Chapter III – Combinatorics

Permutations and Combinations

The Multiplication Principle

Suppose n choices must be made, with m_1 ways to make choice 1, and for each of these ways, m_2 ways to make choice 2, and so on, with m_n ways to make choice n. Then there are $m_1 \cdot m_2 \cdot \cdots \cdot m_n$ different ways to make the entire sequence of choices.

Factorial Notation

For any natural number n,

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

Also, $0! = 1$.

Permutations

A *permutation* of r (where $r \geq 1$) elements from a set of n elements is any specific ordering or arrangement, without repetition, of the r elements. Each rearrangement of the r elements is a different permutation. The number of permutations of n things taken r at a time (with $r \leq n$) is written

$$P(n,r) \text{ or } P_r^n.$$

If $P(n,r)$ (where $r \leq n$) is the number of permutations of n elements taken r at a time, then

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Remark: $P_n^n = n!$

Examples

I) Three married couples have bought six seats in a row for a performance of a musical comedy.

(a) In how many ways can they be seated?

$6! = 720$

(b) In how many ways can they be seated if each couple is to sit together with the husband to the left of his wife?

$3! = 6$

(c) In how many ways can they be seated if each couple is to sit together?

$3! \times 2! \times 2! \times 2! = 48$

(d) In how many ways can they be seated if all the men are to sit together and all the women are to sit together?

$2! \times 3! \times 3! = 72$

II) In how many ways can 8 people *A*, *B*, *C*, *D*, *E*, *F*, *G* and *H* be seated in a row if

(a) there are no restrictions on seating arrangement;

$8! = 40320$

(b) persons A and B must not sit next to each other;

$8! - 7!2! = 30240$

III) In how many ways can six coupons for free lunches at different restaurants be distributed among 10 students

(a) if none is to receive more than one coupon;

$P_6^{10} = 151200$

(b) if there is no restriction on the number of coupons that each student can receive?

$10^6 = 1000000$

Remark: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Combinations

A *combination* of r (where $r \geq 1$) elements from a set of n elements is a subset of r elements without regard to order.

If $C(n,r)$ (or C_r^n) denotes the number of combinations of n elements taken r at a time, where $r \leq n$, then

$$C_r^n = \frac{n!}{(n-r)!r!}.$$

Remark: $C_n^n = 1$ and $C_r^n = C_{n-r}^n$.

http://www.omegamath.com/Data/d2.2.html

Example

For betting on the Mark Six draw,

(a) how many single entries can be split from an 8-number multiple entry?

$C_6^8 = 28$

(b) how many single entries can be split from a 3-banker-and-7-leg-number entry?

$C_3^7 = 35$

Remark: There are C_r^{n+r-1} combinations of r elements from a set of n elements when repetition of elements is allowed.

Example

Suppose there are 1 red ball, 1 blue ball and 1 green ball in a box. Five students are invited to come out one by one to draw a ball from the box and put it back. How many combinations of colors are possible? (Note: “GRBBR” and “RBRGB” are regarded as the same combination.)

$C_5^{3+5-1} = C_5^7 = 21$

Permutations with Indistinguishable Objects

Theorem The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,…, and n_k indistinguishable objects of type k, is $\dfrac{n!}{n_1!n_2!\cdots n_k!}$.

Examples

I) How many strings can be made by reordering the letters of the word "daricks"?

$P_7^7 = 7! = 5040$

II) How many strings can be made by reordering the letters of the word "darickschan"?

$$\frac{11!}{1!\times 2!\times 1!\times 1!\times 2!\times 1!\times 1!\times 1!\times 1!} = \frac{11!}{2!2!} = 9979200$$

III) How many strings can be made by reordering the letters of the word "darickswaihongchan"?

$$\frac{18!}{3!\times 2!\times 2!\times 2!\times 2!} = 66691392768000$$

Theorem The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1,2,\ldots,k$, equals $\dfrac{n!}{n_1!n_2!\cdots n_k!}$

Examples

I) In a class of 20 students, 5 of them will get Grade A, 10 of them Grade B, 3 of them Grade C, and 2 will be fail. How many grade distributions are possible among 20 students?

$$\frac{20!}{5!10!3!2!} = 465585120$$

II) How many ways can we distribute a standard deck of 52 playing cards into 4 sets of 13 cards each?

$$\frac{52!}{13!13!13!13!} = 53644737765488792839237440000 \quad \text{(very large!!!)}$$

The Pigeonhole Principle

Suppose that a flock of pigeons flies into a set of pigeonholes to roost. The *pigeonhole principle* states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

Theorem [The Pigeonhole Principle] If $k+1$ or more objects are placed into k boxes, then there is at least on box containing two or more of the objects.

Proof Suppose contrary that there is at most 1 object in each box. The total number of objects in the k boxes should be less than or equal to k.

Example

If every student in a class will receive a grade from A to E, and there are 6 students in the class, then there should be at least 2 students will receive the same grade.

Theorem [The Generalized Pigeonhole Principle] If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof Suppose contrary that all boxes are containing at most $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most $\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) \times k < \left(\left(\frac{N}{k} + 1\right) - 1\right) \times k < N$.

Example

I) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
$\left\lceil \frac{N}{4} \right\rceil \geq 3 \Rightarrow \frac{N}{4} > 2 \Rightarrow N > 8$; therefore the least N is 9.

II) How many must be selected to guarantee that at least three hearts are selected?
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