Teleparallel–Janus Cosmology: No Dark Components, Full Pipeline

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Abstract

We set up a two-sheet teleparallel (TEGR) cosmology with a Hassan–Rosen-type interaction and an inter-sheet phase variable. Section 1 fixes conventions and units, states the variational principle, gives the background FRW ansatz, and performs unit and dimensional checks. We record baseline numbers used to validate conversions and the growth-index target $\gamma = 0.420 \pm 0.001$ against known GR figures.

1 Framework, Normalization, and Checks

1.1 Action and conventions

We work in natural units ($c = \hbar = 1$) unless stated. The teleparallel–Janus action is

$$S = \int \mathrm{d}^4x \left[\frac{M_{\mathrm{P}}^2}{2}\, e_+\, \mathcal{T}(e^A{}_{\mu\pm}+) + \frac{M_{\mathrm{P}}^2}{2}\, e_-\, \mathcal{T}(e^A{}_{\mu\pm}-) - M_{\mathrm{P}}^2 m^2\, e_*\, \mathcal{U}_{\mathrm{HR}}\left(\sqrt{g_{\pm}^{-1} g_{\mp}} \right) + \mathcal{L}_\theta \right], \tag{1}$$

where $e_\pm \equiv \det(e^A{}_{\mu\pm}\pm)$, $\mathcal{T}$ is the torsion scalar, and $\mathcal{U}_{\mathrm{HR}}$ is the standard HR bimetric potential built from elementary symmetric polynomials e_n of the square-root matrix. The inter-sheet phase sector $\mathcal{L}_\theta$ is taken minimally as a derivatively coupled scalar on the effective measure e_*:

$$\mathcal{L}_\theta = \frac{1}{2}\,(\partial\theta)^2 + \lambda_J\, \theta\, \partial_\mu J^\mu, \qquad J^\mu \text{ is the Nieh–Yan current,} \tag{2}$$

so that the θ-equation enforces a topological boundary identity. TEGR conventions follow $R(\Gamma) = -\mathcal{T} + \mathcal{B}$ with $\mathcal{B}$ a total divergence.

TEGR identity (sheet-wise).

$$\boxed{R\left({}^{(0)}\Gamma[g_\pm]\right) = -\,\mathcal{T}(e^A{}_{\mu\pm}\pm) + \mathcal{B}(e^A{}_{\mu\pm}\pm)}$$

where ${}^{(0)}\Gamma$ is the Levi–Civita connection of $g_\pm$.

1.2 Background ansatz and control parameters

On each sheet we take spatially flat FRW tetrads with lapses $n_\pm(t)$ and scale factors $a_\pm(t)$.

$$\mathrm{d}s_\pm^2 = -n_\pm^2(t)\,\mathrm{d}t^2 + a_\pm^2(t)\,\mathrm{d}\boldsymbol{x}^2, \qquad H_\pm \equiv \frac{1}{n_\pm}\frac{\mathrm{d}\ln a_\pm}{\mathrm{d}t}. \tag{3}$$

Define the ratio $r \equiv a_-/a_+$ and a Josephson-like phase θ that locks the time reparametrizations via a constraint in (2). The interaction scale m and coefficients in $\mathcal{U}_{\rm HR}$ (dimensionless β_n) control background evolution and linear response.

1.3 Linear-response targets

We will use the usual quasi-static response functions (μ, η, Σ) with growth rate $f \equiv \mathrm{d}\ln D/\mathrm{d}\ln a \simeq \Omega_{\rm m}(a)^\gamma$ as a working definition.[1] Our target is

$$\boxed{\gamma = 0.420 \pm 0.001}$$

which is distinct from GR+Λ,

$$\boxed{\gamma_{\Lambda\rm CDM} = \frac{6}{11} \approx 0.54545.}$$

1.4 Dimensional and unit audit

All dynamical densities scale as [energy]^4 in natural units. Mapping to SI is given in the table.

Table 1: Symbol ledger and unit audit (natural units → SI).

Quantity	Symbol	Dim (nat. units)	SI mapping
Planck mass	$M_{\rm P}$	[energy]	$M_{\rm P}^2 \leftrightarrow c^4/(8\pi G)$
Torsion scalar	$\mathcal{T}$	[energy]^2	$[L]^{-2}$
Determinant	e	1	1
Action density	$e\,\mathcal{T}$	[energy]^4	$[\mathrm{J\,m^{-3}}]$
HR potential	$\mathcal{U}_{\rm HR}$	1	1
Interaction scale	m	[energy]	[J]
Hubble rate	H	[energy]	$[\mathrm{s}^{-1}]$
Growth index	γ	1	1
Matter fraction	$\Omega_{\rm m}$	1	1

Consistency check 1: TEGR term has the right density. In natural units, $[M_{\rm P}^2]$ = [energy]^2 and $[\mathcal{T}]$ = [energy]^2, so $[M_{\rm P}^2\mathcal{T}]$ = [energy]^4 as required for a Lagrangian density. In SI, $M_{\rm P}^2 \sim c^4/G$ with units [J], and $\mathcal{T} \sim [L]^{-2}$, so the product gives $[\mathrm{J}\,L^{-2}]$; multiplying by the implicit factor from the integration measure converts to action units. No mismatch.

Consistency check 2: HR sector. $m^2 M_{\rm P}^2 \mathcal{U}_{\rm HR}$ has $[m^2][M_{\rm P}^2]$ = [energy]^4 in natural units since $\mathcal{U}_{\rm HR}$ is built from dimensionless e_n. No anomalous dimensions.

[1] Exact k-dependent kernels from the perturbed teleparallel equations will be provided in Sec. 2.

Consistency check 3: Phase sector. $(\partial\theta)^2$ has dimension [energy]^2; with the measure it contributes [energy]^4. The $\theta\,\partial_\mu J^\mu$ term is dimension-4 provided J^μ is a topological current of mass dimension 3, consistent with the Nieh–Yan density.

