** M3M6: Methods of Mathematical Physics **

Dr. Sheehan Olver ¡br¿ s.olver@imperial.ac.uk

1 Lecture 2: Cauchy's theorem

1.1 Complex-differentiable functions

Definition (Complex-differentiable) Let $D \subset \mathbb{C}$ be an open set. A function $f : D \to \mathbb{C}$ is called *complex-differentiable* at a point $z_0 \in D$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, for any angle of approach to z_0.

1.2 Holomorphic functions

Definition (Holomorphic) Let $D \subset \mathbb{C}$ be an open set. A function $f : D \to \mathbb{C}$ is called *holomorphic* in D if it is complex-differentiable at all $z \in D$.

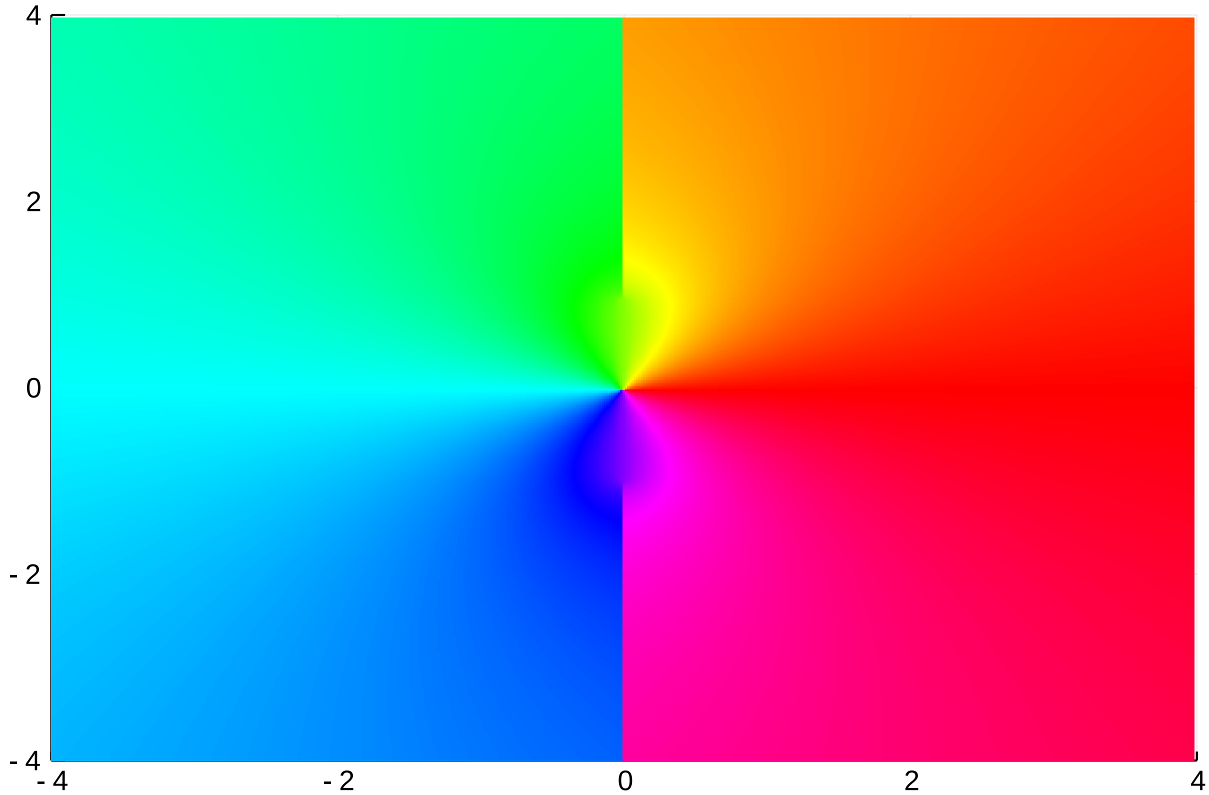
Definition (Entire) A function is *entire* if it is holomorphic in $\mathbb{C}$

Examples

1. $$1$$
   is entire
2. $$z$$
   is entire
3. $$1/z$$
   is holomorphic in $\mathbb{C}\backslash\{0\}$
4. $$\sin z$$
   is entire
5. $$\csc z$$
   is holomorphic in $\mathbb{C}\backslash\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$
6. $$\sqrt{z}$$
   is holomorphic in $\mathbb{C}\backslash(-\infty, 0]$

