Homework 4 for MATH 312

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Matrix substitution The algorithm characterizing substitution in the case of $A = LU$ is essentially solving for y, where $Ly = b$, and recovering the solution to $Ax = b$ as $Ux = y$. Performing this first substitution for the given matrices yields $y = (1, 0, 1, 0)$ and then back substituting yields $x = (2, -1, 1, 0)$.

(Strang 3.1.17) The set of invertible matrices is given by $\mathcal{A} = \{A \mid A, B \in \mathbf{R}^{n\times n} \text{ and } AB = I \text{ and } BA = I_n\}$. The matrices $\mathcal{A}$ trivially do not form a subspace as the zero matrix $\mathbf{0}$ clearly cannot be inverted. The set of singular matrices is given by $\mathcal{A}^c$, the absolute complement of $\mathcal{A}$ in the space of n-dimensional square matrices $\mathbf{R}^{n\times n}$. Clearly $\mathcal{A}^c$ is not a subspace either. Let $\mathbf{diag}\, x \in \mathbf{R}^{n\times n}$ be the matrix formed with the vector x and all zeros otherwise. Then clearly $\mathbf{diag}\,(0, 1)$ and $\mathbf{diag}\,(1, 0) \in \mathcal{A}^c$ and just as clearly $\mathbf{diag}\,(0, 1) + \mathbf{diag}\,(1, 0) \in \mathcal{A}$. Thus $\mathcal{A}^c$ cannot be a subspace given that it isn't closed under addition.

Column space of invertible matrices The invertibility of $A \in \mathbf{R}^{n\times n}$ requires that $Ax = 0$ admit only the trivial solution $x = \mathbf{0}$. Otherwise, $A^{-1}Ax = Ix$ would contradict the fact that $A\,\mathbf{0} = \mathbf{0}$ for any A. Thus $N(A)$ has dimension zero and consists only of the trivially zero vector. It follows that the matrix has column space of dimension n, *e.g.*it is full rank and equivalently $C(A) = \mathbf{R}^n$.

(Strang 3.2.1) A and B reduce to their triangular echelon forms respectively by letting $A_2 \leftarrow A_2 - A_1$, $A_3 \leftarrow A_3 - A_2$ and $B_3 \leftarrow B_3 - 2B_2$ (and dividing by the pivots)

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus A is pivoted by x_1, x_3 and has x_2, x_4, x_5 as free variables. B is pivoted by x_1, x_2 and has x_3 as a free variable.

(Strang 3.2.20) The dimension of the nullspace is given by $n - r$ so both $N(A)$ and $C(A)$ would have unit dimension in the given case. The kernel must be a line so without loss of generality $N(A) = \{(0, x) \mid x \in \mathbf{R}\}$. To satisfy the definition of $N(A)$, let the first column of A be a line, which for simplicity we will take in the direction of a basis vector, and the second column be the zero vector, *e.g.*

$$Ax = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 0 \\ x \end{bmatrix}, \quad Az = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The column space of A is the line traced by Ax and the null space is another line that takes A to the zero vector.

(Strang 3.2.53) Write A and B as

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$$

Each of the combined matrices are trivially rank one, and solve the required by simple matrix addition.