Chapter Content(EAC)

- Inner Products
- Angle and Orthogonality in Inner Product Spaces
- Orthonormal Bases; Gram-Schmidt Process; QR-Decomposition
- Best Approximation; Least Squares

6-3 Orthonormal Basis

- A set of vectors in an inner product space is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.
- An orthogonal set in which each vector has norm 1 is called orthonormal.

- Example 1
  - Let $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 0, -1)$ and assume that R^3 has the Euclidean inner product.
  - It follows that the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0.$$

DEFINITION

A set of vectors in an inner product space is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is called ***orthonormal***.

EXAMPLE 1 **An Orthogonal Set in R^3**

Let

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = (1, 0, 1), \quad \mathbf{u}_3 = (1, 0, -1)$$

and assume that R^3 has the Euclidean inner product. It follows that the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0$.

If $\mathbf{v}$ is a nonzero vector in an inner product space, then by part (*c*) of Theorem 6.2.2, the vector

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has norm 1, since

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

The process of multiplying a nonzero vector $\mathbf{v}$ by the reciprocal of its length to obtain a unit vector is called ***normalizing v***. An orthogonal set of *nonzero* vectors can always be converted to an orthonormal set by normalizing each of its vectors.

6-3 Example 2

- Let $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 0, -1)$
  - The Euclidean norms of the vectors are

$$\|\mathbf{u}_1\| = 1, \quad \|\mathbf{u}_2\| = \sqrt{2}, \quad \|\mathbf{u}_3\| = \sqrt{2}$$

  - Normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = (0,1,0), \quad \mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \mathbf{v}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

  - The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is <u>orthonormal</u> since

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0 \quad \textit{and} \quad \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$$

3. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on R^3?

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

(c) $(1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

ⓐ $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Soln $\langle u_1, u_2 \rangle = \left(\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}}\right) \neq 0$ $\quad \langle u_2, u_3 \rangle = \frac{-1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}} = 0.$

No.

ⓑ $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$\langle u_1, u_2 \rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$

$\langle u_2, u_3 \rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0.$

$\langle u_1, u_3 \rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0.$

$\therefore$ The given set of vectors are orthogonal w.r.t EIP.

c) $(1,0,0)$, $(0, 1/\sqrt{2}, 1/\sqrt{2})$, $(0,0,1)$.

$\langle u_1, u_2 \rangle = 0$; $\langle u_2, u_3 \rangle = \frac{1}{\sqrt{2}} \neq 0$; $\langle u_1, u_3 \rangle = 0$

No

d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) , \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \Rightarrow$ Yes

5. Which of the following sets of polynomials are orthonormal with respect to the inner product on P_2 discussed in Example 8 of Section 6.1?

(a) $\frac{2}{3}-\frac{2}{3}x+\frac{1}{3}x^2, \frac{2}{3}+\frac{1}{3}x-\frac{2}{3}x^2, \frac{1}{3}+\frac{2}{3}x+\frac{2}{3}x^2$

(b) $1, \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}x^2, x^2$

$\langle u, v \rangle =$ ~~...~~ $a_0 b_0 + a_1 b_1 + a_2 b_2$.

(a) $\frac{2}{3} - \frac{2x}{3} + \frac{x^2}{3}$; $\frac{2}{3} + \frac{x}{3} - \frac{2}{3}x^2$; $\frac{1}{3} + \frac{2x}{3} + \frac{2x^2}{3}$.

Soln:- $\langle p_1, p_2 \rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$ $\quad \langle p_2, p_3 \rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$.

$\langle p_1, p_3 \rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$.

$\therefore$ The given set of polynomials are orthogonal.

$\|p_1\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{1} = 1$. $\|p_2\| = 1$ $\|p_3\| = 1$.

$\therefore$ S is orthonormal set.

(b) 1 ; $\frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}$; x^2

Soln $\langle p_1, p_2 \rangle = 0$; $\langle p_2, p_3 \rangle = 0 + 0 + \frac{1}{\sqrt{2}} \neq 0$ $\langle p_1, p_3 \rangle = 0$.

$\therefore$ The given set of polynomials are not orthogonal and hence it is not an orthonormal set.

1) Which of the following set of poly are orthonormal w.r.t i.p

$$\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$$

(i) $P_1 = \frac{2}{3} - \frac{2x}{3} + \frac{x^2}{3}$; $P_2 = \frac{2}{3} + \frac{x}{3} - \frac{2x^2}{3}$, $P_3 = \frac{1}{3} + \frac{2x}{3} + \frac{2x^2}{3}$

$$\langle p_1, p_2\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$\langle p_2, p_3\rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0.$$

$$\langle p_3, p_1\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

They are orthogonal

$\Leftrightarrow \|p_1\| = \langle p_1, p_1 \rangle^{1/2} = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = \sqrt{\frac{9}{9}} = 1$

$\sqrt{a_0^2+a_1^2+a_2^2}$

$\|p_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1$ $\Rightarrow$ Their norms are also equal to 1

$\|p_3\| = 1$

$\Rightarrow$ The given polynomials are Orthonormal

(ii) $P_1 = 1$; $P_2 = \frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}$; $P_3 = x^2$

$\langle P_1, P_2 \rangle = 0.$

$\langle P_2, P_3 \rangle = \frac{1}{\sqrt{2}}$

$\langle P_3, P_1 \rangle = 0$

Not orthogonal ;

but $\|P_1\| = \|P_2\| = \|P_3\| = 1$

$\Rightarrow$ They are not Orthonormal set.

