Extending unified gravity to account for graviton–graviton interaction

Mikko Partanen[1] and Jukka Tulkki[2]

[1] *Photonics Group, Department of Electronics and Nanoengineering, Aalto University, P.O. Box 13500, 00076 Aalto, Finland*

[2] *Engineered Nanosystems Group, School of Science, Aalto University, P.O. Box 12200, 00076 Aalto, Finland*

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Recently, a gauge theory of unified gravity [Rep. Prog. Phys. **88**, 057802 (2025)] has been developed to extend the Standard Model to include gravity. Here we present unified gravity using the ordinary four-vector and tensor field notation of the Standard Model. The main goal of the present work is to extend the original Minkowski spacetime formulation of the theory to account for graviton–graviton interaction. This is a necessary extension for problems involving interactions between gravitational fields, for example, in the propagation of gravitational waves in external gravitational potentials. The $4 \times \mathrm{U}(1)$ gauge invariance of unified gravity is preserved in this extension.

I. INTRODUCTION

A recently introduced quantum field theory, unified gravity (UG) [1], describes gravity by the $4 \times \mathrm{U}(1)$ tensor gauge field. This gauge field is distinct from the metric and appears as an extension of the Standard Model. In the semiclassical limit, it enables dynamical description of the same phenomena, which are calculated through the metric in general relativity (GR) [2, 3]. On the relation between UG and GR, we point out that teleparallel equivalent of GR (TEGR) [4–6] results from one particular geometric condition of UG [1]. However, this geometric condition breaks the $4 \times \mathrm{U}(1)$ gauge symmetry of UG. Therefore, the pertinent geometric condition makes TEGR fundamentally different from the Minkowski spacetime formulation of UG, which is used in the present work.

Instead of the full Standard Model and gravity, here we study the system of the Dirac electron–positron field, the electromagnetic field, and the gravitational field. The original formulation of UG utilized the so-called eight-spinor formalism [1]. In this work, we present UG in the ordinary four-vector and tensor field notation of the Standard Model. Note that in Ref. [1], already many key equations were presented both in the eight-spinor representation and in the standard field representation. These representations are mathematically equivalent and have no influence on the physical interpretation of UG. We also extend the original version of the theory to account for graviton–graviton interaction. This is found to be a necessary extension for problems involving interactions between gravitational fields.

The benchmark effects, the gravitational lensing, the perihelion precession of planetary orbits, and gravitational redshift, are investigated using UG in preprints [7–9]. Accounting for the graviton–graviton interaction, studied in the present work, does not influence the weak-field limit of these effects.

II. THEORETICAL CONCEPTS AND CONVENTIONS

A. Coordinates and index conventions

In this work, UG refers to the Minkowski spacetime formulation of the theory [1]. In UG, one can use arbitrary coordinates of the global Minkowski spacetime. For simplicity, in this work, we assume Cartesian coordinates $x^\nu = (ct, x, y, z)$, where c is the speed of light in vacuum and in zero gravitational potential. The Latin indices of UG are associated with the Cartesian coordinates x^a. Therefore, the choice of using Cartesian coordinates x^ν corresponds to a trivial tetrad, given by the Kronecker delta δ_a^μ.

The components of the diagonal Minkowski metric tensor $\eta_{\mu\nu}$, in the assumed Cartesian coordinates, are given by $\eta_{00} = 1$ and $\eta_{xx} = \eta_{yy} = \eta_{zz} = -1$. In this work, the Einstein summation convention is used for all repeated Greek indices. Below, the Latin indices are, however, not implicitly summed over repeated indices.

B. Spacetime dimension field

The original formulation of UG in Ref. [1] was made using the so-called eight-spinor formalism. This formalism is not necessary and not used in the present work. In the present work, the quantity, called the spacetime dimension field, is defined as

$$I_{\mathrm{g}}^a = \frac{1}{\sqrt{g_{\mathrm{g}}}} e^{-i g_{\mathrm{g}} x_a}. \tag{1}$$

Here g_{g} is called the scale constant of UG and $x_a = (ct, -x, -y, -z)$. In the spacetime dimension field, the Latin indices represent Cartesian spacetime indices also when the theory is written in non-Cartesian coordinates, such as spherical coordinates.

C. Equivalence principle of unified gravity

The Lagrangian density of UG is formulated using the gravitational mass m'_{e}, the inertial mass m_{e}, the scale constant g_{g}, and the coupling constant g'_{g}. The equivalence principles of UG between these quantities are given by [1]

$$m'_{\mathrm{e}} = m_{\mathrm{e}}, \qquad g'_{\mathrm{g}} = g_{\mathrm{g}}. \tag{2}$$

The first relation in Eq. (2) is called the equivalence principle of mass, and the second relation is called the equivalence principle of scale.

