

GATE AE-54(2023)

EE:1205 (Signals Systems)

Indian Institute of Technology, Hyderabad

Md Ayaan Ashraf

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Question

Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \geq 1$$

with initial conditions $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 1$.
The value of y at $x = 2$ is ?

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Solution

Given :

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \geq 1 \quad (1)$$

Using Euler Substitution :

$$x = e^t \quad (2)$$

$$\frac{dy}{dx} = e^{-t} \frac{dy}{dt} \quad (3)$$

$$\frac{d^2 y}{dx^2} = e^{-2t} \frac{d^2 y}{dt^2} - e^{-2t} \frac{dy}{dt} \quad (4)$$

Substituting equations (2),(3),(4) in equation (1):

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \quad (5)$$

Taking Laplace on Both Sides:

$$s^2 Y(s) - sy(1) - y'(1) + 3sY(s) - y(1) + 2Y(s) = 0 \quad (6)$$

We know that:

$$y(1) = 0 \quad (7)$$

$$y'(1) = 1 \quad (8)$$

Substituting equations (7),(8) in (6)

$$s^2 Y(s) + 3sY(s) + 2Y(s) = 1 \quad (9)$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 3s + 2} \quad (10)$$

$$\Rightarrow Y(s) = \frac{1}{s+1} - \frac{1}{s+2} \quad (11)$$

Taking Inverse Laplace:

$$y(t) = e^{-t} - e^{-2t} \quad (12)$$

Substituting $x = e^t$:

$$y(x) = e^{-\ln x} - e^{-2 \ln x}, \quad x \geq 1 \quad (13)$$

$$y(x) = \frac{1}{x} - \frac{1}{x^2}, \quad x \geq 1 \quad (14)$$

$$\therefore y(2) = 0.25 \quad (15)$$

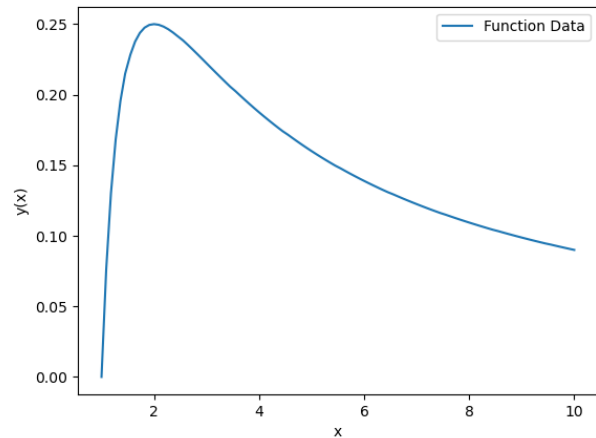


Fig. 1: Plot of $y(x)$ vs x