Example: Solve $2x^3 - 2.5x - 5 = 0$ for the root in the interval $[1, 2]$ by Bisection method.

Sol) Given $f(x) = 2x^3 - 2.5x - 5$ on $[1, 2]$

$f(1) = -5.5 < 0$

$f(2) = 6 > 0$

Bisection Method

iteration no.	a_n	b_n	$\xi_n = \frac{a_n+b_n}{2}$	$f(\xi_n)$
0	1.0000000000	2.0000000000	1.5000000000	-2.0000000000
1	1.5000000000	2.0000000000	1.7500000000	1.3437500000
2	1.5000000000	1.7500000000	1.6250000000	-0.4804687500
3	1.6250000000	1.7500000000	1.6875000000	0.3920898438
4	1.6250000000	1.6875000000	1.6562500000	-0.0538940430
5	1.6562500000	1.6875000000	1.6718750000	0.1666488647
6	1.6562500000	1.6718750000	1.6640625000	0.0557680130
7	1.6562500000	1.6640625000	1.6601562500	0.0007849932
8	1.6562500000	1.6601562500	1.6582031250	-0.0265924782
9	1.6582031250	1.6601562500	1.6591796875	-0.0129132364
10	1.6591796875	1.6601562500	1.6596679688	-0.0060664956
11	1.6596679688	1.6601562500	1.6599121094	-0.0026413449
12	1.6599121094	1.6601562500	1.6600341797	-0.0009283243
13	1.6600341797	1.6601562500	1.6600952148	-0.0000717027
14	1.6600952148	1.6601562500	1.6601257324	0.0003566360
15	1.6600952148	1.6601257324	1.6601104736	0.0001424643
16	1.6600952148	1.6601104736	1.6601028442	0.0000353802
17	1.6600952148	1.6601028442	1.6600990295	-0.0000181614
18	1.6600990295	1.6601028442	1.6601009369	0.0000086094

Regula Falsi Method

Iteration no.	a_n	b_n	W_n	$f(W_n)$
0	1.0000000000	2.0000000000	1.4782608747	-2.2348976135
1	1.4782608747	2.0000000000	1.6198574305	-0.5488323569
2	1.6198574305	2.0000000000	1.6517157555	-0.1169833690
3	1.6517157555	2.0000000000	1.6583764553	-0.0241659321
4	1.6583764553	2.0000000000	1.6597468853	-0.0049594725
5	1.6597468853	2.0000000000	1.6600278616	-0.0010169938
6	1.6600278616	2.0000000000	1.6600854397	-0.0002089010
7	1.6600854397	2.0000000000	1.6600972414	-0.0000432589
8	1.6600972414	2.0000000000	1.6600997448	-0.0000081223

Modified Regula Falsi Method

iteration no.	a_n	b_n	W_n	$f(W_n)$
0	1.0000000000	2.0000000000	1.4782608747	-2.2348976135
1	1.4782608747	2.0000000000	1.7010031939	0.5908976793
2	1.4782608747	1.7010031939	1.6544258595	-0.0793241411
3	1.6544258595	1.7010031939	1.6599385738	-0.0022699926
4	1.6599385738	1.7010031939	1.6602516174	0.0021237291
5	1.6599385738	1.6602516174	1.6601003408	0.0000002435

OPEN METHOD

Secant Method

Secant Method – Derivation

The secant method can also be derived from geometry:

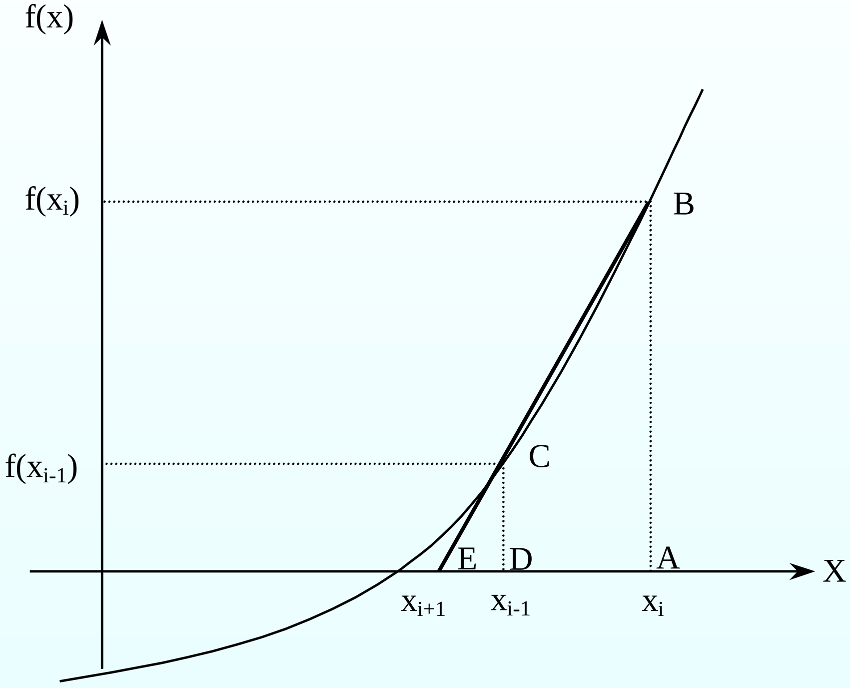


Figure 2 Geometrical representation of the Secant method.

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Algorithm for Secant Method

Step 1

Calculate the next estimate of the root from two initial guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Find the absolute relative approximate error

$$|\in_a| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

Step 2

Find if the absolute relative approximate error is greater than the prespecified relative error tolerance.

If so, go back to step 1, else stop the algorithm.