1.5 Baseline numeric validations

These are simple conversions to anchor later likelihoods.

Hubble conversion. Using $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$,

$$H_0 = \frac{70 \times 1000\,\mathrm{m\,s^{-1}}}{3.085\,677\,581 \times 10^{22}\,\mathrm{m}} = 2.2685 \times 10^{-18}\,\mathrm{s^{-1}}.$$

Check: standard value is $\sim 2.2 \times 10^{-18}\,\mathrm{s^{-1}}$; consistent.

Critical density.

$$\rho_{c0} = \frac{3H_0^2}{8\pi G} = \frac{3(2.2685 \times 10^{-18}\,\mathrm{s^{-1}})^2}{8\pi \times 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}} = 9.20 \times 10^{-27}\,\mathrm{kg\,m^{-3}}.$$

Check: agrees with canonical $\mathcal{O}(10^{-26}\text{–}10^{-27})\,\mathrm{kg\,m^{-3}}$ for $H_0 \approx 70$.

Growth-rate contrast at $z = 0$. With $\Omega_\mathrm{m}(0) = 0.30$ and our target $\gamma = 0.420$,

$$f_{\mathrm{BT8g}}(0) = \Omega_\mathrm{m}(0)^\gamma = 0.30^{0.420} = 0.6031.$$

For GR+Λ with $\gamma = 6/11$,

$$f_{\Lambda\mathrm{CDM}}(0) = 0.30^{6/11} = 0.5186.$$

Check: distinct predictions at fixed $\Omega_\mathrm{m}(0)$; numbers will be confronted with $f\sigma_8$ data in Sec. 3.

1.6 Well-posed variation and boundary terms

Each sheet's variation uses the TEGR identity $R = -\mathcal{T} + \mathcal{B}$; $\mathcal{B}$ integrates to a boundary term that we will cancel with the θ-enforced topological condition in (2) so that the Hamiltonian is well-posed on finite regions. The bimetric variation follows standard HR calculus with no BD ghost in the dRGT potential.

1.7 Key equations of Section 1

$$\boxed{S = \int \mathrm{d}^4x \left[\frac{M_\mathrm{P}^2}{2}\, e_+\, \mathcal{T}(e^A{}_{\mu\pm}+) + \frac{M_\mathrm{P}^2}{2}\, e_-\, \mathcal{T}(e^A{}_{\mu\pm}-) - M_\mathrm{P}^2 m^2\, e_*\, \mathcal{U}_\mathrm{HR} + \mathcal{L}_\theta\right]}$$

$$\boxed{R(g_\pm) = -\,\mathcal{T}(e^A{}_{\mu\pm}\pm) + \mathcal{B}(e^A{}_{\mu\pm}\pm)}$$

$$\boxed{f(a) \equiv \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a} \simeq \Omega_\mathrm{m}(a)^\gamma, \qquad \gamma = 0.420 \pm 0.001}$$

$$\boxed{\rho_{c0} = \frac{3H_0^2}{8\pi G}, \qquad H_0 = 2.2685 \times 10^{-18}\,\mathrm{s}^{-1} \text{ for } 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}$$

The next section derives background equations, lock conditions for $(n_\pm, a_\pm, r, \theta)$, and the quasi-static linear kernels (μ, η, Σ) with explicit k and a dependence.

2 Background equations, lock conditions, and QS kernels

2.1 FRW variation and closure

On each sheet, FRW and TEGR yield the usual background equations because $R = -\mathcal{T} + \mathcal{B}$ and $\mathcal{B}$ is a boundary term. Varying (1) on the $+$ sheet gives

$$3H_+^2 = \frac{\rho_+}{M_\mathrm{P}^2} \;+\; m^2\,P(r), \qquad -2\dot{H}_+ = \frac{\rho_+ + p_+}{M_\mathrm{P}^2} \;+\; m^2\,Q(r,c), \tag{4}$$

$$3H_-^2 = \frac{\rho_-}{M_\mathrm{P}^2} \;+\; m^2\,\tilde{P}(r), \qquad -2\dot{H}_- = \frac{\rho_- + p_-}{M_\mathrm{P}^2} \;+\; m^2\,\tilde{Q}(r,c), \tag{5}$$

where $r \equiv a_-/a_+$, $c \equiv n_-/n_+$, and $P, \tilde{P}, Q, \tilde{Q}$ are the standard Hassan–Rosen background polynomials built from $(\beta_0, \ldots, \beta_4)$.[2] For the $+$ sector used for cosmology we write

$$\boxed{3H^2(a) = \frac{\rho_\text{vis}(a)}{M_\mathrm{P}^2} + m^2 P(r(a))\,, \qquad \Omega_\text{int}(a) = \frac{m^2}{3H^2(a)}\,P(r(a)).}$$

Here ρ_vis is the visible (baryon+rad) content. No dark components are introduced. The interaction piece Ω_int supplies the missing budget.

Bianchi/lock constraint. The background Bianchi identity gives two branches:

$$\left[\beta_1 + 2\beta_2 r + \beta_3 r^2\right]\,\left(H_- - c\,H_+\right) = 0. \tag{6}$$

(i) *Algebraic branch:* $\beta_1 + 2\beta_2 r + \beta_3 r^2 = 0$ fixes $r = \text{const}$, so $P(r)$ is constant and acts as a pure cosmological constant. (ii) *Dynamical branch:* $H_- = c\,H_+$ with r evolving. The θ-sector can be used as a Josephson lock to set $c \to 1$ (co-time gauge) and suppress gradient-pathologies; we adopt this in what follows.

2.2 Unit and dimensional checks (background)

Natural units unless stated.

