Searching for New Bases for Quantum Algorithms

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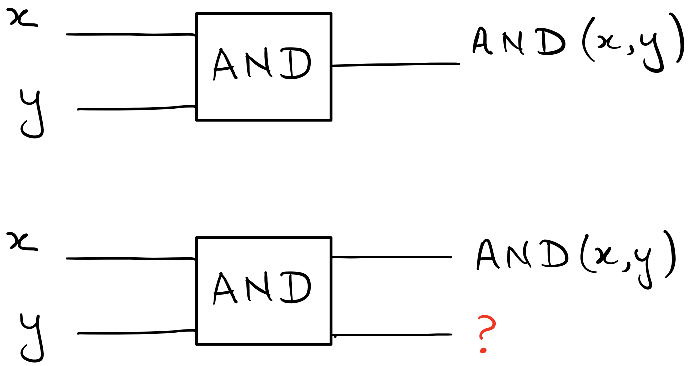
All the Quantum Computing We Need

n-qubit Quantum State $\quad |\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle \; , \quad \sum_{i=0}^{2^n-1} \alpha_i^2 = 1, \; \alpha_i \in \mathbb{R}$

Unitary Evolution $\quad U|\psi\rangle = |\varphi\rangle = \sum_{i=0}^{2^n-1} \beta_i |i\rangle \; , \quad \sum_{i=0}^{2^n-1} \beta_i^2 = 1, \; \beta_i \in \mathbb{R}$

Measurement $\quad$ Probability to observe particular outcome i on measuring $|\psi\rangle$ is given by α_i^2

Circuit Model for Quantum Algorithms



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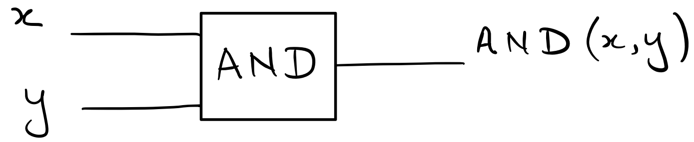
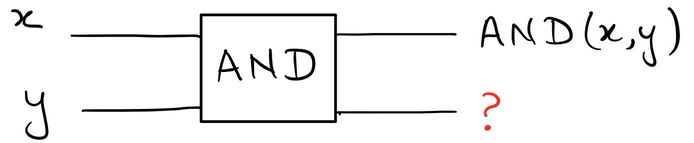
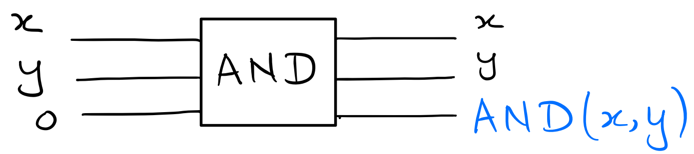
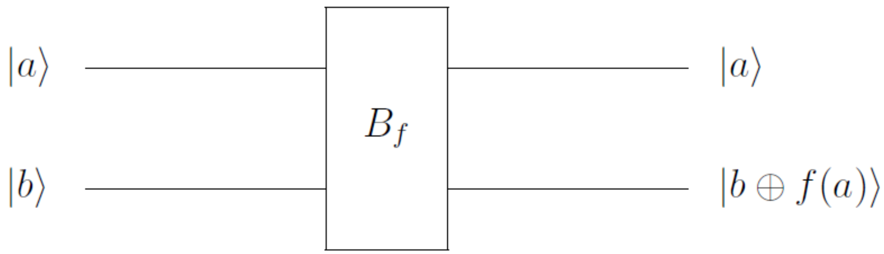






Classical Gates Via Unitaries



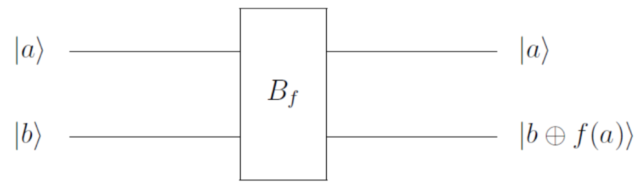
$$B_f : |a\rangle|b\rangle \longrightarrow |a\rangle|b \oplus f(a)\rangle$$

