M3M6: Methods of Mathematical Physics

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1 Lecture 3: Laurent series and residue calculus

This lecture we cover

1. Fourier and Laurent series
2. Contour integrals and Laurent coefficients
3. Isolated singularities
   - Residue at a point
4. Contour integrals in domains with multiple holes
   - The residue theorem
5. Calculated integrals
   - Application: Trigonometric integrals with rational functions

1.1 Fourier and Laurent series

Definition (Fourier series) On $[-\pi, \pi)$, *Fourier series* is an expansion of the form

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{ik\theta}$$

where

$$g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \mathrm{e}^{-ik\theta} d\theta$$

Definition (Laurent series) In the complex plane, Laurent series around z_0 is an expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k$$

Lemma (Fourier series = Laurent series) On a circle in the complex plane

$$\gamma_r = \{z_0 + re^{i\theta} : -\pi \leq \theta < \pi\},$$

Laurent series of $f(z)$ around z_0 converges for $z \in C_r$ if the Fourier series of $g(\theta) = f(z_0 + re^{i\theta})$ converges, and the coefficients are given by

$$f_k = \frac{g_k}{r^k} = \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

Proof This follows immediately from the change of variables $z = re^{i\theta} + z_0$. ■

Interestingly, analytic properties of f can be used to show decaying properties in g:

1.2 Residue on a circle

In this course, we will *always* think of Laurent series living on a circle \$ \gamma*r(z*0) = {z : —z-z_0— = r }\$. That is,

$$f(z) \approx \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k$$

for $z \in \gamma_r(z_0)$.

Proposition (Residue on a circle) Suppose the Laurent series is absolutely summable on γ_r. Then

$$\oint_{\gamma_r} f(z) dz = 2\pi \mathrm{i} f_{-1}$$

We refer to f_{-1} as the residue over γ_r.

Example For all $0 < r < \infty$,

$$\oint_{\gamma_r} \frac{1}{z} dz = 2\pi \mathrm{i}$$

When f is holomorphic in a neighbourhood of the circle, we can extend it to an annulus (like Taylor series and disks):

Proposition (Laurent series in an annulus) Suppose f is holomorphic in an open annulus $A_{\rho R}(z_0) = \{z : \rho < |z - z_0| < R\}$. Then the Laurent series converges uniformly in any closed annulus inside $A_{\rho R}$

Proof *Exercise.* Hint: use the decay in the Laurent coefficients f_k from last lecture.

Proposition (Residue on a circle) holds true regardless of the radius.

1.3 Isolated singularities

Definition (isolated singularity) f has an *isolated singularity at* z_0 if it is holomorphic in an open annulus with inner radius 0:

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}.$$

Definition (Removable singularity) f has a *removable singularity at* z_0 if it has an isolated singularity at z_0 and all negative terms in the Laurent series in $A_{0R}(z_0)$ are zero:

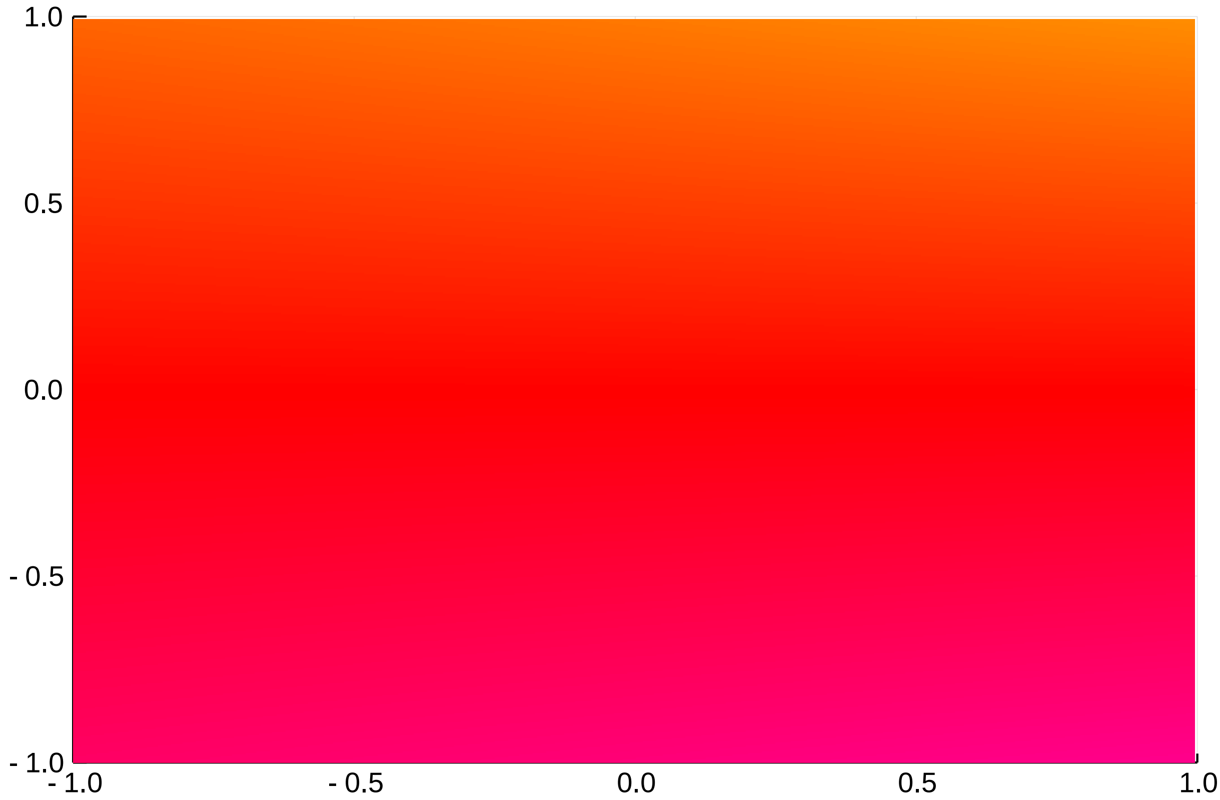
$$f(z) = f_0 + f_1(z - z_0) + f_2(z - z_0)^2 + \cdots$$

```
using Plots, ComplexPhasePortrait
f = z -> (exp(z)-1)/z
f(0.0)
```

```
NaN
```

