Special Functions

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$\int_a^b f(x)\,dx \rightarrow \infty$ — improper

, $\int_a^b f(x)\,dx$ — finite value — proper.

Improper integral:

$\int_a^b f(x)\,dx \longrightarrow$ improper integral if either a is $-\infty$ or b is ∞ or both $a = -\infty$ & $b = \infty$ (or) if a & b are finite but the $f(x)$ is undefined at a or b or in b/w a & b

Ex: $\int_0^\infty x\,dx$ — improper $\qquad \int_{-1}^{1} \frac{1}{x}\,dx$ — improper.

$\int_{-\infty}^{0} x\,dx$ — improper

Improper integral of first kind $\rightarrow \int_{-\infty}^{b} f(x)\,dx$ & $\int_a^\infty f(x)\,dx$.

(or) $\int_{-\infty}^{\infty} f(x)\,dx$.

Improper integral of second kind:

$\int_a^b f(x)\,dx$, a & b are finite but $f(x)$ is undefined either at a or b or in b/w a & b

Ex: $\int_{-1}^{1} \frac{1}{x}\,dx$. (or) $\int_0^1 \frac{1}{x}\,dx$.

$\rightarrow$ Beta function:- (0–1)

An improper integral of the form $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. , $m>0$, $n>0$ is called as Beta function & it is denoted by $\beta(m,n)$ or $B(m,n)$

is called as Beta function & it is denoted by $B(m,n)$

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m>0, n>0}$$

Properties

1) $\beta(m,n) = \beta(n,m)$

sol: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $1-x = y$ $\qquad x \to 0, y \to 1$

$dx = -dy$ $\qquad x \to 1, y \to 0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$\beta(m,n) = \beta(n,m)$$

2) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

sol: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x \to 0, \theta \to 0$

$x \to 1, \theta \to \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

3) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$+ \beta(m, n+1)$

$\Rightarrow \beta(m,n) = \beta(\ldots,\ldots)$

Sol $\beta(m+1,n) + \beta(m,n+1)$

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}\,dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[x + 1 - x\right] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

4) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$

Sol: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \frac{1}{1+y} \Rightarrow dx = -\frac{1}{(1+y)^2}\,dy$

$x \to 0,\ y \to \infty$

$x \to 1,\ y \to 0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right) dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}\,dy}{(1+y)^{m+n}} \qquad y = x$$

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx. = \beta(n,m)$$

$$= \int^\infty \frac{x^{m-1}}{\ldots^{n+m}}\,dx.$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}\, dx.$$

prob

→ show that $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$.

sol:

$$\int_0^{\pi/2} \sin^{m-1}\theta \cos^{n-1}\theta \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} (\cos^2\theta)^{\frac{n-1}{2}} \sin\theta\cos\theta \, d\theta$$

put $\sin^2\theta = x \Rightarrow \cos^2\theta = 1-x$

$\sin\theta\cos\theta\, d\theta = \frac{dx}{2}$

$\theta \to 0 \Rightarrow x \Rightarrow 0$

$\theta \to \frac{\pi}{2} \Rightarrow x \to 1$

$$= \int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} \frac{dx}{2} = \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} dx.$$

$$\int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1} (1-x)^{\frac{n+1}{2}-1} dx$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

→ Express $\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$ in terms of β-function.

sol:

$$\int_0^1 x(1-x^2)^{-1/2} dx. \longleftrightarrow \int_0^1 x^{m-1}(1-x)^{n-1} dx.$$

$x^2 = y$ $\quad \sqrt{y}(1-y)$

$y^{1/2}(1-y)$

let $x^2 = y$

$x\, dx = \frac{dy}{2}$

$x \to 0 \quad y \to 0$

$x \to 1 \quad y \to 1$

$$\int_0^1 (1-y)^{-1/2}\, dy = \int_0^1 y^{1-1} (1-y)^{\frac{1}{2}-1}\, dy$$

$$= \beta\left(1, \tfrac{1}{2}\right).$$

H.W

→ Express $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ in terms of β - function

$\int (9-x^2)^{-1/2}\, dx$

$9^{-1/2}\left(1-\frac{x^2}{9}\right)^{-1/2}$

$\frac{x^2}{9} = y$

Ans : $\frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{2}\right)$

→ $\int_0^a (a-x)^{m-1} x^{n-1}\, dx = a^{m+n-1}\beta(m,n)$

→ $\int_0^1 x^m (1-x^n)^p\, dx = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right).$

