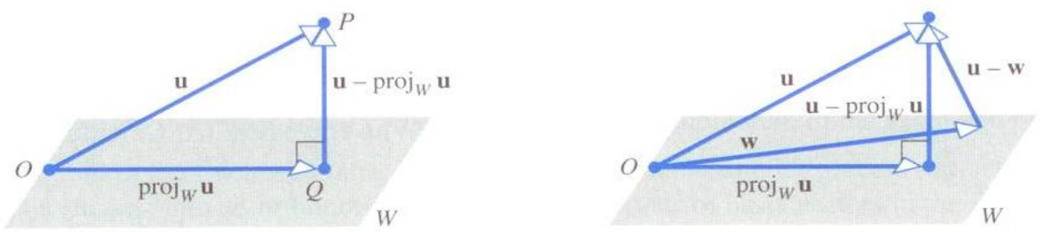
Chapter Content

- Inner Products
- Angle and Orthogonality in Inner Product Spaces
- Orthonormal Bases; Gram-Schmidt Process; QR-Decomposition
- Best Approximation; Least Squares
- Change of Basis

6-4 Orthogonal Projections Viewed as Approximations

- If P is a point in 3-space and W is a plane through the origin, then the point Q in W closest to P is obtained by dropping a perpendicular from P to W.



(a) Q is the point in W closest to P. (b) $\|\mathbf{u} - \mathbf{w}\|$ is minimized by $\mathbf{w} = \text{proj}_W\,\mathbf{u}$.

Figure 6.4.1

- If we let $\mathbf{u} = OP$, the distance between P and W is given by $\|\ \mathbf{u} - \text{proj}_W\mathbf{u}\ \|$.
- In other words, among all vectors $\mathbf{w}$ in W the vector $\mathbf{w} = \text{proj}_W\mathbf{u}$ minimize the distance $\|\ \mathbf{v} - \mathbf{w}\ \|$.

6.4
BEST APPROXIMATION; LEAST SQUARES

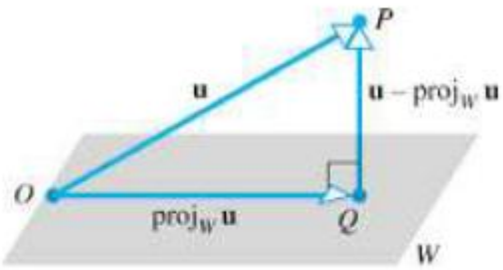
In this section we shall show how orthogonal projections can be used to solve certain approximation problems. The results obtained in this section have a wide variety of applications in both mathematics and science.

Orthogonal Projections Viewed as Approximations

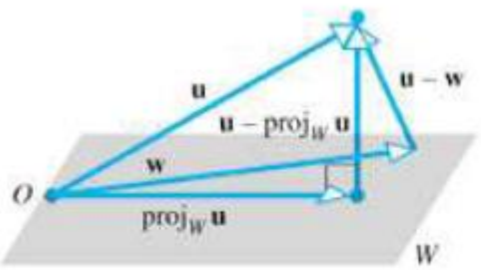
If P is a point in ordinary 3-space and W is a plane through the origin, then the point Q in W that is closest to P can be obtained by dropping a perpendicular from P to W (Figure 6.4.1*a*). Therefore, if we let $\mathbf{u} = \overrightarrow{OP}$, then the distance between P and W is given by

$$\|\mathbf{u} - \text{proj}_W \mathbf{u}\|$$

In other words, among all vectors w in W, the vector $\mathbf{w} = \text{proj}_W \mathbf{u}$ minimizes the distance $\|\mathbf{u} - \mathbf{w}\|$ (Figure 6.4.1*b*).



(*a*) Q is the point in W closest to P.



(*b*) $\|\mathbf{u} - \mathbf{w}\|$ is minimized by $\mathbf{w} = \text{proj}_W \mathbf{u}$.

