Chapter Content

- Inner Products
- Angle and Orthogonality in Inner Product Spaces
- Orthonormal Bases; Gram-Schmidt Process
- Best Approximation; Least Squares

6-1 Inner Product Space

- An inner product on a real vector space V
  - a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.
  - $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
  - $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
  - $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$
  - $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$

  A real vector space with an <u>inner product</u> is called a real inner product space.

6-1 Example

- Euclidean Inner Product on R^n
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_\mathrm{n})$ are vectors in R^n, then the formula

$$\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

  defines $\langle \mathbf{v}, \mathbf{u} \rangle$ to be the Euclidean product on R^n.

- The four inner product axioms hold by Theorem 4.1.2.

6-1 Preview of Inner Product

- Theorem 4.1.2
  - If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k is any scalar, then
    - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
    - $(k\,\mathbf{u}) \cdot \mathbf{v} = k\,(\mathbf{u} \cdot \mathbf{v})$
    - $\mathbf{v} \cdot \mathbf{v} \geq \mathbf{0}$; Further, $\mathbf{v} \cdot \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v} = 0$

- Example
  - $(3\mathbf{u} + 2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u} + \mathbf{v}) + (2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u}) + (3\mathbf{u}) \cdot \mathbf{v} + (2\mathbf{v}) \cdot (4\mathbf{u}) + (2\mathbf{v}) \cdot \mathbf{v}$
    $=12(\mathbf{u} \cdot \mathbf{u}) + 11(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{v})$

6-1 Weighted Euclidean Inner Product

- If $w_1, w_2, \ldots, w_n$ are positive real numbers
  - We call weights
- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the formula

$$\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{u} \cdot \mathbf{v} = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

is called the weighted Euclidean inner product with weights $w_1, w_2, \ldots, w_n$.

1. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on R^2, and let $\mathbf{u} = (3, -2)$, $\mathbf{v} = (4, 5)$, $\mathbf{w} = (-1, 6)$, and $k = -4$. Verify that

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(a) $\langle u, v \rangle = \langle v, u \rangle \Rightarrow \langle u, v \rangle = u_1 v_1 + u_2 v_2 = 3 \times 4 + (-2)(+5)$

$= 12 - 10 = 2$

$\langle v, u \rangle = 2.$

(b) $\boxed{\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle}$

(b) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$\Rightarrow \langle u+v, w \rangle = (u_1 + v_1) w_1 + (u_2 + v_2) w_2$

$= (3+4)(-1) + (-2+5)(6) = -7 + 18$

$= \underline{\underline{11}}$

$\langle u, w \rangle = 3(-1) + (-2)(6) = -3 - 12 = -15$

$\langle v, w \rangle = 4(-1) + (+5)\, 6 = -4 + 30 = 26.$

$\langle u, w \rangle + \langle v, w \rangle = 11$

(c) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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c) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$$\Rightarrow \langle u, v+w \rangle = (3)[4-1] + (-2)[+5+6]$$
$$= 9 - 22 = \underline{-13}$$

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 = (3)(4) + (-2)(+5)$$
$$= 12 - 10 = \underline{2}$$

$$\langle u, w \rangle = (3)(-1) + (-2)(6) = -3 - 12 = \underline{-15}$$

(c) Since $\mathbf{v} + \mathbf{w} = (3, 11)$, we have

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = 3(3) + (-2)(11) = -13$$

On the other hand,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(4) + (-2)(5) = 2$$

and

$$\langle \mathbf{u}, \mathbf{w} \rangle = 3(-1) + (-2)(6) = -15$$

(d) $\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, k\mathbf{v}\rangle$

d) $\langle ku, v\rangle = k\langle u, v\rangle$

$\Rightarrow \langle ku, v\rangle = \langle (-12, 8), (4, +5)\rangle$

$= (-12)(4) + 8(-5) = -48 + 40 = -8.$

$k\langle u, v\rangle = (-4)(2) = -8$

(d) Since $k\mathbf{u} = (-12, 8)$ and $k\mathbf{v} = (-16, -20)$, we have

$$\langle k\mathbf{u}, \mathbf{v}\rangle = (-12)(4) + (8)(5) = -8$$

and

$$\langle \mathbf{u}, k\mathbf{v}\rangle = 3(-16) + (-2)(-20) = -8$$

Since $\langle \mathbf{u},\mathbf{v}\rangle = 2$, $k\langle \mathbf{u}, \mathbf{v}\rangle = -8$.

(e) $\boxed{\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0}$

e) $\langle 0, v\rangle = \langle v, 0\rangle = 0$

2. Repeat Exercise 1 for the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1 v_1 + 5u_2 v_2$.

(a) $\langle u, v \rangle = \langle v, u \rangle$

Sol:

$\langle u, v \rangle = 4u_1v_1 + 5u_2v_2 = 4(3)(4) + 5(-2)(5)$

$= 48 - 50 = -2$

$\langle v, u \rangle = -2$

(b) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$\langle u+v, w \rangle = 4(7)(-1) + 5(3)(6) = -28 + 90$

$= 62$

$\langle u, w \rangle + \langle v, w \rangle = 4 \times 3 \times -1 + 5 \times -2 \times 6 + 4 \times 4 \times -1 + 5 \times 5 \times 6$

$= -12 - 60 - 16 + 150 = 62$

(c) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$\Rightarrow \langle u, v+w \rangle = 4(3)(3) + 5(-2)(11)$

$= 36 - 110 = -74$

d) $\langle ku, v \rangle = k \langle u, v \rangle$

(e) $\langle 0, v \rangle = \langle v, 0 \rangle = 0$

8

* Let $u = (u_1, u_2)$; $v = (v_1, v_2)$ be vectors from R^2.
Determine which of the following are I.P. For those that are not I.P, list the axioms that do not hold true.

