Tutorial Sheet : 1
Course Code : BMAT101L
Course Title : CALCULUS

Vellore Institute of Technology
(Deemed to be University under section 3 of UGC Act, 1956)
CHENNAI

Module 5: Special Functions

EASY

1. Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.

2. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$.

Solution: $\dfrac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2}$.

3. Evaluate $\int_0^\infty \frac{x^c}{c^x} dx$ using Gamma function

Solution: $\frac{\Gamma c+1}{log c(c+1)}$

4. Prove that $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n\, n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. Hence evaluate $\int_0^1 x(\log x)^3\, dx$.

5. Find $\iint_D x^{l-1} y^{m-1} dx dy$, where D is the domain $x \geq 0, y \geq 0$ and $x + y \leq h$

Solution: $\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+n)} h^{l+m}$

MODERATE

6. Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$. (Hint:Put $x = \frac{1}{1+y}$)

7. Show that $\int_0^{\pi/2} \sqrt{(\sin\theta)} d\theta \times \int_0^{\pi/2} \dfrac{d\theta}{\sqrt{(\sin\theta)}} = \pi$.

8. Find $\int_0^\infty xe^{-x^8} dx \int_0^\infty x^2 e^{-x^4} dx$

Solution: $\frac{\pi}{16\sqrt{2}}$

9. Evaluate $\iiint_V x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx dy dz$, where V is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the coordinate planes.

Solution: $\frac{1}{8}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+1)}$

10. Show that the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}=1, \quad \alpha>0, \beta>0$ is given by

$$\frac{ab}{\alpha+\beta}\cdot\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}$$

HARD

11. Show that $\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx=(a-b)^{m+n-1}\beta(m,n)$. (Hint:Put $x=\frac{y-b}{a-b}$)

12. Prove that $\int\limits_0^1 x^m(\log x)^n dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m>-1$. Hence evaluate $\int\limits_0^1 x(\log x)^3 dx$.

Solution: $-\frac{3}{8}$.

13. Find n^{th} derivative of x^k $(\frac{d^n(x^k)}{dx^n})$. Express the result in factorial and generalise using Gamma function. Hence deduce $\frac{d^{5/2}}{dx^{5/2}}x^4$ using Gamma function.(Hard)

Solution: $\frac{32x^{3/2}}{\sqrt{\pi}}$.

14. Hard. Find the mass of an octant the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being $\rho=kxyz$.

Solution: $\frac{ka^2b^2c^2}{48}$

15. Show that $\int\int x^{m-1}y^{n-1}dx\,dy$ over the positive quadrant of the ellipse $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ is

$$\frac{a^m b^n}{2n}\cdot\beta\left(\frac{m}{2},\frac{n}{2}+1\right)$$