$x^n = y$

Hint: $\int a^{m-1}\left(1-\frac{x}{a}\right)^{m-1} x^{n-1}$

$\frac{x}{a} = y$

→ Gamma function:

An improper integral of the form $\int_0^\infty e^{-x} x^{n-1}\, dx,\ n>0$ is called Gamma function and it is denoted by $\Gamma(n)$

Γ — Gamma.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n>0$$

Properties : i) $\Gamma(1) = 1$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$n=1 \Rightarrow \Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = 1$$

2) $\Gamma(n) = (n-1)\Gamma(n-1), \quad n>0$

$$\Gamma(n) = \int^\infty e^{-x} x^{n-1}\, dx.$$

2) $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.$$

$$\left[x^{n-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}(-e^{-x})\,dx$$

$\lim_{x\to a} f(x)g(x) \to 0$ if either $\lim_{x\to a} f(x) \to 0$ (or) $\lim_{x\to a} g(x) \to 0$

$$= (0-0) + (n-1)\int_0^\infty e^{-x} x^{n-2}\,dx$$

$$\Gamma(n) = (n-1)\underbrace{\int_0^\infty e^{-x} x^{(n-1)-1}\,dx.}$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

Note:

$\rightarrow$ $\Gamma(n+1) = n\Gamma(n)$. ✓

$\rightarrow$ $\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$

$(n-1)(n-2)(n-3)\Gamma(n-3)$

$\vdots$

$(n-1)(n-2)(n-3)\cdots\cdots(n-r)\Gamma(n-r)$, $n-r>0$

$\rightarrow$ $\Gamma(n+1) = n!$, n is positive integer.

$\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)(n-2)(n-3)\cdots\cdots(n-(n-1))\Gamma(n-(n-1))$

$= n(n-1)(n-2)\cdots\cdots 1\,\Gamma(1)$

$= 1\cdot 2\cdot\cdots\cdots(n-1)(n-2)n. = n!$ $\boxed{n \geq 0}$

$\rightarrow$ $\Gamma(n+1) = n\Gamma(n)$

$\Rightarrow$ $\Gamma(n) = \frac{1}{n}\Gamma(n+1)$

$n=0$ $\Rightarrow$ $\Gamma(0) = \frac{1}{0}\Gamma(1)$ — undefined

$\Rightarrow \Gamma(0)$ — undefined

$n=-1$ $\Rightarrow$ $\Gamma(-1) = \frac{1}{-1}\Gamma(-1+1) = -\Gamma(0)$ — undefined

$\Gamma(-1) \rightarrow$ undefined.

lly $\Gamma(-2), \Gamma(-3), \Gamma(-4), \cdots\cdots$ are undefined.

Relation b/w Beta and Gamma function:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m>0, n>0$$

proof:

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\, dx.$$

Let $x = yt$. $\qquad x \to 0 \Rightarrow t \to 0$

$dx = y\,dt \qquad x \to \infty \Rightarrow t \to \infty$

$$\Gamma(m) = \int_0^\infty e^{-yt} (yt)^{m-1}\, y\,dt = \int_0^\infty y^m e^{-yt} t^{m-1}\, dt.$$

$$\Rightarrow \Gamma(m) = y^m \int_0^\infty e^{-yt} t^{m-1}\, dt$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yt} t^{m-1}\, dt \qquad t \to x$$

$\int f(x)\,dx$

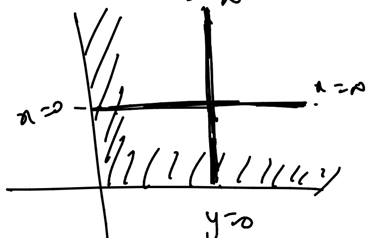
$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1}\, dx \quad \text{--- (1)}$$

$\int f(y)\,dy$

Multiply both sides with $e^{-y} y^{m+n-1}$ & integrate w.r.to 'y' from 0 to ∞.

