M3M6: Methods of Mathematical Physics

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1 Lecture 4: Applications to real integrals

This lecture we discuss applications of residue calculus.

1. Trigonometric integrals
2. Integrals over real lines
   - Principal value integral
   - Cauchy's integral formula and Residue theorem on the real line
3. Oscillatory integrals
   - Jordan's lemma
   - Application: Calculating Fourier tranforms of weakly decaying functions

1.1 Trigonomteric

We can calculate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) \mathrm{d}\theta$$

where $R(x, y)$ is rational by doing the change of variables $z = e^{\mathrm{i}\theta}$ to reduce it to

$$\oint_{C_1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2\mathrm{i}}\right) \frac{\mathrm{d}z}{\mathrm{i}z}$$

This is rational in z hence guaranteed to be amenable to residue calculus, provided we can find the poles of the denominator.

Example Consider

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2\rho\cos\theta + \rho^2}$$

for $0 < \rho < 1$. We need to first locate the poles of

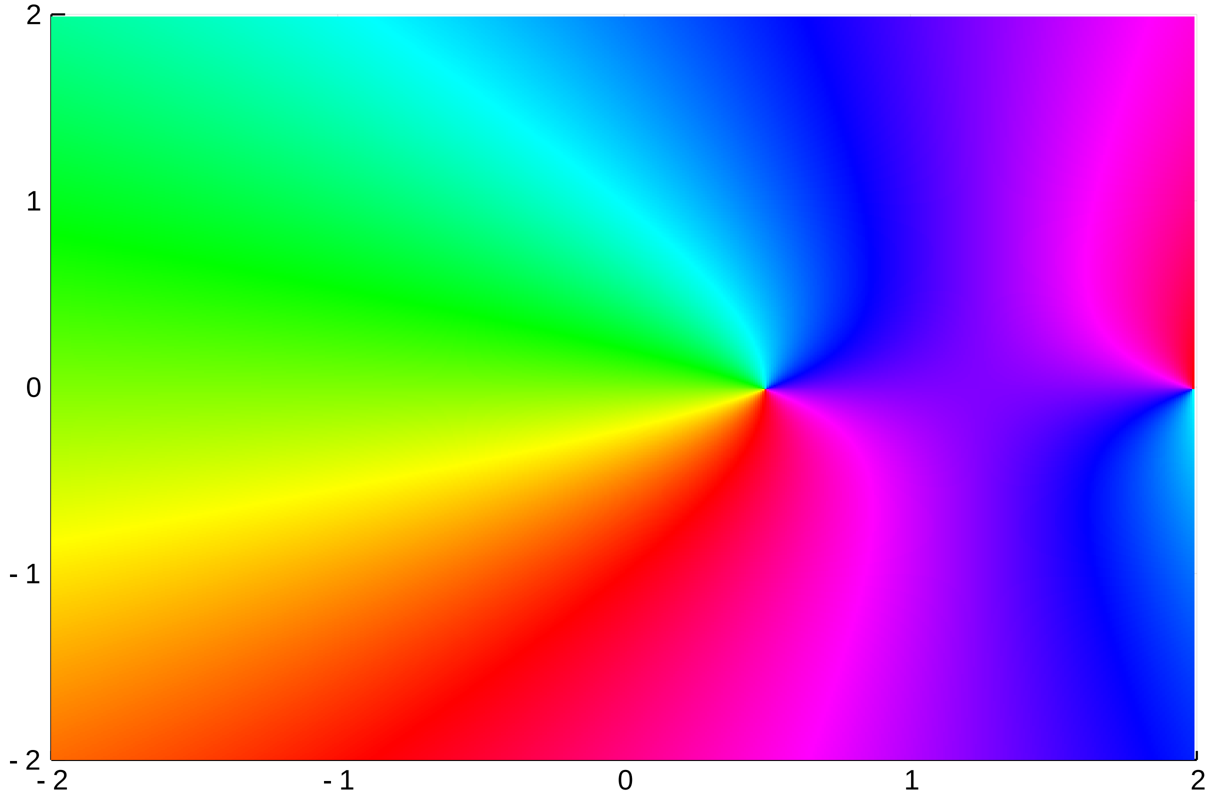
$$f(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2\mathrm{i}}\right) \frac{1}{\mathrm{i}z} = \frac{\mathrm{i}}{\rho z^2 - (1 + \rho^2)z + \rho}$$

