

# 1 M3M6: Methods of Mathematical Physics

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## 2 Lecture 10: Branch cuts

We now discuss functions with branch cuts

1. Logarithm:  $\log z$  with a cut on  $(-\infty, 0]$
2. Powers:  $z^\alpha$  with a cut on  $(-\infty, 0]$
3. Cominations:  $\sqrt{z-1}\sqrt{z+1}$  with a cut on  $[-1, 1]$

This is a step towards a Cauchy transforms on cuts, for recovering a holomorphic function from its behaviour on a cut. This lecture we discuss:

1. Complex logarithm
2. Algebraic powers
3. Inferring analyticity from continuity

### 2.1 Complex logarithm

One way to define the logarithm is as  $\log |z| + i \arg z$ . We find it more convenient in order to understand its behaviour to define it as an integral:

**Definition (Complex Logarithm)**

$$\log z := \int_1^z \frac{1}{\zeta} d\zeta$$

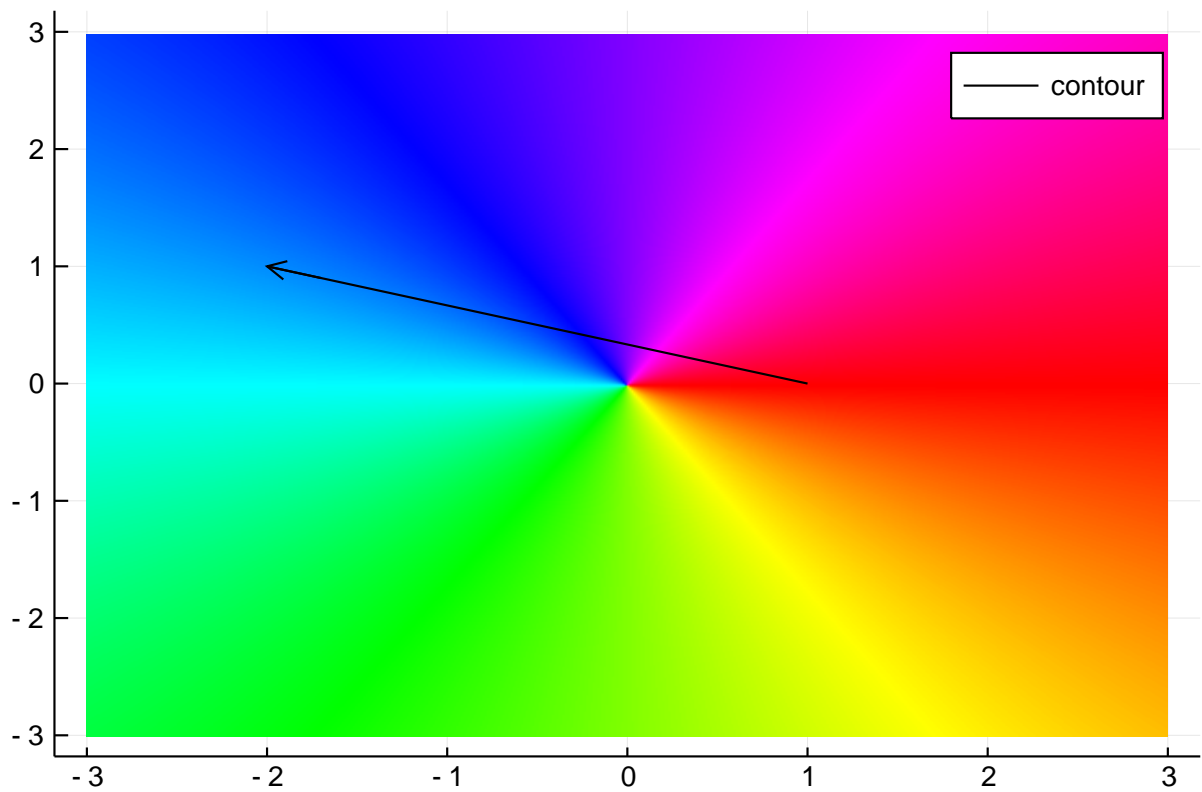
where the integral is understood to be on a straight line segment, that is

$$\log z := \int_{\gamma_z} \frac{1}{\zeta} d\zeta$$

where  $\gamma_z(t) = 1 + (z-1)t$  for  $0 \leq t \leq 1$ .

