Homework y for MATH 312

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October 16, 2017

Log det Using the cofactor expansion on any fixed row i of A, the determinant is

$$\det A = \sum_{j=1}^{n} A_{ij} C_{ij}$$

where C_{ij} is the cofactor matrix, that is $C_{ij} = (-1)^{i+j} \det M_{ij}$ in which M_{ij} is the $(n-1)x(n-1)$ the (i,j) minor of A. To find the gradient of $\log \det A$ apply the chain rule on this definition,

$$\begin{aligned} \nabla(\log \det A) &= (\det A)^{-1} \left(\nabla A_j C_j\right)_i \\ &= (\det A)^{-1} \cdot C_{ij} = A_{ji}^{-1} \quad \text{(for } i,j = 1,\dots,n\text{)} \end{aligned}$$

The first and second equalities consider the vector gradient of $\log x$ before concluding the result using Cramer's rule.

Strang 6.1.27 Clearly $\textbf{rank}\, A = 1$. The expansion in $\det A - \lambda I$ produces only one non-zero cofactor on each row or column, since $\textbf{rank}\, M_{ij} < \textbf{rank}\, M_{ii}$ for $i \neq j$. This applies inductively, so

$$\lambda^3(\lambda - 4) = 0, \quad \lambda = 0, 0, 0, 4$$

C has two independent rows and columns spanning $\mathbb{R}^2 \subseteq \mathbb{R}^4$. Given that $\textbf{Tr}\, C = 4$ and $C(1,1,1,1) = 2(1,1,1,1)$, it follows that $\lambda_1 = 2$ and because the eigenvalues sum to the trace, $\lambda_2 = 2$. (The other eigenvalues are zero since $\textbf{rank}\, C = 2 < 4$.

Strang 6.1.2 Since $A+I$ is singular, we automatically have $\lambda_1 = -1$ for the eigenvector $(2,-1)$. This gives that $\lambda_2 = 5$ (since the product of eigenvalues is the determinant) and the associated eigenvector $(1,1)$. Adding the identity preserves the eigenvectors by definition $(A+I)x = Ax + x = (\lambda+1)x$, which holds generally. Therefore we have, for $A+I$, $\lambda = 0, 6$ with the same eigenvectors.

Change of polynomial basis Let $\mathcal{P}$ and $\mathcal{P}'$ be the polynomial bases given by,

$$\mathcal{P} = \left\{ \begin{bmatrix} x^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{P}' = \left\{ \frac{1}{2} \begin{bmatrix} x^2 \\ x \\ 0 \end{bmatrix}, \begin{bmatrix} x^2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

The change of basis into the v's is simply the underlying matrix W with $V = W^{-1}$ sending the coordinates back to the w's that is,

$$W = \begin{bmatrix} 1/2 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 1/3 & -1/3 & 1 \end{bmatrix}$$

Linear transformations I will use W as basis for the space of $2x2$ matrices:

$$W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Generally e_{ij} representing the all zeros matrix except one at (i, j). We want to describe the matrix form of the transformation $T(A) = A^T + 2A$,

$$T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The range and domain of T_1 are $4x1$ vectors which we understand as corresponding to the basis W matrices in the row-major ordering. Both by definition of $T(A)$ and inspection of the matrix thereof it is clear that $\mathcal{N}(T) = 0$.