Which of the following set of matrices are orthonormal wrt I.P $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ wrt $M_{2\times 2}$

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

$D = \begin{bmatrix} 0 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

a) Sol:

$$\langle A, B\rangle = \cancel{(\pm)}\ Tr(A^T B)$$

$$\langle u, v\rangle = Tr(u^T v)$$

$$\langle A, B\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \cancel{\times} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \langle A, B\rangle = Tr = 0$$

$$\langle B, C\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$\langle C, D\rangle = 0 + \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\langle D, A\rangle = 0$$

$$\|A\| = 1 \; ; \; \|B\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1 \; ; \; \|C\| = 1 \; ; \; \|D\| = 1$$

$\Rightarrow$ They form an Orthonormal set.

(b) sol:- $\langle A, B\rangle = 0 = \langle B, C\rangle = \langle C, D\rangle = \langle D, A\rangle = 0$ //

$\|A\| = 1$

$\|B\| = 1$ $\Rightarrow$ Not an orthonormal set //.

$\|C\| = \sqrt{2}$

$\|D\| = \sqrt{2}$

7. Verify that the given set of vectors is orthogonal with respect to the Euclidean inner product; then convert it to an orthonormal set by normalizing the vectors.

(a) $(-1, 2), (6, 3)$

(b) $(1, 0, -1), (2, 0, 2), (0, 5, 0)$

(c) $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(-\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

a) $(-1,2);(6,3)$

$\langle u_1, u_2\rangle = -6+6=0$

$\|u\| = \sqrt{1+4} = \sqrt{5}$ $\|v\| = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

$\therefore$ The orthonormal set of vectors are $\left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

b) $(1,0,-1)$; $(2,0,2)$; $(0,5,0)$

$\langle u_1, u_2\rangle = 2-2=0$; $\langle u_2, u_3\rangle = 0$; $\langle u_1, u_3\rangle = 0$.

$\|u_1\| = \sqrt{2}$; $\|u_2\| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$; $\|u_3\| = \sqrt{25} = 5$.

$\therefore$ The orthonormal set of vectors are $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),\right.$

$\left.\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(0, 1, 0\right)\right\}$.

(c) $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), \left(-\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

$\langle u_1, u_2 \rangle = -\frac{1}{10} + \frac{1}{10} = 0 \quad \langle u_2, u_3 \rangle = -\frac{1}{6} + \frac{1}{6} = 0 \quad \langle u_1, u_3 \rangle = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0.$

$\|u_1\| = \sqrt{\frac{1}{25} + \frac{1}{25} + \frac{1}{25}} = \sqrt{\frac{3}{25}} = \frac{\sqrt{3}}{5}$; $\|u_2\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \frac{\sqrt{2}}{2}$.

$\|u_3\| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$.

$\therefore$ The orthonormal set of vectors are

$\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)\right\}$.

8. Verify that the set of vectors $\{(1, 0), (0, 1)\}$ is orthogonal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1 v_1 + u_2 v_2$ on R^2; then convert it to an orthonormal set by normalizing the vectors.

Soln:

$$\langle u_1, u_2 \rangle = 4 \times 0 + 0 = 0.$$

$\therefore \{(1,0), (0,1)\}$ is orthogonal w.r.t the IP given.

$$\|u\| = \sqrt{4u_1^2 + u_2^2} = \sqrt{4} = 2 \quad ; \quad \|v\| = \sqrt{4v_1^2 + v_2^2} = \sqrt{0+1} = 1.$$

$\therefore$ The orthonormal set is $\left\{\left(\frac{1}{2}, 0\right), (0,1)\right\}$.

6-3 Orthonormal Basis

- Theorem 6.3.1*
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

- Remark
  - The scalars $\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle$ are the coordinates of the vector $\mathbf{u}$ relative to the orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle)$$

is the coordinate vector of $\mathbf{u}$ relative to this basis

6-3 Example 3

- Let $\mathbf{v}_1 = (0, 1, 0)$, $\mathbf{v}_2 = (-4/5, 0, 3/5)$, $\mathbf{v}_3 = (3/5, 0, 4/5)$.
  It is easy to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for R^3 with the Euclidean inner product.
  Express the vector $\mathbf{u} = (1, 1, 1)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_s$.

- Solution:
  - $\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1$, $\langle \mathbf{u}, \mathbf{v}_2 \rangle = -1/5$, $\langle \mathbf{u}, \mathbf{v}_3 \rangle = 7/5$
  - Therefore, by Theorem 6.3.1 we have $\mathbf{u} = \mathbf{v}_1 - 1/5\ \mathbf{v}_2 + 7/5\ \mathbf{v}_3$
  - That is, $(1, 1, 1) = (0, 1, 0) - 1/5\ (-4/5, 0, 3/5) + 7/5\ (3/5, 0, 4/5)$
  - The coordinate vector of $\mathbf{u}$ relative to S is

$$(\mathbf{u})_s = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle) = (1, -1/5, 7/5)$$

9. Verify that the vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $\mathbf{v}_3 = (0, 0, 1)$ form an orthonormal basis for R^3 with the Euclidean inner product; then use Theorem 6.3.1 to express each of the following as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(a) $(1, -1, 2)$

(b) $(3, -7, 4)$

(c) $\left(\frac{1}{7}, -\frac{3}{7}, \frac{5}{7}\right)$

(a) $(1,-1,2)$ Express as linear Combination of V_1, V_2, V_3 using theorem 6.3.1

$$(u)_s = (\langle u, V_1\rangle V_1, \langle u, V_2\rangle, \langle u, V_3\rangle)$$

$$= \left(\frac{-3}{5} - \frac{4}{5}, \frac{4}{5} - \frac{3}{5}, 2\right) = \left(\frac{-7}{5}, \frac{1}{5}, 2\right)$$

(b) $\left(\frac{1}{7}, \frac{-3}{7}, \frac{5}{7}\right)$

$$(u)_s = (\langle u, V_1\rangle \ \langle u, V_2\rangle \ \langle u, V_3\rangle)$$

$$= \left(\frac{-3}{35} \frac{-12}{35}, \frac{4}{35} \frac{-9}{35}, \frac{5}{7}\right)$$

$$= \left(\frac{-3}{7}, \frac{-1}{7}, \frac{5}{7}\right).$$

11. In each part, an orthonormal basis relative to the Euclidean inner product is given. Use Theorem 6.3.1 to find the coordinate vector of w with respect to that basis.

