Chapter - 1 : Linear Algebra

1.1 Determinants

If a, b, c and d are any four terms then the representation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a determinant and is denoted by D.

A determinant of order 2 is evaluated as follows:

$$\mathbf{D} = \begin{vmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{vmatrix} = \mathbf{ad - bc}$$

A determinant of order 3 can be evaluated as follows:

$$\mathbf{D} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

<u>For example</u>: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$

$$= 1(2 - 3) - 2(4 - 9) + 3(2 - 3) = -1 + 10 - 3 = 6$$

<u>Properties of Determinants</u>

Interchange of rows and columns ($R_i \longleftrightarrow C_i$)

The value of determinant is not affected by changing the rows into the corresponding columns, and the columns into the corresponding rows. Thus

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Identical rows and columns ($R_i = R_j$, $C_i = C_j$).

If two rows or two columns of a determinant are identical, the determinant has the value zero. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \qquad \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_1 & b_3 \\ c_1 & c_1 & c_3 \end{vmatrix} = 0$$

Interchange of two adjacent rows and columns ($R_i \longleftrightarrow R_{i+1}$, $C_i \longleftrightarrow C_{i+1}$)

If two adjacent rows or columns of the determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant. Thus

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Multiplication of row or column by factor [R_i (m), C_j(n)]

If the elements of any row or column are multiplied by the same factor, the value of the determinant so obtained is equal to the value of the original determinant multiplied by that factor. Thus,

$$\begin{vmatrix} ma_1 & ma_2 & ma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Sum of determinants

If any element in any row (or column) consists of the sum of two terms, the determinant can be expressed as the sum of two other determinants whose other rows (or columns) remain the same, while the remaining row (or column) consists of these terms respectively. Thus,

$$\begin{vmatrix} a_1+\alpha_1 & a_2 & a_3 \\ b_1+\beta_1 & b_2 & b_3 \\ c_1+\gamma_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \gamma_1 & c_2 & c_3 \end{vmatrix}$$

Change of row or column by multiples of other rows and columns $R_{ij}(p)$, $C_{ij}(p)$

As the consequence of the properties 5, 4 and 2 we have the result.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pa_2 & a_2+qa_3 & a_3 \\ b_1+pb_2 & b_2+qb_3 & b_3 \\ c_1+pc_2 & c_2+qc_3 & c_3 \end{vmatrix}$$

where care must be taken to leave at least one row or column unaltered in such changes P and q being any positive or negative factors.

Note : **1. Area of a triangle whose vertices are (x_1, y_1) (x_2, y_2) and (x_3, y_3) is given by the absolute value of** $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

2. Area of a quadrilateral can be found by dividing it into two triangles.

3. If the area of a triangle obtained from the three given points is zero, then the three points lie on a line.

$\therefore$ The condition for three points to be collinear is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Solved Example 1 :

Find the area of a triangle whose vertices are (2, 1), (4, –3), (–2, 5).

Solution :

The area of the triangle is

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 4 & -3 & 1 \\ -2 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \text{abs}\,[2(-3-5) - 4(1-5) - 2(1+3)]$$

$$= \frac{1}{2} \times \text{abs}\,[-16 + 16 - 8] = 4 \text{ sq. units}$$

Solved Example 2 :

Find if the three points (–1, –1), (5, 7) and (8, 11) are collinear.

Solution :

If the points are collinear, area of the triangle formed by the three given points should be zero.

$$\text{Area} \quad A = \frac{1}{2}\begin{vmatrix} -1 & -1 & 1 \\ 5 & 7 & 1 \\ 8 & 11 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,[-1(7-11) - 5(-1-11) + 8\,(-1-7)]$$

$$= \frac{1}{2}\,[4 + 60 - 64] = 0$$

The given points are collinear.

1.2 Matrices

A Matrix is a rectangular array of elements written as

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & . & a_{2n} \\ . & . & . & & & & . \\ . & . & & . & & & . \\ . & . & & & . & & . \\ . & . & & & & . & . \\ a_{m_1} & a_{m_2} & . & . & . & . & a_{m_n} \end{bmatrix}$$

The above matrix A has m rows and n columns. So it is a $m \times n$ matrix or it is said that the size of the matrix is $m \times n$.

Types of Matrices

Square Matrix :

It is a Matrix in which number of rows = number of columns

For example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a square matrix of order 3.

Diagonal Matrix :

It is a square matrix in which all non diagonal elements are zero.

For example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Scalar Matrix :

It is a diagonal matrix in which all diagonal elements are equal

For example: $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Unit Matrix :

It is a scalar matrix with diagonal elements as unity. It is also called **Identity Matrix**.

Identity matrix of order 2 is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Identity matrix of order 3 is $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note : For any matrix A, AI = IA = A

Upper Triangular Matrix :

It is a square matrix in which all the elements below the principal diagonal are zero.

For example: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

Lower Triangular Matrix :

It is a square matrix in which all the elements above principal diagonal are zero.

For example: $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$

Column Matrix :

It is a matrix in which there is only one column.

For example: $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Row Matrix :

It is a matrix in which there is only one row.

For example: $[\,2 \quad 3 \quad 4\,]$

Transpose of a Matrix :

It is a matrix obtained by interchanging rows into columns

For example: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 8 \end{bmatrix}$

A' = transpose of $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 5 & 8 \end{bmatrix}$

Symmetric Matrix :

If for a square matrix A, $A = A'$ then A is symmetric

For example: $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 8 \\ 5 & 8 & 3 \end{bmatrix}$

Skew Symmetric matrix :

If for a square matrix A, $A = -A'$ then it is skew – symmetric matrix.

For example: $\begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix}$

Note : For a skew symmetric matrix, diagonal elements are zero.

Orthogonal Matrix :

A square matrix A is orthogonal if $AA' = A'A = I$

For example: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Here $AA' = I$

Note : For orthogonal matrix A, $A^{-1} = A'$

Conjugate of a Matrix :

Let A be a complex matrix of order $m \times n$. Then conjugate of A is the matrix obtained by taking conjugate of every element in the matrix and denoted by $\bar{A}$

For example: if $A = \begin{bmatrix} 7+i & 3-3i & 4 \\ 9+2i & -i & 8-4i \end{bmatrix}$

then conjugate of A = $\bar{A} = \begin{bmatrix} 7-i & 3+3i & 4 \\ 9-2i & i & 8+4i \end{bmatrix}$

Matrix A^θ :

The transpose of the conjugate of a matrix A is denoted by A^θ

For example: Let $A = \begin{bmatrix} 7+i & 2-4i & 4 \\ 3+2i & -i & 1-2i \end{bmatrix}$

then $\bar{A} = \begin{bmatrix} 7-i & 2+4i & 4 \\ 3-2i & i & 1+2i \end{bmatrix}$

and $$A^{\theta} = \left(\overline{A}\right)' = \begin{bmatrix} 7-i & 3-2i \\ 2+4i & i \\ 4 & 1+2i \end{bmatrix}$$

Unitary Matrix :

A square matrix A is said to be unitary if $A^{\theta} A = I$

For example: $$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Here $$A^{\theta}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hermitian Matrix :

A square matrix A is called Hermitian matrix if $a_{ij} = \overline{a}_{ji}$.

For example: $$A = \begin{bmatrix} 4 & 1-i & 2+5i \\ 1+i & 3 & 1-2i \\ 2-5i & 1+2i & 8 \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Hermitian is that $\mathbf{A = A^{\theta}}$.

Skew – Hermitian Matrix :

A square matrix A is skew Hermitian matrix if $a_{ij} = -\overline{a}_{ji}$

For example: $$\begin{bmatrix} 2i & 2-8i & 1-2i \\ -(2+8i) & 0 & 2i \\ -(1+2i) & 2i & -4i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be skew–Hermitian is that $\mathbf{A^{\theta} = -A}$

Note : All the diagonal elements of a skew Hermitian matrix are either zeroes or pure imaginary.

Idempotent Matrix :

Matrix A is called idempotent matrix if $\mathbf{A^2 = A}$

For example: $$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Here $A^2 = A$

Periodic Matrix :

A matrix A is called a periodic matrix. $A^{K+1} = A$ where K is a +ve integer; if K is the least +ve integer for which $A^{K+1} = A$, then K is the period of A.

Note : If K = 1, we get $A^2 = A$ and it is idempotent matrix.

Nil potent Matrix :

A matrix is called a Nilpotent matrix, if $A^K = 0$ where K is a positive integer. If K is the least positive integer for which $A^K = 0$, then K is the index of the nil potent matrix.

For example: $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ has index 2

Involutory Matrix :

A matrix A is called involutory matrix if $A^2 = I$

For example: $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

Here $A^2 = I$

Note : $I^2 = I$. $\therefore$ Identity matrix is always involutory.

Determinant of a square matrix

Let A be a square matrix, then | A | = determinant of A.

For example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 1(2-3) - 2(4-9) + 3(2-3) = -1 + 10 - 3$$

$$= 6$$

- If $|A| \neq 0$ then matrix A is called as non–singular.
- If $|A| = 0$, A is called singular.

Adjoint and Inverse of a Square Matrix

Minor : Consider the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

To find minor leave the row and column passing through the element a_{ij}.

The minor of the element $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of the element $a_{32} = M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

The minor of the element $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor : The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij}.

The co–factor of the element $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

The cofactor of the element $a_{32} = A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

The cofactor of the element $a_{11} = A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

And so on.

Adjoint of a Matrix :

Adjoint of a square matrix A is the transpose of the matrix formed by the cofactors of the elements of the given matrix A.

If $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Inverse of a Square Matrix :

For a non–singular square matrix A

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

where A^{-1} is called the inverse of square matrix.

Note : $AA^{-1} = A^{-1}A = I$

Solved Example 3 :

Calculate the adjoint of A,

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution :

A_{11} = the co–factor of a_{11} in

$$|A| = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3$$

A_{12} = the co–factor of a_{12} in

$$|A| = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9$$

A_{13} = the co–factor of a_{13} in

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

A_{21} = the co–factor of a_{21} in

$$|A| = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$$

A_{22} = the co–factor of a_{22} in

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

A_{23} = the co–factor of a_{23} in

$$|A| = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$$

A_{31} = the co–factor of a_{31}

$$\text{in } |A| = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

A_{32} = the co–factor of a_{32} in

$$|A| = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4$$

A_{33} = the co–factor of a_{33} in

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

Adj (A) = transpose of the matrix formed by co–factor

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Solved Example 4 :

Find the inverse of the matrix by finding its

$$\text{adjoint where } A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution :

$|A| = 1 \neq 0$

A^{-1} exists

$$\text{Now } A' = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{bmatrix}$$

The co–factors of the elements of A′ are

(1) = 7 ; (1) = –3 ; (1) = –3

(3) = –1 ; (4) = 1 ; (3) = 0

(3) = –1 ; (3) = 0 ; (4) = 1

$$\therefore \text{ Adj (A)} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Solved Example 5 :

Find the adjoints of the matrices A and B

$$\text{where} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 4 & 5 \\ -1 & 2 & 3 \\ 1 & -1 & 7 \end{bmatrix}.$$

Verify the formula adj(AB) = (adj B) (adj A)

Solution :

We can find that

$$\text{adj } A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix},$$

$$\text{adj } B = \begin{bmatrix} 17 & -33 & 2 \\ 10 & -5 & -5 \\ -1 & 4 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 5 & 32 \\ 1 & 6 & 42 \\ -1 & 9 & 38 \end{bmatrix}$$

$$(\text{adj } B)\ (\text{adj } A) = \begin{bmatrix} -150 & 98 & 18 \\ -80 & 70 & -10 \\ 15 & -14 & 1 \end{bmatrix},$$

$$\text{adj (AB)} = \begin{bmatrix} -150 & 98 & 18 \\ -80 & 70 & -10 \\ 15 & -14 & 1 \end{bmatrix}$$

Hence we verify (adj B) (adj A) = adj (AB)

Solved Example 6 :

Find the inverse of the matrix finding its adjoint where $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solution :

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 6 \neq 0 \therefore A^{-1} \text{ exists}$$

$$\text{transpose of } A = A' = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

The co–factors of the elements of A′ are

(2) = –1 ; (3) = 3 ; (1) = –1

(1) = –7 ; (1) = 3 ; (2) = –5

(3) = 5 ; (2) = –3 ; (3) = –1

$$\text{adj (A)} = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

$$= \frac{1}{6}\begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Solved Example 7 :

Show that

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ is orthogonal}$$

Solution :

$$A' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly $A'A = I$

Hence $AA' = A'A = I$

$\therefore$ A is orthogonal

Solved Example 8 :

Show that

$$A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \text{ is orthogonal and}$$

find A^{-1}.

Solution :

$$AA' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly $A'A = I$

$\therefore$ A is orthogonal and therefore

$$A^{-1} = A' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

1.3 Rank of a Matrix

Sub-matrix

Any matrix obtained by omitting some rows and columns form a given $m \times n$ matrix A is called a sub-matrix of A.

For example: $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ contains

three 2 x 2 sub-matrices $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \begin{bmatrix} b_1 & c_1 \\ b_2 & c_3 \end{bmatrix}$

two 1×3 sub matrices namely $[a_1\ b_1\ c_1]$ and $[a_2\ b_2\ c_2]$ and

three 2×1 sub matrix namely $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and so on

The **rank of a matrix** is r if :

i) It has atleast one non-zero minor of order r

ii) Every minor of A of order higher than r is zero

The rank of a matrix in Row-Echelon form is equal to the number of non-zero rows.

The rank of a matrix is also given by the number of linearly independent rows.

Note :
1. **If A is zero matrix, then r(A) = 0**
2. **IF A is not a zero matrix, r(A) $\geq$ 1**
3. **IF A is a non–singular n $\times$ n matrix then r(A) = n ($\because$ | A | $\neq$ 0)**
4. **r(I_n) = n**
5. **If A is an m $\times$ n matrix then r(A) $\leq$ minimum of m and n**

Solved Example 9 :

Find the rank of the matrix $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

Solution :

This matrix contains four 3 $\times$ 3 matrices.

$$\begin{bmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{bmatrix}$$

The determinants of all these are zero. Then consider 2^{nd} order sub matrices. It can be seen that $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ has determinant whose value is 5. i.e. not zero. Hence the rank of the matrix is 2.

Solved Example 10 :

Find the rank of the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}.$$

Solution :

The matrix contains four 3 $\times$ 3 matrices

$$\begin{bmatrix} 3 & 2 & -1 \\ 5 & 1 & 4 \\ 1 & -4 & 11 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & 5 \\ 5 & 4 & -2 \\ 1 & 11 & -19 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 5 \\ 5 & 1 & -2 \\ 1 & -4 & -19 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & -2 \\ -4 & 11 & -19 \end{bmatrix}$$

The determinants of all these are zero. Then consider 2^{nd} order sub matrices.

It can be seen that $\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$ has a value -7 i.e. not zero.

Hence the rank of the matrix is 2.

1.4 System of Linear Equations

Consider a set of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The equations can be written in the matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

which is of the form AX = B

Augmented matrix, $[A : B] = \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$

Consistency conditions

After reducing [A : B] to Row-Echelon form, find the ranks of A and [A : B]

Case 1 : **r(A) ≠ r (A : D)**, then the system is **inconsistent**.
i.e. it has **no solution**.

Case 2 : **r(A : D) = r(A)** then the system is **consistent** and if

(i) **r(A : D) = r(A) = Number of unknowns** then the system is consistent and has **unique solution**.

(ii) **r(A : D) = r(A) < Number of unknowns** then the system is consistent and has **infinitely many solutions**.

Solution of Linear Equations

Cramer's Rule

The solutions of the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The solution is given by

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{1}{\Delta} \qquad [\Delta \neq 0]$$

i.e.

$$\mathbf{x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}}$$

Method of Inversion

If the matrix form of the given equations is AX = B and if $|A| \neq 0$, then the solution can be obtained as

$$\mathbf{X = A^{-1}B}$$

Non Zero Solutions of Linear Homogenous Equation

The homogenous equation in x, y, z are

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

A system of simultaneous linear equations is said to have zero or trivial solutions if all the unknowns have zero values, and is said to have non–zero solution if at least one of the unknowns has the non–zero value.

The necessary condition that the equations have non–zero solutions is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Solved Example 11 :

Solve $3x + y = 19$

$3x - y = 23$

Solution :

Here $\Delta = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6$

By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{\begin{vmatrix} 19 & 1 \\ 23 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}}$$

$$= \frac{-19-23}{-3-3} = 7$$

and $y = \dfrac{\Delta y}{\Delta} = \dfrac{\begin{vmatrix} 3 & 19 \\ 3 & 23 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}} = \dfrac{69-57}{-3-3} = -2$

Solved Example 12 :

Solve $3x - 5z = -1$

$2x + 7y = 6$

$x + y + z = 5$

Solution :

Here $\Delta = \begin{vmatrix} 3 & 0 & -5 \\ 2 & 7 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

$= 3(7 - 0) - 0(2 - 0) - 5(2 - 7)$

$= 21 - 0 + 25 = 46$

$\Delta x = \begin{vmatrix} -1 & 0 & -5 \\ 6 & 7 & 0 \\ 5 & 1 & 1 \end{vmatrix}$

$= -1(7 - 0) - 0(6 - 0) - 5(6 - 35)$

$= -7 - 0 + 145 = 138$

$\Delta y = \begin{vmatrix} 3 & -1 & -5 \\ 2 & 6 & 0 \\ 1 & 5 & 1 \end{vmatrix}$

$= 3(6 - 0) + 1(2 - 0) - 5(10 - 6)$

$= 18 + 2 - 20 = 0$

$\Delta z = \begin{vmatrix} 3 & 0 & -1 \\ 2 & 7 & 6 \\ 1 & 1 & 5 \end{vmatrix}$

$= 3(35 - 6) - 0(10 - 6) - 1(2 - 7)$

$= 87 - 0 + 5 = 92$

Now $x = \dfrac{\Delta x}{\Delta} = \dfrac{138}{46} = 3$

$y = \dfrac{\Delta y}{\Delta} = \dfrac{0}{46} = 0$

$z = \dfrac{\Delta z}{\Delta} = \dfrac{92}{46} = 2$

Solved Example 13 :

Test whether the following equations have non–zero solution. If they have such solution obtain the solutions.

$x + y - 3z = 0,\quad 3x - y - z = 0,$

$2x + y - 4z = 0$

Solution :

Now $\begin{vmatrix} 1 & 1 & -3 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{vmatrix} = 0$

and hence the equations have non–zero solution. Solving the first two equations we get

$$\frac{x}{-4} = \frac{-y}{8} = \frac{z}{-4}$$

so that $x = -4\lambda$, $y = -8\lambda$, $z = -4\lambda$, where λ is a non-zero constant. These values satisfy the third equation and hence they are non-zero solutions.

Solved Example 14 :

Test whether the following equations have non zero solution

$$2x + 3y + 4z = 0$$
$$x - 2y - 3z = 0$$
$$3x + y - 8z = 0$$

Solution :

$$\text{Now} \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{vmatrix} = 63 \neq 0$$

Hence the equations have no non–zero solution. The only solution is $x = 0$, $y = 0$, $z = 0$

Solved Example 15 :

Discuss the consistency of

$$x + y + z = 1$$
$$2x + 4y - 3z = 9$$
$$3x + 5y - 2z = 11$$

Solution :

In the matrix form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & -3 \\ 3 & 5 & -2 \end{bmatrix} \text{ and}$$

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & -3 & 9 \\ 3 & 5 & -2 & 11 \end{bmatrix}$$

$$\text{Rank } (A) = 2 \text{ since } \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -3 \\ 3 & 5 & -2 \end{vmatrix} = 0$$

$$\text{and } \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2 \neq 0$$

Rank $[A : B] = 3$

$$\text{since } \begin{vmatrix} 1 & 1 & 1 \\ 4 & -3 & 9 \\ 5 & -2 & 11 \end{vmatrix} = -7 \neq 0$$

$r(A) \neq r(A : B)$ thus the system is inconsistent i.e. it has no solution.

Solved Example 16 :

Examine for consistency

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Solution :

In the matrix form

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Rank of $A = r(A) = 3$ since $|A| = 8 \neq 0$

$$[A : B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

Similarly $r(A : B)$ is also 3.

$r(A) = r(A : B)$

$\therefore$ System is consistent and has unique solution.

Solved Example 17 :

Examine the consistency of

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution :

In the matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

Rank of $A = r(A) = 2$ since $|A| = 0$

and $\begin{vmatrix} 5 & 3 \\ 3 & 26 \end{vmatrix} = 121 \neq 0$

$$[A : B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

[A : B] contains four 3×3 sub matrices:

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \quad \begin{bmatrix} 5 & 7 & 4 \\ 3 & 2 & 9 \\ 7 & 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 & 4 \\ 3 & 26 & 9 \\ 7 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 3 & 7 & 4 \\ 26 & 2 & 9 \\ 2 & 10 & 5 \end{bmatrix}$$

All the above 4 sub matrices have determinant value 0

$\therefore$ r (A) = r(A : B)

= 2 < 3 (number of unknowns)

$\therefore$ System is consistent and has infinitely many solutions.

1.5 Eigen Values and Eigen Vectors

Characteristic Equation

Let A be an $n \times n$ square matrix. Then **[A – λI]** is **characteristic matrix of A**, where I is identity matrix.

|A – λI| is **characteristic polynomial**.

|A – λI| = 0 is **characteristic equation** of A.

Eigen values

The roots of the characteristic equation of a matrix are called its Eigen values.

Eigen vectors

If λ is an Eigen value of A, then a non-zero vector X such that

$AX = \lambda X$ or $[A - \lambda I]\,[X] = 0$ is called the Eigen vector of A corresponding to Eigen value λ.

Properties of Eigen values and Eigen vectors

- Sum of Eigen values of a matrix is equal to the trace of the matrix.
- Product of Eigen values of a matrix is equal to its determinant.
- Eigen values of A and A^T are same.
- If A is a triangular or diagonal matrix, Eigen values are the diagonal elements.
- If an Eigen value of A is λ, then $\frac{1}{\lambda}$ is an Eigen value of A^{-1}.
- If Eigen values of A are λ_1, λ_2….., then Eigen values of A^k are $\lambda_1^k,\ \lambda_2^k$………
- Eigen vectors of a real symmetric matrix corresponding to different Eigen values are orthogonal.
- If X is an Eigen vector of A corresponding to an Eigen value, the kX is also an Eigen vector of A corresponding to Eigen value X, where k is a non-zero scalar.

Cayley – Hamilton Theorem

Every square matrix satisfies its characteristic equation i.e.,

$|A - \lambda I| = 0$

$\Rightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \ldots\ldots.. + a_n = 0$

is satisfied by A,

$\therefore\ A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots\ a_nI = 0$

Solved Example 18 :

1. Find the eigen values of the matrix.

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution :

Characteristic equation of A in λ is

$$| A - \lambda I | = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

Instead of evaluating the determinant directly we use the formula for its expansion which is as follows :

λ^3 – (sum of diagonal elements of A) λ^2 + (sum of minors of diagonal elements of A) $\lambda - |A| = 0$

$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$\therefore\quad (\lambda - 1)(\lambda - 7)(\lambda - 1) = 0$

$\therefore\quad \lambda_1 = 7,\ \ \lambda_2 = 1,\ \ \lambda_3 = 1$

Solved Example 19 :

Find the eigen vectors for the matrix :

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution :

From example (1) we get

$\lambda_1 = 7,\ \ \lambda_2 = 1,\ \ \lambda_3 = 1$

Matrix equation of A in λ is $(A - \lambda I)\, x = 0$

$$\therefore \quad \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (1) for $\lambda_1 = 7$, matrix equation is

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule

$$\therefore \quad \frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$\therefore \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$$

$$\therefore \quad x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Case (2) for $\lambda_2 = 1$ Matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule, we get

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$$\therefore \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But by definition we want nor –zero X_2. So we proceed as follows,

Expanding R_1

$x_1 + x_2 + x_3 = 0$

We assure any element to be zero say x_1 and give any convenient value say 1 to x_2 and find x_3.

Let $x_1 = 0,\ \ x_2 = 1\ \ \therefore x_3 = -1$

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case 3 : for $\lambda_3 = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule we get

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$$\therefore \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Again consider $x_1 + x_2 + x_3 = 0$

Now let $x_2 = 0\ \ x_1 = 1$ and $x_3 = -1$

$$\therefore \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solved Example 20 :

Verify whether the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ satisfies its characteristic equation.

Solution :

$$| A - \lambda I | = \begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 3 \\ 2 & -1 & 1-\lambda \end{vmatrix}$$

$$= \lambda^3 - 2\lambda^2 + 4\lambda - 18$$

Now $A^2 = \begin{vmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{vmatrix}$

$$A^3 = \begin{vmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{vmatrix}$$

$$\therefore\ A^3 - 2A^2 + 4A - 18\,I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.6 Vectors

An ordered set of n number is called an n–vector or a vector of order n.

<u>For example</u>: $X = (x_1, x_2, \ldots x_n)$ is an n–vector.

The numbers $x_1, x_2, \ldots x_n$ are called as the components of X.

The components $x_1, x_2, \ldots x_n$ of a vector may be written in a row or a column.

i.e. $X = (x_1, x_2, x_3, \ldots x_n)$ and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_n \end{bmatrix} \quad \text{are n vectors}$$

<u>Operation on Vectors</u>

Inner product of two vectors

Let $X = (x_1, x_2, \ldots x_n)$ and

$Y = (y_1, y_2, \ldots y_n)$ be two n–vectors

Then the product $XY = x_1y_1 + x_2y_2 + \ldots + x_n y_n$ is called the inner product of two n–vectors

Length of a vector

Let $X = (x_1, x_2, \ldots x_n)$ be a vector. Then the length of a vector is the positive square root of the expression

$$x_1^2 + x_2^2 + \ldots\ldots x_n^2$$

Note : Length of the vector is also called as Norm of the vector

Normal vector

A vector whose length is 1 is called a normal vector

i.e. if $X = (x_1, x_2, \ldots\ldots x_n)$ is a normal vector then $x_1^2 + x_2^2 + \ldots\ldots x_n^2 = 1$

If a vector is not a normal one, then it can be converted to a normal vector as follows.

Let X = (1, 3, − 7) be a 3–vector.

Let $d = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$

Then $\bar{x} = \left(\frac{1}{d}, \frac{3}{d}, \frac{-7}{d}\right)$ is a normal vector.

Orthogonal vector

A vector X is said to be orthogonal to Y if the inner product of X and Y is zero i.e. XY = 0

i.e. if $X = (x_1, x_2, \ldots\ldots x_n)$ and

$Y = (y_1, y_2, \ldots\ldots y_n)$ and if X and Y are orthogonal then

$x_1y_1 + x_2y_2 + \ldots\ldots + x_n y_n = 0$

Linear dependence and independence of vectors

A system of n vectors $x_1, x_2, \ldots x_n$ of the same order are said to be linearly dependent if there exists n numbers $K_1, K_2, \ldots K_n$ (where all of them are not zero) such that

$K_1X_1 + K_2X_2 + \ldots + K_nX_n = 0$

(where 0 is a null vector of the same order)

The vector are linearly independent only if $K_1 = K_2 = \ldots\ldots K_n = 0$

Solved Example 21 :

Examine for linear dependence

$X_1 = (1\ \ 2\ \ 4)^T$, $X_2 = (3\ \ 7\ \ 10)^T$

Solution :

We have

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$$

Let $c_1x_1 + c_2x_2 = 0$

(Note 0 on R. H. S. is Zero Vector)

i.e. $$c_1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 + 7c_2 \\ 4c_1 + 10c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + 3c_2 = 0$$
$$2c_1 + 7c_2 = 0$$
$$4c_1 + 10c_2 = 0$$

Consider first two equations in Matrix form.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|A| = 1 \neq 0$$

$\therefore$ system has zero solution

$\therefore$ $c_1 = c_2 = 0$

$\therefore$ X_1, X_2 are linear independent.

Solved Example 22 :

Show that

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 6 \\ 0 & 4 & 3 \end{bmatrix}$$

are linearly dependent.

Solution :

Consider

$$c_1\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$+c_3\begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 6 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 2c_2 + 0c_1 & 2c_1 + c_2 + 3c_3 \\ 3c_1 + 3c_2 + 3c_3 & 2c_1 + 4c_2 + 0c_3 \\ c_1 + 2c_2 + 0c_3 & 3c_1 + 2c_2 + 4c_3 \end{bmatrix}$$

$$\begin{matrix} 3c_1 + 3c_2 + 3c_3 \\ 4c_1 + 2c_2 + 6c_3 \\ 2c_1 + c_2 + 3c_3 \end{matrix}\Bigg] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $c_1 + 2c_2 = 0$

$2c_1 + c_2 + 3c_3 = 0$

$c_1 + c_2 + c_3 = 0$

Solving we get

$c_1 = -2c_2$ and $c_2 = c_3$

Let $c_2 = c_3 = 1$

$\therefore$ $c_1 = -2$

We see that x_1 x_2 x_3 are linearly dependent

$$-2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 6 \\ 0 & 4 & 3 \end{bmatrix} = 0$$

❑ ❑ ❑ ❑ ❑

Assignment – 1

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is –12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is **[ME – 2014]**

(A) –96 (B) –24
(C) 24 (D) 96

2. Consider a 3 x 3 real symmetric matrix S such that two of its eigen values are $a \neq 0$, $b \neq 0$ with respective eigen vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$ then $x_1y_1 + x_2y_2 + x_3y_3$ equals **[ME – 2014]**

(A) a (B) b
(C) ab (D) 0

3. A real (4 X 4) matrix A satisfies the equation $A^2 = I$, where I is the (4 X 4) identity matrix. The positive eigen value of A is ______. **[EC – 2014]**

4. Which one of the following statements is true for all real symmetric matrices? **[EE – 2014]**

(A) All the eigen values are real.
(B) All the eigen values are positive.
(C) All the eigen values are distinct.
(D) Sum of all the eigenvalues is zero.

5. With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates: $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; and $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to **[CE – 2014]**

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$
(C) $\frac{4}{5}$ (D) $\frac{5}{2}$

6. At least one eigen value of a singular matrix is **[ME – 2015]**

(A) positive (B) zero
(C) negative (D) imaginary

7. We have a set of 3 linear equations in 3 unknowns. 'X $\equiv$ Y' means X and Y are equivalent statements and 'X $\not\equiv$ Y' means X and Y are not equivalent statements. **[EE – 2015]**

P : There is a unique solution.
Q : The equations are linearly independent.
R : All eigen values of the coefficient matrix are nonzero.
S : The determinant of the coefficient matrix is nonzero.

Which one of the following is TRUE?

(A) $P \equiv Q \equiv R \equiv S$

(B) $P \equiv R \not\equiv Q \equiv S$

(C) $P \equiv Q \not\equiv R \equiv S$

(D) $P \not\equiv Q \not\equiv R \not\equiv S$

8. Consider a system of linear equations:

$x - 2y + 3z = -1,$

$x - 3y + 4z = 1$, and

$-2x + 4y - 6z = k.$ **[EC – 2015]**

The value of k for which the system has infinitely many solutions is _____.

