Filter Design

EE23BTECH11041 - Md Ayaan Ashraf

I. Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for given filter number. This is a bandpass filter whose specifications are available below.

II. Filter Specifications

A. The Digital Filter

1) Passband: The passband is from {4 + 0.6(j)}kHz to {4 + 0.6(j+2)}kHz. where

$$j = (r - 11000) \mod \sigma \tag{1}$$

where σ is sum of digits of roll number and r is roll number.

$$r = 11041 \tag{2}$$
$$\sigma = 7 \tag{3}$$
$$j = 6 \tag{4}$$

substituting $j = 6$ gives the passband range for our bandpass filter as 7.6 kHz - 8.8 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 7.6$ kHz and $F_{p2} = 8.8$ kHz.
The corresponding normalized digital filter passband frequencies are

$$\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.32\pi \tag{5}$$
$$\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.37\pi \tag{6}$$

2) Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

3) Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband.

$$F_{s1} = 7.6 - 0.3 = 7.3\text{kHz} \tag{7}$$
$$F_{s2} = 8.8 + 0.3 = 9.1\text{kHz} \tag{8}$$

$$\omega_{s1} = 2\pi\frac{F_{s1}}{F_s} = 0.304\pi \tag{9}$$
$$\omega_{s2} = 2\pi\frac{F_{s2}}{F_s} = 0.379\pi \tag{10}$$
$$\tag{11}$$

B. The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency(ω) :

$$\Omega = \tan\frac{\omega}{2} \tag{12}$$

Using this relation, we obtain the analog passband and stopband frequencies as: $\Omega_{p1} = 0.5497$, $\Omega_{p2} = 0.6568$ and $\Omega_{s1} = 0.5174$, $\Omega_{s2} = 0.6773$ respectively.

III. The IIR Filter Design

We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the Chebyschev approximation to design our bandpass IIR filter.

A. *The Analog Filter*

1) Low Pass Filter Specifications: Let $H_{a,BP}(j\Omega)$ be the desired analog bandpass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{13}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.6008$ and $B = \Omega_{p2} - \Omega_{p1} = 0.1071$.

Substituting Ω_{s1} and Ω_{s2} in (13) we obtain the stopband edges of lowpass filter

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = -1.682 \tag{14}$$

$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = 1.347 \tag{15}$$

And we choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.347. \tag{16}$$

2) <u>The Low Pass Chebyschev Filter Parameters:</u> The magnitude of frequency response of the low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{17}$$

The passband edge of the low pass filter is chosen as $\Omega_{Lp} = 1$. Therefore ,

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{18}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N \cosh^{-1} x), x = \Omega_L \tag{19}$$

$$c_0(x) = 1 \tag{20}$$

$$c_1(x) = x \tag{21}$$

There exists a recurssive relation from which all the polynomials can be found out.

$$c_{N+2} = 2xc_{N+1} - c_N \tag{22}$$

Imposing the band restrictions on (17)

$$|H_{a,LP}(j\Omega_L)|^2 < \delta_2 \text{ for } \Omega_L = \Omega_{Ls} \tag{23}$$

$$1 - \delta_1 < |H_{a,LP}(j\Omega_L)|^2 < 1 \text{ for } \Omega_L = \Omega_{Lp} \tag{24}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1},$$

$$N \ge \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \tag{25}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator .

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 - 8x^2 + 1$

TABLE I
Parameter Table

we get $N \ge 4$ and $0.3268 \le \epsilon \le 0.61$

3) <u>The Low Pass Chebyschev Filter:</u> Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{26}$$

where

$$c_4(x) = 8x^4 - 8x^2 + 1. \tag{27}$$

The poles of the frequency response in (17) lying in the left half plane are in general obtained as

$$p(k) = -\sinh\phi \sin\phi(k) + j\cosh\phi \cos\phi(k) \tag{28}$$

where

$$\phi = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{29}$$

$$\phi(k) = \frac{(2k+1)}{N}\pi \;\; k = 0, \dots, N-1 \tag{30}$$

The following code generates the poles for $N = 4$, stores it in a .txt file and plots the pole-zero plot in Figure 1,

```
https://github.com/Ashraf−1508/Filter−
Design/blob/main/codes/pole_zero.py
```