$$\langle u, v\rangle = u_1 v_1 + u_2 v_2 \longrightarrow \text{Euclidian I.P.}$$

Ax 1:

$$\langle v, u\rangle = v_1 u_1 + v_2 u_2$$
$$= u_1 v_1 + u_2 v_2 = \langle u, v\rangle$$

Ax 2:-

$$\langle u+v, w\rangle = \cancel{\langle (u_1+v_1), w\rangle}$$
$$= (u_1+v_1)w_1 + (u_2+v_2)w_2$$
$$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$$
$$= u_1 w_1 + u_2 w_2 + v_1 w_1 + v_2 w_2$$
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \,//$$

Ax 3:-

$$\langle ku, v\rangle = k u_1 v_1 + k u_2 v_2$$
$$= k[u_1 v_1 + u_2 v_2)$$
$$= k\langle u, v\rangle$$

Ax 4:-

$$\langle u, u\rangle = u_1^2 + u_2^2 \geq 0$$
$$\langle u, u\rangle = u_1^2 + u_2^2 = 0 \text{ iff } u = (0, 0) \,//$$

6-1 Example 2

- Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four product axioms.

- Solution:

  <1> $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

  <2> If $\mathbf{w} = (w_1, w_2)$, then $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  <3> $\langle k\mathbf{u}, \mathbf{v} \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k \langle \mathbf{u}, \mathbf{v} \rangle$

  <4> $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1v_1 + 2v_2v_2 = 3v_1^2 + 2v_2^2$ .Obviously , $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 \geq 0$. Furthermore, $\langle \mathbf{v}, \mathbf{v} \rangle = 3v_1^2 + 2v_2^2 = 0$ if and only if $v_1 = v_2 = 0$, That is , if and only if $\mathbf{v} = (v_1, v_2) = 0$.

8. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Show that the following are inner products on R^2 by verifying that the inner product axioms hold.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 5u_2 v_2$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1 v_1 + u_2 v_1 + u_1 v_2 + 4u_2 v_2$

An 1 : $\langle v, u \rangle = 3v_1 u_1 + 5v_2 u_2$

$= 3u_1 v_1 + 5u_2 v_2$

$= \langle u, v \rangle$

An 2 : $\langle u+v, w \rangle = 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2$

$= 3u_1 w_1 + 3v_1 w_1 + 5u_2 w_2 + 5v_2 w_2$

$= 3u_1 w_1 + 5u_2 w_2 + 3v_1 w_1 + 5v_2 w_2$

$= \langle u, w \rangle + \langle v, w \rangle$

Ans:- $\langle ku, v\rangle = 3ku_1v_1 + 5ku_2v_2$

$= k(3u_1v_1 + 5u_2v_2)$

$= k\langle u, v\rangle$

Ans:- $\langle u, u\rangle = 3u_1^2 + 5u_2^2 > 0$

$\langle u, u\rangle = 0$ if $u = (0, 0)$

9. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on R^3. For those that are not, list the axioms that do not hold.

(a) $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + u_3v_3$

Axiom-1:

$\langle u, v\rangle = u_1v_1 + u_3v_3$

$\langle v, u\rangle = v_1u_1 + v_3u_3$ L.H.S = R.H.S

is $\langle u, v\rangle = \langle v, u\rangle$.

Axiom-2: $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.

L.H.S

$\Rightarrow (u_1+v_1)w_1 + (u_3+v_3)w_3 = u_1w_1 + u_3w_3 + v_1w_1 + v_3w_3$.

$= \langle u, w\rangle + \langle v, w\rangle$.

L.H.S = R.H.S

Axiom-3:

$$\langle ku, v\rangle = (ku_1)v_1 + (ku_3)v_3 = k[u_1v_1 + u_3v_3]$$
$$= k\langle u, v\rangle.$$

Axiom-4:

$$\langle u, u\rangle = u_1^2 + u_3^2 \geq 0.$$

- But $\langle u, u\rangle = 0$ if we have $u = (0, k, 0) \neq 0$

$\therefore$ Axiom-4 fails.

OR

$$\langle u, v\rangle = u_1v_1 + u_3v_3$$

Ax1 $\langle v, u\rangle = v_1u_1 + v_3u_3$
$= u_1v_1 + u_3v_3$
$= \langle u, v\rangle$

Ax2 $\langle u+v, w\rangle = (u_1+v_1)w_1 + (u_3+v_3)w_3$
$= u_1w_1 + u_3w_3 + v_1w_1 + v_3w_3$
$= \langle u, w\rangle + \langle v, w\rangle$

Ax3 $\langle ku, v\rangle = ku_1v_1 + ku_3v_3$
$= k\langle u, v\rangle$

Ax4 $\langle u, u\rangle = u_1^2 + u_3^2 \geq 0$
$= 0$ iff ~~u = 0, 0~~
$(0, u_2, 0)$
$\neq (0, 0, 0)$

so Ax 4 fails

9. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on R^3. For those that are not, list the axioms that do not hold.

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$

Axiom-1: $\langle u, v \rangle = \langle v, u \rangle$

Axiom-2: $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$$\langle u+v, w \rangle = (u_1+v_1)^2 w_1^2 + (u_2+v_2)^2 w_2^2 + (u_3+v_3)^2 w_3^2$$

$$= (u_1^2 + 2u_1v_1 + v_1^2) w_1^2 + (u_2^2 + 2u_2v_2 + v_2^2) w_2^2 + (u_3^2 + 2u_3v_3 + v_3^2) w_3^2$$

$$\neq \langle u, w \rangle + \langle v, w \rangle = u_1^2 w_1^2 + u_2^2 w_2^2 + u_3^2 w_3^2 + (v_1^2 w_1^2 + v_2^2 w_2^2 + v_3^2 w_3^2)$$

Axiom-2 fails

Axiom-3: $\langle ku, v \rangle = k^2 u_1^2 v_1^2 + k^2 u_2^2 v_2^2 + k^2 u_3^2 v_3^2$

$k\langle u, v \rangle = k[u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2]$

Axiom-3 fails.

