$\sum f(a+k_i h_i) = K_{odd} = \sum_{k=1}^{n_i+1} f(a+2kh_i) = \sum_{k=1}^{n_i-1} f(a+kh_{i-1})$

$$I_i = \frac{1}{2} h_{i-1}\left[\frac{1}{2} f(a) + \frac{1}{2} f(b) + \sum_{k=1}^{n_{i-1}-1} f(a+kh_{i-1}) \right.$$

$$\left. + \; h_i \sum_{k\,odd} f(a+kh_i) \right]$$

Adaptive method.

$$I_i = I_{i-1} + h_i \sum_{k\,odd} f(a+kh_i).$$

Simpson

$$\epsilon = \frac{1}{90} h^4 [f'''(a) - f'''(b)] \propto O(h^4)$$ 3

→ Error dependence on the number of steps.

$N_1 \Rightarrow h_1 = \frac{b-a}{N_1} \Rightarrow I_1$

$N_2 = 2N_1 \Rightarrow h_2 = \frac{(b-a)}{N_2} \Rightarrow I_2$

Trapezoidal $O(\epsilon) \approx ch^2$

Simpson $O(\epsilon) \approx ch^4$

$h_1 >>> h_2 \rightarrow$ [Error $|(I_2 - I_1)|$]

→ $I_2 = I_1 + ch_1^2$

$I_3 = I_2 + ch_2^2$

$I_2 - I_1 = c(h_1^2 - h_2^2)$

$(I_2 - I_1) = 3ch_2^2$ $\quad \epsilon = ch_2^2$

$$\epsilon = ch_2^2 = \frac{1}{3}(I_2 - I_1)$$

$h_i = \frac{b-a}{N_i}$

$h_{i-1} = 2h_i$

$N_{i-1} = \frac{N_i}{2}$

Adaptive rules

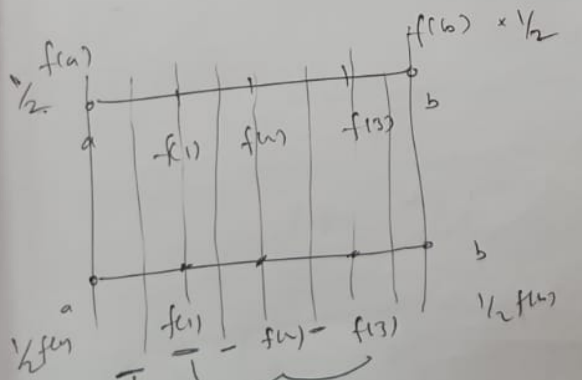
$N_1,\ N_2 = 2N_1,\ N_3 = 2N_2 \ \ldots$

$$\epsilon_i = \frac{1}{3}(I_i - I_{i-1})$$

$I_1 = h_i \sum \frac{1}{2} f(a) + \frac{1}{2} f(b)$

$+ \sum_{k=1}^{N_1-1} f(a + b h_i)$

$+ \sum_{k=1} f(a + b h_i$



These new points are to be calculated.

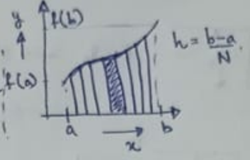
[dp] as this part are already stored.

1. Integration:→

- Numerical:

Trapezoidal Rule:

k^{th} slice area: $\frac{1}{2}[f(a+kh)+f(a+(k-1)h)]h$ ← A_k



$h = \frac{b-a}{N}$.

$$I(a,b) = \sum_{k=1}^{N} A_k = \int_a^b f(x)\,dx$$

$$= \frac{1}{2}h\left[\sum_{k=1}^{N}\left[f(a+kh) + f(a+(k-1)h)\right]\right] = h\left[\frac{1}{2}f(a) + \frac{1}{2}(f(a+h) + f(a+h)) + f(a+2h) + \dots + \frac{1}{2}f(b)\right]$$

$$= h\left[\frac{f(a)}{2} + f(a+h) + f(a+2h) + \dots + \frac{1}{2}f(b)\right]$$

$$= h\left[\frac{f(a)}{2} + \frac{f(b)}{2} + \sum_{k=1}^{N-1} f(a+kh)\right].$$

Ex: $\int_0^2 (x^4 - 2x + 1)\,dx \rightarrow f(x) = x^4 - 2x + 1$

$a=0, b=2$. N: ?? (how to choose)

Code:

```
def f(x):
    return x**4 - 2*x + 1
N = 10
a = 0.0 , b = 2.0
h = (b-a)/N.
s = 0.5 * f(a) + 0.5 * f(b)
for k in range (1, N):
    s += f(a + kh)
print (h*s)
```

$N\uparrow \Rightarrow$ better approx.

