

M3M6: Applied Complex Analysis

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1 Solution Sheet 2

1.1 Problem 1.1

1.1.1 1.

Take as an initial guess

$$\phi_1(z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2i(1+z^2)}$$

This satisfies for $-1 < x < 1$

$$\phi_1^+(x) - \phi_1^-(x) = \frac{\sqrt{1-x^2}}{2(1+x^2)} - \frac{-\sqrt{1-x^2}}{2(1+x^2)} = \frac{\sqrt{1-x^2}}{1+x^2}$$

Further, as $z \rightarrow \infty$,

$$\phi_1(z) \sim \frac{z}{i(1+z^2)} \rightarrow 0$$

The catch is that it has poles at $\pm i$:

$$\begin{aligned}\phi_1(z) &= -\frac{\sqrt{i-1}\sqrt{i+1}}{4} \frac{1}{z-i} + O(1) \\ \phi_1(z) &= \frac{\sqrt{-i-1}\sqrt{-i+1}}{4} \frac{1}{z+i} + O(1)\end{aligned}$$

Thus it follows that

$$\phi(z) = \phi_1(z) + \frac{\sqrt{i-1}\sqrt{i+1}}{4} \frac{1}{z-i} - \frac{\sqrt{-i-1}\sqrt{-i+1}}{4} \frac{1}{z+i}$$

is

1. Analyticity: Analytic at $\pm i$ and off $[-1, 1]$
2. Decay: $\phi(\infty) = 0$
3. Regularity: Has weaker than pole singularities
4. Jump: Satisfies

$$\phi_+(x) - \phi_-(x) = \frac{\sqrt{1-x^2}}{1+x^2}$$

By Plemelj II, this must be the Cauchy transform.

Demonstration We will see experimentally that it correct. First we do a phase plot to make sure we satisfy (Analyticity):

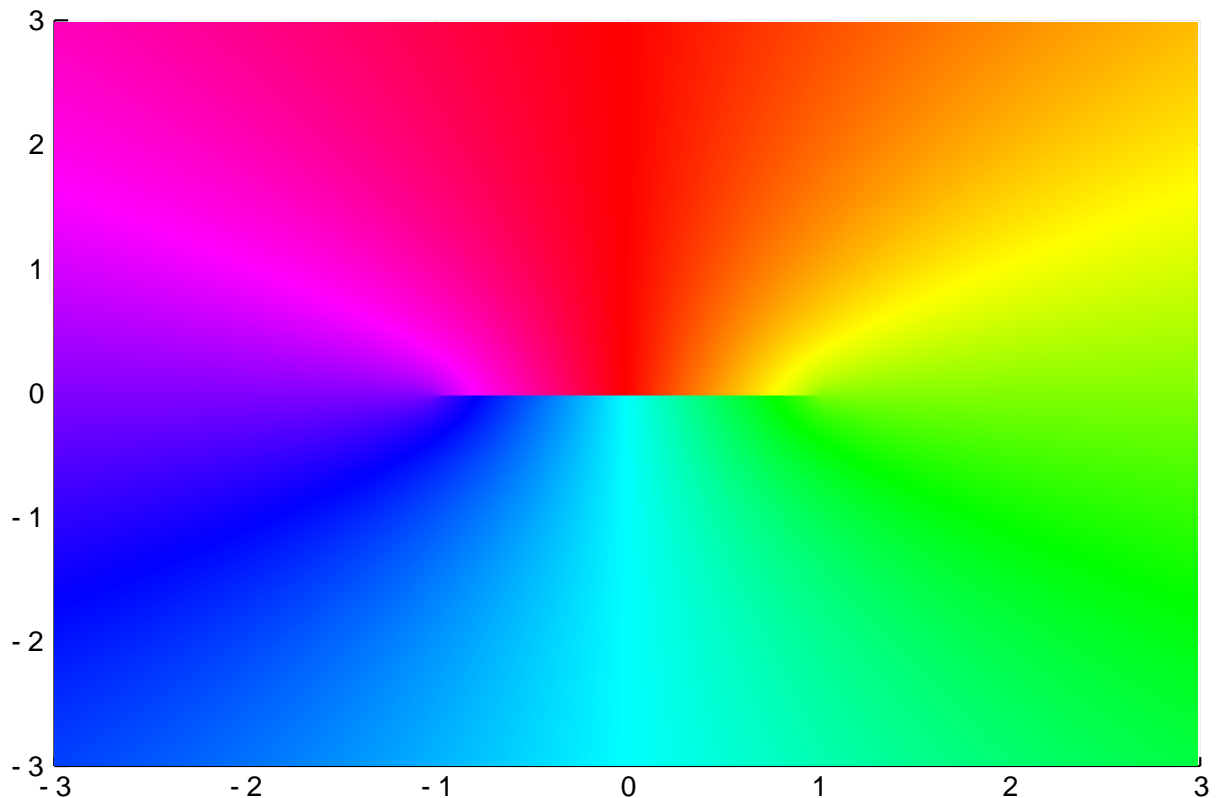
```
using ComplexPhasePortrait, Plots, ApproxFun, SingularIntegralEquations
```

```
H(f) = -hilbert(f)
```

```
H(f,x) = -hilbert(f,x)
```

```
φ = z -> sqrt(z-1)*sqrt(z+1)/(2im*(1+z^2)) +  
          sqrt(im-1)*sqrt(im+1)/4*1/(z-im) -  
          sqrt(-im-1)*sqrt(-im+1)/4*1/(z+im)
```

```
phaseplot(-3..3, -3..3, φ)
```



We can also see from the phase plot (Regularity): we have weaker than pole singularities, otherwise we would have at least a full, counter clockwise colour wheel. We can check decay as well:

```
φ(200.0+200.0im)
```

```
0.0005177682933717976 + 0.000517765612545622im
```

Finally, we compare it numerically to `cauchy(f, z)` which is implemented in `SingularIntegralEquations.jl`:

```
x = Fun()
```

```
φ(2.0+2.0im), cauchy(sqrt(1-x^2)/(1+x^2), 2.0+2.0im)
```

```
(0.05303535516221752 + 0.05036581190871381im, 0.0530353551622175 + 0.050365811908713795im)
```

1.1.2 2.

Recall that

$$\psi(z) = \frac{\log(z-1) - \log(z+1)}{2\pi i}$$

satisfies

$$\psi_+(x) - \psi_-(x) = 1$$

Therefore, consider

$$\phi_1(z) = \frac{\psi(z)}{2+z}$$

This has the right jump, but has an extra pole at $z = -2$: for $x < -1$ we have

$$\phi_1(x) = \frac{\log_+(x-1) - \log_+(x+1)}{2\pi i} \frac{1}{2+x} = \frac{\log(1-x) - \log(-1-x)}{2\pi i} \frac{1}{2+x}$$

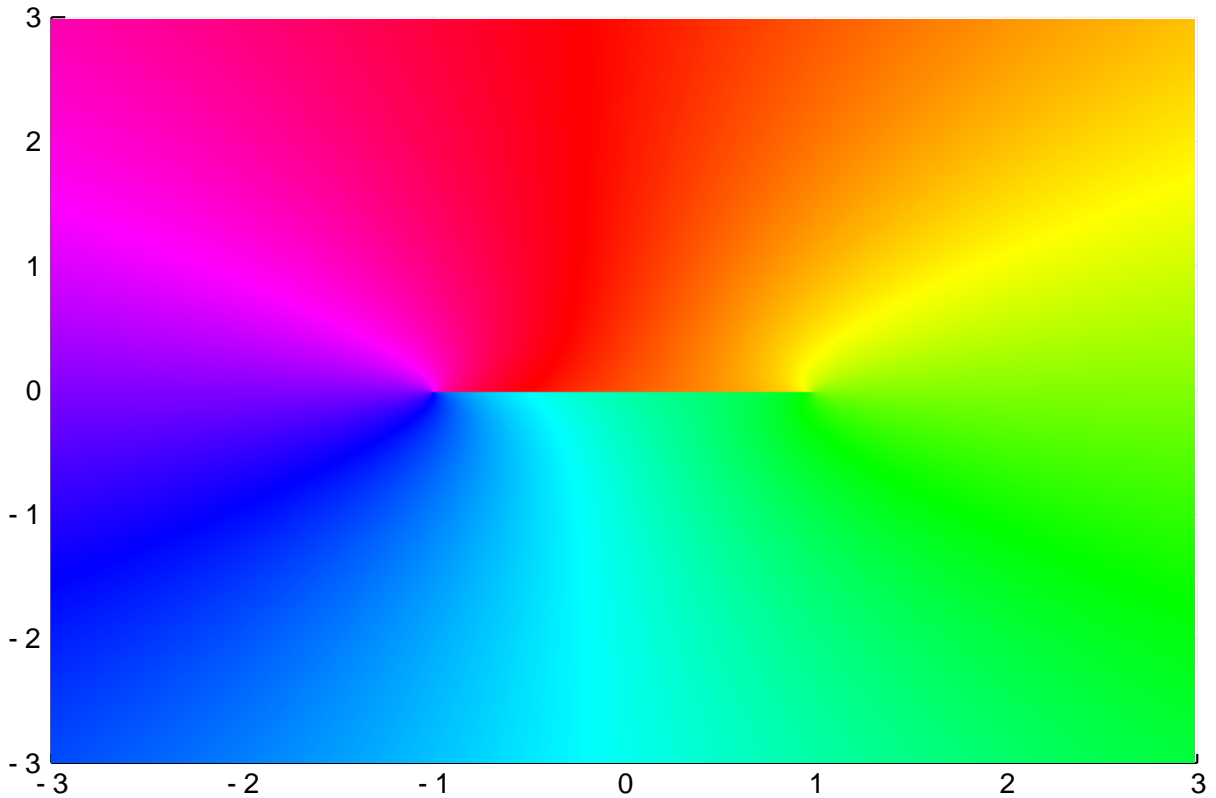
hence we arrive at the solution

$$\phi_1(z) - \frac{\log 3}{2\pi i(2+z)}$$

We can verify that $\phi_1(\infty) = 0$.

