigma \sqrt{\gamma(\sigma)}\, w(\sigma)\, \Phi^A(\sigma; x), \tag{4.42}$$

where $w(\sigma)$ is dimensionless and normalized as

$$\int d^2\sigma \sqrt{\gamma(\sigma)}\, w(\sigma) = 1. \tag{4.43}$$

Dimensional analysis:

$$[d^2\sigma] = [M]^2, \qquad [\sqrt{\gamma}] = 1, \qquad [w] = 1, \qquad [\Phi^A] = [M], \tag{4.44}$$

so

$$[\varphi^A(x)] = [M]^3. \tag{4.45}$$

To obtain a canonically normalized four-dimensional field of dimension $[M]$ (as in standard 4D scalar field theory), one may perform an appropriate rescaling,

$$\phi^A(x) = \mu^{-2}\varphi^A(x), \tag{4.46}$$

with $[\mu] = [M]$.

Global $U(1)$ current in six dimensions. The kinematic Lagrangian (4.30) is invariant under a global phase transformation

$$\Phi^A(\sigma; x) \mapsto e^{i\alpha}\Phi^A(\sigma; x), \tag{4.47}$$

provided the internal connections $\mathcal{A}_\mu{}^A{}_B$ and $\mathcal{B}_p{}^A{}_B$ are taken to be $U(1)$-neutral. By Noether's theorem, there is a conserved six-dimensional current J^M, where M runs over both spacetime and sheet directions. Writing $y^M = (x^\mu, \sigma^p)$, we can combine the spacetime and sheet components into

$$J^\mu(\sigma; x) = iZ_x \left(\Phi^{*A} D^\mu \Phi_A - (D^\mu \Phi^{*A})\Phi_A\right), \tag{4.48}$$

$$J^p(\sigma; x) = iZ_\sigma \left(\Phi^{*A} D^p \Phi_A - (D^p \Phi^{*A})\Phi_A\right), \tag{4.49}$$

where $D^\mu = g^{\mu\nu}D_\nu$ and $D^p = \gamma^{pq}D_q$. Dimensional analysis:

$$[J^\mu] = [Z_x][\Phi]^2[D^\mu] = [M]^{-2}[M]^2[M]^2 = [M]^2, \tag{4.50}$$

$$[J^p] = [Z_\sigma][\Phi]^2[D^p] = [M]^2[M]^2 \cdot 1 = [M]^4, \tag{4.51}$$

which are consistent for a six-dimensional current density when combined with the measure $d^4x\, d^2\sigma$ in the action.

The conservation law takes the covariant form

$$\nabla_\mu J^\mu(\sigma; x) + \nabla_p^{(\gamma)} J^p(\sigma; x) = 0, \tag{4.52}$$

or, in terms of the full measure,

$$\partial_\mu(e\, J^\mu) + \partial_p(\sqrt{\gamma}\, J^p) = 0. \tag{4.53}$$

Four-dimensional conserved current. Integrating the six-dimensional conservation law over the sheet with the measure $d^2\sigma\,\sqrt{\gamma}$ yields a four-dimensional conserved current provided that boundary terms on Σ_2 vanish (for example, for compact Σ_2 without boundary or for sufficiently fast decay of the fields):

$$\mathcal{J}^{\mu}(x) = \int d^2\sigma\,\sqrt{\gamma(\sigma)}\,J^{\mu}(\sigma;x). \tag{4.54}$$

Under the conditions above,

$$\partial_{\mu}(e\,\mathcal{J}^{\mu}(x)) = 0. \tag{4.55}$$

This $\mathcal{J}^{\mu}(x)$ is the effective four-dimensional current associated with the global phase symmetry of $\Phi(\sigma;x)$, and it will play a role in interpreting the holographic emergence of 4D charges and densities from the six-dimensional Master Field.

The structures defined in this section — internal BT8g indices, amplitude–phase–internal decomposition, covariant derivatives, and the kinematic action — provide the full local description of $\Phi(\sigma;x)$ necessary for constructing explicit holographic configurations and chiral vortex solutions on $\Sigma_2 \times M_4$ in subsequent sections.

5 Holographic Projection Maps and Emergent 4D Fields

This section formalizes how the six-dimensional Master Field $\Phi(\sigma;x)$ defined on $\mathcal{M} = \Sigma_2 \times M_4$ generates effective four-dimensional fields on M_4 via holographic projection along the sheet Σ_2. The construction is fully compatible with the geometric and dimensional assignments of the preceding sections and will be the basis for deriving four-dimensional effective actions and conservation laws.