Also check if the number of iterations has exceeded the maximum number of iterations.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

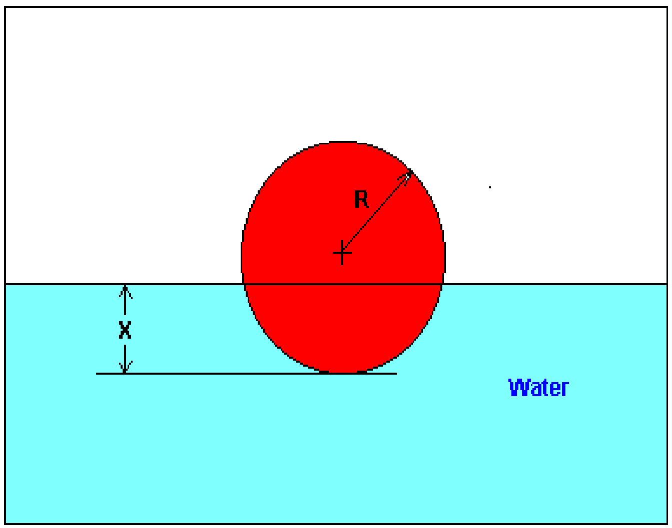


Figure 3 Floating Ball Problem.

Example 1 Cont.

The equation that gives the depth *x* to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

Use the Secant method of finding roots of equations to find the depth *x* to which the ball is submerged under water.

- Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

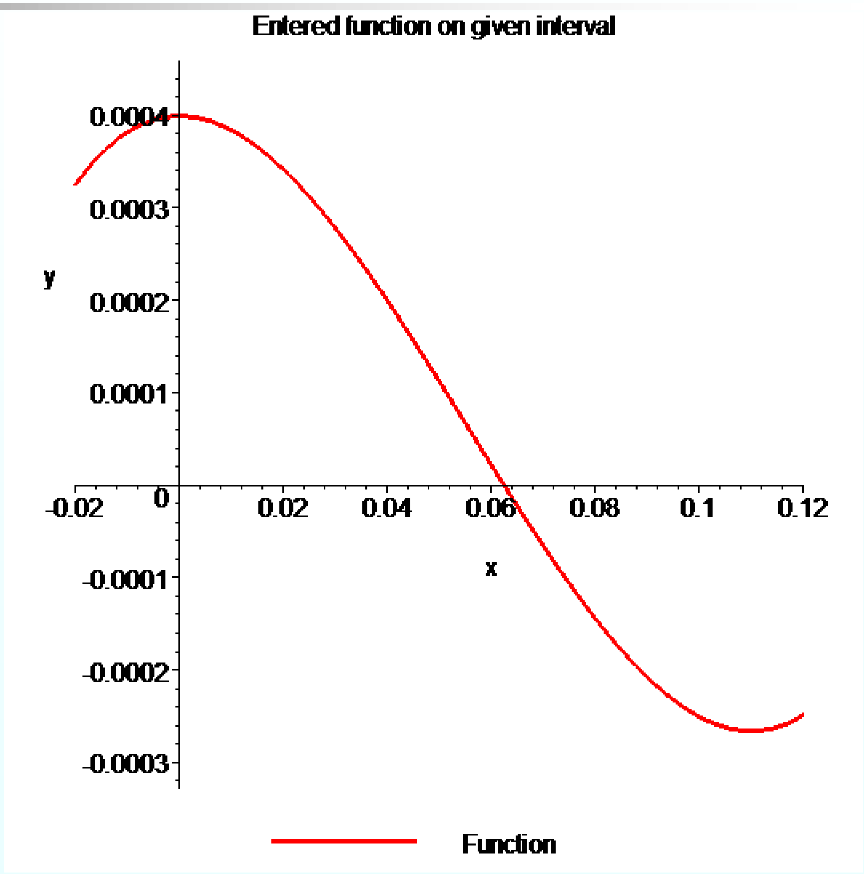


Figure 4 Graph of the function f(x).

Example 1 Cont.

Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05.$

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= 0.05 - \frac{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)(0.05-0.02)}{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right) - \left(0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06461$$

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 1 is

$$\left|\in_a\right| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06461 - 0.05}{0.06461}\right| \times 100$$

$$= 22.62\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for one significant digits to be correct in your result.

Example 1 Cont.

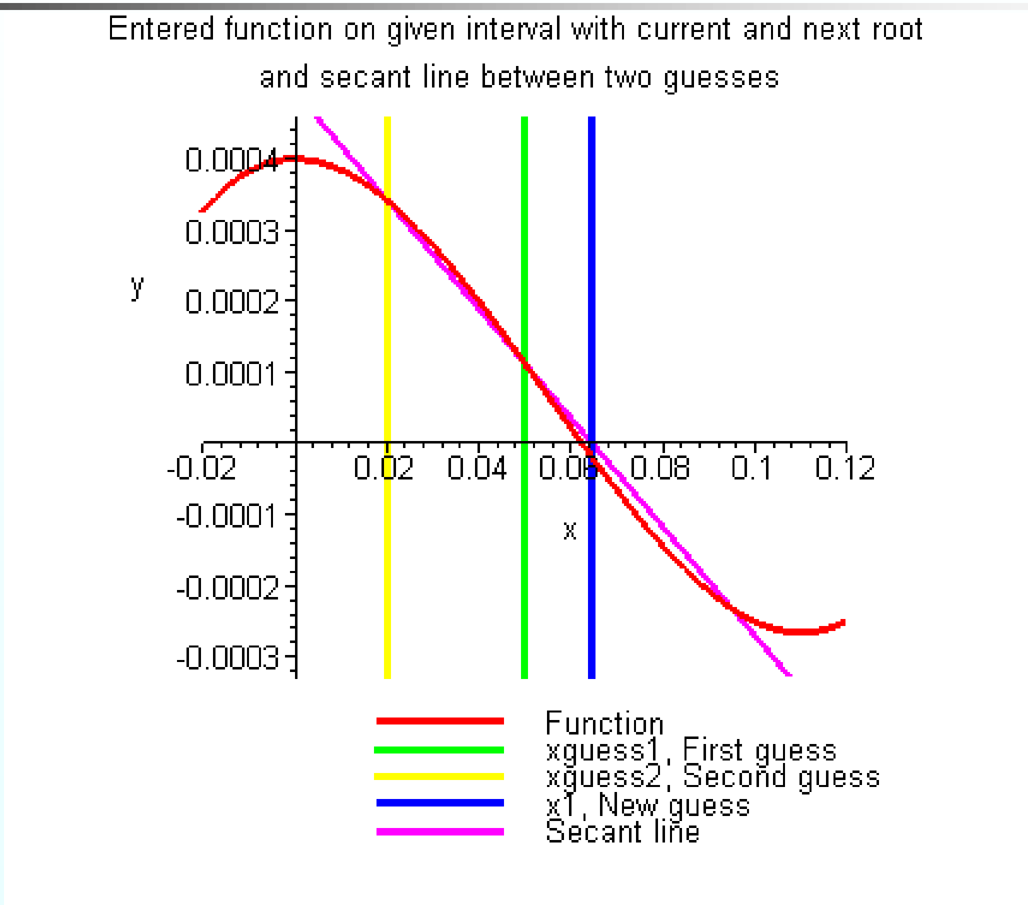


Figure 5 Graph of results of Iteration 1.