Axiom-4:

$\langle u, u \rangle = u_1^4 + u_2^4 + u_3^4 > 0.$

and $\langle u, u \rangle = 0$ iff $u = 0 = (0, 0, 0)$.

OR

$$\langle u,v\rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$$

Ax 1: $\langle v,u\rangle = v_1^2u_1^2 + v_2^2u_2^2 + v_3^2u_3^2$

$= u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 = \langle u,v\rangle$

Ax 2: $\langle u+v,w\rangle = (u_1+v_1)^2w_1^2 + (u_2+v_2)^2w_2^2 + (u_3+v_3)^2w_3^2$

$= u_1^2w_1^2 + v_1^2w_1^2 + 2u_1v_1w_1^2$

$\neq \langle u,w\rangle + \langle v,w\rangle$

Ax 3: $\langle ku,v\rangle = k^2u_1^2v_1^2 + k^2u_2^2v_2^2 + k^2u_3^2v_3^2$

$= \cancel{k} \; k^2\langle u,v\rangle$ Ax 3 fails

Ax 4: $\langle u,u\rangle = u_1^2+u_2^2+u_3^2 \geq 0.$

$= 0$ iff $\{u = (0,0,0)\}$

OR

9. **(b)** Axioms 1 and 4 are easily checked. However, if $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (u_1 + v_1)^2 w_1^2 + (u_2 + v_2)^2 w_2^2 + (u_3 + v_3)^2 w_3^2$$

$$= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + 2u_1 v_1 w_1^2 + 2u_2 v_2 w_2^2 + 2u_3 v_3 w_3^2$$

If, for instance, $\mathbf{u} = \mathbf{v} = \mathbf{w} = (1, 0, 0)$, then Axiom 2 fails.

To check Axiom 3, we note that $\langle k\mathbf{u}, \mathbf{v} \rangle = k^2 \langle \mathbf{u}, \mathbf{v} \rangle$. Thus $\langle k\mathbf{u}, \mathbf{v} \rangle \neq k\langle \mathbf{u}, \mathbf{v} \rangle$ unless $k = 0$ or $k = 1$, so Axiom 3 fails.

9. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on R^3. For those that are not, list the axioms that do not hold.

(c) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + u_2 v_2 + 4u_3 v_3$

(d) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$

(c) $\langle u, v \rangle = 2u_1v_1 + u_2v_2 + 4u_3v_3$ (Yes)

(d) $\langle u, v \rangle = u_1v_1 - u_2v_2 + u_3v_3$. (axiom-4 fails)

(c) (1) Axiom 1 follows from the commutativity of multiplication in R.

(2) If $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\begin{aligned}\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 2(u_1 + v_1)w_1 + (u_2 + v_2)w_2 + 4(u_3 + v_3)w_3 \\ &= 2u_1w_1 + u_2w_2 + 4u_3w_3 + 2v_1w_1 + v_2w_2 + 4v_3w_3 \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

(3) $\langle k\mathbf{u}, \mathbf{v} \rangle = 2(ku_1)v_1 + (ku_2)v_2 + 4(ku_3)v_3 = k\langle \mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle = 2v_1^2 + v_2^2 + 4v_3^2 \geq 0$

$= 0$ if and only if $v_1 = v_2 = v_3 = 0$, or $\mathbf{v} = \mathbf{0}$

Thus this is an inner product for R^3.

6-1 Norm & Length

- If V is an inner product space, then the norm (or length) of a vector $\mathbf{u}$ in V is denoted by $\|\mathbf{u}\|$ and is defined by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2}$$

- The distance between two points (vectors) $\mathbf{u}$ and $\mathbf{v}$ is denoted by $d(\mathbf{u},\mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

6-1 Example 3

- Norm and Distance in R^n
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n with the Euclidean inner product, then

$$\|\mathrm{u}\| = <\mathbf{u},\ \mathbf{u}>^{1/2} = (\mathbf{u},\ \mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

$$d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = <\mathbf{u}-\mathbf{v},\ \mathbf{u}-\mathbf{v}>^{1/2} = \left[(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\right]^{1/2}$$
$$= \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \ldots + (u_n-v_n)^2}$$

6-1 Example 4
(Weighted Euclidean Inner Product)

- The norm and distance depend on the inner product used.
  - For example, for the vectors **u** = (1,0) and **v** = (0,1) in R^2 with the Euclidean inner product, we have

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(1,-1)\| = \sqrt{1^2 - (-1)^2} = \sqrt{2}$$

  - However, if we change to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$, then we obtain

$$\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u} \rangle^{1/2} = (3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0)^{1/2} = \sqrt{3}$$

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \langle (1,-1),(1,-1) \rangle^{1/2} = [3 \cdot 1 \cdot 1 + 2 \cdot (-1) \cdot (-1)]^{1/2} = \sqrt{5}$$

6-1 Unit Circles and Spheres in IPS

- If V is an inner product space, then the set of points in V that satisfy

$$\|\mathbf{u}\| = 1$$

is called the unite sphere or sometimes the unit circle in V. In R^2 and R^3 these are the points that lie 1 unit away form the origin.

