**M3M6: Methods of Mathematical Physics**

Dr. Sheehan Olver

s.olver@imperial.ac.uk

# 1 Lecture 18: Examples of logarithmic singular integrals

This lecture we look at 3 examples of computing logarithmic singular integrals: $x/\sqrt{1-x^2}$, 1, and $1/\sqrt{1-x^2}$

## 1.1 Example 1

Consider $f(x) = x/\sqrt{1-x^2}$ and we want to compute

$$\mathcal{L}f(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{t}{\sqrt{1-t^2}} \log|t-z| \mathrm{d}t = \Re \underbrace{\frac{1}{\pi}\int_{-1}^{1} \frac{t}{\sqrt{1-t^2}} \log(z-t) \mathrm{d}t}_{Mf(z)}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1-x^2} = -f(x)$$

hence

$$F(x) = \int_x^1 f(t)\mathrm{d}t = \sqrt{1-x^2}.$$

and in particular $F(-1) = 0$. Recall that

$$\mathcal{C}F(z) = \frac{\sqrt{z-1}\sqrt{z+1} - z}{2\mathrm{i}}$$

Thus the result from last lecture gives

$$Mf(z) = \frac{1}{\pi}\int_{-1}^{1} f(t)\mathrm{d}t \log(z+1) + 2\mathrm{i}\mathcal{C}F(z) = \sqrt{z-1}\sqrt{z+1} - z$$

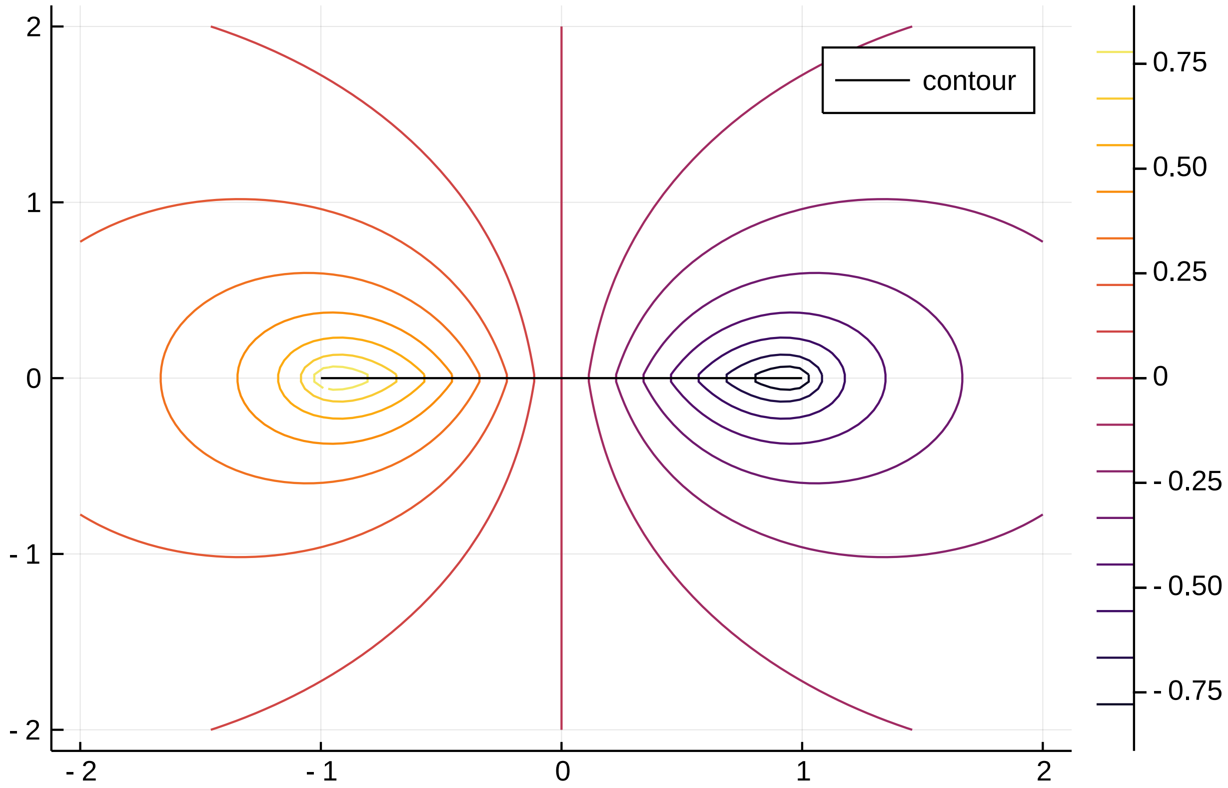
Let's double check the formula:

```
using ApproxFun, SingularIntegralEquations, Plots
x = Fun()
f = x/sqrt(1-x^2)
Mf = z -> sqrt(z-1)sqrt(z+1) - z
Lf = z -> real(Mf(z))
z = 0.1+0.1im
logkernel(f,z),Lf(z)
```

```
(-0.09000049991252067, -0.09000049991252063)
```