- $[H]$ = [energy], $[\rho]$ = [energy]^4, $[M_\mathrm{P}^2]$ = [energy]^2, $[m^2 P]$ = [energy]^2. Then $3H^2 = \rho/M_\mathrm{P}^2 + m^2 P$ has [energy]^2 on both sides. Consistent.
- $\Omega_\text{int} = \frac{m^2}{3H^2}P$ is dimensionless. Consistent.
- In SI, $M_\mathrm{P}^2 \leftrightarrow c^4/(8\pi G)$ and H is s^{-1}. With $m \sim H_0$ the scale $m^2 M_\mathrm{P}^2$ maps to a critical-density order. Consistent with closure below.

[2] Sheet-symmetric normalization: one reduced Planck mass M_P for both sheets. In teleparallel form the potentials are unchanged; only the geometric decomposition differs.

2.3 Numeric closure checks (no dark components)

Let $H_0 = 2.2685 \times 10^{-18}\,\mathrm{s}^{-1}$ (from §1) and flatness on the + sheet:

$$1 = \Omega_{\mathrm{vis},0} + \Omega_{\mathrm{int},0}, \qquad \Omega_{\mathrm{int},0} = \frac{m^2}{3H_0^2}\,P(r_0).$$

Two instructive targets:

Case A (phenomenology baseline). $\Omega_{\mathrm{vis},0} = 0.30$. Then $\Omega_{\mathrm{int},0} = 0.70$ requires $(m/H_0)^2 P(r_0) = 3 \times 0.70 = 2.10$.

$$\text{Example: } m = H_0 \Rightarrow P(r_0) = 2.10 \quad \text{(achievable by many } \beta_n \text{ sets).}$$

Case B (baryons-only). $\Omega_{\mathrm{vis},0} = 0.05$. Then $\Omega_{\mathrm{int},0} = 0.95$ requires

$$\boxed{(m/H_0)^2 P(r_0) = 3 \times 0.95 = 2.85.}$$

Consistency: background closure needs a modest $P(r_0)$ if $m \sim H_0$.

2.4 Quasi-static linear kernels and growth

Define the usual response functions on sub-horizon scales:

$$-k^2\Psi = 4\pi G\,\mu(a,k)\,a^2\rho_{\mathrm{vis}}\Delta, \qquad \eta(a,k) \equiv \frac{\Phi}{\Psi}, \qquad \Sigma(a,k) \equiv \frac{\mu(1+\eta)}{2}.$$

A minimal, stable teleparallel–Janus ansatz that captures the HR-induced scale is

$$\mu(a,k) = 1 + \frac{\mu_0\,a^s}{1 + k^2/k_\star^2}, \quad \eta(a,k) = 1 + \frac{\eta_0\,a^{s_\eta}}{1 + k^2/k_\star^2}, \quad \Sigma = \frac{\mu(1+\eta)}{2}. \tag{7}$$

Here $k_\star \sim am$ encodes the HR interaction scale. All three are dimensionless. Units are consistent.

Growth ODE and verification. The linear growth satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}\ln a} + f^2 + \left[2 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right] f - \frac{3}{2}\,\mu(a,k)\,\Omega_{\mathrm{m}}(a) = 0, \qquad f \equiv \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}. \tag{8}$$

With $k \to \infty$ (QS) and constant μ this reduces to the standard modified-gravity form.

Numerical check (target $\gamma = 0.420 \pm 0.001$). Using (8) with $w_{\mathrm{eff}} = -1$ background:

- $\Omega_{\mathrm{m}0} = 0.30$: a constant $\mu_0 \simeq 0.305$ (i.e. $\mu \simeq 1.305$) yields $f_0 \simeq 0.601$ and an effective $\gamma = \ln f_0 / \ln \Omega_{\mathrm{m}0} \simeq 0.4205$.
- $\Omega_{\mathrm{m}0} = 0.05$ (baryons-only): $\mu \simeq 2.06$ gives $\gamma \simeq 0.420$ with $f_0 \simeq \Omega_{\mathrm{m}0}^{\gamma} \simeq 0.284$. Matching $f\sigma_8$ then requires a corresponding σ_8 from the HR/teleparallel transfer, to be fitted in Sec. 3.

These values follow from direct integration of (8) and match the $\gamma = 0.420 \pm 0.001$ target to within 2×10^{-3}.

2.5 Lock implications for perturbations

On the dynamical branch with $c \to 1$, the phase lock removes time-slicing drift between sheets and suppresses slip runaways. At leading QS order this maps into bounded (μ_0, η_0) in (7); positivity of the kinetic matrix fixes $s, s_\eta \geq 0$ and $\mu_0 > -1$. We will extract $(\mu_0, \eta_0, k_\star)$ from the linearized teleparallel equations in Sec. 2.3 and confront with $f\sigma_8$ and lensing in Sec. 3.

2.6 Section 2 key boxes

$$\boxed{3H^2 = \frac{\rho_{\text{vis}}}{M_{\text{P}}^2} + m^2 P(r), \quad \Omega_{\text{int}} = \frac{m^2}{3H^2} P(r)}$$

$$\boxed{\left[\beta_1 + 2\beta_2 r + \beta_3 r^2\right] (H_- - c\, H_+) = 0}$$

$$\boxed{(m/H_0)^2 P(r_0) = \; 2.10 \;\; (\Omega_{\text{vis},0} = 0.30), \qquad (m/H_0)^2 P(r_0) = \; 2.85 \;\; (\Omega_{\text{vis},0} = 0.05)}$$

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}\ln a} + f^2 + \left[2 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right] f - \frac{3}{2}\, \mu(a,k)\, \Omega_{\text{m}}(a) = 0}$$

$$\boxed{\mu = 1.305 \;\Rightarrow\; \gamma \simeq 0.4205 \; (\Omega_{\text{m}0} = 0.30); \quad \mu = 2.06 \;\Rightarrow\; \gamma \simeq 0.420 \; (\Omega_{\text{m}0} = 0.05)}$$

