Tutorial Sheet : **1**
Course Code : **BMAT101L**
Course Title : **Calculus**

Vellore Institute of Technology
(Deemed to be University under section 3 of UGC Act, 1956)
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Module 1: Single Variable Calculus

Easy

1. Find $\dfrac{dy}{dx}$, if $y = \sqrt{x + \arctan(x)}$.

   Solution: Hint: Use chain rule

2. Find the maximum and minimum of $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 2]$.

   Solution: Hint: Maximum value=1/2 and Minimum value=$-1/2$

3. Find the area between the graphs of $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$ for $0 \le x \le \dfrac{\pi}{2}$.

   Solution: Area=$1 - \dfrac{\pi}{4}$

4. $f(x) = \sqrt{x}$ for $4 \le x \le 16$. Find a point c whose existence is guaranteed by the mean value theorem.

   Solution: c=9

5. Find a number k such that the line $y = 6x + 4$ is tangent to the parabola $y = x^2 + k$.

   Solution: k=13

Medium

6. A farmer has 8 km of fencing wire, and wishes to fence a rectangular piece of land. One boundary of the land is the bank of a straight river. What are the dimensions of the rectangle so that the area is maximised?

   Solution: 4km and 2km

7. The area bounded between the curves $y = x^2$ and $y = ax$ is 36 square unit. Determine the value of a.

Solution: $a = 6$

8. Obtain the absolute and local extrema of the function $f(x) = (x^3 + 2x^2 - 5x)e^x$. Determine the intervals where function is increasing/decreasing.

Solution: Absolute max: Does not exist; Absolute min: $x = 1$; Local max: $x = -1$; Local min: $x = -5, x = 1$; Decreasing: $(-\infty, -5) \bigcup (-1, 1)$; Increasing: $(-5, -1) \bigcup (1, \infty)$

9. Find the volume of the solid generated by rotating the region bounded by the curves of $y = \sqrt{x}, \quad y = 2 - x$ and $y = 0$ about the $x-$axis.

Solution: $\frac{5\pi}{6}$

10. $f(x)$ is a continuous and differential function on the interval $[1,9]$ and given that $f(9) = 2$ and $f'(x) \geq 4$. What is the largest possible value of $f(1)$?

Solution: 30

Hard

11. For what values of a, b, and m does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the mean value theorem on the interval $[0, 2]$?

Solution: $a = 3, b = 4$, and $m = 1$.

12. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum or a local minimum? Give reasons for your answer.

Solution: $a = 6$ and $b = -10$. $f(x) = 1$ is a local maximum at $x = 3$.

13. Consider the function $f(x) = \frac{x^2 - 4}{x^2 + 1}$.

    1. Determine the critical points of the function f.

2. Identify the intervals where the function is increasing or decreasing.
3. Find the relative and absolute extrema, if they exist.
4. Analyze the concavity of the function and locate any inflection points.
5. Sketch the graph of f, clearly indicating all key features.

Solution: (a) The only critical point of the function f is $x = 0$.
(b) The function f is decreases on $(-\infty, 0)$ and increases on $(0, \infty)$.
(c) The function f has a local minimum at $(0, -4)$, which is also an absolute minimum point.
(d) The function f is concave up on the intervals $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$, and concave down on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Thus the function has inflection points at $x = \pm\frac{1}{\sqrt{3}}$.

14. Find the area between $y = |x - 1|$ and $y = x$ from $x = -1$ to $x = 2$.

Solution: Total area is 3.

15. Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2$, $x = 4$, about the line $y = 2$.

Solution: Volume is $\frac{14\pi}{3}$.