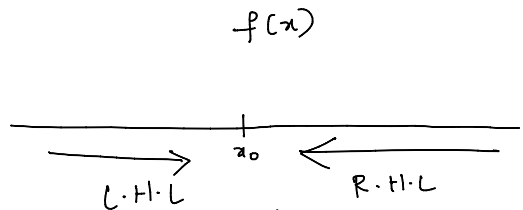
Single Variable Calculus:



$y = f(x)$ — y dependent, x independent

limit of $f(x)$ at x_0:

As x approaches x_0, then $f(x)$ should approach a unique & finite value then we say limit of $f(x)$ exists at x_0.

$$\lim_{x \to x_0} f(x)$$

Continuity:

$f(x)$ at x_0

1) $\lim_{x \to x_0} f(x)$ exists
2) $f(x_0)$ exists
3) $\lim_{x \to x_0} f(x) = f(x_0)$

} f is continuous at x_0.

Derivative: $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists}$$

Differentiability of $f(x)$ on an interval: (a, b) or $[a, b]$

$f(x)$ is differentiable on an open interval (a, b) (finite or infinite) if it has derivative at each point of the interval (a, b).

$f(x)$ is differentiable on $[a, b]$, if it is differentiable at interior of (a, b) & if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{(Right-hand derivative at a)}$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{(Left hand derivative at b).}$$

Exists

Chain rule of differentiation

$y = f(u)$ & $u = g(t)$

$$\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dt}$$

Rolle's theorem:

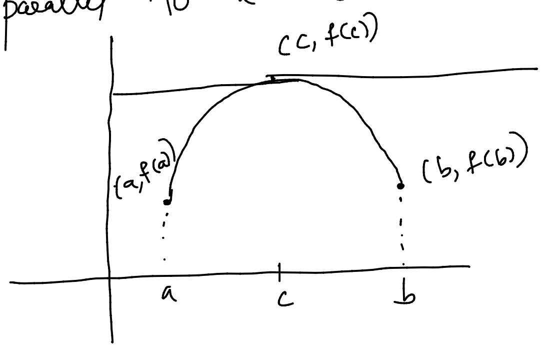
... $[a, b]$ & differentiable on (a, b) & $f(a) = f(b)$

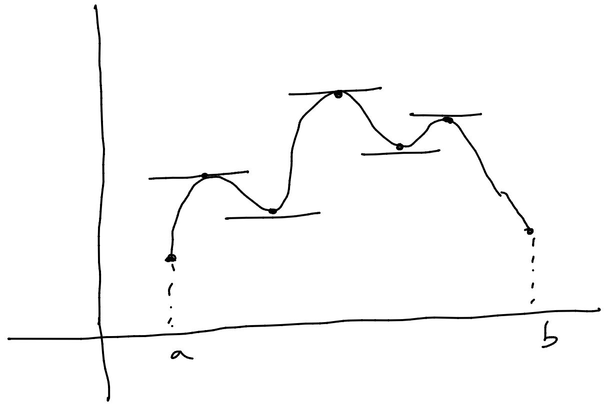
Rolle's theorem:

If $f(x)$ is continuous on $[a,b]$ & differentiable on (a,b) & $f(a)=f(b)$ then $\exists$ atleast one point $c \in (a,b) \ni f'(c)=0$.

Geometrical interpretation:

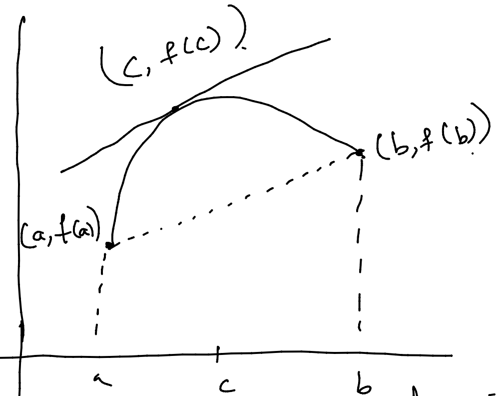
If $f(x)$ is continuous (no breaks) on $[a,b]$ & differentiable on (a,b) (we can draw a unique tangent at each & every point in (a,b)) & $f(a)=f(b)$ [the end points of the curve are at same height from x-axis]. then $\exists$ atleast one c in $(a,b) \ni$ tangent drawn at $(c, f(c))$ is parallel to x-axis.





Lagrange's Mean value theorem (Mean value theorem):

If $f(x)$ is continous on $[a,b]$ & diff on (a,b) then $\exists$ atleast one $c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



Geometrical interpretation

If $f(x)$ is cont in $[a,b]$ & diff in (a,b) then $\exists$ atleast one $c \in (a,b)$ $\ni$ slope of tangent drawn at $(c, f(c))$ is same as slope of straight line (chord) joining the end points of the curve.

① Verify Rolle's theorem for the following functions

i) $f(x) = (x+2)^3(x-3)^4$ in $[-2,3]$

Sol $f(x)$ — polynomial function.

Since every polynomial function is cont & diff Everywhere.

$\therefore$ $f(x)$ is cont in $[-2,3]$ & diff in $(-2,3)$.

$f(-2) = 0$, $f(3) = 0$

$\therefore$ f(x) is cont ...

$f(-2) = 0$, $f(3) = 0$

$f(-2) = f(3)$.

Hence Rolle's theorem is applicable for $f(x)$

By Rolle's theorem $\exists$ atleast one $c \in (-2,3) \ni f'(c) = 0$

$$f'(x) = 3(x+2)^2(x-3)^4 + (x+2)^3 4(x-3)^3$$

$$f'(c) = 0 \Rightarrow 3(c+2)^2(c-3)^4 + 4(c+2)^3(c-3)^3 = 0$$

$$\Rightarrow (c+2)^2(c-3)^3[3(c-3) + 4(c+2)] = 0$$

$$(c+2)^2(c-3)^3[7c-1] = 0$$

$c \in (-2,3)$

$$c = \cancel{-2}, \cancel{3}, \frac{1}{7}$$

$$c = \frac{1}{7} \in (-2,3)$$

Hence Rolle's theorem is verified.

ii) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

iii) $f(x) = \tan x$ in $[0, \pi]$

iv) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$

v) $f(x) = (x-a)^m(x-b)^n$, m, n are +ve integers in (a,b)

vi) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$.

$\rightarrow$ Verify Lagrange's mean value theorem for the following

1) $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Sol: $f(x)$ — polynomial function

$f(x)$ is cont in $[0,4]$ & diff in $(0,4)$

L.M.T is applicable

$\therefore$ By L.M.T $\exists$ atleast one $c \in (0,4) \ni f'(c) = \frac{f(4) - f(0)}{4-0}$

$\Rightarrow f'(x) = 3x^2 - 2x - 5$, $f(4) = 31$

$f(0) = 3$

$$3c^2 - 2c - 5 = \frac{31-3}{4} = 7 \Rightarrow 3c^2 - 2c - 12 = 0$$

$$\ldots \pm \sqrt{4+144} \ldots = \frac{2 \pm 2\sqrt{37}}{\ldots} = \frac{1 \pm \sqrt{37}}{\ldots}$$

$$c = \frac{2 \pm \sqrt{4+144}}{6} = \frac{2 \pm 2\sqrt{37}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

$$c = \frac{1+\sqrt{37}}{3} \in (0,4) \qquad c = \frac{1-\sqrt{37}}{3} \notin (0,4)$$

Hence L.M.T is verified.

2) $f(x) = \log x$ in $[1, e]$

3) $f(x) = x^{1/3}$ in $[-1, 1]$

4) If $a<b$, then prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ using Lagrange's mean value theorem (LMT) & also deduce that

a) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$ $\qquad a<b$

b) $\frac{5\pi+4}{20} < \tan^{-1} 2 < \frac{\pi+2}{4}$

Sol: $a<b$ — (a,b)

Let $f(x) = \tan^{-1} x$ in (a,b)

$f(x)$ is cont & differentiable.

By L.M.T $\exists$ atleast one $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$f'(x) = \frac{1}{1+x^2}, \quad f'(c) = \frac{1}{1+c^2}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \quad \text{— (1)}$$

$$c \in (a,b)$$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

Take $a=1$, $b=\frac{4}{3}$

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{1}{\cancel{3}}\times\frac{\cancel{9}^{3}}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{3\times 2}$$

$$\frac{3}{25}+\frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6}+\frac{\pi}{4}$$

→ for $x>0$, show that $1+x < e^{x} < 1+xe^{x}$ by using L.M.T

Sol $(0,x)$, $f(x)=e^{x}$

→ $f(x) = (x-a)^{m}(x-b)^{n}$, m,n are +ve integers.