This has poles at

$$z = \frac{1 + \rho^2 \pm \sqrt{(1 + \rho^2)^2 - 4\rho^2}}{2} = \rho, \rho^{-1}$$

We can confirm this with a phase plot:

```julia
using Plots, ComplexPhasePortrait
ρ = 0.5
phaseplot(-2..2, -2..2, z ->  1/(1-ρ*(z+(z^(-1))) + ρ^2) * 1/(im*z))
```



Thus we can use either the interior or exterior residue calculus. Since we know then roots of the denominator of f we determine that

$$f(z) = \frac{\mathrm{i}}{(z-\rho)(\rho z - 1)}$$

Thus we find

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2\rho\cos\theta + \rho^2} = 2\pi\mathrm{i}\operatorname*{Res}_{z=\rho} f(z) = -2\pi\mathrm{i}\operatorname*{Res}_{z=1/\rho} f(z) = \frac{2\pi}{1-\rho^2}.$$

1.2 Integrals over the real line

Integrals on the real line are typically viewed as improper integrals:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_0^{\infty} f(x)\mathrm{d}x + \int_{-\infty}^0 f(x)\mathrm{d}x = \lim_{b\to\infty}\int_0^b f(x)\mathrm{d}x + \lim_{a\to-\infty}\int_a^0 f(x)\mathrm{d}x.$$

It is convenient to work with a slightly different notion where we take the limit simealtaneously:

Definition (Principal value integral on the real line) The *(Cauchy) principal value integral on the real line* is defined as

$$⨍_{-\infty}^{\infty} f(x)\mathrm{d}x := \lim_{M\to\infty}\int_{-M}^{M} f(x)\mathrm{d}x$$

This is a weaker concept:

Proposition (Integability $\Rightarrow$ Prinipal value integrability) If $\int_{-\infty}^{\infty} f(x)\mathrm{d}x < \infty$ then

$$\fint_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x.$$

Example Consider integrating $1/(x - \mathrm{i})$ with indefinite integral $\log(x - \mathrm{i})$. We use the Definition

$$\log z = \log |z| + \mathrm{i} \arg z$$

where $-\pi \le \arg z < \pi$. Thus we have

$$\begin{aligned} \lim_{b\to\infty} \log(b - \mathrm{i}) &= \lim_{b\to\infty} \log |b - \mathrm{i}| = \infty \\ \lim_{a\to-\infty} \log(a - \mathrm{i}) = \lim_{a\to-\infty} (\log |a - \mathrm{i}| &+ \mathrm{i} \arg(a - \mathrm{i})) = \infty - \pi\mathrm{i} \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - \mathrm{i}} = \lim_{a\to-\infty} [\log(-\mathrm{i}) - \log(a - \mathrm{i})] + \lim_{b\to\infty} [\log(b - \mathrm{i}) - \log(-\mathrm{i})] = -\infty + \mathrm{i}\pi + \infty$$

is undefined. On the other hand, the Cauchy principal value integral gives

$$\fint_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - \mathrm{i}} = \lim_{M\to\infty} [\log(M - \mathrm{i}) - \log(-M - \mathrm{i})] = \lim_{M\to\infty} [\log |M - \mathrm{i}| - \log |-M - \mathrm{i}|] + \mathrm{i}\pi = \mathrm{i}\pi.$$