$$\int_0^\infty \frac{\Gamma(m)}{y^m} e^{-y} y^{m+n-1}\, dy = \int_0^\infty \left[\int_0^\infty \left(e^{-y} y^{m+n-1} e^{-yx} x^{m-1}\right) dx\right] dy.$$

$$\Gamma(m) \int_0^\infty e^{-y} y^{n-1}\, dy = \int_{y=0}^\infty \left\{ \int_{x=0}^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1}\, dx \right\} dy$$



$$\Gamma(m)\Gamma(n) = \int_{x=0}^\infty \left\{ \int_{y=0}^\infty e^{-y(1+x)} y^{m+n-1}\, dy \right\} x^{m-1}\, dx.$$

($\because$ By changing order of integration).

$$\Gamma(m)\Gamma(n) = \int_{x=0}^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1}\, dx. \quad (\because \text{By (1)})$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n) \int_{x=0}^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx.$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1}\, dx$$

$$\int_0^\infty e^{-yx} x^{m-1}\, dx \quad \Bigg| \quad \int_0^\infty e^{-y(1+x)} y^{m+n-1}\, dy$$

$\downarrow \qquad \downarrow$

y

$x=0 \quad (1+x)^{m+n}$

$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$

$\Rightarrow \beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

t	
x	y
m	$m+n$
y	$1+x$.

$\dfrac{\Gamma(m+n)}{(1+x)^{m+n}}$

$\rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

sol: $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$m=\frac{1}{2}, n=\frac{1}{2} \Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \dfrac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{1}$ — (1)

$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx. = \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$

let $x = \sin^2\theta \Rightarrow 1-x = \cos^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta.$

$x \to 0 \Rightarrow \theta \to 0$

$x \to 1 \Rightarrow \theta \to \frac{\pi}{2}$

$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(\cos^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\,d\theta.$

$= 2\int_0^{\pi/2} (\sin\theta)^{-1}(\cos\theta)^{-1}\sin\theta\cos\theta\,d\theta$

$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = 2\cdot\frac{\pi}{2} = \pi$

(1) $\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$

$\Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

find $\Gamma\left(-\frac{1}{2}\right)$

sol: $\Gamma(n+1) = n\Gamma(n)$

sol: $\Gamma(n+1) = n\Gamma(n)$

$$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$n = -\frac{1}{2} \Rightarrow \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

→ find $\Gamma\left(-\frac{3}{2}\right), \Gamma\left(-\frac{5}{2}\right), \Gamma\left(-\frac{7}{2}\right)$

→ show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

sol: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$

let $n = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2} dx. \Rightarrow \sqrt{\pi} = \int_0^\infty e^{-x} x^{-1/2} dx.$$

let $x = y^2 \Rightarrow dx = 2y\,dy$ $\quad x \to 0 \Rightarrow y \to 0$, $x \to \infty \Rightarrow y \to \infty$

$$\sqrt{\pi} = \int_0^\infty e^{-y^2} (y^2)^{-1/2} (2y\,dy)$$

$$\sqrt{\pi} = 2\int_0^\infty e^{-y^2} dy$$

$$\Rightarrow \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

Note: $\int_{-\infty}^\infty e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

→ find $\Gamma\left(\frac{11}{2}\right)$

sol: $\Gamma(n) = (n-1)(n-2) \cdots (n-r)\Gamma(n-r)$, $n-r > 0$

$n = \frac{11}{2}$

$$\Gamma\left(\frac{11}{2}\right) = \left(\frac{11}{2}-1\right)\left(\frac{11}{2}-2\right)\left(\frac{11}{2}-3\right)\left(\frac{11}{2}-4\right)\left(\frac{11}{2}-5\right)\Gamma\left(\frac{11}{2}-5\right)$$

$$= \frac{9}{2}\,\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\frac{9}{2} \quad \frac{7}{2} \quad \frac{5}{2} \quad \frac{3}{2} \quad \frac{1}{2} \quad \sqrt{\pi}$$

→ $\beta(m+1, n) = \frac{m}{m+n}\beta(m,n) \quad , m>0, n>0$

Sol:
$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\beta(m+1, n) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+1+n)} = \frac{m\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \qquad (\Gamma(n+1) = n\Gamma(n))$$

$$= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$= \frac{m}{m+n}\beta(m,n)$$

→ Evaluate $\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, \; n>0 \longrightarrow \int_0^\infty e^{-x}x^{n-1}dx$

Sol: Let $\log\frac{1}{x} = y \Rightarrow e^{y} = \frac{1}{x}$

$\Rightarrow x = e^{-y}$. $\qquad x \to 0 \Rightarrow y \to \infty$

$dx = -e^{-y}dy \qquad x \to 1 \Rightarrow y \to 0$

$$\int_\infty^0 y^{n-1}(-e^{-y}dy) = \int_0^\infty e^{-y}y^{n-1}dy$$
$$= \Gamma(n).$$

→ Evaluate the following using β and Γ functions.

