Fixed Point and Functional Iteration

Definition *x is called a fixed point of a given function f if $f(x) = x$.*

If $f : \mathbb{R} \to \mathbb{R}$ is a function, then the fixed point of f can be viewed as the intersection of the curve $y = f(x)$ and the straight line $y = x$.

The problem of finding a fixed point for a given function is equivalent to the problem of finding a zero for a nonlinear fucntion. For instance, given a function $f(x)$ and a root x_* such that $f(x_*) = 0$, let $g(x) = x - f(x)$. Then $g(x_*) = x_* - f(x_*) = x_*$. That is, x_* is a fixed point for $g(x)$. Conversely, suppose that $g(x)$ has a fixed point at x_*. Then one may define $f(x) = x - g(x)$ so that $f(x_*) = x_* - g(x_*) = x_* - x_* = 0$, i.e., x_* is a zero of $f(x)$.

Functional Iteration

To determine a fixed point x_* for a continuous function f, we choose an initial point x_0 and generate a sequence of points $\{x_k\}_{k\geq 0}$ by

$$x_{k+1} = f(x_k), \quad k \geq 0.$$

This algorithm is called fixed-point iteation or functional iteration.

say,

$$\lim_{k\to\infty} x_k = x_*,$$

then, since f is continuous,

$$f(x_*) = f(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} x_{k+1} = x_*.$$

That is, x_* is a fixed point of f.

Note that Newton's method for solving $g(x) = 0$

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

is just a special case of functional iteration in which

$$f(x) = x - \frac{g(x)}{g'(x)}.$$

Definition *A function (mapping) f is said to be contractive if there exists a constant $0 \leq \lambda < 1$ such that*

$$|f(x) - f(y)| \leq \lambda |x - y|$$

for all x, y in the domain of f.

Theorem **(Contractive Mapping Theorem)** *Suppose $f : D \to D$, where $D \subseteq \mathbb{R}$ is a closed set, is a contractive mapping. Then f has a unique fixed point in D. Moreover, this fixed point is the limit of every sequence obtained by*

$$x_{k+1} = f(x_k)$$

with any initial point x_0.

Theorem **(Contractive Mapping Theorem)** *Suppose $f : D \to D$, where $D \subseteq \mathbb{R}$ is a closed set, is a contractive mapping. Then f has a unique fixed point in D. Moreover, this fixed point is the limit of every sequence obtained by*

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Proof: We first show that $\lim_{k\to\infty} x_k$ exists. Since

$$x_k = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_k - x_{k-1}) = x_0 + \sum_{i=1}^{k}(x_i - x_{i-1}),$$

$\{x_k\}_{k\geq 0}$ converges if and only if $\sum_{i=1}^{\infty}(x_i - x_{i-1})$ converges and it is sufficient to show $\sum_{i=1}^{\infty}|x_i - x_{i-1}|$ converges.

Since f is contractive, we have

$$\begin{aligned} |x_i - x_{i-1}| &= |f(x_{i-1}) - f(x_{i-2})| \\ &\leq \lambda|x_{i-1} - x_{i-2}| \\ &\leq \lambda^2|x_{i-2} - x_{i-3}| \\ &\vdots \\ &\leq \lambda^{i-1}|x_1 - x_0|. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{\infty}|x_i - x_{i-1}| &\leq \sum_{i=1}^{\infty}\lambda^{i-1}|x_1 - x_0| \\ &= |x_1 - x_0|\sum_{i=1}^{\infty}\lambda^{i-1} \\ &= \frac{1}{1-\lambda}|x_1 - x_0| \qquad \text{since } 0 \leq \lambda < 1. \end{aligned}$$

This show that $\sum_{i=1}^{\infty}|x_i - x_{i-1}|$ is bounded, hence it converges.

the sequence $\{x_k\}_{k\geq 0}$ converges for any initial point x_0..

Let $\lim_{k\to\infty} x_k = x_*$. Then we have showed that x_* would be a fixed point of f. (note: contractivenedd implies continuity)

To prove the uniqueness, let x and y both be fixed points of f. Then

$$|x-y| = |f(x) - f(y)| \leq \lambda |x-y|.$$

Since $\lambda < 1$, this forces $|x-y| = 0$. That is, $x = y$.