1.2.1 Residue theorem on the real line

The real line doesn't have an *inside* and *outside*, rather an *above* and *below*, or *left* and *right*. Thus we get the following two versions of the Residue theorem:

Definition (Upper/lower half plane) Denote the upper/lower half plane by

$$\begin{aligned} \mathbb{H}^+ &= \{z : \Re z > 0\} \\ \mathbb{H}^- &= \{z : \Re z < 0\} \end{aligned}$$

The closure is denoted

$$\begin{aligned} \bar{\mathbb{H}}^+ &= \mathbb{H}^+ \cup \mathbb{R} \cup \{\infty\} \\ \bar{\mathbb{H}}^- &= \mathbb{H}^- \cup \mathbb{R} \cup \{\infty\} \end{aligned}$$

Theorem (Residue theorem on the real line) Suppose $f : \bar{\mathbb{H}}^+ \backslash \{z_1, \ldots, z_r\} \to \mathbb{C}$ is holomorphic in $\mathbb{H}^+ \backslash \{z_1, \ldots, z_r\}$, where $\Re z_k > 0$, and $\lim_{\epsilon\to0} f(x + i\epsilon) = f(x)$ converges uniformly. If

$$\lim_{z\to\infty} z f(z) = 0$$

uniformly for $z \in \bar{\mathbb{H}}^+$, then

$$\fint_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi\mathrm{i}\sum_{k=1}^{r} \operatorname*{Res}_{z=z_k} f(z)$$

Similarly, if the equivalent conditions hold in the lower half plane for $f : \bar{\mathbb{H}}^- \backslash \{z_1, \ldots, z_r\} \to \mathbb{C}$ then

$$\fint_{-\infty}^{\infty} f(x)\mathrm{d}x = -2\pi\mathrm{i}\sum_{k=1}^{r} \operatorname*{Res}_{z=z_k} f(z)$$

Proof This follows by considering the contour $\gamma_R = [-R, R] \cup H_R$ where

$$H_R := \{R\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\},$$

that is, the upper-half circle. Classical residue calculus gives

$$\oint_{\gamma_R} f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{r} \operatorname*{Res}_{z=z_k} f(z).$$

provided R is large enough to contain all singularities. But the decay in f suffices to show that $\int_{H_R} f(z)\mathrm{d}z \to 0$ as $R \to \infty$. The result therefore follows.

