CS182 Homework #1

Shreyas Viswanathan

January 19, 2024

1. *I, Shreyas Viswanathan, affirm that I have not given or received any unauthorized help on this assignment and that this work is my own. What I have submitted is expressed and explained in my own words. I have not used any online websites that provide a solution. I will not post any parts of this problem set to any online platform and doing so is a violation of course policy.*

2. Complete the truth table for the proposition $(p \wedge (q \to r)) \leftrightarrow \neg r$.

p	q	r	$\neg r$	$q \to r$	$p \wedge (q \to r)$	$p \wedge (q \to r) \leftrightarrow \neg r$
F	F	F	T	T	F	F
F	F	T	F	T	F	T
F	T	F	T	F	T	T
F	T	T	F	T	F	T
T	F	F	T	T	T	T
T	F	T	F	T	T	F
T	T	F	T	F	F	F
T	T	T	F	T	T	F

3. Prove if the following statements are a tautology (always True) or a contradiction (always False). For tautologies/contradictions, use equivalences to show it (state the names of the equivalences used in your proof). If it is not a tautology or contradiction, give a counterexample (i.e. show that the statement can be true or false when T or F is assigned to p and q). Do NOT use truth tables.

(a)

$$
\begin{aligned}
&\equiv ((\neg b \wedge \neg c) \vee (b \wedge \neg a)) \vee (c \vee a) && \text{De morgan's and negation law}\\
&\equiv ((\neg b \wedge \neg c) \vee b) \wedge ((\neg b \wedge \neg c) \vee \neg a) \vee (c \vee a) && \text{Distributive law}\\
&\equiv ((b \vee \neg b) \wedge (b \vee \neg c)) \wedge ((\neg a \vee \neg b) \wedge (\neg a \vee \neg c)) \vee (c \vee a) && \text{Distributive and commutative law}\\
&\equiv (b \vee \neg c) \wedge ((\neg a \vee \neg b) \wedge (\neg a \vee \neg c)) \vee (c \vee a) && \text{Negation and identity law}\\
&\equiv (b \vee \neg c) \wedge (\neg a \vee (\neg b \wedge \neg c)) \vee (c \vee a) && \text{Distributive law}\\
&\equiv (b \vee \neg c) \wedge (\neg b \wedge \neg c) \vee (\neg a \vee (c \vee a)) && \text{Commutative and associative law}\\
&\equiv (b \vee \neg c) \wedge (\neg b \wedge \neg c) \vee T && \text{Associative and negation and domination law}\\
&\equiv T && \text{Domination law}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\neg((p \wedge (\neg q \rightarrow \neg p)) \rightarrow q) &\equiv \neg((p \wedge (q \vee \neg p)) \Rightarrow q) && \text{CDE}\\
&\equiv \neg(((p \wedge q) \vee F) \Rightarrow q) && \text{Distributive law and negation law}\\
&\equiv \neg(\neg((p \wedge q) \vee F) \vee q) && \text{CDE}\\
&\equiv \neg(((\neg p \vee \neg q) \wedge T) \vee q) && \text{De morgan's law}\\
&\equiv \neg((\neg p \vee \neg q) \vee q) && \text{Distributive and identity law}\\
&\equiv p \wedge q \wedge \neg q && \text{De morgan's and double negation law}\\
&\equiv p \wedge F && \text{Negation law}\\
&\equiv F && \text{Domination law}
\end{aligned}
$$

(c)

$$
\begin{aligned}
(p \rightarrow q) \vee (\neg q \rightarrow \neg p) &\equiv (\neg p \vee q) \vee (q \vee \neg p) && \text{CDE}\\
&\equiv (\neg p \vee \neg p) \vee (q \vee q) && \text{Associative law}\\
&\equiv \neg p \vee q && \text{Idempotent law}
\end{aligned}
$$

For $\neg p \vee q$:

$$
\begin{aligned}
&1.\text{If } p = \text{T}, q = \text{T}, \text{ then the proposition results in T}\\
&2.\text{If } p = \text{T}, q = \text{F}, \text{ then the proposition results in T}\\
&3.\text{If } p = \text{F}, q = \text{T}, \text{ then the proposition results in T}\\
&4.\text{If } p = \text{F}, q = \text{F}, \text{ then the proposition results in F}
\end{aligned}
$$

4. For the following, use equivalences (state the names of the equivalences used in your proof) and NOT truth tables.

   (a) Show that $(\neg q \Rightarrow \neg p) \vee (\neg q \Rightarrow \neg r) \equiv \neg q \Rightarrow (\neg p \vee \neg r)$

$$\begin{aligned}
(\neg q \Rightarrow \neg p) \vee (\neg q \Rightarrow \neg r) &\equiv (q \vee \neg p) \vee (q \vee \neg r) && \text{CDE} \\
&\equiv (q \vee q) \vee (\neg p \vee \neg r) && \text{Associative law} \\
&\equiv q \vee (\neg p \vee \neg r) && \text{Idempotent law} \\
&\equiv \neg q \Rightarrow (\neg p \vee \neg r) && \text{CDE}
\end{aligned}$$

