Homework 2 for Physics 531

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Problem 1 Cofactor expansion gives that, $\det\{B-\lambda I\} = (b-\lambda)\left|\begin{smallmatrix} -\lambda & -ib \\ ib & -\lambda \end{smallmatrix}\right|$ and therefore that $|K_1\rangle = (1,0,0)$ is clearly a simultaneous eigenket, for B corresponding to the eigenvalue $\lambda = b$. The first principal subminor has characteristic polynomial $p(\lambda) = (\lambda-b)(\lambda+b)$, and thus yields the degenerate spectrum $\{b,b,-b\}$. Clearly $[A,B]=0$ as

$$AB = BA = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & abi \\ 0 & -abi & 0 \end{pmatrix} \tag{1}$$

Using that the degeneracy arises in the first principal subminor of block diagonal B, performing Gramm Schmidt orthogonalization on the resulting eigenvectors of that subminor will yield simultaneous eigenkets of A and B given the commutativity just established.[1] By inspection, the second eigenket corresponding to $\lambda = b$ must be $|K_2\rangle = (1/\sqrt{2})(0,1,i)$ when normalized. Similarly, $|K_3\rangle = (1\sqrt{2})(0,1,-i)$. These simultaneous eigenkets respectively correspond to the values $K_1 = \{a,b\}$, $K_2 = \{-a,b\}$, and $K_3 = \{-a,-b\}$, and the degeneracy is over different states as B is a single permutation from the diagonal.

Problem 2 In the position basis compute that,[2]

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int \langle\psi|x\rangle\ \langle x|P|\psi\rangle\,\mathrm{d}x \tag{2}$$

$$= -i\hbar\int \psi^\star(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\psi\right)\mathrm{d}x = -i\hbar\int \psi\,\mathrm{d}\psi = -i\hbar\left.\frac{\psi^2(x)}{2}\right|_{-\infty}^{\infty} \tag{3}$$

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1 Particularly, $B = \left(\begin{smallmatrix} b & 0 \\ 0 & M_{11} \end{smallmatrix}\right)$ where $M_{11} = \left(\begin{smallmatrix} 0 & -ib \\ ib & 0 \end{smallmatrix}\right)$

2 Note that the postulates lead to $\langle x|P|\psi\rangle = \int \langle x|P|x\rangle \langle x'|\psi\rangle\, dx' = -i\hbar\int \delta'(x-x')\psi(x')dx' = -i\hbar\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}$

Clearly however, (9) must vanish as the expectation value must be real. Consider that in the momentum basis,

$$\frac{c^\star c}{2\pi\hbar}\int\int \exp\left\{\frac{ip(x-x')}{\hbar}\right\} f(x)f(x')\,\mathrm{d}x\,\mathrm{d}x' = \frac{c^\star c}{2\pi\hbar}\int\int \exp\left\{-\frac{ip(x'-x)}{\hbar}\right\} f(x)f(x')\,\mathrm{d}x'\,\mathrm{d}x \tag{4}$$

In other words, that $\left|\langle p|\psi\rangle\right|^2 = \left|\langle -p|\psi\rangle\right|^2$, so that the probability distribution is even about zero, and therefore has vanishing expectation value. Computing for mean momentum when ψ is shifted follows from (3) and (4) like so:

$$\langle P_{op}\rangle = -i\hbar\int \exp\left\{\frac{ip_0x}{\hbar}\right\}\psi^\star(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\exp\left\{\frac{-ip_0x}{\hbar}\right\}\psi(x)\right)\mathrm{d}x \tag{5}$$

$$= -i\hbar\int \exp\left\{\frac{-ip_0x}{\hbar}\right\}\psi^\star(x)\left(\exp\left\{\frac{ip_0x}{\hbar}\right\}\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{ip_0\psi(x)}{\hbar}\exp\left\{\frac{ip_0x}{\hbar}\right\}\right)\mathrm{d}x \tag{6}$$

$$= -i\hbar\int \psi^\star(x)\,\mathrm{d}\psi + p_0\int\left|\psi(x)\right|^2\mathrm{d}x \tag{7}$$

$$= \langle P\rangle + p_0 \tag{8}$$

To evaluate $\langle\beta|x|\alpha\rangle$ in the momentum basis,

$$\langle\beta|x|\alpha\rangle = \int \mathrm{d}p'\int \mathrm{d}p''\,\langle\beta|p'\rangle\,\langle p'|x|p''\rangle\,\langle p''|\alpha\rangle \tag{9}$$

$$= \int \mathrm{d}p'\int \mathrm{d}p''\,\langle\beta|p'\rangle\left(i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\right)\langle p''|\alpha\rangle \tag{10}$$

$$= \int \mathrm{d}p'\,\phi^\star_\beta(p')\,i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p') \tag{11}$$