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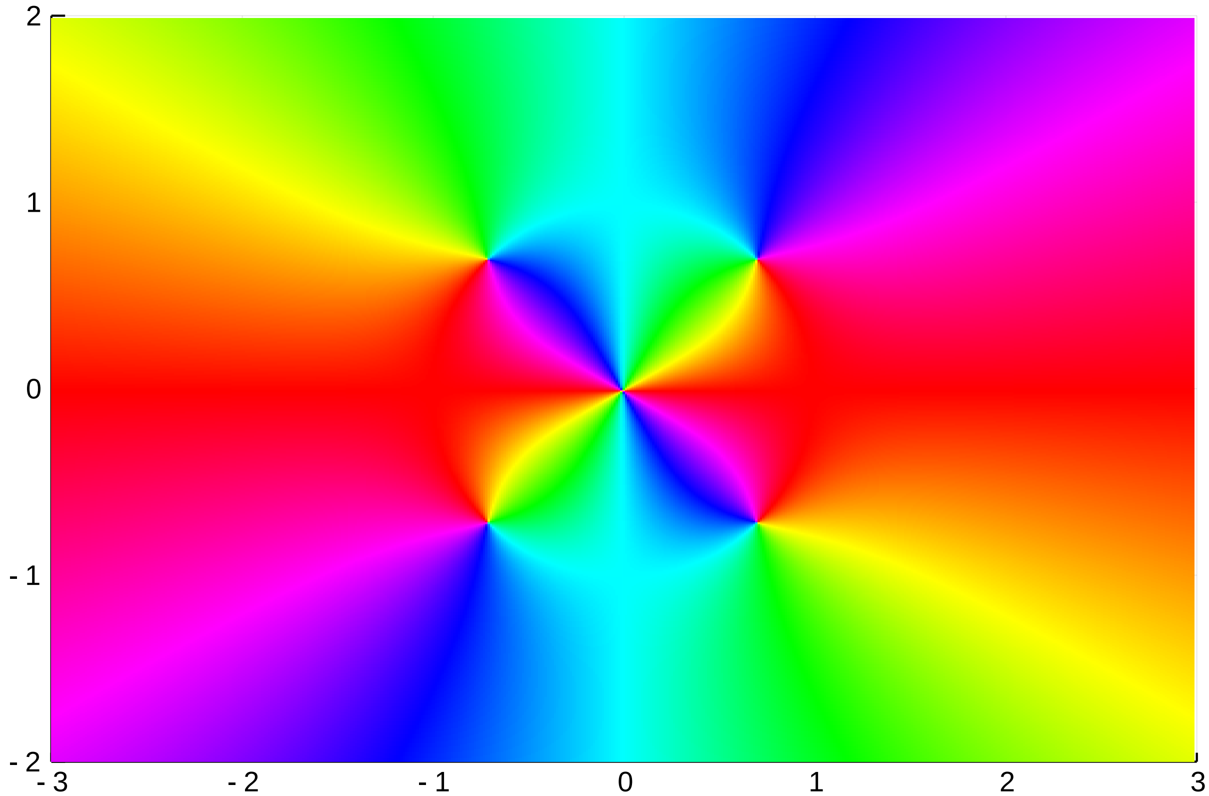
Demonstration

```
f = x -> x^2/(x^4+1)
phaseplot(-3..3, -2..2, f)
```



This function has poles in the upper plane, but has sufficient decay that we can apply Residue theorem:

```
using ApproxFun
z_1,z_2,z_3,z_4 = exp(im*π/4), exp(3im*π/4), exp(5im*π/4), exp(7im*π/4)

res_1 = z_1^2 / ((z_1 - z_2)*(z_1 - z_3)*(z_1 - z_4) )
res_2 = z_2^2 / ((z_2 - z_1)*(z_2 - z_3)*(z_2 - z_4) )

2π*im*(res_1 + res_2), sum(Fun(f, Line()))
```

```
(2.221441469079183 + 3.487868498008632e-16im, 2.221441469084968)
```

We can also apply Resiude theorem in the lower-half plane, and we get the same result:

```
res_3 = z_3^2 / ((z_3 - z_1)*(z_3 - z_2)*(z_3 - z_4) )
res_4 = z_4^2 / ((z_4 - z_1)*(z_4 - z_3)*(z_4 - z_2) )

-2π*im*(res_3 + res_4), sum(Fun(f, Line()))
```

```
(2.221441469079183 + 5.231802747012948e-16im, 2.221441469084968)
```

1.2.2 Cauchy's integral formula on the real line

An immediate consequence of the Residue theorem is Cauchy's integral formula on the real line:

Theorem (Cauchy's integral formula on the real line) Suppose $f : \bar{\mathbb{H}}^+ \to \mathbb{C}$ is holomorphic in $\mathbb{H}^+$, and $\lim_{\epsilon \to 0} f(x + i\epsilon) = f(x)$ converges uniformly. If

$$\lim_{z \to \infty} f(z) = 0$$

uniformly for $z \in \bar{\mathbb{H}}^+$, then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(x)}{x - z} dx$$

for all $z \in \mathbb{H}^+$.