1) $\int_0^\infty x^{n-1}e^{-kx}dx$.

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$\int_0^\infty x^{n-1}e^{-kx}dx$$

Let $kx = t \qquad x \to 0 \Rightarrow t \to 0$

$dx = dt/k \qquad x \to \infty \Rightarrow t \to \infty$

$\infty \qquad \int^{\infty} \qquad \Gamma(n).$

$dx = dt/k \qquad x \to \infty \Rightarrow t \to \infty$

$$\int_0^\infty \left(\frac{t}{k}\right)^{n-1} e^{-t} \frac{dt}{k} = \frac{1}{k^n}\int_0^\infty t^{n-1} e^{-t}\, dt = \frac{1}{k^n}\Gamma(n).$$

2) $\int_0^\infty e^{-x^{\frac{1}{n}}}\, dx,\ n>0 \qquad$ (Hint: $t = x^{\frac{1}{n}}$) $\qquad$ Ans: $n\Gamma(n)$.

3) $\int_0^1 x^5 (1-x)^3\, dx$

Sol: $$\int_0^1 x^{6-1}(1-x)^{4-1}\, dx = \beta(6,4)$$
$$= \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)} = \frac{\Gamma(5+1)\Gamma(3+1)}{\Gamma(9+1)}$$
$$= \frac{5!\,3!}{9!}$$

4) $\int_0^1 x^{5/2}(1-x^2)^{3/2}\, dx.$

Sol: $x^2 = t \Rightarrow x = \sqrt{t}$
$dx = \frac{1}{2\sqrt{t}}\, dt.$

$x \to 0 \Rightarrow t \to 0$
$x \to 1 \Rightarrow t \to 1$

$$\int_0^1 (\sqrt{t})^{5/2}(1-t)^{3/2}\frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^1 t^{5/4}\, t^{-1/2}(1-t)^{3/2}\, dt$$

$$= \frac{1}{2}\int_0^1 t^{3/4}(1-t)^{3/2}\, dt \qquad \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$\frac{3}{4} = m-1 \Rightarrow m = \frac{7}{4}$$
$$\frac{3}{2} = n-1 \Rightarrow n = \frac{5}{2}$$

$$= \frac{1}{2}\int_0^1 t^{7/4-1}(1-t)^{5/2-1}\, dt. = \frac{1}{2}\beta\left(\frac{7}{4},\frac{5}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{2}\right)} = \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{4}\right)\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-2\right)\Gamma\left(\frac{5}{2}-2\right)}{\Gamma\left(\frac{17}{4}\right)}$$

$\int^{\pi/2} \ldots \theta\, d\theta. \qquad \int^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$

$$\rightarrow \int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta.$$

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Sol:

$5 = 2m-1$

$\Rightarrow m = 3$, $\frac{7}{2} = 2n-1 \Rightarrow n = \frac{9}{4}$

$$\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \int_0^{\pi/2} \sin^{2\times 3-1}\theta \cos^{2\cdot\frac{9}{4}-1}\theta \, d\theta = \frac{1}{2}\beta\left(3,\frac{9}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma(\frac{9}{4})}{\Gamma(3+\frac{9}{4})} = \frac{1}{2}\frac{2!\,\Gamma(\frac{9}{4})}{\Gamma(\frac{21}{4})}.$$

$$\rightarrow \int_0^{\pi/2} \sin^7\theta \, d\theta.$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$

Sol: $\int_0^{\pi/2} \sin^7\theta \cos^0\theta \, d\theta$

$2m-1 = 7$, $2n-1 = 0$

$m = 4$, $n = \frac{1}{2}$

$$\int_0^{\pi/2} \sin^{2\times 4-1}\theta \cos^{2\times\frac{1}{2}-1}\theta \, d\theta = \frac{1}{2}\beta\left(4,\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(4+\frac{1}{2})}$$

$$= \frac{1}{2}\frac{3!\sqrt{\pi}}{\Gamma(\frac{9}{2})} = \frac{1}{2}\frac{3!\sqrt{\pi}}{(\frac{9}{2}-1)(\frac{9}{2}-2)(\frac{9}{2}-3)(\frac{9}{2}-4)\Gamma(\frac{9}{2}-4)}$$

$$= \frac{3\sqrt{\pi}}{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{\cancel{3}\cancel{\sqrt{\pi}}}{\frac{7}{2}\frac{5}{2}\frac{\cancel{3}}{2}\frac{1}{2}\cancel{\sqrt{\pi}}} = \frac{16}{35}$$

