Gauss-Legendre Method: Any interval can be written as a definite integral in the range (-1, 1) by applying the following transformation:

$$x = \left(\frac{b-a}{2}\right)t + \left(\frac{b+a}{2}\right)$$

where $(a, b)$ is the range for the actual integration.

The integral can be written ~~be~~ as:

$$\int_{-1}^{1} w(x) f(t)\, dt$$

$$\int_{-1}^{1} w(x) f(x)\, dx = \sum_{i=0}^{n} \lambda_i f_k$$

For Gauss-Legendre method, $w(x) = 1$

(i) ~~For~~ One-point formula ($n=0$): For a single point of integration, the integral can be written as,

$$\int_{-1}^{1} f(t)\, dt = \lambda_0 f(t_0)$$

For a single point, the function $f(t)$ should satisfy:

$$f(t) = 1 \quad , \quad f(t) = t$$

when, $f(t) = 1$

$$\int_{-1}^{1} f(t)\, dt = (1)\Big|_{-1}^{1} = \lambda_0 (1)$$

$$\lambda_0 = 2$$

when $f(t) = t$

$$\int_{-1}^{1} t\, dt = \frac{t^2}{2}\Big|_{-1}^{1} = \left[\frac{1}{2} - \frac{1}{2}\right] = 0 = \lambda_0 t_0$$

As $\lambda_0 \neq 0$, $t_0 = 0$.

Hence, the formula becomes:

$$\int_{-1}^{1} f(t)dt = 2f(0)$$

The error is:

$$~~E=~~ C = \int_{-1}^{1} x^2 dx - 2[0] = \frac{2}{3} - 0 = \frac{2}{3}$$

(ii) Two-point formula ($n=1$): The method has 4 unknowns - $\lambda_0$, $\lambda_1$, ~~$x_0$~~ $t_0$ and $t_1$. Hence, the function $f(t)$ should satisfy

~~$f(t) = 1, x, x^2, x^3$~~ $f(t) = 1, t, t^2, t^3$

when $f(t)=1$

$$\int_{-1}^{1} f(t)dt = 2 = \lambda_0 + \lambda_1 \quad - (I)$$

when $f(t) = t$

$$\int_{-1}^{1} t\,dt = \frac{t^2}{2} = \left[\frac{1}{2} - \frac{1}{2}\right] = 0 = \lambda_0 t_0 + \lambda_1 t_1 \quad - (II)$$

when $f(t) = t^2$

$$\int_{-1}^{1} t^2 dt = \frac{t^3}{3} = \frac{2}{3} = \lambda_0 t_0^2 + \lambda_1 t_1^2 \quad - (III)$$

when $f(t) = t^3$

$$\int_{-1}^{1} t^3 dt = \frac{t^4}{4} = \left[\frac{1}{4} - \frac{1}{4}\right] = 0 = \lambda_0 t_0^3 + \lambda_1 t_1^3 \quad - (IV)$$

From (II),

$$\lambda_0 = -\frac{\lambda_1 t_1}{t_0} \quad - (V)$$

From (IV) and (V)

$$-\frac{\lambda_1 t_1}{t_0} \cdot t_0^3 + \lambda_1 t_1^3 = 0$$

$$\lambda_1 t_1 (t_1^2 - t_0^2) = 0$$

$$\lambda_1 t_1 (t_1 - t_0)(t_1 + t_0) = 0$$

As $\lambda_1 \neq 0$, $t_1 \neq 0$ ~~and~~ and $t_1 \neq t_0$, we can say that:

$$t_0 + t_1 = 0$$

$$t_0 = -t_1 \quad \text{- (VI)}$$

From (II) and (VI)

$$\lambda_0 t_0 - \lambda_1 t_0 = 0$$

$$\lambda_0 - \lambda_1 = 0$$

$$\lambda_0 = \lambda_1 \quad \text{- (VII)}$$

From (I) and (VII)

$$2\lambda_0 = 2$$

$$\lambda_0 = 1$$

$$\lambda_1 = 1$$

Now, in (III),

$$t_0^2 + t_1^2 = \frac{2}{3}$$

$$2t_0^2 = \frac{2}{3}$$

$$t_0^2 = \frac{1}{3}$$

$$t_0 = \pm\frac{1}{\sqrt{3}} \quad , \text{ then, } \quad t_1 = \mp\frac{1}{\sqrt{3}}$$

