$y = f(x)$ — single variable function

$z = f(x,y)$ — function of two variables (Multivariable)

x, y — independent variable.

z — dependent variable.

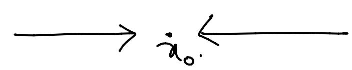
$y = f(x)$ — curve.

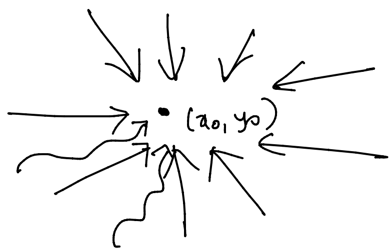
$z = f(x,y)$ — surface.

→ limit of function of two variables:

$f(x,y)$

(x_0, y_0)

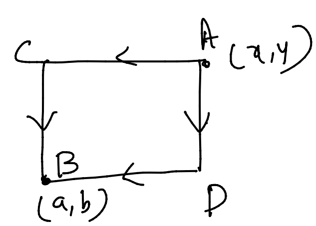


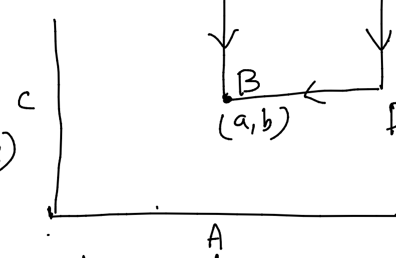


The function $f(x,y)$ is said to a limit 'L' at a (x_0, y_0), if we approach (x_0,y_0) in any direction, then $f(x,y)$ should approach a unique value 'L'

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad (\text{or}) \quad \lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = L.$$

To find the limit of $f(x,y)$ at $(x,y) = (a,b)$, $a \neq 0$, $b \neq 0$





ADB

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = f_1 \text{ (say)}$$

ACB

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right] = f_2 \text{ (say)}$$

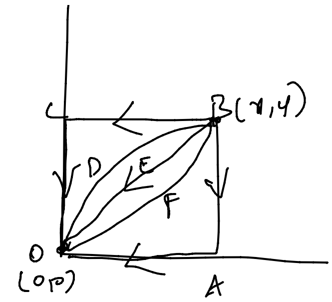
If $f_1 \neq f_2$, then limit does not exists.

If $f_1 = f_2$ then limit exists.

To find limit. of $f(x,y)$ at $(x,y) = (0,0)$

BAO $\lim f(x,y) = \lim \left[\lim f(x,y)\right] = f_1$

BAO $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = f_1$



BCO $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = f_2$

If $f_1 \neq f_2$ then limit does not exists

If $f_1 = f_2$

$f = x^2+y^2$
$f = x^2+m^2x^2$

OEB $y = mx$.

$(x,y)\to(0,0)$, as $x\to 0 \Rightarrow y\to 0$ $(\because y = mx)$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = f_3$

If $f_1 = f_2 \neq f_3$, then limit does not exists.

If $f_1 = f_2 = f_3$

OFB $y = mx^2$

$\boxed{y = mx^n}$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx^2) = f_4$

$n = 2/3$

If $f_1 = f_2 = f_3 \neq f_4$ then limit does not exists

$\boxed{y = mx^n}$

lly we take all the paths $y = mx^n$, $n\in\mathbb{R}$

If in all these paths the limit value is unique

then $f(x,y)$ is said to have limit at $(0,0)$.

→ (1) $\lim_{(x,y)\to(1,2)} \frac{2xy}{x^2+y^2+1}$

sol: $f_1 = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{2xy}{x^2+y^2+1}\right] = \lim_{x\to 1}\left[\frac{4x}{x^2+5}\right] = \frac{4}{6} = \frac{2}{3}$

$f_2 = \lim_{y\to 2}\left[\lim_{x\to 1} \frac{2xy}{x^2+y^2+1}\right] = \lim_{y\to 2}\left[\frac{2y}{y^2+2}\right] = \frac{4}{6} = \frac{2}{3}$

$f_1 = f_2$

$f_1 = f_2$

$\Rightarrow$ The limit exists.