9. The larger of the two eigen values of the matrix $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ is _____.**[CS – 2015]**

10. A real square matrix A is called skew-symmetric if **[ME – 2016]**

(A) $A^T = A$

(B) $A^T = A^{-1}$

(C) $A^T = -A$

(D) $A^T = A + A^{-1}$

Q 11 to Q 20 carry two marks each

11. The system of linear equations

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 14 \end{pmatrix}$$ has

[EC – 2014]

(A) a unique solution

(B) infinitely many solutions

(C) no solution

(D) exactly two solutions

12. A system matrix is given as follows.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigen value to the minimum eigen value is________. **[EE – 2014]**

13. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$. AB^T is equal to **[CE – 2017]**

(A) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$

(C) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$

(D) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

14. The value of x for which all the eigen values of the matrix given below are real is **[EC – 2015]**

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(A) $5 + j$

(B) $5 - j$

(C) $1 - 5j$

(D) $1 + 5j$

15. The two Eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3:1 for p = 2. What is another value of p for which the Eigen values have the same ratio of 3:1? **[CE – 2015]**

(A) –2

(B) 1

(C) 7/3

(D) 14/3

16. Let the eigen values of a 2 x 2 matrix A be 1, -2 with eigen vectors x_1 and x_2 respectively. Then the eigen values and eigen vectors of the matrix $A^2 - 3A + 4I$ would, respectively, be

(A) 2, 14; x_1, x_2 **[EE – 2016]**

(B) 2, 14; $x_1 + x_2$, $x_1 - x_2$

(C) 2, 0; x_1, x_2

(D) 2, 0; $x_1 + x_2$, $x_1 - x_2$

17. Consider the matrix

$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$ whose eigen values are 1, –1 and 3. Then Trace of $(A^3 - 3A^2)$ is _______. **[IN – 2016]**

18. Consider the following system of equations:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0, \\ x_2 - x_3 &= 0, \\ x_1 + x_2 &= 0. \end{aligned}$$

This system has **[ME – 2011]**

(A) a unique solution

(B) no solution

(C) infinite number of solutions

(D) five solutions

19. Given that

$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the value of $\mathbf{A}^3$ is **[EC, EE, IN – 2012]**

(A) 15 **A** + 12 **I**

(B) 19 **A** + 30 **I**

(C) 17 **A** + 15 **I**

(D) 17 **A** + 21 **I**

20. Consider the matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Which one of the following statements about P is INCORRECT? **[ME – 2017]**

(A) Determinant of P is equal to 1

(B) P is orthogonal

(C) Inverse of P is equal to its transpose

(D) All eigen values of P are real numbers

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Assignment – 2

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. The matrix form of the linear system $\frac{dx}{dt} = 3x - 5y$ and $\frac{dy}{dt} = 4x + 8y$ is **[ME – 2014]**

(A) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(B) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(C) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

(D) $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

2. Which one of the following equations is a correct identity for arbitrary 3 x 3 real matrices P, Q and R? **[ME – 2014]**

(A) P(Q + R) = PQ + RP

(B) $(P - Q)^2 = P^2 - 2PQ + Q^2$

(C) det (P + Q) = det P + det Q

(D) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

3. The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is _________. **[EC – 2014]**

4. Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is _______. **[CE – 2014]**

5. The determinant of matrix

$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ______. **[CE – 2014]**

6. The lowest eigen value of the 2 × 2 matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is ______. **[ME – 2015]**

7. For what value of p the following set of equations will have no solution? **[CE – 2015]**

$2x + 3y = 5$

$3x + py = 10$

8. The value of p such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigen vector of the matrix

$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is ______. **[EC – 2015]**

9. The solution to the system of equations $\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ -30 \end{Bmatrix}$ is **[ME – 2016]**

(A) 6, 2 (B) –6, 2
(C) –6, –2 (D) 6, –2

10. Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals: **[EC – 2014]**

(A) M^{4k+1} (B) M^{4k+2}
(C) M^{4k+3} (D) M^{4k}

Q 11 to Q 20 carry two marks each

11. The maximum value of the determinant among all 2 x 2 real symmetric matrices with trace 14 is _______. **[EC – 2014]**

12. The rank of the matrix **[CE – 2014]**

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$ is _________.

13. The smallest and largest eigen values of the following matrix are:

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$ **[CE – 2015]**

(A) 1.5 and 2.5 (B) 0.5 and 2.5
(C) 1.0 and 3.0 (D) 1.0 and 2.0

14. Consider the following 2×2 matrix A where two elements are unknown and are marked by a and b. The eigen values of this matrix are –1 and 7. What are the values of a and b?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$ **[CS – 2015]**

(A) a = 6, b = 4 (B) a = 4, b = 6
(C) a = 3, b = 5 (D) a = 5, b = 3

15. The number of linearly independent eigen vectors of matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ is ______. **[ME – 2016]**

16. Let A be a 4×3 real matrix with rank 2. Which one of the following statement is TRUE? **[EE – 2016]**

(A) Rank of $A^T A$ is less than 2.
(B) Rank of $A^T A$ is equal to 2.
(C) Rank of $A^T A$ is greater than 2.
(D) Rank of $A^T A$ can be any number between 1 and 3.

17. The system of equations

$x + y + z = 6$
$x + 4y + 6z = 20$
$x + 4y + \lambda z = \mu$

has NO solution for values of λ and μ given by **[EC – 2011]**

(A) $\lambda = 6, \mu = 20$ (B) $\lambda = 6, \mu \neq 20$
(C) $\lambda \neq 6, \mu = 20$ (D) $\lambda \neq 6, \mu \neq 20$

18. Let A be the 2×2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Then the eigen values of the matrix A^{19} are **[CS – 2012]**

(A) 1024 and –1024

(B) $1024\sqrt{2}$ and $-1024\sqrt{2}$

(C) $4\sqrt{2}$ and $-4\sqrt{2}$

(D) $512\sqrt{2}$ and $-512\sqrt{2}$

19. Which one of the following does NOT equal **[CS – 2013]**

(A) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$

(B) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(C) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(D) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

20. Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigen vectors corresponding to eigen values λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$, respectively. The value of $x_1^T x_2$ is _________. **[ME – 2017]**

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Assignment – 3

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. One of the eigen vectors of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is **[ME – 2014]**

(A) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (B) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(C) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (D) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

2. For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(A) $(M^T)^T = M$ **[EC – 2014]**

(B) $(cM)^T = c(M)^T$

(C) $(M + N)^T = M^T + N^T$

(D) $MN = NM$

3. Given a system of equations:

$x + 2y + 2z = b_1$

$5x + y + 3z = b_2$

Which of the following is true regarding its solutions? **[EE – 2014]**

(A) The system has a unique solution for any given b_1 and b_2

(B) The system will have infinitely many solutions for any given b_1 and b_2

(C) Whether or not a solution exists depends on the given b_1 and b_2

(D) The system would have no solution for any values of b_1 and b_2

4. The sum of eigen values of the matrix, [M] is **[CE – 2014]**

Where $[M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$

(A) 915 (B) 1355

(C) 1640 (D) 2180

5. If any two columns of a determinant $P = \begin{vmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{vmatrix}$ are interchanged, which one of the following statements regarding the value of the determinant is **CORRECT**? **[ME – 2015]**

(A) Absolute value remains unchanged but sign will change

(B) Both absolute value and sign will change

(C) Absolute value will change but sign will not change

(D) Both absolute value and sign will remain unchanged

6. The eigen value of the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$ are **[IN – 2017]**

(A) $-1, 5, 6$ (B) $1, -5 \pm j6$

(C) $1, 5 \pm j6$ (D) $1, 5, 5$

7. Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and

$Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices.

Then the rank of P + Q is _________.

[CS – 2017]

8. For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is **[EC – 2015]**

(A) $\sec^2 x$ (B) $\cos 4x$

(C) 1 (D) 0

9. Consider the following simultaneous equations (with c_1 and c_2 being constants):

$3x_1 + 2x_2 = c_1$

$4x_1 + x_2 = c_2$

The characteristic equation for these simultaneous equations is **[CE – 2017]**

(A) $\lambda^2 - 4\lambda - 5 = 0$

(B) $\lambda^2 - 4\lambda + 5 = 0$

(C) $\lambda^2 + 4\lambda - 5 = 0$

(D) $\lambda^2 + 4\lambda + 5 = 0$

10. The value of x for which the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$ has zero as an eigen value is _______. **[EC – 2016]**

Q 11 to Q 20 carry two marks each

11. Which one of the following statements is NOT true for a square matrix A?

[EC – 2014]

(A) If A is upper triangular, the eigen values of A are the diagonal elements of it

(B) If A is real symmetric, the eigen values of A are always real and positive

(C) If A is real, the eigen values of A and A^T are always the same

(D) If all the principal minors of A are positive, all the eigen values of A are also positive.

12. For given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$, where $i = \sqrt{-1}$, the inverse of matrix P is **[ME – 2015]**

(A) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$

(B) $\frac{1}{25}\begin{bmatrix} i & 4-3i \\ 4+3i & -i \end{bmatrix}$

(C) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

(D) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

13. If the characteristic polynomial of a 3×3 matrix M over $\mathbb{R}$ (the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$, $a \in \mathbb{R}$, and one eigen value of M is 2, then the largest among the absolute values of the eigen values of M is ________. **[CS – 2017]**

14. Perform the following operations on the matrix $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$.

(i) Add the third row to the second row.

(ii) Subtract the third column from the first column.

The determinant of the resultant matrix is ____________. **[CS – 2015]**

15. The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det (A) = 100 and trace (A) = 14.

The value of $|a - b|$ is ___. **[EC – 2016]**

16. Consider the following linear system.

$x + 2y - 3z = a$

$2x + 3y + 3z = b$

$5x + 9y - 6z = c$

This system is consistent if a, b and c satisfy the equation **[CE – 2016]**

(A) $7a - b - c = 0$

(B) $3a + b - c = 0$

(C) $3a - b + c = 0$

(D) $7a - b + c = 0$

17. Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Which one of the following statements is TRUE for the eigen values and eigen vectors of this matrix? **[CE – 2017]**

(A) Eigen value 3 has a multiplicity of 2, and only one independent eigen vector exists.

(B) Eigen value 3 has a multiplicity of 2, and two independent eigen vectors exist.

(C) Eigenvalue 3 has a multiplicity of 2, and no independent eigen vector exists.

(D) Eigenvalues are 3 and –3, and two independent eigen vectors exist.

18. For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, ONE of the normalized eigen vectors is given as **[ME – 2012]**

(A) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (B) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(C) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (D) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

19. The minimum eigen value of the following matrix is

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$ **[EC – 2013]**

(A) 0 (B) 1

(C) 2 (D) 3

20. The eigen values of the matrix given below are

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$ **[EE – 2017]**

(A) (0, –1, –3) (B) (0, –2, –3)

(C) (0, 2, 3) (D) (0, 1, 3)

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Assignment – 4

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. A square matrix each of whose diagonal elements are '1' and non–diagonal elements are '0' is called
(A) Null matrix
(B) Skew symmetric matrix
(C) Identity matrix
(D) row matrix

2. If product of two matrices AB exists – does BA exists ?
(A) Always exist (B) May exist
(C) Never exist (D) None of these

3. If a square matrix A is real and symmetric, then the eigen values
(A) are always real
(B) are always real and positive
(C) are always real and non-negative
(D) occur in complex conjugate pairs

4. If A and B are two matrices such that AB and A + B are both defined then A, B are
(A) Scalar matrices of same order
(B) square matrices of same order
(C) matrices of different order
(D) cannot predict

5. A matrix A has x rows and x + 5 columns, matrix B has y rows and 11–y columns. Both AB and BA exists. Then values of x and y respectively are
(A) 3, –8 (B) –5, 11
(C) 5, –11 (D) 3, 8

6. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular
(A) 0, 3 (B) 0, –3
(C) –1, 4 (D) 3, 4

Q7 to Q18 carry two marks each

7. Evaluate

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ where ω is one of the imaginary cube root of unity
(A) 0 (B) 1
(C) 2 (D) None of these

8. If a given matrix [A] $m \times n$ has r linearly independent vectors (rows or columns) and the remaining vectors are combination of these r vectors. Then rank of matrix is
(A) m (B) n
(C) r (D) m – n

9. Are following vectors linearly dependent

$x_1 = (3, 2, 7)$ $x_2 = (2, 4, 1)$

$x_3 = (1, -2, 6)$

(A) dependent (B) independent

(C) can't say (D) none of above

10. If A is non–zero column matrix and B is a non–zero row matrix then rank of matrix AB is

(A) always 1

(B) always = No. of elements in row

(C) always = No. of elements in column

(D) does not depends on rows & columns

11. If $A = [x\ \ y\ \ z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$,

$C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Then ABC =

(A) ABC is not possible

(B) $[ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$

(C) $[a^2x + b^2y + c^2z + 2abc + 2xyz + 2fgh]$

(D) $\begin{bmatrix} ax^2 + byz + fcz^2 \\ 2hxy + 2gzx + 2fyz \end{bmatrix}$

12. The inverse of a matrix $\begin{bmatrix} 8 & 7 & 9 \\ 5 & 10 & 15 \\ 1 & 2 & 3 \end{bmatrix}$ is

(A) $\frac{1}{2}\begin{bmatrix} 15 & 7 & 8 \\ 10 & 5 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} -15 & 7 & 8 \\ -10 & -5 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} -15 & 7 & 8 \\ 10 & -5 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

(D) none of these

13. Rank of matrix $\begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$ is

(A) 3 (B) 2

(C) 1 (D) none of these

14. Eigen values of matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

(A) 1, –2, 4 (B) 1, 3, 4

(C) 1, 2, 5 (D) –1, 3, 4

15. Whether the following equations are inconsistent

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$

(A) Yes
(B) No
(C) Can't say
(D) Can't be determined.

16. If $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ -2 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 6 & 3 \\ -2 & 5 & 1 \end{bmatrix}$

& $A + B' - X = 0$ then $X =$

(A) $\begin{bmatrix} 4 & -2 \\ 9 & -7 \\ 1 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 2 \\ 9 & -7 \\ 1 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 4 & 2 \\ 9 & 7 \\ 1 & -4 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & 2 \\ 9 & 7 \\ 1 & 4 \end{bmatrix}$

17. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ &

$B = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ then

$\cos\theta\,(A) + \sin\theta\,(B) =$

(A) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

18. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ & $A^2 = kA + 14I$, then

$k = \ldots.$

(A) 3 (B) 5
(C) 1 (D) –5

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Assignment – 5

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. If A be any matrix, then matrix B if it exists such that AB = BA = I, then it is called

(A) Transpose of A

(B) Inverse of A

(C) Cofactor matrix A

(D) Adjoint of A.

2. The eigen vector corresponding to an eigen values are

(A) different (B) unique

(C) non–unique (D) none of these

3. Rank of matrix which is in echelon form is equal to

(A) No. of non-zero rows in the matrix

(B) No. of columns in the matrix

(C) It is independent of No. of rows and columns

(D) Can't say

4. Cayley Hamilton theorem is

(A) A matrix can be expressed as sum of symmetric and skew–symmetric matrices

(B) Every square matrix satisfies its own characteristic equation.

(C) Inverse of a matrix exists if it is singular

(D) None of these

5. Inverse of a matrix

(A) exist if matrix is singular

(B) is unique

(C) $\frac{|A|}{\text{adj} A}$

(D) is not unique

6. If $P = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, $Q = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}$, then PQ is equal to

(A) $\begin{bmatrix} pr-qs & ps+qr \\ -qr-ps & -qs+pr \end{bmatrix}$

(B) $\begin{bmatrix} pq-rs & pr-qs \\ qr+ps & -qs+pr \end{bmatrix}$

(C) $\begin{bmatrix} pr+qs & ps-qr \\ qr+ps & qs-pr \end{bmatrix}$

(D) $\begin{bmatrix} pq+rs & -pr+qs \\ -qp-rs & qs-pr \end{bmatrix}$

Q7 to Q18 carry two marks each

7. Is the matrix $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ an involutory matrix ?

(A) No

(B) Yes

(C) Can't say

(D) Cannot be determined

8. The inverse of the matrix

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \text{ is}$$

(A) $\begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$

(B) $\begin{bmatrix} 5 & 4 & 3 \\ 10 & 7 & 6 \\ 8 & 6 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 10 & 7 & 6 \\ 8 & 6 & 5 \\ 5 & 4 & 3 \end{bmatrix}$

(D) $\begin{bmatrix} -5 & 4 & -3 \\ 10 & 7 & 6 \\ 8 & 6 & 5 \end{bmatrix}$

9. The rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 5 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

(A) 4 (B) 3

(C) 2 (D) 1

10. The following equations have solutions

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - 2y + 3z &= 2 \\ 3x - y + 2z &= 1 \\ x - y + z &= -1 \end{aligned}$$

(A) $x = -1, y = 4, z = 4$

(B) $x = 4, y = 1, z = 1$

(C) infinite solutions

(D) it is consistent

11. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(A) 2 (B) 4

(C) 6 (D) 8

12. Characteristic root of matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix} \text{ are}$$

(A) –1, –3, –2 (B) 1, 3, 2

(C) –1, –2, 3 (D) –3, 1, 2

13. The eigen values of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are

(A) 0, 0, 0 (B) 0, 0, 1

(C) 0, 0, 3 (D) 1, 1, 1

14. The rank of matrix $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ is

(A) 3 (B) 4

(C) 2 (D) 1

15. The following equations have solutions.

$$x + 2y + 3z = 6$$
$$3x - 2y + z = 2$$
$$4x + 2y + z = 7$$

(A) $x = 2, y = 2, z = \frac{1}{2}$

(B) $x = y = z = 1$

(C) They are inconsistent

(D) Infinite solutions

16. If $A = \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix}$ then the matrix X which satisfies the equation $3A + X = 2B$ is given by X =

(A) $\begin{bmatrix} 23 & 9 \\ 7 & 10 \end{bmatrix}$ (B) $\begin{bmatrix} -23 & 9 \\ -7 & 10 \end{bmatrix}$

(C) $\begin{bmatrix} -23 & 9 \\ -7 & -10 \end{bmatrix}$ (D) None of these

17. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, then the value of k for which $A^2 = 8A + kI$ is . . . .

(A) 5 (B) –5

(C) 7 (D) –7

18. If $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ & $Y - 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, then X = ...

(A) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

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Assignment – 6

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. Eigen values of Matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ are

(A) –6, 1 (B) 6, –1

(C) –6, –1 (D) 6, 1

2. Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\left[AA^T\right]^{-1}$ is

(A) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(B) $\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$

3. If a square matrix A be such that $A^2 = I$ then it is called as

(A) nilpotent matrix

(B) idempotent matrix

(C) involutary matrix

(D) none of these

4. If λ_1 , λ_2 , ……..λ_n are the latent roots of matrix A then A^3 has latent roots

(A) λ_1 …….λ_n

(B) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots\ldots \frac{1}{\lambda_n}$

(C) $\lambda_1^3, \lambda_2^3, \ldots\ldots.\lambda_n^3$

(D) $\frac{1}{\lambda_1^3}, \frac{1}{\lambda_2^3}, \ldots\ldots\ldots \frac{1}{\lambda_n^3}$

5. If $A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, then A^{-1} is equal to

(A) $\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

6. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then value of x is

(A) 1 (B) 2

(C) 1/2 (D) none of these

Q7 to Q18 carry two marks each

7. A matrix 'P' which diagonalises 'A' is called

(A) Spectral matrix (B) Modal matrix

(C) Square matrix (D) None of these

8. If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then A^{-1} is equal to

(A) A^2 (B) $A^2 - 3A + 3I_3$

(C) A (D) $A^2 + 2A - 2I_3$

9. If the rank of the matrix

$A = \begin{bmatrix} -1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$ is 1, then the value of λ is

(A) 1 (B) –1

(C) ± 1 (D) None of these

10. Rank of a unit matrix of order n is

(A) 1 (B) 2

(C) 0 (D) n

11. For what value of λ, the system of equations

$$3x - y + z = 0$$

$$15x - 6y + 5z = 0$$

$$\lambda x - 2y + 2z = 0$$

has non – zero solution.

(A) 1 (B) 2

(C) 6 (D) None of these

12. The eigen value of $A = \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ are

(A) $\pm 3\sqrt{33}$ (B) 3, 1

(C) $-1 \pm 2\sqrt{7}$ (D) None of these

13. Given $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, indicate the statement which is not correct for A,

(A) It is orthogonal

(B) It is non singular

(C) It is singular

(D) A^{-1} exists

14. The eigen values and the corresponding eigen vectors of a 2×2 matrix are given by

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(A) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

15. If A is orthogonal ($A A^T = I = A^T A$) then $|A|$ is

(A) $\neq 0$

(B) 1

(C) 1 or –1

(D) can be any value

16. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and

$A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$

Then (a + b) =

(A) 7/20 (B) 3/20

(C) 19/60 (D) 11/20

17. For a second order matrix A if $A^2 = -I$ then A is equal to

(A) $\begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix}$

(C) $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}$

18. If A = [1, 2, 3], $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

AB =

(A) $\begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix}$ (B) $\begin{bmatrix} -2 & -1 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 4 & -1 & 2 \end{bmatrix}$ (D) None of these

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Assignment – 7

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. Eigen value of an inverse of matrix is
(A) Same as the matrix
(B) Negative of matrix values
(C) Inverse of the matrix values
(D) No any relation between them

2. Rank 'n' of non–zero matrix
(A) may be $n = 0$ (B) may be $n > 1$
(C) may be $n = 1$ (D) may be $n \geq 1$

3. The rank of matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is
(A) 3 (B) 2
(C) 1 (D) 0

4. For what value of λ do the equations $x + 2y = 1$, $3x + \lambda y = 3$
have unique solution
(A) $\lambda = 6$ (B) $\lambda \neq 6$
(C) $\lambda = 5$ (D) $\lambda \neq 5$

5. The formula for A^{-1} is given by
(A) $A^{-1} = \text{Adj } A \mid A \mid$
(B) $A^{-1} = \text{Adj } A - \mid \text{adj } A \mid$
(C) $A^{-1} = \dfrac{|A|}{\text{adj}A}$
(D) None of these

6. Find the eigen values of the matrix and state that whether it is diagonal or not

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

(A) Eigen value 1, 2, 3 & diagonal
(B) Eigen value 1, 2, 3 and non diagonal
(C) Eigen value 1, 1, 3 and non diagonal
(D) Eigen value 1, 1, 3 and diagonal

Q7 to Q18 carry two marks each

7. The inverse of matrix

$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ is

(A) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$

(B) $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1/2 & -1/2 \\ 2 & -1/2 & -3/2 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & -1/2 \\ 1 & 1/2 & -3/2 \end{bmatrix}$

(D) $\begin{bmatrix} -1 & 1 & 2 \\ -1/2 & 1/2 & -1/2 \\ -3/2 & -1/2 & -3/2 \end{bmatrix}$

8. The eigen vectors of the matrix

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha \neq 0 \text{ is (are)]}$$

(i) $(a, 0, \alpha)$ (ii) $(\alpha, 0, 0)$

(iii) $(0, 0, 1)$ (iv) $(0, \alpha, 0)$

(A) (i), (ii) (B) (iii), (iv)

(C) (ii), (iv) (D) (i), (iii)

9. The rank of the following $(n + 1) \times (n + 1)$ matrix, where a is a real number is

$$\begin{bmatrix} 1 & a & a^2 & . & . & . & a^n \\ 1 & a & a^2 & . & . & . & a^n \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 1 & a & a^2 & . & . & . & a^n \end{bmatrix}$$

(A) 1

(B) 2

(C) n

(D) Depends on the value of a

10. The characteristic equation of matrix

$$A = \begin{bmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{bmatrix} \text{ is}$$

(A) $\lambda^3 - \lambda (f^2 + g^2 + h^2) - 2fgh = 0$

(B) $\lambda^2 - \lambda (f^2 + g + h) - 2fgh = 0$

(C) $\lambda^3 + \lambda (f^2 + g^2 + h^2) - 2f^2gh = 0$

(D) $\lambda^2 + \lambda\, cf^2 + g^2 + h^2) - 2fgh = 0$

11. The eigen values of a matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \text{are}$$

(A) 5, 2 (B) 1, 6

(C) 4, 5 (D) 1, 5

12. The rank of matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

13. The following set of equations

$3x + 2y + z = 4$

$x - y + z = 2$

$-2x + 2z = 5$ have

(A) No solution

(B) Unique solution

(C) Multiple solution

(D) An inconsistency

14. The matrix form of quadratic equations $6x_1^2 + 3x_2^2 + 14\, x_3^2 + 4x_2x_3 + 18x_3x_1 + 4x_1x_2$ …is

(A) $\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$

(B) $\begin{bmatrix} 6 & 4 & 18 \\ 4 & 3 & 4 \\ 18 & 4 & 14 \end{bmatrix}$

(C) $\begin{bmatrix} -6 & -2 & -9 \\ -2 & -3 & -2 \\ 9 & 2 & -14 \end{bmatrix}$

(D) $\begin{bmatrix} -6 & 4 & 18 \\ 4 & -3 & 4 \\ 18 & 4 & -14 \end{bmatrix}$

15. If P and Q are both unitary matrices then product PQ is

(A) non–unitary (B) unitary

(C) can't say (D) none of these

16. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj A is

(A) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

17. If $\begin{bmatrix} x & 1 \\ -1 & -y \end{bmatrix} + \begin{bmatrix} y & 1 \\ 3 & x \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then (x, y) = ...

(A) (1, 0) (B) (1, 1)

(C) (0, 1) (D) (2, 1)

18. If $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$, then . . .

(A) $\Delta = 2abc$ (B) $\Delta = 0$

(C) $\Delta = -abc$ (D) $\Delta = a^2 + b^2 + c^2$

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Assignment – 8

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. Matrix A and B are square matrices of order n. Then B is said to be similar to A if there exists a non–singular matrix P such that

(A) $B = PAP^{-1}$ (B) $P'AP$

(C) $B = P^{-1}AP$ (D) $B = PAP'$

2. For m number of equations, n number of columns, let r be rank of $m \times n$ coefficient matrix, then for $r < n$, the solution of matrix is

(A) zero matrix

(B) unit matrix

(C) infinite number of solutions

(D) solution not possible

3. $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ is equal to

(A) 0 (B) 1

(C) a + b + c (D) ab + bc + cd

4. The multiplication of matrix is

(A) always commutative

(B) may be commutative

(C) not always commutative

(D) can't say

5. If the matrix A has inverse, then which is wrong

(A) A is non singular

(B) $|A| \neq 0$

(C) A is any matrix

(D) It's inverse is unique

6. If $\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & -2 \\ 3 & 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8x+3y & 6z & 32 \\ 4 & 12 & 26x-5y \end{bmatrix}$, the values of x, y, z are

(A) x = 3, y = 4, z = 1

(B) x = 0, y = 1, z = 4

(C) x = 1, y = 3, z = 4

(D) x = 1, y = –3, z = –4

Q7 to Q18 carry two marks each

7. Find the rank of matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

(A) 1 (B) 2 (C) 3 (D) 4

8. If the characteristic roots of matrix A are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ldots \lambda_n$, then the characteristic roots of A^2 are

(A) Squares of eigen values of A

(B) Square roots of eigen values of A

(C) Reciprocal of eigen values of A

(D) Does not have any relation.

9. Are following vectors linearly dependent

$x_1 = (1, 1, 1, 3)$, $x_2 = (1, 2, 3, 4)$

$x_3 = (2, 3, 4, 9)$

(A) Yes (B) No
(C) Can't say (D) None of these

10. Consider the system of equations

$x + 2y + z = 6$

$2x + y + 2z = 6$

$x + y + z = 5$

This system has
(A) Unique solution
(B) Infinite no. of solution
(C) No solution
(D) exactly two solutions

11. The eigen value of a matrix

$$\begin{bmatrix} 5 & 8 & 5 \\ 0 & 7 & 12 \\ 0 & 0 & 13 \end{bmatrix} \text{ are}$$

(A) 5, 8, 5 (B) –6, –7, –13
(C) 5, 7, 13 (D) 5, 12, 13

12. Determine the eigen values of the matrix

$$A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$$

(A) –2, 2, (B) 1, –1
(C) 3, –3 (D) 1, 3

13. Find the rank of matrix

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ -1 & -2 & 1 & 2 \end{bmatrix}$$

(A) 1 (B) 2
(C) 3 (D) 4

14. The value of determinant of matrix

$$\begin{bmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{bmatrix} \text{ is}$$

(A) $a^3 . b^3 .c^3 . d^3 - abcd$
(B) $a^2 b^2 c^2 d^2 - abcd$
(C) 0
(D) 1

15. Eigen values of matrix $\begin{bmatrix} 8 & 29 & 18 \\ 0 & 7 & 29 \\ 0 & 0 & 18 \end{bmatrix}$ are

(A) –8, –7, 29 (B) –8, –7, –18
(C) 18, 7, 8 (D) 29, 18, 29

16. $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 4^2 & 3^2 & 2^2 \end{vmatrix} = \ldots\ldots$

(A) 2 (B) –2
(C) 1 (D) 0

17. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ca & c^2-ab \end{vmatrix} = \ldots\ldots$

(A) 1 (B) 0
(C) –1 (D) None of these

18. The value of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & \sec x & \tan x \\ 0 & \tan x & \sec x \end{vmatrix} \text{ is} \ldots\ldots$$

(A) 1 (B) 0
(C) $\tan^2 x$ (D) $\sec^2 x$

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Assignment – 9

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. A square matrix U is called unitary if

(A) $U = U^{-1}$ (B) $U' = U^{-1}$

(C) $U = -U^{-1}$ (D) $U' = -U^{-1}$

2. If A is Hermitian matrix, then IA is

(A) Symmetric

(B) Hermitian

(C) Skew – Hermitian

(D) None of these

3. If A A′ = I, then | A | is equal to

(A) +1 (B) –1

(C) ± 1 (D) 0

4. The matrix A and B are given below :

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. The value of AB is

(A) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$

5. Inverse of a matrix $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is

(A) $\begin{bmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$

(B) $\begin{bmatrix} -\sin\alpha & \sin\alpha \\ \cos\alpha & \cos\alpha \end{bmatrix}$

(C) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(D) $\begin{bmatrix} \sin\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix}$, then value of A + A′

(A) $\begin{bmatrix} 2 & -2 & 8 & 7 \\ -2 & -2 & 8 & -3 \\ 8 & 8 & 14 & -1 \\ 7 & -3 & -1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & -2 & 8 & 7 \\ -2 & 2 & 8 & -3 \\ 8 & 8 & 14 & -1 \\ 7 & -3 & -1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 8 & -2 & 7 \\ -2 & 8 & 2 & -3 \\ 8 & 14 & 8 & -1 \\ 7 & -1 & -3 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 2 & 7 & -2 \\ 2 & -2 & -3 & 2 \\ 14 & 8 & -1 & 8 \\ 0 & 7 & 0 & -3 \end{bmatrix}$

Q7 to Q18 carry two marks each

7. $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary matrix if and only if

(A) $a^2 + b^2 + c^2 = 0$

(B) $b^2 + c^2 + d^2 = 0$

(C) $a^2 + b^2 + c^2 + d^2 = 1$

(D) $a^2 + b^2 + c^2 + d^2 = 0$

8. The quadratic expression in matrix form are

$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$

(A) $\begin{bmatrix} 1 & -2 & 3 & -1 \\ -2 & 4 & -6 & 0 \\ 3 & -6 & 9 & -3 \\ -1 & 0 & -3 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 4 & -6 & 2 \\ 4 & -4 & 12 & 0 \\ -6 & 12 & -9 & 6 \\ 2 & 0 & 6 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 2 & -3 & 1 \\ 2 & -4 & 6 & 0 \\ -3 & 6 & -9 & 3 \\ 1 & 0 & 3 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} -1 & -4 & 6 & -2 \\ -4 & 4 & -12 & 0 \\ 6 & -12 & 9 & -6 \\ -2 & 0 & -6 & 1 \end{bmatrix}$

9. The solution of the equation

$x + 2y + 3z = 0$

$3x + 4y + 4z = 0$

$7x + 10y + 12z = 0$

(A) $x = y = z = 1$

(B) $x = 1, y = z = 0$

(C) $x = y = z = 0$

(D) none of these

10. The matrix form of quadratic expression $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ is

(A) $\begin{bmatrix} 2 & 12 & -4 \\ 12 & 1 & -8 \\ -4 & -8 & -3 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & -6 & -4 \\ -6 & 1 & -4 \\ -4 & -4 & -3 \end{bmatrix}$

(D) $\begin{bmatrix} -2 & -6 & -2 \\ -6 & -1 & -4 \\ -2 & -4 & 3 \end{bmatrix}$

11. Find the rank of matrix

$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 8 & -2 & 3 \end{bmatrix}$

(A) 1 (B) 2

(C) 3 (D) 4

12. Find eigen value of the matrix

$\begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix}$

(A) 1, 5 (B) –1, 5

(C) –1, –5 (D) 1, –5

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, the value

$A^2 - 5A + 7I$

(A) $\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 9 \\ 1 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then value of A^n is

(A) $\begin{bmatrix} n\cos\alpha & n\sin\alpha \\ -n\sin\alpha & n\cos\alpha \end{bmatrix}$

(B) $\begin{bmatrix} -\cos n\alpha & -\sin n\alpha \\ \sin n\alpha & -\cos n\alpha \end{bmatrix}$

(C) $\begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

(D) $\begin{bmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{bmatrix}$

15. Which of the following statement is wrong ?

(A) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(B) $(ABC)^T = C^T A^T B^T$

(C) $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ provided A is square matrix

(D) A = LU,

where A – square matrix,
L – lower triangular matrix
U – Upper triangular matrix

16. T_p, T_q, T_r are the p^{th}, q^{th} & r^{th} terms of an A.P. then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ equals

(A) 1 (B) –1

(C) 0 (D) p + q + r

17. If 1, ω, ω^2 are three cube roots of unity, then $\begin{vmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{vmatrix}$ is equal to

(A) 1 (B) ω

(C) ω^2 (D) zero

18. If 1, ω, ω^2 are three cube roots of unity, then $\begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ has the value....