Hence, the two-point formula becomes:

$$\int_{-1}^{1} f(t)\,dt = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

The error becomes:

$$e = \int_{-1}^{1} t^4 dt - \left[\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{-1}{\sqrt{3}}\right)^4\right]$$

$$= \frac{2}{5} - \frac{2}{9} = \frac{8}{45} = 0.17778$$

(iii) 3-point formula: The 3-point formula will be given as:

$$\int_{-1}^{1} f(t)\,dt = \lambda_0 f(t_0) + \lambda_1 f(t_1) + \lambda_2 f(t_2)$$

The number of unknowns is 6 and hence, the function should satisfy:

$$f(t) = 1, t, t^2, t^3, t^4, t^5$$

when $f(t) = 1$,

$$\int_{-1}^{1} (1)\,dt = t\Big|_{-1}^{1} = 1-(-1) = 2$$

$$\lambda_0(1) + \lambda_1(1) + \lambda_2(1) = 2$$

$$\lambda_0 + \lambda_1 + \lambda_2 = 2 \quad - \text{(viii)}$$

when $f(t) = t$,

$$\int_{-1}^{1} t\,dt = \frac{t^2}{2}\Big|_{-1}^{1} = 0$$

$$\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2 = 0 \quad - \text{(ix)}$$

when $f(t) = t^2$

$$\int_{-1}^{1} t^2\,dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\lambda_0 t_0^2 + \lambda_1 t_1^2 + \lambda_2 t_2^2 = \frac{2}{3} \quad - \text{(x)}$$

when $f(t) = t^3$

$$\int_{-1}^{1} t^3\,dt = \left.\frac{t^4}{4}\right|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

So, $\lambda_0 t_0^3 + \lambda_1 t_1^3 + \lambda_2 t_2^3 = 0$ — (XI)

when $f(t) = t^4$,

$$\int_{-1}^{1} t^4\,dt = \left.\frac{t^5}{5}\right|_{-1}^{1} = \frac{1}{5} - \left(-\frac{1}{5}\right) = \frac{2}{5}$$

So, $\lambda_0 t_0^4 + \lambda_1 t_1^4 + \lambda_2 t_2^4 = \frac{2}{5}$ — (XII)

when $f(t) = t^5$,

$$\int_{-1}^{1} t^5\,dt = \frac{t^6}{6} = \frac{1}{6} - \frac{1}{6} = 0$$

So, $\lambda_0 t_0^5 + \lambda_1 t_1^5 + \lambda_2 t_2^5 = 0$ — (XIII)

From (IX), (XI) and (XIII), we get

$$\lambda_0 = -\frac{[\lambda_1 t_1 + \lambda_2 t_2]}{t_0}$$

$$-\left[\frac{\lambda_1 t_1 + \lambda_2 t_2}{t_0}\right] t_0^3 + \lambda_1 t_1^3 + \lambda_2 t_2^3 = 0$$

$$\lambda_1 t_1 (t_1^2 - t_0^2) + \lambda_2 t_2 (t_2^2 - t_0^2) = 0 \quad \text{— (XIV)}$$

and,

$$\lambda_1 t_1^3 (t_1^2 - t_0^2) + \lambda_2 t_2^3 (t_2^2 - t_0^2) = 0 \quad \text{— (XV)}$$