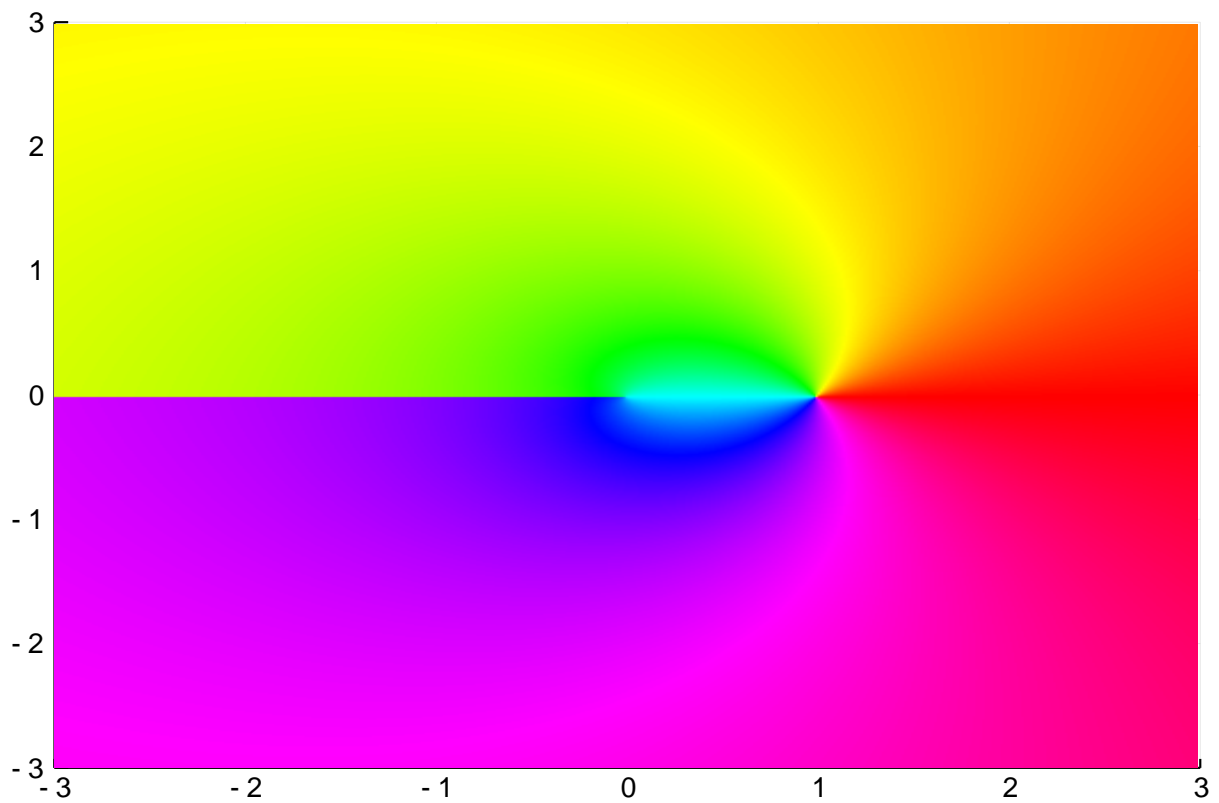
*Demonstration* this shows the integral path for a point  $z$ . We see how the path avoids the pole of  $\zeta^{-1}$  at the origin:

```
using Plots, ComplexPhasePortrait, ApproxFun
z = -2 + 1.0im
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
t = 0:0.1:1
γ = 1 .+ (z-1)*t
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
```



This is well-defined apart from  $z \in (-\infty, 0]$ , where there is a pole on the contour. This induces a *branch cut*: a jump in the value of the function, which can be clearly seen from a phase portrait:

```
phaseplot(-3..3, -3..3, z -> log(z))
```



We see that the limits from above and below exist: we can define

$$\log_+ x := \lim_{\epsilon \rightarrow 0^+} \log(x + i\epsilon)$$

$$\log_- x := \lim_{\epsilon \rightarrow 0^+} \log(x - i\epsilon)$$

By deformation of contours, the value of the integrals is independent of the path. Here we calculate the integral on an arc:

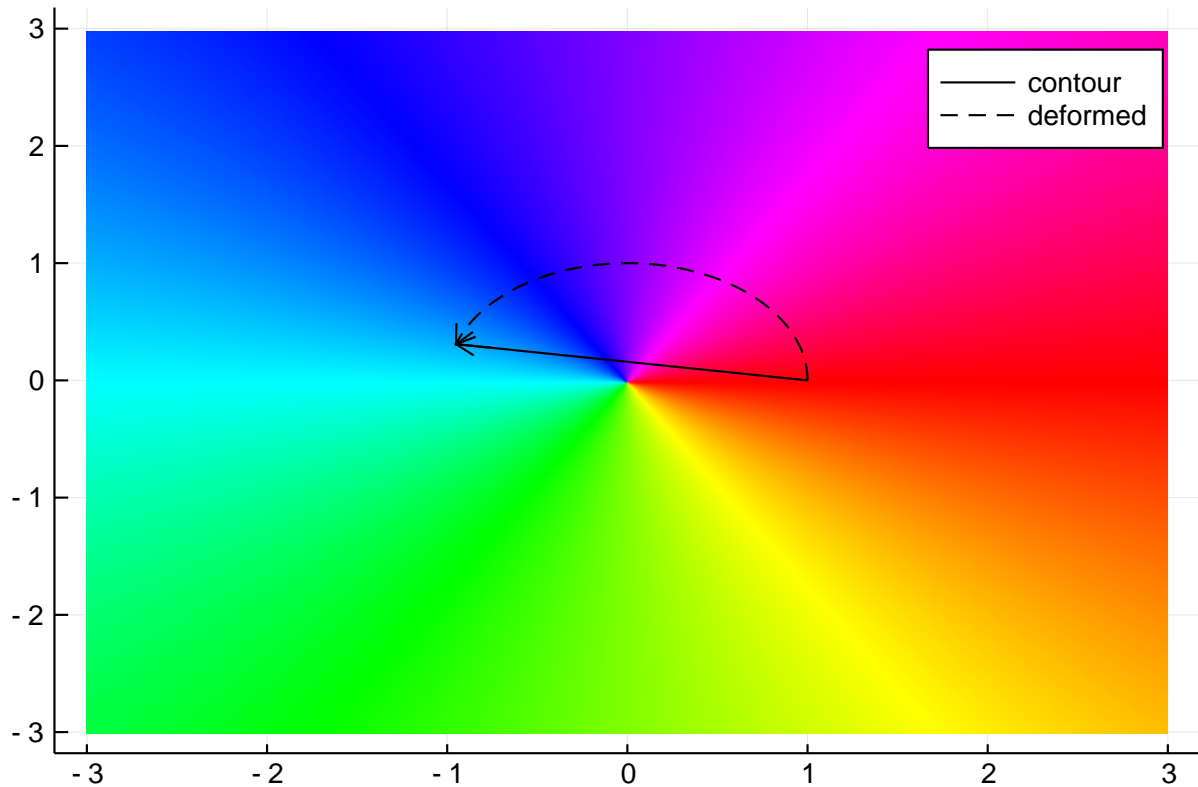
```

θ = range(0, stop=0.9π, length=100)
a = exp.(im*θ)

z = exp(0.9*π*im)

γ = 1 .+ (z-1)*t
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
plot!(real.(a), imag.(a), color=:black, linestyle=:dash, label="deformed", arrow=true)

