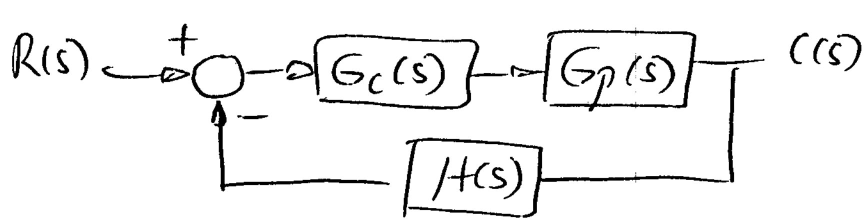
The Routh Hurwitz Stability Criterion

The most common method used for determining the stability of a Nonlinear Time-Varying system (without actually operating the physical system itself) is to observe the output of a Simulation of the system for typical inputs and initial conditions. The output must remain bounded in amplitude for any bounded input and/or initial condition. Simulink may be used for stability testing.

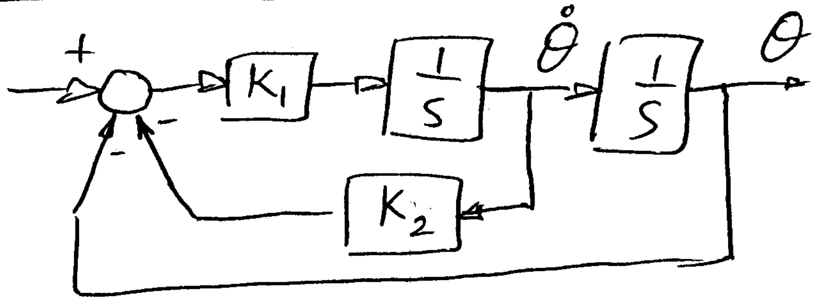
The Routh Hurwitz Stability Criterion is an alternative analytical procedure for determining the stability of a Linear Time-Invariant (LTIV) system. Consider the LTIV closed-loop control system



$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}$$

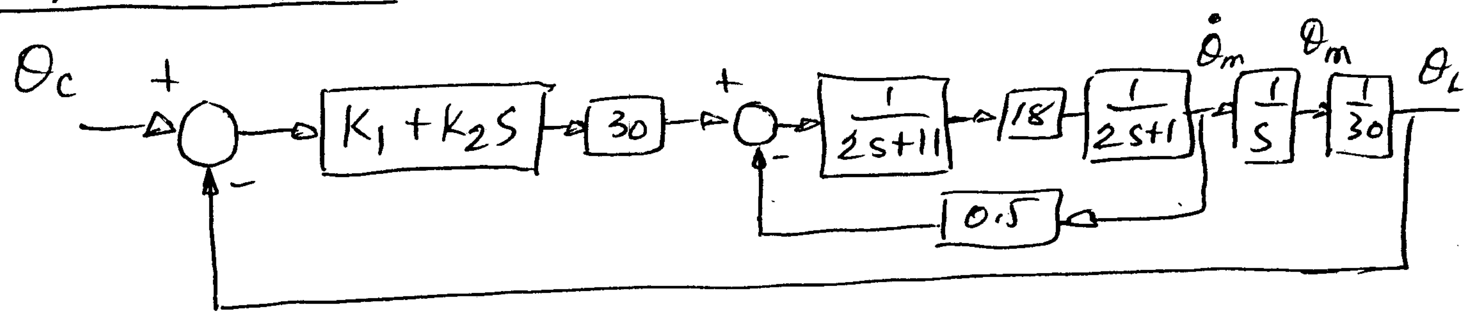
The Routh Hurwitz Stability Criterion may be used to determine whether this system is stable, marginally stable, or unstable. It may also be used to determine the range of certain parameters or gains of the system such that stability is guaranteed.

FOR EXAMPLE, CONSIDER THE SYSTEM FOR CONTROLLING THE ANGLE OF ORIENTATION OF A SATELLITE IN ORBIT —



(SEE FIGURE P4.5, p.145)

OR CONSIDER THE SYSTEM FOR CONTROLLING THE ANGLE OF A ROBOT ARM (SEE FIG. 2.45, p.58, & FIG P6.11 p.208)



FOR EACH OF THESE SYSTEMS THE OBJECTIVE IS TO FIND THE RANGE OF GAINS K_1 & K_2 SUCH THAT THE SYSTEMS REMAIN STABLE.