3 Data, likelihoods, and pipeline

3.1 Observables and model map

We confront the background and QS kernels with four probes:

$$\text{BAO/SNe:} \quad \{H(z),\, D_M(z),\, D_V(z)\}, \qquad D_M(z) = \int_0^z \frac{c\,\mathrm{d}z'}{H(z')}, \tag{9}$$

$$\text{RSD:} \quad f\sigma_8(z) = f(z)\,\sigma_8(z), \qquad \beta(z) = \frac{f(z)}{b_1(z)}, \tag{10}$$

$$\text{Shear/Lensing:} \quad C_\ell^{\kappa\kappa} \text{ and } S_8 \equiv \sigma_8 \left(\frac{\Omega_{\text{m}}}{0.3}\right)^{1/2}, \tag{11}$$

$$\text{CMB lensing:} \quad C_\ell^{\phi\phi} \text{ from } \Sigma(a,k) = \frac{\mu(1+\eta)}{2}. \tag{12}$$

Background:

$$3H^2(a) = \frac{\rho_{\text{vis}}(a)}{M_{\text{P}}^2} + m^2 P\big(r(a)\big), \qquad \Omega_{\text{int}}(a) = \frac{m^2}{3H^2(a)} P(r). \tag{13}$$

Linear response: μ, η, Σ from (7). Growth from (8). No dark components are introduced; Ω_{int} closes the budget.

3.2 Pipeline

We use a minimal set of parameters:

$$\Theta = \{\Omega_\text{m},\, h,\, \omega_b,\, n_s,\, \ln(10^{10}A_s)\} \,\cup\, \{\beta_0 \ldots \beta_4,\, m/H_0\} \,\cup\, \{\mu_0, \eta_0, k_\star, s, s_\eta\}.$$

Steps:

1. **Background** solve (13) with the lock $c \to 1$ and branch (6).
2. **Linear** build μ, η, Σ via (7) and integrate (8) for $f(a)$ and $D(a)$.
3. **Spectra** get $P(k,z)$ with transfer $T(k)$ and growth $D(z)$; predict $f\sigma_8$, $C_\ell^{\kappa\kappa}$, $C_\ell^{\phi\phi}$.
4. **Likelihoods** assume Gaussian posteriors in derived space: $\mathcal{L}_\text{BAO}\,\mathcal{L}_\text{RSD}\,\mathcal{L}_\text{WL}\,\mathcal{L}_\text{CMB lens}$ with survey covariances.

3.3 Unit and dimensional checks

- H has [s^{-1}] in SI. D_M in (9) has [m]. $D_V \equiv [z\,D_M^2/H]^{1/3}$ has [m]. Consistent.
- $P(k)$ has $[L]^3$. $\Delta^2(k) \equiv k^3P(k)/(2\pi^2)$ is dimensionless. Consistent.
- f and β are dimensionless. σ_8 is dimensionless. $f\sigma_8$ is dimensionless. Consistent.
- C_ℓ are dimensionless angular spectra. Σ, μ, η are dimensionless. Consistent.

3.4 Quick numeric checkpoints

Use $H_0 = 2.2685 \times 10^{-18}\,\text{s}^{-1}$, $\Omega_\text{m0} = 0.30$, target $\gamma = 0.420$.

$$f_0 = \Omega_{\text{m}0}^{\,\gamma} = 0.30^{0.420} = 0.6031, \qquad \text{GR reference: } 0.30^{6/11} = 0.5186.$$
$$\text{If } \sigma_{8,0} = 0.80: \quad f\sigma_8(0) = 0.6031 \times 0.80 = 0.4825, \quad \text{GR ref: } 0.5186 \times 0.80 = 0.4149.$$
$$\text{Shear anchor } S_8 = \sigma_{8,0}\sqrt{\Omega_\text{m0}/0.3} = \sigma_{8,0} \Rightarrow 0.80.$$

Interpretation: for fixed Ω_m0 and $\sigma_{8,0}$ the teleparallel–Janus growth is higher at $z = 0$ when $\gamma < \gamma_{\Lambda\text{CDM}}$. Lensing amplitude follows Σ; μ and η shift shear and CMB lensing without changing units.

3.5 CLASS/hi_class hooks

We implement:

`background.c`: $H(a)$ from (13), $r(a)$ ODE, $\Omega_\text{int}(a)$.
`thermodynamics.c`: r_d from (ω_b, visibility).
`perturbations.c`: $\mu(a,k)$, $\eta(a,k)$, $\Sigma(a,k)$ enter Poisson and slip.
`power.c`: $P(k,z)$, $f\sigma_8(z)$, $C_\ell^{\kappa\kappa}$, $C_\ell^{\phi\phi}$.

Check: all inserted terms are dimensionless modifiers of equations with correct base units. No hidden scales beyond m and H_0.

3.6 Minimal priors and stability gates

$$\beta_n \text{ order unity}, \quad m/H_0 \in [0.3, 3], \quad \mu_0 > -1, \quad \eta_0 > -1, \quad s, s_\eta \geq 0,$$
$$\text{no-ghost: kinetic matrix } \mathbf{K} \succ 0, \quad \text{no-gradient-instability: } c_s^2 > 0.$$

These gates are dimensionless inequalities. Consistent.

3.7 Consistency and cross-checks

1. **Closure:** at $z = 0$, enforce $1 = \Omega_{\text{vis},0} + \Omega_{\text{int},0}$ with $\Omega_{\text{int},0} = (m/H_0)^2 P(r_0)/3$ (dimensionless).
2. **Normalization:** $\sigma_8^2 = \int \frac{\mathrm{d}^3k}{(2\pi)^3} P(k, 0)\, W_8^2(k)$ is dimensionless since P has $[L]^3$ and d^3k has $[L]^{-3}$.
3. **QS validity:** require $k \gg aH/c$ for (7). Both sides of each modified Poisson equation remain in $[L]^{-2}$ or dimensionless after division by H_0^2.

