Algorithm for Bisection Method

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Step 1

Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)\, f(x_u) < 0$, or in other words, f(x) changes sign between x_ℓ and x_u. This was demonstrated in Figure 1.

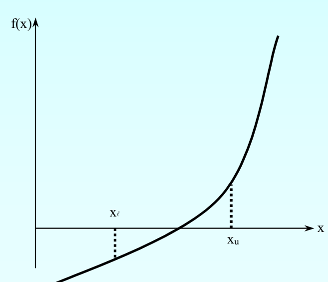

Figure 1

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Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

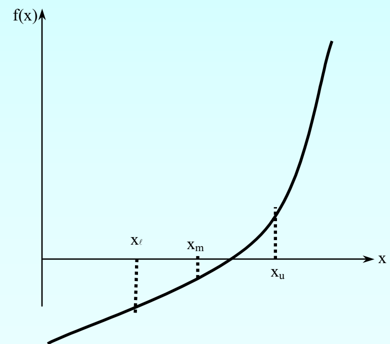

Figure 5 Estimate of x_m

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Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

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Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

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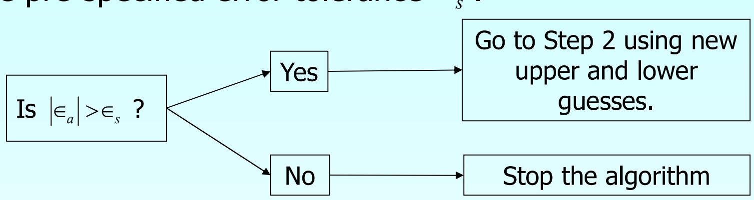
Step 5

Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified error tolerance $\in_s$.



Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

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Example 1a.

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4} = 0 \quad for$$
$$0 \le x \le 0.11$$

- Use the bisection method of finding roots of equation
- Find the absolute relative approximate error at the end of each iteration.

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Example 1 Cont.

Let us assume

$$x_\ell = 0.00$$
$$x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u .

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$
$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

So there is at least on root between x_ℓ and x_u, that is between 0 and 0.11

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Example 1 Cont.

Iteration 1

The estimate of the root is $x_m = \dfrac{x_\ell + x_u}{2} = \dfrac{0 + 0.11}{2} = 0.055$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993 \times 10^{-4} = 6.655 \times 10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = \left(3.993 \times 10^{-4}\right)\left(6.655 \times 10^{-5}\right) > 0$$

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $\left|\in_a\right|$ cannot be calculated as we do not have a previous approximation.

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Example 1 Cont.

Iteration 2

The estimate of the root is $x_m = \dfrac{x_\ell + x_u}{2} = \dfrac{0.055 + 0.11}{2} = 0.0825$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993 \times 10^{-4} = -1.622 \times 10^{-4}$$

$$f(x_l)f(x_m) = f(0.055)f(0.0825) = \left(-1.622 \times 10^{-4}\right)\left(6.655 \times 10^{-5}\right) < 0$$

Hence the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.0825$$

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Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.333\%$$

None of the significant digits are at least correct in the estimate root of $x_m = 0.0825$ because the absolute relative approximate error is greater than 5%.

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Example 1 Cont.

<u>Iteration 3</u>
The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.0825}{2} = 0.06875$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$

$$f(x_l)f(x_m) = f(0.055)f(0.06875) = \left(6.655 \times 10^{-5}\right)\left(-5.563 \times 10^{-5}\right) < 0$$

Hence the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.06875. So, the lower and upper limits of the new bracket are

$$x_l = 0.055, \ x_u = 0.06875$$

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Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100$$

$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.
Seven more iterations were conducted and these iterations are shown in Table 1.

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Table 1 Cont.

Table 1 Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\|\in_a\|\%$	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

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