18. Sketch the unit circle in R^2 using the given inner product.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}u_1 v_1 + \frac{1}{16}u_2 v_2$

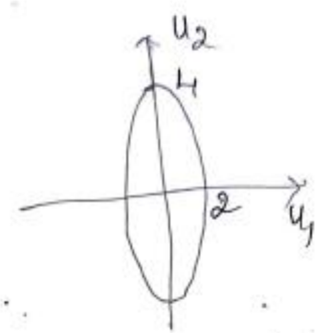
(b) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + u_2 v_2$

(a) $\langle u, v \rangle = \frac{1}{4} u_1 v_1 + \frac{1}{16} u_2 v_2$

solu $\|u\| = \langle u, u \rangle^{1/2} = \sqrt{\frac{u_1^2}{4} + \frac{u_2^2}{16}}$.

unit circle is given by, $\|u\| = 1$.

$\Rightarrow \frac{u_1^2}{(2)^2} + \frac{u_2^2}{(4)^2} = 1$.



Ellipse.

(b) $\langle u, v \rangle = 2u_1 v_1 + u_2 v_2$.

$\|u\| = \langle u, u \rangle^{1/2} = \sqrt{2u_1^2 + u_2^2}$

$1/\sqrt{2}$.

unit circle is $\|u\| \Rightarrow 2u_1^2 + u_2^2 = 1$

$\Rightarrow \frac{u_1^2}{(1/2)} + \frac{u_2^2}{(1)} = 1$

19. Find a weighted Euclidean inner product on R^2 for which the unit circle is the ellipse shown in the accompanying figure.

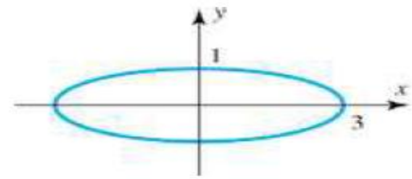
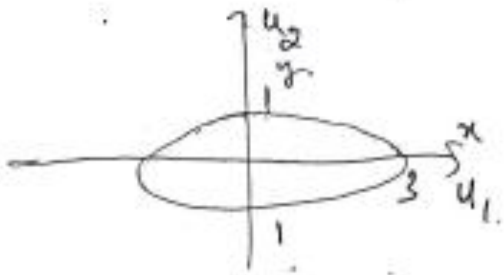


Figure Ex-19



$$\frac{u_1^2}{(3)^2}+\frac{u_2^2}{(1)^2}=1$$

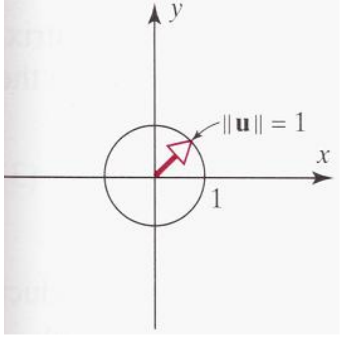
$$\sqrt{\frac{u_1^2}{9}+\frac{u_2^2}{(1)^2}}=1$$

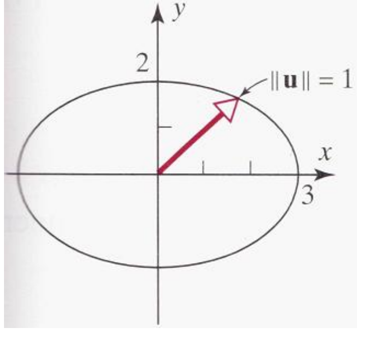
$$\Rightarrow \|u\| = \sqrt{\frac{u_1^2}{9}+\frac{u_2^2}{1}}$$

$$\Rightarrow \langle u, v\rangle = \frac{1}{9}u_1v_1 + u_2v_2$$

6-1 Example 5 (Unit Circles in R^2)

- Sketch the unit circle in an *xy*-coordinate system in R^2 using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2$
- Sketch the unit circle in an *xy*-coordinate system in R^2 using the Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 1/9u_1v_1 + 1/4u_2v_2$
- Solution
  - If $\mathbf{u} = (x,y)$, then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{½} = (x^2 + y^2)^{½}$, so the equation of the unit circle is $x^2 + y^2 = 1$.
  - If $\mathbf{u} = (x,y)$, then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{½} = (1/9x^2 + 1/4y^2)^{½}$, so the equation of the unit circle is $x^2 /9 + y^2/4 = 1$.





6-1 Inner Product Generated by Matrices

- Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in R^n (expressed as $n \times 1$ matrices), and let A be an invertible $n \times n$ matrix.
- If $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on R^n, then the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$$

  defines an inner product; it is called the inner product on R^n generated by A.
- The Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ can be written as the matrix product $\mathbf{v}^T\mathbf{u}$, the above formula can be written in the alternative form $\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{v})^T A\mathbf{u}$, or equivalently,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A\mathbf{u}$$

6-1 Example 6
(Inner Product Generated by the Identity Matrix)

- The inner product on R^n generated by the $n \times n$ identity matrix is the Euclidean inner product: Let $A = I$, we have $\langle \mathbf{u}, \mathbf{v} \rangle = I\mathbf{u} \cdot I\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$
- The weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ is the inner product on R^2 generated by

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

since

$$\langle u, v \rangle = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 3u_1v_1 + 2u_2v_2$$

- In general, the weighted <u>Euclidean inner product</u> $\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \ldots + w_nu_nv_n$ is the inner product on R^n generated by

$$A = \mathrm{diag}(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_n})$$

5.

(a) Use Formula 3 to show that $\langle \mathbf{u}, \mathbf{v} \rangle = 9u_1 v_1 + 4u_2 v_2$ is the inner product on R^2 generated by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) Use the inner product in part (a) to compute $\langle \mathbf{u}, \mathbf{v} \rangle$ if $\mathbf{u} = (-3, 2)$ and $\mathbf{v} = (1, 7)$.