(a) $\mathbf{w}=(3,7);\ \mathbf{u}_1=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\ \mathbf{u}_2=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

(b) $\mathbf{w}=(-1,0,2);\ \mathbf{u}_1=\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right),\ \mathbf{u}_2=\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right),\ \mathbf{u}_3=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$

11)(a) $W=(3,7)$ $\quad u_1=\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)\ u_2=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

orthonormal basis

$(W)_S=(\langle W,u_1\rangle,\langle W,u_2\rangle)=\left(\frac{-4}{\sqrt{2}},\frac{10}{\sqrt{2}}\right)$ //

11)(b) $W=(-1,0,2)$ $\quad u_1=\left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right),\ u_2=\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)\ u_3=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$

$(W)_S=$ ~~(<W~~ $=\left(\frac{-2}{3}+\frac{2}{3},\frac{-2}{3}-\frac{4}{3},\frac{-1}{3}+\frac{4}{3}\right)$

$=(0,-2,1)$ //

12. Let R^2 have the Euclidean inner product, and let $S = \{\mathbf{w}_1, \mathbf{w}_2\}$ be the orthonormal basis with $\mathbf{w}_1 = \left(\frac{3}{5}, -\frac{4}{5}\right)$, $\mathbf{w}_2 = \left(\frac{4}{5}, \frac{3}{5}\right)$.

(a) Find the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that have coordinate vectors $(\mathbf{u})_S = (1, 1)$ and $(\mathbf{v})_S = (-1, 4)$.

(b) Compute $\|\mathbf{u}\|$, $d(\mathbf{u}, \mathbf{v})$, and $\langle \mathbf{u}, \mathbf{v} \rangle$ by applying Theorem 6.3.2 to the coordinate vectors $(\mathbf{u})_S$ and $(\mathbf{v})_S$; then check the results by performing the computations directly on $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

(a) given $\langle u, w_1 \rangle = 1$ & $\langle u, w_2 \rangle = 1$.

$\therefore u = \langle u, w_1 \rangle w_1 + \langle u, w_2 \rangle w_2 = 1(3/5, -4/5) + 1\left(\frac{4}{5}, 3/5\right)$

$u = \left(\frac{7}{5}, -1/5\right)$

$\langle v, w_1 \rangle = -1$; $\langle v, w_2 \rangle = 4$.

$\therefore v = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 = (-1)\left(3/5, \frac{-4}{5}\right) + 4\left(\frac{4}{5}, 3/5\right)$

$= \left(\frac{13}{5}, \frac{16}{5}\right)$.

(b)

$\|u\| = \sqrt{u_1^2 + u_2^2} = \sqrt{2}$;

$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} = \sqrt{(1+1)^2 + (1-4)^2} = \sqrt{4+9}$

$= \sqrt{13}$

& $\langle u, v \rangle = u_1 v_1 + u_2 v_2$

$= (1)(-1) + (1)(4) = 3$.

Theorem 6.3.2

- If S is an orthonormal basis for an n-dimensional inner product space, and if $(\mathbf{u})_s = (u_1, u_2, \ldots, u_n)$ and $(\mathbf{v})_s = (v_1, v_2, \ldots, v_n)$ then:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Example 4 (Calculating Norms Using Orthonormal Bases)
  - u=(1, 1, 1)

EXAMPLE 4 Calculating Norms Using Orthonormal Bases

If R^3 has the Euclidean inner product, then the norm of the vector $\mathbf{u} = (1, 1, 1)$ is

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

However, if we let R^3 have the orthonormal basis S in the last example, then we know from that example that the coordinate vector of u relative to S is

$$(\mathbf{u})_S = \left(1, -\frac{1}{5}, \frac{7}{5}\right)$$

The norm of u can also be calculated from this vector using part (a) of Theorem 6.3.2. This yields

$$\|\mathbf{u}\| = \sqrt{1^2 + \left(-\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{75}{25}} = \sqrt{3}$$

14. In each part, S represents some orthonormal basis for a four-dimensional inner product space. Use the given information to find $\|\mathbf{u}\|$, $\|\mathbf{v}-\mathbf{w}\|$, $\|\mathbf{v}+\mathbf{w}\|$, and $\langle \mathbf{v}, \mathbf{w} \rangle$.

(a) $(\mathbf{u})_S = (-1, 2, 1, 3)$, $(\mathbf{v})_S = (0, -3, 1, 5)$, $(\mathbf{w})_S = (-2, -4, 3, 1)$

(b) $(\mathbf{u})_S = (0, 0, -1, -1)$, $(\mathbf{v})_S = (5, 5, -2, -2)$, $(\mathbf{w})_S = (3, 0, -3, 0)$

ⓐ $(u)_S = (-1, 2, 1, 3)$; $(v)_S = (0, -3, 1, 5)$; $(w)_S = (-2, -4, 3, 1)$

Solution:- $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{1+4+1+9} = \sqrt{15}$.

$\|v-w\| = \|(2, 1, -2, 4)\| = \sqrt{4+1+4+16} = \sqrt{25} = 5$.