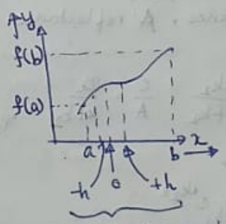
Error: $O(h^2) \rightarrow$ correct upto $O(h)$

$c. f(x) / (1-2)^2$

$N \rightarrow 2N$

2. Simpson's Rule:→ (or one-third rule)

Fit a quadratic curve through three points.



Take three points for each $\frac{1}{3}$rd portion of the actual curve.
(or $\frac{1}{N}$th too!)

Quadratic: $Ax^2 + Bx + C$.

$f(-h) = Ah^2 - Bh + C$.

$f(0) = C$

$f(h) = Ah^2 + Bh + C$

$$A = \frac{1}{h^2}\left[\frac{1}{2}f(-h) - f(0) + \frac{1}{2}f(h)\right]$$

$$B = \frac{1}{2h}\left[f(h) - f(-h)\right], \quad C = f(0).$$

Area under the curve of $f(x)$ from $-h$ to $+h$:

$$\int_{-h}^{h}(Ax^2 + Bx + C)\,dx = \frac{2}{3}Ah^3 + 2Ch = \frac{1}{3}h(f(-h) + 4f(0) + f(h)] \rightarrow 1$$

First three points:

$x = a, a+h, a+2h$.

Then: $x = a+2h, a+3h, a+4h$

$$I(a,b) \cong \frac{1}{3}h\left[f(a) + 4f(a+h) + f(a+2h)\right] + \frac{1}{3}h\left[f(a+2h) + 4f(a+3h) + f(a+4h)\right] + \dots + \frac{1}{3}h\left[f(a+(N-2)h) + 4f(a+(N-1)h) + f(b)\right]$$

Expand

$$\Rightarrow I(a,b) \approx \frac{1}{3}h\left[f(a) + f(b)\right] + \left[4\sum_{\substack{k\in odd \\ k\neq a,b}} f(a+kh) + 2\sum_{\substack{k\in even \\ k\neq a,b}} f(a+kh)\right]$$

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→ Error in numerical calculations. (integration)

$$I(a,b) = \int_a^b f(x)\cdot dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) \cdots f(x_{n-2}) + 2f(x_{n-1}) \right]$$

$$\boxed{h = \frac{b-a}{n}}$$

→ x_{k-1} point

$$f(x) = f(x_{k-1}) + (x - x_{k-1}) f'(x_{k-1}) + \frac{1}{2}(x - x_{k-1})^2 f''(x_{k-1}) + \cdots$$

$$\int_{x_{k-1}}^{x_k} f(x)\cdot dx = f(x_{k-1})\int_{x_{k-1}}^{x_k} dx + f'(x_{k-1})\int_{x_{k-1}}^{x_k} (x - x_{k-1})^2 dx + \frac{1}{2} f''(x_{k-1}) \int_{x_{k-1}}^{x_k} (x - x_{k-1})^2 dx \cdots$$

$x - x_{k-1} = u$ | $\Rightarrow$

$$f(x_{k-1})\int_0^h dx + f'(x_{k-1})\int_0^h (u)\,du + \frac{1}{2} f''(x_{k-1}) \int_0^h u^2 du \cdots$$

$$\int_{x_k} = h f(x_{k-1}) + f'(x_{k-1}) \frac{h^2}{2} + \frac{f''(x_{k-1})}{2} \times \frac{h^3}{6} + O(h^4)$$

$$\int_{x_{k-1}}^{x_k} f(x)\,dx = h f(x_k) - \frac{1}{2} h^2 f'(x_k) + \frac{h^3}{2} f''(x) - O(h^4)$$

$$\int_{x_{k-1}}^{x_k} f(x)\cdot dx = \frac{1}{2}\left[h \left(f(x_{k-1}) + f(x_k) \right) \right] + \frac{1}{4} h^2 \left(f'(x_{k-1}) + f'(x_k) \right) + \frac{h^3}{2}\left[f''(x_{k-1}) + f''(x_k) \right] + O(h^4)$$

$= \frac{1}{2} h \sum_{k=1}^{i} [f(x_{k-1}) + f(x_k)] + \frac{1}{2} h^2 [f'(a) - f''(b)]$

$$+ \frac{1}{12} h^3 [f''(x_{k-1}) - f''(x)] + O(h^4)$$

→ Calculation accurate in order.

