M3M6: Applied Complex Analysis

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1 Solution Sheet 3

1.1 Problem 1.1

We actually start by showing the second properties of Problem 1.2, for all α:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x)\right] &= \frac{1}{n!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\ &= (n+1)x^{\alpha-1}\mathrm{e}^{-x}\frac{x^{1-\alpha}\mathrm{e}^{x}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\ &= (n+1)x^{\alpha-1}\mathrm{e}^{-x}L_{n+1}^{(\alpha-1)}(x).\end{aligned}$$

Expanding out the derivative we see

$$x^{\alpha-1}\mathrm{e}^{-x}\left((\alpha-x)L_n^{(\alpha)}(x) + (L_n^{(\alpha)})'(x)\right) = (n+1)x^{\alpha-1}\mathrm{e}^{-x}L_{n+1}^{(\alpha-1)}(x)$$

or in other words

$$(\alpha-x)L_n^{(\alpha)}(x) + (L_n^{(\alpha)})'(x) = (n+1)L_{n+1}^{(\alpha-1)}(x)$$

By induction with the fact $L_0^{(\alpha)}(x) = 0$, we therefore get

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1-x)L_{n-1}^{(\alpha+1)}(x) + (L_n^{(\alpha+1)})'(x)}{n}$$

is a degree n polynomial. We further have that the leading coefficient is

$$\begin{aligned}L_n^{(\alpha)}(x) &= -\frac{x}{n}L_{n-1}^{(\alpha+1)}(x) + O(x^{n-1}) = \frac{x^2}{n(n-1)}L_{n-2}^{(\alpha+2)}(x) + O(x^{n-1}) = \cdots \\ &= \frac{(-1)^n x^n}{n!}L_0^{(\alpha+n)}(x) + O(x^{n-1}) \\ &= \frac{(-1)^n x^n}{n!} + O(x^{n-1})\end{aligned}$$

We now show orthogonality with lower degree polynomials using integration by parts:

$$\int_0^\infty L_n^{(\alpha)}(x)p_m(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = \int_0^\infty \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+n}\mathrm{e}^{-x}\right]p_m(x)\mathrm{d}x = (-1)^n\int_0^\infty \frac{1}{n!}\left[x^{\alpha+n}\mathrm{e}^{-x}\right]p_m^{(n)}(x)\mathrm{d}x = 0$$

since $p_m^{(n)}(x) = 0$. Note we use the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] = O(x^{\alpha+n-k})$$

hence vanishes at zero to ignore the boundary terms in integration by parts.

1.2 Problem 1.2

We showed the second property as part of 1.1. For the first part, it is clear that we have the correct constant. Now we show orthogonality with all degree $m < n-1$ polynomials (using the fact that $x^{\alpha+1}\mathrm{e}^{-x}$ is zero at $x = 0$):

$$\int_0^\infty \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x} p_m(x) x^{\alpha+1}\mathrm{e}^{-x}\mathrm{d}x = -\int_0^\infty L_n^{(\alpha)}(x)(xp'_m(x) + (\alpha+1)p_m - xp_m)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = 0$$

since $(xp'_m(x) + (\alpha+1)p_m - xp_m)$ is degree $m+1 < n$.

For the third part, use the product rule on the last derivative:

$$\begin{aligned}
(n+1)L_{n+1}^{(\alpha)}(x) &= \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+n+1}\mathrm{e}^{-x}\right] \\
&= \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(\alpha+n+1)x^{\alpha+n}\mathrm{e}^{-x} - x^{\alpha+n+1}\mathrm{e}^{-x}\right] \\
&= (\alpha+n+1)L_n^{(\alpha)}(x) - xL_n^{(\alpha+1)}(x)
\end{aligned}$$

For the last result, we apply the product rule n times:

$$\begin{aligned}
L_n^{(\alpha+1)}(x) &= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\mathrm{d}}{\mathrm{d}x}\left[xx^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}x\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{2}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{3}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}x\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&\vdots \\
&= \frac{n}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= L_{n-1}^{(\alpha+1)}(x) + L_n^{(\alpha)}(x)
\end{aligned}$$