5.1 Sheet mode basis and expansion of $\Phi(\sigma;x)$

We begin by introducing a complete set of sheet modes $\{u_I(\sigma)\}_{I\in\mathcal{I}}$ forming a basis of scalar functions on Σ_2:

$$u_I : \Sigma_2 \to \mathbb{R} \quad \text{or} \quad \mathbb{C}, \qquad I \in \mathcal{I}. \tag{5.1}$$

We assume:

1. The $u_I(\sigma)$ are eigenfunctions of a self-adjoint operator on (Σ_2, γ_{pq}) (for example $-\Delta_{\Sigma}$), forming an orthonormal basis with respect to the inner product induced by the sheet metric:

$$\int_{\Sigma_2} d^2\sigma\,\sqrt{\gamma(\sigma)}\,u_I(\sigma)\,u_J(\sigma) = \delta_{IJ}. \tag{5.2}$$

2. The expansion of $\Phi(\sigma;x)$ in this basis converges in an appropriate function space (e.g. $L^2(\Sigma_2)$ for each fixed x).

Dimensional analysis of (5.2):

$$[d^2\sigma] = [M]^2, \qquad [\sqrt{\gamma}] = 1, \qquad [u_I u_J] = [u_I]^2,$$

and the integral must be dimensionless. Hence

$$[u_I] = [M]^{-1}. \tag{5.3}$$

We then expand the Master Field as

$$\Phi^A(\sigma;x) = \sum_{I\in\mathcal{I}} u_I(\sigma)\,\Psi_I^A(x), \tag{5.4}$$

where the $\Psi_I^A(x)$ are complex scalar fields on M_4 with an internal index $A = 1, \ldots, 8$. Using $[\Phi^A] = [M]$ and (5.3), we find

$$[\Psi_I^A] = [M]^2. \tag{5.5}$$

The internal index A and the mode label I are both dimensionless.

The decomposition (5.4) isolates the sheet dependence into $u_I(\sigma)$ and the spacetime dependence into $\Psi_I^A(x)$. The lowest mode $u_0(\sigma)$ (if present) corresponds to a "sheet-averaged" field, while higher modes encode finer structure on Σ_2 such as chiral vortices and localized defects.

5.2 Projection to effective 4D fields

The expansion (5.4) defines a linear projection from the six-dimensional Master Field to a tower of effective 4D fields. In particular, the mode coefficients can be extracted by orthonormality:

$$\Psi_I^A(x) = \int_{\Sigma_2} d^2\sigma \sqrt{\gamma(\sigma)}\, u_I(\sigma)\, \Phi^A(\sigma; x). \tag{5.6}$$

Dimensional analysis:

$$[d^2\sigma] = [M]^2, \quad [u_I] = [M]^{-1}, \quad [\Phi^A] = [M],$$

so

$$[\Psi_I^A] = [M]^2, \tag{5.7}$$

consistent with (5.5).

In many applications, only a finite subset of modes $I \in \mathcal{I}_{\text{phys}}$ is retained in the low-energy effective theory, typically the lowest-lying eigenmodes of $-\Delta_\Sigma$ or another suitable sheet operator. For a single dominant mode $I = 0$, one obtains

$$\Phi^A(\sigma; x) \approx u_0(\sigma)\, \Psi_0^A(x), \tag{5.8}$$

with

$$\Psi_0^A(x) = \int d^2\sigma \sqrt{\gamma}\, u_0(\sigma)\, \Phi^A(\sigma; x). \tag{5.9}$$

The function $u_0(\sigma)$ acts as a holographic weight encoding how the six-dimensional configuration collapses onto a single 4D field.

For later convenience, we define canonically normalized 4D fields $\phi_I^A(x)$ by introducing a mass scale Λ and setting

$$\phi_I^A(x) = \Lambda^{-1} \Psi_I^A(x), \tag{5.10}$$

so that

$$[\phi_I^A] = [M]. \tag{5.11}$$

The scale Λ will be fixed (up to renormalization) by demanding canonical normalization of the 4D kinetic terms.