Figure 6.4.1

6-4 Best Approximation

- Remark
  - Suppose $\mathbf{u}$ is a vector that we would like to approximate by a vector in W.
  - Any approximation $\mathbf{w}$ will result in an "error vector" $\mathbf{u} - \mathbf{w}$ which, unless $\mathbf{u}$ is in W, cannot be made equal to $\mathbf{0}$.
  - However, by choosing $\mathbf{w} = \text{proj}_W\mathbf{u}$ we can make the length of the error vector
    $\| \mathbf{u} - \mathbf{w} \| = \| \mathbf{u} - \text{proj}_W\mathbf{u} \|$ as small as possible.
  - Thus, we can describe $\text{proj}_W\mathbf{u}$ as the "*best approximation*" to $\mathbf{u}$ by the vectors in W.

THEOREM 6.4.1

Best Approximation Theorem

If W is a finite-dimensional subspace of an inner product space V, and if $\mathbf{u}$ is a vector in V, then $\text{proj}_W \mathbf{u}$ is the best approximation to $\mathbf{u}$ from W in the sense that

$$\|\mathbf{u} - \text{proj}_W \mathbf{u}\| < \|\mathbf{u} - \mathbf{w}\|$$

for every vector $\mathbf{w}$ in W that is different from $\text{proj}_W \mathbf{u}$.

Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns, find a vector $\mathbf{x}$, if possible, that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on R^m. Such a vector is called a ***least squares solution*** of $A\mathbf{x} = \mathbf{b}$.

To solve the least squares problem, let W be the column space of A. For each $n \times 1$ matrix $\mathbf{x}$, the product $A\mathbf{x}$ is a linear combination of the column vectors of A. Thus, as $\mathbf{x}$ varies over R^n, the vector $A\mathbf{x}$ varies over all possible linear combinations of the column vectors of A; that is, $A\mathbf{x}$ varies over the entire column space W. Geometrically, solving the least squares problem amounts to finding a vector $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is the closest vector in W to $\mathbf{b}$ (Figure 6.4.2).

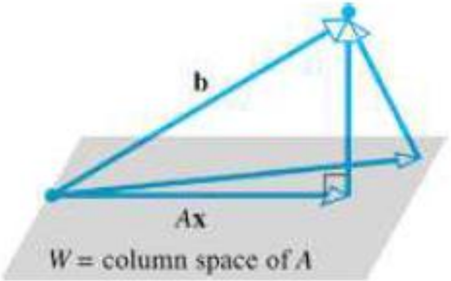


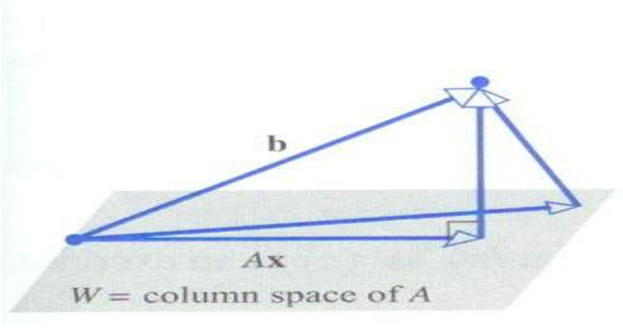
Figure 6.4.2 A least squares solution $\mathbf{x}$ produces the vector $A\mathbf{x}$ in W closest to $\mathbf{b}$.

Theorem 6.4.1 (Best Approximation Theorem)

- If W is a finite-dimensional subspace of an inner product space V, and if $\mathbf{u}$ is a vector in V,
  - then $\text{proj}_W\mathbf{u}$ is the best approximation to $\mathbf{u}$ form W in the sense that

$$\| \mathbf{u} - \text{proj}_W\mathbf{u} \| < \| \mathbf{u} - \mathbf{w} \|$$

for every vector $\mathbf{w}$ in W that is different from $\text{proj}_W\mathbf{u}$.



6-4 Least Square Problem

- Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns
  - find a vector $\mathbf{x}$, if possible, that minimize $\| A\mathbf{x} - \mathbf{b} \|$ with respect to the Euclidean inner product on R^m.
  - Such a vector is called a least squares solution of $A\mathbf{x} = \mathbf{b}$.

