Homework 1 for Physics 503

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Problem 1

(a) To obtain the metric components $g_{\mu\nu}$ apply the transformation rule for components of dual vectors ((1.53) in Carroll) which is,

$$\frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial \phi}{\partial x^{\mu}}$$

These follow from the defined spherical coordinates stated in inverse,

$$\begin{aligned}
dx &= d(r\cos\varphi\sin\theta) \\
&= \cos\phi\sin\theta \cdot dr + r(\cos\phi\cos\theta\cdot d\theta - \sin\phi\sin\theta\cdot d\phi) \\
dy &= d(r\sin\varphi\sin\theta) \\
&= \sin\phi\sin\theta\cdot dr + r(\sin\phi\cos\theta\cdot d\theta + \cos\phi\sin\theta\cdot d\phi) \\
dz &= d(r\cos\theta) \\
&= \cos\theta\cdot dr - r\sin\theta\cdot d\theta
\end{aligned}$$

The transformed components are found by substitution into $ds^2 = dx^2 + dy^2 + dz^2$ which will be in the form $ds^2 = c_1(r,\theta,\varphi)dr^2 + c_2(r,\theta,\varphi)d\theta^2 + c_3(r,\theta,\varphi)d\varphi^2$. Collecting terms for the coefficients,

$$\begin{aligned}
c_1 &= \sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta = 1 \\
c_2 &= r^2(\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\varphi) = r^2 \\
c_3 &= r^2\sin^2\theta(\sin^2\phi + \cos^2\phi) = r^2\sin^2\theta
\end{aligned}$$

Lower order differential $drd\phi$ and $d\theta d\phi$ cancel when adding dx^2 with dy^2 since the cross terms cancel leaving only the sum of squares. Likewise for $drd\theta$ when adding dz^2. This leaves,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$$

(b) A rotation by θ in $\mathbf{R}^2$ is given by,

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The spacetime metric in the rotating coordinates can be found by noting $R(\theta)R(-\theta) = I$ and that only x, y are transformed so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega t' & -\sin\omega t' \\ \sin\omega t' & \cos\omega t' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

given that the rotation ωt is counterclockwise. It is straightforward that $dt' = dt'$ and $dz' = dz$. Use the change of basis matrix to calculate that

$$\begin{aligned} dx &= -\omega(x'\sin\omega t' + y'\cos\omega t')dt' + \cos\omega t' dx' - \sin\omega t' dy' \\ dy &= \omega(x'\cos\omega t' - y'\sin\omega t')dt' + \sin\omega t' dx' + \cos\omega t' dy' \end{aligned}$$

To compute the metric, first assure that the cross terms more or less vanish (so that the rest is easy). These amount to,

$$\begin{aligned} dx^2 &= 2\omega\left[\sin\omega t'\cos\omega t'(y'dy' - x'dx') + x'\sin^2\omega t' dy' - y'\cos^2\omega t' dx'\right] - 2\sin\omega t'\cos\omega t' dx'dy' + (\cdots) \\ dy^2 &= 2\omega\left[\sin\omega t'\cos\omega t'(x'dx' - y'dy') - y'\sin^2\omega t' dx' + x'\cos^2\omega t' dy'\right] + 2\sin\omega t'\cos\omega t' dx'dy' + (\cdots) \end{aligned}$$

The cancellations are straightforward so that what remains from the cross terms is,

$$dx^2 + dy^2 = 2\omega(x'dy' - y'dx')dt' + (\cdots)$$

The squared terms combine into unit coefficient on dx', dy'. The coefficient on dt'^2 is $\omega^2(x'^2 + y'^2)$. So $g_{\mu\nu}$ in the "rotating coordinates" may be written,

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= [(-1 + \omega^2(x'^2 + y'^2))]dt'^2 + dx'^2 + dy'^2 + dz'^2 + 2\omega(x'dy' - y'dx')dt' \end{aligned}$$

Problem 2

(a) Parametrize the path by $(x, y) = (t, 2t)$. To find the line element,

$$\begin{aligned} ds^2 &= (1 + x^2)^2(dx^2 + dy^2) \\ &= 5(1 + t^2)^2(dt^2) \\ ds &= \sqrt{5}(1 + t^2)dt \end{aligned}$$

