Gauss-Jordan Elimination & The number of solutions

Warm up: *Row reduce the following matrix to find its rank:*

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

What's the rank for this one?

$$\begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \end{bmatrix}$$

1. *Use Gauss-Jordan elimination to solve the system* $\begin{vmatrix} 2x + 4y - 2z = -10 \\ 3x + 6y \qquad = -12 \end{vmatrix}$.

   Solution. The augmented matrix of the system is $\left[\begin{array}{ccc:c} 2 & 4 & -2 & -10 \\ 3 & 6 & 0 & -12 \end{array}\right]$. To solve the system, we start by row-reducing the augmented matrix:

$$\left[\begin{array}{ccc:c} 2 & 4 & -2 & -10 \\ 3 & 6 & 0 & -12 \end{array}\right] \begin{array}{c} \div 2 \\ \\ \end{array} \to \left[\begin{array}{ccc:c} 1 & 2 & -1 & -5 \\ 3 & 6 & 0 & -12 \end{array}\right] \begin{array}{c} \\ -3(\mathrm{I}) \end{array}$$
$$\to \left[\begin{array}{ccc:c} 1 & 2 & -1 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{c} \\ \div 3 \end{array}$$
$$\to \left[\begin{array}{ccc:c} 1 & 2 & -1 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} +(\mathrm{II}) \\ \\ \end{array}$$
$$\to \left[\begin{array}{ccc:c} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

   This corresponds to the system

$$\begin{vmatrix} \boxed{x} + 2y \qquad = -4 \\ \qquad\qquad \boxed{z} = \ \ 1 \end{vmatrix}.$$

   The boxed variables (x, z) are leading variables, and the unboxed variables (y) are free variables. So, y can be any real number; let's say $y = t$. Then, we can solve for the leading variables: $x = -2t - 4$ and $z = 1$. Therefore, our solutions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t - 4 \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

   where t is any real number. (Note that you can easily check that these really are solutions by plugging back in to the original equation.)

2. *Use Gauss-Jordan elimination to solve the system*

$$\begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.$$

Solution. First, $\vec{x}$ must be a vector in $\mathbb{R}^5$ in order for the product $\begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix} \vec{x}$ to make sense. If we write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, then we can multiply $\begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix} \vec{x}$ out to rewrite the given system as

$$\left| \begin{array}{rcrcrcrcrcr} & & x_2 & + & 2x_3 & + & 2x_4 & - & 2x_5 & = & 1 \\ x_1 & & & + & 3x_3 & & & + & 4x_5 & = & 5 \\ -x_1 & + & 3x_2 & + & 3x_3 & & & - & 10x_5 & = & 4 \end{array} \right|.$$

The augmented matrix of this system is

$$\left[\begin{array}{rrrrr:r} 0 & 1 & 2 & 2 & -2 & 1 \\ 1 & 0 & 3 & 0 & 4 & 5 \\ -1 & 3 & 3 & 0 & -10 & 4 \end{array} \right].$$

To solve the system, we row-reduce the augmented matrix:

$$\begin{aligned}
\left[\begin{array}{rrrrr:r} 0 & 1 & 2 & 2 & -2 & 1 \\ 1 & 0 & 3 & 0 & 4 & 5 \\ -1 & 3 & 3 & 0 & -10 & 4 \end{array} \right] \begin{array}{l} \\ \text{swap with (I)} \\ \\ \end{array} &\to \left[\begin{array}{rrrrr:r} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ -1 & 3 & 3 & 0 & -10 & 4 \end{array} \right] \begin{array}{l} \\ \\ +\text{(I)} \end{array} \\
&\to \left[\begin{array}{rrrrr:r} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 3 & 6 & 0 & -6 & 9 \end{array} \right] \begin{array}{l} \\ \\ -3\text{(II)} \end{array} \\
&\to \left[\begin{array}{rrrrr:r} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & -6 & 0 & 6 \end{array} \right] \begin{array}{l} \\ \\ \div -6 \end{array} \\
&\to \left[\begin{array}{rrrrr:r} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right] \begin{array}{l} \\ -2\text{(III)} \\ \\ \end{array} \\
&\to \left[\begin{array}{rrrrr:r} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right]
\end{aligned}$$

This corresponds to the equations

$$\left| \begin{array}{lllll} \boxed{x_1} & & +\,3x_3 & & +\,4x_5 = 5 \\ & \boxed{x_2} & +\,2x_3 & & -\,2x_5 = 3 \\ & & & \boxed{x_4} & = -1 \end{array} \right|$$

The boxed variables are leading variables, and the other variables (x_3 and x_5) are free variables. The free variables can be any real numbers, so we can take $x_3 = s$ and $x_5 = t$ where s and t are any real numbers. Solving for each of the leading variables in terms of the free variables, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5-3s-4t \\ 3-2s+2t \\ s \\ -1 \\ t \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } s \text{ and } t \text{ are any real numbers}}$$

3. *If you perform Gauss-Jordan elimination on an inconsistent system, how will you recognize that the system is inconsistent?*

   Solution. When we use the method of elimination, we recognize that a system is inconsistent when we get to an equation that says $0 =$ something nonzero, like 6. In the augmented matrix, that would correspond to a row of the form $\left[\begin{array}{cccc:c} 0 & 0 & \cdots & 0 & \text{something nonzero} \end{array}\right]$. (If we continued row-reducing the matrix, we would divide the row by the nonzero scalar to get $\left[\begin{array}{cccc:c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$.)