Examples Here is a simple example of $f(x) = \frac{x^2}{(x+i)^3}$, which is analytic in the upper half plane:

```
f = x -> x^2/(x+im)^3
z = 2.0+2.0im
sum(Fun(x-> f(x)/(x - z), Line()))/(2π*im) - f(z)
```

```
2.69638478211931e-13 - 3.032296636007459e-13im
```

Evaluating in lower half plane doesn't work because it has a pole there:

```
f = x -> x^2/(x+im)^3
z = 2.0-2.0im
sum(Fun(x-> f(x)/(x - z), Line()))/(2π*im) , f(z)
```

```
(2.2331966377567175e-13 + 2.0218605967538834e-14im, 0.7040000000000001 - 0.
128im)
```

But does for a function analytic in the lower half plane (with a minus sign):

```
f = x -> x^2/(x-im)^3
z = 2.0-2.0im
-sum(Fun(x-> f(x)/(x - z), Line()))/(2π*im) , f(z)
```

```
(0.03277196176631425 + 0.16750113791564233im, 0.03277196176604461 + 0.16750
11379153391im)
```

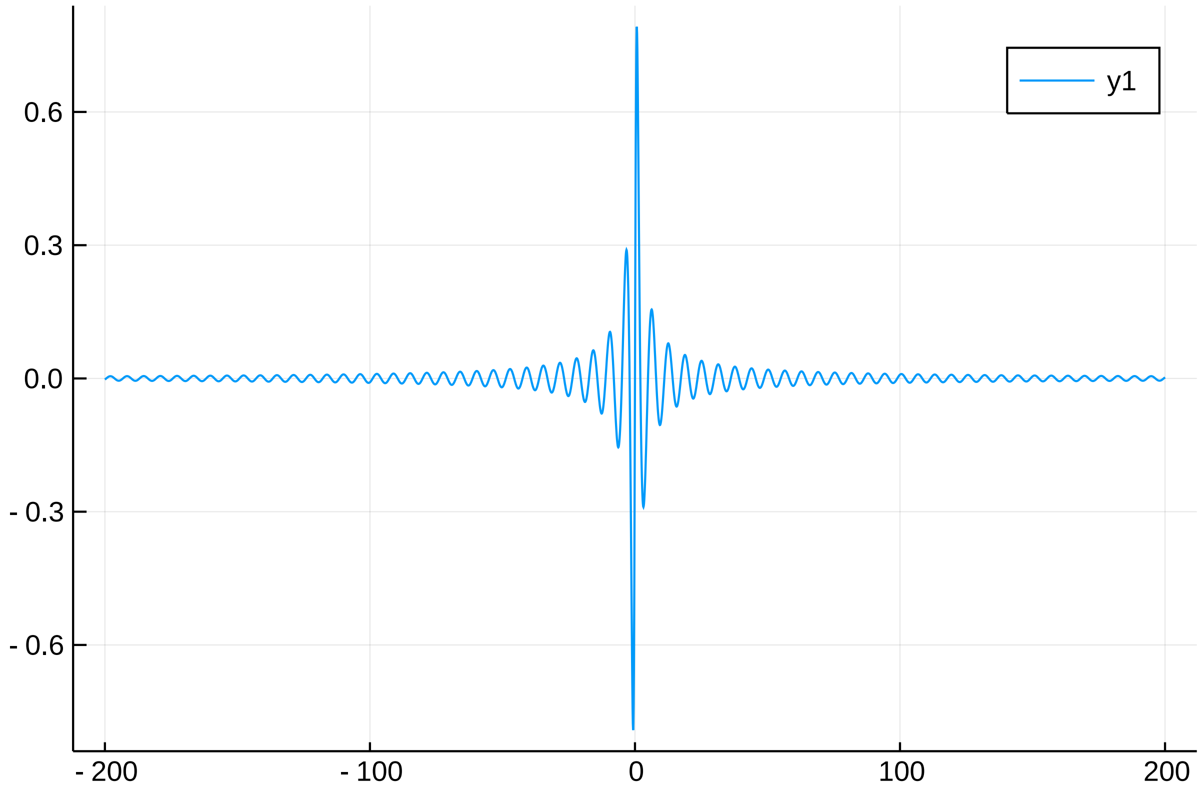
It also works for functions with exponential decay in the upper-half plane:

```
f = x -> exp(im*x)/(x+im)
z = 2 + 2im
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2π*im) - f(z)
```

```
4.501316791527543e-9 + 5.93330299732131e-7im
```