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{2x-y}$

Sol:

$$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\frac{x+y}{2x-y}\right] = \lim_{x\to 0}\left[\frac{x}{2x}\right] = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\frac{x+y}{2x-y}\right] = \lim_{y\to 0}\left[\frac{y}{-y}\right] = \lim_{y\to 0}(-1) = -1$$

$$f_1 \neq f_2$$

Hence limit does not exists.

(3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$

Sol:

$$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2y}{x^4+y^2}\right] = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\frac{x^2y}{x^4+y^2}\right] = 0$$

$$f_1 = f_2$$

$y = mx$

$$f_3 = \lim_{x\to 0}\frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0}\frac{mx^3}{x^4+m^2x^2} = \lim_{x\to 0}\frac{\cancel{x^2}(mx)}{\cancel{x^2}(x^2+m^2)} = 0$$

$$f_1 = f_2 = f_3$$

$y = mx^2$

$$f_4 = \lim_{x\to 0}\frac{x^2(mx^2)}{x^4+(mx^2)^2} = \lim_{x\to 0}\frac{mx^4}{x^4+m^2x^4} = \lim_{x\to 0}\frac{m}{1+m^2} = \frac{m}{1+m^2}$$

$m = 1$ — $y = x^2$ — $1/1+1^2$

$= 2$ — $y = 2x^2$ — $2/1+2^2$

Along $y = mx^2$, the limit of the function is depending on 'm'

for different 'm' values we get different limits

Along $y = mx^2$, the limit of ...
for different 'm' values we get different 'limits'
Hence limit of $f(x,y)$ at $(0,0)$ does not exist.

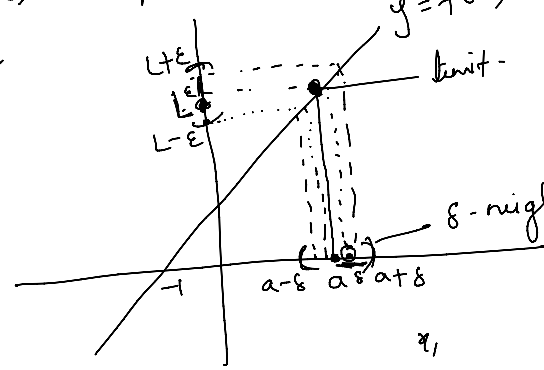
→ $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2 - x^2}{x^2+y^2}$$

→ $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 - y^3}{x^2+y^2}$$

Epsilon-delta definition for limit:

A function $f(x)$ is said to have a limit 'L' (finite) at a point 'a' if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $0 < |x-a| < \delta$ then
$$|f(x) - L| < \varepsilon.$$

$|x-a| < \delta$
$\Rightarrow -\delta < x - a < \delta$
$a - \delta < x < a + \delta$



x_1

$|x_1 - a| < \delta$

Ex

$$\boxed{\lim_{x\to 2} x^3 = 8}$$

$f(x) = x^3$, $L = 8$

for $\varepsilon > 0$ $\exists$ $\delta > 0$ → $0 < |x-2| < \delta$
$\Rightarrow |f(x) - 8| < \varepsilon$

$|xy| = |x||y|$

ε – very small
$\frac{\varepsilon}{3}$ – δ – even very small

$|f(x) - 8| < \varepsilon$

$|x^3 - 8| < \varepsilon$

$|(x-2)(x^2+2x+4)| < \varepsilon$

$|x-2||x^2+2x+4| < \varepsilon$

$|(x+1)^2+3| \geq 3$

$$|x-2| < \frac{\varepsilon}{|x^2+2x+4|} = \frac{\varepsilon}{|(x+1)^2+3|} < \frac{\varepsilon}{3} = \delta$$

