

大数据分析

Linear Algebra
Preliminaries

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Vectors

- $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ (each x_i is a component)
 - A point in d-dimensional space
- Norm or magnitude $\|\mathbf{x}\| = (\mathbf{x}^T \mathbf{x})^{1/2} = (x_1^2 + x_2^2 + \dots + x_d^2)^{1/2}$
 - Length of the vector (Pythagorean theorem)
- Zero vector (norm zero), unit vector (norm one)
- Inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \dots + x_d y_d$
 - Result is a scalar
 - $\|\mathbf{x}\| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}$
 - $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ implies $\mathbf{x} \perp \mathbf{y}$

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Vector spaces

- Space where vectors live
- Formally, a collection of vectors which is closed under linear combination
 - If $\{\mathbf{x}, \mathbf{y}\}$ are in the space, so is $a\mathbf{x} + b\mathbf{y}$ for any scalars $a, b \in \mathbb{R}$
 - Should always contain zero vector
- Examples: $\{0\}$, \mathbb{R}^d , the line $x = 3y$ in \mathbb{R}^2

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Span and basis

- A set of vectors is said to span a vector space if one can write any vector in the vector space as a linear combination of the set
- $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ span the space $\{\sum a_i \mathbf{x}_i \mid a_i \in \mathbb{R}\}$
- This set is called the basis set
- Examples
 - The vectors $\{(0,1), (1,0)\}$ span \mathbb{R}^2
 - $\{(1, 1)\}$ spans $x=y$ which is a subspace of \mathbb{R}^2
 - The vector $\{(0,1), (0,1), (1,1)\}$ also span \mathbb{R}^2

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Linear independence and orthonormality

- Linear independence – a notion to remove redundancy in the basis
 - $\{x_1, x_2, \dots, x_n\}$ are linearly independent iff the only solution to $\sum a_i x_i = 0$ is $a_1 = a_2 = \dots = a_n = 0$.
 - Cannot express any vector x_i as a linear combination of the others
- Dimensionality of a vector space is the maximum number of linearly independent basis vectors
- Orthonormal basis
 - $\{x_1, x_2, \dots, x_n\}$ is orthonormal basis if $\langle x_i, x_j \rangle = 1$ if $i=j$ and 0 otherwise
 - Coordinate axes for the vector space
- Example: The basis $\{(0, 1), (1, 1)\}$ for \mathbb{R}^2 is linear independent but not orthonormal.

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Matrices

- Operator which transforms vectors from one vector space to another
 - $y = Ax$
- The operator is linear, that is

$$A(ax + by) = a(Ax) + b(Ay)$$
- The result of applying the operator is a linear combination of the column vectors
 - Thus, $Ax = b$ has an exact solution iff b is in the column space of A
- Eigen vectors of A are the special vectors x which satisfy

$$Ax = \lambda x \text{ for some } \lambda$$
 - λ is called the eigen value and x is the eigen vector
- How do we visualize the transformation geometrically?

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Visualizing the matrix operator – special cases

- Identity matrix
 - Square matrix with diagonal elements 1 and non-diagonal elements 0
 - The transformed vector Ax is same x
- Diagonal matrix
 - Square matrix with non-diagonal elements 0
 - i^{th} component in Ax is a scaled version of x_i (scaling = A_{ii})
- Orthonormal (or rotation) matrix
 - Matrix whose columns $\{a_1, a_2, \dots, a_n\}$ are such that $\langle a_i, a_j \rangle = 1$ if $i=j$ and 0 otherwise. That is, $A^T A = I$
 - Rotates the vector
 - Preserves norms $\|Ax\| = \|x\|$ (why?)

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Thank you!

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