III. LAGRANGIAN DENSITY

A. Generating Lagrangian density of gravity

The generating Lagrangian density of gravity is the Lagrangian density of the theory at zero gravity gauge field. It is then equal to the Lagrangian density of QED. This Lagrangian density equals Eq. (35) of Ref. [1], written without the eight-spinor notation as

$$\begin{aligned}&\mathcal{L}|_{H=0}\\ &= -i\sum_a T_{\mathrm{m}}^{a\nu} I_{\mathrm{g}}^{a*}\partial_\nu I_{\mathrm{g}}^a + (2m'_{\mathrm{e}} - m_{\mathrm{e}})c^2\bar{\psi}\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{3}$$

Here $T_{\mathrm{m}}^{a\nu}$ is the stress-energy-momentum (SEM) tensor of the Dirac and electromagnetic fields, m_{e} is the inertial mass of the electron, m'_{e} is the gravitational mass of the electron, ψ is the Dirac spinor, $\bar{\psi}$ is the Dirac adjoint, $F_{\mu\nu}$ is the electromagnetic field tensor, and μ_0 is the permeability of vacuum.

The electromagnetic field-strength tensor is given in terms of the electromagnetic four-potential A^μ by the conventional expression as [10, 11]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4}$$

The SEM tensors of the Dirac field and the electromagnetic gauge field are given by [1, 12]

$$\begin{aligned}
T_{\mathrm{m}}^{\mu\nu} &= T_{\mathrm{D}}^{\mu\nu} + T_{\mathrm{em}}^{\mu\nu},\\
T_{\mathrm{D}}^{\mu\nu} &= \frac{c}{2}P^{\mu\nu,\rho\sigma}[i\hbar\bar{\psi}(\boldsymbol{\gamma}_\rho\vec{D}_\sigma - \overleftarrow{D}_\rho\boldsymbol{\gamma}_\sigma)\psi - m'_{\mathrm{e}}c\eta_{\rho\sigma}\bar{\psi}\psi],\\
T_{\mathrm{em}}^{\mu\nu} &= \frac{1}{\mu_0}\Big(F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big)\\
&= \frac{1}{2\mu_0}P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_\rho A_\sigma\partial_\eta A_\lambda.
\end{aligned} \tag{5}$$

Here $\hbar$ is the reduced Planck constant and $\boldsymbol{\gamma}^\mu$ are the conventional 4×4 Dirac gamma matrices. The right and left electromagnetic gauge-covariant derivatives are given by [13–15]

$$\vec{D}_\nu = \vec{\partial}_\nu + i\frac{q_{\mathrm{e}}}{\hbar}A_\nu, \qquad \overleftarrow{D}_\nu = \overleftarrow{\partial}_\nu - i\frac{q_{\mathrm{e}}}{\hbar}A_\nu. \tag{6}$$

Here q_{e} is the electric charge. The constant coefficients $P^{\mu\nu,\rho\sigma}$ and $P^{\mu\nu,\rho\sigma,\eta\lambda}$ in Eq. (5) are given by [1, 7]

$$P^{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta^{\mu\sigma}\eta^{\rho\nu} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}), \tag{7}$$

$$\begin{aligned}
P^{\mu\nu,\rho\sigma,\eta\lambda} &= \eta^{\eta\sigma}\eta^{\lambda\mu}\eta^{\nu\rho} - \eta^{\eta\mu}\eta^{\lambda\sigma}\eta^{\nu\rho} - \eta^{\eta\rho}\eta^{\lambda\mu}\eta^{\nu\sigma}\\
&+ \eta^{\eta\mu}\eta^{\lambda\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\lambda}\eta^{\rho\eta} + \eta^{\mu\sigma}\eta^{\nu\eta}\eta^{\rho\lambda} + \eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\eta}\\
&- \eta^{\mu\rho}\eta^{\nu\eta}\eta^{\sigma\lambda} - \eta^{\mu\nu}\eta^{\eta\sigma}\eta^{\lambda\rho} + \eta^{\mu\nu}\eta^{\eta\rho}\eta^{\lambda\sigma}.
\end{aligned} \tag{8}$$

B. Four U(1) gauge symmetries of gravity and the conservation law of the SEM tensor

It is straightforward to observe that the generating Lagrangian density of gravity in Eq. (3) satisfies the following four U(1) symmetries globally:

$$I_{\mathrm{g}}^a \to U_a I_{\mathrm{g}}^a, \qquad U_a = e^{i\phi_a}. \tag{9}$$

Here ϕ_a are the symmetry transformation parameters, which are constant for a global symmetry.

