HW3

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1. (a) see code

(b) ① 4-d:

```
%% HW2 4_d
A = [10.^-20, 1;
      1, 1];
b = [1;
      2];

[U, x] = gausselim(A,b);
error = norm(x - A\b);

% Display solution
disp("x");
disp(x);
disp("Error");
disp(error);
```

```
>> HW2_Prom4d
x
     1
     1

Error
     0
```

- Solution x is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ now and error is zero
- Result in HW2 was $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with error of 1
- This indicates that pivoting help avoid error generated by float-point calculations.

② 4-e

```
%% Problem 4_e
A = 2*rand(1000)-1;
A = A/norm(A);
xtrue = 2*rand(1000,1)-1;
xtrue = xtrue/norm(xtrue);
b = A*xtrue;

[U,x] = gausselim(A,b);

error = norm(x - A\b);

disp("Partial Pivoting Error")
disp(error);

[L, U, z] = lu_nopivot(A, b);
y = forwardsub(L, b);
x = backsub(U, y);
error = norm(x - A\b);

disp("LU Error")
disp(error);
```

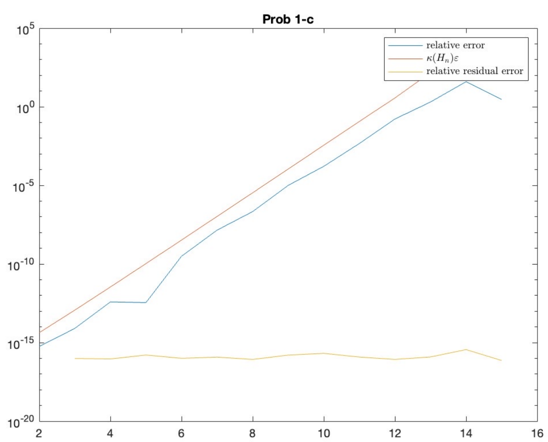
```
>> HW2_Prom4e
Partial Pivoting Error
   2.2174e-13

LU Error
   3.0147e-10
```

```
>> HW2_Prom4e
Partial Pivoting Error
   2.0736e-13

LU Error
   9.2729e-10
```

```
>> HW2_Prom4e
Partial Pivoting Error
   3.3826e-13

LU Error
   8.3628e-09
```

- In multiple trials, the errors of partial pivoting gausselim are still significant, but each of them is smaller than error of non-pivoting LU decomposition
- This indicates that the result's error is only partly generated by rounding error, Other factors also lead to the error generated.

(c)



- As the graph above indicate, relative error is smaller than bound error, but both increase as the size of matrix increases. In contrast, residual error remains stable as size of matrix increases
- This indicate that though Gaussia elimination is backward stable, there exists accumulation of rounding errors

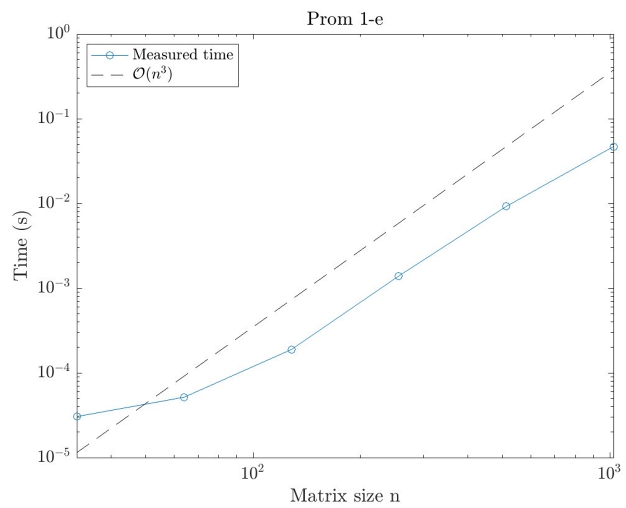
(d)

```
>> Prob1_d
Relative Error:
    0.6643

Condition Number:
   44.8023

Bound Error:
   9.9481e-15

Relative Residual Error:
    0.6675
```

- As the outputs indicate, Gaussian elimination here is no longer backward stable. error bound is now smaller than relative error & residual error.

- Since the Gaussian elimination is no longer backward stabe, the relative error can be larger than ε, therefore bound error can no longer be estimited using $\kappa(A)\cdot\varepsilon$; Instead, it should use relative residual error.
  $\Rightarrow$ cond_num * res_err = 29.9046, significantly larger.
  Therefore, this does NOT violate what we learned.

(e)



slope of $O(n^3)$ is greater than slope of measured time when size of matrix is large enough. complexity is $O(n^3)$

2. (a) $Ax = b$ Prove $(A+\delta)x = b+\delta x$

$(A+\delta)\hat{x} = \hat{b}$

$$\Rightarrow \begin{cases} (A+\delta)x = b+\delta x \\ (A+\delta)\hat{x} = \hat{b} \end{cases}$$

$$\Rightarrow (A+\delta)(x-\hat{x}) = b+\delta x-\hat{b} \quad ①$$

$$\frac{\|x-\hat{x}\|}{\|x\|} = \|(A+\delta)^{-1}\| \cdot \frac{1}{\|x\|} \cdot \|b-\hat{b}+\delta x\|$$

$$= \|(A+\delta)^{-1}\| \cdot \frac{\|b-\hat{b}\|}{\|x\|} + \|(A+\delta)^{-1}\| \cdot \frac{\|\delta x\|}{\|x\|}$$

