Chapter Content

- Inner Products
- Angle and Orthogonality in Inner Product Spaces
- Orthonormal Bases; Gram-Schmidt Process;
- Best Approximation; Least Squares

Theorems 6.2.1 & 6.2.2

- Theorem 6.2.1 (Cauchy-Schwarz Inequality)
  - If **u** and **v** are vectors in a real inner product space, then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$$

- Theorem 6.2.2 (Properties of Length)
  - If **u** and **v** are vectors in an inner product space V, and if k is any scalar, then :
    - $\| \mathbf{u} \| \ge 0$
    - $\| \mathbf{u} \| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $\| k\mathbf{u} \| = | k | \, \| \mathbf{u} \|$
    - $\| \mathbf{u} + \mathbf{v} \| \le \| \mathbf{u} \| + \| \mathbf{v} \|$ (Triangle inequality)

12. In each part, verify that the Cauchy–Schwarz inequality holds for the given vectors using the Euclidean inner product.

(a) $\mathbf{u} = (3, 2), \mathbf{v} = (4, -1)$

a) $u = (3, 2)$, $v = (4, -1)$

$\|u\| = \sqrt{9+4} = \sqrt{13}$ $\|v\| = \sqrt{16+1} = \sqrt{17}$

$\|u\|\,\|v\| = 14.866$.

$\langle u, v\rangle = 3\times 4 - 2 = 12 - 2 = 10$. $\therefore |\langle u, v\rangle| \leq \|u\|\,\|v\|$

ⓑ $U = (-3, 1, 0)$, $V = (2, -1, 3)$

$\|u\| = \sqrt{9+1} = \sqrt{10}$, $\|v\| = \sqrt{4+1+9} = \sqrt{14}$.

$\|u\|\|v\| = 11.83$

$|\langle u, v\rangle| = |-6 - 1 + 0| = 7$.

$\Rightarrow |\langle u, v\rangle| \leq \|u\|\,\|v\|$

ⓒ $u = (-4, 2, 1)$, $v = (8, -4, -2)$

$\|u\| = \sqrt{16+4+1} = \sqrt{21}$; $\|v\| = \sqrt{64+16+4}$

$= \sqrt{84}$

$= 2\sqrt{21}$

$\|u\|\,\|v\| = 42$

$|\langle u, v\rangle| = |-32 - 8 - 2|$

$= 42$

Theorem 6.2.3 (Properties of Distance)

- If **u**, **v**, and **w** are vectors in an inner product space V, and if k is any scalar, then:
  - $d(\mathbf{u}, \mathbf{v}) \geq 0$
  - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
  - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
  - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (Triangle inequality)

6-2 Angle Between Vectors

- The Cauchy-Schwarz inequality for R^n (Theorem 4.1.3) follows as a special case of Theorem 6.2.1 by taking $\langle \mathbf{u}, \mathbf{v} \rangle$ to be the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$.

- The angle between vectors in general inner product spaces can be defined as

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad \text{and} \quad 0 \le \theta \le \pi$$

- Example 2
  - Let R^4 have the Euclidean inner product. Find the cosine of the angle θ between the vectors $\mathbf{u} = (4, 3, 1, -2)$ and $\mathbf{v} = (-2, 1, 2, 3)$.

Solution

We leave it for the reader to verify that

$$\|\mathbf{u}\| = \sqrt{30}, \quad \|\mathbf{v}\| = \sqrt{18}, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle = -9$$

so that

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = -\frac{9}{\sqrt{30}\sqrt{18}} = -\frac{3}{2\sqrt{15}}$$

5. Let R^2, R^3, and R^4 have the Euclidean inner product. In each part, find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) $\mathbf{u} = (1, -3)$, $\mathbf{v} = (2, 4)$

$u = (1, -3)$, $v = (2, 4)$

$\langle u, v \rangle = u_1 v_1 + u_2 v_2 = 2 - 12 = -10.$ $\quad \|u\| = \sqrt{1+9} = \sqrt{10}$ $\quad \|v\| = \sqrt{4+16} = \sqrt{20}.$

$\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\|v\|} = \dfrac{-10}{\sqrt{10} \cdot \sqrt{20}} = -0.707$ or $\dfrac{-1}{\sqrt{2}}$

(b) $\mathbf{u} = (-1, 0)$, $\mathbf{v} = (3, 8)$

b) $u = (-1, 0)$, $v = (3, 8)$

$\langle u, v \rangle = -3$ $\quad \|u\| = 1$ $\quad \|v\| = \sqrt{9+64} = \sqrt{73}.$