Example 1 Cont.

<u>Iteration 2</u>

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 0.06461 - \frac{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)(0.06461 - 0.05)}{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06241$$

Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$
\begin{aligned}
|\in_a| &= \left|\frac{x_2 - x_1}{x_2}\right| \times 100 \\
&= \left|\frac{0.06241 - 0.06461}{0.06241}\right| \times 100 \\
&= 3.525\%
\end{aligned}
$$

The number of significant digits at least correct is 1, as you need an absolute relative approximate error of 5% or less.

Example 1 Cont.

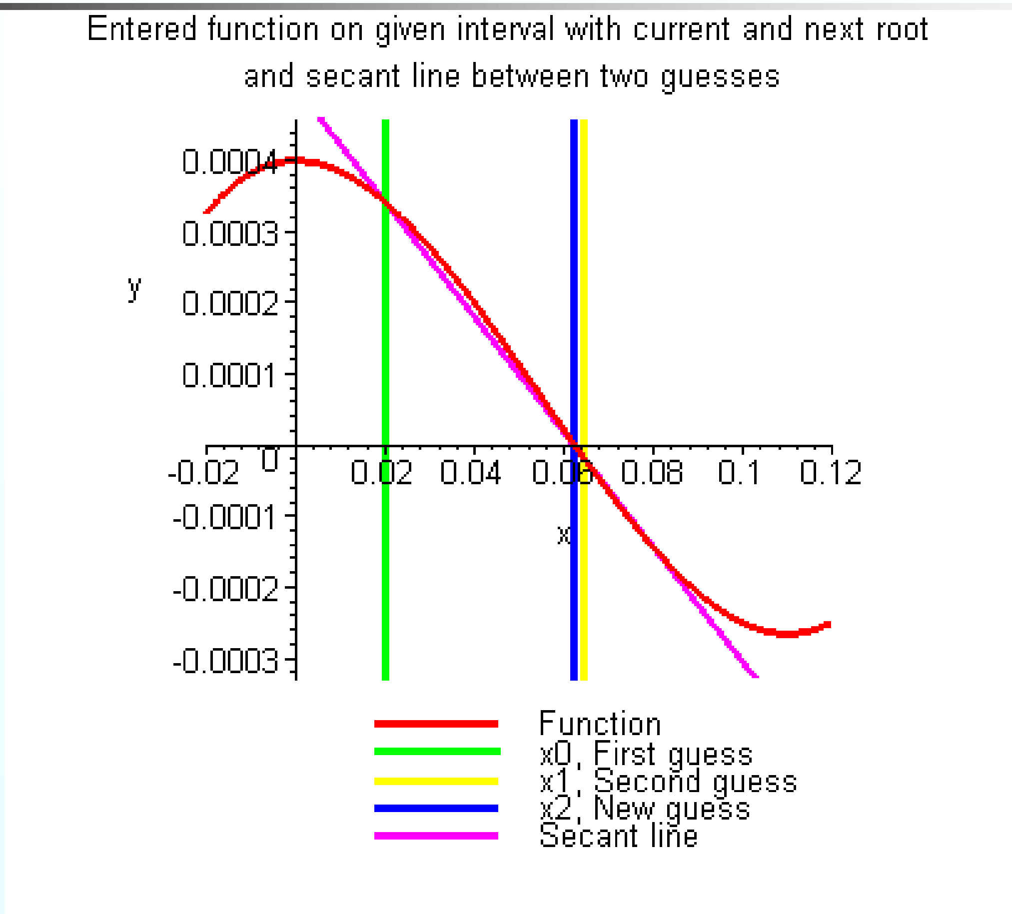


Figure 6 Graph of results of Iteration 2.

Example 1 Cont.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 0.06241 - \frac{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right)(0.06241 - 0.06461)}{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06238$$

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_3 - x_2}{x_3}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06241}{0.06238}\right| \times 100$$

$$= 0.0595\%$$

The number of significant digits at least correct is 5, as you need an absolute relative approximate error of 0.5% or less.