(A) 1 (B) ω^2

(C) ω (D) zero

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Assignment – 10

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. Rank of matrix $\begin{bmatrix} 3 & 2 & -9 \\ -6 & -4 & 18 \\ 12 & 8 & -36 \end{bmatrix}$ is

(A) 2 (B) 3

(C) $\sqrt{2}$ (D) 1

2. The value of the determinant for an upper triangular matrix is equal to

(A) sum of elements along principal diagonal

(B) product of the elements along principal diagonal

(C) sum of all elements

(D) product of all elements

3. The skew matrix always have rank 'r' equal to

(A) r = 1 (B) r > 1

(C) r = 0 (D) can't say

4. The quadratic from of the symmetrical matrix

diag. $[\lambda_1, \lambda_2, \ldots \lambda_n]$ is

(A) $\lambda_1 + \lambda_2 + \ldots. \lambda_n$

(B) $\lambda_1^2 x_1 + \lambda_2^2 x_2 + \ldots. \lambda_n^2 x_n$

(C) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots. + \lambda_n x_n^2$

(D) $\lambda_1 x_1 + \lambda_2 x_2 + \ldots. + \lambda_n x_n$

5. $P = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$ then

$Q = \begin{vmatrix} 1 & 3 & 2 \\ -3 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = ?$

(A) 8 (B) –8

(C) 16 (D) –16

6. The rank of matrix

$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

Q7 to Q18 carry two marks each

7. The matrix $\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is

(A) orthogonal (B) not orthogonal

(C) singular (D) none of these

8. The eigen values of matrix

$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ are

(A) 2, 2, 1 (B) –2, –2, –1

(C) 3, 4, –1 (D) –3, –4, 1

9. Write quadratic equation in matrix form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$

(A) $\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 12 & -4 \\ 12 & 1 & -8 \\ -4 & -8 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & -6 & 2 \\ -6 & -1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

(D) $\begin{bmatrix} -2 & -12 & 4 \\ -12 & -1 & 8 \\ 4 & 8 & 3 \end{bmatrix}$

10. The characteristic roots of the matrix

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are

(A) –2, 0, 8 (B) 2, 2, 8

(C) 2, 2, –8 (D) 0, 2, 8

11. The rank of the matrix

$\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

12. The inverse of a matrix $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ is

(A) $\frac{1}{37}\begin{bmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix}$

(B) $-\frac{1}{37}\begin{bmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix}$

(C) $\frac{1}{36}\begin{bmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix}$

(D) $\frac{1}{36}\begin{bmatrix} 1 & -4 & 6 \\ -4 & 16 & 13 \\ 6 & 13 & -1 \end{bmatrix}$

13. For $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$ matrix which of following statement is false

(A) A is square matrix

(B) A is skew symmetric matrix

(C) A is symmetric matrix

(D) $|A| = 0$

14. The rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 3 & 2 & 1 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

15. The value of the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{vmatrix} \text{ is}$$

(A) 1, 00, 000 (B) 10,000

(C) 1,000 (D) 0

16. If a + b + c = 0, one root of the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is . . . .

(A) $x = 1$ (B) $x = 2$

(C) $x = 0$ (D) $x = a^2 + b^2 + c^2$

17. The roots of the equation

$$\begin{vmatrix} a-x & b & c \\ 0 & b-x & a \\ 0 & 0 & c-x \end{vmatrix} = 0 \text{ are}$$

(A) a & b (B) b & c

(C) a & c (D) a, b & c

18. If $\begin{vmatrix} 1-x & 2 & 3 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$, then its roots are . . . .

(A) 1 only (B) 0 only

(C) 0 & 1 (D) 0, 1 & –1

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Assignment – 11

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. If $x\begin{bmatrix}1\\1\\1\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}17\\15\\13\end{bmatrix}$,

then $\begin{bmatrix}x\\y\\z\end{bmatrix}$ = ...

(A) $\begin{bmatrix}4\\11\\2\end{bmatrix}$ (B) $\begin{bmatrix}2\\4\\11\end{bmatrix}$

(C) $\begin{bmatrix}11\\4\\2\end{bmatrix}$ (D) None of these

2. If A & B are matrices such that A + B and AB are both defined then :

(A) A & B can be any matrices

(B) A & B are square matrices, not necessarily of the same order

(C) A & B are square matrices of the same order

(D) No. of columns of A = no. of rows of B.

3. If the matrix AB = 0, then we must have

(A) Both A & B should be zero

(B) At least one of A & B should be zero

(C) Neither A nor B may be zero

(D) A + B = 0

4. If A & B are arbitrary square matrices of the same order, then

(A) $(AB)' = A'B'$

(B) $(A')'\,(B')' = B'\,A'$

(C) $(A + B)' = A' - B'$

(D) $(AB)' = B'\,A'$

5. Which of the following matrices is not invertible

(A) $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ (B) $\begin{bmatrix}-1 & -1\\-1 & 2\end{bmatrix}$

(C) $\begin{bmatrix}2 & 3\\4 & 6\end{bmatrix}$ (D) $\begin{bmatrix}2 & -2\\1 & 1\end{bmatrix}$

6. If A is a square matrix, then $AA' + A'A$ is a

(A) Unit matrix

(B) Null matrix

(C) Symmetric matrix

(D) Skew–symmetric matrix

Q7 to Q18 carry two marks each

7. $A = (a_{ij})$ is a 3×2 matrix, whose elements are given by

$a_{ij} = 2i - j$ if $i > j$

$= 2j - i$ if $i \le j$

Then the matrix A is

(A) $\begin{bmatrix}1 & 0\\3 & 2\\-1 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & 3\\3 & 2\\-1 & 1\end{bmatrix}$

(C) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$

8. If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ & $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then the matrices X & Y are

(A) $X = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, Y = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$

(B) $X = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$

(C) $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(D) $X = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}, Y = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$

9. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ &

$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then B equals

(A) $I \cos\theta + J \sin\theta$

(B) $I \sin\theta + J \cos\theta$

(C) $I \cos\theta - J \sin\theta$

(D) $-I \cos\theta + J \sin\theta$

10. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \ldots\ldots$

(A) $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

(D) None of these

11. The value of $\begin{vmatrix} p & 0 & 0 & 0 \\ a & q & 0 & 0 \\ b & c & r & 0 \\ d & e & f & s \end{vmatrix}$ is . . .

(A) $p + q + r + s$ (B) 1

(C) $ab + cd + ef$ (D) pqrs

12. The determinant

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0 \text{ if}$$

(A) a, b, c are in A.P.

(B) a, b, c are in G.P. or $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

(C) a, b, c are in H.P.

(D) α is a root of $ax^2 + 2bx + c = 0$

13. If $a_1, a_2, \ldots$ form a G.P. & $a_i > 0$ for all $i \geq 1$, then

$$\begin{vmatrix} \log a_m & \log a_{m+1} & \log a_{m+2} \\ \log a_{m+3} & \log a_{m+4} & \log a_{m+5} \\ \log a_{m+6} & \log a_{m+7} & \log a_{m+8} \end{vmatrix}$$

is equal to

(A) $\log a_{m+8} - \log a_m$

(B) $\log a_{m+8} + \log a_m$

(C) zero

(D) $\log {a_{m+4}}^2$

14. If α, β, γ are real numbers, then

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} \text{ is}$$

equal to

(A) -1

(B) $\cos\alpha \cos\beta \cos\gamma$

(C) $\cos\alpha + \cos\beta + \cos\gamma$

(D) None of these

15. The characteristic equation of the matrix $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$ is . . .

(A) $x^3 + 6x^2 + 11x - 6 = 0$

(B) $x^3 - 6x^2 + 11x - 6 = 0$

(C) $x^3 + 6x^2 + 11x + 6 = 0$

(D) None of these

16. The characteristic equation of the matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has a repeated root.

The eigen vector corresponding to this eigen value is . . .

(A) $\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$ (B) $\begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$

(C) $\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$ (D) None of these

17. If $\Delta_1 = \begin{vmatrix} 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \\ 4 & 4^2 & 4^3 \end{vmatrix}$ & $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix}$,

then

(A) $\Delta_1 = 2\Delta$ (B) $\Delta_1 = 3\Delta$

(C) $\Delta_1 = 4\Delta$ (D) $\Delta_1 = 24\Delta$

18. Let a, b, c, d u, v be integers. If the system of equations ax + by = u, cx + dy = v has a unique solution in integers, then

(A) ad – bc = 1

(B) ad – bc = –1

(C) ad – bc = ±1

(D) ad – bc need not be equal to ±1

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Test Paper – 1

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. A square matrix having determinant equal to zero is called
(A) scalar matrix
(B) singular matrix
(C) non–singular matrix
(D) skew symmetric matrix

2. The values of x, y, z and a which satisfies the matrix equations

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$ are

(A) $x = 3, y = -2, z = 4, a = 3$
(B) $x = -3, y = -2, z = -4, a = 1/3$
(C) $x = -3, y = -2, z = 4, a = 3$
(D) $x = 3, y = 2, z = 4, a = -1/3$

3. Transpose of the product of two matrices is
(A) product of matrices in reverse order
(B) product of transposes of matrices in same order
(C) product of transpose of matrices in reverse order
(D) product of matrices in same order

4. Any matrix is said to be orthogonal if its determinant is
(A) 0 (B) ± 1
(C) ± 2 (D) None of these

5. For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, the eigen values are
(A) 3, –3 (B) –3, –5
(C) 3, 5 (D) 5, 0

Q6 to Q15 carry two marks each

6. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$ then

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \ldots.$$

(A) $(I+A)^2$ (B) $(I-A)^2$
(C) $I-A^2$ (D) $\dfrac{I+A}{I-A}$

7. The relationship between following vectors
$x_1 = (1, 3, 4, 2)$, $x_2 = (3, -5, 2, 2)$,
$x_3 = (2, -1, 3, 2)$ is
(A) $2x_1 + x_2 - 2x_3 = 0$
(B) $x_4 + x_2 + x_3 = 0$
(C) $x_1 + x_2 - 2x_3 = 0$
(D) $x_1 - 2x_2 + x_3 = 0$

8. The points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear if and only if the rank of matrix

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$ is

(A) equal to 3
(B) less than 3
(C) equal to or less than 3
(D) can't say

9. Consider the following equation

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

(A) They have a unique solution
(B) They posses multiple solutions
(C) They have no solution
(D) The coefficient matrix is orthogonal

10. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$,

$B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$ Then AB is

(A) $\begin{bmatrix} 2bc & abc & abc \\ abc & -abc & -ac \\ abc & -ac & abc \end{bmatrix}$

(B) $\begin{bmatrix} -2bc & ab & ac \\ ab & -2ac & ab \\ ac & ab & -2ba \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & ab & ac \\ ab & 0 & bc \\ ac & bc & 0 \end{bmatrix}$

11. Given that $f(x) = x^2 - 5x + 6$,

$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then f(A) = ?

(A) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

12. Is the given matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ idempotent matrix ?

(A) Yes
(B) No
(C) Can't say
(D) Cannot be determined

13. Inverse of a matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(A) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

14. The eigen values of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are

(A) 0, 0, 0 (B) 0, 0, 1

(C) 0, 0, 3 (D) 1, 1, 1

15. The rank of matrix $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

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Test Paper – 2

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. Every square matrix can be represented as

(A) sum of lower and upper triangular matrix

(B) sum of symmetric and skew symmetric matrix

(C) product of symmetric and skew symmetric matrix.

(D) product of Hermitian & skew hermitian matrix.

2. Sum of eigen values of a matrix is equal to

(A) sum of all elements

(B) sum of principal elements

(C) sum of diagonal elements

(D) none of these

3. If $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 15 & 14 \end{bmatrix}$

then

(A) x = 2, y = 9 (B) x = 2, y = 5

(C) x = 4, y = 6 (D) None of these

4. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & 5 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 4 & 6 \end{bmatrix}$, then 2A + 3B will be

(A) $\begin{bmatrix} 4 & 9 & 12 \\ 8 & 8 & 28 \end{bmatrix}$

(B) $\begin{bmatrix} -4 & 9 & 12 \\ 8 & -8 & 28 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 9 & 12 \\ 8 & 8 & 28 \end{bmatrix}$

(D) $\begin{bmatrix} 4 & 9 & 8 \\ 8 & -8 & 28 \end{bmatrix}$

5. Consider the following determinant

$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ which of the following is a factor of Δ ?

(A) a + b (B) a – b

(C) a + b + c (D) abc

Q6 to Q15 carry two marks each

6. Consider the following equations

$2x + 6y + 11 = 0$

$6x + 20y - 6z + 3 = 0$

$6y - 18z + 1 = 0$

(A) They have a unique solution

(B) No solution

(C) They have multiple solutions

(D) The coefficient matrix is idempotent

7. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

$B = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$. Then AB = ?

(A) $\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ \sin(\theta+\phi) & -\cos(\theta+\phi) \end{bmatrix}$

(B) $\begin{bmatrix} \cos(\theta-\phi) & \sin(\theta-\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

(C) $\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix}$

(D) $\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

8. Given matrix $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index . . .

(A) less than 3

(B) 3

(C) higher than 3

(D) can not be determined

9. The rank of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ is

(A) 3 (B) 2

(C) 1 (D) 0

10. Which of following is correct ?

(A) $(AB)^{-1} = A^{-1} B^{-1}$

(B) $(AB)^t = A^t B^t$

(C) $(A^{-1})^{-1} = A^t$

(D) adj(AB) = (adj B) (adj A)

11. Given $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Indicate the statement which is not correct for A.

(A) It is orthogonal

(B) It is non singular

(C) It is singular

(D) A^{-1} exists

12. If $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, the eigen values of the matrix $I + A + A^2$ are

(A) 1, 2, 3 (B) 3, 7, 13

(C) 1, 7, 13 (D) 1, 1, 1

13 The matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplication

(A) if $a = b\theta^2$, $\theta = n\pi$, in integer

(B) always

(C) if a = b

(D) never

14. The rank of the matrix given below is

$$\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 21 \end{bmatrix}$$

(A) 3 (B) 1

(C) 2 (D) 4

15. For a square matrix A, if $A^2 = A$ then A is called

(A) idempotent matrix

(B) nilpotent matrix

(C) involuntary matrix

(D) none of above

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Test Paper – 3

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. For triangular matrix eigen values are
(A) same as that of row elements
(B) same as of column elements
(C) same as of that principal diagonal elements
(D) can not predict

2. Rank of a matrix A is said to be ‘r’ if
(A) atleast one square matrix of A of order ‘r’ whose determinant is not equal to zero.
(B) the matrix A contains any square sub–matrix of order r + 1, then the determinant of every square sub matrix of order r + 1 should be zero.
(C) both (A) and (B) true
(D) only (B) true

3. If A is $m \times n$ matrix and B is $\ell \times n$ matrix, then product AB is
(A) $m \times n$ matrix
(B) $n \times \ell$ matrix
(C) $n \times n$ matrix
(D) product not possible

4. The equations are said to be inconsistent if
(A) the equations have no solution
(B) the solution is zero vector
(C) there are infinite number of solutions
(D) solution is possible is in the form of complex number

5. Consider the following four possible properties of a matrix A.
(i) It is square matrix
(ii) $a_{ij} = +a_{ji}$
(iii) $a_{ij} = -a_{ji}$ (iv) All leading diagonal elements are zero.
If A is skew-symmetric, which of the following is True ?
(A) (i), (ii) (iv) (B) (iii) & (iv)
(C) (i) , (iii), (iv) (D) None of these

Q6 to Q15 carry two marks each

6. If a_{ij} = 1 for all i , j then rank (A) is
(A) 1 (B) 0
(C) n (D) anything

7. | Adj A | is equal to
(A) $|A|^{n}$ (B) $|A|^{n-1}$
(C) $|A|^{n+1}$ (D) $|A^{-1}|$
where n is the order of the matrix

8. A and B are idempotent then AB is also idempotent if
(A) $A = B^T$
(B) A and B are non–singular
(C) AB is symmetric
(D) AB = BA

9. The rank of $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 \end{bmatrix}$ is

(A) 0 (B) 1

(C) 2 (D) 3

10. What are values of λ and μ so that the equations have no solution

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

(A) $\mu = 9, \lambda = 5$ (B) $\lambda = 5, \mu = 9$

(C) $\mu \neq 9, \lambda = 5$ (D) $\mu \neq 9, \lambda \neq 5$

11. The rank of matrix $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

12. The characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ is

(A) $\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$

(B) $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

(C) $2\lambda^3 + \lambda^2 - 6\lambda + 7 = 0$

(D) $2\lambda^3 - \lambda^2 + 6\lambda - 7 = 0$

13. Consider system of equations

$$2x + y + 2z = 1$$
$$x + y = 0$$
$$x - ky + 6z = 3$$

for the system to have unique solution, the value of k is

(A) $k = 2$ (B) $k \neq 2$

(C) $k = 1$ (D) $k = 0$

14. The adjoint of matrix

$A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ is

(A) $\begin{bmatrix} 7 & 8 & 6 \\ -11 & -14 & -13 \\ -5 & -5 & -5 \end{bmatrix}$

(B) $\begin{bmatrix} 7 & -4 & -5 \\ -5 & -14 & 6 \\ -5 & -11 & 8 \end{bmatrix}$

(C) $\begin{bmatrix} 7 & -11 & -5 \\ 8 & -14 & -5 \\ 6 & -13 & -5 \end{bmatrix}$

(D) $\begin{bmatrix} 7 & -11 & -13 \\ 5 & 9 & 11 \\ 2 & 14 & 20 \end{bmatrix}$

15. Find the value of x if

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$$

(A) $-1, -1, -2$ (B) $1, -1, 2$

(C) $1, 2$ (D) $1, 1$

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Test Paper – 4

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. The matrix A is said to be skew Hermitian if

(A) $A' = \bar{A}$ (B) $A' = -\bar{A}$

(C) $A' = A^{-1}$ (D) $A' = \text{Adj } A$

2. If λ is eigen value of a matrix and also $1/\lambda$ is eigen value of same matrix then that matrix is

(A) Skew – Hermitian matrix

(B) Hermitian matrix

(C) Orthogonal matrix

(D) None of these

3. If matrix A has rank 'r', then A′ (transpose of A) will have rank

(A) r + 1 (B) r – 1

(C) r (D) can't say

4. A square matrix A be such that $A^K = 0$ then it is called

(A) Hermitian matrix

(B) idempotent matrix for index K

(C) nilpotent for index K

(D) singular matrix for index K

5. For $x_1^2 - 18x_1x_2 + 5x_2^2$ quadratic form, matrix is

(A) $\begin{bmatrix} 1 & -9 \\ 9 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -9 \\ -9 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 18 \\ 18 & -5 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -18 \\ -18 & 5 \end{bmatrix}$

Q6 to Q15 carry two marks each

6. Find the rank of matrix

$$A = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

(A) 1 (B) 2

(C) 3 (D) 4

7. If A & B are 3×3 matrices then AB = 0 implies :

(A) Both A = 0 and B = 0

(B) | A | = 0 and | B | = 0

(C) Either | A | = 0 or | B | = 0

(D) Either A = 0 or B = 0

8. The characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ is}$$

(A) $6\lambda^3 + 9\lambda^2 - 4\lambda + 1 = 0$

(B) $\lambda^3 + 6\lambda^2 - 9\lambda - 4 = 0$

(C) $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

(D) $6\lambda^3 + 9\lambda^2 + 4\lambda - 1 = 0$

9. If I_n is the identity matrix of order n, then $(I_n)^{-1}$

(A) does not exist (B) $= I_n$

(C) $= 0$ (D) $= nI_n$

10. If A′ is the transpose of a square matrix A, then

(A) | A′ | ≠ | A |

(B) | A′ | = |A |

(C) | A | + | A ′ | = 0

(D) | A′ | = | A | only when A is symmetric

11. Inverse of matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is

(A) $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(B) $2\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(C) $\frac{1}{2}\begin{bmatrix} -1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(D) $2\begin{bmatrix} -1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

12. If a matrix is singular then the characteristic root of the matrix is

(A) 0

(B) diagonal matrix

(C) can't say

(D) not possible

13. Find the rank of matrix

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

(A) 1 (B) 2

(C) 3 (D) 4

14. If A = $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, then A is

(A) Singular (B) Non–singular

(C) I (D) $6\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

15. If A is 3 × 4 matrix and B is a matrix such that both A′B & BA′ are defined, then B is a . . .

(A) 4 × 3 matrix (B) 3 × 4 matrix

(C) 3 × 3 matrix (D) 4 × 4 matrix

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Test Paper – 5

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. If for a square matrix A, $A^t = A$, then the matrix A is

(A) Symmetric
(B) Orthogonal
(C) Hermitian
(D) Skew symmetric

2. The values of 'a', & 'b' from the matrix equation

$$\begin{bmatrix} a+3 & b^2+2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2a+1 & 3b \\ 0 & b^2-3b \end{bmatrix}$$

are

(A) a = 1, b = 1 (B) a = 2, b= 1
(C) a = 2, b = –2 (D) a = 1, b = 2

3. Find the index for which the matrix $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent

(A) 3
(B) 2
(C) 1
(D) It is not nil potent

4. λ is characteristic root of matrix A if and only if these exists a non–zero vector X such that

(A) $A^{-1} \times A = \lambda X$ (B) $AX = \lambda X$
(C) $XA = \lambda X$ (D) $A \times A^{-1} = XX$

5. If the eigen values of an $n \times n$ matrix are all distinct then it is always similar to a

(A) square matrix
(B) triangular matrix
(C) diagonal matrix
(D) none of these

Q6 to Q15 carry two marks each

6. If A & B are square matrices of order 3 such that | A | = –1, | B | = 3 then determinant of 3AB is equal to

(A) –9 (B) –27
(C) –81 (D) 81

7. Find the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

(A) 1 (B) 2
(C) 3 (D) 4

8. Eigen values of the matrix

$$A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ are}$$

(A) a, b, c (B) 0, a, b
(C) b, a, g (D) h, b, g

9. If following vectors are linearly dependent then what is relation between x_1, x_2, x_3, x_4 ?

$x_1 = (1, 2, 4)$; $x_2 = (2, -1, 3)$;

$x_3 = (0, 1, 2)$; $x_4 = (-3, 7, 2)$

(A) $9x_1 + 12x_2 + 15x_3 + x_4 = 0$

(B) $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

(C) $9x_1 + 12x_2 + 5x_3 + 5x_4 = 0$

(C) None of these

10. If A is a non–singular matrix & A′ is the transpose of A then

(A) $|A . A'| \neq |A^2|$

(B) $|A' . A| \neq |A'|^2$

(C) $|A| + |A'| = 0$

(D) $|A| = |A'|$

11. The inverse of the matrix

$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ if $a^2 + b^2 + c^2 + d^2 = 1$

is

(A) $\begin{bmatrix} a+ib & c+id \\ c-id & a-ib \end{bmatrix}$

(B) $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

(C) $\begin{bmatrix} a+ib & c+id \\ c+id & a+ib \end{bmatrix}$

(D) $\begin{bmatrix} a+ib & -c+id \\ c+id & a-ib \end{bmatrix}$

12. Is the given matrix $\begin{bmatrix} 3 & 4 & 2 \\ 4 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ idempotent ?

(A) No

(B) Yes

(C) Can't say

(D) Insufficient Data

13. The rank of the matrix

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 2 & 5 \end{bmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

14. The determinant of the matrix

$\begin{bmatrix} 6 & 0 & 0 & 0 \\ -8 & 2 & 0 & 0 \\ 1 & 4 & 4 & 0 \\ 1 & 6 & 8 & -1 \end{bmatrix}$ is

(A) 11 (B) 24

(C) –48 (D) 0

15. If $a \neq b \neq c$, one value of x which satisfies the equation

$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ is given by

(A) $x = a$ (B) $x = b$

(C) $x = c$ (D) $x = 0$

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Test Paper – 6

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. The value of the determinant of matrix

$$A = \begin{bmatrix} a & h & g & f \\ 0 & b & c & e \\ 0 & 0 & d & k \\ 0 & 0 & 0 & \ell \end{bmatrix} \text{ is}$$

(A) $-abd\ell$ (B) $abd\ell$

(C) $abck - abde$ (D) $abd\ell - abge$

2. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, then the product BA is

(A) $\begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$ (B) $\begin{bmatrix} -2 & 3 \\ -5 & 8 \\ -8 & 7 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (D) does not exist.

3. Let $A = \begin{bmatrix} 1+i & 1 & 2-3i \\ -2+i & 2i & 1+i \\ -3i & 0 & 3+4i \end{bmatrix}$, then the matrix $\bar{A}$ is given by

(A) $\begin{bmatrix} i+1 & 1 & 2-3i \\ -2-i & 2i & 1+i \\ -3i & 0 & 3-4i \end{bmatrix}$

(B) $\begin{bmatrix} 1-i & 1 & 2+3i \\ -2-i & -2i & 1-i \\ +3i & 0 & 3-4i \end{bmatrix}$

(C) $\begin{bmatrix} 1-i & 1 & 2+3i \\ -2+i & 2i & 1-i \\ 3i & 0 & 3+4i \end{bmatrix}$

(D) $\begin{bmatrix} 1-i & -1 & 2+3i \\ -2+i & -2i & 1-i \\ 3i & 0 & 3-4i \end{bmatrix}$

4. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ Then rank of matrix A is

(A) 1 (B) 2

(C) 3 (D) 0

5. Eigen value of matrix $\begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$ are

(A) 6, –4 (B) 5, 3

(C) –3, 5 (D) –4, –6

Q6 to Q15 carry two marks each

6. The value of determinant

$$\begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix} \text{ is}$$

(A) –1 (B) 1

(C) 0 (D) 4

7. The rank of matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \text{ is}$$

(A) 1 (B) 2

(C) 3 (D) 4

8. Adjoint of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ is

(A) $\begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 6 & 5 \end{bmatrix}$

(B) $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & -3 & 5 \\ -10 & 6 & 0 \\ 0 & 15 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 15 & -10 & 0 \\ -3 & 6 & 15 \\ 5 & 0 & 0 \end{bmatrix}$

9. The characteristic roots of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{ are}$$

(A) $1, 2 \pm \sqrt{3}$ (B) $3, 1 \pm \sqrt{2}$

(C) $2, -1 \pm \sqrt{3}$ (D) None of these

10. Determine eigen values of matrix

$$A = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ are}$$

(A) g, b, c (B) g, h, c

(C) a, b, g (D) b, a, c

11. If A + B + C = π, then the value of

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix} \text{ is}$$

equal to

(A) 0

(B) 1

(C) 2 sin B tan A cos C

(D) None of these

12. The rank of matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

is

(A) 1 (B) 2

(C) 3 (D) 4

13. If $a\mu^3 + b\mu^2 + c\mu + d$

$$= \begin{vmatrix} 3\mu & \mu+1 & \mu-1 \\ \mu-3 & -2\mu & \mu+2 \\ \mu+3 & \mu-4 & 5\mu \end{vmatrix} \text{ be an identity}$$

in μ, where a, b, c, d are constants, then the value of d is

(A) 5 (B) –6

(C) 9 (D) 0

14. If A = $\begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ & B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$ then

AB = $B'\,A'$

(A) Yes

(B) No

(C) Can't say

(D) Insufficient Data

15. If the system of equations

$x + ay + az = 0$, $bx + y + bz = 0$, $cx + cy + z = 0$ where a, b, c are non zero & non unity has a non trivial solution then the value of

$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}$ is

(A) 0

(B) 1

(C) –1

(D) $\frac{abc}{a^2 + b^2 + c^2}$

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Chapter - 2 : Calculus

2.1 Function of single variable

A real valued function y = f(x) of a real variable x is a mapping whose domain S and co-domain R are sets of real numbers. The range of the function is the set $\{y = f(x) : x \in R\}$, which is a subset of R.

2.2 Limit of a function

The function f is said to tend to the limit ℓ as $x \to a$, if for a given positive real number $\varepsilon > 0$ we can find a real number $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Symbolically we write $\lim_{x \to a} f(x) = \ell$

Left Hand and Right Hand Limits

Let x < a and $x \to a$ from the left hand side.

If $|f(x) - \ell_1| < \varepsilon$, $a - \delta < x < a$ or $\lim_{x \to a^-} f(x) = \ell_1$

then ℓ_1 is called the left hand limit.

Let x > a and $x \to a$ from the right hand side.

If $|f(x) - \ell_2| < \varepsilon$, $a < x < a + \delta$ or $\lim_{x \to a^+} f(x) = \ell_2$

then ℓ_2 is called the right hand limit.

If $\ell_1 = \ell_2$ then $\lim_{x \to a} f(x)$ exists. If the limit exists then it is unique.