$\varphi = z \rightarrow (\log(z-1) - \log(z+1)) / ((2\pi i) * (2+z)) - \log(3) / (2\pi i * (2+z))$

`phaseplot(-3..3, -3..3, φ)`



1.1.3 3.

We first calculate the Cauchy transform of $f(x) = x/\sqrt{1-x^2}$:

$$\phi(z) = \frac{iz}{2\sqrt{z-1}\sqrt{z+1}} - \frac{i}{2}$$

This vanishes at ∞ and has the correct jump. We then have

$$i\mathcal{H}f(x) = \phi^+(x) + \phi^-(x) = -i$$

This implies that (note the sign)

$$\int_{-1}^1 \frac{t}{(t-x)\sqrt{1-t^2}} dt = -\pi\mathcal{H}f(x) = -\pi$$

```
f = x/sqrt(1-x^2)
-pi*H(f, 0.1)
```

3.141592653589793

1.2 Problem 1.2

1.2.1 1.2.1

From Problem 1.1 part 4, we have a solution:

$$\phi(z) = -\frac{z}{2\sqrt{z-1}\sqrt{z+1}} + \frac{1}{2}$$

All other solutions are then of the form:

$$\phi(z) + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

```
C = randn()
phi = z -> -z/(2*sqrt(z-1)*sqrt(z+1))+1/2 + C/(sqrt(z-1)*sqrt(z+1))
phi(0.1+0.0im)+phi(0.1-0.0im), phi(1E8)
```

(1.0 + 0.0im, 1.3672454908531775e-8)

1.2.2 1.2.2

$$\psi(z) = \phi(z) - 1$$

satisfies

$$\psi_+(x) + \psi_-(x) = -2, \psi(\infty) = 0$$

hence we know that $\psi(z) = \frac{z}{\sqrt{z-1}\sqrt{z+1}} - 1 + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$, giving

$$\psi(z) = \frac{z}{\sqrt{z-1}\sqrt{z+1}} + \frac{C}{\sqrt{z-1}\sqrt{z+1}} + 1$$

```
C = randn()
φ = z -> z/(sqrt(z-1)*sqrt(z+1)) + C/(sqrt(z-1)*sqrt(z+1))

φ(0.1+0.0im)+φ(0.1-0.0im),φ(1E9)

(0.0 + 0.0im, 1.00000000019499917)
```

1.2.3 1.2.3

For $f(x) = \sqrt{1-\diamond^2}$, we use the formula

$$\phi(z) = \frac{i}{\sqrt{z-1}\sqrt{z+1}}\mathcal{C}[\sqrt{1-\diamond^2}f](z) + \frac{C}{\sqrt{z-1}\sqrt{z+1}} = \frac{i}{\sqrt{z-1}\sqrt{z+1}}\mathcal{C}[1-\diamond^2](z) + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

We already know $\mathcal{C}1(z)$, and we can deduce $\mathcal{C}[\diamond^2]$ as follows: try

$$\phi_1(z) = z^2\mathcal{C}1(z) = z^2 \frac{\log(z-1) - \log(z+1)}{2\pi i}$$

this has the right jump, but blows up at ∞ like:

$$x^2(\log(x-1) - \log(x+1)) = x^2(\log(1-1/x) - \log(1+1/x)) = -2x + O(x^{-1})$$

using

$$\log z = (z-1) - \frac{1}{2}(z-1)^2 + O(z-1)^3$$

Thus we have

$$\mathcal{C}[\diamond^2](z) = \frac{z^2(\log(z-1) - \log(z+1)) + 2z}{2\pi i}$$

and

$$\phi(z) = \frac{i}{\sqrt{z-1}\sqrt{z+1}} \frac{(1-z^2)(\log(z-1) - \log(z+1)) - 2z}{2\pi i} + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

Demonstration Here we see that the Cauchy transform of x^2 has the correct formula:

```
z = 2.0+2.0im
cauchy(x^2, z), (z^2*(log(z-1)-log(z+1))+2z)/(2π*im)

(0.0283222937395961 + 0.024377589786690298im, 0.028322293739596032 + 0.02437758978669024im)
```

We now see that ϕ has the right jumps:

```
C = randn()
φ = z -> im/(sqrt(z-1)*sqrt(z+1)) * ((1-z^2)*(log(z-1)-log(z+1))-2z)/(2π*im) +
C/(sqrt(z-1)sqrt(z+1))
```

```
φ(0.1+0.0im) + φ(0.1-0.0im) - sqrt(1-0.1^2)
```

```
1.1102230246251565e-16 + 0.0im
```

Finally, it vanishes at infinity:

```
φ(1E5)
```

```
2.9107584856300957e-5 - 0.0im
```

1.2.4 1.2.4

Let $f(x) = \frac{1}{1+x^2}$. From Problem 1.1 part 1 we know

$$\mathcal{C}[\sqrt{1-\diamond^2}]f(z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2i(1+z^2)} + \frac{\sqrt{i-1}\sqrt{i+1}}{4} \frac{1}{z-i} - \frac{\sqrt{-i-1}\sqrt{-i+1}}{4} \frac{1}{z+i}$$

hence from the solution formula we have

$$\phi(z) = \frac{1}{2(1+z^2)} + \frac{\sqrt{i-1}\sqrt{i+1}i}{4\sqrt{z-1}\sqrt{z+1}} \frac{1}{z-i} - \frac{\sqrt{-i-1}\sqrt{-i+1}i}{4\sqrt{z-1}\sqrt{z+1}} \frac{1}{z+i} + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

But we want something stronger: that $\phi(z) = O(z^{-2})$. To accomplish this, we need to choose C . Fortunately, I made the problem easy as every term apart from the last one is already $O(z^{-2})$, so choose $C = 0$:

```
φ = z -> 1/(2*(1+z^2)) +
sqrt(im-1)sqrt(im+1)*im/(4sqrt(z-1)sqrt(z+1))*1/(z-im) -
sqrt(-im-1)sqrt(-im+1)*im/(4sqrt(z-1)sqrt(z+1))*1/(z+im)
```

```
φ(1E5)*1E5^2
```

```
-0.2071067812011923 + 0.0im
```

We see also that it has the right jump:

```
φ(0.1+0.0im) + φ(0.1-0.0im), 1/(1+0.1^2)
```

```
(0.9900990099009901 + 0.0im, 0.9900990099009901)
```

1.3 Problem 1.3

1. From the Hilbert formula, we know that the general solution of $\mathcal{H}u = f$ is

$$u(x) = \frac{-1}{\sqrt{1-x^2}} \mathcal{H} \left[f(\diamond) \sqrt{1-\diamond^2} \right] (x) - \frac{C}{\sqrt{1-x^2}}$$

Plugging in $f(x) = x/\sqrt{1-x^2}$ means we need to calculate

$$\mathcal{H}[\diamond](x)$$

We do so by first finding the Cauchy transform. Consider

$$\phi_1(z) = z\mathcal{C}1(z) = z \frac{\log(z-1) - \log(z+1)}{2\pi i}$$

This has the right jump:

$$\phi_1^+(x) - \phi_1^-(x) = x$$

but doesn't decay at ∞ :

$$\begin{aligned} x \frac{\log(x-1) - \log(x+1)}{2\pi i} &= x \frac{\log x + \log(1-1/x) - \log x - \log(1+1/x)}{2\pi i} \\ &= -x \frac{2}{x2\pi i} = -\frac{1}{i\pi} \end{aligned}$$

But this means that

$$\phi(z) = \phi_1(z) + \frac{1}{i\pi} = z \frac{\log(z-1) - \log(z+1)}{2\pi i} + \frac{1}{i\pi}$$

Decays and has the right jump, hence is $\mathcal{C}[\diamond](z)$.

```
t = Fun()
z = 2.0+2.0im
cauchy(t, z), z*(log(z-1)-log(z+1))/(2π*im) + 1/(im*π)

(0.013174970881571602 - 0.00098617598822645im, 0.013174970881571569 - 0.000
9861759882264232im)
```

Therefore, we have

$$\mathcal{H}[\diamond](x) = -i(\mathcal{C}^+ + \mathcal{C}^-) \diamond(x) = -x \frac{\log(1-x) - \log(1+x)}{\pi} - \frac{2}{\pi}$$

```
x = 0.1
H(t,x), -x*(log(1-x)-log(1+x))/(π) - 2/π

(-0.6302322257442835, -0.6302322257442834)
```

Therefore, we get

$$u(x) = -\frac{x(\log(1-x) - \log(1+x)) + 2}{\pi\sqrt{1-x^2}} - \frac{C}{\sqrt{1-x^2}}$$

This can be verified in Mathematica via

```
NIntegrate[-((
  x (Log[1 - x] - Log[1 + x]) +
  2)/(Pi Sqrt[1 - x^2] (x - 0.1))), {x, -1, 0.1, 1},
PrincipalValue -> True]/Pi
```

