

M3M6: Applied Complex Analysis

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1 Lecture 4: Applications to real integrals

This lecture we discuss applications of residue calculus.

1. Trigonometric integrals
2. Integrals over real lines
 - Principal value integral
 - Cauchy's integral formula and Residue theorem on the real line
3. Oscillatory integrals
 - Jordan's lemma
 - Application: Calculating Fourier transforms of weakly decaying functions

1.1 Trigonometric

We can calculate integrals of the form

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta$$

where $R(x, y)$ is rational by doing the change of variables $z = e^{i\theta}$ to reduce it to

$$\oint_{C_1} R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz}$$

This is rational in z hence guaranteed to be amenable to residue calculus, provided we can find the poles of the denominator.

Example Consider

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\rho \cos \theta + \rho^2}$$

for $0 < \rho < 1$. We need to first locate the poles of

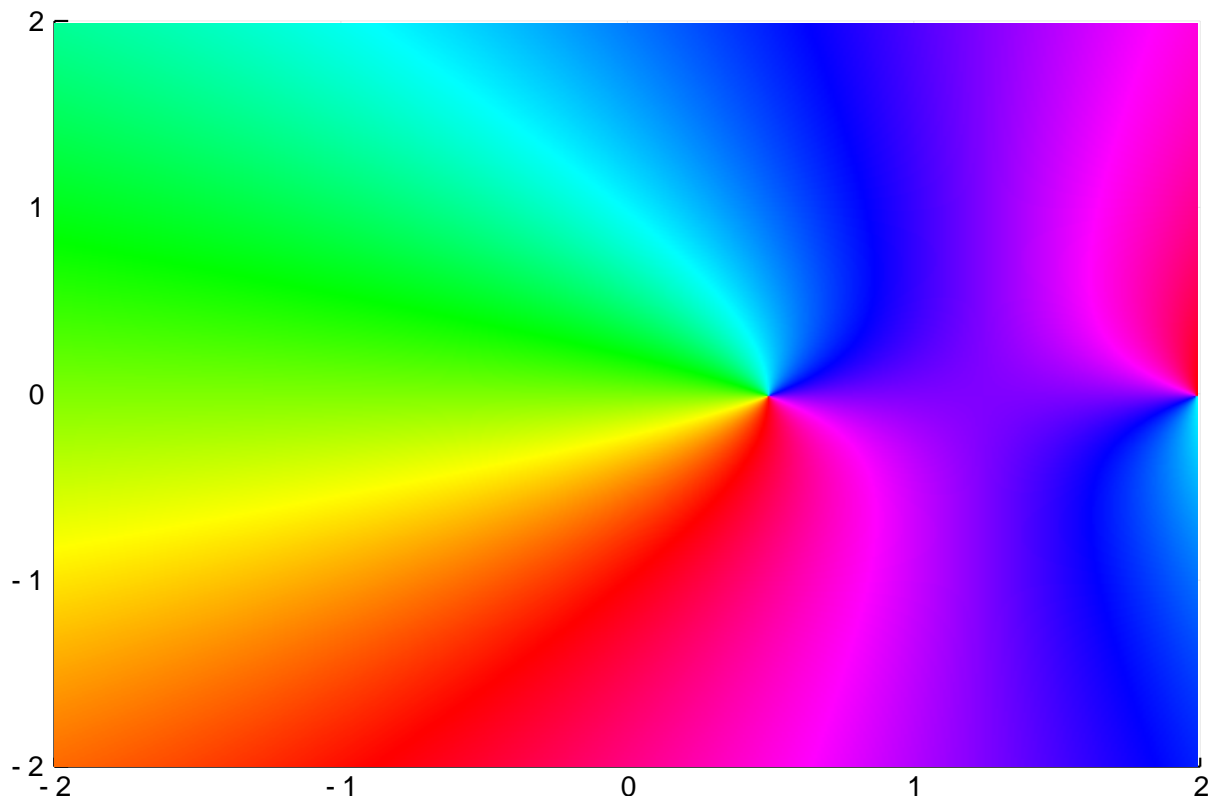
$$f(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{1}{iz} = \frac{i}{\rho z^2 - (1 + \rho^2)z + \rho}$$

This has poles at

$$z = \frac{1 + \rho^2 \pm \sqrt{(1 + \rho^2)^2 - 4\rho^2}}{2} = \rho, \rho^{-1}$$