From (XIV)

$$t_1^2 - t_0^2 = \frac{-\lambda_2 t_2 (t_2^2 - t_0^2)}{\lambda_1 t_1} \quad \text{— (XVI)}$$

From (XV) and (XVI),

$$\lambda_2 t_2^3 (t_2^2 - t_0^2) - \lambda_2 t_2 t_1^2 (t_2^2 - t_0^2) = 0$$

$$\lambda_2 t_2 (t_2 - t_0)(t_2 + t_0)(t_2 - t_1)(t_2 + t_1) = 0$$

As all the points are distinct,

$$\lambda_2 t_2 (t_2 + t_0)(t_2 + t_1) = 0 \quad - \text{(XVII)}$$

As $\lambda_2 \neq 0$ and let $t_2 \neq 0$, then, either, $t_2 = -t_0$ or $t_2 = -t_1$.

Let $t_2 = -t_0$. Then, from (IX) and (XI),

$$(\lambda_2 - \lambda_0) t_0 + \lambda_1 t_1 = 0 \quad - \text{(XVIII)}$$

$$(\lambda_2 - \lambda_0) t_0^3 + \lambda_1 t_1^3 = 0 \quad - \text{(XIX)}$$

From (XVIII),

$$\lambda_2 - \lambda_0 = \frac{-\lambda_1 t_1}{t_0} \quad - \text{(XX)}$$

From (XX) and (XIX),

$$\lambda_1 t_1 (t_1^2 - t_0^2) = 0$$

$$\lambda_1 t_1 (t_1 - t_0)(t_1 + t_0) = 0$$

~~As $\lambda_1 \neq 0$, ~~$t_0$~~ and let $t_1 \neq 0$ and $t_0, t_1$ are distinct, we can say:~~

~~$$t_1 + t_0 = 0$$~~

~~$$t_1 = -t_0 \quad - \text{(XXI)}$$~~

~~Then, from (XVIII) and (XXI),~~

~~$$(\lambda_2 - \lambda_0 - \lambda_1) t_0 = 0$$~~

~~$$\lambda_2 - \lambda_0 - \lambda_1 = 0 \quad - \text{(XXII)}$$~~

Any integral can be written in

~~From (VIII) and (XXIV),~~

On adding

~~As $\lambda_1 \neq 0$, $x_1 \neq x_0$, $x_1$~~

As $\lambda_1 \neq 0$, $t_1 \neq t_0$, $t_1 \neq -t_0$, we get

$$x_1 = 0$$

Eq (XVIII) then becomes

$$(\lambda_0 - \lambda_2)\, t_0 = 0$$

So, $\lambda_0 = \lambda_2$

Now, (X) and (XII) become:

$$2\lambda_0 t_0^2 = \frac{2}{3} \quad \text{and} \quad 2\lambda_0 t_0^4 = \frac{2}{5}$$

Dividing, we get.

$$t_0^2 = \frac{3}{5}$$

$$t_0 = \pm\sqrt{\frac{3}{5}}$$

$$t_2 = \mp\sqrt{\frac{3}{5}}$$

$$\lambda_0 \left(\frac{3}{5}\right) = \frac{1}{3}$$

$$\lambda_0 = \frac{5}{9}$$

So, $\lambda_2 = \frac{5}{9}$

From (VIII),

$$\lambda_1 = 2 - 2\left(\frac{5}{9}\right) = 2 - \frac{10}{9} = \frac{8}{9}$$

The three-point formula now becomes:

$$\int_{-1}^{1} f(t)\,dt = \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right)$$

$$= \frac{1}{9}\left[5 f\left(\sqrt{\frac{3}{5}}\right) + 8 f(0) + 5 f\left(-\sqrt{\frac{3}{5}}\right)\right]$$

