System of Linear Equations

To Solve the system of Linear Equations of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{cases}$$

This is a linear system of n equations in n unknowns $x_1, x_2, \ldots, x_n$.

The system may be rewritten in the form:

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The system has a unique solution when the co-efficient matrix is non-singular

$$x = A^{-1}b$$

Method of Solution

- Direct Methods
  - Gaussian Elimination Method
    - ✓ Thomas Method
    - ✓ Pivoting
  - Factorization Methods
    - ✓ LU
    - ✓ Crouts
- Iterative methods
  - Gauss Jacobi Method
  - Gauss Seidel Method
  - SOR Method
  - CGM
  - Etc.

Direct Methods

Diagonal System

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

All the non-Zero entries of A are on the main diagonal

In this case, computing the solution x is trivial

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix},$$

provided that all $a_{ii} \neq 0$.

If $a_{ii} = 0$ and $b_i = 0$ for some index i, then x_i can be any real number. If $a_{ii} = 0$ but $b_i \neq 0$, no solution of the system exists.

Triangular Systems

When a linear system $Lx = b$ is lower triangular of the form

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

For solution, we will go for FORWARD SUBSTITUION METHOD

Forward Substitution

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where all diagonals $\ell_{ii} \neq 0$, x_i can be obtained by the following procedure

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

Forward Substitution

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

The general formulation for computing x_i is

$$x_i = \left(b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j\right) \Big/ \ell_{ii}, \qquad i = 1, 2, \ldots, n.$$

The general formulation for computing x_i is

$$x_i = \left(b_i - \sum_{j=1}^{i-1} \ell_{ij} x_j \right) \Big/ \ell_{ii}, \qquad i = 1, 2, \ldots, n.$$

Algorithm 3.1 (Forward Substitution: Row Version) *Suppose that $L \in \mathbb{R}^{n\times n}$ is nonsingular lower triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Lx = b$ using row-oriented procedure.*

for $i = 1, \ldots, n$ **do**
  $tmp = 0.0$
  for $j = 1, \ldots, i-1$ **do**
    $tmp = tmp + L(i,j) * x(j)$
  end for
  $x(i) = (b(i) - tmp)/L(i,i)$
end for

LOWER TRIANGULAR SYSTEM

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

Forward Substitution Method

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

Algorithm 3.1 (Forward Substitution: Row Version) *Suppose that $L \in \mathbb{R}^{n\times n}$ is nonsingular lower triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Lx = b$ using row-oriented procedure.*

for $i = 1, \ldots, n$ **do**
  $tmp = 0.0$
  for $j = 1, \ldots, i-1$ **do**
    $tmp = tmp + L(i,j) * x(j)$
  end for
  $x(i) = (b(i) - tmp)/L(i,i)$
end for

Solve 1: $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Answer: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0.3333 \end{pmatrix}$

Solve 2: $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 4 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$

Answer:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5.25 \\ 2.5 \end{pmatrix}$$

Forward Substitution

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

The general formulation for computing x_i is

$$x_i = \left(b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j \right) \Big/ \ell_{ii}, \qquad i = 1, 2, \ldots, n.$$

Complexity of Forward Substitution

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

for $i = 1, \ldots, n$ **do**
  $tmp = 0.0$
  for $j = 1, \ldots, i-1$ **do**
    $tmp = tmp + L(i,j) * x(j)$
  end for
  $x(i) = (b(i) - tmp)/L(i,i)$
end for

Any idea on Algorithmic Complexity ?

$$\sum_{i=1}^{n} [2(i-1) + 2] = n^2 + n.$$

Hence the forward substitution algorithm is an $O(n^2)$ algorithm.

