M3M6: Methods of Mathematical Physics

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1 Inverting Log kernels

2. Solving a logarithmic singular integral equation

3. Application: electrostatic potentials in 2D

   – Potential arising from a point charge and a single plate

1.0.1 Solving logarithmic singular integral equations

Consider the problem of calculating $u(x)$ such that

$$\frac{1}{\pi}\int_a^b u(t)\log|x-t|\mathrm{d}t = f(x) \qquad \text{for} \qquad a < x < b$$

To tackle this problem, recall that $Lu(z) = \Re Mu(z)$ for

$$Mu(z) = \frac{1}{\pi}\int_a^b \log|z-t|u(t)\mathrm{d}t = -2\mathrm{i}\int_c^z \mathcal{C}u(\zeta)\mathrm{d}\zeta + Mu(c)$$

where c is arbitrary, in fact we can choose it to be b. Then we have

$$Lu(x) = \Re M^+u(x) = 2\Im\int_b^x \mathcal{C}^+u(t)\mathrm{d}t = \int_b^x Hu(t)\mathrm{d}t$$

where

$$\mathcal{H}u(x) = \frac{1}{\pi}⨍_a^b \frac{u(t)}{t-x}\mathrm{d}t$$

is the Hilbert transform. In other words, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}Lu(x) = \mathcal{H}u(x).$$

Differentiating both sides of the equation for u therefore gives

$$\mathcal{H}u(x) = f'(x).$$

recall, we can express the solution as

$$u = \frac{-\mathcal{H}[\sqrt{b-\diamond}\sqrt{\diamond-a}f'] + D}{\sqrt{b-x}\sqrt{x-a}}$$

Here the constant D is not arbitrary: recall that

$$\mathcal{L}[1/\sqrt{1-\diamond^2}](x) = -\log 2$$

hence varying D will vary u by a constant. To choose D, we use the fact that

$$f(c) = \frac{1}{\pi} \int_a^b u(t) \log |t - c| \mathrm{d}t$$

for any choice of point c.

Demonstration

Let's do a numerical example:

```
using ApproxFun, SingularIntegralEquations, Plots
H(f) = -hilbert(f) # fix normalisation
H(f,x) = -hilbert(f,x)


x = Fun()
f = exp(x)
D = randn()

u_1 = -H(sqrt(1-x^2)*f')/sqrt(1-x^2)

u = u_1 + D/sqrt(1-x^2)

@show H(u, 0.1) + f(0.1)
```

```
H(u, 0.1) + f(0.1) = 2.2103418361512954
```

```
@show logkernel(u,0.1) - f(0.1)  # didn't work
```

```
logkernel(u, 0.1) - f(0.1) = -0.0941709295301647
```

```
@show logkernel(u,0.2) - f(0.2);  # but we are only off by a constant
```

```
logkernel(u, 0.2) - f(0.2) = -0.09417092953016448
```

We thus need to choose the constant

```
# choose C so that
# logkernel(u_1, 0) + C*logkernel(1/sqrt(1-x^2), 0) == f(0)
C = (f(0) - logkernel(u_1, 0))/logkernel(1/sqrt(1-x^2), 0)
u = u_1 + C/sqrt(1-x^2)

@show hilbert(u, 0.1) + f(0.1)
```

```
hilbert(u, 0.1) + f(0.1) = -6.661338147750939e-16
```

```
@show logkernel(u,0.1) - f(0.1)  # Works!
```

```
logkernel(u, 0.1) - f(0.1) = -2.220446049250313e-16
```

```
@show logkernel(u,0.2) - f(0.2);  # And at all x!
```

```
logkernel(u, 0.2) - f(0.2) = 0.0
```

Example 1 We now do an example which can be solved by hand. Find $u(x)$ so that:

$$\mathcal{L}u(x) = x$$

for $-1 < x < 1$. Differentiating his becomes

$$\mathcal{H}u(x) = 1$$

which we solve via

$$u(x) = \frac{-\mathcal{H}[\sqrt{1-\diamond^2}](x) + D}{\sqrt{1-x^2}} = \frac{D-x}{\sqrt{1-x^2}}$$