This is difficult as a real integral as the integrand is very oscillatory:

```
xx = -200:0.1:200
plot(xx,real.(f.(xx)))
```



An equivalent result holds in the negative real axis, but be careful:

```
z = -2-im
f = x -> exp(im*x)/(x+im)
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2π*im)
```

```
-4.529525111429054e-9 + 5.865568293152012e-7im
```

```
z = -2-im
f = x -> exp(im*x)/(x-im)
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2π*im), f(z)
```

```
(0.09196985577264773 - 0.09196926921581833im, 0.9007327639404081 + 0.335130
5720620013im)
```

```
z = -2-im
f = x -> exp(-im*x)/(x-im)
-sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2π*im), f(z)
```

```
(-0.045354995402086956 - 0.1219015148055592im, -0.04535499089125899 - 0.121
90092372837213im)
```

1.2.3 Fourier transforms and Jordan's lemma

The case of calculating

$$\int_{-\infty}^{\infty} e^{i\omega x} g(x) dx$$

is important because it is the Fourier transform of $g(x)$. Provided g is defined in the upper half plane and $\omega > 0$, $f(z) = e^{i\omega z} g(z)$ has exponential decay. If g decays fast enough we can use residue calculus as before.

Example Consider the Fourier transform of $g(x) = 1/(x^2 + 1)$. This decays fast enough to use residue calculus so we have for $\omega > 0$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2 + 1} \mathrm{d}x = 2\pi \mathrm{i} \operatorname*{Res}_{z=\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}\omega z}}{z^2 + 1} = \pi \mathrm{e}^{-\omega}.$$

On the other hand if $\omega < 0$ we have exponential decay in lower half plane and use corresponding residues to determine

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{x^2 + 1} \mathrm{d}x = -2\pi \mathrm{i} \operatorname*{Res}_{z=-\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}\omega z}}{z^2 + 1} = \pi \mathrm{e}^{\omega}.$$

That is, the Fourier transform of $g(x) = 1/(x^2 + 1)$ is $\pi \mathrm{e}^{-|\omega|}$.

Exponential decay actually gives us sharper results:

Lemma (Jordan) Assume $\omega > 0$. If $g(z)$ is continuous in on the half circle $H_R = \{Re^{i\theta} : 0 \le \theta \le \pi\}$ then

$$\left| \int_{H_R} g(z) e^{\mathrm{i}\omega z} dz \right| \le \frac{\pi}{\omega} M$$