Problem 3 Position and momentum are incompatible observables only when they are not orthogonal, that is $[x_i, p_j] = i\hbar\delta_{ij}$. Thus, the calculation of the desired commutation relation between x_i, p_i and $G(\vec{\mathbf{p}})$, $F(\vec{\mathbf{x}})$ respectively simplifies to one taking G and F to be functions of p_i and x_i modulo scalar constants that will cancel. Before evaluating the power series, I will establish that $[x_i, p_i^n] = ni\hbar p_i^{n-1} = i\hbar\frac{\partial p_i^n}{\partial p_i}$ by induction (for $n=1$ this is trivial),

$$[x_i, p_i^n p_i] = [x_i, p_i^n]p_i + p_i^n[x_i, p_i] \tag{12}$$

$$= [n(i\hbar p_i^{n-1})]p_i \tag{13}$$

$$= (n+1)i\hbar p_i^n \tag{14}$$

Applying this result and linearity of the commutator to compute that

$$\left[x_i, G(p_i)\right] = \sum_{i=1}^{\infty} a_n[x_i, p_i^n] = i\hbar\frac{\partial}{\partial p_i}\sum_{i=1}^{n} a_n p_i^n = i\hbar\frac{\partial G}{\partial p_i} \tag{15}$$

The corresponding relation $\left[p_i, F(\vec{\mathbf{x}})\right] = -i\hbar\frac{\partial F}{\partial x_i}$ follows exactly as above by observing the anti-commutativity of the commutator on the inductive claim, *e.g.* that $[x_i, p_i] = -[p_i, x_i]$ and proceeding precisely as above (beyond which step no property unique to either operator was used). Calculating $\left[x^2, p^2\right]$,

$$\begin{aligned}\left[x^2, p^2\right] &= [xx, p]p + p[xx, p] && (16)\\ &= \big([x,p]x + x[x,p]\big)\, p + p\,\big([x,p]x + x[x,p]\big) && (17)\\ &= 2i\hbar\{x, p\} && (18)\end{aligned}$$

This compares with the classical Poisson bracket,[3]

$$\left[x^2, p^2\right]_{\text{classical}} \to \frac{\left[x^2, p^2\right]_{\text{quantum}}}{i\hbar} = \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} = 4xp \tag{19}$$

Problem 4 Compute the area under the Lorentzian

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon}{x^2+\epsilon^2}\,\mathrm{d}x &= \frac{1}{\pi}\lim_{j\to\infty}\left(\int_{-j}^{j}\frac{1}{(x^2/\epsilon)+\epsilon}\,\mathrm{d}x\right) && (20)\\ &= \frac{1}{\pi}\left[\pi\sqrt{\frac{1}{\epsilon^2}}\epsilon + \mathcal{O}\left(\frac{\epsilon}{j}\right)\right] && (21)\\ &\approx 1 && (22)\end{aligned}$$

Thus, so long as $(\epsilon/j) \to 0$, both $\epsilon \to 0$ and $j \to 0$ thereby concentrating measure around the center peak of the Lorentzian so as to replicate the properties of the Dirac function. Similarly, in terms of the Gaussian consider that,

$$\delta(x - x') = \lim_{\sigma\to 0}\left(\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-x')^2}{2\sigma^2}\right\}\right) \tag{23}$$

Consider that the concentration of measure within half a standard deviation from the mean limits towards the inscribing rectangle as $\sigma \to 0$

$$\lim_{\sigma\to 0}\left[\int_{x-\sigma/2}^{x+\sigma/2}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-x')^2}{2\sigma^2}\right\}\mathrm{d}x'\right] \approx \frac{1}{\pi}\frac{\sigma}{\sqrt{\sigma^2}} \tag{24}$$

Since the area does not vary with σ, it follows that towards the limit the measure of the distribution can be concentrated in an arbitrarily tall and narrow band around x' such that, like above, the necessary properties are

