GATE AE-54(2023)

EE:1205 ($Signals Systems$)

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Question

Consider the differential equation

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \quad \text{for } x\geq 1$$

with initial conditions $y=0$ and $\frac{dy}{dx}=1$ at $x=1$. The value of y at $x=2$ is ?

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Solution

Given :

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \quad \text{for } x\geq 1 \tag{1}$$

Using Euler Substitution :

$$x=e^t \tag{2}$$

$$\frac{dy}{dx}=e^{-t}\frac{dy}{dt} \tag{3}$$

$$\frac{d^2y}{dx^2}=e^{-2t}\frac{d^2y}{dt^2}-e^{-2t}\frac{dy}{dt} \tag{4}$$

Substituting equations (2),(3),(4) in equation (1):

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0 \tag{5}$$

Taking Laplace on Both Sides:

$$s^2Y(s)-sy(1)-y'(1)+3sY(s)-y(1)+2Y(s)=0 \tag{6}$$

We know that:

$$y(1)=0 \tag{7}$$

$$y'(1)=1 \tag{8}$$

Substituting equations (7),(8) in (6)

$$s^2Y(s)+3sY(s)+2Y(s)=1 \tag{9}$$

$$\implies Y(s)=\frac{1}{s^2+3s+2} \tag{10}$$

$$\implies Y(s)=\frac{1}{s+1}-\frac{1}{s+2} \tag{11}$$

Taking Inverse Laplace:

$$y(t)=(e^{-t}-e^{-2t})u(t) \tag{12}$$

Substituting $x=e^t$:

$$y(x)=e^{-\ln x}-e^{-2\ln x}, \quad x\geq 1 \tag{13}$$

$$y(x)=\frac{1}{x}-\frac{1}{x^2}, \quad x\geq 1 \tag{14}$$

$$\therefore y(2)=0.25 \tag{15}$$

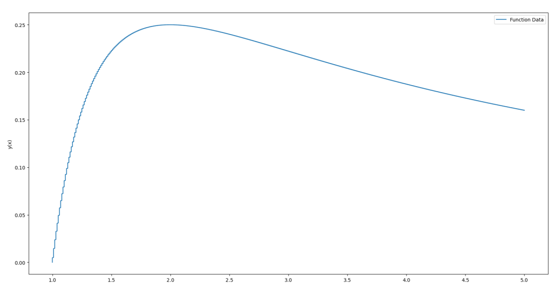

Fig. 1: Plot of $y(x)$ vs x