$\cos\theta = \dfrac{-3}{\sqrt{73}}$

(c) $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = (2, 4, -9)$

(d) $\mathbf{u} = (4, 1, 8)$, $\mathbf{v} = (1, 0, -3)$

(e) $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{v} = (-3, -3, -3, -3)$

(f) $\mathbf{u} = (2, 1, 7, -1)$, $\mathbf{v} = (4, 0, 0, 0)$

(c) $u = (-1, 5, 2)$

$v = (2, 4, -9)$

$\langle u, v \rangle = -2 + 20 - 18$

$= 0$

$\cos\theta = \underline{0}$

(d) $u = (4, 1, 8)$, $v = (1, 0, -3)$

Ans $= \dfrac{-20}{9\sqrt{10}}$

(e) $u = (1, 0, 1, 0)$; $v = (-3, -3, -3, -3)$

$\langle u, v \rangle = -3 + 0 - 3 + 0 = -6$.

$\|u\| = \sqrt{2}$ $\|v\| = \sqrt{9+9+9+9}$

$= \sqrt{36} = 6$.

$\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \, \|v\|} = \dfrac{-6}{\sqrt{2} \times 6} = \dfrac{-1}{\sqrt{2}}$

(f) $u = (2, 1, 7, -1)$

$v = (4, 0, 0, 0)$

Ans $= 2/\sqrt{55}$

5. Let R^2, R^3, and R^4 have the Euclidean inner product. In each part, find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) $\mathbf{u} = (1, -3)$, $\mathbf{v} = (2, 4)$

(e) $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{v} = (-3, -3, -3, -3)$

(c) $\mathbf{u} = (-1, 5, 2)$, $\mathbf{v} = (2, 4, -9)$

5. (a) $$\cos\theta = \frac{\langle (1,-3),(2,4)\rangle}{\|(1,-3)\|\,\|(2,4)\|} = \frac{2-12}{\sqrt{10}\sqrt{20}} = \frac{-1}{\sqrt{2}}$$

(c) $$\cos\theta = \frac{\langle (-1,5,2),\,(2,4,-9)\rangle}{\|(-1,5,2)\|\,\|(2,4,-9)\|} = \frac{-2+20-18}{\sqrt{30}\,\sqrt{101}} = 0$$

(e) $$\cos\theta = \frac{\langle (1,0,1,0),(-3,-3,-3,-3)\rangle}{\|(1,0,1,0)\|\,\|(-3,-3,-3,-3)\|} = \frac{-3-3}{\sqrt{2}\sqrt{36}} = \frac{-1}{\sqrt{2}}$$

5.)

$u = (1, -3)$ $v = (2, 4)$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{2-12}{\sqrt{10}\cdot\sqrt{20}} = \frac{-10}{\sqrt{200}} = \frac{-1}{\sqrt{2}}$$

$$= \frac{3}{4}\pi.$$

(c) $u = (-1, 5, 2)$ $v = (2, 4, -9)$

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{-2+20-18}{\sqrt{30}\cdot\sqrt{56}} = 0 = 90^\circ //$$

6. Let P_2 have the inner product in Example 8 of Section 6.1. Find the cosine of the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$.

(a) $\mathbf{p} = -1 + 5x + 2x^2$, $\mathbf{q} = 2 + 4x - 9x^2$

(b) $\mathbf{p} = x - x^2$, $\mathbf{q} = 7 + 3x + 3x^2$

(a) $p = -1 + 5x + 2x^2$, $q = 2 + 4x - 9x^2$

$\langle p, q \rangle = -2 + 20 - 18 = 0$

$\cos\theta = 0$.

(b) $p = x - x^2$, $q = 7 + 3x + 3x^2$

$\langle p, q \rangle = 0 + 3 - 3 = 0$.