   (b) Show that $q \Rightarrow (\neg p \Rightarrow \neg r) \equiv \neg p \Rightarrow (\neg q \vee \neg r)$

$$\begin{aligned}
q \Rightarrow (\neg p \Rightarrow \neg r) &\equiv q \Rightarrow (p \vee \neg r) && \text{CDE} \\
&\equiv \neg q \vee (p \vee \neg r) && \text{CDE} \\
&\equiv (p \vee \neg r) \vee \neg q && \text{Commutative law} \\
&\equiv p \vee (\neg r \vee \neg q) && \text{Associative law} \\
&\equiv p \vee (\neg q \vee \neg r) && \text{Commutative law} \\
&\equiv \neg p \Rightarrow (\neg q \vee \neg r) && \text{CDE}
\end{aligned}$$

5. Let P(x) denote the statement “x is a CS major,” Q(x) denote the statement “x has taken a course in discrete math,” and R(x, y) denote the statement “x studies at least y hours per week.” Take the universal set U to be all students at Purdue University. Express the following statements in terms of quantifiers and predicates.

   (a) Each CS major has taken a course in discrete math and studies at least 5 hours per week: $\forall x(P(x) \Rightarrow (Q(x) \wedge R(x, 5)))$

   (b) Some students have not taken a course in discrete math and study at least 8 hours per week: $\exists x(\neg Q(x) \wedge R(x, 8))$

   (c) No CS major that has taken a course in discrete math studies less than 2 hours per week: $\forall x((P(x) \wedge Q(x)) \Rightarrow R(x, 2))$

6. Students were told, “If you work hard, then you will get good grades. If you don’t work hard, then you will not get good grades.” Can they conclude that they will get good grades if and only if they work hard? Explain using mathematical notation.

   Let p = “You work hard”
   Let q = “You will get good grades”
   Given: $p \Rightarrow q =$ T and $\neg p \Rightarrow \neg q = T$
   Prove that $q \leftrightarrow p = T$

$$\begin{aligned} q \leftrightarrow p &\equiv (q \Rightarrow p) \wedge (p \Rightarrow q) && \text{Expansion of } q \leftrightarrow p \\ &\equiv T \wedge T && \text{Inference law} \\ &\equiv T && \text{Idempotent law} \end{aligned}$$

7. Let $p = \neg[(a \vee \neg c) \wedge (a \vee d) \vee (c \leftrightarrow \neg d)] \vee \neg[b \vee ((\neg c \to d) \wedge (\neg c \wedge \neg d))]$. Suppose we know that a is true and also resulting p is false. Show that b must be true. Do NOT use truth tables. [Hint: Show that it is not possible for b to be false]

$$
\begin{aligned}
F &\equiv \neg[b \vee ((c \vee d) \wedge (\neg c \wedge \neg d))] && \text{Both propositions need to equal F} \\
F &\equiv \neg b \wedge \neg((c \vee d) \wedge (\neg c \wedge \neg d)) && \text{De morgan's law} \\
F &\equiv \neg b \wedge (\neg(c \vee d) \vee \neg(\neg c \wedge \neg d)) && \text{De morgan's law} \\
F &\equiv \neg b \wedge ((\neg c \wedge \neg d) \vee (c \vee d)) && \text{De morgan's and double negation law} \\
F &\equiv \neg b \wedge (((\neg c \wedge \neg d) \vee c) \vee d) && \text{Associative law} \\
F &\equiv \neg b \wedge (((\neg c \vee c) \wedge (\neg d \vee c)) \vee d) && \text{Distributive law} \\
F &\equiv \neg b \wedge ((\neg d \vee c) \vee d) && \text{Negation and identity law} \\
F &\equiv \neg b \wedge T && \text{Commutative and negation and domination law}
\end{aligned}
$$

If $b = \text{F}$, then $\neg b \wedge \text{T} \neq \text{F} \therefore B \neq \text{F}$

8. Let a, b, and c be well defined propositions. A group of students wants to prove $a \leftrightarrow b \leftrightarrow c$. They have shown that the following propositions are true:

$$\begin{aligned} a &\to b \\ b &\to c \\ \neg a &\to \neg c \end{aligned}$$

Have they proven $a \leftrightarrow b \leftrightarrow c$? Explain.

The students have not proven that $a \leftrightarrow b \leftrightarrow c = T$ since they haven't shown that the converses of the propositions are true which are that $b \to a = T$, $c \to b = T$ and $\neg c \to \neg a = T$, which is needed to show that a bidirectional proposition is true. To elaborate further:

$$\begin{aligned} a \leftrightarrow b \leftrightarrow c &\equiv (a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow a) && \text{Inference law} \\ &\equiv ((a \to b) \wedge (b \to a)) \wedge ((b \to c) \wedge (c \to b)) \wedge ((c \to a) \wedge (a \to c)) && \text{Expansion} \\ &\equiv (T \wedge (b \to a)) \wedge (T \wedge (c \to b)) \wedge ((c \to a) \wedge (a \to c)) && \text{Plugging in} \\ &\equiv (b \to a) \wedge (c \to b) \wedge ((c \to a) \wedge (a \to c)) && \text{Identity law} \end{aligned}$$

The above statement can only be true when $(b \to a) = T$, $(c \to b) = T$, $(a \to c) = T$, *and* $(c \to a) = T$ and since none of these propositions have been proved to be true, the students have not proved that $a \leftrightarrow b \leftrightarrow c = T$.