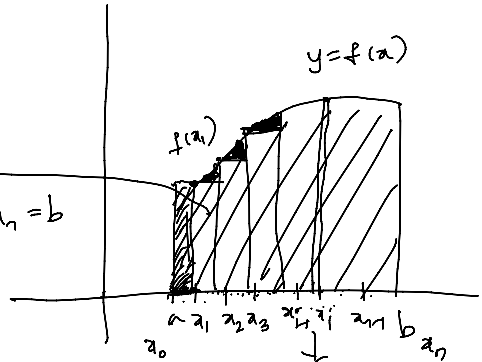
Multiple integrals

09 November 2021 09:13 AM

$$\int_a^b f(x)\,dx.$$



$$x_i - x_{i-1} = \delta x_i$$

$[a, b] \to \quad a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

Consider the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \cdots + f(x_n)\,\delta x_n$$

$$\sum_{i=1}^{n} f(x_i)\,\delta x_i$$

Now as $n \to \infty$ such that $\delta x_i \to 0$

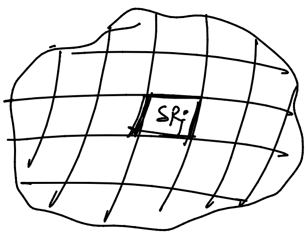
If the above sum tends to a finite value (limit) as $n \to \infty$ such that $\delta x_i \to 0$ then this limit is called as definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i = \int_a^b f(x)\,dx$$

→ Double integral:

Consider a function $f(x, y)$ defined in a region R.

R.



Let the region R be divided into n-subregions of area $\delta R_1, \delta R_2, \delta R_3 \ldots \delta R_n$ respectively

Let (x_i, y_i) be any point in δR_i

Consider

$$f(x_1, y_1)\,\delta R_1 + f(x_2, y_2)\,\delta R_2 + \cdots + f(x_n, y_n)\,\delta R_n$$

If the above sum tends to a finite limit as $n \to \infty$ such that $\delta R_i \to 0$, then this limit is called as double integral of $f(x, y)$ over the region R & it is denoted by $\iint_R f(x, y)\,dR$ (or) $\iint_R f(x, y)\,dx\,dy$.

→ $\iint f(x, y)\,dx\,dy$ represents volume if $f(x, y) \neq 1$

→ $\iint f(x, y)\,dx\,dy$ represent area if $f(x, y) = 1$

Ex:

$\iint 2\,dx\,dy$ — area.

$\iint x\,dx\,dy$ — volume.

$$\iint x\,dx\,dy \text{ — volume.}$$

→ **Properties:**

Let f & g be two functions defined in a region R.

1) $$\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$$

2) $$\iint_R k f\,dx\,dy = k\iint_R f\,dx\,dy, \quad k \text{ is constant}$$

3) If R is divided into n- sub regions $R_1, R_2, \ldots R_n$

then $$\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy + \cdots + \iint_{R_n} f\,dx\,dy.$$

→ **Evaluation of double integrals:**

1) $$\iint_R f(x,y)\,dx\,dy \qquad R: a \le x \le b, \quad y_1(x) \le y \le y_2(x)$$

$$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right]dx$$

2) If $R: c \le y \le d, \quad x_1(y) \le x \le x_2(y)$

$$\int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy.$$

3) If $R: a \le x \le b, \quad c \le y \le d.$

then $$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx \quad (\text{or}) \quad \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

→ Evaluation of double integrals:-

1) Consider $\iint_R f(x,y)\,dx\,dy$, $R: a \le x \le b, \; y_1(x) \le y \le y_2(x)$

$$\int^{b}\int^{y_2(x)} f(x,y)\,dy\,dx = \int^{b}\left[\int^{y_2(x)} f(x,y)\,dy\right]dx$$

$$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx = \int_{x=a}\left[\int_{y=y_1(x)} f(x,y)\,dy\right]dx$$

integrate w.r.to y first by keeping x as constant & then later we integrate w.r.to x with in the limits a & b.

2). $\iint_R f(x,y)\,dx\,dy$ R: $c \le y \le d$, $x_1(y) \le x \le x_2(y)$

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy. = \int_{y=c}^{d}\left[\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy$$

first we integrate w.r.to 'x' by keeping x as constant & then we integrate w.r.to y from c to d.

3) $\iint_R f(x,y)\,dx\,dy$, R: $a \le x \le b$, $c \le y \le d$.

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

(or)

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy.$$

Examples:

1) $\int_0^2\int_0^3 xy\,dx\,dy$

Sol: $\int_{y=0}^{2}\int_{x=0}^{3} xy\,dx\,dy = \int_{y=0}^{2}\left[\int_{x=0}^{3} xy\,dx\right]dy$

$$= \int_{y=0}^{2} y\left.\frac{x^2}{2}\right|_{x=0}^{3} dy = \frac{1}{2}\int_{y=0} y\left[9-0\right]dy = \frac{9}{2}\int_{y=0}^{2} y\,dy$$

$$\frac{9}{2}\left[\frac{y^2}{2}\right]_0^2 = \frac{9}{4}\left[4-0\right] = 9$$

2) $\int_0^2 \int_0^x y\,dy\,dx$

sol $\int_{x=0}^{2} \int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2} \frac{y^2}{2}\Big|_{y=0}^{x} dx = \frac{1}{2}\int_{x=0}^{2} x^2\,dx$

$= \frac{1}{2}\cdot \frac{x^3}{3}\Big|_0^2 = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$

3) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx.$ H.W

4) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy.$

sol: $\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\right] dy$

let $a^2-y^2 = p^2$

$= \int_{y=0}^{a}\left[\int_{x=0}^{p} \sqrt{p^2-x^2}\,dx\right] dy$

$\boxed{\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}}$

$= \int_{y=0}^{a} \frac{x}{2}\sqrt{p^2-x^2} + \frac{p^2}{2}\sin^{-1}\frac{x}{p}\Big|_{x=0}^{p} dy = \int_{y=0}^{a}\left[\frac{p}{2}(0) + \frac{p^2}{2}\cdot\frac{\pi}{2} - 0 - \frac{p^2}{2}(0)\right] dy$

$= \int_{y=0}^{a} \frac{p^2\pi}{4}\,dy = \frac{\pi}{4}\int_{y=0}^{a}(a^2-y^2)\,dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$

$\frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{4}\left[\frac{2a^3}{3}\right] = \frac{\pi a^3}{6}.$

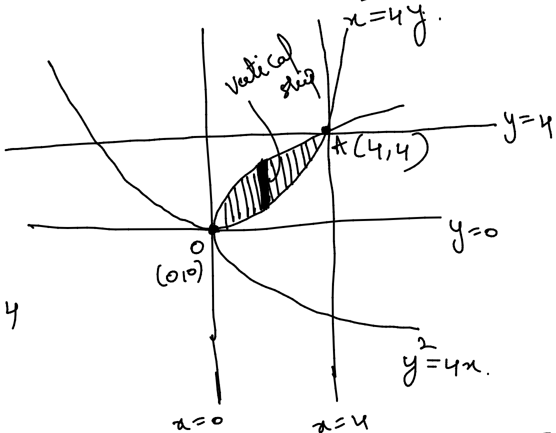
→ $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ H.W

$\rightarrow \displaystyle\int_0^2 \int_0^x e^{x+y}\, dy\, dx.$ H.W

$\rightarrow \displaystyle\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dy\, dx.$ H.W

$\rightarrow$ Evaluate $\displaystyle\iint_R y\, dx\, dy$ where R is bounded by the curves

$x^2 = 4y$ & $y^2 = 4x$.



Sol: sub $y = \dfrac{x^2}{4}$ in $y^2 = 4x$.

$\left(\dfrac{x^2}{4}\right)^2 = 4x. \Rightarrow x^4 = 64x.$

$x(x^3 - 64) = 0 \Rightarrow x = 0, 4$

$x = 0,\ y = 0 \Rightarrow (0,0)$

$x = 4,\ y = 4 \Rightarrow (4,4)$

$y = \pm 2\sqrt{x}$

$y = +2\sqrt{x}$.

plot the given curves & identify the region b/w the curves

Draw st lines parallel to x-axis & y-axis at the points of intersection.

Fix any one of the variable as constant & the other as varying.

Let us consider x as fixed & y as varying.

Now consider vertical or horizontal strips inside the region depending the the variable that is varying & slide those ~~strips through~~ out the region to cover the entire region of integration.

