(I have taken reference from wiki page on RK)

Prob 13

RK4 can be generalised as:

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + c_2 h, y_n + (a_{21} k_1) h)$$

$$k_3 = f(t_n + c_3 h, y_n + (a_{31} k_1 + a_{32} k_2) h)$$

$$\vdots$$

$$k_s = f(t_n + c_s h, y_n + (a_{s1} k_1 + a_{s2} k_2 + \dots a_{s,s-1} k_{s-1}) h)$$

$b_i \rightarrow$ weights

$c_i \rightarrow$ nodes

The a_{ij}'s form a strictly lower triangular matrix.

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
1	1				
$\vdots$	$\vdots$				
c_s	a_{s1}	a_{s2}	$\cdots$	$a_{s,s-1}$	
	b_1	b_2	$\cdots$	b_{s-1}	b_s

All the weights b_i have to add up to one.

$$\sum_{i=1}^{s} b_i = 1$$

Also for RK6, order is 6, so the min. no. of stages (s) required is 7.

$s = 7$ for RK6.

→ We can use the Butcher conditions to get more relations b/w them coefficients and the solve them to get a working Butcher Tableau for RK6 (not unique).

Eg :-

0	0						
1/3	1/3	0					
2/3	0	2/3	0				
1/3	1/12	1/3	−1/12	0			
1/2	−1/16	9/8	−3/16	−3/8	0		
1/2	0	9/8	−3/8	−3/4	1/2	0	
1	9/44	−9/11	63/44	18/11	0	−16/11	0
	11/120	0	27/40	27/40	−4/15	−4/15	11/120