Module 6 - Vector calculus – Tutorial sheet

Easy

1. Find the directional derivative of div$\vec{A}$ at the point (4, 4, 2) in the direction of the corresponding outer normal of the sphere $x^2 + y^2 + z^2 = 36$, where $\vec{A} = xz\boldsymbol{i} + yx\boldsymbol{j} + zy\boldsymbol{k}$.

   Ans: 5/3
2. Find the directional derivative of $\phi = 4\, e^{2x-y+z}$ at the point (1,1,-1) in the directional towards the point (3,-5,6).
3. Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1,2,-1)
4. Find the divergence and curl of the vector $\vec{V} = xyz\,\vec{\imath} + 3x^2y\vec{\jmath} + (xz^2 - y^2\mathrm{z})\ \vec{k}$ at the point (2,-1,1)
5. Find the value of a so that the vector $\vec{F} = (x+3y)\vec{\imath} + (y-2z)\vec{\jmath} + (x+az)\vec{k}$ is solenoidal. [Ans: -2]

Medium

6. Show that the vector $\vec{F}$ is given by $\vec{F} = (x^2 - yz)\vec{\imath} + (y^2 - zx)\vec{\jmath} + (z^2 - xy)\vec{k}$ is irrotational. Find its scalar potential φ. [Ans: $\varphi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$]
7. Prove that $\vec{F} = (y^2\cos x + z^3)\vec{\imath} + (2y\sin x - 4)\vec{\jmath} + (3xz^2)\vec{k}$ is irrational and find its scalar potential.
8. Prove that $\vec{\mathrm{F}} = \mathrm{r}^2\vec{\mathrm{r}}$ is conservative and find the scalar potential $\emptyset$ such that $\vec{\mathrm{F}} = \nabla\emptyset$
9. If grad(f)=$2xyzi + (x^2z + z^2)j + (x^2y + 2yz + 3z^2)k$ and f(1,0,-1)=11, then find f(x,y,z).

   Ans:$f(x,y,z) = x^2yz + yz^2 + z^3 + 12$
10. Find $f(r)$ if the vector $f(r)\ \bar{r}$ is both solenoidal and irrotational. Ans : c/r^3

Hard

11. Find the directional derivative of the function $\varphi = x^2 - y^2 + 2z^2$ at the point (1,2,3) in the direction of the line PQ where Q is the point (5,0,4) Ans : $\frac{28}{\sqrt{21}}$
12. Prove that $\nabla^2\mathrm{f(r)} = \mathrm{f}''(\mathrm{r}) + \frac{2}{\mathrm{r}}\mathrm{f}'(\mathrm{r})$.
13. If the directional derivative of $\phi = ax^2y + by^2z + c\,z^2x$ at the point (1,1,1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{(y-3)}{-2} = \frac{z}{1}$, find the values of a, b and c

Ans : $a = \pm\frac{20}{9}, \quad b = \pm\frac{55}{9} \quad \textbf{and} \quad c = \pm\frac{50}{9}$

14. If *r* is a position vector of the point (x,y,z) with respect to the origin, prove that

    (i) $\nabla\cdot\left(\frac{1}{r}\bar{r}\right)=\frac{2}{r}$ and (ii) $\nabla\left[\nabla\cdot\left(\frac{1}{r}\bar{r}\right)\right]=-\frac{2}{r^3}$

15. If a is a constant vector and r is a position vector of (x,y,z) with respect to origin, prove that $\nabla\times[(\bar{a}\cdot\bar{r})\bar{r}]=\bar{a}\times\bar{r}$.