Sol: $f(x)$ is cont in $[a,b]$ & diff in (a,b)

$f(a)=0$, $f(b)=0$ ⇒ $f(a)=f(b)$

Rolle's theorem is applicable.

Then ∃ atleast one c in (a,b) → $f'(c)=0$

$$f'(x) = (x-a)^{m}n(x-b)^{n-1} + (x-b)^{n}m(x-a)^{m-1}\cdot 1$$

$$f'(c)=0 \Rightarrow n(c-a)^{m}(c-b)^{n-1} + m(c-b)^{n}(c-a)^{m-1} = 0$$

$$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}\left[n(c-a)+m(c-b)\right]=0$$

$$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}\left[(m+n)c-(mb+na)\right]=0$$

$c=a$, $c=b$, $c=\frac{mb+na}{m+n}$

× ×

$c=\frac{mb+na}{m+n}$ ⇒ c is dividing a & b internally in the ratio $m:n$

⇒ $c\in(a,b)$

Hence Rolle's theorem is verified.

Increasing and decreasing functions

Increasing and decreasing functions

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).

- If $f'(x)>0$ at each point $x\in(a,b)$, then f is increasing on $[a,b]$.

$f(x)$ (a,b)

$f'(x)>0$ ∀ $x\in(a,b)$ → f ↑

$f'(x)<0$ ∀ $x\in(a,b)$ → f ↓

- If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

$f'(x) > 0 \ \forall \ x \in (a,b) \rightarrow f \uparrow$

$f'(x) < 0 \ \forall \ x \in (a,b) \rightarrow f \downarrow$

$f'(x) = 0$ for some 'x' $\downarrow$ critical point.

Critical point

Critical point is a point where $f'(x) = 0$

1. Find the critical points of the function $f(x) = x^3 - 12x - 5$ and find the intervals in which $f(x)$ is increasing or decreasing.
2. Determine the intervals in which the function $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$ is increasing or decreasing. H.W

(1) $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$f'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

$-\infty \quad -2 \quad 2 \quad \infty$

$(-\infty, -2) \quad (-2, 2) \quad (2, \infty)$

In $(-\infty, -2)$, $f'(x) > 0 \Rightarrow f$ is increasing on $(-\infty, -2)$

In $(-2, 2)$, $f'(x) < 0 \Rightarrow f$ is decreasing on $(-2, 2)$

In $(2, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing on $(2, \infty)$.

$f'(x) = 3x^2 - 12$

$f'(-5) = 3(25) - 12 > 0$

$f'(-9) = 3(81) - 12 > 0$

$f'(0) = -12 < 0$

$f'(-1) = 3 - 12 < 0$

$f'(3) = 3(9) - 12 > 0$

$f'(4) = 3(16) - 12 > 0$

Contd..

First derivative test for local extrema

Let $x = c$ be the critical point of $f(x)$.

- If $f'(x)$ changes sign from -ve to +ve at $x = c$, then $f(x)$ is said to have local minimum. i,e if $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $f(x)$ is said to have local minimum or relative minimum.
- If $f'(x)$ changes sign from +ve to -ve at $x = c$, then $f(x)$ is said to have local maximum. i,e if $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $f(x)$ is said to have local maximum or relative maximum.
- If $f'(x)$ does not change sign at c, then f has no local extremum at c.

$x = c \ — \ f(x)$

$f'(x) < 0 \ \forall \ x < c \quad f'(x) > 0 \ \forall \ x > c$

$x < c$ c $x > c$

$f'(x) < 0$ (-ve) $f'(x) > 0$ (+ve)

$x < c$ c $x > c$

$f'(x) > 0$ (+ve) $f'(x) < 0$ (-ve)

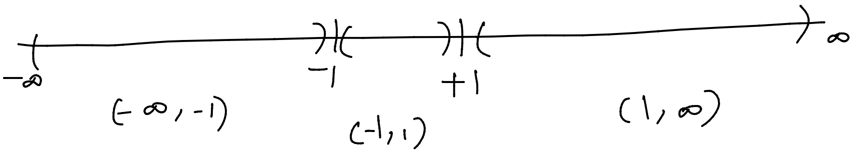
- Examine the function $f(x) = x^3 - 3x + 3$ for relative maximum and minimum values.

Sol $f(x) = x^3 - 3x + 3$

$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ are critical points.

Sol $f(x) = x^3 - 3x + 3$

$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ are critical points.



$f'(x) = 3x^2 - 3$

$f'(-2) = 3(4) - 3 > 0$

$f'(-4) = 3(16) - 3 > 0$

$f'(0) = -3 < 0$

$f'(\frac{1}{2}) = 3(\frac{1}{4}) - 3 < 0$

$f'(2) = 3(4) - 3 > 0$

for $x < -1$, $f'(x) > 0 \Rightarrow f$ is increasing in $(-\infty,-1)$

for $x > -1$ & $x < 1$, $f'(x) < 0 \Rightarrow f$ is decreasing in $(-1,1)$

for $x > 1$, $f'(x) > 0 \Rightarrow f$ is increasing in $(1,\infty)$

$x = -1$ for $x < -1$, $f'(x) > 0$ (+ve) & for $x > -1$, $f'(x) < 0$ (−ve)

$\therefore$ $f(x)$ has local maximum at $x = -1$

$x = 1$ for $x < 1$, $f'(x) < 0$ (−ve) & for $x > 1$, $f'(x) > 0$ (+ve)

$\therefore$ $f(x)$ has local minimum at $x = 1$

local maximum value is $f(-1) = 5$

" minimum value is $f(1) = 1$

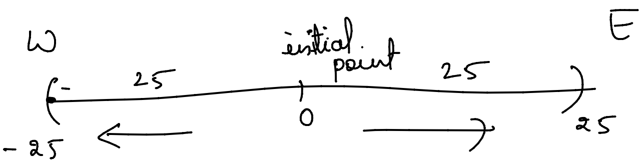
Contd..

(2) The population of rabbits(in hundreds) after t years in a certain area is given by $P(t) = t^2 \ln(3t) + 6$. Determine the population ever decreases in first two years.

(3) Suppose that the elevation above sea level of a road is given by the function $E(x) = 500 + \cos\frac{x}{4} + \sqrt{3}\sin\frac{x}{4}$. Here x is in miles. Assume that if x is +ve we are to the east of the initial point of measurement & if x is - ve we are to the west of the initial point of measurement. If we start 25 miles west to the initial point of measurement and drive until we are 25 miles to the east of the initial point of measurement. How many miles of our drive were we driving up an incline.

$E(x) = 500 + \cos\frac{x}{4} + \sqrt{3}\sin\frac{x}{4}$

$x > 0 \longrightarrow E$



(2) Sol $p(t) = t^2\ln(3t) + 6$, in $[0,2]$

$p'(t) = 0 \Rightarrow 2t\ln(3t) + t^{\cancel{2}}\frac{1}{\cancel{3t}}\cdot\cancel{3} = 0$

$$\Rightarrow \quad 2t\ln(3t) + t = 0$$

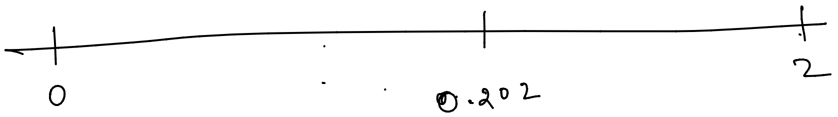
$$\Rightarrow \quad t\left[2\ln(3t) + 1\right] = 0$$

$$\Rightarrow \quad t = 0, \quad \ln(3t) = -\tfrac{1}{2}$$

$$3t = e^{-1/2}$$

$$t = 0, \quad t = \tfrac{1}{3}e^{-1/2} = \frac{0.606}{3} = 0.202$$

$t = 0,\ 0.202$ are critical values



In $(0, 0.202)$, $p'(t) < 0 \Rightarrow p(t)$ is decreasing in (0.202)

$p'(t) = 2t\ln(3t) + t$

$p'(0.1) = 2(0.1)\ln(0.3) + 0.1 < 0$

In $(0.202, 2)$, $p'(t) > 0 \Rightarrow p(t)$ is increasing in $(0.202, 2)$

$p'(1) = 2\ln(3) + 1 > 0$

The population of the rabbits decreases for a short period i.e $(0, 0.202)$.

& continues to increase forever.

contd..

Def:

Let $f(x)$ is differentiable on an open interval I, then

- $f(x)$ is **Concave upward** on an open interval I, if $f'(x)$ is increasing on I.
- $f(x)$ is **Concave downward** on an open interval I, if $f'(x)$ is decreasing on I.

Second derivative test for concavity

Let $f(x)$ be twice differentiable on an open interval I.