Theorem 6.4.2

- For *any* linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

is *consistent*, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$.

Moreover, if W is the column space of A, and $\mathbf{x}$ is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W\mathbf{b} = A\mathbf{x}$$

(or you can treat it as $A\mathbf{x} - \text{proj}_W\mathbf{b} = \mathbf{0}$)

Theorem 6.4.3

- If A is an $m \times n$ matrix, then the following are equivalent.
  - A has linearly independent column vectors.
  - $A^T A$ is invertible.

Theorem 6.4.4

- If A is an $m \times n$ matrix with linearly independent column vectors,
  - then for every $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.
  - This solution is given by

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

  - Moreover, if W is the column space of A, then the orthogonal projection of $\mathbf{b}$ on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x} = A(A^T A)^{-1} A^T \mathbf{b}$$

6-4 Example 1 (Least Squares Solution)

- Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} x_1 - x_2 &= 4 \\ 3x_1 + 2x_2 &= 1 \\ -2x_1 + 4x_2 &= 3 \end{aligned}$$

and find the orthogonal projection of **b** on the column space of A.

- Solution:

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

  - Observe that <u>A has linearly independent column vectors, so we know in advance that there is a unique least squares solution</u>.

6-4 Example 1

- We have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $A^{\mathrm{T}}A\mathbf{x} = A^T\mathbf{b}$ in this case is $\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

- Solving this system yields the least squares solution

$$x_1 = 17/95,\ x_2 = 143/285$$

- The orthogonal projection of $\mathbf{b}$ on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 17/95 \\ 143/285 \end{bmatrix} = \begin{bmatrix} -92/285 \\ 439/285 \\ 94/57 \end{bmatrix}$$

01) Find the normal system associated with the given linear system. (after Thm-6.4.2)

ⓐ $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Solu: The associated normal system is given by,

$A^T A X = A^T b$. $A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}$

$A^T A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}} = \begin{bmatrix} 1+4+16 & -1+6+20 \\ -1+6+20 & 1+9+25 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+2+20 \\ -2-3+25 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$

$\therefore$ The normal system is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

2.)

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$A \qquad x \quad = \quad b$$

$$A^T A = \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix} \quad ; \quad A^T b = \begin{pmatrix} -1 \\ 9 \\ 13 \end{pmatrix}$$

$$\Rightarrow \quad A^T A\, X = A^T b$$

$$\Rightarrow \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} -1 \\ 9 \\ 13 \end{pmatrix} \Rightarrow X = \begin{pmatrix} -0.34 \\ -0.57 \\ -0.707 \end{pmatrix}$$

2. In each part, find $\det(A^T A)$, and apply Theorem 6.4.3 to determine whether A has linearly independent column vectors.

(a) $$A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \\ 4 & -5 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$$

$\det[A^T A] = 0 \Rightarrow$ Column vectors of A are not L.I

3. Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of b onto the column space of A.

(a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$

Solution

3 a)

Sol:- First Check whether the Column Vectors are linearly Independent or not.

$\Rightarrow \quad \rho(A) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \simeq 2 =$ no. of Column vectors.

$\Rightarrow$ Linearly Independent.

least square solution of the system $Ax = b$ is given by the solution of the normalized system $A^TAX = A^Tb$.

$$A^TA = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^Tb = \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

The least square solutions are $x_1 = 5$, $x_2 = \frac{1}{2}$

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

3. **(c)** The associated normal system is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

This system has solution $x_1 = 12$, $x_2 = -3$, $x_3 = 9$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of $\mathbf{b}$ on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

which can be written as (3, 3, 9, 0).

3) (d) $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$

For L. Independency find Rank or Null$(A^T A)$.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 3 =$ No. of Column Vectors.