The infinitesimal variations of the components of the spacetime dimension field with respect to the symmetry transformation parameters ϕ_a are given by

$$\delta I_{\mathrm{g}}^a = iI_{\mathrm{g}}^a\delta\phi_a. \tag{10}$$

When the generating Lagrangian density of gravity in Eq. (3) is varied with respect to all ϕ_a and these variations are summed over, we obtain

$$\delta\mathcal{L}|_{H=0} = \frac{1}{g_{\mathrm{g}}}T_{\mathrm{m}}^{\mu\nu}\partial_\nu\delta\phi_\mu \tag{11}$$

This relation is the origin for our definition of the spacetime dimension field and the corresponding representation of the generating Lagrangian density of gravity in Eq. (3). The variation is analogous to the variation of the Lagrangian density of QED at zero electromagnetic four-potential, given by $\delta\mathcal{L}_{\mathrm{QED}}|_{A=0} = -\frac{\hbar}{e}J_{\mathrm{e}}^\nu\partial_\nu\delta\theta$, where θ is the U(1) gauge transformation parameters of QED and $J_{\mathrm{e}}^\nu = q_{\mathrm{e}}c\bar{\psi}\boldsymbol{\gamma}^\nu\psi$ is the electric four-current density.

The variation of the action integral with respect to the four U(1) gauge transformation parameters of gravity at zero gravity gauge field becomes

$$\begin{aligned}
\delta S|_{H=0} &= \int \delta\mathcal{L}|_{H=0}d^4x\\
&= \int \frac{1}{g_{\mathrm{g}}}T_{\mathrm{m}}^{\mu\nu}\partial_\nu\delta\phi_\mu d^4x\\
&= -\int \frac{1}{g_{\mathrm{g}}}(\partial_\nu T_{\mathrm{m}}^{\mu\nu})\delta\phi_\mu d^4x
\end{aligned} \tag{12}$$

In the second equality, we have used Eq. (11). In the third equality, we have applied partial differentiation and set the total divergence term to zero by assuming that the fields in $T_{\mathrm{m}}^{\mu\nu}$ vanish at distant boundary.

The last form of equation (12) shows that the variation of the action integral vanishes for arbitrary $\delta\phi_\mu$ when

$$\partial_\nu T_{\mathrm{m}}^{\mu\nu} = 0. \tag{13}$$

This is the well-known conservation law of the SEM tensor in Cartesian coordinates at zero gravitational field.

C. Gauge-covariant derivative

The generating Lagrangian density of gravity in equation (3) is globally gauge invariant in the symmetry transformation of Eq. (9). In the global symmetry transformation, the values of ϕ_a are constant. To promote the global symmetry to a local symmetry, we allow ϕ_a to depend on the spacetime coordinates x^μ. Following conventional gauge theory [14, 16], the generating Lagrangian density of gravity in Eq. (3) can be made locally gauge invariant in the symmetry transformation of equation (9) when we generalize the partial derivative that acts on I_{g}^a into a gauge-covariant derivative $\mathcal{D}_\nu$, defined as

$$\mathcal{D}_\nu I_{\mathrm{g}}^a = (\partial_\nu - i g_{\mathrm{g}}' H_{a\nu}) I_{\mathrm{g}}^a. \tag{14}$$

Here g_{g}' is the coupling constant of UG and $H_{a\nu}$ is the gravity gauge field [1]. The gauge transformation of $H_{a\nu}$ is given by

$$H_{a\nu} \to H_{a\nu} + \frac{1}{g_{\mathrm{g}}'}\partial_\nu \phi_a. \tag{15}$$

This transformation of $H_{a\nu}$ makes $\mathcal{D}_\nu I_{\mathrm{g}}^a$ gauge invariant when I_{g}^a is transformed according to Eq. (9). Substituting $\mathcal{D}_\nu I_{\mathrm{g}}^a$ in Eq. (14) in place of $\partial_\nu I_{\mathrm{g}}^a$ in the generating Lagrangian density of gravity in Eq. (3) makes the Lagrangian density locally gauge invariant with respect to the gauge transformations in Eqs. (9) and (15).

D. Gravity gauge field strength tensor

To construct the full gauge-invariant Lagrangian density, we need to add a term that involves only the gauge field $H_{a\nu}$, and this term must itself be gauge invariant. Gauge theory provides a well-defined way to do this by utilizing the commutator of the gauge-covariant derivatives [14, 16]. Then, we obtain a unique expression for the antisymmetric gravity gauge-field-strength tensor $H_{a\mu\nu}$ as

$$\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu] I_{\mathrm{g}}^a &= -i g_{\mathrm{g}}' H_{a\mu\nu} I_{\mathrm{g}}^a, \\
H_{a\mu\nu} &= \partial_\mu H_{a\nu} - \partial_\nu H_{a\mu}, \\
H_{\rho\mu\nu} &= \delta_\rho^a H_{a\mu\nu}.
\end{aligned} \tag{16}$$

In the last form of Eq. (16), we use the trivial tetrad related to our choice of using the Cartesian coordinates. Since UG is an Abelian gauge theory, the gauge-field-strength tensor $H_{a\mu\nu}$ is invariant in the gauge transformation, given in Eq. (15).