- If $f''(x) > 0\ \forall\ x \in I$, then the graph of f over I is **Concave upward**.
- If $f''(x) < 0\ \forall\ x \in I$, then the graph of f over I is **Concave downward**.

open interval I

$f'(x)$ is $\uparrow$ I — concave $\uparrow$

$f'(x)$ is $\downarrow$ I $\rightarrow$ concave $\downarrow$

$(\ \ ) \rightarrow f'$ is increasing

$f''(x) > 0$

$f \rightarrow f'(x) > 0$

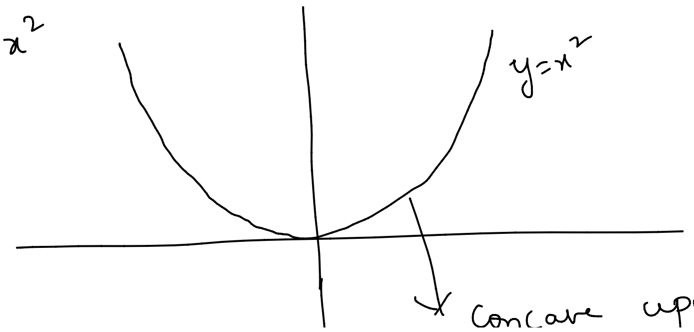
$f''(x) > 0 \rightarrow f'$ is increasing

$I \rightarrow$ concave upward

$f''(x) < 0 \rightarrow$ concave downward

I

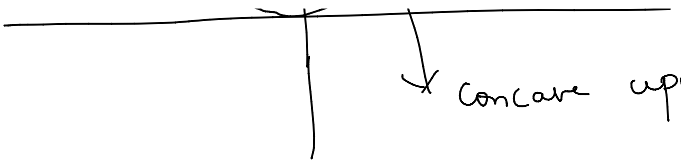
① $f(x) = x^2$



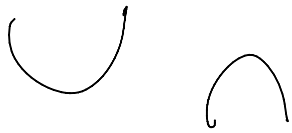
Concave upward.

$f'(x) = 2x$

$f''(x) = 2 > 0\ \forall\ x$



Concave upward.



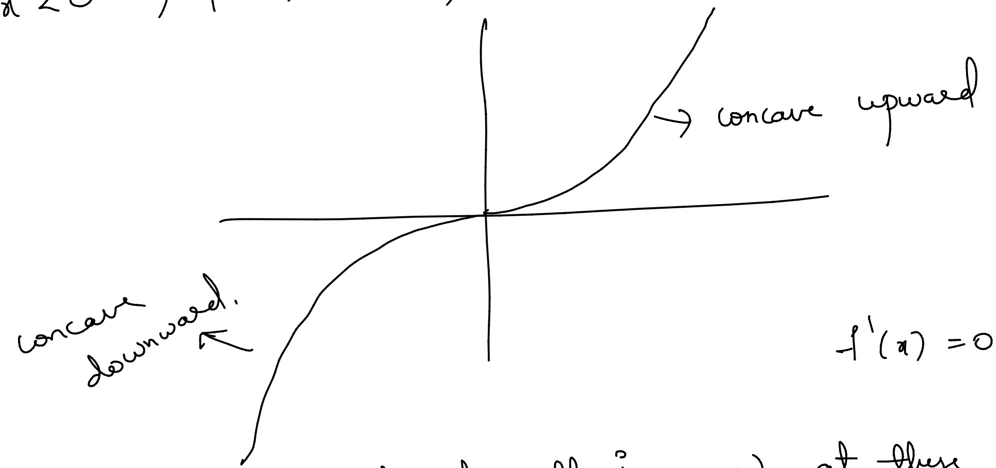
(2) $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

for $x > 0$, $f''(x) > 0 \Rightarrow f$ is concave upward in $(0, \infty)$

for $x < 0$, $f''(x) < 0 \Rightarrow f$ is " downward in $(-\infty, 0)$



$f'(x) = 0 \rightarrow x$ – critical

for $f''(x) = 0$, x — points of inflexion $\Rightarrow$ at these points f is neither concave upward nor concave downward.

Contd

Second derivative test for local extrema

Let $f''(x)$ is continuous on an open interval that contains $x = c$, a critical point

- If $f''(c) < 0$, then f has a local maximum at $x = c$.
- If $f''(c) > 0$, then f has a local minimum at $x = c$.
- If $f''(c) = 0$, then the test fails. The function f may have a local maximum or local minimum or neither.

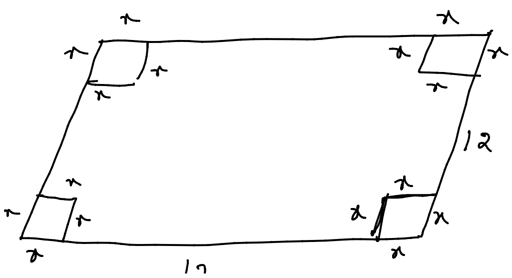
$f''(x)$ $\quad$ $x = c$

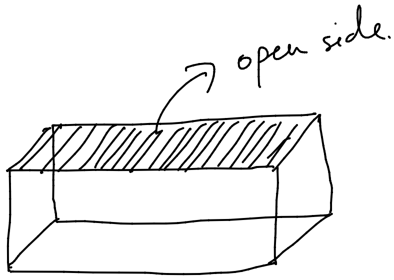
$f''(c) < 0, \rightarrow$ local max at c

$f''(c) > 0, \rightarrow$ " min at c.

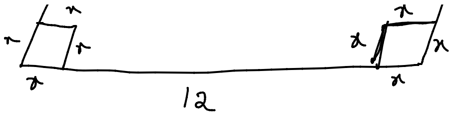
$f''(c) = 0 \rightarrow$

- An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?





sheet



Let 'x' be the dimension of square, cut from the sheet

length of the box = $12-2x$

width of the box = $12-2x$

height " = x.

$$V = lbh = (12-2x)(12-x)\,x = (12-2x)^2 x.$$

Here we find 'x' such that V is max.

$$V' = 0 \Rightarrow (12-2x)^2 \cdot 1 + 2x(12-2x)(-2) = 0$$

$$(12-2x)\left[12-2x-4x\right] = 0$$

$$\Rightarrow 12-2x=0\ ,\quad 12-6x=0$$

$$x = 6\ ,\quad x = 2.$$

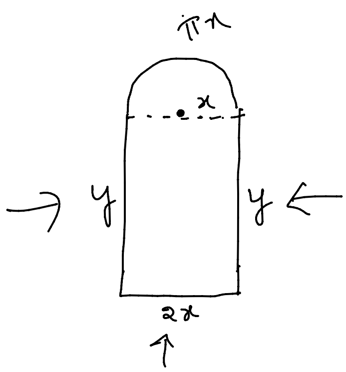
with $x = 6$ we cannot form a box.

$x = 2$ is the dimension of the square that is cut from the sheet to make open top box.

Contd..

1. ■ A window has the form of a rectangle surmounted by a semicircle. If the perimeter of the window is 40 ft, then find its dimensions so that greatest amount of light is admitted through it.
2. ■ A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions? H-W
3. ■ Find the local extrema for $f(x) = \sin x(1+\cos x)$ in $(0, \pi)$. H-W

1 Sol



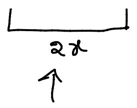
Perimeter of window is 40 ft

$$2x + 2y + \pi x = 40 \quad \text{—(1)}$$

We get more light when area of window is max.

A = Area of rectangle + area of semi circle.

is max.



A = Area of rectangle + area of semi circle.

$A = y \times 2x + \frac{1}{2}\pi x^2 = 2xy + \frac{1}{2}\pi x^2$ — (2)

from (1) $2y = 40 - \pi x - 2x$. — (3)

sub (3) in (2)

$A = x(40 - \pi x - 2x) + \frac{1}{2}\pi x^2$.

$A = 40x - \pi x^2 - 2x^2 + \frac{1}{2}\pi x^2$

$A' = 40 - 2\pi x - 4x + \pi x$.

$A' = 0 \Rightarrow 40 - \pi x - 4x = 0$

$\Rightarrow x(\pi + 4) = 40$

$x = \frac{40}{\pi+4}$ → critical point

(1) $\Rightarrow 2y = 40 - \pi x - 2x = 40 - x(\pi + 2) = 40 - \frac{(\pi+2)40}{\pi+4}$

$2y = 40\left[\frac{\cancel{\pi}+4-\cancel{\pi}-2}{\pi+4}\right] = \frac{80}{\pi+4}$

$y = \frac{40}{\pi+4}$

$x = \frac{40}{\pi+4}$.