We can confirm this with a phase plot:

```
using Plots, ComplexPhasePortrait
ρ = 0.5
phaseplot(-2..2, -2..2, z -> 1/(1-ρ*(z+(z^(-1)))) + ρ^2 * 1/(im*z))
```



Thus we can use either the interior or exterior residue calculus. Since we know then roots of the denominator of f we determine that

$$f(z) = \frac{i}{(z - \rho)(\rho z - 1)}$$

Thus we find

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\rho \cos \theta + \rho^2} = 2\pi i \operatorname{Res}_{z=\rho} f(z) = -2\pi i \operatorname{Res}_{z=1/\rho} f(z) = \frac{2\pi}{1 - \rho^2}.$$

1.2 Integrals over the real line

Integrals on the real line are typically viewed as improper integrals:

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} f(x) dx + \int_{-\infty}^0 f(x) dx = \lim_{b \rightarrow \infty} \int_0^b f(x) dx + \lim_{a \rightarrow -\infty} \int_a^0 f(x) dx.$$

It is convenient to work with a slightly different notion where we take the limit simultaneously:

Definition (Principal value integral on the real line) The *(Cauchy) principal value integral on the real line* is defined as

$$\oint_{-\infty}^{\infty} f(x) dx := \lim_{M \rightarrow \infty} \int_{-M}^M f(x) dx$$

This is a weaker concept:

Proposition (Integability \Rightarrow Prinipal value integrability) If $\int_{-\infty}^{\infty} f(x)dx < \infty$ then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Example Consider integrating $1/(x - i)$ with indefinite integral $\log(x - i)$. We use the Definition

$$\log z = \log |z| + i \arg z$$

where $-\pi \leq \arg z < \pi$. Thus we have

$$\begin{aligned} \lim_{b \rightarrow \infty} \log(b - i) &= \lim_{b \rightarrow \infty} \log |b - i| = \infty \\ \lim_{a \rightarrow -\infty} \log(a - i) &= \lim_{a \rightarrow -\infty} (\log |a - i| + i \arg(a - i)) = \infty - \pi i \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{x - i} = \lim_{a \rightarrow -\infty} [\log(-i) - \log(a - i)] + \lim_{b \rightarrow \infty} [\log(b - i) - \log(-i)] = -\infty + i\pi + \infty$$

is undefined. On the other hand, the Cauchy principal value integral gives

$$\int_{-\infty}^{\infty} \frac{dx}{x - i} = \lim_{M \rightarrow \infty} [\log(M - i) - \log(-M - i)] = \lim_{M \rightarrow \infty} [\log |M - i| - \log |-M - i|] + i\pi = i\pi.$$

1.2.1 Residue theorem on the real line

The real line doesn't have an *inside* and *outside*, rather an *above* and *below*, or *left* and *right*. Thus we get the following two versions of the Residue theorem:

Definition (Upper/lower half plane) Denote the upper/lower half plane by

$$\begin{aligned} \mathbb{H}^+ &= \{z : \Re z > 0\} \\ \mathbb{H}^- &= \{z : \Re z < 0\} \end{aligned}$$

The closure is denoted

$$\begin{aligned} \bar{\mathbb{H}}^+ &= \mathbb{H}^+ \cup \mathbb{R} \cup \{\infty\} \\ \bar{\mathbb{H}}^- &= \mathbb{H}^- \cup \mathbb{R} \cup \{\infty\} \end{aligned}$$

Theorem (Residue theorem on the real line) Suppose $f : \bar{\mathbb{H}}^+ \setminus \{z_1, \dots, z_r\} \rightarrow \mathbb{C}$ is holomorphic in $\mathbb{H}^+ \setminus \{z_1, \dots, z_r\}$, where $\Re z_k > 0$, and $\lim_{\epsilon \rightarrow 0} f(x + i\epsilon) = f(x)$ converges uniformly. If

$$\lim_{z \rightarrow \infty} z f(z) = 0$$

uniformly for $z \in \bar{\mathbb{H}}^+$, then

$$\oint_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{k=1}^r \operatorname{Res}_{z=z_k} f(z)$$

Similarly, if the equivalent conditions hold in the lower half plane for $f : \bar{\mathbb{H}}^- \setminus \{z_1, \dots, z_r\} \rightarrow \mathbb{C}$ then

$$\oint_{-\infty}^{\infty} f(x) dx = -2\pi i \sum_{k=1}^r \operatorname{Res}_{z=z_k} f(z)$$

