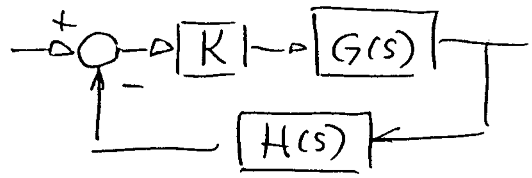
ROOT LOCUS CONSTRUCTION

CONSIDER THE FOLLOWING CLOSED LOOP CONTROL SYSTEM:



OBJECTIVE: DETERMINE GRAPHICAL RULES FOR THE CONSTRUCTION OF THE ROOT LOCUS.

THE CLOSED LOOP TRANSFER FUNCTION IS GIVEN BY

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)}.$$

THE CLOSED LOOP TRANSFER FUNCTION POLES ARE OBTAINED BY SETTING THE DENOMINATOR TO ZERO:

$$1+KG(s)H(s)=0$$

THIS EQUATION IS CALLED THE CHARACTERISTIC EQUATION. (THUS THE CHARACTERISTIC EQUATION ROOTS ARE THE SAME AS THE POLES OF THE CLOSED LOOP TRANSFER FUNCTION.)

ROOT LOCUS: THE ROOT LOCUS IS DEFINED AS A PLOT SHOWING THE MOVEMENT OR MIGRATION OF THE CLOSED LOOP POLES AS THE GAIN K INCREASES FROM 0 TO ∞. THE DIRECTION OF THE ROOT LOCUS IS THE DIRECTION OF INCREASING VALUES OF THE GAIN K. AS SHOWN BELOW, ROOT LOCUS CONSTRUCTION IS BASED UPON USE OF THE OPEN LOOP TRANSFER FUNCTION, KG(s)H(s).

ROOT LOCUS: A PLOT OF THE ROOTS OF $1+KG(s)H(s)=0$ AS K VARIES FROM 0 TO ∞.

FROM THE CHARACTERISTIC EQUATION WE OBTAIN

$$G(s)H(s) = -\frac{1}{K}$$

AND SINCE K IS ASSUMED TO BE ON THE RANGE $0<K<\infty$, IE, REAL AND POSITIVE, WE OBTAIN WHAT IS REFERRED TO AS THE ANGLE CRITERION:

SEE p. 235 FOR ROOT LOCUS CONSTRUCTION RULES

$$\angle G(s)H(s) = r180, \quad r=\pm 1, \pm 3, \pm 5, \cdots$$

ANY VALUE OF s SATISFYING THE ANGLE CRITERION WILL BE ON THE ROOT LOCUS, IE, THAT VALUE OF s WILL BE A POLE OF THE CLOSED LOOP TRANSFER FUNCTION. THE VALUE OF GAIN THAT YIELDS THIS POLE CAN BE DETERMINED BY

$$K = -\frac{1}{G(s)H(s)}.$$

SINCE K IS REAL AND POSITIVE, IT CAN ALSO BE EXPRESSED IN TERMS OF THE MAGNITUDE CRITERION:

$$K = \left|\frac{1}{G(s)H(s)}\right|.$$

NOTE: COMPLEMENTARY ROOT LOCUS: $-\infty < K < 0$

$$1 + KG(s)H(s) = 0 \Rightarrow G(s)H(s) = -\frac{1}{K}$$

$$\Rightarrow \angle G(s)H(s) = 360r, \quad r = 0, \pm 1, \pm 2, \cdots$$

SEE p. 270 FOR ROOT LOCUS CONSTRUCTION RULES FOR $K<0$.

<u>POLES AND ZEROS OF A TRANSFER FUNCTION</u>: GIVEN A TRANSFER FUNCTION, A <u>POLE</u> OF A TRANSFER FUNCTION IS ANY VALUE OF s, EITHER FINITE OR INFINITE, THAT CAUSES THE FUNCTION TO BE INFINITE. A <u>ZERO</u> OF THE TRANSFER FUNCTION IS ANY VALUE OF s, EITHER <u>FINITE</u> OR <u>INFINITE</u>, THAT CAUSES THE FUNCTION TO BE ZERO. <u>FOR EXAMPLE</u>, GIVEN THE TRANSFER FUNCTION

$$KG(s)H(s) = \frac{K(s+1)}{s(s+2)}$$

THERE ARE TWO FINITE POLES, $s=0$ AND $s=-2$. THE FINITE VALUE OF s THAT CAUSES H(s) TO BE ZERO IS $s=-1$. THUS H(s) HAS A FINITE ZERO AT $s=-1$. BUT NOTICE THAT WHEN s IS VERY LARGE, WE HAVE

$$\lim_{s\ large} KG(s)H(s) = \frac{\cancel{1}\,K}{s}$$

THUS AN INFINITE VALUE OF s WILL CAUSE H(s) TO BE ZERO. IN THIS EXAMPLE, WE SAY THAT WE HAVE <u>TWO FINITE POLES</u>, <u>ONE FINITE ZERO</u>, AND <u>ONE ZERO AT INFINITY</u>.