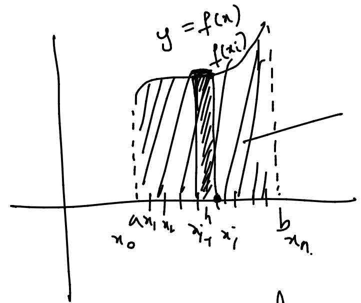
$A'' = -\pi - 4 < 0$

A is max. at $x = \frac{40}{\pi+4}$

Contd..

Def:
If $y = f(x)$ is non-negative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)dx.$$



$\int_a^b f(x)dx.$

$x_i - x_{i-1} = h. = \delta x_i$

$\boxed{f(x_i)\,\delta x_i}$

$f(x_1)\delta x_1 + f(x_2)\delta x_2) + \cdots + f(x_n)\delta x_n)$

$\sum^n f(x_i)\delta x_i \qquad n \to \infty$

Def:
If f is integrable on $[a, b]$, then its average value on $[a, b]$, also called its mean, is

$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots$

$\sum_{i=1}^{n} f(x_i)\delta x_i \qquad n \to \infty$

$\int_a^b f(x)\,dx.$

Def:

If f is integrable on $[a,b]$, then its average value on $[a,b]$, also called its mean, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad ✓$$

$T(t)$

(1) ■ Determire the average value of the function

$$f(x) = x^2 - 5x + 6\cos \pi x \text{ in } \left[-1, \frac{5}{2}\right]$$

H-W

(2) ■ Find the average value of the function

$$f(x) = \sin 2x e^{1-\cos 2x} \text{ in } [-\pi, \pi]$$

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. = \frac{1}{\pi+\pi}\int_{-\pi}^{\pi} \sin 2x \, e^{1-\cos 2x}\,dx.$$

Let $1-\cos 2x = t.$

$2\sin 2x = dt \Rightarrow \sin 2x = \frac{dt}{2}$

$x \to \pi, \quad 1-\cos 2\pi = t \Rightarrow t = 0$

$x \to -\pi, \quad 1-\cos(-2\pi) \Rightarrow t = 0$

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx.$$

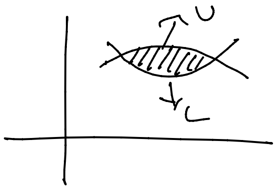
$$\text{av}(f) = 0$$

→ <u>Area between the curves</u>:

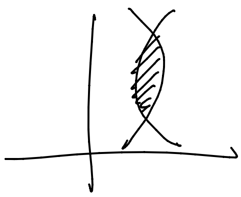
Let $y_1 = f(x)$ & $y_2 = g(x)$, $f(x) \geq g(x)$ in $[a,b]$

then the area b/w the curves $f(x)$ & $g(x)$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$



$$A = \int_a^b [(\text{upper curve}) - (\text{lower curve})]\,dx$$



$$A = \int_a^b (\text{Right curve}) - (\text{left curve})]\,dx.$$

(1) find the area of the region enclosed by the parabola $y = 2 - x^2$

and the line $y = -x$

1) find the area of the region enclosed by

& line $y = -x$

Sol: point of intersection of $y = 2 - x^2$ & $y = -x$

$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0$$

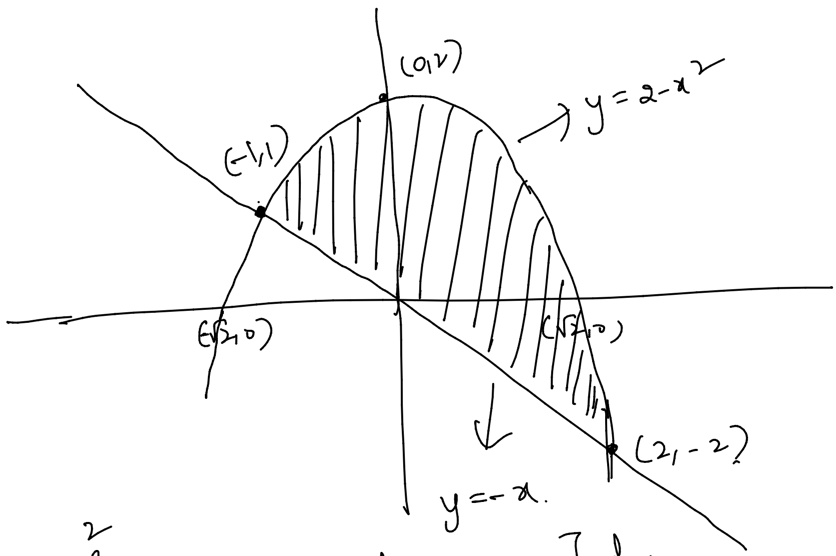
$$\Rightarrow x = \frac{1 \pm \sqrt{1+8}}{2} = 2, -1$$

$x = 2,\ y = -x \Rightarrow y = -2$ $\quad$ $(2,-2)$ & $(-1,1)$ are pts of intersection of curves.

$x = -1 \Rightarrow y = 1$

At $x = 0$, $y = 2$ $\quad$ $(0,2)$ is pt of inter with y-axis of $y = 2 - x^2$

At $y = 0$, $x = \pm\sqrt{2}$, $(\sqrt{2}, 0)$ & $(-\sqrt{2}, 0)$ are pts of inter of $y = 2 - x^2$ with x-axis.



$$A = \int_{x=-1}^{2} [\text{upper curve} - \text{lower curve}]\, dx$$

$$A = \int_{-1}^{2} (2 - x^2) - (-x)\, dx. = \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$A = \frac{9}{2} \text{ sq. units}$$

2) find the area of the region bounded by $y = \sqrt{x}$, $y = x - 2$, x-axis in first quadrant.

Sol: $y = \sqrt{x} \Rightarrow x = y^2$ — (1)

$y = x - 2 \rightarrow$ — (2)

x-axis $\rightarrow y = 0$ — (3)

$\left.\begin{array}{l} x = y^2 \\ y = x - 2 \end{array}\right\}$ $y = y^2 - 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1, 2.$

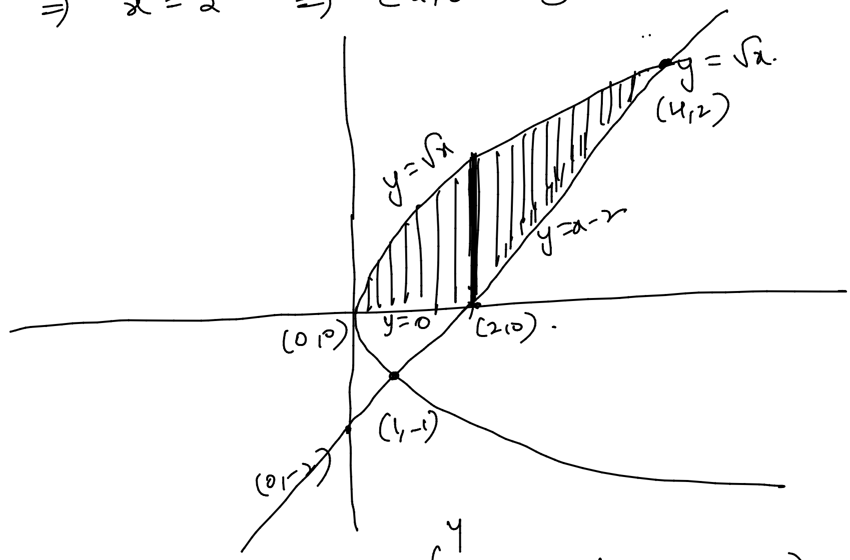
for $y = -1 \Rightarrow x = 1 \Rightarrow (1,-1)$

for $y = 2 \Rightarrow x = 4 \Rightarrow (4,2)$ $\Big\}$ Points of intersection of (1) & (2)

$y=\sqrt{x}\,,\quad x=0 \Rightarrow y=0 \Rightarrow (0,0)$

$y=x-2,\quad x=0\,, \Rightarrow y=-2 \Rightarrow (0,-2)$ } points of intersection of ② with

$y=0 \Rightarrow x=2 \Rightarrow (2,0)$ } x & y axis



$y=x-2$

$\downarrow$

$x=y+2$

$y=\sqrt{x}$

$x=y^2$

$$A=\int_{x=0}^{2}(\text{upper}-\text{lower curve})\,dx+\int_{x=2}^{4}(\text{upper}-\text{lower curve})$$

$$=\int_{x=0}^{2}(\sqrt{x}-0)\,dx+\int_{x=2}^{4}(\sqrt{x}-(x-2))\,dx=\frac{10}{3}\text{ sq.units.}$$

Also we can take

$$A=\int_{y=0}^{2}(\text{Right}-\text{left curve})\,dy=\int_{y=0}^{2}\left[(y+2)-y^2\right]dy=\frac{10}{3}\text{ sq.units}$$

$\rightarrow$ 3) $y=xe^{-x^2}$, $y=x+1$, $x=2$ and y-axis

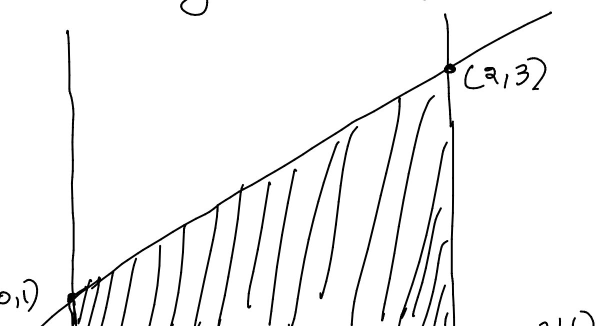
sol: $y=x+1\,,\quad x=0 \Rightarrow y=1 \Rightarrow (0,1)$ } Points of intersection of $y=x+1$

$\quad\quad , y=0 \Rightarrow x=-1 \Rightarrow (-1,0)$ } with x & y axis.