E. Lagrangian density of the gravity gauge field

In analogy with the gauge theories in the Standard Model [14], the Lagrangian density for the gravity gauge field strength is not uniquely fixed by gauge invariance alone. In the Standard Model, the form of the gauge field Lagrangian is further constrained by requirements such as parity and time-reversal symmetries, as well as renormalizability [14]. As detailed in Ref. [1], the Lagrangian density of the gravity gauge field strength is given by

$$\mathcal{L}_{\mathrm{g,kin}} = \frac{1}{4\kappa} H_{\rho\mu\nu} S^{\rho\mu\nu}. \tag{17}$$

Here $S^{\rho\mu\nu}$ is the superpotential and $\kappa = 8\pi G/c^4$ is Einstein's constant, in which G is the gravitational constant. The prefactor of equation (17) has been determined by comparison of the weak field limit of UG with Newton's law of gravitation [1]. The superpotential is given by

$$S^{\rho\mu\nu} = \frac{1}{2}(H^{\nu\mu\rho} + H^{\mu\rho\nu} - H^{\rho\nu\mu}) + \eta^{\rho\mu} H^{\sigma\nu}{}_{\sigma} - \eta^{\rho\nu} H^{\sigma\mu}{}_{\sigma}. \tag{18}$$

F. Accounting for graviton–graviton interaction

Above, we have followed the original formulation of UG in Ref. [1], but presented it in the standard field formulation. Next, we present an extension that accounts for the graviton–graviton interaction. The SEM tensor of the gravity gauge field did not appear as a source term of the gravitational field in the Minkowski spacetime formulation in Ref. [1]. The gauge theory gives us the freedom to introduce an additional gauge-invariant Lagrangian density term, which depends on the spacetime dimension field and the gravity gauge field strength, given by

$$\mathcal{L}_{\mathrm{gg,int}} = -i \sum_a T_{\mathrm{g}}^{a\nu} I_{\mathrm{g}}^{a*} \mathcal{D}_\nu I_{\mathrm{g}}^a. \tag{19}$$

As will be explained in detail below, adding this term enables us to establish a more general conservation law of the SEM tensor in comparison with the conservation law at zero gravitational field in Eq. (13). In Eq. (19), $T_{\mathrm{g}}^{a\nu}$ is the gauge-invariant SEM tensor of the gravity gauge field, written as

$$\begin{aligned}
T_{\mathrm{g}}^{\mu\nu} &= \frac{1}{\kappa}\Big(H_{\rho\sigma}{}^{\mu} S^{\rho\sigma\nu} - \frac{1}{4}\eta^{\mu\nu} H_{\rho\sigma\lambda} S^{\rho\sigma\lambda}\Big) \\
&= \frac{1}{2\kappa} P^{\mu\nu,\rho\sigma\lambda,\alpha\beta\gamma} \partial_\rho H_{\sigma\lambda} \partial_\alpha H_{\beta\gamma}.
\end{aligned} \tag{20}$$

This SEM tensor is formed from the second-order terms of the derivatives of the gravity gauge field. It is traceless since gravitons are massless. The constant coefficients

$P^{\mu\nu,\rho\sigma\lambda,\alpha\beta\gamma}$ in Eq. (20) are given by

$$\begin{aligned}P^{\mu\nu,\sigma\rho\lambda,\beta\alpha\gamma} &= C^{\lambda\mu,\rho\sigma\nu,\alpha\beta\gamma} - C^{\lambda\mu,\rho\sigma\nu,\alpha\gamma\beta}\\ &\quad - C^{\sigma\mu,\rho\lambda\nu,\alpha\beta\gamma} + C^{\sigma\mu,\rho\lambda\nu,\alpha\gamma\beta},\\ C^{\lambda\mu,\rho\sigma\nu,\alpha\beta\gamma} &= \eta^{\lambda\mu}D^{\rho\sigma\nu,\alpha\beta\gamma} - \frac{1}{4}\eta^{\mu\nu}D^{\rho\sigma\lambda,\alpha\beta\gamma},\\ D^{\rho\mu\nu,\alpha\beta\gamma} &= \eta^{\nu\alpha}\eta^{\mu\beta}\eta^{\rho\gamma} + \eta^{\mu\alpha}\eta^{\rho\beta}\eta^{\nu\gamma} - \eta^{\rho\alpha}\eta^{\nu\beta}\eta^{\mu\gamma}\\ &\quad + 2\eta^{\rho\mu}\eta^{\alpha\gamma}\eta^{\nu\beta} - 2\eta^{\rho\nu}\eta^{\alpha\gamma}\eta^{\mu\beta}.\end{aligned} \tag{21}$$

The main motivation for the definition of $\mathcal{L}_{\mathrm{gg,int}}$ in Eq. (19) is that it leads to the total SEM tensor $T^{\mu\nu}$ of the Dirac electron–positron field, the electromagnetic gauge field, and the gravity gauge field, given by the sum of the SEM tensors in Eqs. (5) and (20) as

$$T^{\mu\nu} = T_{\mathrm{m}}^{\mu\nu} + T_{\mathrm{g}}^{\mu\nu}. \tag{22}$$

In analogy with Eq. (11), we can calculate the variation of the gauge-invariant Lagrangian density, to be given below, with respect to the gauge symmetry transformation parameters in a nonzero gravity gauge field. Applying the gauge transformation to the spacetime dimension field and preserving the gravity gauge field fixed in the variation, we obtain

$$\delta\mathcal{L}|_H = \frac{1}{g_{\mathrm{g}}}T^{\mu\nu}\partial_\nu\delta\phi_\mu. \tag{23}$$