1.3 Problem 1.3

Note that relationship 3 above did not depend on $\alpha > -1$. We therefore have from 1.2, comibing property (3) and (4),

$$\begin{aligned}
xL_n^{(\alpha)}(x) &= -(n+1)L_{n+1}^{(\alpha-1)}(x) + (n+\alpha)L_n^{(\alpha-1)}(x) \\
&= -(n+1)L_{n+1}^{(\alpha)}(x) + (n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \\
&= -(n+\alpha)L_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)
\end{aligned}$$

The Jacobi operator therefore has the form

$$x\begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha+1 & -1 & & & & \\ -1-\alpha & \alpha+3 & -2 & & & \\ & -2-\alpha & \alpha+5 & -3 & & \\ & & -3-\alpha & \alpha+7 & -4 & \\ & & & -4-\alpha & \alpha+9 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}\begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \end{pmatrix}$$

1.4 Problem 2.1

Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x/2}u(x) = \mathrm{e}^{-x/2}(-\frac{u(x)}{2} + u'(x))$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x/2}u(x) = \mathrm{e}^{-x/2}(u'(x) - \frac{u(x)}{2} - xu(x))$$

We have the derivative operator from $L_k(x)$ to $L_k^{(1)}(x)$ as:

$$D = \begin{pmatrix} 0 & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$$

and the Multiplication operator for $\alpha = 0$ (from Problem 1.3)

$$J^\top = \begin{pmatrix} 1 & -1 & & \\ -1 & 3 & -2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

and the conversion operator (from Problem 1.2 property 4)

$$S = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \end{pmatrix}$$

We thus have multiplication by x from basis to the other as

$$SJ^\top = \begin{pmatrix} 2 & -4 & 2 & & & \\ -1 & 5 & -7 & 3 & & \\ & -2 & 8 & -10 & 4 & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Putting everything together, we get the operator

```
using ApproxFun, SingularIntegralEquations, Plots, LinearAlgebra, SpecialFunctions
import ApproxFunOrthogonalPolynomials: Recurrence

D = Derivative() :  Laguerre(0) → Laguerre(1)
```

```
S = I : Laguerre(0) → Laguerre(1)

Jt = Recurrence(Laguerre(0))

D - S/2 -S*Jt

PlusOperator : ApproxFunOrthogonalPolynomials.Laguerre{Int64,ApproxFunOrtho
gonalPolynomials.Ray{false,Float64}}(0,  0.0,∞ ) → ApproxFunOrthogonalPolyn
omials.Laguerre{Int64,ApproxFunOrthogonalPolynomials.Ray{false,Float64}}(1,
 0.0,∞ )
 -2.5 + 0.0im   3.5 + 0.0im  -2.0 + 0.0im  …            ⋅              ⋅
 ⋅
  1.0 + 0.0im  -5.5 + 0.0im   6.5 + 0.0im              ⋅              ⋅
 ⋅
       ⋅          2.0 + 0.0im  -8.5 + 0.0im             ⋅              ⋅
 ⋅
       ⋅             ⋅          3.0 + 0.0im             ⋅              ⋅
 ⋅
       ⋅             ⋅             ⋅                ⋅              ⋅
 ⋅
       ⋅             ⋅             ⋅         …         ⋅              ⋅
 ⋅
       ⋅             ⋅             ⋅             -8.0 + 0.0im          ⋅
 ⋅
       ⋅             ⋅             ⋅             24.5 + 0.0im   -9.0 + 0.0im
 ⋅
       ⋅             ⋅             ⋅            -26.5 + 0.0im   27.5 + 0.0im
 ⋱
       ⋅             ⋅             ⋅              9.0 + 0.0im  -29.5 + 0.0im
 ⋱
       ⋅             ⋅             ⋅         …         ⋅              ⋱
 ⋱
```

1.5 Problem 2.2

We have

$$\frac{\mathrm{e}^x}{x^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+1}\mathrm{e}^{-x}\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x}\right] = -\frac{\mathrm{e}^x}{x^\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+1}\mathrm{e}^{-x}\frac{L_{n-1}^{(\alpha+1)}}{\mathrm{d}x}\right]$$
$$= -nL_n^{(\alpha)}(x)$$

Therefore $\lambda_n = -n$. This can be expanded in the form:

$$x\frac{\mathrm{d}^2 L_n^{(\alpha)}}{\mathrm{d}x^2} + (\alpha+1-x)\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = -nL_n^{(\alpha)}(x)$$