Note that

$$\mathcal{L}u(0) = 0$$

A symmetry argument shows that

$$\mathcal{L}[\diamond/\sqrt{1-\diamond^2}](x) = \frac{1}{\pi}\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \log|x| \mathrm{d}x = 0$$

which imples that $D = 0$ and we have

$$u(x) = \frac{-x}{\sqrt{1-x^2}}$$

We can confirm this:

```
x = Fun()
logkernel(-x/sqrt(1-x^2), 0.1) # approximately 0.1
```

```
0.10000000000000006
```

Example 2 Find $u(x)$ so that:

$$\frac{1}{\pi}\int_{-1}^{1} u(t)\log|x-t|\mathrm{d}t = 1.$$

Differentiating, we know that

$$\int_{-1}^{1} \frac{u(t)}{x-t}\mathrm{d}t = 0$$

hence $u(x)$ must be of the form $D/\sqrt{1-x^2}$. Since we know

$$Ł[1/\sqrt{1-\diamond^2}](x) = -\log 2$$

we have $D = -1/\log 2$. Let's check:

```
x = Fun()
u = -1/(log(2)*sqrt(1-x^2))
logkernel(u, 0.1) # approximately 0.1
```

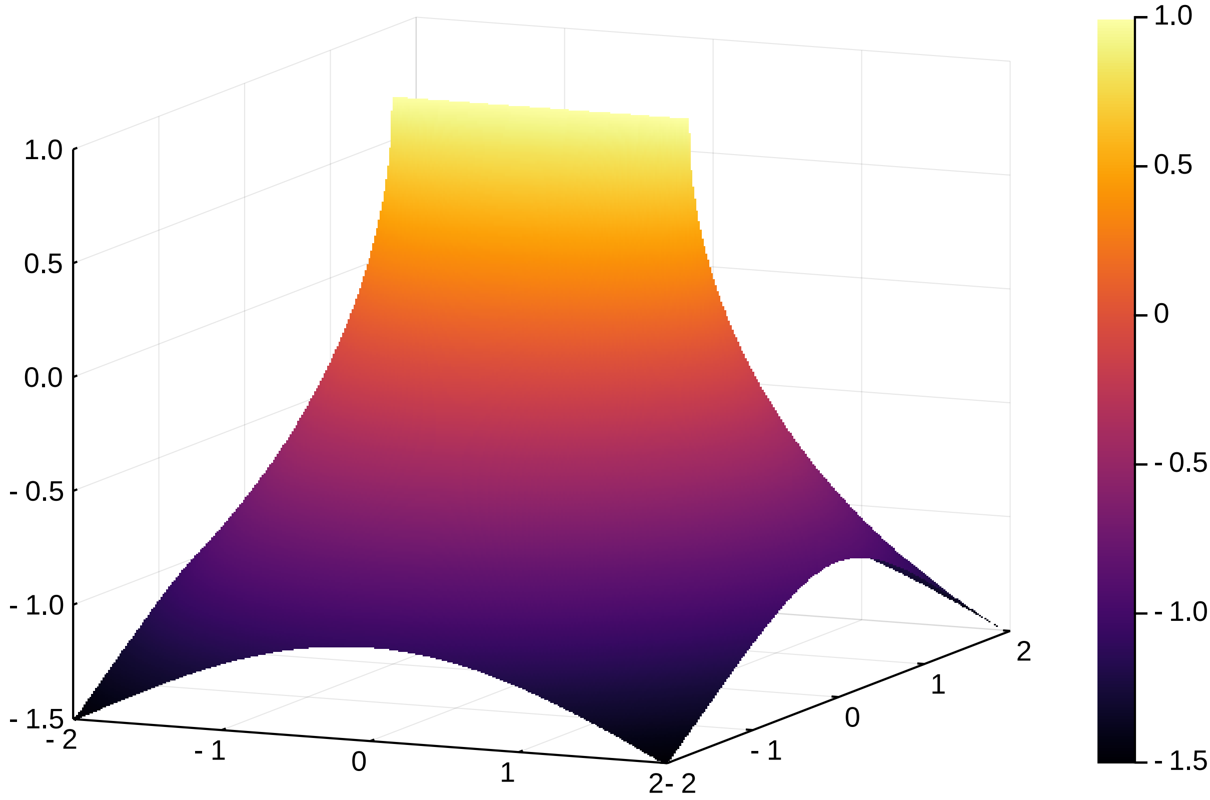
```
1.0
```

Physically, this solution gives us the potential field \$ \LL u(z) \$ corresponding to holding a metal plate at constant potential:

```
v = z -> logkernel(u, z)

xx = yy = -2:0.01:2
V = v.(xx' .+ im*yy)

surface(xx, yy, V)
```



1.1 Application: Potential arising from a point charge and a single plate

Now imagine we put a point source at $x = 2$, and a metal plate on $[-1, 1]$. We know the potential on the plate must be constant, but we don't know what constant. This is equivalent to the following problem (see e.g. [Chapman, Hewett & Trefethen 2015]):

$$\begin{aligned}
v_{xx} + v_{yy} &= 0 && \text{for} && z \text{ off } [-1,1] \text{ and } 2 \\
v(z) &\sim \log|z-2| + O(1) && \text{for} && z \to 2 \\
v(z) &\sim \log|z| + o(1) && \text{for} && z \to \infty \\
v(x) &= \kappa && \text{for} && -1 < x < 1
\end{aligned}$$

where κ is an unknown constant. We write the solution as

$$v(z) = \frac{1}{\pi}\int_{-1}^{1} u(t)\log|t-z|\mathrm{d}t + \log|z-2|$$

for a to-be-determined u. On $-1 < x < 1$ this satisfies

