**M3M6: Applied Complex Analysis**

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# 1 Lecture 23: Orthogonal polynomials and singular integrals

This lecture we do the following:

1. Cauchy transforms of weighted orthogonal polynomials
   - Three-term recurrence and calculation
   - Hilbert transform of weighted orthogonal polynomials
   - Hilbert transform of weighted Chebyshev polynomials
2. Log transform of weighted *classical* orthogonal polynomials

## 1.1 Cauchy transforms of orthogonal polynomials

Given a family of orthogonal polynomials $p_k(x)$ with respect to the weight $w(x)$ on $(a,b)$, we always know it satisfies a three-term recurrence:

$$\begin{aligned} xp_0(x) &= a_0 p_0(x) + b_0 p_1(x) \\ xp_k(x) &= c_k p_{k-1}(x) + a_k p_k(x) + b_k p_{k+1}(x) \end{aligned}$$

Consider now the Cauchy transform of the weighted orthogonal polynomial:

$$C_k(z) := \mathcal{C}_{(a,b)}[p_k w](z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{p_k(x) w(x)}{x - z} \mathrm{d}x$$

**Theorem (Three-term recurrence Cauchy transform of weighted OPs)** $C_k(z)$ satisfies the same recurrence relationship as $p_k(x)$ for $k = 1, 2, \ldots$:

$$\begin{aligned} zC_0(z) &= a_0 C_0(z) + b_0 C_1(z) - \frac{1}{2\pi \mathrm{i}} \int_a^b w(x) \mathrm{d}x \\ zC_k(z) &= c_k C_{k-1}(z) + a_k C_k(z) + b_k C_{k+1}(z) \end{aligned}$$

**Proof**

$$\begin{aligned} zC_k(z) &= \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{z p_k(x) w(x)}{x - z} \mathrm{d}x = \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{(z - x) p_k(x) w(x)}{x - z} \mathrm{d}x + \int_a^b \frac{x p_k(x) w(x)}{x - z} \mathrm{d}x \\ &= -\frac{1}{2\pi \mathrm{i}} \int_a^b p_k(x) w(x) \mathrm{d}x + \int_a^b \frac{(c_k p_{k-1}(x) + a_k p_k(x) + b_k p_{k+1}(x) w(x)}{x - z} \mathrm{d}x \\ &= -\frac{1}{2\pi \mathrm{i}} \int_a^b p_k(x) w(x) \mathrm{d}x + c_k C_{k-1}(z) + a_k C_k(z) + b_k C_{k+1}(z) \end{aligned}$$

when $k > 0$, the integral term disappears. ■

This gives a convenient way to calculate the Cauchy transforms: if we know $C_0(z) = \mathcal{C}w(z)$ and $\int_a^b w(x)\mathrm{d}x$, solve the lower triangular system:

$$\begin{pmatrix} 1 & & & & \\ a_0 - z & b_0 & & & \\ c_1 & a_1 - z & b_1 & & \\ & c_2 & a_2 - z & b_2 & \\ & & c_3 & a_3 - z & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} C_0(z) \\ C_1(z) \\ C_2(z) \\ C_3(z) \\ \vdots \end{pmatrix} = \begin{pmatrix} C_0(z) \\ \frac{1}{2\pi\mathrm{i}} \int_a^b w(x)\mathrm{d}x \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

**Example (Chebyshev Cauchy transform)**

Consider the Chebyshev case $w(x) = \frac{1}{\sqrt{1-x^2}}$, which satisfies $\int_{-1}^1 w(x)\mathrm{d}x = \pi$. Recall that

$$C_0(z) = \mathcal{C}w(z) = \frac{\mathrm{i}}{2\sqrt{z-1}\sqrt{z+1}}$$

Further, we have

$$xT_0(x) = T_1(x)$$
$$xT_k(x) = \frac{T_{k-1}(x)}{2} + \frac{T_{k+1}(x)}{2}$$

hence

$$zC_0(z) = C_1(z) - \frac{1}{2\mathrm{i}}$$
$$zC_k(z) = \frac{C_{k-1}(z)}{2} + \frac{C_{k+1}(z)}{2}.$$

In other words, we want to solve

$$\begin{pmatrix} 1 & & & & \\ -z & 1 & & & \\ 1/2 & -z & 1/2 & & \\ & 1/2 & -z & 1/2 & \\ & & 1/2 & -z & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} C_0(z) \\ C_1(z) \\ C_2(z) \\ C_3(z) \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{i}}{2\sqrt{z-1}\sqrt{z+1}} \\ \frac{1}{2\mathrm{i}} \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

with forward substitution.