No singularity appears when plotting `f`:

```
phaseplot(-1..1, -1..1, f)
```

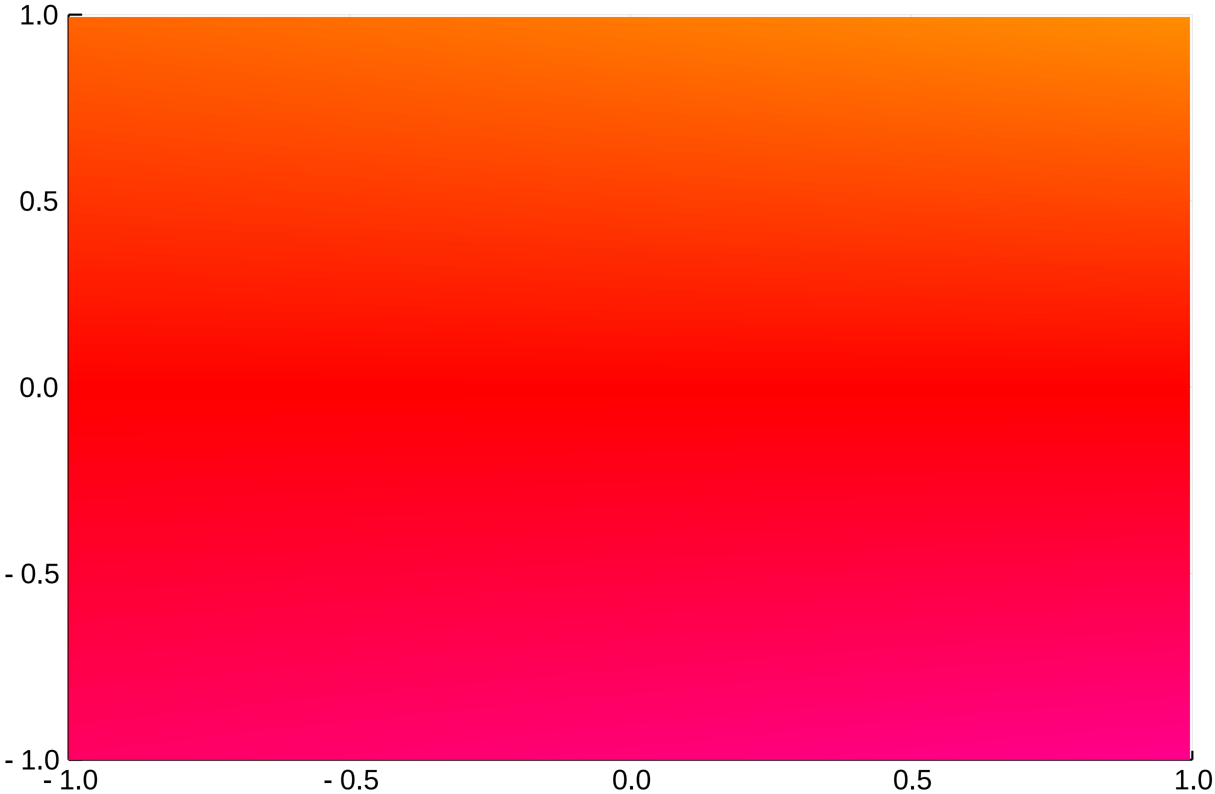


Proposition (Removing a removable singularity) If f has a removable singularity at z_0, then

$$\tilde{f}(z) = \begin{cases} f_0 & z = z_0 \\ f(z) & 0 < |z - z_0| < R \end{cases}$$

is analytic in the disk $B_R(z_0) = \{z : |z - z_0| < R\}$, with a convergent Taylor series. Hence the name.

```
g = z -> z ≈ 0 ? 1 : f(z)
phaseplot(-1..1, -1..1, g)
```



Definition (simple pole) f has a *simple pole at* z_0 if it is holomorphic in

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only one negative term in the Laurent series in $A_{0R}(z_0)$:

$$f(z) = \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

where $f_{-1} \neq 0$.

Definition (higher order pole) f has a *pole of order* N *at* z_0 if it is holomorphic in

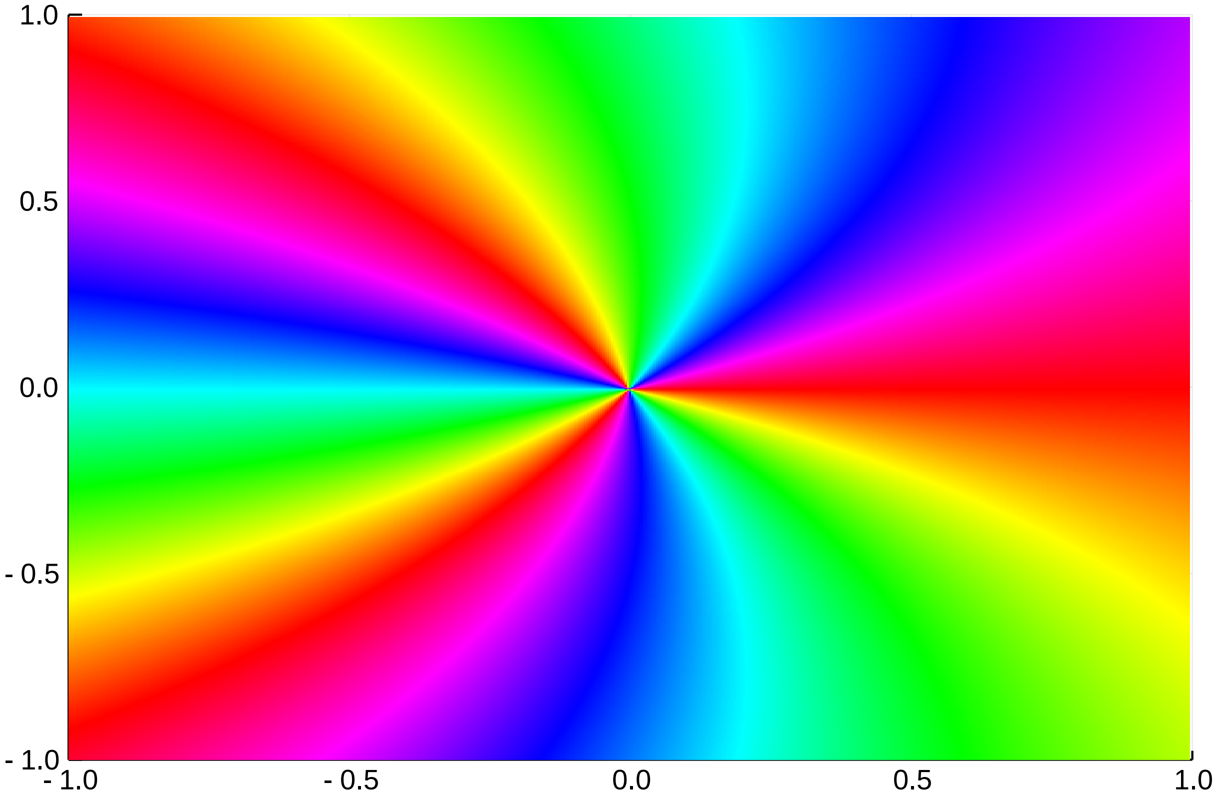
$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only N negative coefficients in the Laurent series:

$$f(z) = \frac{f_{-N}}{(z - z_0)^N} + \frac{f_{1-N}}{(z - z_0)^{N-1}} + \cdots + \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