```

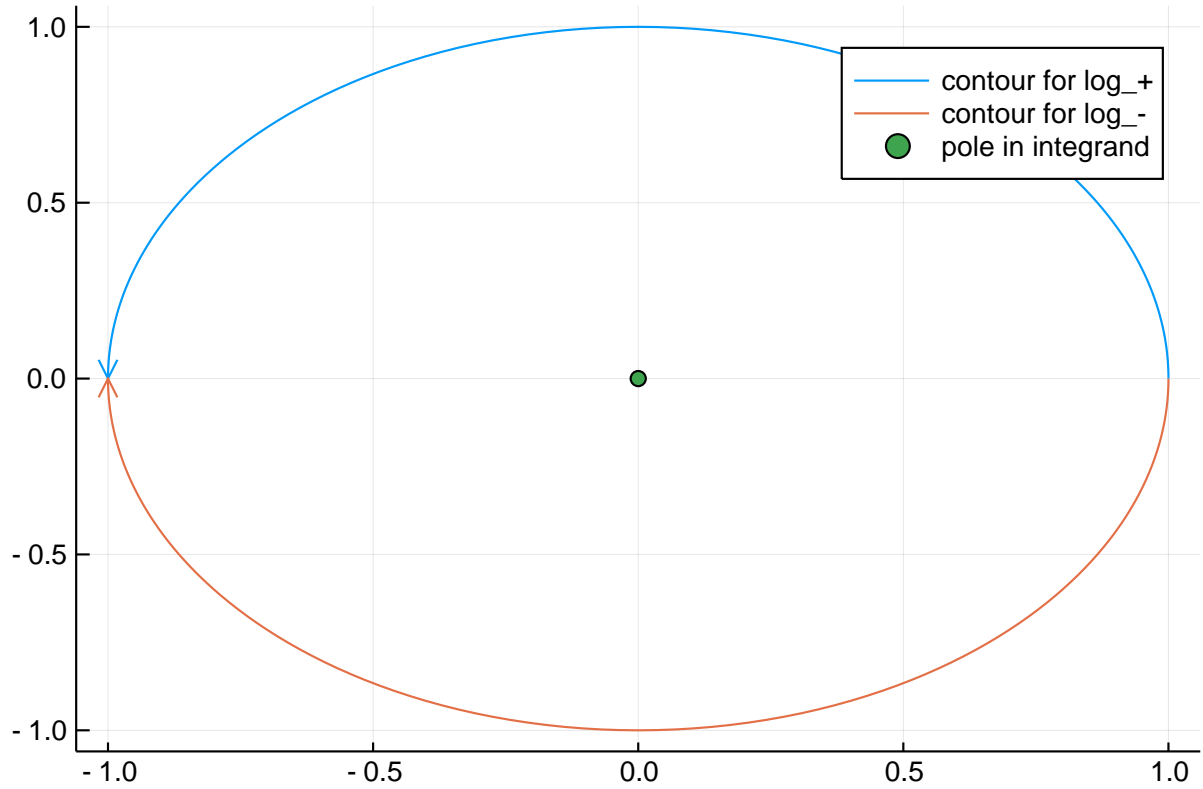


This works all the way to the negative real axis. Thus we can calculate  $\log_{\pm} 1$  using integrals over half circles:

```

plot(Arc(0.,1.,(π,0)); label="contour for log_+", arrow=true)
plot!(Arc(0.,1.,(-π,0)); label="contour for log_-", arrow=true)
scatter!([0],[0]; label="pole in integrand")

```



Combining the two contours we have the *subtractive jump* (for any  $x < 0$ )

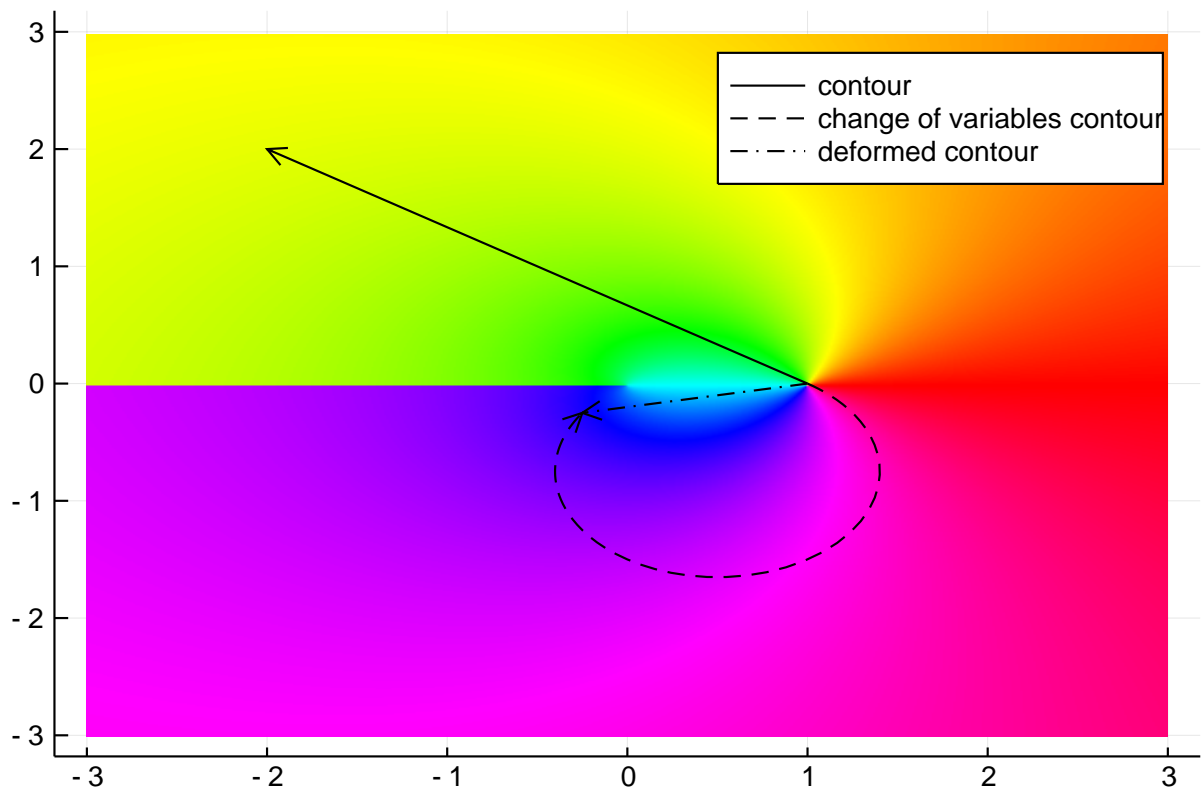
$$\log_+ x - \log_- x = \oint \frac{d\zeta}{\zeta} = 2\pi i$$