Here is a plot of the solution:

```
xx = yy = range(-2,2;length=100)
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z)
plot!(-1..1; color=:black, label="contour")
```




## 1.2 Example 2

Consider $f(x) = 1$ and we want to compute

$$Lf(z) = \frac{1}{\pi}\int_{-1}^{1} \log|t-z|\mathrm{d}t = \Re \underbrace{\frac{1}{\pi}\int_{-1}^{1} \log(z-t)\mathrm{d}t}_{Mf(z)}$$

We have

$$F(x) = \int_x^1 f(t)\mathrm{d}t = 1 - x.$$

We can determine the Cauchy transform by considering $1-z$ times the known Cauchy transform

$$\mathcal{C}1(z) = \frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} = \frac{\mathrm{i}}{\pi z} + O(z^{-3})$$

so that

$$\mathcal{C}F(z) = (1-z)\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} + \frac{\mathrm{i}}{\pi}$$

Thus the result from last lecture gives

$$M1(z) = \frac{1}{\pi}\int_{-1}^{1} \mathrm{d}t \log(z+1) + 2\mathrm{i}\mathcal{C}F(z) = \frac{2}{\pi}\log(z+1) + (1-z)\frac{\log(z-1) - \log(z+1)}{\pi} - \frac{2}{\pi}$$
$$= \frac{(1-z)\log(z-1) + (1+z)\log(z+1) - 2}{\pi}$$

For $-1 < x < 1$ real this gives a particularly simple formula:

$$L1(x) = \Re M^{+}1(x) = \frac{(1-x)\log(1-x) + (1+x)\log(x+1) - 2}{\pi}$$

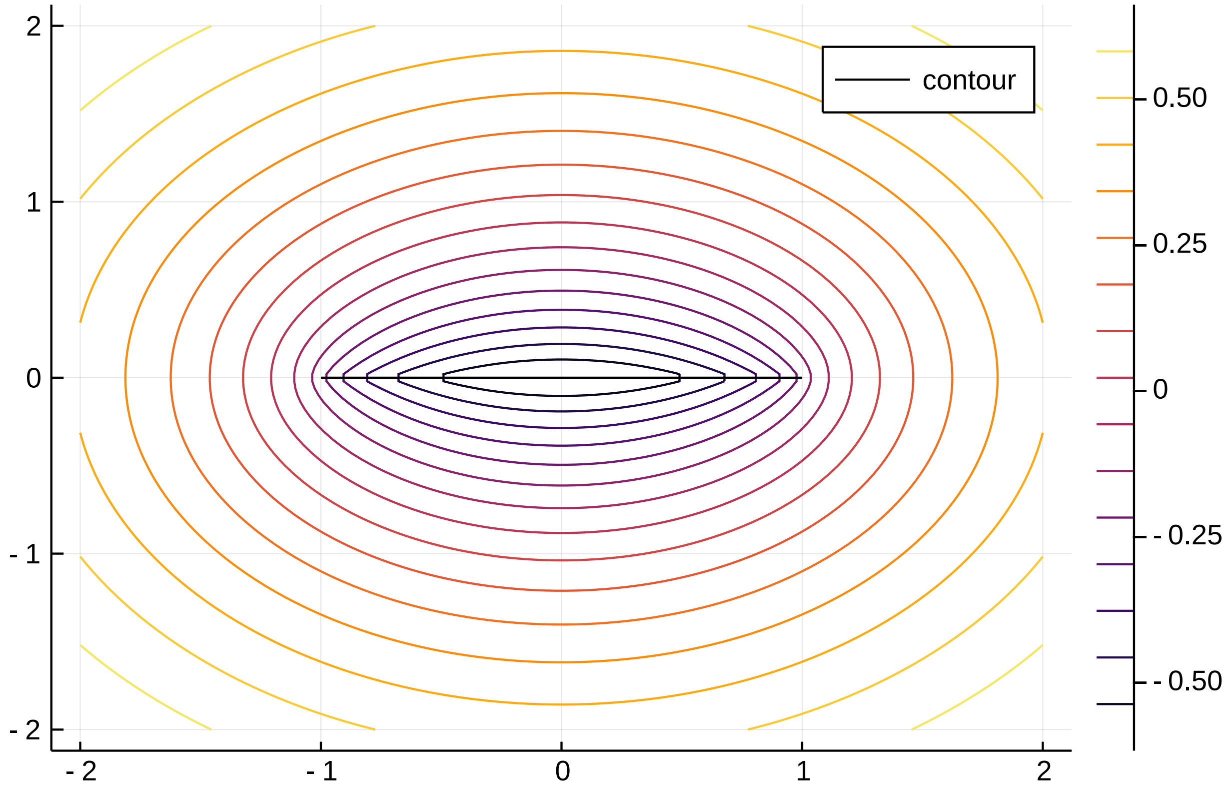
We can double check this:

```
f = Fun(1,-1..1)
x = 0.1
logkernel(f,x), ((1-x)*log(1-x) + (1+x)*log(x+1) - 2)/π
```

```
(-0.6334313470059203, -0.6334313470059203)
```

