Problem sets for MATH 312

Ashok M. Rao

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1 Homework 1

1.1 (Strang 1.1.2)

See figure 1. (I could not make the arrows in time but they are radially outward).

1.2 (Strang 1.1.18)

Let me pick e_1 and e_2 in $\mathbf{R}^2$. Linear combinations over positive coefficients would then yield the square shaded over $\{0,1\} \times \{0,1\}$. See figure 2.

1.3 (Strang 1.2.7)

The inner product between two vectors returns the (normalized) cosine of the angle subtended thereof. So

1. $\theta_a = \arccos 1/2$, 60 degrees.
2. $\theta_b = \arccos 0$, 90 degrees.
3. $\theta_c = \arccos 1/2$, 60 degrees.
4. $\theta_d = \arccos 1/\sqrt{2}$, 135 degrees.

1.4 (Strang 1.2.13)

Flipping bits across the z-axis, vectors $(0,1,0)$ and $(1,0,-1)$ are orthogonal to each other and the given vector $(1,0,1)$.

1.5 (Strang 1.2.14)

Flipping bits like above, so that there are an even number of zero coordinates on each vector, we first have $(1,1,-1,-1)$, and then $(0,0,1,-1)$ and $(1,-1,0,0)$, as possibilities.

1.6 Linear independence of three vectors

No. $\mathbf{u} + \mathbf{w} = 2\mathbf{v}$.

Figure 1: Vectors as labeled

Figure 2: The unshaded area is the span of linear combinations

1.7 (Strang 1.3.14)

This follows from cross multiplication. Whenever (a, b) is a multiple of (c, d) we have $a/c = b/d$ and thus it must be that $a/b = c/d$.

2 Homework 2

2.1 (Strang 2.1.16)

A matrix R rotates a vector $x \in \mathbf{R}^2$ by θ degrees if $\langle x, Rx \rangle = (x^2 + y^2)\cos\theta$. Not quite guessing possible candidates in the dark, let $x_r = x\cos\theta - y\sin\theta$ and $y_r = x\sin\theta + y\cos\theta$, *e.g.*

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Clearly now $\langle x, Rx \rangle = (x^2 + y^2)\cos\theta$ providing the desired result. In clockwise terms then,

$$R(-90 \text{ degrees}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R(180 \text{ degrees}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The latter result is more obviously interpreted as $-I$.

2.2 Triangular elimination

The system of equations to be solved is,

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 10x_1 + 9x_1 &= 11 \end{aligned}$$

so $A = \begin{bmatrix} 2 & 3 \\ 10 & 9 \end{bmatrix}$ and from $r_2 \leftarrow r_2 - 5r_1$ we get the system,

$$\begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

yielding $x = [2 \quad -1]^T$.

2.3 (Strang 2.2.13)

The triangular matrix is

$$\begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

following row operations

$$\begin{aligned} r_2' &\leftarrow r_2 - 2r_1 \\ r_3' &\leftarrow r_3 - r_1 - 2r_2' \end{aligned}$$

which gives $x = (3, 1, 0)$ as the solution.

2.4 (Strang 2.3.17)

The constraints here define the system

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}4\\8\\14\end{bmatrix}$$

With row operations,

$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\end{bmatrix}\begin{bmatrix}4\\8\\14\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&3\\0&2&8\end{bmatrix}\begin{bmatrix}4\\4\\10\end{bmatrix}\to\begin{bmatrix}1&1&1\\0&1&3\\0&0&2\end{bmatrix}\begin{bmatrix}4\\4\\2\end{bmatrix}$$

And so $a=2, b=1$ and $c=1$.

2.5 (Strang 2.3.25)

In the given matrix $A_3 = A_1 + A2$ and quite obviously $b_1 + b_2 \neq b_3$. Thus row operations would get you as far as $0 = 3$. Letting $b_3' = b_1 + b_2$ would reduce the system to the rectangular matrix defined on any two of the three rows, with infinitely many solutions.

2.6 (Strang 2.3.28)

We have $AB = I$ and $BC = I$. By the definition of I, $A = A(BC)$ and $C = (AB)C$ and thus $A = C$ holds by the associative property.

3 Homework 3

3.1 (Strang 2.4.15)

1. True, otherwise rows and columns don't match or commute.
2. False, AB need not equal BA and they may have different sizes. ($m \times m$ and $n \times n$).
3. True, since inner dimensions agree on both.
4. False, let $B = 0$ as counterexample.

3.2 (Strang 2.5.10)

For A the inverse follows just by logic of matrix multiplication and B is block diagonal,

$$A^{-1}=\begin{bmatrix}0&0&0&1/5\\0&0&1/4&0\\0&1/3&0&0\\1/2&0&0&0\end{bmatrix},\quad B^{-1}=\begin{bmatrix}3&-2&0&0\\-4&3&0&0\\0&0&6&5\\0&0&7&6\end{bmatrix}$$

3.3 (Strang 2.5.25)

For B notice that the all-ones vector is in the nullspace and thus no inverse exists. To invert A, first swap rows 1 and 2 and then eliminate

$$\begin{bmatrix}1&2&1&0&1&0\\0&-1&1&0&-1&1\\0&0&-4&1&1&-3\end{bmatrix},\quad\begin{bmatrix}1&0&0&3/4&-1/4&-1/4\\0&1&0&-1/4&3/4&-1/4\\0&0&1&-1/4&-1/4&3/4\end{bmatrix}$$

Thus the answer becomes

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

3.4 (Strang 2.6.13)

By inspection, L is a lower triangular matrix of ones and U is upper triangular of a, then $b-a$, then $c-b$, and finally $d-c$. Specifically,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

3.5 (Strang 2.7.16)

For A, we swap rows 1 and 2,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

And for B we swap rows 2 and 3,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3.6 Question about linearly independent vectors

Yes. The matrix formed by these vectors are diagonally-dominant and thus invertible.