M3M6: Applied Complex Analysis

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1 Solution Sheet 3

1.1 Problem 1.1

Since $\int_{-1}^1 x\mathrm{d}x = 0$, we have no logarithmic growth at infinity. Thus by the formula relating log to Cauchy transforms we find for

$$F(x) = \int_x^1 t\mathrm{d}t = \frac{1-x^2}{2}$$

that

$$\int_{-1}^1 \log|z-x| x\mathrm{d}x = \Re\left(-2\mathrm{i}\pi \mathcal{C}F(z)\right)$$

So we just have to work out the Cauchy transform. Try as an ansatz

$$\frac{1-z^2}{2}\frac{\log(z-1)-\log(z+1)}{2\pi\mathrm{i}}$$

We only need to remove the growth at infinity. We do so via:

$$\log(z-1)-\log(z+1) = -\frac{2}{z} + O(z^{-3})$$

telling us

$$\mathcal{C}F(z) = \frac{1-z^2}{2}\frac{\log(z-1)-\log(z+1)}{2\pi\mathrm{i}} + \frac{z}{2\pi\mathrm{i}}$$

and

$$\int_{-1}^1 \log|z-x| x\mathrm{d}x = \Re\left(\frac{1-z^2}{2}(\log(z-1)-\log(z+1)) - z\right)$$

Verification

We can confirm the formula:

```
using ApproxFun, SingularIntegralEquations, Plots, QuadGK, LinearAlgebra,
SpecialFunctions
import ApproxFunOrthogonalPolynomials: Recurrence

H(f,x) = -hilbert(f,x)
H(f) = -hilbert(f)

z = 2+im

x = Fun()
π*logkernel(x,z) , real((1-z^2)/2 * (log(z-1)-log(z+1)) - z)
```

```
(-0.2679858257813376, -0.26798582578133745)
```

1.2 Problem 1.2

From the hint and the fact that $\operatorname{asin} 1 = \pi/2$ we have

$$F(x) = \frac{\pi}{4} - \frac{x\sqrt{1-x^2} + \operatorname{asin} x}{2} = -\frac{\operatorname{acos} x + x\sqrt{1-x^2}}{2}$$

and (for $f(x) = \sqrt{1-x^2}$)

$$\int_{-1}^{1} f(x)\mathrm{d}x = F(-1) = \frac{\pi}{2}$$

Thus we need to dermine the Cauchy transform of $F(x)$. Using the usual techniques of subtracting out the growth at infinity

$$\mathcal{C}[x\sqrt{1-x^2}](z) = \frac{z\sqrt{z-1}\sqrt{z+1} - z^2 + 1/2}{2\mathrm{i}}$$

Using the hint allows us to related the Cauchy transform of $\operatorname{asin} x$ to that of $\operatorname{acos} x$. Recall from lectures

$$\mathcal{C}\operatorname{acos}(z) = \frac{\log(\sqrt{z-1} + \sqrt{z+1})}{\mathrm{i}} - \frac{\log(z+1)}{2\mathrm{i}} + \mathrm{i}\log 2$$

Thus we have

$$\mathcal{C}F(z) = -\frac{\log(\sqrt{z-1} + \sqrt{z+1})}{2\mathrm{i}} + \frac{\log(z+1)}{4\mathrm{i}} - \mathrm{i}\frac{\log 2}{2} - \frac{z\sqrt{z-1}\sqrt{z+1} - z^2 + 1/2}{4\mathrm{i}}$$

And

$$Mf(z) = \frac{\log(z+1)}{2} - 2\mathrm{i}\mathcal{C}F(z) = \log(\sqrt{z-1} + \sqrt{z+1}) - \log 2 - \frac{z\sqrt{z-1}\sqrt{z+1} - z^2 + 1/2}{2}$$

taking the real part and multiplying by π gives the answer.

Verification

```
x = Fun()
f = sqrt(1-x^2)

z = 1+im
π*logkernel(f, z), π*real(log(sqrt(z-1)+sqrt(z+1))-log(2) -
(z*sqrt(z-1)sqrt(z+1)-z^2+1/2)/2)
```

```
(0.556055856991731, 0.556055856991731)
```

1.3 Problem 1.3

We want to solve

$$\int_{-1}^{1} \log|x-t| u(t)\mathrm{d}t = \frac{1}{x^2+1}$$

differentiating and multiplying though by $1/\pi$ we have

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{u(t)}{x-t}\mathrm{d}t = -\frac{2x}{\pi(1+x^2)^2}$$

This is an inverse Hilbert problem so we know

$$u(x) = -\frac{1}{\sqrt{1-x^2}}H[\sqrt{1-t^2}f(t)](x) + \frac{C}{\sqrt{1-x^2}}$$

for

$$f(x) = -\frac{2x}{\pi(1+x^2)^2}.$$

We determine the Cauchy transform of $\sqrt{1-x^2}f(x)$ using the usual methods: start with the ansatz

$$\phi(z) = -\frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}}\frac{2z}{\pi(1+z^2)^2}$$