1.6 Problem 3.1

Define $C_k^{(\alpha)}(z) = \mathcal{C}[L_k^{(\alpha)} \diamond^\alpha \mathrm{e}^{-\diamond}](z)$ and recall that

$$
\begin{aligned}
C_1(z) &= \frac{\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x + (z-a_0)C_0(z)}{b_0} = -\frac{1}{2\pi\mathrm{i}} - (z-1)C_0(z) \\
&= \frac{(z-1)\mathrm{e}^{-z}\mathrm{Ei}\, z - 1}{2\pi\mathrm{i}}
\end{aligned}
$$

Here we double check the formula, noting that $L_1(x) = \mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x\mathrm{e}^{-x} = 1 - x$:

```
const ei_-_1 = let ζ = Fun(-100 .. -1)
    sum(exp(ζ)/ζ)
end
function ei(z)
    ζ = Fun(Segment(-1 , z))
    ei_-_1 + sum(exp(ζ)/ζ)
end

x = Fun(0..10)
w = exp(-x)
z = 1+im
cauchy((1-x)*w, z),((z-1)*exp(-z)*ei(z)-1)/(2π*im)
```

```
(0.018684644298457898 + 0.04836133565334999im, 0.018683923002628996 + 0.048
368487990893105im)
```

We now use these to determine the results with $\alpha = 1$. Note that:

$$
C_0^{(1)}(z) = \mathcal{C}[\diamond \mathrm{e}^{-\diamond}](z) = C_0(z) - C_1(z) = \frac{\mathrm{e}^{-z}\mathrm{Ei}\, z - (z-1)\mathrm{e}^{-z}\mathrm{Ei}\, z + 1}{2\pi\mathrm{i}}
$$

```
cauchy(x*w, z),(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1)/(2π*im)
```

```
(0.09210173751684986 - 0.02967669135489 21im, 0.09210253209837324 - 0.029684
56498826411im)
```

Therefore, we have

$$
\begin{aligned}
C_1^{(1)}(z) &= \frac{\frac{1}{2\pi\mathrm{i}}\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x + (z-a_0^{(1)})C_0^{(1)}(z)}{b_0^{(1)}} \\
&= \frac{\frac{1}{2\pi\mathrm{i}} + (z-2)C_0^{(1)}(z)}{-1} \\
&= \frac{1 + (z-2)(\mathrm{e}^{-z}\mathrm{Ei}\, z - (z-1)\mathrm{e}^{-z}\mathrm{Ei}\, z + 1)}{-2\pi\mathrm{i}}
\end{aligned}
$$

Let's check the result using

$$
L_1^{(1)}(x) = x^{-1}\mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x^2 \mathrm{e}^{-x} = 2 - x
$$

```
cauchy((2-x)*x*w, z),(1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π*im)
```

```
(0.0624250461619579 + 0.03729703236453841im, 0.06241796711010913 + 0.037367
846005258im)
```

1.7 Problem 3.2

We have

$$\int_x^\infty L_2(x)\mathrm{e}^{-x}\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{-x}L_1^{(1)}(x)\mathrm{e}^{-x}$$

Thus from lectures we have

$$\frac{1}{2\pi\mathrm{i}}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log(z-x)\mathrm{d}x = \frac{1}{2}\mathcal{C}[\diamond\mathrm{e}^{-\diamond}L_1^{(1)}](z)$$

and therefore

$$\frac{1}{\pi}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log|z-x|\mathrm{d}x = -\Im\mathcal{C}[\diamond\mathrm{e}^{-\diamond}L_1^{(1)}](z) = \Re\frac{1+(z-2)(\mathrm{e}^{-z}\mathrm{Ei}\,z-(z-1)\mathrm{e}^{-z}\mathrm{Ei}\,z+1)}{-2\pi}$$