The error constant is:

$$C = \int_{-1}^{1} x^6\,dx - \frac{1}{9}\left[5 \times \frac{27}{125} + 8(0) + 5\left(\frac{27}{125}\right)\right]$$

$$= \frac{2}{7} - \frac{6}{25} = \frac{8}{175} = 0.04571$$

·) Gauss - Chebyshev method: Any integral can be written in the form:

$$\int_{-1}^{1} \omega(x)\, f(x)\, dx = \sum_{i=0}^{n} \lambda_i f_i$$

For this method, the weight is written as:

$$\omega(x) = \frac{1}{\sqrt{1-x^2}}$$

(i) One - point formula: The integral can be written as:

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \lambda_0 f_0$$

As only $\lambda_0$ and $t_0$ are the unknowns, $f(t)$ must satisfy:

$f(t) = 1$
$f(t) = t$

When $f(t) = 1$

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Therefore, $\lambda_0(1) = \pi$
$\lambda_0 = \pi$

When $f(t) = t$

$$I = \int_{-1}^{1} \frac{t\, dt}{\sqrt{1-t^2}}$$

Taking $1-t^2 = z$
$-2t\,dt = dz$

$$t\,dt = -\frac{dz}{2}$$

The limits become:

$$z(1) = 1-(1)^2 = 0$$

$$z(-1) = 1-(-1)^2 = 0$$

The integration reduces for only point. Hence,

$$\lambda_0 t_0 = 0$$

or, $t_0 = 0$ [as $\lambda_0 = \pi$]

The one-point formula now becomes:

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} = \pi f(0)$$

~~The~~ The error constant is:

$$C = \int_{-1}^{1} \frac{x^2\,dx}{\sqrt{1-x^2}} - \pi(0)$$

$$= 2\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}}$$

Putting $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$C = 2\int_0^{\pi/2} \sin^2\theta\, d\theta = \cancel{2/2\left[\frac{1-\cos 2\theta}{2}\right]_0^{\pi/2}}$$

$$\cancel{= \frac{1-(-1)+(1-(1))}{2}}$$

$$= \theta - \frac{\sin 2\theta}{2}\Big|_0^{\pi/2} = \frac{\pi}{2} = 1.57.$$

(ii) Two-point formula: The integral can be written as:

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} = \lambda_0 f_0 + \lambda_1 f_1$$

As there are 4 unknowns, $f(t)$ should satisfy the following:

$$f(t)=1, \quad f(t)=t, \quad f(t)=t^2, \quad f(t)=t^3$$

when $f(t)=1$

$$\int_{-1}^{1} \frac{(1)\,dt}{\sqrt{1-t^2}} = \sin^{-1} t \Big|_{-1}^{1} = \pi$$

So, $\lambda_0(1) + \lambda_1(1) = \pi$

$\lambda_0 + \lambda_1 = \pi$ — (I)

when $f(t) = t$

$$\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}} = 0 \quad \text{[ As it is an odd function]}$$

So, $\lambda_0 t_0 + \lambda_1 t_1 = 0$ — (II)

when $f(t) = t^2$

$$\int_{-1}^{1} \frac{t^2\,dt}{\sqrt{1-t^2}} = 2\int_{0}^{\pi/2} \sin^2\theta\, d\theta = \pi/2$$

So, $\lambda_0 t_0^2 + \lambda_1 t_1^2 = \pi/2$ — (III)

when $f(t) = t^3$

$$\int_{-1}^{1} \frac{t^3\,dt}{\sqrt{1-t^2}} = 0 \quad \text{[ As it is an odd function]}$$

So, $\lambda_0 t_0^3 + \lambda_1 t_1^3 = 0$ — (IV)