BEFORE PRESENTING THE ROUTH HURWITZ STABILITY CRITERION, WHICH GIVES NECESSARY & SUFFICIENT CONDITIONS FOR STABILITY, WE PRESENT AN INITIAL SCREENING PROCEDURE THAT GIVES NECESSARY (BUT NOT SUFFICIENT) CONDITIONS FOR STABILITY.

CONSIDER THE 3RD ORDER POLYNOMIAL HAVING REAL a_i, $i=0,1,2$.

$$Q(s) = s^3 + a_2 s^2 + a_1 s + a_0 = (s-p_1)(s-p_2)(s-p_3)$$
$$= s^3 \underbrace{-(p_1+p_2+p_3)}_{a_2} s^2 + \underbrace{(p_1p_2+p_1p_3+p_2p_3)}_{a_1} s \underbrace{-p_1p_2p_3}_{a_0}$$

THE ROOTS p_K, $K=1,2,\dots,n$ ARE DENOTED BY

$$p_K = \beta_K + j\delta_K$$

(1) IF ANY COEFFICIENT a_i, $i=0,1,2$ IS NEGATIVE, AT LEAST ONE ROOT WILL BE IN RHP.

EXAMPLE: $s^3 - 2s^2 - 5s + 6 = (s-1)(s-3)(s+2)$

NEGATIVE COEFFICIENTS

2 RHP ROOTS (AT $s=1,3$).

IF ALL THE ROOTS ARE IN THE LHP, NO COEFFICIENT CAN BE ZERO.

EXAMPLE: $\underbrace{(s+1)(s+2)(s+3)}_{\text{ROOTS AT } s=-1,-2,-3} = \underbrace{s^3 + 6s^2 + 11s + 6}_{\text{NO ZERO COEFFICIENT.}}$

CONSIDER THE CONVERSE OF THE STATEMENT "IF ALL ROOTS ARE IN THE LHP, NO COEFFICIENT CAN BE ZERO." IT IS:

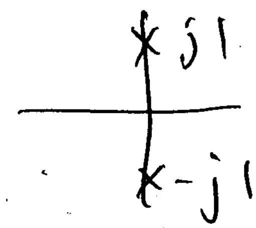
(2) IF ONE OR MORE ZERO COEFFICIENTS EXIST, THEN NOT ALL ROOTS WILL BE IN THE LHP.

ALTHOUGH THE STATEMENTS (1) & (2) ABOVE WERE BASED ON THE 3^{RD} ORDER POLYNOMIAL, THEY APPLY EQUALLY TO THE GENERAL n^{TH} ORDER POLYNOMIAL WITH REAL COEFFICIENTS a_i, $i = 0, 1, 2, \ldots, n-1$.

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0$$

WHERE n IS AN INTEGER GREATER THAN ZERO.

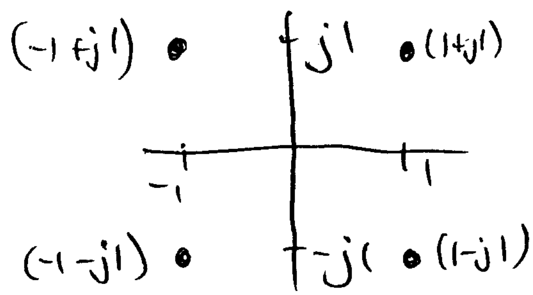
EXAMPLE: $\underbrace{s^2 + 0s + 1}_{\text{HAS ONE ZERO COEFFICIENT}} = \underbrace{(s+j1)(s-j1)}_{\text{ROOTS ON } j\omega \text{ AXIS}}$



(A SYSTEM HAVING A TRANSFER FUNCTION WITH IMAGINARY AXIS POLES WOULD BE MARGINALLY STABLE)

EXAMPLE: $s^4 + 4 = s^4 + 0s^3 + 0s^2 + 0s + 4 = (s^2+2s+2)(s^2-2s+2)$

$$= (s+1+j1)(s+1-j1)(s-1+j1)(s-1-j1)$$



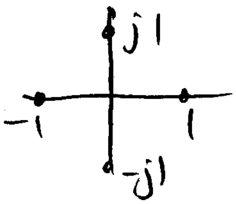
A CONTROL SYSTEM WITH THIS DISTRIBUTION OF POLES WOULD BE UNSTABLE

NOTE: COMPLEX ROOTS OF AN EVEN POLYNOMIAL, SUCH AS s^4+4, OCCUR IN GROUPS OF 4. THE ROOTS HAVE QUADRANTAL SYMMETRY, I.E., THEY ARE SYMMETRICAL WITH RESPECT TO BOTH THE REAL & IMAGINARY AXES. EQUIVALENTLY, THE ROOTS OCCUR IN PAIRS THAT ARE EQUAL IN MAGNITUDE & OPPOSITE IN SIGN.