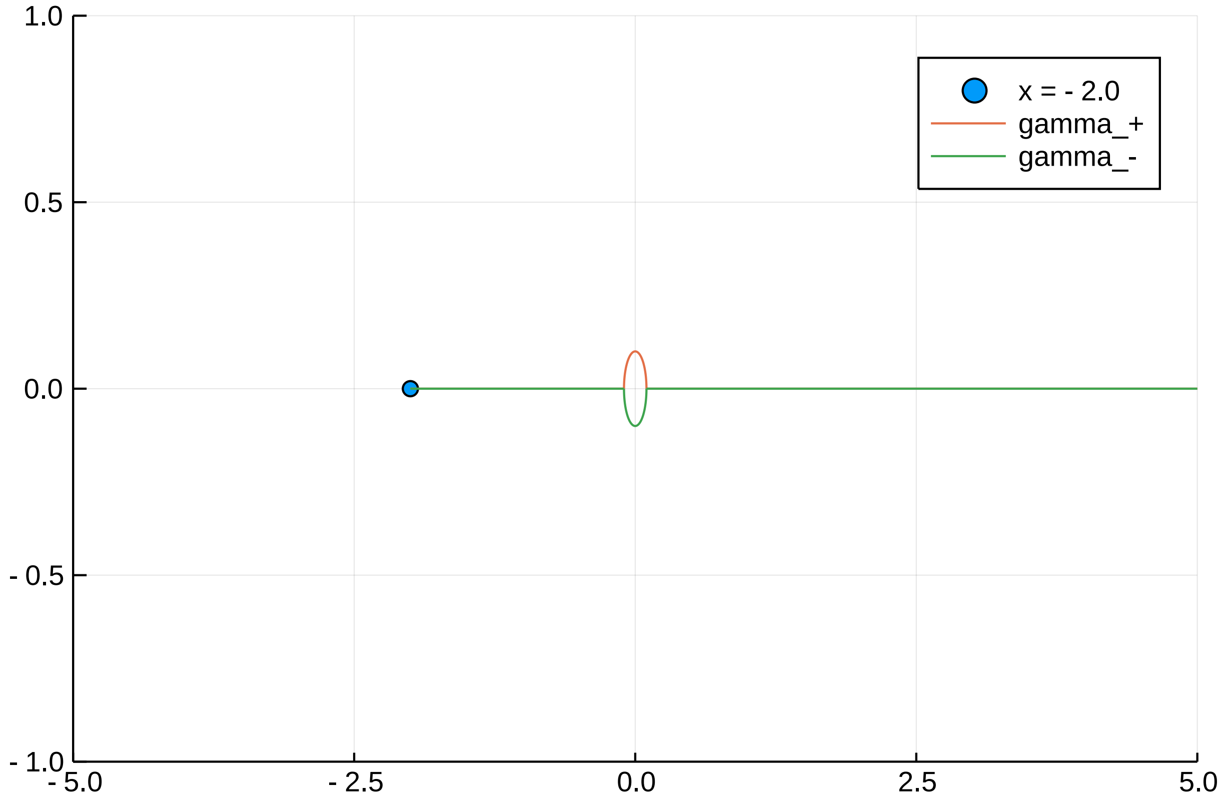
Let's check the result:

```
x = Fun(0 .. 100)
w = exp(-x)
z = 2+im

-sum(1/2*(2 - 4x + x^2)*w*log(abs(z-x)))/π,imag(sum(1/2*(2 - 4x +
x^2)*w*log(z-x))/(π*im))

(-0.06972323453771544, -0.06972323453771496)

-imag(cauchy((2-x)*x*w,z)),real((1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π))

(-0.0697232345406222, -0.06972323454061682)
```

1.7.1 Problem 4.1

Consider integration contours γ_{+x} and γ_{-x} that avoid 0 above and below:

```
x = -2.0
r = 0.1
γ_+_𝑥 = Segment(-2.0 , -r) ∪ Arc(0.,r, (π,0)) ∪ Segment(r , 100)
γ_-_𝑥 = Segment(-2.0 , -r) ∪ Arc(0.,r, (-π,0)) ∪ Segment(r , 100)
scatter([x],[0.0];label="x = $x")
plot!(γ_+_𝑥 ; xlims=(-5,5), ylims=(-1,1), label="gamma_+")
plot!(γ_-_𝑥; xlims=(-5,5), ylims=(-1,1), label="gamma_-")
```



So that

$$\Gamma_{\pm}(\alpha, x) = \int_{\gamma_{\pm x}} \zeta^{\alpha-1} \mathrm{e}^{-\zeta} \mathrm{d}\zeta$$