The variation in Eq. (23) depends on the gravity gauge field through $T_{\mathrm{g}}^{\mu\nu}$, which is part of $T^{\mu\nu}$. This is associated with the energy content of the gravitational field itself. In contrast, the variation of the Lagrangian density of QED at constant electromagnetic four-potential, given by $\delta\mathcal{L}_{\mathrm{QED}}|_A = -\frac{\hbar}{e}J_{\mathrm{e}}^\nu\partial_\nu\delta\theta$, does not depend on the value of the four-potential. This is because the four-potential does not have electric charge so that it could contribute to the electric four-current density.

The variation of the action integral corresponding to the variation in Eq. (23) becomes, in analogy with Eq. (12), zero when the total SEM tensor in Eq. (22) satisfies the conservation law, given by

$$\partial_\nu T^{\mu\nu} = 0. \tag{24}$$

This is the generalization of the conservation law in Eq. (13) for nonzero gravitational fields.

G. Gauge-invariant Lagrangian density

Using the gauge-covariant derivative in Eq. (14) and adding the terms associated with the gravity gauge field in Eqs. (17) and (19) to the generating Lagrangian density of gravity in Eq. (3), we obtain the locally gauge-invariant Lagrangian density. This Lagrangian density satisfies locally the electromagnetic [U(1)] gauge-invariance and the gravity [4×U(1)] gauge-invariance, and it is given by

$$\begin{aligned}\mathcal{L} &= -i\sum_a T^{a\nu}I_{\mathrm{g}}^{a*}\mathcal{D}_\nu I_{\mathrm{g}}^a + (2m_{\mathrm{e}}' - m_{\mathrm{e}})c^2\bar{\psi}\psi\\ &\quad - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu}.\end{aligned} \tag{25}$$

The Lagrangian density in Eq. (25) now includes the effects of the graviton–graviton interaction through $\mathcal{L}_{\mathrm{gg,int}}$, which was not accounted for in Ref. [1]. This term makes the total SEM tensor $T^{a\nu}$ to appear in the gauge-invariant Lagrangian density in Eq. (25).

H. Faddeev–Popov gauge-fixed Lagrangian density

It is well known that the formulation of a gauge field theory requires fixing the gauge [14, 15]. This necessity arises because gauge theories describe physical configurations by equivalence classes defined under gauge transformations. These classes reflect the presence of redundant degrees of freedom in the gauge fields. Therefore, one must eliminate this redundancy through a process called gauge fixing. In UG, we follow the well-known Faddeev–Popov method [14, 15, 17]. The Faddeev–Popov gauge-fixed Lagrangian density of UG is given by [1]

$$\mathcal{L}_{\mathrm{FP}} = \mathcal{L} + \mathcal{L}_{\mathrm{em,gf}} + \mathcal{L}_{\mathrm{em,ghost}} + \mathcal{L}_{\mathrm{g,gf}} + \mathcal{L}_{\mathrm{g,ghost}}. \tag{26}$$

Here the electromagnetic and gravity gauge-fixing and ghost Lagrangian densities are given by [1]

$$\mathcal{L}_{\mathrm{em,gf}} = -\frac{1}{2\mu_0\xi_{\mathrm{e}}}[C_{\mathrm{em}}(A)]^2 = -\frac{1}{2\mu_0\xi_{\mathrm{e}}}(\partial_\nu A^\nu)^2, \tag{27}$$

$$\mathcal{L}_{\mathrm{em,ghost}} = \hbar c\bar{c}_{\mathrm{em}}\partial^2 c_{\mathrm{em}}, \tag{28}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{g,gf}} &= \frac{1}{4\kappa\xi_{\mathrm{g}}}C_{\mathrm{g}}^\mu(H)C_{\mathrm{g}\mu}(H)\\ &= \frac{1}{\kappa\xi_{\mathrm{g}}}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_\lambda H_{\alpha\beta}\partial_\eta H_{\rho\sigma},\end{aligned} \tag{29}$$

$$\mathcal{L}_{\mathrm{g,ghost}} = -\hbar c\bar{c}_{\mathrm{g}}\partial^2 c_{\mathrm{g}}. \tag{30}$$

Here ξ_{e} and ξ_{g} are the electromagnetic and gravity gauge-fixing parameters, respectively.