$\|v+w\| = \|(-2, -7, 4, 6)\| = \sqrt{4+49+16+36} = \sqrt{105}$.

$\langle v, w \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$.

$= 0 + 12 + 3 + 5 = 20$

ⓑ $(u)_S = (0, 0, -1, -1)$; $(v)_S = (5, 5, -2, -2)$; $(w)_S = (3, 0, -3, 0)$

$\|u\| = \sqrt{2}$; $\|v-w\| = \sqrt{34}$; $\|v+w\| = \sqrt{118}$;

$\langle v, w \rangle = 21$.

6-3 Coordinates Relative to Orthogonal Bases

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space V, then normalizing each of these vectors yields the <u>orthonormal basis</u>

$$S' = \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \cdots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\}$$

- Thus, if $\mathbf{u}$ is any vector in V, it follows from theorem 6.3.1 that

$$\mathbf{u} = \left\langle \mathbf{u}, \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \right\rangle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \left\langle \mathbf{u}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\rangle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} + \cdots + \left\langle \mathbf{u}, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\rangle \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$$

  or

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n$$

- The above equation expresses $\mathbf{u}$ as a linear combination of the vectors in the orthogonal basis S.

Theorem 6.3.3

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space
  - then S is linearly independent.

- Remark
  - By working with orthonormal bases, the computation of general norms and inner products can be reduced to the computation of Euclidean norms and inner products of the coordinate vectors.

Theorem 6.3.4 (Projection Theorem)

- If W is a finite-dimensional subspace of an product space V,
  - then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

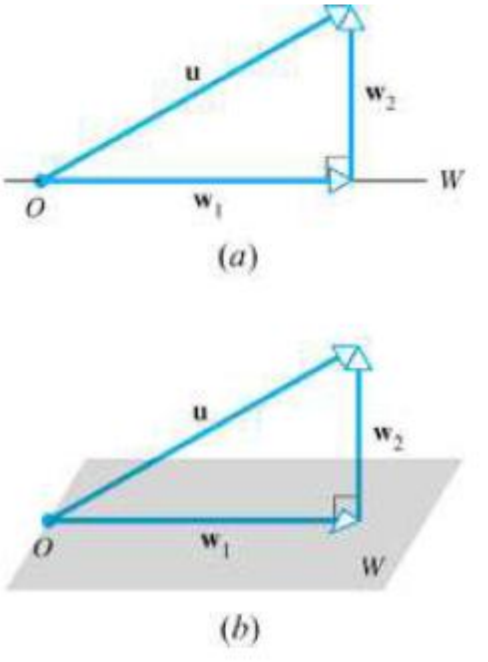


Figure 6.3.1

THEOREM 6.3.4

Projection Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \tag{3}$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$.

The vector $\mathbf{w}_1$ in the preceding theorem is called the ***orthogonal projection of u on W*** and is denoted by $\text{proj}_W \mathbf{u}$. The vector $\mathbf{w}_2$ is called the ***component of u orthogonal to W*** and is denoted by $\text{proj}_{W^\perp}\mathbf{u}$. Thus Formula 3 in the Projection Theorem can be expressed as

$$\mathbf{u} = \text{proj}_W \mathbf{u} + \text{proj}_{W^\perp} \mathbf{u} \tag{4}$$

Since $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ it follows that

$$\text{proj}_{W^\perp} \mathbf{u} = \mathbf{u} - \text{proj}_W \mathbf{u}$$

so Formula 4 can also be written as

$$\mathbf{u} = \text{proj}_W \mathbf{u} + (\mathbf{u} - \text{proj}_W \mathbf{u}) \tag{5}$$

(Figure 6.3.2).

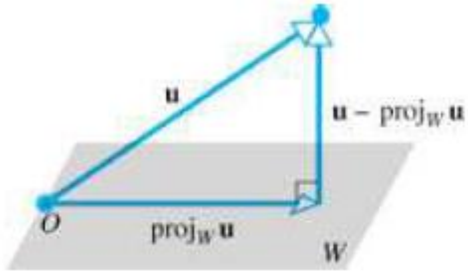


Figure 6.3.2

THEOREM 6.3.5

Let W be a finite-dimensional subspace of an inner product space V.

(a) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *is an orthonormal basis for W, and* **u** *is any vector in V, then*

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r\rangle \mathbf{v}_r \tag{6}$$

(b) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *is an orthogonal basis for W, and* **u** *is any vector in V, then*

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r\rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{7}$$

Theorem 6.3.5

- Let W be a finite-dimensional subspace of an inner product space V.
  - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W\mathbf{u} = \langle \mathbf{u},\mathbf{v}_1\rangle\, \mathbf{v}_1 + \langle \mathbf{u},\mathbf{v}_2\rangle\, \mathbf{v}_2 + \ldots + \langle \mathbf{u},\mathbf{v}_r\rangle\, \mathbf{v}_r$$

  - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W\,\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r\rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

Need Normalization

6-3 Example 6

- Let R^3 have the Euclidean inner product, and let W be the subspace spanned by the <u>orthonormal</u> vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-4/5, 0, 3/5)$.
- From the above theorem, the orthogonal projection of $\mathbf{u} = (1, 1, 1)$ on W is $\text{proj}_w \mathbf{u} = <\mathbf{u}, \mathbf{v}_1> \mathbf{v}_1 + <\mathbf{u}, \mathbf{v}_2> \mathbf{v}_2$

$$= (1)(0,\ 1,\ 0) + (-\frac{1}{5})(-\frac{4}{5},\ 0,\ \frac{3}{5}) = (\frac{4}{25},\ 1,\ -\frac{3}{25})$$

- The component of $\mathbf{u}$ orthogonal to W is

$$\text{proj}_{w^\perp} \mathbf{u} = \mathbf{u} - \text{proj}_w \mathbf{u} = (1,\ 1,\ 1) - (\frac{4}{25},\ 1,\ -\frac{3}{25}) = (\frac{21}{25},\ 0,\ \frac{28}{25})$$

- Observe that $\text{proj}_{W^\perp}\mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

1) The subspace of R^3 spanned by vectors $u_1 = \left(\frac{4}{5}, 0, \frac{-3}{5}\right)$, $u_2 = (0, 1, 0)$ is a plane passing through origin. Express vector $V = (1, 2, 3)$ in the form of $w_1 + w_2$, where w_1 lies in the plane and w_2 is $\perp$er to plane.

