① Prove:

$$m^2, (m+1)^2, \dots, n^2 = \frac{n(n+1)(2n+1) - [m(m-1)(2m-1)]}{6}$$

proof needed: $\frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + \dots + n^2$

base case: $n = 1$

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$$

Induction: ↓ assume true

$$\frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + 3^2 + \dots + n^2$$

$$\underbrace{1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2}_{} + (n+1)^2$$

$$\frac{n(2n+1)(n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$\frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$\frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$\frac{(n+1)(2n^2 + n + 6n + 6)}{6}$$

$$\frac{(n+1)(n+2)(2n+3)}{6}$$

looking for $= \frac{(n+1)(n+2)(2n+3)}{6}$

$$\therefore 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Q.E.F.

(2)

$n=5 \quad \dfrac{n(n+1)(2n+1)}{6} - \dfrac{m(m-1)(2m-1)}{6}$

$m=2$

$n=1$

$$\frac{5(\cancel{6})(9)}{\cancel{6}} - \frac{m(m-1)(2m-1)}{6}$$

$$45 - \frac{2(1)(3)}{6}$$

proof needed $m(m-1)(2m-1) = (m-1)((m-1)+1)(2(m-1)+1)$

$$m(m-1)(2m-1) = (m-1)(m)(2m-2+1)$$

$$\therefore m(m-1)(2m-1) = m(m-1)(2m-1)$$

Q.E.F

$$\Rightarrow \frac{m(m-1)(2m-1)}{6} = \frac{(m-1)((m-1)+1)(2(m-1)+1)}{6}$$

!

$$\therefore \frac{n(n+1)(2n+1) - m(m-1)(2m-1)}{6}$$

$$= (1^2+2^2+3^2+\dots+n^2) - (1^2+2^2+3^2+\dots+(m-1)^2)$$

$$\Rightarrow \frac{n(n+1)(2n+1) - m(m-1)(2m-1)}{6}$$

$$= m^2+(m+1)^2+(m+2)^2+(m+3)^2+\dots+n^2$$

$$\forall \{\text{integers } m,n : m \le n\}$$

Q.E.D