3_1证明拉普拉斯算子具有理论上的旋转不变性

证明式中的拉普拉斯变换是各向同性的（旋转不变）。需要下列轴旋转 θ 角的坐标方程：

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

其中 (x, y) 为非旋转坐标，而 (x', y')为旋转坐标。

证：

对f求x'的偏导：

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$$

对f求x‘的二阶偏导：

$$\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}'^2} = \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2}\cos^2(\theta) + \frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{f}}{\partial \mathrm{y}}\right)\sin(\theta)\cos(\theta) + \frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right)\sin(\theta)\cos(\theta) + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\sin^2(\theta)$$

求f对y‘的偏导：

$$\frac{\partial \mathrm{f}}{\partial \mathrm{y}'} = \frac{\partial \mathrm{f}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{y}'} + \frac{\partial \mathrm{f}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{y}'} = -\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\sin(\theta) + \frac{\partial \mathrm{f}}{\partial \mathrm{y}}\cos(\theta)$$

求f对y‘的二阶偏导：

$$\frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}'^2} = \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2}\sin^2(\theta) - \frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial \mathrm{f}}{\partial \mathrm{y}}\right)\sin(\theta)\cos(\theta) \quad - \frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right)\sin(\theta)\cos(\theta) + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\cos^2(\theta)$$

将两项二阶偏导项相加得：

$$\begin{aligned}\frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}'^2} + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}'^2} &= \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2}\sin^2(\theta) + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2}\cos^2(\theta) + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\cos^2(\theta) + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\sin^2(\theta) \\ &= \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{f}}{\partial \mathrm{y}^2}\end{aligned}$$

即拉普拉斯算子具有旋转不变性得证。