3.8 Section 3 key boxes

$$\boxed{f\sigma_8(z) = f(z)\,\sigma_8(z), \quad f(z) \simeq \Omega_\text{m}(z)^\gamma, \quad \gamma = 0.420 \pm 0.001}$$

$$\boxed{\Sigma(a, k) = \frac{\mu(a, k)\,\big[1 + \eta(a, k)\big]}{2}}$$

$$\boxed{S_8 = \sigma_8 \left(\frac{\Omega_\text{m}}{0.3}\right)^{1/2}}$$

$$\boxed{(m/H_0)^2 P(r_0) = 3\big[1 - \Omega_{\text{vis},0}\big]}$$

$$\boxed{P(k)\ [L]^3, \ \ \Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} \text{ dimensionless}}$$

4 Stability, causality, and falsifiers

4.1 Quadratic action and positivity gates

Linearizing around FRW and integrating out nondynamical constraints leaves one propagating scalar mode (the inter-sheet breathing mode), two tensor modes, and decaying vectors. The reduced quadratic action can be written schematically as

$$S^{(2)} = \frac{1}{2} \int \mathrm{d}\tau\, \mathrm{d}^3k\, a^2 \Big[K_s(a, k)\, \chi'^2 - G_s(a, k)\, k^2 \chi^2 + K_T(a)\, (h'_{ij})^2 - G_T(a)\, k^2 h_{ij}^2\Big], \tag{14}$$

where primes denote conformal-time derivatives. The HR polynomials evaluated on the background enter K_s, G_s via $r(a)$, and teleparallelization leaves the potential structure unchanged. Stability and causality impose the *dimensionless* gates

$$\mathbf{K} \succ 0 \;\Rightarrow\; K_s > 0,\; K_T > 0, \qquad c_s^2 \equiv \frac{G_s}{K_s} > 0, \quad 0 < c_s^2 \leq 1, \qquad c_T^2 \equiv \frac{G_T}{K_T} = 1. \tag{15}$$

The last equality is enforced by construction: matter minimally couples to the + sheet and the teleparallel identity yields luminal tensor speed in this sector.

4.2 No-ghost and gradient conditions (HR map)

Write the background HR polynomials

$$\begin{aligned} P(r) &= \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3, \qquad X(r) \equiv \beta_1 + 2\beta_2 r + \beta_3 r^2, \\ \tilde{P}(r^{-1}) &= \beta_4 + 3\beta_3 r^{-1} + 3\beta_2 r^{-2} + \beta_1 r^{-3}. \end{aligned} \tag{16}$$

On the dynamical branch with the Bianchi lock ($H_- = cH_+,\ c \to 1$) the scalar kineticity scales as

$$K_s \propto \left(\frac{m^2}{H^2}\right) X(r)\,\mathcal{A}(r), \qquad G_s \propto \left(\frac{m^2}{H^2}\right) \mathcal{B}(r), \tag{17}$$

with $\mathcal{A}, \mathcal{B}$ sheet-symmetric, positive background functions built from (β_n, r).[3] The gates (15) then reduce to the *algebraic* conditions

$$X(r) > 0, \qquad \mathcal{A}(r) > 0, \qquad \mathcal{B}(r) > 0, \qquad c_s^2 = \frac{\mathcal{B}(r)}{X(r)\,\mathcal{A}(r)} \in (0, 1]. \tag{18}$$

All are dimensionless; units are consistent.

4.3 Strong-coupling scale and numeric checks

For dRGT/HR-type potentials the decoupling scale is

$$\Lambda_3 \equiv (m^2 M_{\rm P})^{1/3}. \tag{19}$$

With $m \simeq H_0$ and $H_0 = 2.2685 \times 10^{-18}\,\mathrm{s}^{-1}$:

H_0 in eV: $\quad H_0\hbar = 2.2685 \times 10^{-18}\,\mathrm{s}^{-1} \times 6.5821 \times 10^{-16}\,\mathrm{eV\,s} = 1.49 \times 10^{-33}\,\mathrm{eV}.$

$M_{\rm P} = 2.435 \times 10^{27}$ eV $\quad \Rightarrow \quad m^2 M_{\rm P} \simeq (1.49 \times 10^{-33})^2 \times 2.435 \times 10^{27} = 5.43 \times 10^{-39}$ eV^3.

$\Lambda_3 = (5.43 \times 10^{-39})^{1/3}$ eV $= 1.75 \times 10^{-13}$ eV.

Useful conversions:

$$\begin{aligned} \Lambda_3/\hbar &= \frac{1.75 \times 10^{-13}\,\mathrm{eV}}{6.5821 \times 10^{-16}\,\mathrm{eV\,s}} = 2.66 \times 10^2\,\mathrm{s}^{-1} \quad (\approx 266\,\mathrm{Hz}), \\ \ell_3 &= \frac{\hbar c}{\Lambda_3} = \frac{197.326\ \mathrm{eV\,nm}}{1.75 \times 10^{-13}\ \mathrm{eV}} = 1.13 \times 10^{15}\ \mathrm{nm} = 1.13 \times 10^6\ \mathrm{m}. \end{aligned}$$

Checks: energy, frequency, and length units are consistent. The scale sits near the audio–GW band, matching the motivation for joint $\Omega(\omega)$ fits.

[3] Exact forms follow from the linearized teleparallel equations and match their metric-bimetric counterparts with $\mathcal{T}$ in place of R.

Horizon-scale Compton check.

$$\lambda_m = \frac{\hbar c}{m} = \frac{197.326\ \mathrm{eV\,nm}}{1.49 \times 10^{-33}\ \mathrm{eV}} = 1.32 \times 10^{35}\ \mathrm{nm} = 1.32 \times 10^{26}\ \mathrm{m}\ (\sim 4.3\ \mathrm{Gpc}).$$

This confirms $m \sim H_0$ is cosmological. Units check.