And the poles are stored into the following .txt file,

```
https://github.com/Ashraf−1508/Filter−
Design/blob/main/codes/poles.txt
```

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form (Only the poles on the left side of the $j\omega$ axis would be considered to ensure stability of the filter)

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - p(1))(s_L - p(2))(s_L - p(3))(s_L - p(4))} \tag{31}$$

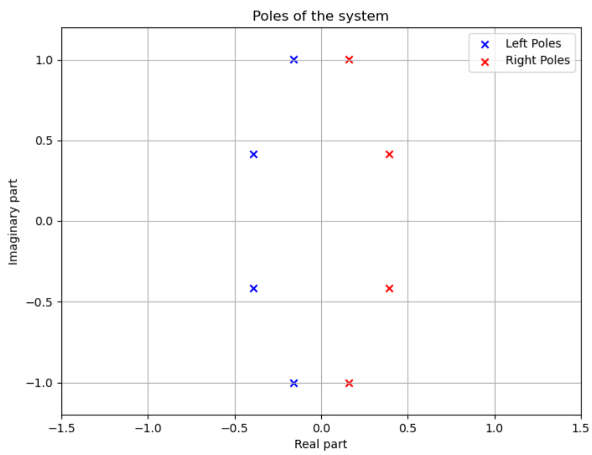

Fig. 1. pole-zero plot

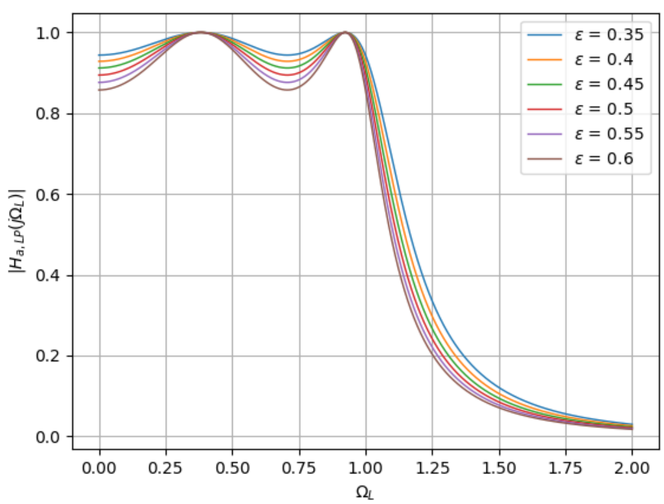

Fig. 2. The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

Substituting $N = 4$, $\epsilon = 0.4$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{32}$$

In Figure 3 we plot $|H(j\Omega)|$ using (26) and (32), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

4) The Band Pass Chebyschev Filter: The analog bandpass filter is obtained from (32) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{33}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{4.3489 \times 10^{-5} s^4}{s^8 + 0.1179s^7 + 1.4320s^6 + 0.1262s^5 + 0.7625s^4 + 0.0446s^3 + 0.1789s^2 + 0.0052s + 0.0156} \tag{34}$$

Where,

$$G_{BP} = 1.0788 \tag{35}$$

The above substitution is done by the following code,

```
https://github.com/Ashraf−1508/Filter−
    Design/blob/main/codes/coeff_analog
    .py
```

And the coefficients are stored into the .txt file,

```
https://github.com/Ashraf−1508/Filter−
    Design/blob/main/codes/
    coefficients_analog.txt
```

In Figure 4, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

B. The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{36}$$

where G is the gain of the digital filter. From (34) and (36), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{37}$$

where $G = 4.3489 \times 10^{-5}$,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{38}$$

and

$$\begin{aligned} D(z) = 3.6830 - 13.7277z^{-1} + 33.2138z^{-2} - 51.2028z^{-3} + 59.5578z^{-4} \\ -49.0243z^{-5} + 30.4476z^{-6} - 12.0480z^{-7} + 3.0950z^{-8} \end{aligned} \tag{39}$$