Proof This follows by considering the contour $\gamma_R = [-R, R] \cup H_R$ where

$$H_R := \{R e^{i\theta} : 0 \leq \theta \leq \pi\},$$

that is, the upper-half circle. Classical residue calculus gives

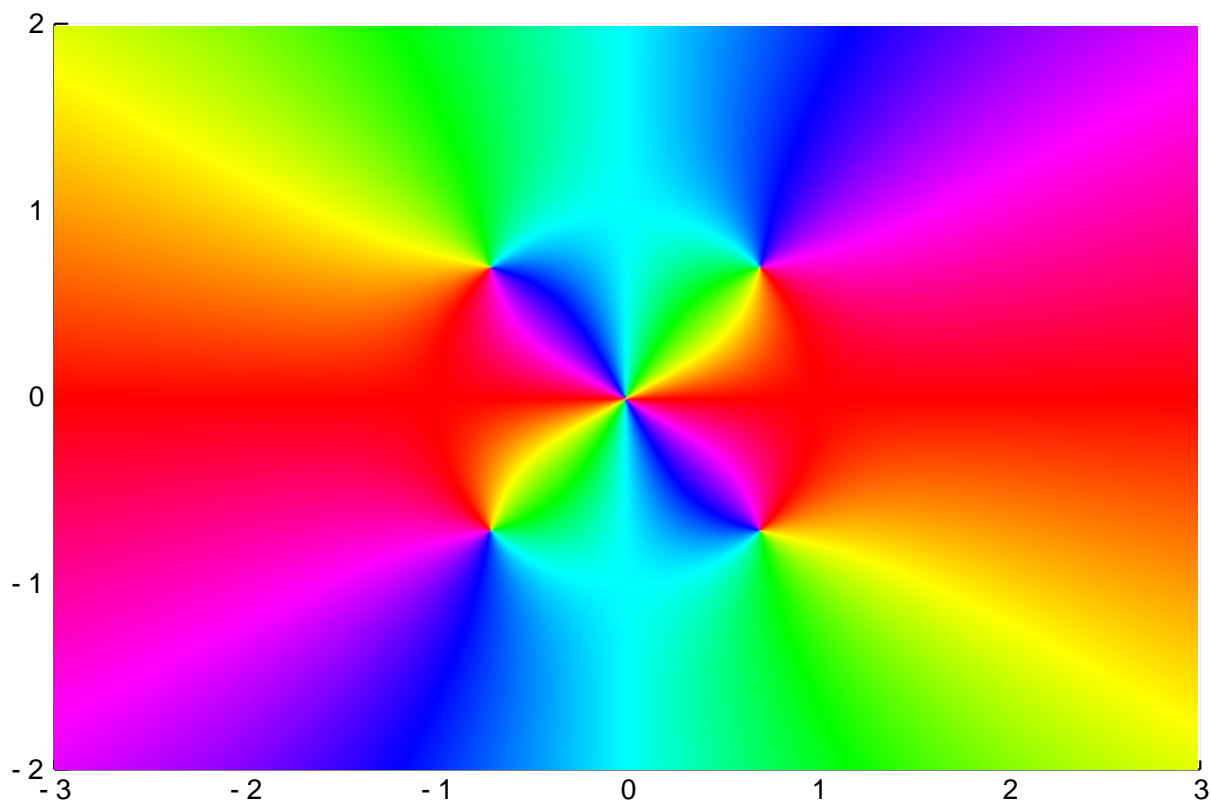
$$\oint_{\gamma_R} f(z) dz = 2\pi i \sum_{k=1}^r \operatorname{Res}_{z=z_k} f(z).$$

provided R is large enough to contain all singularities. But the decay in f suffices to show that $\int_{H_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$. The result therefore follows.

