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Approaching the Gaussian channel capacity with APSK constellations

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***Abstract*—We consider the Gaussian channel with power constraint P. A gap exists between the channel capacity and the highest achievable rate of equiprobable uniformly spaced signal. Several approaches enable to overcome this limitation such as constellations with non-uniform probability or constellation shaping. In this letter, we focus on constellation shaping. We give a construction of amplitude and phase-shift keying (APSK) constellations with equiprobable signaling that achieve the Gaussian capacity as the number of constellation points goes to infinity.**

I. Introduction

We consider the additive white Gaussian noise (AWGN) channel with signal-to-noise ratio (SNR) P/N_0, where P is the average power constraint of the input signal and N_0 the noise variance. Let $\mathrm{snr} = P/N_0$, the Gaussian channel capacity is

$$C = \frac{1}{2}\log_2(1+\mathrm{snr}) \text{ bits/dimension}, \tag{1}$$

and the optimal input distribution is Gaussian with zero mean and variance P [1].

In digital communications systems, approaching the Gaussian capacity is a challenging problem. Indeed the optimal input distribution is continuous while in practice the signal is chosen from a finite constellation of points. Several approaches enable to overcome this limitation such as constellation shaping, constellations with non-uniform probabilities [2] or lattice encoding and decoding [3]. In this paper, we focus on the former solution as most systems use *equiprobable signaling*.

Sun and van Tilborg were the first to present a sequence of random variables equiprobably distributed over a finite support that achieves the Gaussian capacity (in one dimension) as the signal set cardinality tends to infinity [4]. Until then, the traditional rule of designing constellations for the AWGN channel was to maximize the squared Euclidean distance between the signal points under the power constraint. In one dimension, this results in uniformly spaced signal sets that, when combined with equiprobable signaling, exhibit a gap of $\frac{\pi e}{6} \approx 1.56$ dB with the Gaussian capacity at large SNR [5]. Sun and van Tilborg shows how to close this gap in one dimension. In higher dimensions, this shaping gain can be obtained by choosing the signal points on an N-dimensional rectangular lattice from within an N-sphere rather than an N-cube [5]. However this is impractical when N increases.

The result in [4] was extended by Schwarte that provided sufficient conditions for uniform input distributions with finite support to approach the Gaussian capacity in any dimension [6]. Similar conditions are given in [7, Theorem 9].

A recent work by Wu and Verdú studied the AWGN channel capacity with finite signal set [8]. They showed that as the input signal cardinality grows, the constellation capacity approaches the Gaussian capacity exponentially fast. They also introduced a family of constellation, based on the Hermite polynomials roots, achieving exponential convergence. The resulting modulations combine constellation shaping with non-uniform probabilities.

If the channel inputs are also subject to peak power constraints, the capacity and the optimal input distribution were studied for scalar and quadrature channels in [9] and [10], respectively. In two dimensions, the optimal distribution is discrete along the radial direction and continuous in the angular direction. This proves the advantages of circular APSK constellations under those power constraints. Even if we do not consider peak power constraints in our work, the relevance of designing APSK signals is obvious.

In this work, our main contribution (presented in Section II) is the construction of APSK modulations with equiprobable signaling that achieve the Gaussian capacity as the number of constellation points goes to infinity. To that end, we rely on the Box-Muller theorem [11]. Designing APSK signals is an important topic in communications. For instance, De Gaudenzi *et al.* addressed the optimization of APSK modulations for nonlinear channel in [12]. The research results may benefit some practical systems as the digital video broadcasting (DVB) standards that implement APSK constellations with large cardinality, up to 256 [13].