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

Phase Kickback

Recall

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$



Consider $B_f |a\rangle |-\rangle$

$$= B_f |a\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\left(B_f |a\rangle|0\rangle - B_f|a\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|a\rangle|0 \oplus f(a)\rangle - |a\rangle|1 \oplus f(a)\rangle\right)$$

if $f(a) = 0 \longrightarrow |0\rangle - |1\rangle$

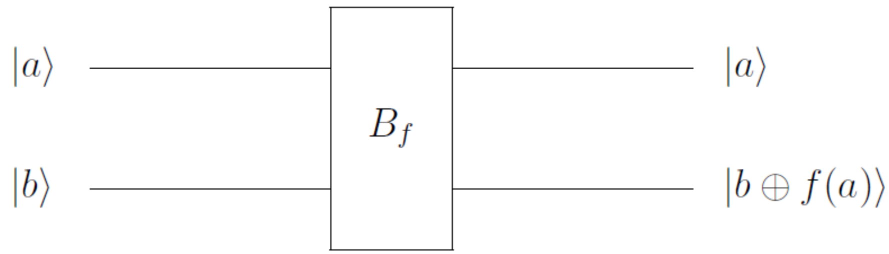
$f(a) = 1 \longrightarrow |1\rangle - |0\rangle$

$\Big\}\ (-1)^{f(a)}\left(|0\rangle - |1\rangle\right)$

$$= (-1)^{f(a)} |a\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$B_f : |a\rangle|-\rangle \longrightarrow (-1)^{f(a)} |a\rangle |-\rangle$$

Classical Gates Via Unitaries



$$B_f : |a\rangle|b\rangle \longrightarrow |a\rangle|b \oplus f(a)\rangle$$

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

$$B_f : |a\rangle|-\rangle \longrightarrow (-1)^{f(a)}|a\rangle|-\rangle$$

Hadamard on n Qubits

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

$$(H \otimes H)|00\rangle = |+\rangle \otimes |+\rangle$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$H^{\otimes n}|00\ldots0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

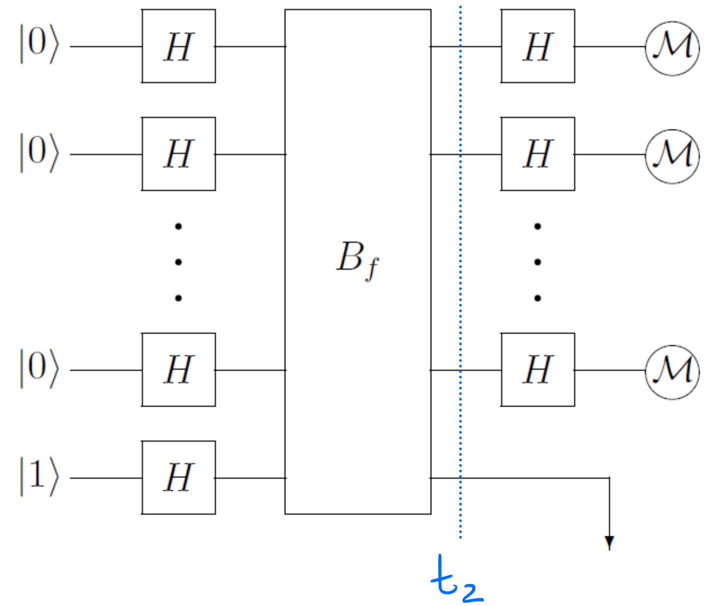
Bernstein-Vazirani Problem

Suppose a function $f : \{0,1\}^n \to \{0,1\}$ is given as a black-box in the usual way, i.e., as a unitary transformation B_f that acts as follows for all $x \in \{0,1\}^n$ and $y \in \{0,1\}$:

$$B_f : |x\rangle\,|y\rangle \mapsto |x\rangle\,|y \oplus f(x)\rangle\,.$$

This time you are promised that there exists some string $s \in \{0,1\}^n$ such that $f(x) = s \cdot x$ for all $x \in \{0,1\}^n$, where

$$s \cdot x = \sum_{i=1}^{n} s_i x_i \pmod 2.$$



$$H^{\otimes n}\,|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n}\,|y\rangle$$