H.W

$$\rightarrow \int_0^1 x^4\left(\log\frac{1}{x}\right)^3 dx.$$

$$\rightarrow \int_0^\infty 3^{-4x^2} dx.$$

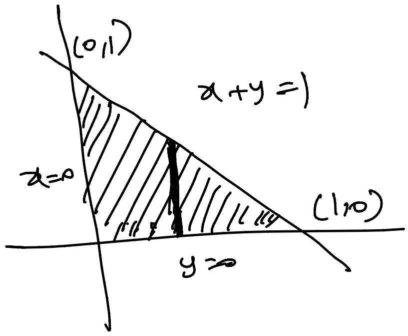
[Hint: $e^{\log 3^{-4x^2}} = 3^{-4x^2}$

$\quad = e^{-4x^2\log 3}$ $\qquad 4x^2\log 3 = t$.]

$\rightarrow$ Prove that $\iint x^p y^q \, dx\, dy = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+3)}$ where $p>0, q>0$

→ Prove that $\iint_D x^p y^q \, dx\, dy = \dfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+3)}$ where $p>0, q>0$

& $D: x \geq 0,\ y \geq 0,\ x+y \leq 1$



Sol: $y=0$ to $y=1-x$ slides from $x=0$ to $x=1$

$$\iint_D x^p q^q \, dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} x^p y^q \, dy\, dx$$

$$= \int_{x=0}^{1} x^p \left.\frac{y^{q+1}}{q+1}\right|_{y=0}^{1-x} dx = \frac{1}{q+1}\int_{x=0}^{1} x^p (1-x)^{q+1}\, dx$$

$p = m-1$

$m = p+1$

$$= \frac{1}{q+1}\int_0^1 x^{p+1-1}(1-x)^{q+2-1}\, dx$$

$$= \frac{1}{q+1}\beta(p+1, q+2) = \frac{1}{q+1}\frac{\Gamma(p+1)\Gamma(q+2)}{\Gamma(p+1+q+2)}$$

$$= \frac{1}{q+1}\frac{\Gamma(p+1)\Gamma(q+1+1)}{\Gamma(p+q+3)} = \frac{1}{\cancel{q+1}}\frac{\Gamma(p+1)\cancel{(q+1)}\Gamma(q+1)}{\Gamma(p+q+3)}$$

$$= \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+3)}.$$

→ Evaluate $\iint_D x^{l-1} y^{m-1} \, dx\,dy$, $D: x \geq 0, y \geq 0, x+y \leq h$.

$\boxed{x+y \leq 1}$

Sol: let $x = Xh$, $y = Yh$ ✓

$x \geq 0 \Rightarrow Xh \geq 0 \Rightarrow X \geq 0$

$y \geq 0 \Rightarrow Yh \geq 0 \Rightarrow Y \geq 0$

$x+y \leq h \Rightarrow Xh + Yh \leq h \Rightarrow X+Y \leq 1$

The region D is changed to $D' : X \geq 0, Y \geq 0, X+Y \leq 1$

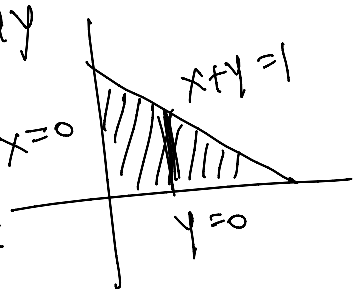
$(x, y) \to (X, Y)$

The region D is changed

$$(x,y) \to (X,Y)$$

$$J = \frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} h & 0 \\ 0 & h \end{vmatrix} = h^2$$

$$\iint_D x^{l-1} y^{m-1}\,dx\,dy = \iint_{D'} (Xh)^{l-1}(Yh)^{m-1}\,|J|\,dX\,dY$$



$$= \int_{X=0}^{1}\int_{Y=0}^{1-X} h^{l-1+m-1} X^{l-1} Y^{m-1} h^2\,dY\,dX$$

$$= h^{l+m}\int_{X=0}^{1} X^{l-1}\left.\frac{Y^m}{m}\right|_{Y=0}^{1-X} dX$$

$$= \frac{h^{l+m}}{m}\int_{X=0}^{1} X^{l-1}(1-X)^m\,dX$$

$$= \frac{h^{l+m}}{m}\beta(l, m+1)$$

$$= \frac{h^{l+m}}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{h^{l+m}}{\cancel{m}}\frac{\Gamma(l)\,\cancel{m}\,\Gamma(m)}{\Gamma(l+m+1)}$$