II. APSK signals approach the Gaussian capacity

The Box-Muller transform is a method for generating pairs of independent normally distributed random numbers [11]. The method proposed by Box and Muller works as follows: let U and V be two independent random variables that are uniformly distributed in the interval $(0, 1)$. Consider the random variables

$$X = \sqrt{-2\log_e U}\cos(2\pi V) \tag{2}$$

and

$$Y = \sqrt{-2\log_e U}\sin(2\pi V), \tag{3}$$

then we have the following result [11]:

Theorem (Box-Muller). *X and Y are independent random variables with a standard normal distribution.*

Based on the Box-Muller transform, we show how to design APSK constellations that achieve the Gaussian capacity

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as the number of constellation points goes to infinity. Our construction comes from the following intuition: consider two *discrete random variables* that converge weakly to the uniform distribution over $(0,1)$. Applying a Box-Muller transform to that pairs of random variables will generate a set of two-dimensional points with a Gaussian shape, the key ingredient to approach the Gaussian capacity. As we will discuss in Section III, several constructions based on the Box-Muller transform are possible. The solution presented here facilitates the proof of the Gaussian capacity achievability.

Let $n \geqslant 1$ be an integer and $\mathbb{N}_n = \{0, 1, 2, \ldots, n-1\}$ the set of integers from 0 to $n-1$. We consider the discrete set

$$\mathcal{S}_n = \left\{ \frac{1}{2n} + \frac{k}{n} \mid k \in \mathbb{N}_n \right\}, \quad (4)$$

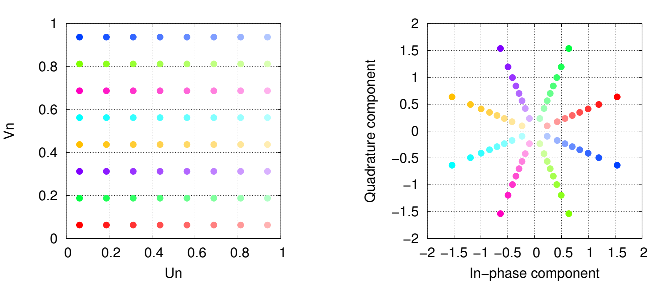
that verifies $|\mathcal{S}_n| = n$ and $\mathcal{S}_n \subset (0,1)$. We introduce U_n and V_n two *discrete* random variables uniformly distributed on $\mathcal{S}_n$. The sequences $(U_n)_{n\geqslant 1}$ and $(V_n)_{n\geqslant 1}$ converge weakly to the uniform distribution over $(0,1)$ (consider the characteristic function of each random variable).

Then we define the *shaping function* $\varphi\colon (0,1)^2 \to \mathbb{R}^2$ by

$$\varphi(x,y) = \left(\sqrt{-P\log_{\mathrm{e}} x}\,\cos(2\pi y), \sqrt{-P\log_{\mathrm{e}} x}\,\sin(2\pi y)\right),$$

where P is the power constraint of the input signal. Compared to (2) and (3), there is a scaling factor of $\sqrt{P/2}$ as the optimal input distribution is Gaussian with variance $P/2$ per dimension. The shaping function objective is to give the Gaussian shape to our constellation.

Finally, we introduce the random vector $W_n = \varphi(U_n, V_n)$. By construction, W_n is a random vector *uniformly distributed* on a set $\mathcal{C}_n$ of n^2 points in $\mathbb{R}^2$. The points in $\mathcal{C}_n$ exhibit a Gaussian shape and are distributed on n circles, each circle containing n points. The set $\mathcal{S}_n$ ensures that U_n is never equal to zero, avoiding problem with the logarithm, and also that the constellation points are uniformly distributed on each circle. Fig. 1 illustrates the construction of the 64 APSK ($n=8$ and $P=1$): from a set of equiprobable and uniformly distributed points in $(0,1)^2$, the shaping function generates a Gaussian-shaped constellation. The random variable U_n controls the magnitude of the signal points, while V_n affects the argument.



(a) All possible (U_n, V_n) pairs

(b) Gaussian-shaped 64 APSK

Fig. 1: 64 APSK construction: (a) depicts the (U_n, V_n) pairs, a discrete and equiprobable set of points in $(0,1)^2$; (b) represents the points after applying the shaping function φ. There is a one-to-one mapping between the points in (a) and (b).