$\cos\theta = 0$

6) (a) $p = -1 + 5x + 2x^2$, $q = 2 + 4x - 9x^2$

$\cos\theta = \dfrac{\langle p, q \rangle}{\|p\|\,\|q\|} = \dfrac{-2+20-18}{\sqrt{30}\,\sqrt{56}}$

(b) $x - x^2 = p$; $q = 7 + 3x + 3x^2$

$\cos\theta = \dfrac{3-3}{\sqrt{2}\cdot\sqrt{68}} = 0$

8. Let M_{22} have the inner product in Example 7 of Section 6.1. Find the cosine of the angle between A and B.

(a) $A=\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

(b) $A=\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}, B=\begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

(a) $A=\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

$\langle A, B\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

$= 6 + 12 + 1 - 0 = 19.$

$\|A\| = \sqrt{4+36+1+9} = \sqrt{50}$ $\quad \|B\| = \sqrt{9+4+1+0} = \sqrt{14}.$

$\cos\theta = \dfrac{\langle A, B\rangle}{\|A\|\,\|B\|} = \dfrac{19}{\sqrt{50}\cdot\sqrt{14}} = \underline{0.718}$ or $\dfrac{19}{10\sqrt{7}}.$

(b) $A=\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$\langle A, B\rangle = -6+4-4+6 = 0.$

$\cos\theta = 0.$

6-2 Orthogonality

- Two vectors **u** and **v** in an <u>inner product space</u> are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
- Example 3
  - If M_{22} has the inner project defined previously, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal, since $\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$.

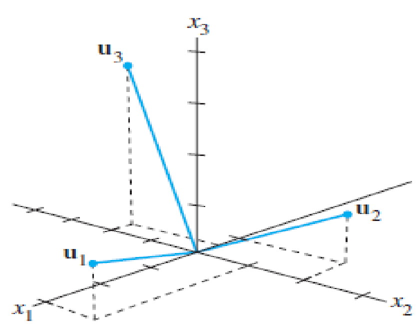

FIGURE 1

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

EXAMPLE 1 Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

SOLUTION Consider the three possible pairs of distinct vectors, namely, $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_3\}$, and $\{\mathbf{u}_2, \mathbf{u}_3\}$.

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 3(-1) + 1(2) + 1(1) = 0$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = 3\left(-\tfrac{1}{2}\right) + 1(-2) + 1\left(\tfrac{7}{2}\right) = 0$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = -1\left(-\tfrac{1}{2}\right) + 2(-2) + 1\left(\tfrac{7}{2}\right) = 0$$

Each pair of distinct vectors is orthogonal, and so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. See Fig. 1; the three line segments there are mutually perpendicular. ■

In each part, determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

1.

(a) $\mathbf{u} = (-1, 3, 2)$, $\mathbf{v} = (4, 2, -1)$

(b) $\mathbf{u} = (-2, -2, -2)$, $\mathbf{v} = (1, 1, 1)$

(c) $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (0, 0, 0)$

(d) $\mathbf{u} = (-4, 6, -10, 1)$, $\mathbf{v} = (2, 1, -2, 9)$

(e) $\mathbf{u} = (0, 3, -2, 1)$, $\mathbf{v} = (5, 2, -1, 0)$

(f) $\mathbf{u} = (a, b)$, $\mathbf{v} = (-b, a)$

(a) $u=(-1,3,2)$ $v=(4,2,-1)$

$\langle u,v\rangle = -4+6-2=0$

Yes

(b) $u=(-2,-2,-2)$

$v=(1,1,1)$

No.

(c) $u=(u_1,u_2,u_3)$, $v=(0,0,0)$ (Yes)

(d) $u=(-4,6,-10,1)$

$v=(2,1,-2,9)$

No

(e) $u=(0,3,-2,1)$, $v=(5,2,-1,0)$ No.

(f) $u=(a,b)$, $v=(-b,a)$

Yes

10. Let R^3 have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal?

(a) $\mathbf{u} = (2, 1, 3)$, $\mathbf{v} = (1, 7, k)$

(b) $\mathbf{u} = (k, k, 1)$, $\mathbf{v} = (k, 5, 6)$

(a) $u = (2,1,3)$, $v = (1,7,k)$

$\langle u, v \rangle = 0 \Rightarrow 2 + 7 + 3k = 0.$

$3k = -9$

$\boxed{k = -3}$

(b) $u = (k,k,1)$, $v = (k,5,6)$

$\langle u, v \rangle = 0 \Rightarrow k^2 + 5k + 6 = 0.$

$\boxed{k = -2, -3}$

2. Do there exist scalars k, l such that the vectors $\mathbf{u} = (2, k, 6)$, $\mathbf{v} = (l, 5, 3)$, and $\mathbf{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

$u = (2,k,6)$, $v = (l,5,3)$, $w = (1,2,3)$

$\langle u, w \rangle = 0 \Rightarrow 2 + 2k + 18 = 0 \Rightarrow k = -10$

$\langle v, w \rangle = 0 \Rightarrow l + 10 + 9 = 0 \Rightarrow l = -19.$

$\langle u, v \rangle = 0 \Rightarrow 2l + 5k + 18 = 0.$

$\therefore$ No scalars k, l do not exist such that all are mutually orthogonal.