2. Following the procedure of multiplying $\mathcal{C}[\sqrt{1-\diamond^2}](z)$ by $1/(2+z)$ and subtracting off the pole at $z = -2$, we first find:

$$\mathcal{C}\left[\frac{\sqrt{1-\diamond^2}}{2+\diamond}\right](z) = \frac{\sqrt{z-1}\sqrt{z+1}-z}{2i(2+z)} - \frac{\sqrt{-2-1}\sqrt{-2+1}+2}{2i(2+z)} = \frac{\sqrt{z-1}\sqrt{z+1}-z}{2i(2+z)} + \frac{\sqrt{3}-2}{2i(2+z)}$$

```
t = Fun()
z = 2.0+2.0im
cauchy(sqrt(1-t^2)/(2+t), z), (sqrt(z-1)sqrt(z+1)-z)/(2im*(2+z)) +
(sqrt(3)-2)/(2im*(z+2))

(0.03230545315801244 + 0.032449695183223826im, 0.032305453158012455 + 0.032
449695183223784im)
```

Therefore, calculating $-i(\mathcal{C}^+ + \mathcal{C}^-)$ we find that

$$\mathcal{H}\left[\frac{\sqrt{1-\diamond^2}}{2+\diamond}\right](z) = \frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x}$$

```
x = 0.1
H(sqrt(1-t^2)/(2+t), x), x/(2+x)-(sqrt(3)-2)/(2+x)

(0.17521390115767738, 0.17521390115767752)
```

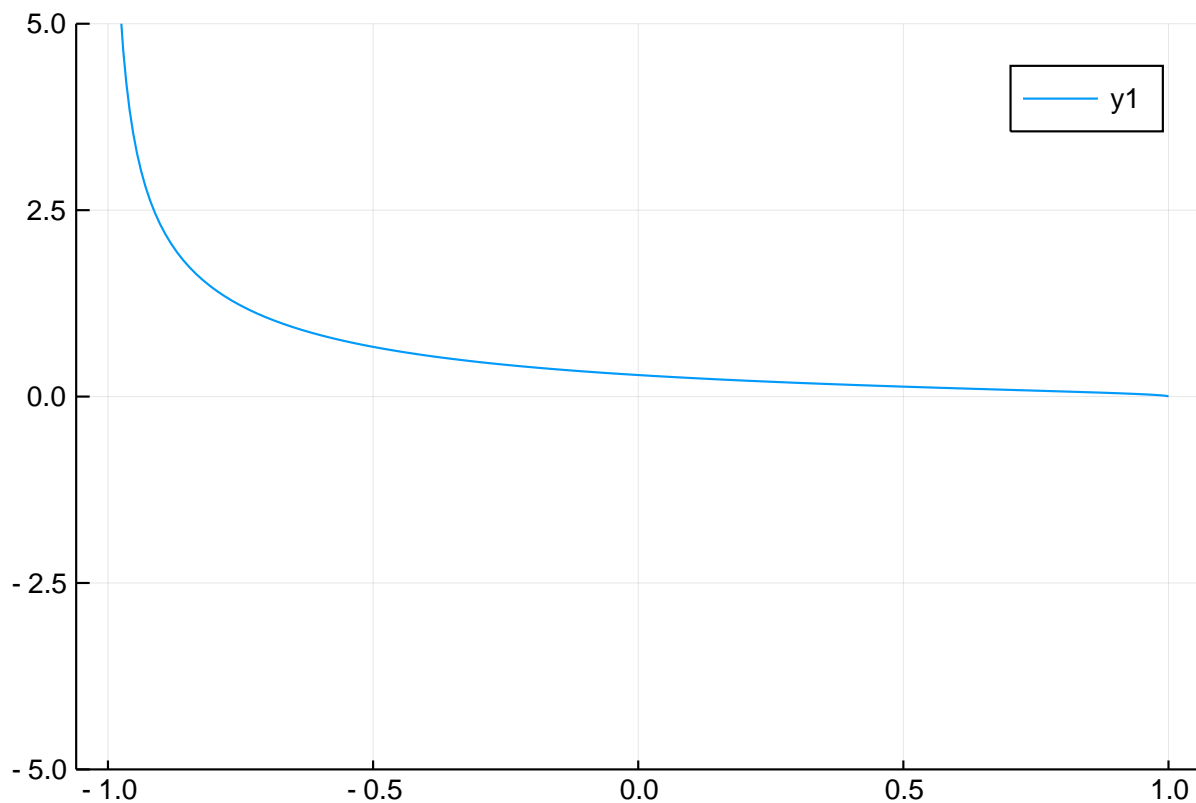
Thus the general solution is

$$u(x) = -\frac{1}{\sqrt{1-x^2}} \left(\frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x} + C \right)$$

We need to choose C so this is bounded at the right-endpoint: In other words,

$$u(x) = -\frac{1}{\sqrt{1-x^2}} \left(\frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x} + \frac{\sqrt{3}-3}{3} \right)$$

```
u = -(t/(2+t)-(sqrt(3)-2)/(2+t)+(sqrt(3)-3)/3)/sqrt(1-t^2)
plot(u; ylims=(-5,5))
```

```
x = 0.1
H(u,x) , 1/(2+x)

(0.47619047619047544, 0.47619047619047616)
```

1.4 Problem 2.1

Doing the change of variables $\zeta = bs$ we have

$$\log(ab) = \int_1^{ab} \frac{d\zeta}{\zeta} = \int_{1/b}^a \frac{ds}{s}$$

if γ does not surround the origin, we have

$$0 = \oint_{\gamma} \frac{ds}{s} = \left[\int_1^{1/b} + \int_{1/b}^a + \int_a^1 \right] \frac{ds}{s}$$

which implies

$$\log(ab) = \left[-\int_a^1 - \int_1^{1/b} \right] \frac{ds}{s} = \log a - \log \frac{1}{b} = \log a + \log b$$

Here's a picture:

```
a = 1.0+2.0im
b = -1.0-2.0im

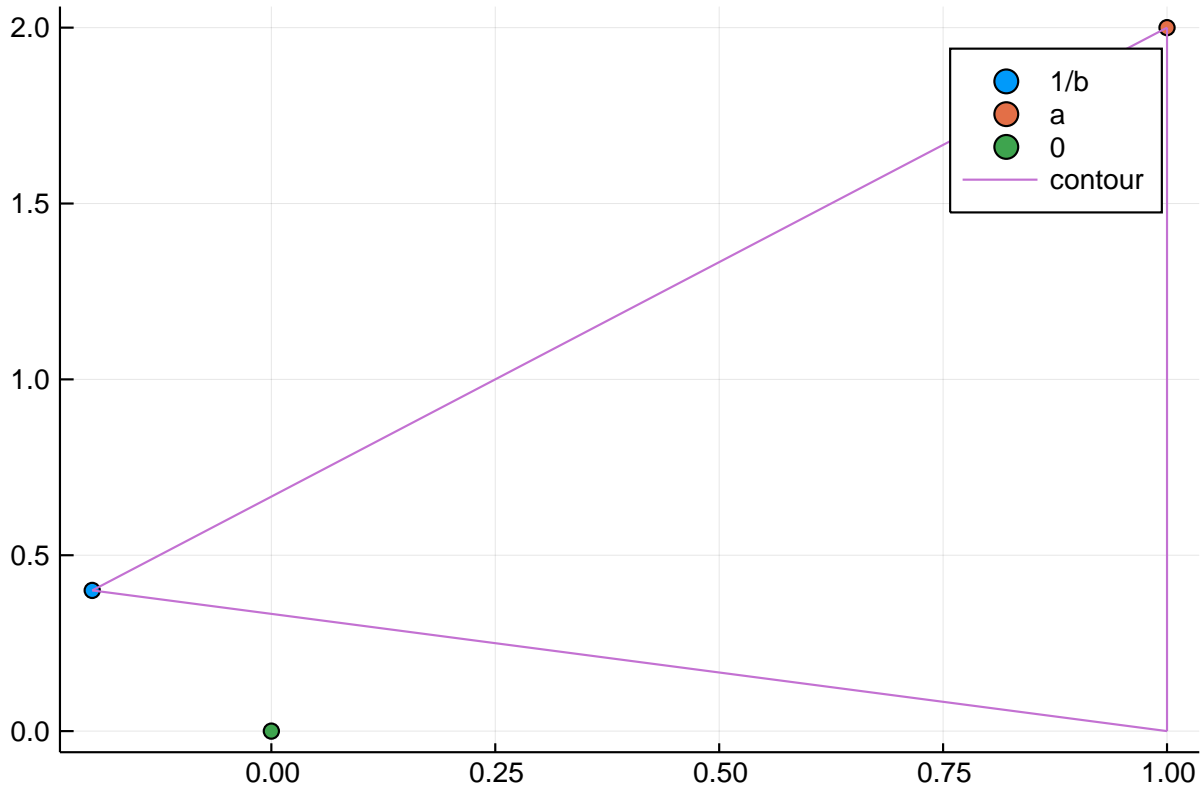
@show log(a*b)

log(a * b) = 1.6094379124341003 - 0.9272952180016122im
```

```
@show log(a) + log(b)
```

```
log(a) + log(b) = 1.6094379124341003 - 0.9272952180016123im
```

```
scatter([real(1/b)], [imag(1/b)]; label="1/b")
scatter!([real(a)], [imag(a)]; label="a")
scatter!([0.0], [0.0]; label="0")
plot!(Segment(1, 1/b) ∪ Segment(1/b, a) ∪ Segment(a, 1); label="contour")
```