Before stating our result, we present a lemma that will be helpful to prove that $\mathcal{C}_n$ satisfies the average power constraint.

Lemma. *For any integer $k \geqslant 1$,*

$$k\log_{\mathrm{e}} k - k \leqslant \sum_{j=0}^{k-1} \log_{\mathrm{e}}\left(j + \frac{1}{2}\right). \quad (5)$$

Proof: Let $t \geqslant 1/2$ be a real number, it is easy to verify that $\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \log_{\mathrm{e}} u\mathrm{d}u \leqslant \log_{\mathrm{e}} t$. This leads to

$$\begin{aligned} k\log_{\mathrm{e}} k - k &= \int_0^k \log_{\mathrm{e}} u\mathrm{d}u \\ &= \sum_{j=0}^{k-1} \int_j^{j+1} \log_{\mathrm{e}} u\mathrm{d}u \\ &\leqslant \sum_{j=0}^{k-1} \log_{\mathrm{e}}\left(j+\frac{1}{2}\right). \end{aligned} \quad (6)$$

■

The following theorem is the main result of the paper:

Theorem. *The APSK constellations previously designed achieve the Gaussian channel capacity as $n \to \infty$. More formally,*

$$I(W_n; W_n + N) \underset{n\to\infty}{\longrightarrow} \log_2(1 + \mathrm{snr}), \quad (7)$$

where $N \sim \mathcal{N}(0, N_0)$ is the AWGN with variance N_0 ($N_0/2$ on each dimension).

Proof: The idea is to show that the sequence $(W_n)_{n\geqslant 1}$ satisfies the two conditions given in [6, Corollary 7].

Condition 1: First, we prove that the constellation $\mathcal{C}_n$ satisfies the power constraint for $n \geqslant 1$. The input signal energy is

$$\begin{aligned} \mathbb{E}\left[W_n^2\right] &= \frac{1}{n^2} \sum_{\mathbf{w}\in\mathcal{C}_n} \|\mathbf{w}\|^2 \\ &= \frac{1}{n}\sum_{k=0}^{n-1} -P\log_{\mathrm{e}}\left(\frac{1}{2n} + \frac{k}{n}\right) \\ &= -\frac{P}{n}\left(-n\log_{\mathrm{e}} n + \sum_{k=0}^{n-1}\log_{\mathrm{e}}\left(k+\frac{1}{2}\right)\right) \\ &\leqslant P, \end{aligned} \quad (8)$$

where the last inequality follows from the previous lemma. Thus the constellation $\mathcal{C}_n$ verifies the power constraint.

Condition 2: Next, we show that the distribution of $W_n = \varphi(U_n, V_n)$ converges (weakly) to the distribution of a Gaussian variable, denoted W^*, with variance $P/2$ on each dimension. To that end, we study Φ_{W_n} the characteristic function of W_n. By definition,

$$\Phi_{W_n}(\mathbf{t}) = \mathbb{E}\left[\mathrm{e}^{i\langle \mathbf{t}|W_n\rangle}\right], \quad (9)$$

where $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $\langle\cdot|\cdot\rangle$ is the scalar product. Introducing the function

$$\psi(x,y) = \mathrm{e}^{i\sqrt{-P\log_{\mathrm{e}} x}\,(t_1\cos(2\pi y) + t_2\sin(2\pi y))}, \quad (10)$$

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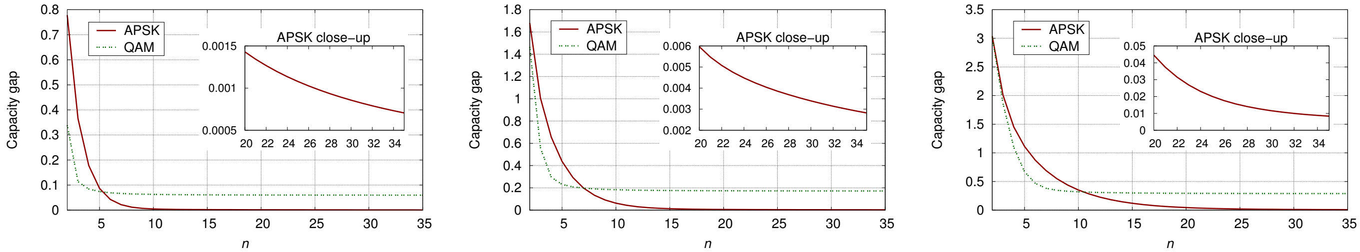