$$\frac{1}{\pi}\int_{-1}^{1} u(t)\log|t-x|\mathrm{d}t = \kappa - \log(2-x)$$

We can solve this equation explicitly. By differentiating we see that u satisfies the following:

$$\mathcal{H}u(x) = f'(x) = \frac{1}{2-x}$$

We now use the inverse Hilbert formula to determine:

$$u(x) = \frac{1}{\sqrt{1-x^2}}\mathcal{H}[\frac{1}{\diamond - 2}\sqrt{1-\diamond^2}](x) + \frac{D}{\sqrt{1-x^2}}$$

Using the usual procedure of taking an obvious ansatz and subtracting off the singularities at poles and ∞ we find that:

$$\mathcal{C}[\frac{\sqrt{1-\diamond^2}}{x-2}](z) = \underbrace{\frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(z-2)}}_{\text{ansatz}} - \underbrace{\frac{\sqrt{3}}{2\mathrm{i}(z-2)}}_{\text{remove pole near } z=2} - \underbrace{\frac{1}{2\mathrm{i}}}_{\text{remove constant at } \infty}$$

This implies that

$$\mathcal{H}[\frac{1}{\diamond-2}\sqrt{1-\diamond^2}](x) = -\mathrm{i}(C^+ + C^-)[\frac{1}{\diamond-2}\sqrt{1-\diamond^2}](x) = \frac{\sqrt{3}}{x-2} + 1$$

in other words,

$$u = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{D}{\sqrt{1-x^2}}$$

We still need to determine D. This is chosen so that we tend to $\log|z|$ near ∞. In particular, we know that

$$\frac{1}{\pi}\int_{-1}^{1}\log|z-x|u(x)\mathrm{d}x \sim \frac{\int_{-1}^{1} u(x)\mathrm{d}x}{\pi}\log|z|$$

so we need to choose D so that $\int_{-1}^{1} u(x)\mathrm{d}x = 0$ and only the $\log|z-2|$ singularity appears near ∞. We first note that

$$C[\frac{1}{(\diamond-2)\sqrt{1-\diamond^2}}](z) = \underbrace{\frac{\mathrm{i}}{2\sqrt{z-1}\sqrt{z+1}(z-2)}}_{\text{ansatz}} - \underbrace{\frac{\mathrm{i}}{2\sqrt{3}(z-2)}}_{\text{remove pole}}$$
$$= -\frac{\mathrm{i}}{2\sqrt{3}z} + O(z^{-2})$$

near ∞. Since

$$Cw(z) = \frac{\int_{-1}^{1} w(x)\mathrm{d}x}{-2\pi\mathrm{i}z} + O(z^{-2})$$

we can infer the integral from the Cauchy transform and find that

$$\int_{-1}^{1} \frac{1}{(x-2)\sqrt{1-x^2}} \mathrm{d}x = -\frac{\pi}{\sqrt{3}}$$

