Homework 1 for Physics 531

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Problem 1

(a) Expand the operator as $\left\langle (AB)^\dagger a' \middle| a'' \right\rangle = \left\langle a' \middle| ((AB)^\dagger)^\dagger \middle| a'' \right\rangle$ then conclude

$$\left\langle a' \middle| AB \middle| a'' \right\rangle = \left\langle B^\dagger A^\dagger a' \middle| a'' \right\rangle$$

So $(AB)^\dagger = B^\dagger A^\dagger$.

(b) Likewise

$$\begin{aligned}
\operatorname{Tr} AB &= \sum_{a'} \left\langle a' \middle| AB \middle| a' \right\rangle \\
&= \sum_{a',a''} \left\langle a' \middle| A \middle| a'' \right\rangle \left\langle a'' \middle| B \middle| a' \right\rangle \\
&= \sum_{a',a''} \left\langle a'' \middle| B \middle| a' \right\rangle \left\langle a' \middle| A \middle| a'' \right\rangle \\
&= \sum_{a''} \left\langle a'' \middle| BA \middle| a'' \right\rangle
\end{aligned}$$

which evaluates to $\operatorname{Tr} BA$.

(c) Using either of the above

$$\begin{aligned}
\operatorname{Tr} U^\dagger A U &= \operatorname{Tr} U U^\dagger A \\
&= \operatorname{Tr} A
\end{aligned}$$

(d) Express the operator function as $f(A) = \sum_{a'} f(a') \left| a \right\rangle \left\langle a \right|$ using completeness.

Problem 2

(a) By inspection it is clear that $\operatorname{Tr} \sigma_i = 0$ and $\det\{\sigma_i\} = -1$ for each Pauli matrix σ_i. Given that each is two-dimensional it is easy to conclude that all eigenvalues are like $\lambda_{ii} = \pm 1$.

(b) Compute that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0$$

Directly, $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ follows To find the commutator and anticommutator compute that

$$\begin{aligned}
\left[\sigma_i, \sigma_j\right] &= \delta_{ij} + i\epsilon_{ijk}\sigma_k - \delta_{ji} + i\epsilon_{ijk}\sigma_k \\
&= 2i\epsilon_{ijk}\sigma_k \\
\left\{\sigma_i, \sigma_j\right\} &= \delta_{ij} + i\epsilon_{ijk}\sigma_k + \delta_{ji} - i\epsilon_{ijk} \\
&= 2I\delta_{ij}
\end{aligned}$$

(c) Expand and calculate

$$
\begin{aligned}
\exp\{i\mathbf{a}\cdot\vec{\sigma}\} &= \sum_{n=0}^{\infty}\frac{(-1)^n(\mathbf{a}\cdot\vec{\sigma})^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n(\mathbf{a}\cdot\vec{\sigma})^{2n}\cdot(\mathbf{a}\cdot\vec{\sigma})}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n|\mathbf{a}|^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n|\mathbf{a}|^{2n+1}}{(2n+1)!}\left(\frac{\mathbf{a}\cdot\vec{\sigma}}{|\mathbf{a}|}\right) \\
&= \cos|\mathbf{a}| + i\frac{\mathbf{a}\cdot\vec{\sigma}}{|\mathbf{a}|}\sin|\mathbf{a}|
\end{aligned}
$$

wherein (b) is used to simplify $(\mathbf{a}\cdot\vec{\sigma})^{2n}$ since the cross term in $(\mathbf{a}\cdot\vec{\sigma})^2 = \sum_{i,j}\mathbf{a}_i\mathbf{a}_j\sigma_i\sigma_j = \sum_{i,j}\delta_{ij}\mathbf{a}_i\mathbf{a}_j$ due to the fact that $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$

Problem 3

(a) Simply decomposing H (without assuming anything further), note that

$$
H = \frac{1}{2}\left[(h_{11}+h_{22})I + (h_{12}+h_{21})\sigma_x + i(h_{12}-h_{21})\sigma_y + (h_{11}-h_{22})\sigma_z\right]
$$

Since we have not assumed anything about H, the coefficients may be complex. Noting that the trace of $\sigma_i = 0$ and for convenience letting $\sigma_0 = I$,

$$
a_k = \frac{1}{2}\operatorname{Tr}(\sigma_k H)
$$

Yet comparing this with the decomposition above, it's clear that Hermiticity of H is one and the same as requiring real a_0 and $\mathbf{a}$.