EXAMPLE: $s^4 - 1 = s^4 + 0s^3 + 0s^2 + 0s - 1$

THE ZERO COEFFICIENTS MEAN THAT NOT ALL ROOTS WILL BE IN LHP
THE NEGATIVE COEFFICIENT MEANS THAT AT LEAST ONE ROOT WILL BE IN THE RHP.

$$s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s-1)(s+1)(s+j1)(s-j1)$$



← QUADRANTAL SYMMETRY

THE CONDITION $a_i > 0$ FOR ALL i IS A NECESSARY CONDITION BUT NOT A SUFFICIENT CONDITION FOR ALL ROOTS OF THE POLYNOMIAL TO BE IN LHP. FOR EXAMPLE,

$$\underbrace{s^3 + 2s^2\ 2s + 4}_{\text{ALL POSITIVE COEFFICIENTS}} = (s+4)(s^2 - 2s + 10)$$
$$= (s+4)\underbrace{(s-1+j3)(s-1-j3)}_{\text{2 RHP ROOTS}}$$

THE ROUTH ARRAY: BEFORE INTRODUCING THE ROUTH HURWITZ STABILITY CRITERION WE SET UP THE ROUTH ARRAY USING THE CHARACTERISTIC POLYNOMIAL $Q(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \cdots + a_1 s + a_0$.
(IF $a_0 = 0$, DIVIDE $Q(s)$ BY s AND FORM ARRAY BASED UPON RESULTING POLYNOMI

$$\begin{array}{lllll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_1 & b_2 & \cdots & \\
s^{n-3} & c_1 & c_2 & \cdots & \\
\vdots & & & & \\
s^2 & k_1 & k_2 & & \\
s^1 & l_1 & & & \\
s^0 & m_1 & & &
\end{array}$$

THE ELEMENTS OF THE 3RD ROWS ARE FORMED USING THE ELEMENTS OF THE FIRST 2 ROWS. THE ELEMENTS OF SUCCEEDING ROWS ARE FORMED BASED UPON THE ELEMENTS OF THE PREVIOUS 2 ROWS.

$$b_1 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{1}{a_{n-1}}\left(a_{n-1}a_{n-2} - a_n a_{n-3}\right.$$

$$b_2 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{1}{a_{n-1}}\left(a_{n-1}a_{n-4} - a_n a_{n-5}\right.$$

THE ROUTH HURWITZ STABILITY CRITERION: THE NUMBER OF ROOTS OF THE CHARACTERISTIC POLYNOMIAL THAT ARE IN THE RHP IS EQUAL TO THE NUMBER OF SIGN CHANGES IN THE FIRST COLUMN OF THE ARRAY.

THERE ARE 3 CASES TO CONSIDER:

CASE I: NO ROW OF THE ARRAY IS MADE UP OF ALL ZERO ELEMENTS. THIS MEANS THAT THERE IS NO POSSIBILITY OF IMAGINARY AXIS ROOTS. (ROOTS WOULD BE EITHER IN LHP, THE RHP, OR BOTH LHP & RHP.)

EXAMPLE: $Q(s) = 1s^3 + 3s^2 + 3s + 1 = (s+1)^3$

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & 1 \\ s^1 & 8/3 & \\ s^0 & 1 & \end{array}$$

SCANNING THE FIRST COLUMN, THERE ARE NO SIGN CHANGES. ALSO, NO ROW CONSISTS OF ALL ZERO ELEMENTS.

$\therefore$ NO RHP ROOTS & NO $j\omega$ AXIS ROOTS IE ALL ROOTS ARE IN LHP.

EXAMPLE: $Q(s) = s^3 + s^2 + 2s + 8 = (s+2)\left(s - \frac{1}{2} + j\frac{\sqrt{15}}{2}\right)\left(s - \frac{1}{2} - j\frac{\sqrt{15}}{2}\right)$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 8 \\ s^1 & -6 & \\ s^0 & 8 & \end{array}$$

THERE ARE 2 SIGN CHANGES (FROM 1 TO −6 AND FROM −6 TO +8), THUS 2 RHP ROOTS. NO ROW CONSISTS OF ALL ZERO ELEMENTS. THUS NO $j\omega$ AXIS ROOTS. THUS THE 3RD ROOT MUST BE IN THE LHP.