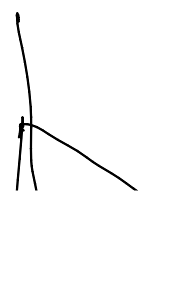
$$= h^{l+m}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}.$$

→ Dirichlet integral:

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

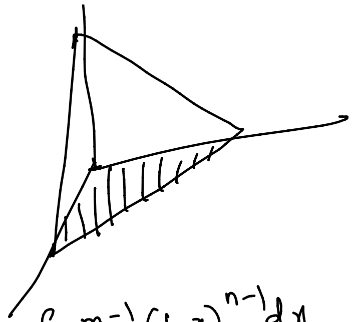
$V: x \geq 0,\ y \geq 0,\ z \geq 0\ \&\ x+y+z \leq 1$

$$\int^{1}\int^{1-x}\int^{1-x-y}$$



Sol:

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\, dx.$$



$\int x^{m-1}(1-x)^{n-1}dx$

$$x+y+z \le 1$$
$$y+z \le 1-x = h \text{ (say)}$$
$$y+z \le h$$
$$z \le h-y.$$

$\boxed{y+z \le h}$

$\boxed{y \ge 0}$

$$\int_{x=0}^{1}\left[\int_{y=0}^{h}\int_{z=0}^{h-y} x^{l-1} y^{m-1} z^{n-1}\, dz\, dy\right] dx$$

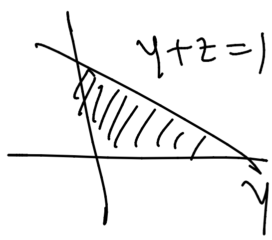
$y=0$ to $y=1$

$z=0$ to $1-y$. $y+z \le 1$

$y=0$
$\Rightarrow Y=0$
$y=h$
$\Rightarrow Y=1$
$z=0 \Rightarrow Z=0$
$z=y-h$
$\Rightarrow Z=1-Y$

Let $y = Yh$, $z = Zh$

$$J = \frac{\partial(y,z)}{\partial(Y,Z)} = h^2$$



$y \ge 0 \Rightarrow Yh \ge 0 \Rightarrow Y \ge 0$

$z \ge 0 \Rightarrow Zh \ge 0 \Rightarrow Z \ge 0$

$y+z \le h \Rightarrow Yh+Zh \le h \Rightarrow Y+Z \le 1$

$$\int_{x=0}^{1}\left[\int_{Y=0}^{1}\int_{Z=0}^{1-Y} x^{l-1}(Yh)^{m-1}(Zh)^{n-1}|J|\, dZ\, dY\right] dx.$$

$y = Yh$, $z = Zh$
$z = h-y$
$Zh = h - Yh$
$Z = 1-Y$

$$\int_{x=0}^{1}\left[\int_{Y=0}^{1}\int_{Z=0}^{1-Y} Y^{m-1} Z^{n-1}\, dZ\, dY\right] h^{m+n} x^{l-1}\, dx.$$

$$\int_{x=0}^{1}\left[\int_{Y=0}^{1} Y^{m-1}\left.\frac{Z^n}{n}\right|_{Z=0}^{1-Y} dY\right] h^{m+n} x^{l-1}\, dx$$

$$\frac{1}{n}\int^{1}\left[\int^{1} Y^{m-1}(1-Y)^n\, dY\right] h^{m+n} x^{l-1}\, dx$$

$$\frac{1}{n}\int_{x=0}^{1}\left[\underbrace{\int_{y=0}^{1} y^{m-1}(1-y)^{n}\,dy}\right] h^{m+n}\,x^{l-1}\,dx$$