Iteration #3

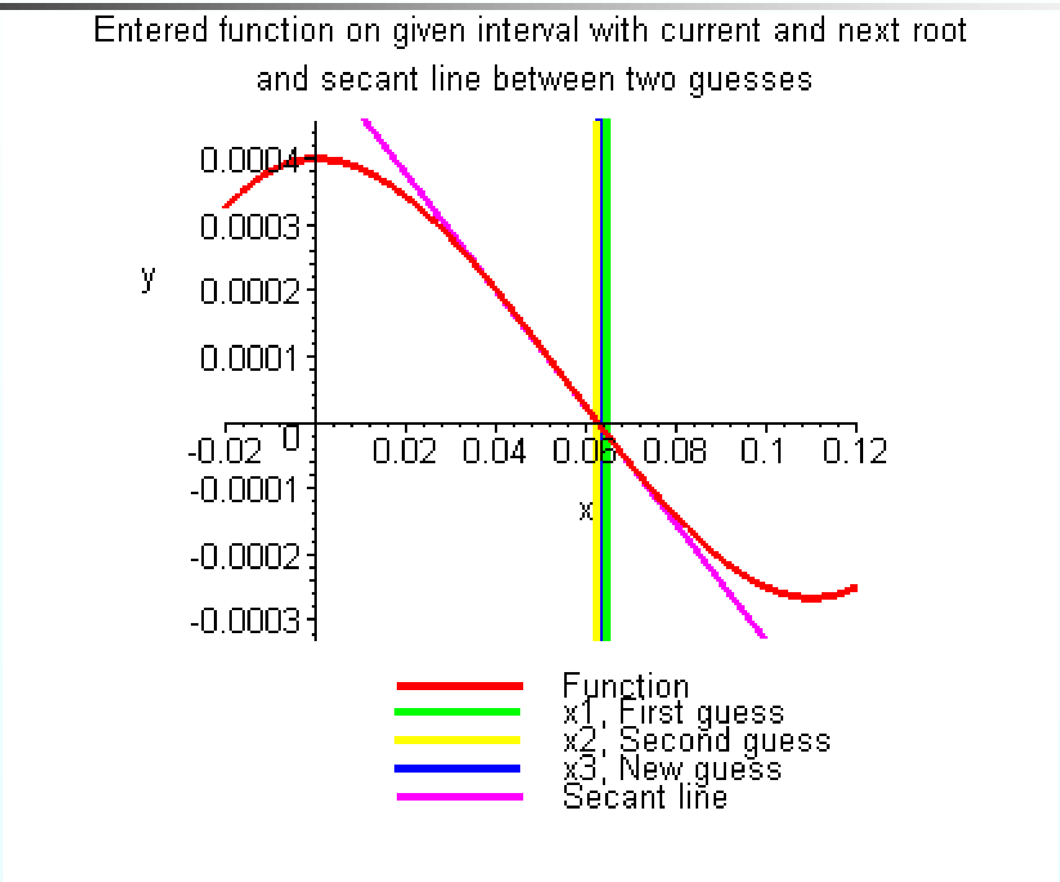


Figure 7 Graph of results of Iteration 3.

Advantages

- Converges fast, if it converges
- Requires two guesses that do not need to bracket the root

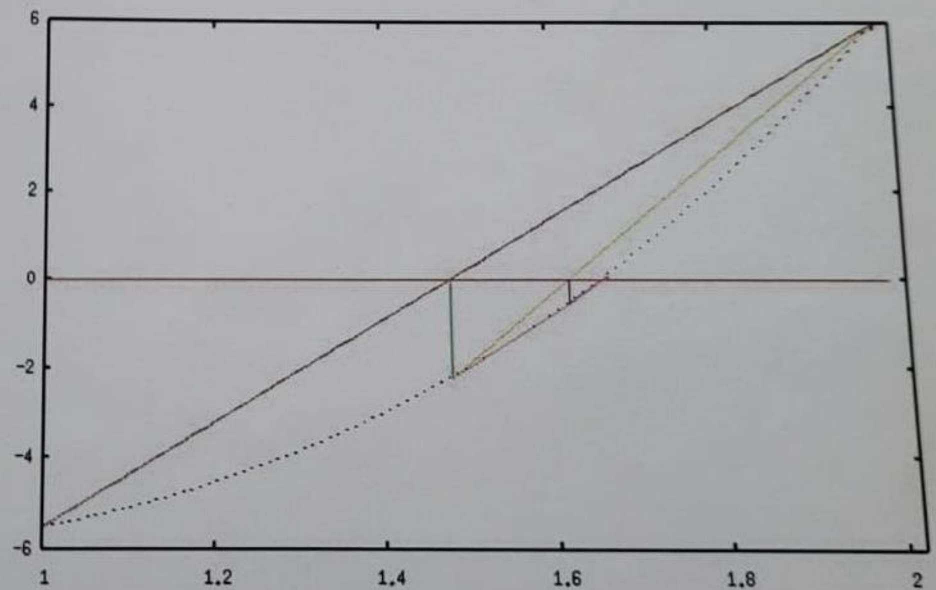
Example: Solve $2x^3 - 2.5x - 5 = 0$ for the root in the interval $[1, 2]$ by Bisection method.

Sol) Given $f(x) = 2x^3 - 2.5x - 5$ on $[1, 2]$

$f(1) = -5.5 < 0$

$f(2) = 6 > 0$

Geometrical visualisation of the root tracking procedure by secant method for the above example:

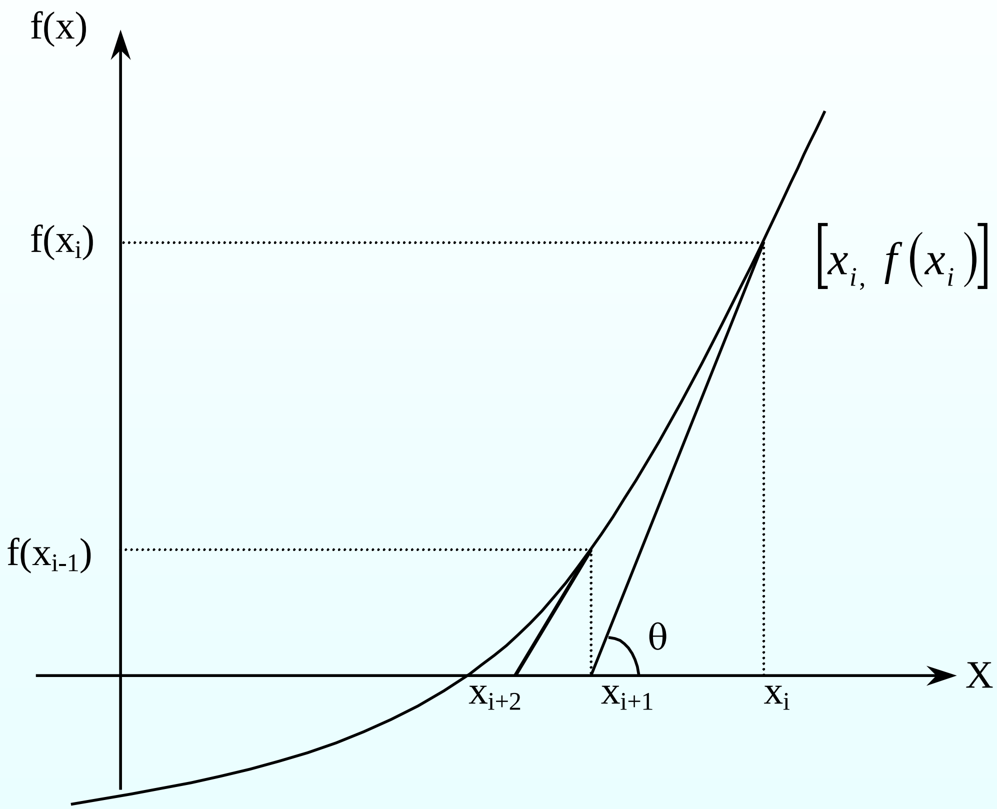


Secant Method

iteration no.	x_{n-1}	x_n	x_{n+1}	$f(x_{n+1})$
0	1.0000000000	2.0000000000	1.4782608747	-2.2348976135
1	2.0000000000	1.4782608747	1.6198574305	-0.5488323569
2	1.4782608747	1.6198574305	1.6659486294	0.0824255496
3	1.6198574305	1.6659486294	1.6599303484	-0.0023854144
4	1.6659486294	1.6599303484	1.6600996256	-0.0000097955