## 1.2 Hilbert transform of weighted orthogonal polynomials

Now consider the Hilbert transform of weighted orthogonal polynomials:

$$H_k(x) = \mathcal{H}_{(a,b)}[p_k w](x) = \frac{1}{\pi}\int_a^b \frac{p_k(t)w(t)}{x-t}\mathrm{d}t$$

Just like Cauchy transforms, the Hilbert transforms have

**Corollary (Hilbert transform recurrence)**

$$
\begin{aligned}
xH_0(x) &= a_0 H_0(x) + b_0 H_1(x) + \frac{1}{\pi}\int_a^b w(x)\mathrm{d}x \\
xH_k(x) &= c_k H_{k-1}(x) + a_k H_k(x) + b_k H_{k+1}(x)
\end{aligned}
$$

**Proof** Recall

$$
\mathcal{C}^+ f(x) + \mathcal{C}^- f(x) = \mathrm{i}\mathcal{H} f(x)
$$

Therefore, we have

$$
C_k^+(x) + C_k^-(x) = \mathrm{i}\mathcal{H}[wp_k](x)
$$

hence we have

$$
\begin{aligned}
xH_0(x) = -\mathrm{i}x(C_0^+(x) + C_0^-(x)) &= -\mathrm{i}\left[a_0(C_0^+(x) + C_0^-(x)) + b_0(C_1^+(x) + C_1^-(x)) - \frac{1}{\pi \mathrm{i}}\int_a^b w(x)\mathrm{d}x\right] \\
&= a_0 H_0(x) + b_0 H_1(x) + \frac{1}{\pi}\int_a^b w(x)\mathrm{d}x
\end{aligned}
$$

Other $k$ follows by a similar argument.

■

### 1.2.1 Example 1: weighted Chebyshev T

For

$$
H_k(x) := \mathcal{H}[T_k/\sqrt{1-x^2}](x) = \frac{1}{\pi}\int_{-1}^1 \frac{T_k(t)}{(x-t)\sqrt{1-t^2}}\mathrm{d}t
$$

The recurrence gives us

$$
\begin{aligned}
xH_0(x) &= H_1(x) + 1 \\
xH_k(x) &= \frac{H_{k-1}(x)}{2} + \frac{H_k(x)}{2}
\end{aligned}
$$

In this case, we have $H_0(x) = \mathcal{H}[w](x) = 0$. Therefore, we can rewrite this recurrence as

$$
\begin{aligned}
H_1(x) &= -1 \\
xH_1(x) &= \frac{H_2(x)}{2} \\
xH_k(x) &= \frac{H_{k-1}(x)}{2} + \frac{H_{k+1}(x)}{2}
\end{aligned}
$$

This is precisely the three-term recurrence satisfied by $-U_{k-1}$! We therefore have

$$H_k(x) = -U_{k-1}(x)$$

This gives a very easy way to compute Hilbert transforms: if

$$f(x) = \sum_{k=0}^{\infty} f_k T_k(x)$$

then

$$\mathcal{H}\left[\frac{f}{\sqrt{1-\diamond^2}}\right](x) = -\sum_{k=0}^{\infty} f_{k+1} U_k(x)$$

### 1.2.2 Example 2: weighted Chebyshev U

For

$$H_k(x) := \mathcal{H}[U_k\sqrt{1-x^2}](x) = \frac{1}{\pi}\int_{-1}^{1} \frac{U_k(t)\sqrt{1-t^2}}{x-t}\mathrm{d}t$$

The recurrence gives us

$$\begin{aligned} xH_0(x) &= H_1(x) + 1/2 \\ xH_k(x) &= \frac{H_{k-1}(x)}{2} + \frac{H_k(x)}{2} \end{aligned}$$