3. Let R^3 have the Euclidean inner product. Let $\mathbf{u} = (1, 1, -1)$ and $\mathbf{v} = (6, 7, -15)$. If $\|k\mathbf{u} + \mathbf{v}\| = 13$, what is k?

3. We have $k\mathbf{u} + \mathbf{v} = (k + 6, k + 7, -k - 15)$, so

$$\begin{aligned}\|k\mathbf{u} + \mathbf{v}\| &= \langle (k\mathbf{u} + \mathbf{v}), (k\mathbf{u} + \mathbf{v})\rangle^{1/2} \\ &= \left[(k + 6)^2 + (k + 7)^2 + (-k - 15)^2\right]^{1/2} \\ &= (3k^2 + 56k + 310)^{1/2}\end{aligned}$$

Since $\|k\mathbf{u} + \mathbf{v}\| = 13$ exactly when $\|k\mathbf{u} + \mathbf{v}\|^2 = 169$, we need to solve the quadratic equation $3k^2 + 56k + 310 = 169$ to find k. Thus, values of k that give $\|k\mathbf{u} + \mathbf{v}\| = 13$ are $k = -3$ or $k = -47/3$.

7. Show that $\mathbf{p} = 1 - x + 2x^2$ and $\mathbf{q} = 2x + x^2$ are orthogonal with respect to the inner product in Exercise 6.

Solution:

$\langle p, q \rangle = 0 - 2 + 2 = 0$.

Yes.

9. Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Which of the following matrices are orthogonal to A with respect to the inner product in Exercise 8?

(a) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

9. (b) $\left\langle \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\rangle = (2)(1) + (1)(1) + (-1)(0) + (3)(-1) = 0$

Thus the matrices are orthogonal.

(d) $\left\langle \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \right\rangle = 4 + 1 - 5 + 6 = 6 \neq 0$

Thus the matrices are not orthogonal.

6-2 Example 4 (Orthogonal Vectors in P_2)

- Let P_2 have the inner product $<\mathbf{p},\ \mathbf{q}>=\int_{-1}^{1} p(x)q(x)dx$ and let $\mathbf{p}$ = x and $\mathbf{q}$ = x^2.

- Then

$$\|\mathbf{p}\| =<\mathbf{p},\ \mathbf{p}>^{1/2}=\left[\int_{-1}^{1} xxdx\right]^{1/2} =\left[\int_{-1}^{1} x^2dx\right]^{1/2} =\sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| =<\mathbf{q},\ \mathbf{q}>^{1/2}=\left[\int_{-1}^{1} x^2x^2dx\right]^{1/2} =\left[\int_{-1}^{1} x^4dx\right]^{1/2} =\sqrt{\frac{2}{5}}$$

$$<\mathbf{p},\mathbf{q}>=\int_{-1}^{1} xx^2dx=\int_{-1}^{1} x^3dx=0$$

because $\langle \mathbf{p}, \mathbf{q}\rangle = 0$, the vectors $\mathbf{p}$ = x and $\mathbf{q}$ = x^2 are orthogonal relative to the given inner product.

Theorem 6.2.4 (Generalized Theorem of Pythagoras)

- If **u** and **v** are orthogonal vectors in an inner product space, then

$$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$$

6-2 Example 5

- Since **p** = x and **q** = x^2 are orthogonal relative to the inner product $<\mathbf{p}, \mathbf{q}> = \int_{-1}^{1} p(x)q(x)dx$ on P_2.
- It follows from the Theorem of Pythagoras that

$$\| \mathbf{p} + \mathbf{q} \|^2 = \| \mathbf{p} \|^2 + \| \mathbf{q} \|^2$$

- Thus, from the previous example:

$$\|\mathbf{p}+\mathbf{q}\|^2 = (\sqrt{\frac{2}{3}})^2 + (\sqrt{\frac{2}{5}})^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

- We can check this result by direct integration:

$$\|\mathbf{p}+\mathbf{q}\|^2 = <\mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q}> = \int_{-1}^{1} (x+x^2)(x+x^2)dx$$

$$= \int_{-1}^{1} x^2 dx + 2\int_{-1}^{1} x^3 dx + \int_{-1}^{1} x^4 dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}$$

6-2 Orthogonality

- Let W be a subspace of an inner product space V.
  - A vector **u** in V is said to be orthogonal to W if it is orthogonal to every vector in W, and
  - the set of all vectors in V that are orthogonal to W is called the orthogonal complement of W.