This decays at infinity so we just need to remove the poleas at $\pm\mathrm{i}$. Here we determine:

$$\phi(z) = -\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{\pi}\left(-\frac{1}{4(z-\mathrm{i})^2}+\frac{\mathrm{i}}{8(z-\mathrm{i})}+O(1)\right)$$

and

$$\phi(z) = -\frac{\sqrt{-\mathrm{i}-1}\sqrt{1-\mathrm{i}}}{\pi}\left(\frac{1}{4(z+\mathrm{i})^2}+\frac{\mathrm{i}}{8(z+\mathrm{i})}+O(1)\right)$$

Telling us that

$$\begin{aligned}C[\sqrt{1-t^2}f(t)](z) = -\frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}}\frac{2z}{\pi(1+z^2)^2} + \frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{\pi}\left(-\frac{1}{4(z-\mathrm{i})^2}+\frac{\mathrm{i}}{8(z-\mathrm{i})}\right)\\ + \frac{\sqrt{-\mathrm{i}-1}\sqrt{1-\mathrm{i}}}{\pi}\left(\frac{1}{4(z+\mathrm{i})^2}+\frac{\mathrm{i}}{8(z+\mathrm{i})}\right)\end{aligned}$$

Thus we have

$$\begin{aligned}H[\sqrt{1-t^2}f](x) &= -\mathrm{i}(C^+ + C^-)[\sqrt{1-t^2}f](x)\\ &= -\frac{2\mathrm{i}\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{\pi}\left(-\frac{1}{4(x-\mathrm{i})^2}+\frac{2\mathrm{i}}{8(x-\mathrm{i})}\right)\\ &\quad - \frac{2\mathrm{i}\sqrt{-\mathrm{i}-1}\sqrt{1-\mathrm{i}}}{\pi}\left(\frac{1}{4(x+\mathrm{i})^2}+\frac{\mathrm{i}}{8(x+\mathrm{i})}\right)\end{aligned}$$

In other words, for

$$\tilde{u}(x) = \frac{2\mathrm{i}\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{\pi\sqrt{1-x^2}}\left(-\frac{1}{4(x-\mathrm{i})^2} + \frac{2\mathrm{i}}{8(x-\mathrm{i})}\right) - \frac{2\mathrm{i}\sqrt{-\mathrm{i}-1}\sqrt{1-\mathrm{i}}}{\pi\sqrt{1-x^2}}\left(\frac{1}{4(x+\mathrm{i})^2} + \frac{\mathrm{i}}{8(x+\mathrm{i})}\right)$$

we have

$$u(x) = \tilde{u}(x) + \frac{C}{\sqrt{1-x^2}}.$$

To find C we impose the condition that

$$\int_{-1}^{1} \log|t| u(t) \mathrm{d}t = 1$$

We thus need to determine C. Recall that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\log(z-x)}{\sqrt{1-x^2}} \mathrm{d}x = 2\log(\sqrt{z-1} + \sqrt{z+1}) - 2\log 2$$

Thus for $x \in [-1,1]$ we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-t|}{\sqrt{1-t^2}} \mathrm{d}t = 2\Re(\log(\mathrm{i}\sqrt{1-x} + \sqrt{x+1}) - 2\log 2)$$

or in particular

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\log|x|}{\sqrt{1-x^2}} \mathrm{d}x = \log(1/2)$$

Thus we want to solve

$$\int_{-1}^{1} \log|t| \tilde{u}(t) \mathrm{d}t + C\frac{\log(1/2)}{\pi} = 1$$

i.e.

$$C = (1 - \int_{-1}^{1} \log|t| \tilde{u}(t) \mathrm{d}t)\frac{1}{\pi\log(1/2)}$$

Check derivation

Let's check the derivation. First we can calculate u numerically:

```
L = SingularIntegral(0) : JacobiWeight(-0.5,-0.5,Chebyshev())

x = Fun()
g = 1/(x^2+1)
u = (π*L) \ g

π*logkernel(u, 0.1) , g(0.1)
```

```
(0.9900990099009912, 0.9900990099009914)
```

Differentiating it satisfies $Hu = g'/(\pi)$:

```
H(u,0.1) , g'(0.1)/(π) , -(2*0.1)/(π*(1+0.1^2)^2)
```

```
(-0.062407584782646346, -0.062407584782646346, -0.062407584782627326)
```

The Cauchy transform also works:

```
f = g'/(π)

z = 1+im
ψ = z -> -sqrt(z-1)sqrt(z+1)/(2im) * 2z/(π*(1+z^2)^2) +
               sqrt(im-1)sqrt(im+1)/π * (-1/(4*(z-im)^2) + im/(8(z-im))) +
               sqrt(-im-1)sqrt(1-im)/π * (1/(4*(z+im)^2) + im/(8(z+im)))

cauchy(sqrt(1-x^2)*f, z) , ψ(z)
```

```
(-0.009150867763797383 + 0.0018378879027817 1im, -0.009150867763797393 + 0.0
018378879027817im)
```