In this case, we have $H_0(x) = \mathcal{H}[w](x) = x$. Therefore, we can rewrite this recurrence as

$$\begin{aligned} H_{-1}(x) &:= 1 \\ xH_{-1}(x) &= H_0(x) \\ xH_0(x) &= \frac{H_{-1}(x)}{2} + \frac{H_1(x)}{2} \\ xH_k(x) &= \frac{H_{k-1}(x)}{2} + \frac{H_{k+1}(x)}{2} \end{aligned}$$

This is precisely the three-term recurrence satisfied by $T_{k+1}$! We therefore have

$$H_k(x) = T_{k+1}(x)$$

## 1.3 Log transforms of weighted orthogonal polynomials

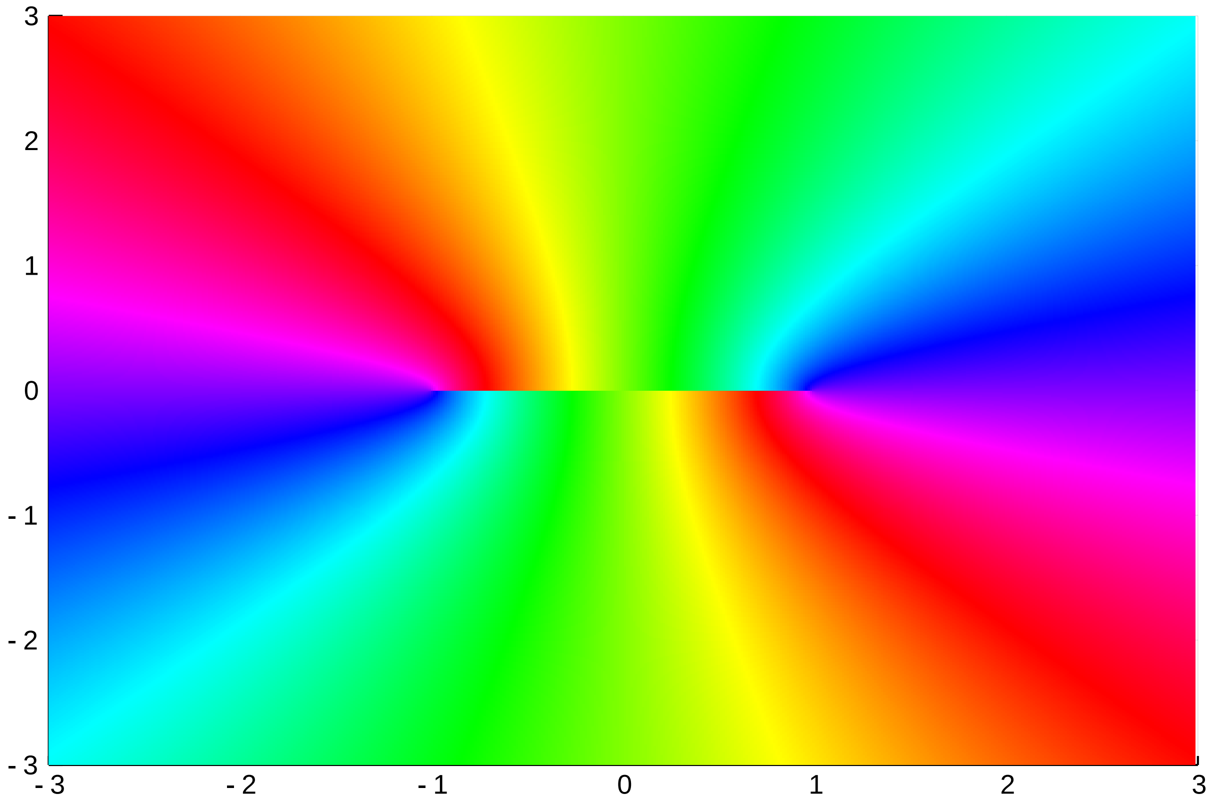
Now consider $\frac{1}{\pi}\int_a^b p_k(x)w(x)\log|z-x|\mathrm{d}x$, which we write in terms of the real part of

$$L_k(z) = L[p_k w](z) = \frac{1}{\pi}\int_a^b p_k(x)w(x)\log(z-x)\mathrm{d}x$$

For $k > 0$ we have $\int_a^b p_k(x)w(x)dx = 0$ due to orthogonality, and hence we actually have no branch cut:

```
using ApproxFun, SingularIntegralEquations, LinearAlgebra, Plots, ComplexPhasePortrait
x = Fun()
T_5 = Fun(Chebyshev(), [zeros(2);1])
w = 1/sqrt(1-x^2)
L_5 = z->cauchyintegral(w*T_5, z)

phaseplot(-3..3, -3..3, L_5)
```




## 1.4 Weighted Chebyshev log transform

For classical orthogonal polynomials we can go a step further and relate the indefinite integrals to other orthogonal polynomials.