We can establish some properties. First we show that  $\log z = -\log \frac{1}{z}$  by considering the change of variables  $\zeta = \frac{1}{s}$ . Because  $\gamma_z(t)^{-1}$  stays uniformly in the lower-half plane, we can deform it to a straight contour, which gives us the result:

$$\log z = \int_{\gamma_z} \frac{d\zeta}{\zeta} = - \int_{\frac{1}{\zeta} \circ \gamma_z} \frac{ds}{s} = - \int_{\gamma_{z^{-1}}} \frac{ds}{s} = -\log z^{-1}$$

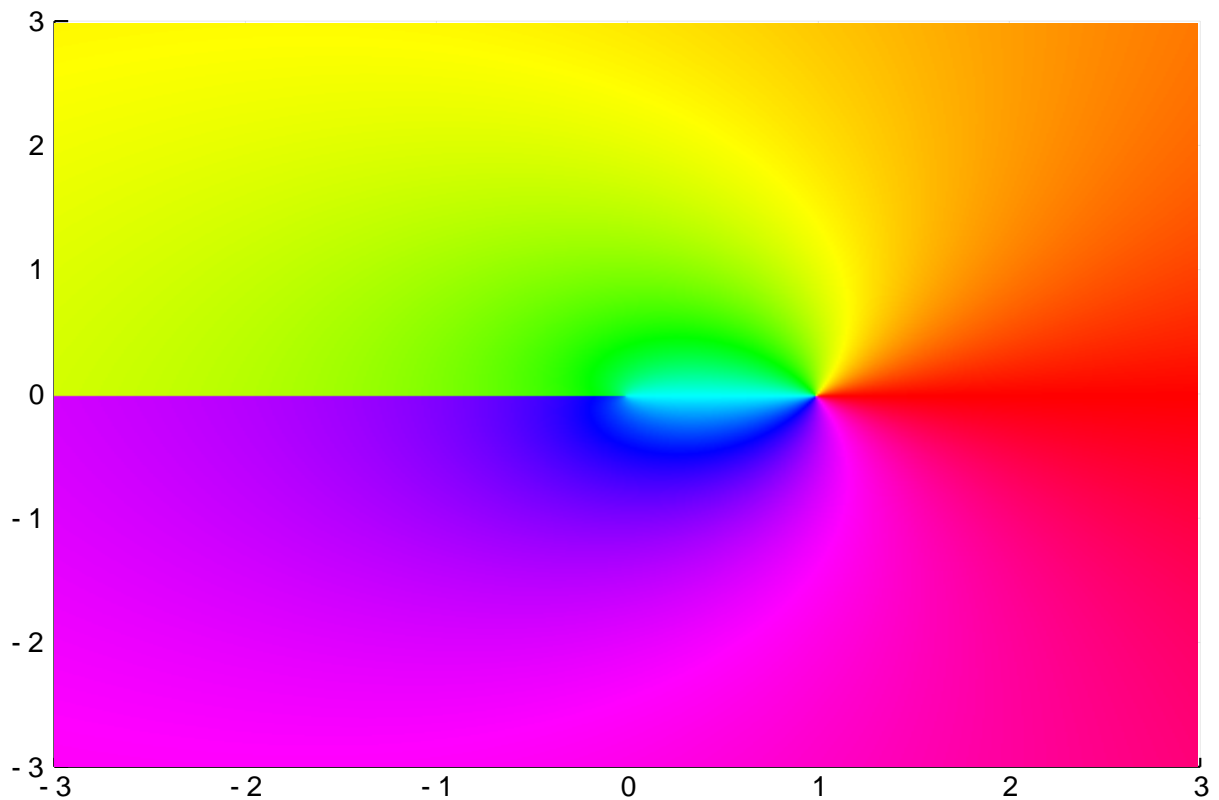
Here's a plot of the relevant contours:

```
phaseplot(-3..3, -3..3, z -> log(z))
z = -2 + 2im
γ = (z,t) -> 1 + t*(z-1)
tt = range(0,stop=1,length=100)
plot!(real.(γ.(z,tt)), imag.(γ.(z,tt)); color=:black, label="contour", arrow=true)
plot!(real.(1 ./ γ.(z,tt)), imag.(1 ./ γ.(z,tt)); color=:black, linestyle=:dash,
arrow=true, label="change of variables contour")
plot!(real.(γ.(1/z,tt)), imag.(γ.(1/z,tt)); color=:black, linestyle=:dashdot,
arrow=true, label="deformed contour")
```



Here we see by looking at the phase plot that the two functions match:

```
phaseplot(-3..3, -3..3, z -> -log(1/z))
```



## 2.2 Algebraic powers

We define algebraic powers in terms of logarithms, which gives us what we need to know about their jumps.