Sol:-

$$V = (1, 2, 3)$$

$$V = w_1 + w_2$$

$$= \text{Proj}_W(V) + \text{Proj}_{W^\perp}(V)$$

$$\langle u_1, u_2 \rangle = 0$$
$$\|u_1\| = \|u_2\| = 0$$
} orthonormal basis

$$\text{Proj}_W(V) = \cancel{\langle} \langle V, u_1 \rangle u_1 + \langle V, u_2 \rangle u_2 +$$

$$= \left[\left(\frac{4}{5} - \frac{9}{5}\right)\left(\frac{4}{5}, 0, \frac{-3}{5}\right)\right] + \left[(2)(0, 1, 0)\right]$$

$$= \left(\frac{-4}{5}, 2, \frac{3}{5}\right)$$

$$\text{Proj}_{W^\perp}(V) = (1, 2, 3) - \left(\frac{-4}{5}, 2, \frac{3}{5}\right)$$

$$= \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

2) $u_1 = (1,1,2)$; $u_2 = (2,0,-1)$; $v = (1,2,3)$

$\Rightarrow$ They are orthogonal But not orthonormal

$\cancel{v = \mathrm{Proj}_W(u) + \mathrm{Proj}_{W^\perp}(u)}$

$$\mathrm{Proj}_W(v) = \frac{\langle v, u_1\rangle u_1}{\|u_1\|^2} + \frac{\langle v, u_2\rangle u_2}{\|u_2\|^2}$$

$$= \frac{9}{6}(1,1,2) + \frac{-1}{5}(2,0,-1)$$

$\frac{3}{2} - \frac{2}{5}$

$$= \left(\frac{11}{10}, \frac{3}{2}, \frac{16}{5}\right)$$

$$\mathrm{Proj}_{W^\perp}(v) = (1,2,3) - \left(\frac{11}{10}, \frac{3}{2}, \frac{16}{5}\right)$$

$$= \left(\frac{-1}{10}, \frac{1}{2}, \frac{-1}{5}\right)$$

6-3 Finding Orthogonal/Orthonormal Bases

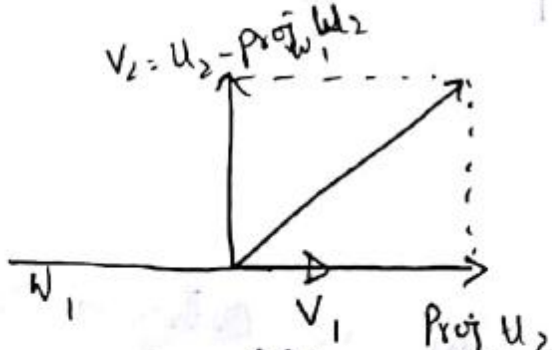
- Theorem 6.3.6
  - Every nonzero finite-dimensional inner product space has an orthonormal basis.

- Remark
  - The step-by-step construction for converting an arbitrary basis into an orthogonal basis is called the Gram-Schmidt process.

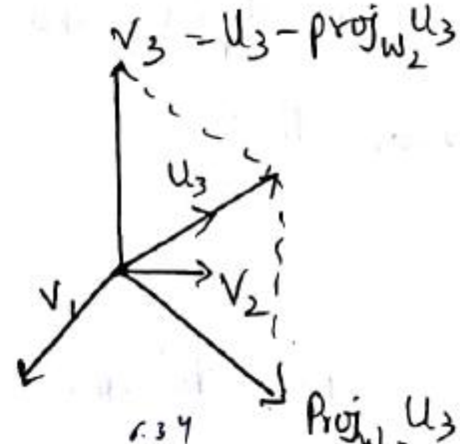
Theorem 6.3.6

Every non zero finite dimensional inner product space has an orthonormal basis.

Proof: Let V be any non zero finite dimensional inner product space and suppose that $\{u_1, u_2, u_3 \dots, u_n\}$ is any basis for V. It suffices to show that V has an orthogonal basis, since the vectors in the orthogonal basis can be normalized to produce an orthonormal basis for V. The following sequence of steps will produce an orthogonal basis $\{v_1, v_2 \dots v_n\}$ for V.



6.3.3



6.3.4

<u>Step 1</u> :- Let $V_1 = U_1$; $W_1 = \{V_1\}$

<u>Step 2</u> :- As illustrated in 6.3.3, we can obtain a vector V_2 that is obtained by Construction Component of U_2 that is Orthogonal to the space W_1

$$V_2 = U_2 - proj_{W_1} U_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{\|V_1\|^2} V_1$$

of Course, if $V_2 = 0$ then V_2 is not a basis vector.