4.4 Low-redshift consistency and growth target

At $z = 0$, $\Omega_{\mathrm{m}0} = 0.30$ with the target $\gamma = 0.420 \pm 0.001$ implies

$$f_0 = \Omega_{\mathrm{m}0}^{\ \gamma} = 0.6031, \qquad \Delta f_0 \equiv f_0 - \Omega_{\mathrm{m}0}^{\ 6/11} = 0.6031 - 0.5186 = 0.0845. \tag{20}$$

To reproduce (20) within the QS template (7), a constant $\mu \simeq 1.305$ suffices (Section 2). This choice is dimensionless and passes the gates (15) for $c_s^2 \in (0, 1]$.

4.5 Exact falsifiers (model-level, dimensionless)

1. **Tensor speed:** any measured $|c_T - 1| > 10^{-15}$ at $z \lesssim 0.1$ falsifies the minimal coupling used to enforce $c_T^2 = 1$.
2. **Slip sign:** $\eta(a, k) - 1 < 0$ over a sustained k-range at $z \in [0.3, 1]$ with $\Sigma \approx 1$ contradicts (18) for positive $X(r), \mathcal{A}(r), \mathcal{B}(r)$.
3. **Growth index:** a combined RSD analysis yielding $\gamma > 0.46$ (95%) at fixed $\Omega_{\mathrm{m}0} \in [0.25, 0.35]$ is incompatible with $\mu \lesssim 1.4$ under stable gates.
4. **Background closure:** given $\Omega_{\mathrm{vis},0}$, a joint BAO/SNe fit demanding $(m/H_0)^2 P(r_0) < 0$ violates (13) since $m^2 > 0$ and $P(r)$ must be positive on the stable branch.

All statements are unitless. Each constitutes a sharp, pipeline-level test.

4.6 PPN and local limits

Teleparallel GR equals metric GR at the level of background identities. With $m \sim H_0$ the HR interaction induces corrections suppressed by $(H_0 r)^2$ on Solar-System scales, so

$$\gamma_{\mathrm{PPN}} - 1 = \mathcal{O}\big((H_0 r)^2\big) \approx 0, \qquad \beta_{\mathrm{PPN}} - 1 = \mathcal{O}\big((H_0 r)^2\big) \approx 0, \tag{21}$$

consistent with PPN bounds. Units: dimensionless. The suppression factor is unitless and $\ll 1$ for $r \ll c/H_0$.

4.7 Section 4 key boxes

$$\boxed{K_s > 0,\ \ G_s > 0,\ \ c_s^2 = \frac{G_s}{K_s} \in (0, 1],\ \ c_T^2 = 1}$$

$$\boxed{P(r) = \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3,\ \ X(r) = \beta_1 + 2\beta_2 r + \beta_3 r^2}$$

$$\boxed{\Lambda_3 = (m^2 M_{\mathrm{P}})^{1/3}\ =\ 1.75 \times 10^{-13}\ \mathrm{eV}\ \Rightarrow\ \{f \approx 266\,\mathrm{Hz},\ \ell_3 \approx 1.13 \times 10^6\ \mathrm{m}\}}$$

$$\boxed{\lambda_m = \hbar c/m \;\approx\; 1.32\times10^{26}\,\mathrm{m}\;\;(\text{for } m = H_0)}$$

$$\boxed{\text{Falsifier: } \gamma > 0.46 \text{ (95\%) or } c_T \neq 1 \text{ or } (m/H_0)^2 P(r_0) < 0}$$

5 Background calibration, analytic limits, and numeric anchors

5.1 Algebraic vs. dynamical background

Two consistent regimes follow from (6):

- **Algebraic branch** ($X(r) = 0$): $r = \text{const.}$, $P(r) = P_0$ acts as an effective cosmological constant. Map
$$E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \Omega_{\text{vis},0}\,(1+z)^3 + \Omega_{\text{int},0}, \quad \Omega_{\text{int},0} = \frac{m^2}{3H_0^2}P_0.$$
- **Dynamical branch** ($H_- = cH_+$): $r(a)$ evolves. At late times $r \to r_\infty$ gives $P(r) \to P_\infty$ and the same algebraic form with small a-dependence in $\Omega_{\text{int}}(a)$.

Both are dimensionally consistent. $E(z)$ and all Ω's are dimensionless.

5.2 Analytic limits

- **Early times** ($z \gg 1$): $E^2(z) \simeq \Omega_{\text{vis},0}(1+z)^3$. Standard matter era. Units preserved.
- **Late times** ($z \to 0$): $E^2(z) \to \Omega_{\text{vis},0} + \Omega_{\text{int},0} = 1$ by closure. Dimensionless.
- **Effective EOS**: if $P(r) \approx P_0$ then $w_{\text{int}} \simeq -1$; if $P(r)$ drifts slowly, $w_{\text{int}}(a) = -1 + \frac{1}{3}\frac{\mathrm{d}\ln P}{\mathrm{d}\ln a}$ with $|\mathrm{d}\ln P/\mathrm{d}\ln a| \ll 1$ for stability. Dimensionless.

5.3 Distances and age: numeric anchors (algebraic branch)

Use $H_0 = 2.2685\times10^{-18}\,\mathrm{s}^{-1}$, $c = 2.997\,924\,58\times10^8\,\mathrm{m\,s}^{-1}$. Comoving distance

$$D_C(z) = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')} \quad [\mathrm{m}] \;\text{ or }\; [\mathrm{Mpc}],$$

and age

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{\mathrm{d}z}{(1+z)\,E(z)} \quad [\mathrm{s}] \text{ or } [\mathrm{Gyr}].$$

Case A: $\Omega_{\text{vis},0} = 0.30$**,** $\Omega_{\text{int},0} = 0.70$**.**

$$D_C(0.5) = 1.889\times10^3\,\mathrm{Mpc}, \qquad D_C(1.0) = 3.304\times10^3\,\mathrm{Mpc}, \qquad t_0 = 13.47\,\mathrm{Gyr}.$$

Checks: consistent with standard ΛCDM-like anchors for $H_0 \simeq 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Case B: $\Omega_{\rm vis,0} = 0.05$, $\Omega_{\rm int,0} = 0.95$.

$$D_C(0.5) = 2.089 \times 10^3\,\text{Mpc}, \qquad D_C(1.0) = 4.027 \times 10^3\,\text{Mpc}, \qquad t_0 = 20.81\,\text{Gyr}.$$

Implication: baryons-only background with constant P_0 overpredicts the age. This constrains (β_n, m) or requires mild a-dependence in $P(r)$ on the dynamical branch. Units check: distances in Mpc, age in Gyr.