**Definition (algebraic power)**

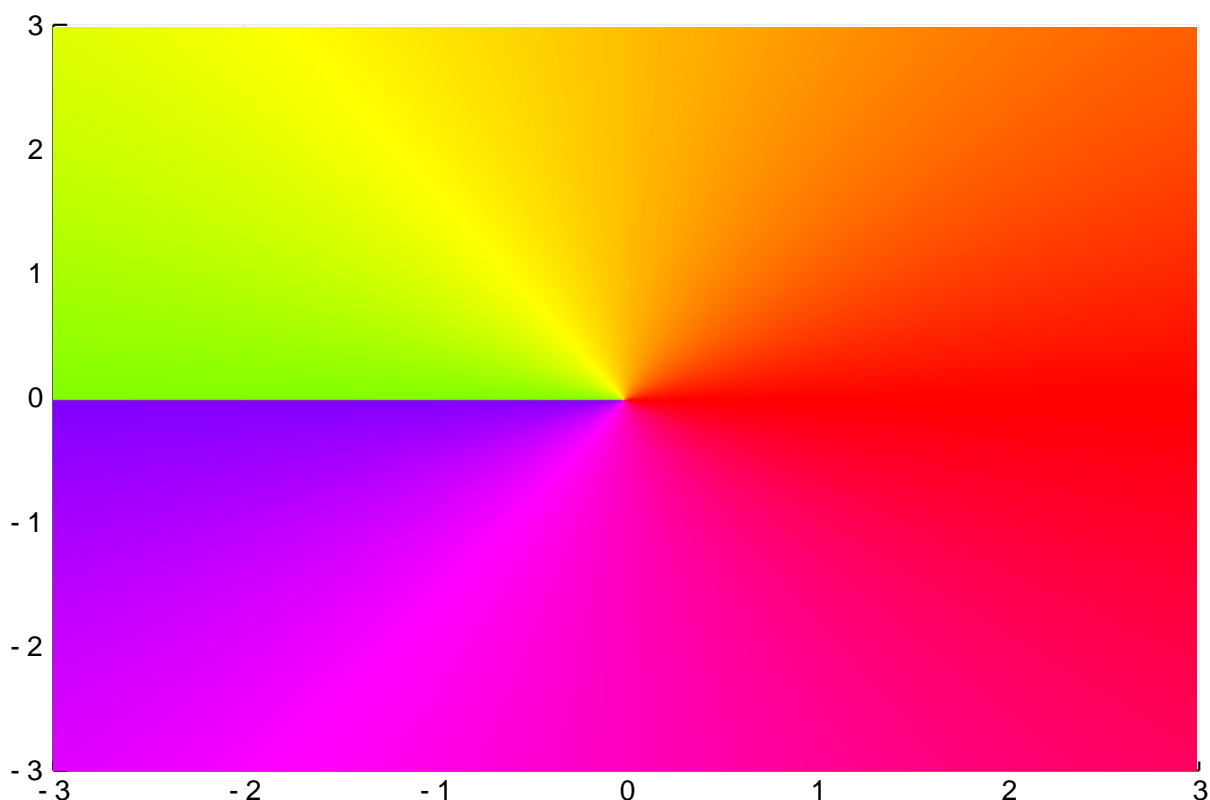
$$z^\alpha := e^{\alpha \log z}$$

Note, for example, when  $\alpha = 1/2$ ,  $\sqrt{z} \equiv z^{1/2}$  is only one solution to  $y^2 = z$ .

Here's a phase plot showing that  $\sqrt{z}$  also has a branch cut on  $(-\infty, 0]$ :

$\alpha = 0.5$

`phaseplot(-3..3, -3..3, z -> z^alpha)`



On the branch cut along  $(-\infty, 0]$  it has the jump:

$$\frac{x_+^\alpha}{x_-^\alpha} = e^{\alpha(\log_+ x - \log_- x)} = e^{2\pi i \alpha}$$

In particular,

$$\sqrt{x}_+ = -\sqrt{x}_- = i\sqrt{|x|}$$

These are *multiplicative jumps*. We also have a *subtractive jumps*:

$$\begin{aligned} x_+^\alpha - x_-^\alpha &= e^{\alpha \log_+ x} - e^{\alpha \log_- x} = e^{\alpha \log(-x) + i\pi\alpha} - e^{\alpha \log(-x) - i\pi\alpha} \\ &= 2i(-x)^\alpha \sin \pi\alpha \end{aligned}$$

and an *additive jump*:

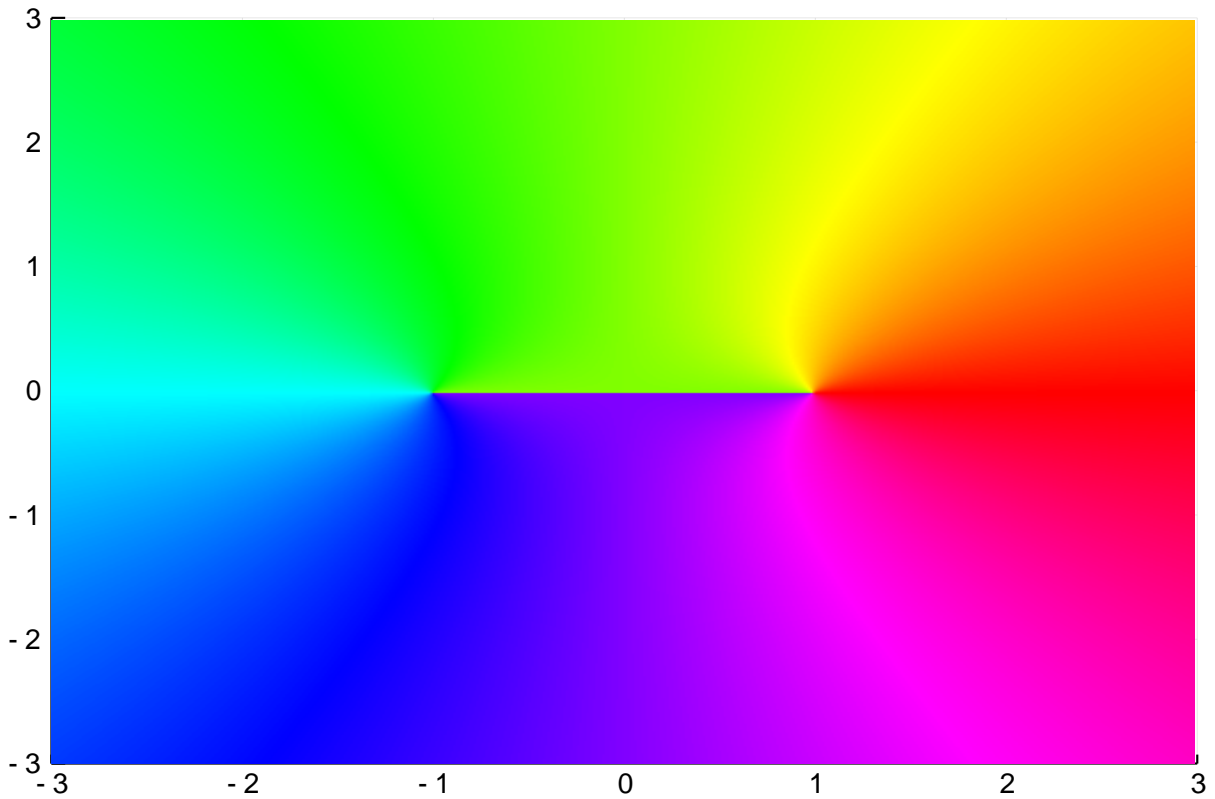
$$x_+^\alpha + x_-^\alpha = 2(-x)^\alpha \cos \pi\alpha$$

In particular, for  $x < 0$ ,

$$\begin{aligned}\sqrt{x}_+ - \sqrt{x}_- &= 2i\sqrt{-x} \\ \sqrt{x}_+ + \sqrt{x}_- &= 0\end{aligned}$$

Let's look at another example:  $\varphi(z) = \sqrt{z-1}\sqrt{z+1}$ . Each square root term induces a jump: one on  $(-\infty, -1]$  and one on  $(-\infty, 1]$ . Surprisingly these jumps cancel out, in fact (as we explain below)  $\varphi$  is analytic off  $[-1, 1]$ , as can be seen from the phase portrait:

```
 $\varphi = z \rightarrow \text{sqrt}(z-1)*\text{sqrt}(z+1)$   
 $\text{phaseplot}(-3..3, -3..3, \varphi)$ 
```



For  $-1 < x < 1$  we have the multiplicative jump:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1} = -\sqrt{x-1}_- \sqrt{x+1} = -\varphi_-(x)$$

which gives the additive jump

$$\varphi_+(x) + \varphi_-(x) = 0$$

But we also have a *subtractive jump*:

$$\varphi_+(x) - \varphi_-(x) = (\sqrt{x-1}_+ - \sqrt{x-1}_-) \sqrt{x+1} = 2i\sqrt{1-x}\sqrt{x+1} = 2i\sqrt{1-x^2}$$

For  $x < -1$  we actually have continuity:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1}_+ = (-\sqrt{x-1}_-)(-\sqrt{x+1}_-) = \varphi_-(x)$$

This feature is what we use to show analyticity.

### 3 Inferred analyticity

Here we review properties of inferring analyticity from continuity.

**Theorem (continuity on a curve implies analyticity)** Let  $D$  be a domain and  $\gamma \subset D$  a contour. Suppose  $f$  is analytic in  $D \setminus \gamma$ , and continuous on the interior of  $\gamma$ . Then  $f$  is analytic in  $D \setminus \{\gamma(a), \gamma(b)\}$ .

#### Sketch of Proof

For simplicity, suppose  $D$  is a circle of radius 2 and  $\gamma$  is the interval  $[-1, 1]$ . For any  $z$  off the interval, we can write

