BMAT101L - CALCULUS

Problem Sheet - Module 3

EASY

1. Expand $f(x, y) = e^{xy}$ in Taylor series at $(1, 1)$ upto second degree.

Solution. $f(x, y) = e + \frac{1}{1!}[(x-1)e + (y-1)e] + \frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)(2e) + (y-1)^2(e)] + ...$

2. Construct the Taylor series through the 2nd order for $f(x, y) = x^2y + y^2$ at $(x, y) = (1, 3)$.

Solution. $f(x, y) = 12 + 6(x-1) + 7(y-3) + \frac{1}{2}(2(x-1)^2 + 4(x-1)(y-3) + 2(y-3)^2) +$

3. Find the area of the rectangle with the sides measuring x and y, if the perimeter is 14.

Solution. $x = y = \frac{7}{2}$, $\frac{49}{4}$

4. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Solution. $(x, y, z) = (3, 3, 3)$**, minimum value** $= 27$**.**

5.For a rectangle whose perimeter is 20 m, find the dimensions that will maximize the area.

Solution. The maximum area occurs for a rectangle whose width and height both are 5 m.

MODERATE

6. Expand $e^x log(1+y)$ in powers of x and y upto terms of third degree.

Solution. $f(x, y) = y + \frac{2xy - y^2}{2!} + \frac{3x^2y - 3xy^2 + 2y^3}{3!} + ...$

7. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ upto the third degree term.

Solution. $-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$

8. Find the points on the circle $x^2 + y^2 = 80$ which are closest to and farthest from the point $(1, 2)$.

Solution. $(4,8)$ **is the point on the circle closest to** $(1,2)$ **and** $(-4,-8)$ **is the farthest from** $(1,2)$

Hints: The distance d **from any point** (x,y) **to the point** $(1,2)$ **is**
d=$\sqrt{(x-1)^2+(y-2)^2}$
and minimizing the distance is equivalent to minimizing the square of the distance. Thus the problem can be stated as:
Maximize (and minimize) : $f(x,y)=(x-1)^2+(y-2)^2$
given : $g(x,y)=x^2+y^2=80$**.**

9. Find the dimensions of the box with largest volume if the total surface area is 64 cm^2.

Solution. $x=y=z=3.266$**.**
Hints: We want to find the largest volume and so the function that we want to optimize is given by,

$f(x,y,x)=xyz$**.**

Next, we know that the surface area of the box must be a constant 64. So this is the constraint. The surface area of a box is simply the sum of the areas of each of the sides so the constraint is given by,
$2xy+2yz+2xz=64$**.**

10. Expand $sin(xy)$ **in power of** $(x-1)$**and** $(y-\pi/2)$ **upto second degree terms by using Taylor's series.**

Solution. $1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)(y-\frac{\pi}{2})-\frac{1}{2}(y-\frac{\pi}{2})^2$.

HARD

11. A rectangular box open at the top is to have a volume of 32**cc. Find the dimensions of the box that requires the least material for its construction.**

Solution. $x=4, y=4, z=2$

12. Find the point on the plane $ax+by+cz=p$ **at which** $f=x^2+y^2+z^2$ **has a stationary value and find the stationary value of** f **using Lagrange's method of multipliers.**

Ans. $x=\frac{ap}{a^2+b^2+c^2}$**,** $y=\frac{bp}{a^2+b^2+c^2}$**,** $z=\frac{cp}{a^2+b^2+c^2}$**; stationary value of** $f=\frac{p^2}{a^2+b^2+c^2}$

13. Find the extreme distance of a variable point on the parabola$y^2=4ax$ **from the fixed point** $B(0,b)$**.**

$\boxed{\textbf{Ans. The minimum value is } 2\sqrt{a(b-a)}.}$

Hints: Let $R(x, y)$ be any pointon the parabola. If we assume the length from of the linr from B as r, then $\mathbf{r}^2 = (x-b)^2 + y^2$. The Lagrangian function is $\mathbf{L} = (\mathbf{x}\ \mathbf{-b})^2 + y^2 + \lambda(y^2 - 4ax)$

14. If $f(x, y) = (2x - 3y + 1)(x + y) - 8x + 2y - 9$, find its stationary points. Show that the both the stationary points are saddle points.

15. If $f(x, y, z) = (ax + by + cz)e^{-(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)}$, then the maximum and minimum values of f are $\pm\sqrt{\frac{\frac{1}{2}\{(\frac{a}{\alpha})^2+(\frac{b}{\beta})^2+(\frac{c}{\gamma})^2\}}{e}}$

Hints: Let $f(x, y, z) = \sum ax.e^{\sum \alpha^2x^2} = uv$ where $u = \sum ax$ and $v = \sum \alpha^2x^2$. Then $\mathbf{f}_x = (a - 2\alpha^2xu)v$ $\mathbf{f}_y = (b - 2\beta^2yu)v$ $\mathbf{f}_z = (a - 2\gamma^2zu)v$