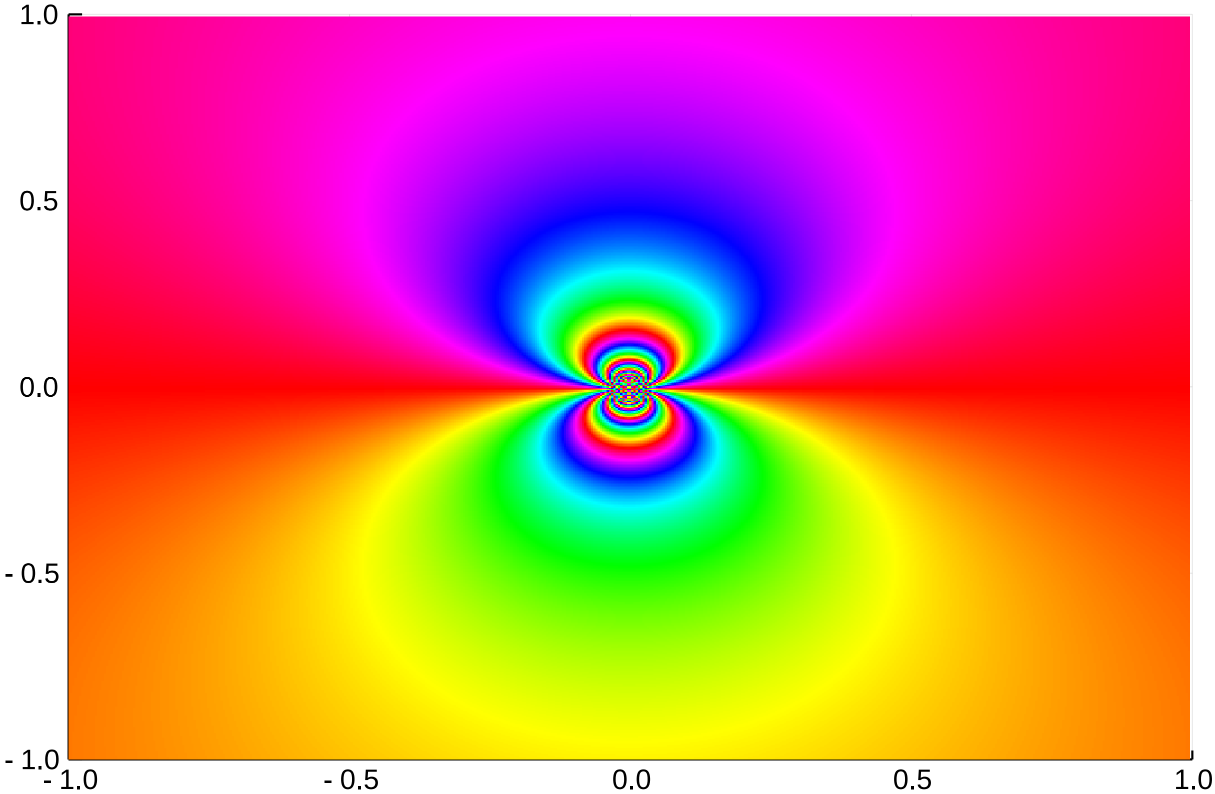
where $f_{-N} \neq 0$.

```
phaseplot(-1..1, -1..1, z -> exp(z)/z^3)
```



Definition (essential singularity) f has an *essential singularity at* z_0 if it is holomorphic in $A_{0R}(z_0)$ and has an infinite number of negative Laurent coefficients.

```
phaseplot(-1..1, -1..1, z -> exp(1/z))
```



Even though the singlarity is essential, we can still calculate the integrals using Residue calculus:

```
using ApproxFun
sum(Fun(z -> exp(1/z), Circle())),
2π*im*Fun(z -> exp(1/z), Laurent(Circle())).coefficients[2]
```

```
(-8.078182973046723e-16 + 6.283185307179586im, -8.078182973046723e-16 + 6.2
83185307179586im)
```

1.3.1 Residue at a point

Definition (Residue at a point) Suppose f has an isolated singularity at z_0, and is analytic in the annulus $A_{0R}(z_0)$ for some $R > 0$. Then we define the *residue at* z_0 as

$$\operatorname*{Res}_{z=z_0} f(z) = f_{-1}$$

where f_{-1} is the first negative coefficent of the Laurent series in $A_{0R}(z_0)$.