5. (a) By Formula (4),

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} 9v_1 & 4v_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= 9u_1 v_1 + 4u_2 v_2
\end{aligned}$$

(b) We have $\langle \mathbf{u}, \mathbf{v} \rangle = 9(-3)(1) + 4(2)(7) = 29$.

6.

(a) Use Formula 3 to show that $\langle \mathbf{u}, \mathbf{v} \rangle = 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$ is the inner product on R^2 generated by

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

(b) Use the inner product in part (a) to compute $\langle \mathbf{u}, \mathbf{v} \rangle$ if $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

6.

(a)

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$$\langle u, v \rangle = Au \cdot Av$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1 + u_2 \\ -u_1 + 3u_2 \end{bmatrix} \cdot \begin{bmatrix} 2v_1 + v_2 \\ -v_1 + 3v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2v_1 + v_2 & -v_1 + 3v_2 \end{bmatrix} \begin{bmatrix} 2u_1 + u_2 \\ -u_1 + 3u_2 \end{bmatrix}$$

$$= (2v_1 + v_2)(2u_1 + u_2) + (-v_1 + 3v_2)(-u_1 + 3u_2)$$

$$= \underline{4u_1v_1} + 2u_1v_2 + 2u_2v_1 + u_2v_2$$
$$+ \underline{u_1v_1} - 3u_1 \cdot v_2 - 3u_2v_1 + 9u_2v_2$$

$$= 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$$

Thus proved.

(b) Use the inner product in part (a) to compute $\langle u, v \rangle$ if $u = (0, -3)$ and $v = (6, 2)$.

$$\langle u, v \rangle = 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$$
$$= 5(0)(6) - (0)(2) - (-3)(6) + 10(-3)(2)$$
$$= 0 - 0 + 18 - 60 = -42$$

7. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. In each part, the given expression is an inner product on R^2. Find a matrix that generates it.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 6u_2v_2$

(a) $\langle u, v \rangle = 3u_1v_1 + 5u_2v_2$

$w_1 = 3 \quad w_2 = 5$

$$\therefore A = \begin{bmatrix} \sqrt{w_1} & 0 \\ 0 & \sqrt{w_2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

(b) $\langle u, v \rangle = 4u_1v_1 + 6u_2v_2$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

7. (a) By Formula (4), we have $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A \mathbf{u}$ where

$$A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

10. In each part, use the given inner product on R^2 to find $\|\mathbf{w}\|$, where $\mathbf{w} = (-1, 3)$.

(a) the Euclidean inner product

(b) the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$

(c) the inner product generated by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

(a) the Euclidean inner product.

$\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$

$\|w\| = \langle w, w \rangle^{1/2} = \sqrt{w_1^2 + w_2^2} = \sqrt{1+9} = \sqrt{10}$

(b) The weighted Euclidean innerproduct $\langle u, v \rangle = 3u_1 v_1 + 2u_2 v_2$, where $u = (u_1, u_2)$ and $v = \langle v_1, v_2 \rangle$.

$\|w\| = \langle w, w \rangle^{1/2} = \sqrt{3w_1^2 + 2w_2^2} = \sqrt{3 + 2(9)} = \sqrt{3+18} = \sqrt{21}$

Ⓒ The inner product generated by the matrix.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

$$\langle u, v \rangle = v^T A^T A u = [v_1 \; v_2] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= [v_1 \; v_2] \begin{bmatrix} 2 & -1 \\ -1 & 13 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [2v_1 - v_2 \quad -v_1 + 13v_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= 2u_1v_1 - u_1v_2 - u_2v_1 + 13u_2v_2$$

$$\|w\| = \langle w, w \rangle^{1/2} = \sqrt{2w_1^2 - 2w_1w_2 + 13w_2^2} = \sqrt{2+6+117}$$

11. Use the inner products in Exercise 10 to find $d(\mathbf{u}, \mathbf{v})$ for $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$.

(a) $\langle u, v \rangle = u_1 v_1 + u_2 v_2$

$d(u,v) = \langle u - v, u - v \rangle^{1/2} = \langle (-3, -3), (-3, -3) \rangle^{1/2}$

$= \sqrt{(-3)(-3) + (-3)(-3)}$

$= \sqrt{18} = 3\sqrt{2}$

(b) $\langle u, v \rangle = 3u_1 v_1 + 2u_2 v_2$

$d(u,v) = \langle u - v, u - v \rangle^{1/2} = \sqrt{3(u_1 - v_1)(u_1 - v_1) + 2(u_2 - v_2)(u_2 - v_2)}$

$= \sqrt{3(-3)(-3) + 2(-3)(-3)}$

$= \sqrt{27 + 18}$

$= \sqrt{45} = 3\sqrt{5}$

(c) $\langle u, v \rangle = 2u_1 v_1 - u_1 v_2 - u_2 v_1 + 13 u_2 v_2$

$d(u,v) = \cancel{\sqrt{2(u_1 - v_1)^2 - \ldots}}$

$= \langle (-3, -3), (-3, -3) \rangle^{1/2} = \sqrt{2(-3)(-3) - (-3)(-3) - (-3)(-3) + 13(-3)(-3)}$

$= \sqrt{18 - 9 - 9 + 117}$

$= \sqrt{117} = 3\sqrt{13}$

11. Use the inner products in Exercise 10 to find $d(\mathbf{u}, \mathbf{v})$ for $\mathbf{u} = (-1, 2)$ and $\mathbf{v} = (2, 5)$.

