Optional Homework for MATH 312

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Strang 4.1.10 The nullspace of a matrix is the orthogonal complement of its row space, $e.g. N(A) \perp C(A^T)$. For symmetric matrices $A = A^T$ this also implies that $N(A) \perp C(A)$. By implication, if $Ax = 0$ and $Az = 5z$, then $x \in N(A)$ and $z \in C(A^T)$, so $x^T z = 0$.

Strang 4.2.13 The projection matrix P associated with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ brings an $x \in \mathbf{R}^4$ to $\mathbf{R}^3$ so

$$\mathbf{Proj}(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \end{bmatrix}^T$$

Strang 4.2.17 Since $P = PP$,

$$(I - P)^2 = I^2 + P^2 - 2PI = I - P$$

For some matrix A, any vector b is a combination of $x_p \in C(A)$ and $x_n \in N(A^T)$, that is $b = x_p + x_n$. So we have $(I - P)b = x_n - Px_n = (I - P)x_n$. In words, $(I - P)$ projects b onto the left nullspace $N(A^T)$.

Strang 4.3.17 We need to fit the data,

$$\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix}^T \begin{bmatrix} C \\ D \end{bmatrix}$$

Now to compute the estimator, we calculate $\hat{\beta} = X(X^T X)^{-1} X^T y$,

$$\begin{aligned} (X^T X)^{-1} X^T &= \left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 8 & 4 & 2 \\ -5 & 1 & 4 \end{bmatrix} \end{aligned}$$

Applying this to y gives,

$$\hat{\beta} = \frac{1}{14} \begin{bmatrix} 8 & 4 & 2 \\ -5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

So $C = 9$ and $D = 4$. Therefore the least square solutions $(\hat{t}, \hat{b})$ are $(-1, 5), (1, 13)$, and $(2, 17)$. This fit and the data are plotted in Figure 1 overleaf.

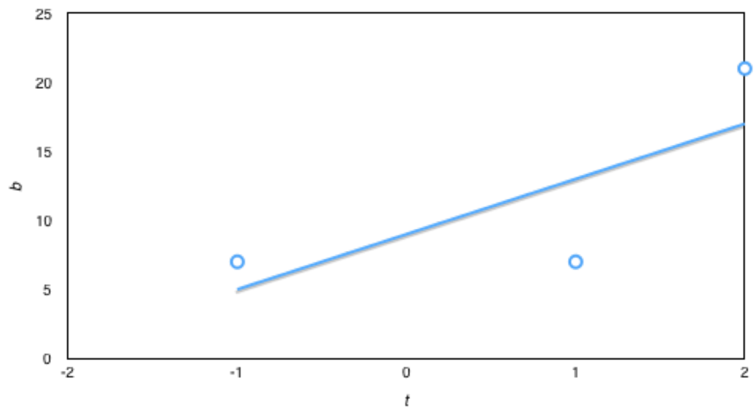

Figure 1: OLS fit for Strang 4.3.17

Strang 4.4.21 First obtain A^TA (since we care about the column space) and then implement the Gram-Schmidt process by performing elimination as follows:

$$
\begin{aligned}
[A^TA \mid A^T] &= \left[\begin{array}{cc|cccc} 4 & 2 & 1 & 1 & 1 & 1 \\ 2 & 14 & -2 & 0 & 1 & 3 \end{array}\right] \\
&\to \left[\begin{array}{cc|cccc} 1 & 1/2 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 13 & -5/2 & -1/2 & 1/2 & 5/2 \end{array}\right] \\
&\to \left[\begin{array}{cc|cccc} 1 & 1/2 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & -5/26 & -1/26 & 1/26 & 5/26 \end{array}\right]
\end{aligned}
$$

Normalizing, this yields

$$e_1 = \frac{1}{2}(1,1,1,1), \quad e_2 = \frac{1}{\sqrt{52}}(-5,-1,1,5)$$

Now obtain the projection,

$$\mathbf{Proj}\,(b) = (e_1^T b)e_1 + (e_2^T b)e_2 = (-1/2)(7,3,1,-3)$$

Both the basis vectors are clearly orthonormal. Further the part of the vector excluding the projection, *e.g.* $b - \mathbf{Proj}\,(b)$ is also orthogonal to the basis.