[3] In the classical limit it holds that $\{x, p\} = 2xp$

met. A similar representation also obtains by taking the dual of a Fourier transform, that would be,

$$f(x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{ikx'\} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-ikx\} f(x)\, \mathrm{d}x \right) \mathrm{d}k \tag{25}$$

$$= \int_{-\infty}^{\infty} \exp\{ikx'\} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-ikx\} f(x)\, \mathrm{d}x\, \mathrm{d}k \tag{26}$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{ik(x' - x)\}\, \mathrm{d}k \right\} f(x)\, \mathrm{d}x \tag{27}$$

wherein $\delta(x)$ is clearly represented by the term in the braces, given that this shows

$$f(x') = \int_{-\infty}^{\infty} \delta(x' - x) f(x)\, \mathrm{d}x \tag{28}$$

which is a property that characterizes the function. To relate this to the completeness and orthonormality of the momentum space, calculate that,

$$\int \mathrm{d}p' \left|p'\right\rangle \left\langle p'\right| = \int \mathrm{d}p' \int \mathrm{d}x' \int \mathrm{d}x'' \int \left|x'\right\rangle \left\langle x'|p'\right\rangle \left\langle p'|x''\right\rangle \left\langle x''\right| \tag{29}$$

$$= \int \mathrm{d}p' \int \mathrm{d}x' \int \mathrm{d}x'' \int \left|x'\right\rangle \left(\frac{1}{2\pi\hbar} \exp\left\{ \frac{ip'(x' - x'')}{\hbar} \right\} \right) \left\langle x''\right| \tag{30}$$

To recover the Fourier transform in (27), let $k = p/\hbar$ so that the expression becomes

$$\int \mathrm{d}p' \left|p'\right\rangle \left\langle p'\right| = \int \hbar\, \mathrm{d}k' \int \mathrm{d}x' \int \mathrm{d}x'' \int \left|x'\right\rangle \left(\frac{1}{2\pi\hbar} \exp\{ik(x' - x'')\} \right) \left\langle x''\right| \tag{31}$$

$$= \int \mathrm{d}x' \int \mathrm{d}x'' \int \left|x'\right\rangle \int \delta(x' - x'') \left\langle x''\right| \tag{32}$$

$$= \int \mathrm{d}x' \left|x'\right\rangle \left\langle x'\right| \tag{33}$$

$$= 1 \tag{34}$$

This proves that the momentum space is complete, and also gives the infinite-dimensional characterization of orthonormality used in the final line, which is that $\int \mathrm{d}x' \left|x'\right\rangle \left\langle x'\right| = 1$. Note the following property to compute higher derivatives of the Delta function,

$$\delta'(x - x') = \frac{\mathrm{d}}{\mathrm{d}x} \delta(x - x') = \delta(x - x') \frac{\mathrm{d}}{\mathrm{d}x'} \tag{35}$$

Now identify the linear differential operator as a matrix element $\delta'(x-x') = \langle x|D|x'\rangle$ noticing that $D^\dagger = -\delta(x-x')$.[4] This anticommutative property holds for for further derivatives, of course, allowing us to deduce that

$$\int \mathrm{d}x\, \delta^{(n)}(x-a)f(x) = (-1)^n f^{(n)}(a) \tag{36}$$

Compute inductively that

$$\begin{aligned}\int \mathrm{d}x'\, \delta'(x-x') \int \mathrm{d}x''\, \delta^{(n)}(x'-x'')f(x'') &= (-1)^n \int \mathrm{d}x'\, \frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\frac{\mathrm{d}^n}{\mathrm{d}x'^n} \int \mathrm{d}x''\, \delta(x'-x'')f(x'') \quad (37)\\ &= (-1)^n \int \mathrm{d}x'\, \frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\frac{\mathrm{d}^n}{\mathrm{d}x'^n} f(x') \quad (38)\\ &= (-1)^{n+a} \int \mathrm{d}x'\, \delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x'}\frac{\mathrm{d}^n}{\mathrm{d}x'^n} f(x') \quad (39)\\ &= (-1)^{n+1} \int \mathrm{d}x'\, \delta(x-x')\frac{\mathrm{d}^{n+1}f}{\mathrm{d}x^{n+1}} \quad (40)\end{aligned}$$

This corresponds to the infinite-dimensional linear differential operator.[5] It is trivial that the given property holds for $n=1$, and thus applying the induction on (36) gives the desired result. This formulation also makes it clear that $d^{(n)}(x) = 0$ for $x \neq 0$ as the integrand vanishes with the original delta function. Graphs are found in Figure 1. To show that $\delta\left(g(x)\right) = \sum_i \frac{\delta(x-x_i)}{\left|g'(x_i)\right|}$ provide the Taylor expansion, taking the function to linear order only

$$\begin{aligned}g(x) &= g(x_i) + g'(x_i)(x-x_i) + \frac{1}{2!}g''(x_i)(x-x_i)^2 + \mathcal{O}(x^3) \quad (41)\\ &= g'(x_i)(x-x_i) \quad (42)\end{aligned}$$