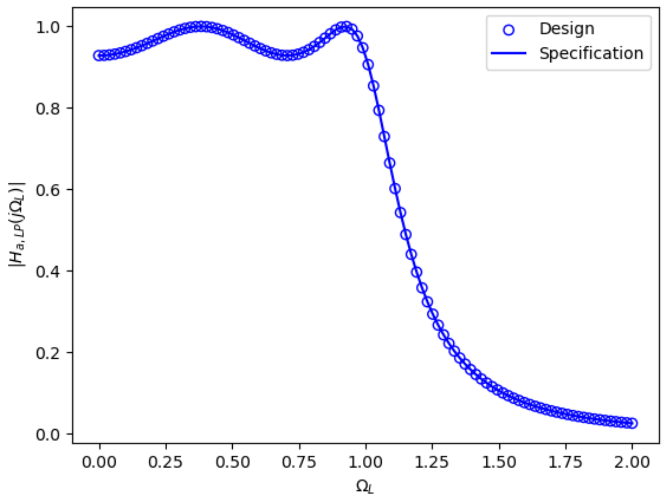

Fig. 3. The magnitude response plots from the specifications in Equation 26 and the design in Equation 32

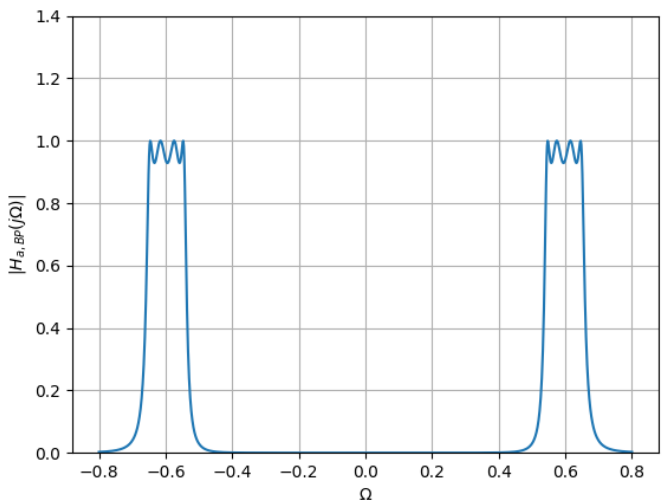

Fig. 4. The analog bandpass magnitude response plot from Equation 34

The substitution is done by the code,

https://github.com/Ashraf−1508/Filter−Design/blob/main/codes/coeff_digital.py

And the the coefficients are then stored in this .txt file,

https://github.com/Ashraf−1508/Filter−Design/blob/main/codes/coefficients_digital.txt

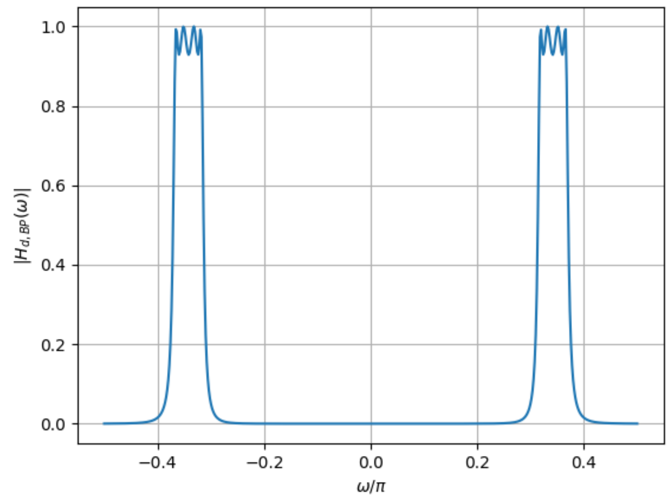

Fig. 5. The magnitude response of the bandpass digital filter designed to meet the given specifications

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Figure 5. Again we find that the passband and stopband frequencies meet the specifications well enough.

IV. The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

A. The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1) The *passband frequency* ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.
2) *The impulse response* $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \tag{40}$$

where $w(n)$ is the Kaiser window obtained from the design specifications.