$\left.\begin{array}{l}y=x+1\\x=2\end{array}\right\}$ $x=2,\ y=3 \Rightarrow (2,3)$ is point of intersection of $y=x+1$ & $x=2$

$\left.\begin{array}{l}y=xe^{-x^2}\\x=2\end{array}\right\}$ $x=2\,,\ y=2e^{-4} \Rightarrow y=0.036\,,\quad (2,0.036)$

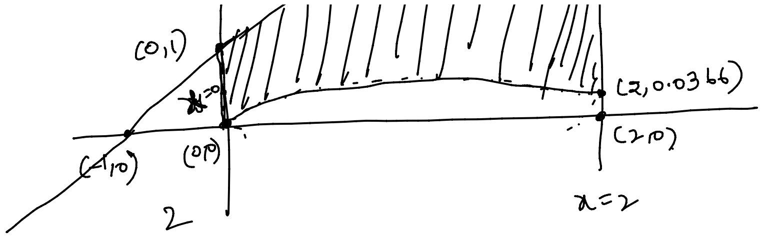
$y=xe^{-x^2}$ passes through $(0,0)$



$x=1$ in $y=xe^{-x^2}$

$y=e^{-1}=0.37$

$y''>0$



$y = e = 0\cdot 37$

$y'' > 0$

$y'' < 0$

$$A = \int_{x=0}^{2} (\text{upper} - \text{lower curve})\, dx.$$

$$A = \int_{x=0}^{2} \left[(x+1) - xe^{-x^2}\right] dx. = \frac{7}{2} + \frac{1}{2e^4} \text{ sq. units.}$$

→ 4) $y = 2x^2+10$, $y = 4x+16$, $x = -2$ & $x = 5$
(1) (2) (3) (4)

Sol: $\left.\begin{array}{l} y = 2x^2+10 \\ y = 4x+16 \end{array}\right\}$ $2x^2+10 = 4x+16 \Rightarrow x^2-2x-3 = 0 \Rightarrow x = 3, -1$

$x = 3 \Rightarrow y = 28$ $(3,28), (-1,12)$ are points of intersection

$x = -1 \Rightarrow y = 12$ of (1) & (2)

$y = 2x^2+10$, $x=0 \Rightarrow y = 10 \Rightarrow (0,10)$ is p-o-i of (1) & y-axis.

$y = 2x^2+10$, $x = -2 \Rightarrow y = 18 \Rightarrow (-2,18)$ " " (1) & (3)

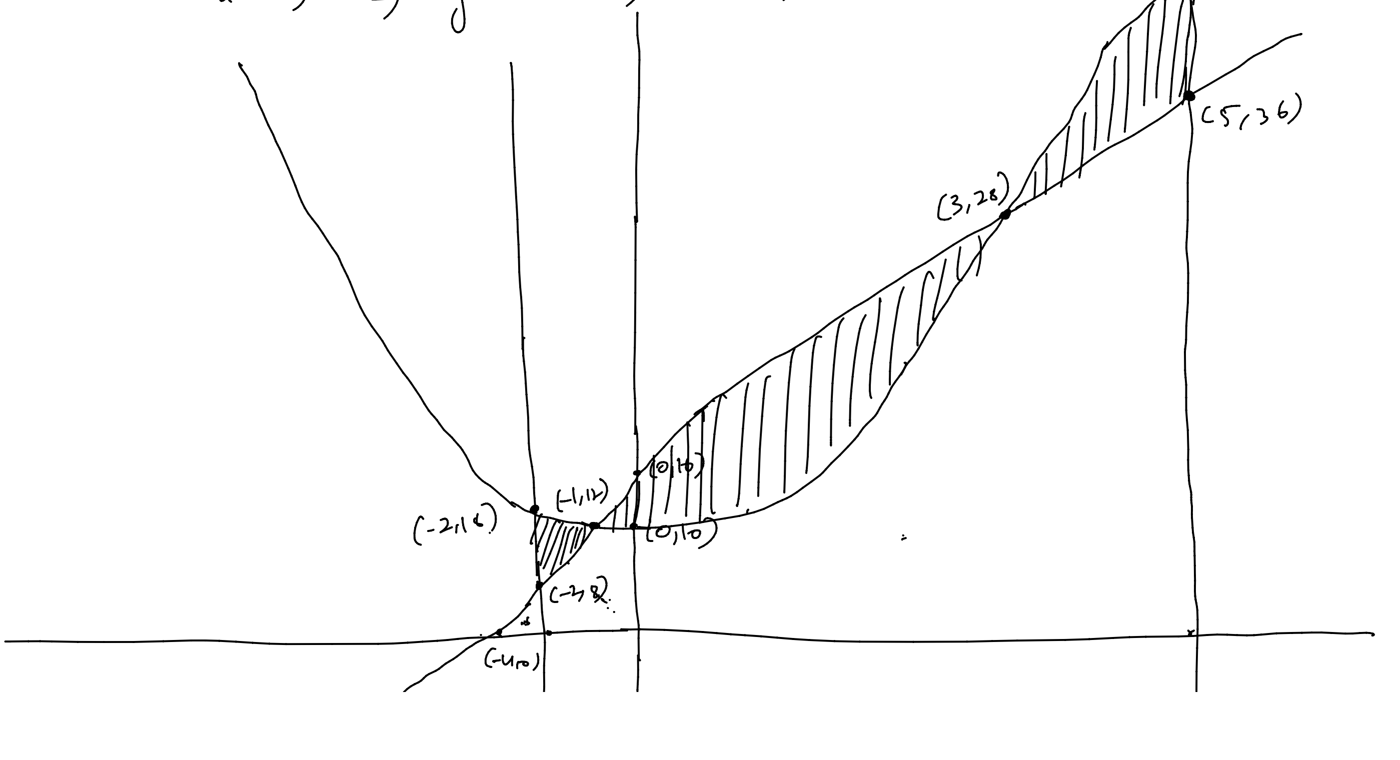
$y = 2x^2+10$, $x = 5 \Rightarrow y = 60 \Rightarrow (5,60)$ " " (1) & (4)

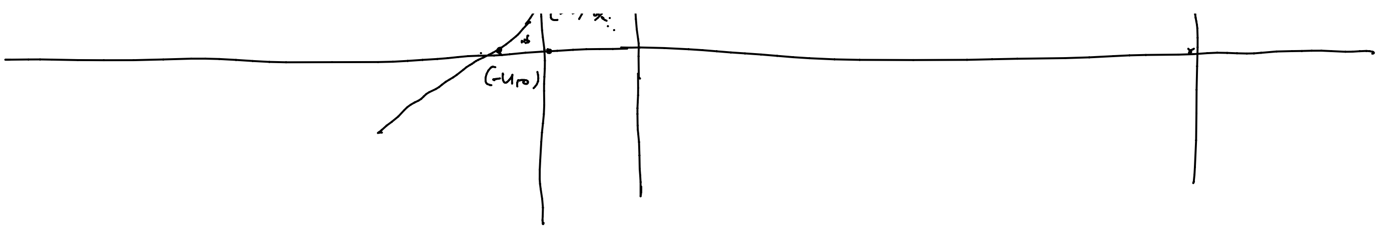
$y = 4x+16$, $x = 0 \Rightarrow y = 16 \Rightarrow (0,16)$ " " (2) & y-axis

