Convergence Analysis

Definition *Let $\{x_n\}$ be a sequence of real numbers that converges to x^*. We say that the rate of convergence is*

1. *linear if there exist a constant $0 < c < 1$ and an integer $N > 0$ such that*

$$|x_{n+1} - x^*| \leq c|x_n - x^*|, \quad \forall\, n \geq N;$$

2. *superlinear if there exist a sequence $\{c_n\}$, $c_n \to 0$ as $n \to \infty$, and an integer $N > 0$ such that*

$$|x_{n+1} - x^*| \leq c_n|x_n - x^*|, \quad \forall\, n \geq N,$$

or, equivalently,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = 0;$$

Convergence Analysis…

Definition *Let $\{x_n\}$ be a sequence of real numbers that converges to x^*. We say that the rate of convergence is*

3. *quadratic if there exist a constant $c > 0$ (not necessarily less than 1) and an integer $N > 0$ such that*

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^2, \quad \forall\, n \geq N.$$

In general, if there are positive constants c and α and an integer $N > 0$ such that

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^\alpha, \quad \forall\, n \geq N,$$

then we say the rate of convergence is of order α.

Convergence Analysis…

Definition *Suppose $\{\beta_n\}$ is a sequence known to converge to zero and $\{x_n\}$ converges to x^*. If there exist a postive constant c and an integer $N > 0$ such that*

$$|x_n - x^*| \leq c|\beta_n|, \quad \forall\, n \geq N,$$

then we say $\{x_n\}$ converges to x^* with rate of convergence $O(\beta_n)$, and write $x_n = x^* + O(\beta_n)$.

Example *Compare the convergence behavior of the sequences $\{x_n\}$ and $\{y_n\}$, where*

$$x_n = \frac{n+1}{n^2}, \quad \textit{and} \quad y_n = \frac{n+3}{n^3}.$$

Sol: Note that both

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = 0.$$

Let $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{n^2}$. Then

$$\begin{aligned} |x_n - 0| &= \frac{n+1}{n^2} \leq \frac{n+n}{n^2} = \frac{2}{n} = 2\alpha_n, \\ |y_n - 0| &= \frac{n+3}{n^3} \leq \frac{n+3n}{n^3} = \frac{4}{n^2} = 4\beta_n. \end{aligned}$$

Hence

$$x_n = 0 + O(\frac{1}{n}) \quad \text{and} \quad y_n = 0 + O(\frac{1}{n^2}).$$

This shows that $\{y_n\}$ converges to 0 much faster than $\{x_n\}$. ■

Bisection Method

Theorem *Let $[a_0, b_0], [a_1, b_1], \ldots$ denote the intervals produced by the bisection algorithm. Then $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, are equal, and represent a zero of $f(x)$. If $z = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and $c_n = \frac{1}{2}(a_n + b_n)$, then*

$$|z - c_n| \leq \frac{1}{2^{n+1}}(b_0 - a_0).$$

Proof:

Let $[a_0, b_0], [a_1, b_1], \ldots$ denote the successive intervals produced by the bisection algorithm. Then

$$\begin{aligned} a &= a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_0 = b \\ b &= b_0 \geq b_1 \geq b_2 \geq \cdots \geq a_0 = a. \end{aligned}$$

This means that the sequences $\{a_n\}$ and $\{b_n\}$ are bounded.

Hence $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Since

$$\begin{aligned} b_1 - a_1 &= \frac{1}{2}(b_0 - a_0) \\ b_2 - a_2 &= \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0) \\ &\vdots \\ b_n - a_n &= \frac{1}{2^n}(b_0 - a_0) \end{aligned}$$

hence

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \equiv z.$$

Since f is a continuous function

$lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(z)$ and $lim_{n\to\infty} f(b_n) = f(\lim_{n\to\infty} b_n) = f(z)$.

The bisection method ensures that $f(a_n)f(b_n) \leq 0$.

This implies that $\lim_{n\to\infty} f(a_n)f(b_n) = f^2(z) \leq 0.$,

and, consequently, $f(z) = 0$.

That is, the limit of the sequences $\{a_n\}$ and $\{b_n\}$ is a zero of f in $[a, b]$.