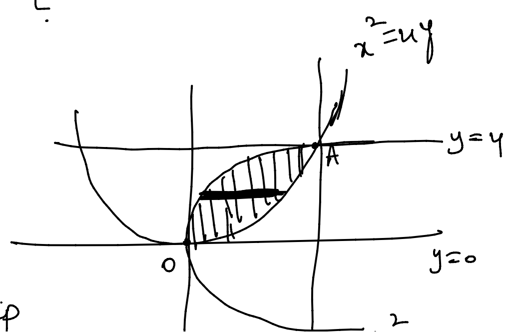
Consider a vertical strip $\left(y = \dfrac{x^2}{4}\right.$ to $\left.y = 2\sqrt{x}.\right)$ is slided from $x=0$ to $x=4$ to cover the total region

$y = \dfrac{x^2}{4}$: lower edge of vertical strip

$y = 2\sqrt{x}$: upper edge of strip

$$\iint_R y\, dx\, dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\, dy\, dx. = ?$$

(or)



If you fix y & vary 'x'

we have to consider horizontal strip

if you fix y & vary x
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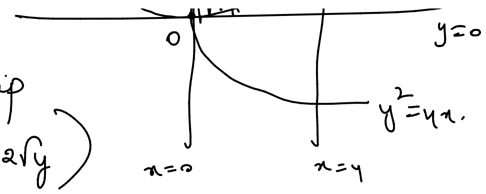
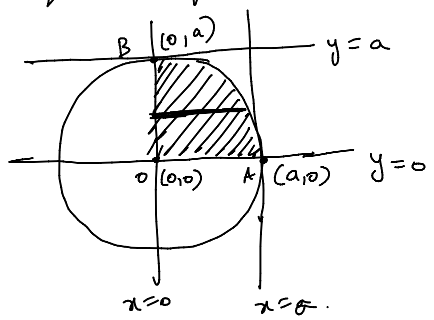


The horizontal strip ($x = \frac{y^2}{4}$ to $x = 2\sqrt{y}$)
left edge right edge
is slided from $y=0$ to $y=4$ to cover the total region

$$\iint_R y\,dx\,dy = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{2\sqrt{y}} y\,dx\,dy = ?$$

Ans $\frac{48}{5}$

→ Evaluate $\iint_R xy\,dx\,dy$ over the positive quadrant of $x^2+y^2=a^2$



Sol: Let us fix y and vary 'x'
So, a horizontal strip is considered
The horizontal strip ($x=0$ to $x=\sqrt{a^2-y^2}$)
slides from $y=0$ to $y=a$.

$x = \pm\sqrt{a^2-y^2}$
$x = \sqrt{a^2-y^2}$.

$$\iint_R xy\,dx\,dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\,dy = ?$$

Ans $\frac{a^4}{8}$

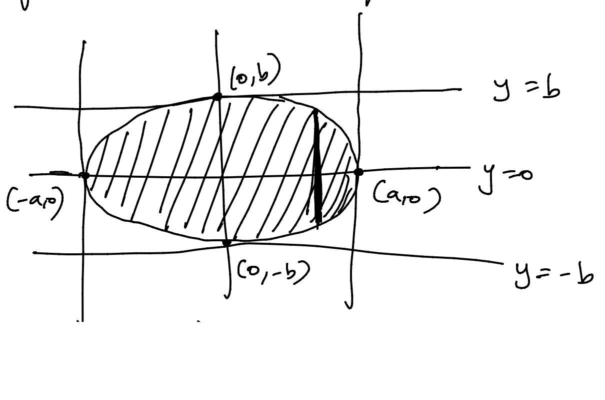
→ $\iint xy(x+y)\,dx\,dy$ over the region bounded by $y=x$ & $y=x^2$.
H-W

Ans : $\frac{3}{56}$

→ $\iint_R xy\,dx\,dy$, R: x-axis , $x=2a$ & $x^2=4ay$. H-W

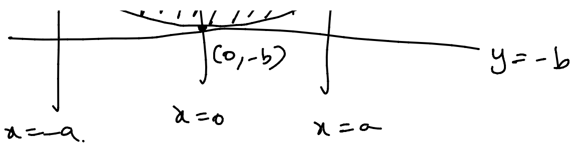
Ans : $\frac{a^4}{3}$

→ $\iint_R (x^2+y^2)\,dx\,dy$ over the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

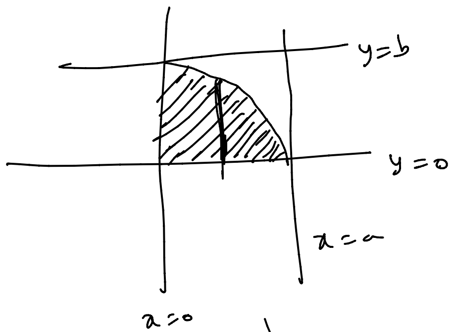


$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y = \pm b\sqrt{1-x^2}$



$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$



vertical strip $\rightarrow$ $\left.\begin{array}{l} y = -b\sqrt{1-\frac{x^2}{a^2}} \\ y = b\sqrt{1-\frac{x^2}{a^2}} \end{array}\right\}$

$x=-a$ to $x=a$.

$\left.\begin{array}{l} y=0 \text{ to } y = b\sqrt{1-\frac{x^2}{a^2}} \\ x=0 \text{ to } x=a. \end{array}\right\}$

$$4\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} (x^2+y^2)\,dy\,dx.$$

$\rightarrow$ Change of order of integration

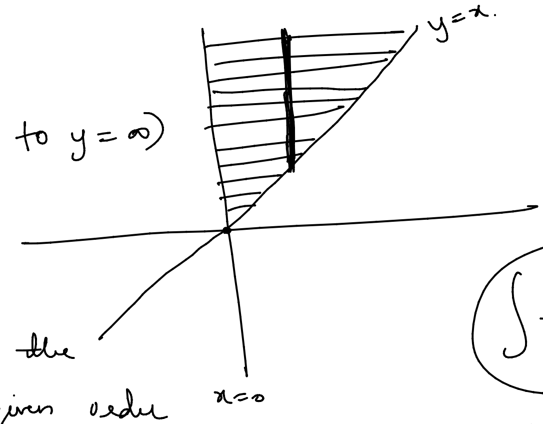
$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx.$$

$\rightarrow$ Evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx.$ by changing the order of integration

Sol:
$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dy\,dx.$$

$x=0 \quad x=\infty$

$y=x \quad y=\infty$



e^{-y} $\quad$ $\frac{1}{y}$

The vertical strip ($y=x$ to $y=\infty$) slides from $x=0$ to $x=\infty$

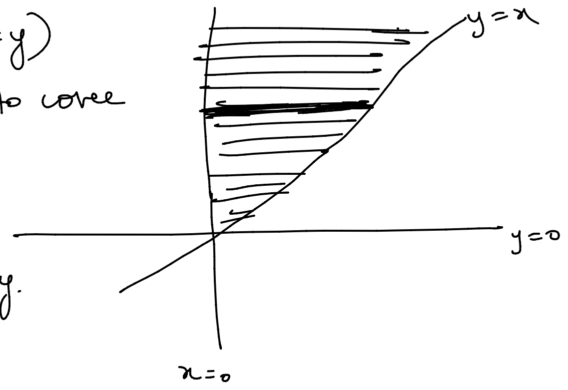
$$\int \frac{e^{-y}}{y}\,dy$$

As we can't evaluate the given integral in the given order i.e we can't integrate $\frac{e^{-y}}{y}$ w.r.to y we change the order of integration

Change the order of integration by taking a horizontal strip order of integration

Change the order of integration by taking a horizontal strip

The horizontal strip ($x=0$ to $x=y$) slides from $y=0$ to $y=\infty$ to cover the entire region.



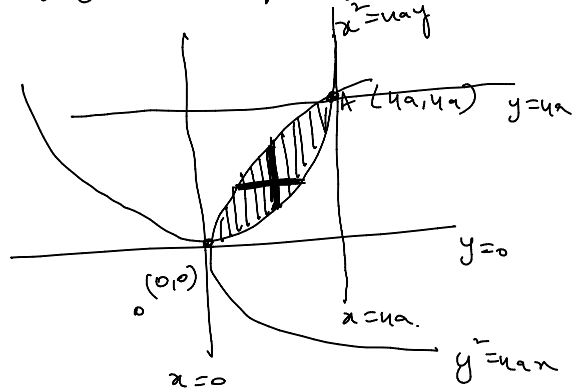
$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy.$$

$$= \int_{y=0}^{\infty}\left[\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\right] dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}\,(x)_{x=0}^{y}\, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}}\, \cancel{y}\, dy = \int_{y=0}^{\infty} e^{-y}\, dy. = \left(\frac{e^{-y}}{-1}\right)_0^{\infty}$$

$$-(0-1) = 1$$

→ Evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$ by changing order of integration.



Sol

$x=0 \quad x=4a$

$y = \frac{x^2}{4a} \quad y = 2\sqrt{ax}$

$\downarrow \qquad \downarrow$

$x^2 = 4ay \quad y^2 = 4ax.$

$y = \frac{x^2}{4a} \rightarrow y^2 = 4ax$

$\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow x^4 - 64a^3x = 0$

$x = 0, 4a$

vertical strip $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$.

$x=0$ to $x=4a$.

$x=0 \Rightarrow y=0 \Rightarrow (0,0)$

$x=4a \Rightarrow y=4a. \quad (4a,4a)$

According to the given problem we have y varying i.e a vertical strip

So to change order of integration we take a horizontal strip

The horizontal strip $x = y^2$ to $x = 2\sqrt{ay}$ slides from $y=0$ to $y=4a$.