But this cannot happen it would then follow from the preceding formula V_2 that.

$$u_2 = \frac{\langle u_2, V_1 \rangle}{\|V_1\|^2} V_1 = \frac{\langle u_2 \; V_1 \rangle}{\|u_1\|^2} u_1$$

which says u_2 is a multiple of u_1 Contradicting the linear independency of basis $S = \{u_1, u_2 \dots u_n\}$

Step 3: To construct a vector V_3 that is orthogonal to both V_1 and V_2 we . component of u_3 orthogonal to the space W_2 spanned by ~~V_1, V_2~~ ~~an~~ V_1 and V_2

$$V_3 = u_3 - \text{proj}_{W_2} u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} V_1 - \frac{\langle u_3, V_2 \rangle}{\|V_2\|^2} V_2$$

As in step(2) the linear independence of $\{u_1, u_2 \cdots u_n\}$ ensures that step 4: To determine a vector V_4 that is orthogonal to V_1, V_2 and V_3 we Component of U_4 orthogonal to the space W_3 spanned by V_1, V_2 and V_3

$$V_4 = U_4 - \text{proj}_{W_3} U_4 = U_4 - \frac{\langle U_4, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle U_4, V_2\rangle}{\|V_2\|^2} V_2 - \frac{\langle U_4, V_3\rangle}{\|V_3\|^2} V_3$$

Continuing this way, we will obtain after n steps an orthogonal set $\{v_1, v_2 \dots v_n\}$. Since V is n dimensional and every orthogonal set is linear independent $\{v_1, v_2 \dots v_n\}$ is orthogonal basis.

The preceding step by step construction for converting an arbitrary basis orthogonal basis is called Gram Schmidt process

IIly

$$V_n = u_n - \frac{\langle u_n, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_n, v_2 \rangle}{\|v_2\|^2} v_2 - \dots - \frac{\langle u_n, v_{n-1} \rangle}{\|v_{n-1}\|^2} \cdot v_{n-1}$$

6-3 Example 7(Gram-Schmidt Process)

- Consider the vector space R^3 with the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1, 1, 1),\ \mathbf{u}_2 = (0, 1, 1),\ \mathbf{u}_3 = (0, 0, 1)$$

  into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

- Solution:
  - Step 1: Let $\mathbf{v}_1 = \mathbf{u}_1$.That is, $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$
  - Step 2: Let $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2$. That is,

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{<\mathbf{u}_2,\ \mathbf{v}_1>}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$= (0,\ 1,\ 1) - \frac{2}{3}(1,\ 1,\ 1) = (-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3})$$

6-3 Example 7(Gram-Schmidt Process)

We have two vectors in W_2 now!

- Step 3: Let $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3$. That is,

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{<\mathbf{u}_3, \mathbf{v}_1>}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{<\mathbf{u}_3, \mathbf{v}_2>}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$= (0, 1, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (0, -\frac{1}{2}, \frac{1}{2})$$

- Thus, $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (-2/3, 1/3, 1/3)$, $\mathbf{v}_3 = (0, -1/2, 1/2)$ form an orthogonal basis for R^3. The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{3}, \ \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \ \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for R^3 is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \ \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = (-\frac{2}{\sqrt{6}}, \frac{1}{6}, \frac{1}{\sqrt{6}}),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = (0, -\frac{1}{\sqrt{2}}, \frac{1}{2})$$

17/7/18

1. Let R^2 have a E.I.P. Use Gram Smith process to transform the basis $S = \left\{ \underset{\tilde{u}_1}{\begin{pmatrix} 4 \\ 2 \end{pmatrix}} \underset{\tilde{u}_2}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} \right\}$ into an orthogonal basis.

Sol:-

Step 1:- $v_1 = u_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$; $W_1 = \{ v_1 \}$

$$\|v_1\|^2 = 20$$

Step 2:- $v_2 = u_2 - \operatorname{proj}_{W_1} u_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{10}{20} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \|v_2\|^2 = 5$$

$W_2 = \{ v_1 \, v_2 \} \longrightarrow$ Orthogonal

$\langle v_1, v_2 \rangle = 0$

2.) Let R^3 have an euclidian Inner Product. Use Gramsmith process to transform $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ into orthogonal basis.

Step 1:- $V_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $W_1 = \{V_1\}$

$\Rightarrow \|V_1\|^2 = 3$

Step 2:- $V_2 = u_2 - \text{proj}_W u_2 = u_2 - \dfrac{\langle u_2 ; V_1 \rangle}{\|V_1\|^2} V_1$

$= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \dfrac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix}$

$\|V_2\|^2 = \dfrac{8}{3}$

$W_2 = \{V_1, V_2\}$

Step 3: $V_3 = U_3 - \mathrm{proj}_W U_3 = U_3 - \frac{\langle U_3, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle U_3, V_2\rangle}{\|V_2\|^2} V_2$

$$= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$W_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \langle V_1, V_2\rangle = \langle V_2, V_3\rangle = \langle V_3, V_1\rangle = 0.$$

1) Let R^3 have E.I.P use Gramsmith process to transform the given vector $\{(0,0,1)\ (0,1,1)\ (1,1,1)\}$ into a Ortho normal basis.

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Step 1:- $V_1 = u_1 = (0,0,1) \Rightarrow W_1 = \{V_1\}$

Step 2:- $V_2 = u_2 - \text{proj}_W(\cancel{v}u_2) = u_2 - \frac{\langle u_2\ v_1\rangle}{\|V_1\|^2} V_1$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{(1)}{1}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$W_2 = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

Step 3:- $V_3 = \cancel{u_3 - \text{proj}(u_3\ V_1) - \text{proj}(u_3}$

$u_3 - \text{proj}_W u_3$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$W_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow \cancel{W_3 = \langle v} \quad \langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_3, v_2 \rangle = 0 \,//$$

17. Let R^3 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

(a) $\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (-1, 1, 0), \mathbf{u}_3 = (1, 2, 1)$

(b) $\mathbf{u}_1 = (1, 0, 0), \mathbf{u}_2 = (3, 7, -2), \mathbf{u}_3 = (0, 4, 1)$

(a) $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$; $u_3 = (1,2,1)$

solution: Step-1: Let $v_1 = u_1 = (1,1,1)$ $\qquad W_1 = \{v_1\}$.