Newton-Raphson Method

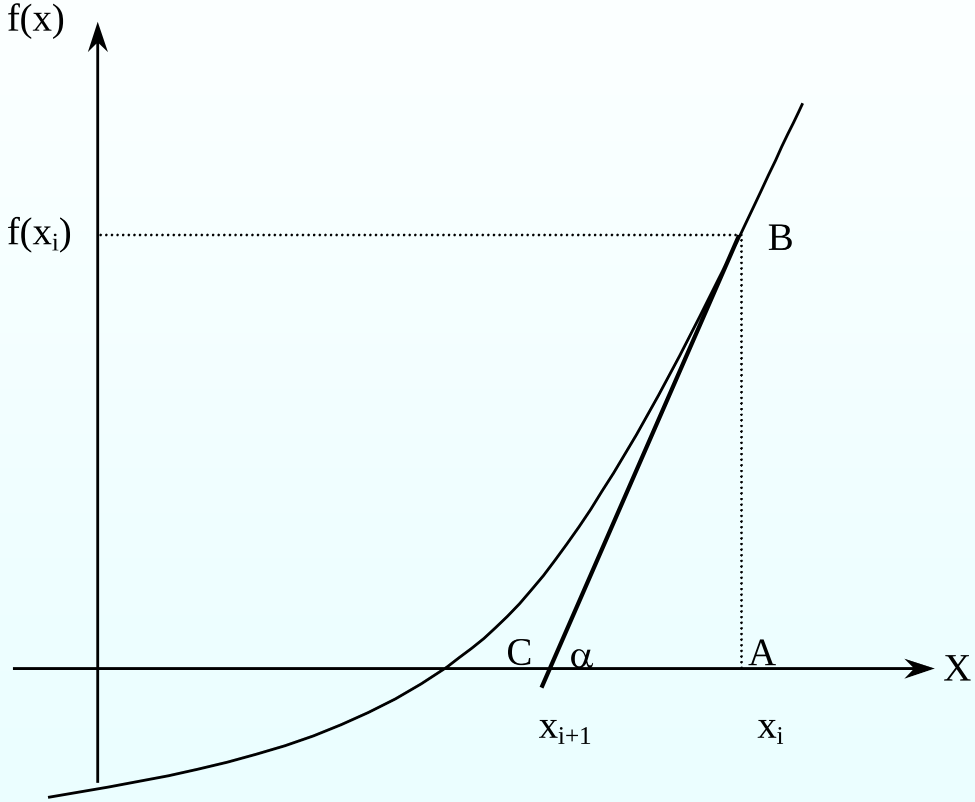
Newton-Raphson Method



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 1 Geometrical illustration of the Newton-Raphson method.

Derivation



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure 2 Derivation of the Newton-Raphson method.

Algorithm for Newton-Raphson Method

Step 1

Evaluate $f'(x)$ symbolically.

Step 2

Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

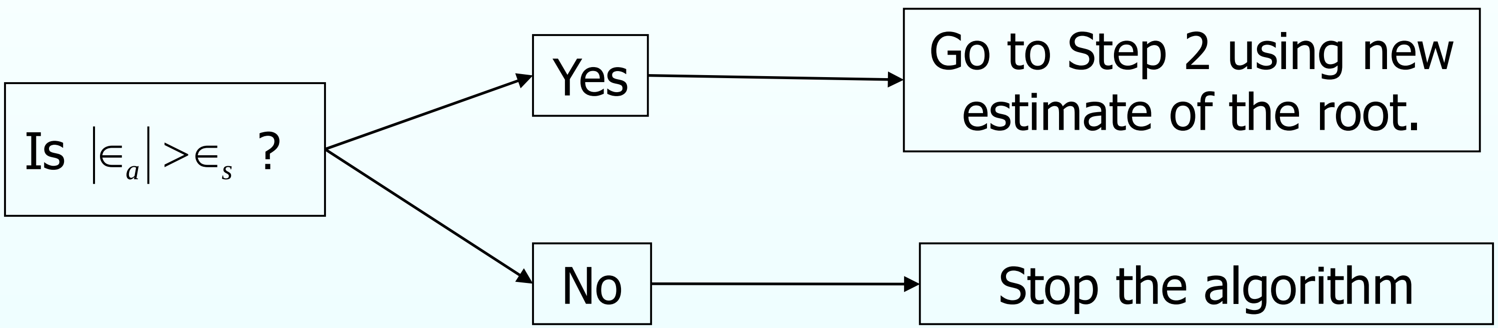
Step 3

Find the absolute relative approximate error $|\in_a|$ as

$$|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

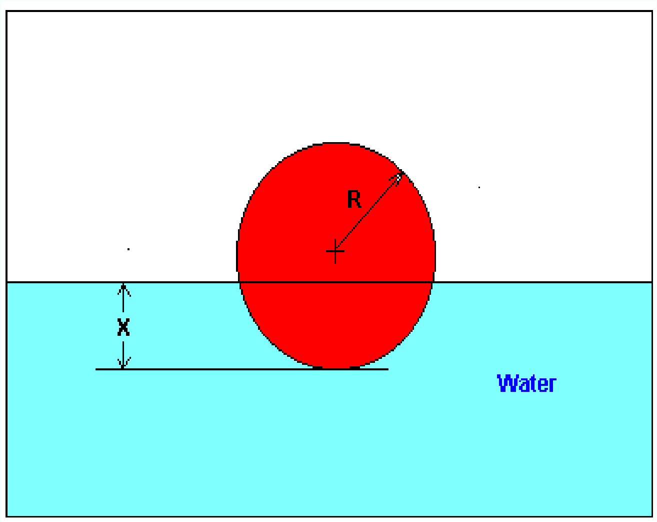


Figure 3 Floating ball problem.