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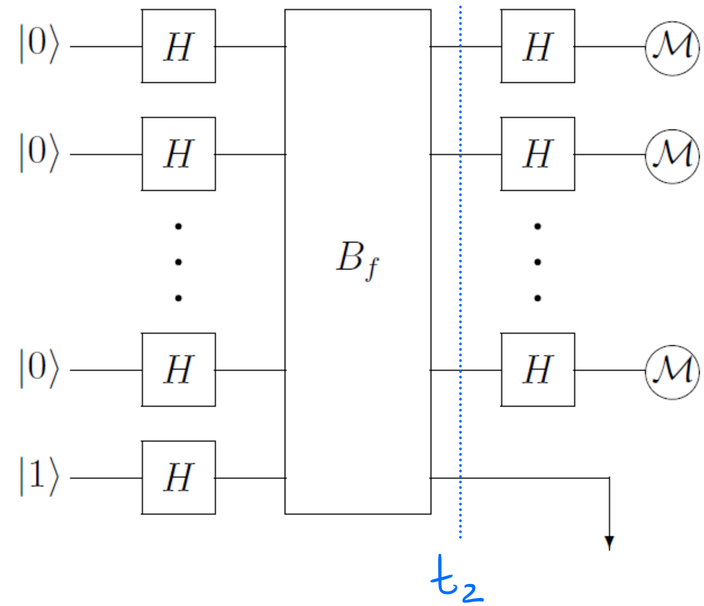


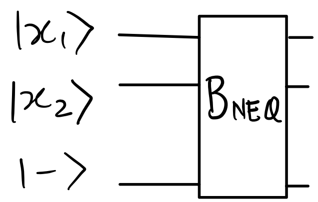
$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{s \cdot x} |x\rangle \right) |-\rangle$$

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n} |y\rangle$$

An Example

Consider $NEQ(x_1, x_2)$



x_1	x_2	$NEQ(x_1,x_2)$
0	0	0
0	1	1
1	0	1
1	1	0

$$B_{NEQ} : |x_1 x_2\rangle|-\rangle \longrightarrow (-1)^{f(x_1,x_2)}|x_1 x_2\rangle|-\rangle$$

$$B_{NEQ}|00\rangle = |00\rangle$$

$$B_{NEQ}|01\rangle = -|01\rangle$$

$$B_{NEQ}|10\rangle = -|10\rangle$$

$$B_{NEQ}|11\rangle = |11\rangle$$

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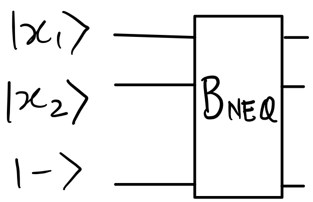

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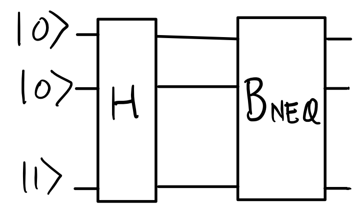
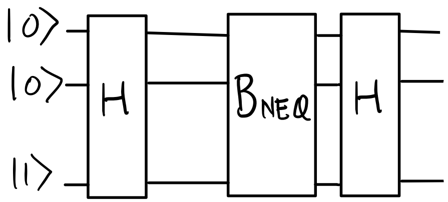


$$\frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

An Example



x_1	x_2	$NEQ(x_1, x_2)$
0	0	0
0	1	1
1	0	1
1	1	0

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

What if we apply Hadamard again

$$\frac{1}{2}(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle) = |11\rangle$$

What is the amplitude of $|11\rangle \longrightarrow$ $\frac{1}{2}\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)\right)$

$= 1$

An Example

$$H^{\otimes 3} |110\rangle = |{-}{-}{+}\rangle = \frac{1}{\sqrt{8}} (|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{8}} \sum_{s \in \{0,1\}^3} \; ? \; |s\rangle$$

What is the sign in front of $|s\rangle$?

$$|s\rangle = \begin{array}{lll} 000 & +++ & = + \\ 001 & +++ & = + \\ 010 & +-+ & = - \\ \vdots & \vdots & \vdots \\ 111 & & \end{array}$$

An Example

$$H^{\otimes 3}|110\rangle = |--+\rangle = \frac{1}{\sqrt{8}}\underbrace{(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)}$$

$$= \frac{1}{\sqrt{8}}\sum_{s\in\{0,1\}^3} \; ? \; |s\rangle$$

What is the sign in front of $|s\rangle$?