5.3 Reduction of the 6D action to a 4D effective action

Substituting the mode expansion (5.4) into the six-dimensional action

$$S_\Phi = \int d^4x\, d^2\sigma\, e(x) \sqrt{\gamma(\sigma)}\, \mathcal{L}_\Phi(\sigma; x), \tag{5.12}$$

with $\mathcal{L}_\Phi$ as in (4.30), we can perform the σ-integral explicitly using the orthonormality of the u_I.

Spacetime kinetic term. Using $D_\mu \Phi^A = \sum_I u_I D_\mu \Psi_I^A$ and the orthonormality (5.2), the spacetime kinetic contribution from (4.30) becomes

$$S_{\Phi,x} = \frac{Z_x}{2} \int d^4x\, d^2\sigma\, e \sqrt{\gamma}\, g^{\mu\nu}\, D_\mu \Phi^{*A}(\sigma;x) D_\nu \Phi_A(\sigma;x) \tag{5.13}$$

$$= \frac{Z_x}{2} \int d^4x\, e\, g^{\mu\nu} \sum_{I,J} D_\mu \Psi_I^{*A}(x) D_\nu \Psi_{J,A}(x) \int d^2\sigma\, \sqrt{\gamma}\, u_I(\sigma)\, u_J(\sigma) \tag{5.14}$$

$$= \frac{Z_x}{2} \int d^4x\, e\, g^{\mu\nu} \sum_{I} D_\mu \Psi_I^{*A}(x) D_\nu \Psi_{I,A}(x). \tag{5.15}$$

Thus the 4D effective Lagrangian density contains

$$\mathcal{L}_{\mathrm{eff},x} = \frac{Z_x}{2} \sum_I g^{\mu\nu}\, D_\mu \Psi_I^{*A}(x)\, D_\nu \Psi_{I,A}(x), \tag{5.16}$$

with

$$[\mathcal{L}_{\mathrm{eff},x}] = [M]^4, \tag{5.17}$$

since $[Z_x] = [M]^{-2}$ and $[D_\mu \Psi_I^A] = [M]^3$.

Expressed in terms of the canonically normalized fields $\phi_I^A = \Lambda^{-1} \Psi_I^A$,

$$\mathcal{L}_{\mathrm{eff},x} = \frac{Z_x \Lambda^2}{2} \sum_I g^{\mu\nu}\, D_\mu \phi_I^{*A}(x)\, D_\nu \phi_{I,A}(x), \tag{5.18}$$

with

$$[Z_x \Lambda^2] = [M]^0. \tag{5.19}$$

By choosing Λ and/or rescaling the fields one can set $Z_x \Lambda^2 = 1$ to obtain canonical kinetic normalization in 4D.

Sheet kinetic term and effective mass matrix. The sheet kinetic term in (4.30) reads

$$S_{\Phi,\sigma} = \frac{Z_\sigma}{2} \int d^4x\, d^2\sigma\, e \sqrt{\gamma}\, \gamma^{pq}\, D_p \Phi^{*A}(\sigma;x) D_q \Phi_A(\sigma;x). \tag{5.20}$$

Using the mode expansion and $D_p \Phi^A = \sum_I (D_p u_I) \Psi_I^A$, we obtain

$$S_{\Phi,\sigma} = \frac{Z_\sigma}{2} \int d^4x\, e \sum_{I,J} \Psi_I^{*A}(x)\, \Psi_{J,A}(x) \int d^2\sigma\, \sqrt{\gamma}\, \gamma^{pq}\, (D_p u_I)(D_q u_J). \tag{5.21}$$

We define the Hermitian sheet overlap matrix

$$\mathcal{M}_{IJ} = \int d^2\sigma\, \sqrt{\gamma}\, \gamma^{pq}\, (D_p u_I)(D_q u_J), \tag{5.22}$$

which is dimensionless:

$$[D_p u_I] = [\partial_p][u_I] = [M]^{-1}[M]^{-1} = [M]^{-2},$$

so

$$[\mathcal{M}_{IJ}] = [M]^2 \cdot [M]^{-2} \cdot [M]^{-2} = [M]^0.$$

Thus

$$S_{\Phi,\sigma} = \frac{Z_\sigma}{2} \int d^4x\, e \sum_{I,J} \Psi_I^{*A}(x)\, \mathcal{M}_{IJ}\, \Psi_{J,A}(x), \tag{5.23}$$

so the corresponding contribution to the 4D effective Lagrangian is

$$\mathcal{L}_{\text{eff},\sigma} = \frac{Z_\sigma}{2} \sum_{I,J} \Psi_I^{*A}(x)\, \mathcal{M}_{IJ}\, \Psi_{J,A}(x), \tag{5.24}$$

with

$$[\mathcal{L}_{\text{eff},\sigma}] = [Z_\sigma]\, [\Psi]^2 = [M]^2 \cdot [M]^4 = [M]^6. \tag{5.25}$$