<u>Properties of Limits</u>

Let f and g be two functions defined over S and let a be any point, not necessarily in S

And if $\lim_{x \to a} f(x) = \ell_1$ and $\lim_{x \to a} g(x) = \ell_2$ exist, then

- $\lim_{x \to a} [Cf(x)] = C \lim_{x \to a} f(x) = C\ell_1$ C a real constant.
- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell_1 \pm \ell_2$
- $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = \ell_1 \ell_2$

- $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \left[\lim\limits_{x\to a} f(x) \Big/ \lim\limits_{x\to a} g(x)\right] = \dfrac{\ell_1}{\ell_2}$ $\quad\quad \ell_2 \neq 0$
- $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)} = \left(\ell_1\right)^{\ell_2}$

Standard Formulae

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ ...x in radians	$\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \ell n\, a \ (a > 0)$
$\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1$ ...x in radians	$\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = 1$
$\lim\limits_{x\to 0}\cos x = 1$	$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
$\lim\limits_{x\to a}\dfrac{x^n - a^n}{x - a} = na^{n-1}$	$\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x} = 1$

Solved Example 1 :

Show that $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist.

Solution :

For different values of x in the interval

$0 < | x | < \delta$ the function $\sin\left(\dfrac{1}{x}\right)$ takes

values between –1 and 1.

Since $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ is not unique limit does

not exist.

Solved Example 2 :

Show that $\lim\limits_{x\to 4}\left[x^2 + 1\right]$ does not exist,

where [ ] is the greatest integer function.

Solution :

Let h > 0, we have

$\lim\limits_{h\to 0} f(x+h) = \left[(4+h)^2 + 1\right]$

$= \left[17 + h(h+8)\right]$

$= 17 \quad\quad$ if $h(h+8) < 1$

$\therefore \quad (h+4)^2 < 17 \quad \Rightarrow \quad h < \sqrt{17} - 4$

and $\lim\limits_{h\to 0} f(x-h) = \left[(4-h)^2 + 1\right]$

$= \left[17 + h(h-8)\right]$

$= 16 \quad\quad$ if $h(h-8) > -1$

or $\quad (h-4)^2 > 15 \quad$ or $\quad h > 4 - \sqrt{15}$

$\therefore \quad \lim\limits_{x\to 4^+} f(x) = 17$ and $\lim\limits_{x\to 4^-} f(x) = 16$

The limit does not exist.

2.3 Continuity

Let f be a real valued function of the real variable x. Let x_0 be a point in the domain of f and let f be defined in some neighbourhood of the point x_0. The function f is said to be continuous at $x = x_0$ if

i) $\lim_{x \to x_0} f(x) = \ell$ exists and

ii) $\lim_{x \to x_0} f(x) = f(x_0)$

Types of discontinuity

A point at which f is not continuous is called a point of discontinuity.

If $\lim_{x \to x_0} f(x) = \ell$ exists, but $\lim_{x \to x_0} f(x) \neq f(x_0)$

then x_0 is called the point of removable discontinuity.

In this case we can redefine f(x), such that $f(x_0) = \ell$, so that the new function is continuous at $x = x_0$.

For example :

The function, $f(x) = \begin{cases} (\sin x)/x & ; \quad x \neq 0 \\ 4 & ; \quad x = 0 \end{cases}$

has a removable discontinuity at x = 0, since $\lim_{x \to 0} f(x) = \ell$ and a new function can be defined as

$$f(x) = \begin{cases} (\sin x)/x & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$$

If $\lim_{x \to x_0} f(x)$ does not exist, then x_0 is called the point of irremovable discontinuity.

For Example :

$f(x) = \frac{1}{x}$ at x = 0

Here $\lim_{x \to 0} f(x)$ does not exist. Also f(0) is not defined.

$\therefore$ $f(x) = \frac{1}{x}$ has an irremovable discontinuity at x = 0.

Note : In case of an irremovable discontinuity, it does not matter whether or how the function is defined of the point of discontinuity.

Properties of continuous functions

- Let the functions f and g be continuous at a point $x = x_0$ Then,
  i) cf, $f \pm g$ and $f \cdot g$ are continuous at $x = x_0$, where c is any constant.
  ii) f/g is continuous at $x = x_0$, if $g(x_0) \neq 0$
- If f is continuous at $x = x_0$ and g is continuous at $f(x_0)$, then the composite function g(f(g)) is continuous at $x = x_0$.
- A function f is continuous in a closed interval [a, b] if it is continuous at every point in (a, b)

  $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$
- If f is continuous at an interior point c of a closed interval [a, b] and $f(c) \neq 0$, then there exists a neighbourhood of c, throughout which f(x) has the same sign as f(c).
- If f is continuous in a closed interval [a, b] then it is bounded there and attains its bounds at least once in [a, b].
- If f is continuous in a closed interval [a, b], and if f(a) f(b) are of opposite signs, then there exists at least one point $c \in [a, b]$ such that f(c) = 0.
- If f is continuous in a closed interval [a, b] and $f(a) \neq f(b)$ then it assumes every value between f(a) and f(b).

Solved Example 3 :

Determine the point of discontinuity of the function $f(x) = \dfrac{x}{|x|}$

Solution :

For $x < 0$, $\dfrac{x}{|x|} = -1$

For $x > 0$, $\dfrac{x}{|x|} = +1$

$\therefore$ f(–0) = –1

and f(+0) = +1

Also the function is not defined at x = 0

$\therefore$ The function $f(x) = \dfrac{x}{|x|}$ is discontinuous at x = 0.

Note : |x| = x when x > 0 and |x| = –x when x < 0.

Solved Example 4 :

Find the limit if it exists, as x approaches zero for the function f(x) given by

if $x < 0$, $f(x) = x$

$x = 0$, $f(x) = 1$

$x > 0$, $f(x) = x^2$

Solution :

If $x < 0$, $f(-0) = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$

If x > 0, f(+0) = $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0$

Thus $f(-0) = f(+0)$ each being zero.

Since the limit on the left = the limit on the right at x = 0 the function f(x) has limit 0 at x = 0.

Note : The limit at x = 0 is not equal to f(0) = 1, and hence the function is discontinuous at x = 0.

Solved Example 5 :

Let f(x) = $\begin{cases} x^3 + x^2 - 16x + 20 \Big/ (x-2)^2 \\ k \qquad \text{if } x = 2 \end{cases}$

if $x \neq 2$

Find k if f(x) is continuous at x = 2.

Solution :

For $x \neq 2$ we have

$$f(x) = \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$$

$$= \frac{(x-2)^2 (x+5)}{(x-2)^2} = x + 5$$

$\therefore \quad \lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}(x+5) = 7$

We have f(2) = k

$\therefore$ f(x) is continuous at x = 2.

Now $\lim\limits_{x\to 2} f(x) = f(2)$

i.e. 7 = k

Solved Example 6 :

For the function f(x) = $x + \dfrac{x+2}{|x+2|}$ find the point of discontinuity and determine the jump of the function at this point.

Solution :

$\lim\limits_{x\to -2^-} f(x) = -2 + (-1) = -3 \qquad$ for x < –2,

$\dfrac{x+2}{|x+2|} = -1$

$\lim\limits_{x\to -2^+} f(x) = -2 + (+1) = -1 \qquad$ for x > –2,

$\dfrac{x+2}{|x+2|} = 1$

$\therefore$ The one sided limits do not coincide. Thus the function is not defined at x = –2.

$\therefore$ x = –2 is the only point of discontinuity.

The jump at this point = –1 – (–3) = 2

Solved Example 7 :

Show that f(x) = $\begin{cases} x+2 & \text{for} & x < 2 \\ x^2 - 1 & \text{for} & x > 2 \end{cases}$ is discontinuous at x = 2 and determine the jump of the function at this point.

Solution :

$\lim\limits_{x\to 2^-} (x+2) = 4$

$\lim\limits_{x\to 2^+} \left(x^2 - 1\right) = 3$

$\therefore$ The limit on the left $\neq$ the limit on the right.

$\therefore$ The function has discontinuity at the point x = 2

$\therefore$ The jump of the function at x = 2 is

f(2 + 0) – f(2 – 0) = 3 – 4 = –1

Note : If $f(x_0 - 0) \neq f(x_0 + 0)$, then $f(x_0 + 0) - f(x_0 - 0)$ is called a jump discontinuity of the function f(x) at x_0.

2.4 Differentiability

Let a real valued function f(x) be defined on an I and let x_0 be a point in I. Then, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists and is equal to ℓ, then f(x) is said to be differentiable at x_0 and ℓ is called the derivative of f(x) at $x = x_0$.

If f(x) is differentiable at every point in the interval (a, b) then f(x) is said to be differentiable in (a, b).

<u>Properties of differentiation</u>

- Let the functions f and g be differentiable at a point x_0. Then
  - i) $(cf')\,x_0 = cf'(x_0)$, c any constant.
  - ii) $(f \pm g)'x_0 = f'(x_0) \pm g'(x_0)$
  - iii) $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$
  - iv) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}, \qquad g(x_0) \neq 0$
- If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g(f(x))$ is differentiable at x_0 and
  $h'(x_0) = g'(f(x_0))\, f'(x_0)$
- If the function $y = f(x)$ is represented in the parametric form as
  $x = \phi(t)$ and $y = \psi(t)$, and if $\phi'(t)$, $\psi'(t)$ exist, then
  $f'(x) = \dfrac{dy/dt}{dx/dt} = \dfrac{\psi'(t)}{\phi'(t)} \qquad \phi'(t) \neq 0$

<u>Note :</u> If a function is differentiable at $x = x_0$, then it is continuous at $x = x_0$. However the converse need not be true.

Standard formulae

Function	Derivative	Function	Derivative
k(constant)	0	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
x^n	nx^{n-1}	$\cos^{-1}\frac{x}{a}$	$\frac{-1}{\sqrt{a^2-x^2}}$
logx	1/x	$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
e^x	e^x	$\cot^{-1}\frac{x}{a}$	$\frac{-a}{a^2+x^2}$
a^x	$a^x \log a$	$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
sin x	cos x	$\operatorname{cosec}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{x^2-a^2}}$
cos x	– sin x	sin h x	cos h x
tan x	$\sec^2 x$	cos h x	sin h x
cot x	$-\operatorname{cosec}^2 x$	$\sin h^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
sec x	sec x tan x	$\cos h^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
cosec x	– cosec x cot x	$\tan h^{-1} x$	$\frac{1}{x^2-1}$

Solved Example 8 :

Show that the function

$$f(x) = \begin{cases} x^2\cos(1/x), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at x = 0 but $f'(x)$ is not continuous at x = 0.

Solution :

We have $\lim\limits_{x\to 0} f(x) = 0 = f(0)$

$\therefore$ f(x) is continuous at x = 0

Now $f'(0) = \lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}$

$$= \lim_{x\to 0}\left[x\cos\left(\frac{1}{x}\right)\right] = 0$$

Hence f(x) is differentiable at x = 0 and $f'(0) = 0$

for $x \neq 0$ we have

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left[-\sin\left(\frac{1}{x}\right)\right]\left[-\frac{1}{x^2}\right]$$

$$= 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

Now $\lim\limits_{x\to 0} f'(x)$ does not exist as $\lim\limits_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. Therefore, $f'(x)$ is not continuous at $x = 0$.

Solved Example 9 :

$y = \sqrt[3]{x}$ is defined and continuous for all x investigate whether this function has a derivative at x = 0.

Solution :

$\Delta y = \sqrt[3]{x + \Delta x} - \sqrt[3]{x}$ at $x = 0$

$\Delta y = \sqrt[3]{\Delta x}$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\sqrt[3]{\Delta x}}{\Delta x} = \frac{1}{\sqrt[3]{(\Delta x)^2}}$$

$$\therefore \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{(\Delta x)^2}} = +\infty$$

$\therefore$ There is no finite derivative.

Solved Example 10 :

Investigate the function f(x) = | x | for differentiability at the point x = 0

Solution :

$y = f(x) = |x|$

$\Delta y = |x + \Delta x| - |\Delta x|$

At x = 0, we have $\Delta y = |\Delta x|$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{|\Delta x|}{\Delta x}$$

$$\lim_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x} = -1$$

$$\lim_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x} = 1$$

Since the right and the left side derivative are not equal, the function f(x) = |x| is not differentiable at the point zero.

Solved Example 11 :

Show that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

Solution :

Let $y = x^n$

Let x receive a small increment Δx and let the corresponding increment in y be Δy then

$$y + \Delta y = (x + \Delta x)^n$$

By subtraction, $\Delta y = (x + \Delta x)^n - x^n$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{x + \Delta x - x}$$

$$\therefore \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{x+\Delta x\to x}\frac{(x + \Delta x)^n - x^n}{x + \Delta x - x} = nx^{n-1}$$

$$\therefore \quad \frac{dy}{dx} = nx^{n-1} \text{ or } \frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

2.5 Mean Value Theorems

Rolle's Theorem

The theorem states that

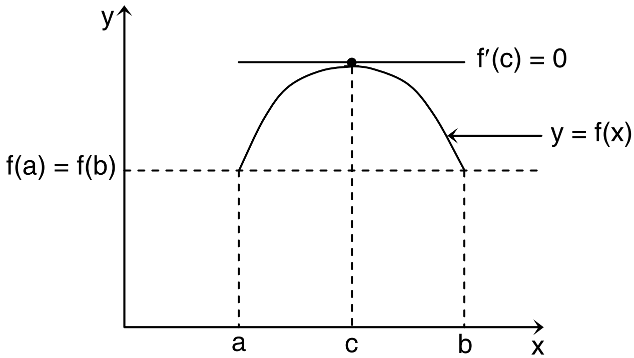
– if f(x) is continuous in the closed interval [a, b] and

– if $f'(x)$ exists in open interval (a, b) and

– if f(a) = f(b)

Then there exists at least one value c in (a, b) such that $f'(c) = 0$

Geometric Interpretation



There exists at last one point at which slope of the tangent is 0 or the tangent is parallel to x-axis

Lagrange's Mean Value Theorem

The theorem states that

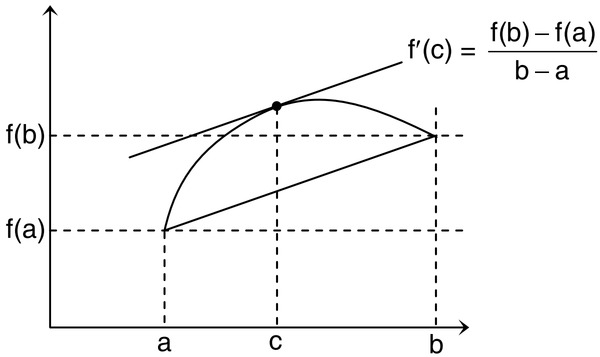
– if f(x) is continuous in the closed interval [a, b] and

– if $f'(x)$ exists in open interval (a, b)

Then there exists one value c such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Geometric Interpretation



There exists at last one point at which the tangent is parallel to the secant through the end points.

Cauchy's Mean Value Theorem

The theorem states that

– if f(x) and g(x) are both continuous in closed interval [a, b] and

– if f′(x) and g(x) both exist in open interval (a, b)

Then there exists at least one value c such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

2.6 Maxima and Minima

A function f(x) is said to have a maximum value at x = a if f(a) is larger than any other values of f(x) in the immediate neighbourhood of 'a'. It has a minimum value if f(a) is less than any other value of f(x) sufficiently near 'a'.

Finding Maximum and Minimum values of y = f(x)

- Get $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Solve $\dfrac{dy}{dx}$ = 0 and consider its roots. These are the values of x which make $\dfrac{dy}{dx} = 0$.
- For each of these values of x, calculate the corresponding value of y and examine the sign of $\dfrac{d^2y}{dx^2}$.
- If the sign is –ve the corresponding value of y is a maximum.
  If the sign is +ve, the corresponding value of y is minimum.

> **Note :** **Maximum and minimum values of a function occur alternately in a continuous function. (The graph of a continuous function has no break gaps)**
> **A maximum is not necessarily the greater value and a minimum is not necessarily the least value of the function.**

Stationary Points and Turning Points

A stationary point on a graph is any point at which $\frac{dy}{dx} = 0$. A value of x for which $\frac{dy}{dx} = 0$ does not necessarily give either a maximum or minimum. A turning point is a maximum or minimum point.

Solved Example 12 :

Find the maximum and minimum points on the curve $y = 2x^3 + 5x^2 - 4x + 7$

Solution :

Differentiating the equation we have

$$\frac{dy}{dx} = 6x^2 + 10x - 4$$

$$\frac{d^2y}{dx^2} = 12x + 10$$

Now

$$\frac{dy}{dx} = 0 \quad \text{if} \quad 6x^2 + 10x - 4 = 0$$

or $\quad 3x^2 + 5x - 2 = 0$

i.e. $(3x - 1)(x + 2) = 0$

i.e. $\frac{dy}{dx} = 0, \text{ when } x = \frac{1}{3} \text{ or } -2$

for $x = \frac{1}{3}, \frac{d^2y}{dx^2} = 14$ and $\therefore$ is positive

for $x = -2, \frac{d^2y}{dx^2} = -14$ and $\therefore$ is negative

Hence $x = \frac{1}{3}$ gives a minimum value of y

and $\quad x = -2$ gives a maximum value of y

for $x = \frac{1}{3}, \; y = 6\frac{8}{27}$; for $x = -2, \; y = 19$

$\therefore \left(\frac{1}{3}, 6\frac{8}{27}\right)$ is a minimum point

(–2, 19) is a maximum point.

Solved Example 13 :

Find the maximum and minimum values of the function $\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$

Solution :

Let $\quad y = \frac{x^2 - 2x + 4}{x^2 + 2x + 4}$

Then $\quad \frac{dy}{dx} = \frac{\left\{\begin{matrix}(x^2 + 2x + 4)(2x - 2) \\ -(x^2 - 2x + 4)(2x + 2)\end{matrix}\right\}}{(x^2 + 2x + 4)^2}$

$\therefore \quad \frac{dy}{dx} = 0$ when $x^2 = 4$

or $\quad x = \pm 2$

In this case it is much more convenient to find the changes of sign of $\frac{dy}{dx}$ than to work out the value of $\frac{d^2y}{dx^2}$.

The denominator must be positive and therefore only the sign of the numerator need be examined.

Near x = 2

If x is slightly less than 2, $x^2 < 4$;

$\frac{dy}{dx}$ is –ve

If x is slightly greater than 2, $x^2 > 4$;

$\frac{dy}{dx}$ is +ve

Here $\frac{dy}{dx}$ changes from –ve to +ve

Near x = –2

If x is slightly less than –2, $x^2 > 4$;

$\frac{dy}{dx}$ is +ve

If x is slightly greater than –2, $x^2 < 4$;

$\frac{dy}{dx}$ is –ve

Here $\frac{dy}{dx}$ changes from +ve to –ve

$\therefore$ x = 2 gives a minimum value of y, $y = \frac{1}{3}$;

and x = –2 gives a maximum value of y, y = 3

This the maximum value is 3 and the minimum is $\frac{1}{3}$.

Note : If the $\frac{dy}{dx}$ sign changes from + to – the point is a maximum point; if the sign changes from – to +, the point is a minimum point.

Solved Example 14 :

Find the maximum and minimum values of the function 4 cos x – 3 sin x.

Solution:

Let $y = 4\cos x - 3\sin x$

Then $\frac{dy}{dx} = -4\sin x - 3\cos x$

$\frac{d^2y}{dx^2} = -4\cos x + 3\sin x$

$\frac{dy}{dx} = 0$ when $-4\sin x - 3\cos x = 0$

or $\tan x = \frac{-3}{4}$

and then (i) $\sin x = \frac{3}{5}$, $\cos x = \frac{-4}{5}$

(ii) $\sin x = \frac{-3}{5}$, $\cos x = \frac{4}{5}$

$\because$ $\tan\theta$ is negative.

In the first case, $\frac{d^2y}{dx^2} = \frac{16}{5} + \frac{9}{5} = 5$

and $\therefore$ is +ve

In the second case, $\frac{d^2y}{dx^2} = \frac{-16}{5} - \frac{9}{5} = -5$

and $\therefore$ is –ve

$\therefore$ The minimum value

$$= 4\left(\frac{-4}{5}\right) - 3\left(\frac{3}{5}\right) = -5$$

and the maximum value

$$= 4\left(\frac{4}{5}\right) - 3\left(\frac{-3}{5}\right) = 5$$

Solved Example 15 :

Find the maximum and minimum values of $f(x, y) = 7x^2 + 8xy + y^2$ where x, y are connected by the relation $x^2 + y^2 = 1$.

Solution :

Since $x^2 + y^2 = 1$,

we can put $x = \cos\theta$, $y = \sin\theta$

$f(x, y) = 7\cos^2\theta + 8\cos\theta\sin\theta + \sin^2\theta = F(\theta)$

$$F(\theta) = 7\frac{1+\cos 2\theta}{2} + 4\sin 2\theta + \frac{1-\cos 2\theta}{2}$$

$$= 4 + 3\cos 2\theta + 4\sin 2\theta$$

$$= 4 + 5\sin \times \cos 2\theta + 5\cos \times \sin 2\theta$$

$$= 4 + 5\sin(2\theta + \alpha)$$

Since sine functions takes its maximum and minimum value as +1 and –1 respectively.

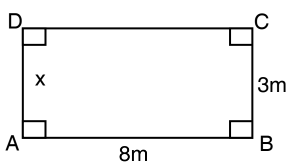
$F_{max} = 4 + 5$; $F_{min} = 4 - 5$

$\therefore F_{max} = 9$ $F_{min} = -1$

Solved Example 16 :

A rectangular sheet of a metal is 8 meters by 3 meters; equal squares are cut out at each of the corners and the flaps are then folded upto form an open rectangular box. Find its maximum volume.

Solution:



Let ABCD by rectangular sheet and let a square of edge x m be cut from each corner.

When the flaps are folded up, the dimensions of the rectangular box obtained are 8 – 2x, 3 – 2x and x meters.

Let V cubic meters be the volume of the box then

$$V = (8-2x)(3-2x)x$$

$$= 4x^3 - 22x^2 + 24x;$$

$$\frac{dV}{dx} = 12x^2 - 44x + 24$$

and $\frac{d^2V}{dx^2} = 24x - 44$

$$\frac{dV}{dx} = 0 \text{ if } 4\left(3x^2 - 11x + 6\right) = 0$$

(3x – 2) (x – 2) = 0

i.e. when x = 2/3 or 3

for $x = \frac{2}{3}, \frac{d^2v}{dx^2} = -28$ and

$\therefore$ is negative

Hence $x = \frac{2}{3}$ gives a maximum value of V

For $x = \frac{2}{3}$, $V = 7\frac{11}{27}$

Thus the maximum volume is $7\frac{11}{27}$ cm.

x = 3 is inadmissible, since the breadth itself of the sheet is 3m.

2.7 Integration

Integrand and element of integration

The function under the sign of integration is called integrand. For e.g. in $\int x^3\,dx$; x^3 is called integrand. In the integral $\int f(x)\,dx$, dx is known as the element of integration and it indicates the variable with respect to which the given function is to be integrated.

Constant of integration :

We know that

$$\frac{d}{dx}(x^2) = 2x \qquad \int 2x\,dx = x^2$$

Also $\quad \frac{d}{dx}(x^2+c) = 2x \quad \therefore \quad \int 2x\,dx = x^2 + c$, where c is any constant

So we notice that x^2 is an integral of 2x, then x^2 + c is also an integral of 2x. In general if

$$\int f(x)\,dx = \phi(x) \text{ then}$$

$$\int f(x)\,dx = \phi(x) + c$$

Standard formulae

$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ (n \neq -1)$	$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$
$\int \frac{1}{x} dx = \log x + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = -\cos^{-1}\frac{x}{a} + c$
$\int e^x dx = e^x + c$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
$\int a^x dx = \frac{a^x}{\log_e a} + c$	$\int \frac{dx}{a^2+x^2} = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$
$\int \cos x\,dx = \sin x + c$	$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$
$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$	$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{-1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$
$\int \sin x\,dx = -\cos x + c$	$\int \sinh x\,dx = \cosh x + c$

$\int \sec x \tan x\, dx = \sec x + c$	$\int \cosh x\, dx = \sin h\ x + c$
$\int \sec^2 x\ dx = \tan x + c$	$\int \frac{dx}{\sqrt{x^2+1}} = \sin h^{-1}\ x + c$
$\int \cos ec^2 x\, dx\ = -\cot x + c$	$\int \frac{dx}{\sqrt{x^2-1}} = \cos h^{-1}\ x + c$
$\int \cot x dx\ = \log \sin x + c$	$\int \frac{dx}{x^2-1}\ = \tan h^{-1}\ x + c$
$\int \tan x dx\ = \log \sec x + c$	$\int \tan x dx\ = -\log(\cos x) + c$
$\int \sec x dx = \log(\sec x + \tan x) + c$	$\int \sec x dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$
$\int \mathrm{cosec}\, x\, dx = -\log(\mathrm{cosec} x + \cot x) + c$	$\int \mathrm{cosec}\, x\, dx = \log\left(\tan\frac{x}{2}\right) + c$

Important Trigonometric Identities

- $\sin^2 A + \cos^2 A = 1$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
- $\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$
- $\cos^2 A - \sin^2 B = \cos(A + B) \cos(A - B)$
- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
- $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

- $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$
- $2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2} = \sin C + \sin D$
- $2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2} = \sin C - \sin D$
- $2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2} = \cos C + \cos D$
- $2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2} = \cos C - \cos D$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\sin 2A = 2 \sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - 4\tan^3 A}{1-3\tan^2 A}$

Note : Integration of $\int \sin^m x \cos^n x\,dx$ where m and n are positive integers

(i) If m be odd and n be even, for integration put t = cos x

(ii) If m be even and n be odd, for integration put t = sin x

(iii) If m and n are odd, for integration put either t = cos x or sin x

(iv) If m and n are even, for integration put either t = cos x or sin x

Solved Example 17 :

Evaluate $\int \sin^3 x\,dx$

Solution :

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$4 \sin^3 x = 3 \sin x - \sin 3x$

$$\therefore\ 4\int \sin^3 x\,dx = \int (3\sin x - \sin 3x)\,dx$$

$$= 3\int \sin x\,dx - \int \sin 3x\,dx$$

$$\int \sin^3 x\,dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

Solved Example 18 :

Evaluate $\int \sin 3x \cos 2x\, dx$

Solution :

$$\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int (\sin 5x + \sin x)\, dx$$

$$= \frac{-1}{10}\cos 5x - \frac{1}{2}\cos x$$

Solved Example 19 :

Evaluate $\int \sin 2x \sin 3x\, dx$

Solution :

$$\int \sin 2x \sin 3x\, dx = \frac{1}{2}\int \cos x - \cos 5x\, dx$$

$$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x$$

Solved Example 20 :

Integrate $\int \frac{dx}{\cos^2 x \sin^2 x}$

Solution :

Now, $\frac{1}{\cos^2 x \sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x}$

$$= \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

$$= \operatorname{cosec}^2 x + \sec^2 x$$

$[\because 1 = \sin^2 x + \cos^2 x]$

$\therefore \int \frac{dx}{\cos^2 x \sin^2 x}$

$$= \int (\operatorname{cosec}^2 x + \sec^2 x)\, dx$$

$$= \int \operatorname{cosec}^2 x\, dx + \int \sec^2 x\, dx$$

$$= -\cot x + \tan x + c$$

Solved Example 21 :

Evaluate $\int \sin^3 x \cos^3 x\, dx$

Solution :

We have

$\sin^3 x \cos^3 x = (\sin x \cos x)^3$

$$= \frac{1}{8}(2 \sin x \cos x)^3$$

$$= \frac{1}{8}\sin^3 2x$$

$$= \frac{1}{8} \times \frac{1}{4} \quad (3 \sin 2x - \sin 6x)$$

$[\because 4 \sin^3 x = 3 \sin x - \sin 3x]$

$$= \frac{1}{32} \quad (3 \sin 2x - \sin 6x)$$

$\therefore \int \sin^3 x \cos^3 x\, dx$

$$= \frac{1}{32}\int (3 \sin 2x - \sin 6x)\, dx$$

$$= \frac{3}{32}\int \sin 2x\, dx - \frac{1}{32}\int \sin 6x\, dx$$

$$= \frac{3}{32}\left(\frac{-\cos 2x}{2}\right) - \frac{1}{32}\left(\frac{-\cos 6x}{6}\right) + c$$

Hence $\int \sin^3 x \cos^3 x\, dx$

$$= \frac{-3}{64}\cos 2x + \frac{1}{192}\cos 6x + c$$

Solved Example 22 :

Evaluate $\int \frac{1}{1+\sin x}\, dx$

Solution :

$$\frac{1}{1+\sin x} = \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

$$= \frac{1-\sin x}{1-\sin^2 x}$$

$$= \frac{1-\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}$$

or $\frac{1}{1+\sin x}$ = $\sec^2 x - \tan x \sec x$

$\therefore \int \frac{1}{1+\sin x} dx$

$$= \int (\sec^2 x - \tan x \sec x) dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + c$$

Solved Example 23 :

Evaluate $\int \sin^7 x \cos^6 x\, dx$

Solution :

$$\int \sin^7 x \cos^6 x\, dx$$

$$= \int \sin^6 x \cos^6 x \sin x dx$$

$$= \int (1-\cos^2 x)^3 \cos^6 x \sin x\, dx$$

$$= \int (1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x)$$

$$\cos^6 x \sin x dx$$

$$= \int (\cos^6 x - 3\cos^8 x + 3\cos^{10} x - \cos^{12} x)$$

$$\sin x dx$$

$$= -\int (t^6 - 3t^8 + 3t^{10} - t^{12}) dt$$

by putting t = cos x

$$= -\frac{t^7}{7} + 3\frac{t^9}{9} - 3\frac{t^{11}}{11} + \frac{t^{13}}{13}$$

$$= -\frac{1}{7}\cos^7 x + \frac{1}{3}\cos^9 x - \frac{3}{11}\cos^{11} x$$

$$+ \frac{1}{13}\cos^{13} x$$

Solved Example 24 :

Evaluate $\int \sin^6 x \cos^3 x\, dx$

Solution :

$$\int \sin^6 x \cos^3 x\, dx = \int \sin^6 x (1 - \sin^2 x) \cos x dx$$

$$= \int (\sin^6 x - \sin^8 x) \cos x\, dx$$

$$= \int (t^6 - t^8) dt$$

$\because$ by putting

$$\sin x = t = \frac{1}{7}t^7 - \frac{1}{9}t^9 = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x$$

Solved Example 25 :

Evaluate $\int \sin^4 x \cos^2 x\, dx$

Solution :

$\sin^4 x \cos^2 x$

$$= \frac{1}{8}(2\sin^2 x)^2 (1+\cos 2x)$$

$$= \frac{1}{8}(1 - \cos 2x - \cos^2 2x + \cos^3 2x)$$

$$= \frac{1}{8}\left(1 - \cos 2x - \frac{1+\cos 4x}{2} + \frac{\cos 6x + 3\cos 2x}{4}\right)$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{4}\cos 2x - \frac{1}{2}\cos 4x + \frac{1}{4}\cos 6x\right)$$

$$= \frac{1}{16}\left(1 - \frac{1}{2}\cos 2x - \cos 4x + \frac{1}{2}\cos 6x\right)$$

$$\int \sin^4 x \cos^2 x dx$$

$$= \frac{1}{16}\left(1 - \frac{1}{2}\cos 2x - \cos 4x + \frac{1}{2}\cos 6x\right) dx$$

$$= \frac{1}{16}\left(x - \frac{1}{4}\sin 2x - \frac{1}{4}\sin 4x + \frac{1}{12}\sin 6x\right)$$