11. We have $\mathbf{u} - \mathbf{v} = (-3, -3)$.

(b) $d(\mathbf{u}, \mathbf{v}) = \|(-3, -3)\| = [3(9) + 2(9)]^{1/2} = \sqrt{45} = 3\sqrt{5}$

(c) From Problem 10(c), we have

$$[d(\mathrm{u}, \mathrm{v})]^2 = \begin{bmatrix} -3 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 13 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix} = 117$$

Thus

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{117} = 3\sqrt{13}$$

6-1 Example 7 (An Inner Product on M_{22})

- If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$
  are any two 2×2 matrices, then
  $$\langle U, V \rangle = \mathrm{tr}(U^T V) = \mathrm{tr}(V^T U) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$
  defines an inner product on M_{22}
- For example, if $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$
  then $\langle U, V \rangle = 16$
- The norm of a matrix U relative to this inner product is
  $$\|U\| = < U,\ U >^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$
  and the unit sphere in this space consists of all 2×2 matrices U whose entries satisfy the equation $\|U\| = 1$, which on squaring yields $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$

3. Compute $\langle \mathbf{u}, \mathbf{v} \rangle$ using the inner product in Example 7.

(a) $\mathbf{u} = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

(b) $\mathbf{u} = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

(a) $u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$$\begin{aligned} \langle u, v \rangle &= \text{trace}(U^T V) \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \\ &= (3)(-1) + (-2)(3) + (4)(1) + 8(1) \\ &= -3 - 6 + 4 + 8 \\ &= 3 \end{aligned}$$

(b) $u = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$ $v = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

$$\begin{aligned} \langle u, v \rangle &= 4 + 12 + 0 + 40 \\ &= 56 \end{aligned}$$

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 3(-1) - 2(3) + 4(1) + 8(1) = 3$

13. Let M_{22} have the inner product in Example 7. In each part, find $\|A\|$.

(a) $A = \begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

a) $A = \begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$

$\|A\| = \langle A, A \rangle^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$

$= \sqrt{4 + 25 + 9 + 36}$

$= \sqrt{74}$

(b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\|A\| = 0$

13. **(a)** $\|A\| = [(-2)^2 + (5)^2 + (3)^2 + (6)^2]^{1/2} = \sqrt{74}$

15. Let M_{22} have the inner product in Example 7. Find $d(A, B)$.

(a) $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 6 & 2 \end{bmatrix}$

(a) $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$

$d(A,B) = \langle A-B, A-B \rangle^{1/2} = \sqrt{(6)^2 + (-1)^2 + (8)^2 + (-2)^2}$

$A-B = \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix}$ $\qquad = \sqrt{36 + 1 + 64 + 4}$

$= \sqrt{105}$

(b) $A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 6 & 2 \end{bmatrix}$

$A-B = \begin{bmatrix} 3 & 3 \\ -5 & -2 \end{bmatrix}$ $\quad d(A,B) = \langle A-B, A-B \rangle^{1/2}$

$= \sqrt{9 + 9 + 25 + 4}$

$= \sqrt{47}$

15. Let M_{22} have the inner product in Example 7. Find $d(A, B)$.

(a) $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 1 \\ 6 & 2 \end{bmatrix}$

15. **(a)** Since $A - B = \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix}$, we have

$$d(A, B) = \langle A - B, A - B \rangle^{1/2} = \left[6^2 + (-1)^2 + 8^2 + (-2)^2\right]^{1/2} = \sqrt{105}$$

6-1 Example 8 (An Inner Product on P_2)

- If $\mathbf{p} = a_0 + a_1x + a_2x^2$ and $\mathbf{q} = b_0 + b_1x + b_2x^2$ are any two vectors in P_2,
  - An inner product on P_2:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

- The norm of the polynomial $\mathbf{p}$ relative to this inner product is $\|\mathbf{p}\| = <\mathbf{p},\ \mathbf{p}>^{1/2} = \sqrt{a_0^2 + a_1^2 + a_2^2}$
- The unit sphere in this space consists of all polynomials $\mathbf{p}$ in P_2 whose coefficients satisfy the equation $\| \mathbf{p} \| = 1$, which on squaring yields

$$a_0^2 + a_1^2 + a_2^2 = 1$$

4. Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ using the inner product in Example 8.

(a) $\mathbf{p} = -2 + x + 3x^2, \quad \mathbf{q} = 4 - 7x^2$

(b) $\mathbf{p} = -5 + 2x + x^2, \quad \mathbf{q} = 3 + 2x - 4x^2$

(a) $p = -2 + x + 3x^2$, $q = 4 - 7x^2$

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 = (-2)(4) + (1)(0) + (3)(-7)$

$= -8 - 21 = -29$

(b) $p = -5 + 2x + x^2$, $q = 3 + 2x - 4x^2$

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 = (-5)(3) + (2)(2) + (1)(-4)$

$= -15 + 4 - 4 = -15$

12. Let P_2 have the inner product in Example 8. In each part, find $\|\mathbf{p}\|$

(a) $\mathbf{p} = -2 + 3x + 2x^2$

(b) $\mathbf{p} = 4 - 3x^2$

(a) $p = -2+3x+2x^2$

$\|p\| = \sqrt{(-2)^2+(3)^2+(2)^2}$

$= \sqrt{4+9+4}$

$= \sqrt{17}$

(b) $p = 4-3x^2$

$\|p\| = \sqrt{16+9} = \sqrt{25} = 5$

14. Let P_2 have the inner product in Example 8. Find $d(\mathbf{p}, \mathbf{q})$.

$$\mathbf{p} = 3 - x + x^2, \quad \mathbf{q} = 2 + 5x^2$$

$\langle p,q \rangle = a_0b_0 + a_1b_1 + a_2b_2 \qquad p-q = 1 - x - 4x^2$

$d(p,q) = \langle p-q, p-q \rangle^{1/2} = \sqrt{(1)^2+(-1)^2+(-4)^2} = \sqrt{2+16}$

$= \sqrt{18}$

$= 3\sqrt{2}$.