■

Demonstration

```
f = x -> x^2/(x^4+1)
phaseplot(-3..3, -2..2, f)
```



This function has poles in the upper plane, but has sufficient decay that we can apply Residue theorem:

```
using ApproxFun
z_1,z_2,z_3,z_4 = exp(im*π/4), exp(3im*π/4), exp(5im*π/4), exp(7im*π/4)

res_1 = z_1^2 / ((z_1 - z_2)*(z_1 - z_3)*(z_1 - z_4) )
res_2 = z_2^2 / ((z_2 - z_1)*(z_2 - z_3)*(z_2 - z_4) )

2π*im*(res_1 + res_2), sum(Fun(f, Line()))

(2.221441469079183 + 3.487868498008632e-16im, 2.221441469084968)
```

We can also apply Residue theorem in the lower-half plane, and we get the same result:

```
res_3 = z_3^2 / ((z_3 - z_1)*(z_3 - z_2)*(z_3 - z_4) )
res_4 = z_4^2 / ((z_4 - z_1)*(z_4 - z_3)*(z_4 - z_2) )

-2π*im*(res_3 + res_4), sum(Fun(f, Line()))

(2.221441469079183 + 5.231802747012948e-16im, 2.221441469084968)
```

1.2.2 Cauchy's integral formula on the real line

An immediate consequence of the Residue theorem is Cauchy's integral formula on the real line:

Theorem (Cauchy's integral formula on the real line) Suppose $f : \bar{\mathbb{H}}^+ \rightarrow \mathbb{C}$ is holomorphic in \mathbb{H}^+ , and $\lim_{\epsilon \rightarrow 0} f(x + i\epsilon) = f(x)$ converges uniformly. If

$$\lim_{z \rightarrow \infty} f(z) = 0$$

uniformly for $z \in \bar{\mathbb{H}}^+$, then

$$f(z) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \frac{f(x)}{x - z} dx$$

for all $z \in \mathbb{H}^+$.

Examples Here is a simple example of $f(x) = \frac{x^2}{(x+i)^3}$, which is analytic in the upper half plane:

```
f = x -> x^2/(x+im)^3
z = 2.0+2.0im
sum(Fun(x-> f(x)/(x - z), Line()))/(2*pi*im) - f(z)

2.69638478211931e-13 - 3.032296636007459e-13im
```

Evaluating in lower half plane doesn't work because it has a pole there:

```
f = x -> x^2/(x+im)^3
z = 2.0-2.0im
sum(Fun(x-> f(x)/(x - z), Line()))/(2*pi*im) , f(z)

(2.2331966377567175e-13 + 2.0218605967538834e-14im, 0.7040000000000001 - 0.128im)
```

But does for a function analytic in the lower half plane (with a minus sign):

```
f = x -> x^2/(x-im)^3
z = 2.0-2.0im
-sum(Fun(x-> f(x)/(x - z), Line()))/(2*pi*im) , f(z)

(0.03277196176631425 + 0.16750113791564233im, 0.03277196176604461 + 0.1675011379153391im)
```

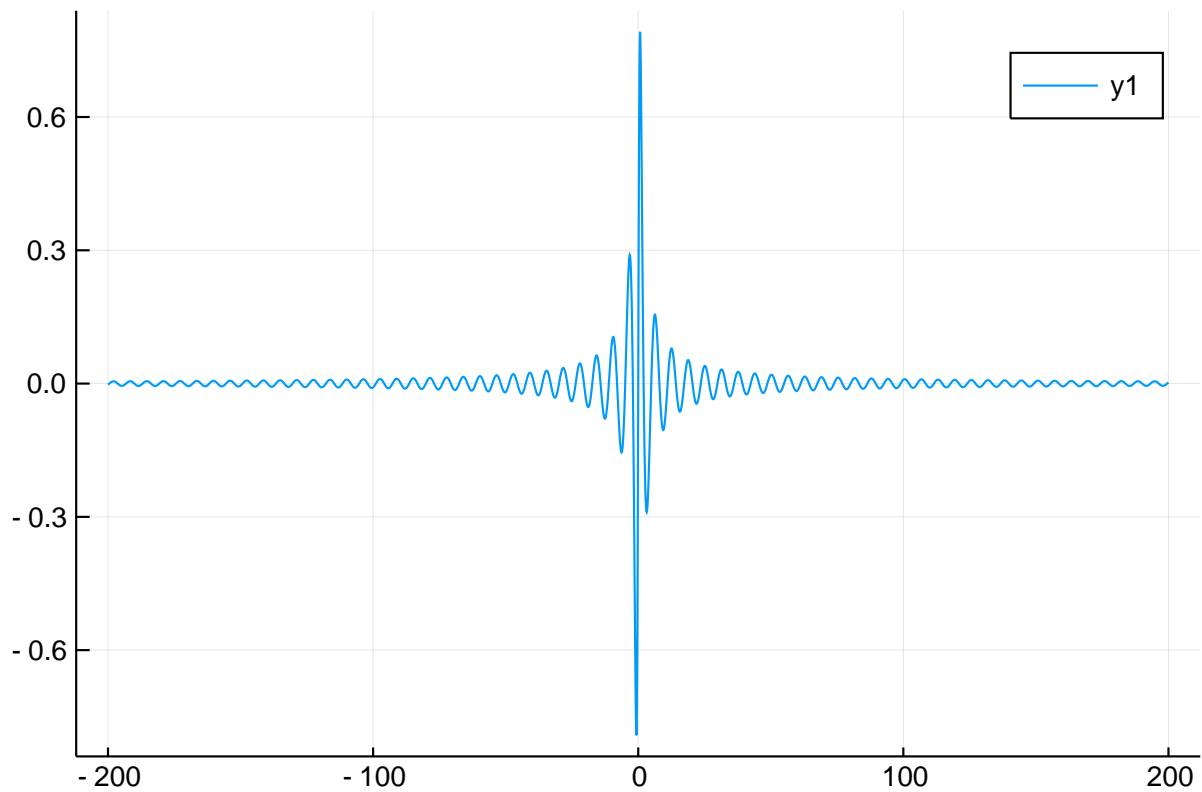
It also works for functions with exponential decay in the upper-half plane:

```
f = x -> exp(im*x)/(x+im)
z = 2 + 2im
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2*pi*im) - f(z)

4.501316791527543e-9 + 5.93330299732131e-7im
```