Here is a plot of the solution:

```
xx = yy = range(-2,2;length=100)
Lf = z -> real((1-z)*log(z-1) + (1+z)*log(z+1) - 2)/π
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z)
plot!(-1..1; color=:black, label="contour")
```




## 1.3 Example 3

Consider $f(x) = 1/\sqrt{(1-x^2)}$. We have

$$F(x) = \operatorname{acos} x = 2\operatorname{atan} \frac{\sqrt{1-x}}{\sqrt{1+x}}$$

where the second version can be verified by differentiation, using $\operatorname{atan}' x = 1/(1+x^2)$. One can determine $M$ by indefinite integration of

$$\mathcal{C}f(z) = \frac{\mathrm{i}}{2\sqrt{z-1}\sqrt{z+1}}$$

but we prefer to start with an ansatz and verify the solution. Namely, consider

$$\phi(z) := \frac{\log(\sqrt{z-1} + \sqrt{z+1})}{\mathrm{i}}.$$

First, this is analytic off $(-\infty, 1]$, as for $z$ in the upper-half plane we have $\sqrt{z-1}$ and $\sqrt{z+1}$ are in the upper-right quadrant: we never cross the branch cut of $\log z$. Similar argument holds for $z$ in the lower-half plane. On $x \in (-\infty, -1]$ it has the jump

$$\begin{aligned}
\phi_+(x) - \phi_-(x) &= \frac{\log(\mathrm{i}(\sqrt{1-x} + \sqrt{-x-1})) - \log(-\mathrm{i}(\sqrt{1-x} + \sqrt{-x-1}))}{\mathrm{i}} \\
&= 2\arg(\mathrm{i}(\sqrt{1-x} + \sqrt{-x-1})) = \pi
\end{aligned}$$

For $x \in (-1, 1)$ we have

$$\begin{aligned}
\phi_+(x) - \phi_-(x) &= \frac{\log(\mathrm{i}\sqrt{1-x} + \sqrt{1-x}) - \log(-\mathrm{i}\sqrt{1-x} + \sqrt{1-x})}{\mathrm{i}} \\
&= 2\arg(\mathrm{i}\sqrt{1-x} + \sqrt{-x-1}) = 2\operatorname{atan}\frac{\sqrt{1-x}}{\sqrt{1+x}} = F(x).
\end{aligned}$$