The horizontal strip $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ slides from $y=0$ to $y=4a$.

$$\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy. = ? \qquad \underline{\underline{\text{Ans}}} \ \frac{16a^2}{3}$$

3) Evaluate the following by change of order integrations.

1) $\displaystyle\int_0^a \int_{x/a}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dy\,dx.$

$\underline{\underline{\text{sol}}}$: $x=0$, $x=a$, $y=\frac{x}{a}$, $y=\sqrt{\frac{x}{a}}$

$\Rightarrow x = ay \quad y^2 = \frac{x}{a} \longrightarrow y^2 = 4ax$

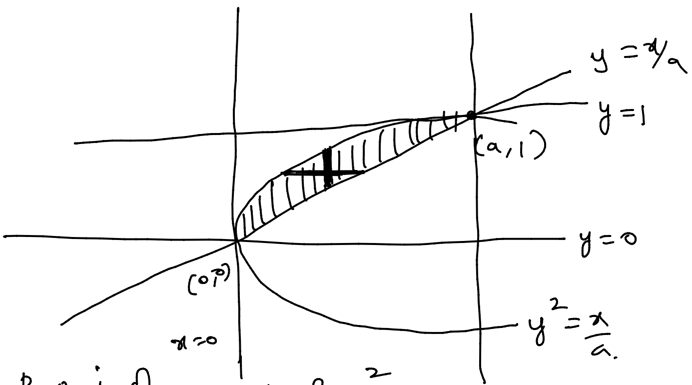
$y = \frac{x}{a} \quad y^2 = \frac{x}{a}.$

$\frac{x^2}{a^2} = \frac{x}{a}.$

$\Rightarrow x^2 - ax = 0$

$x(x-a) = 0$

$x = 0, a.$



$x=0 \Rightarrow y=0 \quad (0,0)$
$x=a \Rightarrow y=1 \quad (a,1)$ $\Big\}$ p.o.i of $y=\frac{x}{a}$ & $y^2=\frac{x}{a}$

According to the given limits y is varying, hence a vertical strip is given.

Now to change the order we take a horizontal strip

The horizontal strip $x=ay^2$ to $x=ay$ slides from $y=0$ to $y=1$

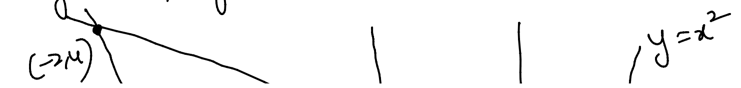
$$\int_{x=0}^{a}\int_{y=\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dy\,dx = \int_{y=0}^{1}\int_{x=ay^2}^{ay} (x^2+y^2)\,dy\,dx. = ?$$

$\underline{\underline{\text{H.W}}}$

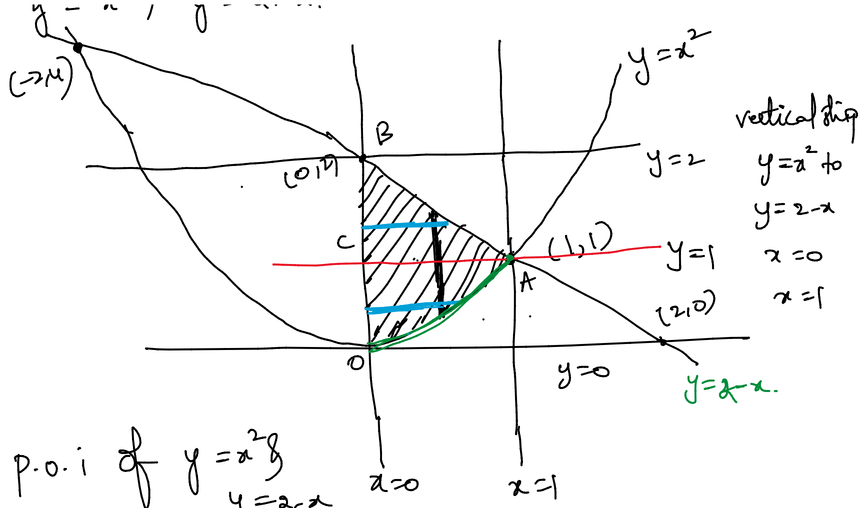
2) $\displaystyle\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy.$

3) Evaluate $\displaystyle\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx.$

$\underline{\underline{\text{sol}}}$: $x=0$, $x=1$, $y=x^2$, $y=2-x.$



Sol: $x=0, x=1, \ y = \ldots, \ y = \ldots$



$y = x^2, \ y = 2-x$

$x^2 = 2-x.$

$x^2+x-2=0$

$x = 1, -2.$

$x=1 \Rightarrow y=1$

$x=-2 \Rightarrow y=4$

$(1,1)$ & $(-2,4)$ are p.o.i of $y=x^2$ & $y=2-x$

According to the given limits we have y varying so a vertical strip is given.

We change the order by taking a horizontal strip in it

When the horizontal strip slided from $y=0$ to $y=2$, the left edge remains on a single curve ($x=0$) but the right edge is changing from ($y=x^2$ to $y=2-x$) . at the point $(1,1)$

We divide the whole region in two sub regions at $(1,1)$

The whole region OABCO is divided in to two sub regions OACO & ABCA.

In the subregion OACO, the horizontal strip $x=0$ to $x=\sqrt{y}$ slides from $y=0$ to $y=1$

In the subregion ABCA, the horizontal strip $x=0$ to $x=2-y$ slides from $y=1$ to $y=2$

$$\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\,dx = \iint_{OABCO} xy\,dy\,dx = \iint_{OACO} xy\,dy\,dx + \iint_{ABCA} xy\,dy\,dx.$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} y\,\frac{x^2}{2}\Big|_{x=0}^{\sqrt{y}} dy + \int_{y=1}^{2} y\,\frac{x^2}{2}\Big|_{x=0}^{2-y} dy$$

$$\frac{1}{2}\int_{y=0}^{1} y^2\,dy + \frac{1}{2}\int_{y=1}^{2} y(2-y)^2\,dy$$

$$\frac{1}{2}\int_{y=0} y\,dy + \frac{1}{2}\int_{y=1} y(2-y)\,dy$$

$$= \frac{1}{2}\left.\frac{y^3}{3}\right|_0^1 + \frac{1}{2}\int_1^2 (y^3 - 4y^2 + 4y)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{y^4}{4} - \frac{4y^3}{3} + 2y^2\right)_1^2$$

$$\frac{1}{6} + \frac{1}{2}\left(\frac{16}{4} - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2\right)$$

$$= ? \qquad \text{Ans } \frac{3}{8}$$

H-W

$$\rightarrow \int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx. \qquad \text{Ans} = \frac{47a^5}{120}$$

→ **Evaluation of double integrals in polar coordinates:**

$$\iint_R f(r,\theta)\,dr\,d\theta \rightarrow \text{double integral in polar coordinates.}$$

1) $R: \quad \theta_1 \le \theta \le \theta_2, \quad r_1 \le r \le r_2.$

2) $R: \quad \theta_1 \le \theta \le \theta_2, \quad r_1(\theta) \le r \le r_2(\theta)$

→ Evaluate the following

1) $$\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$$

Sol:
$$\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi} \left.\frac{r^2}{2}\right|_{r=0}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi - 0\right] = \frac{\pi a^2}{4}.$$

→ 2) $$\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr.$$

Sol
$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r\,d\theta\,dr = \int_{r=0}^{\infty} e^{-r^2} r\,(\theta)_{\theta=0}^{\pi/2}\,dr$$

∞

$$= \frac{\pi}{2}\int_{r=0} e^{-r^2}\, r\,dr.$$

Let $r^2 = t$

$r\,dr = \frac{dt}{2}$

$r = 0 \Rightarrow t = 0$

$r \to \infty \Rightarrow t \to \infty$

$$= \frac{\pi}{2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}$$

$$= \frac{\pi}{4}\int_0^{\infty} e^{-t}\,dt = \frac{\pi}{4}\left(\frac{e^{-t}}{-1}\right)_0^{\infty}$$

$$= -\frac{\pi}{4}(0-1) = \frac{\pi}{4}$$

H.W

→ 3) $$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta.$$

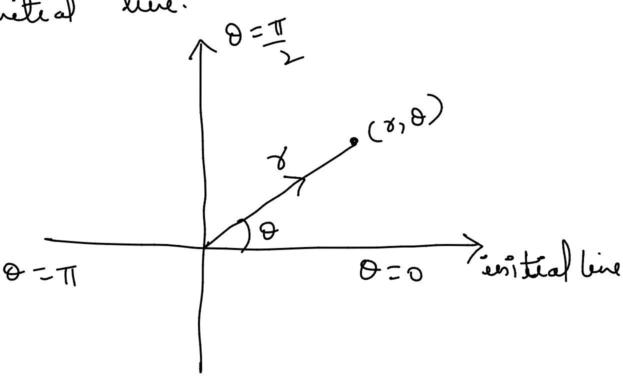
4) $$\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}.$$

→ 5) Evaluate $\iint_R r^2\sin\theta\,dr\,d\theta$ where R is the semicircle $r = 2a\cos\theta$ above the initial line.