(a) snr $= 5$ dB (b) snr $= 10$ dB (c) snr $= 15$ dB

Fig. 2: Gap to Gaussian channel capacity for the regular QAM (with equiprobable signaling) and the proposed APSK

we can write (9) as

$$\Phi_{W_n}(\mathbf{t}) = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\psi\left(\frac{1}{2n}+\frac{k}{n},\frac{1}{2n}+\frac{l}{n}\right).$$

Moreover, it results from the Box-Muller theorem that

$$\Phi_{W^*}(\mathbf{t}) = \iint_{(0,1)^2}\psi(x,y)\mathrm{d}x\mathrm{d}y. \tag{11}$$

Now, we consider the sequence $(\psi_n)_{n\geqslant 1}$ defined by

$$\psi_n(x,y) = \psi\left(\frac{1}{2n}+\frac{\lfloor nx\rfloor}{n},\frac{1}{2n}+\frac{\lfloor ny\rfloor}{n}\right), \tag{12}$$

where $\lfloor\cdot\rfloor$ is the floor function. The sequence $(\psi_n)_{n\geqslant 1}$ converges pointwise to ψ. Over $(0,1)^2$, $|\psi_n|$ is dominated by the constant function equals to 1 (for all n). Applying the Lebesgue's dominated convergence theorem [14, Theorem 16.4], we obtain

$$\iint_{(0,1)^2}\psi_n(x,y)\mathrm{d}x\mathrm{d}y \underset{n\to\infty}{\longrightarrow} \iint_{(0,1)^2}\psi(x,y)\mathrm{d}x\mathrm{d}y. \tag{13}$$

Moreover,

$$\begin{aligned}\iint_{(0,1)^2}\psi_n(x,y)\mathrm{d}x\mathrm{d}y &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_{\frac{l}{n}}^{\frac{l+1}{n}}\psi_n(x,y)\mathrm{d}x\mathrm{d}y\\ &= \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\psi\left(\frac{1}{2n}+\frac{k}{n},\frac{1}{2n}+\frac{l}{n}\right)\\ &= \Phi_{W_n}(\mathbf{t}). \end{aligned}\tag{14}$$

Combining (11), (13) and (14), we obtain

$$\Phi_{W_n}(\mathbf{t}) \underset{n\to\infty}{\longrightarrow} \Phi_{W^*}(\mathbf{t}). \tag{15}$$

Finally, it follows from the continuity theorem that W_n converges weakly to W^* [14, Theorem 26.3].

Conclusion: The constellation uniformly distributed over $\mathcal{C}_n$ satisfies both conditions in [6, Corollary 7]. We conclude that the constellations approach the Gaussian channel capacity with power constraint P as $n\to\infty$. ■

To illustrate the previous theorem, Fig. 2 depicts the gap to Gaussian capacity for the regular QAM and the proposed APSK constellations for various SNR values. As expected, the capacity gap vanishes for the APSK as $n\to\infty$. However the convergence speed remains an open question.

III. Concluding Remarks

Our APSK design enables to achieve the Gaussian capacity as the constellation size grows to infinity. In that sense, APSK modulations are better than regular QAM. However, non-uniform QAM with equiprobable signaling may also reach the Gaussian capacity. Indeed, a way to construct such modulations is to consider a capacity-achieving signal in one dimension (for instance [4] or [6]) and transmit such signals on quadrature carriers.