Therefore the Hilbert transform is given by:

```
Hf = x-> - 2im*sqrt(im-1)sqrt(im+1)/π * (-1/(4*(x-im)^2) + im/(8(x-im))) -
           2im*sqrt(-im-1)sqrt(1-im)/π * (1/(4*(x+im)^2) + im/(8(x+im)))

H(sqrt(1-x^2)*f)(0.1) , Hf(0.1)
```

```
(0.21402480802677823, 0.21402480802676035 + 0.0im)
```

We thus have an expression for u, we are just missing the constant:

```
ũ = x -> -Hf(x)/sqrt(1-x^2)

C = u(0.0) + Hf(0.0)
u(0.1) ≈ ũ(0.1) + C/sqrt(1-0.1^2)
```

```
true
```

We can find C interms of the relevant integral, which we call D:

```
D = quadgk(x -> x == 0 ? 0.0 : ũ(x)*log(abs(x)),-1,1)[1]
C, (1-D)/(π*log(1/2))
```

```
(-0.3247204711377926 + 0.0im, -0.32472047136213555 + 0.0im)
```

1.4 Problem 2

1.4.1 Problem 2.1

From Lecture 19 we know that we want to solve

1. $$v_{xx} + v_{yy} = 0$$

   for $z \notin [-1, 1] \cup \mathrm{i}$

2. $$v(z) \sim \log |z|$$

   as $z \to \infty$

3.
$$v(z) \sim \log|z - \mathrm{i}|$$

as $z \to \mathrm{i}$

4.
$$v(x, 0) = D$$

for some unknown constant D on $[-1, 1]$.

1.4.2 Problem 2.2

We write

$$v(x, y) = \int_{-1}^{1} u(t) \log|z - t| \mathrm{d}t + \log|z - \mathrm{i}|$$

The behaviour at infinity requires that $\int_{-1}^{1} u(t)\mathrm{d}t = 0$. We further have the singular integral equation