Finally we have

$$\phi(z) = \frac{\log z}{2\mathrm{i}} - \mathrm{i}\log 2 + o(1)$$

We therefore have from Plemelj that

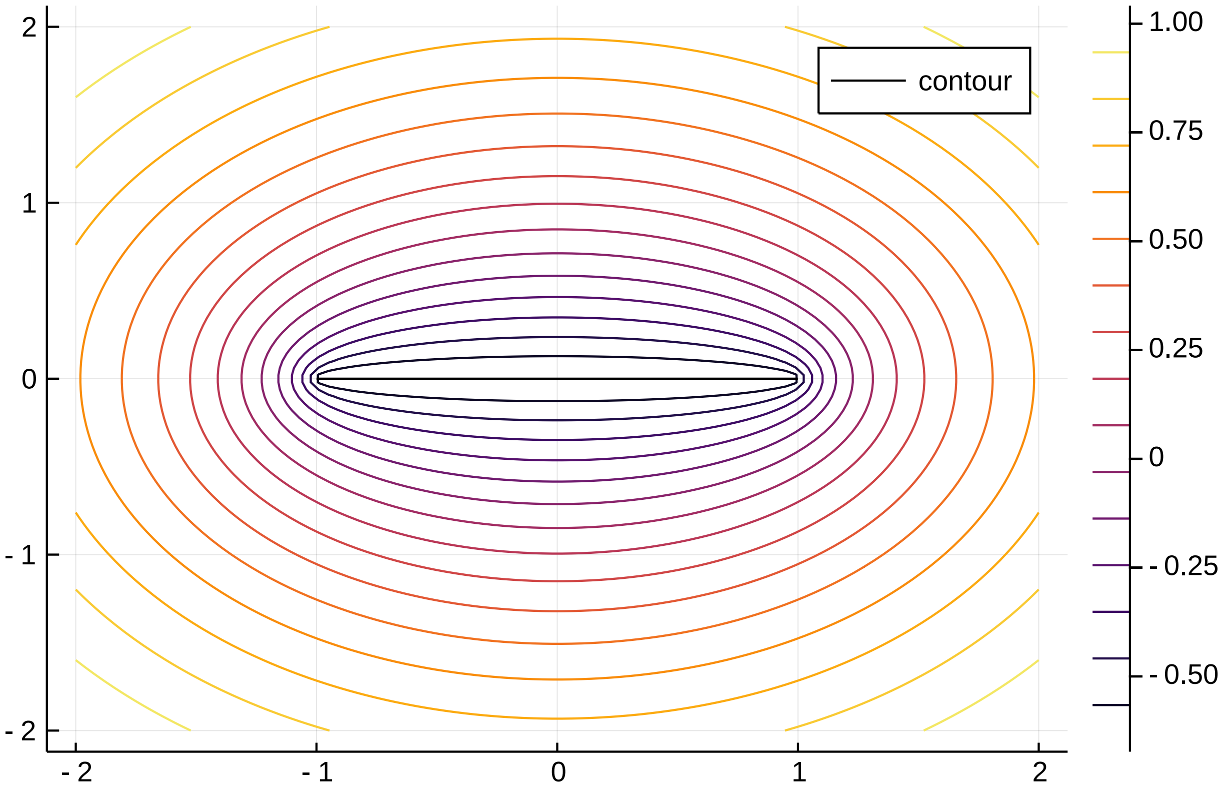
$$\mathcal{C}F(z) = \phi(z) - \frac{\log(z+1)}{2\mathrm{i}} + \mathrm{i}\log 2$$

and

$$Mf(z) = \frac{F(-1)}{\pi}\log(z+1) + 2\mathrm{i}\mathcal{C}F(z) = 2\log(\sqrt{z-1} + \sqrt{z+1}) - 2\log 2$$

Here is a plot of the solution:

```
xx = yy = range(-2,2;length=100)
Lf = z -> real(2log(sqrt(z-1) + sqrt(z+1)) - 2log(2))
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z)
plot!(-1..1; color=:black, label="contour")
```




Note that it is clearly visable that it is constant on the contour, which follows from:

$$Lf(x) = \Re M^f(x) = 2\Re \log(\mathrm{i}\sqrt{1-x} + \sqrt{1+x}) - 2\log 2 = 2\log\sqrt{1-x+1+x} - 2\log 2 = -\log 2$$