ZEROS (FINITE AND AT INFINITY) =

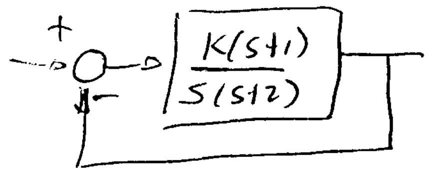
POLES (FINITE AND AT INFINITY)

THE ANGLE CRITERION CAN BE EQUIVALENTLY EXPRESSED AS

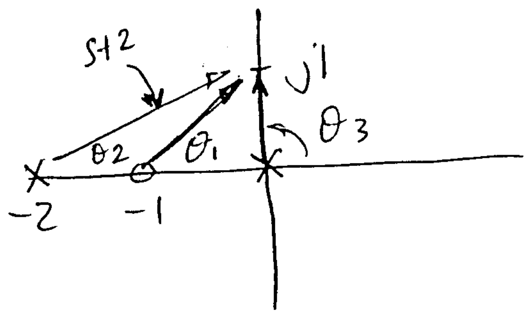
$\sum$ (ALL ANGLES FROM FINITE ZEROS)-$\sum$ (ALL ANGLES FROM FINITE POLES) = $r(180^\circ)$, $r=\pm 1, \pm 3, \ldots$

GIVEN THE TRANSFER FUNCTION

$$KG(s)H(s)=\frac{K(s+1)}{s(s+2)}$$



WE CAN DETERMINE WHETHER A POINT IN THE s-PLANE, SAY s=j1, IS ON THE ROOT LOCUS BY APPLYING THE ANGLE CRITERION. WE CAN DEPICT EACH FACTOR AS A VECTOR IN THE s-PLANE,

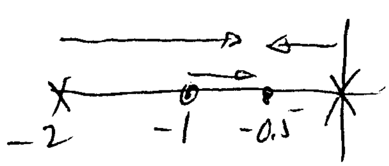


THE ANGLE CRITERION is THEN

$$\theta_1-\theta_2-\theta_3= \pm 180,\ \pm 540,\ \cdots$$

IN THIS EXAMPLE, $\theta_1=45^\circ$, $\theta_2=26.6^\circ$, $\theta_3=90^\circ$, THUS $\theta_1-\theta_2-\theta_3=-71.6^\circ$. CONCLUDE THAT THE POINT s=j1 IS NOT ON THE ROOT LOCUS.

Is $s=-0.5$ ON THE ROOT LOCUS?



$\theta_1=0^\circ$, $\theta_2=0^\circ$, $\theta_3=180^\circ$

$\theta_1-\theta_2-\theta_3=-180^\circ$

$\therefore$ $s=-0.5$ IS ON ROOT LOCUS.

THE FOLLOWING ROOT LOCUS CONSTRUCTION RULES CAN BE DERIVED FROM THE CHARACTERISTIC EQUATION:

(1) THE ROOT LOCUS IS SYMMETRICAL WITH RESPECT TO THE REAL AXIS, BECAUSE COMPLEX ROOTS OCCUR IN CONJUGATE PAIRS.

(2) WHERE DOES THE ROOT LOCUS BEGIN, IE, WHERE ARE THE CHARACTERISTIC EQUATION ROOTS WHEN K=0? CONSIDER A GENERAL EXPRESSION FOR THE OPEN LOOP TRANSFER FUNCTION:

$$KG(s)H(s)=K\frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

WHERE $\alpha=n-m$ (THE POLE ZERO EXCESS).

THE CORRESPONDING CHARACTERISTIC EQUATION IS

$$1+KG(s)H(s)=1+K\frac{b_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}=0$$

OR

$$(s-p_1)(s-p_2)\cdots(s-p_n)+Kb_m(s-z_1)(s-z_2)\cdots(s-z_m)=0$$

→ WHEN K=0, ROOTS ARE THE SAME AS THE POLES OF THE OPEN LOOP TRANSFER FUNCTION. ROOT LOCUS BEGINS AT THE OPEN LOOP TRANSFER FUNCTION POLES.

(3) WHERE DOES THE ROOT LOCUS TERMINATE, IE, WHEN $K \to \infty$, WHERE ARE THE CHARACTERISTIC EQUATION ROOTS? 45

DIVIDING THE ABOVE EXPRESSION BY K, WE OBTAIN

$$\frac{(s-p_1)(s-p_2)\cdots(s-p_n)}{K} + b_m(s-z_1)(s-z_2)\cdots(s-z_m) = 0$$

IF $K \to \infty$, THE FIRST TERM DISAPPEARS. THE RESULTING ROOTS OF THE CHARACTERISTIC EQUATION ARE THE ZEROS OF THE OPEN LOOP TRANSFER FUNCTION, IE ROOT LOCUS TERMINATES AT THE OPEN LOOP TRANSFER FUNCTION ZEROS

(4) FOR $\alpha \geq 1$, THE ASYMPTOTES TOWARD WHICH THE BRANCHES OF THE ROOT LOCUS APPROACH AS $K \to \infty$ DEFINED BY THE ANGLE OF THE ASYMPTOTE AND THE ORIGIN OF THE ASYMPTOTE,

$$\theta = \frac{r180}{\alpha}, \quad r = \pm 1, \ \pm 3, \ \cdots; \quad \sigma_a = \frac{\Sigma \text{ pole locations} - \Sigma \text{ zero locations}}{\alpha}$$

(5) ON THE REAL AXIS, THE ROOT LOCUS EXISTS TO THE LEFT OF AN ODD NUMBER OF SINGULARITIES (WHERE A SINGULARITY CAN BE A POLE OR A ZERO).

(6) BREAKAWAY OF THE ROOT LOCUS FROM (OR RE-ENTRY TO) THE REAL AXIS OF THE s-PLANE OCCURS AT A RELATIVE MAXIMUM (MINIMUM) OF K.

THE NUMBER OF BRANCHES OF THE ROOT LOCUS =

THE NUMBER OF ROOTS OF THE CHARACTERISTIC EQUATION =

THE NUMBER OF POLES OF CLOSED LOOP TRANSFER FUNCTION =

THE NUMBER OF POLES OF OPEN LOOP TRANSFER FUNCTION.