If it surrounds the origin counter-clockwise, that is, it has positive orientation, we have $2\pi i = \oint_{\gamma} \frac{ds}{s}$, which shows that

$$\log(ab) = 2\pi i - \left[\int_a^1 + \int_1^{1/b} \right] \frac{ds}{s} = \log a + \log b + 2\pi i$$

and a similar result when counter clockwise.

```
a = 1.0+2.0im
b = -2.0+2.0im
```

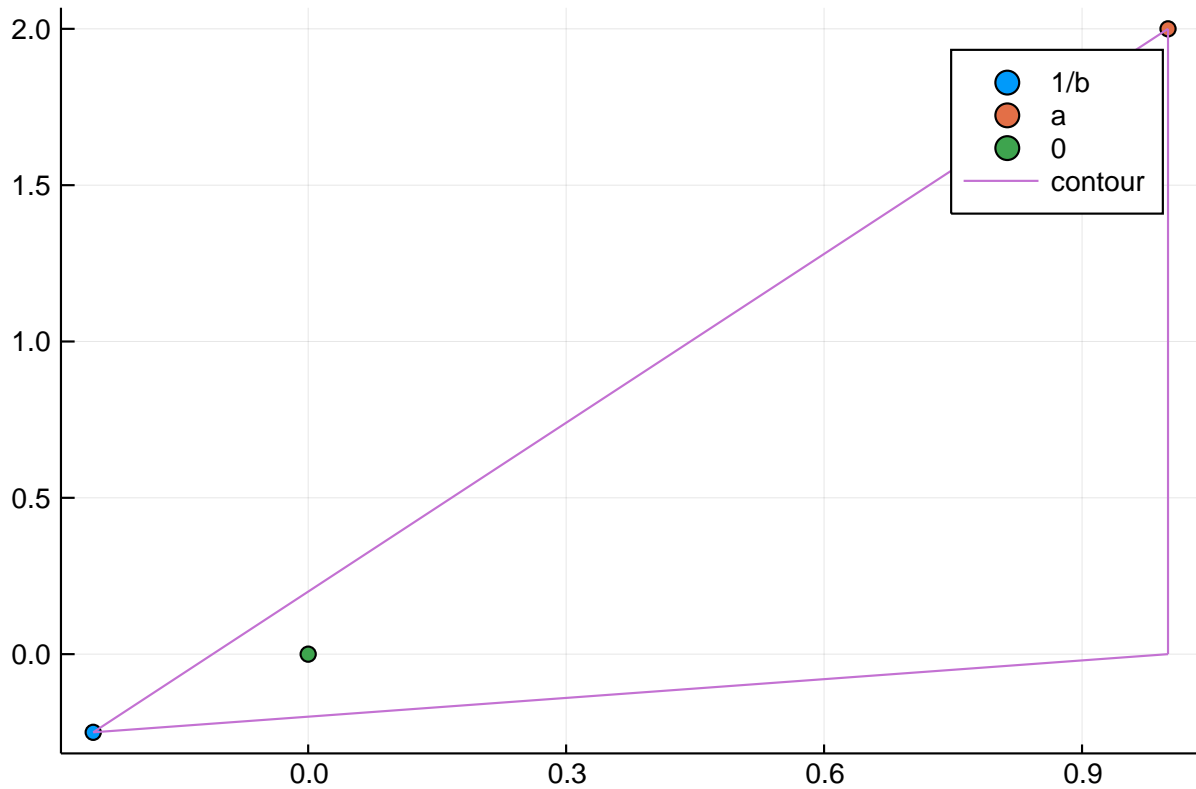
```
@show log(a*b)
```

```
log(a * b) = 1.8444397270569681 - 2.819842099193151im
```

```
@show log(a) + log(b) - 2π*im
```

```
(log(a) + log(b)) - (2π) * im = 1.844439727056968 - 2.819842099193151im
```

```
scatter([real(1/b)], [imag(1/b)]; label="1/b")
scatter!([real(a)], [imag(a)]; label="a")
scatter!([0.0], [0.0]; label="0")
plot!(Segment(1, 1/b) ∪ Segment(1/b, a) ∪ Segment(a, 1); label="contour")
```



If the contour passes through the origin, there are three possibility:

1. $[a, 1]$ contains zero, hence $a < 0$

2.

$$[1, 1/b]$$

contains zero, hence $b < 0$

3.

$$[1/b, a]$$

contains zero, which can only be true if $ab < 0$ by considering the equation of the line segment.

1. In the case where $a < 0$ and $b < 0$ (and hence $ab > 0$), perturbing a above and b below or vice versa avoids γ winding around zero, so we have

$$\log(ab) = \log_+ a + \log_- b = \log_- a + \log_+ b = \log_+ a + \log_+ b - 2\pi i = \log_- a + \log_- b + 2\pi i$$

`a = -2.0`

`b = -3.0`

`@show log(a*b)`

`log(a * b) = 1.791759469228055`

`@show log(a+0.0im) + log(b-0.0im)`

```
log(a + 0.0im) + log(b - 0.0im) = 1.791759469228055 + 0.0im
```

```
@show log(a-0.0im) + log(b+0.0im)
```

```
log(a - 0.0im) + log(b + 0.0im) = 1.791759469228055 + 0.0im
```

```
@show log(a-0.0im) + log(b-0.0im) + 2π*im
```

```
log(a - 0.0im) + log(b - 0.0im) + (2π) * im = 1.791759469228055 + 0.0im
```

```
@show log(a+0.0im) + log(b+0.0im) - 2π*im;
```

```
(log(a + 0.0im) + log(b + 0.0im)) - (2π) * im = 1.791759469228055 + 0.0im
```

In the case where $a < 0$ and $b > 0$, then $ab < 0$, but we can perturb a above/below to get

$$\log_{\pm}(ab) = \log_{\pm} a + \log b$$

(and by symmetry, the equivalent holds for $b < 0$ and $a > 0$.)

```
a = -2.0
```

```
b = 3.0
```

```
@show log(a*b +0.0im)
```

```
log(a * b + 0.0im) = 1.791759469228055 + 3.141592653589793im
```

```
@show log(a+0.0im) + log(b);
```

```
log(a + 0.0im) + log(b) = 1.791759469228055 + 3.141592653589793im
```

```
@show log(a*b -0.0im)
```

```
log(a * b - 0.0im) = 1.791759469228055 - 3.141592653589793im
```

```
@show log(a-0.0im) + log(b);
```

```
log(a - 0.0im) + log(b) = 1.791759469228055 - 3.141592653589793im
```

In the case where $a < 0$, if $\Im b > 0$ we can perturb a below so that γ does not contain zero, giving us

$$\log(ab) = \log_- a + \log b$$

similarly, if $\Im b < 0$ we can perturb a above.

```
a = -2.0
```

```
b = 3.0 + im
```

```
@show log(a*b)
```

```
log(a * b) = 1.8444397270569681 - 2.819842099193151im
```

```
@show log(a-0.0im) + log(b);
```

```
log(a - 0.0im) + log(b) = 1.8444397270569683 - 2.819842099193151im
```

```
b = 3.0 + im;
```

```
@show log(a*b)
```

```
log(a * b) = 1.8444397270569681 - 2.819842099193151im
```

```
@show log(a+0.0im) + log(b);
```

```
log(a + 0.0im) + log(b) = 1.8444397270569683 + 3.4633432079864352im
```

2. In this case, swap the role of a and b and use the answers for $a < 0$.

3. Finally, we have the case $ab < 0$ and neither a nor b is real. Note that

$$ab = (a_x + ia_y)(b_x + ib_y) = a_x b_x - a_y b_y + i(a_x b_y + a_y b_x)$$

It follows if $b_x > 0$ we have

$$(ab)_+ = a_+ b$$

and if $b_x < 0$ we have

$$(ab)_+ = a_- b$$

```
a = 1.0 + 1.0im
```

```
b = 1.0 + 1.0im
```

```
@show log(a*b + eps()im)
```

```
log(a * b + eps() * im) = 0.6931471805599453 + 1.5707963267948966im
```

```
@show log((a+eps()im)*b)
```

```
log((a + eps() * im) * b) = 0.6931471805599453 + 1.5707963267948968im
```

```
a = 1.0 + 1.0im
```

```
b = -1.0 + 1.0im
```

```
@show log(a*b + eps()im)
```

```
log(a * b + eps() * im) = 0.6931471805599453 + 3.141592653589793im
```

```
@show log((a-eps()im)*b)
```

```
log((a - eps() * im) * b) = 0.6931471805599452 + 3.141592653589793im
```

```
0.6931471805599452 + 3.141592653589793im
```

We can use this perturbation to reduce to the previous cases. For example, if $a = 1 + i$ and $b = -1 + i$, perturbing ab above causes a to be perturbed above, which causes the contour to surround the origin clockwise, hence we have

$$\log_+(ab) = \log(a)_+ b = \log ab - 2\pi i$$

```
a = 1.0 + 1.0im
```

```
b = -1.0 + 1.0im
```

```
@show log(a*b - eps()im)
```

```
log(a * b - eps() * im) = 0.6931471805599453 - 3.141592653589793im
```

```
@show log(a)+log(b)-2π*im;
```

```
(log(a) + log(b)) - (2π) * im = 0.6931471805599453 - 3.141592653589793im
```