$$\int_{-1}^{1} u(t) \log|x - t| \mathrm{d}t = C - \log|x - \mathrm{i}|$$

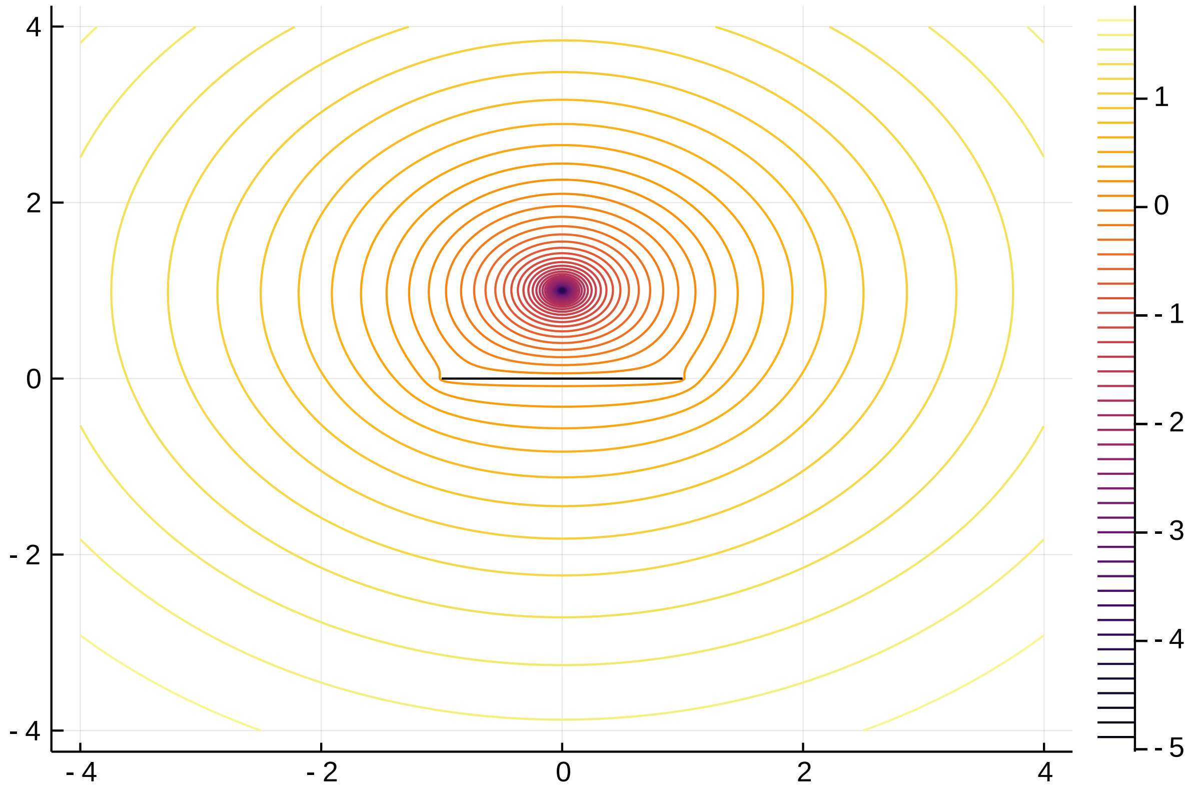
which follows from $D = v(x, 0)$

We can actually solve this numerically:

```
w = SingularIntegral(0) \ (-log(abs(x-im))/π)
u = w - sum(w)/(π*sqrt(1-x^2)) # ensure integrates to zero
v = z -> π*logkernel(u, z) + log(abs(z-im))
D = v(0)

xx = yy = -4:0.011:4
V = v.(xx' .+ im*yy)

contour(xx, yy, V; nlevels=50)
plot!(domain(x); color=:black, legend=false)
```



Problem 2.3 As usual for logarithmic singular integral equations we want to solve

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{u(t)}{x-t}\mathrm{d}t=\frac{f'(x)}{\pi}$$

we need to be a bit careful differentiating f:

$$\frac{\mathrm{d}}{\mathrm{d}x}\log|x-\mathrm{i}|=\frac{\mathrm{d}}{\mathrm{d}x}\log\sqrt{x^2+1}=\frac{x}{x^2+1}$$

Now we do our usual game and solve for u using the inverse Hilbert transform formula. That is, first calculate

$$C[\sqrt{1-x^2}\frac{x}{x^2+1}](z)=\frac{z\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(z^2+1)}-\frac{1}{2\mathrm{i}}+\frac{\mathrm{i}\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4(z-\mathrm{i})}+\frac{\mathrm{i}\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4(z+\mathrm{i})}$$

Therefore, we know that

$$u(x)=\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{2\pi(x-\mathrm{i})\sqrt{1-x^2}}+\frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{2\pi(x+\mathrm{i})\sqrt{1-x^2}}+\frac{C}{\sqrt{1-x^2}}$$

We can show (e.g. by finding its Cauchy transform and lookin at the asymptotic behaviour) that

$$\int_{-1}^{1}\left[\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{2\pi(x-\mathrm{i})\sqrt{1-x^2}}+\frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{2\pi(x+\mathrm{i})\sqrt{1-x^2}}\right]=-1$$

Combined with the fact that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} = \pi$$

we find that $C = 1/\pi$.

Verification Let's check our work. First we see that the Hilbert transform of u does indeed satisfy the specified equation (using the numerical `u` as calculated above):

```
fp = x -> -x/(x^2+1)
H(u, 0.1) , fp(0.1)/π
```

```
(-0.03151583031523278, -0.031515830315226805)
```

And the Cauchy transform of $\sqrt{1-x^2}f'(x)$ satisfies the derived formula:

```
Cf = z -> z * sqrt(z-1)sqrt(z+1)/(2im*(z^2+1)) - 1/(2im) +
im*sqrt(im-1)sqrt(im+1)/(4(z-im)) +
         im*sqrt(-im-1)sqrt(-im+1)/(4(z+im))
-cauchy(sqrt(1-x^2)fp(x), z) , Cf(z)
```

```
(0.02014804460385935 - 0.004468734515943387im, 0.02014804460385937 - 0.0044
68734515943457im)
```

Therefore we can invert to the Hilbert transform for f':

```
w = -H(sqrt(1-x^2)fp(x))/sqrt(1-x^2)
w2 = sqrt(im-1)sqrt(im+1)/(2(x-im) * sqrt(1-x^2)) +
         sqrt(-im-1)sqrt(-im+1)/(2(x+im) * sqrt(1-x^2))
H(w,0.1) , H(w2,0.1) , fp(0.1)
```

```
(-0.09900990099009971, -0.09900990099010008 - 0.0im, -0.09900990099009901)
```

And we have recovered u up to $C/\sqrt{1-x^2}$:

```
C = 1/π

u(0.1) , w2(0.1)/π + C/sqrt(1-0.1^2)
```

```
(-0.1280330340643996, -0.12803303406431615 + 0.0im)
```

1.5 Problem 3.1

We actually start by showing the second properties of Problem 3.2, for all α:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha}\mathrm{e}^{-x}L_n^{(\alpha)}(x)\right] &= \frac{1}{n!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\ &= (n+1)x^{\alpha-1}\mathrm{e}^{-x}\frac{x^{1-\alpha}\mathrm{e}^{x}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\ &= (n+1)x^{\alpha-1}\mathrm{e}^{-x}L_{n+1}^{(\alpha-1)}(x).\end{aligned}$$