Orthogonal Complements

If V is a plane through the origin of R^3 with the Euclidean inner product, then the set of all vectors that are orthogonal to every vector in V forms the line L through the origin that is perpendicular to V (Figure 6.2.2). In the language of linear algebra we say that the line and the plane are *orthogonal complements* of one another. The following definition extends this concept to general inner product spaces.

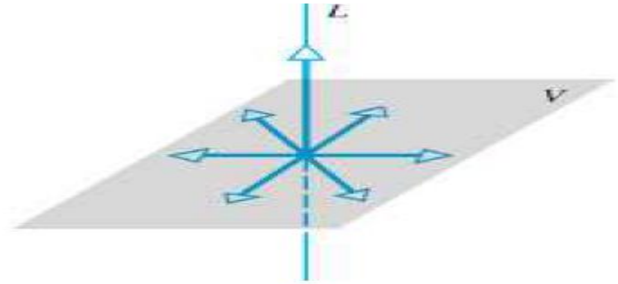

Figure 6.2.2 Every vector in L is orthogonal to every vector in V.

Orthogonal Complements

To provide practice using inner products, we introduce a concept here that will be of use in Section 6.3 and elsewhere in the chapter. If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply "W perp").

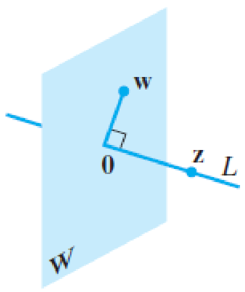

FIGURE 7
A plane and line through $\mathbf{0}$ as orthogonal complements.

EXAMPLE 6 Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. If $\mathbf{z}$ and $\mathbf{w}$ are nonzero, $\mathbf{z}$ is on L, and $\mathbf{w}$ is in W, then the line segment from $\mathbf{0}$ to $\mathbf{z}$ is perpendicular to the line segment from $\mathbf{0}$ to $\mathbf{w}$; that is, $\mathbf{z}\cdot\mathbf{w} = 0$. See Fig. 7. So each vector on L is orthogonal to every $\mathbf{w}$ in W. In fact, L consists of *all* vectors that are orthogonal to the $\mathbf{w}$'s in W, and W consists of all vectors orthogonal to the $\mathbf{z}$'s in L. That is,

$$L = W^\perp \quad \text{and} \quad W = L^\perp$$ ■

1. A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Theorem 6.2.5
(Properties of Orthogonal Complements)

- If W is a subspace of a finite-dimensional inner product space V, then:
  - $W^\perp$ is a subspace of V.
  - The only vector common to W and $W^\perp$ is $\mathbf{0}$;
    that is , $W \cap W^\perp = \mathbf{0}$.
  - The orthogonal complement of $W^\perp$ is W;
    that is , $(W^\perp)^\perp = W$.

A Geometric Link between Nullspace and Row Space

The following fundamental theorem provides a geometric link between the nullspace and row space of a matrix.

Theorem 6.2.6

- If A is an $m \times n$ matrix, then:
  - The nullspace of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.
  - The nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product.

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

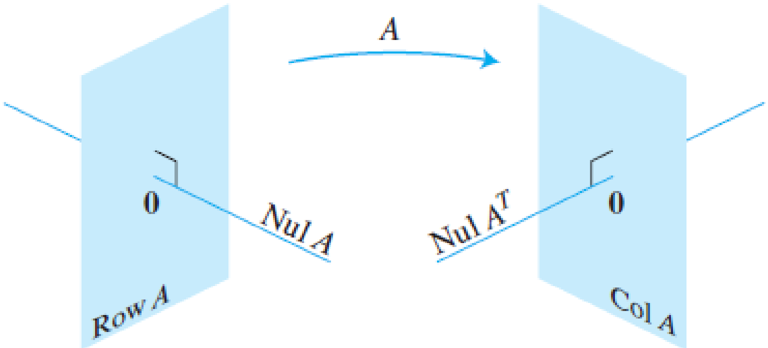

FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

6-2 Example 6 (Basis for an Orthogonal Complement)