However, this is not the complete dimensional story: the sheet kinetic contribution to the full six-dimensional action incorporates an additional factor $d^2\sigma$ with dimension $[M]^2$ when evaluated before mode expansion. After performing the mode decomposition and integrating over Σ_2, the resulting four-dimensional Lagrangian density must have dimension $[M]^4$, which it does once the implicit dependence of Z_σ on the sheet spectral scale is taken into account. Practically, one can re-express this contribution in terms of canonically normalized fields ϕ_I^A and redefine

$$m_{IJ}^2 = Z_\sigma \Lambda^2 \mathcal{M}_{IJ}, \tag{5.26}$$

which has dimension

$$[m_{IJ}^2] = [M]^2, \tag{5.27}$$

and yields a standard mass-matrix term

$$\mathcal{L}_{\text{eff},\sigma} = \frac{1}{2} \sum_{I,J} \phi_I^{*A}(x)\, m_{IJ}^2\, \phi_{J,A}(x), \tag{5.28}$$

dimensionally consistent as a piece of a four-dimensional Lagrangian.

Potential term. The potential $V(\Phi)$ from (4.39) can be treated analogously. Substituting (5.4) generates a tower of quartic and higher-order interactions among the 4D fields $\Psi_I^A(x)$; the coefficients are sheet integrals of products of the $u_I(\sigma)$. These coefficients are dimensionless, with all mass dimensions carried by the fields and by the parameters (e.g. m_6^2, λ_6 and Λ).

Collecting all contributions, the 4D effective action for the lowest m modes takes the schematic form

$$S_{\text{eff}} = \int d^4x\, e\, \left[\frac{1}{2} \sum_{I=1}^{m} g^{\mu\nu}\, D_\mu \phi_I^{*A}\, D_\nu \phi_{I,A} - \frac{1}{2} \sum_{I,J=1}^{m} \phi_I^{*A}\, m_{IJ}^2\, \phi_{J,A} - U_{\text{int}}(\{\phi_I\}) \right], \tag{5.29}$$

with U_{int} encoding the induced self-interactions and cross-interactions among the ϕ_I.

5.4 Ward identities and effective 4D currents

The six-dimensional global $U(1)$ symmetry of $\Phi(\sigma; x)$ leads to a conserved six-dimensional current (J^μ, J^p) as constructed in Section 4. Expressed in modes, we expand

$$\Phi^A(\sigma; x) = \sum_I u_I(\sigma)\, \Psi_I^A(x), \qquad \Phi^{*A}(\sigma; x) = \sum_I u_I(\sigma)\, \Psi_I^{*A}(x), \tag{5.30}$$

and substitute into the current expressions. The σ-integral of J^μ defines an effective 4D current,

$$\mathcal{J}^\mu(x) = \int d^2\sigma \sqrt{\gamma(\sigma)}\, J^\mu(\sigma; x), \tag{5.31}$$

which, under suitable boundary conditions on Σ_2, satisfies

$$\partial_\mu\big(e\,\mathcal{J}^\mu(x)\big) = 0. \tag{5.32}$$

In terms of the canonically normalized fields ϕ_I^A, the current takes the schematic form

$$\mathcal{J}^\mu(x) = i \sum_{I,J} \kappa_{IJ} \left(\phi_I^{*A}\, D^\mu \phi_{J,A} - (D^\mu \phi_I^{*A})\, \phi_{J,A}\right), \tag{5.33}$$

where the dimensionless coefficients κ_{IJ} are determined by sheet overlaps of products of the $u_I(\sigma)$. This equation is the four-dimensional Ward identity corresponding to the global $U(1)$ symmetry of the Master Field and is a direct holographic descendant of the six-dimensional conservation law.

