Homework 8 for MATH 312

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October 16, 2017

6.2.15 A diagonalizable A^k approaches the zero matrix if and only if every eigenvalue satisfies $-1 < \lambda_i < 1$. The columns of A_1 sum to 1 so it is Markov and has at least one eigenvalue $\lambda = 1$. A_2 is like a Markov chain without full probability, so we can guess that it's bound to disappear and this is seen by the eigenvalues as well: the characteristic function gives $(\lambda - 0.6)^2 = 0.09$ and so the eigenvalues have absolute value below one, and $A_2^\infty \to 0$.

6.2.18 The eigenvalues for this A are the roots of

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$$

so $\lambda = 1, 3$. By inspection, the eigenvector corresponding to $\lambda = 1$ is $x_1 = (1, 1)$ and the eigenvector corresponding to $\lambda = 3$ is $x_2 = (1, -1)$. This gives,

$$A^k = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

with $k = 1$ giving the diagonalization.

Strang 6.2.34 A is the rotation matrix so we have

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - 2\lambda\cos\theta + (\cos^2\theta + \sin^2\theta) && (1) \\ &= \lambda^2 - 2\lambda\cos\theta + 1 && (2) \\ &= (\lambda - \cos\theta)^2 + \sin^2\theta && (3) \end{aligned}$$

So completing the square gives two roots: $\lambda = \cos\theta \pm i\sin\theta$.To find the eigenvectors see that in $(A - \lambda I)$ only the imaginary term is left on the diagonal. For $\lambda_1 = \cos\theta + i\sin\theta$ multiplying the right column by $-i$ eliminates the left. For the conjugate λ_2, multiplying the same column by i works. Thus we have,

$$A^n = \frac{1}{2i}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \text{cis}(\theta) & 0 \\ 0 & \text{cis}(-\theta) \end{bmatrix}^n\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

This uses the fact that $\cos\theta$ is even and $\sin\theta$ is odd so cis$(-\theta)$ returns the conjugate.

Eigenvectors of "nearly symmetric" matrices The given matrix has the characteristic polynomial

$$\det(A - \lambda I) = \lambda^2 - (2 + \epsilon)\lambda + (\epsilon + 1) \implies (\lambda - 1)^2 = (\lambda - 1)\epsilon$$

so we have $\lambda = 1, \epsilon + 1$ as the two eigenvalues. These correspondingly give the eigenvectors $(1, 0)$ and $(1, 1)$. The angle between these vectors is $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45$ degrees, which is not orthogonal.

6.4.13 Using the characteristic equations for S and B,

$$\det(S - \lambda I) = (\lambda - 4)(\lambda - 2), \quad \det(B - \lambda I) = \lambda(\lambda - 25)$$

it is clear that S has eigenvectors $(1, 1)$ and $(-1, 1)$ for $\lambda = 4, 2$ respectively and similarly that B has eigenvectors $(3, 4)$ and $(-4, 3)$ for $\lambda = 25, 0$. Thus,

$$S = (4/2)(1, 1)(1, 1)^T + (2/2)(-1, 1)(-1, 1)^T, \quad B = (25/25)(3, 4)(3, 4)^T$$

The fractional coefficients are left unsimplified to highlight the distinction between the eigenvalue numerators and normalizing denominators (the eigenvector corresponding with the zero eigenvalue for B is removed). This gives the $Q\Lambda Q^T$ form of S and B since the eigenvectors are orthogonal and may be added as shown to recompose into the original matrix. Since Q is symmetric we use that $QQ^T = I$.