, $y = 0 \Rightarrow x = -4 \Rightarrow (-4,0)$ " " (2) & x-axis

, $x = -2 \Rightarrow y = 8 \Rightarrow (-2,8)$ " " (2) & ~~(3)~~

$x = 5 \Rightarrow y = 36 \Rightarrow (5,36)$ " " (2) & (4) — (5,60)





$$A = \int_{x=-2}^{-1} (\text{upper} - \text{lower})\,dx + \int_{x=-2}^{3} (\text{upper} - \text{lower})\,dx + \int_{3}^{5} (\text{upper} - \text{lower})\,dx$$

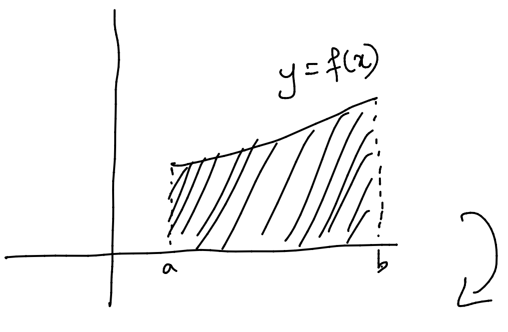
$$\int_{x=-2}^{-1} \left[(2x^2+10) - (4x+16)\right]dx + \int_{-1}^{3} \left[(4x+16) - (2x^2+10)\right]dx + \int_{3}^{5} \left[(2x^2+10) - (4x+16)\right]dx$$

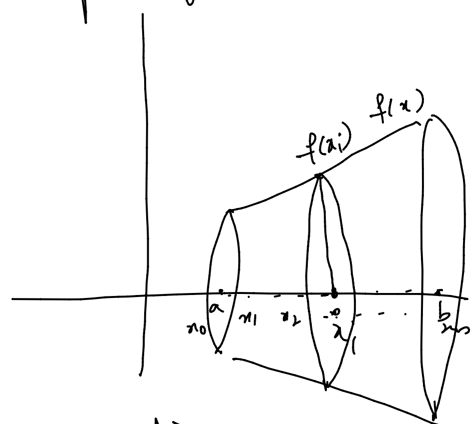
$A = ?$

5) $x = \frac{1}{2}y^2 - 3$, $y = x - 1 \longrightarrow A = 18$ sq. units

6) $x = -y^2 + 10$, $x = (y-2)^2 \rightarrow A = \frac{64}{3}$ sq. units

→ Volume of solid of revolution: (Method of rings / disks)





The volume of the solid formed by rotating a curve or a region about an axis is called volume of solid of revolution.

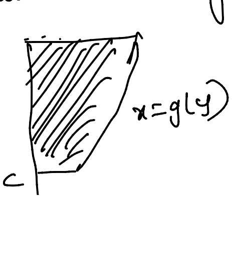
$A = \pi [f(x_i)]^2$

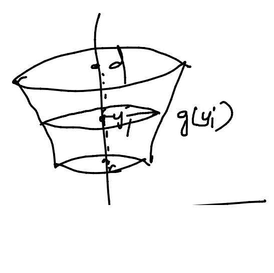
$\pi [f(x_1)]^2 + \pi [f(x_2)]^2 + \dots + \pi [f(x_n)]^2$

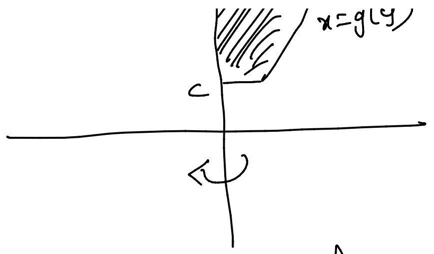
$V = \int_a^b \pi [f(x)]^2\,dx = \sum_{i=1}^{n} \pi [f(x_i)]^2$, $n \to \infty$

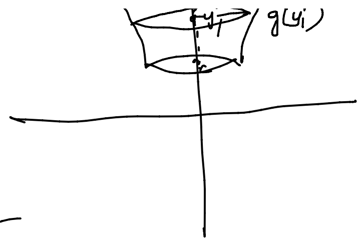
↓

Volume of the solid formed when $y = f(x)$ rotated about x-axis









$$A = \pi [g(y_i)]^2$$

Method of disks (or) Method of rings

$$V = \int_c^d \pi [g(y)]^2 \, dy.$$

$\downarrow$ volume of the solid formed when $x = g(y)$ rotated about y-axis

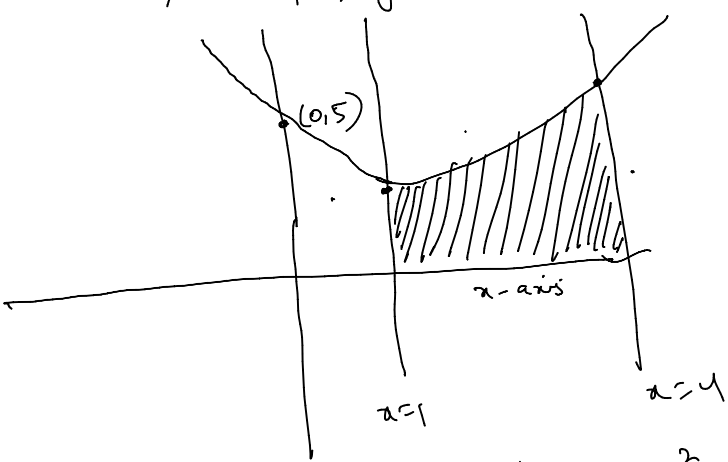
$\rightarrow$ 1) Determine the volume of the solid formed by the following regions about the given line.

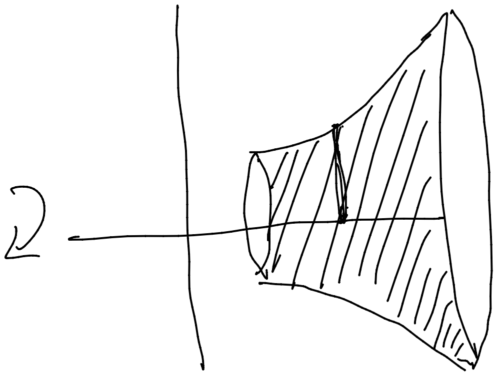
1) $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, x-axis about the x-axis.

sol: $y = x^2 - 4x + 5$, $x = 0 \Rightarrow y = 5$, $(0,5)$ is the p.o.i of $y = x^2 - 4x + 5$ with y-axis

, $x = 1 \Rightarrow y = 2$, $(1,2)$ is the p.o.i of $y = x^2 - 4x + 5$ with $x = 1$

, $x = 4 \Rightarrow y = 5$, $(4,5)$ " " " " with $x = 4$





radius $= x^2 - 4x + 5$

$$\text{Area} = \pi (r)^2 = \pi (x^2 - 4x + 5)^2$$

$$\text{Volume} = \int_{x=1}^{4} \pi (x^2 - 4x + 5)^2 \, dx. =$$

2) $y = 3\sqrt{x}$, $y = \frac{x}{4}$ that lies in the first quadrant about y-axis.

sol $y = x^{1/3}$, $x = 0 \Rightarrow y = 0 \Rightarrow y = x^{1/3}$ is passing through the origin.

3 . . (2 (u) = 0 $\Rightarrow$ $x = 0$, $x = \pm 8$

Sol $y = x^{1/3}$, $x = 0 \Rightarrow y = 0 \Rightarrow y = x^{1/3}$ is pass...

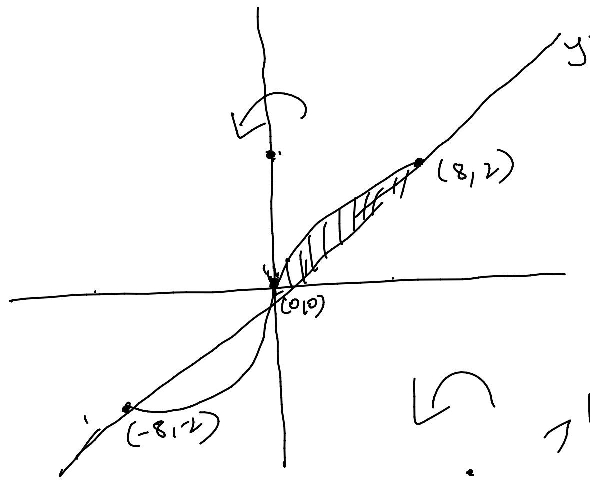
$\left.\begin{array}{l} y = x^{1/3} \\ \& \; y = \frac{x}{4} \end{array}\right\}$ $x^{1/3} = \frac{x}{4} \Rightarrow 64x = x^3 \Rightarrow x(x^2 - 64) = 0 \Rightarrow x = 0, x = \pm 8$

$x = 0,\ y = 0 \quad (0,0)$

$x = 8,\ y = 2 \quad (8,2)$ $\Big\}$ p.o.i of $y = \sqrt[3]{x}$ & $y = \frac{x}{4}$