- Let W be the subspace of R^5 spanned by the vectors $\mathbf{w}_1$=(2, 2, -1, 0, 1), $\mathbf{w}_2$=(-1, -1, 2, -3, 1), $\mathbf{w}_3$=(1, 1, -2, 0, -1), $\mathbf{w}_4$=(0, 0, 1, 1, 1). Find a basis for the orthogonal complement of W.
- Solution
  - The space W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ is the same as the row space of the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution

The space W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ is the same as the row space of the matrix

$$A=\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and by part (a) of Theorem 6.2.6, the nullspace of A is the orthogonal complement of A. In Example 4 of Section 5.5 we showed that

$$\mathbf{v}_1=\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2=\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for this nullspace. Expressing these vectors in the same notation as $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$, we conclude that the vectors

$$\mathbf{v}_1=(-1, 1, 0, 0, 0) \quad \text{and} \quad \mathbf{v}_2=(-1, 0, -1, 0, 1)$$

form a basis for the orthogonal complement of W. As a check, the reader may want to verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ by calculating the necessary dot products.

4. Let R^4 have the Euclidean inner product, and let $\mathbf{u} = (-1, 1, 0, 2)$. Determine whether the vector u is orthogonal to the subspace spanned by the vectors $w_1 = (0, 0, 0, 0)$, $w_2 = (1, -1, 3, 0)$, and $w_3 = (4, 0, 9, 2)$

Solu: The subspace W spanned by the vectors w_1, w_2 & w_3 is the same as the row space of the matrix,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 4 & 0 & 9 & 2 \end{bmatrix}.$$

To check whether 'u' is orthogonal to the subspace W, we need to show that $Au = 0$. (i.e. 'u' is the null space of A).

From Thm-6.2.6 i.e, we will show that vector 'u' is orthogonal to every vector in the row space.

$\therefore$ $\langle u, w_1 \rangle = 0$, $\langle u, w_2 \rangle = -2$; $\langle u, w_3 \rangle = 0$.

Since u is not orthogonal to the row vector w_2, we conclude 'u' is not orthogonal to the subspace W.

14. Let W be the line in R^2 with equation $y = 2x$. Find an equation for $W^{\perp}$.

Solution:

Method-1:

$2x - y = 0$

$m_1 = \frac{dy}{dx} = 2$

$m_1 m_2 = -1$

$m_2 = -\frac{1}{2} \Rightarrow \frac{dy_2}{dx} = -\frac{1}{2} \Rightarrow \boxed{y_2 = \frac{-x}{2}}$

$\Rightarrow x + 2y = 0$

Method-2:

$2x - y = 0 \longrightarrow (1)$

Null space

$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

The $W^{\perp}$ is the row space with $V = [2 \ -1]$

$t\vec{V} = [2t, -t] \rightarrow$ parametric eqn line

$x = 2t \ ; \ y = -t$

$x = -2y \Rightarrow x + 2y = 0 \longrightarrow W^{\perp}$

15.

(a) Let W be the plane in R^3 with equation $x-2y-3z=0$. Find parametric equations for $W^{\perp}$.

(b) Let W be the line in R^3 with parametric equations

$$x=2t, \quad y=-5t, \quad z=4t \quad (-\infty < t < \infty)$$

Find an equation for $W^{\perp}$.

(c) Let W be the intersection of the two planes

$$x+y+z=0 \quad \text{and} \quad x-y+z=0$$

in R^3. Find an equation for $W^{\perp}$.

Solution:
a)

$x-2y-3z=0.$

$$\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

The Compliment of W is vector $V=[1,-2,-3]$

$t\vec{V}=[t,-2t,-3t]$

$x=t$; $y=-2t$; $z=-3t$

Solution:
b)

$t[2,-5,4]$

$V \rightarrow$ Row space (W) then $W^{\perp} \rightarrow$ Null space

$$\begin{bmatrix} 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \Rightarrow W^{\perp} = \{(x,y,z) \,/\, 2x-5y+4z=0\}$$

15. (a) Here $W^{\perp}$ is the line which is normal to the plane and which passes through the origin. By inspection, a normal vector to the plane is $(1, -2, -3)$. Hence this line has parametric equations $x=t$, $y=-2t$, $z=-3t$.

Solution:c)

Solve it using the similar method applied for next problem.

Let W be the subspace of R^5 spanned by vectors $u=[1,2,3,-1,2]; v=[2,4,7,2,-1]$. Find a basis for orthogonal compliment $W^{\perp}$ of the subspace W.

