1 M3M6: Applied Complex Analysis

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2 Solution Sheet 1

2.1 Problem 1.1

2.1.1 1.

Use fundamental theorem of algebra: a polynomial is a constant times a product of terms like $z - \lambda_k$, where λ_k are the roots. In this case, the roots are a times the quartic-root of -1, hence this gives us:

$$z^4 + a^4 = (z - a\mathrm{e}^{\mathrm{i}\pi/4})(z - a\mathrm{e}^{3\mathrm{i}\pi/4})(z - a\mathrm{e}^{5\mathrm{i}\pi/4})(z - a\mathrm{e}^{7\mathrm{i}\pi/4})$$

We are only interested in the root $a\mathrm{e}^{\mathrm{i}\pi/4}$, thus we simplify the expression

$$\begin{aligned}\frac{z^3 \sin z}{z^4 + a^4} &= \frac{z^3 \sin z}{(z - a\mathrm{e}^{3\mathrm{i}\pi/4})(z - a\mathrm{e}^{5\mathrm{i}\pi/4})(z - a\mathrm{e}^{7\mathrm{i}\pi/4})} \frac{1}{z - a\mathrm{e}^{\mathrm{i}\pi/4}} \\ &= \frac{a^3 \mathrm{e}^{3\mathrm{i}\pi/4} \sin(a\mathrm{e}^{\mathrm{i}\pi/4})}{a^3(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{3\mathrm{i}\pi/4})(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{5\mathrm{i}\pi/4})(\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{e}^{7\mathrm{i}\pi/4})} \frac{1}{z - a\mathrm{e}^{\mathrm{i}\pi/4}} + O(1)\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z = a\mathrm{e}^{\mathrm{i}\pi/4}} \frac{z^3 \sin z}{z^4 + a^4} = \frac{\mathrm{e}^{3\mathrm{i}\pi/4} \sin(a\mathrm{e}^{\mathrm{i}\pi/4})}{\mathrm{e}^{3\mathrm{i}\pi/4}(1 - \mathrm{e}^{\frac{\mathrm{i}\pi}{2}})(1 - \mathrm{e}^{\mathrm{i}\pi})(1 - \mathrm{e}^{3\mathrm{i}\pi/2})} = \frac{\sin(a\mathrm{e}^{\mathrm{i}\pi/4})}{(1 - \mathrm{i})(2)(1 + \mathrm{i})} = \frac{\sin(a\mathrm{e}^{\mathrm{i}\pi/4})}{4}$$

Let's check our work: we compare the numerically calculated residue to the formula we have derived:

```
using ApproxFun, Plots, ComplexPhasePortrait, LinearAlgebra, DifferentialEquations
a = 2.0
γ = Circle(a*exp(im*π/4), 0.1)
f = Fun(z -> z^3*sin(z)/(z^4+a^4), γ)
sum(f)/(2π*im), sin(a*exp(im*π/4))/4
```

```
(0.5378838853348212 + 0.07544036746694016im, 0.5378838853348215 + 0.0754403
674669402im)
```

2. We have

$$(z^2 - 1)^2 = (z - 1)^2 (z + 1)^2$$

Thus this is a slightly more challenging since it has a double pole. But we can expand using Geometric series:

$$\frac{z+1}{(z^2-1)^2} = \frac{1}{(z-1)^2}\frac{1}{2-(1-z)} = \frac{1}{(z-1)^2}\frac{1}{2}(1+(1-z)/2+O(1-z)^2) = \frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} + O(1)$$

Thus the residue is the negative-first Laurent coefficient, namely $-\frac{1}{4}$.

We again check our work:

```
γ = Circle(1, 0.1)
f = Fun(z -> (z+1)/(z^2-1)^2, γ)
sum(f)/(2π*im) # almost equals -1/4
```

```
-0.2500000000000023 - 1.1916485920484316e-16im
```

3.

$$\frac{z^2 \mathrm{e}^z}{z^3 - a^3} = \frac{z^2 \mathrm{e}^z}{(z-a)(z^2+az+a^2)}$$

We thus need only evaluate the extra term at $z = a$:

$$\operatorname*{Res}_{z=a} \frac{z^2 \mathrm{e}^z}{z^3 - a^3} = \frac{\mathrm{e}^a}{3}$$

Let's check:

```
a = 2.0

γ = Circle(a, 0.1)
f = Fun(z -> z^2*exp(z)/(z^3-a^3), γ)
sum(f)/(2π*im), exp(a)/3
```

```
(2.4630186996435506 + 4.738982805534393e-16im, 2.46301869964355)
```

2.2 Problem 1.2

2.2.1 1.

Change of variables $z = \mathrm{e}^{\mathrm{i}\theta}$, $\mathrm{d}z = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}z\mathrm{d}\theta$, $\cos\theta = \frac{z+z^{-1}}{2}$ gives

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5-4\cos\theta} = -\mathrm{i}\oint \frac{\mathrm{d}z}{5z-2z^2-2} = \mathrm{i}\oint \frac{\mathrm{d}z}{(z-2)(2z-1)} = -\pi \operatorname*{Res}_{z=1/2} \frac{1}{(z-2)(z-1/2)} = \frac{2}{3}\pi$$

```
θ = Fun(0 .. 2π)
sum(1/(5-4cos(θ))) , 2π/3
```

```
(2.0943951023931975, 2.0943951023931953)
```

2.2.2 2.

Use $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+z^{-2}}{2}$ to get

$$\int_0^{2\pi} \frac{\cos 2\theta \mathrm{d}\theta}{5+4\cos\theta} = -\frac{\mathrm{i}}{4}\oint \frac{(z^4+1)\mathrm{d}z}{z^2(z+1/2)(z+2)} = \frac{\pi}{2}\left(\operatorname*{Res}_{z=-1/2} + \operatorname*{Res}_{z=0}\right)\frac{z^4+1}{z^2(z+2)(z+1/2)} = \frac{\pi}{6}$$

```
θ = Fun(0 .. 2π)
sum(cos(2θ)/(5+4cos(θ))), π/6
```

```
(0.5235987755982996, 0.5235987755982988)
```