let $\delta = \frac{\varepsilon}{3}$

$|x-2| < \delta$

$|f(x) - 8| = |x^3 - 8|$

$$|f(x)-8| = |x^3-8|$$
$$= |(x-2)(x^2+2x+4)|$$
$$= |x-2||(x^2+2x+4)|$$
$$= |(x-2)||(x+1)^2+3| > |(x-2)|\,3$$
$$< 3\delta$$
$$< 3\cdot\frac{\varepsilon}{3}$$
$$\boxed{|f(x)-8| < \varepsilon}$$

$$\delta = \frac{\varepsilon}{3}$$

$$\Rightarrow \lim_{x\to 2} f(x) = 8.$$

$\varepsilon-\delta$ method for limit of

$\rightarrow$ Two Variables

P.T $\boxed{\lim_{x\to 2} x^2 - 4 = 0}$

A function $f(x,y)$ is said to have a limit at (a,b) is L (finite)

$\varepsilon > 0 \;\exists\; \delta > 0 \rightarrow |f(x,y)-L| < \varepsilon$ whenever

there exist $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$

$(x-a) < \delta$ then $|f(x)-L| < \varepsilon$

$|f(x)-L| < \varepsilon$ whenever $|x-a| < \delta$

$(x,y) \quad (a,b)$

$\sqrt{(x-a)^2+(y-b)^2}$

1) $$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^4} = 0 \qquad \sqrt{(x-0)^2+(y-0)^2} < \delta$$

$$|f(x,y) - L|$$

$$\left|\frac{x^3y}{x^2+y^4} - 0\right| = \left|\frac{x^3y}{x^2+y^4}\right|$$

$$= |y|\left|\frac{x^3}{x^2+y^4}\right| \leq |y|\left|\frac{x^3}{x^2}\right|$$

$$\leq |y||x|.$$

$$|x^2+y^4| \geq x^2$$

$$x < \sqrt{x^2+y^2}.$$

$$|x| = \sqrt{x^2} = \sqrt{(x-0)^2} < \sqrt{(x-0)^2+(y-0)^2} < \delta$$

$$|y| = \sqrt{y^2} = \sqrt{(y-0)^2} < \sqrt{(x-0)^2+(y-0)^2} < \delta$$

$|x| = \sqrt{x^2} = \sqrt{(x-0)^2} < \sqrt{\quad} < \delta$

$|y| = \sqrt{y^2} = \sqrt{(y-0)^2} < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$$\left| f(x,y) - L \right| < \delta^2 \to \varepsilon$$

$\varepsilon = \delta^2$

$\delta = \sqrt{\varepsilon}$ $\qquad$ $\delta = \sqrt{\varepsilon}.$ $\to$ $\delta = \sqrt{\varepsilon}.$

$$\to \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

sol: $f_1 = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^3-y^3}{x^2+y^2}\right] = \lim_{x\to 0}\left[x\right] = 0$

$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^3-y^3}{x^2+y^2}\right] = \lim_{y\to 0}\left[-y\right] = 0$

$0 < \sqrt{(x-0)^2+(y-0)^2} < \delta \ldots$

$$\left| f(x) - L \right| = \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right|$$

$$= \left| \frac{x^3-y^3}{x^2+y^2} \right| = \left| \frac{(x-y)(x^2+xy+y^2)}{x^2+y^2} \right|$$

$$= |x-y| \left| \frac{x^2+xy+y^2}{x^2+y^2} \right| \geq |x-y|$$

?

$x = r\cos\theta$, $y = r\sin\theta$

$$\left| f(x,y) - L \right| = \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \left| \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \right|$$

$$|f(x,y)-L| = \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \left| \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \right|$$

$$= \left| r(\cos^3\theta - \sin^3\theta) \right|$$

$|a-b| < |a|+|b|$

$$= |r|\,|\cos^3\theta - \sin^3\theta|$$

$|\cos\theta| < 1$

$$\leq |r|\left(|\cos^3\theta| + |\sin^3\theta|\right)$$

$|\sin\theta| < 1$

$$|f(x,y)-L| < 2|r| \quad \text{—— (1)}$$

$$|r| = \sqrt{x^2+y^2} = \sqrt{(x-0)^2+(y-0)^2} < \delta$$

$$|f(x,y)-L| < 2\delta \rightarrow \varepsilon \text{ (say)}$$

$$|f(x,y)-L| < \varepsilon \quad \text{where} \quad \varepsilon = 2\delta$$

Hence for the given ε, We can find $\boxed{\delta = \frac{\varepsilon}{2}}$

$\delta = \frac{\varepsilon}{2}$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0.$$