for the 2nd order derivative

$$\int_a^b f''(x) \cdot dx = \frac{1}{2} \left[\sum_{k=1}^{N} [(f''(x_{k-1}) + f''(x_k)] + O(h^2) \right]$$

$$= \frac{1}{12} h^3 \left[\sum_{k=1}^{N} (f''(x_{k-1}) + f''(x_k) \right]$$

$$= \frac{1}{6} h^2 \int_a^b f''(x) \cdot dx$$

$$= \frac{1}{h} \cdot \frac{1}{6} h^2 [f'(b) - f'(a)] + O(h^4)$$

$$\left[\int_a^b f(x)\, dx = \frac{1}{2} h \sum_{k=1}^{N} (f(x_{k-1}) + f(x_k)) + \frac{1}{12} h^2 (f'(a) - f'(b)) + O(h^4) \right]$$

$$E = \frac{1}{12} h^2 [f'(a) - f'(b)] \approx O(h^4)$$

→ ~~error~~ depends on? ~~what~~ what?

first order approximate error?

1. Romberg Integration:→

Practical error estimation: $N \to 2N$, check $I_2 - I_1$. If it is less than 10^{-8}, just stop.

← helps to get higher accuracy even while using Trapezoidal rule.

Trapezoidal: Accurate upto $O(h^2) \approx Ch^2$ ⇝ Only accurate upto first-order.

Simpson: Accurate upto $O(h^3)$.

$$ch_i^2 = \frac{1}{3}(I_i - I_{i-1}) \Rightarrow I = I_i + ch_i^2 + O(h^4)$$

$$= \boxed{I_i + \frac{1}{3}(I_i - I_{i+1})} + O(h^4) \longrightarrow (A)$$

Integral is now accurate upto third-order, by adding a few terms.

Generalisation: (only for Trapezoidal)

$$R_{i,1} = I_i \,.\quad R_{i,2} = I_i + \frac{1}{3}(I_i - I_{i-1}).$$

$$\Rightarrow R_{i,2} = R_{i,1} + \frac{1}{3}(R_{i,1} - R_{i-1,1})$$

(A): $$I = R_{i,2} + c_2 h_i^4 + O(h_i^6) \left\{= R_{i-1,2} + 16c_2 h_i^4 + O(h_i^6)\right\}$$

Pf: $$I = R_{i-1,2} + c_2 h_{i-1}^4 + O(h_{i-1}^6) = R_{i-1,2} + 16c_2 h_i^4 + O(h_i^6).$$

$$\Rightarrow c_2 h_i^4 = \frac{1}{15}(R_{i,2} - R_{i-1,2}) + O(h_i^6).$$

$$\boxed{I = R_{i,2} + \frac{1}{15}(R_{i,2} - R_{i-1,2}) + O(h_i^6)}$$ → Accurate in 5th order.

With an error of order h^{2m},

$$I = R_{i,m} + c_m h_i^{2m} + O(h_i^{2m+2})$$

Also, $$I = R_{i-1,m} + c_m h_{i-1}^{2m} + O(h_{i-1}^{2m+2}) = R_{i-1,m} + 4^m c_m h_i^{2m} + O(h_i^{2m+2})$$

$$c_m h_i^{2m} = \frac{1}{4^m - 1}(R_{i,m} - R_{i-1,m}) + O(h_i^{2m+2}).$$

$$\therefore\, I = R_{i,m+1} + O(h_i^{2m+2})\,;\quad R_{i,m+1} = R_{i,m} + \frac{1}{4^m - 1}(R_{i,m} - R_{i-1,m}).$$

Algorithm:→

~~$I_1 - R_{1,1}$, $I_2 = R_{2,1} \to R_{2,2}$~~
~~$I_3 =$~~

$$\begin{tikzcd}
I_1 = R_{1,1} \arrow[dr] & & & \\
I_2 = R_{2,1} \arrow[r] \arrow[dr] & R_{2,2} \arrow[dr] & & \\
I_3 = R_{3,1} \arrow[r] \arrow[dr] & R_{3,2} \arrow[r] \arrow[dr] & R_{3,3} \arrow[dr] & \\
I_4 = R_{4,1} \arrow[r] & R_{4,2} \arrow[r] & R_{4,3} \arrow[r] & R_{4,4}
\end{tikzcd}$$