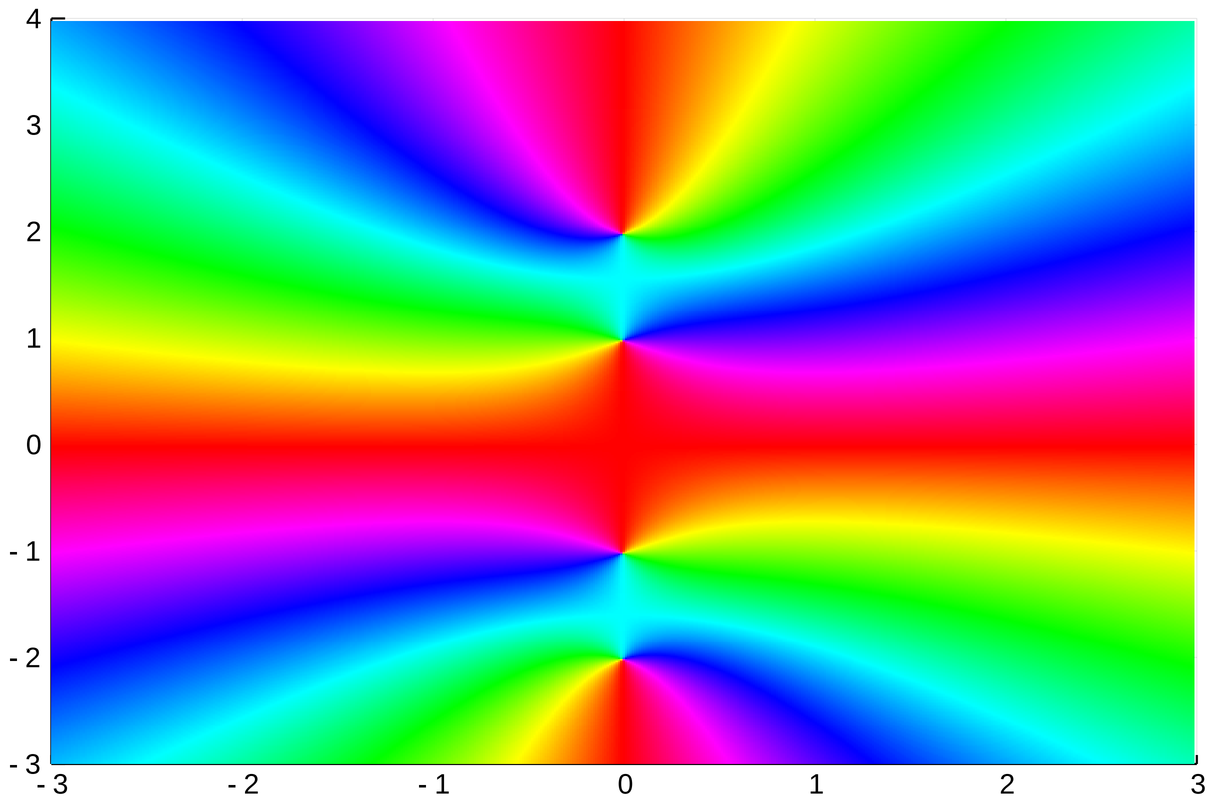
2.2.3 3.

Because the integrand is analytic and $O(z^{-2})$ in the upper half plane, we can use the residue theorem in the upper half plane using

$$\frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+\mathrm{i})(z-\mathrm{i})(z+2\mathrm{i})(z-2\mathrm{i})}$$

This has two poles in the upper half plane:

```
phaseplot(-3..3, -3..4, z-> 1/((z^2+1)*(z^2+4)))
```



$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \mathrm{d}x &= 2\pi\mathrm{i} \left(\mathop{\mathrm{Res}}_{z=\mathrm{i}} + \mathop{\mathrm{Res}}_{z=2\mathrm{i}} \right) \frac{1}{(z^2+1)(z^2+4)} \\
&= 2\pi\mathrm{i} \left(\frac{1}{2\mathrm{i}3\mathrm{i}(-\mathrm{i})} + \frac{1}{3\mathrm{i}\mathrm{i}4\mathrm{i}} \right) = \pi/6
\end{aligned}
$$

We can check the result numerically:

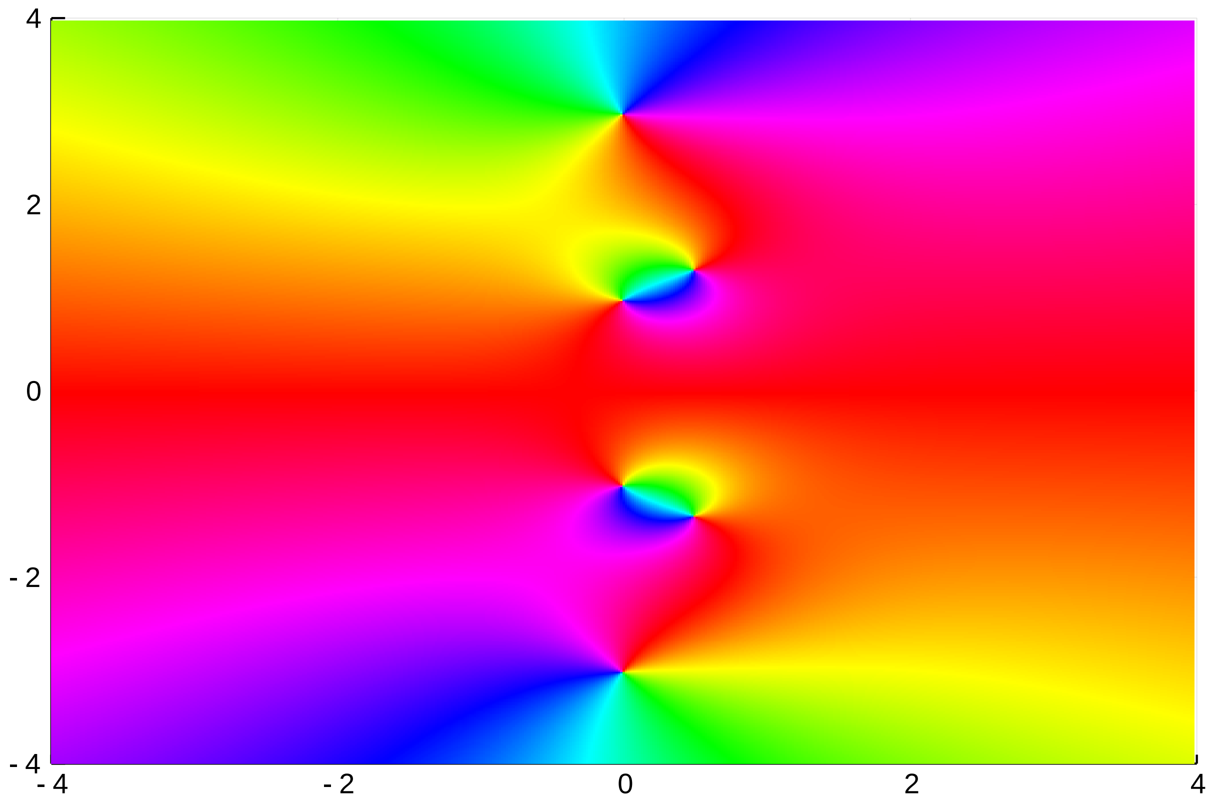
```
x = Fun( Line())
sum(1/((x^2+1)*(x^2+4))), π/6
```

```
(0.523598775598299, 0.5235987755982988)
```

2.2.4 4.