[4] Perhaps see this through the the commutation relations

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}, \delta(x)\right] = 2\delta'(x), \quad \left\{\frac{\mathrm{d}}{\mathrm{d}x}, \delta(x)\right\} = 0$$

[5] For example see this as a matrix element by chaining the relation,

$$\begin{aligned}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &= \int \mathrm{d}x'\, \langle x|D|x'\rangle \left\langle x' \middle| \int \mathrm{d}x''\, \langle x'|D|x''\rangle \langle x''|f\rangle \right\rangle\\ &= \int \mathrm{d}x'\, \langle x|D|x'\rangle \left\langle x' \middle| \int \mathrm{d}x''\, \delta(x'-x'')\frac{\mathrm{d}}{\mathrm{d}x''} \langle x''|f\rangle \right\rangle\\ &= \int \mathrm{d}x'\, \delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x'} \left\langle x' \middle| \frac{\mathrm{d}}{\mathrm{d}x'} \langle x'|f\rangle \right\rangle\end{aligned}$$

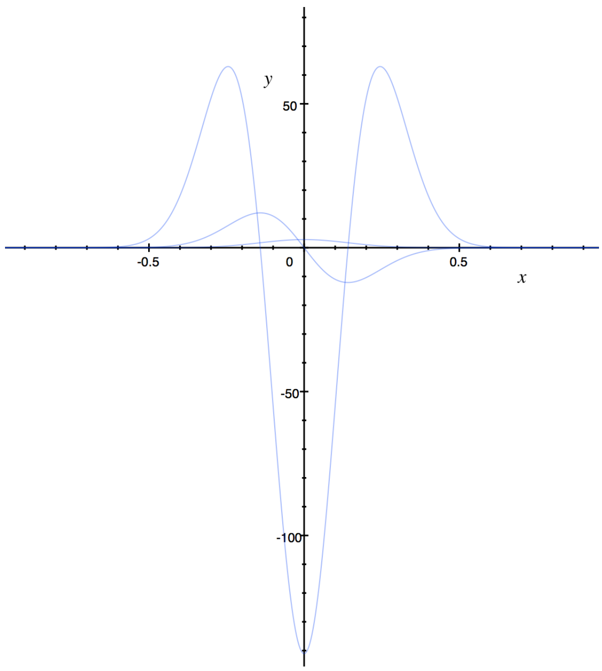

Figure 1: Gaussian representation of $\delta(x)$ and its derivatives ($\Delta = 0.2$)

Before continuing note that,

$$\int_{-\infty}^{\infty} (ax)\,\mathrm{d}x = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(|a|x)\,\mathrm{d}|a|x \tag{43}$$

This property is an exercise from Shankar, and in particular demonstrates that $\delta(ax) = \frac{\delta(x)}{|a|}$, by the chain rule. Represent (42) as an infinite, convergent sum,

$$\int_{-\infty}^{\infty} \varphi(x)\delta(g(x))\,\mathrm{d}x = \sum_i \int_{\mathcal{B}_\epsilon(x_i)} \varphi(x)\delta(g(x))\,\mathrm{d}x \tag{44}$$

$$= \sum_i \int_{\mathcal{B}_\epsilon(x_i)} \varphi(x)\delta\left(g'(x_i)(x - x_i)\right)\mathrm{d}x \tag{45}$$

$$= \sum_i \int_{\mathcal{B}_\epsilon(x_i)} \varphi(x)\frac{\delta(x - x_i)}{g'(x)}\,\mathrm{d}x \tag{46}$$

where the final line uses the result shown above using $\varphi(x)$ is a smooth function with compact support as in a test function from the previous parts. Note the following fact about the delta function,

$$\delta(x - x') = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x} \delta(x - x')\,\mathrm{d}x = \begin{cases} 0 & \text{if } x - x' < 0 \\ 1 & \text{if } x - x' > 0 \end{cases} \tag{47}$$

By the fundamental theorem of calculus,

$$\theta(x - x') = \int_{-\infty}^{x} \delta(x - x') \, \mathrm{d}x \tag{48}$$

So the required property is obtained,

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta(x - y)}{\mathrm{d}x} f(x) && (49) \\ &= \int_{-\infty}^{\infty} f(x)\delta(x - y) \, \mathrm{d}x && (50) \\ &= f(y) && (51) \end{aligned}$$

Although it should be mentioned that in (47) the integral value at $x = x'$ may not be well-defined, though this is more for notational convenience than final result (that is to say, taking limits as the quantity vanishes would work).