

## M3M6: Methods of Mathematical Physics

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# 1 Lecture 3: Laurent series and residue calculus

This lecture we cover

1. Fourier and Laurent series
2. Contour integrals and Laurent coefficients
3. Isolated singularities
  - Residue at a point
4. Contour integrals in domains with multiple holes
  - The residue theorem
5. Calculating integrals
  - Application: Trigonometric integrals with rational functions

## 1.1 Fourier and Laurent series

**Definition (Fourier series)** On  $[-\pi, \pi)$ , *Fourier series* is an expansion of the form

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k e^{ik\theta}$$

where

$$g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-ik\theta} d\theta$$

**Definition (Laurent series)** In the complex plane, Laurent series around  $z_0$  is an expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k$$

**Lemma (Fourier series = Laurent series)** On a circle in the complex plane

$$C_r = \{z_0 + re^{i\theta} : -\pi \leq \theta < \pi\},$$

Laurent series of  $f(z)$  around  $z_0$  converges for  $z \in C_r$  if the Fourier series of  $g(\theta) = f(z_0 + re^{i\theta})$  converges, and the coefficients are given by

$$f_k = \frac{g_k}{r^k} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

**Proof** This follows immediately from the change of variables  $z = re^{i\theta} + z_0$ . ■

Interestingly, analytic properties of  $f$  can be used to show decaying properties in  $g$ :

## 1.2 Residue on a circle

In this course, we will *always* think of Laurent series living on a circle  $C_r(z_0) = \{z : |z - z_0| = r\}$ . That is,

$$f(z) \approx \sum_{k=-\infty}^{\infty} f_k(z - z_0)^k$$

for  $z \in C_r(z_0)$ .

**Proposition (Residue on a circle)** Suppose the Laurent series is absolutely summable on  $C_r(z_0)$ . Then

$$\oint_{C_r(z_0)} f(z) dz = 2\pi i f_{-1}$$

We refer to  $f_{-1}$  as the residue over  $C_r(z_0)$ :

$$\text{Res}_{C_r(z_0)} f(z) := f_{-1} = \frac{1}{2\pi i} \int_{C_r} f(z) dz.$$

*Example* For all  $0 < r < \infty$ ,

$$\oint_{C_r} \frac{1}{z} dz = 2\pi i$$

When  $f$  is holomorphic in a neighbourhood of the circle, we can extend it to an annulus (like Taylor series and disks):

**Proposition (Laurent series in an annulus)** Suppose  $f$  is holomorphic in an open annulus  $A_{\rho R}(z_0) = \{z : \rho < |z - z_0| < R\}$ . Then the Laurent series converges uniformly in any closed annulus inside  $A_{\rho R}$

*Proposition (Residue on a circle)* holds true regardless of the radius, that is, the definition of  $f_{-1}$  only depends on the annulus of analyticity:

$$\text{Res}_{A_{\rho R}(z_0)} f(z) := f_{-1} = \frac{1}{2\pi i} \int_{C_r} f(z) dz.$$

for any  $\rho < r < R$ .

## 1.3 Isolated singularities

**Definition (isolated singularity)**  $f$  has an *isolated singularity* at  $z_0$  if it is holomorphic in an open annulus with inner radius 0:

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}.$$

**Definition (Removable singularity)**  $f$  has a *removable singularity* at  $z_0$  if it has an isolated singularity at  $z_0$  and all negative terms in the Laurent series in  $A_{0R}(z_0)$  are zero:

$$f(z) = f_0 + f_1(z - z_0) + f_2(z - z_0)^2 + \cdots$$

Equivalently,  $f$  has a removable singularity at  $z_0$  if it is bounded as  $z \rightarrow z_0$ .