$\leftarrow$ Erwin Kreyszig

B S Grewal

$$\rightarrow \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

$$\rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \checkmark$$

$\rightarrow$ Continuity of $f(x,y)$ at (a,b)

$f(x,y)$ is said to be continuous at a point (a,b) if

1) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

2) $f(a,b)$ is well defined

3) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

... indicated

2) $f(a,b)$

3) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

→ Check the continuity of the following functions at the indicated points

1) $f(x,y) = \begin{cases} \dfrac{x^2-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Sol: $(a,b) = (0,0)$

f at $(a,b) = 0 \Rightarrow f(0,0) = 0$

from above example $\lim_{(x,y)\to(0,0)} f(x,y)$ exists. $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Hence. f is continuous at $(0,0)$.

2) $f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y}, & (x,y) \neq (2,-2) \\ 4, & (x,y) = (2,-2). \end{cases}$

Sol: f is defined at $(2,-2)$ i.e $f(2,-2) = 4$.

$f(x,y) = \dfrac{x^2+xy+x+y}{x+y} = \dfrac{x(x+y)+x+y}{x+y} = \dfrac{(x+y)(x+1)}{x+y}$

$f(x,y) = x+1$

$f_1 = \lim_{x\to 2}\left[\lim_{y\to -2} f(x,y)\right] = \lim_{x\to 2}\left[\lim_{y\to -2} x+1\right] = \lim_{x\to 2}(x+1) = 3$

for $\varepsilon > 0 \ \exists \ \delta > 0 \rightarrow 0 < \sqrt{(x-2)^2+(y+2)^2} < \delta$ when

$|f(x,y) - L| < \varepsilon$.

$|x| = \sqrt{x^2}$

$|f(x,y) - L| = |x+1-3|$

$= |x-2|$

$|x-2| = \sqrt{(x-2)^2} < \sqrt{(x-2)^2+(y+2)^2} < \delta \Rightarrow |x-2| < \delta$

$|x-2| = \sqrt{(x-2)^2} < \sqrt{(x-2)^2+(y+2)^2} < \delta \Rightarrow |x-2| < \delta$

$\Rightarrow |f(x,y)-L| < \delta = \varepsilon$ (say)

for a given ε, we can find $\boxed{\delta = \varepsilon}$ $\Rightarrow$ limit exists

$$\lim_{(x,y)\to(2,-2)} f(x,y) = 3$$

but $f(2,-2) = 4$

$$\lim_{(x,y)\to(2,-2)} f(x,y) \neq f(2,-2)$$

Hence f is not continuous at $(2,-2)$.

→ $f(x,y) = \begin{cases} 2x^2+y, & (x,y) \neq (1,2) \\ 0, & (x,y) = (1,2) \end{cases}$

→ $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

→ Partial derivative:

Let $z = f(x,y)$

If $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x, y)-f(x,y)}{\Delta x}$ exists, then this limit is called and we denote it as as partial derivative of $f(x,y)$ w.r.to 'x'

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y)-f(x,y)}{\Delta x} = f_x \to \frac{\partial f}{\partial x}.$$

lly $$\frac{\partial f}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y)-f(x,y)}{\Delta y} = f_y \to \frac{\partial f}{\partial y}.$$

if $z = f(x_1, x_2, \ldots x_n)$.

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i\to 0} \frac{f(x_1, x_2, \ldots x_i+\Delta x_i, \ldots x_n) - f(x_1, x_2, \ldots x_n)}{\Delta x_i}$$

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \dots x_i + \Delta x_i, \dots)\ \ -- }{\Delta x_i}$$

If $z = f(x,y)$, then we represent.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = p \ , \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = q,$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = r \ , \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = s$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = t.$$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ when $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continous.