Complexity of Forward Substitution

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

$$\begin{aligned} x_1 &= b_1/\ell_{11} \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn} \end{aligned}$$

Row Version

for $i = 1, \ldots, n$ **do**
  $tmp = 0.0$
  for $j = 1, \ldots, i-1$ **do**
    $tmp = tmp + L(i,j) * x(j)$
  end for
  $x(i) = (b(i) - tmp)/L(i,i)$
end for

Column Version

for $i = 1, \ldots, n$ **do**
  $x(i) = b(i)$
end for
for $j = 1, \ldots, n$ **do**
  $x(j) = x(j)/L(j,j)$
  for $i = (j+1), \ldots, n$ **do**
    $x(i) = x(i) - L(i,j) * x(j)$
  end for
end for

$$
\begin{aligned}
x_1 &= b_1/\ell_{11} \\
x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\
x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33} \\
&\vdots \\
x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn}
\end{aligned}
$$

Algorithm **(Forward Substitution: Column version)** *Suppose that* $L \in \mathbb{R}^{n\times n}$ *is nonsingular lower triangular and* $b \in \mathbb{R}^n$. *This algorithm computes the solution of* $Lx = b$ *using column-oriented procedure.*

```
for i = 1, ..., n do
  x(i) = b(i)
end for
for j = 1, ..., n do
  x(j) = x(j)/L(j, j)
  for i = (j + 1), ..., n do
    x(i) = x(i) − L(i, j) * x(j)
  end for
end for
```

Back Substitution

The analogous algorithm for upper triangular system $Ux = b$ of the form

$$
\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

is called back substitution.

The solution x_i are computed in a reversed order by

$$
\begin{aligned}
x_n &= b_n/u_{nn} \\
x_{n-1} &= (b_{n-1} - u_{n-1,n}x_n)/u_{n-1,n-1} \\
x_{n-2} &= (b_{n-2} - u_{n-2,n-1}x_{n-1} - u_{n-2,n}x_n)/u_{n-2,n-2} \\
&\vdots \\
x_1 &= (b_1 - u_{12}x_2 - u_{13}x_3 - \cdots - u_{1n}x_n)/u_{11}
\end{aligned}
$$

provided that all $u_{ii} \neq 0$.

The general formulation is $$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \Big/ u_{ii}, \qquad i = n, n-1, \ldots, 1.$$

$$\begin{aligned}
x_n &= b_n/u_{nn} \\
x_{n-1} &= (b_{n-1} - u_{n-1,n}x_n)/u_{n-1,n-1} \\
x_{n-2} &= (b_{n-2} - u_{n-2,n-1}x_{n-1} - u_{n-2,n}x_n)/u_{n-2,n-2} \\
&\vdots \\
x_1 &= (b_1 - u_{12}x_2 - u_{13}x_3 - \cdots - u_{1n}x_n)/u_{11}
\end{aligned}$$

Algorithm **(Back Substitution: Row Version)** *Suppose that $U \in \mathbb{R}^{n\times n}$ is non-singular upper triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Ux = b$ using row-oriented procedure.*

$x(n) = b(n)/U(n,n)$
for $i = (n-1), \ldots, 1$ **do**
  $tmp = 0.0$
  for $j = i+1 : n$ **do**
    $tmp = tmp + U(i,j) * x(j)$
  end for
  $x(i) = (b(i) - tmp)/U(i,i)$
end for

Algorithm **(Back Substitution: Column Version)** *Suppose that $U \in \mathbb{R}^{n\times n}$ is nonsingular upper triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Ux = b$ using row-oriented procedure.*

for $i = 1, \ldots, n$ **do**
  $x(i) = b(i)$
end for
for $j = n, \ldots, 1$ **do**
  $x(j) = x(j)/U(j,j)$
  for $i = 1 : (j-1)$ **do**
    $x(i) = x(i) - U(i,j) * x(j)$
  end for
end for

Once again the memory storage for b can be overwritten by x. Back substitution requires $n^2 + O(n)$ flops.

Solve 1:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & 2 \\ 0 & 0 & -\frac{11}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -\frac{11}{5} \end{pmatrix}$$

Answer:

by back substitution to obtain $x_1 = 1, x_2 = 0, x_3 = 1$.

Gaussian Elimination

Given a system of linear equations to be solved, it can be transformed into a simpler equivalent system by the following three types of elementary operations:

1. Interchange two equations in the system (or equivalently, interchange two rows in A):
$$\mathcal{E}_i \leftrightarrow \mathcal{E}_j;$$
2. Multiply an equation by a non-zero constant (multiply one row of A by a non-zero constant):
$$\mathcal{E}_i \leftarrow \lambda \mathcal{E}_i.$$
3. Add to an equation a multiple of some other equation (add to a row a multiple of some other row):
$$\mathcal{E}_i \leftarrow \mathcal{E}_i + \lambda \mathcal{E}_j.$$

Here $\mathcal{E}_i$ denotes the i-th equation in the system.