B. The Kaiser Window

The Kaiser window is defined as

$$
\begin{aligned}
w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \quad -N \le n \le N, \quad \beta > 0 \\
&= 0 \qquad \text{otherwise} \qquad (41)
\end{aligned}
$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1) N is chosen according to

$$N \ge \frac{A-8}{4.57\Delta\omega}, \qquad (42)$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \ge 48$.

2) β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \qquad (43)$$

In our design, we have $A = 16.4782 < 21$. Hence, from (43) we obtain $\beta = 0$.

3) We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (41) gives us the rectangular window

$$
\begin{aligned}
w(n) &= 1, \quad -100 \le n \le 100 \\
&= 0 \quad \text{otherwise} \qquad (44)
\end{aligned}
$$

From (40) and (44), we obtain the desired low-pass filter impulse response

$$
\begin{aligned}
h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \quad -100 \le n \le 100 \\
&= 0, \qquad \text{otherwise} \qquad (45)
\end{aligned}
$$

The response of the filter in (45) is shown in Figure 6.

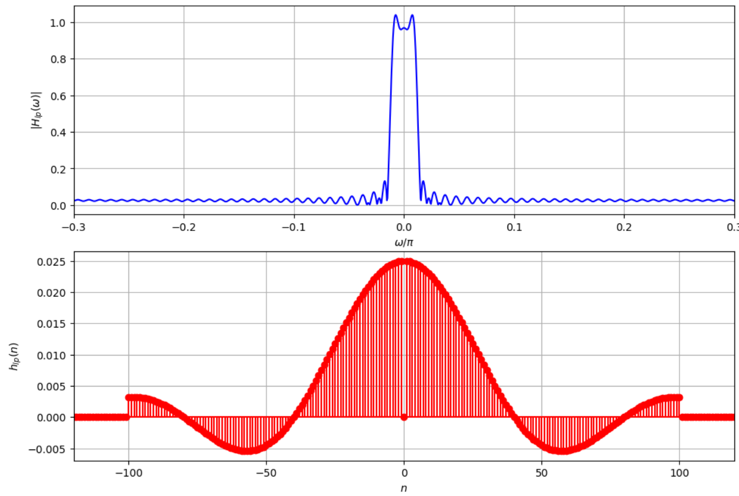

Fig. 6. The frequency and the impulse response of the FIR lowpass digital filter designed to meet the given specifications

C. The FIR Bandpass Filter

The centre of the passband of the desired band-pass filter was found to be $\omega_c = 0.275\pi$ in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)cos(n\omega_c) \qquad (46)$$

Thus, from (45), we obtain

$$
\begin{aligned}
h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(0.342n\pi)}{n\pi} \quad -100 \le n \le 100 \\
&= 0, \qquad \text{otherwise} \qquad (47)
\end{aligned}
$$

The frequency response of the FIR bandpass filter designed to meet the given specifications is plotted in Figure 7.

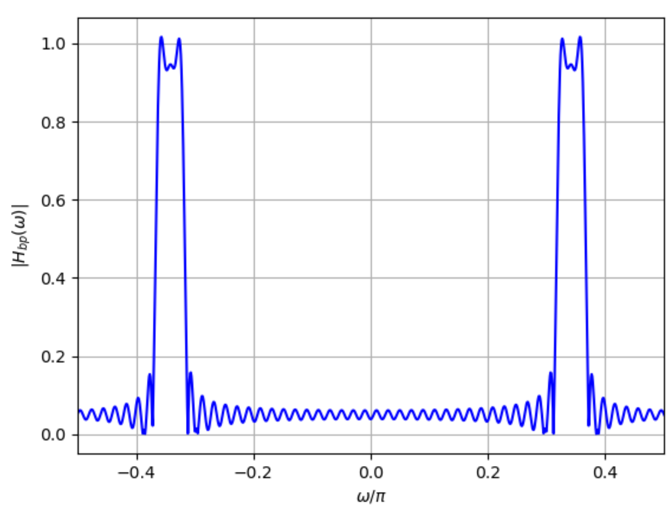

Fig. 7. The frequency response of the FIR bandpass digital filter designed to meet the given specifications