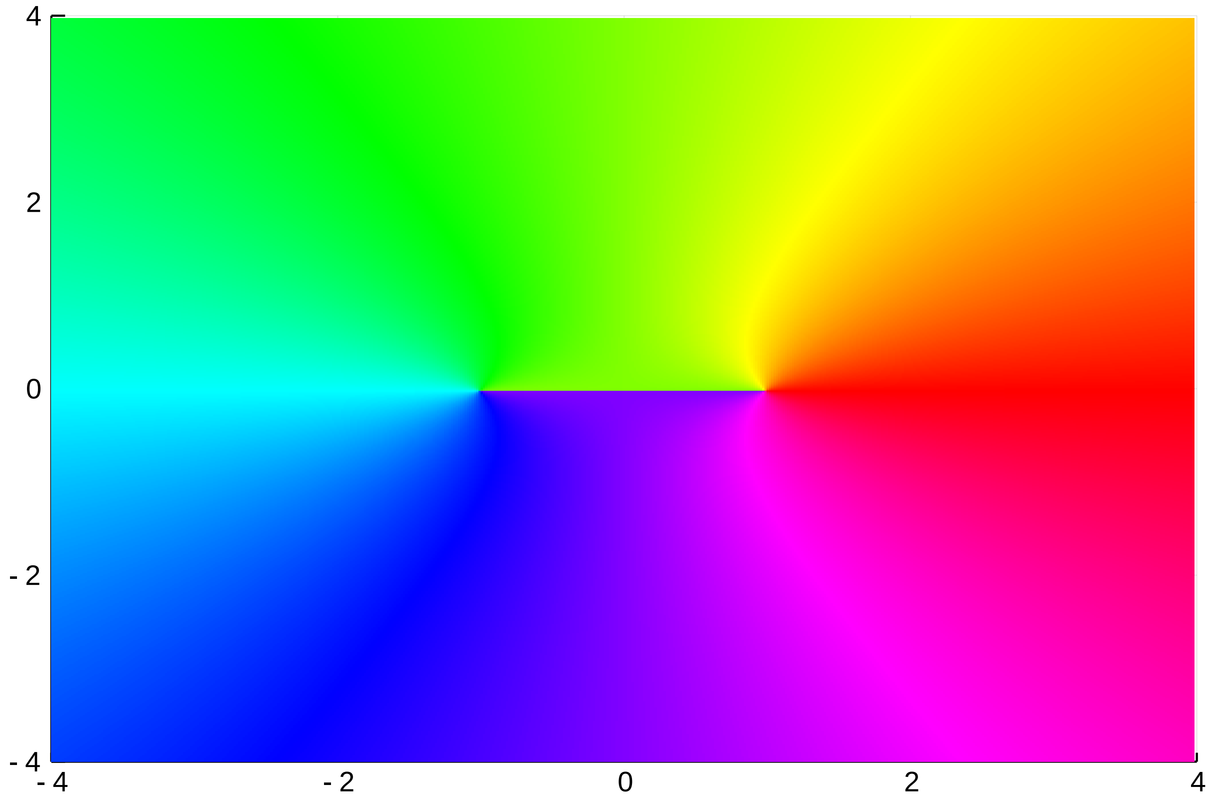
We can usually infer the domain where a function is holomorphic from a phase portrait, here we see that $\operatorname{arcsinh} z$ has cuts on $[\mathrm{i}, \mathrm{i}\infty)$ and $[-\mathrm{i}, -\mathrm{i}\infty)$, and a zero (red–green–blue–red) at zero, hence we can infer that it is holomorphic in $\mathbb{C}\backslash([\mathrm{i}, \mathrm{i}\infty) \cup [-\mathrm{i}, -\mathrm{i}\infty))$.

```
using Plots, ComplexPhasePortrait, SpecialFunctions, ApproxFun
phaseplot(-4..4, -4..4, z -> asinh(z))
```



The following example $\sqrt{z-1}\sqrt{z+1}$ is analytic in $\mathbb{C}\backslash[-1,1]$ and will be returned to:

```
phaseplot(-4..4, -4..4, z -> sqrt(z-1)sqrt(z+1))
```



1.3 Contours

Definition (Contour) A *contour* is a continuous & piecewise-continuously differentiable function $\gamma : [a, b] \to \mathbb{C}$.

Definition (Simple) A *simple contour* is a contour that is 1-to-1.