$$\frac{1}{n}\int_{x=0}^{1}\beta(m,n+1)\,h^{m+n}\,x^{l-1}\,dx.$$

$$=\frac{\beta(m,n+1)}{n}\int_{x=0}^{1}(1-x)^{m+n}\,x^{l-1}\,dx.$$

$$=\frac{1}{n}\,\beta(m,n+1)\,\beta(l,m+n+1)$$

$$=\frac{1}{n}\,\frac{\Gamma(m)\Gamma(n+1)}{\cancel{\Gamma(m+n+1)}}\;\frac{\Gamma(l)\,\cancel{\Gamma(m+n+1)}}{\Gamma(l+m+n+1)}$$

$$=\frac{1}{\cancel{n}}\,\frac{\Gamma(m)\,\cancel{n}\,\Gamma(n)\Gamma(l)}{\Gamma(l+m+n+1)}=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}.$$

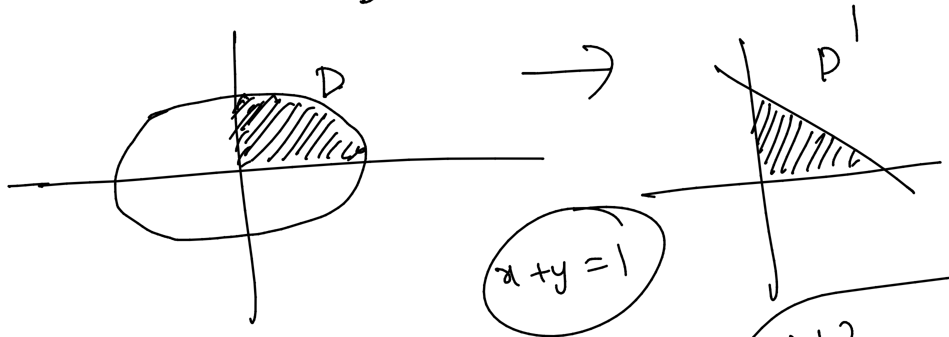
H.W

→ Evaluate $\iiint xyz\,dx\,dy\,dz$ taken over the volume of the tetrahedron given by $x\geq 0,\ y\geq 0,\ z\geq 0,\ x+y+z\leq 1$.

→ Find the value of $\iint x^{m-1}y^{n-1}\,dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of Gamma function.

$\rightarrow \begin{cases} x\geq 0,\ y\geq 0 \\ x+y\leq 1 \end{cases}$

Sol:



$\rightarrow \begin{cases} x\geq 0,\ y\geq 0,\ z\geq 0 \\ x+y+z\leq 1 \end{cases}$

let $\frac{x^2}{a^2}=X$ & $\frac{y^2}{b^2}=Y$

$$\cancel{\frac{X^2h^2}{a^2}+\frac{Y^2h^2}{b^2}=1}$$

Let $\frac{x^2}{a^2} = X$ & $\frac{y^2}{b^2} = Y$ $\qquad \left(\frac{x^n}{a^2} + \frac{y^n}{b^2} = 1 \right)$

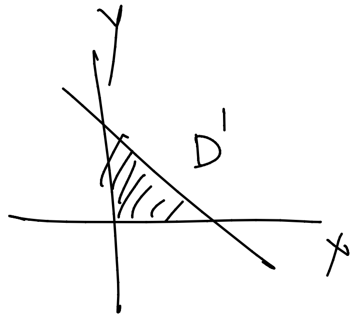
$\Rightarrow \frac{x}{a} = \sqrt{X}$, $\frac{y}{b} = \sqrt{Y}$

D: , $x \geq 0$, $y \geq 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

D$'$: $x \geq 0 \Rightarrow a\sqrt{X} \geq 0 \Rightarrow X \geq 0$

$y \geq 0 \Rightarrow b\sqrt{Y} \geq 0 \Rightarrow Y \geq 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow X + Y = 1$