$$= \|(A+\delta)^{-1}\| \cdot \frac{\|b-\hat{b}\|}{\cancel{\|b\|}} \cdot \frac{\cancel{\|b\|}}{\|x\|} + \|\delta\| \cdot \|(A+\delta)^{-1}\|$$

$$\leq \|(A+\delta)^{-1}\| \cdot \frac{\|b-\hat{b}\|}{\|b\|} \cdot \|A\| + \|\delta\| \cdot \|(A+\delta)^{-1}\|$$

$$= \|(A+\delta)^{-1}\| \cdot \|A\| \cdot \left(\frac{\|b-\hat{b}\|}{\|b\|} + \frac{\|\delta\|}{\|A\|}\right)$$

$$\therefore \frac{\|x-\hat{x}\|}{\|x\|} \leq \|(A+\delta)^{-1}\| \cdot \|A\| \cdot \left(\frac{\|b-\hat{b}\|}{\|b\|} + \frac{\|\delta\|}{\|A\|}\right)$$

(b) $$(A+\delta)^{-1} = (I+A^{-1}\delta)^{-1}A^{-1}$$

$$= \left(\sum_{k=0}^{\infty}(-1)^k \cdot (A^{-1}\delta)^k\right) \cdot A^{-1}$$

$$= \left(I + \sum_{k=1}^{\infty}(-1)^k(A^{-1}\delta)^k\right) \cdot A^{-1}$$

$$= A^{-1} + \left(\sum_{k=1}^{\infty}(-1)^k(A^{-1}\delta)^k\right) \cdot A^{-1}$$

$$\therefore \|(A+\delta)^{-1}\| \leq \|A^{-1}\| + \|A^{-1}\| \sum_{k=1}^{\infty}\|A^{-1}\delta\|^k$$

$$= \|A^{-1}\|\left(1+\sum_{k=1}^{\infty}\|A^{-1}\delta\|^k\right)$$

$$\therefore \|(A+\delta)^{-1}\| \leq \|A^{-1}\|\left(1+\sum_{k=1}^{\infty}\|A^{-1}\delta\|^k\right)$$

(c) From (b):

$$\|(A+\delta)^{-1}\| \leq \|A^{-1}\|\left(1+\sum_{k=1}^{\infty}\|A^{-1}\delta\|^k\right)$$

$$= \|A^{-1}\|\left(1+\frac{\|A^{-1}\delta\|}{1-\|A^{-1}\delta\|}\right)$$

$$\leq \|A^{-1}\|\left(1+\frac{\|A^{-1}\|\,\|\delta\|}{1-\|A^{-1}\|\,\|\delta\|}\right)$$

(d) $$\|(A+\delta)^{-1}\| \leq \|A^{-1}\|\left(1+\sum_{k=1}^{\infty}\|A^{-1}\delta\|^k\right)$$

$$\leq \|A^{-1}\|\left(1+\frac{\|A^{-1}\|\,\|\delta\|}{1-\|A^{-1}\|\,\|\delta\|}\right)$$

$$= \|A^{-1}\|\left(1+\frac{(\|A^{-1}\|\|\delta\|-1)+1}{1-\|A^{-1}\|\|\delta\|}\right)$$

$$= \|A^{-1}\|\left(1+(-1)+\frac{1}{1-\|A^{-1}\|\|\delta\|}\right)$$

$$= \|A^{-1}\|\left(\frac{1}{1-\|A^{-1}\|\|\delta\|}\right)$$

$$\therefore \frac{\|x-\hat{x}\|}{\|x\|} \leq \|(A+\delta)^{-1}\| \cdot \|A\| \cdot \left(\frac{\|b-\hat{b}\|}{\|b\|} + \frac{\|\delta\|}{\|A\|}\right)$$

$$= \frac{1}{1-\|A^{-1}\|\|\delta\|} \cdot \left(\frac{\|b-\hat{b}\|}{\|b\|} + \frac{\|\delta\|}{\|A\|}\right)$$

$$\leq \frac{\kappa(A)}{1-\kappa(A)\|\delta\|/\|A\|}\left(\frac{\|b-\hat{b}\|}{\|b\|} + \frac{\|\delta\|}{\|A\|}\right)$$

3. (a) $f(x) = \cos^3(\sin x)$

① $f(-x) = \cos^3(\sin(-x))$

$= \cos^3(-\sin x)$ ∵ $\cos x = \cos(-x)$

$= \cos^3(\sin x)$

∴ $f(x) = f(-x) = \cos^3(\sin x)$

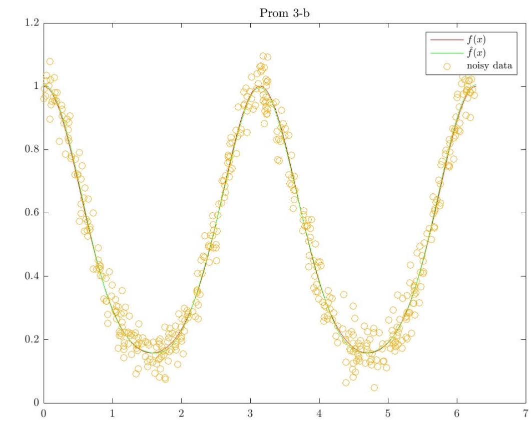
$f(x)$ is even

② ∵ $\sin x$, $\cos x$ are all 2π-periodic.

∴ $f(x)$ is 2π-periodic

∴ $f(x)$ is even and 2π-periodic

(b)




```
>> prob3
max error:
     1.035877565898013e-02
```