1 M3M6: Methods of Mathematical Physics

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2 Lecture 10: Branch cuts

We now discuss functions with branch cuts

1. Logarithm: $\log z$ with a cut on $(-\infty, 0]$
2. Powers: z^α with a cut on $(-\infty, 0]$
3. Cominations: $\sqrt{z-1}\sqrt{z+1}$ with a cut on $[-1, 1]$

This is a step towards a Cauchy transforms on cuts, for recovering a holomorphic function from its behaviour on a cut.

1. Complex logarithm
2. Algebraic powers

We begin with $\log z$.

2.1 Complex logarithm

One way to define the logarithm is as $\log |z| + \mathrm{i} \arg z$. We find it more convenient in order to understand its behaviour to define it as an integral:

Definition (Complex Logarithm)

$$\log z := \int_1^z \frac{1}{\zeta} \mathrm{d}\zeta$$

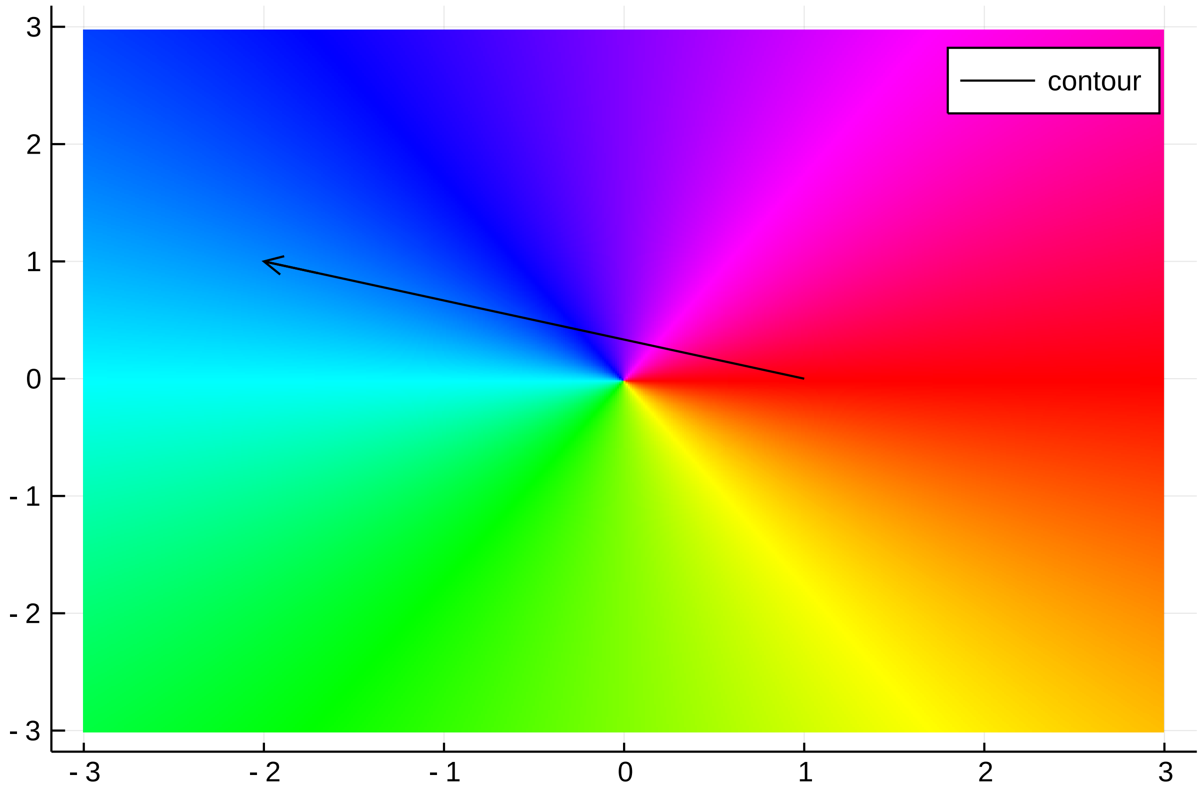
where the integral is understood to be on a straight line segment, that is

$$\log z := \int_{\gamma_z} \frac{1}{\zeta} d\zeta$$

where $\gamma_z(t) = 1 + (z-1)t$ for $0 \leq t \leq 1$.