Proposition (Residue of ratio of analytic functions with simple pole) Suppose

$$f(z) = \frac{A(z)}{B(z)}$$

and A, B are analytic/holomorphic in a disk of radius R around z_0 and that B has only a single zero at z_0:

$$\begin{aligned} A(z) &= A_0 + A_1(z - z_0) + \cdots \\ B(z) &= B_1(z - z_0) + \cdots \end{aligned}$$

Then $\operatorname*{Res}_{z=z_0} f(z) = \frac{A_0}{B_1}$

Exercise (Residue of ratio of analytic functions with higher order poles) What is the residue at z_0 if B has a higher order zero: $B(z) = B_N(z - z_0)^N + \cdots$?

1.4 Contour integrals on domains with multiple holes

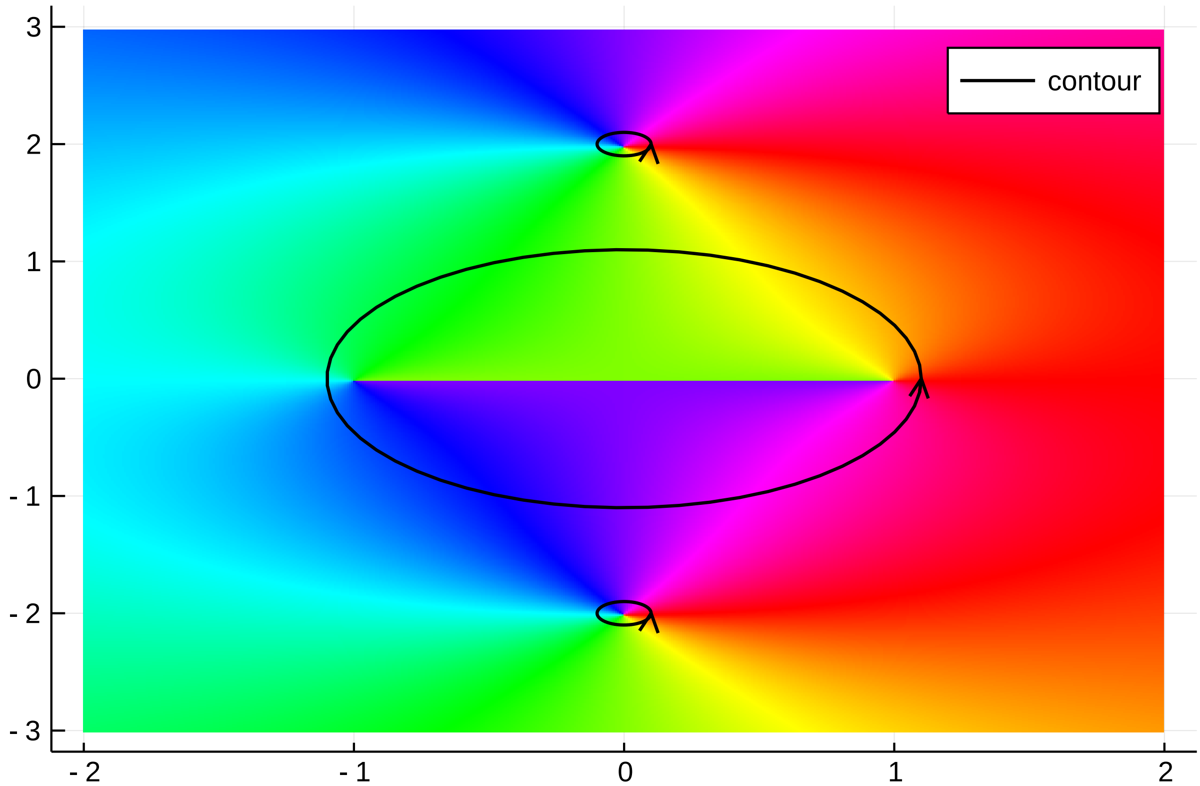
Consider the following example:

$$\frac{\sqrt{z-1}\sqrt{z+1}}{z^2+4}$$

We still have the contour integral over a circle, and so *Proposition (Residue on a circle)* still holds true for $r > 2$. But we can also deform the contour into three contours:

```
f = z -> sqrt(z-1)sqrt(z+1)/(z^2+4)

Γ = Circle(1.1) ∪ Circle(2.0im,0.1) ∪ Circle(-2.0im,0.1)
phaseplot(-2..2, -3..3, f)
plot!(Γ; color=:black, label=:contour, arrow=true, linewidth=1.5)
```




```
sum(Fun(f, Circle(2.1))), sum(Fun(f, Γ))
```

```
(-8.270426731549189e-16 + 6.283185307179587im, -1.00975500904025e-15 + 6.28
3185307179586im)
```

Thus we can sum over three residues.

1.4.1 Residue theorem

Theorem (Cauchy's Residue Theorem) Let f be holomprohic inside and on a simple closed, positively oriented contour γ except at isolated points $z_1, \dots, z_r$ inside γ. Then

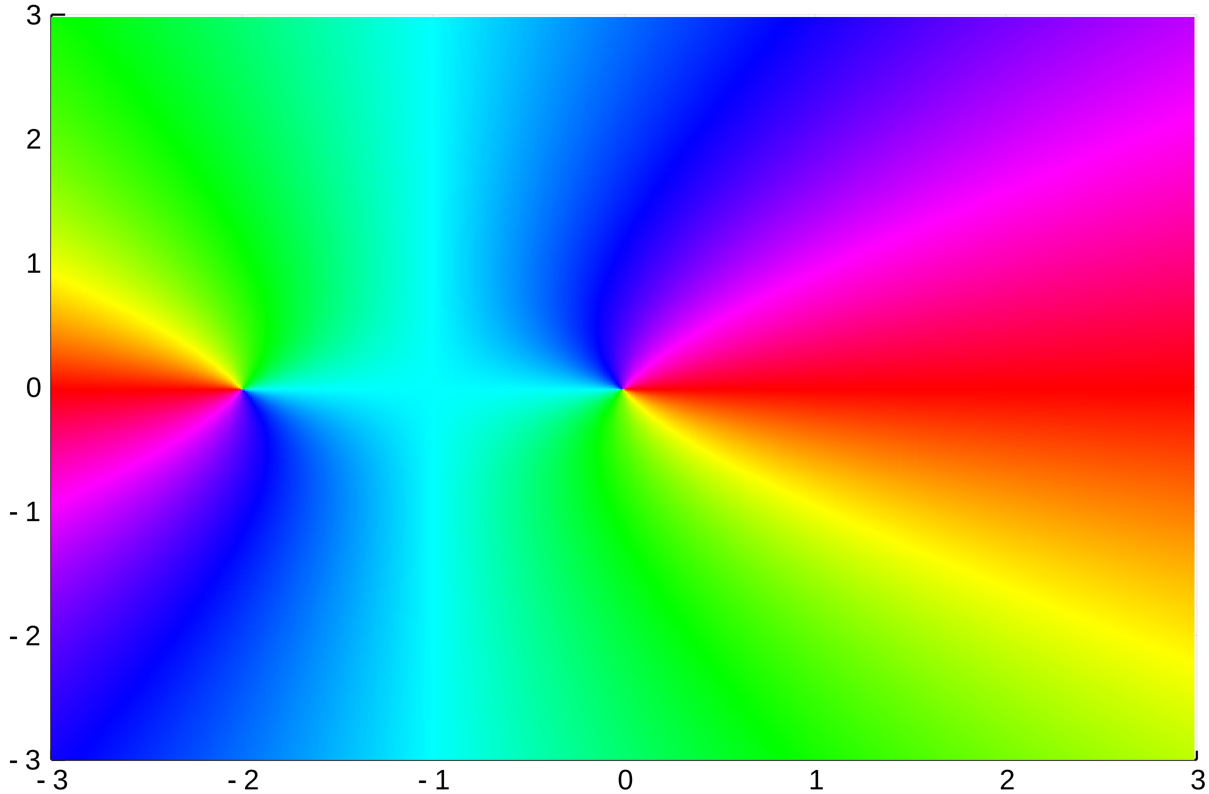
$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{r} \operatorname*{Res}_{z=z_j} f(z)$$

1.5 Calculating integrals

We can use the Residue theorem to calculate "hard" integrals.