Step-2: $v_2 = u_2 - \text{proj}_{W_1} u_2$ $\qquad \langle u_2, v_1\rangle = -1+1+0 = 0$

$\|v_1\| = \sqrt{3}$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 = (-1,1,0) - 0.$$

$\Rightarrow v_2 = (-1,1,0)$ $\qquad \therefore W_2 = \{v_1, v_2\}$

Step-3: $v_3 = u_3 - \text{proj}_{W_2} u_3$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$\langle u_3, v_1\rangle = 1+2+1 = 4$

$$v_3 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$\langle u_3, v_2\rangle = -1+2+0 = 1.$

$$= \left(1 - \frac{4}{3} + \frac{1}{2},\ 2 - \frac{4}{3} - \frac{1}{2},\ 1 - \frac{4}{3} - 0\right)$$

$\|v_2\| = \sqrt{2}$

$$v_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

∴ The set $v_1 = (1,1,1)$, $v_2 = (-1,1,0)$ $v_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$ is an orthogonal set.

Since R^3 is 3-D & every orthogonal set is L.I.

$\Rightarrow \{v_1, v_2, v_3\}$ is an orthogonal basis for V.

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) ; \quad q_2 = \frac{1}{\sqrt{2}}(-1,1,0)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{6}\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right).$$

$\|v_3\| = \sqrt{\frac{1}{36} + \dots}$

$= \sqrt{\frac{2}{36} + \dots}$

$= \sqrt{\frac{2+\dots}{36}}$

$= \sqrt{\frac{6}{36}}$

$= \sqrt{\frac{1}{6}}$

$= \frac{1}{\sqrt{\dots}}$

∴ $\{q_1, q_2, q_3\}$ is an orthonormal basis for R^3.

17. Let R^3 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

(a) $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

(b) $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (3, 7, -2)$, $\mathbf{u}_3 = (0, 4, 1)$

17. (a) Let

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Since $\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = 0$, we have

$$\mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Since $\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = \frac{4}{\sqrt{3}}$ and $\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = \frac{1}{\sqrt{2}}$, we have

$$\mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2$$

$$= (1, 2, 1) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

This vector has norm $\left\|\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)\right\| = \frac{1}{\sqrt{6}}$. Thus

$$\mathbf{v}_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$$

and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the desired orthonormal basis.

17.

(a) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$

(b) $(1, 0, 0), \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right), \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$

THEOREM 6.3.7

QR-Decomposition

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Problem If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix with orthonormal column vectors that results from applying the Gram–Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and the orthonormal column vectors of Q are $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$; thus

$$A = [\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_n] \quad \text{and} \quad Q = [\mathbf{q}_1 | \mathbf{q}_2 | \cdots | \mathbf{q}_n]$$

It follows from Theorem 6.3.1 that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are expressible in terms of the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ as

$$\begin{aligned}
\mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_1, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_1, \mathbf{q}_n \rangle \mathbf{q}_n \\
\mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_2, \mathbf{q}_n \rangle \mathbf{q}_n \\
&\ \vdots \\
\mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n
\end{aligned}$$

Recalling from Section 1.3 that the jth column vector of a matrix product is a linear combination of the column vectors of the first factor with coefficients coming from the jth column of the second factor, it follows that these relationships can be expressed in matrix form as

$$[\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_n] = [\mathbf{q}_1 | \mathbf{q}_2 | \cdots | \mathbf{q}_n] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n \rangle & \langle \mathbf{u}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}$$

or more briefly as

$$A = QR \tag{8}$$

However, it is a property of the Gram–Schmidt process that for $j \geq 2$, the vector $\mathbf{q}_j$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}$; thus, all entries below the main diagonal of R are zero,

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix} \tag{9}$$

Theorem 6.3.7 (*QR*-Decomposition)

- If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

EXAMPLE 8 *QR*-Decomposition of a 3×3 Matrix

Find the QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution from examples 7

EXAMPLE 7 Using the Gram–Schmidt Process

Consider the vector space R^3 with the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_1 = (0, 0, 1)$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

Solution

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

Step 2.

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$
$$= (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3.

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$
$$= (0, 1, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Thus

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

form an orthogonal basis for R^3. The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for R^3 is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \qquad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$
$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

and from 9 the matrix R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Thus the QR-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}} \underset{R}{\begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}}$$

6-3 Example 8
(QR-Decomposition of a 3×3 Matrix)

- Find the *QR*-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- Solution:
  - The column vectors A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

  - Applying the Gram-Schmidt process with subsequent normalization to these column vectors yields the orthonormal vectors

$$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix},\ \mathbf{q}_2 = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix},\ \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \Rightarrow \quad Q$$

6-3 Example 8

- The matrix R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

- Thus, the QR-decomposition of A is

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}} \underset{R}{\begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}}$$

1) $A = \begin{array}{c} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \end{array}$

2. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

1) Sol:-

$V_1 = u_1 = (-1, 1, -1, 1)$

$V_2 = u_2 - \text{Proj}_W u_2$

$= (-1, 3, -1, 3) - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$= (-1, 3, -1, 3) - 2(-1, 1, -1, 1)$