Again, decays like $O(z^{-2})$ in upper half plane so we can use residue calculus. This integrand has poles at $z = \mathrm{i}$ and $z = 3\mathrm{i}$:

```
phaseplot(-4..4, -4..4, z-> (z^2 - z + 2) / (z^4 + 10z^2 +9))
```



The residues are $(-1-\mathrm{i})/16$ and $(3-7\mathrm{i})/48$ giving the answer

$$
\frac{5\pi}{12}
$$

2.2.5 5.

$$
\int_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x
$$

Trick question: it's undefined because the integral doesn't decay fast enough. But what if I had asked for

$$\fint_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x?$$

We can't use residue theorem since it doesn't decay fast enough, but we can use, with a contour $C_R = \{R\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\}$

$$\oint_{[-M,M]\cup C_R} \frac{1}{z+\mathrm{i}} = 0$$

Further, by direct substitution, we have

$$\int_{C_R} \frac{1}{z+\mathrm{i}} \mathrm{d}z = \mathrm{i} \int_0^{\pi} R \frac{\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}} \mathrm{d}\theta$$

Letting $R \to \infty$, the integrand tends to one uniformly hence

$$\int_{C_R} \frac{1}{z+\mathrm{i}} \mathrm{d}z \to \mathrm{i} \int_0^{\pi} \mathrm{d}\theta = \mathrm{i}\pi.$$

Therefore, we have

$$\fint_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x = \lim_{M\to\infty} \int_{-M}^{M} \frac{1}{x+\mathrm{i}} \mathrm{d}x = -\mathrm{i}\pi.$$

`Indeed:`

```
x = Fun(-1000 .. 1000)
sum(1/(x+im))
```

```
-8.881784197001252e-16 - 3.13959265425659im
```

2.2.6 6.

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1} \mathrm{d}x = \Im \int_{-\infty}^{\infty} \frac{\mathrm{e}^{2\mathrm{i}x}}{x^2+x+1} \mathrm{d}x$$

This can be deformed in the upper half plane with a pole at $\frac{-1+\mathrm{i}\sqrt{3}}{2}$, using residue calculus gives us

$$-\frac{2\pi}{\sqrt{3}} \frac{\sin 1}{\mathrm{e}^{\sqrt{3}}}$$

```
x = Fun(-100 .. 100)
sum(sin(2x)/(x^2+x+1)),-2π/sqrt(3) * sin(1)/exp(sqrt(3))
```

```
(-0.5400548830723279, -0.5400553569742235)
```

2.2.7 7.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} \mathrm{d}x = \Re \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2+4} \mathrm{d}x$$

and residue calculus gives $\frac{\pi}{2\mathrm{e}^2}$

```
M = 200
x = Fun(-M .. M)
sum(cos(x)/(x^2+4)),π/(2*exp(2)) # converges if we make M even bigger
```

```
(0.2125402683670092, 0.21258416579381814)
```

2.2.8 8.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} \mathrm{d}x = \frac{\pi}{\mathrm{e}}$$

using Residue calculus. You need to appeal to Jordan's lemma to argue that it can still be done even with only $O(x^{-1})$ decay.

```
M = 2000
x = Fun(-M .. M)
sum(x*sin(x)/(x^2+1)),π/e # Converges if we make M even bigger
```

```
(1.156094344065755, 1.1557273497909217)
```

2.2.9 9.

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x \qquad \text{where} \qquad a, b > 0$$

We have for $f(x) = \frac{\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx}}{x^2}$

$$\Re f(x) = \frac{\cos ax - \cos bx}{x^2}$$

Note that, since $\cos x = 1 + x^2/2 + O(x^4)$, the integrand is fine near zero:

$$\frac{\cos ax - \cos bx}{x^2} = \frac{(a-b)}{2} + O(x^2)$$

But $f(x)$ has a pole:

$$\frac{\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx}}{x^2} = \frac{\mathrm{i}(a-b)}{x} + O(1)$$

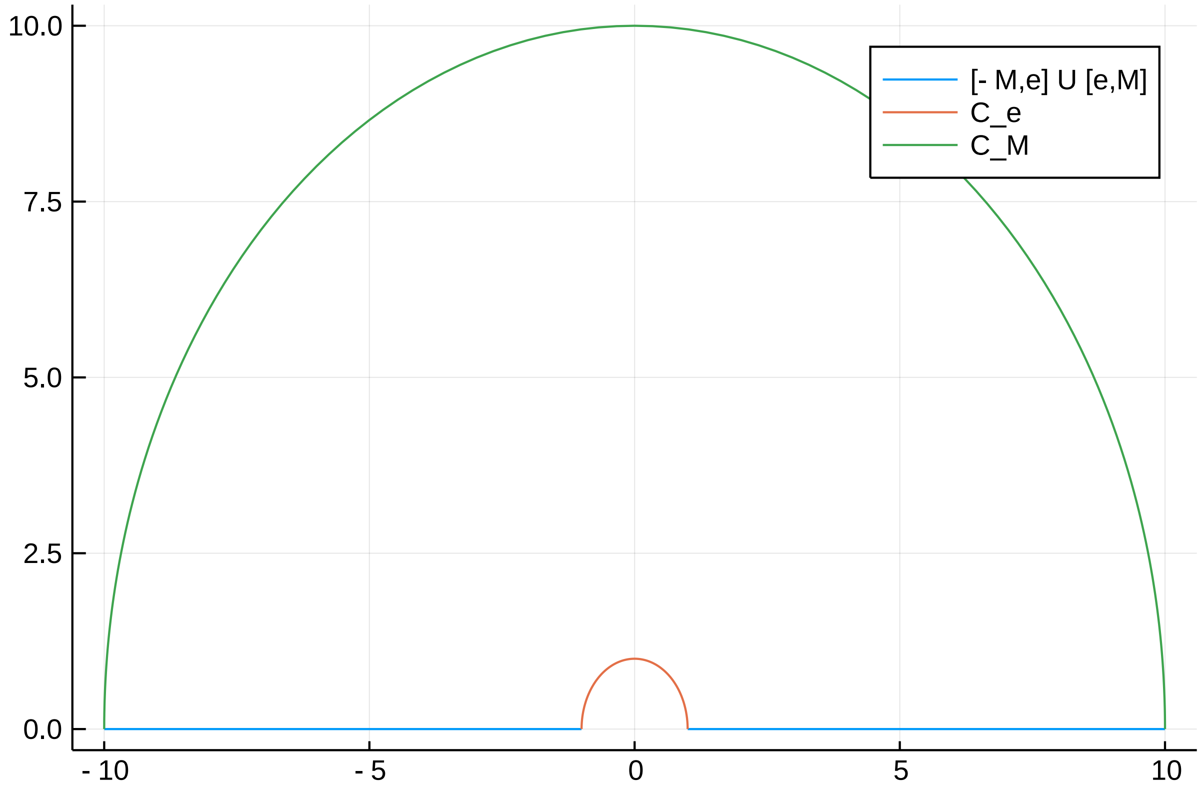
To rectify this, we need to be a bit more careful. First note that

$$\int_{\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x = \Re ⨍_{-\infty}^{\infty} f(x) \mathrm{d}x$$