This is difficult as a real integral as the integrand is very oscillatory:

```
xx = -200:0.1:200
plot(xx, real.(f.(xx)))
```



An equivalent result holds in the negative real axis, but be careful:

```

z = -2-im
f = x -> exp(im*x)/(x+im)
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2*pi*im)

-4.529525111429054e-9 + 5.865568293152012e-7im

z = -2-im
f = x -> exp(im*x)/(x-im)
sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2*pi*im), f(z)

(0.09196985577264773 - 0.09196926921581833im, 0.9007327639404081 + 0.335130
5720620013im)

z = -2-im
f = x -> exp(-im*x)/(x-im)
-sum(Fun(x-> f(x)/(x - z), -500 .. 500))/(2*pi*im), f(z)

(-0.045354995402086956 - 0.1219015148055592im, -0.04535499089125899 - 0.121
90092372837213im)

```

1.2.3 Fourier transforms and Jordan's lemma

The case of calculating

$$\int_{-\infty}^{\infty} e^{i\omega x} g(x) dx$$

is important because it is the Fourier transform of $g(x)$. Provided g is defined in the upper half plane and $\omega > 0$, $f(z) = e^{i\omega z} g(z)$ has exponential decay. If g decays fast enough we can use residue calculus as before.

Example Consider the Fourier transform of $g(x) = 1/(x^2 + 1)$. This decays fast enough to use residue calculus so we have for $\omega > 0$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 + 1} dx = 2\pi i \operatorname{Res}_{z=i} \frac{e^{i\omega z}}{z^2 + 1} = \pi e^{-\omega}.$$

On the other hand if $\omega < 0$ we have exponential decay in lower half plane and use corresponding residues to determine

$$\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2 + 1} dx = -2\pi i \operatorname{Res}_{z=-i} \frac{e^{i\omega z}}{z^2 + 1} = \pi e^{\omega}.$$

That is, the Fourier transform of $g(x) = 1/(x^2 + 1)$ is $\pi e^{-|\omega|}$.

Exponential decay actually gives us sharper results:

Lemma (Jordan) Assume $\omega > 0$. If $g(z)$ is continuous in on the half circle $H_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$ then

