Homework 5 for MATH 312

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(Strang 3.3.4) To find the general solution we first reduce to $Rx = d$ to find the particular solution:

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here x_1, x_3 are pivots with x_2, x_4 free. Solving for d the particular solution is $x_p = (1/2, 0, 1/2, 0)$. $N(R)$ is spanned by vectors $(-3, 1, 0, 0)$ and $(0, 0, -2, 1)$, found by setting x_2 or x_4 to 0. The general solution is thus $x = (1/2, 0, 1/2, 0) + c_1(-3, 1, 0, 0) + c_2(0, 0, -2, 1)$.

Rank of a matrix Reduce A by subtracting A_1 from A_2 and A_3 and A_2 from A_3 resulting in R (below). If $q = 2$ then the final row is redundant and $r = 2$; the nullspace solution formed on the line $c(1, 1, -1)$. Otherwise the matrix is full rank with $n = m = r = 3$ with one solution via substitution.

$$R = \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & q-2 & b_3 - b_2 \end{array}\right]$$

(Strang 3.3.24) The matrices A, B, C, D below solve the respective conditions. For A, b has a solution if and only if $b_1 = b_2$. B admits infinitely many solutions on the line $x_1 + x_2 = b$. C admits any $x \in \mathbf{R}^n$ if $b = 0$ and nothing otherwise. D is always solved by $(b_1, b_2/2)$.

$$Ax = \begin{bmatrix} a & a \end{bmatrix}^T \begin{bmatrix} x \end{bmatrix}, \quad Bx = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \quad Cx = \mathbf{0}\, x, \quad Dx = \begin{bmatrix} e_1^T & 2e_2^T \end{bmatrix} \begin{bmatrix} x_1 x_2 \end{bmatrix}^T$$

(Strang 3.3.30) The particular solution follows from the Gauss-Jordan form on the right. By substitution and a zero free variable, $x_p = (-4, 3, 0, 2)$ is obtained as a solution. The homogenous solutions are obtained in the nullspace with $x_3 = 1$ yielding $x_n = c(-2, 0, 1, 2)$ for $x = (-4, 3, 0, 2) + c(-2, 0, 1, 2)$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

(Strang 3.4.13) The row spaces of A and U are equivalent and 2 dimensional, since they are each spanned by $(1, 1, 0)$ and $(0, 2, 1)$. The respective column spaces share one basis vector, $(1, 2, 0)$, and also span a 2 dimensional space since $a_1^T + 2a_3^T = a_2^T$ and $u_1^T + u_3^T = u_2^T$.

(Strang 3.4.33) The functions $y_1 = \exp\{2x\}$ and $y_2 = x$ span the solution spaces for $y'/y = 2$ and $y'/y = 1/x$ respectively. These equations have unit dimensional solution spaces as they are first order.

(Strang 3.4.34) $y_1(x), y_2(x)$, and $y_3(x)$ may be polynomials. If they each have degree one, like $y_i(x) = c_i x$, their span has dimension 1. If otherwise any two have degree 1 and the other degree 2 (like ax, bx, and cx^2) then the span has dimension 2. If all three have unique degrees 1,2, and 3 (like x, x^2, x^3) then the span has dimension 3.