First, two trivial examples:

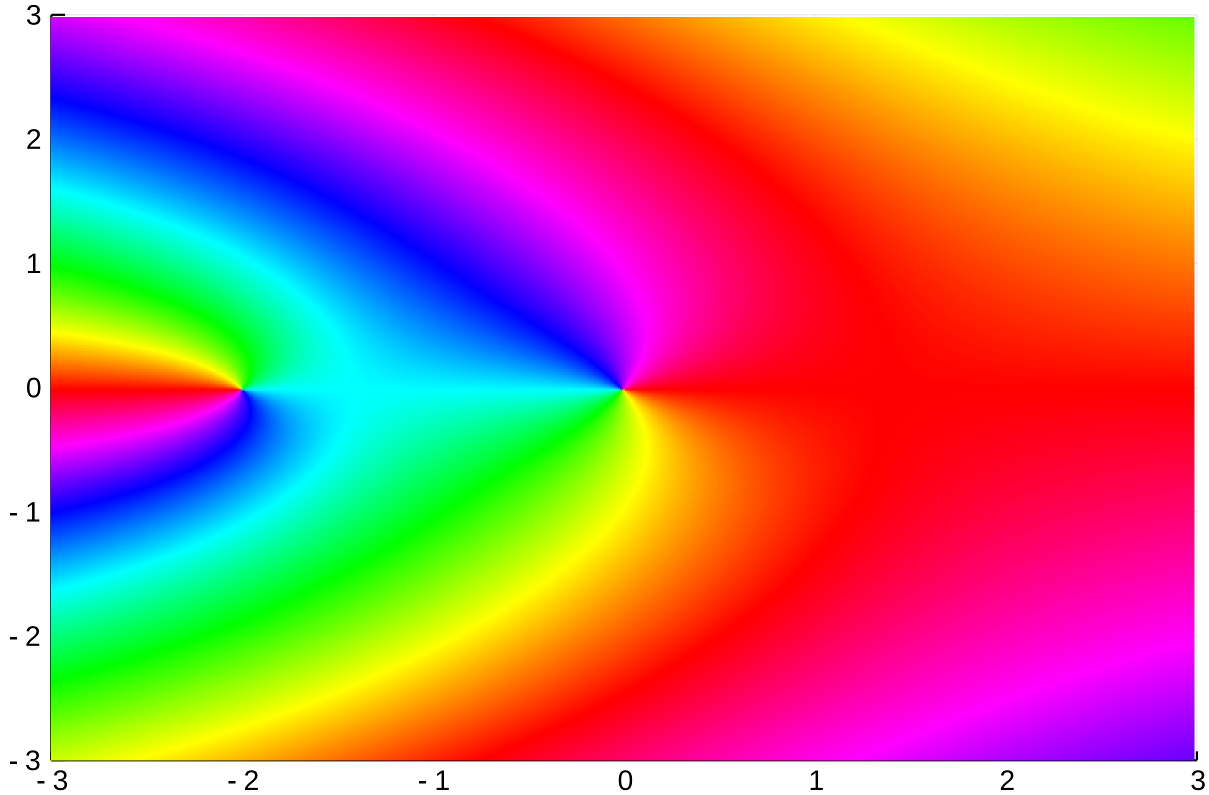
```
f = z -> 1/(z*(z+2))
phaseplot(-3..3, -3..3, f)
```




```
sum(Fun(f, Circle(3.0)))
```

```
7.874234295592502e-19 - 9.742139082662117e-17im
```

```
f = z -> exp(z)/(z*(z+2))
phaseplot(-3..3, -3..3, f)
```



```
sum(Fun(f, Circle(3.0)))
```

```
-5.073166565789438e-16 + 2.716424322002157im
```

```
2π*im*(1/2 - exp(-2)/2)
```

```
0.0 + 2.716424322002157im
```

```
sum(Fun(z -> exp(z)/(z^2*(z+2)), Circle(3.0)))
```

```
-2.313334476762615e-16 + 1.7833804925887144im
```

```
2*pi*im * (1/4 + exp(-2)/4)
```

```
0.0 + 1.783380492588715im
```

1.6 Application: Integrals on the real line of rational functions

We can calculate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$$

where $R(x, y)$ is rational by doing the change of variables $z = e^{i\theta}$ to reduce it to

$$\oint_{\gamma_1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz}$$

Example Consider

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\rho\cos\theta + \rho^2}$$

for $0 < \rho < 1$.

```
ρ = 0.5
plot(Fun(θ -> 1/(1-2ρ*cos(θ) + ρ^2), 0 .. 2π))
```

```
phaseplot(-2..2, -2..2, z ->  1/(1-ρ*(z+(z^(-1))) + ρ^2) * 1/(im*z))
```

```
sum(Fun(θ -> 1/(1-2ρ*cos(θ) + ρ^2), 0 .. 2π)), 2π /(1-ρ^2)
```

```
(8.377580409572783, 8.377580409572781)
```