$= (1, 1, 1, 1)$ //

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ 0 & 0 & -4 \\ 0 & 2 & 8 \end{bmatrix} \Rightarrow$ L.Indepe

$$V_3 = \cancel{u_3} \; u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (1,3,5,7) - 1(-1,1,-1,1) - 4(1,1,1,1)$$

$$= (-2,-2,2,2)_{//}$$

$$\langle V_1, V_2 \rangle = \langle V_2, V_3 \rangle = \langle V_3, V_1 \rangle = 0_{//} \rightarrow \text{Orthogonal basis}$$

$$q_1 = \frac{V_1}{\|V_1\|} = \left(\frac{-1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{-1}{\sqrt{4}}, \frac{1}{\sqrt{4}} \right) = \left(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2} \right)$$

$$q_2 = \frac{V_2}{\|V_2\|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).$$

$$q_3 = \frac{V_3}{\|V_3\|} = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

$$Q = [q_1 \;\; q_2 \;\; q_3] = \cancel{-\tfrac{1}{2}} \; \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{4\times 3}$$

$$\cancel{A} = R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$$

$$A = QR = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{4\times 3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & -2 & 2 \\ 2 & 6 & 6 \\ -2 & -2 & 10 \\ 2 & 6 & 14 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

2) $A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ u_1 \; u_2 \; u_3 \end{array} \cong \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow$ L. Independent.

Sol:-

$$V_1 = u_1 = (1,1,0) \Rightarrow W_1 = \{V_1\}$$

$$V_2 = u_2 - \text{Proj}_W u_2 = u_2 - \frac{\langle u_2, V_1 \rangle}{\|V_1\|^2} V_1$$

$$= (1,0,1) - \frac{1}{2}(1,1,0) = \left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$V_3 = u_3 - \text{Proj}_W u_3 = u_3 - \frac{\langle u_3, V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle u_3, V_2 \rangle}{\|V_2\|^2} V_2$$

$$= (0,1,1) - \frac{1}{2}(1,1,0) - \frac{\cancel{2}}{\cancel{2} \times 3}\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$= (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(\frac{1}{6}, \frac{-1}{6}, \frac{1}{3}\right)$$

$$= \left(\frac{-2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$A \cdot A^T = I$
only if the

Only if they are orthogonal

$$q_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1,0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$q_2 = \frac{V_2}{\|V_2\|} = \frac{\left(\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{3}{2}}} = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$q_3 = \frac{V_3}{\|V_3\|} = \frac{\left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)}{\frac{2}{\sqrt{3}}} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$Q = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$$

$$Q = [q_1 \quad q_2 \quad q_3]$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{6} & -1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{3\times 3}$$

24) (c) $\begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 1 \end{bmatrix} = A$ L.I.

$\quad\ \ \tilde{u}_1 \quad \tilde{u}_2$

Sol:

$$V_1 = U_1 = \cancel{(1, -2, 2)} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \Rightarrow W = \{V_1\}$$

$$V_2 = U_2 - \text{proj}_W U_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\langle U_2\, V_1 \rangle}{\|V_1\|^2} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{9} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 8 \\ 11 \\ 7 \end{pmatrix}$$

$$\begin{array}{r} 64 \\ 121 \\ 49 \\ \hline 234 \end{array}$$

$$Q = [q_1 \quad q_2] \quad : \quad q_1 = \frac{V_1}{\|V_1\|} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$q_2 = \frac{V_2}{\|V_2\|} = \cancel{\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \frac{1}{9\sqrt{234}} \begin{pmatrix} 8 \\ 11 \\ 7 \end{pmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{8}{\sqrt{234}} \\ -\frac{2}{3} & \frac{11}{\sqrt{234}} \\ \frac{2}{3} & \frac{7}{\sqrt{234}} \end{bmatrix} \quad ; \quad R = \left[\begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{bmatrix} \right] = \begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{\sqrt{26}}{3} \end{bmatrix}$$

24 b

Find the ~~pure~~ QR decomposition of the following matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$$
$\quad u_1 \quad u_2$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}$$

$\rho(A) = 2$. no. of variable

$\{u_1, u_2\} \rightarrow L \cdot I.$

GSP

Step 1: Let $V_1 = u_1 = (1, 0, 1)$; $W_1 = \{V_1\}$

$\|V_1\|^2 = 2$

Step 2: $V_2 = u_2 - \text{proj}_W u_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$$= (2,1,4) - \frac{2+0+4}{2}(1,0,1)$$

$$= (2,1,4) - (3,0,3) = (-1,1,1)$$

$$\|V_2\|^2 = 3$$

$\{V_1, V_2\} \rightarrow$ Orthogonal basis

$$\left\{ q_1 = \frac{V_1}{\|V_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad q_2 = \frac{V_2}{\|V_2\|} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \right\}$$

$$\text{Orthonormal basis} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}_{3\times 2} = Q[q_1, q_2]$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}_{3\times 2} \begin{bmatrix} \sqrt{3} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}_{2\times 2}$$

24. Find the QR-decomposition of the matrix, where possible.

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

24.

(a) $$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

(b) $$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

(c) $$\begin{bmatrix} \frac{1}{3} & \frac{8}{\sqrt{234}} \\ -\frac{2}{3} & \frac{11}{\sqrt{234}} \\ \frac{2}{3} & \frac{7}{\sqrt{234}} \end{bmatrix}\begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{\sqrt{26}}{3} \end{bmatrix}$$

(d) $$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}$$

(e) $$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

(f) Columns not linearly independent

Assignments(EAC)

- Exercise-6.3: Examples 1 – 8 (Pages 478-490)
- Exercise-6.3: Problems 1– 24 (Pages 490-495)