Sol $r = 2a\cos\theta$.

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$.



$r = 2a\cos\theta$

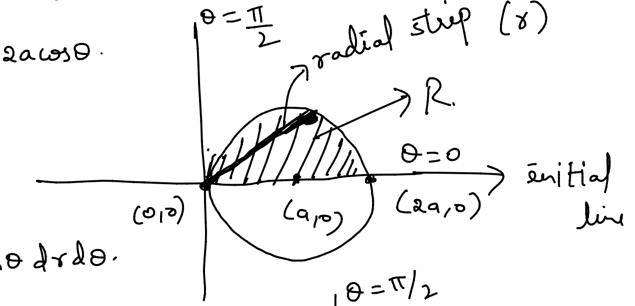
$r = 2a\frac{x}{r}$

$\Rightarrow r^2 = 2ax.$

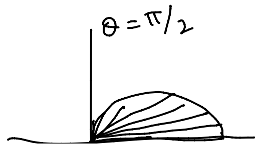
$\Rightarrow x^2+y^2 = 2ax. \Rightarrow x^2+y^2-2ax = 0$ → a circle with centre $(a,0)$ & radius 'a' & it will pass through origin

The radial strip $r = 0$ to $r = 2a\cos\theta$.

slides $\theta = 0$ to $\theta = \frac{\pi}{2}$.

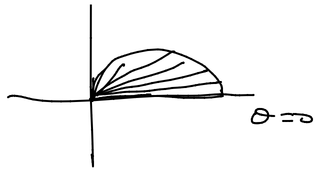


$$\iint_R r^2\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta.$$



$\pi/2$ $2a\cos\theta$

R $\theta=0$ $r=0$



$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_{r=0}^{2a\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin\theta\left(8a^3\cos^3\theta\right)d\theta = \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta\cos^3\theta\, d\theta.$$

$$\boxed{\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta}$$

Let $\cos\theta = t$

$\sin\theta\, d\theta = -dt.$

$\theta = 0 \Rightarrow t = 1$

$\theta = \frac{\pi}{2} \Rightarrow t = 0$

$$= \frac{8a^3}{3}\int_{t=1}^{0} t^3(-dt) = \frac{8a^3}{3}\int_{t=0}^{1} t^3 dt = \frac{8a^3}{3}\left.\frac{t^4}{4}\right|_0^1$$

$$= \frac{2a^3}{3}.$$

→ Evaluate $\iint r^3 dr d\theta$ over the area included b/w the circles

$r = 2\sin\theta$, $r = 4\sin\theta$.

Sol: $r = 2\sin\theta$. $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2=r^2$.

$r = 2\frac{y}{r}$

$\Rightarrow r^2 = 2y$

$\Rightarrow x^2+y^2 = 2y$

$r = 4\sin\theta$

$\Rightarrow r^2 = 4y$

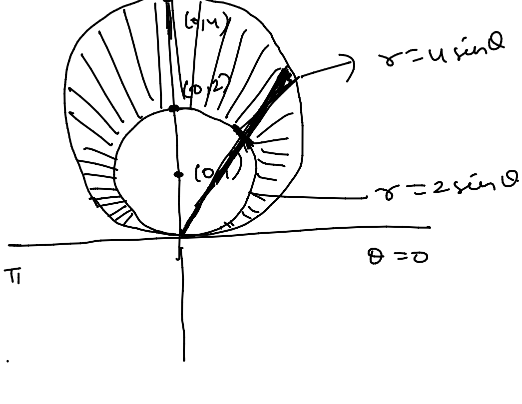
$x^2+y^2 = 4y$

$x^2+y^2-2y=0 \rightarrow$ a circle with centre $(0,1)$ & radius '1' & it pass through origin

$x^2+y^2-4y=0 \rightarrow$ " " " $(0,2)$ & radius '2' "

The radial strip $r=2\sin\theta$ to $r=4\sin\theta$

slides from $\theta=0$ to $\theta=\pi$



$$\iint_R r^3 dr d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3 dr\, d\theta.$$

π $4\sin\theta$ π

R

$$= \int_{\theta=0}^{\pi} \left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{4}\int_0^{\pi} \left(256\sin^4\theta - 16\sin^4\theta\right) d\theta$$

$\theta = 0 \quad r = 2\sin\theta$

$$= 60\int_0^{\pi} \sin^4\theta \, d\theta.$$

$$\int_0^{\pi/2} \sin^m\theta \, d\theta = \frac{(m-1)(m-3)\cdots 1}{m(m-2)(m-4)\cdots 2}\,\frac{\pi}{2}$$

$$= 60\times 2\int_0^{\pi/2} \sin^4\theta \, d\theta.$$

$$60\times 2\cdot\frac{(4-1)(4-3)}{4(4-2)}\,\frac{\pi}{2} = \frac{\overset{30}{\cancel{60}}\times\cancel{2}\times 3\times\pi}{4\times\cancel{2}\times\cancel{2}}$$

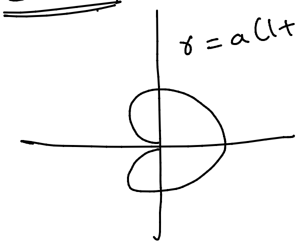
$$= \frac{90\pi}{4} = \frac{45\pi}{2}.$$

H.W

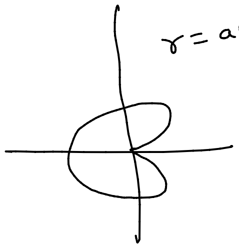
→ $\iint r^3 \, dr\, d\theta$ over the area b/w the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

→ Cardiod

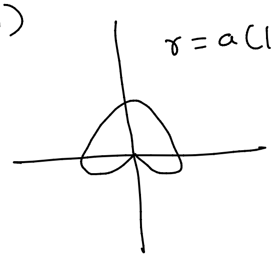
$r = a(1+\cos\theta)$



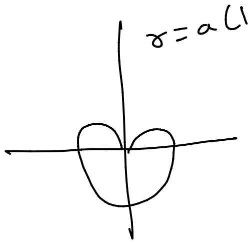
$r = a(1-\cos\theta)$



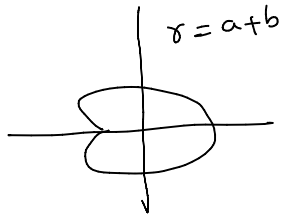
$r = a(1+\sin\theta)$



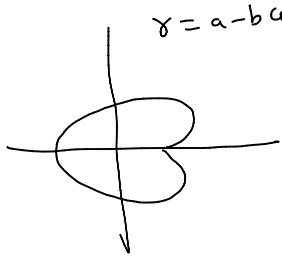
$r = a(1-\sin\theta)$



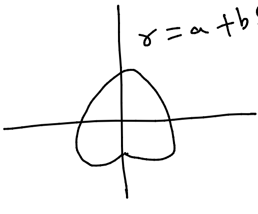
$r = a + b\cos\theta$



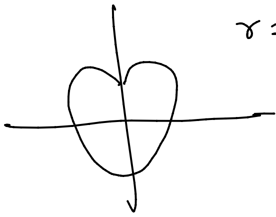
$r = a - b\cos\theta$



$r = a + b\sin\theta$

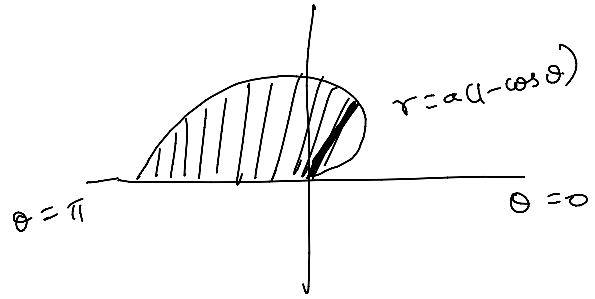


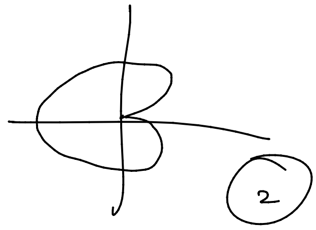
$r = a - b\sin\theta$.



→ Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line.

Sol





$r = 0$ to $r = a(1-\cos\theta)$ slides from $\theta = 0$ to π.

→ Change of variables :-

let $x = f(u,v)$ & $y = g(u,v)$

The above relation represents transformation (change) of variables from x, y to u, v

The Jacobian of the transformation is $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

Now the double integral $\iint_R f(x,y)\,dx\,dy$ after change of variables

is $\iint_{R'} F(u,v)\,|J|\,du\,dv$.