Definition (Closed) A *closed contour* is a contour such that $\gamma(a) = \gamma(b)$

Examples of contours

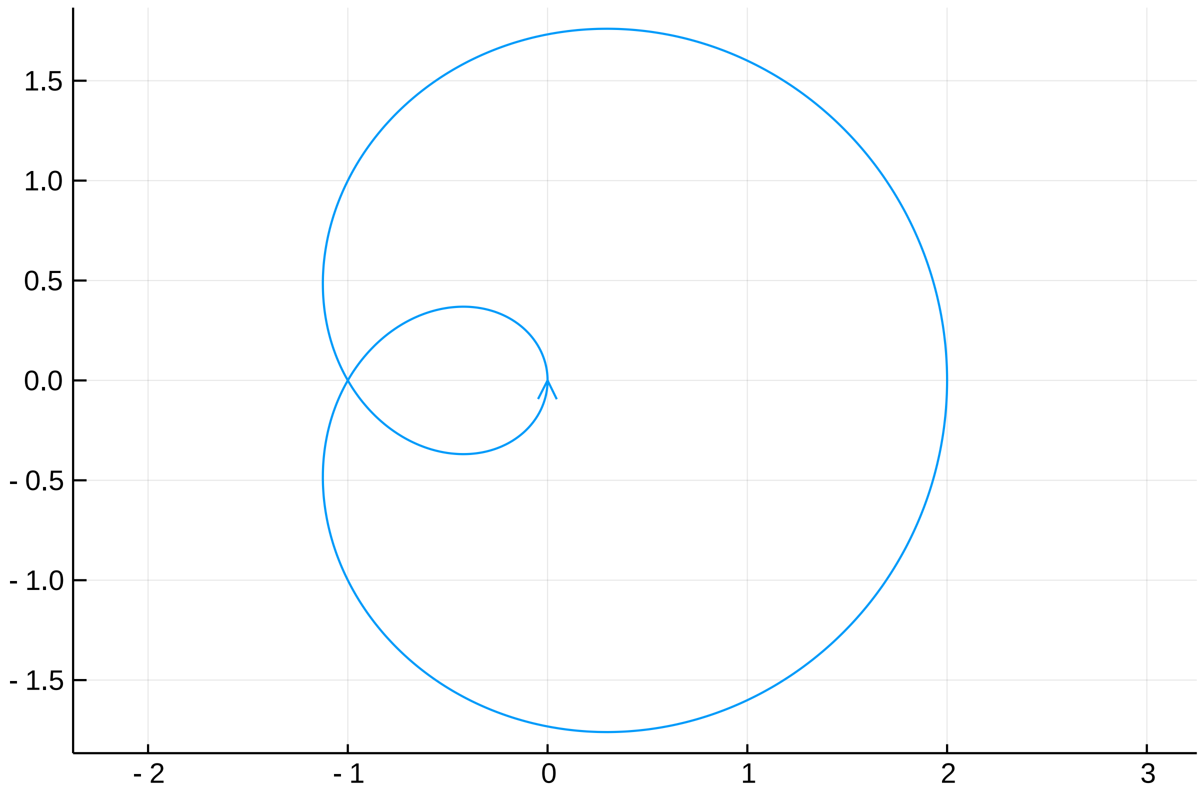
1. Line segment $[a, b]$ is a simple contour, with $\gamma(t) = t$
2. Arc from re^{ia} to re^{ib} is a simple contour, with $\gamma(t) = re^{it}$
3. Circle of radius r is a closed simple contour, with $\gamma(t) = re^{it}$ and $a = -\pi$, $b = \pi$
4. $$\gamma(t) = \cos(t + i)^2$$
   defines a contour that is not simple or closed
5. $$\gamma(t) = e^{it} + e^{2it}$$
   for $[a, b] = [-\pi, \pi]$ defines a contour that is closed but not simple

Here's an example of a closed contour that is not simple:

```
a,b = -π, π
tt = range(a, stop=b, length=1000)

γ = t -> exp(im*t) +exp(2im*t)

plot(real.(γ.(tt)), imag.(γ.(tt)); ratio=1.0, legend=false, arrow=true)
```