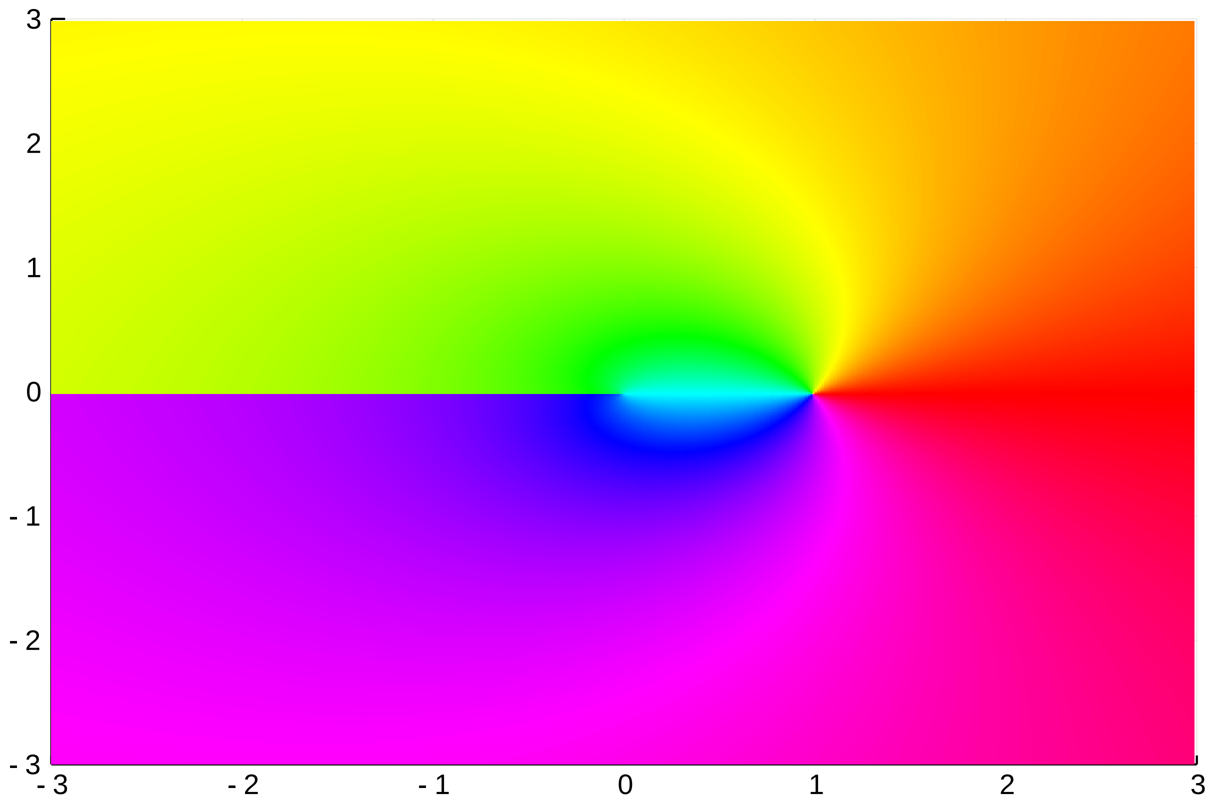
Demonstration this shows the integral path for a point z. We see how the path avoids the pole of ζ^{-1} at the origin:

```
using Plots, ComplexPhasePortrait, ApproxFun
z = -2 + 1.0im
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
t = 0:0.1:1
γ = 1 .+ (z-1)*t
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
```



This is well-defined apart from $z \in (-\infty, 0]$, where there is a pole on the contour. This induces a *branch cut*: a jump in the value of the function, which can be clearly seen from a phase portrait:

```
phaseplot(-3..3, -3..3, z -> log(z))
```