Change of variables to polar coordinates:

The relation b/w cartesian & polar coordinates is

$x = r\cos\theta$, $y = r\sin\theta$. $\quad x, y \to r, \theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\boxed{J = r}$$

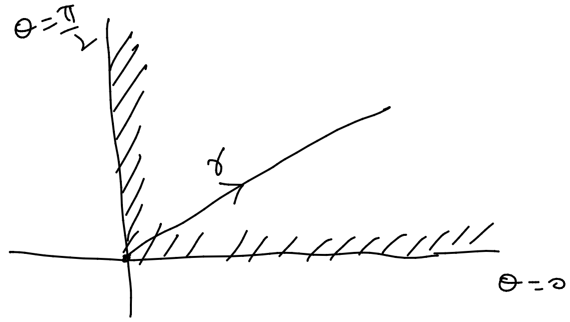
$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta.$$

$$= \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

→ Evaluate $\displaystyle\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ changing variables to polar coordinates.

$\theta = \frac{\pi}{2}$,

→ Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ changing ... coordinates.



Sol: $x=0 \quad x=\infty$

$y=0 \quad y=\infty$

The region is entire first quadrant.

Change the variables to polar by taking $x=r\cos\theta,\ y=r\sin\theta$

$x^2+y^2=r^2$

$J=r$

$r=0$ to $r=\infty$, $\theta=0$, to $\theta=\frac{\pi}{2}$

$$\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)}\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}|J|\,dr\,d\theta$$

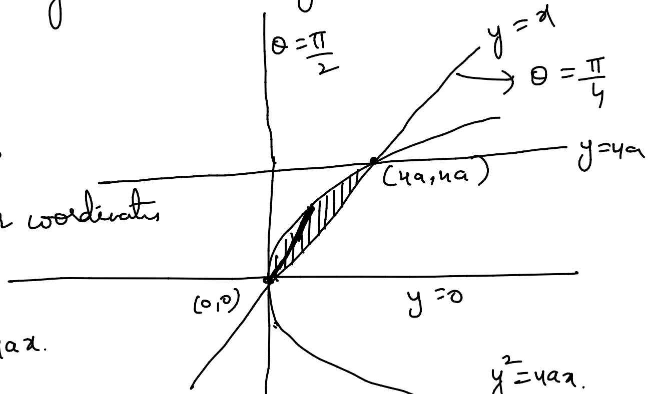
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta = ?$$

2) $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ Ans $\frac{3\pi}{2}$

3) $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

Sol: $x=\frac{y^2}{4a}$ to $x=y$, $y=0$ to $y=4a$

$\Rightarrow y^2=4ax$



Take $x=r\cos\theta,\ y=r\sin\theta$ to change variables to polar coordinates

$x^2+y^2=r^2$

$J=r$

$r=0$ to $r=\frac{4a\cos\theta}{\sin^2\theta}$.

slides from $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$y^2=4ax$.

$r^2\sin^2\theta = 4ar\cos\theta$

$r=\frac{4a\cos\theta}{\sin^2\theta} \rightarrow y^2=4ax$.

$$\int^{4a}\int^{y} \frac{x^2-y^2}{...}\,dx\,dy = \int^{\pi/2}\int^{\frac{4a\cos\theta}{\sin^2\theta}} r^2\cos^2\theta - r^2\sin^2\theta\ |J|\,dr\,d\theta.$$

$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}\,|J|\,dr\,d\theta.$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, r\,dr\,d\theta.$$

$$\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos^2\theta - \sin^2\theta)\,\frac{r^2}{2}\Big|_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}}\,dr\,d\theta.$$

$$= \frac{1}{2}\int_{\theta=\frac{\pi}{4}}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(16\,a^2\frac{\cos^2\theta}{\sin^4\theta} - 0\right)d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\pi/2} (\cot^4\theta - \cot^2\theta)\,d\theta.$$

$$\boxed{\int \cot^m\theta\, d\theta}$$

2.

→ By using the transformations $x+y=u$ and $y=uv$, evaluate

$$\int_0^1\int_0^{1-x} e^{y/x+y}\,dy\,dx.$$

$$(x,y) \longrightarrow (u,v)$$

Sol

$$x+y=u$$

$$x+uv=u \implies x = u-uv$$

$$y = uv$$

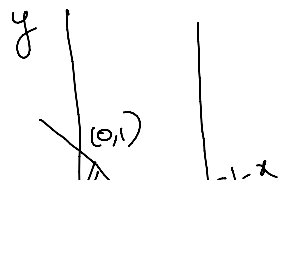
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u-\cancel{uv}+\cancel{uv}$$

$$= u$$

$$\boxed{J=u}$$

$$x=0,\quad x=1$$

$$y=0,\quad y=1-x$$



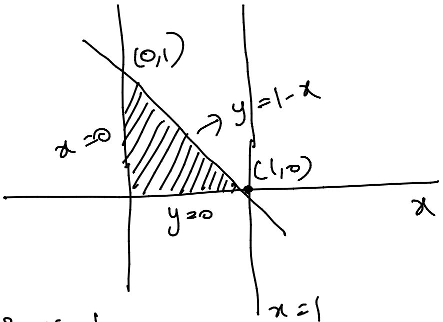
$y = 0 \quad , y = 1 - x$

$x + y = u \quad , y = uv$

$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0$ or $v = 0$

$y = 1 - x \Rightarrow x + y = 1 \Rightarrow u = 1$

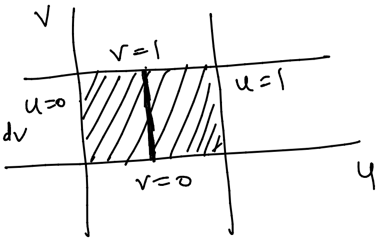
$x = 0 \Rightarrow u(1-v) = 0 \Rightarrow u = 0$ or $v = 1$



$x = 1 \Rightarrow u - uv = 1$

$\underline{u = 0 , u = 1 , v = 0 , v = 1}$

$$\int_0^1 \int_0^{1-x} e^{y/x+y}\, dy\, dx = \int_{v=0}^{1} \int_{u=0}^{1} e^{uv/u} |J|\, du\, dv$$



$$= \int_{v=0}^{1} \int_{u=0}^{1} e^{v} u\, du\, dv$$

$$= \int_{v=0}^{1} e^{v} \left.\frac{u^2}{2}\right|_0^1 dv = \frac{1}{2} \int_{v=0}^{1} e^{v} dv = \frac{1}{2} \left(e^{v}\right)_0^1$$

$$= \frac{1}{2}(e - 1).$$

Evaluate

$\rightarrow \iint \frac{x^2 y^2}{x^2 + y^2}\, dx\, dy$ over the annular region of $x^2 + y^2 = a^2$ & $x^2 + y^2 = b^2$, $b > a$, by changing to polar coordinates.

sol

$x = r\cos\theta \qquad y = r\sin\theta$

$J = r$

$r = a$ to $r = b$

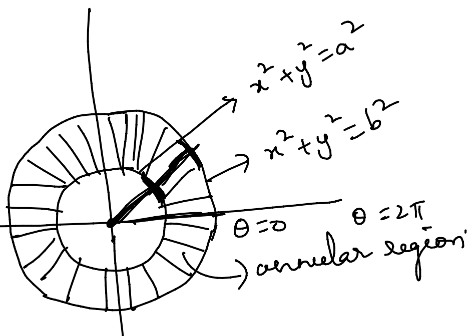
slides from $\theta = 0$ to 2π

$x^2 + y^2 = a^2$

$r^2 = a^2$

$r = a$

$r = b$



$$\iint \frac{x^2 y^2}{x^2 + y^2}\, dx\, dy = \int_{\theta=0}^{2\pi} \int_{r=a}^{b} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2} |J|\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3 \sin^2\theta \cos^2\theta\, dr\, d\theta.$$

$$= \frac{1}{4} \int_{\theta=0}^{2\pi} \int_{r=a}^{b} r^3 (2\sin\theta\cos\theta)^2\, dr\, d\theta.$$

$$= \frac{1}{4}\int_{\theta=0}\int_{r=a}$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3 \sin^2 2\theta \, dr\, d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi} \sin^2 2\theta \left.\frac{r^4}{4}\right|_a^b d\theta = \frac{1}{16}\int_0^{2\pi} (b^4-a^4)\sin^2 2\theta \, d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi} \frac{1-\cos 4\theta}{2}\, d\theta = \frac{b^4-a^4}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{32}\left[2\pi - 0\right]$$

$$\frac{\pi(b^4-a^4)}{16}.$$

→ Triple integrals :-

$f(x)$ — $[a,b]$

$f(x,y)$ — R

$f(x,y,z)$ — V

Let $f(x,y,z)$ be defined over a volume V.

Now divide the volume in to sub volumes

$$\delta v_1, \delta v_2, \delta v_3, \ldots \delta v_n$$

Let (x_i, y_i, z_i) in δv_i

Consider the sum

$$f(x_1,y_1,z_1)\delta v_1 + f(x_2,y_2,z_2)\delta v_2 + \cdots\cdots + f(x_n,y_n,z_n)\delta v_n$$

If the above sum tends to a finite limit as $n\to\infty$ such $\delta v_i \to 0$ then that limit is called triple integral of $f(x,y,z)$ over V & denoted by $\iiint_V f(x,y,z)\,dv$ (or) $\iiint_V f(x,y,z)\,dx\,dy\,dz$

→ If $f(x,y,z) = 1$, $\iiint f(x,y,z)\,dx\,dy\,dz$ represents volume

→ If $f(x,y,z) \neq 1$, $\iiint f(x,y,z)\,dx\,dy\,dz$ represents hyper volume.