From (II),

$$\lambda_0 = -\frac{\lambda_1 t_1}{t_0} \quad \text{-(VI)}$$

From (VI) and (IV),

$$-\lambda_1 t_1 t_0^2 + \lambda_1 t_1^3 = 0$$

$$\lambda_1 t_1 (t_1^2 - t_0^2) = 0$$

$$\lambda_1 t_1 (t_1 - t_0)(t_1 + t_0) = 0$$

As $\lambda_1 \neq 0$ and $t_1 \neq 0$, and $t_0, t_1$ are distinct, we say:

$$t_1 + t_0 = 0$$

$$t_0 = -t_1$$

(II) now becomes:

$$\lambda_0 t_0 - \lambda_1 t_0 = 0$$

$$\lambda_0 - \lambda_1 = 0$$

$$\lambda_0 = \lambda_1$$

From (I)

$$\lambda_0 = \lambda_1 = \pi/2$$

From (III),

$$2\lambda_0 t_0^2 = \pi/2$$

$$t_0^2 = \frac{\pi}{2 \times 2 \lambda_0} = \frac{1}{2}$$

$$t_0^2 = \frac{1}{2}$$

$$t_0 = \pm\frac{1}{\sqrt{2}}$$

and $t_1 = \mp \frac{1}{\sqrt{2}}$

The two-point formula is:

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} = \pi/2 \left[ f\left(\frac{1}{\sqrt{2}}\right) + f\left(-\frac{1}{\sqrt{2}}\right) \right]$$

The error constant is:

$$C = \int_{-1}^{1} \frac{x^4\,dx}{\sqrt{1-x^2}} - \frac{\pi}{2}\left[\frac{1}{4} + \frac{1}{4}\right]$$

$$= 2\int_0^{\pi/2} \sin^4\theta\, d\theta - \frac{\pi}{4}$$

$$= 2\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) - \frac{\pi}{4} = \frac{\pi}{8} = 0.3925$$

(iii). 3-point formula: The integral can be written as:

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} = \lambda_0 f(t_0) + \lambda_1 f(t_1) + \lambda_2 f(t_2)$$

As there are 6 unknowns, the function $f(t)$ must satisfy:

$$f(t)=1,\quad f(t)=t,\quad f(t)=t^2,\quad f(t)=t^3,\quad f(t)=t^4,\quad f(t)=t^5$$

when $f(t)=1$

$$\int_{-1}^{1} \frac{(1)\,dt}{\sqrt{1-t^2}} = \pi$$

So, $\lambda_0 + \lambda_1 + \lambda_2 = \pi$ - (vi)

when $f(t)=t$

$$\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}} = 0 \quad [\text{ As it is an odd function}]$$

$\lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2 = 0$ - (vii)

when $f(t)=t^2$

$$\int_{-1}^{1} \frac{t^2\,dt}{\sqrt{1-t^2}} = \frac{\pi}{2}$$

So, $\lambda_0 t_0^2 + \lambda_1 t_1^2 + \lambda_2 t_2^2 = \frac{\pi}{2}$ - (viii)

When $f(t) = t^3$

$$\int_{-1}^{1} \frac{t^3 dt}{\sqrt{1-t^2}} = 0 \quad \text{[As it is an odd function]}$$

$$\lambda_0 t_0^3 + \lambda_1 t_1^3 + \lambda_2 t_2^3 = 0 \quad - \text{(IX)}$$

When $f(t) = t^4$

$$\int_{-1}^{1} \frac{t^4 dt}{\sqrt{1-t^2}} = 2\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{3\pi}{8}$$

$$\lambda_0 t_0^4 + \lambda_1 t_1^4 + \lambda_2 t_2^4 = \frac{3\pi}{8} \quad - \text{(X)}$$

When $f(t) = t^5$

$$\int_{-1}^{1} \frac{t^5 dt}{\sqrt{1-t^2}} = 0 \quad \text{[As it is an odd function]}$$

$$\lambda_0 t_0^5 + \lambda_1 t_1^5 + \lambda_2 t_2^5 = 0 \quad - \text{(XI)}$$

From (VII),

$$\lambda_0 = \frac{-[\lambda_1 t_1 + \lambda_2 t_2]}{t_0}$$

Now, (IX) becomes

$$-[\lambda_1 t_1 + \lambda_2 t_2] t_0^2 + \lambda_1 t_1^3 + \lambda_2 t_2^3 = 0$$

$$\lambda_1 t_1 (t_1^2 - t_0^2) + \lambda_2 t_2 (t_2^2 - t_0^2) = 0 \quad - \text{(XII)}$$

and (XI) becomes

$$\lambda_1 t_1^3 (t_1^2 - t_0^2) + \lambda_2 t_2^3 (t_2^2 - t_0^2) = 0 \quad - \text{(XIII)}$$