$\Rightarrow$ Column Vectors are L.I

$$A^T A = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Normalized system : $A^T A x = A^T b$

$$= \begin{bmatrix} 9 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 6 \end{pmatrix}$$

Least square solution $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 30 \\ 26 \end{pmatrix}$

orthogonal Projection = $\mathrm{Proj}_W b = AX = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 14 \\ 30 \\ 26 \end{bmatrix}$

$$\Rightarrow Ax = \begin{pmatrix} 2 \\ 6 \\ -2 \\ 4 \end{pmatrix} //$$

Solution
3 b)

(b)

$$x_1 = \frac{3}{7}, x_2 = -\frac{2}{3}; \begin{bmatrix} \frac{46}{21} \\ -\frac{5}{21} \\ \frac{13}{21} \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$ $b = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Soln:- Given $AX = \mathbf{b}$

$A^TA = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}$ $A^Tb = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

$\therefore$ The normalised system is $(A^TA)x = A^Tb$.

$$\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$\therefore$ Least square Solutions are $x_1 = +\frac{3}{7}$ $x_2 = -\frac{2}{3}$

The orthogonal projection of 'b' on the column space of 'A' is.

$$\operatorname{proj}_W(b) = AX = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3/7 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 46/21 \\ -5/21 \\ 13/21 \end{bmatrix}.$$

6-4 Example 2 (Orthogonal Projection on a Subspace)

- Find the orthogonal projection of the vector **u** = (-3,-3,8,9) on the subspace of R^4 spanned by the vectors

$$\mathbf{u}_1 = (3,1,0,1),\ \mathbf{u}_2 = (1,2,1,1),\ \mathbf{u}_3 = (-1,0,2,-1)$$

- Solution:
  - The subspace spanned by $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, is the column space of

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

  - If **u** is expressed as a column vector, we can find the orthogonal projection of **u** on W by finding a least squares solution of the system $A\mathbf{x} = \mathbf{u}$.
  - $\text{proj}_W\mathbf{u} = A\mathbf{x}$ from the least square solution.

6-4 Example 2

- From Theorem 6.4.4, the least squares solution is given by

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{u}$$

- That is,

$$\mathbf{x} = \left(\begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

- Thus, $\text{proj}_W\mathbf{u} = A\mathbf{x} = [-2\ 3\ 4\ 0]^T$

$$\text{proj}_W\mathbf{u} = A\mathbf{x} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

or, in horizontal notation (which is consistent with the original phrasing of the problem), $\text{proj}_W\mathbf{u} = (-2, 3, 4, 0)$.

5. Find the orthogonal projection of $\boldsymbol{u}$ onto the subspace of R^4 spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(a) $\mathbf{u} = (6, 3, 9, 6)$; $\mathbf{v}_1 = (2, 1, 1, 1)$, $\mathbf{v}_2 = (1, 0, 1, 1)$, $\mathbf{v}_3 = (-2, -1, 0, -1)$

(b) $\mathbf{u} = (-2, 0, 2, 4)$; $\mathbf{v}_1 = (1, 1, 3, 0)$, $\mathbf{v}_2 = (-2, -1, -2, 1)$, $\mathbf{v}_3 = (-3, -1, 1, 3)$

Solution

5 a)

$Ax = u$ (write all the Vectors as Column Vectors)

Though given spanned, we have to Check for L.I

? more reason.

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} X = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$$

$$\det(A^TA) = 7(9) - 4(6) - 6(6) = 63-24-36 = 63-60 = 3 \neq 0$$

$\Rightarrow$ Linearly Independent Column vectors.

$$A^Tb = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$$

Normalized system $\quad A^TAx = A^Tb \Rightarrow \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$\text{Projection}_W\, b = Ax = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}$$

6. Find the orthogonal projection of $\mathbf{u} = (5, 6, 7, 2)$ onto the solution space of the homogeneous linear system

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

Solun. $W \rightarrow$ Solun. space of homogeneous linear system.

Find the bases for the system of

$x_1 = -(x_2 + x_3)$ $\Rightarrow x_1 = -\left[\frac{-x_3}{2} + x_3 - \frac{x_4}{2}\right]$

$x_2 = \frac{-1}{2}(x_3 + x_4)$

$\Rightarrow x_1 = -\left[\frac{x_3}{2} - \frac{x_4}{2}\right]$

Let $x_3 = t$, $x_4 = r$.