→ If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$, then find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Sol: $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}(x^2+y^2+z^2) = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}x)$$

lly $\frac{\partial u}{\partial y} = -(x^2+y^2+z^2)^{-3/2}y$ & $\frac{\partial u}{\partial z} = -(x^2+y^2+z^2)^{-3/2}z$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[\frac{-3x}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

lly $\frac{\partial^2 u}{\partial y^2} = -\left[-3y(x^2+y^2+z^2)^{-5/2}y + (x^2+y^2+z^2)^{-3/2}\right]$

$$\frac{\partial^2 u}{\partial z^2} = -\left[-3z(x^2+y^2+z^2)^{-5/2}z + (x^2+y^2+z^2)^{-3/2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2\right] + 3(x^2+y^2+z^2)^{-3/2}.$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2} = 0.$$

→ If $u = \log(x^3+y^3+z^3-3xyz)$, then find $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$$(3x^2+0+0-3yz) = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

→ If $u = \log(\dots)$

Sol $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2+0+0-3yz) = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$

IIly $\frac{\partial u}{\partial y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \longrightarrow (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2-yz)+3(y^2-zx)+3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{\cancel{(x^2+y^2+z^2-xy-yz-zx)}(x+y+z)} = \frac{3}{x+y+z}.$$

$$(1) \Rightarrow \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

$x\log x + y\log y + z\log z = \log_e e$
$= 1$

H.W

→ If $x^x y^y z^z = e$, then prove that $\frac{\partial^2 z}{\partial x \partial y} = (-x\log x)^{-1}$ at $x=y=z$.

→ If $u = \log(x^2+y^2+z^2)$, then find $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$.

→ Chain rule of Partial differentiation

If $y = f(x)$, $x = x(t)$
then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

If $z = f(x,y)$ & $x = \phi(u,v)$ & $y = \psi(u,v)$

If $z = f(x,y)$ & $x = \phi(u,v)$ & y

then
$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\}$$
Chain rule of Partial differentiation of two variables

If $s = f(x,y,z)$, $x = \phi(u,v,w)$, $y = \psi(u,v,w)$, $z = \xi(u,v,w)$.

then
$$\left.\begin{aligned} \frac{\partial s}{\partial u} &= \frac{\partial s}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial s}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial s}{\partial v} &= \frac{\partial s}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial s}{\partial z}\frac{\partial z}{\partial v} \\ \frac{\partial s}{\partial w} &= \frac{\partial s}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial s}{\partial z}\frac{\partial z}{\partial w} \end{aligned}\right\}$$
chain rule of Partial differentiation of three variables

→ If $z = f(x,y)$, $x = \phi(t)$, $y = \psi(t)$

then $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ → Total differential coefficient.

$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$ → Total derivative (or) Total differential

→ If $s = f(x,y,z)$, then total derivative of 's' is

$ds = \frac{\partial s}{\partial x}dx + \frac{\partial s}{\partial y}dy + \frac{\partial s}{\partial z}dz$.

→ If $u = f(y-z, z-x, x-y)$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$f(x,y,z)$

sol: let $y - z = r$
$z - x = s$ $\Rightarrow u = f(r,s,t)$
$x - y = t$

$r = y - z \Rightarrow \frac{\partial r}{\partial x} = 0$, $\frac{\partial r}{\partial y} = 1$, $\frac{\partial r}{\partial z} = -1$

$r = y - z \Rightarrow \frac{\partial r}{\partial x} = 0, \quad \frac{\partial r}{\partial y} = 1, \quad \frac{\partial r}{\partial z} = -1$

$s = z - x \Rightarrow \frac{\partial s}{\partial x} = -1, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial s}{\partial z} = 1$

$t = x - y. \Rightarrow \frac{\partial t}{\partial x} = 1, \quad \frac{\partial t}{\partial y} = -1, \quad \frac{\partial t}{\partial z} = 0$

$u = f(r, s, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = 0 + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

H.W

→ If $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

H.W

→ If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

→ $r = e^{y-z}$, $s = e^{z-x}$, $t = e^{x-y}$. $e^{y} \cdot e^{-z}$

$\frac{\partial r}{\partial x} = 0$, $\frac{\partial r}{\partial y} = e^{y-z}$, $\frac{\partial r}{\partial z} = -e^{y-z}$.