→ Evaluation of triple integrals :

1) $\iiint_V f(x,y,z)\,dx\,dy\,dz$, V : $a_1 \le x \le a_2$, $b_1 \le y \le b_2$, $c_1 \le z \le c_2$.

then in this case, we can either integrate w.r.to 'x' or 'y' or 'z' first by keeping the other variables fixed.

first by keeping the other variables fixed.

2) If $V: a_1 \le x \le a_2,\ y_1(x) \le y \le y_2(x),\ z_1(x,y) \le z \le z_2(x,y)$

then $\iiint f(x,y,z)\,dx\,dy\,dz = \int_{x=a_1}^{a_2}\int_{y=y_1(x)}^{y_2(x)}\int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$

integrate w.r.to z first by keeping x & y fixed then integrate w.r.to 'y' by keeping x fixed & then at last we integrate w.r.to 'x'

→ $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy.$

sol: $\int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^x} \log z\, dz\, dx\, dy = \int_{y=1}^{e}\int_{x=1}^{\log y} (z\log z - z)_{z=1}^{e^x} dx\,dy$

$\int uv = u\int v - \int u'\int v$

$\int \log z \cdot 1\, dz$ ($u = \log z$, $v = 1$)

$(\log z)z - \int \frac{1}{z} z$

$z\log z - z$

$\log e^x = x\log_e e$

$e^{\log y} = y$

$= \int_{y=1}^{e}\int_{x=1}^{\log y} (e^x x - e^x - 0 + 1)\,dx\,dy$

$= \int_{y=1}^{e} \left(xe^x - e^x - e^x + x\right)_{x=1}^{\log y} dy$

$= \int_{y=1}^{e} \left[(y\log y - y - y + \log y) - (\cancel{e} - e - \cancel{e} + 1)\right] dy$

$\int_{y=1}^{e} (y\log y - 2y + \log y + e - 1)\,dy$

$\left[\log y \frac{y^2}{2} - \frac{y^2}{4} - y^2 + y\log y - y + ey - y\right]_{y=1}^{e}$

$\left[\frac{y^2}{2}\log y - \frac{5y^2}{4} + y\log y - 2y + ey\right]_1^e$

$\left[\frac{e^2}{2} - \frac{5e^2}{4} + e - 2e + e^2 - 0 + \frac{5}{4} - 0 + 2 - e\right]$

$= ?$ Ans $\frac{1}{4}\left[e^2 - 8e + 13\right].$

→ $\int_0^1 \int_0^{\sqrt{1-x^2}} \int^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx.$

$x = -1$ to 1

$y = -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$z = -\sqrt{1-x^2-y^2}$ to

$z = \sqrt{1-x^2-y^2}$

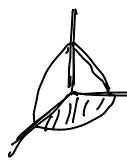
$x^2+y^2+z^2 = 1$

$$\rightarrow \int_0 \int_0 \int_0 xyz\, dz\, dy\, dx.$$

$y = -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$z = -\sqrt{1-x^2-y^2}$ to $\sqrt{1-x^2-y^2}$

$x^2+y^2+z^2=1$



$\rho \to 0$ to 1

$\theta \to 0$ to $\frac{\pi}{2}$

$\phi = 0$ to $\frac{\pi}{2}$

Sol:
$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy \left.\frac{z^2}{2}\right|_{z=0}^{\sqrt{1-x^2-y^2}} dy\,dx = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\, dy\, dx.$$

$$= \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (xy - x^2y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left(\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right)_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2\right] dx$$

$$= \frac{1}{4}\int_{x=0}^{1} x(1-x^2)\, dx - \frac{1}{4}\int_{x=0}^{1} x^3(1-x^2)\, dx - \frac{1}{8}\int_{x=0}^{1} x(1-x^2)^2\, dx.$$

$1-x^2 = t$ put $x = \sin\theta$ $\int \sin^m\theta \cos^n\theta\, d\theta$

Ans: $\frac{1}{48}$

H.W $\int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\, dy\, dx.$ Ans $\frac{1}{6}$

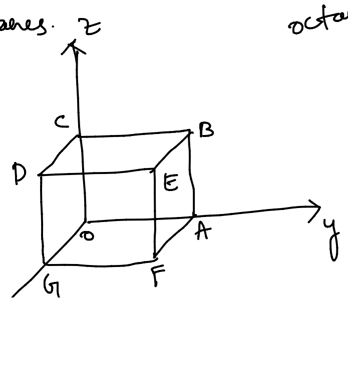
→ Evaluate $\iiint (xy+yz+zx)\, dx\, dy\, dz$ over the volume bounded by $x=0$, $y=0$, $z=0$, $x=1$, $y=1$ & $z=1$

Sol: $x=0$ — yz plane
$y=0$ — xz plane
$z=0$ — xy plane
} coordinate planes.

cube in first octant

$x=1$ — ||el to yz plane

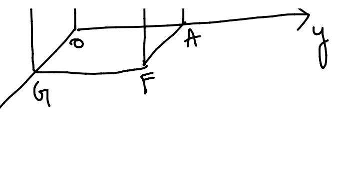
$y=1$ — ||el to xz plane



$x = 1$ — ||el to yz plane

$y = 1$ — ||el to xz plane

$z = 1$ — ||el to xy plane



OABC — $x = 0$

DEFG — $x = 1$

OCDG — $y = 0$

ABEF — $y = 1$

OAFG — $z = 0$

BCDE — $z = 1$

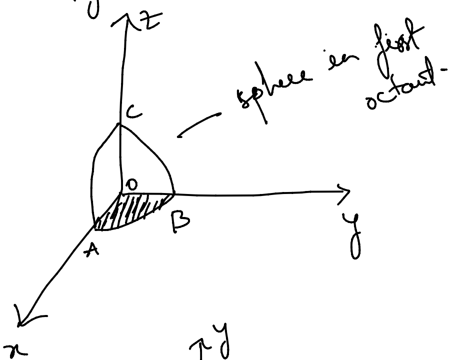
$$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (xy + yz + zx)\, dx\,dy\,dz = ?$$

→ Evaluate $\iiint xyz\, dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2 = a^2$.

Sol:

$z = 0$, $x^2+y^2+z^2 = a^2$

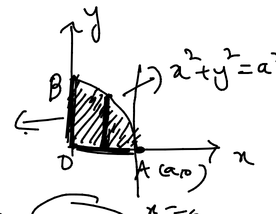
$x^2+y^2 = a^2$



$x = 0$ to a

$y = 0$ to $\sqrt{a^2-x^2}$

$z = 0$ to $\sqrt{a^2-x^2-y^2}$



$x^2+y^2+z^2 = a^2 \Rightarrow z^2 = \sqrt{a^2-x^2-y^2}$

$z = 0$ to $\sqrt{a^2-x^2-y^2}$

$z = 0 \Rightarrow x^2+y^2 = a^2$

$y^2 = \sqrt{a^2-x^2}$

$y = 0$ to $\sqrt{a^2-x^2}$

$z = 0, y = 0$

$x^2 = a^2 \Rightarrow x = a$

$x = 0$ to $x = a$

vertical strip $\{ y = 0$, $y = \sqrt{a^2-x^2}$, $x = a$.

$x = 0$ to $x = a$

$z = 0$, $z = \sqrt{a^2-x^2-y^2}$

$\rho = 0$ to a

$\theta = 0$ to $\frac{\pi}{2}$

$\phi = 0$ to $\frac{\pi}{2}$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\, dz\,dy\,dx = ? \qquad \underline{\text{Ans}}: \frac{a^6}{48}.$$

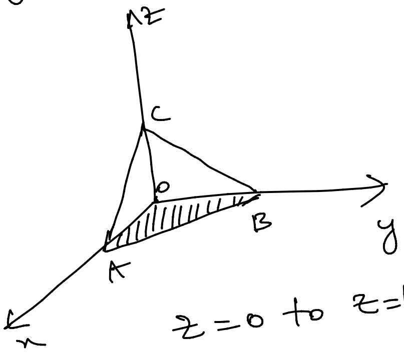
H.W

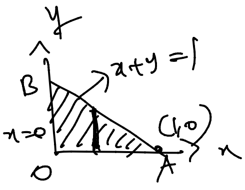
→ $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the volume bounded by tetrahedron $x=0$, $y=0$, $z=0$ & $x+y+z=1$.

→ $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ over the volume bounded by ...

$x=0,\ y=0,\ z=0$ & $x+y+z=1$.