Solved Example 26 :

Evaluate $\int \sin^3 x \cos^5 x \, dx$

Solution :

$$\int \sin^3 x \cos^5 x \, dx$$

$$= \int \sin^3 x (\cos^4 x) \cos x \, dx$$

$$= \int \sin^3 x (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int t^3 (1 - t^2) \, dt$$

by putting $\sin x = t$

$$= \int (t^3 - 2t^5 + t^7) \, dt$$

$$= \frac{t^4}{4} - \frac{t^6}{3} + \frac{t^8}{8}$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8}$$

Integration by Substitution

Some of the general substitutions are

Type	**Substitution**
$a^2 - x^2$	$x = a \sin \theta$ or $a \cos \theta$
$a^2 + x^2$	$x = a \tan \theta$ or $a \cot \theta$ or $a \sin h\, \theta$
$x^2 - a^2$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$ or $a \cos h\, \theta$

By means of above substitution some standard integrals are

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \text{ or } -\cos^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a} \right) \text{ or } \log(x + \sqrt{x^2 + a^2}) + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a} \right) \text{ or } \log(x + \sqrt{x^2 - a^2}) + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \text{ or } \frac{-1}{a} \cot^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \sqrt{a^2 + x^2} \, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\frac{a+x}{a-x} + c$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a} + c$$

Solved Example 27 :

Integrate $\int \frac{\cos x - \sin x}{1+\sin 2x}dx$

Solution :

We know that

$1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$= (\sin x + \cos x)^2$

Let $I = \int \frac{\cos x - \sin x}{1+\sin 2x}dx$

$= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}dx$

Let $\cos x + \sin x = u$

$\therefore (\cos x - \sin x)\, dx = du$

Then $I = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{\cos x + \sin x} + c$

$\therefore \quad \int \frac{\cos x - \sin x}{1+\sin 2x}dx = \frac{-1}{\cos x + \sin x} + c$

Solved Example 28 :

Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Solution :

Put $\sqrt{x} = \sin\theta$ or $x = \sin^2\theta$

$\therefore \quad dx = 2 \sin\theta \cos\theta\, d\theta$

$I = \int \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\, .\, 2 \sin\theta \cos\theta\, d\theta$

$\int \frac{1-\sin\theta}{\cos\theta}.2\sin\theta\cos\theta\, d\theta$

$= \int (2\sin\theta + \cos 2\theta - 1)d\theta$

$= -2\cos\theta + \frac{\sin 2\theta}{2} - \theta + c$

$= -2\cos\theta + \sin\theta\cos\theta - \theta + c$

$= -2\sqrt{1-\sin^2\theta} + \sin\theta\sqrt{1-\sin^2\theta} - \theta + c$

$= -2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$

$= (\sqrt{x} - 2)\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$

Solved Example 29 :

Find the value of the integral

$\int \frac{\sin x \cos x\, dx}{a^2\sin^2 x + b^2\cos^2 x}$

Solution :

Put $a^2\sin^2 x + b^2\cos^2 x = t$

$2(a^2 \sin x \cos x)\, dx + 2b^2 \cos x\, (-\sin x)\, dx = dt$

$\sin x \cos x\, dx = \frac{dt}{2(a^2 - b^2)}$

$\therefore \quad \int \frac{\sin x \cos x dx}{a^2\sin^2 x + b^2\cos^2 x}$

$$= \int \frac{dt}{2(a^2 - b^2)t}$$

$$= \frac{1}{2(a^2 - b^2)} \int \frac{dt}{t} = \frac{1}{2(a^2 - b^2)} \log t + c$$

$$= \frac{1}{2(a^2 - b^2)} \log\,(a^2 \sin^2 x + b^2 \cos^2 x) + c$$

Solved Example 30 :

Evaluate $\int \frac{d\theta}{\sin^4\theta + \cos^4\theta}$

Solution :

$$I = \int \frac{d\theta}{\sin^4\theta + \cos^4\theta} = \int \frac{\sec^4\theta}{\tan^4\theta + 1} d\theta$$

Dividing numerator and denominator by $\cos^4\theta$

$$I = \int \frac{\sec^2\theta \sec^2\theta}{1 + \tan^4\theta} d\theta$$

$$= \int \frac{(1 + \tan^2\theta)\sec^2\theta}{1 + \tan^4\theta} d\theta$$

Put $\tan\theta = x$ so that $\sec^2\theta \, d\theta = dx$

$$\therefore \quad I = \int \frac{(1 + x^2)dx}{1 + x^4} = \int \frac{\frac{1}{x^2} + 1}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx$$

Put $x - \frac{1}{x} = t$, $\quad \therefore \left(1 + \frac{1}{x^2}\right) dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + c$$

$$= \frac{1}{\sqrt{2}} \tan\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2\theta - 1}{\sqrt{2}\tan\theta}\right) + c$$

Solved Example 31 :

Integrate $\int \frac{\cos x}{1 + \sin^2 x} dx$

Solution :

Let $\sin x = z$ $\quad \therefore \cos x \, dx = dz$

$$\text{Now} \int \frac{\cos x}{1 + \sin^2 x} dx = \int \frac{dz}{1 + z^2}$$

$$= \tan^{-1} z + c$$

$$= \tan^{-1} (\sin x) + c$$

Solved Example 32 :

Integrate $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$

Solution :

Let $x = \sin\theta$ $\quad \therefore dx = \cos\theta \, d\theta$

$$\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx = \int \frac{\sin^{-1}(\sin\theta)\cos\theta}{\sqrt{1 - \sin^2\theta}} d\theta$$

$$= \int \theta d\theta = \frac{\theta^2}{2} + c$$

$$= \frac{(\sin^{-1} x)^2}{2} + c$$

Integration by parts

Integral of the product of two functions

$$\int u\,v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

Rule to choose the factor of differentiation or the 1st function

Of the two functions the first function u is selected based on the following preference order

1. Inverse function
2. Logarithmic function
3. Algebraic function
4. Trigonometric function
5. Exponential function

Solved Example 33 :

Integrate $\int x\cos x dx$

Solution :

Here F. D. $= x$ F. I. $= \cos x$

Applying the rules of integration by parts.

$\int x\cos x dx$

$= x\int \cos x dx - \int\left[\frac{dx}{dx}\int \cos x dx\right]dx$

$= x \sin x - \int 1.\sin x dx$

$= x \sin x + \cos x + c$

Solved Example 34 :

Integrate $\int \sin^{-1} x\,dx$

Solution :

Here F. I. $= 1$ and F. D. $= \sin^{-1} x$

Now $\int 1.\sin^{-1} x = \sin^{-1} x\left(\int 1 dx\right)$

$$-\int\left[\frac{d}{dx}\sin^{-1} x)\int 1.dx\right]dx$$

$\therefore \int \sin^{-1} x dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$

Put $1 - x^2 = z$, $\therefore$ $-2x\,dx = dz$

$\therefore \int \sin^{-1} x dx = x\sin^{-1} x + \frac{1}{2}\int\frac{dz}{\sqrt{z}}$

$= x\sin^{-1} x + \sqrt{z}$

$\therefore \int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2}$

Solved Example 35 :

Find $\int \log x dx$

Solution :

We have $\int \log x dx = \int 1\cdot\log x dx$

Integrating by parts,

$\int \log x dx = \log x\int 1dx - \int\left[\frac{d}{dx}(\log x)\int 1dx\right]dx$

$= x\log x - \int \frac{1}{x}\cdot x dx$

$= x\log x - x + c = x(\log x - 1) + c$

$$= x\left(\log x - \log e\right) + c$$

$(\because\ 1 = \log e)$

Solved Example 36 :

Evaluate $\int e^{ax}\sin bx\,dx$

Solution :

$$I = \int e^{ax}\sin bx\,dx$$

$$= \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax} b\cos bx}{a}dx$$

$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx dx$$

$$= \frac{e^{ax}\sin bx}{a}$$

$$-\frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx dx\right]$$

$$= \frac{e^{ax}\sin bx}{a}$$

$$-\frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

$$\therefore I = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2} - \frac{b^2}{a^2}I$$

$$\therefore I\left(1 + \frac{b^2}{a^2}\right) = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2}$$

$$\therefore \quad I = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2 + b^2}$$

2.8 Definite Integration

The integral $\int_a^b f(x)dx$ is called definite integral and is defined to be $\phi(b) - \phi(a)$ where $\phi(x)$ is the indefinite integral of $\int f(x)dx$. It is read as integral from a to b of f(x), a is called the lower limit and b is called the upper limit.

$$\boldsymbol{\int_a^b f(x)dx = \left|\phi(x)\right|_{x=a}^{x=b} = \phi(b) - \phi(a)}$$

This is also called the theorem of calculus.

Properties of definite integral

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ f(–x) = f(x) i.e. it is an even function

  $= 0$ f(–x) = –f(x) i.e. it is an odd function
- $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ f(2a – x) = f(x)

  $= 0$ f(2a – x) = –f(x)

Solved Example 37 :

Integrate $\int_0^{\pi/2} \cos x dx$

Solution :

We know that $\int \cos x dx = \sin x + c$

$$\therefore \int_0^{\pi/2} \cos x dx = \left|\sin x\right|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Solved Example 38 :

Integrate $\int_0^1 \frac{dx}{1+x^2}$

Solution :

We know that $\int \frac{dx}{1+x^2} = \tan^{-1}x + c$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \left|\tan^{-1}x + c\right|_0^1$$

$$= \left[\tan^{-1}1 + c\right] - \left[\tan^{-1}0 + c\right]$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Solved Example 39 :

Evaluate $\int_a^b \frac{1}{x}dx$

Solution :

We know that $\int \frac{dx}{x} = \log x$

$$\therefore \int_a^b \frac{dx}{x} = \left|\log x\right|_a^b = \log b - \log a = \log\frac{b}{a}$$

Solved Example 40 :

Evaluate $\int_0^{\pi/2} \left(1-\cos 2x\right)^{1/2} dx$

Solution :

Let $I = \int_0^{\pi/2} \left(1-\cos 2x\right)^{1/2} dx$

$$= \int_0^{\pi/2} \sqrt{1-\left(1-2\sin^2 x\right)}dx$$

$$= \int_0^{\pi/2} \sqrt{2}\sin x dx = \sqrt{2}\int_0^{\pi/2} \sin x$$

$$= \sqrt{2}\left|-\cos x\right|_0^{\pi/2}$$

$$= \sqrt{2}\left(\cos 0 - \cos\frac{\pi}{2}\right)$$

$$= \sqrt{2}\left(1-0\right) = \sqrt{2}$$

Solved Example 41 :

Evaluate the integral $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x}dx$

Solution :

Put sin x = y then cos x dx = dy

when x = 0, then y = 0 and

when x = π/3 then $y = \frac{\sqrt{3}}{2}$

$$\text{Let } I = \int_0^{\pi/3} \frac{\cos x dx}{3+4\sin x} = \int_0^{\sqrt{3}/2} \frac{dy}{3+4y}$$

$$= \frac{1}{4}\left|\log(3+4y)\right|_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{4}\left\{\log 3 + 4\frac{\sqrt{3}}{2} - \log 3\right\}$$

$$= \frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$$

Improper Integrals

- The integral $\int_a^b f(x)dx$ is said to be an improper integral of the first kind, if one or both the limits of integration are infinite.
- The integral $\int_a^b f(x)dx$ is said to be improper integral of second kind, if f(x) becomes unbounded at one or more points in the interval of integration [a, b].

Limit comparison Test for Convergence and Divergence of Improper Integrals :

If f(x) and g(x) are positive functions and if

$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L,\ 0 < L < \infty$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

<u>**Note :**</u> **1.** $\int_1^\infty \frac{dx}{1+x^2}$ **converges because** $\frac{1}{1+x^2} < \frac{1}{x^2}$ **and** $\int_1^\infty \frac{1}{x^2}dx$ **converges.**

2. $\int_1^\infty \frac{1}{e^{2x}}dx$ **converges because** $\frac{1}{e^{2x}} < \frac{1}{e^x}$ **and** $\int_1^\infty \frac{1}{e^x}dx$ **converges.**

3. $\int_1^\infty \frac{dx}{\sqrt{x}}$ **diverges because** $\frac{1}{\sqrt{x}} > \frac{1}{x}$ **for x > 1 and** $\int_1^\infty \frac{1}{x}dx$ **diverges**

4. $\int_1^\infty \left(\frac{1}{x}+\frac{1}{x^2}\right)dx$ **diverges because** $\frac{1}{x} < \left(\frac{1}{x}+\frac{1}{x^2}\right)$ **and** $\int_1^\infty \frac{1}{x}dx$ **diverges.**

Solved Example 42 :

Solve $\int_0^1 \sqrt{\frac{1+x}{1-x}}dx$

Solution :

The integrand becomes infinite at x = 1.

The area under the curve y = f(x) = $\sqrt{\frac{1+x}{1-x}}$

between the lines x = 0 and x = 1 is not well defined since the curve extends to infinity as x tends to 1 from the left. Nevertheless, we can define the area from x = 0 to x = t where 0 < t < 1.

Then the integral $\int_0^1 f(x)dx = \lim_{t\to1^-}\int_0^t f(x)dx$

We also say that the improper integral converges.

$$\lim_{t\to1^-}\int_0^t \sqrt{\frac{1+x}{1-x}}dx = \lim_{t\to1^-}\int_0^t \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \lim_{t\to1^-}\left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^t$$

$$= \lim_{t\to1^-}\left[\sin^{-1}t - \sqrt{1-t^2} + 1\right]$$

$$= \sin^{-1}1 - 0 + 1 = \frac{\pi}{2} + 1$$

Solved Example 43 :

Solve $\int_0^1 \frac{dx}{x}$.

Solution :

For some positive number

t < 1, $\int_t^1 \frac{dx}{x} = [\ell n\, x]_t^1$

$$= \ell n 1 - \ell n t = \ell n\left(\frac{1}{t}\right)$$

$$\therefore \quad \lim_{t\to0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to0^+}\left(\ell n\frac{1}{t}\right) = +\infty$$

The given integral diverges.

Solved Example 44 :

Solve $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution :

The function f(x) = $\frac{1}{(x-1)^{2/3}}$ becomes infinite at x = 1 which lies between the limits of integration.

For the integral to exist, we must have the convergence of the following integrals.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} \text{ and } \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

It can be easily verified the convergence of these two integrals are +3 and $3\left(\sqrt[3]{2}\right)$ respectively.

Hence the given improper integral converges.

Solved Example 45 :

Compare $\int_1^\infty \frac{dx}{x^2}$ and $\int_1^\infty \frac{dx}{1+x^2}$ with limit comparison test.

Solution :

f(x) = $\frac{1}{x^2}$ and $g(x) = \frac{1}{1+x^2}$

$$\lim_{x\to\infty}\frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}=\lim_{x\to\infty}\frac{1+x^2}{x^2}$$

$$=\lim_{x\to\infty}\left(\frac{1}{x^2}+1\right)$$

= 1 a positive finite limit.

Hence $\int_1^\infty \frac{dx}{x^2}$ converges, therefore $\int_1^\infty \frac{dx}{1+x^2}$ converges.

Solved Example 46 :

Solve $\int_1^\infty \frac{3}{e^x+5}dx$

Solution :

$\int_1^\infty \frac{3}{e^x+5}dx$ converges because $\int_1^\infty \frac{1}{e^x}dx$ converges

Now $\lim_{x\to\infty}\frac{3}{e^x+5}\cdot\frac{e^x}{1}=\lim_{x\to\infty}\frac{3}{1+5e^{-x}}$

= 3 a positive finite limit.

Solved Example 47 :

Solve $\int_1^\infty \frac{1}{e^{2x}-10e^x}dx$

Solution :

$\int_1^\infty \frac{1}{e^{2x}-10e^x}dx$ converges because

$\int_1^\infty \frac{1}{e^{2x}}dx$ converges and

$\lim_{x\to\infty}\frac{1}{e^{2x}}\left(e^{2x}-10e^x\right)$

$=\lim_{x\to\infty}\left(1-\frac{10}{e^x}\right)=1$, a positive finite limit

Solved Example 48 :

Solve $\int_1^\infty \frac{1}{\sqrt{x}}dx$

Solution :

$\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges because $\int_1^\infty \frac{1}{x}dx$ diverges

and $\frac{1}{\sqrt{x}}>\frac{1}{x}$ for x > 1

2.9 Double Integrals

The double integral of a function f(x, y) over a region D is denoted by $\iint_D f(x,y)\,dxdy$.

Let f(x, y) be a continuous function defined on a closed rectangle

R = { (x, y) / a ≤ x ≤ b and c ≤ y ≤ d }

For continuous functions f(x, y) we have

$$\iint_R f(x,y)\,dxdy=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

Note : 1. $\iint\limits_S dxdy$ represent the area of the region S.

2. In an iterated integral the limits in the first integral are constants and if the limits in the second integral are functions of x then we must first integrate w.r.t. y and the integrand will become a function of x alone. This is integrated w.r.t. x.

3. If R cannot be written in neither of the above two forms we divide R into finite number of subregions such that each of the subregions can be represented in one of the above forms and we get the double integral over R by adding the integrals over these subregions.

Solved Example 49 :

Evaluate $I = \int_0^1 \int_0^2 (x+2)dydx$

Solution :

$$\int_0^1 [xy + 2y]_0^2 dx = \int_0^1 (2x+4)dx$$

$$= \left[\frac{2x^2}{2}\right]_0^1 + 4[x]_0^1$$

$$= 1 + 4 = 5$$

Solved Example 50 :

Evaluate $\int_0^1 \int_x^1 (x^2 + y^2)dydx$

Solution :

$$\int_0^1 \int_x^1 (x^2 + y^2)dydx$$

$$= \int_0^1 \left[\int_x^1 (x^2 + y^2)dy\right]dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^1 dx$$

$$= \int_0^1 \left[\left(x^2 + \frac{1}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right]dx$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} - \frac{4x^3}{3}\right)dx$$

$$= \left[\frac{x^3}{3} + \frac{x}{3} - \frac{x^4}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

Solved Example 51 :

Evaluate $\int_2^3 \int_1^2 \frac{dxdy}{xy}$

Solution :

$$\int_2^3 \int_1^2 \frac{dxdy}{xy} = \int_2^3 \frac{dy}{y} \int_1^2 \frac{dx}{x}$$

$$= \int_2^3 \frac{dy}{y} [\log x]_1^2$$

$$= (\log 2 - \log 1)\int_2^3 \frac{dy}{y}$$

$$= \log 2 [\log y]_2^3$$

$$= \log 2[\log 3 - \log 2]$$

$$= \log 2 \ \log\frac{3}{2}$$

Solved Example 52 :

Evaluate the double integral $\iint\limits_R e^{x^2} dxdy$ where the region R is given by

R : $2y \le x \le 2$ and $0 \le y \le 1$.

Solution :

Integrating with respect to y first, we get

$$I = \int_0^2 \left[\int_0^{x/2} e^{x^2} dy\right] dx = \int_0^2 \left[ye^{x^2}\right]_0^{x/2} dx$$

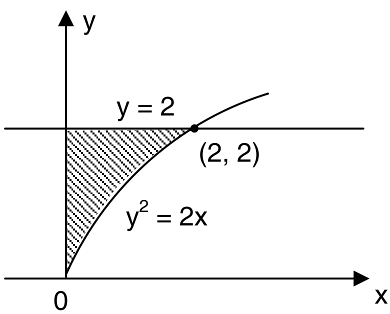
$$= \frac{1}{2}\int_0^2 xe^{x^2} dy = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}\left(e^4 - 1\right)$$

Solved Example 53 :

Evaluate the integral

$$\int_0^2 \int_0^{\frac{y^2}{2}} \frac{y}{\sqrt{x^2 + y^2 + 1}} dxdy$$

Solution :



It would be easier to integrate it first with respect to y.

The region of integration $0 \le y \le 2$ and $0 \le x \le \frac{y^2}{2}$ can also be written as

$0 \le x \le 2$ and $\sqrt{2x} \le y \le 2$

$$\therefore \quad I = \int_0^2 \left[\int_{\sqrt{2x}}^2 \frac{y}{\sqrt{x^2 + y^2 + 1}} dy\right] dx$$

$$= \int_0^2 \left[\sqrt{x^2 + y^2 + 1}\right]_{\sqrt{2x}}^2 dx$$

$$= \int_0^2 \left[\sqrt{x^2 + 5} - (x + 1)\right] dx$$

$$= \left[\frac{x\sqrt{x^2 + 5}}{2} + \frac{5}{2}\log\left(x + \sqrt{x^2 + 5}\right) - \frac{1}{2}(x + 1)^2\right]_0^2$$

$$= 3 + \frac{5}{2}\left(\log 5 - \log\sqrt{5}\right) - \frac{1}{2}(9 - 1)$$

$$= \frac{5}{4}\log 5 - 1$$

Solved Example 54 :

The cylinder $x^2 + z^2 = 1$ is cut by the planes y = 0, z = 0 and x = y. Find the volume of the region in the first octant.

Solution :

In the first octant we have $z = \sqrt{1 - x^2}$. The projection of the surface in the x – y plane is bounded by x = 0, x = 1, y = 0 and y = x.

$$\therefore \quad V = \iint\limits_R zdxdy = \int_0^1 \left[\int_0^x \sqrt{1 - x^2} dy\right] dx$$

$$= \int_0^1 \sqrt{1 - x^2}\,[y]_0^x\, dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx$$

$$= \frac{-1}{3}\left[\left(1-x^2\right)^{3/2}\right]_0^1 = \frac{1}{3} \text{ cubic units.}$$

Solved Example 55 :

Evaluate $I = \int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$

Solution :

$$I = \int_0^{\pi}\sin\theta\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta\sin\theta\,d\theta$$

$$= \frac{-a^2}{2}\int_0^{\pi}\cos^2\theta\,d(\cos\theta)$$

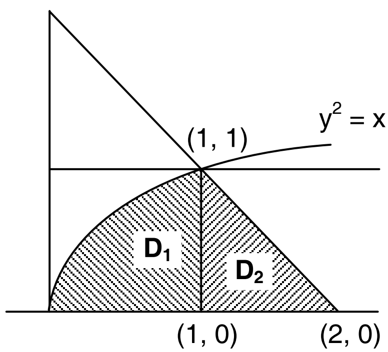
$[d(\cos\theta) = \sin\theta\,d\theta]$

$$= \frac{-a^2}{6}\left[\cos^3\theta\right]_0^{\pi} = \frac{a^2}{3}$$

Solved Example 56 :

Evaluate $I = \iint_D xy\,dy\,dx$ where D is the region bounded by the curve $x = y^2$, $x = 2 - y$, $y = 0$ and $y = 1$.

Solution :



In this region x varies from 0 to 2 when $0 \le x \le 1$ for fixed x, y varies from 0 to $\sqrt{x}$. When $1 \le x \le 2$, y varies from 0 to $2 - x$.

$\therefore$ The region D can be subdivided into 2 regions D_1 and D_2 as shown.

$$\therefore \iint_D xy\,dy\,dx = \iint_{D_1} xy\,dy\,dx + \iint_{D_2} xy\,dy\,dx$$

In the region D_1 for fixed x, y varies from $y = 0$ to $y = \sqrt{x}$ and for fixed y, x varies from $x = 0$ to $x = 1$.

Similarly for D_2 the limit of integration for y is $y = 0$ to $y = 2 - x$

$$\iint_D xy\,dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{x}} xy\,dy\,dx + \int_1^2\int_0^{2-x} xy\,dy\,dx$$

$$= \int_0^1\left[\frac{xy^2}{2}\right]_0^{\sqrt{x}} dx + \int_1^2\left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^2 x(2-x)^2\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^1 + \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(4) + \frac{16}{4} - 4\frac{(8)}{3}\right] - \frac{1}{2}\left[2 + \frac{1}{4} - \frac{4}{3}\right]$$

$$= \frac{1}{6} + \frac{4}{6} - \frac{11}{24} = \frac{9}{24}$$

Solved Example 57 :

Change the order of integration in the integral $I = \int_1^4\int_{y/2}^{y} f(x,y)\,dx\,dy$

Solution :

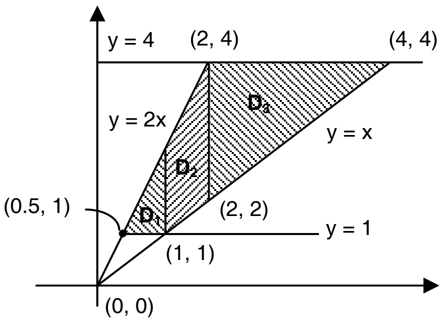
The region of integration D is bounded by

$x = y/2$

$x = y$

$y = 1$

$y = 4$



Here, x varies from 0.5 to 4

when $\frac{1}{2} \le x \le 2$, y varies from 1 to 2x

when $1 \le x \le 2$, y varies from x to 2x

when $2 \le x \le 4$, y varies from x to 4

For changing the order of integration we must divide D into sub regions D_1, D_2, D_3.

$$\therefore \quad I = \iint_D f(x,y)\,dxdy$$

$$= \iint_{D_1} f(x,y)\,dxdy + \iint_{D_2} f(x,y)\,dxdy$$

$$+ \iint_{D_3} f(x,y)\,dxdy$$

$$= \int_{1/2}^{1}\int_{1}^{2x} f(x,y)\,dydx + \int_{1}^{2}\int_{x}^{2x} f(x,y)\,dydx$$

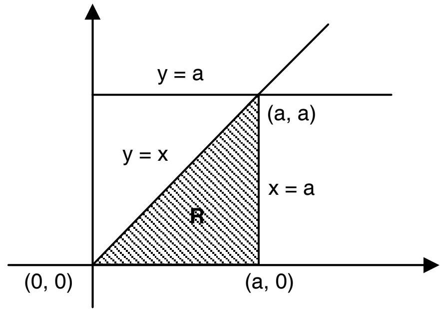
$$+ \int_{2}^{4}\int_{x}^{4} f(x,y)\,dydx$$

Solved Example 58 :

Evaluate $\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}$

by changing the order of integration.

Solution :



Region R of integration

is bounded by

$x = y$

$x = a$

$y = 0$

$y = a$

We have to change the order of integration as dy dx.

Here in the region R, x varies from 0 to a and for fixed x, y varies from 0 to x.

$$\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2} = \int_0^a \int_0^x \frac{x}{x^2+y^2}\,dydx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

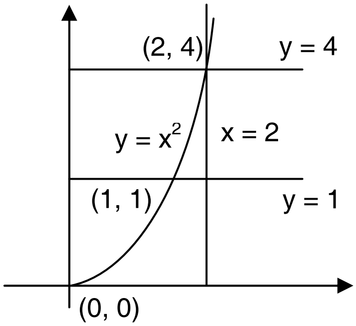
$$= \int_0^a \left[\tan^{-1}1 - \tan^{-1}0\right]dx$$

$$= \frac{\pi}{4}\int_0^a dx = \frac{\pi}{4}[x]_0^a = \frac{a\pi}{4}$$

Solved Example 59 :

Evaluate $\int_{1}^{4}\int_{\sqrt{y}}^{2}\left(x^2+y^2\right)dxdy$ by changing the order of integration.

Solution :



The region of integration is bounded by

$y = 1$

$y = 4$

$x^2 = y$

$x = 2$

We change the order of integration as dy dx. In the region x varies from 1 to 2 and for fixed x, y varies from 1 to x^2.

$$\int_{1}^{4}\int_{\sqrt{y}}^{2}\left(x^2+y^2\right)dxdy$$

$$= \int_{1}^{2}\int_{1}^{x^2}\left(x^2+y^2\right)dydx$$

$$= \int_{1}^{2}\left[x^2y+\frac{y^3}{3}\right]_1^{x^2} dx$$

$$= \int_{1}^{2}\left(x^4+\frac{x^6}{3}-x^2-\frac{1}{3}\right)dx$$

$$= \left[\frac{x^5}{5}+\frac{x^7}{21}-\frac{x^3}{3}-\frac{x}{3}\right]_1^2$$

$$= \left[\frac{32}{5}+\frac{128}{21}-\frac{8}{3}-\frac{2}{3}\right]-\left[\frac{1}{5}+\frac{1}{21}-\frac{1}{3}-\frac{1}{3}\right]$$

$$= \frac{1006}{105}$$

Solved Example 60 :

Evaluate $\int_{0}^{3}\int_{1}^{\sqrt{4-y}}\left(x+y\right)dxdy$ by changing the order of integration.

Solution :

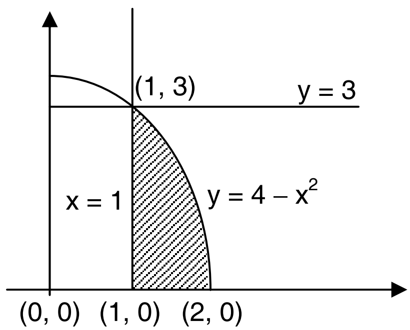
The region is bounded by

$y = 0$

$y = 3$

$x = 1$

$x^2 = 4 - y$



We have to change the order of integration as dydx.

In this region x varies from 1 to 2

and y varies from 0 to $4 - x^2$

$$\int_{0}^{3}\int_{1}^{\sqrt{4-y}}\left(x+y\right)dxdy$$

$$= \int_{1}^{2}\int_{0}^{4-x^2}\left(x+y\right)dydx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} dx$$

$$= \int_1^2 \left[x\left(4 - x^2\right) + \frac{\left(4 - x^2\right)^2}{2} \right] dx$$

$$= \int_1^2 \left[\frac{x^4}{2} - x^3 - 4x^2 + 4x + 8 \right] dx$$

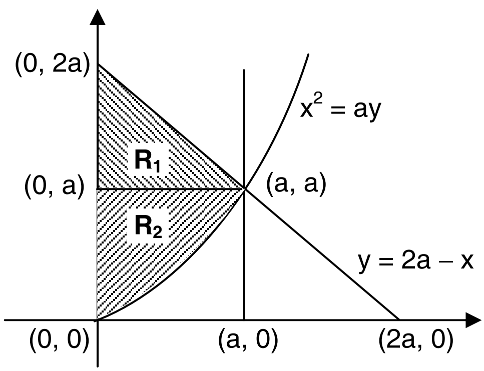
$$= \left[\frac{x^5}{10} - \frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 + 8x \right]_1^2$$

$$= \frac{241}{60}$$

Solved Example 61 :

Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xydydx$ by changing the order of integration

Solution :



The region R is bounded by

$x = 0$

$x = a$

$y = 2a - x$

$y = \frac{x^2}{a}$

We divide the region into two subregions R_1 and R_2

$$\therefore \quad \iint_R xydydx = \iint_{R_1} xydydx + \iint_{R_2} xydydx$$

We have to change the order of integration as dxdy in R_1 and R_2.