Expanding out the derivative we see

$$x^{\alpha-1}\mathrm{e}^{-x}\left((\alpha-x)L_n^{(\alpha)}(x) + (L_n^{(\alpha)})'(x)\right) = (n+1)x^{\alpha-1}\mathrm{e}^{-x}L_{n+1}^{(\alpha-1)}(x)$$

or in other words

$$(\alpha - x)L_n^{(\alpha)}(x) + (L_n^{(\alpha)})'(x) = (n+1)L_{n+1}^{(\alpha-1)}(x)$$

By induction with the fact $L_0^{(\alpha)}(x) = 0$, we therefore get

$$L_n^{(\alpha)}(x) = \frac{(\alpha + 1 - x)L_{n-1}^{(\alpha+1)}(x) + (L_n^{(\alpha+1)})'(x)}{n}$$

is a degree n polynomial. We further have that the leading coefficient is

$$\begin{aligned} L_n^{(\alpha)}(x) &= -\frac{x}{n}L_{n-1}^{(\alpha+1)}(x) + O(x^{n-1}) = \frac{x^2}{n(n-1)}L_{n-2}^{(\alpha+2)}(x) + O(x^{n-1}) = \cdots \\ &= \frac{(-1)^n x^n}{n!}L_0^{(\alpha+n)}(x) + O(x^{n-1}) \\ &= \frac{(-1)^n x^n}{n!} + O(x^{n-1}) \end{aligned}$$

We now show orthogonality with lower degree polynomials using integration by parts:

$$\int_0^\infty L_n^{(\alpha)}(x)p_m(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = \int_0^\infty \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+n}\mathrm{e}^{-x}\right]p_m(x)\mathrm{d}x = (-1)^n\int_0^\infty \frac{1}{n!}\left[x^{\alpha+n}\mathrm{e}^{-x}\right]p_m^{(n)}(x)\mathrm{d}x = 0$$

since $p_m^{(n)}(x) = 0$. Note we use the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] = O(x^{\alpha+n-k})$$

hence vanishes at zero to ignore the boundary terms in integration by parts.

1.6 Problem 3.2

We showed the second property as part of 3.1. For the first part, it is clear that we have the correct constant. Now we show orthogonality with all degree $m < n-1$ polynomials (using the fact that $x^{\alpha+1}\mathrm{e}^{-x}$ is zero at $x = 0$):

$$\int_0^\infty \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x}p_m(x)x^{\alpha+1}\mathrm{e}^{-x}\mathrm{d}x = -\int_0^\infty L_n^{(\alpha)}(x)(xp_m'(x) + (\alpha+1)p_m - xp_m)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = 0$$

since $(xp_m'(x) + (\alpha+1)p_m - xp_m)$ is degree $m + 1 < n$.

For the third part, use the product rule on the last derivative:

$$\begin{aligned} (n+1)L_{n+1}^{(\alpha)}(x) &= \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+n+1}\mathrm{e}^{-x}\right] \\ &= \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(\alpha+n+1)x^{\alpha+n}\mathrm{e}^{-x} - x^{\alpha+n+1}\mathrm{e}^{-x}\right] \\ &= (\alpha+n+1)L_n^{(\alpha)}(x) - xL_n^{(\alpha+1)}(x) \end{aligned}$$

For the last result, we apply the product rule n times:

$$
\begin{aligned}
L_n^{(\alpha+1)}(x) &= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\mathrm{d}}{\mathrm{d}x}\left[x x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}x\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{2}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{3}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}x\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&\vdots \\
&= \frac{n}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= L_{n-1}^{(\alpha+1)}(x) + L_n^{(\alpha)}(x)
\end{aligned}
$$

1.7 Problem 3.3

Note that relationship 3 above did not depend on $\alpha > -1$. We therefore have from 3.2, comibing property (3) and (4),

$$
\begin{aligned}
xL_n^{(\alpha)}(x) &= -(n+1)L_{n+1}^{(\alpha-1)}(x) + (n+\alpha)L_n^{(\alpha-1)}(x) \\
&= -(n+1)L_{n+1}^{(\alpha)}(x) + (n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \\
&= -(n+\alpha)L_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)
\end{aligned}
$$

The Jacobi operator therefore has the form

$$
x\begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha+1 & -1 & & & & \\ -1-\alpha & \alpha+3 & -2 & & & \\ & -2-\alpha & \alpha+5 & -3 & & \\ & & -3-\alpha & \alpha+7 & -4 & \\ & & & -4-\alpha & \alpha+9 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}\begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \end{pmatrix}
$$

1.8 Problem 4.1

Note that

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x/2}u(x) = \mathrm{e}^{-x/2}(-\frac{u(x)}{2} + u'(x))
$$

Thus

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x/2}u(x) = \mathrm{e}^{-x/2}(u'(x) - \frac{u(x)}{2} - xu(x))
$$