EXAMPLE 9 An Inner Product on $C[a, b]$

Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be two functions in $C[a, b]$ and define

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx \tag{6}$$

This is well-defined since the functions in $C[a, b]$ are continuous. We shall show that this formula defines an inner product on $C[a, b]$ by verifying the four inner product axioms for functions $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$, and $\mathbf{s} = s(x)$ in $C[a, b]$:

1. $$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle \mathbf{g}, \mathbf{f} \rangle$$

   which proves that Axiom 1 holds.

2. $$\begin{aligned} \langle \mathbf{f} + \mathbf{g}, \mathbf{s} \rangle &= \int_a^b (f(x) + g(x))s(x)\,dx \\ &= \int_a^b f(x)s(x)\,dx + \int_a^b g(x)s(x)\,dx \\ &= \langle \mathbf{f}, \mathbf{s} \rangle + \langle \mathbf{g}, \mathbf{s} \rangle \end{aligned}$$

   which proves that Axiom 2 holds.

3. $$\langle k\mathbf{f}, \mathbf{g} \rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle \mathbf{f}, \mathbf{g} \rangle$$

   which proves that Axiom 3 holds.

4. If $\mathbf{f} = f(x)$ is any function in $C[a, b]$, then $f^2(x) \geq 0$ for all x in $[a, b]$; therefore,

$$\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b f^2(x)dx \geq 0$$

Further, because $f^2(x) \geq 0$ and $\mathbf{f} = f(x)$ is continuous on $[a, b]$, it follows that $\int_a^b f^2(x)dx = 0$ if and only if $f(x) = 0$ for all x in $[a, b]$. Therefore, we have $\langle \mathbf{f}, \mathbf{f} \rangle = \int_a^b f^2(x)dx = 0$ if and only if $\mathbf{f} = \mathbf{0}$. This proves that Axiom 4 holds.

EXAMPLE 10 Norm of a Vector in $C[a, b]$

If $C[a, b]$ has the inner product defined in the preceding example, then the norm of a function $\mathbf{f} = f(x)$ relative to this inner product is

$$\|\mathbf{f}\| = \langle \mathbf{f}, \mathbf{f} \rangle^{1/2} = \sqrt{\int_a^b f^2(x)dx} \tag{7}$$

and the unit sphere in this space consists of all functions f in $C[a, b]$ that satisfy the equation $\|\mathbf{f}\| = 1$, which on squaring yields

$$\int_a^b f^2(x)dx = 1$$

17. **(For Readers Who Have Studied Calculus)**

Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

(a) Find $\|\mathbf{p}\|$ for $\mathbf{p} = 1$, $\mathbf{p} = x$, $\mathbf{p} = x^2$.

(b) Find $d(\mathbf{p}, \mathbf{q})$ if $\mathbf{p} = 1$ and $\mathbf{q} = x$.

(a) Find $\|p\|$ for $p=1$, $p=x$ and $p=x^2$.

If $p=1$, $\langle p,p\rangle = \sqrt{\int_{-1}^{1} p^2\,dx}$

$= \sqrt{\int_{-1}^{1} dx} = \sqrt{[x]_{-1}^{1}} = \sqrt{2}$

If $p=x$, $\langle p,p\rangle = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\left(\frac{x^3}{3}\right)_{-1}^{1}} = \sqrt{\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$

If $p=x^2$, $\langle p,p\rangle = \sqrt{\int_{-1}^{1} x^4\,dx} = \sqrt{\left(\frac{x^5}{5}\right)_{-1}^{1}} = \sqrt{\frac{1}{5}+\frac{1}{5}} = \sqrt{\frac{2}{5}} = \frac{1}{5}\sqrt{10}$

(b) $d(p,q)$ if $p=1$ and $q=x$.

$d(p,q) = \langle p-q, p-q\rangle^{1/2} = \sqrt{\int_{-1}^{1} (1-x)^2\,dx}$

$= \sqrt{\int_{-1}^{1} (1+x^2-2x)\,dx} = \sqrt{\left[x+\frac{x^3}{3}-\frac{2x^2}{2}\right]_{-1}^{1}}$

$= \sqrt{\left(1+\frac{1}{3}-1\right)-\left(-1-\frac{1}{3}-1\right)}$

$= \sqrt{2+\frac{2}{3}} = \sqrt{8/3} = \frac{2}{3}\sqrt{6}$

17. (For Readers Who Have Studied Calculus)

Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$$

(a) Find $\|\mathbf{p}\|$ for $\mathbf{p} = 1$, $\mathbf{p} = x$, $\mathbf{p} = x^2$.

(b) Find $d(\mathbf{p}, \mathbf{q})$ if $\mathbf{p} = 1$ and $\mathbf{q} = x$.

17. **(a)** For instance, $\|x\| = \left(\int_{-1}^{1} x^2 dx \right)^{1/2} = \left(\left. \frac{x^3}{3} \right]_{-1}^{1} \right)^{1/2} = \left(\frac{2}{3} \right)^{1/2}$

(b) We have

$$\begin{aligned} d(\mathbf{p}, \mathbf{q}) &= \|\mathbf{p} - \mathbf{q}\| \\ &= \|1 - x\| \\ &= \left(\int_{-1}^{1} (1-x)^2 dx \right)^{1/2} \\ &= \left(\int_{-1}^{1} (1 - 2x + x^2) dx \right)^{1/2} \\ &= \left(\left. \left(x - x^2 + \frac{x^3}{3} \right) \right]_{-1}^{1} \right)^{1/2} \\ &= 2\left(\frac{2}{3} \right)^{1/2} = \frac{2}{3}\sqrt{6} \end{aligned}$$

27. (For Readers Who Have Studied Calculus)

Use the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$$

to compute $\langle \mathbf{p}, \mathbf{q} \rangle$, for the vectors $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ in P_3.