→ If $u = f(x^2 + 2yz, y^2 + 2zx)$, then find

$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$.

Sol: $u = f(x^2 + 2yz, y^2 + 2zx)$

let $x^2 + 2yz = r$, $y^2 + 2zx = s$

$u = f(r, s)$, $\frac{\partial r}{\partial x} = 2x$, $\frac{\partial r}{\partial y} = 2z$, $\frac{\partial r}{\partial z} = 2y$.

$$\frac{\partial s}{\partial x} = 2z, \quad \frac{\partial s}{\partial y} = 2y, \quad \frac{\partial s}{\partial z} = 2x.$$

$u = f(r, s)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = 2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = 2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = 2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s}$$

A.W

→ If $\phi(cx - az, cy - bz) = 0$, ~~then~~ show that $ap + bq = c$ where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

→ If $u = r^m$ & $r^2 = x^2 + y^2 + z^2$, then find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Sol:- $u = r^m$

$r^2 = x^2 + y^2 + z^2 \longrightarrow (1)$

Diff (1) partially w.r.to 'x'

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Diff (1) w.r.to 'y' partially then we get $\frac{\partial r}{\partial y} = \frac{y}{r}$

" " 'z' " " " $\frac{\partial r}{\partial z} = \frac{z}{r}$

$u = r^m$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = mr^{m-1}\cdot\frac{x}{r} = mr^{m-2}x.$$

|||y $\frac{\partial u}{\partial y} = mr^{m-2}y$ & $\frac{\partial u}{\partial z} = mr^{m-2}z$.

$$\frac{\partial^2 u}{} = \frac{\partial}{}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[mr^{m-2}x\right] = m\left[r^{m-2}\cdot 1 + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[m r^{m-2} x\right] = m\left[\quad\right]$$

$$= m\left[r^{m-2} + (m-2) r^{m-3} x \cdot \frac{x}{r}\right] = m\left[r^{m-2} + (m-2) r^{m-4} x^2\right].$$

$$|||^{ly}\ \frac{\partial^2 u}{\partial y^2} = m\left[r^{m-2} + (m-2) r^{m-4} y^2\right] \ \& \ \frac{\partial^2 u}{\partial z^2} = m\left[r^{m-2} + (m-2) r^{m-4} z^2\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m\left[3 r^{m-2}\right] + m(m-2) r^{m-4}(x^2+y^2+z^2)$$

$$= 3m r^{m-2} + m(m-2) r^{m-4} r^2$$

$$= 3m r^{m-2} + m(m-2) r^{m-2}$$

$$m r^{m-2}\left[3 + m-2\right] = r^{m-2} m (m+1)$$

→ If $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

→ If $u = x\log xy$ & $\underbrace{x^3+y^3+3xy = 1}$, then find $\underbrace{\frac{du}{dx}}_{\text{total derivative}}$

sol: $Z = f(x,y)$, $x = x(t)$, $y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$$u = x\log xy$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy}(y) + \log xy \cdot 1$$

$$\frac{\partial u}{\partial x} = 1 + \log xy.$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{\cancel{x}y} \cdot \cancel{x}. = \frac{x}{y}.$$

$$x^3 + y^3 + 3xy = 1$$

Diff w.r.to 'x'

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left[x\frac{dy}{dx} + y\right) = 0$$

$$3x^2 + (3y^2+3x)\frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx} = \frac{-(3x^2+3y)}{3y^2+3x}.$$

$$\ldots \quad x\left(-\frac{(x^2+y)}{\ldots}\right)$$

$\frac{\partial}{\partial y}$ $\quad xy$ $\quad ()$

$$\frac{dy}{dx} = (1+\log xy)\cdot 1 + \frac{x}{y}\left(-\frac{(x^2+y)}{y^2+x}\right).$$