$$\left| \int_{H_R} g(z) e^{i\omega z} dz \right| \leq \frac{\pi}{\omega} M$$

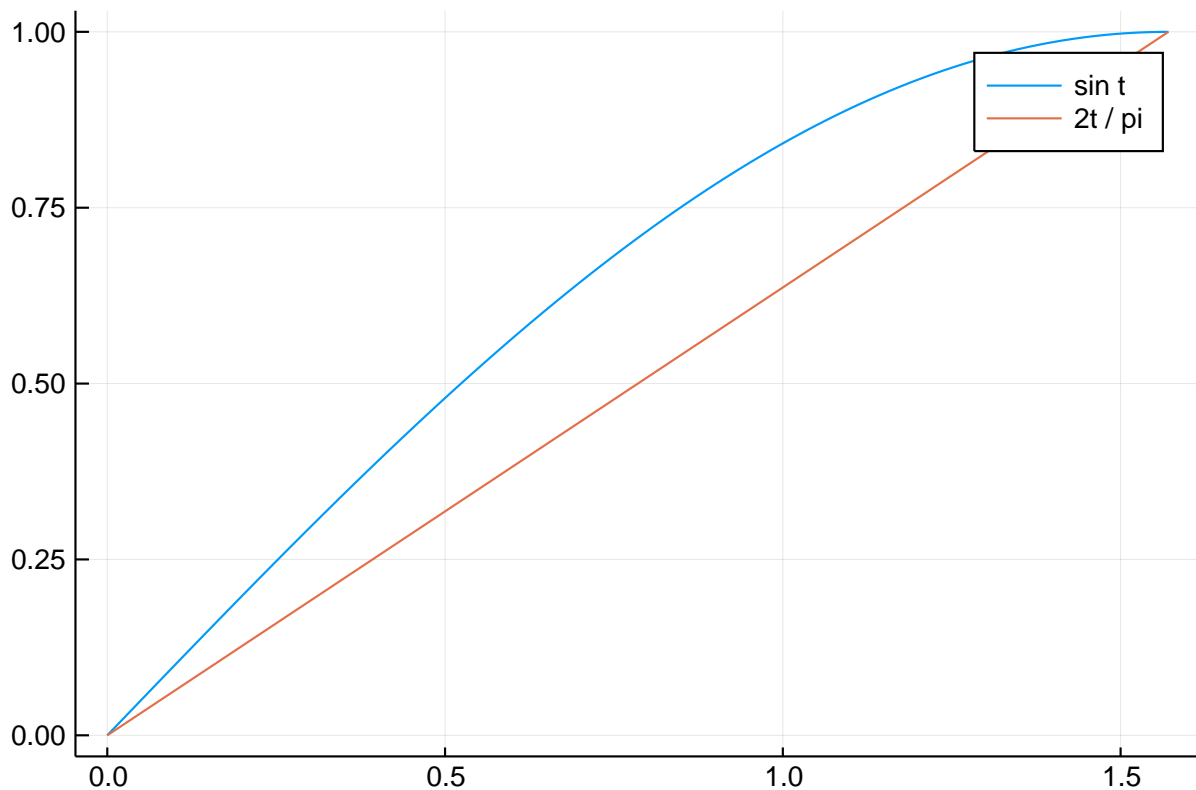
where $M = \sup_{z \in H_R} |g(z)|$.

Sketch of proof We have

$$\left| \int_{H_R} g(z) e^{i\omega z} dz \right| \leq R \int_0^\pi |g(Re^{i\theta}) e^{i\omega Re^{i\theta}} e^{i\theta}| d\theta \leq MR \int_0^\pi e^{-\omega R \sin \theta} d\theta = 2MR \int_0^{\frac{\pi}{2}} e^{-\omega R \sin \theta} d\theta$$

But we have $\sin \theta \geq \frac{2\theta}{\pi}$:

```
theta = range(0; stop=pi/2, length=100)
plot(theta, sin.(theta); label="sin t")
plot!(theta, 2theta/pi; label = "2t / pi")
```

Hence

$$\left| \int_{H_R} g(z) e^{i\omega z} dz \right| \leq 2MR \int_0^{\frac{\pi}{2}} e^{-\frac{2\omega R\theta}{\pi}} d\theta = \frac{\pi}{\omega} (1 - e^{-\omega R}) M \leq \frac{\pi M}{\omega}.$$

1.3 Application: Calculating Fourier integrals of weakly decaying functions

Why is this useful? We can use it to apply Residue theorem to functions that only have z^{-1} decay. Such functions do not decay absolutely but do converge in a principal value sense:

```
f = x -> exp(im*x)*x/(x^2+1)
```

```
sum(Fun(f, -30000 .. 30000))
```

```
-6.776263578034403e-21 + 1.1557671135433842im
```

Thus we can construct a Residue theorem for calculating

$$\int_{-\infty}^{\infty} g(x) e^{i\omega x} dx$$

provided that $g(z) \rightarrow 0$ and is analytic in the upper-half plane.

```
2*pi*im*exp(-1)*im/(im+im) # 2*pi*im* residue of g(z)exp(im*z) at z = im
```

```
0.0 + 1.1557273497909217im
```