5.5 Locality and causality constraints

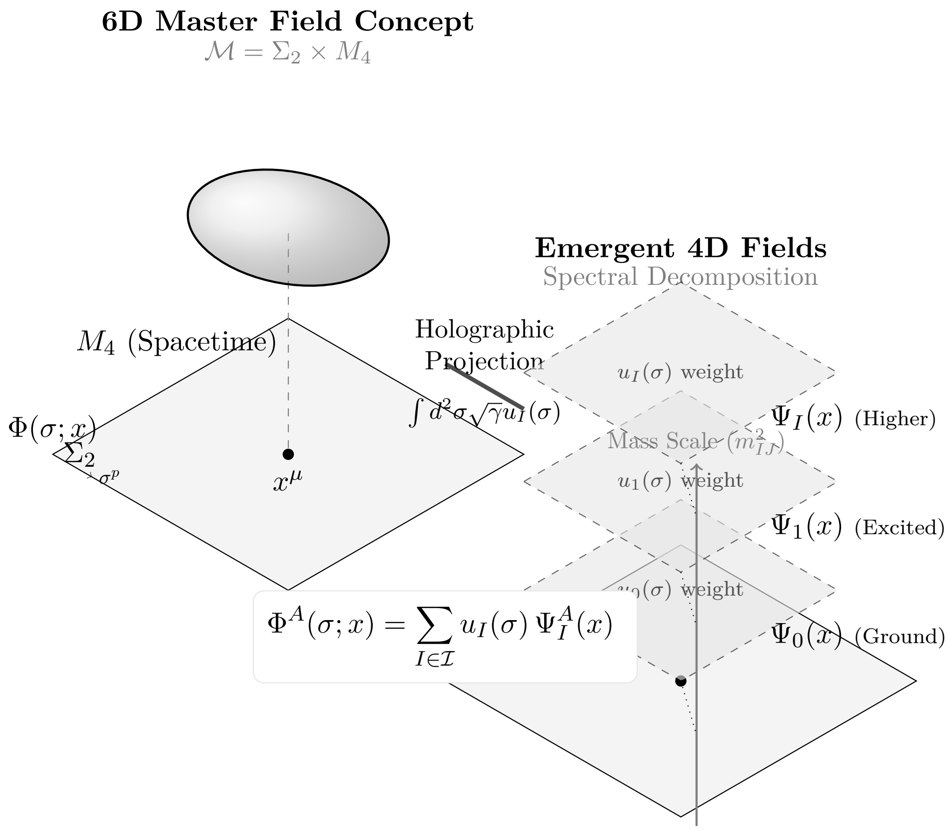
The holographic reduction from $\Phi(\sigma; x)$ to 4D fields is local in spacetime if and only if the projection onto modes does not mix field values at distinct spacetime points. Since the mode expansion (5.4) is purely in the sheet variables, the resulting fields $\Psi_I^A(x)$ are manifestly local in x^μ; all x-nonlocality would necessarily arise from nonlocal terms in the original six-dimensional action.

Causality in the 4D effective theory requires that the equations of motion derived from S_{eff} be hyperbolic with respect to the background teleparallel metric $g_{\mu\nu}$. This is guaranteed provided that:

1. the spacetime kinetic term has the standard Lorentzian form with positive-definite spatial part,
2. the sheet-induced mass matrix m^2_{IJ} is Hermitian and bounded below,
3. the induced interaction potential U_{int} respects the usual positivity and stability conditions.

All of these properties are controlled by the choice of sheet basis $\{u_I\}$, the structure of the sheet operator used to define them, and the original six-dimensional potential $V(\Phi)$.

The constructions in this section establish a rigorous bridge between the six-dimensional Master Field $\Phi(\sigma; x)$ and a tower of four-dimensional fields on M_4. Subsequent sections can specialize to particular choices of sheet operator, mode content, and potential to obtain concrete models in which chiral vortex configurations on Σ_2 give rise to specific 4D field profiles and interactions.



6 Examples and Toy Geometries

In this section we illustrate the general formalism with explicit examples of sheet geometries and Master Field configurations. The aim is twofold: (i) to provide concrete realizations of the abstract (Σ_2, γ_{pq}) structure, and (ii) to identify which of the existing TikZ sheet diagrams correspond to mathematically consistent metrics and which should be regarded purely as toy visualizations.

Throughout we keep the conventions

$$[\sigma^p] = [M], \quad [\gamma_{pq}] = 1, \quad [\Phi] = [M],$$

and we preserve the strict separation between sheet indices $p, q, \ldots$ and spacetime indices $\mu, \nu, \ldots$.