Example 1 Cont.

The equation that gives the depth x in meters to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

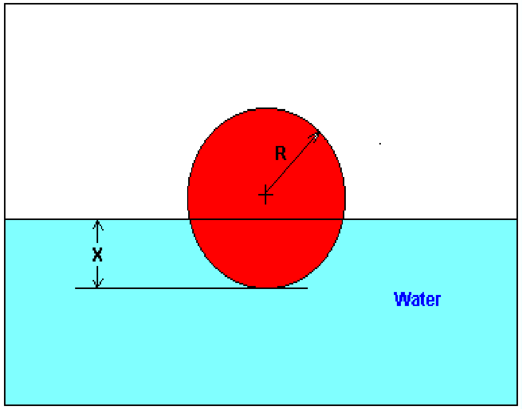


Figure 3 Floating ball problem.

Use the Newton's method of finding roots of equations to find

a) the depth 'x' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.
b) The absolute relative approximate error at the end of each iteration, and
c) The number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times 10^{-4}$$

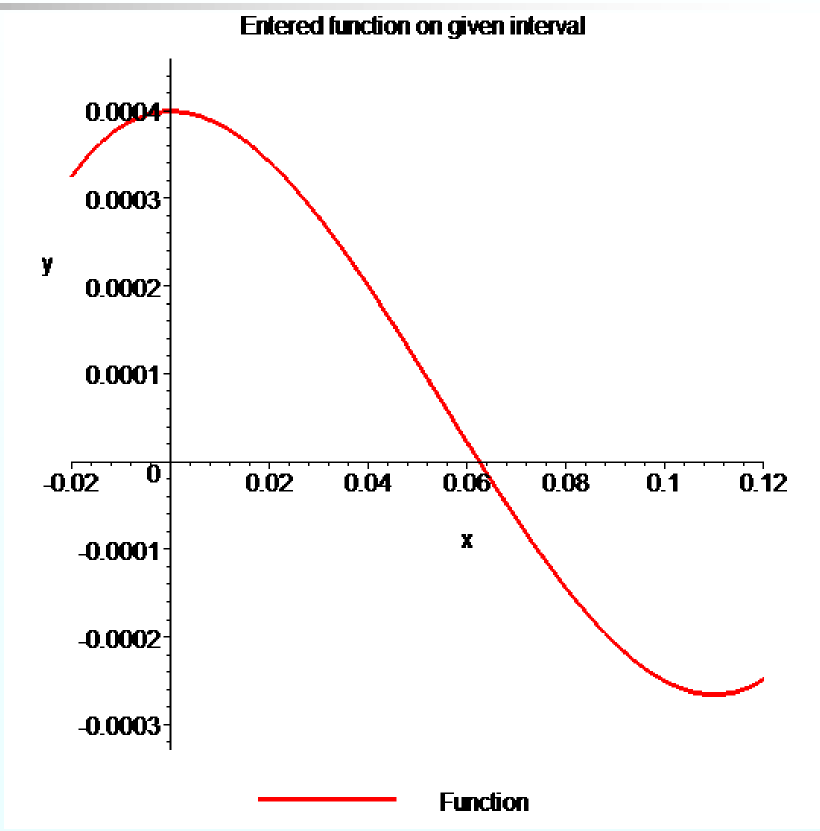


Figure 4 Graph of the function f(x)

Example 1 Cont.

Solve for $f'(x)$

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of $f(x) = 0$ is $x_0 = 0.05\text{m}$. This is a reasonable guess (discuss why $x = 0$ and $x = 0.11\text{m}$ are not good choices) as the extreme values of the depth x would be 0 and the diameter (0.11 m) of the ball.

Example 1 Cont.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993\times10^{-4}}{3(0.05)^2 - 0.33(0.05)}$$

$$= 0.05 - \frac{1.118\times10^{-4}}{-9\times10^{-3}}$$

$$= 0.05 - (-0.01242)$$

$$= 0.06242$$

Example 1 Cont.

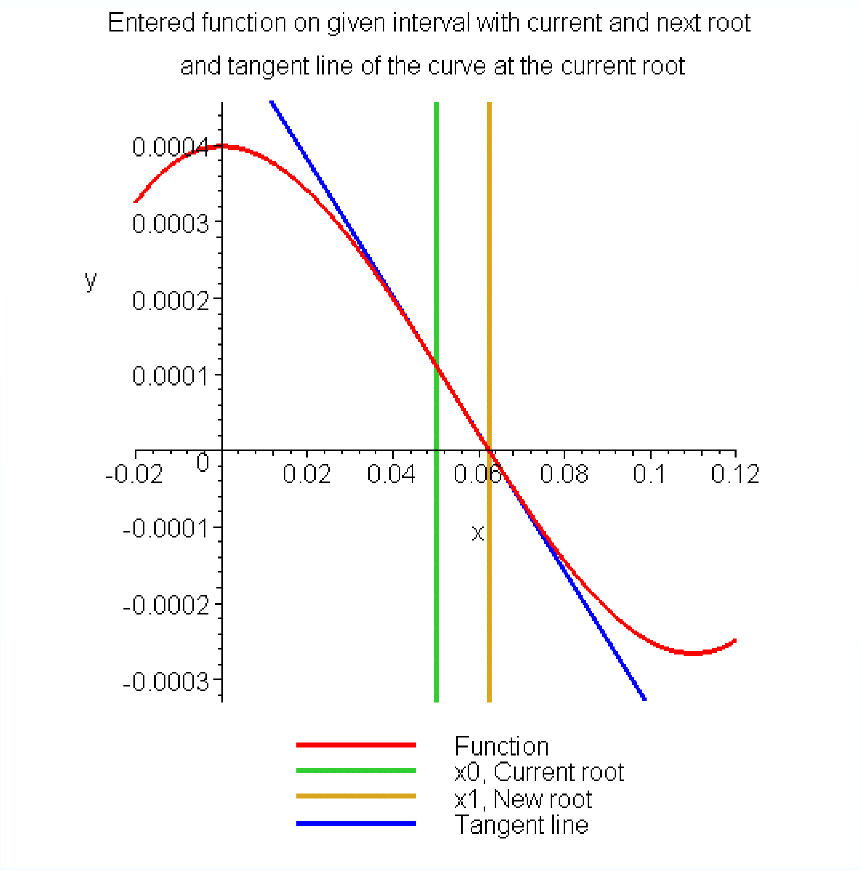


Figure 5 Estimate of the root for the first iteration.

Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100$$

$$= \left|\frac{0.06242 - 0.05}{0.06242}\right| \times 100$$

$$= 19.90\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digits to be correct in your result.

Example 1 Cont.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993\times10^{-4}}{3(0.06242)^2 - 0.33(0.06242)}$$

$$= 0.06242 - \frac{-3.97781\times10^{-7}}{-8.90973\times10^{-3}}$$

$$= 0.06242 - \left(4.4646\times10^{-5}\right)$$

$$= 0.06238$$

Example 1 Cont.

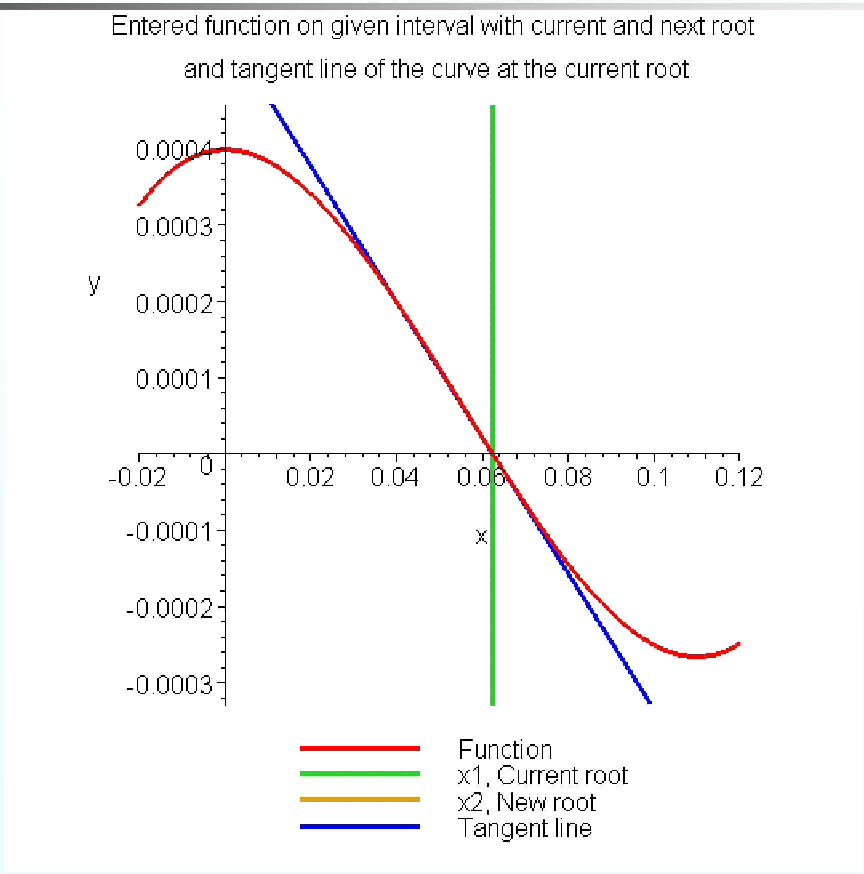


Figure 6 Estimate of the root for the Iteration 2.

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06242}{0.06238}\right| \times 100$$

$$= 0.0716\%$$

The maximum value of m for which $\left|\in_a\right| \leq 0.5 \times 10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

Example 1 Cont.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f\left(x_2\right)}{f\,'\left(x_2\right)}$$

$$= 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993\times10^{-4}}{3(0.06238)^2 - 0.33(0.06238)}$$

$$= 0.06238 - \frac{4.44\times10^{-11}}{-8.91171\times10^{-3}}$$

$$= 0.06238 - \left(-4.9822\times10^{-9}\right)$$

$$= 0.06238$$

Example 1 Cont.

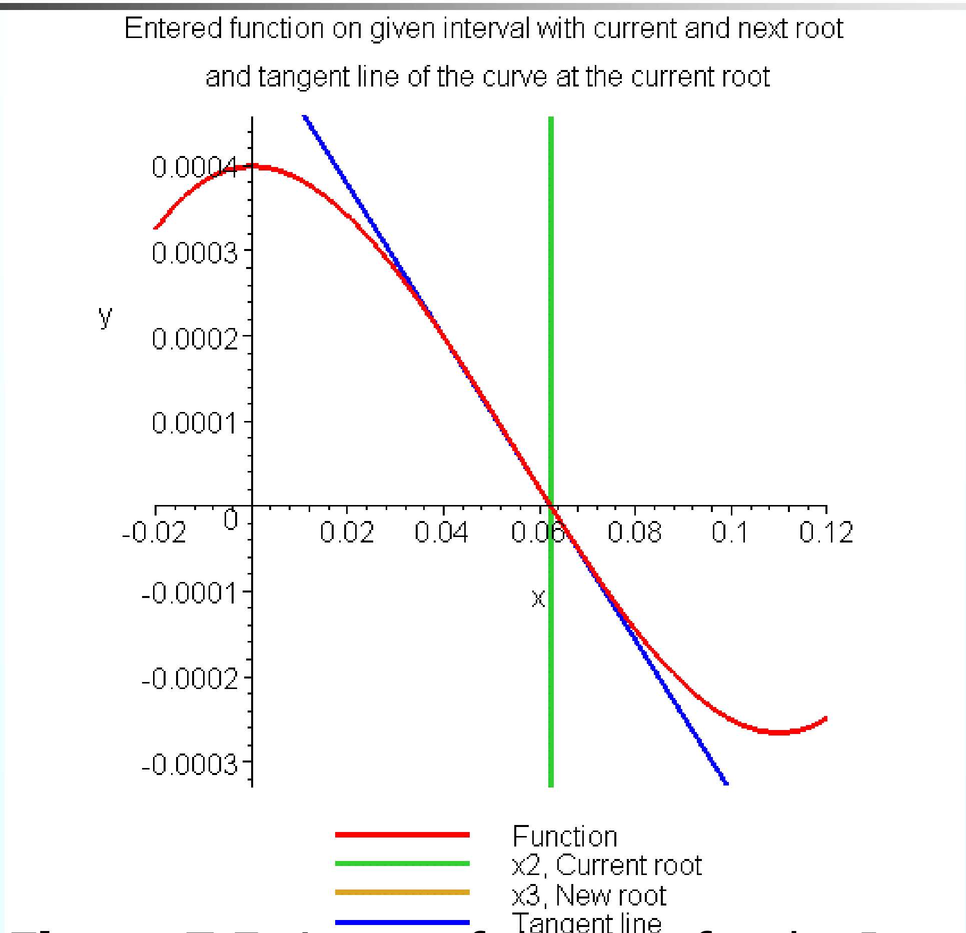


Figure 7 Estimate of the root for the Iteration 3.