1.5 Problem 2.2

Use the contour $\gamma(t) = 1 + t(1 - z)$ to reduce it to a normal integral: $\overline{\log z} = \overline{\int_1^z \frac{1}{\zeta} d\zeta} = \int_0^1 \frac{(\bar{z}-1)}{1+(\bar{z}-1)t} dt = \int_0^1 \frac{(\bar{z}-1)}{1+(\bar{z}-1)t} dt = \int_1^{\bar{z}} \frac{d\zeta}{\zeta} = \log \bar{z}$. We then have, since the contour from 1 to $1/(\bar{z})$ to z never surrounds the origin since both $\Im z$ and $\Im 1/(\bar{z})$ have the same sign, we have

$$2\Re \log z = \log z + \overline{\log z} = \log z + \log \bar{z} = \log z\bar{z} = \log |z|^2 = 2 \log |z|$$

On the other hand, we have, where the contour of integration is chosen to be to the right of zero and then we do the change of variables $\zeta = |z|e^{i\theta}$

$$2\Im \log z = \log z - \log \bar{z} = \int_{\bar{z}}^z \frac{d\zeta}{\zeta} = i \int_{-\arg z}^{\arg z} d\theta = 2i \arg z$$

1.6 Problem 2.3

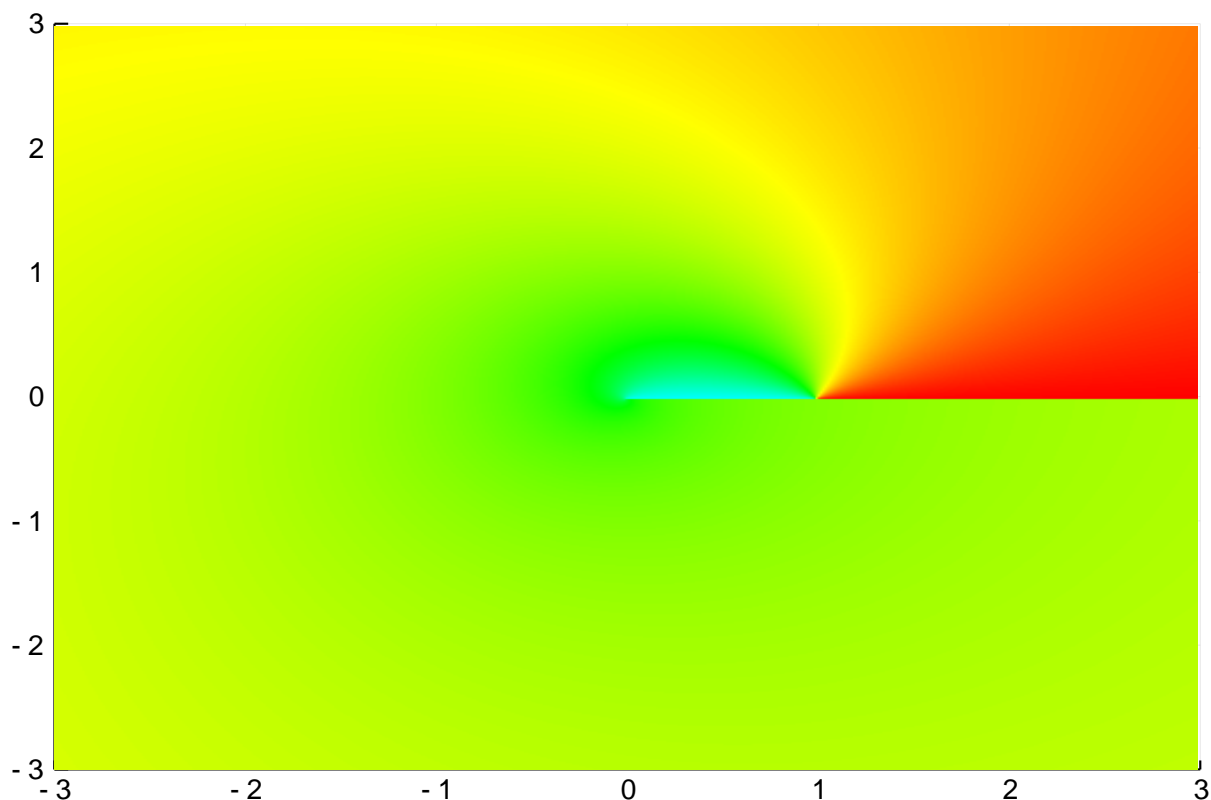
We first show that it is analytic on $(-\infty, 0)$. To do this, we need to show that the limit from above equals the limit from below: for $x < 0$ we have $\log_1^+ x - \log_1^- x = \log_+ x - \log_- x + 2\pi i = 0$. Then for $x > 0$ and using $\log_1^\pm(x) = \lim_{\epsilon \rightarrow 0} \log(x \pm i\epsilon)$ we find

$$\log_1^+(x) - \log_1^- x = \log x - \log x - 2\pi i = -2\pi i$$

Demonstration Here we see that the following is the analytic continuation:

```
log1 = z -> begin
  if imag(z) > 0
    log(z)
  elseif imag(z) == 0 && real(z) < 0
    log(z + 0.0im)
  elseif imag(z) < 0
    log(z) + 2*pi*im
  else
    error("log1 not defined on real axis")
  end
end

phaseplot(-3..3, -3..3, log1)
```



1.7 Problem 3.1

1. Because it's absolutely integrable, we can exchange derivatives and integrals to determine

$$\frac{d\mathcal{C}f}{dz} = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

2. There are two different possible approaches:

– the subtract and add back in technique: since f is analytic for z near ζ , we can write

$$\mathcal{C}f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta) - f(z)}{(\zeta - z)^2} d\zeta + f(z)\mathcal{C}1(z)$$

Therefore

$$\mathcal{C}^+f(\zeta) - \mathcal{C}^-f(\zeta) = f(\zeta)(\mathcal{C}^+1(\zeta) - \mathcal{C}^-1(\zeta))$$

But we know (using Cauchy's integral formula / Residue calculus)

$$\mathcal{C}1(z) = \begin{cases} 1 & |z| < 1 \\ 0 & |z| > 1 \end{cases}$$

hence $(\mathcal{C}^+ - \mathcal{C}^-)1(\zeta) = 1$

- Since f is analytic, we have for any radius $R > 1$ but inside the annulus

$$\mathcal{C}^+ f(\zeta) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Similarly, for $\mathcal{C}^- f(\zeta)$ with any radius $r < 1$ but inside the annulus. Therefore,

$$\mathcal{C}^+ f(\zeta) - \mathcal{C}^- f(\zeta) = \frac{1}{2\pi i} \left[\oint_{|z|=R} - \oint_{|z|=r} \right] \frac{f(\zeta)}{\zeta - z} d\zeta$$

Deforming the contour and using Cauchy-integral formula gives the result.

3. This follow since $\frac{1}{\zeta - z} \rightarrow 0$ uniformly.

1.8 Problem 3.2

Suppose we have another solution ϕ and consider $\psi(z) = \phi(z) - \mathcal{C}f(z)$. Then on the circle we have

$$\psi_+(\zeta) - \psi_-(\zeta) = \phi_+(\zeta) - \mathcal{C}_+ f(\zeta) - \phi_-(\zeta) + \mathcal{C}_- f(\zeta) = f(\zeta) - f(\zeta) = 0$$

Thus ψ is entire, and since it decays at infinity, it must be zero by Liouville's theorem.

1.9 Problem 3.3

When $k \geq 0$, we have from 3.1 and 3.2

$$\mathcal{C}[\diamond^k](z) = \begin{cases} z^k & |z| < 0 \\ 0 & |z| > 0 \end{cases}$$

when $k < 0$ since $\mathcal{C}[\diamond^k]^+(\zeta) - \mathcal{C}[\diamond^k]^-(\zeta) = \zeta^k - 0 = \zeta^k$. we similarly have

$$\mathcal{C}[\diamond^k](z) = \begin{cases} 0 & |z| < 0 \\ -z^k & |z| > 0 \end{cases}$$

Therefore,

$$\Im \mathcal{C}^-[\diamond^k](\zeta) = \begin{cases} 0 & k \geq 0 \\ -\frac{\zeta^k - \zeta^{-k}}{2i} & k < 0 \end{cases}$$

and

$$\Re \mathcal{C}^-[\diamond^k](\zeta) = \begin{cases} 0 & k \geq 0 \\ -\frac{\zeta^k + \zeta^{-k}}{2} & k < 0 \end{cases}$$

1.10 Problem 3.4

Express the solution outside the circle as

$$v(x, y) = \Im(e^{-i\theta} z + \mathcal{C}f(z))$$

for a to-be-determined f . On the circle, this reduces to

$$\Im \mathcal{C}^- f(\zeta) = -\cos \theta \frac{\zeta - \zeta^{-1}}{2i} + \sin \theta \frac{\zeta + \zeta^{-1}}{2}$$