The first step in the Gaussian elimination process consists of performing, for each $i = 2, 3, \ldots, n$, the elementary operations

$$\mathcal{E}_i \leftarrow (\mathcal{E}_i - m_{i,1}\mathcal{E}_1), \qquad \text{where} \quad m_{i,1} = \frac{a_{i1}}{a_{11}}.$$

These operations transform the system into one in which all the entries in the first column below the diagonal are zero. Then the process is repeated on the resulting equations $\mathcal{E}_2, \ldots, \mathcal{E}_n$, and so on.

$$A = A^{(1)} \rightarrow A^{(2)} \rightarrow \cdots \rightarrow A^{(n)},$$

$$A = A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)},$$

where $A^{(n)}$ is upper triangular. At the conclusion of step $k-1$, the matrix $A^{(k)}$ is constructed from $A^{(k-1)}$ by elementary operations similar to (3.10), and has the following form:

$$A^{(k)} = \left[\begin{array}{ccc|c|ccccc}
a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\
\vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\
\hline
0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\
\hline
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)}
\end{array}\right]$$

$$A^{(k)} = \left[\begin{array}{ccc|c|ccccc}
a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\
\vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\
\hline
0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\
\hline
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)}
\end{array}\right]$$

In the k-th step, $a_{kk}^{(k)}$ is used as a pivot element and the k-th row as pivot row, and elementary operations are applied to rows $k+1$ through n so that zeros are produced in column k below the diagonal. That is, $A^{(k+1)}$ is obtained from $A^{(k)}$ in which $a_{k+1,k}^{(k+1)}, \cdots, a_{nk}^{(k+1)}$ are zero, row $k+1$ through n are modified but row 1 through row k are unchanged when compared to $A^{(k)}$. More precisely, the entries of $A^{(k+1)}$ are produced by the formula

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)}, & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n; \\ 0, & \text{for } i = k+1, \ldots, n, \text{ and } j = 1, \ldots, k; \\ a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} * a_{kj}^{(k)}, & \text{for } i = k+1, \ldots, n, \text{ and } j = k+1, \ldots, n. \end{cases}$$

If we collect all the multipliers

$$\ell_{ik} = \begin{cases} 0, & \text{if } i < k; \\ 1, & \text{if } i = k; \\ \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, & \text{if } i > k, \end{cases}$$

and let $L = [\ell_{ik}]$ and $U = A^{(n)}$, then L is unit lower triangular, U is upper triangular, and later we shall show that matrix A has the factorization $A = LU$.

Algorithm **(Gaussian elimination)** *Given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, this algorithm implements the Gaussian elimination procedure to reduce A to upper triangular and modify the entries of b accordingly.*

```
for k = 1, ..., n-1 do
   for i = k+1, ..., n do
      t = A(i,k)/A(k,k)
      A(i,k) = 0
      b(i) = b(i) - t * b(k)
      for j = k+1, ..., n do
         A(i,j) = A(i,j) - t * A(k,j)
      end for
   end for
end for
```

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$A = A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)},$$

$$A^{(k)} = \left[\begin{array}{ccc|c|ccccc} a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\ \hline 0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\ \hline \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right]$$

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)}, & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n; \\ 0, & \text{for } i = k+1, \ldots, n, \text{ and } j = 1, \ldots, k; \\ a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} * a_{kj}^{(k)}, & \text{for } i = k+1, \ldots, n, \text{ and } j = k+1, \ldots, n. \end{cases}$$

for $k = 1, \ldots, n-1$ **do**
  for $i = k+1, \ldots, n$ **do**
    $t = A(i,k)/A(k,k)$
    $A(i,k) = 0$
    $b(i) = b(i) - t * b(k)$
    for $j = k+1, \ldots, n$ **do**
      $A(i,j) = A(i,j) - t * A(k,j)$
    end for
  end for
end for

$$A^{(k)} = \left[\begin{array}{ccc|c|ccccc} a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\ \hline 0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\ \hline \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right]$$

Forward Elimination

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1
For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}.