In region R_1, y varies from a to 2a for a fixed y, x varies from 0 to $2a - y$

In the region R_2, y varies from 0 to a and for a fixed y, x varies from 0 to $\sqrt{ay}$.

$$\therefore \quad \int_0^a \int_{x^2/a}^{2a-x} xydydx$$

$$= \iint_{R_1} xydydx + \iint_{R_2} xydydx$$

$$= \int_a^{2a} \int_0^{2a-y} xydxdy + \int_0^a \int_0^{\sqrt{ay}} xydxdy$$

$$= \int_a^{2a} \left[\frac{x^2 y}{2} \right]_0^{2a-y} dy + \int_0^a \left[\frac{x^2 y}{2} \right]_0^{\sqrt{ay}} dy$$

$$= \int_a^{2a} \frac{y}{2}(2a - y)^2 dy + \int_0^a \frac{ay^2}{2} dy$$

$$= \int_a^{2a} \left(2a^2 y + \frac{y^3}{2} - 2ay^2 \right) dy + \frac{a}{2} \int_0^a y^2 dy$$

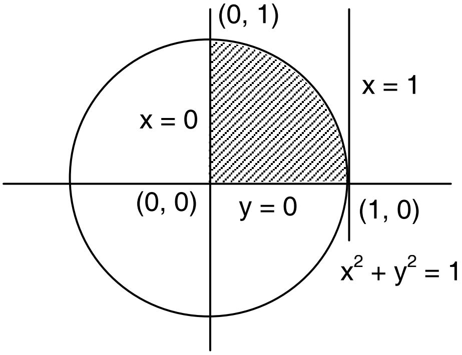
$$= \left[a^2 y^2 + \frac{y^4}{8} - \frac{2ay^3}{3} \right]_a^{2a} + a\left[\frac{y^3}{6} \right]_0^a$$

$$= \frac{5a^4}{24} + \frac{a^4}{6} = \frac{9a^4}{24}$$

Solved Example 62 :

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} ydydx$ by interchanging the order of integration.

Solution :



The region is bounded by

$y = 0$ (x – axis)

$y = \sqrt{1-x^2}$ or $x^2 + y^2 = 1$

$x = 0$ (y – axis)

$x = 1$

Here y varies from 0 to 1 and for fixed y

x varies from 0 to $\sqrt{1-y^2}$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 dxdy$$

$$= \int_0^1 y^2 [x]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} dy$$

$$= \int_0^{\pi/2} \sin^2\theta \cos\theta d\theta$$

$$\begin{bmatrix} \text{Putting } y = \sin\theta & \\ \text{when } y = 0 & \theta = 0 \\ y = 1 & \theta = \pi/2 \end{bmatrix}$$

again put $t = \sin\theta$

$\therefore \quad dt = \cos\theta \, d\theta$

When $\theta = 0 \qquad t = 0$

$\theta = \pi/2 \qquad t = 1$

$$\therefore \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx = \frac{1}{3}$$

2.10 Triple Integrals

A triple integral of a function defined over a region R is denoted by

$$\iiint_R f(x,y,z)dxdydz \quad \text{or} \quad \iiint f(x,y,z)dV \quad \text{or} \quad \iiint_R f(x,y,z)d(x,y,z)$$

Evaluation of Triple Integrals :

If the region R can be described by

$$x_1 \le x \le x_2,\ y_1(x) \le y \le y_2(x),\ z_1(x,y) \le z \le z_2(x,y)$$

then we evaluate triple integral as

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dzdydx = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right] dy \right] dx$$

Note : **There are six possible ways in which a triple integral can be evaluated (order of variables of integration). We choose the one which is simple to use.**

Solved Example 63 :

Evaluate $\int_0^2\int_1^3\int_1^2 xy^2 z\,dz\,dy\,dx$

Solution :

$$\int_0^2\int_1^3\int_1^2 xy^2 z\,dz\,dy\,dx = \int_0^2\int_1^3\left[\frac{xy^2z^2}{2}\right]_1^2 dydx$$

…(Integrating w.r.t. z keeping x, y constants)

$$= \int_0^2\int_1^3\left[2xy^2 - \frac{xy^2}{2}\right]dydx$$

$$= \frac{3}{2}\int_0^2\int_1^3 xy^2 dydx$$

$$= \frac{3}{2}\int_0^2\left[\frac{xy^3}{3}\right]_1^3 dx$$

…(integrating w.r.t. y keeping x constant)

$$= \frac{1}{2}\int_0^2(27x - x)dx$$

$$= 13\int_0^2 xdx$$

$$= 13\left[\frac{x^2}{2}\right]_0^2 = 26$$

Solved Example 64 :

Evaluate $I = \int_0^a\int_0^x\int_0^y xyz\,dz\,dy\,dx$

Solution :

$$I = \int_0^a\int_0^x\left[\frac{xyz^2}{2}\right]_0^y dydx$$

$$= \frac{1}{2}\int_0^a\int_0^x xy^3 dydx = \frac{1}{2}\int_0^a\left[\frac{xy^4}{4}\right]_0^x dx$$

$$= \frac{1}{8}\int_0^a x^5 dx = \frac{1}{8}\left[\frac{x^6}{6}\right]_0^a = \frac{a^6}{48}$$

Solved Example 65 :

Evaluate : $I = \int_0^{\pi}\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta\,d\phi$

Solution :

$$I = \int_0^{\pi}\int_0^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^1 d\theta d\phi$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\pi/2}\sin\theta\,d\theta\,d\phi = \frac{1}{3}\int_0^{\pi}\left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \frac{1}{3}\int_0^{\pi} d\phi = \frac{1}{3}\left[\phi\right]_0^{\pi} = \frac{\pi}{3}$$

Solved Example 66 :

Evaluate $I = \int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}dzdydr$

Solution :

$$I = \int_0^{\log a}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+y} dydx$$

$$= \int_0^{\log a}\int_0^{x}\left(e^{2(x+y)} - e^{x+y}\right)dydx$$

$$= \int_0^{\log a}\left[\frac{1}{2}e^{2(x+y)} - e^{x+y}\right]_0^x dx$$

$$= \int_0^{\log a}\left(\frac{1}{2}e^{4x} - \frac{3}{2}e^{2x} + e^x\right)dx$$

$$= \left[\frac{1}{8}e^{4x} - \frac{3}{4}e^{2x} + e^x\right]_0^{\log a}$$

$$= \frac{1}{8}a^4 - \frac{3}{4}a^2 + a - \frac{3}{8}$$

Solved Example 67 :

Find the volume of the solid in the first octant bounded by the paraboloid.

$z = 36 - 4x^2 - 9y^2$

Solution :

We have $\quad I = \iiint_R dz\,dy\,dx$

The projection of the paraboloid (in the first octant) in the x – y plane is the region in the first quadrant of the ellipse

$4x^2 + 9y^2 = 36$

$\therefore$ The region R is

$$0 \le z \le 36 - 4x^2 - 9y^2$$

$$0 \le y \le \frac{1}{3}\sqrt{36 - 4x^2}$$

$$0 \le x \le 3$$

$$\therefore \quad I = \int_0^3\left[\int_0^{2\sqrt{9-\frac{x^2}{3}}}\left(36 - 4x^2 - 9y^2\right)dy\right]dx$$

$$= \int_0^3\left[4\left(9 - x^2\right)y - 3y^3\right]_0^{2\sqrt{9-\frac{x^2}{3}}} dx$$

$$= \int_0^3\left[\frac{8}{3}\left(9 - x^2\right)^{3/2} - \frac{8}{9}\left(9 - x^2\right)^{3/2}\right]dx$$

$$= \frac{16}{9}\int_0^3\left(9 - x^2\right)^{3/2} dx$$

Substituting x = 3 sin θ we get

$$I = \frac{16}{9}\int_0^{\pi/2}\left(27\cos^3\theta\right)\left(3\cos\theta\right)d\theta$$

$$= 144\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= 144\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = 27\pi \text{ cubic units}$$

2.11 Change of Variables in Double and Triple Integrals and Jacobians

Consider the transformation given by the equation

$x = x(u, v, w)$;

$y = y(u, v, w)$;

$z = z(u, v, w)$

Where the functions x, y, z have continuous first order partial derivatives.

The Jacobian J of the transformation is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The Jacobian is also denoted by $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

For a transformation in two variables x = x(u, v) and y = y(u, v) the Jacobian is given by a determinant of order two. Hence $J = \frac{\partial(x,y)}{\partial(u,v)}$

Solved Example 68 :

The transformation from cartesian coordinates (x, y) to polar co–ordinates (r, θ) is given by

$x = r\cos\theta \qquad y = r\sin\theta$

Solution :

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

and

$$\iint_R f(x,y)\,dxdy = \iint_{R'} f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

Solved Example 69 :

The transformation from Cartesian co–ordinates (x, y, z) to spherical polar co–ordinates

(r, θ, ϕ) is given by

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Solution :

Here $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= r^2 \sin\theta$

and $\iiint_R f(x,y,z)\,dxdydz$

$$= \iiint_{R'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\sin\theta)$$

$$r^2 \sin\phi\, dr\, d\theta\, d\phi$$

Solved Example 70 :

The transformation from Cartesian coordinates (x, y, z) to cylindrical co–ordinates

(r, θ, z) is :

$x = r\cos\theta,\ y = \sin\theta,\ z = z$

Solution :

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

and $\iiint\limits_R f(x,y,z)\,dxdydz$

$$= \iiint\limits_{R'} f(r\cos\theta, r\sin\theta, z)\,r dr\, d\theta\, dz$$

Solved Example 71 :

If $u = x^2 - y^2$ and $v = 2xy$

Prove that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4\left(x^2 + y^2\right)$$

Solution :

We have $\left(x^2 + y^2\right) = \left(x^2 - y^2\right) + (2xy)^2$

$$= u^2 + v^2$$

$$\therefore \quad \left(x^2 + y^2\right) = \sqrt{u^2 + v^2}$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = 4\sqrt{u^2 + v^2}$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{4\sqrt{u^2 + v^2}}$$

Solved Example 72 :

Evaluate the integral

$\iint\limits_R (x-y)^2 \cos^2(x+y)\,dxdy$, where R is the parallelogram with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$

Solution :

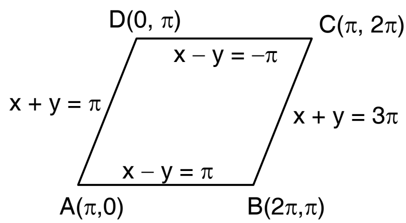
The equations are

$AB \Rightarrow x - y = \pi$

$BC \Rightarrow x + y = 3\pi$

$CD \Rightarrow x - y = -\pi$

$DA \Rightarrow x + y = \pi$



Substitute $y - x = u$ and $y + x = v$

Then $\quad -\pi \le u \le \pi$ and $\pi \le v \le 3\pi$

$$\therefore \quad x = \frac{v-u}{2} \text{ and } y = \frac{v+u}{2}$$

and

$$I = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\therefore \quad |J| = \frac{1}{2}$$

$$\therefore \quad I = \iint\limits_R (x-y)^2 \cos^2(x+y)\,dxdy$$

$$= \frac{1}{2}\int_{\pi}^{3\pi}\int_{-\pi}^{\pi} u^2 \cos^2 v\,dudv$$

$$= \frac{\pi^3}{3}\int_{\pi}^{3\pi} \cos^2 v\,dv$$

$$= \frac{\pi^3}{6}\int_{\pi}^{3\pi} (1 + \cos 2v)\,dv = \frac{\pi^4}{3}$$

Solved Example 73 :

Evaluate $I = \iint_R \sqrt{x^2+y^2}\,dxdy$ by changing to polar co–ordinates, where R is the region in the x – y plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution :

Using $x = r\cos\theta$

$y = r\sin\theta$

We get $dx\,dy = r\,dr\,d\theta$ and

$$I = \int_0^{2\pi}\int_2^3 r(r\,dr\,d\theta) = \int_0^{2\pi}\left[\frac{r^3}{3}\right]_2^3 d\theta$$

$$= \frac{19}{3}\int_0^{2\pi} d\theta = \frac{38\pi}{3}$$

Solved Example 74 :

Evaluate the improper integral

$$I = \int_0^\infty e^{-x^2}dx$$

Solution :

$I^2 = I \cdot I$

$$= \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

Put $x = r\cos\theta$

$y = r\sin\theta$

Hence $J = r$

The region of integration is the entire first quadrant. Hence r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$\therefore \quad I^2 = \int_0^\infty\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$$

$$= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr = \frac{\pi}{2}\int_0^\infty -\frac{1}{2}e^{-r^2}d(-r^2)$$

$$= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$$

$$\therefore \quad I = \frac{\sqrt{\pi}}{2}$$

Solved Example 75 :

Evaluate the integral $\iiint_T z\,dxdydz$, where T is the hemisphere of radius a, $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Solution :

Changing to spherical co–ordinates substitute

$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi$

$0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$

We get,

$dx\,dy\,dz = r^2 \sin\phi\,dr\,d\theta\,d\phi$

$$\therefore \quad I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (r\cos\phi)r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi}\int_0^{\pi/2} \sin 2\phi\,d\phi\,d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi}\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi} d\theta$$

$$= \frac{\pi a^4}{4}$$

Solved Example 76 :

Using triple integral find the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Solution :

$V = 8\iiint\limits_D dxdydz$ where D is the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$ in the first quadrant.

Substitute $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$

Jacobian $|J| = r^2 \sin\theta$

and the limits r = 0 to a; $0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$

$$\therefore \quad V = 8\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} |J|\, d\theta d\phi dr$$

$$= 8\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r^2 \sin\theta d\theta d\phi dr$$

$$= 8\int_0^a\int_0^{\pi/2} r^2 \left[-\cos\theta\right]_0^{\pi/2} d\phi dr$$

$$= 8\int_0^a\int_0^{\pi/2} r^2 d\phi\, dr = 8\int_0^a r^2 \left[\phi\right]_0^{\pi/2} dr$$

$$= 4\pi\int_0^a r^2 dr = 4\pi\left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{4\pi a^3}{3}$$

Solved Example 77 :

Find by triple integral the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution :

The projection of the given region on the x − y plane is the triangle bounded by the lines

x = 0;

y = 0 and $\frac{x}{a} + \frac{y}{b} = 1$ $\qquad 0 \le x \le a$

In this region x varies from 0 to a.

For fixed x, y varies from 0 to $\left(1 - \frac{x}{a}\right)b$.

i.e. $0 \le y \le \left(1 - \frac{x}{a}\right)b$

For fixed (x, y), z varies from 0 to $\left(1 - \frac{x}{a} - \frac{y}{b}\right)c$

$$\left[0 \le z \le \left(1 - \frac{x}{a} - \frac{y}{b}\right)c\right]$$

$$\therefore \quad I = \iiint\limits_D dxdydz$$

$$= \int_0^a \int_0^{\left(1-\frac{x}{a}\right)b} \int_0^{\left(1-\frac{x}{a}-\frac{x}{b}\right)c} dzdydx$$

$$= c\int_0^a \int_0^{\left(1-\frac{x}{a}\right)b} \left(1 - \frac{x}{a} - \frac{y}{b}\right)dydx$$

$$= c\int_0^a \left[\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{\left(1-\frac{x}{a}\right)b} dx$$

$$= bc\int_0^a \left[\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2}\left(1 - \frac{x}{a}\right)^2\right]dx$$

$$= \frac{bc}{2}\int_0^a \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[-\frac{a}{3}\left(1 - \frac{x}{a}\right)^3\right]_0^a = \frac{abc}{6}$$

Solved Example 78 :

Evaluate $I = \iiint_R xyzdxdydz$ where D is the positive octant of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Solution :

Substitute $x = au$,

$y = bv$

$z = cw$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Let R be the positive octant of the sphere $u^2 + v^2 + w^2 = 1$

$$\therefore \quad I = \iiint_{R'} (abc)(uvw)(abc)dudvdw$$

$$= a^2b^2c^2 \iiint_{R'} uvwdudvdw$$

Now substitute

$u = r \sin\theta \cos\phi$

$v = r \sin\theta \sin\phi$

$w = r \cos\theta$

Then $J = r^2 \sin\theta$

$$I = a^2b^2c^2 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^5 \sin^3\theta \cos\theta \cos\phi \sin\phi \, d\phi \, d\theta \, dr$$

$$= a^2b^2c^2 \int_0^1 r^5 dr \int_0^{\pi/2} \sin^3\theta \cos\theta d\theta \int_0^{\pi/2} \sin\phi \cos\phi d\phi$$

$$= a^2b^2c^2 \left[\frac{1}{6}r^6\right]_0^1 \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2} \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2}$$

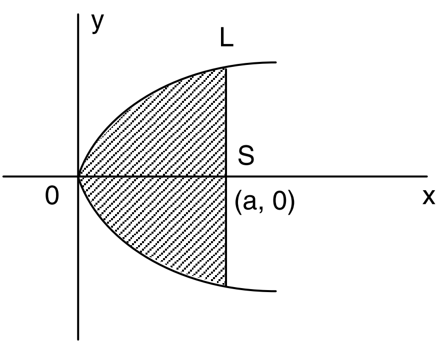
$$= \frac{a^2b^2c^2}{48}$$

2.12 Application of Integration

Solved Example 79 :

Calculate the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Solution :



The latus rectum of the parabola $y^2 = 4ax$ is a line parallel to y − axis. The shaded portion is the desired area. As the parabola is symmetric about the x–axis, therefore area

OLL′ = 2 × Area OSL.

$$= 2\int_0^a ydx = 2\int_0^a \sqrt{4ax}dx = 4\sqrt{a}\int_0^a x^{1/2}dx$$

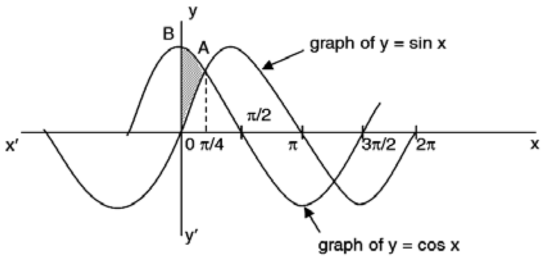
$$= \left|4\sqrt{a}\,\frac{x^{3/2}}{3/2}\right|_{x=0}^{a} = \frac{8}{3}\sqrt{a}\left|a^{3/2}-0\right|$$

$$= \frac{8a^2}{3}$$

Solved Example 80 :

Find the area of the region bounded by the graph of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi/4$.

Solution :



$$\therefore \text{ Required area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= \left|(\sin x + \cos x)\right|_0^{\pi/4}$$

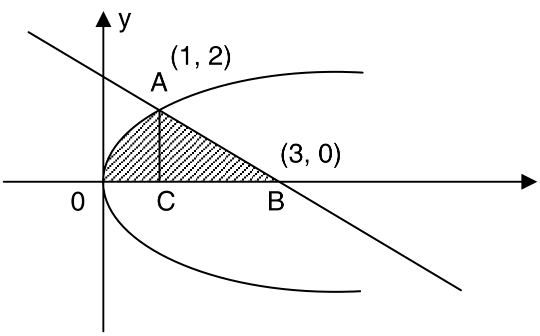
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1)$$

$$= \left(\sqrt{2} - 1\right) \text{ sq. units.}$$

Solved Example 81 :

Find the area above the x–axis bounded by $y^2 = 4x$ and the line $x + y = 3$

Solution :



Required area = area OCA + area ACB

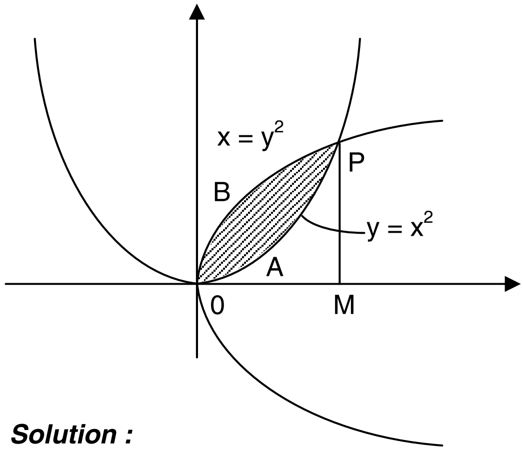
$$= \int_0^1 \sqrt{4x}\,dx + \int_1^3 (3-x)\,dx$$

$$= \left[2\cdot\frac{2}{3}x^{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^3$$

$$= \frac{4}{3} + 2 = \frac{10}{3} \text{ sq. units.}$$

Solved Example 82 :

Find the area of the region bounded by the two parabolas $y = x^2$ and $x = y^2$.



Solution :

Let us first find the points of intersection solving $y = x^2$ and $x = y^2$

$$y = y^4$$

$$\therefore \quad y(y^3 - 1) = 0$$

$\therefore \quad y = 0 \quad \Rightarrow \quad x = 0$

$\quad\quad y = 1 \quad \Rightarrow \quad x = 1$

For the curve $x = y^2$

$$\text{Now the area OBPM} = \int_0^1 y\,dx = \int_0^1 \sqrt{x}\,dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3}\text{ sq.units}$$

For the curve $y = x^2$

Area OAPM = $\int_0^1 y\,dx$

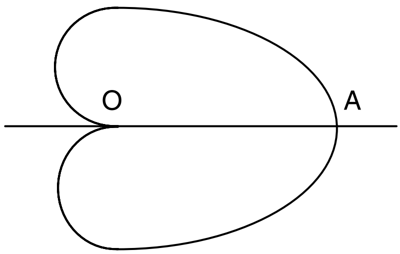
$$= \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \text{ sq. units.}$$

The required area = $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ sq. units.

Solved Example 83 :

Find the area bounded by the cardioid $r = a(1 + \cos\theta)$

Solution :



A cardioid is symmetrical about the initial line.

Hence the required area = $2\int_0^{\pi} \frac{1}{2} r^2 d\theta$ since θ varies from 0 to π for the upper part.

$$= \int_0^{\pi} a^2 (1+\cos\theta)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} d\theta = 4a^2 \int_0^{\pi/2} \cos^4 t2dt$$

(putting $\frac{\theta}{2} = t$)

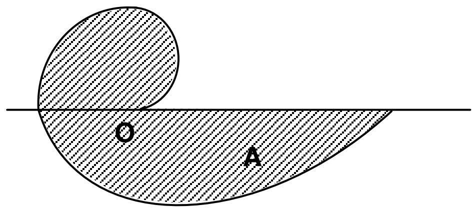
$$= 8a^2 \int_0^{\pi/2} \cos^4 t dt = 8a^2 \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$= \frac{3\pi a^2}{2}$$

Solved Example 84 :

Find the area of one turn of the archimedan spiral $r = a\theta$.

Solution :



The area is $A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \theta^2 d\theta$

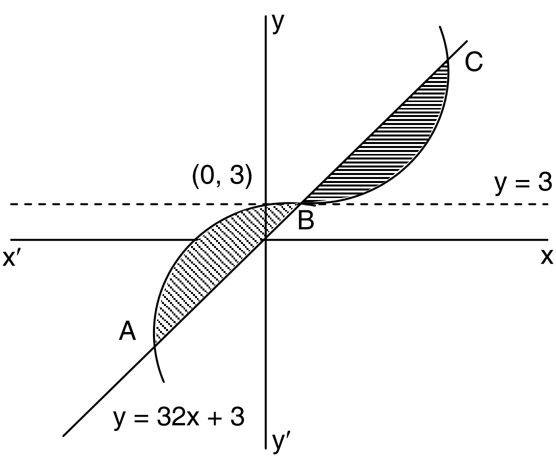
$$= \frac{a^2}{2}\frac{(2\pi)^3}{3} = \frac{4}{3}\pi^3 a^2$$

Solved Example 85 :

Find the area bounded by the curves :

$y = 2x^5 + 3$ and $y = 32x + 3$

Solution :



Two curves will intersect when

$$2x^5 + 3 = 32x + 3 \Rightarrow 2x\left[x^4 - 16\right] = 0$$

In the interval (–2, 0) ; $2x^5 + 3 \geq 32x + 3$

and in the interval (0, 2) ; $32x + 3 \geq 2x^5 + 3$

Thus the required area is given by

$$A = \int_{-2}^{0}\left(2x^5 + 3 - 32x - 3\right)dx$$

$$+\int_{0}^{2}\left(32x + 3 - 2x^5 - 3\right)dx$$

$$= \left|\left(\frac{x^6}{3} - 16x^2\right)\right|_{-2}^{0} + \left|\left(16x^2 - \frac{x^6}{3}\right)\right|_{x=0}^{2}$$

$$= \frac{128}{3} + \frac{128}{3} = \frac{256}{3} \text{ sq. units.}$$

2.13 Partial and Total Derivatives

Functions of two variables

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called the function of two variables x and y and this is denoted by $z = f(x, y)$

Partial derivatives of first order

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary then z becomes a function of x only. The derivative of z with respect to x, treating y as constant, is called partial derivative of z with respect to x and is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0}\frac{f(x+h, y) - f(x, y)}{h}$$

Similarly the derivative of z with respect to y, treating x as constant, is called partial derivative of z with respect to y and is denoted by

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

Thus $\dfrac{\partial z}{\partial y} = \lim_{k \to 0}\dfrac{f(x, y+k) - f(x, y)}{k}$

$\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are called first order partial derivatives of z

Note : **(i) If z = u + v where u = f(x, y) , v ≠ φ (x, y) , then z is a function of x and y.**

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad ; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

(ii) If z = uv, where u = f(x, y) , v = φ(x, y)

$$\text{then } \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(iii) If $z = \frac{u}{v}$**, where u = f(x, y), v =φ(x, y)**

$$\text{then } \frac{\partial z}{dx} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

(iv) If z = f(u) where u = φ (x, y)

$$\text{then } \frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} \quad ; \quad \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y}$$

Partial derivatives of higher order

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}$$

In general $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ or $f_{xy} = f_{yx}$

Solved Example 86 :

First order partial derivative of $u = y^x$ is

Solution :

$u = y^x$

Treating y as constant

$$\frac{\partial u}{\partial x} = y^x \log y$$

treating x as constant

$$\frac{\partial u}{\partial y} = x\, y^{x-1}$$

Solved Example 87 :

Find first order partial derivative of

$$u = \tan^{-1} \frac{x^2+y^2}{x+y}$$

Solution :

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2+(x^2+y^2)^2}$$

$$\times \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2)-(x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2+(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2}$$

Similarly $\dfrac{\partial u}{\partial y} = \dfrac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$

Solved Example 88 :

If u = log (tan x + tan y) show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$$

Solution :

u = log(tan x + tan y) [u is symmetrical with respect to x and y]

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x+\tan y} \cdot \frac{\partial}{\partial x}(\tan x+\tan y)$$

$$= \frac{\sec^2 x}{\tan x+\tan y}$$

Similarly $\dfrac{\partial u}{\partial y}$

$$= \frac{\sec^2 y}{\tan x+\tan y}$$

$$= \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y}$$

$$= \frac{1}{\tan x+\tan y}\left[\sin 2x \sec^2 x + \sin 2y \sec^2 y\right]$$

$$= \frac{1}{\tan x+\tan y}$$

$$\left[2\sin x\cos x.\frac{1}{\cos^2 x}+2\sin y\cos y\frac{1}{\cos^2 y}\right]$$

$$= 2\frac{(\tan x+\tan y)}{\tan x+\tan y} = 2$$

Solved Example 89 :

If $u = x^y$ show that $\dfrac{\partial^3 y}{\partial x^2 \partial y} = \dfrac{\partial^3 y}{\partial x \partial y \partial x}$

Solution :

$u = x^y$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 x}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = yx^{y-1}\log x + x^y \frac{1}{x}$$

$$= x^{y-1} (y \log x + 1)$$

$$\frac{\partial^2 y}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = x^{y-1} + yx^{y-1}\log x$$

$$= x^{y-1} (y \log x + 1)$$

$$\frac{\partial^3 y}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 x}{\partial x \partial y}\right) = \frac{\partial}{\partial x}[x^{y-1}(y\log x + 1)] \quad ...(1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^3 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left[x^{y-1}(y\log x + 1)\right] \quad ...(2)$$

$$\frac{\partial u}{\partial x} = y\, x^{y-1}$$

from (1) and (2) $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$

Euler's theorem on Homogenous functions

If u is a homogenous function of degree n in x and y then

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\, u}$$

Since u is a homogenous function of degree n in x and y it can be expressed as

$$u = x^n\, f$$

$$\therefore \quad \frac{\partial u}{\partial x} = n\, x^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = n\, x^n\, f\left(\frac{y}{x}\right) - x^{n-1}\, y f'\left(\frac{y}{x}\right) \quad ...(1)$$

Also $\dfrac{\partial u}{\partial y} = x^n\, f'\left(\dfrac{y}{x}\right).\dfrac{1}{x} = x^{n-1}\, f'\left(\dfrac{y}{x}\right)$

$$\therefore\ y\frac{\partial u}{\partial y} = x^n\, y\, f'\left(\frac{y}{x}\right) \quad ...(2)$$

Adding (1) and (2) we get

$$x\frac{\partial u}{\partial x} + y\, \frac{\partial u}{\partial y} = n\, x^n\, f\left(\frac{y}{x}\right) = nu$$

> **<u>Note :</u> Euler's theorem can be extended to a homogenous function of any number of variables. Thus if u is a homogenous function of degree n in x, y and z then**
>
> $$\mathbf{x\frac{\partial u}{\partial x} + y\, \frac{\partial u}{\partial y} + z\, \frac{\partial u}{\partial z} = nu}$$

Solved Example 90 :

If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0$

Solution :

$$f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x / y}{x^2 + y^2}$$

$$f(tx, ty) = \frac{1}{t^2x^2} + \frac{1}{t^2xy} + \frac{\log\frac{tx}{ty}}{t^2x^2 + t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{\log\frac{x}{y}}{x^2 + y^2}\right]$$

$$= t^{-2} f(x, y)$$

f (x, y) is a homogenous function of degree –2 in x and y

By Euler's theorem, we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2 f(x, y)$$

Solved Example 91 :

If $u = \frac{x^2y^2}{x + y}$ show that

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}$$

Solution :

$u = \frac{x^2y^2}{x + y}$ is a homogenous function of degree 3 in x, y

$\therefore$ By Euler's theorem we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$$

Differentiating (1) partially with respect to x

$$x\frac{\partial^2 u}{\partial x^2} + 1.\frac{\partial u}{\partial x} + y\frac{\partial^2 y}{\partial x \partial y} = 3\frac{\partial u}{\partial x}$$

or $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}$

Composite functions

(i) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$.

(ii) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

then z is called a composite function of (2 variables) u & v so that we can find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$

Total derivative of composite functions

If u is a composite function of t, defined by the relations

$u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Note : 1. **If u = f (x, y, z) and x, y, z are functions of t then u is a composite function of t and**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

2. **If z = f (x, y) and x, y are functions of u and v, then**

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

3. **If u = f (x, y) where y = ϕ(x)**

then since x = ψ(x), u is a composite function of x

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$\frac{du}{dx}$ is called the total differential co–efficient of u, to distinguish it from its partial derivative $\frac{\partial u}{\partial x}$.

Solved Example 92 :

Find $\frac{du}{dt}$ when u = x when $u = xy^2 + x^2 y$, $x = at^2$, $y = 2at$.