An example would be a function like  $f(z) = (e^z - 1)/z$ , which is analytic for  $z \neq 0$  but is not defined for  $z = 0$ . However, this singularity is artificial: we have from its Taylor series that

$$f(z) = (e^z - 1)/z = (z + z^2/2! + \cdots)/z = 1 + z/2! + \cdots$$

hence  $f(z) \rightarrow 1$ . We can therefore remove the singularity by taking  $f_0$ , the zeroth Laurent coefficient:

$$\tilde{f}(z) = \begin{cases} (e^z - 1)/z & z \neq 0 \\ 1 & z = 0 \end{cases}$$

This construction is general:

**Proposition (Removing a removable singularity)** If  $f$  has a removable singularity at  $z_0$  and  $f_0$  is the zeroth Laurent coefficient

$$f_0 := \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(z)}{z} dz$$

for  $r$  sufficiently small, then

$$\tilde{f}(z) = \begin{cases} f_0 & z = z_0 \\ f(z) & 0 < |z - z_0| < R \end{cases}$$

is analytic in the disk  $B_R(z_0) = \{z : |z - z_0| < R\}$ , with a convergent Taylor series. Hence the name.

**Definition (simple pole)**  $f$  has a *simple pole* at  $z_0$  if it is holomorphic in

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only one negative term in the Laurent series in  $A_{0R}(z_0)$ :

$$f(z) = \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

where  $f_{-1} \neq 0$ .

**Definition (higher order pole)**  $f$  has a *pole of order  $N$*  at  $z_0$  if it is holomorphic in

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only  $N$  negative coefficients in the Laurent series:

$$f(z) = \frac{f_{-N}}{(z - z_0)^N} + \frac{f_{1-N}}{(z - z_0)^{N-1}} + \cdots + \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

where  $f_{-N} \neq 0$ .

**Definition (essential singularity)**  $f$  has an *essential singularity* at  $z_0$  if it is holomorphic in  $A_{0R}(z_0)$  and has an infinite number of negative Laurent coefficients.

An essential singularity is complicated but isolated, hence we can still calculate the integrals using Residue calculus.

### 1.3.1 Residue at a point

**Definition (Residue at a point)** Suppose  $f$  has an isolated singularity at  $z_0$ , and is analytic in the annulus  $A_{0R}(z_0)$  for some  $R > 0$ . Then we define the *residue at  $z_0$*  as

$$\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}_{A_{0R}} f(z) = f_{-1}$$

where  $f_{-1}$  is the first negative coefficient of the Laurent series in  $A_{0R}(z_0)$ , that is, integrating over  $C_r$  for any  $0 < r < R$ .

**Proposition (Residue of ratio of analytic functions with simple pole)** Suppose

$$f(z) = \frac{A(z)}{B(z)}$$

and  $A, B$  are analytic/holomorphic in a disk of radius  $R$  around  $z_0$  and that  $B$  has only a single zero at  $z_0$ :

$$\begin{aligned} A(z) &= A_0 + A_1(z - z_0) + \cdots \\ B(z) &= B_1(z - z_0) + \cdots \end{aligned}$$

Then  $\operatorname{Res}_{z=z_0} f(z) = \frac{A_0}{B_1}$

**Exercise (Residue of ratio of analytic functions with higher order poles)** What is the residue at  $z_0$  if  $B$  has a higher order zero:  $B(z) = B_N(z - z_0)^N + \cdots$ ?

## 1.4 Contour integrals on domains with multiple holes

Consider the following example:

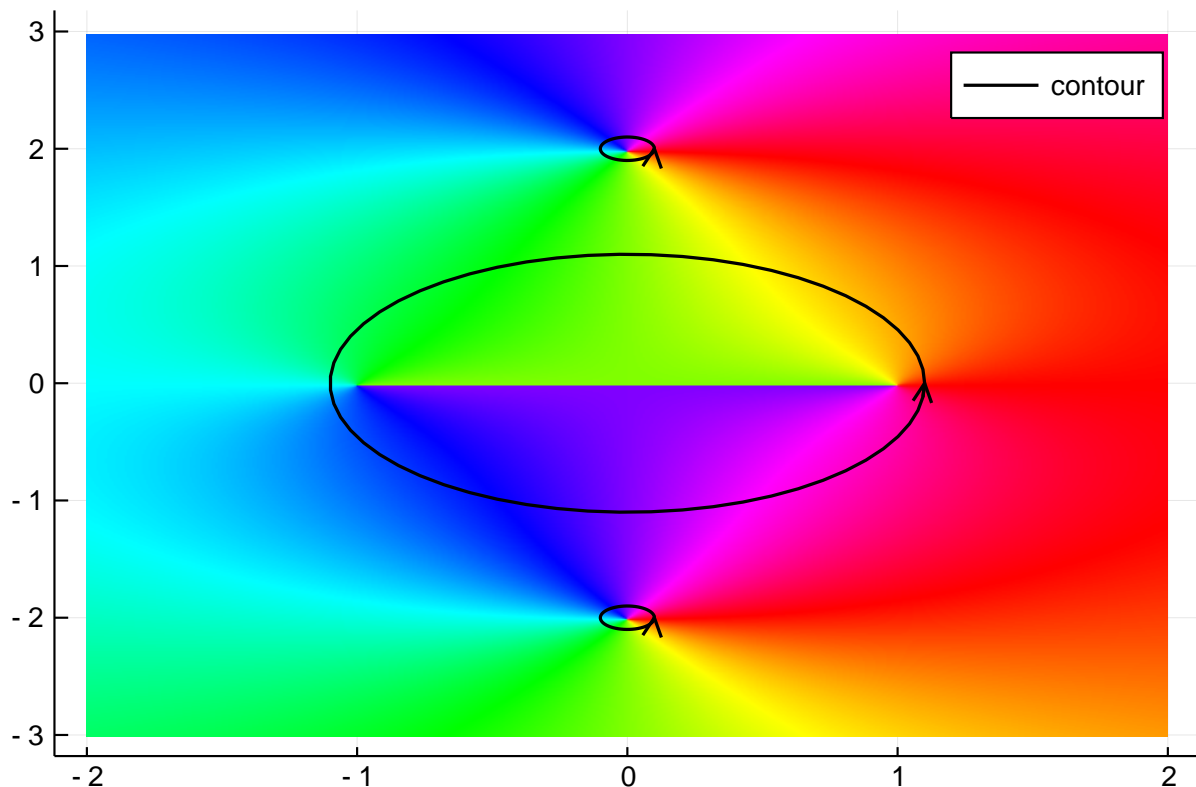
$$\frac{\sqrt{z-1}\sqrt{z+1}}{z^2+4}$$

We still have the contour integral over a circle, and so *Proposition (Residue on a circle)* still holds true for  $r > 2$ . But we can also deform the contour into three contours:

```
using ApproxFun, Plots, ComplexPhasePortrait
f = z -> sqrt(z-1)sqrt(z+1)/(z^2+4)

Γ = Circle(1.1) ∪ Circle(2.0im,0.1) ∪ Circle(-2.0im,0.1)
```

```
phaseplot(-2..2, -3..3, f)
plot!(Γ; color=:black, label=:contour, arrow=true, linewidth=1.5)
```



```
sum(Fun(f, Circle(2.1))), sum(Fun(f, Γ))
```

```
(-8.270426731549189e-16 + 6.283185307179587im, -1.00975500904025e-15 + 6.283185307179586im)
```

Thus we can sum over three residues.

### 1.4.1 Residue theorem

**Theorem (Cauchy's Residue Theorem)** Let  $f$  be holomorphic inside and on a simple closed, positively oriented contour  $\gamma$  except at isolated points  $z_1, \dots, z_r$  inside  $\gamma$ . Then

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^r \operatorname{Res}_{z=z_j} f(z)$$