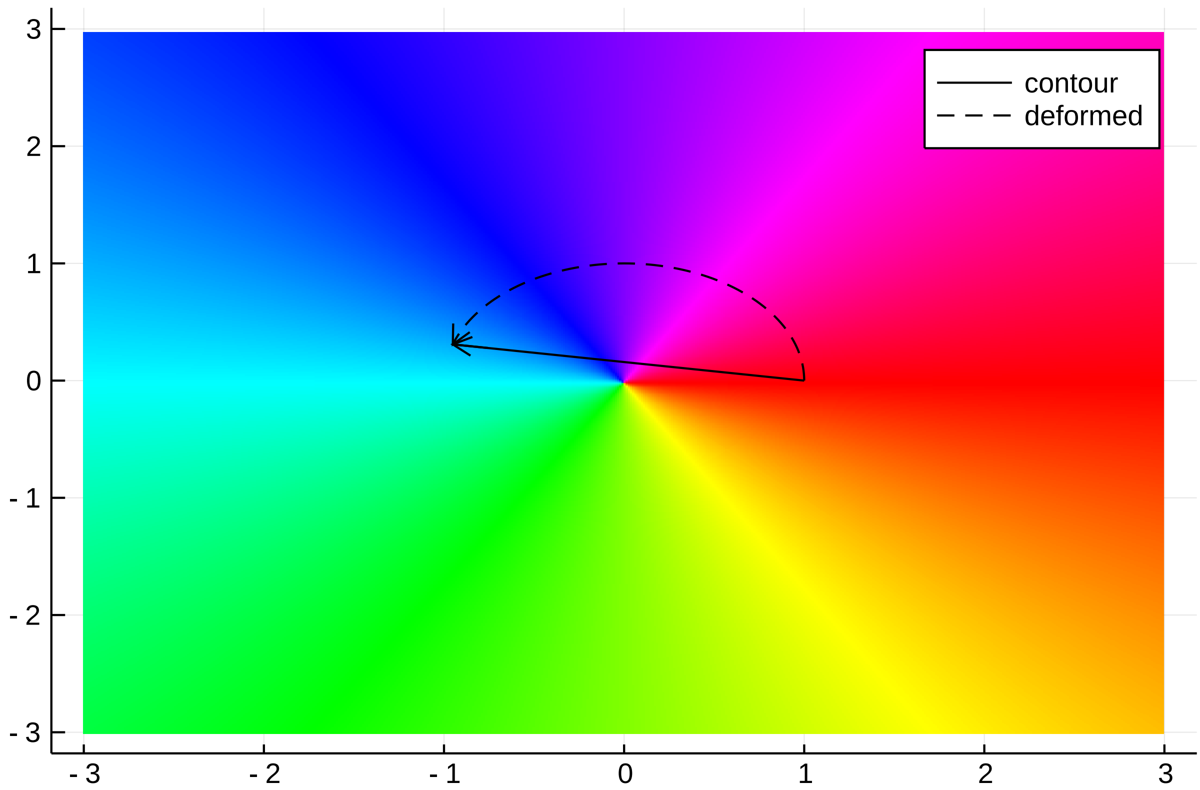


We see that the limits from above and below exist: we can define

$$\log_+ x := \lim_{\epsilon \to 0^+} \log(x + \mathrm{i}\epsilon)$$
$$\log_- x := \lim_{\epsilon \to 0^+} \log(x - \mathrm{i}\epsilon)$$