4. *If $\vec{b} \in \mathbb{R}^2$ and A is a 2×2 matrix such that* $\operatorname{rref}(A)$ *has two leading 1s, what can you say about the number of solutions of the system $A\vec{x} = \vec{b}$?*

   Solution. To solve $A\vec{x} = \vec{b}$, we row-reduce the augmented matrix $\left[\begin{array}{c:c} A & \vec{b} \end{array}\right]$. We'll get something that looks like $\left[\begin{array}{c:c} \operatorname{rref}(A) & \vec{c} \end{array}\right]$. Since A is a 2×2 matrix and rref A has two leading ones, $\operatorname{rref}(A)$ must be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by **??**. So, row-reducing $\left[\begin{array}{c:c} A & \vec{b} \end{array}\right]$ gives something of the form

   $$\left[\begin{array}{cc:c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right].$$

   This has exactly one solution.

5. *True or false: if A is any matrix, the system $A\vec{x} = \vec{0}$ is consistent.*

   Solution. This is $\boxed{\text{true}}$ because $\vec{0}$ is always a solution of the system $A\vec{x} = \vec{0}$. (After all, $A\vec{0} = \vec{0}$.)

6. *True or false: If A is a 4×3 matrix and the linear system $A\vec{x} = \vec{b}$ has exactly one solution, then the linear system $A\vec{x} = \vec{c}$ has exactly one solution for all vectors $\vec{c}$ in $\mathbb{R}^4$.*

   Solution. The statement is $\boxed{\text{false}}$, and a counterexample is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   Note: If you mistakenly thought the statement was true, work through your reasoning with the counterexample above and see where it breaks down. This will help you understand your misconceptions better, and it's why we give you counterexamples in the solutions!

7. *If A is an $n \times m$ matrix such that $A\vec{x} = \vec{b}$ is consistent for* every $\vec{b} \in \mathbb{R}^n$*, what can you say about* rref A*?*

   Solution. The fact that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^n$ means that, no matter what $\vec{b}$ is, when you row-reduce the augmented matrix $\left[\begin{array}{c:c} A & \vec{b} \end{array}\right]$, you don't get any rows of the form $\left[\begin{array}{ccc:c} 0 & \cdots & 0 & 1 \end{array}\right]$.

That means that rref A $\boxed{\text{cannot have a row of all zeros}}$. (If it did, there would be at least one vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ was inconsistent.)

8. *Find all values of a for which the system* $\begin{bmatrix} 2 & a \\ 3 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is consistent.*

Solution. Let's use Gauss-Jordan to start solving the system. Normally, you'd probably have your first step be dividing the first row by 2, but let's instead start by swapping the two rows; this pushes the variable a further down the system, which is useful because it's easier to to work with definite numbers (as opposed to variables) when we're row reducing.[1]

$$\left[\begin{array}{cc:c} 2 & a & 1 \\ 3 & 6 & 0 \end{array}\right] \text{ swap with (I)} \rightarrow \left[\begin{array}{cc:c} 3 & 6 & 0 \\ 2 & a & 1 \end{array}\right] \div 3$$
$$\rightarrow \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & a & 1 \end{array}\right] -2(\text{I})$$
$$\rightarrow \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & a-4 & 1 \end{array}\right]$$

At this point, what happens next depends on the value of a. If $a = 4$, then the second row is $\left[\begin{array}{cc:c} 0 & 0 & 1 \end{array}\right]$, which means the system is inconsistent. If $a \neq 4$, we can divide the second row by $(a-4)$ to get a leading 1, and then the system must be consistent. So, we conclude that the system is consistent for $\boxed{\text{all } a \neq 4}$.

- A unique solution to $A\vec{x} = \vec{b}$ if and only if rref(A) has a leading 1 in every column.
- No solution if and only if rref$(A|\vec{b}\,)$ has a leading 1 in the last column.
- In all other cases we have infinitely many solutions.

A system which has 1 or ∞ solution is called **consistent**. Otherwise it is called **inconsistent**

[1] The issue that can come up with variables is being careful not to divide by zero. For instance, if you want to divide a row by a, that only works if $a \neq 0$, so you then need to split your work into two cases: what happens when $a = 0$ and what happens when $a \neq 0$.