Sol: Ans: $\frac{3}{16}+4\log 2$





Hint: $x+y+z=1$ in xy-plane is $x+y=1$ ($\because z=0$)

$z=0$ to $z=1-x-y$

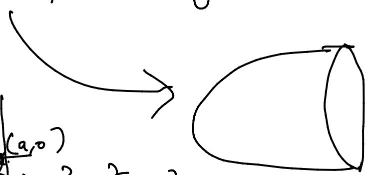
$y=0$

$y=1-x$

$x=0$ to $x=1$

→ $\iiint z^2\,dx\,dy\,dz$ taken over the volume bounded by the surfaces $x^2+y^2=a^2$ & $x^2+y^2=z$ & $z=0$

$x^2+y^2=a^2$ → cylinder; $x^2+y^2=z$ → paraboloid; $z=0$ → xy-plane.



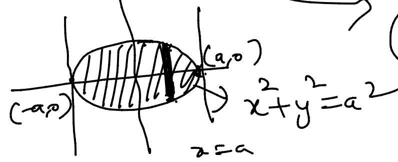
Sol:

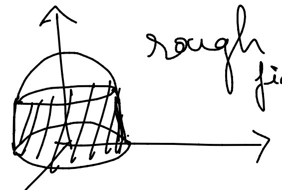
$z=0$ to $z=x^2+y^2$

$x^2+y^2=a^2$

$y=-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$ ✓

$x=-a$ to a. ✓





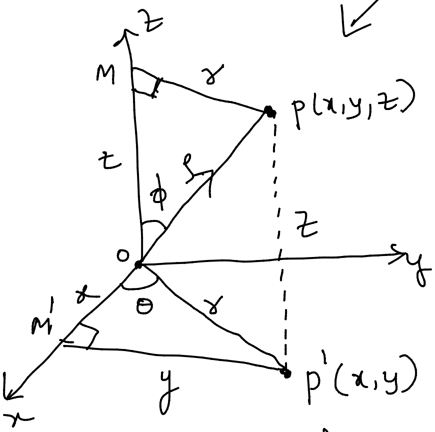
rough figure.

?

change of variables to spherical coordinates:

spherical coordinates — (ρ, θ, ϕ)

$(x,y,z) \longrightarrow (\rho, \theta, \phi)$



ϕ is angle made by ρ with z-axis

so max range of ϕ is π

so $0 \le \phi \le \pi$

θ is angle made by r in xy plane.

so max range of θ is 2π

so $0 \le \theta \le 2\pi$

In Δ^{le} OMP

$\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho\sin\phi$ — (a)

$\cos\phi = \frac{z}{\rho} \Rightarrow z = \rho\cos\phi$ — (b)

In Δ^{le} $OP'M'$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$x = r\cos\theta$

$x = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

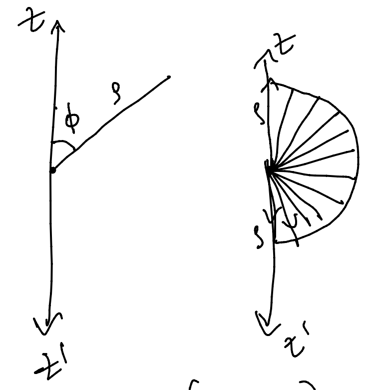
$z = \rho\cos\phi$ → relation b/w cartesian

$x = \rho \sin\phi \cos\theta$

$y = r\sin\theta = \rho \sin\phi \sin\theta$

$z = \rho\cos\phi$

$$\boxed{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho\cos\phi \end{aligned}} \longrightarrow \text{relation b/w cartesian \& spherical}$$



$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$\rho \geq 0$ — radius of sphere.

$(x, y, z) \longrightarrow (\rho, \theta, \phi)$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$J = \cos\phi\left[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right] - 0 - \rho\sin\phi\left[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right]$$

$$= -\rho^2\sin\phi\cos^2\phi\left[\sin^2\theta + \cos^2\theta\right] - \rho^2\sin^3\phi\left[\cos^2\theta + \sin^2\theta\right]$$

$$-\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi = -\rho^2\sin\phi\left[\cos^2\phi + \sin^2\phi\right]$$

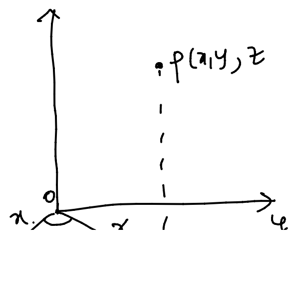
$$\boxed{J = -\rho^2\sin\phi}$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V F(\rho,\theta,\phi)\,|J|\,d\rho\,d\theta\,d\phi.$$

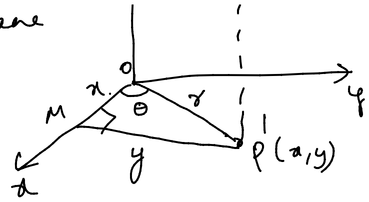
→ <u>change</u> of <u>variables</u> to <u>cylindrical</u> <u>coordinates</u>:

<u>cylindrical</u> coordinates — (r, θ, z).

θ is angle made by 'r' in xy plane



θ is angle made by 'r' in xy plane



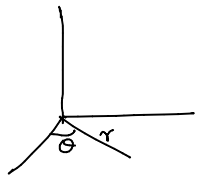
So max range of 'r' is 2π

So, $0 \le \theta \le 2\pi$

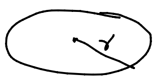
In $OP'M$ $\quad \sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}} \longrightarrow \text{relation b/w cartesian \& cylindrical coordinates}$$



$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$



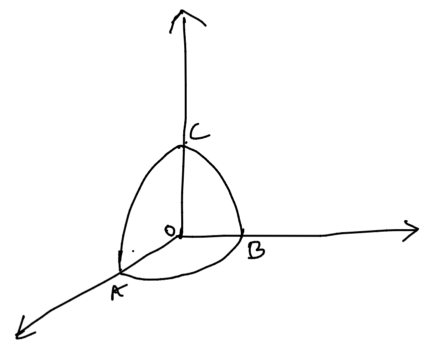
$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{J = r}$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} F(r,\theta,z)\,|J|\,dr\,d\theta\,dz$$

→ Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical coordinates.



Sol:

$x = 0 \quad x = 1$

$y = 0 \quad y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$

$z = 0 \quad z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 1$

We change (x,y,z) to (ρ,θ,ϕ) by

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi.$

$J = -\rho^2 \sin\phi$

$\rho \to 0$ to 1 (radius of sphere)

$\theta \to 0$ to $\frac{\pi}{2}$

$\phi \to 0$ to $\frac{\pi}{2}$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{1}{\sqrt{1-(x^2+y^2+z^2)}} |J| \, d\phi \, d\theta \, d\rho.$$

$$x^2+y^2+z^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2\cos^2\phi = \rho^2 \sin^2\phi + \rho^2 \cos^2\phi = \rho^2$$

$$= \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{\rho^2 \sin\phi \, d\phi\, d\theta\, d\rho}{\sqrt{1-\rho^2}}$$

$$= \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \frac{\rho^2 \left(-\cos\phi\right)_{\phi=0}^{\pi/2}}{\sqrt{1-\rho^2}} d\theta\, d\rho = \int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \frac{\rho^2}{\sqrt{1-\rho^2}} d\theta\, d\rho$$

$$= \cancel{\int_{\rho=0}^{1}}$$

$$= \int_{\rho=0}^{1} \frac{\rho^2}{\sqrt{1-\rho^2}} (\theta)_{\theta=0}^{\pi/2} d\rho$$

$$\frac{\pi}{2} \int_{\rho=0}^{1} \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho = \frac{\pi}{2} \int_{\rho=0}^{1} \frac{\rho^2 - 1 + 1}{\sqrt{1-\rho^2}} d\rho = -\frac{\pi}{2} \int_{\rho=0}^{1} \frac{1-\rho^2-1}{\sqrt{1-\rho^2}} d\rho$$

$$= -\frac{\pi}{2} \int_{\rho=0}^{1} \frac{(1-\rho^2)}{\sqrt{1-\rho^2}} d\rho + \frac{\pi}{2} \int_{\rho=0}^{1} \frac{1}{\sqrt{1-\rho^2}} d\rho$$

$$= -\frac{\pi}{2} \int_{\rho=0}^{1} \sqrt{1-\rho^2}\, d\rho + \frac{\pi}{2} \int_{\rho=0}^{1} \frac{1}{\sqrt{1-\rho^2}} d\rho$$

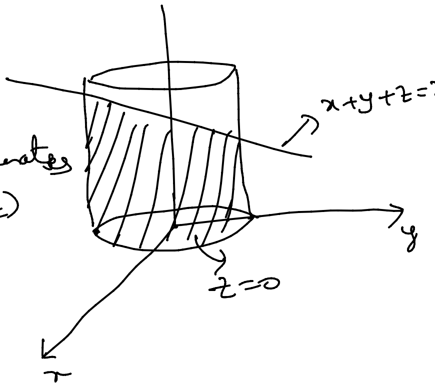
$\int \sqrt{a^2-x^2}\, dx.$ $\int \frac{dx}{\sqrt{a^2-x^2}}$