A SYSTEM HAVING $Q(s) = s^3 + s^2 + 2s + 8$ AS ITS DENOMINATOR WOULD BE UNSTABLE.

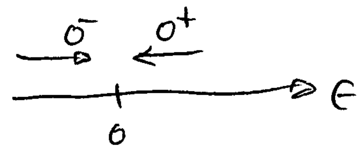
CASE 2: THE FIRST ELEMENT OF A ROW IS ZERO WITH AT LEAST ONE NON-ZERO ELEMENT IN THE SAME ROW. RHP ROOTS WILL ALWAYS EXIST IN THIS CASE.

EXAMPLE: $Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11
s^4	2	4	10
s^3	$\cancel{0}\ \epsilon$	6	0
s^2	$\frac{4\epsilon-12}{\epsilon}$	10	
s^1	6	0	
s^0	10		

REPLACE THE FIRST ZERO ELEMENT OF THE ROW BY ZERO & CONTINUE WITH ARRAY FORMATION.

TO FIND THE NUMBER OF SIGN CHANGES IN FIRST COLUMN, LET $\epsilon \to 0^+$ OR LET $\epsilon \to 0^-$, I.E.,



AS $\epsilon \to 0^+$, THE SIGN OF ϵ IS POSITIVE.

AS $\epsilon \to 0^+$, THE SIGN OF $\frac{4\epsilon - 12}{\epsilon}$ IS NEGATIVE.

THUS AS THE FIRST COLUMN IS SCANNED, 2 SIGN CHANGES ARE ENCOUNTERED. $\Rightarrow$ 2 RHP ROOTS.

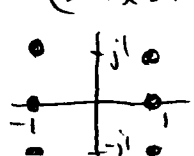
CASE 3: ALL ELEMENTS IN A ROW OF THE ARRAY ARE ZERO.

IN THIS CASE, $Q(s)$ WILL CONTAIN AN EVEN POLYNOMIAL CALLED THE "AUXILIARY" POLYNOMIAL. ROOTS OF THE AUXILIARY POLYNOMIAL COULD BE ON THE IMAGINARY AXIS.

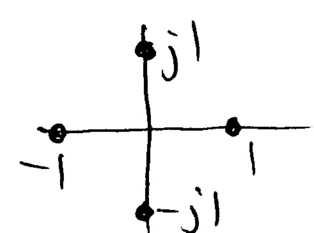
BEFORE GIVING EXAMPLES OF CASE 3, CONSIDER POSSIBLE AUXILIARY POLYNOMIALS.

s^2+1
$=(s+j1)(s-j1)$

$s^6 - 4s^4 + 4s^2 - 4$
$= (s^2-1)(s^4+4)$
$= (s-1)(s+1)(s^2+2s+2)(s^2-2s+2)$
$= (s-1)(s+1)(s+1+j1)(s+1-j1)(s-1+j1)(s-1-j1)$



$s^4 - 1 = (s^2-1)(s^2+1)$
$= (s-1)(s+1)(s+j1)(s-j1)$



EXAMPLE: $Q(s) = s^5 + 3s^4 + 4s^3 + 4s^2 + 3s + 1 = (s+1)^3(s^2+1)$

s^5	1	4	3
s^4	3	4	1
s^3	$\cancel{8/3}$ 1	$\cancel{8/3}$ 1	0
s^2	1	1	
s^1	$\cancel{0}$ 2	$\cancel{0}$ 0	
s^0	1		

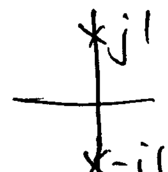
← THE ELEMENTS OF ANY ROW CAN BE MULTIPLIED BY A POSITIVE CONSTANT (IN THIS CASE 3/8) WITHOUT CHANGING THE RESULT OF THE ANALYSIS.

→ AUXILIARY POLYNOMIAL: $1s^2 + 1s^0$

TAKE ITS DERIVATIVE: $2s + 0$

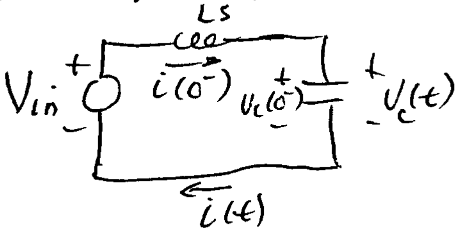
REPLACE THE ROW OF ZEROS IN THE ARRAY WITH THE ABOVE VALUES.