Solution:

A · Rowspace

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$W^{\perp}$ Null space

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0.$$

$$x_3 + 4x_4 - 5x_5 = 0.$$

$$x_1 + 2x_2 + 3x_3 - x_4 + 2x_5 = 0. \Rightarrow x_1 = -2x_2 + 13x_4 + 17x_5$$

16. Let

$$A=\begin{bmatrix}1 & 2 & -1 & 2\\ 3 & 5 & 0 & 4\\ 1 & 1 & 2 & 0\end{bmatrix}$$

(a) Find bases for the row space and nullspace of A.

(b) Verify that every vector in the row space is orthogonal to every vector in the nullspace (as guaranteed by Theorem 6.2.6*a*).

Find bases for row space and Null space of A. Verify that every vector in the row space is orthogonal to vector in the N.S.

Sol: $AX = 0$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & -2 \\ 0 & -1 & 3 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & +1 & -3 & +2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The basis for row space is $(1, 2, -1, 2)$, $(0, 1, -3, 2)$.

$\therefore \quad A.X = 0 \Rightarrow \quad x_1 + 2x_2 - x_3 + 2x_4 = 0$

$x_2 - 3x_3 + 2x_4 = 0$

$x_2 = 3x_3 - 2x_4 \quad x_1 = x_3 - 2[3x_3 - 2x_4] - 2x_4$

$\Rightarrow x_1 = -5x_3 + 2x_4$.

$\therefore$ let $x_3 = t \quad x_4 = r$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5t + 2r \\ 3t - 2r \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis for Nullspace is $X_1 = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

$\langle r_1, X_1 \rangle = 0 \quad , \langle r_2, X_1 \rangle = 0 , \quad \langle r_3, X_1 \rangle = 0$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & -2 \\ 0 & -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Row space $= \{[1,2,-1,2],[0,1,-3,2]\}$

Null Space: $AX = 0$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

No. of parameters = No. of variables − No. of eq^n

$= 4 - 2 = 2$

$$x_1 = -5x_3 + 2x_4 \qquad Nullspace = X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$
$$x_2 = 3x_3 - 2x_4$$

Basis for $Nullspace = \left\{ \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for Row space $= \{[1,2,-1,2],[0,1,-3,2]\}$

Now check whether the vectors in Row space basis is orthogonal to vectors in Null space.

Let A be the matrix in Exercise 16.

17.

(a) Find bases for the column space of A and nullspace of A^T.

(b) Verify that every vector in the column space of A is orthogonal to every vector in the nullspace of A^T (as guaranteed by Theorem 6.2.6*b*).

17. **(a)** The subspace of R^3 spanned by the given vectors is the row space of the matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \quad \text{which reduces to} \quad \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

The space we are looking for is the nullspace of this matrix. From the reduced form, we see that the nullspace consists of all vectors of the form $(16, 19, 1)t$, so that the vector $(16, 19, 1)$ is a basis for this space.

Alternatively the vectors $\mathbf{w}_1 = (1, -1, 3)$ and $\mathbf{w}_2 = (0, 1, -19)$ form a basis for the row space of the matrix. They also span a plane, and the orthogonal complement of this plane is the line spanned by the normal vector $\mathbf{w}_1 \times \mathbf{w}_2 = (16, 19, 1)$.

Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & -2 \\ 0 & -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ c_1' & c_2' \end{array}$$

$\Rightarrow$ Column space of A = $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$

Null space of A^T =

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -3 \\ 2 & 2 \end{bmatrix} \times \quad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ -1 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ No of parameter = 1

let $x_3 = a$

$\Rightarrow x_2 = -a \quad \Rightarrow x_1 + 3x_2 + x_3 = 0$

$\Rightarrow x_1 = -(-3a + a) = 2a$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2a \\ -a \\ a \end{pmatrix} = a \underbrace{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}_{w_1}$$

(ii) Null space is $\perp$r to basis Column space

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 2 - 3 + 1 = 0 \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = 4 - 5 + 1 = 0$$

Ans

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's in Row reduced echelon form matrix R are $c_1' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $c_2' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

The corresponding column vectors in 'A' are

$c_1 = (1, 3, 1)^T \quad c_2 = (2, 5, 1)^T$.