On the other hand,

$$\int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi$$

Thus we require $D = 1$ and have the solution:

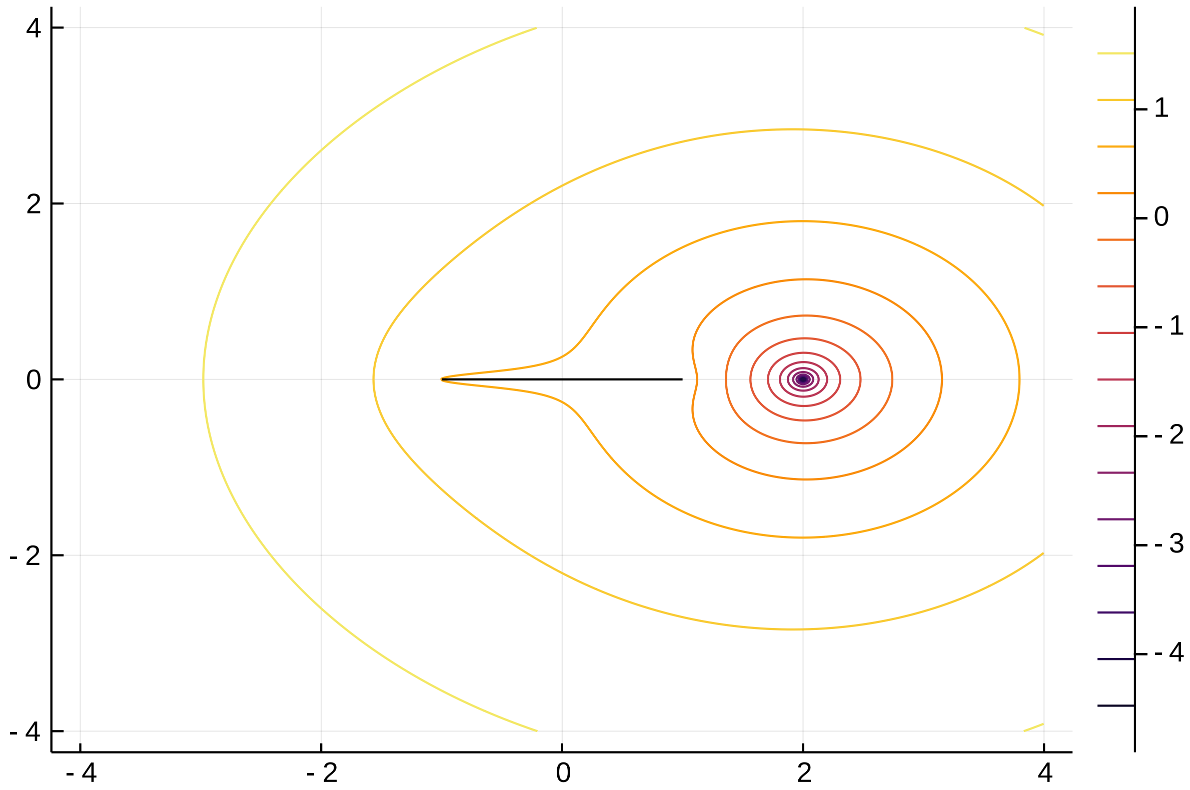
$$u(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

Demonstration

```
x = Fun()
u = sqrt(3)/((x-2)*sqrt(1-x^2)) + 1/sqrt(1-x^2)
v = z -> logkernel(u, z) + log(abs(z-2))

xx = yy = -4:0.011:4
V = v.(xx' .+ im*yy)

contour(xx, yy, V)
plot!(domain(x); color=:black, legend=false)
```



We can compute κ numerically:

```
v(0.2)
```

```
0.6238107163648714
```