We have the derivative operator from $L_k(x)$ to $L_k^{(1)}(x)$ as:

$$D = \begin{pmatrix} 0 & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$$

and the Multiplication operator for $\alpha = 0$ (from Problem 3.3)

$$J^\top = \begin{pmatrix} 1 & -1 & & \\ -1 & 3 & -2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

and the conversion operator (from Problem 3.2 property 4)

$$S = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \end{pmatrix}$$

We thus have multiplication by x from basis to the other as

$$SJ^\top = \begin{pmatrix} 2 & -4 & 2 & & & \\ -1 & 5 & -7 & 3 & & \\ & -2 & 8 & -10 & 4 & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Putting everything together, we get the operator

```
D = Derivative() :  Laguerre(0) → Laguerre(1)
S = I : Laguerre(0) → Laguerre(1)

Jt = Recurrence(Laguerre(0))

(D - S/2 -S*Jt)[1:10,1:10]
```

```
10×10 BandedMatrices.BandedMatrix{Complex{Float64},Array{Complex{Float64},2
},Base.OneTo{Int64}}:
 -2.5+0.0im   3.5+0.0im  -2.0+0.0im       ⋅      …        ⋅            ⋅

  1.0+0.0im  -5.5+0.0im   6.5+0.0im  -3.0+0.0im          ⋅            ⋅

      ⋅       2.0+0.0im  -8.5+0.0im   9.5+0.0im          ⋅            ⋅

      ⋅           ⋅       3.0+0.0im  -11.5+0.0im         ⋅            ⋅

      ⋅           ⋅           ⋅        4.0+0.0im         ⋅            ⋅

      ⋅           ⋅           ⋅            ⋅      …      ⋅            ⋅

      ⋅           ⋅           ⋅            ⋅          -8.0+0.0im       ⋅

      ⋅           ⋅           ⋅            ⋅          24.5+0.0im   -9.0+0.0
im
      ⋅           ⋅           ⋅            ⋅         -26.5+0.0im   27.5+0.0
im
      ⋅           ⋅           ⋅            ⋅           9.0+0.0im  -29.5+0.0
im
```

1.9 Problem 4.2

We have

$$\frac{e^x}{x^\alpha}\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{dL_n^{(\alpha)}}{dx}\right] = -\frac{e^x}{x^\alpha}\frac{d}{dx}\left[x^{\alpha+1}e^{-x}\frac{L_{n-1}^{(\alpha+1)}}{dx}\right]$$
$$= -nL_n^{(\alpha)}(x)$$

Therefore $\lambda_n = -n$. This can be expanded in the form:

$$x\frac{d^2L_n^{(\alpha)}}{dx^2} + (\alpha+1-x)\frac{dL_n^{(\alpha)}}{dx} = -nL_n^{(\alpha)}(x)$$

1.10 Problem 5.1

Define $C_k^{(\alpha)}(z) = \mathcal{C}[L_k^{(\alpha)} \diamond^\alpha e^{-\diamond}](z)$ and recall that

$$C_1(z) = \frac{\frac{1}{2\pi i}\int_0^\infty e^{-x}dx + (z-a_0)C_0(z)}{b_0} = -\frac{1}{2\pi i} - (z-1)C_0(z)$$
$$= \frac{(z-1)e^{-z}\operatorname{Ei} z - 1}{2\pi i}$$

Here we double check the formula, noting that $L_1(x) = e^x\frac{d}{dx}xe^{-x} = 1-x$:

```
const ei_−_1 = let ζ = Fun(-100 .. -1)
    sum(exp(ζ)/ζ)
end
function ei(z)
    ζ = Fun(Segment(-1 , z))
    ei_−_1 + sum(exp(ζ)/ζ)
end

x = Fun(0..10)
w = exp(-x)
z = 1+im
cauchy((1-x)*w, z),((z-1)*exp(-z)*ei(z)-1)/(2π*im)
```

```
(0.018684644298457904 + 0.04836133565335002im, 0.01868392300262891 + 0.0483
6848799089318im)
```

We now use these to determine the results with $\alpha = 1$. Note that:

$$C_0^{(1)}(z) = \mathcal{C}[\diamond e^{-\diamond}](z) = C_0(z) - C_1(z) = \frac{e^{-z}\operatorname{Ei} z - (z-1)e^{-z}\operatorname{Ei} z + 1}{2\pi i}$$

```
cauchy(x*w, z),(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1)/(2π*im)
```

```
(0.09210173751684987 - 0.029676691354892096im, 0.09210253209837325 - 0.0296
8456498826427im)
```

Therefore, we have

$$C_1^{(1)}(z) = \frac{\frac{1}{2\pi \mathrm{i}}\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x + (z - a_0^{(1)})C_0^{(1)}(z)}{b_0^{(1)}}$$
$$= \frac{\frac{1}{2\pi \mathrm{i}} + (z-2)C_0^{(1)}(z)}{-1}$$
$$= \frac{1 + (z-2)(\mathrm{e}^{-z}\mathrm{Ei}\, z - (z-1)\mathrm{e}^{-z}\mathrm{Ei}\, z + 1)}{-2\pi \mathrm{i}}$$