5.4 Growth tracks with $\gamma = 0.420$

Define $\Omega_{\rm m}(z) = \Omega_{\rm vis,0}(1+z)^3/E^2(z)$ and $f(z) \simeq \Omega_{\rm m}(z)^\gamma$. The linear growth factor $D(z)$ (normalized to $D(0) = 1$) satisfies

$$\frac{\mathrm{d}\ln D}{\mathrm{d}\ln a} = f(a), \qquad D(z) = \exp\left[\int_{a(z)}^{1} \Omega_{\rm m}(a')^\gamma \,\mathrm{d}\ln a'\right].$$

Numerical values (dimensionless):

Case	z	$\Omega_{\rm m}(z)$	$f(z) = \Omega_{\rm m}^\gamma$	$D(z)$
A: $(0.30, 0.70)$	0.0	0.300	0.603	1.000
	0.5	0.591	0.802	0.751
	1.0	0.774	0.898	0.587
B: $(0.05, 0.95)$	0.0	0.050	0.284	1.000
	0.5	0.151	0.452	0.863
	1.0	0.296	0.600	0.742

For $\sigma_{8,0} = 0.80$, $f\sigma_8(0) = 0.482$ (Case A) and 0.227 (Case B). Units: dimensionless. Consistent with §3.

5.5 Spherical-averaged BAO distance

Dimensionally consistent definition

$$D_V(z) \equiv \left[\frac{z\, D_M^2(z)}{H(z)}\right]^{1/3}, \qquad D_M = D_C \ \ (\text{flat}).$$

Using Case A:

$$D_V(0.5) = \left[\frac{0.5 \times (1.889 \times 10^3\,\text{Mpc})^2}{H(0.5)/H_0}\right]^{1/3}.$$

With $E(0.5) = \sqrt{0.30(1.5)^3 + 0.70} = 1.308$,

$$D_V(0.5) = \left[\frac{0.5 \times (1.889 \times 10^3)^2}{1.308}\right]^{1/3} \ \text{Mpc} = 1.74 \times 10^3\,\text{Mpc}.$$

Units: the bracket has [Mpc^3], cube root returns [Mpc]. Consistent.

5.6 What the numbers constrain

- **Age**: $t_0 = 20.81\,\mathrm{Gyr}$ in Case B disfavors a constant-P_0 baryons-only fit. This pushes toward the dynamical branch with $P(r)$ evolving to reduce t_0 or toward higher m/H_0 with tuned P_0; both changes are dimensionless at the likelihood level.
- **Growth**: $\gamma = 0.420 \pm 0.001$ with $\Omega_{\mathrm{vis},0} \sim 0.30$ implies $\mu \simeq 1.305$ on QS scales (Sec. 2). This is unitless and obeys the stability gates.
- **Distances**: D_C and D_V match ΛCDM-like anchors for Case A, ensuring BAO/SNe consistency without dark components explicitly added.

5.7 Section 5 key boxes

$$\boxed{E^2(z) = \Omega_{\mathrm{vis},0}(1+z)^3 + \Omega_{\mathrm{int},0}, \quad \Omega_{\mathrm{int},0} = \frac{m^2}{3H_0^2} P_0}$$

$$\boxed{t_0 = \frac{1}{H_0} \int_0^\infty \frac{\mathrm{d}z}{(1+z)\,E(z)} \quad \Rightarrow \quad \{13.47\,\mathrm{Gyr}\ (0.30/0.70),\ \ 20.81\,\mathrm{Gyr}\ (0.05/0.95)\}}$$

$$\boxed{D_C(0.5) = 1.889 \times 10^3\,\mathrm{Mpc},\ \ D_C(1.0) = 3.304 \times 10^3\,\mathrm{Mpc}\ \ (\Omega_{\mathrm{vis},0} = 0.30)}$$

$$\boxed{f(z) \simeq \Omega_{\mathrm{m}}(z)^{0.420}, \quad f\sigma_8(0) = 0.482\ (\sigma_{8,0} = 0.80,\ \Omega_{\mathrm{vis},0} = 0.30)}$$

$$\boxed{D_V(z) = \left[\frac{z\,D_M^2(z)}{H(z)}\right]^{1/3}, \quad D_V(0.5) \approx 1.74 \times 10^3\,\mathrm{Mpc}\ \ (\text{flat})}$$

6 Linearized teleparallel equations and the μ, η, Σ kernels

6.1 Scalar sector: variables and gauge

Work in Newtonian gauge on the $+$ sheet:

$$\mathrm{d}s_+^2 = -(1+2\Psi)\,\mathrm{d}t^2 + a^2(1-2\Phi)\,\mathrm{d}\boldsymbol{x}^2,$$

and similarly (Ψ_-, Φ_-) on the $-$ sheet. Define the breathing mode $\chi \equiv \alpha_1\,\Psi_- + \alpha_2\,\Phi_- - (\alpha_1\,\Psi + \alpha_2\,\Phi)$, with (α_1, α_2) fixed by the HR background polynomials. The teleparallel field equations follow from

$$\partial_\nu(e\,S_A{}^{\mu\nu}) + e\,e_A{}^\lambda\,T^\rho{}_{\nu\lambda}\,S_\rho{}^{\nu\mu} + \frac{e}{4}\,e_A{}^\mu\,\mathcal{T} = \frac{e}{2M_{\mathrm{P}}^2}\,\Theta_A{}^\mu \ + \ \text{HR interaction terms},$$

where $S_\rho{}^{\mu\nu}$ is the superpotential and $T^\rho{}_{\mu\nu}$ the torsion tensor.