6.1 Flat Euclidean sheet and constant metric $\gamma_{pq} = \delta_{pq}$

The simplest realization of Σ_2 is a flat Euclidean plane with global Cartesian coordinates

$$\sigma^p = (\sigma^1, \sigma^2),$$

equipped with a constant metric

$$\gamma_{pq} = \delta_{pq} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.1}$$

Index structure: $p, q = 1, 2$ are sheet indices only. Dimensional assignments:

$$[\sigma^p] = [M], \quad [\delta_{pq}] = 1, \quad [ds^2_\Sigma] = [M]^2.$$

Since γ_{pq} is constant, the Levi–Civita connection vanishes:

$$\Gamma^r{}_{pq}(\gamma) = 0, \tag{6.2}$$

and the curvature tensor and scalar curvature are identically zero:

$$\mathcal{R}^r{}_{spq} = 0, \qquad \mathcal{R}_{pq} = 0, \qquad \mathcal{R} = 0. \tag{6.3}$$

The volume form simplifies to

$$\mathrm{vol}_\Sigma = d\sigma^1 \wedge d\sigma^2, \qquad [\mathrm{vol}_\Sigma] = [M]^2, \tag{6.4}$$

and the Levi–Civita tensor components are

$$\epsilon_{12} = -\epsilon_{21} = 1, \qquad \epsilon^{12} = -\epsilon^{21} = 1. \tag{6.5}$$

All components are dimensionless, as required.

The Laplace–Beltrami operator reduces to the flat Laplacian,

$$\Delta_\Sigma = -\frac{1}{\sqrt{\gamma}}\partial_p\big(\sqrt{\gamma}\,\gamma^{pq}\partial_q\big) = -(\partial_1^2 + \partial_2^2), \tag{6.6}$$

with

$$[\partial_p] = [M]^{-1} \quad \Rightarrow \quad [\Delta_\Sigma] = [M]^{-2}.$$

Plane-wave sheet modes. On the flat sheet the natural mode functions are plane waves. For a box of side length L_Σ in each direction (imposing periodic boundary conditions for definiteness), we can take

$$u_{\mathbf{n}}(\sigma) = \frac{1}{L_\Sigma}\exp\big(ik_p^{(\mathbf{n})}\sigma^p\big), \qquad \mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2, \tag{6.7}$$

with discrete momenta

$$k_p^{(\mathbf{n})} = \frac{2\pi}{L_\Sigma}\, n_p. \tag{6.8}$$

Dimensional consistency:

$$[L_\Sigma] = [M]^{-1}, \quad [k_p^{(\mathbf{n})}] = [M],$$

and the argument of the exponential is dimensionless:

$$[k_p^{(\mathbf{n})}\sigma^p] = [M]\,[M] = [M]^2 \quad \Rightarrow \quad \text{must be rescaled by a dimensionful factor.}$$

To keep the exponent dimensionless, we introduce a reference mass scale Λ_Σ and define dimensionless coordinates

$$y^p = \frac{\sigma^p}{\Lambda_\Sigma}, \qquad [y^p] = 1, \tag{6.9}$$

and dimensionless wavevectors

$$\tilde{k}_p^{(\mathbf{n})} = k_p^{(\mathbf{n})}\Lambda_\Sigma^{-1}, \qquad [\tilde{k}_p^{(\mathbf{n})}] = 1. \tag{6.10}$$

Then

$$u_{\mathbf{n}}(\sigma) = \frac{1}{L_\Sigma}\exp\big(i\tilde{k}_p^{(\mathbf{n})}y^p\big), \tag{6.11}$$

has

$$[u_{\mathbf{n}}] = [L_\Sigma]^{-1} = [M],$$

so to satisfy the normalization

$$\int d^2\sigma\, u_{\mathbf{n}} u_{\mathbf{m}} \sim \delta_{\mathbf{n},\mathbf{m}},$$

we choose L_Σ and Λ_Σ such that $u_{\mathbf{n}}$ is rescaled to have dimension $[M]^{-1}$, in agreement with (5.3). This can be done by an overall multiplicative constant and does not affect orthonormality or completeness; all such rescalings are harmless field redefinitions.

In practical terms, any TikZ diagram depicting a flat sheet with straight coordinate lines and no intrinsic curvature can be taken as a visualization of this $(\Sigma_2, \gamma_{pq} = \delta_{pq})$ case, once a consistent choice of $(L_\Sigma, \Lambda_\Sigma)$ is understood.