Let's check the result using

$$L_1^{(1)}(x) = x^{-1}\mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x^2 \mathrm{e}^{-x} = 2 - x$$

```
cauchy((2-x)*x*w, z),(1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π*im)
```

```
(0.06242504616195792 + 0.037297032364538574im, 0.06241796711010899 + 0.0373
6784600525782im)
```

1.11 Problem 5.2

We have

$$\int_x^\infty L_2(x)\mathrm{e}^{-x}\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{-x}L_1^{(1)}(x)\mathrm{e}^{-x}$$

Thus from lectures we have

$$\frac{1}{2\pi \mathrm{i}}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log(z-x)\mathrm{d}x = \frac{1}{2}\mathcal{C}[\diamond \mathrm{e}^{-\diamond} L_1^{(1)}](z)$$

and therefore

$$\frac{1}{\pi}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log|z-x|\mathrm{d}x = -\Im\mathcal{C}[\diamond \mathrm{e}^{-\diamond} L_1^{(1)}](z) = \Re\frac{1 + (z-2)(\mathrm{e}^{-z}\mathrm{Ei}\, z - (z-1)\mathrm{e}^{-z}\mathrm{Ei}\, z + 1)}{-2\pi}$$

Let's check the result:

```
x = Fun(0 .. 100)
w = exp(-x)
z = 2+im
```

```
-sum(1/2*(2 - 4x + x^2)*w*log(abs(z-x)))/π,imag(sum(1/2*(2 - 4x +
x^2)*w*log(z-x))/(π*im))
```

```
(-0.06972323453971352, -0.06972323453971373)
```

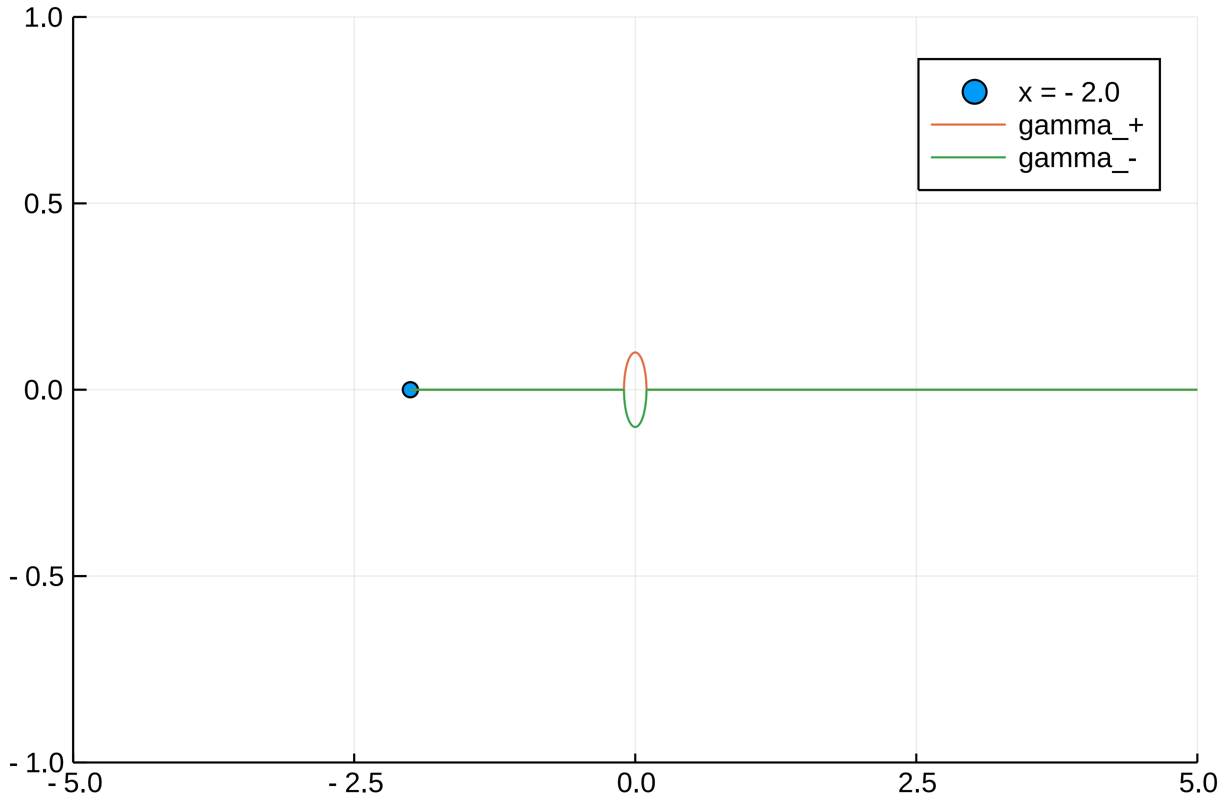
```
-imag(cauchy((2-x)*x*w,z)),real((1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π))
```

```
(-0.06972323454064278, -0.06972323454061675)
```