~~From (XII)~~

$$\cancel{(t_1^2 - t_0^2) = \frac{-\lambda_2 t_2 (t_2^2 - t_0^2)}{\lambda_1 t_1}}$$

From (XII),

$$\lambda_2 t_2 = \frac{-\lambda_1 t_1 (t_1^2 - t_0^2)}{(t_2^2 - t_0^2)}$$

(XIII) now becomes,

$$\lambda_1 t_1^3 (t_1^2 - t_0^2) - \lambda_1 t_1 t_2^2 (t_1^2 - t_0^2) = 0$$

$$\lambda_1 t_1 (t_1^2 - t_0^2)(t_2^2 - t_1^2) = 0$$

$$\lambda_1 t_1 (t_1 - t_0)(t_2 - t_1)(t_1 + t_0)(t_2 + t_1) = 0$$

As $t_0$, $t_1$ and $t_2$ are distinct

$$\lambda_1 t_1 (t_1 + t_0)(t_2 + t_1) = 0 \quad \text{- (XIV)}$$

let $t_1 = 0$, then, (XIII) becomes:

$$\cancel{\lambda_2} \; \lambda_2 t_2 (t_2^2 - t_0^2) = 0$$

$$\lambda_2 t_2 (t_2 - t_1)(t_2 + t_0) = 0$$

$$\lambda_2 t_2 (t_2 + t_0) = 0$$

As $\lambda_2 \neq 0$, $t_2 \neq 0$ [as $t_1 = 0$],

$$t_2 + t_0 = 0$$

$$t_2 = -t_0 \quad \text{- (XV)}$$

(VII) now becomes,

$$\lambda_0 t_0 + \lambda_1 (0) + \lambda_2 (-t_0) = 0$$

$$(\lambda_0 - \lambda_2) t_0 = 0$$

$$\lambda_0 = \lambda_2$$

(VIII) becomes

$$2\lambda_0 t_0^2 = \pi/4 \quad \text{- (XVI)}$$

(X) becomes

$$2\lambda_0 t_0^4 = 3\pi/8 \quad \text{- (XVII)}$$

Dividing (XVI) by (XVI),

$$t_0^2 = \frac{3}{4}$$

$$t_0 = -\frac{\sqrt{3}}{2} \qquad t_2 = +\frac{\sqrt{3}}{2}$$

From (XVI)

$$2 \times \lambda_0 \times \frac{3}{4} = \frac{\pi}{2}$$

$$\lambda_0 = \frac{\pi}{3} = \lambda_2$$

From (VI),

$$\lambda_1 + \frac{2\pi}{3} = \pi$$

$$\lambda_1 = \frac{\pi}{3}$$

~~Hence,~~

Hence, the three point formula becomes:

$$\int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} = \frac{\pi}{3}\left[f\left(\frac{\sqrt{3}}{2}\right) + f(0) + f\left(-\frac{\sqrt{3}}{2}\right)\right]$$

The error constant is:

$$C = \int_{-1}^{1} \frac{x^6\,dx}{\sqrt{1-x^2}} - \frac{\pi}{3}\left[\frac{2\times 27}{32}\right]$$

$$= \int_{-\pi/2}^{\pi/2} \sin^6\theta - \frac{9\pi}{16} = 2\left(\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) - \frac{9\pi}{16}$$

$$= \frac{\pi}{32} = 0.098125$$