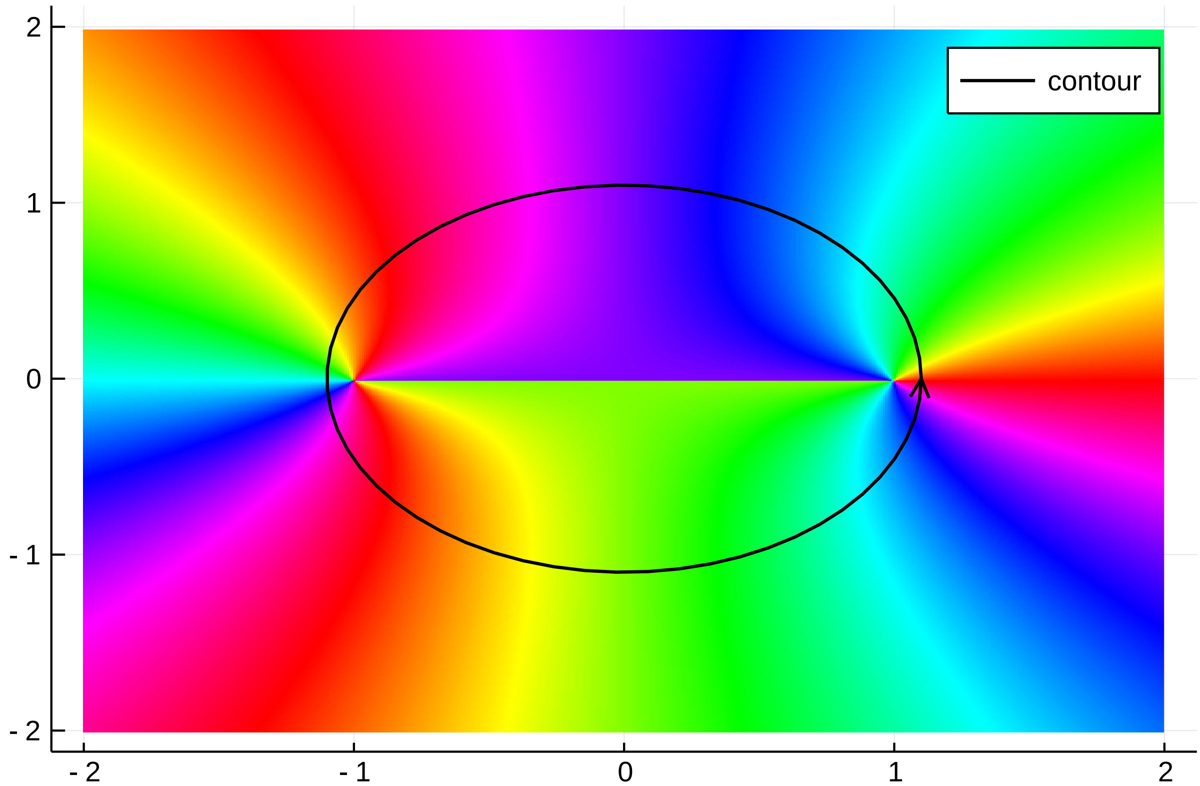
1.7 Demonstration

Example This works for functions not analytic:

$$\oint_{\gamma_1} (\sqrt{z-1}\sqrt{z+1})^3 dz$$

```
f = z -> (sqrt(z-1)*sqrt(z+1))^3
```

```
phaseplot(-2..2, -2..2, f)
plot!(Circle(1.1); color=:black, label="contour", linewidth=1.5, arrow=true)
```




```
@show sum(Fun(f, Laurent(Circle(1.1))))  # integral over circle
```

```
sum(Fun(f, Laurent(Circle(1.1)))) = -1.2751605303035503e-16 + 2.35619449019
23444im
```

```
f_−_1 = Fun(f, Laurent(Circle(1.1))).coefficients[2] # numerical Laurent coefficient
@show 2π*im*f_−_1;
```

```
(2π) * im * f_−_1 = -1.1592368457305e-16 + 2.141994991083949im
```