Solution :

The given equations define u as a composite function of t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$= (y^2 + 2xy).\ 2at + (2xy + x^2)\ .\ 2a$

$= (4a^2t^2 + 2at^2\ .\ 2at)\ 2at$

$\quad + (2at^2 \cdot 2at + a^2\ t^4).2a$

$= 8a^3t^3 + 8a^3\ t^4 + 8a^3t^3 + 2a^3\ t^4$

$= 2a^3t^3(5t + 8)$

Also $u = xy^2 + x^2y = at^2\ .\ 4a^2t^2 + a^2\ t^4.\ 2at$

$= 4a^3t^4 + 2a^3\ t^5$

$$\frac{du}{dt} = 16a^3\ t^3 + 10a^3\ t^4 = 2a^3\ t^3\ (5t + 8)$$

Solved Example 93 :

If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$ show that

$$\frac{du}{dt} = \frac{3}{\sqrt{1 - t^2}}$$

Solution :

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)}}.3$$

$$+\frac{1}{\sqrt{1-(x-y)^2}}\ (-1)\ .\ 12\,t^{2}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^2)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)\,(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

Solved Example 94 :

If z is a function of x and y

where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$,

show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution :

Here z is a composite function of u and v

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}.\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}.\dfrac{\partial y}{\partial v}$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$

subtracting

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{-v})\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solved Example 95 :

If ϕ (x, y, z) = 0 show that

$$\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$$

Solution :

The given relation defines y as a function of x and z. Treating x as constant.

$$\left(\frac{\partial y}{\partial z}\right)_x = \frac{\frac{-\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$$

The given relation defines z as a function of x and y. Treating y as constant.

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\frac{-\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$$

Similarly $\left(\dfrac{\partial x}{\partial y}\right)_z = \dfrac{\frac{-\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$

Multiplying we get the desired result.

Solved Example 96 :

If u = f (y – z, z – x, x–y) prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution :

Here u = f (X, Y, Z)

where X = y –z, Y = z – x, Z = x–y

$\therefore$ u is a composite function of X, Y and Z

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

2.14 Taylor's Series and Maclaurin's Series

Taylor's series is a representation of a function as an infinite sum of terms that are calculated from the values of the functions derivatives at a single point.

$$\mathbf{f(x) = f(a) + \frac{f'(a)\,(x-a)}{1!} + f''(a)\,.\,\frac{(x-a)^2}{2!} + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}}$$

Note : Putting x = a + h, Taylor's series becomes

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f'(a) + ... + \frac{h^n}{n!}f^n(a) + ...}$$

If the Taylor's series is centered at zero, that series is also called Maclaurin's series.

If a = 0 in Taylor series of f(a + h), then the series becomes

$$\mathbf{f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + ... + \frac{h^n}{n!}f^n(0) + ...}$$

This is called as Maclaurin's series about a = 0.

Taylor's series expansion of some functions

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
- $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = 2.7182....$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

- $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+....\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}x^{2n}$
- $\tan x = x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+....$

Taylor's series in two variables

Taylor's series of a function in two variables, f(x, y) about (a, b) is

$$\mathbf{f(x, y) = f(a, b) + (x - a)\, f_x (a, b) + (y - b)\, f_y (a, b)}$$
$$\mathbf{+\ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)\,f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+....}$$

Solved Example 97 :

Expand cos z in a Taylor's series about $z = \pi/4$

Solution :

$f(z) = \cos z$
$f'(z) = -\sin z$
$f''(z) = -\cos z$
$f'''(z) = \sin z$

$f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$f'\left(\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$

$f''\left(\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$

$f'''\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

By using Taylor's series

$$f(z) = f\left(\frac{\pi}{4}\right)+\left(z-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right)$$
$$+\frac{(z-\pi/4)^2}{2!}f''\left(\frac{\pi}{4}\right)+\ldots$$
$$= \frac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2\ldots\right]$$

Solved Example 98 :

Obtain the Taylor's series for sin x.

Solution :

$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin z$
$f'''(x) = -\cos z$

By Maclaurin's series

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)$$
$$+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\ldots$$

Solved Example 99 :

Obtain the Taylor's series for cos hx.

Solution :

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$
$$e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$$
$$\cosh x = \frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+....$$

Solved Example 100 :

Find the Taylor's series expansion for the function $f(x) = 1 - 2x + 2x^2 + 3x^3$ about the point x = 2.

Solution :

$f(x) = 1 - 2x + 2x^2 + 3x^3$ $\quad f(2) = 29$

$f'(x) = -2 + 4x + 9x^2$ $\quad f'(2) = 42$

$f''(x) = 4 + 18x$ $\quad f''(2) = 40$

$f'''(x) = 18$

$$f(x) = f(2) + hf'(2) + \frac{h^2}{2!}f''(2) + \frac{h^3}{3!}f'''(2)$$

$$= 29 + h \cdot 42 + \frac{h^2}{2}(40) + \frac{h^3}{6}(18)$$

$$f(x) = 29 + 42h + 20h^2 + 3h^3$$

where h = x – 2

2.15 Fourier Series

Periodic function :

A function f(x) is said to be periodic if f(x + T) = f(x) for all values of x where T is the same positive number.

For Example:

1. sin x = sin (x + 2π)

   = sin (x + 4π) and so on.

   sin x is periodic in x with period 2π.

2. The period of tan x is π.

Fourier series (General form)

Let f(x) be defined in (–L, L) then Fourier series is given by

$$\mathbf{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Similarly if f(x) is defined in (0, 2L) then Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx$$

Odd and Even Functions :

Let f(x) be defined in (–C, C)

1. If f(–x) = f(x), for all x in (–C, C) then f(x) is even function.
2. If f(–x) = –f(x), for all x in (–C, C) then f(x) is an odd function.

Note : **Unless '0' is midpoint of the given interval, we cannot talk of the function being even or odd.**

For Example: f(x) = x^2 in (0, 3) is neither even nor odd, since if x $\in$ (0, 3) then $-x \notin$ (0, 3) and hence f(–x) is not defined.

But if, f(x) = x^2 in (–3, 3) then f(x) is even function.

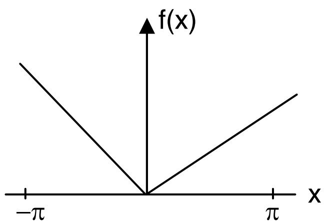
Graphical representation of even and odd function :

i) If f(x) is even, then its graph is symmetrical about f(x) axis

For Example: If f(x) = – x, $\quad -\pi < x < 0$

$\qquad\qquad\qquad\qquad$ = x $\quad 0 < x < \pi$

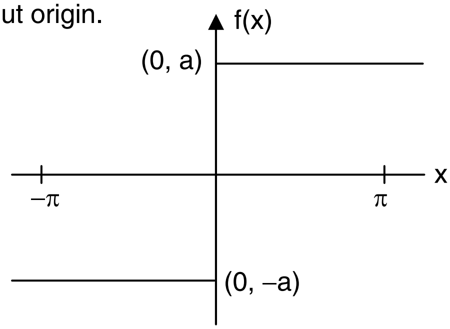
then,



ii) If f(x) is odd then its graph is symmetrical about origin.

For Example: if $f(x) = -a$, if $-\pi < x < 0$

$= a$, if $0 < x < \pi$



Fourier expansion of an even function

Let f(x) be defined in $(-\pi, \pi)$ and let f(x) be even function. Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$ f(x) is even

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$ [$\because$ f(x) and cos nx are even functions]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ [f(x) sin nx is odd]

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ [Half range cosine series in $(0, \pi)$]

Fourier expansion of an odd function

Let f(x) be an odd function in $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ f(x) is odd function

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0$ f(x) cos nx is odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx$ {Half range sine series in $(0, \pi)$}

Solved Example 101 :

Find the Fourier series representing

f(x) = x, 0 < x < 2π and draw its graph from

x = –4π to x = 4π.

Solution :

$$f(x) = x = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\frac{4\pi^2}{2} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nxdx$$

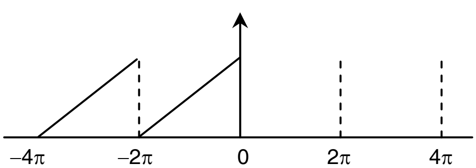
$$= \frac{1}{\pi}\left[\left[\frac{x\sin nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{n}\sin nxdx\right]$$

$$= \frac{1}{n\pi}\left[x\sin nx + \frac{\cos nx}{n}\right]_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nxdx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\times\frac{-2\pi}{n} = \frac{-2}{n}$$



Hence $x = \pi - 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]$

Solved Example 102 :

Find Fourier Series for f(x) = x^2 in $(-\ell, \ell)$ and show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$$

Solution :

We have f(x) = x^2 is an even function in $(-\ell, \ell)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

i) $a_0 = \frac{1}{\ell}\int_0^{\ell} x^2 dx = \frac{\ell^2}{3}$

ii) $a_n = \frac{2}{\ell}\int_0^{\ell} x^2 \cos\frac{n\pi x}{\ell}dx$

Integrating by parts

$$= \frac{2}{\ell}\left[x^2\left(\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right)\right.$$

$$\left. -2x\left(\frac{-\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}\right) + 2\left(\frac{-\ell^3}{n^3\pi^3}\sin\frac{n\pi x}{\ell}\right)\right]_0^{\ell}$$

$$= \frac{2}{\ell}\left[\frac{2\ell^3}{n^2\pi^2}(-1)^n\right] = \frac{4\ell^2}{n^2\pi^2}(-1)^n$$

$$\therefore \quad f(x) = \frac{\ell^2}{3} + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{\ell}\right) \ldots(1)$$

putting $x = 0$ in equation (1) we get

$$0 = \frac{\ell^2}{3} + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

Solved Example 103 :

Find a Fourier series to represent

$f(x) = \pi - x$ for $0 < x < 2\pi$.

Solution :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)dx$$

$$= \frac{1}{\pi}\left[[\pi x]_0^{2\pi} - \left(\frac{x^2}{2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[2\pi^2 - 2\pi^2\right] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\cos nx dx$$

$$= \frac{1}{\pi}\left[\pi\frac{\sin nx}{n} - \frac{x\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

Integrating by parts.

$$= \frac{1}{\pi}\times 0 = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos nx}{n} + \frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi\cos 2n\pi}{n}\right) = \frac{2}{n}$$

Solved Example 104 :

Find a Fourier expansion for

$f(x) = x^2$ for $-\pi < x < \pi$

Solution :

x^2 is an even function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{3\pi}\pi^3 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx dx$$

The indefinite integral

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \int\frac{\sin nx}{n}2x dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x}{n}\frac{\cos nx}{n} - \int\frac{\cos nx}{n^2}\cdot 2dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]$$

$$\therefore \quad a_n = \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{4}{n^2\pi}[\pi\cos n\pi] = \frac{4}{n^2}\cos n\pi$$

$b_n = 0$

$$a_1 = \frac{-4}{1^2},\ a_2 = \frac{4}{2^2},\ a_3 = \frac{-4}{3^2}\ \text{etc}$$

Hence

$$x^2 = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \ldots$$

Solved Example 105:

Find the Fourier series for f(x) = sin h ax in $(-\pi, \pi)$

Solution :

Clearly f(x) is an odd function of x

$f(-x) = \sin h\, a(-x) = -\sin h\, ax = -f(x)$

Let, $\sin h\, ax = \sum_{n=1}^{\infty} b_n \sin nx$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} \sinh ax \cdot \sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{e^{ax}-e^{-ax}}{2}\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} e^{ax}\sin nx\, dx - \int_0^{\pi} e^{-ax}\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{a^2+n^2}e^{ax}(a\sin nx - n\cos nx)\right.$$

$$\left. - \frac{1}{a^2+n^2}e^{-ax}(-a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}(-n)(-1)^n + ne^{-a\pi}(-1)^n\right]$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right)$$

$$= \frac{-2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \quad \left(\because \sinh a\pi = \frac{e^{a\pi}-e^{-a\pi}}{2}\right)$$

$$= \frac{2n(-1)^{n-1}\sinh a\pi}{\pi(a^2+n^2)}$$

$$\left[\therefore -(-1)^n = (-1)(-1)^n = (-1)^{n+1} = (-1)^{n-1}\right]$$

$$\therefore \quad f(x) = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n-1}\sin nx}{a^2+n^2}$$

Solved Example 106 :

Find the Fourier series for

$f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in $-\pi < x < \pi$ and show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} .... = \frac{\pi^2}{12}$

Solution :

Note that f(x) is an even function of x

$$\because \quad f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(z)$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{\pi^2}{4}\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{12}x - \frac{x^3}{12}\right]_0^{\pi} = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx dx$$

Integrating by parts

$$= \frac{2}{\pi}\left[\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right)\right.$$

$$\left. - \left(\frac{-x}{2}\right)\left(\frac{-\cos nx}{n^2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2n^2}\cos n\pi\right] = \frac{-(-1)^n}{n^2} = \frac{(-1)^{n-1}}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} - \frac{x^2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\cos nx$$

Put x = 0 we get

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right]$$

List of Formulae

- **Limits**

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \ldots x \text{ in radians}$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1 \quad \ldots x \text{ in radians}$$

$$\lim_{x\to 0}\cos x = 1$$

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \ell n\, a \quad (a > 0)$$

$$\lim_{x\to 0}\frac{e^x - 1}{x} = 1$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1$$

- **Continuity**

The function f is said to be continuous at $x = x_0$ if

i) $\lim_{x\to x_0} f(x) = \ell$ exists and

ii) $\lim_{x\to x_0} f(x) = f(x_0)$

- **Derivatives**

$$\frac{d}{dx}k(\text{constant}) = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\log x = \frac{1}{x}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \log_e a$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\cos^{-1}\frac{x}{a} = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2}$$

$$\frac{d}{dx}\cot^{-1}\frac{x}{a} = \frac{-a}{a^2 + x^2}$$

$$\frac{d}{dx}\sec^{-1}\frac{x}{a} = \frac{a}{x\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\operatorname{cosec}^{-1}\frac{x}{a} = \frac{-a}{x\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\cos hx = \sin hx$$

$$\frac{d}{dx}\sin hx = \cos hx$$

$$\frac{d}{dx}\sin h^{-1}x = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{x^2-1}$$

- **Integration**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c\,,\ (n \neq -1)$$

$$\int \frac{1}{x}dx = \log x + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\log_e a} + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \sec^2 x\, dx = \tan x + c$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\int \cot x dx = \log \sin x + c$$

$$\int \tan x dx = \log \sec x + c$$

$$\int \sec x dx = \log(\sec x + \tan x) + c$$

$$\int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = -\cos^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{-1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$$

$$\int \sinh x dx = \cos h\, x + c$$

$$\int \cosh x dx = \sin h\, x + c$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \sin h^{-1} x + c$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cos h^{-1} x + c$$

$$\int \frac{dx}{x^2-1} = \tan h^{-1} x + c$$

$$\int \tan x dx = -\log(\cos x) + c$$

$$\int \sec x dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$$

$$\int \operatorname{cosec} x\, dx = \log\left(\tan\frac{x}{2}\right) + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} \text{ or } -\cos^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

or $\log(x + \sqrt{x^2+a^2}) + c$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

or $\log(x + \sqrt{x^2-a^2}) + c$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

or $\frac{-1}{a}\cot^{-1}\left(\frac{x}{a}\right)+c$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right)+c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}+c$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}+c$$

Integration by parts

$$\int u\,v\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

Of the two functions the first function u is selected based on the following preference order

1. Inverse function
2. Logarithmic function
3. Algebraic function
4. Trigonometric function
5. Exponential function

Some Important Substitutions In Integration

Rule 1: If the numerator is equal to or a multiple of the derivative of the denominator, then put the denominator equal to some other variable say z. It is then reduced to standard form after simplification.

Rule 2: If there are two factors in the integrand one of which is of the type ϕ [f(x)] and other of the type f ′(x) then put f(x) = z.

Rule 3: If the integrand is of the type $\sqrt{a^2-x^2}$, then put $x = a\sin\theta$ or $x = a\cos\theta$

Rule 4: If the integrand is of the type $x^2 + a^2$ or some power of it then put $x = a\tan\theta$ or $x = a\cot\theta$

Rule 5: If the integrand is of the type $\sqrt{x^2-a^2}$, then put $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$

Rule 6: If the integrand is of the type $\sqrt{\frac{a-x}{a+x}}$, then put $x = a\cos 2\theta$ and if it is of the type $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$, then put $x^2 = a^2\cos 2\theta$

Rule 7: The presence of $\sqrt{a+x}$ suggests the substitution of $a + x = t^2$ or $x = a\tan^2\theta$ or $x = a\cos 2\theta$.

Rule 8: The presence of $\sqrt{2ax - x^2}$ suggests the substitution

$x = a\,(1 - \cos\theta)$

- **Important Trigonometric Identities**

$\sin^2 A + \cos^2 A = 1$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$

$\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$

$2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$2\cos A \sin B = \sin(A + B) - \sin(A - B)$

$2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$2\sin A \sin B = \cos(A - B) - \cos(A + B)$

$$2\sin\frac{C + D}{2}\cos\frac{C - D}{2} = \sin C + \sin D$$

$$2\cos\frac{C + D}{2}\sin\frac{C - D}{2} = \sin C - \sin D$$

$$2\cos\frac{C + D}{2}\cos\frac{C - D}{2} = \cos C + \cos D$$

$$2\sin\frac{C + D}{2}\sin\frac{D - C}{2} = \cos C - \cos D$$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

$$= 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - 4\tan^3 A}{1 - 3\tan^2 A}$$

- **Definite Integration**

$$\int_a^b f(x)dx = \left|\phi(x)\right|_{x=a}^{x=b} = \phi(b) - \phi(a)$$

Properties of definite integral

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx$$

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \quad \ldots f(-x) = f(x)$$

$$= 0 \quad \ldots f(-x) = -f(x)$$

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \quad \ldots f(2a - x) = f(x)$$

$$= 0 \quad \ldots f(2a - x) = -f(x)$$

- **Euler's theorem on Homogenous functions**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$$

- **Total derivative of composite functions**

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

- **Taylor's series**

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!}$$

$$+ f''(a).\frac{(x-a)^2}{2!} + ...$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f(x, y) = f(a, b) + (x-a) f_x(a, b)$$

$$+ (y-b) f_y(a, b)$$

$$+ \frac{1}{2!}\left\{\begin{array}{l} (x-a)^2 f_{xx}(a,b) \\ + 2(x-a)(y-b) f_{xy}(a,b) \\ + (y-b)^2 f_{yy}(a,b) \end{array}\right\} + ...$$

- **Maclaurin's Series**

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + ...$$

$$+ \frac{h^n}{n!} f^n(0) + ...$$

- **Fourier Series**

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

❑ ❑ ❑ ❑ ❑

Assignment – 1

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. $\underset{x \to 0}{\text{Lt}} \dfrac{x - \sin x}{1 - \cos x}$ is **[ME – 2014]**

(A) 0 (B) 1
(C) 3 (D) not defined

2. If a function is continuous at a point,
(A) the limit of the function may not exist at the point
(B) the function must be derivable at the point
(C) the limit of the function at the point tends to infinity
(D) the limit must exist at the point and the value of limit should be same as the value of the function at that point **[ME – 2014]**

3. The value of $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$ is **[EC – 2014]**

(A) $\ell n2$ (B) 1.0
(C) e (D) ∞

4. The series $\sum_{n=0}^{\infty} \dfrac{1}{n!}$ converges to **[ME – 2014]**

(A) 2 ln 2 (B) $\sqrt{2}$
(C) 2 (D) e

5. The value of $\lim_{x \to 0} \dfrac{1 - \cos(x^2)}{2x^4}$ is **[ME – 2015]**

(A) 0 (B) $\dfrac{1}{2}$
(C) $\dfrac{1}{4}$ (D) undefined

6. If a continuous function f(x) does not have a root in the interval [a, b], then which one of the following statements is TRUE? **[EE – 2015]**

(A) $f(a) \cdot f(b) = 0$ (B) $f(a) \cdot f(b) < 0$
(C) $f(a) \cdot f(b) > 0$ (D) $f(a) / f(b) \le 0$

7. The double integral $\int_0^a \int_0^y f(x, y)\, dx\, dy$ is equivalent to **[IN – 2015]**

(A) $\int_0^x \int_0^y f(x, y)\, dx\, dy$
(B) $\int_0^a \int_x^y f(x, y)\, dx\, dy$
(C) $\int_0^a \int_x^a f(x, y)\, dy\, dx$
(D) $\int_0^a \int_0^a f(x, y)\, dx\, dy$

8. The value of $\sum\limits_{n=0}^{\infty} n\left(\dfrac{1}{2}\right)^n$ is __________. **[EC – 2015]**

9. $\underset{x \to 0}{\text{Lt}} \dfrac{\log_e(1 + 4x)}{e^{3x} - 1}$ is equal to **[ME – 2016]**

(A) 0 (B) $\dfrac{1}{12}$
(C) $\dfrac{4}{3}$ (D) 1

10. The value of $\lim_{x\to 0} \dfrac{x^3 - \sin(x)}{x}$ is **[ME – 2017]**

(A) 0 (B) 3
(C) 1 (D) –1

Q 11 to Q 20 carry two marks each

11. The minimum value of the function

$f(x) = \dfrac{1}{3}x(x^2 - 3)$ in the interval

$-100 \le x \le 100$ occurs at

x = ________. **[EC – 2017]**

12. The maximum value of

$f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval

$0 \le x \le 3$ is _____. **[EC – 2014]**

13. Consider a spatial curve in three-dimensional space given in parametric form by **[ME – 2015]**

$x(t) = \cos t$, $y(t) = \sin t$,

$z(t) = \dfrac{2}{\pi}t$, $0 \le t \le \dfrac{\pi}{2}$.

The length of the curve is __________

14. $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}dx$ = ______. **[CS – 2015]**

15. The region specified by

$\{(\rho, \varphi, z): 3 \le \rho \le 5, \dfrac{\pi}{8} \le \varphi \le \dfrac{\pi}{4},$

$3 \le z \le 4.5\}$ in cylindrical coordinates has volume of _______ **[EC – 2016]**

16. The area between the parabola $x^2 = 8y$ and the straight line y = 8 is _______.

[CE – 2016]

17. A political party orders an arch for the entrance to the ground in which the annual convention is being held. The profile of the arch follows the equation $y = 2x - 0.1x^2$, where y is the height of the arch in meters. The maximum possible height of the arch is

[CS, ME-2012]

(A) 8 meters (B) 10 meters
(C) 12 meters (D) 14 meters

18. Consider the function f (x) = |x| in the interval $-1 \le x \le 1$. At the point x = 0, f (x) is **[ME – 2012]**

(A) Continuous and differentiable
(B) Non-continuous and differentiable
(C) Continuous and non-differentiable
(D) Neither continuous nor differentiable

19. The maximum value of

$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval

[1, 6] is **[EE – 2012]**

(A) 21 (B) 25
(C) 41 (D) 46

20. A parametric curve defined by $x = \cos\left(\dfrac{\pi u}{2}\right)$, $y = \sin\left(\dfrac{\pi u}{2}\right)$ in the range $0 \le u \le 1$ is rotated about the X-axis by 360 degrees. Area of the surface generated is **[ME – 2017]**

(A) $\dfrac{\pi}{2}$ (B) π
(C) 2π (D) 4π

❑ ❑ ❑ ❑ ❑

Assignment – 2

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. $\underset{x\to 0}{\text{Lt}}\left(\dfrac{e^{2x}-1}{\sin(4x)}\right)$ is equal to **[ME – 2014]**

(A) 0 (B) 0.5 (C) 1 (D) 2

2. The value of the integral

$$\int_0^2 \frac{(x-1)^2 \sin(x-1)}{(x-1)^2+\cos(x-1)}\,dx \text{ is}$$

(A) 3 (B) 0 **[ME – 2014]**

(C) –1 (D) –2

3. The maximum value of the function f(x) = ln(1 + x) – x (where x > –1) occurs at x = _________. **[EC – 2014]**

4. Let $f(x) = x\,e^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

(A) e^{-1} (B) e **[EE – 2014]**

(C) $1 - e^{-1}$ (D) $1 + e^{-1}$

5. At x = 0, the function f(x) = |x| has

(A) a minimum **[ME – 2015]**

(B) a maximum

(C) a point of inflexion

(D) neither a maximum nor minimum

6. While minimizing the function f(x), necessary and sufficient conditions for a point, x_0 to be a minima are:

[CE – 2015]

(A) $f'(x_0) > 0$ and $f''(x_0) = 0$

(B) $f'(x_0) < 0$ and $f''(x_0) = 0$

(C) $f'(x_0) = 0$ and $f''(x_0) < 0$

(D) $f'(x_0) = 0$ and $f''(x_0) > 0$

7. A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval [–1, 1]. The value of x, in the open interval (–1, 1) for which the mean value theorem is satisfied, is **[EC – 2015]**

(A) –1/2 (B) –1/3 (C) 1/3 (D) 1/2

8. Consider the function $f(x) = 2x^3 - 3x^2$ in the domain [–1, 2]. The global minimum of f(x) is ______ **[ME – 2016]**

9. Given the following statements about a function f: $\mathbb{R} \to \mathbb{R}$, select the right option: **[EC – 2016]**

P: If f(x) is continuous at $x = x_0$, then it is also differentiable at $x = x_0$.

Q: If f(x) is continuous at $x = x_0$, then it may not be differentiable at $x = x_0$.

R: If f(x) is differentiable at $x = x_0$, then it is also continuous at $x = x_0$.

(A) P is true, Q is false, R is false

(B) P is false, Q is true, R is true

(C) P is false, Q is true, R is false

(D) P is true, Q is false, R is true

10. The integral $\int_0^1 \dfrac{dx}{\sqrt{(1-x)}}$ is equal to ____

[EC – 2016]

Q 11 to Q 20 carry two marks each

11. The value of the integral is

$\int_0^2 \int_0^x e^{x+y} dy\, dx$ is **[ME – 2014]**

(A) $\frac{1}{2}(e-1)$ (B) $\frac{1}{2}(e^2-1)^2$

(C) $\frac{1}{2}(e^2-e)$ (D) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$

12. The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval [–3, 3] is **[EE – 2014]**

(A) 20 (B) 28 (C) 16 (D) 32

13. Consider an ant crawling along the curve $(x-2)^2 + y^2 = 4$, where x and y are in meters. The ant starts at the point (4, 0) and moves counter-clockwise with a speed of 1.57 meters per second. The time taken by the ant to reach the point (2, 2) is (in seconds) ____ **[ME – 2015]**

14. $\lim_{x\to\infty} \sqrt{x^2+x-1} - x$ is **[ME – 2016]**

(A) 0 (B) ∞ (C) 1/2 (D) $-\infty$

15. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is **[CE – 2016]**

(A) $\frac{59}{6}$ (B) $\frac{9}{2}$ (C) $\frac{10}{3}$ (D) $\frac{7}{6}$

16. Let $f : [-1, 1] \to \mathbb{R}$,

where $f(x) = 2x^3 - x^4 - 10$. The minimum value of f(x) is _____. **[IN – 2016]**

17. At x = 0, the function $f(x) = x^3 + 1$ has

(A) A maximum value **[ME – 2012]**

(B) A minimum value

(C) A singularity

(D) A point of inflection

18. $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$ is **[ME – 2012]**

(A) 1/4 (B) 1/2 (C) 1 (D) 2

19. Which one of the following functions is continuous at x = 3? **[CS – 2013]**

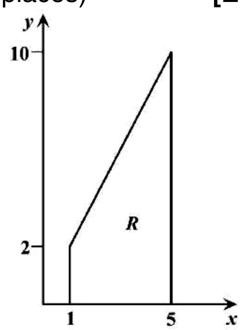
(A) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(B) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8-x & \text{if } x \neq 3 \end{cases}$

(C) $f(x) = \begin{cases} x+3, & \text{if } x \leq 3 \\ x-4, & \text{if } x > 3 \end{cases}$

(D) $f(x) = \frac{1}{x^3-27}$, if $x \neq 3$

20. Let $I = c\iint_R xy^2\, dxdy$, where R is the region shown in the figure and $c = 6 \times 10^{-4}$. The value of I equals _______. (Give the answer up to two decimal places) **[EE – 2017]**



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Assignment – 3

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. Consider a function f(x,y,z) given by

$$f(x,y,z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$$

The partial derivative of this function with respect to x at the point, x = 2, y = 1 and z = 3 is _____. **[EE – 2017]**

2. For $0 \le t < \infty$, the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

(A) $t = \log_e 4$ (B) $t = \log_e 2$
(C) $t = 0$ (D) $t = \log_e 8$

[EC – 2014]

3. If $z = xy \ln(xy)$, then **[EC – 2014]**

(A) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$

(B) $y\dfrac{\partial z}{\partial x} = x\dfrac{\partial z}{\partial y}$

(C) $x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$

(D) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 0$

4. $\lim_{x\to\infty}\left(\dfrac{x+\sin x}{x}\right)$ equals to **[CE – 2014]**

(A) $-\infty$ (B) 0
(C) 1 (D) ∞

5. The value of $\lim_{x\to 0}\left(\dfrac{-\sin x}{2\sin x + x\cos x}\right)$ is ___________ **[ME – 2015]**

6. Given, $i = \sqrt{-1}$, the value of the definite integral, $I = \int_0^{\pi/2}\dfrac{\cos x + i\sin x}{\cos x - i\sin x}dx$ is: **[CE – 2015]**

(A) 1 (B) –1
(C) i (D) –i

7. The contour on the x-y plane, where the partial derivative of $x^2 + y^2$ with respect to y is equal to the partial derivative of 6y + 4x with respect to x, is **[EC – 2015]**

(A) y = 2 (B) x = 2
(C) x + y = 4 (D) x – y = 0

8. The values of x for which the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2 + 3x - 4}$ is NOT continuous are **[ME – 2016]**

(A) 4 and –1 (B) 4 and 1
(C) –4 and 1 (D) –4 and –1

9. As x varies from –1 to + 3, which one of the following describes the behaviour of the function $f(x) = x^3 - 3x^2 + 1$? **[EC – 2016]**

(A) f(x) increases monotonically.
(B) f(x) increases, then decreases and increases again.

(C) f(x) decreases, then increases and decreases again.

(D) f(x) increases and then decreases.

