Chapter Content

- Real Vector Spaces
- Subspaces
- Linear Independence
- Basis and Dimension
- Row Space, Column Space, and Nullspace
- Rank and Nullity

5-6 Four Fundamental Matrix Spaces

- Consider a matrix A and its transpose A^T together, then there are six vector spaces of interest:
  - row space of A, row space of A^T
  - column space of A, column space of A^T
  - null space of A, null space of A^T
- However, the fundamental matrix spaces associated with A are
  - row space of A, column space of A
  - null space of A, null space of A^T

5-6 Four Fundamental Matrix Spaces

- If A is an $m \times n$ matrix
  - the row space of A and nullspace of A are subspaces of R^n
  - the column space of A and the nullspace of A^T are subspace of R^m
- What is the relationship between the dimensions of these four vector spaces?

5-6 Dimension and Rank

- Theorem 5.6.1
  - If A is any matrix, then the row space and column space of A have the same dimension.

- Definition
  - The common dimension of the row and column space of a matrix A is called the rank of A and is denoted by rank(A).
  - The dimension of the nullspace of a is called the nullity of A and is denoted by nullity(A).

5-6 Example 1 (Rank and Nullity)

- Find the rank and nullity of the matrix

$$A=\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Solution:

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \mathrm{I}$$

Since there are two non-zero rows (two leading 1's), the row space and column space are both 2-Dimensional, so rank $(A)=2$.

5-6 Example 1 (Rank and Nullity)

- Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Solution:
  - The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Since there are two nonzero rows, the row space and column space are both two-dimensional, so rank(A) = 2.

5-6 Example 1 (Rank and Nullity)

- The corresponding system of equations will be

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16\,x_5 + 5\,x_6 = 0$$

- It follows that the general solution of the system is

$$x_1 = 4r + 28s + 37t - 13u,\ x_2 = 2r + 12s + 16t - 5u,$$
$$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Thus, nullity(A) = 4.

5-6 Theorems

- Theorem 5.6.2
  - If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

- Theorem 5.6.3 (Dimension Theorem for Matrices)
  - If A is a matrix with n columns, then $\underline{\text{rank}(A) + \text{nullity}(A) = n}$.

- Theorem 5.6.4
  - If A is an $m \times n$ matrix, then:
    - $\text{rank}(A)$ = Number of leading variables in the solution of $A\mathbf{x} = \mathbf{0}$.
    - $\text{nullity}(A)$ = Number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

5-6 Example 2 (Sum of Rank and Nullity)

- The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\text{rank}(A) + \text{nullity}(A) = 6$$

- This is consistent with the previous example, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4$$

1) Find the rank and nullity of the matrix and verify Rank nullity theorem.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{Rank} = 1$$

$$\rho(A) = 1$$

No. of Columns of A = 3

<u>Nullity</u>: $2x - z = 0$

$\Rightarrow z = 2x$

Let $y = b$

$x = a$

$\Rightarrow \boxed{z = 2a}$

$$\Rightarrow \quad X = \begin{pmatrix} a \\ b \\ 2a \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \text{Null space} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow \quad \boxed{\text{Nullity } (A) = 2}$$

$$\Rightarrow \quad \text{No. of Columns} = \rho(A) + \text{Nullity}(A)$$

$$3 = 1 + 2 = 3$$

5-6 Example

- Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.

- Solution:
  - $\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4$
  - Thus, there are four parameters.

5-6 Dimensions of Fundamental Spaces

- Suppose that A is an $m\times n$ matrix of rank r, then
  - A^T is an $n\times m$ matrix of rank r by Theorem 5.6.2
  - nullity(A) = $n - r$, nullity(A^T) = $m - r$ by Theorem 5.6.3

Fundamental Space	Dimension
Row space of A	r
Column space of A	r
Nullspace of A	$n - r$
Nullspace of A^T	$m - r$

5-6 Maximum Value for Rank

- If A is an $m \times n$ matrix
  $\Rightarrow$ The row vectors lie in R^n and the column vectors lie in R^m.
  $\Rightarrow$ The row space of A is at most n-dimensional and the column space is at most m-dimensional.

- Since the row and column space have the same dimension (the rank A), we must conclude that if $m \neq n$, then the rank of A is at most the smaller of the values of m or n.
- That is,

$$\text{rank}(A) \leq \min(m, n)$$

5-6 Example 4

- If A is a 7×4 matrix,
  - the rank of A is at most 4
  - the seven row vectors must be linearly dependent
- If A is a 4×7 matrix,
  - the rank of A is at most 4
  - the seven column vectors must be linearly dependent

Theorem 5.6.5 (The Consistency Theorem)

- If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.
  - $A\mathbf{x} = \mathbf{b}$ is consistent.
  - $\mathbf{b}$ is in the column space of A.
  - The coefficient matrix A and the augmented matrix $[A \mid \mathbf{b}]$ have the same rank.