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$- \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$\mathcal{E}_i \leftarrow \mathcal{E}_i + \lambda\mathcal{E}_j.$

or $$a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad \vdots \qquad & \quad \vdots \\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}$$

$\mathcal{E}_i \leftarrow \mathcal{E}_i + \lambda\mathcal{E}_j.$

$\lambda = \lambda_i = -\frac{a_{i1}}{a_{11}}, \mathrm{i} = 2, .., \mathrm{n}$

j=1

End of Step 1

Forward Elimination

Step 2
Repeat the same procedure for the 3rd term of Equation 3.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

$$\mathcal{E}_i \leftarrow \mathcal{E}_i + \lambda \mathcal{E}_j.$$

$$\lambda = \lambda_i = -\frac{a'_{2i}}{a'_{22}}, \mathrm{i} = 3, \ldots, \mathrm{n}$$

j=2

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$A = A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)},$$

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)}, & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n; \\ 0, & \text{for } i = k+1, \ldots, n, \text{ and } j = 1, \ldots, k; \\ a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} * a_{kj}^{(k)}, & \text{for } i = k+1, \ldots, n, \text{ and } j = k+1, \ldots, n. \end{cases}$$

$$A^{(k)} = \left[\begin{array}{ccc|c|ccccc} a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\ \hline 0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\ \hline \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)}, & \text{for } i = 1,\ldots,k, \text{ and } j = 1,\ldots,n; \\ 0, & \text{for } i = k+1,\ldots,n, \text{ and } j = 1,\ldots,k; \\ a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} * a_{kj}^{(k)}, & \text{for } i = k+1,\ldots,n, \text{ and } j = k+1,\ldots,n. \end{cases}$$

$$A = A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)},$$

$$A^{(k)} = \left[\begin{array}{ccc|ccccc} a_{11}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1k}^{(k)} & \cdots & a_{1j}^{(k)} & \cdots & a_{1n}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,j}^{(k)} & \cdots & a_{k-1,n}^{(k)} \\ \hline 0 & \cdots & 0 & a_{kk}^{(k)} & \cdots & a_{kj}^{(k)} & \cdots & a_{kn}^{(k)} \\ \hline \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ik}^{(k)} & \cdots & a_{ij}^{(k)} & \cdots & a_{in}^{(k)} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nk}^{(k)} & \cdots & a_{nj}^{(k)} & \cdots & a_{nn}^{(k)} \end{array}\right]$$

```
for k = 1,...,n-1 do
  for i = k+1,...,n do
    t = A(i,k)/A(k,k)
    A(i,k) = 0
    b(i) = b(i) - t * b(k)
    for j = k+1,...,n do
      A(i,j) = A(i,j) - t * A(k,j)
    end for
  end for
end for
```

Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{n}x_n &= b''_3 \\
\vdots \qquad & \quad \vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Algorithm 3.4 (Back Substitution: Column Version) *Suppose that $U \in \mathbb{R}^{n\times n}$ is nonsingular upper triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Ux = b$ using row-oriented procedure.*

```
for i = 1, ..., n do
   x(i) = b(i)
end for
for j = n, ..., 1 do
   x(j) = x(j)/U(j, j)
   for i = 1 : (j − 1) do
      x(i) = x(i) − U(i, j) ∗ x(j)
   end for
end for
```

Example 1

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time, $t\,(\mathrm{s})$	**Velocity,** $v\,(\mathrm{m/s})$
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \;, \qquad 5 \le \mathrm{t} \le 12.$$

Find the velocity at t=6 seconds .

Example 1 Cont.

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3 \;, \quad 5 \le t \le 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is $(n-1)=(3-1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25}=2.56$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by −4.8 and multiply it by −16.8, $\frac{-16.8}{-4.8} = 3.5$.

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 & \vdots & 335.968 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Naïve Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Example 1 Cont.

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \le t \le 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

Example *This example illustrates the application of Gaussian elimination in solving system of linear equations.*

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 34 \\ 27 \\ -38 \end{bmatrix}$$

Sol:

1^{st} **step** Use 6 as pivot element, the first row as pivot row, and multipliers $2, \frac{1}{2}, -1$ are produced to reduce the system to

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 21 \\ -26 \end{bmatrix}$$

2^{nd} **step** Use -4 as pivot element, the second row as pivot row, and multipliers $3, -\frac{1}{2}$ are computed to reduce the system to

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ -9 \\ -21 \end{bmatrix}$$

3^{rd} **step** Use 2 as pivot element, the third row as pivot row, and multipliers 2 is found to reduce the system to

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ -9 \\ -3 \end{bmatrix}$$

Now the solution can be obtained by solving this triangular system with back substitution.

