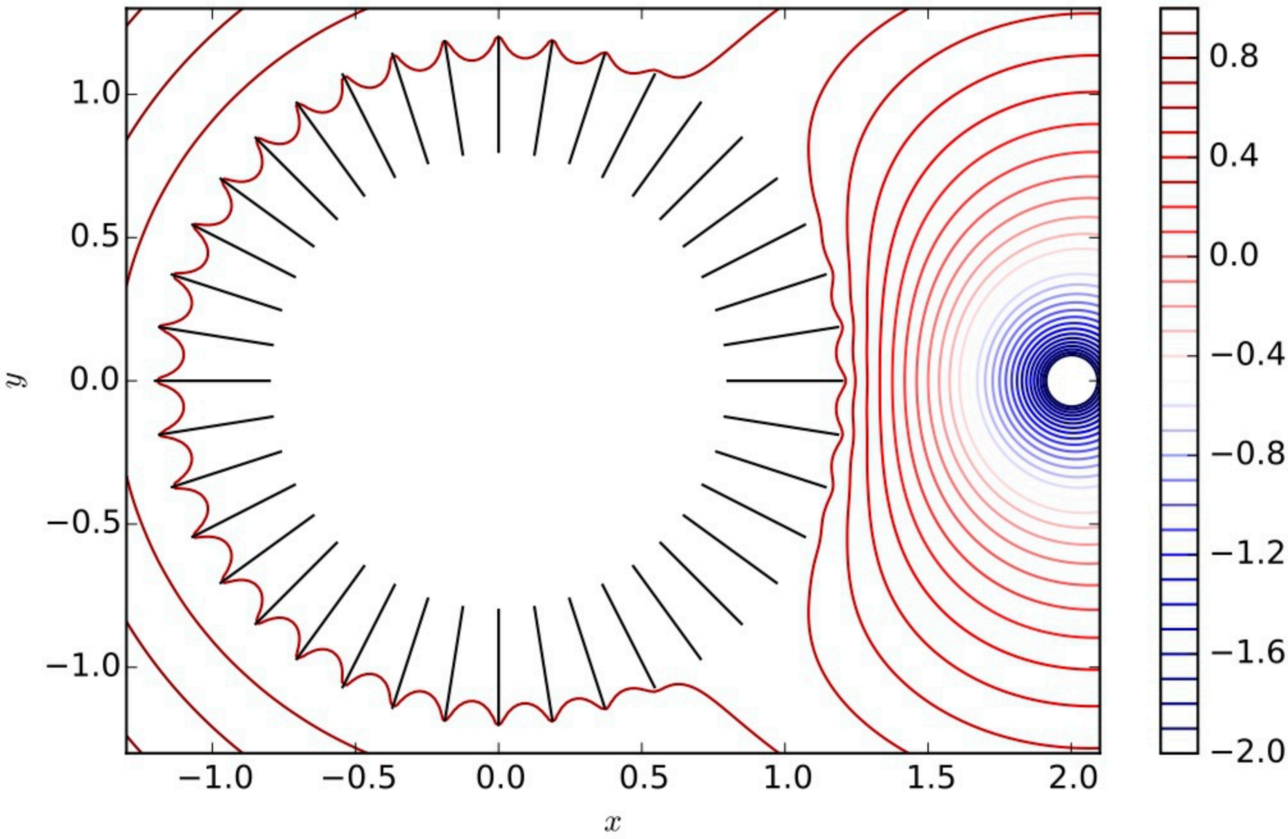

Figure 1: Faradaycage

M3M6: Methods of Mathematical Physics

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1 Lecture 17: Logarithmic singular integrals

1. Evaluating logarithmic singular integrals
2. Logarithmic singular integrals and orthogonal polynomials

The motivation behind this lecture is to calculate electrostatic potentials. An example is the Faraday cage: imagine a series of metal plates connected together so that they have the same charge. If configured to surround a region, this configuration will shield the interior from an external charge, see figure. Here, the coloured lines are equipotential lines, and there is a point source at $x = 2$, which corresponds to a forcing of $\log \|(x, y) - (2, 0)\| = \log |z - 2|$ where $z = x + \mathrm{i}y$.