Then we construct a contour avoiding zero as follows:

```
M = 10
ε = 1.0

plot(Segment(-M, -ε) ∪ Segment(ε, M);label="[-M,e] U [e,M]")
plot!(Arc(0.,ε, (π,0.)); label="C_e")
plot!(Arc(0., M, (0,π)); label = "C_M")
```



Note that $\oint_\gamma f(z)\mathrm{d}z = 0$,

$$\int_{C_\epsilon} \frac{\mathrm{e}^{iaz} - \mathrm{e}^{ibz}}{z^2}\mathrm{d}z = \int_\pi^0 \frac{(b-a)\mathrm{e}^{i\theta} + O(\epsilon)}{\mathrm{e}^{i\theta}}\mathrm{d}\theta \to (a-b)\pi$$

Also, as the integrand is $O(z^{-2})$ the integral over C_M vanishes as $M \to \infty$. We therefore get

$$⨍ f(x)\mathrm{d}x = (b-a)\pi$$

```
ε =0.001
M = 600.0
x = Fun(Segment(-M , -ε) ∪ Segment(ε, M))
a = 2.3; b = 3.8
sum((cos(a*x) - cos(b*x))/x^2),π*(b-a) # Converges if we make M bigger
```

```
(4.703235784666415, 4.71238898038469)
```

2.2.10 10.

Use binomial formula

$$
\begin{aligned}
\int_0^{2\pi} (\cos\theta)^n \mathrm{d}\theta &= \frac{1}{2^n \mathrm{i}} \oint (z+z^{-1})^n \frac{\mathrm{d}z}{z} \\
&= \frac{1}{2^n \mathrm{i}} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \oint z^k z^{k-n} \frac{\mathrm{d}z}{z} \\
&= \frac{1}{2^n \mathrm{i}} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \oint z^{2k-n-1} \mathrm{d}z
\end{aligned}
$$

We only have a residue of $2k-n-1=-1$, that is, if $2k=n$. If n is odd, this can't happen (duh! the integral is symmetric with respect to θ). If it's even, then we have

$$
\int_0^{2\pi} (\cos\theta)^n \mathrm{d}\theta = \frac{\pi}{2^{n-1}} \frac{n!}{2(n/2)!}
$$

```
θ = Fun(0 .. 2π)
n = 4;
sum(cos(θ)^n), π*factorial(1.0n)/(2^(n-1)*2*factorial(n/2))
```

```
(2.3561944901923493, 2.356194490192345)
```

2.3 Problem 2.1

By integrating around a rectangular contour with vertices at $\pm R$ and $\pi\mathrm{i} \pm R$ and letting $R \to \infty$, show that:

$$
\int_0^{\infty} \operatorname{sech} x \mathrm{d}x = \frac{\pi}{2}
$$

where $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$.

Recall $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$. This shows that $\operatorname{sech}(-x) = \operatorname{sech} x$ But we also have

$$
\operatorname{sech}(x+\mathrm{i}\pi) = \frac{2}{\mathrm{e}^{-x-\mathrm{i}\pi}+\mathrm{e}^{x+\mathrm{i}\pi}} = \frac{2}{-\mathrm{e}^{-x}-\mathrm{e}^{x}} = -\operatorname{sech} x
$$

Thus we have

$$
4\int_0^{\infty} \operatorname{sech} x \mathrm{d}x = \left[\int_{-\infty}^{\infty} + \int_{\infty+\mathrm{i}\pi}^{-\infty+\mathrm{i}\pi}\right] \operatorname{sech} z \mathrm{d}z
$$

We can approximate this using

$$
\left[\int_{-R}^{R} + \int_{R}^{R+\mathrm{i}\pi} + \int_{R+\mathrm{i}\pi}^{-R+\mathrm{i}\pi} + \int_{-R+\mathrm{i}\pi}\right] \operatorname{sech} z \mathrm{d}z = 2\pi\mathrm{i}\operatorname*{Res}_{z=\frac{\mathrm{i}\pi}{2}} \operatorname{sech} z = 2\pi
$$

since, for $z_0 = \frac{\mathrm{i}\pi}{2}$, we have

$$
\operatorname{sech} z = \frac{1}{\cos \mathrm{i}z} = \frac{1}{-\mathrm{i}\sin \mathrm{i}z_0(z-z_0)+O(z-z_0)^2} = -\frac{\mathrm{i}}{(z-z_0)} + O(1)
$$

Finally, we need to show that the limit as $R \to \infty$ tends to the right value. In this case, it follows since

$$\left|\int_R^{R+\mathrm{i}\pi} \operatorname{sech} z \mathrm{d}z\right| \le \frac{2\pi\mathrm{e}^{-R}}{1-\mathrm{e}^{-2R}} \to 0$$

(and by symmetry for $\int_{-R+\mathrm{i}\pi}^{-R}$.)

2.4 Problem 2.2

Show that the Fourier transform of $\operatorname{sech} x$ satisfies

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\operatorname{sech} x \mathrm{d}x = \pi\operatorname{sech}\frac{\pi k}{2}$$

Define

$$f(z) = \mathrm{e}^{\mathrm{i}kz}\operatorname{sech} z = \frac{2\mathrm{e}^{(1+\mathrm{i}k)z}}{\mathrm{e}^{2z}+1}$$

In this case, we have the symmetry

$$f(x+\mathrm{i}\pi) = -\mathrm{e}^{-k\pi}\mathrm{e}^{\mathrm{i}kx}\operatorname{sech} x = -\mathrm{e}^{-k\pi} f(x)$$

```
k = 2.0
f = z -> exp(im*k*z)*sech(z)
-exp(-k*π)f(2.0),f(2.0+im*π)
```

```
(0.00032444937189257726 + 0.00037565438782217884im, 0.0003244493718925772 +
0.00037565438782217884im)
```

In other words, we have

$$(1+\mathrm{e}^{-k\pi})\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \left(\int_{-\infty}^{\infty} + \int_{\infty+\mathrm{i}\pi}^{-\infty+\mathrm{i}\pi}\right) f(z)\mathrm{d}z$$

By similar logic as above, we can show that the integral over the rectangular contour converges to this.