Unlike the real case, we can include imaginary coefficients, thus the solution is

$$f(\zeta) = (\cos \theta + i \sin \theta) \zeta^{-1}$$

and thus the full solution is

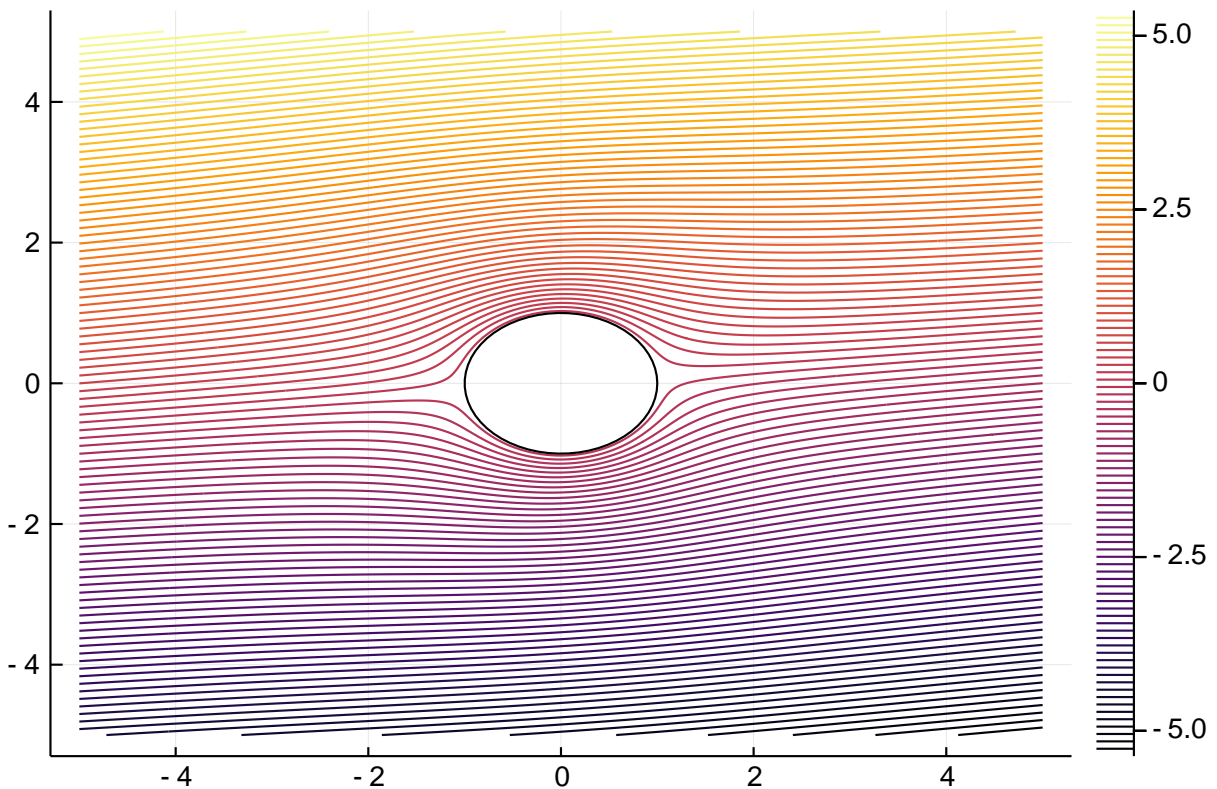
$$v(x, y) = \Im(e^{-i\theta} z - e^{i\theta} z^{-1})$$

$\theta = 0.1$

$v = (x, y) \rightarrow x^2 + y^2 < 1 ? 0 : \text{imag}(\exp(-i\theta) * (x+im*y) + \exp(i\theta) * (x+im*y)^{-1})$

$xx = yy = -5:0.01:5$

`contour(xx, yy, v.(xx', yy); nlevels=100)`
`plot!(Circle(); color=:black, legend=false)`



Problem 4.1

$$z^\alpha$$

has the limits $z_\pm^\alpha = e^{\pm i\pi\alpha} |z|^\alpha$, thus choose $\alpha = -\frac{\theta}{2\pi}$ where if we take $0 < \theta < 2\pi$ we have $0 < \alpha < 1$ (the case $\theta = 0$ and $\theta = \pi$ are covered by the Cauchy transform, that is). Then consider

$$\kappa(z) = (z-1)^{-\alpha}(z+1)^{\alpha-1}$$

which has weaker than pole singularities and satisfies $\kappa(z) \sim z^{-1}$. For $-1 < x < 1$ it has the right jump

$$\kappa_+(x) = (x-1)_+^\alpha (x+1)^{1-\alpha} = e^{i\pi\alpha} (1-x)^\alpha (x+1)^{1-\alpha} = e^{2i\pi\alpha} (x-1)_-^\alpha (x+1)^{1-\alpha} = e^{2i\pi\alpha} \kappa_-(x) = e^{i\theta} \kappa_-(x)$$

and for $x < -1$ it has the jump

$$\kappa_+(x) = (x-1)_+^\alpha (x+1)_+^{1-\alpha} = e^{i\pi\alpha} e^{i\pi(1-\alpha)} (1-x)^\alpha (-1-x)^{1-\alpha} = \kappa_-(x)$$

hence κ is analytic.

We need to show this times a constant spans the entire space. Suppose we have another solution $\tilde{\kappa}$ and consider $r(z) = \frac{\tilde{\kappa}(z)}{\kappa(z)}$. Note by construction that κ has no zeros. Then

$$r_+(x) = \frac{\tilde{\kappa}_+(x)}{\kappa_+(x)} = \frac{\tilde{\kappa}_-(x)}{\kappa_-(x)} = r_-(x)$$

hence r is analytic on $(-1, 1)$. It has weaker than pole singularities because $\kappa(z)^{-1}$ is actually bounded at ± 1 . Therefore r is bounded and entire, and thus must be a constant $r(z) \equiv r$, and thence $\tilde{\kappa}(z) = r\kappa(z)$.

```

theta = 2.3
alpha = -theta/(2*pi)
kappa = z -> (z-1)^(-alpha)*(z+1)^(alpha-1)
kappa(0.1+0.0im) - exp(im*theta)*kappa(0.1-0.0im), kappa(100.0)

(0.0 + 1.1102230246251565e-16im, 0.00982876598532333)

phaseplot(-3..3, -3..3, kappa)

```



1.11 Problem 4.2

We want to mimic the solution of $\phi_+(x) + \phi_-(x)$. So take

$$\phi(z) = \kappa(z) \mathcal{C} \left[\frac{f}{\kappa_+} \right] (z) = e^{-i\theta/2} (z-1)^{-\alpha} (z+1)^{\alpha-1} \mathcal{C}[f(1-x)^\alpha (1+x)^{1-\alpha}](z)$$

This has the jump

$$\begin{aligned} \phi_+(x) - e^{i\theta} \phi_-(x) &= \kappa_+(z) \mathcal{C}_+ \left[\frac{f}{\kappa_+} \right] (x) - e^{i\theta} \kappa_-(z) \mathcal{C}_- \left[\frac{f}{\kappa_+} \right] (x) \\ &= \kappa_+(x) \left(\mathcal{C}_+ \left[\frac{f}{\kappa_+} \right] (x) - \mathcal{C}_- \left[\frac{f}{\kappa_+} \right] (x) \right) = f \end{aligned}$$

Thus the general solution is $\phi(z) + C\kappa(z)$.

$\theta = 2.3$

$\alpha = -\theta/(2\pi)$

$\kappa = z \rightarrow (z-1)^{-\alpha} (z+1)^{\alpha-1}$

$x = \text{Fun}()$

$\kappa_+ = \exp(i\theta/2) (1-x)^{-\alpha} (x+1)^{\alpha-1}$

$f = \text{Fun}(\exp)$

$z = 2+im$

$\varphi = z \rightarrow \kappa(z) * \text{cauchy}(f/\kappa_+, z)$

$\varphi(0.1+0.0im) - \exp(i\theta) * \varphi(0.1-0.0im) - f(0.1)$

-1.7763568394002505e-15 - 4.440892098500626e-16im

1.12 Problem 4.3

Note for $x < 0$

$$x_+^{i\beta} = e^{i\beta \log_+ x} = e^{i\beta \log_- x - 2\pi y} = e^{-2\pi\beta} x_-^{i\beta}$$

```

β = 2.3;
x = -2.0
(x+0.0im)^(im*β) - exp(-2π*β)*(x-0.0im)^(im*β)

-3.3881317890172014e-21 + 1.0842021724855044e-19im

```

We actually have bounded (oscillatory) growth near zero since

$$|e^{i\beta \log z}| = |e^{i\beta \log |z|} e^{-\beta \arg z}| = e^{-\beta \arg z}$$

Thus if we write $c = re^{i\theta}$ for $0 < \theta < 2\pi$ and define $\alpha = -\frac{\theta}{2\pi} + i\frac{\log r}{2\pi}$ we can write the solution to 4.1 as

$$\kappa(z) = (z-1)^{-\alpha}(z+1)^{\alpha-1}$$

The same arguments as before then proceed. and the solution to 4.3 is

$$\phi(z) = \kappa(z) \mathcal{C} \left[\frac{f}{\kappa_+} \right] (z) + C\kappa(z)$$

```

θ = 2.3
α = -θ/(2π)
κ = z -> (z-1)^(-α)*(z+1)^(α-1)
κ(0.1+0.0im) - exp(im*θ)*κ(0.1-0.0im)

0.0 + 1.1102230246251565e-16im

```

```

r = 2.4
θ = 2.1
c = r*exp(im*θ)

```

```

α = -θ/(2π) + im*log(r)/(2π)

```

```

κ = z -> (z-1)^(-α)*(z+1)^(α-1)

```

```

κ(0.1+0.0im)-c*κ(0.1-0.0im)