(b) Without loss of generality express H as

$$
H = \begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix}
$$

Solving the characteristic polynomial,

$$
\begin{aligned}
\det\{H-\lambda I\} &= \lambda^2 - 2a_0\lambda + \left[(a_0+a_3)(a_0-a_3) - a_1^2 - a_2^2\right] \\
&= \lambda^2 - \lambda\operatorname{Tr}H + \det\{H\}
\end{aligned}
$$

Solving the quadratic,

$$
\begin{aligned}
\lambda &= \frac{1}{2}\left(\operatorname{Tr}H \pm \sqrt{\operatorname{Tr}H^2 - 4\det\{H\}}\right) \\
&= a_0 \pm \sqrt{\mathbf{a}\cdot\mathbf{a}}
\end{aligned}
$$

(c) In the additive form $H = a_0 I + \mathbf{a}\cdot\vec{\sigma}$, it is clear that the eigenvectors of H are the same as those for $\mathbf{a}\cdot\vec{\sigma}$ since the first term is just an additive shift. The magnitude on $\sqrt{\mathbf{a}\cdot\mathbf{a}}$ may be similarly disregarded for convenience so that the norm goes to unity. We solve for

$$
|H;\pm\rangle = \pm\begin{pmatrix} a_3 & a_1-ia_2 \\ a_1+ia_2 & a_3 \end{pmatrix}|H;\pm\rangle
$$

Solve the problem and this may be written as

$$
\begin{aligned}
|H;\pm\rangle &= \left(\pm\sqrt{a_1^2+a_2^2}, a_1+ia_2\right) \\
&= \left(\pm\sqrt{(1+a_3)(1-a_3)}, a_1+ia_2\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{1\pm a_3}, \pm\sqrt{1\mp a_3}\cdot(a_1+ia_2)\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{1\pm a_3}, \pm\sqrt{1\mp a_3}\exp\{i\alpha\}\right)
\end{aligned}
$$

Further, from the eigenvalue condition we have that $a_3 = \pm(1 + \sqrt{a_1^2 + a_2^2})$ or equivalently $a_1^2 + a_2^2 = (1 - a_3^2)\cos^2\beta + (1 - a_3^2)\sin^2\beta$ giving

$$|H;+\rangle = \begin{pmatrix} \cos(\beta/2) \\ \sin(\beta/2) \cdot \exp\{i\alpha\} \end{pmatrix}, \quad |H;-\rangle = \begin{pmatrix} \sin(\beta/2) \\ -\cos(\beta/2) \cdot \exp\{i\alpha\} \end{pmatrix}$$

Clearly these are orthogonal and stated both in terms of a_k as well as angles to the axes.

(d) This was done above simply to handle automatically the singularity case where $1 + a_3 = 0$ though as $\beta = \theta$ and $\alpha = \phi$. The eigenvalues are as before though by a shift,

$$\lambda = a_0 \pm \sqrt{A^2(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})} = a_0 \pm A$$

Problem 4

(a) The set of eigenvalues is $\{-1, 0, 1\}$ which are the only observable measurements.

(b) Calculate that

$$\begin{aligned} \langle L_x \rangle &= \langle L_z = 1 | L_x | L_z = 1 \rangle = 0 \\ \left\langle L_x^2 \right\rangle &= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 01 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \end{aligned}$$

This gives a calculation for the variance

$$\left\langle (\Delta L_x)^2 \right\rangle = \left\langle L_x^2 \right\rangle - \langle L_x \rangle^2 = \frac{1}{2}$$

so that the standard deviation is $\Delta L_x = \frac{1}{\sqrt{2}}$.