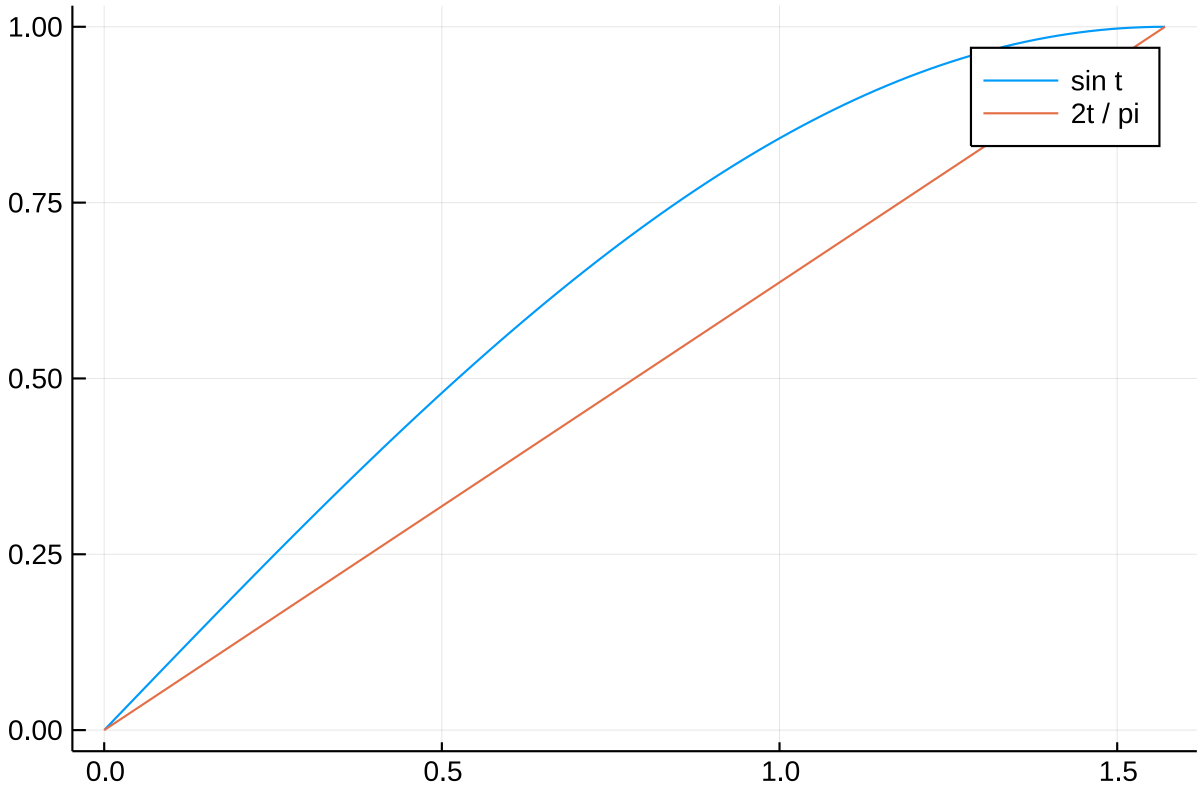
where $M = \sup_{z \in H_R} |g(z)|$.

Sketch of proof We have

$$\left| \int_{H_R} g(z) e^{\mathrm{i}\omega z} dz \right| \le R \int_0^{\pi} \left| g(Re^{i\theta}) e^{\mathrm{i}\omega Re^{i\theta}} e^{i\theta} \right| d\theta \le MR \int_0^{\pi} e^{-\omega R \sin\theta} d\theta = 2MR \int_0^{\frac{\pi}{2}} e^{-\omega R \sin\theta} d\theta$$

But we have $\sin\theta \ge \frac{2\theta}{\pi}$:

```
θ = range(0; stop=π/2, length=100)
plot(θ, sin.(θ); label="sin t")
plot!(θ, 2θ/π; label = "2t / pi")
```



Hence

$$\left|\int_{H_R} g(z)e^{i\omega z}dz\right| \leq 2MR\int_0^{\frac{\pi}{2}} e^{-\frac{2\omega R\theta}{\pi}}d\theta = \frac{\pi}{\omega}(1-e^{-\omega R})M \leq \frac{\pi M}{\omega}.$$

1.3 Application: Calculating Fourier integrals of weakly decaying functions

Why is this useful? We can use it to apply Residue theorem to functions that only have z^{-1} decay. Such functions do not decay absolutely but do converge in a principal value sense:

```
f = x -> exp(im*x)*x/(x^2+1)

sum(Fun(f, -30000 .. 30000))

-6.776263578034403e-21 + 1.1557671135433842im
```

Thus we can construct a Residue theorem for calculating

$$⨍_{-\infty}^{\infty} g(x)e^{i\omega x}dx$$

provided that $g(z) \to 0$ and is analytic in the upper-half plane.

```
2π*im*exp(-1)*im/(im+im)   # 2π*im* residue of g(z)exp(im*z) at z = im

0.0 + 1.1557273497909217im
```