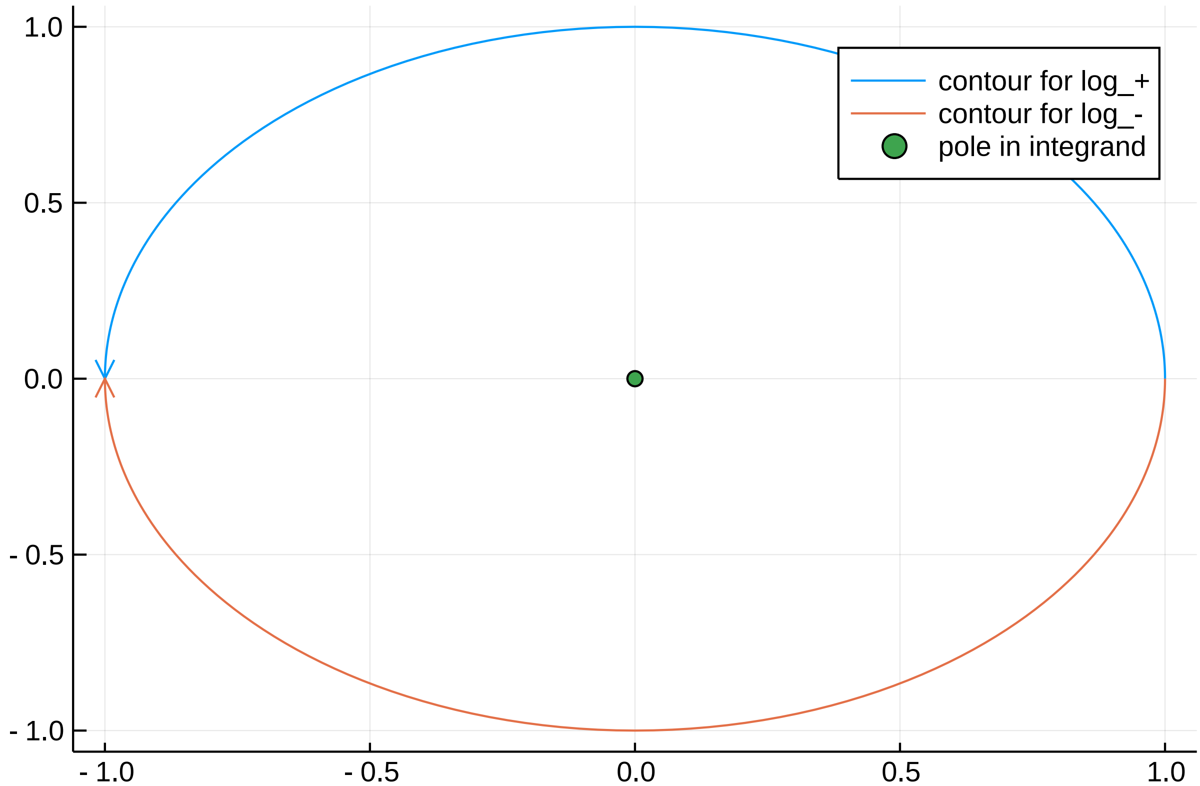
By deformation of contours, the value of the integrals is independent of the path. Here we calculate the integral on an arc:

```
θ = range(0, stop=0.9π, length=100)
a = exp.(im*θ)

z = exp(0.9*π*im)

γ = 1 .+ (z-1)*t
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
plot!(real.(a), imag.(a), color=:black, linestyle=:dash, label="deformed", arrow=true)
```



This works all the way to the negative real axis. Thus we can calculate $\log_\pm 1$ using integrals over half circles:

```
plot(Arc(0.,1.,(π,0)); label="contour for log₊", arrow=true)
plot!(Arc(0.,1.,(-π,0)); label="contour for log₋", arrow=true)
scatter!([0],[0]; label="pole in integrand")
```



Combining the two contours we have the *subtractive jump* (for any $x < 0$)

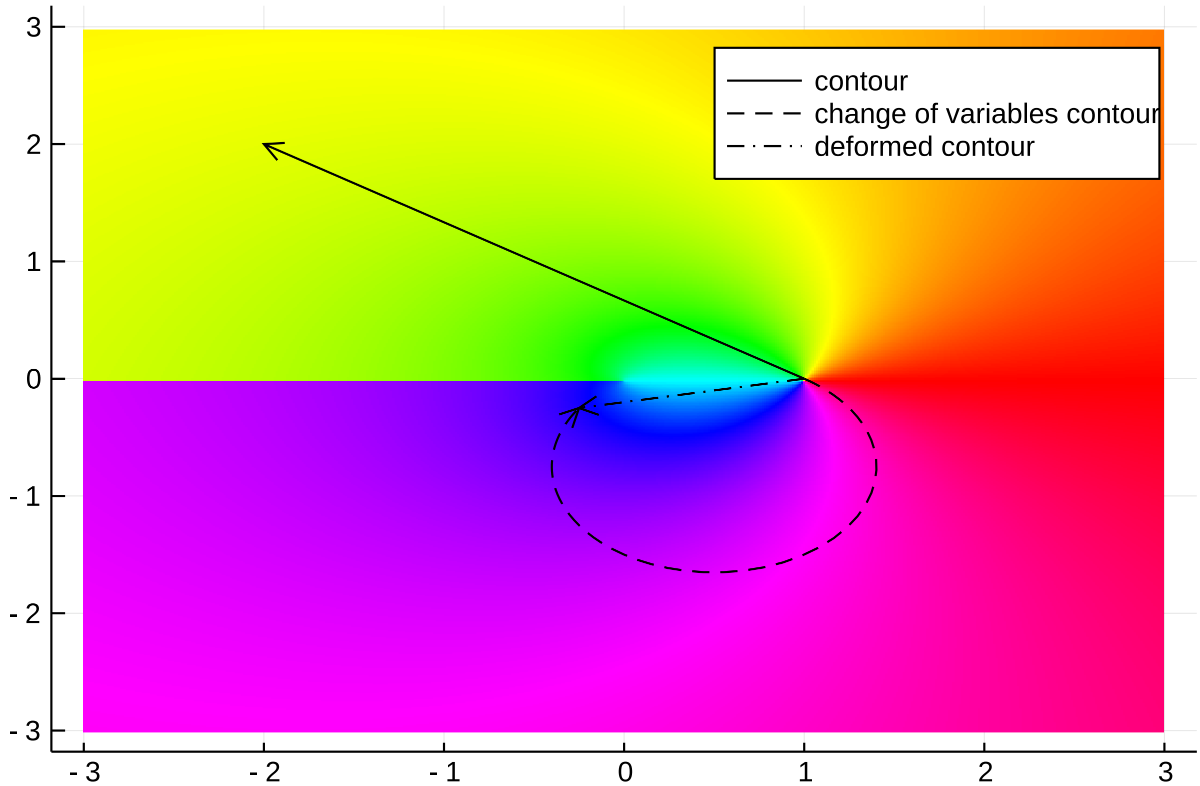
$$\log_+ x - \log_- x = \oint \frac{\mathrm{d}\zeta}{\zeta} = 2\pi \mathrm{i}$$