The DVB-S2X standard implements various APSK modulations, but no justification about the constellation design is provided [13]. When the constellation cardinality is of the form 2^{2k} (i.e., $n = 2^k$ with the previous notations), the standard splits the points on $n/2$ circles where each circle contains $2n$ points uniformly distributed as depicted in Fig. 3. The proof in Section II may be adapted to show that such constellations also reach the Gaussian capacity as $n\to\infty$. Fig. 4 compares this constellation with the one proposed in Section II in terms of achievable rates. Even if both constellations asymptotically achieve the Gaussian capacity, they exhibit different performance for signal sets with finite cardinality. Moreover the DVB-S2X design reduces the peak-to-average power ratio, a suitable property for practical systems.

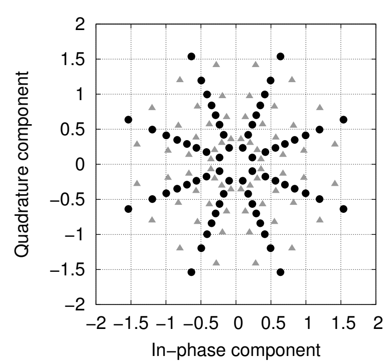

Fig. 3: 64 APSK constellations from Section II (black circles) or inspired from DVB-S2X (gray triangles)

The DVB-S2X example shows that several constructions based on the Box-Muller transform are possible. We now propose a more general framework. Let $m\geqslant 1$ be an integer, consider r_m and θ_m two integers such as $r_m\theta_m = m$. A

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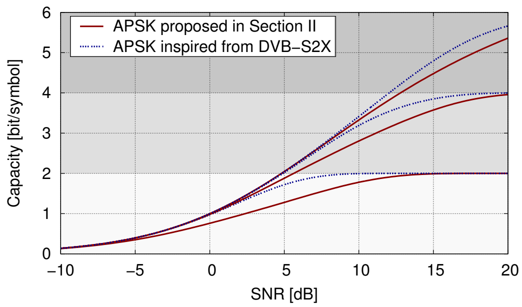

Fig. 4: Achievable rates for the 4, 16 and 64 APSK signals proposed in Section II or derived from DVB-S2X

slight modification of the APSK design in Section II enables to design constellation with cardinality m where the points are distributed on r_m circles, each circle containing θ_m points. Including these conditions in the design, we consider the discrete random variables U_{r_m} and V_{θ_m} uniformly distributed on

$$\mathcal{S}_{r_m} = \left\{ \frac{1}{2r_m} + \frac{k}{r_m} \mid k \in \mathbb{N}_{r_m} \right\} \tag{16}$$

and

$$\mathcal{S}_{\theta_m} = \left\{ \frac{1}{2\theta_m} + \frac{k}{\theta_m} \mid k \in \mathbb{N}_{\theta_m} \right\}, \tag{17}$$

respectively. The shaping function remains the same. With these notations, the design in Section II corresponds to $m = n^2$ with $r_m = \theta_m = \sqrt{m}$. We would like to point out that all constellations constructed in this manner do not achieve the Gaussian channel capacity as $m \to \infty$ (for instance consider the case $r_m = 1$ and $\theta_m = m$). We believe that if $\mathcal{S}_{r_m}$ and $\mathcal{S}_{\theta_m}$ ensure that $(U_{r_m})_{m\geqslant 1}$ and $(V_{\theta_m})_{m\geqslant 1}$ converge weakly to the uniform distribution on the interval $(0,1)$, then the proof in Section II may be adapted to show that the corresponding constellations reach the Gaussian capacity as $m \to \infty$.

Future work will focus on three main directions. First, we plan to study the convergence speed of the proposed APSK. For the scalar AWGN channel with input cardinality m, Wu and Verdú showed that the achievable rate approaches exponentially fast the Gaussian capacity as m grows [8]. Second, we will seek the optimal constellations in terms of capacity (for a given SNR) or convergence speed. Finally, we will investigate if the (U_{r_m}, V_{r_m}) representation of the constellation may accelerate decoding.

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