For example, recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sqrt{1-x^2}U_n(x)] = -\frac{n+1}{\sqrt{1-x^2}}T_{n+1}(x)$$

in other words,

$$\int_x^1 \frac{T_k(t)}{\sqrt{1-t^2}}\mathrm{d}t = -\frac{\sqrt{1-x^2}U_{k-1}(x)}{k}$$

Thus for $k = 1, 2, \ldots,$

$$L_k(z) = -\frac{1}{n+1}\mathcal{C}[\sqrt{1-\diamond^2}U_{k-1}](z)$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_k(x)}{\sqrt{1-x^2}}\log(z-x)\mathrm{d}x = \frac{2\mathrm{i}}{k}\mathcal{C}[\sqrt{1-\diamond^2}U_{k-1}](z)$$

As we saw above, Cauchy transforms of OPs satisfy simple recurrences, and this relationship renders log transforms equally calculable.

## 1.5 Demonstration

We first see how the Cauchy transform can be computed using forward recurrence if we know $Cw(z)$. Here we consider the Chebyshev $T$ case:

```
x = Fun()
w = 1/sqrt(1-x^2)
z = 0.1+0.1im

n = 10
L = zeros(ComplexF64,n,n)
L[1,1] = 1
L[2,1] = -z
L[2,2] = 1
for k=3:n
    L[k,k-1] = -z
    L[k,k-2] = L[k,k] = 1/2
end

C = L \ [ im/(2sqrt(z-1)sqrt(z+1)); 1/(2im); zeros(n-2)]

T_5 = Fun(Chebyshev(), [zeros(5);1])
cauchy(T_5*w,z) , C[6]

(0.14734333381379638 - 0.26445831594251407im, 0.14734333381379644 - 0.26445
83159425141im)
```

**Warning** This fails for large $n$ or large $z$:

```
x = Fun()
w = 1/sqrt(1-x^2)
z = 5+6im

n = 100
L = zeros(ComplexF64,n,n)
L[1,1] = 1
L[2,1] = -z
L[2,2] = 1
for k=3:n
    L[k,k-1] = -z
    L[k,k-2] = L[k,k] = 1/2
end

C = L \ [ im/(2sqrt(z-1)sqrt(z+1)); 1/(2im); zeros(n-2)]

T_20 = Fun(Chebyshev(), [zeros(20);1])

C[21], cauchy(T_20*w, z)
```

```
(-2.591846965643171e6 - 48792.874011516105im, 0.0 + 8.834874115176436e-18im
)
```

Get around it by dropping the first row:

```
L[2:end,1:end-1]

C = L[2:end,1:end-1]\ [1/(2im); zeros(n-2)]

C[6]- cauchy(T_5*w, z)

-3.072062106493749e-17 + 4.410018806364391e-17im
```

We can also use this technique for computing Hiblert transforms:

```
x = 0.1

T = Fun(Chebyshev(),[zeros(n);1])
hilbert(w*T,x) , Fun(Ultraspherical(1), [zeros(n-1);1])(x)

(0.5608031061203765, 0.5608031061203765)
```

And Log transforms:

```
T_5 = Fun(Chebyshev(), [zeros(5);1])
U_4 = Fun(Ultraspherical(1), [zeros(4);1])
x = Fun()
L_5 = z->sum(T_5/sqrt(1-x^2) * log(z-x))/π

L_5(z), 2im*cauchy(sqrt(1-x^2)*U_4,z)/5

(6.576121814966233e-8 - 2.0389718362285204e-7im, 6.576121814966217e-8 - 2.0
389718362285196e-7im)
```