$= ?$

$\int \sqrt{a^2-x^2}\,dx$

$= ? \quad \frac{\pi}{8}$

→ Evaluate $\iiint x\,dv$ over the region b/w the surfaces $x+y+z=2$, $z=0$ & $x^2+y^2=1$



sol: $z=0$ to $z=2-x-y$.

we transform (x,y,z) to cylindrical coordinates (r,θ,z)

$x = r\cos\theta,\ y = r\sin\theta,\ z=z$

$J = r$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$z=0$ to $z = 2-x-y = 2-r\cos\theta - r\sin\theta.$

$$\iiint x\,dv = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{2-r\cos\theta-r\sin\theta} x|J|\,dz\,d\theta\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{2-r\cos\theta-r\sin\theta} r\cos\theta\,(r)\,dz\,d\theta\,dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\cos\theta\,(z)\Big|_{z=0}^{2-r\cos\theta-r\sin\theta} d\theta\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\cos\theta\,(2-r\cos\theta-r\sin\theta)\,d\theta\,dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\left(2r^2\cos\theta - r^3\cos^2\theta - r^3\sin\theta\cos\theta\right)d\theta\,dr$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\left(2r^2\cos\theta - r^3\frac{(1+\cos2\theta)}{2} - \frac{r^3}{2}\sin2\theta\right)d\theta\,dr$$

$$= \int_{r=0}^{1}\left(2r^2\sin\theta - \frac{r^3}{2}\left(\theta+\frac{\sin2\theta}{2}\right) + \frac{r^3}{2}\frac{\cos2\theta}{2}\right)\Bigg|_{\theta=0}^{2\pi} dr$$

$$= \int_{r=0}^{1}\left(2r^2(0) - \frac{r^3}{\cancel{2}}(\cancel{2}\pi+0) + \cancel{\frac{r^3}{4}}(1) - 0 - 0 - \cancel{\frac{r^3}{4}}\right)dr$$

$$= -\pi\int_{r=0}^{1} r^3\,dr = -\pi\,\frac{r^4}{4}\Bigg|_0^1 = -\frac{\pi}{4}.$$

H.W

→ using spherical coordinates, evaluate $\iiint xyz\,dx\,dy\,dz$ taken ... bounded by the sphere $x^2+y^2+z^2=1$ in first ...

#

→ using spherical coordinates, evaluate $\iiint xyz\, dx\,dy\,dz$ over the volume bounded by the sphere $x^2+y^2+z^2=1$ in first octant.

$\rho \to 0$ to 1

$\theta \to 0$ to $\frac{\pi}{2}$

$\phi \to 0$ to $\frac{\pi}{2}$

→ using cylindrical coordinates, evaluate $\iiint (x^2+y^2+z^2)\, dx\,dy\,dz$ over the region bounded by $0 \le z \le x^2+y^2 \le 1$

$z=0$ to $z = x^2+y^2$ — r^2

$x^2+y^2=1$

$r = 0$ to 1

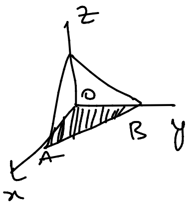
$\theta = 0$ to 2π

$z = 0$ to r^2

→ find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

sol: volume — $V = \iint f(x,y)\,dx\,dy$ (or) $V = \iiint 1\cdot dx\,dy\,dz$

Evaluation of volume by double integral



$V = \iint f(x,y)\,dx\,dy$ $\quad \boxed{z = f(x,y)}$

$V = \iint c\left(1-\frac{x}{a}-\frac{y}{b}\right)dx\,dy$

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

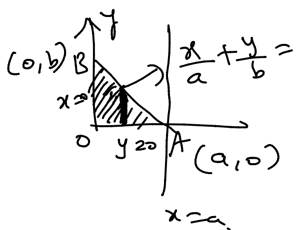
$\Rightarrow \frac{z}{c} = 1-\frac{x}{a}-\frac{y}{b}$

$\Rightarrow z = c\left(1-\frac{x}{a}-\frac{y}{b}\right) \to f(x,y)$

In xy-plane $z=0$

so $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ becomes

$\frac{x}{a}+\frac{y}{b}=1$



$y=0$ to $y = b\left(1-\frac{x}{a}\right)$

$x=0$ to $x=a$

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$V = c\int_{x=0}^{a}\left(y-\frac{x}{a}y-\frac{1}{b}\frac{y^2}{2}\right)_{y=0}^{b\left(1-\frac{x}{a}\right)}dx$$

$$V = c\int^{a}\left[b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

$$V = c\int_{x=0}^{a}\left[b\left(1-\frac{x}{a}\right) - \frac{x}{a}b\left(1-\frac{x}{a}\right) - \frac{1}{2b}b^2\left(1-\frac{x}{a}\right)^2\right]dx$$

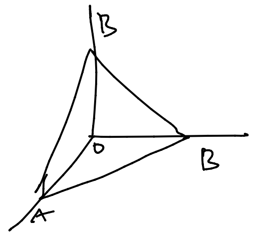
$$V = c\int_{x=0}^{a}\left[b\left(1-\frac{x}{a}\right) - \frac{b}{a}\left(x - \frac{x^2}{a}\right) + \frac{b}{2}\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\right]dx$$

$$= c\left[bx - \frac{bx^2}{2a} - \frac{bx^2}{2a} + \frac{b}{a^2}\frac{x^3}{3} + \frac{b}{2}x + \frac{b}{2a^2}\frac{x^3}{3} - \frac{b}{a}\frac{x^2}{2}\right]_0^a$$

$$= c\left[\cancel{ab} - \frac{\cancel{ab}}{2} - \frac{\cancel{ab}}{2} + \frac{ab}{3} + \frac{ab}{2} + \frac{ab}{6} - \frac{ab}{2}\right]$$

$$V = c\left[\frac{3ab}{6}\right] = \frac{abc}{2} \text{ cubic units.}$$

Evaluation by triple integral



$$V = \iiint dx\,dy\,dz$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

$$z = 0 \text{ to } c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$y = 0 \text{ to } b\left(1-\frac{x}{a}\right), \quad x = 0 \text{ to } a$$

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$= ?$

$\frac{abc}{3}$ cubic units.

→ Find the volume bounded by xy-plane, cylinder $x^2+y^2=1$ & the plane $x+y+z=3$

Sol: xy-plane — $z=0$

$\Rightarrow z = 3-x-y$

Sol:

xy-plane — $z=0$

$x+y+z=3 \Rightarrow z=3-x-y$

$z=0$ to $3-x-y$.

$$V=\iiint dx\,dy\,dz$$

$r=0$ to 1

$\theta=0$ to 2π

$z=0$ to $3-x-y$

$x=r\cos\theta$

$y=r\sin\theta$

$z=z \qquad J=r$

$$V=\iiint |J|\,dr\,d\theta\,dz=\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{3-r\cos\theta-r\sin\theta} r\,dz\,d\theta\,dr$$

$$V=\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r(z)\Big|_{z=0}^{3-r\cos\theta-r\sin\theta} d\theta\,dr=\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r(3-r\cos\theta-r\sin\theta)\,d\theta\,dr$$

$$V=\int_{r=0}^{1}\Big[3r(\theta)-r^2\sin\theta-r^2(-\cos\theta)\Big]_{\theta=0}^{2\pi}dr$$

$$V=\int_{r=0}^{1}\Big[3r(2\pi)-r^2(0)+\cancel{r^2(1)}-0-0-\cancel{r^2}\Big]dr$$

$$V=6\pi\int_{r=0}^{1} r\,dr=6\pi\frac{r^2}{2}\Big|_0^1$$

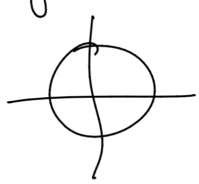
$= 3\pi$ cubic units.

(or)

$$V=\iint f(x,y)\,dx\,dy \qquad x+y+z=3$$

$z=3-x-y=f(x,y)$

$$V=\iint(3-x-y)\,dx\,dy$$



$x=r\cos\theta$

$y=r\sin\theta \quad , J=r$

$r=0$ to 1

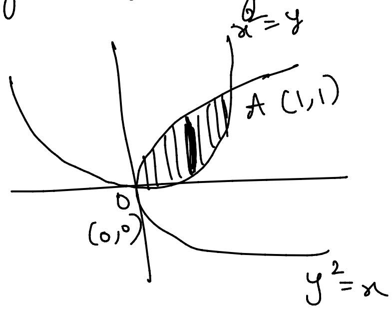
$\theta=0$ to 2π

$$=\int_{r=0}^{1}\int_{\theta=0}^{2\pi}(3-r\cos\theta-r\sin\theta)|J|\,d\theta\,dr.$$

$= 3\pi$ cubic units.

$y^2=x$ & $x^2=y$.

→ find the area enclosed by $y^2 = x$ & $x^2 = y$.



Sol: vertical strip: $y = x^2$ to $y = \sqrt{x}$

$x = 0$ to $x = 1$

$$A = \iint dx\,dy$$

$$A = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} dy\,dx.$$