Note that

$$\int_{x}^{-r} (\zeta_{+}^{\alpha-1} - \zeta_{-}^{\alpha-1}) \mathrm{e}^{-\zeta} \mathrm{d}\zeta = 0$$

since $\zeta_{+}^{\alpha-1} = \mathrm{e}^{\pi \mathrm{i}(\alpha-1)} |\zeta|^{\alpha-1} = \mathrm{e}^{2\mathrm{i}\pi\alpha} \zeta_{-}^{\alpha-1}$. Furthermore, the integrals over the arcs tend to zero as $r \to 0$:

$$|\mathrm{i} r^{\alpha} \int_{0}^{\pi} \mathrm{e}^{-r\mathrm{e}^{\mathrm{i}\theta}} \mathrm{e}^{\mathrm{i}\theta\alpha} \mathrm{d}\theta| \leq r^{\alpha} \pi \mathrm{e}^{r} \to 0$$

and similarly on the lower arc. Thus we have

$$\begin{aligned} \Gamma_{+}(\alpha, x) - \mathrm{e}^{2\mathrm{i}\pi\alpha} \Gamma_{-}(\alpha, x) &= \lim_{r \to 0} \left(\int_{\gamma_{+x}} - \mathrm{e}^{2\mathrm{i}\pi\alpha} \int_{\gamma_{-x}} \right) \zeta^{\alpha-1} \mathrm{e}^{-\zeta} \mathrm{d}\zeta \\ &= (1 - \mathrm{e}^{2\mathrm{i}\pi\alpha}) \int_{0}^{\infty} x^{\alpha-1} \mathrm{e}^{-x} \mathrm{d}x = (1 - \mathrm{e}^{2\mathrm{i}\pi\alpha}) \Gamma(\alpha) \end{aligned}$$

1.8 Problem 4.2

Note that, for $0 < \alpha < 1$,

$$\psi(z) = z^{-\alpha} \mathrm{e}^{z} \Gamma(\alpha, z)$$

has the following properties:

1.
$$\psi(z)$$
decays as $z \to \infty$, via integration by parts:
$$z^{-\alpha}\mathrm{e}^{z}\int_z^\infty \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta = z^{-1}\mathrm{e}^{z} + z^{-\alpha}\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta$$
and we have assuming z is bounded away from the negative real axis:
$$\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta| \leq \int_z^\infty |\zeta|^{\alpha-2}\mathrm{d}\zeta = \int_0^\infty |x+z|^{\alpha-2}\mathrm{d}x < \infty$$
(otherwise one would use a deformed contour).

2. We have the subtractive jump:

$$\begin{aligned}\psi_+(x) - \psi_-(x) &= \mathrm{e}^{x}(x_+^{-\alpha}\Gamma_+(\alpha,x) - \Gamma_-(\alpha,x)) \\ &= \mathrm{e}^{x}|x|^{\alpha}(\mathrm{e}^{-\mathrm{i}\pi\alpha}\Gamma_+(\alpha,z) - \mathrm{e}^{\mathrm{i}\pi\alpha}\Gamma_-(\alpha,x)) \\ &= \mathrm{e}^{x}|x|^{\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha}(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\end{aligned}$$

We use these properties to verify that
$$\mathcal{C}[\diamond^{\alpha}\mathrm{e}^{-\diamond}](z) = \frac{1}{\Gamma(-\alpha)}\frac{(-z)^{\alpha}\mathrm{e}^{-z}\Gamma(-\alpha,-z)}{\breve{a}\mathrm{e}^{-\mathrm{i}\pi\alpha} - \mathrm{e}^{\mathrm{i}\pi\alpha}}$$
via Plemelj.

```
x = Fun(0 .. 20.0)
α = -0.1
z = 2.0+im
cauchy(x^α*exp(-x), z)

Γ = (α,z) -> let ζ = z + Fun(0 .. 500.0)
    linesum(ζ^(α-1)*exp(-ζ))
end

-(-z)^α*exp(-z)Γ(-α,-z)/(gamma(-α)*(exp(im*π*α)-exp(-im*π*α)))

0.07199876331505142 + 0.05850612396048861im
```