Let $c_n = \frac{1}{2}(a_n + b_n)$. Then

$$|z - c_n| = |\lim_{n\to\infty} a_n - \frac{1}{2}(a_n + b_n)| \leq |b_n - \frac{1}{2}(a_n + b_n)| = \frac{1}{2}|b_n - a_n| = \frac{1}{2^{n+1}}|b_0 - a_0|.$$

This proves the following theorem.

Newton's Method

Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f \in C^2[a, b]$, i.e., f" exists and is continuous.

Newton-Raphson's method,

starts with an initial approximation x_0 and generates the sequence $\{x_k\}_{k=0}^{\infty}$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Convergence Analysis

Suppose that f'' is continuous and x_* is a simple zero of f, i.e., $f(x_*) = 0$ but $f'(x_*) \neq 0$. Choose $\delta > 0$ and let

$$D = \{x \mid |x - x_*| \leq \delta\}$$

and

$$\gamma = \frac{1}{2} \cdot \frac{\max_{x \in D} |f''(x)|}{\min_{x \in D} |f'(x)|}.$$

Note that it is possible to choose δ such that $\rho = \delta\gamma < 1$ since f'' is continuous and when $\delta \to 0$, $x \to x_*$ and $\gamma \to \frac{1}{2} \cdot \frac{f''(x_*)}{f'(x_*)}$.

Now suppose we start Newton's iteration with x_0 satisfying $|e_0| = |x_0 - x_*| \leq \delta$. By Taylor's theorem

$$0 = f(x_*) = f(x_0) + f'(x_0)(x_* - x_0) + \frac{1}{2} f''(\xi_0)(x_* - x_0)^2,$$

where ξ_0 is between x_* and x_0.

Consequently $|\xi_0 - x_*| \leq \delta$ and

$$-f(x_0) - f'(x_0)(x_* - x_0) = \frac{1}{2} f''(\xi_0)(x_* - x_0)^2.$$

One iteration of Newton's algorithm gives

$$x_1 = x_0 - \frac{f(x_0}{f'(x_0)}.$$

Hence

$$\begin{aligned}
e_1 = x_1 - x_* &= x_0 - \frac{f(x_0)}{f'(x_0)} - x_* \\
&= \frac{1}{f'(x_0)} \left(-f(x_0) - x_* f'(x_0) + x_0 f'(x_0)\right) \\
&= \frac{1}{f'(x_0)} \left(-f(x_0) - f'(x_0)(x_* - x_0)\right) \\
&= \frac{1}{f'(x_0)} \cdot \frac{1}{2} \cdot f''(\xi_0)(x_* - x_0)^2 \\
&= \frac{f''(\xi_0)}{2f'(x_0)} e_0^2,
\end{aligned}$$

and

$$|e_1| \leq \gamma |e_0|^2 \leq \gamma\delta |e_0| = \rho |e_0|.$$

Repeat the argument, we have, in general,

$$|e_k| \leq \rho |e_{k-1}| \leq \cdots \leq \rho^k |e_0|.$$

Since $\rho < 1$, $|e_k| \to 0$ as $k \to 0$, that is, $x_k \to x_*$.

In summary, Newton's method will generate a seqence of numbers $\{x_k\}_{k\geq 0}$ that converges to the zero, x_*, of f if

1. f'' is continuous;
2. x_* is a simple zero of f; and
3. x_0 is close enough to x_*.

To investigate the convergence rate, we start with

$$x_{k+1} = x_{k+1} - x_* = x_k - \frac{f(x_k)}{f'(x_k)} - x_* = \frac{f'(x_k)e_k - f(x_k)}{f'(x_k)}.$$

Using Taylor's theorem

$$0 = f(x_*) = f(x_k - e_k) = f(x_k) - f'(x_k)e_k + \frac{1}{2}f''(\xi_k)e_k^2,$$

where ξ_k is between x_k and x_*, one has

$$f'(x_k)e_k - f(x_k) = \frac{1}{2}f''(\xi_k)e_k^2.$$

Hence

$$|e_{k+1}| = \frac{|f''(\xi_k)|}{2|f'(x_k)|}|e_k|^2 \approx \frac{|f''(x_*)|}{2|f'(x_*)|}|e_k|^2 \equiv C|e_k|^2.$$

This shows that Newton's method is quadratic convergent.