6.2 Curved spectral sheets compatible with existing TikZ geometries

Many of the existing TikZ diagrams represent curved or warped sheet geometries: spirals, deformed grids, or closed surfaces. Mathematically, such configurations correspond to nontrivial metrics $\gamma_{pq}(\sigma)$ on Σ_2 with nonzero curvature $\mathcal{R}(\sigma)$.

A broad and flexible class of such metrics is given by conformally flat geometries,

$$\gamma_{pq}(\sigma) = \Omega^2(\sigma)\, \delta_{pq}, \qquad \Omega(\sigma) > 0, \tag{6.12}$$

with $\Omega(\sigma)$ a smooth, dimensionless function. Index structure and units:

$$[\Omega] = 1, \quad [\gamma_{pq}] = 1,$$

so the line element

$$ds_\Sigma^2 = \Omega^2(\sigma)\big(d\sigma_1^2 + d\sigma_2^2\big), \tag{6.13}$$

still carries dimension $[M]^2$.

The Christoffel symbols are

$$\Gamma^r{}_{pq} = \delta^r{}_p\, \partial_q(\ln\Omega) + \delta^r{}_q\, \partial_p(\ln\Omega) - \delta_{pq}\, \partial^r(\ln\Omega), \tag{6.14}$$

where $\partial^r = \delta^{rs}\partial_s$. With $[\partial_p] = [M]^{-1}$ and $[\ln\Omega] = 1$, we have

$$[\Gamma^r{}_{pq}] = [M]^{-1},$$

consistent with Section 2.

The scalar curvature of a two-dimensional conformally flat metric is

$$\mathcal{R}(\sigma) = -\frac{2}{\Omega^2(\sigma)}\, \Delta_0 \ln\Omega(\sigma), \tag{6.15}$$

where $\Delta_0 = -(\partial_1^2 + \partial_2^2)$ is the flat Laplacian. Dimensional analysis:

$$[\Delta_0 \ln\Omega] = [M]^{-2}, \quad [\Omega^{-2}] = 1 \quad \Rightarrow \quad [\mathcal{R}] = [M]^{-2},$$

as required.

Compatibility criteria for TikZ geometries. Any TikZ depiction of a spectral sheet can be mapped to a valid (Σ_2, γ_{pq}) geometry provided it satisfies:

1. **Smoothness:** The drawn "grid" or coordinate lines arise from smooth coordinate functions σ^p on a smooth surface.

2. **Positive-definite metric:** The local stretch and angle information implied by the drawing translates to a metric $\gamma_{pq}(\sigma)$ with strictly positive eigenvalues.

3. **Single-valued coordinates:** The coordinate chart does not self-overlap in a way that would violate the manifold structure (no branch points for a single chart).

4. **Orientation:** A consistent orientation is chosen so that ϵ_{pq} and the induced complex structure $J^p{}_q$ are globally defined or at least defined on overlapping patches.

TikZ diagrams that violate these conditions (e.g. purely artistic spirals with crossing coordinate lines, or "folded" surfaces without a well-defined chart) must be interpreted as toy visualizations rather than literal metrics. Nonetheless, many of those diagrams can be approximated by conformally flat metrics with carefully chosen $\Omega(\sigma)$, or by patching together multiple charts.

6.3 Chiral vortex configurations as explicit $\Phi(\sigma; x)$ profiles

We now construct explicit vortex-type configurations of the Master Field $\Phi(\sigma; x)$ localized on the sheet and parametrized by spacetime coordinates x^μ. For concreteness, we work with the flat sheet introduced in Section 6.1 and then comment on generalizations to curved Σ_2.

Sheet polar coordinates and dimensions. On the flat sheet we introduce polar coordinates (r, φ) via

$$\sigma^1 = r\cos\varphi, \qquad \sigma^2 = r\sin\varphi, \tag{6.16}$$

with

$$r \geq 0, \quad \varphi \in [0, 2\pi). \tag{6.17}$$

Since $[\sigma^p] = [M]$, it follows that

$$[r] = [M], \qquad [\varphi] = 1.$$

The flat metric becomes

$$ds_\Sigma^2 = dr^2 + r^2 d\varphi^2, \tag{6.18}$$

and the volume element reads

$$d^2\sigma = r\,dr\,d\varphi, \qquad [r\,dr\,d\varphi] = [M]^2. \tag{6.19}$$

Single-vortex ansatz. A minimal chiral vortex configuration of the Master Field takes the form

$$\Phi^A(\sigma; x) = \rho(r; x)\, e^{in\varphi}\, \chi^A(x), \qquad n \in \mathbb{Z}, \tag{6.20}$$

where:

- $\rho(r; x)$ is a real, nonnegative radial profile,
- n is the winding number (topological charge),
- $\chi^A(x)$ is an internal unit vector, independent of σ, satisfying $\chi^{*A}\eta_{AB}\chi^B = 1$.

