Homework 9 for MATH 312

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Strang 6.5.22 For S_1 the characteristic function is $p(\lambda) = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1)$ giving $\lambda = 1, 9$. For S_2 we have $p(\lambda) = \lambda^2 - 20\lambda + 64 = (\lambda - 4)(\lambda - 16)$ so $\lambda = 4, 16$. By diagonalizing, the square root falls on the eigenvalues giving for S_1

$$A_1^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

And for S_2,

$$A_2^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The roots normalize QQ^T and it is clear that in both cases $S = A^T A$, in fact $S = A^2$ since $A = A^T$ in this case.

Strang 6.5.35 For positive definite S with eigenvalues indexed in descending order, the matrix $(\lambda_1 I - S)$ will have eigenvalues $\lambda_1 - \lambda_i \geq 0$ so the matrix is positive semidefinite. (These are the eigenvalues because the characteristic function $p(\lambda + \lambda_1)$ is just shifted across).

Strang 7.1.6 A has eigenvalues $\lambda = 0, 4$ either by a trivial characteristic function or that these are the only values so that $\det A = 0$ while $\mathbf{Tr}\, A = 4$. Very similarly, $A^T A$ has eigenvalues 0 and 25, giving $\sigma_1 = 5$ and $\sigma_2 = 0$. corresponding to eigenvectors $v_1 = (2, 1)/\sqrt{5}$ and $v_2 = (1, -2)/\sqrt{5}$. These are orthogonal (orthonormal, now) as they must be since $A^T A$ is symmetric. AA^T has the same eigenvalues as $A^T A$ but with eigenvectors $u_1 = (1, 2)/\sqrt{5}$ and $u_2 = (2, -1)/\sqrt{5}$.

SVD example To find the SVD we first calculate $A^T A$ and AA^T for the singular values and orthogonal bases,

$$A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}, \quad AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

It seems easier to find eigenvalues from AA^T, which are shared with $A^T A$ (except for $\lambda = 0$ in the nullspace). This gives $det(AA^T - \lambda I) = (\lambda - 9)(\lambda - 25)$ and so we have $\sigma_1, \sigma_2 = 5, 3$. These correspond to eigenvectors for $A^T A$ which are respectively $v_1 = (1, 1, 0)$ and $v_2 = (1, -1, 4)$. Likewise for AA^T we have $u_1 = (1, 1)$ and $u_2 = (-1, 1)$. By the fact that $Av_i = \sigma_i u_i$ and normalizing, we can get the SVD

$$A = \frac{5}{\sqrt{2}}(1, 1)\frac{1}{\sqrt{2}}(1, 1, 0)^T + \frac{3}{\sqrt{2}}(-1, 1)\frac{1}{\sqrt{18}}(1, -1, 4)^T = \frac{5}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 & 1 & -4 \\ 1 & -1 & 4 \end{bmatrix}$$

Or put in the form $A = U\Sigma V^T$,

$$U\Sigma\, V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 & 3 & 0 \\ -1 & 1 & -4 \\ -2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \end{bmatrix}$$

Strang 7.4.11 For $A = (3, 4, 0)^T$,

$$A^T A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AA^T = [25], \quad U\Sigma V^T = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 1 \\ -4 & 3 & 0 \end{bmatrix}$$

Singular value $\sigma_1 = 5$ corresponds to the vector $v_1 = (1/5)(3, 4)$ and $\sigma_2 = 0$ corresponds to $v_2 = (1/5)(4, -3)$. The pseudoinverse can now be calculated like $A^+ = V\Sigma^+U^T$,

$$A^+ = \frac{1}{5} \begin{bmatrix} 3 & 0 & -4 \\ 4 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/25 \\ 4/25 \\ 0 \end{bmatrix}, \quad A^+A = \begin{bmatrix} 9/25 & 12/25 & 0 \\ 12/25 & 16/25 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AA^+ = \begin{bmatrix} 1 \end{bmatrix}$$