We can establish some properties. First we show that $\log z = -\log \frac{1}{z}$ by considering the change of variables $\zeta = \frac{1}{s}$. Because $\gamma_z(t)^{-1}$ stays uniformly in the lower-half plane, we can deform it to a straight contour, which gives us the result:

$$\log z = \int_{\gamma_z} \frac{\mathrm{d}\zeta}{\zeta} = -\int_{\frac{1}{\zeta}\circ\gamma_z} \frac{\mathrm{d}s}{s} = -\int_{\gamma_{z^{-1}}} \frac{\mathrm{d}s}{s} = -\log z^{-1}$$

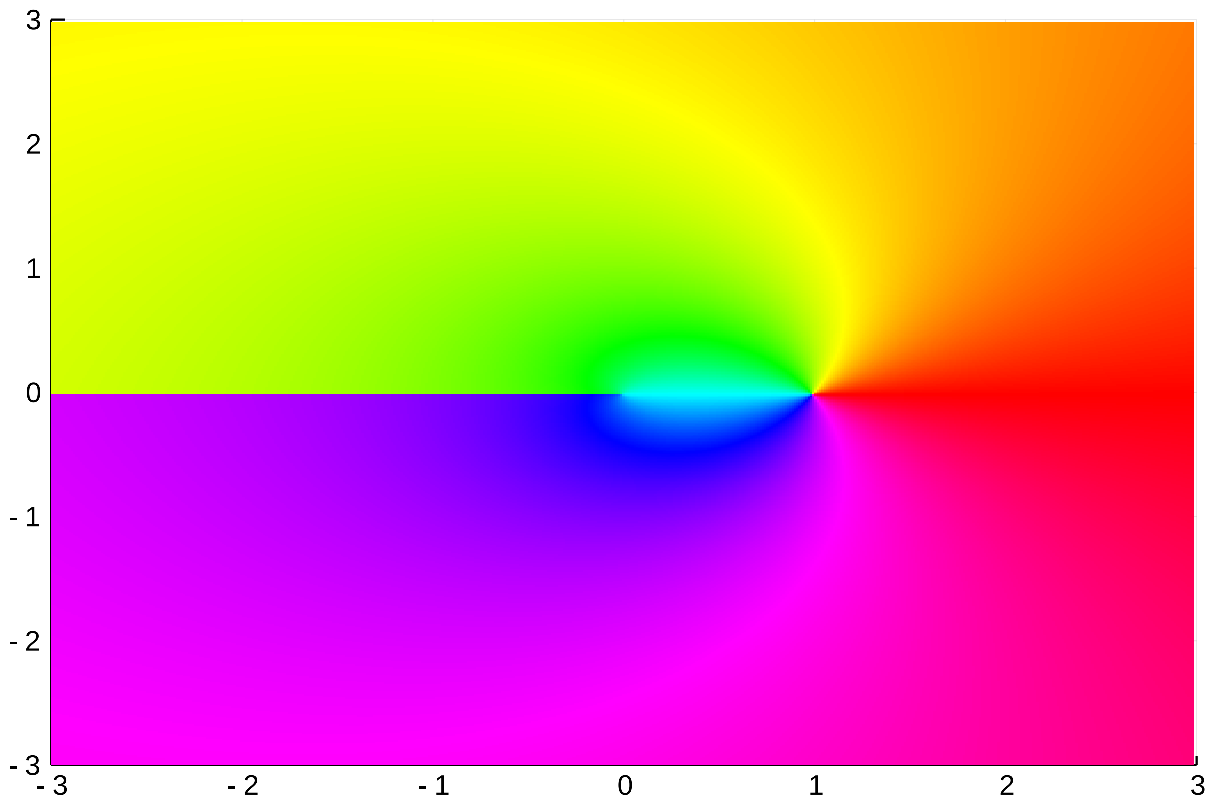
Here's a plot of the relevant contours:

```
phaseplot(-3..3, -3..3, z -> log(z))
z = -2 + 2im
γ = (z,t) -> 1 + t*(z-1)
tt = range(0,stop=1,length=100)
plot!(real.(γ.(z,tt)), imag.(γ.(z,tt)); color=:black, label="contour", arrow=true)
plot!(real.(1 ./ γ.(z,tt)), imag.(1 ./ γ.(z,tt)); color=:black, linestyle=:dash,
arrow=true, label="change of variables contour")
plot!(real.(γ.(1/z,tt)), imag.(γ.(1/z,tt)); color=:black, linestyle=:dashdot,
arrow=true, label="deformed contour")
```