$\therefore$ The basis for column space is $\{c_1, c_2\}$

$= \{(1, 3, 1)^T, (2, 5, 1)^T\}$

$$A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ -1 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix} \quad 4-6$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 + x_3 = 0$

$x_2 + x_3 = 0$

$x_2 = -x_3$

$$\therefore X = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix}1\\3\\1\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}2 & -1 & 1\end{bmatrix}\begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$= 2-3+1=0$$

$$\begin{bmatrix}2\\5\\1\end{bmatrix}\cdot\begin{bmatrix}2 & -1 & 1\end{bmatrix} = 4-5+1=0$$

$$\begin{bmatrix}2\\4\\0\end{bmatrix}\begin{bmatrix}2 & -1 & 1\end{bmatrix} = \begin{bmatrix}4-4\end{bmatrix}.$$

$$\begin{bmatrix}-1\\0\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\end{bmatrix} = -2\ 0+2$$

18. Find a basis for the orthogonal complement of the subspace of R^n spanned by the vectors.

(a) $\mathbf{v}_1 = (1, -1, 3)$, $\mathbf{v}_2 = (5, -4, -4)$, $\mathbf{v}_3 = (7, -6, 2)$

(b) $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, -2)$

(c) $\mathbf{v}_1 = (1, 4, 5, 2)$, $\mathbf{v}_2 = (2, 1, 3, 0)$, $\mathbf{v}_3 = (-1, 3, 2, 2)$

(d) $\mathbf{v}_1 = (1, 4, 5, 6, 9)$, $\mathbf{v}_2 = (3, -2, 1, 4, -1)$, $\mathbf{v}_3 = (-1, 0, -1, -2, -1)$, $\mathbf{v}_4 = (2, 3, 5, 7, 8)$

(a) $v_1 = (1, -1, 3)$ $v_2 = (5, -4, -4)$, $v_3 = (7, -6, 2)$.

$Ax = 0 \Rightarrow$ Basis Sol: $(16, 19, 1)$.

(b) $v_1 = (2, 0, -1)$ $v_2 = (4, 0, -2)$. $(0, 1, 0)$ $(1/2, 0, 1)$

(c) $v_1 = (1, 4, 5, 2)$, $v_2 = (2, 1, 3, 0)$ $v_3 = (-1, 3, 2, 2)$

$(-1, -1, 1, 0)$ $(2/7, -4/7, 0, 1)$

(d) $v_1 = (1, 4, 5, 6, 9)$, $(3, -2, 1, 4, -1)$, $v_3 = (-1, 0, -1, -2, -1)$

$v_4 = (2, 3, 5, 7, 8)$

$(-1, -1, 1, 0, 0)$, $(-2, -1, 0, 1, 0)$, $(-1, -2, 0, 0, 1)$

(a) $\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$ If $x_3 = k$, then $x_1 = 16k$; $x_2 = 19k$

$X = k \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

11. Let R^4 have the Euclidean inner product. Find two unit vectors that are orthogonal to the three vectors $\mathbf{u} = (2, 1, -4, 0)$, $\mathbf{v} = (-1, -1, 2, 2)$, and $\mathbf{w} = (3, 2, 5, 4)$.

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0.$$

Find (u_1, u_2, u_3, u_4) s.t $\|u\| = 1$.

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 - 3R_1$

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 1 & 22 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & -4 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 22 & 12 \end{bmatrix}$$

$2u_1 + u_2 - 4u_3 = 0$

$-u_2 + 4u_4 = 0$. $\quad u_2 = 4u_4$

$22u_3 + 12u_4 = 0$. $\quad u_3 = -\frac{6}{11}u_4$.

$2u_1 = 4u_3 - 2u_2$ $\Rightarrow 2u_1 = \frac{4 \times -6u_4}{11} - 4u_4$

$2u_1 = \frac{-24u_4 - 44u_4}{11}$

$2u_1 = \frac{-68}{11}u_4$

$u_1 = \frac{-34}{11}u_4$

$$\boxed{u_2 = 4u_4;\ u_3 = \frac{-6u_4}{11};\ u_1 = \frac{-34}{11}u_4.}$$

Let $u_4 = 11$.

$$\Rightarrow u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -34 \\ 44 \\ -6 \\ 11 \end{bmatrix}.$$

Given $\|u\| = 1$.

$\sqrt{3249} = 57$.

we need two vectors of norm 1.

$\therefore u = \pm \frac{1}{57}(-34, 44, -6, 11)$

Theorem 6.2.7 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The range of T_A is R^n.
  - T_A is one-to-one.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.
  - The orthogonal complement of the nullspace of A is R^n.
  - The orthogonal complement of the row of A is $\{\mathbf{0}\}$.

Assignments

- Exercise-6.2: Examples 1 – 6 (Pages 461-468)
- Exercise-6.2: Problems 1– 18 (Pages 470-474)