1.1 Logarithmic singular integrals

From the Green's function of the Laplacian, it is natural to consider logarithmic singular integrals over a contour

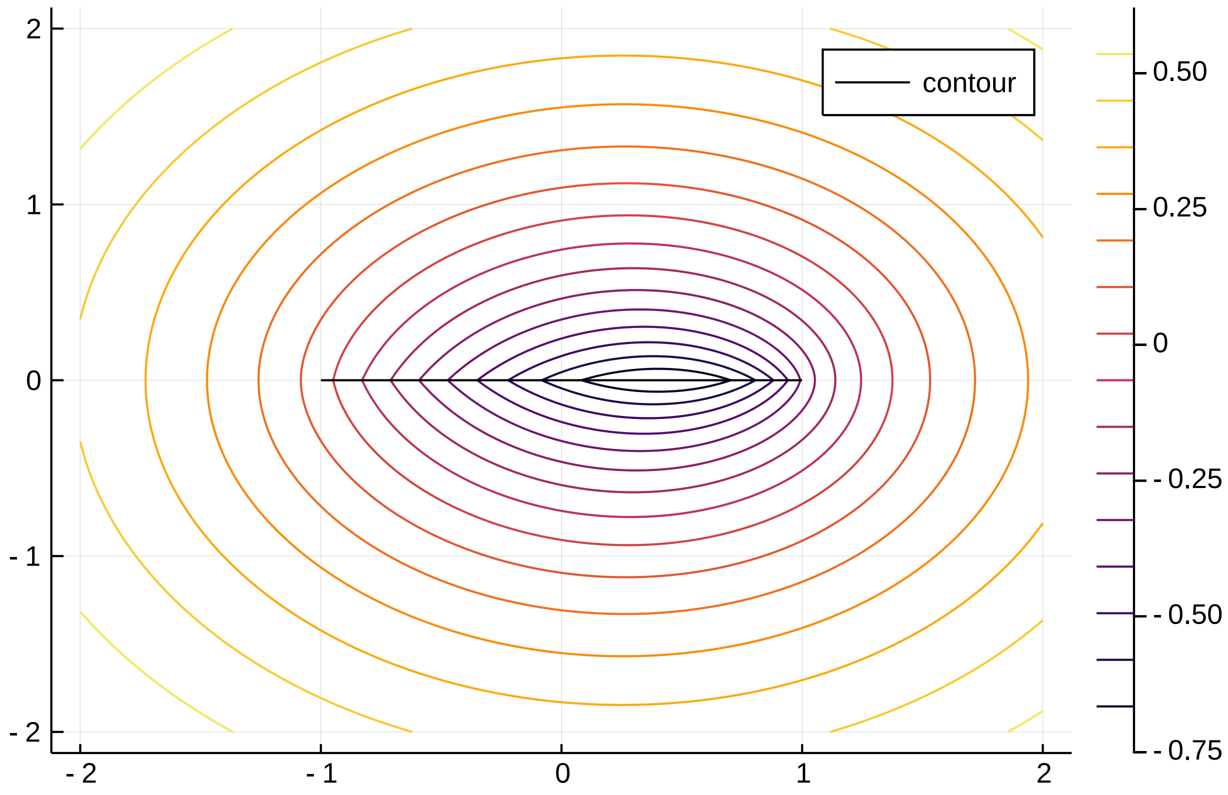
$$L_\gamma f(z) = \frac{1}{\pi}\int_\gamma f(\zeta)\log|\zeta - z|\mathrm{d}s$$

where $\mathrm{d}s$ is the arclength differential (as in line integrals, not complex analysis) and f is real-valued. We focus on intervals where the arclength differential is just the standard one:

$$L_{a,b} f(z) = \frac{1}{\pi}\int_a^b f(t)\log|z - t|\mathrm{d}t$$

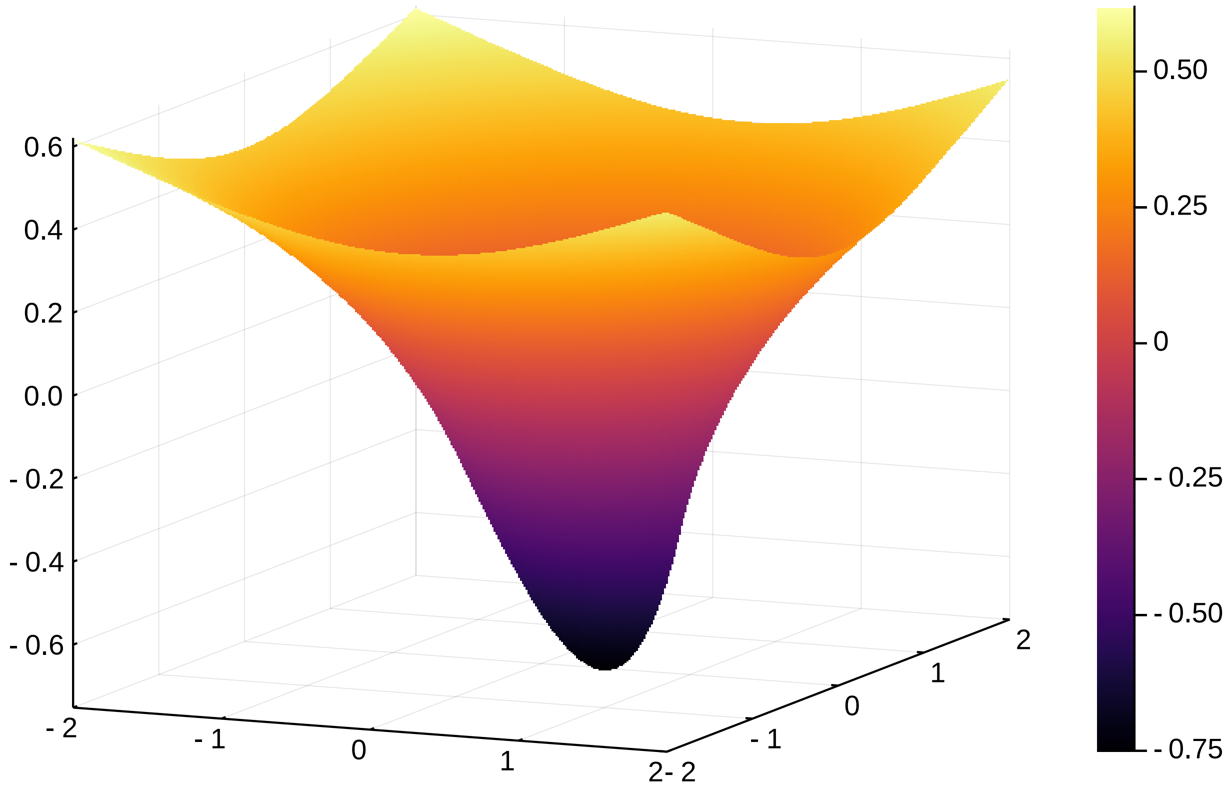
Note that off $[a,b]$, $u(x,y) \equiv u(x+\mathrm{i}y) = Lf(x+\mathrm{i}y)$ solves Laplace's equation, and as the integrand has an integrable singularity it is continuous on $[a,b]$:

```
using ApproxFun, SingularIntegralEquations, Plots
t = Fun()
f = sqrt(1-t^2)*exp(t)
u = z -> logkernel(f, z)  # logkernel(f,z) calculates 1/π * \int f(t)*log|t-z| dt

xx = yy = -2:0.01:2
U = u.(xx' .+ im*yy)

contour(xx, yy, U)
plot!(domain(t); color=:black, label="contour")
```



Or this can also be seen from a surface plot:

```
surface(xx, yy, U)
```



For $z \notin (-\infty, b]$ the fact that u is harmonic (solves Laplace's equation) can also be seen since u is the real part of an analytic function:

$$u(z) = \Re M f(z) \qquad \text{for} \qquad M f(z) := \frac{1}{\pi} \int_a^b f(t) \log(z - t) \mathrm{d}t$$

Note that the integrand avoids the branch cut of $\log z$. To extend this to $z \in (-\infty, a]$ (or more generally, $z \notin [a, \infty)$), we can use the alternative expression

$$u(z) = \Re \tilde{M} f(z) \qquad \text{for} \qquad \tilde{M} f(z) := \frac{1}{\pi} \int_a^b f(t) \log(t - z) \mathrm{d}t$$

which follows from $\log |z - t| = \log |t - z|$.