```

```

2.220446049250313e-16 + 2.220446049250313e-16im

```

1.13 Problem 5.1

1. It is a product of functions analytic off $(-\infty, 1]$ hence is analytic off $(-\infty, 1]$, and we just have to check that it has no jump on $(-\infty, -1)$ and $(-a, a)$. This follows via, for $x < -1$:

$$\kappa_+(x) = \frac{1}{i^4 \sqrt{1-x} \sqrt{-1-x} \sqrt{a-x} \sqrt{-a-x}} = \frac{1}{(-i)^4 \sqrt{1-x} \sqrt{-1-x} \sqrt{a-x} \sqrt{-a-x}} = \kappa_-(x)$$

and for $-a < x < a$ we have

$$\kappa_+(x) = \frac{1}{i^2 \sqrt{1-x} \sqrt{x+1} \sqrt{a-x} \sqrt{x+a}} = \frac{1}{(-i)^2 \sqrt{1-x} \sqrt{1+x} \sqrt{a-x} \sqrt{-a-x}} = \kappa_-(x)$$

2. This follows via the usual arguments: for $a < x < 1$ we have:

$$\kappa_+(x) = \frac{1}{i \sqrt{1-x} \sqrt{x+1} \sqrt{x-a} \sqrt{x+a}} = -\kappa_-(x)$$

and for $-1 < x < -a$ we have

$$\kappa_+(x) = \frac{1}{i^3 \sqrt{1-x} \sqrt{x+1} \sqrt{x-a} \sqrt{x+a}} = -\kappa_-(x)$$

3. This has at most square singularities which are weaker than poles

4.

$$\kappa(z) = \frac{1}{z^2 \sqrt{1-1/z} \sqrt{1-a/z} \sqrt{1+a/z} \sqrt{1+1/z}} \sim \frac{1}{z^2} \rightarrow 0$$

1.14 Problem 5.2

Ah, this is a trick question! Note that $z\kappa(z) \sim z^{-1} = O(z)$ and satisfies all the other properties. Thus consider any other solution $\tilde{\kappa}(z)$ and write

$$r(z) = \frac{\tilde{\kappa}(z)}{\kappa(z)}$$

This has trivial jumps and hence is entire: for example, on $(a, 1)$ we have

$$r_+(x) = \frac{\tilde{\kappa}_+(x)}{\kappa_+(x)} = \frac{-\tilde{\kappa}_-(x)}{-\kappa_-(x)} = r_-(x)$$

But since $\kappa \sim O(z^{-2})$ we only know that κ has at most $O(z)$ growth, hence it can be any first degree polynomial. Therefore, the space of all solutions is in fact two-dimensional: $\psi(z) = (A + Bz)\kappa(z)$.

1.15 Problem 5.3

Here we mimick the usual solution techniques and propose:

$$\phi(z) = \kappa(z) \mathcal{C} \left[\frac{f}{\kappa_+} \right] (z) + (A + Bz)\kappa(z)$$

A quick check confirms it has the right jumps:

$$\phi_+(x) = \kappa_+(x) \mathcal{C}_+ \left[\frac{f}{\kappa_+} \right] (x) + (A+Bx) \kappa_+(x) = \kappa_+(x) \left(\frac{f(x)}{\kappa_+(x)} + \mathcal{C}_- \left[\frac{f}{\kappa_+} \right] (x) \right) - (A+Bx) \kappa_-(x) = -\phi_-(x) +$$

1.16 Problem 6.1

Let's first do a plot and histogram. Here we see the right scaling is $N^{1/4}$, using a simplified model without the second derivative (we expect this to go to the same distribution):

```
using DifferentialEquations
```

```
V = x -> x^4
```

```
Vp = x -> 4x^3
```

```
N = 50
```

```
λ_0 = randn(N) # initial location
```

```
prob = ODEProblem(function(λ,_,t)
```

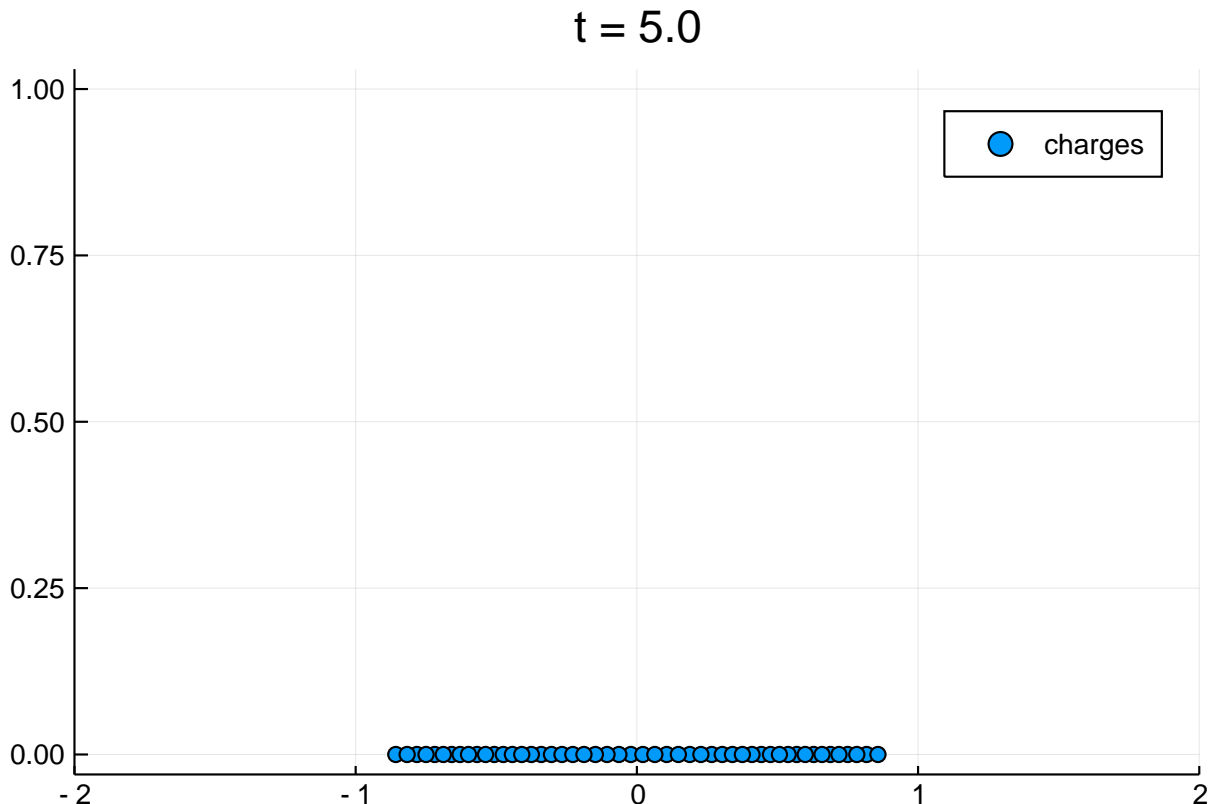
```
    [sum(1 ./ (λ[k] .- λ[[1:k-1;k+1:end]])) - Vp(λ[k]) for k=1:N]
```

```
end, λ_0, (0.0, 5.0))
```

```
λ = solve(prob; reltol=1E-6);
```

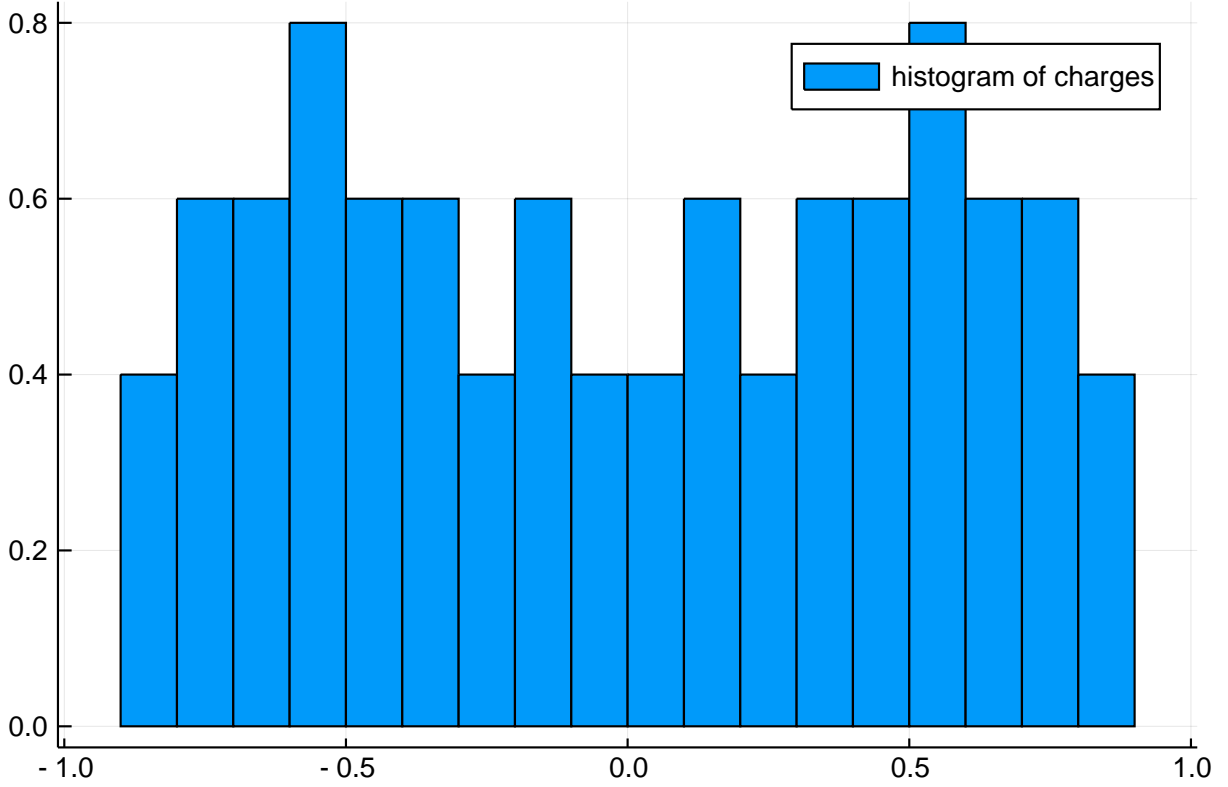
```
t = 5.0
```

```
scatter(λ(t)/N^(1/4), zeros(N); label="charges", xlims=(-2,2), title="t = $t")
```