→ If $z = u^2+v^2$, $u = at^2$, $v = 2at$, find $\frac{dz}{dt}$

Sol: $\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$, $\frac{\partial z}{\partial u} = 2u$, $\frac{\partial z}{\partial v} = 2v$

$= (2u)(2at) + (2v)(2a)$ $\qquad \frac{du}{dt} = 2at$, $\frac{dv}{dt} = 2a$

$= 4uat + 4av$

→ If $z = f(x,y)$ & $\boxed{z = 0}$ → $f = 0$

$dz = 0$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

$$0 = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \Rightarrow \frac{\partial z}{\partial x}dx = -\frac{\partial z}{\partial y}dy$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}}.$$

→ If $x^y = y^x$ then find $\frac{dy}{dx}$. $\qquad$ | $\quad z = x^2+y^2$

Sol: $x^y - y^x = 0$

let $f = x^y - y^x = 0$ —— (1)

$\Rightarrow f = 0$

then $\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$

$\frac{\partial f}{\partial x} = yx^{y-1} - y^x\log y.$

$\frac{\partial f}{\partial y} = x^y\log x - xy^{x-1}$

$$\Rightarrow \frac{dy}{dx} = -\frac{(yx^{y-1} - y^x\log y)}{x^y\log x - xy^{x-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(y x^{y-1} - y^x \log y)}{x^y \log x - x y^{x-1}}$$

→ If $(\sin y)^x = (\cos x)^y$ then find $\frac{dy}{dx}$.

→ <u>Jacobian</u> :-

Let $u = u(x,y)$ & $v = v(x,y)$. Then these relations constitute a transformation from (u,v) to (x,y)

Then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ (or) $\begin{vmatrix} u_x & u_y \\ v_x & v_x \end{vmatrix}$

is called as Jacobian of u,v w.r.to x,y.

(or) $J\left[\frac{u,v}{x,y}\right]$ or $\frac{\partial(u,v)}{\partial(x,y)}$ → Jacobian of u,v w.r.to x,y.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

If $u = u(x,y,z)$, $v = v(x,y,w)$ & $w = w(x,y,z)$ then Jacobian of u,v,w w.r.to x,y,z is

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

$$\boxed{\iint \left[\frac{x^3-y^3}{x^2+y^2}\right] dx\,dy}$$

$$\int \frac{x^3-y^3}{x^2+y^2}$$

$x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\cancel{r^3}(\cos^3\theta - \sin^3\theta)}{\cancel{r^2}}$$

$r(\cos^3\theta - \sin^3\theta)\, dr\,d\theta$

$x = x(r,\theta)$ $y = y(r,\theta)$

<u>Properties</u> :

1) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$.

2) If u,v are functions of r,s and r,s are functions of x,y

then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)}$.

→ If $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$

A. $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

→ If

Sol: $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\frac{\partial(r,\theta)}{\partial(x,y)}$$

$x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\, 2x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} = \frac{1}{(x^2+y^2)^{1/2}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1$$

→ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

Sol: $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$, $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$, $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$\partial z = \cos\theta$, $\frac{\partial z}{} = -r\sin\theta$, $\frac{\partial z}{\partial \phi} = 0$

$\dfrac{\partial z}{\partial r} = \cos\theta$, $\quad \dfrac{\partial z}{\partial \theta} = -r\sin\theta$, $\quad \dfrac{\partial z}{\partial \phi} = 0$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right] + r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right] + 0$$

$$= r^2\cos\theta\sin\theta\cos\theta\left[\cos^2\phi+\sin^2\phi\right] + r\sin\theta\, r\sin^2\theta\left[\cos^2\phi+\sin^2\phi\right]$$

$$r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta\left[\cos^2\theta+\sin^2\theta\right]$$

$$= r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

then $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}}$ $\quad (\because$ property 1$)$

$$\Rightarrow \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta} = \frac{\operatorname{cosec}\theta}{r^2}.$$