$x = -8,\ y = -2 \quad (-8,-2)$

$\left[(-1)^{3}\right]^{5/3}$



$y = x^{1/3}$

$y' = \frac{1}{3} x^{-2/3}$

$y'' = \frac{-2}{9} x^{-5/3} = \frac{-2}{9x^{5/3}}$

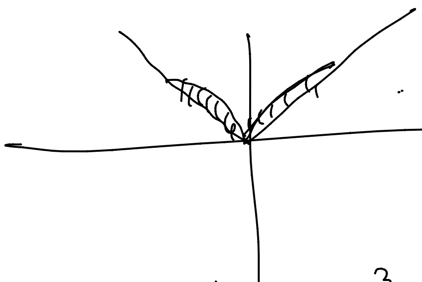
$y''(-1) = \frac{-2}{9(-1)^{5/3}} = \frac{-2}{-9} = \frac{2}{9} > 0$

$y'' > 0$ for $x \in (-8, 0)$

$\Rightarrow y$ is concave upward in $(-8, 0)$

hallow space

$y''(1) = \frac{-2}{9(1)^{5/3}} = -\frac{2}{9} < 0$



> Inner radius
outer radius

$y'' < 0$ in $x \in (0, 8)$

y is concave downward in $(0, 8)$

$y = \sqrt[3]{x}$ $\quad \boxed{x = y^3}$

$y = \frac{x}{4} \Rightarrow \boxed{x = 4y}$

Inner radius $= y^3$

outer radius $= 4y$

$$V = \int_{y=0}^{2} \pi \left[[\text{outer radius}]^2 - [\text{Inner radius}]^2 \right] dy$$

$$\int_{y=0}^{2} \pi \left[(4y)^2 - (y^3)^2 \right] dy = \pi \int_{y=0}^{2} (16y^2 - y^6)\, dy = ?$$

(3) $y = x^2 - 2x$, $y = x$ about the line $y = 4$

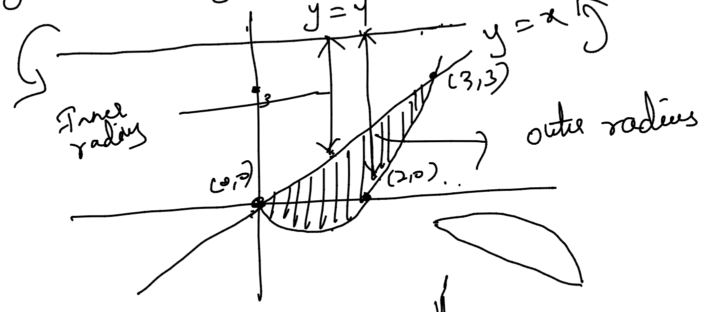
Sol: $\left.\begin{array}{l} y = x^2 - 2x \\ y = x \end{array}\right\}$ $x^2 - 2x = x \Rightarrow x = 0, 3$

$x = 0,\ y = 0$ & $x = 3,\ y = 3$

p.o.i of $y = x^2 - 2x$ & $y = x$ are $(0,0)$ & $(3,3)$

$x^2 - 2x = 0 \Rightarrow x = 0, 2 \Rightarrow (0,0)$ & $(2,0)$ are p.o.i p.o.i of $y = x^2 - 2x$ & $y = x$

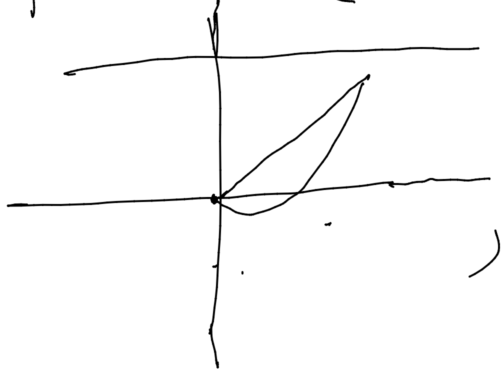
$y = x^2 - 2x$, $y = 0$ $\Rightarrow$ $x^2 - 2x = 0$ $\Rightarrow$ $x = 0, 2$ $\Rightarrow$ $(0,0)$ & $(2,0)$ are p.o.i of $y = x^2 - 2x$ with x-axis



$\boxed{y'' = 2 > 0}$ $\Rightarrow$ y is concave upward in $(0,2)$

Inner radius $= 4 - x$

Outer radius $= 4 - (x^2 - 2x)$



hollow space.

$$V = \pi \int_{x=0}^{3} \left([\text{outer radius}]^2 - [\text{inner radius}]^2 \right) dx = ?$$

$\rightarrow$ 4) $y = 2\sqrt{x-1}$, $y = x - 1$ about the line $x = -1$.

sol:

$\rightarrow$ Volume of solid of revolution by method of shells / cylinders

1) $y = (x-1)(x-3)^2$ & x-axis ($y = 0$) about y-axis.

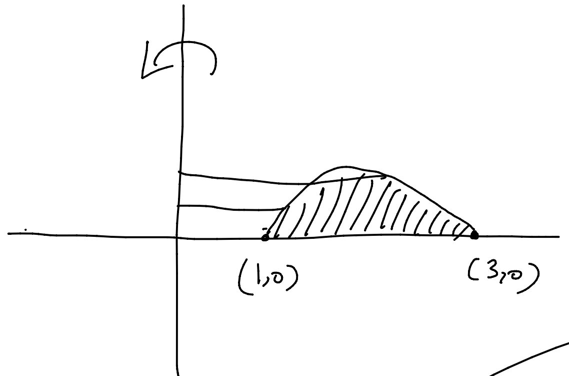
sol: $(x-1)(x-3)^2 = 0$ $\Rightarrow$ $x = 1$ & 3

$(1,0)$ & $(3,0)$ are p.o.i of $y = (x-1)(x-3)^2$ with x-axis.

$y'' < 0$ for $x \in (1,3)$

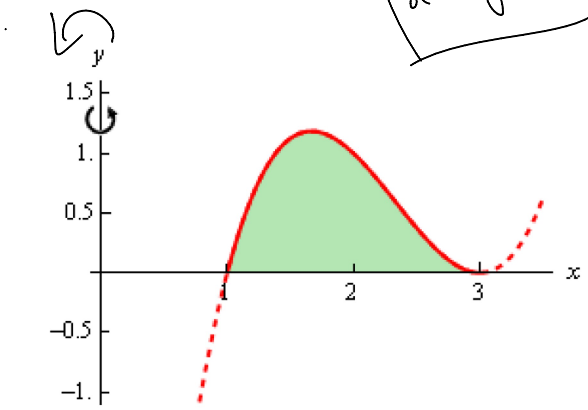
$(x^2 - 6x + 9)(x-1) = y$

$x^3 - x^2 - 6x^2 + 6x + 9x - 9 = y$

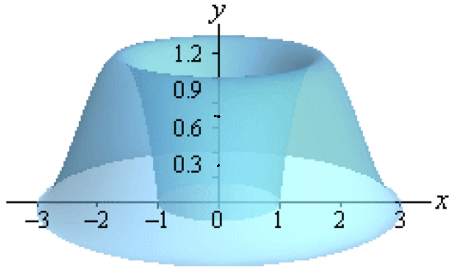


$\boxed{x = g(y)}$

$x = g(y)$ $\boxed{x = g(y)}$



$x = g(y)$



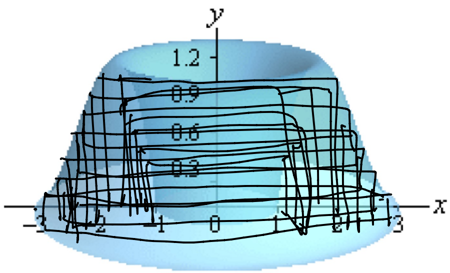
$y = f(x)$

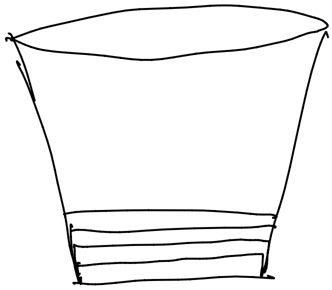
$(x-1)(x-3)^2 - 0 =$ inner ra.