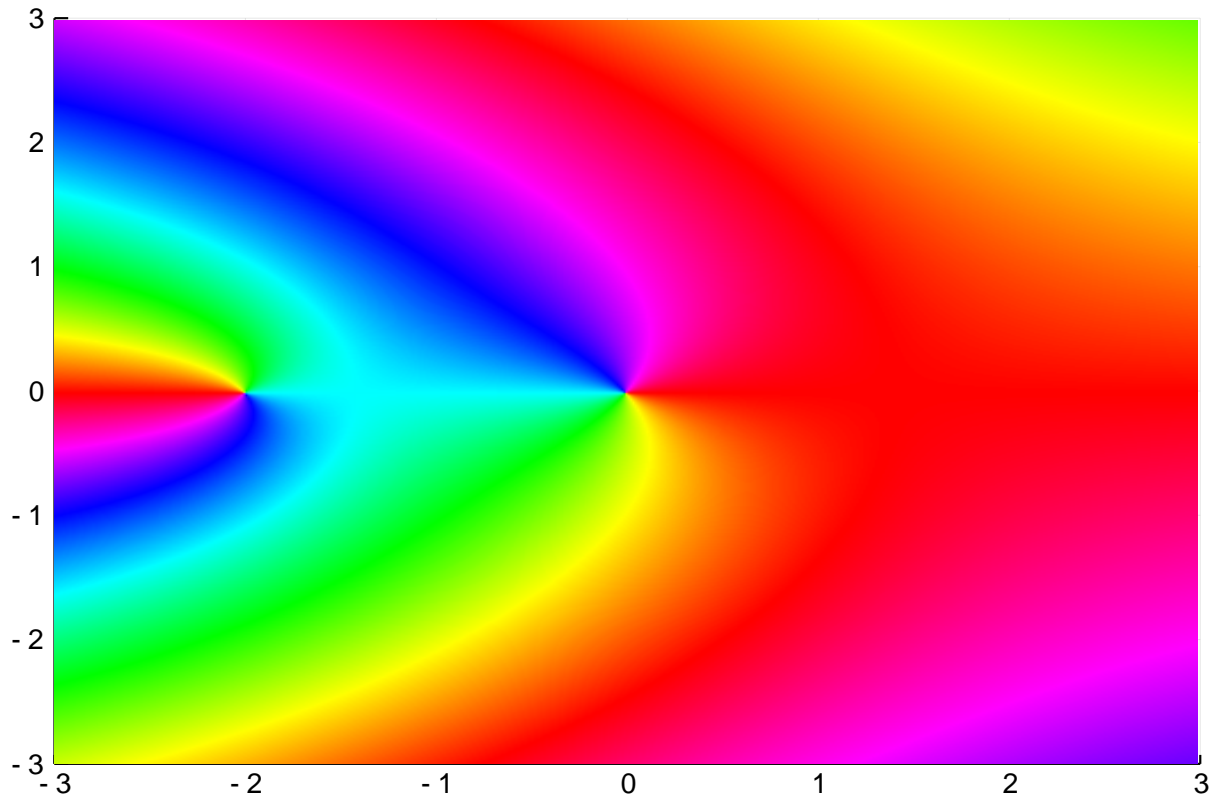
## 1.5 Calculating integrals

We can use the Residue theorem to calculate "hard" integrals. First, two trivial examples: consider

$$f(z) = \frac{e^z}{z(z+2)}$$

This has two simple poles, one at  $z = 0$  and one at  $z = -2$ , as seen clearly from the phase portrait:

```
f = z -> exp(z)/(z*(z+2))
phaseplot(-3..3, -3..3, f)
```



Consider integrating over  $C_3$ , a circle of radius 3 centred at the origin. The residues are

$$\begin{aligned}\operatorname{Res}_{z=0} f(z) &= \operatorname{Res}_{z=0} \frac{1}{z} \frac{e^z}{z+2} = \frac{1}{2} \\ \operatorname{Res}_{z=-2} f(z) &= \operatorname{Res}_{z=-2} \frac{1}{z+2} \frac{e^z}{z} = -\frac{e^{-2}}{2}\end{aligned}$$

Thus the integral must be equal to  $2\pi i(1/2 - e^{-2}/2)$ . This matches a numerical approximation of the integral

```
sum(Fun(f, Circle(3.0))), 2*pi*im*(1/2 - exp(-2)/2)
```

```
(-5.073166565789438e-16 + 2.716424322002157im, 0.0 + 2.716424322002157im)
```

A more complicated example is

$$f(z) = \frac{e^z}{z^2(z+2)}$$

which has a double pole at  $z = 0$ . We find the residue by expanding in Taylor series:

$$\frac{1}{z^2} \frac{e^z}{z+2} = \frac{1}{2z^2} (1 + z + O(z^2))(1 - z/2 + O(z^2)) = \frac{1}{2z^2} + \frac{1}{4z} + O(1)$$

That is,

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{4} \quad \text{and} \quad \operatorname{Res}_{z=-2} f(z) = \frac{e^{-2}}{4}$$

Again, residue calculus matches the numerical computation:

```
sum(Fun(z -> exp(z)/(z^2*(z+2)), Circle(3.0))), 2*pi*im * (1/4 + exp(-2)/4)
(-2.313334476762615e-16 + 1.7833804925887144im, 0.0 + 1.783380492588715im)
```