Note that if we collect all the multipliers and let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

then one can verify that $LU = A$.

Naïve Gauss Elimination Pitfalls

Pitfall#1. Division by zero

$$10x_2 - 7x_3 = 3$$
$$6x_1 + 2x_2 + 3x_3 = 11$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$12x_1 + 10x_2 - 7x_3 = 15$$
$$6x_1 + 5x_2 + 3x_3 = 14$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

Is division by zero an issue here? YES

$$12x_1 + 10x_2 - 7x_3 = 15$$
$$6x_1 + 5x_2 + 3x_3 = 14$$
$$24x_1 - x_2 + 5x_3 = 28$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix} \longrightarrow \begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 12 & -21 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}$$

Division by zero is a possibility at any step of forward elimination

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

Another Case

Another difficulty will arise when a pivot element encountered is a small number ε different from zero. For example, the simple Gaussian elimination algorithm would produce relatively large error on the system

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where $\varepsilon < \epsilon_M$

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{where } \varepsilon < \epsilon_M$$

By the Gaussian Elimination step we get

$$\begin{bmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{bmatrix} \Longrightarrow \begin{bmatrix} \varepsilon & 1 \\ 0 & -\frac{1}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{\varepsilon} \end{bmatrix}$$

since in the computer, if ε is small enough, $1-\frac{1}{\varepsilon}$ will be computed to be the same as $-\frac{1}{\varepsilon}$. Likewise, $2-\frac{1}{\varepsilon}$ will be computed to be the same as $-\frac{1}{\varepsilon}$.

Hence, under these circumstances, back substitution would produce

$$x_2 = \frac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon}} = 1 \quad \text{and} \quad x_1 = \frac{1-1}{\varepsilon} = 0.$$

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{where } \varepsilon < \epsilon_M$$

$$x_2 = \frac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon}} = 1 \quad \text{and} \quad x_1 = \frac{1-1}{\varepsilon} = 0.$$

The computed solution is accurate for x_2 but is extremely inaccurate for x_1 since

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

But actually $x_1 = x_2 = 1$ would be a much better solution since

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon \\ 2 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

On the other hand, it would produce a much better result if we interchange the rows since Gaussian elimination would compute

$$\begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-2\varepsilon \end{bmatrix},$$

and the back substitution will give

$$x_1 = \frac{1-2\epsilon}{1-\varepsilon} \approx 1 \quad \text{and} \quad x_2 = 2 - x_1 \approx 2 - 1 = 1.$$

The strategy of interchange rows/columns as described above is called "pivoting". And we have seen that pivoting is necessary for some systems.

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Gauss Elimination with Partial Pivoting

Pitfalls of Naïve Gauss Elimination

- Possible division by zero
- Large round-off errors

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

What is Different About Partial Pivoting?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

If the maximum of the values is $|a_{pk}|$

in the p^{th} row, $k \le p \le n,$ then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}' & a_{23}' & \cdots & a_{2n}' \\ 0 & a_{32}' & a_{33}' & \cdots & a_{3n}' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}' & a_{n3}' & a_{n4}' & a_{nn}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

Example (2nd step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -7 & 6 & 1 & 2 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -17 & 12 & 11 & 43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 8 \\ 9 \\ 3 \end{bmatrix}$$

Which two rows would you switch?

Example (2nd step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps
1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as naïve Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}' & a_{23}' & \cdots & a_{2n}' \\ 0 & a_{32}' & a_{33}' & \cdots & a_{3n}' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}' & a_{n3}' & a_{n4}' & a_{nn}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}' & a_{23}' & \cdots & a_{2n}' \\ 0 & 0 & a_{33}'' & \cdots & a_{3n}'' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3'' \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution Starting Eqns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

Gauss Elimination with Partial Pivoting Example

Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is $(n-1)=(3-1)=2$

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.

$$|25|,|64|,|144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64, $\frac{64}{144} = 0.4444$.