(a) $\mathbf{p} = 1 - x + x^2 + 5x^3, \quad \mathbf{q} = x - 3x^2$

(a) $p = 1 - x + x^2 + 5x^3 \quad q = x - 3x^2$

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$$

$$= \int_{-1}^{1} (1 - x + x^2 + 5x^3)(x - 3x^2)\, dx$$

$$= \int_{-1}^{1} (x - x^2 + x^3 + 5x^4 - 3x^2 + 3x^3 - 3x^4 - 15x^5)\, dx$$

$$= \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)\, dx$$

$$= \left[\cancel{\frac{x^2}{2}}^{\,0} - \frac{4x^3}{3} + \cancel{\frac{4x^4}{4}}^{\,0} + \frac{2x^5}{5} - \cancel{\frac{15x^6}{6}}^{\,0} \right]_{-1}^{1}$$

$$= -\frac{4}{3}[1 + 1] + \frac{2}{5}[1 + 1]$$

$$= -\frac{8}{3} + \frac{4}{5} = \frac{-40 + 12}{15} = \frac{-28}{15}$$

27. (For Readers Who Have Studied Calculus)

Use the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$$

to compute $\langle \mathbf{p}, \mathbf{q} \rangle$, for the vectors $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ in P_3.

(b) $\mathbf{p} = x - 5x^3, \quad \mathbf{q} = 2 + 8x^2$

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx = \int_{-1}^{1} (x - 5x^3)(2 + 8x^2)\, dx$$

$$= \int_{-1}^{1} (2x - 10x^3 + 8x^3 - 40x^5)\, dx$$

$$= \int_{-1}^{1} (2x - 2x^3 - 40x^5)\, dx$$

$$= \left[x^2 - \frac{2}{4}x^4 - \frac{40}{6}x^6 \right]_{-1}^{1} = 0$$

28. (For Readers Who Have Studied Calculus)

In each part, use the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)\,dx$$

to compute $\langle \mathbf{f}, \mathbf{g} \rangle$, for the vectors $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ in $C[0, 1]$.

(a) $\mathbf{f} = \cos 2\pi x, \quad \mathbf{g} = \sin 2\pi x$

(b) $\mathbf{f} = x, \quad \mathbf{g} = e^x$

(c) $\mathbf{f} = \tan \frac{\pi}{4}x, \quad \mathbf{g} = 1$

ⓐ $f = \cos(2\pi x)$, $g = \sin(2\pi x)$

$\langle f, g \rangle = \int_0^1 \frac{1}{2} \sin(4\pi x)\,dx$

$= 0$

ⓑ $f = x$, $g = e^x$

$= \int_0^1 x e^x\,dx$

$= [x e^x]_0^1 - \int_0^1 e^x\,dx$

$= [e - 0] - [e - 1]$

$= 1$

ⓒ $f = \tan\left(\frac{\pi x}{4}\right)$, $g = 1$

$\frac{2\ln(2)}{\pi}$

Theorem 6.1.1 (Properties of Inner Products)

- If **u**, **v**, and **w** are vectors in a real inner product space, and k is any scalar, then:
  - $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$
  - $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
  - $\langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$
  - $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$
  - $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$

6-1 Example 11

- $\langle \mathbf{u} - 2\mathbf{v}, 3\mathbf{u} + 4\mathbf{v} \rangle$
  $= \langle \mathbf{u}, 3\mathbf{u} + 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u}+4\mathbf{v} \rangle$
  $= \langle \mathbf{u}, 3\mathbf{u} \rangle + \langle \mathbf{u}, 4\mathbf{v} \rangle - \langle 2\mathbf{v}, 3\mathbf{u} \rangle - \langle 2\mathbf{v}, 4\mathbf{v} \rangle$
  $= 3 \langle \mathbf{u}, \mathbf{u} \rangle + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{v}, \mathbf{u} \rangle - 8 \langle \mathbf{v}, \mathbf{v} \rangle$
  $= 3 \| \mathbf{u} \|^2 + 4 \langle \mathbf{u}, \mathbf{v} \rangle - 6 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \| \mathbf{v} \|^2$
  $= 3 \| \mathbf{u} \|^2 - 2 \langle \mathbf{u}, \mathbf{v} \rangle - 8 \| \mathbf{v} \|^2$

16. $\langle \mathbf{u}, \mathbf{v} \rangle = 2, \quad \langle \mathbf{v}, \mathbf{w} \rangle = -3, \quad \langle \mathbf{u}, \mathbf{w} \rangle = 5, \quad \|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = 2, \quad \|\mathbf{w}\| = 7$

Evaluate the given expression.

(a) $\langle \mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w} \rangle$

(b) $\langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle$

(c) $\langle \mathbf{u} - \mathbf{v} - 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle$

(d) $\|\mathbf{u} + \mathbf{v}\|$

(e) $\|2\mathbf{w} - \mathbf{v}\|$

(f) $\|\mathbf{u} - 2\mathbf{v} + 4\mathbf{w}\|$

Assignments

- Exercise-6.1: Examples 1 – 11 (Pages 444-453)
- Exercise-6.1: Problems 1– 28 (Pages 453-458)