Thus the distance along the linear trajectory between $(-1, -2)$ and $(1, 2)$ corresponds to,

$$\sqrt{5}\int_{-1}^{1}(1 + t^2)dt = \frac{8\sqrt{5}}{3}$$

(b) Consider a more general form of the above, which is a trajectory taking the straight path to the y-axis at $(0, -k)$, climbing vertically until $(0, k)$ and then taking the straightest path again to the endpoint. It is clear that the metric is symmetric so the distances of the first and final leg are equal. A natural parametrization is given by $(x, y) = (t, (2-k)t - k)$. The line element (using the same method as above) is given by

$$ds = \sqrt{1 + (2-k)^2}(1 + t^2)dt$$

The second trajectory has the traditional Euclidean metric as it is on the y-axis, so that length is clearly $2k$. Compute the overall length as a function of k:

$$\begin{aligned} f(k) &= 2\left(k + \sqrt{1 + (2-k)^2}\int_{-1}^{0}(1 + t^2)dt\right) \\ &= 2\left(k + \frac{4}{3}\sqrt{1 + (2-k)^2}\right) \end{aligned}$$

Check that this reduces to the answer in (a) for $k = 0$. To find a shorter triplet of linear segments, minimize over k,

$$\frac{d}{dk}f(k) = 2 - \frac{8(2-k)}{3\sqrt{1 + (2-k)^2}}$$

The first order condition on $k^\star$ is

$$\begin{aligned} (2 - k^\star)^2 &= \frac{9}{7} \\ k^\star &= 2 - \frac{3}{\sqrt{7}} \end{aligned}$$

The three segments defined by this $k^\star = 2 - 3/\sqrt{7}$ give a final distance of approximately 5.76 or about 5 percent shorter than the straight line.

I suspect the geodesic somewhat resembles the `arctan`$(\cdot)$ curve (drawn below). The idea is that for $x \ll 0$ the trajectory should be horizontally towards the y-axis. For $x \to 0$, the trajectory should be close to the straight line path, since the metric is Euclidean to first-order.

Problem 3 For the given definitions x, y calculate that

$$\begin{aligned}
dx &= \cos\phi d\theta - \theta\sin\phi d\phi \\
dy &= \sin\phi d\theta + \theta\cos\phi d\phi \\
dx^2 &= \cos^2\phi d\theta^2 + \theta^2\sin^2\phi d\phi^2 - 2\theta\cos\phi\sin\phi d\theta d\phi \\
dy^2 &= \sin^2\phi d\theta^2 + \theta^2\cos^2\phi d\phi^2 + 2\theta\cos\phi\sin\phi d\theta d\phi
\end{aligned}$$

to observe that

$$dx^2 + dy^2 = d\theta^2 + \theta^2 d\phi^2$$

Thus the error according to the Euclidean distance without correction would be,

$$ds^2 - (dx^2 + dy^2) = (\sin^2\theta - \theta^2)d\phi^2$$

To find the value of this term in x, y calculate that

$$\begin{aligned}
\sec^2\phi d\phi &= \frac{xdy - ydx}{x^2} \\
d\phi^2 &= \left[\cos^2\phi\left(\frac{xdy - ydx}{x^2}\right)\right]^2 \\
&= \left[\frac{x^2}{\theta^2}\left(\frac{xdy - ydx}{x^2}\right)\right]^2 \\
&= \left(\frac{xdy - ydx}{\theta^2}\right)^2
\end{aligned}$$

The final two lines follow from the definition of x in the given coordinates. Denoting the correction on the length as $d\tilde{s}^2$, resubstitute the $d\phi^2$ term to obtain,

$$\begin{aligned}
d\tilde{s}^2 &= (\sin^2\theta - \theta^2)\frac{(xdy - ydx)^2}{\theta^4} \\
&= \left(\frac{\sin^2\theta - \theta^2}{\theta^4}\right)(xdy - ydx)^2
\end{aligned}$$

Express the coefficient as a correction factor $\varepsilon(\theta)$ wherein

$$\begin{aligned}
\varepsilon(\theta) &= \frac{\sin^2\theta - \theta^2}{\theta^4} \\
&= \frac{\sin^2\sqrt{x^2 + y^2} - (x^2 + y^2)}{(x^2 + y^2)^2}
\end{aligned}$$

where the expression in terms of x, y uses the fact that $\theta^2 = x^2 + y^2$. The exact metric would then be

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + \frac{\sin^2\sqrt{x^2+y^2} - (x^2+y^2)}{(x^2+y^2)^2}(xdy - ydx)^2 \\
&= \left(1 - y^2\frac{\sin^2\sqrt{x^2+y^2} - (x^2+y^2)}{(x^2+y^2)^2}\right)dx^2 \\
&+ \left(1 - x^2\frac{\sin^2\sqrt{x^2+y^2} - (x^2+y^2)}{(x^2+y^2)^2}\right)dy^2 \\
&- \left(\frac{\sin^2\sqrt{x^2+y^2} - (x^2+y^2)}{(x^2+y^2)^2}\right)2xydxdy
\end{aligned}$$