I. BRST invariance

The Faddeev–Popov Lagrangian density of UG in Eq. (26) exhibits an exact global symmetry known as BRST invariance in analogy with the gauge theories of the Standard Model [14, 15, 18–20]. In the BRST formalism, the local U(1) gauge parameter of QED is replaced by $\theta = \theta' c_{\mathrm{em}}$, where θ' is a constant anticommuting Grassmann number with $\theta'^2 = 0$. Similarly, the four gravity U(1) gauge parameters are replaced by $\phi_\mu = \phi' c_{g\mu}$, where ϕ' is a constant Grasmann number with $\phi'^2 = 0$. Under these substitutions, the Lagrangian density in Eq. (26) remains invariant under the BRST transformations for electromagnetism, given by [1, 15]

$$\begin{aligned}
\psi &\to e^{i\theta' c_{\mathrm{em}} Q}\psi, \\
A_\nu &\to A_\nu - \frac{\hbar}{e}\theta'\partial_\nu c_{\mathrm{em}}, \\
\bar{c}_{\mathrm{em}} &\to \bar{c}_{\mathrm{em}} - \frac{1}{\mu_0 c e \xi_{\mathrm{e}}}\theta' C_{\mathrm{em}}(A), \\
c_{\mathrm{em}} &\to c_{\mathrm{em}},
\end{aligned} \tag{31}$$

and under the BRST transformations associated with gravity, given by [1]

$$\begin{aligned}
I_{\mathrm{g}}^a &\to e^{i\phi' c_{\mathrm{g}a}} I_{\mathrm{g}}^a, \\
H_{a\nu} &\to H_{a\nu} + \frac{1}{g_{\mathrm{g}}'}\phi'\partial_\nu c_{\mathrm{g}a}, \\
\bar{c}_{\mathrm{g}}^a &\to \bar{c}_{\mathrm{g}}^a - \frac{1}{\kappa\hbar c g_{\mathrm{g}}' \xi_{\mathrm{g}}}\phi' C_{\mathrm{g}}^a(H), \\
c_{\mathrm{g}}^a &\to c_{\mathrm{g}}^a.
\end{aligned} \tag{32}$$

BRST symmetry is known to hold at all loop orders in the path integral formalism [15]. Its presence in UG strongly indicates that the theory is renormalizable, like the gauge theories of the Standard Model. This is further supported by the successful one-loop renormalization of UG in Ref. [1]. Unlike conventional gravity theories, where gauge generators are field-dependent, UG features constant generators, allowing BRST symmetry to apply directly, without requiring the more general Batalin–Vilkovisky formalism [16, 21–25].

J. Reduced form of the Lagrangian density

Next, we present the reduced form of the Lagrangian density of UG by writing the spacetime dimension field explicitly using Eq. (1). Furthermore, in this Lagrangian density, we drop out the ghost-field terms of the Lagrangian density, which do not participate in the dynamics of fields in Abelian gauge theories, such as UG. By applying the expression of the spacetime dimension field in Eq. (1), we obtain an identity

$$I_{\mathrm{g}}^{a*}\mathcal{D}_\nu I_{\mathrm{g}}^a = -i\delta_a^\mu\Big(\eta_{\mu\nu} + \frac{g_{\mathrm{g}}'}{g_{\mathrm{g}}}H_{\mu\nu}\Big). \tag{33}$$

Using the identity in Eq. (33) in the Faddeev–Popov Lagrangian density in Eq. (26) and dropping out the ghost-field terms, the reduced form of the Lagrangian density of UG becomes

$$\begin{aligned}
\mathcal{L}_{\mathrm{UG}} = {} & \frac{i\hbar c}{2}\bar{\psi}(\boldsymbol{\gamma}^\nu\vec{\partial}_\nu - \overleftarrow{\partial}_\nu\boldsymbol{\gamma}^\nu)\psi - m_{\mathrm{e}}c^2\bar{\psi}\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \\
& + \frac{1}{4\kappa}H_{\rho\mu\nu}S^{\rho\mu\nu} - J_{\mathrm{e}}^\nu A_\nu - T^{\mu\nu}H_{\mu\nu} - \frac{1}{2\mu_0\xi_{\mathrm{e}}}(\partial_\nu A^\nu)^2 \\
& + \frac{1}{\kappa\xi_{\mathrm{g}}}\eta_{\gamma\delta}P^{\alpha\beta,\lambda\gamma}P^{\rho\sigma,\eta\delta}\partial_\lambda H_{\alpha\beta}\partial_\eta H_{\rho\sigma}.
\end{aligned} \tag{34}$$

IV. DYNAMICAL EQUATIONS

Dynamical equations of all fields in the Lagrangian density can be straightforwardly derived through the Euler–Lagrange equations. Here we use the reduced form of the Lagrangian density of UG in Eq. (34). In the derivation of the dynamical equations, we also apply the equivalence principles of UG, given in Eq. (2).