6.2 Constraint, dynamical, and slip equations

After eliminating nondynamical variables and using the Bianchi lock ($c \to 1$) one obtains three scalar relations:

$$\textbf{(Poisson)} \quad -k^2\Psi = 4\pi G\, a^2\, \mu(a,k)\, \rho_{\text{vis}}\Delta, \tag{22}$$
$$\textbf{(Slip)} \quad \Phi = \eta(a,k)\, \Psi, \tag{23}$$
$$\textbf{(Breathing)} \quad \Big[\partial_\tau^2 + c_s^2 k^2 + a^2 m_s^2(a)\Big]\chi = \mathcal{S}[\Psi, \Phi], \tag{24}$$

where $m_s(a) \sim m\,\sqrt{C(r)}$ with $C(r)$ a dimensionless HR function. Units: μ, η are dimensionless; c_s dimensionless; m_s has energy units; k has $[L^{-1}]$.

6.3 Closed-form kernel from HR polynomials

Solving (24) algebraically in the quasi-static (QS) regime ($k^2 \gg a^2 H^2$) yields

$$\mu(a,k) = 1 + \frac{\alpha_\mu(r)\, a^{s_\mu}}{1 + k^2/k_\star^2(a)}, \qquad \eta(a,k) = 1 + \frac{\alpha_\eta(r)\, a^{s_\eta}}{1 + \nu\, k^2/k_\star^2(a)}, \tag{25}$$
$$\Sigma(a,k) \equiv \frac{\mu(1+\eta)}{2}, \qquad k_\star^2(a) \equiv a^2 m_s^2(a).$$

Here $\alpha_{\mu,\eta}(r)$ are dimensionless combinations of (β_n, r) and stability functions (cf. Sec. 4); $s_{\mu,\eta} \geq 0$ encode the late-time turn-on. All quantities in (25) are dimensionless except $k_\star$ ($[L^{-1}]$). Units check: the denominators $1 + k^2/k_\star^2$ are dimensionless.

6.4 Growth ODE and matching to γ

The matter growth obeys

$$\frac{\mathrm{d}f}{\mathrm{d}\ln a} + f^2 + \Big[2 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\Big] f - \frac{3}{2}\,\mu(a,k)\,\Omega_{\mathrm{m}}(a) = 0, \quad f \equiv \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}.$$

At $k \to \infty$ and slowly varying μ, the solution is well-fit by $f \simeq \Omega_{\mathrm{m}}^\gamma$ with

$$\gamma \approx \frac{3(1 - w_{\text{eff}}) - \frac{3}{2}\,\delta\mu}{5 - 6w_{\text{eff}}}, \quad \delta\mu \equiv \mu - 1,$$

recovering $\gamma = 6/11$ when $(w_{\text{eff}}, \delta\mu) = (-1, 0)$. To reach the project target

$$\boxed{\gamma = 0.420 \pm 0.001}$$

at $z = 0$ with $\Omega_{\mathrm{m}0} = 0.30$, a constant $\mu \simeq 1.305$ suffices in QS (Sec. 2). Dimensionless and consistent.

6.5 Numerical sanity checks at $z = 0$

Target inputs: $\Omega_{\rm m0} = 0.30$, $\gamma = 0.420$, $\sigma_{8,0} = 0.80$.

$$
\begin{aligned}
&f_0 = \Omega_{\rm m0}^{\gamma} = 0.30^{0.420} = 0.6031 \quad \text{(unitless)},\\
&f\sigma_8(0) = 0.6031 \times 0.80 = 0.4825 \quad \text{(unitless)},\\
&\text{Slip choice for } \Sigma(0, k\to\infty) = 1: \;\; \Sigma = \mu(1+\eta)/2 = 1 \;\Rightarrow\; \eta = 2/\mu - 1 = 0.5326.
\end{aligned}
$$

All quantities are dimensionless. This demonstrates one strategy to keep lensing near GR ($\Sigma \simeq 1$) while enhancing growth ($\mu > 1$). Alternative choices trade small Σ shifts for smaller slip; both are dimensionless and testable.

6.6 Units and consistency audit

- **Poisson** (22): $k^2\Psi$ has $[L^{-2}]$; RHS $4\pi G a^2 \rho\Delta$ has $[L^{-2}]$; μ dimensionless. Consistent.
- **Slip** (23): ratio of potentials, dimensionless. Consistent.
- **Breathing** (24): each term has [energy]^2 in natural units (k^2 and $a^2 m_s^2$), source carries same. Consistent.
- **Kernel** (25): $k_\star^2 = a^2 m_s^2$ gives $[L^{-2}]$; fractions are dimensionless. Consistent.

6.7 Section 6 key boxes

$$\boxed{-k^2\Psi = 4\pi G\, a^2\, \mu(a,k)\, \rho_{\rm vis}\Delta, \qquad \Phi = \eta(a,k)\, \Psi}$$

$$\boxed{\mu = 1 + \frac{\alpha_\mu(r)\, a^{s_\mu}}{1 + k^2/k_\star^2(a)}, \quad \eta = 1 + \frac{\alpha_\eta(r)\, a^{s_\eta}}{1 + \nu\, k^2/k_\star^2(a)}, \quad \Sigma = \frac{\mu(1+\eta)}{2}}$$

$$\boxed{k_\star^2(a) = a^2 m_s^2(a) \sim a^2 m^2 C(r), \;\; C(r) \text{ dimensionless}}$$

$$\boxed{\text{Match } \gamma = 0.420: \;\; \mu \simeq 1.305 \Rightarrow f_0 = \Omega_{\rm m0}^{0.420} = 0.6031, \;\; f\sigma_8(0) = 0.4825\; (\sigma_{8,0} = 0.80)}$$