Sign contribution from expansion

$(x = 110)$

$$|s\rangle = \begin{array}{lll} 000 & \underbrace{+++} & = + \\ 001 & +++ & = + \\ 010 & +-+ & = - \\ \vdots & \vdots & \vdots \\ 111 & & \end{array}$$

An Example

$$H^{\otimes 3}|110\rangle = |{-}{-}{+}\rangle = \frac{1}{\sqrt{8}}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)$$

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What is the sign in front of $|s\rangle$?

$$|s\rangle = \begin{array}{c} 000 \\ 001 \\ 010 \\ \vdots \\ 111 \end{array}$$

In general if $\underline{s_i = 0}$ sign is always $+$

if $\underline{s_i = 1}$, sign is $\begin{cases} - & \text{if } x_i = 1 \\ + & \text{if } x_i = 0 \end{cases}$

An Example

$$H^{\otimes 3}|110\rangle = |--+\rangle = \frac{1}{\sqrt{8}}\underbrace{(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle + |1\rangle)}$$

$$= \frac{1}{\sqrt{8}}\sum_{s\in\{0,1\}^3} \underline{\qquad} |s\rangle$$

So, sign in front of $|s\rangle$ is given by

$$\prod_{i:s_i=1} (-1)^{x_i} \qquad (-1)^{\sum_{i:s_i=1} x_i \pmod 2}$$

In general if $s_i = 0$ sign is always +

if $s_i = 1$, sign is $\begin{cases} - & \text{if } x_i = 1 \\ + & \text{if } x_i = 0 \end{cases}$

$$= (-1)^{XOR_s(x)} = (-1)^{s\cdot x} \qquad (-1)^{x_1+x_2}$$

x_1	x_1	NEQ
0	0	0
0	1	1
1	0	1
1	1	0

$XOR_s(x) \longrightarrow$ XOR of bits x_i where $\underline{s_i} = 1$

$\longrightarrow |11\rangle$

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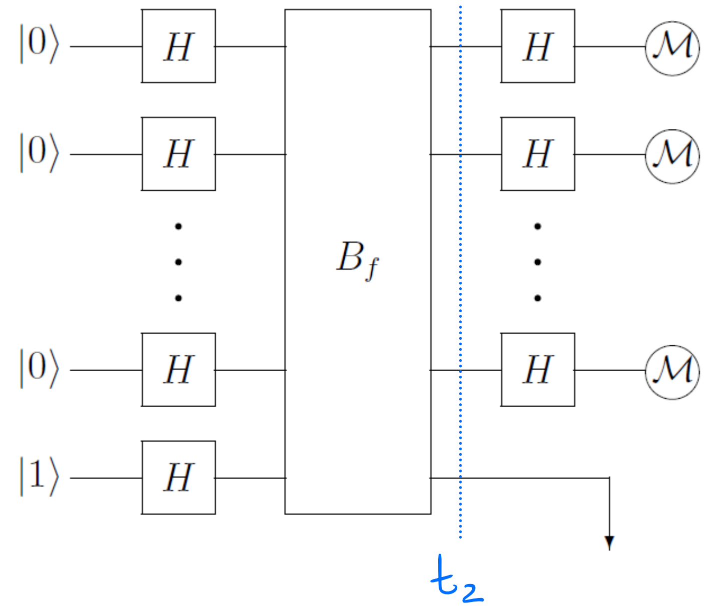


$$H^{\otimes n}\,|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x_1 y_1 + \cdots + x_n y_n}\,|y\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{f(x)}\,|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{\sqrt{2^n}} \sum_{x} (-1)^{s \cdot x}\,|x\rangle\right)|-\rangle$$

$$\xrightarrow{H^{\otimes n} \otimes \mathbb{1}} |s\rangle\,|-\rangle$$

Finding Patterns

① Make superposition of all inputs

$$H^{\otimes n}|0^n\rangle = \frac{1}{\sqrt{N}}\sum_{x\in\{0,1\}^n}|x\rangle$$

② Get answers in the amplitude

B_f gives $\frac{1}{\sqrt{N}}\sum_x (-1)^{f(x)}|x\rangle$

Call $F(x) = (-1)^{f(x)}$

$F: \{0,1\}^n \longrightarrow \{\pm 1\}$

$0 \longrightarrow 1$

$1 \longrightarrow -1$

③ Create interference

$H^{\otimes n}$ again

$$H^{\otimes n}\left(\frac{1}{\sqrt{N}}\sum_x F(x)|x\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\sum_x F(x)\, H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}}\sum_s \,?\, |s\rangle$$

Loading up data in the vector

$$\frac{1}{\sqrt{N}}\begin{bmatrix} F(00..0) \\ F(00...1) \\ \vdots \\ F(11...1) \end{bmatrix}$$