1. Dynamical equation of the gravitational field

Using the Euler–Lagrange equations, written for $H_{\mu\nu}$, the dynamical equation of the gravity gauge field, in the harmonic gauge with $\xi_{\mathrm{g}} = 1$, becomes

$$P^{\mu\nu,\rho\sigma}\partial^2 H_{\rho\sigma} - P^{\sigma\lambda,\rho\mu\nu,\alpha\beta\gamma}\partial_\rho(H_{\sigma\lambda}\partial_\alpha H_{\beta\gamma}) = -\kappa T^{\mu\nu}. \tag{35}$$

This equation is different from the dynamical equation of gravity obtained in Ref. [1] by the nonlinear terms, i.e., the second term on the left and the SEM tensor of gravity, $T_{\mathrm{g}}^{\mu\nu}$, which is part of the total SEM tensor $T^{\mu\nu}$ on the right. Both these terms follow from the Lagrangian density term $\mathcal{L}_{\mathrm{gg,int}}$ in Eq. (19), which was not included in Ref. [1].

2. Dynamical equation of the electromagnetic field

Using the Euler–Lagrange equation, written for A_σ, we obtain the dynamical equation of the electromagnetic four-potential, in the Feynman gauge with $\xi_{\mathrm{e}} = 1$, given by [1, 12]

$$\partial^2 A^\sigma + P^{\mu\nu,\rho\sigma,\eta\lambda}\partial_\rho(H_{\mu\nu}\partial_\eta A_\lambda) = \mu_0 J_{\mathrm{e,tot}}^\sigma. \tag{36}$$

Here $J_{\mathrm{e,tot}}^\mu$ is the total electric four-current density in the presence of the gravity gauge field, given by

$$J_{\mathrm{e,tot}}^\mu = J_{\mathrm{e}}^\rho - P^{\mu\nu,\rho\sigma}J_{\mathrm{e}\sigma}H_{\mu\nu}. \tag{37}$$

In the presence of gravitational interaction, the total electric four-current density $J_{\mathrm{e,tot}}^\mu$ in Eq. (37) satisfies the conservation law, given by $\partial_\nu J_{\mathrm{e,tot}}^\nu = 0$. The conservation law does not hold for $J_{\mathrm{e}}^\nu = q_{\mathrm{e}}c\bar{\psi}\boldsymbol{\gamma}^\nu\psi$. This is discussed in Ref. [9].

3. Dynamical equation of the Dirac field

Using the Euler–Lagrange equation, written for $\bar{\psi}$, the dynamical equation of the Dirac field becomes [1]

$$i\hbar c\gamma^{\rho}\vec{\partial}_{\rho}\psi - m_{\mathrm{e}}c^{2}\psi = q_{\mathrm{e}}c\gamma^{\rho}\psi A_{\rho} + P^{\mu\nu,\rho\sigma}\Big(i\hbar c\gamma_{\sigma}\vec{\partial}_{\rho}\psi - \frac{m_{\mathrm{e}}c^{2}}{2}\eta_{\rho\sigma}\psi + \frac{i\hbar c}{2}\gamma_{\sigma}\psi\vec{\partial}_{\rho} - q_{\mathrm{e}}c\gamma_{\sigma}\psi A_{\rho}\Big)H_{\mu\nu}. \quad (38)$$

V. OBTAINING TEGR FROM UG

The starting point for obtaining TEGR from UG in Ref. [1] is the gauge-invariant Lagrangian density. In the present work, the gauge-invariant Lagrangian density is given in Eq. (25). We also assume the equivalence principles of UG, given in Eq. (2). The formulation of TEGR is considered in the Weitzenböck gauge, where the teleparallel spin connection vanishes [4, 5].

Since UG is formulated in the global Minkowski spacetime, its gravity gauge field has no relation to the metric or tetrad. Therefore, there is no equivalence transformation from UG to TEGR, where the fundamental field is the spacetime-dependent tetrad field, denoted by $\overset{\bullet}{e}{}^{a}{}_{\mu}$. However, it is found that the gauge-invariant Lagrangian density of UG in Eq. (25) reproduces the Lagrangian density of TEGR, if the following substitutions are made:

$$I_{\mathrm{g}}^{a*}\mathcal{D}_{\nu}I_{\mathrm{g}}^{a} \longrightarrow -i\overset{\bullet}{e}_{a\nu}, \quad (39)$$

$$\delta_{\mu}^{a} \longrightarrow \overset{\bullet}{e}{}^{a}{}_{\mu}, \quad (40)$$

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu} = \eta_{ab}\,\overset{\bullet}{e}{}^{a}{}_{\mu}\overset{\bullet}{e}{}^{b}{}_{\nu}, \quad (41)$$

$$d^{4}x \longrightarrow \sqrt{-\det(g_{\mu\nu})}d^{4}x. \quad (42)$$

In TEGR, the Latin indices are raised and lowered by the Minkowski metric, and the Greek indices are raised and lowered by the spacetime-dependent metric in Eq. (41). This fundamentally differs from UG, which uses the spacetime-independent tetrad, which is trivially δ_{μ}^{a} when the Cartesian spacetime coordinates are assumed.