Again, the only pole inside is at $z = \frac{\mathrm{i}\pi}{2}$, where the residue is $-\mathrm{i}\mathrm{e}^{\frac{-\pi k}{2}}$. Thus we have

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \frac{2\pi\mathrm{e}^{\frac{-\pi k}{2}}}{1+\mathrm{e}^{-k\pi}} = \pi\operatorname{sech}\frac{\pi k}{2}$$

2.5 Problem 3.1

We have

$$\rho(A) \subseteq B(1,3) \cup B(2,3) \cup B(4,1)$$

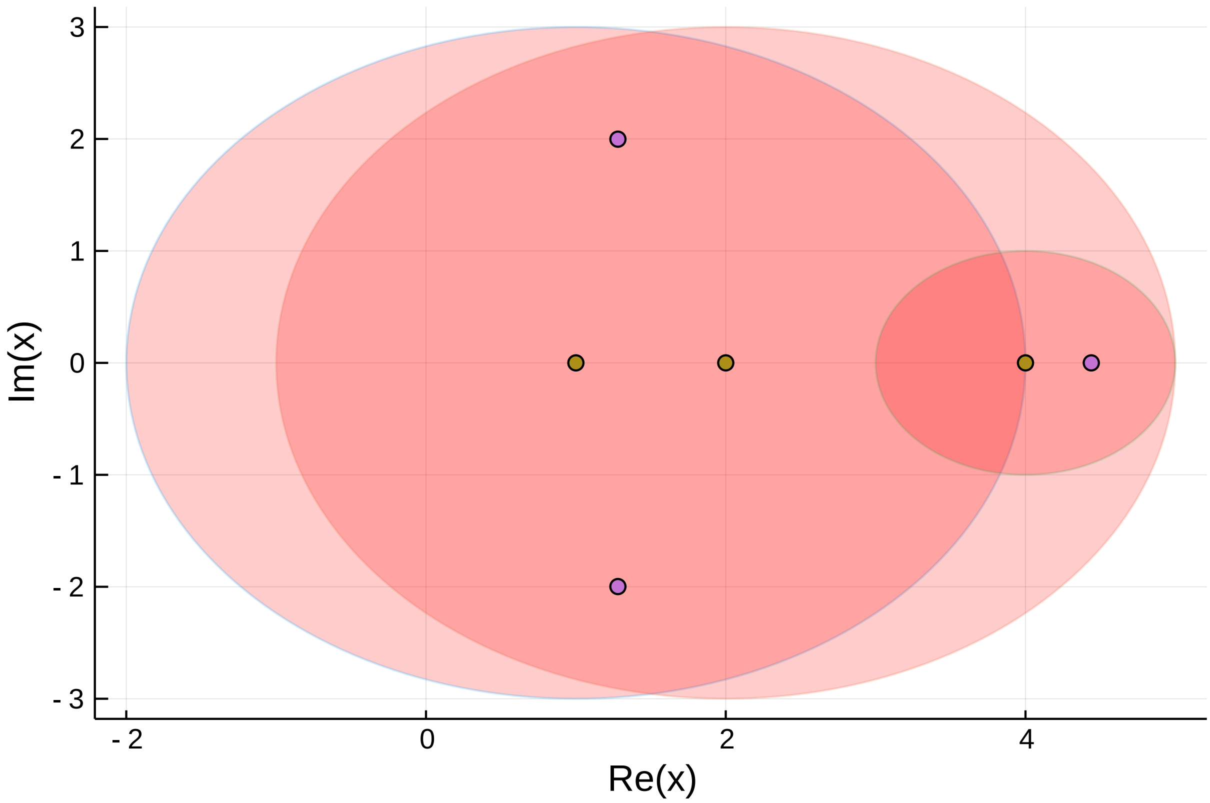
where $B(z_0, r)$ is the ball of radius r around z_0.

Here's a depiction:

```
drawdisk!(z0, R) = plot!(θ-> real(z0) + R[1]*cos(θ), θ-> imag(z0) + R[1]*sin(θ), 0, 2π,
fill=(0,:red), α = 0.2, legend=false)

A = [1 2 -1; -2 2 1; 0 1 4]

λ = eigvals(A)
p = plot()
drawdisk!(1,3)
drawdisk!(2,3)
drawdisk!(4,1)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals")
p
```

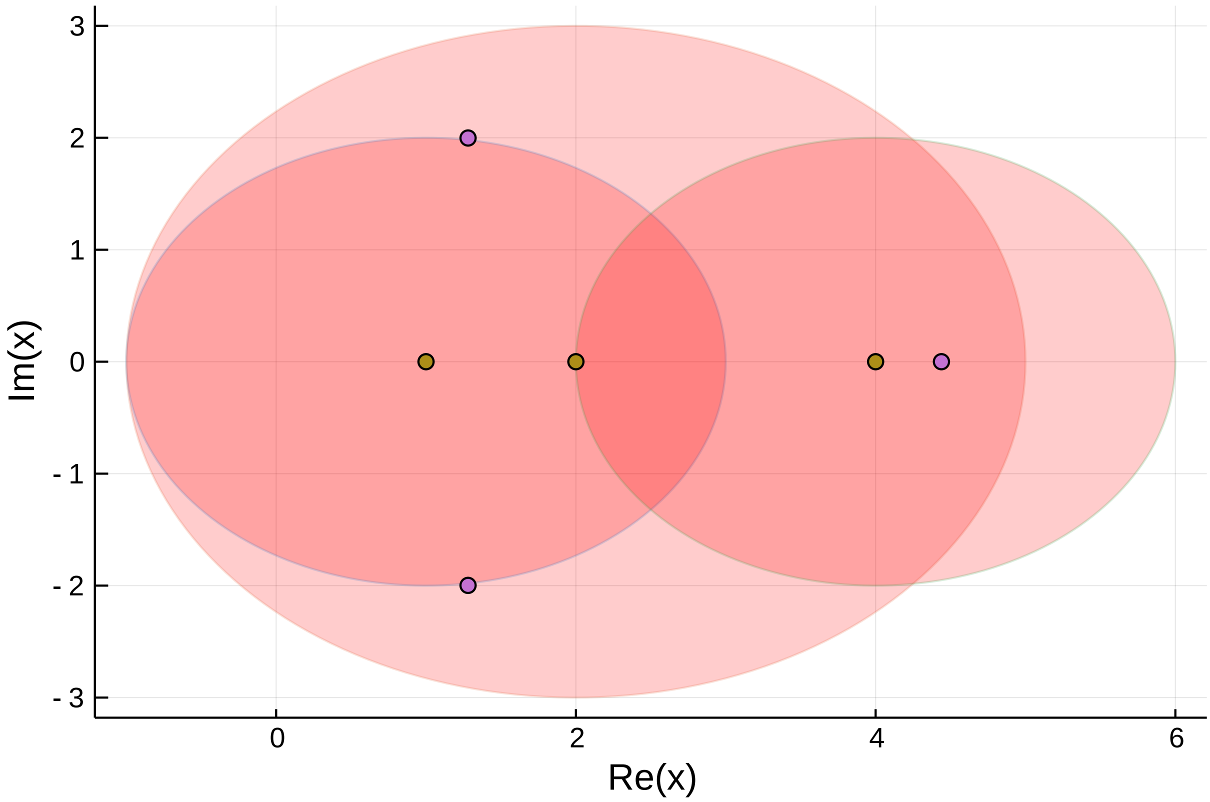


2.6 Problem 3.2

We get

$$\rho(A) \subseteq B(1,2) \cup B(2,3) \cup B(4,2)$$

```
λ = eigvals(A)
p = plot()
drawdisk!(1,2)
drawdisk!(2,3)
drawdisk!(4,2)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals")
p
```