$x_1 = \frac{x_4}{2} - \frac{x_3}{2}$

$\therefore x_2 = -\frac{1}{2}[t + r]$ $x_1 = \frac{1}{2}[r - t]$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(r-t) \\ -\frac{1}{2}(t+r) \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

$\therefore W \rightarrow$ ~~span~~ bases: $\{X_1, X_2\} = \left\{\left(\frac{-1}{2}, \frac{-1}{2}, 1, 0\right), \left(\frac{1}{2}, \frac{-1}{2}, 0, 1\right)\right\}$

$\therefore AX = u$

where: $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} \qquad A^Tu = \begin{bmatrix} -\frac{1}{2} & -1/2 & 1 & 0 \\ 1/2 & -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

$$\therefore (A^TA)X = (A^Tu) \Rightarrow \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$$

$\therefore$ Solu^n is $x_1 = 1$; $x_2 = 1$.

$$\therefore \text{proj}_W u = AX = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

6-4 Orthogonal Projection

- If W is a subspace of R^m,
  - then the transformation $P: R^m \rightarrow W$ that maps each vector $\mathbf{x}$ in R^m into its orthogonal projection $\text{proj}_W\mathbf{x}$ in W is called orthogonal projection of R^m on W.

- The orthogonal projections are linear operators
  - The standard matrix for the orthogonal projection of R^m on W is $[P] = A(A^TA)^{-1}A^T$

where A is constructed using any basis for W as its column vectors

10. Let W be the plane with equation $5x - 3y + z = 0$.

(a) Find a basis for W.

(b) Use Formula 6 to find the standard matrix for the orthogonal projection onto W.

(a) Find a basis for W

$$5x - 3y + z = 0 \Rightarrow \text{let } x = t \;;\; y = r$$

$$\Rightarrow z = 3r - 5t$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ r \\ 3r-5t \end{bmatrix} = t\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + r\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$\Rightarrow$ W $\to$ null space basis

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\} \Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}$$

(b) Find standard matrix for the Orthogonal projection

here we have to find P only.

$$P = A(A^TA)^{-1}A^T.$$

$$A^TA = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}_{3\times2} = \begin{bmatrix} 26 & -15 \\ -15 & 16 \end{bmatrix}$$

$$A A^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$$

$$(A^TA)^{-1} = \begin{bmatrix} 16 & 15 \\ 15 & 26 \end{bmatrix} \frac{1}{191}$$

$$A(A^TA)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 16 & 15 \\ 15 & 26 \end{bmatrix} \frac{1}{191}$$

Find $P = A(A^TA)^{-1}A^T.$

11. Let W be the line with parametric equations

$$x = 2t, \quad y = -t, \quad z = 4t \quad (-\infty < t < \infty)$$

(a) Find a basis for W.

(b) Use Formula 6 to find the standard matrix for the orthogonal projection onto W.

11. **(a)** The vector $\mathbf{v} = (2, -1, 4)$ forms a basis for the line W.

(b) If we let $A = [\mathbf{v}^T]$, then the standard matrix for the orthogonal projection on W is

$$[P] = A(A^T A)^{-1} A^T = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} \frac{1}{21} \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

6-4 Example 3

- Verifying formula $[P] = A(A^TA)^{-1}A^T$
- The standard matrix for the orthogonal projection of R3 on the xy-plane :

EXAMPLE 3 Verifying Formula (6)

In Table 5 of Section 4.2 we showed that the standard matrix for the orthogonal projection of R^3 on the xy-plane is

$$[P] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

To see that this is consistent with Formula 6, take the unit vectors along the positive x and y axes as a basis for the xy-plane, so that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We leave it for the reader to verify that A^TA is the 2×2 identity matrix; thus Formula 6 simplifies to

$$[P] = AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which agrees with 7.