The extension term of the Lagrangian density to account for the graviton–graviton interaction, given in Eq. (19), does not contribute to the Lagrangian density of TEGR, obtained in Ref. [1]. For this term, using Eq. (39), we obtain

$$\mathcal{L}_{\mathrm{gg,int}} = -i\sum_{a}T_{\mathrm{g}}^{a\nu}I_{\mathrm{g}}^{a*}\mathcal{D}_{\nu}I_{\mathrm{g}}^{a} \longrightarrow -T_{\mathrm{g}}^{a\nu}\overset{\bullet}{e}_{a\nu} = -T_{\mathrm{g}}{}^{\nu}{}_{\nu} = 0. \quad (43)$$

The last equality of Eq. (43) gives zero, since the SEM tensor of the gravity gauge field in Eq. (20) is traceless. Therefore, the approach of obtaining TEGR from UG works the same way as studied in Ref. [1]. We point out that the relations in Eqs. (39)–(42) break the $4 \times \mathrm{U}(1)$ gauge symmetry of UG, and TEGR is fundamentally different from the Minkowski spacetime formulation of UG.

VI. CONCLUSION

We have presented UG using a four-vector and tensor formalism. We have also introduced an extension of the original Minkowski spacetime formulation of UG to account for graviton–graviton interaction. This extension was obtained by adding the gauge-invariant SEM tensor of the gravity gauge field to the SEM tensors of the other fields in the Lagrangian density of UG. Therefore, the $4 \times \mathrm{U}(1)$ gauge invariance of UG is preserved. The gauge-fixed Lagrangian density of UG also satisfies the global BRST invariance. The extension of UG presented in this work is necessary for correct description of gravitational interaction in problems involving interactions between gravitational fields, for example, in the propagation of gravitational waves in external gravitational potentials. Regarding the Feynman diagrams of UG, our extension introduces the triple-graviton vertex, whose implications to the renormalization of UG are left as a topic of further work. We have also discussed the relation between UG and TEGR.

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[1] M. Partanen and J. Tulkki, "Gravity generated by four one-dimensional unitary gauge symmetries and the Standard Model," *Rep. Prog. Phys.* **88**, 057802 (2025).

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Freeman, New York (1973).

[3] T. A. Moore, *A General Relativity Workbook*, University Science Books, Mill Walley, CA (2013).

[4] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. L. Said, J. Mifsud, and E. D. Valentino, "Teleparallel

gravity: from theory to cosmology," *Rep. Prog. Phys.* **86**, 026901 (2023).

[5] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction*, Springer, Dordrecht (2012).

[6] J. W. Maluf, "The teleparallel equivalent of general relativity," *Ann. Phys.* **525**, 339 (2013).

[7] M. Partanen and J. Tulkki, "Light deflection in unified gravity and measurable deviation from general relativity," arXiv:2505.14446 (2025).

[8] M. Partanen and J. Tulkki, "Perihelion precession of planetary orbits solved from quantum field theory," arXiv:2506.14447 (2025).

[9] M. Partanen and J. Tulkki, "Atomic Dirac energy-level dynamics and redshift in the $4\times\mathrm{U}(1)$ gravity gauge field," arXiv:2506.22057 (2025).

[10] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon, Oxford (1989).

[11] J. D. Jackson, *Classical Electrodynamics*, Wiley, New York (1999).

[12] M. Partanen and J. Tulkki, "Corrigendum: Gravity generated by four one-dimensional unitary gauge symmetries and the Standard Model (2025 Rep. Prog. Phys. 88 057802)," *Rep. Prog. Phys.* **88**, 069501 (2025).

[13] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, Pergamon, Oxford (1982).

[14] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, CRC Press, Boca Raton, FL (2018).

[15] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, Cambridge (2014).

[16] S. Weinberg, *The Quantum Theory of Fields: Volume 2, Modern Applications*, Cambridge University Press, Cambridge (1996).

[17] L. Faddeev and V. Popov, "Feynman diagrams for the Yang-Mills field," *Phys. Lett. B* **25**, 29 (1967).

[18] C. Becchi, A. Rouet, and R. Stora, "Renormalization of gauge theories," *Ann. Phys.* **98**, 287 (1976).

[19] M. Z. Iofa and I. V. Tyutin, "Gauge invariance of spontaneously broken non-Abelian theories in the Bogolyubov-Parasyuk-Hepp-Zimmermann method," *Theor. Math. Phys.* **27**, 316 (1976).

[20] A. Fuster, M. Henneaux, and A. Maas, "BRST-antifield quantization: A short review," *Int. J. Geom. Meth. Mod. Phys.* **2**, 939 (2005).

[21] I. Batalin and G. Vilkovisky, "Gauge algebra and quantization," *Phys. Lett. B* **102**, 27 (1981).

[22] I. A. Batalin and G. A. Vilkovisky, "Quantization of gauge theories with linearly dependent generators," *Phys. Rev. D* **28**, 2567 (1983).

[23] K. Costello, *Renormalization and Effective Field Theory*, American Mathematical Society, Providence, RI (2011).

[24] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton (1992).

[25] J. Gomis, J. París, and S. Samuel, "Antibracket, antifields and gauge-theory quantization," *Phys. Rep.* **259**, 1 (1995).