1.1.1 Evaluating logarithmic singular integrals

To evaluate $Lf(z)$ we evaluate $Mf(z)$. Following our claim that the right way to represent analytic functions is by their behaviour at singularitites/branch cuts, we will accomplish this this by investigating the jumps of $Mf(z)$. This allows us to reduce it to computing a Cauchy transform.

Lemma (Jumps of M) Suppose f satisfies the conditions of Plemelj. Then $Mf(z)$ satisfies the following

1. Analyticity: M is analytic off $(-\infty, b]$
2. Regularity: M has weaker than pole singularities
3. Asymptotics:

$$
Mf(z) = \frac{1}{\pi}\int_a^b f(t)\mathrm{d}t \log z + o(1) \qquad \text{as} \qquad z \to \infty
$$

4. Jump: for $x < a$ we have

$$
M^+f(x) - M^-f(x) = 2\mathrm{i}\int_a^b f(t)\mathrm{d}t
$$

and for $a < x < b$ we have

$$
M^+f(x) - M^-f(x) = 2\mathrm{i}\int_x^b f(t)\mathrm{d}t
$$

Proof

Analyticity property (1) follows immediately from definition: we have for $z \notin (-\infty, b]$

$$
\frac{\mathrm{d}}{\mathrm{d}z}Mf(z) = \frac{1}{\pi}\int_{-1}^1 f(t)\frac{1}{z-t}\mathrm{d}t = -2\mathrm{i}\mathcal{C}f(z)
$$

The regularity property (2) follows from the expression as an antiderivative: we know $\mathcal{C}f(z)$ has weaker than pole singularities and integrating only makes them better. To derive the asymptotics (3) we use

$$
\log(z-t) = \log z + \log(1 - t/z)
$$

Finally we get to the jumps. For any point $c > b$ we write

$$
Mf(z) = -2\mathrm{i}\int_c^z \mathcal{C}f(\zeta)\mathrm{d}\zeta + Mf(c)
$$

For $x < a$ we can deform the contour just above/below the interval giving us

$$
M^{\pm}f(x) = Mf(c) - 2\mathrm{i}\int_c^b \mathcal{C}f(t)\mathrm{d}t - 2\mathrm{i}\int_b^a \mathcal{C}^{\pm}f(t)\mathrm{d}t - 2\mathrm{i}\int_a^x \mathcal{C}f(t)\mathrm{d}t
$$

so that

$$
(M^+ - M^-)f(x) = -2\mathrm{i}\int_b^a (\mathcal{C}^+ - \mathcal{C}^-)f(t)\mathrm{d}t = 2\mathrm{i}\int_a^b f(t)\mathrm{d}t
$$

Similarly for $a < x < b$ we have

$$
M^{\pm}f(x) = Mf(c) - 2\mathrm{i}\int_c^b \mathcal{C}f(t)\mathrm{d}t - 2\mathrm{i}\int_b^x \mathcal{C}^{\pm}f(t)\mathrm{d}t
$$

which gives

$$
(M^+ - M^-)f(x) = -2\mathrm{i}\int_b^x (\mathcal{C}^+ - \mathcal{C}^-)f(t)\mathrm{d}t = 2\mathrm{i}\int_x^b f(t)\mathrm{d}t.
$$

■

We can construct a function that satisfies the same 4 properties using the Cauchy transform. By uniqueness they must be the same.

Theorem (Log kernel as Cauchy) Suppose f satisfies the conditions of Plemelj and define the (negative) indefinite integral of f via

$$F(x) := \int_x^b f(t)\mathrm{d}t.$$

Then

$$Mf(z) = \frac{\log(z-a)}{\pi} \int_a^b f(t)\mathrm{d}t + 2\mathrm{i}\mathcal{C}F(z)$$

Proof

Define

$$\phi(z) := \frac{\log(z-a)}{\pi} \int_a^b f(t)\mathrm{d}t + 2\mathrm{i}\mathcal{C}F(z)$$

This satisfies conditions (1–4) above. Therefore $\phi(z) - Mf(z)$ is continuous hence analytic on $(-\infty, a)$ and (a, b), has weaker than pole singularities at a and b so analytic there too: it is entire. The logarithmic growth at infinity cancels therefore it is zero by Liouville.

■