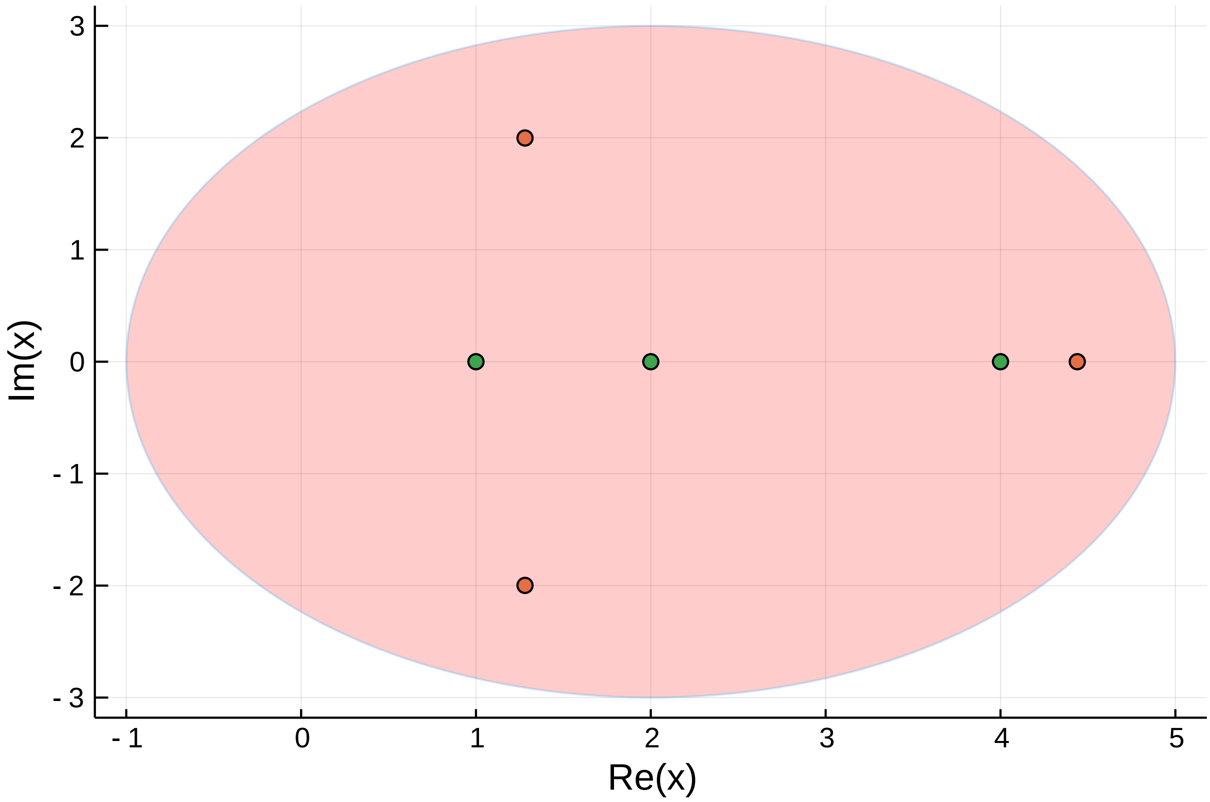


2.7 Problem 3.3

Because the spectrum live in the intersection of the two estimates, the sharpest bound is

$$\rho(A) \subseteq B(2,3)$$

```
λ = eigvals(A)
p = plot()
drawdisk!(2,3)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals")
p
```



Thus we can take $2 + 3e^{i\theta}$ as the contour.

2.8 Problem 4.1

Note that in the scalar case $u'' = au$ we have the solution

$$u(t) = u_0 \cosh \sqrt{a}t + v_0 \frac{\sinh \sqrt{a}t}{\sqrt{a}}$$

Write

$$A = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^\top$$

where $\lambda_k > 0$ and then the solution has the form, where γ is a contour surrounding the eigenvalues and to the right of zero:

$$
\begin{aligned}
\mathbf{u}(t) &= Q\begin{pmatrix}\cosh\sqrt{\lambda_1}t & & \\ & \ddots & \\ & & \cosh\sqrt{\lambda_n}t\end{pmatrix}Q^\top\mathbf{u}_0 + Q\begin{pmatrix}\frac{\sinh\sqrt{\lambda_1}t}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{\sinh\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}\end{pmatrix}Q^\top\mathbf{v}_0 \\
&= \frac{1}{2\pi\mathrm{i}}Q\begin{pmatrix}\oint_\gamma\frac{\cosh\sqrt{z}t\mathrm{d}z}{z-\lambda_1} & & \\ & \ddots & \\ & & \oint_\gamma\frac{\cosh\sqrt{z}t\mathrm{d}z}{z-\lambda_n}\end{pmatrix}Q^\top\mathbf{u}_0 \\
&\quad + \frac{1}{2\pi\mathrm{i}}Q\begin{pmatrix}\oint_\gamma\frac{\sinh\sqrt{z}t}{\sqrt{z}}\frac{\mathrm{d}z}{z-\lambda_1} & & \\ & \ddots & \\ & & \oint_\gamma\frac{\sinh\sqrt{z}t}{\sqrt{z}}\frac{\mathrm{d}z}{z-\lambda_n}\end{pmatrix}Q^\top\mathbf{v}_0 \\
&= \frac{1}{2\pi\mathrm{i}}\oint_\gamma\cosh\sqrt{z}tQ\begin{pmatrix}(z-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (z-\lambda_n)^{-1}\end{pmatrix}Q^\top\mathbf{u}_0\mathrm{d}z \\
&\quad + \frac{1}{2\pi\mathrm{i}}\oint_\gamma\frac{\sinh\sqrt{z}t}{\sqrt{z}}Q\begin{pmatrix}(z-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (z-\lambda_n)^{-1}\end{pmatrix}Q^\top\mathbf{v}_0\mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}}\oint_\gamma\cosh\sqrt{z}t(zI-A)^{-1}\mathbf{u}_0\mathrm{d}z + \frac{1}{2\pi\mathrm{i}}\oint_\gamma\frac{\sinh\sqrt{z}t}{\sqrt{z}}(zI-A)^{-1}\mathbf{v}_0\mathrm{d}z
\end{aligned}
$$

Here we verify the formulae numerically:

```
n = 5
A = randn(n,n)
A = A+ A' + 10I

λ, Q = eigen(A)

norm(A - Q*Diagonal(λ)*Q')
```

```
1.4628458888680505e-14
```

Time-stepping solution:

```
u_0 = randn(n)
v_0 = randn(n)
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0; v_0], (0.,2.));
reltol=1E-10);

t = 2.0
uv(t)[1:n]
```

```
5-element Array{Float64,1}:
 -153.50980866015774
  309.83541224867355
 -285.549833914172
  275.7821685681412
 -384.4859869664274
```