The limiting distribution has the following form:

```
histogram(λ(t)/N^(1/4); nbins=30, normalize=true, label="histogram of charges")
```



We want to solve

$$\frac{d^2\lambda_k}{dt^2} + \gamma \frac{d\lambda_k}{dt} = \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\lambda_k - \lambda_j} - 4\lambda_k^3$$

Rescale via $\mu_k = \frac{\lambda_k}{N^{1/4}}$ gives

$$0 = N^{1/4} \left[\frac{d^2\lambda_k}{dt^2} + \gamma \frac{d\lambda_k}{dt} \right] = N^{-1/4} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\mu_k - \mu_j} - 4N^{3/4} \mu_k^3$$

or in other words

$$0 = N^{-1/2} \left[\frac{d^2\lambda_k}{dt^2} + \gamma \frac{d\lambda_k}{dt} \right] = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\mu_k - \mu_j} - 4\mu_k^3$$

We can now formally let $N \rightarrow \infty$ to get our equation

$$\oint_{-b}^b \frac{w(t)}{x - t} dt = 4x^3$$

where I've used symmetry to assume that the interval is symmetric. We want to find w and b so that this equations holds true and w is a bounded probability density:

1.

$$w(x) > 0$$

for $-b < x < b$

2.

$$\int w(x)dx = 1$$

3.

$$w$$

is bounded

Our equation is equivalent to

$$\mathcal{H}_{[-b,b]}w(x) = \frac{4x^3}{\pi}$$

recall the inversion formula

$$u(x) = \frac{-1}{\sqrt{b^2 - x^2}} \mathcal{H} \left[f(\diamond) \sqrt{b^2 - \diamond^2} \right] (x) - \frac{C}{\sqrt{b^2 - x^2}}$$

In our case $f(x) = \frac{4x^3}{\pi}$ and we use

$$\sqrt{z-b}\sqrt{z+b} = z\sqrt{1-b/z}\sqrt{1+b/z} = z - \frac{b^2}{2z} - \frac{b^4}{8z^3} + O(z^{-4})$$

to determine

$$2i\mathcal{C} \left[\diamond^3 \sqrt{b^2 - \diamond^2} \right] (x) = z^3(\sqrt{z-b}\sqrt{z+b} - z + \frac{b^2}{2z} + \frac{b^4}{8z^3}) = z^3\sqrt{z-b}\sqrt{z+b} - z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{8}$$

```
b = 5
x = Fun(-b .. b)
(2im)cauchy(x^3*sqrt(b^2-x^2),z)

71.6687198577313 + 40.090510492433154im
```

Therefore,

$$u(x) = \frac{4i}{\pi\sqrt{b^2 - x^2}} (C^+ + C^-) \left[\diamond^3 \sqrt{b^2 - \diamond^2} \right] (x) - \frac{C}{\sqrt{b^2 - x^2}} \\ \frac{4}{\pi\sqrt{b^2 - x^2}} (-x^4 + \frac{b^2 x^2}{2} + \frac{b^4}{8}) - \frac{C}{\sqrt{b^2 - x^2}}$$

We choose C so this is bounded, in particular, we get the solution

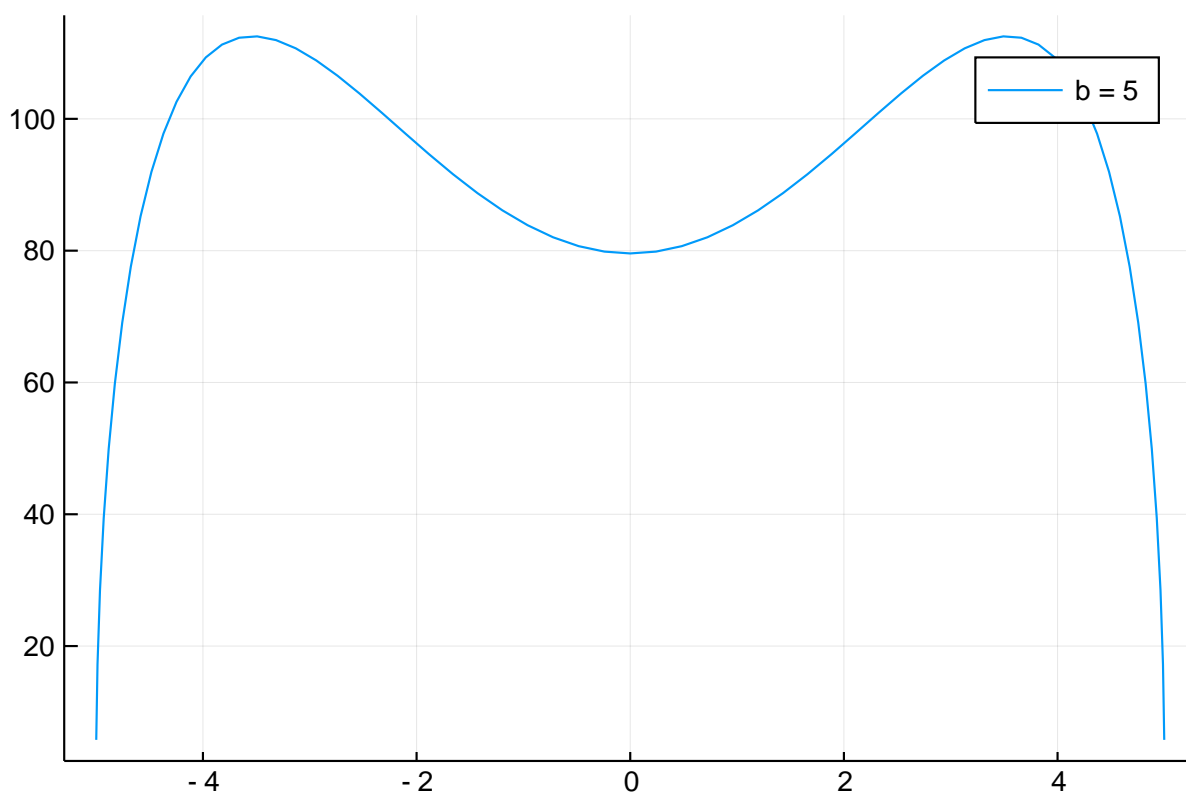
$$u(x) = \frac{4}{\pi\sqrt{b^2 - x^2}} (-x^4 + \frac{b^2 x^2}{2} + \frac{b^4}{2})$$

```
u = 4/(pi*sqrt(b^2-x^2))*(-x^4 + b^2*x^2/2 + b^4/2)
H(u, 0.1),4*0.1^3/pi
```

```
(0.0012732395447351292, 0.001273239544735163)
```

At least it looks right, we just need to get the right b :


```
plot(u; label = "b = $b")
```



We want to choose b now so that this integrates to 1. For example, this choice of b is horrible:

```
sum(u)
```

937.5

There's a nice trick: If $-2\pi i \mathcal{C}u(z) \sim \frac{1}{z}$ then $\int_{-b}^b u(x) dx = 1$. We know since

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \frac{3z^2}{8} - \frac{5z^3}{16} + O(z^4)$$

$$\begin{aligned} \frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}} &= \frac{-z^3 + \frac{b^2 z}{2} + \frac{b^4}{2z}}{\sqrt{1-b/z}\sqrt{1+b/z}} \\ &= (-z^3 + \frac{b^2 z}{2})(1 + \frac{b}{2z} + \frac{3b^2}{8z^2} + \frac{5b^3}{16z^3})(1 - \frac{b}{2z} + \frac{3b^2}{8z^2} - \frac{5b^3}{16z^3}) + O(z^{-1}) \\ &= -z^3 + \left(\frac{b^2}{2} + \frac{b^2}{4} + \frac{b^2}{2} - \frac{3b^2}{8} - \frac{3b^2}{8}\right)z + O(z^{-1}) \\ &= -z^3 + O(z^{-1}) \end{aligned}$$

Thus we know

$$\mathcal{C}u(z) = \frac{2i}{\pi} z^3 + \frac{2i}{\pi} \frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}}$$

```
cauchy(u, z)

z = 100.0im;
2im/π*z^3 + 2im/π*(-z^4 + b^2*z^2/2 + b^4/2)/(sqrt(z-b)sqrt(z+b))

1.4908359390683472 - 0.0im
```

taking this one term further we find

$$\frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}} = -z^3 + \frac{3b^4}{8z} + O(z^{-2})$$

Hence we want to choose b so that

$$-2i\pi\mathcal{C}u(z) = \frac{3b^4}{2z} \sim \frac{1}{z}$$

in other words, $b = (\frac{2}{3})^{1/4}$

```
b = (2/3)^(1/4)
x = Fun(-b .. b)
u = 4/(π*sqrt(b^2-x^2))*(-x^4 + b^2*x^2/2 + b^4/2)
sum(u)

0.9999999999999997
```

And it worked!

```
histogram(λ(5.0)/N^(1/4); nbins=25, normalize=true, label="histogram of charges")
plot!(u; label="u",xlims=(-2,2))
```