The Taylor approximation of $\varepsilon(\theta)$ around $\theta = 0$ is given by,

$$\tilde{\varepsilon}(\theta) = -\frac{1}{3} + \frac{2\theta^2}{45} - \frac{\theta^4}{315} + O(\theta^6)$$

The first correction term above corresponds to the suggested metric. The next order correction includes the additional term,

$$\epsilon(x, y) = \frac{2(x^2+y^2)}{45}(xdy - ydx)^2$$

which yields the modified metric ds'^2 expanded as,

$$ds'^2 = \left(1 - \frac{17y^2 + 2x^2}{45}\right)dx^2 + \left(1 - \frac{17x^2 + 2y^2}{45}\right)dy^2 + \frac{2}{3}\left(1 - \frac{2(x^2+y^2)}{15}\right)(xydxdy)$$

Problem 4 Let $R = 1$ to simplify the equations. Based on the given coordinates,

$$\begin{aligned}
dx^2 &= d\phi^2 \\
dy^2 &= \csc(x)^2 d\theta^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\
&= \sin^2(\theta)\left(\frac{d\theta^2}{\sin^2\theta} + d\phi^2\right) \\
&= \sin^2(\theta)(\csc^2\theta d\theta^2 + d\phi^2)
\end{aligned}$$

Next,

$$e^y = \frac{1}{\tan(\theta/2)}$$
$$\theta = 2\cot^{-1}(e^{-y})$$
$$\Omega^2(x,y) = \sin^2\left(2\cot^{-1}(e^{-y})\right)$$

Wolfram Alpha says this becomes $\Omega^2(x,y) = sech^2(y)$.

Problem 5 I was using various online lecture notes to teach myself the essentials of variational calculus, and some parts of this example were derived within (of at least the Beltrami identity, I avoided reading anything on the bead problem).

(a) Calculate the total derivative given that $L(x,\dot{x})$ has no time dependence,

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{\partial}{\partial t} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial t}$$
$$= \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x}$$

Substituting the Euler-Lagrange equation that $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial L}$ obtain:

$$\frac{dL}{dt} = \frac{d}{dt}\frac{\partial L}{\partial L}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x}$$
$$= \frac{d}{dt}\left(\frac{dx}{dt}\frac{\partial L}{\partial \dot{x}}\right)$$

The last line is by the product rule. Collecting all terms into the LHS yields the required identity,

$$\frac{d}{dt}\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right) = 0$$

(b) Without friction, potential energy is converted into kinetic energy so that at each y,

$$\frac{1}{2}mv^2 = mgy \implies v = \sqrt{2gy}$$

We integrate over the path of the wire, and by dimensional requirements the total time is given as $T = \int_A^B v^{-1}ds$ where ds gives the line element, and can be found as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + y'^2}dx$$

The integral then becomes,

$$T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1+y'^2}{y}} dx$$

Earlier we were integrating over arc length and now that is reduced to an integral over horizontal distance; both of these quantities move forward with time. Also note that the integrand excludes any direct dependence on x, so the conditions for the Beltrami identity in (a) are met giving:

$$\begin{aligned}
\sqrt{\frac{1+y'^2}{2gy}} - y' \frac{\partial\left(\sqrt{\frac{1+y'^2}{2gy}}\right)}{\partial y'} &= C \\
\sqrt{\frac{1+y'^2}{2gy}} - \frac{y'^2}{\sqrt{(2gy)(1+y'^2)}} &= C \\
\frac{1}{1+y'^2} &= 2C^2 gy \\
y(1+y'^2) &= \frac{1/g}{2C^2}
\end{aligned}$$

This gives the necessary differential equation.

(c) Using the equations given,

$$\begin{aligned}
dx &= \frac{C}{2}(d\theta - \cos\theta d\theta) \\
d\theta &= \frac{2}{C}\frac{dx}{1-\cos\theta} \\
\frac{dy}{dx} &= \frac{\sin\theta}{1-\cos\theta} \\
&= \cot(\theta/2)
\end{aligned}$$

Now express the differential equation from (b) as

$$\begin{aligned}
y' &= \sqrt{\frac{C}{y} - 1} \\
&= \sqrt{\frac{2}{1-\cos\theta} - 1} \\
&= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \\
\frac{dy}{dx} &= \cot(\theta/2)
\end{aligned}$$

Clearly the conditions are equivalent.