Theorems 5.6.6

- If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following are equivalent.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $m\times 1$ matrix $\mathbf{b}$.
  - The column vectors of A span R^m.
  - $\mathrm{rank}(A) = m$.

5-6 Overdetermined System

- A linear system with more equations than unknowns is called an overdetermined linear system.

- If $A\mathbf{x} = \mathbf{b}$ is an overdetermined linear system of m equations in n unknowns (so that $m > n$), then the column vectors of A <u>cannot</u> span R^m.

- Thus, <u>the overdetermined linear system $A\mathbf{x} = \mathbf{b}$ cannot be consistent for *every* possible $\mathbf{b}$</u>.

Theorem 5.6.7

- If $A\mathbf{x} = \mathbf{b}$ is consistent linear system of m equations in n unknowns, and if A has rank r,
  - then the general solution of the system contains $n - r$ parameters.

Theorem 5.6.8

- If A is an $m \times n$ matrix, then the following are equivalent.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The column vectors of A are linearly independent.
  - $A\mathbf{x} = \mathbf{b}$ has at most one solution (0 or 1) for every $m \times 1$ matrix $\mathbf{b}$.

5.6 Example 7

- Number of Parameters in a General Solution:
  - If A is a 5×7 matrix with rank 4, and if $A\mathbf{x} = \mathbf{b}$ is a consistent linear system
    - the general solution of the system contains $7 - 4 = 3$ parameters

- An Undetermined System
  - If A is a 5×7 matrix,
    - for every 7×1 matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ is undetermined.
    - $A\mathbf{x} = \mathbf{b}$ must be consistent for some $\mathbf{b}$, and for each such $\mathbf{b}$ the general solution must have $7 - r$ parameters, where r is the rank of A.

Theorem 5.6.9 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \rightarrow R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The range of T_A is R^n.
  - T_A is one-to-one.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy Formula 4 of the Dimension Theorem.

(b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $N = 2,\ R = 1,\ n = 3.$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ $N = 2,\ R = 2,\ n = 4.$

d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$ $N = 3,\ r = 2,\ n = 5$

4. In each part, use the information in the table to find the dimension of the row space, column space, and nullspace of A, and of the nullspace of A^T.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
Rank(A)	3	2	1	2	2	0	2

Soln

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	$\overset{m}{3}\times\overset{n}{3}$	3×3	3×3	5×9	9×5	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Dim(A)	↓	↓	↓	↓	↓	↓	↓
Row space →	3	2	1	2	2	0	2
column space →	3	2	1	2	2	0	2
Null space(A) →	3−3=0	1	2	7	3	4	0
Null space (AT) →	0	1	2	3	7	4	4.

5. In each part, find the largest possible value for the rank of A and the smallest possible value for the nullity of

(a) A is 4×4

(b) A is 3×5

(c) A is 5×3

a) A is 4×4.

Solu: Rank $= 4$, Nullity $= 0$

(b) A is 3×5

Rank $= 3$, nullity $= 2$

(c) A is 5×3.

Rank $= 3$, nullity $= 0$

Note: $\text{Rank}(A) \le \min(m,n)$

6. If A is an $m \times n$ matrix, what are the largest possible value for its rank and the smallest possible value for its nullity?

Hint See Exercise 5.

Solu: $\text{Rank} = \min(m,n)$, $\text{nullity} = n - \min(m,n)$.

7. In each part, use the information in the table to determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If so, state the number of parameters in its general solution.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank $[A \mid \mathbf{b}]$	3	3	1	2	3	0	2

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank [A|b]	3	3	1	2	3	0	2
Consistent →	Yes	No	Yes	Yes	No	Yes	Yes
Nullity	0	1	2	7	7	4	0

$n - \text{rank} = (3-1)$

8. For each of the matrices in Exercise 7, find the nullity of A, and determine the number of parameters in the general solution of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

7. Use Theorems 5.6.5 and 5.6.7.

(a) The system is consistent because the two ranks are equal. Since $n = r = 3$, $n - r = 0$ and therefore the number of parameters is 0.

(b) The system is inconsistent because the two ranks are not equal.