Secant Method

Secant Method

Like the Regula Falsi method and the Bisection method this method also requires two initial estimates x_{-1}, x_0 of the root of f(x)=0 but unlike those earlier methods it gives up the demand of bracketing the root. Like in the Regula Falsi method, this method too retains the use of secants throughout while tracking the root of f(x)=0. The secant joining the points $(x_{-1}, f(x_{-1}))$, $(x_0, f(x_0))$ is given by

$$y = \frac{f(x_0) - f(x_{-1})}{(x_0 - x_{-1})}x + \frac{f(x_{-1})x_0 - f(x_0)x_{-1}}{(x_0 - x_{-1})}$$

Say it intersects with x-axis at $x = x_1$, then

$$x_1 = \frac{f(x_0)x_{-1} - f(x_{-1})x_0}{f(x_0) - f(x_{-1})}$$

If $|f(x_i)| > \epsilon = 10^{-6}$ (say) then replace $(x_{-1}, f(x_{-1})), (x_0, f(x_0))$ with $(x_0, f(x_0)), (x_1, f(x_1))$ and repeat the process to get x_2 and so on . The method is algorithmically described below:

Algorithm:
Given a $f(x)$, two initial points a, b and ε the required level of accuracy carry out the following steps to find the root ξ of f(x)=0.

(1) Set $x_{-1} = a$, $x_0 = b$

(2) For n=0,1,2... until convergence criteria is satisfied, do:
Compute

$$x_{n+1} = \frac{[f(x_n)x_{n-1} - f(x_{n-1})x_n]}{[f(x_n) - f(x_{n-1})]}$$

Convergence of secant method:

Definition: Say, $x_n = \xi + e_n$, $x_{n+1} = \xi + e_{n+1}$ where ξ is the root of $f(x) = 0$, e_n, e_{n+1} are the errors at nth and (n+1)th iterations and x_n, x_{n+1} are the approximations of ξ at n^{th}, (n+1)th iterations. If $e_{n+1} = K e_n^p$ where K is a constant, then the rate of convergence of the method by which $\{x_n\}$ is generated is p.

Claim: Secant method has super linear convergence.
Proof: The iteration scheme for the secant method is given by

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{(f(x_n) - f(x_{n-1}))} \qquad (i)$$

$$= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad (ii)$$

Say $f(\xi) = 0$ and $e_n = (\xi - x_n)$ i.e the error in the nth iteration in estimating ξ.

$$\left.\begin{aligned} x_{n+1} &= e_{n+1} + \xi \\ x_n &= e_n + \xi \\ x_{n-1} &= e_{n-1} + \xi \end{aligned}\right\} (iii)$$

Using (iii) in (ii) we get

$$e_{n+1} = \frac{e_{n-1}f(x_n) - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})} \qquad (iv)$$

By Mean value Theorem, $\exists$ a η_n in the interval x_n and ξ s.t.

By Mean value Theorem, $\exists$ a η_n in the interval x_n and ξ s.t.

$$f'(\eta_n) = \frac{f(x_n) - f(\xi)}{x_n - \xi}$$

$$\because \quad f(\xi) = 0, \quad x_n - \xi = e_n$$

We get

$$f'(\eta_n) = \frac{f(x_n)}{e_n} \quad \text{i.e.} \quad f(x_n) = e_n f'(\eta_n) \qquad (v)$$

Using (iii)above, we get

$$f(x_{n-1}) = e_{n-1} f'(\eta_{n-1}) \qquad (vi)$$

$\therefore$ using (v), (vi), in (iv) we get

$$e_{n+1} = e_n\, e_{n-1} \frac{f'(\eta_n) - f'(\eta_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$i.e. \quad e_{n+1} \propto e_n\, e_{n-1} \qquad (vii)$$

By def of rate of convergence , the method is of order p if

$$\left.\begin{array}{l} e_n \propto e_{n-1}^p \\ i.e. \quad e_{n+1} \propto e_n^p \end{array}\right\} (viii)$$

>From (vii) and (viii) we get

$$\text{i.e} \quad e_n^p \propto e_{n-1}\, e_{n-1}$$

$$\text{i.e} \quad e_n \propto e_{n-1}^{(p+1)/p} \qquad (ix)$$

From (viii), (ix) we get

$$p = (p+1)/p$$

$$\text{i.e.} \quad p^2 - p - 1 = 0$$

$$\Rightarrow p = \frac{1 \pm \sqrt{5}}{2}$$

$$\because p > 0, \quad p = 1.618$$

$$\therefore \quad e_{n+1} \propto e_n^{1.618}$$

Hence the convergence is superlinear.