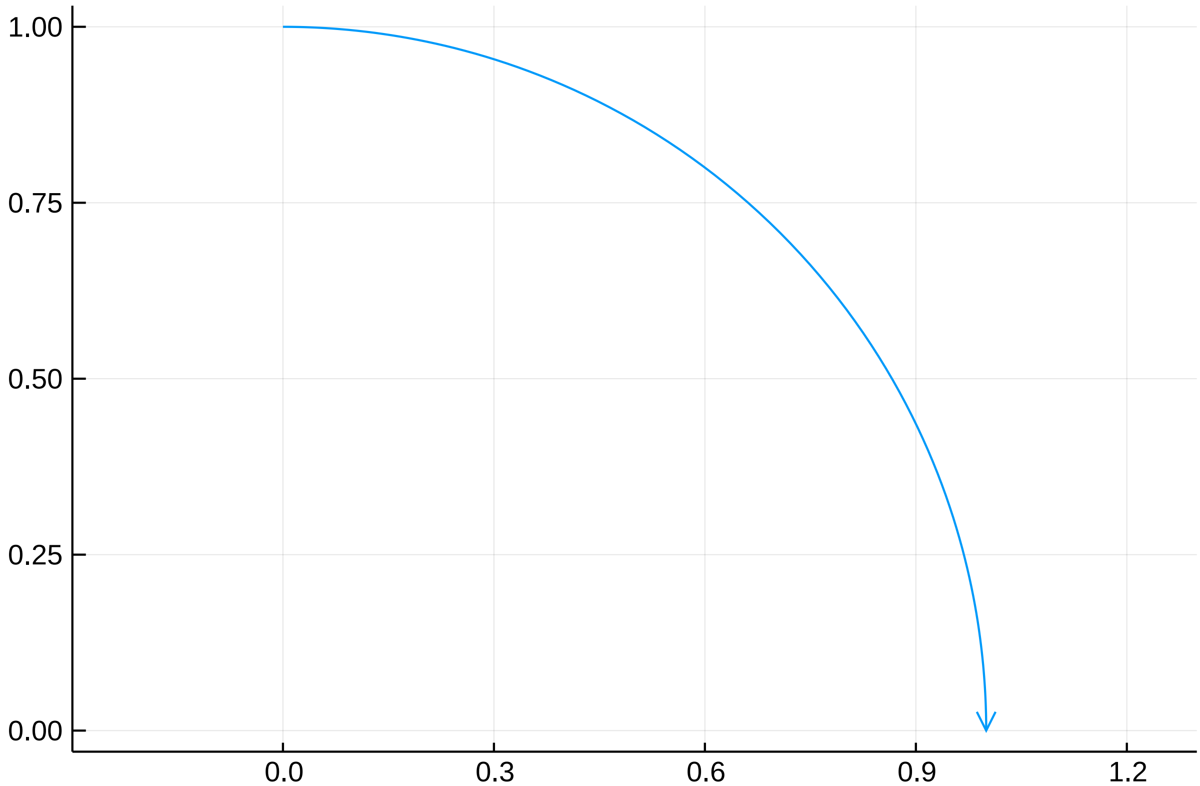
1.4 Contour integrals

Definition (Contour integral) The *contour integral* over γ is defined by

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

```
f = Fun( z -> real(exp(z)), Arc(0.,1.,(0,π/2)))  # Not holomorphic!

plot(domain(f); legend=false, ratio=1.0, arrow=true)
```




```
sum(f)  #this means contour integral
```

```
-1.2485382363935424 + 1.949326343919058im
```

```
g = im*Fun(t-> f(exp(im*t))*exp(im*t), 0 .. π/2)
sum( g )  # this is standard integral
```

```
-1.2485382363935429 + 1.9493263439190578im
```

An important property of a contour is its *arclength*:

Definition (Arclength) The *arclength* of γ is defined as

$$\mathcal{L}(\gamma) := \int_a^b |\gamma'(t)|dt$$

A very useful result is that we can use the maximum and arclength to bound integrals:

Proposition (ML) Let $f : \gamma \to \mathbb{C}$ and

$$M = \sup_{z\in\gamma} |f(z)|$$

Then

$$\left|\int_\gamma f(z)dz\right| \leq M\mathcal{L}(\gamma)$$

1.5 Cauchy's theorem

Proposition If $f(z)$ is holomorphic on γ, then $\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$

Theorem (Cauchy) If f is holomorphic inside and on a closed contour γ, then $\oint_\gamma f(z)dz = 0$

1.5.1 Demonstration

The following show that the contour of integration does not affect the integral for holomorphic functions:

```
f = Fun( z -> exp(z), Arc(0.,1.,(0,π/2))) #integrate over an arc
sum(f)  , f(im)-f(1)
```

```
(-2.1779795225909058 + 0.8414709848078968im, -2.177979522590906 + 0.8414709
848078968im)
```

```
f = Fun( z -> exp(z), Segment(1,im)) # integrate over a line segment
sum(f)  , f(im)-f(1)
```

```
(-2.1779795225909053 + 0.8414709848078966im, -2.177979522590907 + 0.8414709
848078968im)
```

The following demonstrates Cauchy's theorem, integration over a closed contour equals zero:

```
f = Fun( z -> exp(z), Circle())  #  Holomorphic!
sum(f)
```

```
2.9644937254112756e-17 - 1.4872544363724962e-16im
```

If `f` is not holomorphic, it doesn't apply:

```
f = Fun( z -> exp(z)/z, Circle())  # Not holomorphic at zero
sum(f)
```

```
-9.96316757509274e-16 + 6.283185307179588im
```

```
f = Fun( z -> real(exp(z)), Circle())  # Not holomorphic anywhere!
sum(f)
```

```
-3.3420237696193494e-16 + 3.141592653589793im
```