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06238}{0.06238}\right| \times 100$$

$$= 0\%$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through all the calculations.

Advantages

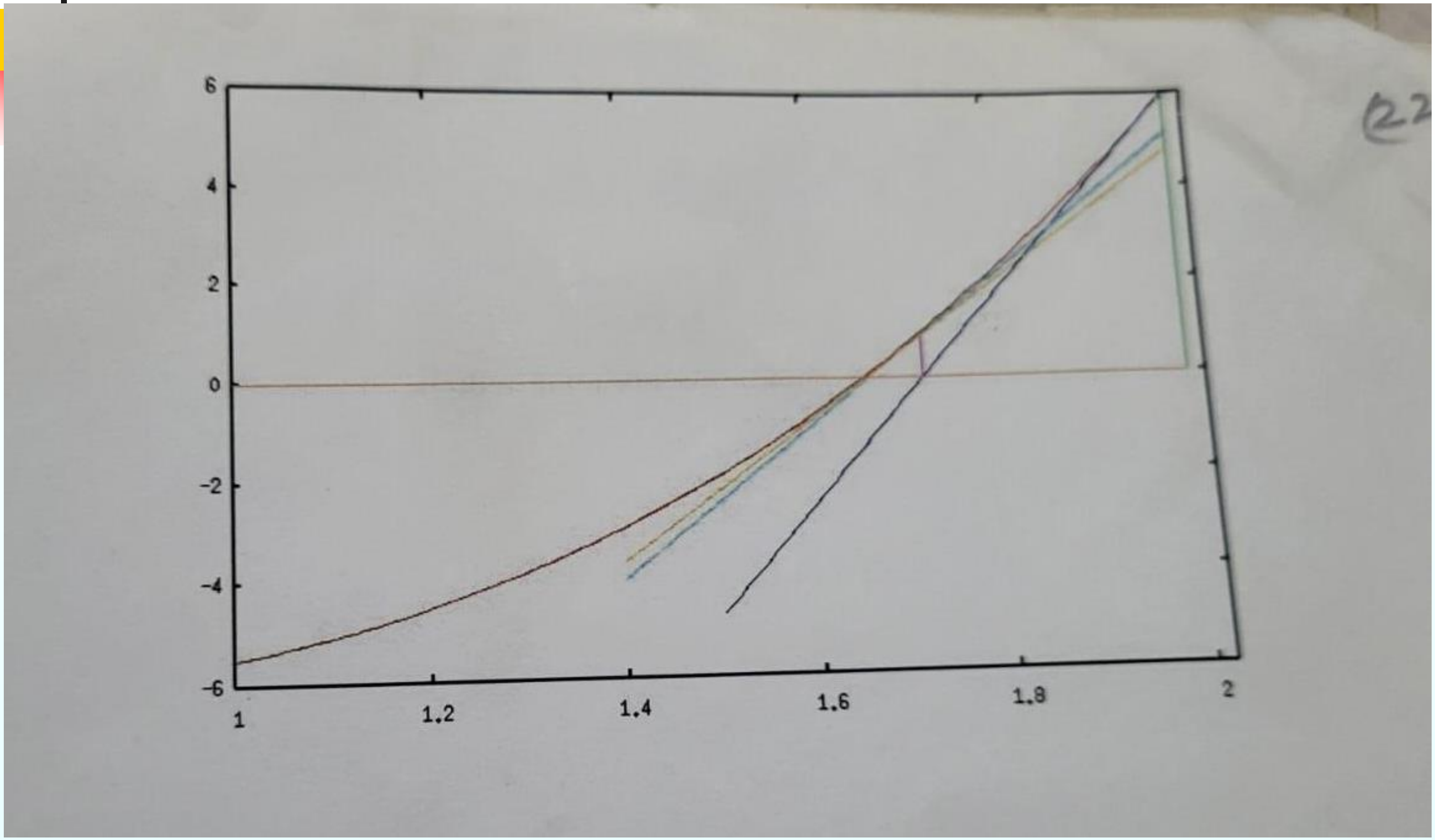
- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Example: Solve $2x^3 - 2.5x - 5 = 0$ for the root in the interval $[1,2]$ by Bisection method.

Sol) Given $f(x) = 2x^3 - 2.5x - 5$ on $[1,2]$

$f(1) = -5.5 < 0$

$f(2) = 6 > 0$



6
4
2
0
-2
-4
-6
1
1.2
1.4
1.6
1.8
2

iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example: Solve $2x^3 - 2.5x - 5 = 0$ for the root in $[1, 2]$ by Newton Raphson method.

Sol) Given $f(x) = 2x^3 - 2.5x - 5 = 0$

$f'(x) = 8x^2 - 2.5.$

Take $x_0 = 2$ and $\epsilon = 0$

$\therefore f(x_0) = 6$; $f'(x_0) = 29.5$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{6.0}{29.5} = 1.7209302187$

$|f(x_1)| = 0.8910911679 > 10^{-6}$. $\therefore$ repeat the process.

Results are tabulated below:

Newton Raphson Method

Iteration no. (n)	x_n	x_{n+1}	$\lvert f(x_{n+1}) \rvert$
0	2.0000000000	1.7209302187	0.8910911679
1	1.7209302187	1.6625729799	0.0347661413
2	1.6625729799	1.6601046324	0.0000604780
3	1.6601046324	1.6601003408	0.0000002435