$\cancel{V = 0}$

$(x-1)(x-3)^2 - 0$

$y = (x-1)(x-3)^2$ $\quad$ $\boxed{x = g(y)}$

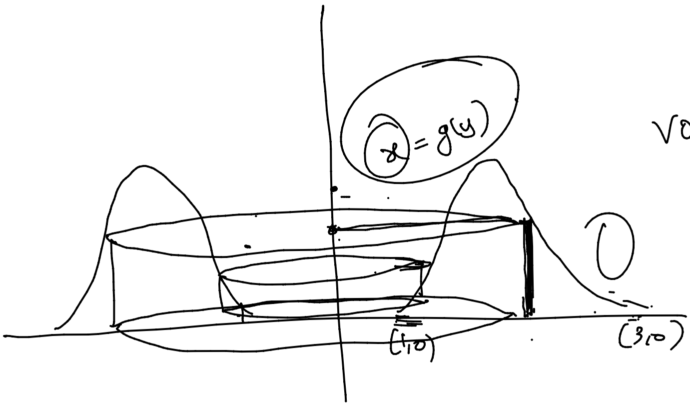




Surface area of cylinder — $2\pi r h$

volume of solid $= \int 2\pi r h$ $\quad$ $f(x) - y$

$g(y) - x$.



radius $= x \quad (x = g(y))$

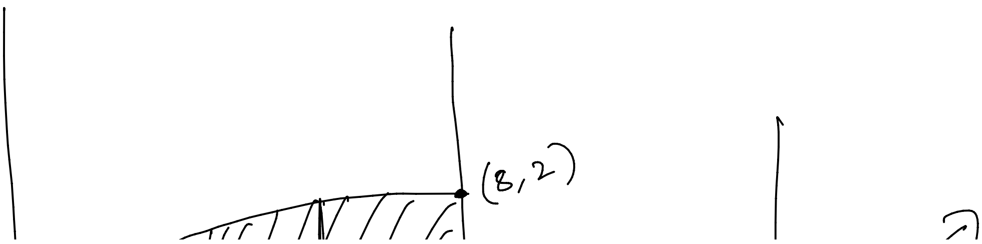
height $= (x-1)(x-3)^2 \quad (y = f(x))$

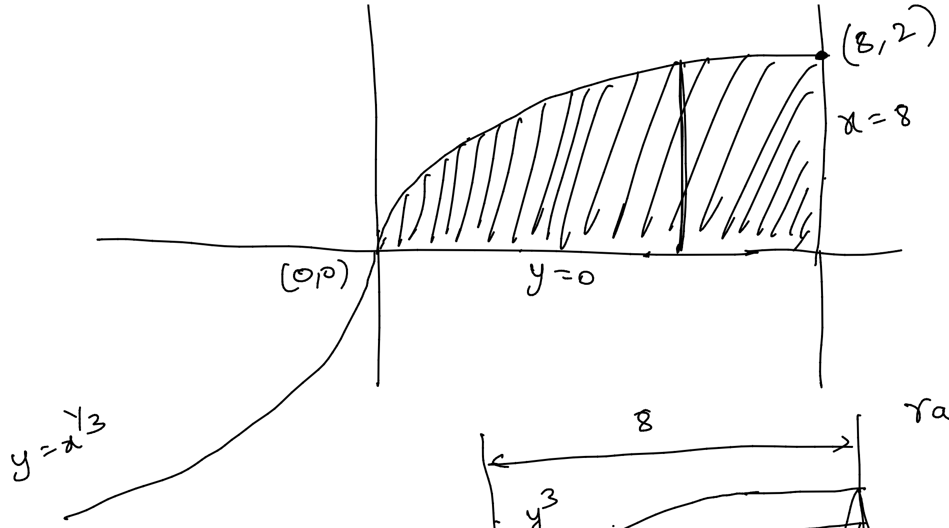
$$V = \int_{x=1}^{3} 2\pi r h \, dx$$

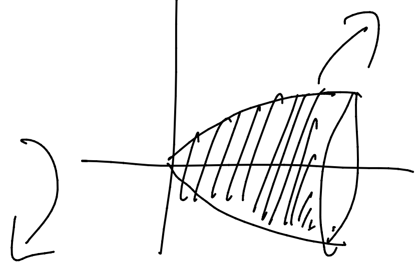
$$V = 2\pi \int_1^3 x(x-1)(x-3)^2 \, dx = ?$$

2) $y = \sqrt[3]{x}$, $x = 8$, x-axis about x-axis

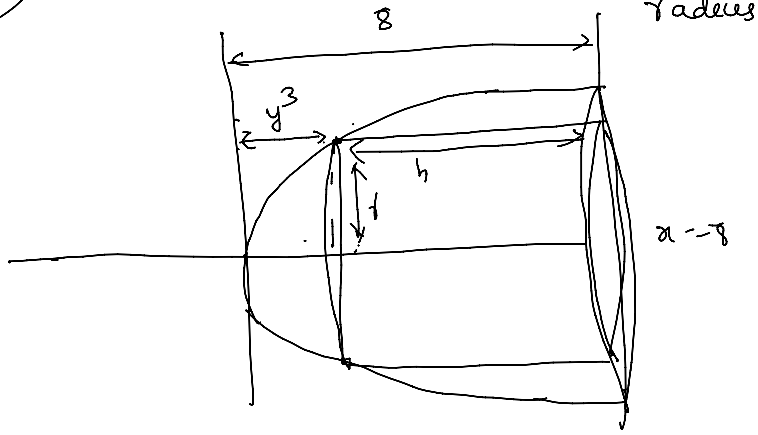
Sol: $y = x^{1/3}$







radius $= \left(x^{1/3} - 0\right) = x^{1/3}$



radius $= y$

height $= 8 - y^3$

$$V = \int_{y=0}^{2} 2\pi (y)(8 - y^3)\, dy.$$

→ 3) $y = 2\sqrt{x-1}$, $y = x-1$ about $x = 6$

sol: $2\sqrt{x-1} = x-1 \Rightarrow 4(x-1) = (x-1)^2 \Rightarrow x^2 + 1 - 2x - 4x + 4 = 0$

$\Rightarrow x^2 - 6x + 5 = 0$

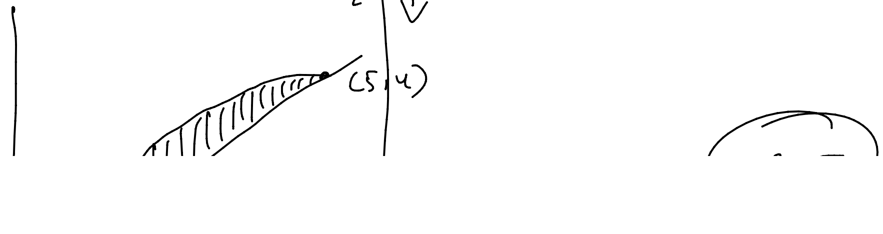
$x = 1, 5$

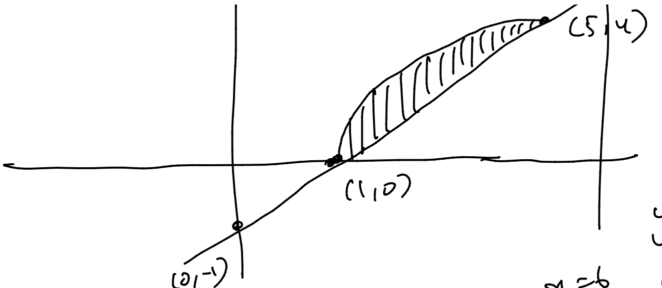
$x = 1$, $y = 0$ $\quad$ $(1,0)$ & $(5,4)$ are p.o.i of $y = 2\sqrt{x-1}$ & $y = x-1$

$x = 5$, $y = 4$

$y = 2\sqrt{x-1}$, $x = 0 \Rightarrow$ imaginary for $y \Rightarrow y = 2\sqrt{x-1}$ donot intersect y-axis.

$y = x-1$, $(0,-1)$, $(1,0)$ are p.o.i of $y = x-1$ with x & y-axis.





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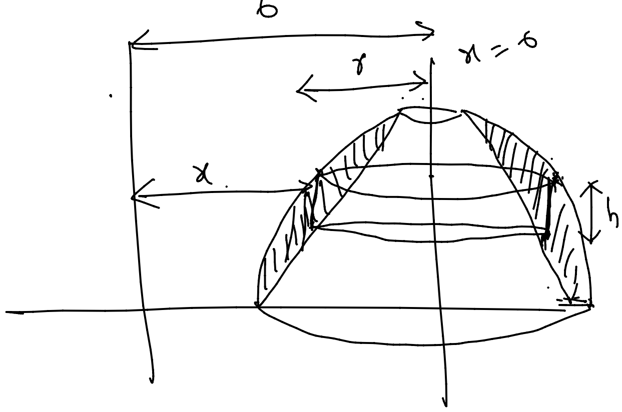
$y(1) = 0$

$y(2) = 2\sqrt{2-1} = > 0$



radius $= 6 - x$

height $= (2\sqrt{x-1}) - (x-1)$

$2\sqrt{x} - 1 - x + 1$