(d) The system is consistent because the two ranks are equal. Here $n = 9$ and $r = 2$, so that $n - r = 7$ parameters will appear in the solution.

(f) Since the ranks are equal, the system is consistent. However A must be the zero matrix, so the system gives no information at all about its solution. This is reflected in the fact that $n - r = 4 - 0 = 4$, so that there will be 4 parameters in the solution for the 4 variables.

9. What conditions must be satisfied by b_1, b_2, b_3, b_4, and b_5 for the overdetermined linear system

$$
\begin{aligned}
x_1 - 3x_2 &= b_1 \\
x_1 - 2x_2 &= b_2 \\
x_1 + x_2 &= b_3 \\
x_1 - 4x_2 &= b_4 \\
x_1 + 5x_2 &= b_5
\end{aligned}
$$

to be consistent?

Solve: The given system is not consistent for all possible values of b_1, b_2, b_3, b_4, b_5.
Exact conditions under which the system is consistent can be obtained by solving the syst by Gauss-Jordan elimination

$$
\left[\begin{array}{cc|c}
1 & -3 & b_1 \\
1 & -2 & b_2 \\
1 & 1 & b_3 \\
1 & -4 & b_4 \\
1 & 5 & b_5
\end{array}\right]
\longrightarrow
\left[\begin{array}{cc|c}
1 & -3 & b_1 \\
0 & 1 & b_2 - b_1 \\
0 & 4 & b_3 - b_1 \\
0 & -1 & b_4 - b_1 \\
0 & 8 & b_5 - b_1
\end{array}\right]
$$

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 4b_2 + 3b_1 \\ 0 & 0 & b_4 + b_2 - 2b_1 \\ 0 & 0 & b_5 - 8b_2 + 7b_1 \end{array}\right]$$

The system is consistent iff $b_1, b_2, \ldots, b_5$ satisfy the conditions.

$$b_3 - 4b_2 + 3b_1 = 0 \qquad b_3 = 4b_2 - 3b_1$$

$$-2b_1 + b_4 + b_2 = 0 \qquad b_4 = 2b_1 - b_2$$

$$b_5 - 8b_2 + 7b_1 = 0 \qquad b_5 = 8b_2 - 7b_1$$

$\Rightarrow$ Let $b_1 = r$, $b_2 = s$,

$\Rightarrow$ $b_3 = 4s - 3r$, $b_4 = 2r - s$,

$b_5 = 8s - 7r$.

9. The system is of the form $A\mathbf{x} = \mathbf{b}$ where rank$(A) = 2$. Therefore it will be consistent if and only if rank$([A|\mathbf{b}]) = 2$. Since $[A|\mathbf{b}]$ reduces to

$$\begin{bmatrix} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 4b_2 + 3b_1 \\ 0 & 0 & b_4 + b_2 - 2b_1 \\ 0 & 0 & b_5 - 8b_2 + 7b_1 \end{bmatrix}$$

the system will be consistent if and only if $b_3 = 4b_2 - 3b_1$, $b_4 = -b_2 + 2b_1$, and $b_5 = 8b_2 - 7b_1$, where b_1 and b_2 can assume any values.

11. Suppose that A is a 3×3 matrix whose nullspace is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

Answer

11. If the nullspace of A is a line through the origin, then it has the form $x = at$, $y = bt$, $z = ct$ where t is the only parameter. Thus nullity(A) = 3 − rank(A) = 1. That is, the row and column spaces of A have dimension 2, so neither space can be a line. Why?

12. Discuss how the rank of A varies with

(a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 1 & 1 & t \\ 0 & (t-1) & (1-t) \\ 0 & 0 & \boxed{(1-t)(t+2)} \end{bmatrix}$$

Rank	if	t
1	if	$t=1$
2	if	$t=-2$
3	if	$t \neq 1, -2$

12. Discuss how the rank of A varies with

(b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

Rank	if	t
2		$t = 1, 3/2$
3	if	$t \neq 1, 3/2$

13. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1 or 2? If so, find those values.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r-2) & 2 \\ 0 & 0 & (r^2-4) \\ 0 & 0 & 2(s-1) \end{bmatrix}$$

Rank = 2, if $r = 2, s = 1$

Rank is never 1.

13. Call the matrix A. If $r = 2$ and $s = 1$, then clearly rank(A) = 2. Otherwise, either $r - 2$ or $s - 1 \neq 0$ and rank(A) = 3. Rank(A) can never be 1.

Assignments

- Exercise-5.6: Examples 1 – 7 (Pages 421-430)
- Exercise-5.6: Problems 1– 13 (Pages 431-434)