1.11.1 Problem 6.1

Consider integration contours γ_{+x} and γ_{-x} that avoid 0 above and below:

```
x = -2.0
r = 0.1
γ_+_x = Segment(-2.0 , -r) ∪ Arc(0.,r, (π,0)) ∪ Segment(r , 100)
γ_-_x = Segment(-2.0 , -r) ∪ Arc(0.,r, (-π,0)) ∪ Segment(r , 100)
scatter([x],[0.0];label="x = $x")
plot!(γ_+_x ; xlims=(-5,5), ylims=(-1,1), label="gamma_+")
plot!(γ_-_x; xlims=(-5,5), ylims=(-1,1), label="gamma_-")
```



So that

$$\Gamma_\pm(\alpha, x) = \int_{\gamma_{\pm x}} \zeta^{\alpha-1} \mathrm{e}^{-\zeta} \mathrm{d}\zeta$$

Note that

$$\int_x^{-r} (\zeta_+^{\alpha-1} - \zeta_-^{\alpha-1}) \mathrm{e}^{-\zeta} \mathrm{d}\zeta = 0$$

since $\zeta_+^{\alpha-1} = \mathrm{e}^{\pi \mathrm{i}(\alpha-1)} |\zeta|^{\alpha-1} = \mathrm{e}^{2\mathrm{i}\pi\alpha} \zeta_-^{\alpha-1}$. Furthermore, the integrals over the arcs tend to zero as $r \to 0$:

$$|\mathrm{i}r^\alpha \int_0^\pi \mathrm{e}^{-r\mathrm{e}^{\mathrm{i}\theta}} \mathrm{e}^{\mathrm{i}\theta\alpha} \mathrm{d}\theta| \leq r^\alpha \pi \mathrm{e}^r \to 0$$

and similarly on the lower arc. Thus we have

$$
\begin{aligned}
\Gamma_+(\alpha, x) - \mathrm{e}^{2\mathrm{i}\pi\alpha}\Gamma_-(\alpha, x) &= \lim_{r\to 0}\left(\int_{\gamma_{+x}} - \mathrm{e}^{2\mathrm{i}\pi\alpha}\int_{\gamma_{-x}}\right)\zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta \\
&= (1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x = (1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\Gamma(\alpha)
\end{aligned}
$$

1.12 Problem 6.2

Note that, for $0 < \alpha < 1$,

$$\psi(z) = z^{-\alpha}\mathrm{e}^z\Gamma(\alpha, z)$$

has the following properties:

1. $$\psi(z)$$

   decays as $z \to \infty$, via integration by parts:

   $$z^{-\alpha}\mathrm{e}^z\int_z^\infty \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta = z^{-1}\mathrm{e}^z + z^{-\alpha}\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta$$

and we have assuming z is bounded away from the negative real axis:

$$\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta| \leq \int_z^\infty |\zeta|^{\alpha-2}\mathrm{d}\zeta = \int_0^\infty |x+z|^{\alpha-2}\mathrm{d}x < \infty$$

(otherwise one would use a deformed contour).

2. We have the subtractive jump:

$$
\begin{aligned}
\psi_+(x) - \psi_-(x) &= \mathrm{e}^x(x_+^{-\alpha}\Gamma_+(\alpha, x) - \Gamma_-(\alpha, x)) \\
&= \mathrm{e}^x|x|^\alpha(\mathrm{e}^{-\mathrm{i}\pi\alpha}\Gamma_+(\alpha, z) - \mathrm{e}^{\mathrm{i}\pi\alpha}\Gamma_-(\alpha, x)) \\
&= \mathrm{e}^x|x|^\alpha\mathrm{e}^{-\mathrm{i}\pi\alpha}(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})
\end{aligned}
$$

We use these properties to verify that

$$\mathcal{C}[\diamond^\alpha\mathrm{e}^{-\diamond}](z) = \frac{1}{\Gamma(-\alpha)}\frac{(-z)^\alpha\mathrm{e}^{-z}\Gamma(-\alpha, -z)}{\mathrm{e}^{-\mathrm{i}\pi\alpha} - \mathrm{e}^{\mathrm{i}\pi\alpha}}$$

via Plemelj.

```
x = Fun(0 .. 20.0)
α = -0.1
z = 2.0+im
cauchy(x^α*exp(-x), z)

Γ = (α,z) -> let ζ = z + Fun(0 .. 500.0)
    linesum(ζ^(α-1)*exp(-ζ))
end

-(-z)^α*exp(-z)Γ(-α,-z)/(gamma(-α)*(exp(im*π*α)-exp(-im*π*α)))

0.0719987633150514 + 0.05850612396048858im
```