Here we see by looking at the phase plot that the two functions match:

```
phaseplot(-3..3, -3..3, z -> -log(1/z))
```



2.2 Algebraic powers

We define algebraic powers in terms of logarithms, which gives us what we need to know about their jumps.

Definition (algebraic power)

$$z^\alpha := \mathrm{e}^{\alpha \log z}$$

Note, for example, when $\alpha = 1/2$, $\sqrt{z} \equiv z^{1/2}$ is only one solution to $y^2 = z$.

Here's a phase plot showing that $\sqrt{z}$ also has a branch cut on $(-\infty, 0]$:

```
α = 0.5
phaseplot(-3..3, -3..3, z -> z^α)
```

On the branch cut along $(-\infty, 0]$ it has the jump:

$$\frac{x_+^\alpha}{x_-^\alpha} = \mathrm{e}^{\alpha(\log_+ x - \log_- x)} = \mathrm{e}^{2\pi \mathrm{i}\alpha}$$

In particular,

$$\sqrt{x}_+ = -\sqrt{x}_- = i\sqrt{|x|}$$

These are *multiplicative jumps*. We also have a *subtractive jumps*:

$$\begin{aligned} x_+^\alpha - x_-^\alpha &= \mathrm{e}^{\alpha \log_+ x} - \mathrm{e}^{\alpha \log_- x} = \mathrm{e}^{\alpha \log(-x) + \mathrm{i}\pi\alpha} - \mathrm{e}^{\alpha \log(-x) - \mathrm{i}\pi\alpha} \\ &= 2\mathrm{i}(-x)^\alpha \sin \pi\alpha \end{aligned}$$

and an *additive jump*:

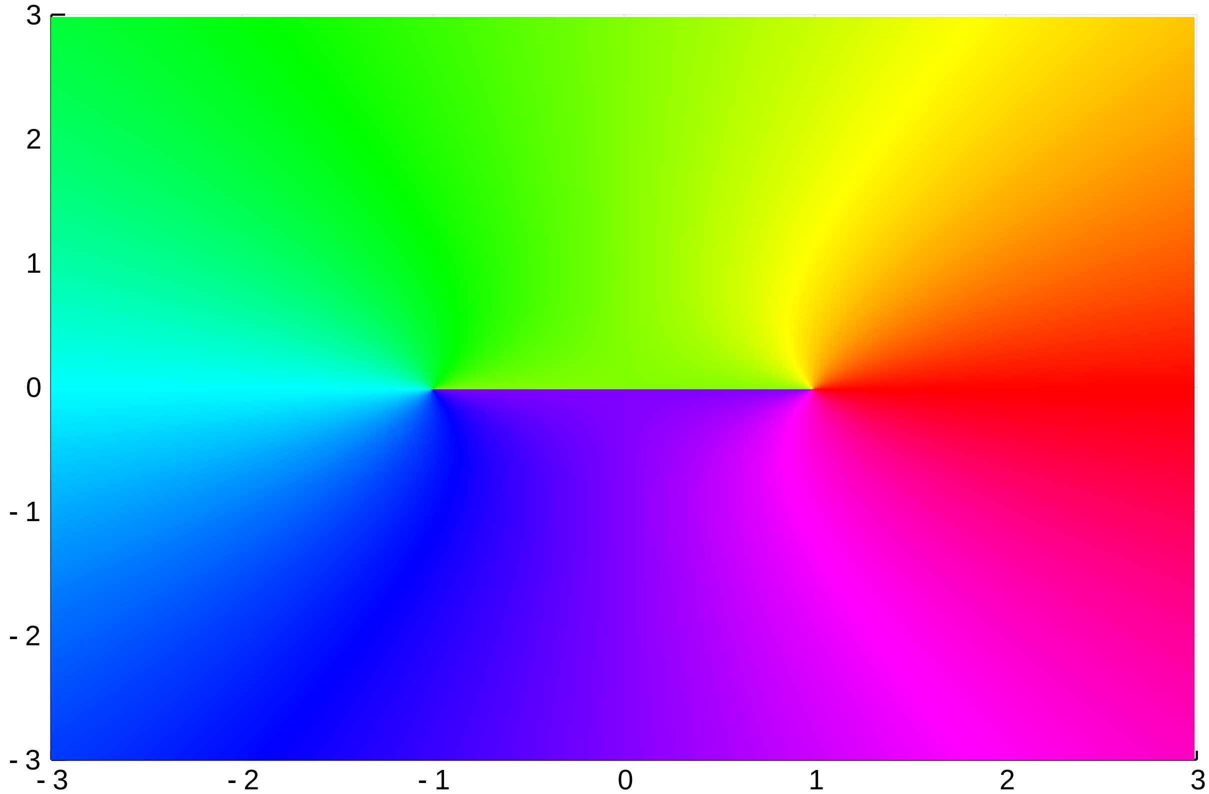
$$x_+^\alpha + x_-^\alpha = 2(-x)^\alpha \cos \pi\alpha$$

In particular, for $x < 0$,

$$\begin{aligned}\sqrt{x}_+ - \sqrt{x}_- &= 2\mathrm{i}\sqrt{-x} \\ \sqrt{x}_+ + \sqrt{x}_- &= 0\end{aligned}$$

Let's look at another example: $\varphi(z) = \sqrt{z-1}\sqrt{z+1}$. Each square root term induces a jump: one on $(-\infty, -1]$ and one on $(-\infty, 1]$. Suprisingly these jumps cancel out, in fact (as we explain next lecture) φ is analytic off $[-1, 1]$, as can be seen from the phase portrait:

```
φ = z -> sqrt(z-1)*sqrt(z+1)
phaseplot(-3..3, -3..3, φ)
```



For $-1 < x < 1$ we have the multiplicative jump:

$$\varphi_+(x) = \sqrt{x-1}_+\sqrt{x+1} = -\sqrt{x-1}_-\sqrt{x+1} = -\varphi_-(x)$$

which gives the additive jump

$$\varphi_+(x) + \varphi_-(x) = 0$$

But we also have a *subtractive jump*:

$$\varphi_+(x) - \varphi_-(x) = (\sqrt{x-1}_+ - \sqrt{x-1}_-)\sqrt{x+1} = 2\mathrm{i}\sqrt{1-x}\sqrt{x+1} = 2\mathrm{i}\sqrt{1-x^2}$$

For $x < -1$ we actually have continuity:

$$\varphi_+(x) = \sqrt{x-1}_+\sqrt{x+1}_+ = (-\sqrt{x-1}_-)(-\sqrt{x+1}_-) = \varphi_-(x)$$

This feature is what we use to show analyticity.