GATE AE-62 (2022)

EE:1205 (*SignalsSystems*)

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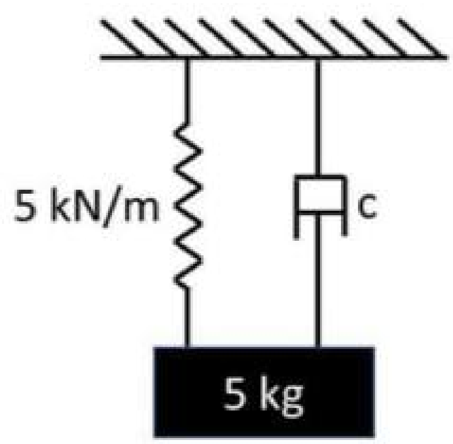
Question

A damper with damping coefficient, c, is attached to a mass of 5 kg and spring of stiffness 5 kN/m as shown in figure. The system undergoes under-damped oscillations. If the ratio of the 3^{rd} amplitude to the 4^{th} amplitude of oscillations is 1.5, the value of c is ?



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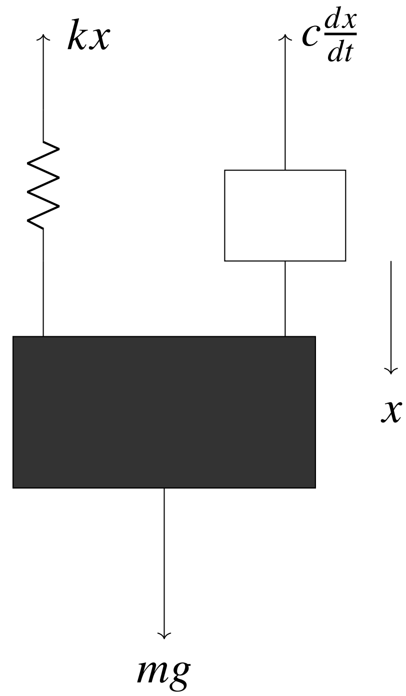
Solution:

Parameter	Value	Description
c	?	Damping Coefficient
k	5 kN/m	Stiffness
r	1.5	Ratio of 3^{rd} amplitude to 4^{th} amplitude of oscillations

TABLE 1
Parameter Table (GATE AE-62)



Now, as the oscillation begins, from the Fig. we write net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{1}$$

$$\implies m\frac{d^2x(t)}{dt^2} = -kx(t) - c\frac{dx(t)}{dt} + mgu(t) \tag{2}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{c}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = gu(t) \tag{3}$$

Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{c}{m}sX(s) + \frac{k}{m}X(s) = \frac{g}{s} \tag{4}$$

$$\implies X(s) = \frac{g}{s\left((s^2 + \frac{c}{m}s + \frac{k}{m}\right)} \tag{5}$$

$$\implies X(s) = \frac{g}{s(s-s_1)(s-s_2)} \tag{6}$$

Where

$$s_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{7}$$

$$s_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{8}$$

From (6) we get,

$$\begin{aligned} \implies X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1(s - s_1)} - \frac{1}{s_2(s - s_2)}\right] \\ - \frac{g}{s_1 s_2}\left(\frac{1}{s}\right) \end{aligned} \tag{9}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = -\frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1}e^{s_1 t} - \frac{1}{s_2}e^{s_2 t}\right]u(t) \tag{10}$$

$$\begin{aligned} \implies x(t) = -\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-ct/2m}u(t) \\ \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}{gc}\right)\right) \\ - \frac{mg}{k}u(t) \end{aligned} \tag{11}$$

(Substituting the values of s_1 and s_2 from (7) and (8))

From (11), we have the ratio of 3^{rd} to 4^{th} amplitude,

$$\begin{aligned} -\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-3cT/2m} = \\ -\frac{3}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-4cT/2m} \end{aligned} \tag{12}$$

$$\implies e^{\pi c/\sqrt{mk}} = \frac{3}{2} \tag{13}$$

$$\implies c = \frac{\sqrt{mk}\ln\frac{3}{2}}{\pi} \tag{14}$$