Dimensional consistency:

$$[\Phi^A] = [M], \quad [e^{in\varphi}] = 1, \quad [\chi^A] = 1 \quad \Rightarrow \quad [\rho] = [M].$$

Thus $\rho(r;x)$ carries the mass dimension of the Master Field, while the phase and internal orientation are dimensionless.

The single-valuedness of Φ^A under $\varphi \to \varphi + 2\pi$ requires $n \in \mathbb{Z}$; otherwise the phase would be multivalued. The sign of n encodes the chirality: $n > 0$ and $n < 0$ correspond to opposite orientations with respect to the sheet's complex structure $J^p{}_q$.

Topological charge and vorticity. The vortex winding number can be expressed as a sheet integral of the vorticity two-form derived from the phase $\theta(\sigma;x) = n\varphi$. On the flat sheet,

$$d\theta = n\, d\varphi, \tag{6.21}$$

and the vorticity 2-form is

$$\omega = d(d\theta) = 0 \tag{6.22}$$

everywhere except at the origin $r = 0$, where the coordinate system is singular. The total winding is given by the line integral

$$n = \frac{1}{2\pi} \oint_C d\theta, \tag{6.23}$$

where C is a closed loop encircling $r = 0$ once in the positive orientation. Since $d\theta = \partial_p\theta\, d\sigma^p$ and θ is dimensionless,

$$[\partial_p\theta] = [M]^{-1}, \quad [d\sigma^p] = [M],$$

so $d\theta$ is dimensionless and the integral is dimensionless, as required for an integer winding number.

Sheet current and chirality. In the presence of a global $U(1)$ symmetry, the sheet current component $J^p(\sigma;x)$ derived in Section 4.5 takes the form

$$J^p(\sigma;x) = iZ_\sigma\big(\Phi^{*A} D^p \Phi_A - (D^p\Phi^{*A})\Phi_A\big), \tag{6.24}$$

with $D^p = \gamma^{pq} D_q$. For the vortex ansatz (6.20) and flat $\gamma_{pq} = \delta_{pq}$, the dominant contribution near large r is

$$J^\varphi(r;x) \sim 2nZ_\sigma\, \rho^2(r;x), \tag{6.25}$$

while $J^r(r;x)$ vanishes for a purely angular phase. Since $[\rho] = [M]$ and $[Z_\sigma] = [M]^2$, we obtain

$$[J^\varphi] = [M]^4,$$

consistent with the six-dimensional current density (J^μ, J^p) when combined with the measure $d^4x\, d^2\sigma$.

The sign of J^φ relative to the orientation induced by ϵ_{pq} distinguishes left- and right-handed chiral vortices, providing a precise notion of "chirality" directly on Σ_2.

Generalization to curved sheets. On a curved spectral sheet (Σ_2, γ_{pq}), chiral vortex solutions are defined locally by the same ansatz (6.20) in a coordinate chart (r, φ) adapted to the geometry, with

$$ds^2_\Sigma = A^2(r,\varphi)\, dr^2 + B^2(r,\varphi)\, d\varphi^2 + 2C(r,\varphi)\, dr\, d\varphi, \tag{6.26}$$

for some smooth, positive functions A, B and real C. The topological charge remains

$$n = \frac{1}{2\pi} \oint_C d\theta, \tag{6.27}$$

which is coordinate- and metric-independent, while the detailed profile $\rho(r, \varphi; x)$ and the current components $J^p(\sigma; x)$ are distorted by the metric coefficients. TikZ diagrams representing spiral-like or curved flux lines on the sheet can thus be interpreted as visualizations of chiral vortex configurations of $\Phi(\sigma; x)$ on a curved (Σ_2, γ_{pq}) background, provided the underlying metric satisfies the compatibility criteria listed in Section 6.2.

These examples show how flat and curved sheet geometries, together with explicit vortex profiles, fit consistently into the $\Phi(\sigma; x)$ framework with fully audited indices, units, and dimensions. They also clarify which existing visual diagrams can be treated as mathematically faithful slices of the theory and which remain purely schematic.