H.W

→ If $u = \dfrac{x+y}{1-xy}$, $\theta = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u,\theta)}{\partial(x,y)}$

→ If $x+y+z=u$, $y+z=uv$, $z=uvw$ then find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: $\quad x+y+z=u$ — ①

$x+y+z=u$

$x+uv=u$ $(\because$ from ②$)$

Sol: $x+y+z=u \quad -(1)$

$y+z=uv \quad -(2)$

$z=uvw \quad -(3)$

$x+y+z=u$

$x+uv=u$ ($\because$ from (2))

$\Rightarrow x=u-uv$

$z=uvw$

$y+z=uv$

$y+uvw=uv$ ($\because$ from (3))

$\Rightarrow y=uv-uvw$

$$\left.\begin{aligned} x &= u-uv \\ y &= uv-uvw \\ z &= uvw. \end{aligned}\right\}$$

$\frac{\partial x}{\partial u}=1-v$, $\frac{\partial x}{\partial v}=-u$

$\frac{\partial x}{\partial w}=0$, $\frac{\partial y}{\partial u}=v-vw$

$\frac{\partial y}{\partial v}=u-uw$, $\frac{\partial y}{\partial w}=-uv$

$\frac{\partial z}{\partial u}=vw$, $\frac{\partial z}{\partial v}=uw$, $\frac{\partial z}{\partial w}=uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)\left[(u-uw)uv+(uv)(uw)\right]+ u\left[(v-vw)uv+uv(vw)\right]$$

$$=(1-v)\left[u^2v-\cancel{u^2vw}+\cancel{u^2vw}\right]+u\left[uv^2-\cancel{uv^2w}+\cancel{uv^2w}\right]$$

$$\Rightarrow u^2v(1-v)+u^2v^2 = u^2v\left[1-v+v\right] = u^2v.$$

→ If $u=x^2-y^2$, $v=2xy$, $x=r\cos\theta$, $y=r\sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}=?$

→ If $x=e^r\sec\theta$, $y=e^r\tan\theta$, show that $\frac{\partial(x,y)}{\partial(r,\theta)}\times\frac{\partial(r,\theta)}{\partial(x,y)}=1.$

→ If $x=\frac{u^2}{v}$ & $y=\frac{v^2}{u}$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

→ Functional dependence.

Let $u=f(x,y)$ & $v=g(x,y)$ are two differentiable functions

... by a relation $F(u,v)=0$

→ Functional dependence:
Let $u = f(x,y)$ & $v = g(x,y)$ are two differentiable functions.
Suppose these functions are connected by a relation $F(u,v) = 0$
Then u & v are functionally dependent or functionally related on
one another if u_x, u_y, v_x, v_y are all not zero at a time.

→ If $u = f(x,y)$ & $v = g(x,y)$ then u & v are functionally dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

$u = 2v$

$\boxed{u + 3v = 0}$

→ If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$, then u, v are not functionally dependent.

→ check whether the following functions are functionally dependent or not, if so find the relation b/w them.

① If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.

sol

$$u_x = \frac{(1-xy) - (x+y)(0-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}.$$

$$u_y = \frac{(1-xy) - (x+y)(0-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$v_x = \frac{1}{1+x^2} + 0, \quad v_y = 0 + \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

⇒ u and v are functionally dependent

$\tan^{-1}x + \tan^{-1}y$

$\Rightarrow$ u and v

$$u = \frac{x+y}{1-xy}, \quad v = \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\boxed{v = \tan^{-1}u}$$

2) $u = e^x \sin y$, $v = e^x \cos y$.

Sol: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix} = -e^{2x}\sin^2 y - e^{2x}\cos^2 y$$

$$= -e^{2x}(\sin^2 y + \cos^2 y)$$

$$= -e^{2x} \neq 0$$

$\Rightarrow$ u & v are not functionally dependent.

3) $u = \dfrac{x}{y}$, $v = \dfrac{x+y}{x-y}$

4) $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$.