07) Use formula $[P] = A(A^TA)^{-1}A^T$ and the method of Example-3 to find the standard matrix for the orthogonal projection $P: R^2 \to R^2$ onto.

(a) the x-axis (b) the y-axis.

Solution:-

(a) Take the unit vector along x-axis as a bases. for this, i.e. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A^T = [1 \ \ 0]$.

$$P = AA^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ \ 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Take the unit vector along y-axis as a bases is $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A^T = [0 \ \ 1]$.

$$P = AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

08) Use formula $[P] = A(A^TA)^{-1}A^T$ and the method of Example -3 to find the standard matrix for the orthogonal projection $P: R^3 \to R^3$ onto

(a) the xz-plane.

Solve. Take the unit vectors along x and z-axis (Bases for XZ-plane)

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) the yz-plane $\longrightarrow$ $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

09) Let W be the plane with equation $5x-3y+z=0$.

ⓐ Find a basis for W

ⓑ Use formula $[P]=A(A^TA)^{-1}A^T$ to find the standard matrix for the orthogonal projection onto W.

ⓒ Use the matrix obtained ⓑ to find the orthogonal projection of a point $P_0(x_0,y_0,z_0)$ on W.

ⓓ Find the distance b/n the point $P_0(1,-2,4)$ and the plane W, and check your result using Th 3.5.2

Sol

ⓐ $x=\frac{1}{5}[3y-z]=\frac{1}{5}[3t-r]$.

$$X=\begin{bmatrix}\frac{3t}{5}-\frac{r}{5}\\ t\\ r\end{bmatrix}=t\begin{bmatrix}3/5\\1\\0\end{bmatrix}+r\begin{bmatrix}-1/5\\0\\1\end{bmatrix} \text{ or}$$

$z=3y-5x \quad \therefore X=\begin{bmatrix}s\\t\\3t-5s\end{bmatrix}=s\begin{bmatrix}1\\0\\-5\end{bmatrix}+t\begin{bmatrix}0\\1\\3\end{bmatrix}$

$\therefore X_1=(1,0,-5) \quad X_2=(0,1,3)$

$$\therefore A=\begin{bmatrix}1&0\\0&1\\-5&3\end{bmatrix}$$

(b) $[P] = A(A^TA)^{-1}A^T$

$$= \begin{bmatrix} 10/35 & 15/35 & -5/35 \\ 15/35 & 26/35 & 3/35 \\ -5/35 & 3/35 & 34/35 \end{bmatrix}$$

(c) Orthogonal projection of P_0 is $= P \cdot P_0$

$$= \frac{1}{35}\begin{bmatrix} 10x_0 + 15y_0 - 5z_0 \\ 15x_0 + 26y_0 + 3z_0 \\ -5x_0 + 3y_0 + 34z_0 \end{bmatrix}$$

(2)

$$PP_0 = \frac{1}{35}\begin{bmatrix} -40 \\ -25 \\ 125 \end{bmatrix}$$

$$\therefore \| P_0 - PP_0 \| = \left\| \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} - PP_0 \right\| = \left\| \begin{matrix} 15/7 \\ -9/7 \\ 3/7 \end{matrix} \right\|$$

$$= \sqrt{\left(\frac{15}{7}\right)^2 + \left(-9/7\right)^2 + \left(3/7\right)^2} = 3\sqrt{35}/7$$

$$= 15/\sqrt{35}$$

Theorem 6.4.5 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The range of T_A is R^n.
  - T_A is one-to-one.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.

Theorem 6.4.5 (Equivalent Statements)

- The column vectors of A span R^n.
- The row vectors of A span R^n.
- The column vectors of A form a basis for R^n.
- The row vectors of A form a basis for R^n.
- A has rank n.
- A has nullity 0.
- The orthogonal complement of the nullspace of A is R^n.
- The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
- A^TA is invertible.

Assignments

- Exercise-6.4: Examples 1 – 4 (Pages 498-504)
- Exercise-6.4: Problems 1– 11 (Pages 505-509)