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma_x} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where  $\Gamma_x$  is a simple closed contour that surrounds  $z$  and passes through  $x$  in both directions:

```

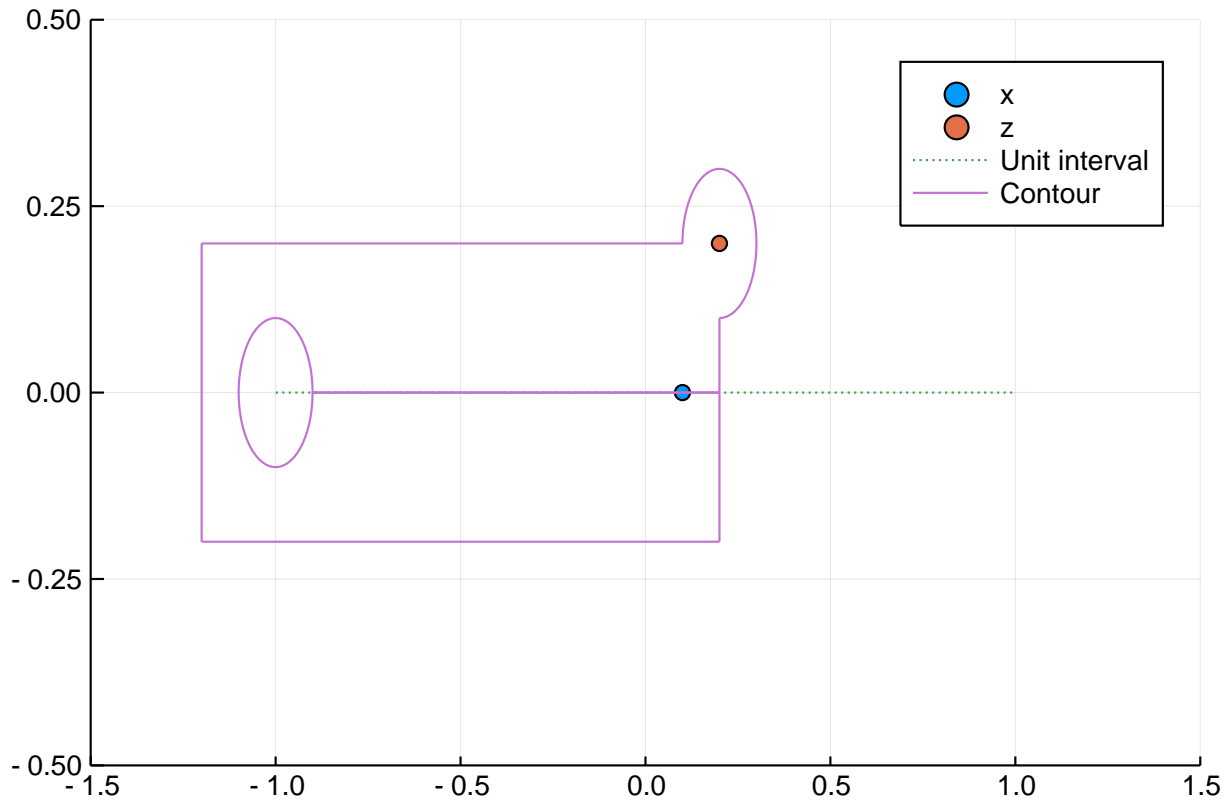
z = 0.2+0.2im
x = 0.1
ε = 0.001
scatter([x],[0.]; label="x", xlims=(-1.5,1.5), ylims=(-0.5,0.5))
scatter!([real(z)],[imag(z)]; label="z")
plot!(-1..1; label="Unit interval", linestyle=:dot)

Γ_x = Arc(z, 0.1, (-π/2,π)) ∪ Segment(0.2+0.1im,0.2+0.0im) ∪ Segment(0.2+0.0im,
-0.9+0.0im) ∪
    Circle(-1.0, 0.1) ∪ Segment(-0.9-0.0im, 0.2-0.0im) ∪ Segment(0.2-0.0im, 0.2-
0.2im) ∪
    Segment(0.2-0.2im, -1.2-0.2im) ∪ Segment(-1.2-0.2im, -1.2+0.2im) ∪
    Segment(-1.2+0.2im, 0.1+0.2im)

plot!(Γ_x; label="Contour")

```





Because  $f$  is continuous at  $x$ , we have

$$f_+(x) = f_-(x) = f(x)$$

where

$$f_{\pm}(x) = \lim_{\epsilon \rightarrow 0} f(x \pm i\epsilon)$$

Therefore, the two integrals along  $[-1, 1]$  cancel out and we get:

$$f(z) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where  $\tilde{\Gamma}_x$  is  $\Gamma_x$  with the contour on the interval removed:

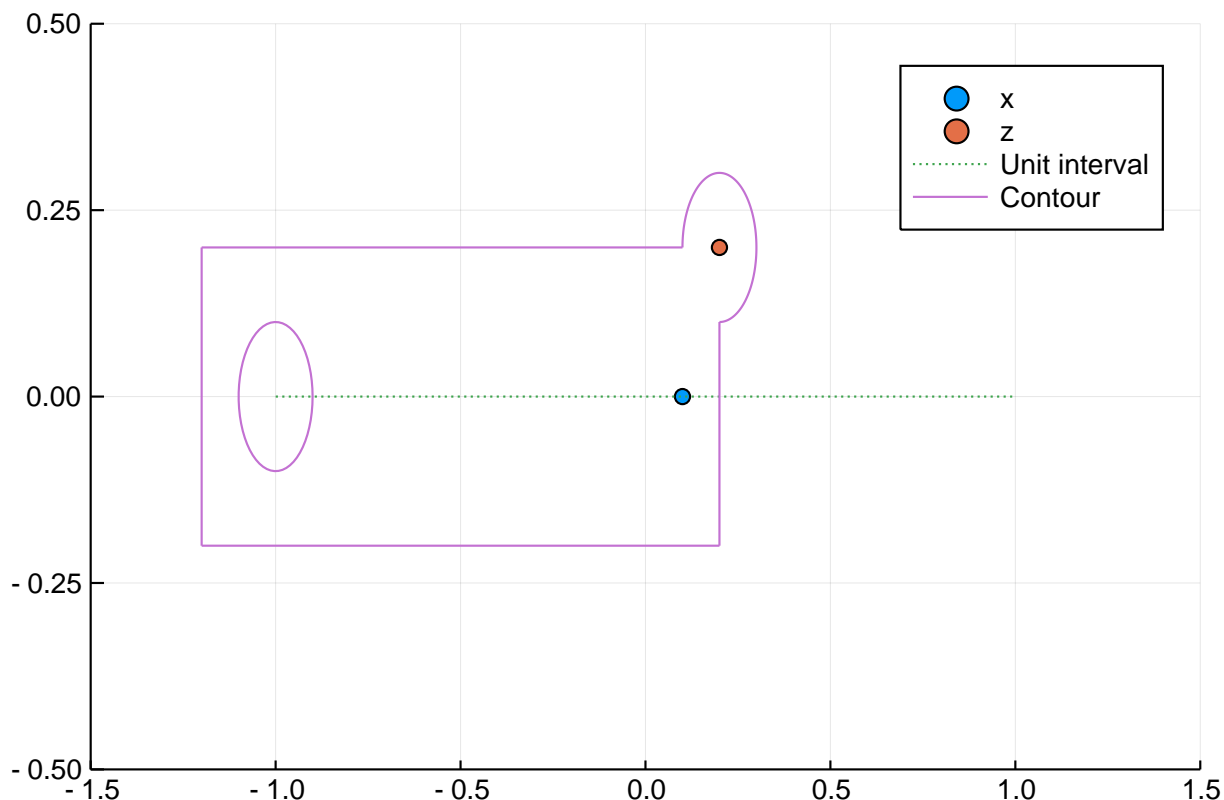
```

z = 0.2+0.2im
x = 0.1
ε = 0.001
scatter([x],[0.]; label="x", xlims=(-1.5,1.5), ylims=(-0.5,0.5))
scatter!([real(z)],[imag(z)]; label="z")
plot!(-1..1; label="Unit interval", linestyle=:dot)

Γt_x = Arc(z, 0.1, (-π/2,π)) ∪ Segment(0.2+0.1im,0.2-0.2im) ∪ Circle(-1.0, 0.1) ∪
Segment(0.2-0.2im, -1.2-0.2im) ∪ Segment(-1.2-0.2im, -1.2+0.2im) ∪
Segment(-1.2+0.2im, 0.1+0.2im)

plot!(Γt_x; label="Contour")