Solution via diagonalization:

```
Q*Diagonal(cosh.(sqrt.(λ) .* t))*Q'*u_0 + Q*Diagonal(sinh.(sqrt.(λ) .* t) ./
sqrt.(λ))*Q'*v_0
```

```
5-element Array{Float64,1}:
 -153.50980862230952
  309.835412190866
 -285.54983388698184
  275.7821685190415
 -384.48598690664744
```

Solution via elliptic integrals. We chose the ellipse to surround all the spectrum of our particular A with eigenvalues:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)

function ellipse_rule(n, a, b)
    θ = periodic_rule(n)[1]
    a*cos.(θ) + b*im*sin.(θ), 2π/n*(-a*sin.(θ) + im*b*cos.(θ))
end

function ellipse_f(f, A, n, z_0, a, b)
    z,w = ellipse_rule(n,a,b)
    z .+= z_0

    ret = zero(A)
    for j=1:n
        ret += w[j]*f(z[j])*inv(z[j]*I - A)
    end
    ret/(2π*im)
end

n = 50
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 10.0, 7.0, 2.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 10.0, 7.0, 2.0)*v_0
```

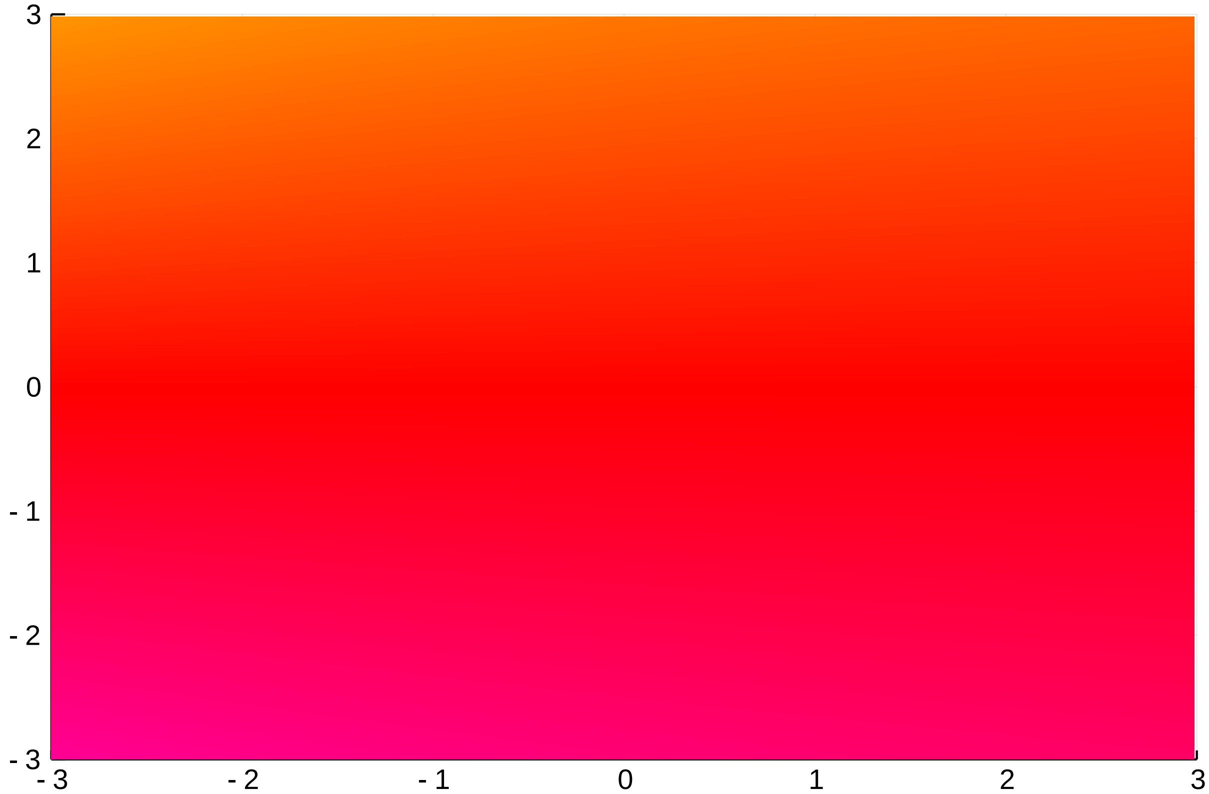
```
5-element Array{Complex{Float64},1}:
 -153.5097646890772 - 1.0797634887558185e-14im
   309.835562148416 - 3.466591845940483e-14im
 -285.5495578350004 - 6.741120797714376e-14im
  275.7822632436917 + 1.613926530107778e-14im
 -384.4857701659812 + 3.5415399749803617e-14im
```

2.9 Problem 4.2

I put the restriction in because of the $\sqrt{z}$ term, which look like it is not analytic on $(-\infty, 0]$. However, this restriction was NOT necessary, since in fact $\cosh \sqrt{z}t$ and $\frac{\sinh \sqrt{z}t}{\sqrt{z}}$ are entire:

```
phaseplot(-6..6, -3..3, z -> cosh(sqrt(z)))
```


```
phaseplot(-3..3, -3..3, z -> sinh(sqrt(z))/sqrt(z))
```



This follows from Taylor series, though I prefer the following argument: we have

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

hence $\cosh it = \cos t = \cosh -it$. Therefore, on the possible branch cut $\cosh \sqrt{z}$ is continuous (hence analytic):

$$\cosh \sqrt{x}_+ = \cosh i\sqrt{|x|} = \cosh -i\sqrt{|x|} = \cosh \sqrt{x}_-$$

Similarly,

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

implies $\sinh it = i \sin t = -i \sin(-t) = -\sinh -it$, which gives us continuity:

$$\frac{\sinh \sqrt{x}_+}{\sqrt{x}_+} = \frac{\sinh i\sqrt{|x|}}{i\sqrt{|x|}} = \frac{\sinh(-i\sqrt{|x|})}{-i\sqrt{|x|}} = \frac{\sinh \sqrt{x}_-}{\sqrt{x}_-}$$

furthermore, they are both bounded at zero, hence analytic there too.

Here's a numerical example:

```
n = 5
A = randn(n,n)
λ, V = eigen(A)

norm(A - V*Diagonal(λ)*inv(V))

6.092286018832885e-15

u_0 = randn(n)
v_0 = randn(n)
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0; v_0], (0.,2.));
reltol=1E-10);

t = 2.0
uv(t)[1:n]

5-element Array{Float64,1}:
 -7.647162935805305
  1.5930861950559003
 -9.51180491790177
  1.972611874650823
  0.6472429738562755

V*Diagonal(cosh.(sqrt.(λ) .* t))*inv(V)*u_0 +
   V*Diagonal(sinh.(sqrt.(λ) .* t) ./ sqrt.(λ))*inv(V)*v_0

5-element Array{Complex{Float64},1}:
 -7.647162935799476 + 6.533336022400347e-16im
 1.5930861947980839 - 3.6988406432835196e-16im
 -9.511804918142495 - 3.347090759974068e-16im
 1.9726118744356422 - 5.891704561258427e-16im
 0.6472429739714145 + 5.162911561487323e-16im
```

Here's the solution using an elliptic integral:

```
n = 100
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 0.0, 3.0, 3.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 0.0, 3.0, 3.0)*v_0
```

```
5-element Array{Complex{Float64},1}:
 -7.647162935799477 - 6.240280857226813e-16im
 1.5930861947980917 + 2.9342463447408167e-16im
 -9.511804918142495 - 1.1383743937886197e-15im
 1.9726118744356462 - 1.349517594358261e-16im
 0.6472429739714101 + 5.071767322649408e-16im
```

2.10 Problem 5.1

We have

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) = \sum_{k=0}^{\infty} g_k \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j}$$

Define the n-th root of unity as $\omega = e^{\frac{2\pi \mathrm{i}}{n}}$ (that is $\omega^n = 1$), and simplify

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} e^{\frac{2\pi jik}{n}} = \sum_{j=0}^{n-1} \omega^{kj} = \sum_{j=0}^{n-1} (\omega^k)^j$$

If k is a multiple of n, then $\omega^k = 1$, and this sum is equal to n. If k is not a multiple of n, use Geometric series:

$$\sum_{j=0}^{n-1} (\omega^k)^j = \frac{\omega^{nk}-1}{\omega^k - 1} = \frac{1^k - 1}{\omega^k - 1} = 0$$

2.11 Problem 5.2

From lecture 4, we have $|f_k| \leq M_r r^{-|k|}$ for any $1 \leq r < R$, where M_r is the supremum of f in an annulus $\{z : r^{-1} < |z| < r\}$. Thus from the previous part we have (using geometric series)

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{d}\theta\right| \leq \sum_{K=1}^{\infty} |f_{Kn}| + |f_{-Kn}| \leq 2M \sum_{K=1}^{\infty} r^{-Kn} = 2M_r \frac{r^{-n}}{1 - r^{-n}}.$$

This is an upper bound that decays exponentially fast.

2.12 Problem 5.3

Note that $f(z) = 2z/(4z - z^2 - 1)$ satisfies $f(\mathrm{e}^{\mathrm{i}\theta}) = g(\theta)$. This has two poles at $2 \pm \sqrt{3}$:

```
f = z -> 2z/(4z-z^2-1)
f(exp(0.1im)) - 1/(2-cos(0.1))
```

```
1.1102230246251565e-16 + 0.0im
```

Here's a phase plot showing the location of poles:

```
phaseplot(-5..5, -5..5, f)
```

```
2+sqrt(3), 2- sqrt(3), 1/(2+sqrt(3))
```

```
(3.732050807568877, 0.2679491924311228, 0.2679491924311227)
```

Note that $2+\sqrt{3} = 1/(2-\sqrt{3})$ so in the previous result, we take $R = 2+\sqrt{3}$. For any $1 \le r < 2+\sqrt{3}$ we have

$$M_r = \frac{2}{4 - r - r^{-1}}$$

Thus we get the upper bounds

$$\frac{4}{4 - r - r^{-1}} \frac{r^{-n}}{1 - r^{-n}}$$

Let's see how sharp it is:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)
```

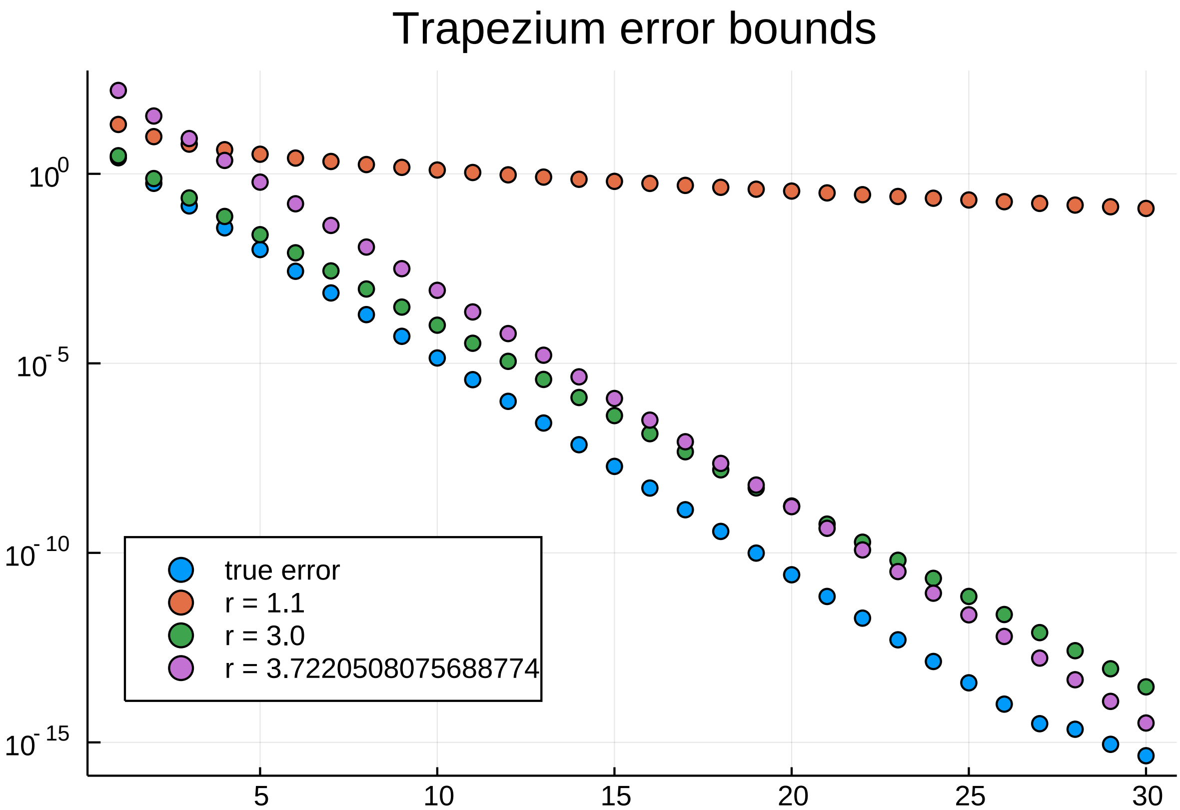
```
g = θ -> 1/(2 - cos(θ))
```

```
Q = sum(Fun(g, 0 .. 2π))
```

```
err = Float64[ (
            (θ, w) = periodic_rule(n);
            sum(w.*g.(θ)) - Q
        ) for n=1:30];
```

```
N = length(err)
```

```
scatter(abs.(err), yscale=:log10, label="true error", title="Trapezium error bounds",
legend=:bottomleft)
r = 1.1
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label = "r = $r")
r = 3.0
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label = "r = $r")
r = 2+sqrt(3)  - 0.01
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label = "r = $r")
```


Trapezium error bounds
true error
r = 1.1
r = 3.0
r = 3.7220508075688774
10^0
10^-5
10^-10
10^-15
5
10
15
20
25
30