$X = 0$ to 1

$Y = 0$ to $1 - X$

$$J = \frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{a}{2\sqrt{X}} & 0 \\ 0 & \frac{b}{2\sqrt{Y}} \end{vmatrix} = \frac{ab}{4\sqrt{XY}}$$

$$\iint_D x^{m-1} y^{n-1}\, dx\, dy = \iint_{D'} (a\sqrt{X})^{m-1} (bY)^{n-1} |J|\, dX\, dY$$

$$= \int_{X=0}^{1} \int_{Y=0}^{1-X} a^{m-1} b^{n-1} X^{\frac{m-1}{2}} Y^{\frac{n-1}{2}} \frac{ab}{4\sqrt{XY}}\, dY\, dX$$

$$= \frac{a^m b^n}{4} \int_{X=0}^{1} \int_{Y=0}^{1-X} X^{\frac{m}{2}-1} Y^{\frac{n}{2}-1}\, dY\, dX$$

$$\frac{a^m b^n}{4} \int_{X=0}^{1} X^{\frac{m}{2}-1} \left. \frac{Y^{\frac{n}{2}}}{\frac{n}{2}} \right|_{Y=0}^{1-X} \cdot dX = \frac{a^m b^n}{4} \times \frac{2}{n} \int_{X=0}^{1} X^{\frac{m}{2}-1} Y^{\frac{n}{2}+1-1}\, dX$$

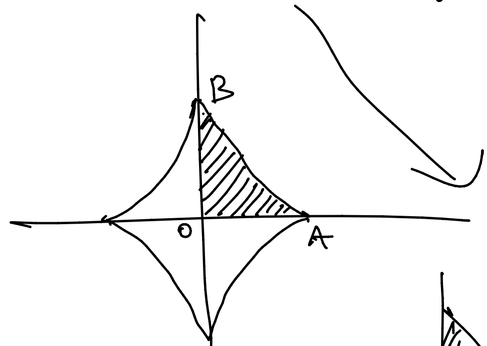
$$= \frac{a^m b^n}{2n} \beta\left(\frac{m}{2}, \frac{n}{2}+1\right)$$

$$= \frac{a^m b^n}{2n} \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$

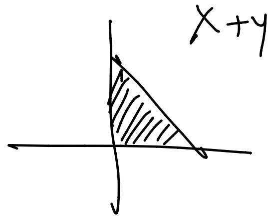
$$= \frac{a^m b^n \Gamma\left(\frac{m}{2}\right) \frac{\cancel{n}}{2}\Gamma\left(\frac{n}{2}\right)}{2\cancel{n}\, \Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)} = \frac{a^m b^n}{4} \frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}.$$

H.W

→ find the area of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ using Gamma function



Sol $\boxed{A = 4\iint\limits_{OAB} dx\,dy}$



$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$$\left(\frac{x}{a}\right)^{2/3} = X \qquad \left(\frac{y}{a}\right)^{2/3} = Y.$$

$= ?$

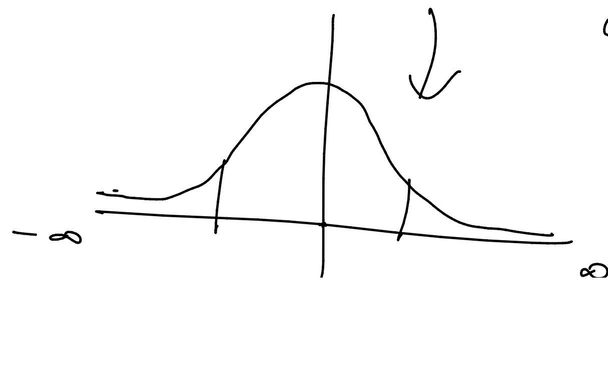
Ans: $\frac{3\pi a^2}{8}$.

→ Error function:

The error function of $f(x)$ is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

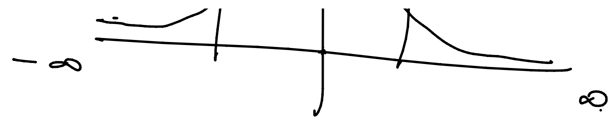
e^{-t^2} → Gaussian curve.



→ Complimentary error function

The complimentary error function of $f(x)$ is defined as

The complementary error function of $f(x)$ is defined as



$$erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - erf(x)$$

properties

1) $erf(0) = 0$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$x = 0 \Rightarrow erf(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2} dt = 0$$

2) $erf(\infty) = 1$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$x \to \infty \Rightarrow erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt. = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

3) $erf(-x) + erf(x) = 0$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{--- (1)}$$

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt.$$

let $t = -u$ $\Rightarrow t \to 0 \Rightarrow u \to 0$

$dt = -du$ $\quad t \to -x \Rightarrow u \to x$

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} (-du) = -\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

$$erf(-x) = -\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{--- (2)}$$

$\int f(x)dx \quad \int f(t)dt$

$1 + 2 = 0.$