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GATE AE-54(2023)

EE:1205 (SignalsSystems)
Indian Institute of Technology, Hyderabad

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Question

Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \ge 1$$

with initial conditions y = 0 and $\frac{dy}{dx} = 1$ at x = 1. The value of y at x = 2 is ?

(GATE AE 2023)

Solution

Given:

$$x^{2} \frac{d^{2}y}{dx^{2}} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \ge 1$$
 (1)

Using Euler Substitution:

$$x = e^t \tag{2}$$

$$\frac{dy}{dx} = e^{-t} \frac{dy}{dt} \tag{3}$$

$$\frac{d^2y}{dx^2} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$
 (4)

Substituting equations (2),(3),(4) in equation (1):

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0\tag{5}$$

Taking Laplace on Both Sides:

$$s^{2}Y(s) - sy(1) - y'(1) + 3sY(s) - y(1) + 2Y(s) = 0$$
(6)

We know that:

$$y(1) = 0 \tag{7}$$

$$y'(1) = 1 \tag{8}$$

Substituting equations (7),(8) in (6)

$$s^{2}Y(s) + 3sY(s) + 2Y(s) = 1 (9)$$

$$\implies Y(s) = \frac{1}{s^2 + 3s + 2} \tag{10}$$

$$\implies Y(s) = \frac{1}{s+1} - \frac{1}{s+2} \tag{11}$$

Taking Inverse Laplace:

$$y(t) = e^{-t} - e^{-2t} (12)$$

Substituting $x = e^t$:

$$y(x) = e^{-\ln x} - e^{-2\ln x}, \quad x \ge 1$$
 (13)

$$y(x) = \frac{1}{x} - \frac{1}{x^2}, \quad x \ge 1 \tag{14}$$

$$y(2) = 0.25$$
 (15)

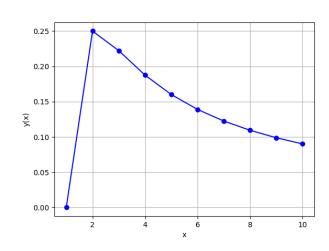


Fig. 1: Plot of y(x) vs x