Boolean Fourier Transform

Decompose $g : \{0,1\}^n \longrightarrow \mathbb{C}$ into basis of XOR functions

$$\chi_s : \{0,1\}^n \longrightarrow \{\pm 1\}$$
$$x \longmapsto (-1)^{XOR_s(x)} \quad , s \in \{0,1\}^n$$

$$(-1)^{s.x}$$

Build χ for $n=1$, $N=2$

$$XOR_s(x) = \sum_{s_i = 1} x_i \bmod 2$$

$$\begin{array}{c} \\ |x\rangle \left\{ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \right. \end{array} \begin{array}{c} \chi_{s=0} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix} \end{array} \quad \begin{array}{c} \chi_{s=1} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix} \end{array}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Similarly, for $n=2$

$$\begin{array}{c|cccc} |x\rangle \backslash |s\rangle & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & +1 & +1 & +1 & +1 \\ |01\rangle & +1 & -1 & +1 & -1 \\ |10\rangle & +1 & +1 & -1 & -1 \\ |11\rangle & +1 & -1 & -1 & +1 \end{array}$$

$$H \otimes H$$

Simple Search

f_{00}

input	output
00	1
01	0
10	0
11	0

f_{01}

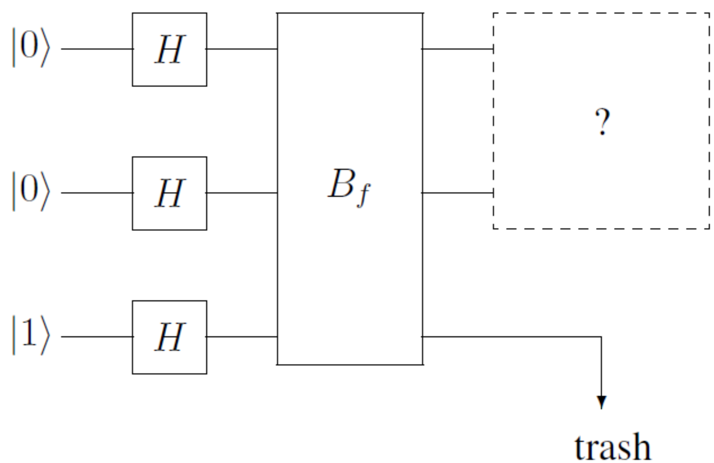
input	output
00	0
01	1
10	0
11	0

f_{10}

input	output
00	0
01	0
10	1
11	0

f_{11}

input	output
00	0
01	0
10	0
11	1



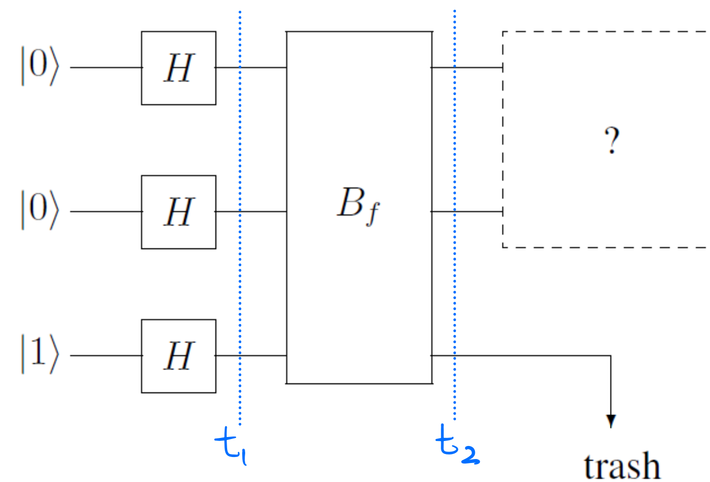
Simple Search

$$|\psi_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Determine $|\psi_2\rangle$ via phase kickback

$$B_f|\psi_1\rangle =$$

$$\frac{1}{2}\left((-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle\right)$$



Simple Search

$$\frac{1}{2}\left((-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle\right)$$

$$f = f_{00} \Rightarrow \frac{1}{2}\left(-|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$f = f_{01} \Rightarrow \frac{1}{2}\left(+|00\rangle - |01\rangle + |10\rangle + |11\rangle\right)$$

$$f = f_{10} \Rightarrow \frac{1}{2}\left(+|00\rangle + |01\rangle - |10\rangle + |11\rangle\right)$$

$$f = f_{11} \Rightarrow \frac{1}{2}\left(+|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$