(c) Since L_x and L_z are compatible operators, they share eigenvalues. The corresponding eigenvectors, in the order of $\{-1, 0, 1\}$ can be found by inspection, and then normalizing gives:

$$\lambda_1 = \left(1/2, -1/\sqrt{2}, 1/2\right), \quad \lambda_2 = \left(1/\sqrt{2}, 0, -1/\sqrt{2}\right), \quad \lambda_3 = \left(1/2, 1/\sqrt{2}, 1/2\right)$$

(d) The probability of each outcome would be given by $\left|\langle L_x | L_z \rangle\right|^2$. By symmetry we can conclude that

$$\left|\langle L_x = - | L_z = - \rangle\right|^2 = \left|\langle L_x = + | L_z = - \rangle\right| = \left(\frac{1}{2}\right)^2$$

Thus measuring the positive and negative state both have probability $(1/4)$ and the remaining density falls on measuring $L_x = 0$ which has probability $(1/2)$.

(e) L_x^2 is diagonal and can be multiplied by inspection, and also it is straightforward that it has eigenvalues $\{0, 1\}$. Thus it must be that the state after the state is $|L_z = \pm 1\rangle$. Calculate the operator as

(f)

$$\left(|L_z = 1\rangle \langle L_z = 1| + |L_z = -1\rangle \langle L_z = -1|\right) \cdot |\psi\rangle$$

Using the solution from (b) calculate the result to be as $\left(1/2, 0, 1/\sqrt{2}\right)$ or which normalizes to $|\psi\rangle = \left(2/2\sqrt{3}, 0, 2/\sqrt{6}\right)$. Thus the probability of measuring the two possibilities $L_z = 1$ and $L_z = -1$ is $1/3$ and $2/3$ respectively.

(g) Let $\left|\langle L_z|\psi\rangle\right|$ be coefficients based on the stipulated probabilities. In particular, $\psi = \sum_{a'} a' \,|L_z\rangle$ so that we can write the coefficients respectively as $|a_-|^2 = |a_+|^2 = 1/4$ (as from above) and likewise $|a_0|^2 = 1/2$. Taking square roots as in $\sqrt{|a_{+,-,-}|^2}$ yields the coefficients for the expansion on the problem set as required. Finally calculate that

(h) Resubstitute from (b) into (g) so that

$$\begin{aligned}\left|\langle L_x = 0|\psi\rangle\right|^2 &= \frac{1}{2}\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}^T \begin{pmatrix}(1/2)\exp\{i\delta_1\}\\(1/\sqrt{2})\exp\{i\delta_2\}\\(1/2)\exp\{i\delta_3\}\end{pmatrix}\right)^2\\ &= \frac{1}{8}\left|\exp\{i\delta_i\} - \exp\{i\delta_3\}\right|2\end{aligned}$$

Apply the identity that $\left|e^{ix} - e^{iy}\right|^2 = \left[\mathrm{Im}\{e^{ix}\} - \mathrm{Im}\{e^{iy}\}\right]^2 + \left[\mathrm{Re}\{e^{ix}\} - \mathrm{Re}\{e^{iy}\}\right]^2$ to conclude:

$$P(L_x = 0) = \frac{1}{4}\left(1 + \mathrm{Re}\Big\{\exp\{i(\delta_1 - \delta_3)\}\Big\}\right) = \frac{1 + \cos(\delta_1 - \delta_3)}{4}$$

where the middle result follows from applying the Pythagorean identity on the squared terms, and double-angle identities on the cross terms. In fact, δ_2 is irrelevant though the phase difference between δ_1 and δ_3 affects the final outcome. In particular, the probability $P(L_x = 0) = 1/2$ is the maximum and attained when $\delta_1 = \delta_3$. Contrarily when $\delta_1 = \delta_3 + \frac{\pi}{2}$ the probability is minimized to $P(L_x = 0) = 0$, thus the result is entirely determined by relative not absolute phase.