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.4444 = [63.99 \quad 5.333 \quad 0.4444 \quad \vdots \quad 124.1]$$

Subtract the result from Equation 2

$$\begin{array}{rcccl} & [64 & 8 & 1 & \vdots \quad 177.2] \\ - & [63.99 & 5.333 & 0.4444 & \vdots \quad 124.1] \\ \hline & [0 & 2.667 & 0.5556 & \vdots \quad 53.10] \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 25, $\frac{25}{144} = 0.1736$.

$$[144 \quad 12 \quad 1 \quad \vdots \quad 279.2] \times 0.1736 = [25.00 \quad 2.083 \quad 0.1736 \quad \vdots \quad 48.47]$$

Subtract the result from Equation 3

$$\begin{array}{rcccl} & [25 & 5 & 1 & \vdots \quad 106.8] \\ - & [25 & 2.083 & 0.1736 & \vdots \quad 48.47] \\ \hline & [0 & 2.917 & 0.8264 & \vdots \quad 58.33] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Forward Elimination: Step 2 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide Equation 2 by 2.917 and multiply it by 2.667,

$$\frac{2.667}{2.917} = 0.9143.$$

$$\begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \times 0.9143 = \begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \\ -\begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_3

$$-0.2a_3 = -0.23$$

$$a_3 = \frac{-0.23}{-0.2}$$

$$= 1.15$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$\begin{aligned} a_2 &= \frac{58.33 - 0.8264a_3}{2.917} \\ &= \frac{58.33 - 0.8264 \times 1.15}{2.917} \\ &= 19.67 \end{aligned}$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$\begin{aligned} a_1 &= \frac{279.2 - 12a_2 - a_3}{144} \\ &= \frac{279.2 - 12 \times 19.67 - 1.15}{144} \\ &= 0.2917 \end{aligned}$$

Gaussian Elimination with Partial Pivoting Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

Gauss Elimination with Partial Pivoting Another Example

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Algorithm for Gaussian Elimination with Partial Pivoting

for $i = 1, \ldots, n$ **do**
  $p(i) = i$ % initialize the pivoting vector
end for
for $k = 1, \ldots, n-1$ **do**
  $m = k$
  for $i = k+1, \ldots, n$ **do**
    if $|A(p(m), k)| < |A(p(i), k)|$ **then**
      $m = i$
    end if
  end for
  $\ell = p(k)$
  $p(k) = p(m)$
  $p(m) = \ell$
  for $i = k+1, \ldots, n$ **do**
    $A(p(i), k) = A(p(i), k)/A(p(k), k)$
    for $j = k+1, \ldots, n$ **do**
      $A(p(i), j) = A(p(i), j) - A(p(i), k) * A(p(k), j)$
    end for
  end for
end for

Algorithm for Forward substitute for partial pivoting case

$y(p(1)) = b(p(1))/A(p(1), 1)$
for $i = 1, \ldots, n$ **do**
  $t = 0$
  for $j = 1, \ldots, i-1$ **do**
    $t = t + A(p(i), j) * y(p(j))$
  end for
  $y(p(i)) = b(p(i)) - t$
end for

Determinant of a Square Matrix Using Naïve Gauss Elimination

Example

Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then det(A)=det(B)

Theorem of Determinants

The determinant of an upper triangular matrix $[A]_{nxn}$ is given by

$$\det(\mathrm{A}) = a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.

$$[A]_{n\times n} \rightarrow [U]_{n\times n}$$

$$\det(A) = \det(U)$$

Example

Using naïve Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 \end{bmatrix} \\ -\begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by −4.8 and multiply it by −16.8, $\frac{-16.8}{-4.8} = 3.5$.

$$([0 \quad -4.8 \quad -1.56]) \times 3.5 = [0 \quad -16.8 \quad -5.46]$$

Subtract the result from Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76] \\ -[0 \quad -16.8 \quad -5.46] \\ \hline [0 \quad 0 \quad 0.7] \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \to \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \det(\mathrm{A}) &= u_{11} \times u_{22} \times u_{33} \\ &= 25 \times (-4.8) \times 0.7 \\ &= -84.00 \end{aligned}$$

Summary

-Forward Elimination

-Back Substitution

-Pitfalls

-Improvements

-Partial Pivoting

-Determinant of a Matrix