NO SIGN CHANGES ∴ <u>NO RHP ROOTS</u>, HOWEVER, AUXILIARY POLYNOMIAL YIELD <u>$j\omega$ AXIS ROOTS</u>: $s^2 + 1 = 0 \Rightarrow s = \pm j1$



ANY SYSTEM HAVING TRANSFER FUNCTION POLES AT $\pm j\omega_1$ WILL, IN THE STEADY STATE, OSCILLATE AT A FREQUENCY OF ω_1 RAD/SEC IN RESPONSE TO A STEP OR INITIAL CONDITION.

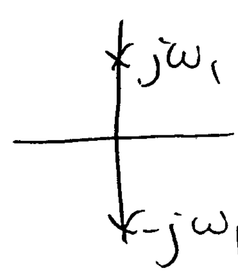
FOR EXAMPLE,



$$G(s) = \frac{V_c(s)}{V_{in}(s)} = \frac{\frac{1}{Cs}}{Ls + \frac{1}{Cs}} = \frac{1/LC}{s^2 + 1/LC}$$

$$G(s) = \frac{\omega_1^2}{s^2 + \omega_1^2}$$

WHERE $\omega_1^2 = \frac{1}{LC}$



ASSUME $V_{in} = 0$. FIND THE RESPONSE TO INITIAL CONDITIONS $i(0^-)$ & $V_c(0^-)$.

$$L\frac{di}{dt} + V_c(t) = 0, \quad i(t) = C\frac{dV_c(t)}{dt} \Rightarrow LC\frac{d^2V_c(t)}{dt^2} + V_c(t) = 0$$

TAKE LAPLACE TRANSFORM: $LC[s^2V_c(s) - sV_c(0^-) - \dot{V}_c(0^-)] + V_c(s) = 0$

$\dot{V}_c(0^-) = \frac{i(0^-)}{C}$

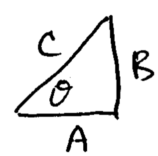
SOLVE FOR $V_c(s)$:

$$V_c(s) = \frac{sV_c(0^-) + \dot{V}_c(0^-)}{s^2 + 1/LC} = \frac{sV_c(0^-) + \dot{V}_c(0^-)}{s^2 + \omega_1^2}$$

$$V_c(s) = V_c(0^-)\frac{s}{s^2+\omega_1^2} + \frac{\dot{V}_c(0^-)}{\omega_1}\frac{\omega_1}{s^2+\omega_1^2}$$

$$V_c(t) = \underbrace{V_c(0^-)}_{A}\cos\omega_1 t + \underbrace{\frac{\dot{V}_c(0^-)}{\omega_1}}_{B}\sin\omega_1 t$$

$$V_c(t) = A\cos\omega_1 t + B\sin\omega_1 t$$



$C = \sqrt{A^2+B^2}$

$\theta = \tan^{-1}\frac{B}{A}$

$$= C\left[\frac{A}{C}\cos\omega_1 t + \frac{B}{C}\sin\omega_1 t\right]$$

$$= C\left[\cos\theta\cos\omega_1 t + \sin\theta\sin\omega_1 t\right]$$

$$V_c(t) = C\cos(\omega_1 t - \theta)$$

EXAMPLE: A COMBINATION OF CASES (2) AND (3):

$$Q(s) = s^5 + s^4 + 0s^3 + 0s^2 + 4s + 4 = (s+1)(s^4+4)$$

THE ZERO COEFFICIENTS MEANS THAT NOT ALL ROOTS ARE IN LHP.

s^5	1	0	4
s^4	1	0	4
s^3	$\cancel{0}\ 4$	0	0
s^2	$\cancel{0}\ \epsilon$	4	
s^1	$-\frac{16}{\epsilon}$		
s^0	4		

→ AUXILIARY POLYNOMIAL IS $s^4 + 0s^2 + 4s^0$

TAKING ITS DERIVATIVE: $4s^3 + 0$

ASSUMING THAT EITHER $\epsilon \to 0^-$ OR $\epsilon \to 0^+$, THERE ARE 2 SIGN CHANGES, ⇒ 2 RHP ROOTS.

KNOWING THAT WE HAVE 2 RHP ROOTS AND AN EVEN POLYNOMIAL HAVING QUADRANTAL SYMMETRY (s^4+4) AS A FACTOR OF $Q(s)$, WE INFER THAT THERE WILL BE 2 SYMMETRICAL LHP ROOTS. THE 5^{TH} ROOT WILL BE IN THE LHP.