```



This integral expression holds for all  $z$  inside the contour  $\tilde{\Gamma}_x$  but off the interval. But it therefore holds true for  $f(x) = f_+(x) = f_-(x)$  by taking limits. Thus  $f(x) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - x} d\zeta$  hence  $f$  is analytic at  $x$ .

■

**Theorem (weaker than pole singularity implies analyticity)** Suppose  $f$  is analytic in  $D \setminus \{z_0\}$  and has a weaker than pole singularity at  $z_0$ :

$$\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$$

holds uniformly. Then  $f$  is analytic at  $z_0$ . (More precisely:  $f$  can be analytically continued to  $z_0$ .)

### Proof

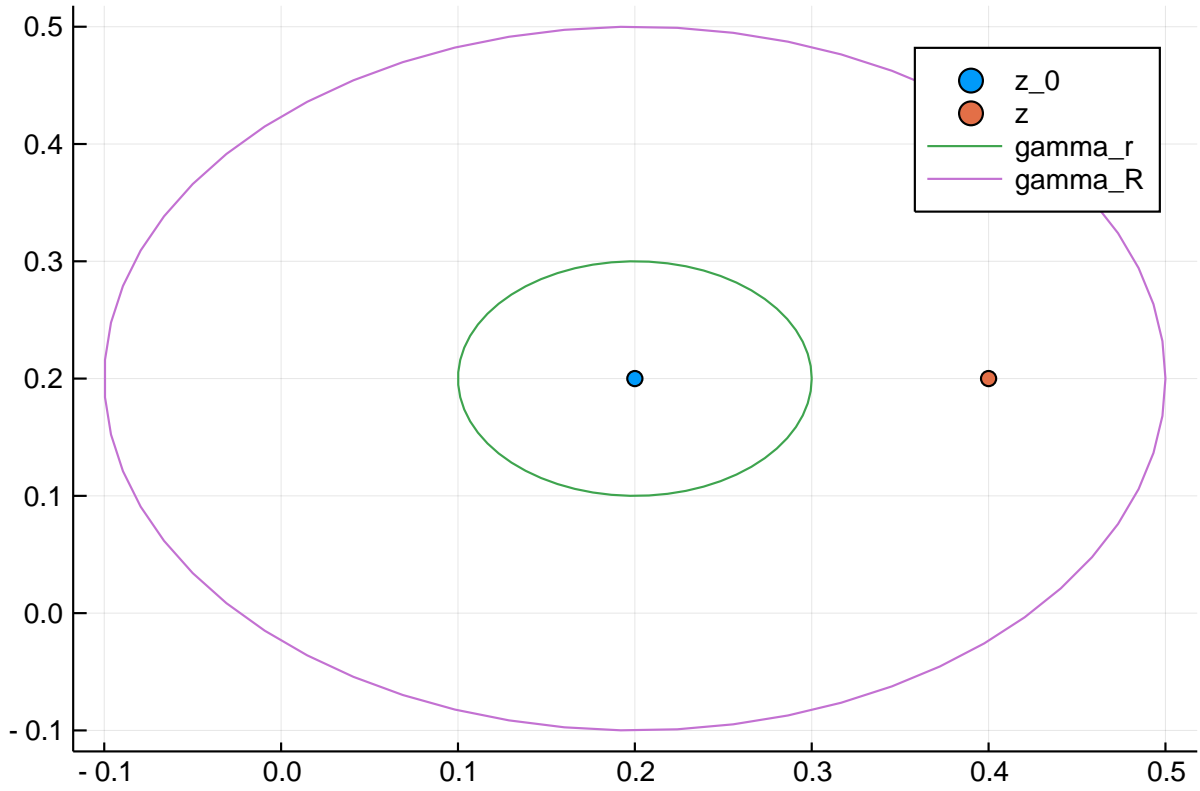
Around  $z_0$  is an annulus  $A_{R0}$  inside which  $f$  is analytic. Consider  $z$  in  $A_{R0}$  and a positively oriented circle  $\gamma_r$  of radius  $r$  with  $|r| < |z - z_0|$ . Then we have

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

here's a plot:

```
z_0 = 0.2 + 0.2im
z = 0.4 + 0.2im
```

```
scatter([real(z_0)], [imag(z_0)]; label="z_0")
scatter!([real(z)], [imag(z)]; label="z")
plot!(Circle(z_0, 0.1); label="gamma_r")
plot!(Circle(z_0, 0.3); label="gamma_R")
```



But we have

$$\left| \oint_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq 2\pi r \sup_{\zeta \in \gamma_r} \left| \frac{f(\zeta)}{\zeta - z} \right| \leq 2\pi \frac{1}{|z_0 - z| - r} \sup_{\zeta \in \gamma_r} |f(\zeta)|$$

which tends to zero as  $r \rightarrow 0$ .

■

**Corollary (Weaker than linear growth implies analyticity at infinity)** If

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 0,$$

then  $f$  is analytic at infinity.

From these result we can infer that

$$\phi(z) = \sqrt{z-1}\sqrt{z+1}$$

is analytic on  $\mathbb{C} \setminus [-1, 1]$ , and  $\phi(z) \sim_{z \rightarrow \infty} z$ .