10. The maximum value attained by the function $f(x) = x(x - 1)(x - 2)$ in the interval [1, 2] is _____. **[EE – 2016]**

Q 11 to Q 20 carry two marks each

11. The Taylor series expansion of $3 \sin x + 2 \cos x$ is **[EC – 2014]**

(A) $2 + 3x - x^2 - \frac{x^3}{2} + ...$

(B) $2 - 3x + x^2 - \frac{x^3}{2} + ...$

(C) $2 + 3x + x^2 + \frac{x^3}{2} + ...$

(D) $2 - 3x - x^2 + \frac{x^3}{2} + ...$

12. The expression $\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha}$ is equal to **[CE – 2014]**

(A) log x (B) 0

(C) x log x (D) ∞

13. The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2 + 4y^2 = 1$ is _________.

[EC – 2015]

14. The integral $\frac{1}{2\pi}\iint_D (x + y + 10)\,dx\,dy$, where D denotes the disc: $x^2 + y^2 \le 4$, evaluates to ________ **[EC – 2014]**

15. The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is

[CE – 2016]

(A) 0° (B) 30°

(C) 45° (D) 90°

16. The cost function for a product in a firm is given by $5q^2$, where q is the amount of production. The firm can sell the product at a market price of Rs. 50 per unit. The number of units to be produced by the firm such that the profit is maximized is **[CS, ME – 2012]**

(A) 5 (B) 10

(C) 15 (D) 25

17. The area enclosed between the straight line y = x and the parabola $y = x^2$ in the x – y plane is **[ME – 2012]**

(A) 1/6 (B) 1/4

(C) 1/3 (D) 1/2

18. Consider the function f(x) = sin(x) in the interval $x \in [\pi/4, 7\pi/4]$. The number and location (s) of the local minima of this function are **[CS – 2012]**

(A) One, at $\pi/2$

(B) One, at $3\pi/2$

(C) Two, at $\pi/2$ and $3\pi/2$

(D) Two, at $\pi/4$ and $3\pi/2$

19. The value of the definite integral

$\int_1^e \sqrt{x}\,\ln(x)dx$ is **[ME – 2013]**

(A) $\frac{4}{9}\sqrt{e^3}+\frac{2}{9}$ (B) $\frac{2}{9}\sqrt{e^3}-\frac{4}{9}$

(C) $\frac{2}{9}\sqrt{e^3}+\frac{4}{9}$ (D) $\frac{4}{9}\sqrt{e^3}-\frac{2}{9}$

20. Consider the following definite integral:

$$I = \int_0^1 \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx$$ **[CE – 2017]**

The value of the integral is

(A) $\frac{\pi^2}{24}$ (B) $\frac{\pi^3}{12}$

(C) $\frac{\pi^3}{48}$ (D) $\frac{\pi^3}{64}$

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Assignment – 4

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ equals to

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{1}{4}$ (D) $\dfrac{1}{5}$

2. What is $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ equal to?

(A) θ (B) $\sin\theta$

(C) 0 (D) 1

3. The value of $\lim\limits_{x\to 8}\dfrac{x^{1/3}-2}{(x-8)}$ is

(A) $\dfrac{1}{16}$ (B) $\dfrac{1}{12}$

(C) $\dfrac{1}{8}$ (D) $\dfrac{1}{4}$

4. $\lim\limits_{x\to\infty}\dfrac{ax+b}{cx}$ is

(A) 0 (B) a

(C) $\dfrac{a}{b}$ (D) $\dfrac{a}{c}$

5. The discontinuity of the function $f(x)\ \dfrac{e^x-1}{x}$ at the point x = 0 is known as

(A) first kind of discontinuity

(B) second kind of discontinuity

(C) mixed discontinuity

(D) removable discontinuity

6. $\lim\limits_{x\to 0}\dfrac{e^x-\left(1+x+\dfrac{x^2}{2}\right)}{x^3}=$

(A) 1/2 (B) 1/6

(C) 1/3 (D) 1

Q7 to Q18 carry two marks each

7. Which of the following is incorrect statement ?

(A) A stationary value is always an extreme value

(B) A maximum value is always a stationary value

(C) A minimum value is always a stationary value

(D) A minimum value may be greater than maximum value

8. Which of the following is correct statement ?

(A) A stationary point on a graph is any point at which $\dfrac{dy}{dx}=0$

(B) A stationary point on a graph is any point at which $\dfrac{dy}{dx}=\infty$

(C) A stationary point on a graph is any point at which there are gaps or breaks

(D) None of these

9. First rule for maximum and minimum values of y = f(x) are as follows

Get $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Solve $\frac{dy}{dx} = 0$ and consider its roots. These are the value of x which make $\frac{dy}{dx} = 0$. For each of these values of x, calculate the corresponding value of y, and examine the sign of $\frac{d^2y}{dx^2}$. Now pick the correct statement from the following alternatives.

(A) If the sign is +, the corresponding value of y is a maximum
(B) If the sign is –, the corresponding value of y is a minimum
(C) If the sign is –, the corresponding value of y is a maximum
(D) None of these

10. Let $f(x) = x^3 - 6x^2 + 24x + 4$. Then f(x) has

(A) a maximum value at x = 2
(B) a maximum value at x = 5
(C) a minimum value at x = 5
(D) neither maximum nor minimum at any point

11. Let a number 30 be divided into two parts such that cube of one multiplied with the other is maximum, then the parts are

(A) 12.4, 17.6 (B) 13, 17
(C) 10.5, 19.5 (D) 22.5, 7.5

12. $f(x) = (x-2)^3$ is

(A) maximum at x = 2
(B) minimum at x = 2
(C) neither maximum nor minimum at x = 2
(D) none of these

13. The function sin x (1 + cos x) is maximum in the interval (0, π), when

(A) $x = \frac{\pi}{4}$ (B) $x = \frac{\pi}{2}$
(C) $x = \frac{\pi}{3}$ (D) $x = \frac{2\pi}{3}$

14. The absolute minimum of $f(x) = 4x^3 - 8x^2 + 1$ in [–1, 1] is at

(A) 1 (B) –3
(C) –1 (D) –8

15. A spherical snowball is melting at rate of 4π cubic centimeter per hour. How fast is the diameter changing when it is 20 centimeters ?

(A) 0.1 cm/hr (B) 0.02 cm/hr
(C) 0.04 cm /hr (D) 0.01 cm/hr

16. If a mothball evaporates at a rate proportional to its surface area $4\pi r^2$, then its radius

(A) decreases at constant rate
(B) decreases logarithmically
(C) decreases exponentially
(D) None of these

17. The absolute maximum of $f(x) = x^4 - 2x^3 - x^2 - 4x + 3$ on the interval [0, 4] is.

(A) 80 (B) 16

(C) 28 (D) None of these

18. The absolute minimum of

$f(x) = \dfrac{x^3}{(x+2)}$ on the interval [−1, 1] is at

(A) 0 (B) −1

(C) 1/3 (D) None of these

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Assignment – 5

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - \sqrt{x+1})$ equals to

(A) 0 (B) e
(C) 1 (D) ∞

2. It is given that

$f(x) = \dfrac{ax+b}{x+1}, \lim\limits_{x\to 0} f(x) = 2$ and

$\lim\limits_{x\to\infty} f(x) = 1$, then value of f (– 2) is

(A) 0 (B) 1
(C) e (D) ∞

3. The limiting value of

$\dfrac{1^2}{n^3} + \dfrac{2^2}{n^3} + \dfrac{3^2}{n^3} + \ldots\ldots\ldots\ldots + \dfrac{n^2}{n^3}$ as $n \to \infty$

is

(A) 1 (B) $\dfrac{1}{2}$
(C) $\dfrac{1}{3}$ (D) $\dfrac{1}{4}$

4. What is the value of $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{2n}$?

(A) 0 (B) e^{-2}
(C) $e^{-1/2}$ (D) 1

5. The value of $\lim\limits_{x\to(\pi/2)} \dfrac{1+\cos 2x}{(\pi-2x)^2}$ is

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{3}$
(C) $\dfrac{1}{2}$ (D) 0

6. The value of $\lim\limits_{\theta\to(\pi/2)} \dfrac{\cot\theta}{(\pi/2)-\theta}$ is

(A) 1 (B) 2
(C) 0 (D) $\dfrac{1}{2}$

Q7 to Q18 carry two marks each

7. If y = 7x – x^3 and x increase at rate of 4 units/sec how fast is slope of graph changing when x = 3 to

(A) –72 (B) –80
(C) –64 (D) 64

8. An object moves on the parabola $3y = x^2$ when x = 3, the x coordinate of the object is increasing at the rate of 1 foot/min. How fast is y coordinate increasing of the moment

(A) 1 (B) 2
(C) 3 (D) 4

9. A snowball is increasing in volume of the rate of 10 cm^3/h. How fast is the surface area growing at the moment where radius of snowball is 5 cm ?

(A) 4 (B) 2
(C) 12 (D) 16

10. The height h and radius r of cylinder of greatest volume that can be cut from a sphere of radius b is

(A) $h = \dfrac{2b}{\sqrt{3}}, r = b\sqrt{\dfrac{2}{3}}$

(B) $h = \frac{b}{\sqrt{3}}, r = b\sqrt{\frac{2}{3}}$

(C) $h = \frac{b}{2\sqrt{3}}, r = \frac{b}{\sqrt{6}}$

(D) $h = \frac{2b}{\sqrt{3}}, r = b\sqrt{\frac{1}{3}}$

11. A pipe with length 3m and radius of 10cm has an outer layer of ice that is melting at the rate of $2\pi cm^3$ /min. How fast is the thickness of ice decreasing when ice is 2cm thick?

(A) $\frac{1}{200}$ cm / min

(B) $\frac{1}{800}$ cm / min

(C) $\frac{1}{1800}$ cm / min

(D) $\frac{1}{3600}$ cm / min

12. The maximum area of any rectangle which may be inscribed in a circle of radius 1 is

(A) 1 (B) 2
(C) 3 (D) 4

13. A point on a curve is said to be an extremum if it is a local minimum or a local maximum. The number of distinct extrema for the curve

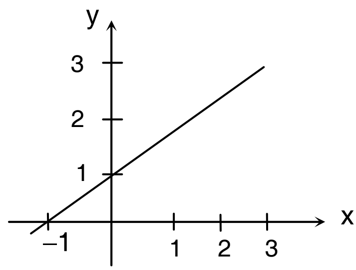
$3x^4 - 16x^3 + 24x^2 + 37$ is

(A) 0 (B) 1
(C) 2 (D) 3

14. $\int_0^{\pi/4} (1-\tan x)/(1+\tan x)\, dx$ evaluates to

(A) 0 (B) 1
(C) ln 2 (D) ½ ln 2

15. The following plot shows a function y which varies linearly with x. The value of the integral $I = \int_1^2 y\, dx$ is



(A) 1.0 (B) 2.5
(C) 4.0 (D) 5.0

16. The value of the quantity P, where

$P = \int_0^1 xe^x dx,$ is equal to

(A) 0 (B) 1
(C) e (D) 1/e

17. The sum of the infinite series,

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$is

(A) π (B) infinity
(C) 4 (D) $\frac{\pi^2}{4}$

18. The value of the integral $\int_0^{\infty}\int_0^{\infty} e^{-x^2} e^{-y^2}$ dx dy is

(A) $\sqrt{\frac{\pi}{2}}$ (B) $\sqrt{\pi}$ (C) π (D) $\frac{\pi}{4}$

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Assignment – 6

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. The discontinuity of the function

$f(x) = \dfrac{|x|}{x}, \quad x \neq 0$

$\quad = 0, \quad x = 0$

at the origin is known as

(A) Discontinuity of 1^{st} kind

(B) Discontinuity of 2^{nd} kind

(C) Mixed discontinuity

(D) None of these

2. The value of the coefficients 'a' such that the function f defined as

$f(x) = ax^2 - 2 \quad x \leq 1$

$\quad = -1 \quad x \geq 1$

continuous is

(A) a = 1 (B) a = 2

(C) a = 0 (D) a = 3

3. If f (x) = 1, if x < 3

= ax + b, if $3 \leq x < 5$

= 7 if $5 \leq x$

Determine the value of 'a' and 'b' so that f (x) is continuous.

(A) a = 4, b = – 7

(B) a = 3, b = – 8

(C) a = 1, b = – 9

(D) a = 2, b = 3

4. The maximum value of

$f(x) = 2|x - 1| + 3|x - 2|$

for all $x \in R$ is

(A) 2 (B) 4

(C) 1 (D) None of these

5. Pick out the correct statement

(A) logarithmic function is decreasing wherever it is defined

(B) logarithmic function is increasing whenever it is defined

(C) logarithmic function is neither increasing nor decreasing

(D) None of these

6. f (x) = log sin x. This function is

(A) increasing on $\left[0, \dfrac{\pi}{2}\right]$ and decreasing on $\left[\dfrac{\pi}{2}, \pi\right]$

(B) increasing on $\left[\dfrac{\pi}{2}, \pi\right]$ and decreasing on $\left[0, \dfrac{\pi}{2}\right]$

(C) increasing on $[0, 0]$ and decreasing on $[\pi, \pi]$

(D) increasing on $[\pi, 2\pi]$ and decreasing on $[0, 0]$

Q7 to Q18 carry two marks each

7. The series $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+.....$is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

8. The series 1 + 2 + 3 + …… is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

9. The series 1 – 2 + 3 – 4 + 5 – 6 + ... is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

10. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.......$is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

11. The series $\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+....$ is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

12. Which of the following to true?

(A) $1+\frac{1}{2^{1/3}}+\frac{1}{3^{1/3}}+\frac{1}{4^{1/3}}$ is convergent

(B) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ is convergent

(C) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}$ is convergent

(D) $1-\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+$ is divergent

13. The series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+.........$ is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

14. The series whose n^{th} term is

$\sqrt{n^3+1}-\sqrt{n^3}$ is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

15. The series whose n^{th} term is $\sin\frac{1}{n}$ is

(A) convergent (B) divergent

(C) oscillatory (D) None of these

16. The series

$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{n+1}{n^2}+........$is

(A) convergent (B) divergent

(C) oscillatory (D) none of these

17. The interval in which Lagrange's theorem is applicable for function f(x) = 1/x is

(A) [–3, 3] (B) [–2, 2]

(C) [2, 3] (D) [–1, 1]

18. For $f(x) = x^3$ suppose

$f(b) – f(a) = (b – a) f'(c)$ holds then c is

(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{a+b}{2}$

(C) $\sqrt{\frac{a^2+b^2+ab}{3}}$ (D) $\sqrt{\frac{a^2+b^2-ab}{3}}$

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Assignment – 7

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. Function f (x) = $e^{1/x}$; x > 0 is a

(A) increasing function

(B) decreasing function

(C) neither increasing nor decreasing

(D) none of these

2. f (x) = $x+\frac{1}{x}$ is

(A) maximum at x = 1

(B) maximum at x = 2

(C) maximum at x = – 1

(D) maximum at x = – 2

3. f (x) = $-x^2 + 2x + 1$ is

(A) maximum at x = 1

(B) maximum at x = 2

(C) maximum at x = – 1

(D) maximum at x = – 2

4. The maximum value of 5 cos θ + 3 cos $\left(\theta+\frac{\pi}{3}\right)$ + 3 is

(A) 5 (B) 10

(C) 11 (D) – 1

5. f (x) = $x^3 - 12x^2 + 45x + 11$ is

(A) minimum at x = 5

(B) minimum at x = – 5

(C) minimum at x = 4

(D) minimum at x = – 4

6. The maximum of f(x) = (x + 2)/(x – 1) is

(A) 4 (B) 1

(C) 2 (D) does not exist

Q7 to Q18 carry two marks each

7. Use Lagrange's mean value theorem for f(x) = e^x . Which of the following is true?

(A) $1 + x < e^x < 1 + xe^x$

(B) $1 + xe^x < e^x < 1 + x$

(C) $e^x < 1 + x\,e^x < 1 + x$

(D) none of these

8. $x^{1/X}$ is decreasing function if

(A) x < e (B) x > e

(C) x = e (D) x > 1/e

9. The Maclaurin series of e^x is

(A) $\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}$ (B) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

(C) $\sum_{n=1}^{\infty}\frac{x^n}{n!}$ (D) none of these

10. Taylor series for sin x about π/4 is

(A) $\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(x-\pi/4)^n}{n!}$

(B) $\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-\pi/4)^n}{n!}$

(C) $\frac{1}{\sqrt{2}}\left[\frac{1+(x-\pi/4)}{1!}-\frac{(x-\pi/4)^2}{2!}...\right]$

(D) none of these

11. The Maclaurin series for ln (1 – x) is

(A) $-x\frac{-x^2}{2}-\frac{x^3}{3}.....$

(B) $-x\frac{+x^2}{2}-\frac{x^3}{3}+.....$

(C) $x-\frac{-x^2}{2}+\frac{x^3}{3}.....$

(D) none of these

12. The Taylor series for 1/x about 1 is

(A) $\sum_{n=0}^{\infty}(-1)^{n+1}(x-1)^n$

(B) $\sum_{n=0}^{\infty}(-1)^{n}(x-1)^n$

(C) $\sum_{n=1}^{\infty}(-1)^{n+1}(x-1)^n$

(D) none of these

13. The first non zero terms of Maclaurin series for sec x is

(A) $1+\frac{1}{2}x^2+\frac{5x^4}{24}$

(B) $x+\frac{x^2}{3}+\frac{5^4}{16}x^3$

(C) $1-\frac{x^2}{2}+\frac{5x^4}{16}x^4$

(D) $x+\frac{x^2}{2}+\frac{5^4}{24}x^4$

14. If f(x) = $\sum_{n=0}^{x}2^n x^n$, then $f^{33}(0)$ is

(A) (33!) 3^{33} (B) (32!) 3^{33}

(C) (3!) 2^{32} (D) (33!) 2^{33}

15. If f(x) = $\tan^{-1}x$, then f^{99} (0) is

(A) 97! (B) –98!

(C) 99! (D) none of these

16. The n^{th} non zero term of Maclaurin series for ln (1 + x^2) is

(A) $\frac{(-1)^{n+1}x^{2n}}{n}$ (B) $(-1)^n\frac{x^{2n}}{n+1}$

(C) $(-1)^{n+1}\frac{x^{2n}}{n+1}$ (D) $(-1)^n\frac{x^{2n+1}}{n}$

17. If f(x) = $\frac{1}{x^2+x+1}$, then $f^{36}(0)$ is

(A) –36! (B) 36!

(C) 2^{36} (D) none of these

18. The first three nonzero terms of Maclaurin series for tan x are

(A) $x+\frac{x^3}{3}+\frac{2}{15}x^5+.....$

(B) $1+\frac{x^2}{2}+\frac{4}{15}x^3+.....$

(C) $1-\frac{x^2}{2}+\frac{2}{15}x^5+.....$

(D) none of these

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Assignment – 8

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. For what value of k will f(x) = $x - kx^{-1}$ have a relative maximum of x = –2

(A) –2 (B) –1
(C) –4 (D) –8

2. A wire 16 feet long is bent to form a rectangle. If the area of the rectangle is to be maximized, the dimensions of the rectangle should be

(A) 2, 6 (B) 5, 3
(C) 4, 4 (D) None of these

3. $\lim\limits_{x\to 2}\frac{|x-2|}{x-2}$ is equal to

(A) 1 (B) –1
(C) 0 (D) does not exit

4. If $\lim\limits_{x\to 1}\frac{f(x)-2}{f(x)+2}=0$ then $\lim\limits_{x\to 1}f(x)$ is equal to

(A) 1 (B) –1
(C) –2 (D) 2

5. Which of the following limit exist ?

(A) $\lim\limits_{x\to 0}e^{1/x}$ (B) $\lim\limits_{x\to 0}\frac{1}{1-e^{1/x}}$
(C) $\lim\limits_{x\to 0}\frac{1}{x}$ (D) $\lim\limits_{x\to 0}1/x^2$

6. Evaluate $\lim\limits_{x\to 0}e^{-n\log n}$

(A) 1 (B) 0
(C) ∞ (D) does not exist

Q7 to Q18 carry two marks each

7. The first three nonzero terms of Taylors series for $\ell n x$ around 2 is

(A) $\ell n2 + (x-2) - \frac{1}{8}(x-2)^2$

(B) $\ell n2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$

(C) $\ell n2 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2$

(D) $\ell n2 + (x-2) - \frac{1}{4}(x-2)^2$

8. The Maclaurin series for 2^x is

(A) $\sum\limits_{n=0}^{x}\frac{(\ell n2)^n}{(n+1)!}x^n$ (B) $\sum\limits_{n=0}^{x}\frac{(\ell n2)^n}{n!}x^{n+1}$

(C) $\sum\limits_{n=0}^{x}\frac{(\ell n2)^n}{n!}x^n$ (D) None of these

9. If f(x) = e^{x^2} then $f^{100}(0)$ is

(A) 100! (B) 2^{100}
(C) $\frac{100!}{50!}$ (D) $100! \times 50!$

10. If f(x) satisfies $f(0) = 2$,

$f'(0) = 1,\ f''(0) = 4, f'''(0) = 12$

and $f^n(0) = 0$ for $n > 3$

then f(x) is

(A) $x + 2x^3$

(B) $2 + 2x^2$

(C) $2 + x + x^2$

(D) $2 + x + 2x^2 + 2x^3$

11. The Taylor series for $f(x) = 2x^2 + 4x - 3$ about 1 is

(A) $3 + 8(x - 1) + 2(x - 1)^2$

(B) $2 + 4x + 2x^2$

(C) $2 + 4(x - 1) + 4(x - 1)^2$

(D) None of these

12. Evaluate $I = \int_0^{\pi} \frac{\sin 2kx}{\sin x} dx$

(A) 0 (B) 1/2

(C) -1 (D) $\pi/2$

13. Evaluate $I = \int_0^{\pi} \log(1 + \cos\theta) d\theta$

(A) $\frac{-\pi}{2}\log 2$ (B) $-\pi\log 2$

(C) $\pi\log 2$ (D) $\pi/4 \log 2$

14. Evaluate $I = \int_0^{\pi/2} \sin 2x \log\tan x\, dx$

(A) 0 (B) 1

(C) $\pi/2$ (D) $\pi/2 \log 2$

15. The area bounded by curve $x^2 = 4y$ and straight line $x = 4y - 2$ is

(A) 1/8 (B) 3/8

(C) 9/8 (D) 2

16. The area bounded by x axis, $y = 1 + \frac{8}{x^2}$ ordinate at $x = 2$ and $x = 4$ is

(A) 2 (B) 4

(C) 8 (D) 1

17. The area bounded by curve $y = 2x + x^2 - x^3$ the x axis and lines $x = -1$ and $x = 1$

(A) 1/2 (B) 1

(C) 3/2 (D) 2

18. Evaluate $I = \int \frac{x - \sin x}{1 - \cos x} dx$

(A) $-x + \cot\frac{x}{2} + C$ (B) $-x\cot\frac{x}{2} + C$

(C) $x\cot\frac{x}{2} + C$ (D) $x + \tan\frac{x}{2} + C$

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Assignment – 9

Duration : 45 Min. Max. Marks : 30

Q1 to Q6 carry one mark each

1. $\lim\limits_{x\to\infty}\left(\frac{x}{1+x}\right)^{2x}$ is

(A) e (B) $\frac{1}{e^2}$

(C) $\frac{1}{e}$ (D) 2e

2. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$ is equal to

(A) 0 (B) 1

(C) –1 (D) 0

3. $\lim\limits_{x\to 0}(\text{cosec}x)^{x}$ is equal to

(A) 1 (B) e

(C) r^{-1} (D) 0

4. $\lim\limits_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots\ldots+\frac{1}{n}\right]$

(A) 0 (B) 1

(C) ½ (D) ∞

5. The function f(x) = $\begin{cases} ax^2+b & x>2 \\ 2 & x=2 \\ 2ax-b & x<2 \end{cases}$

is continuous at x = 2 if

(A) a = 0, b = 1/2

(B) a = 1/2 , b = 0

(C) a = –1/2, b = 0

(D) b = –1/2, a = 0

6. The limiting value of

$\frac{1}{n}+\frac{2}{n}+\ldots\ldots+\frac{n}{n}$ as $n\to\infty$ is

(A) 2/3 (B) 1/2

(C) 1/3 (D) does not exist

Q7 to Q18 carry two marks each

7. $I = \int_0^{\pi/2}\cos^4 x\sin^3 dx$

(A) $\frac{1}{35}$ (B) $\frac{2}{35}$

(C) $\frac{3}{35}$ (D) $\frac{4}{35}$

8. Evaluate I = $\int\frac{1+x^2}{\sqrt{1-x^2}}dx$

(A) $\frac{3}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+C$

(B) $\frac{1}{2}x\sqrt{1-x^2}+C$

(C) $\sin^{-1}+\sqrt{1-x^2}+C$

(D) none of these

9. Evaluate I = $\int\frac{dx}{\cos x+\sin x}$

(A) $\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$

(B) $\log\cot\frac{x}{2}+C$

(C) log cot x + C

(D) log tan x + C

10. Evaluate $I = \int \frac{dx}{1+3\sin^2 x}$

(A) $\frac{1}{2}\tan^{-1}(\tan x)+C$

(B) $\tan^{-1}(\tan x) + C$

(C) $\frac{1}{2}\tan^{-1}(2\tan x)+C$

(D) None of these

11. Evaluate $I = \int \frac{d\theta}{\sin^4\theta+\cos^4\theta}$

(A) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right)$

(B) $\tan^{-1}\left(\frac{\tan^2\theta-1}{\tan\theta}\right)$

(C) $\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}\tan\theta}\right)$

(D) None of these

12. $I = \int \frac{\sin x}{\sin 3x}dx$

(A) $\log\left(\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right)+C$

(B) $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+C$

(C) $\frac{1}{\sqrt{3}}\log\left(\sqrt{3}+\tan x\right)+C$

(D) $\frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3}}{\sqrt{3}-\tan x}\right)+C$

13. $I = \int_0^{\pi/2} x^2 \sin x\, dx$

(A) $\frac{\pi+2}{2}$ (B) $2\pi - 2$

(C) $\pi + 2$ (D) $\pi - 2$

14. The value of $\lim_{x\to 0}\left[\frac{n\sin x}{x}\right]$, where [.] is the greatest integer function

(A) $n + 1$ (B) n

(C) $n - 1$ (D) None of these

15. Let $\overline{f} = (xy, -yz, z)$ and

$C : \left(\sin(2\pi t), t, t^2\right)$.

Then the value of the line integral

$\int_C \overline{f}\, d\overline{r}$ from (0, –1, 1) to (0, 1, 1) is

(A) –1 (B) 1

(C) 0 (D) None of these

16. If $I_N = \int_0^\infty e^{-x}x^{n-1}\log x\, dx$, then

(A) $I_{n+2} + (2n + 1)\, I_{n+1} + n^2 I_n = 0$

(B) $I_{n+2} - (2n+1)\, I_{n+1} - n^2 I_n = 0$

(C) $I_{n+2} + (2n + 1)\, I_{n+1} - n^2 I_n = 0$

(D) $I_{n+2} - (2n+1)\, I_{n+1} + n^2 I_n = 0$

17. The function $f(x) = \sqrt{1-\sqrt{1-x^2}}$ is

(A) not continuous of $x = 0$

(B) is continuous but not differentiable at $x = 0$

(C) differentiable at $x = 0$

(D) None of these

18. The function

$z = 5xy - 4x^2 + y^2 - 2x - y + 5$ has at

$x = \frac{1}{41}, y = \frac{18}{41}$

(A) maximum (B) saddle point

(C) minimum (D) None of these

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Test Paper – 1

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. $\lim\limits_{x\to 0} \dfrac{\sin 3x - \sin x}{\sin x}$ is equal to

(A) 0 (B) 2
(C) 1 (D) 3

2. $\lim\limits_{x\to 0} \sin\dfrac{1}{x}$ is

(A) ∞ (B) 0
(C) 3 (D) does not exist

3. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{5x}$ is equal to

(A) $\dfrac{3}{5}$ (B) $\dfrac{5}{3}$
(C) $\dfrac{1}{5}$ (D) $\dfrac{1}{3}$

4. If [x] denotes the greatest integer not greater than x, then $\lim\limits_{x\to 2}[x]$ is

(A) 0 (B) 1
(C) 2 (D) does not exist

5. $\lim\limits_{x\to 0} \dfrac{(1+x)^n - 1}{x}$ is equal to

(A) n (B) n – 1
(C) 0 (D) does not exist

Q6 to Q15 carry two marks each

6. $\int_1^{\infty} \dfrac{1}{x^2}dx$

(A) Converges
(B) Diverges
(C) Converges and diverges
(D) None of these

7. $\int_0^{\infty} \dfrac{dx}{e^x + e^{-x}}$ is equal to

(A) $\pi/2$ (B) $\pi/4$
(C) $\pi/3$ (D) π

8. $\int_2^{\infty} \dfrac{dx}{x\,\ell n^2 x}$ is equal to

(A) $\dfrac{1}{\ell n 4}$ (B) $\dfrac{1}{\ell n 3}$
(C) $\dfrac{2}{\ell n 2}$ (D) $\dfrac{1}{\ell n 2}$

9. $\int_1^{5} \dfrac{dx}{\sqrt[3]{x-2}}$ is equal to

(A) $\dfrac{\sqrt[3]{9}+1}{2}$ (B) $\dfrac{3}{2}\left(\sqrt[3]{9}-1\right)$
(C) $3\left(\sqrt[3]{9}+1\right)$ (D) None of these

10. The area under curve $y = \dfrac{x}{\sqrt{1-x^2}}$ for $0 \le x \le 1$ is

(A) 1/4 (B) 1/2

(C) 1 (D) 16

11. The volume of solid obtained by revolving area under $y = e^{-2x}$ about x axis is

(A) $\pi/2$ (B) $\pi/4$

(C) 2π (D) π

12. $\displaystyle\int_0^9 \frac{dx}{(x-1)^{2/3}}$ is equal to

(A) 3 (B) 6

(C) 7 (D) 9

13. The area between the curves $y = 1/x$ and $y = \dfrac{1}{x+1}$ to right of line $x = 1$ is

(A) $\ell n\ 3$ (B) $\ell n\ 2$

(C) $2\ \ell n\ 3$ (D) $2\ \ell n\ 2$

14. $\displaystyle\int_1^3 \frac{dx}{x-2}$

(A) converges

(B) diverges

(C) both

(D) neither of the above

15. $\displaystyle\int_e^\infty \frac{dx}{(\ell n x)^p}$ is __________ for $p \ge 1$

(A) convergent

(B) divergent

(C) convergent as well as divergent

(D) None of these

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Test Paper – 2

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. Let $f(x) = 2x \quad \text{if } x < 2$
$= 2 \quad \text{if } x = 2$
$= x^2 \quad \text{if } x > 2$

the discontinuity of f (x) at x = 2 is known as

(A) first kind of discontinuity
(B) second kind of discontinuity
(C) mixed discontinuity
(D) removable discontinuity at x = 2

2. If $f(x) = x - 1 \qquad 1 \le x \le 2$
$= 2x - 3, \qquad 2 \le x \le 3$

Tick the following alternative which is appropriate for the above function.

(A) continuous at x = 1
(B) continuous at x = 2
(C) discontinuous at x = 1
(D) discontinuous at x = 2

3. Consider the following statement

Assertion (A) :

The function $f(x) = x - [x]$, $x \in z$ is discontinuous at x = 1

Reason (R) :

Left $\lim\limits_{x\to 1} f(x) \ne$ Right $\lim\limits_{x\to 1} f(x)$

Of these statements

(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true and R is not the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

4. If a function 'f' is defined on R by

f (x) = 1 for x a rational number
= – 1 for x an irrational number, then

(A) f (x) is continuous at x = 1
(B) f (x) is continuous at x = – 1
(C) f (x) is continuous at x = 0
(D) f (x) is not continuous at any point

5. If $f(x) = 4x + 3, \qquad x \ne 4$
$= 3x + 7, \qquad x = 4$

the function 'f' is

(A) continuous at x = 4
(B) discontinuous at x = 4
(C) continuous at x = – 4
(D) discontinuous at x = – 4

Q6 to Q15 carry two marks each

6. $I = \int_0^{\pi/2} \cot x\,dx$ is

(A) 2 (B) $\pi/4$
(C) π (D) None of these

7. The area in first quadrant under curve

$y = \frac{1}{\left(x^2 + 6x + 10\right)}$ is

(A) $\pi/2$ (B) $\frac{\pi}{4} - 2\tan^{-1} 3$

(C) $\frac{\pi}{2} - \tan^{-1} 3$ (D) $\frac{\pi}{2} + \tan^{-1} 3$

8. The area under $y = \frac{1}{\left(x^2 - a^2\right)}$

for $x \geq a + 1$ is

(A) $\frac{1}{a}\ell n(a+1)$ (B) $\frac{1}{2a}\ell n(a+1)$

(C) $\frac{1}{2}\ell n(a+1)$ (D) None of these

9. For what values of k, with $k \neq 1$ and

$k > 0$ does $\int_0^1 \frac{1}{x^k} dx$ converges ?

(A) 2 (B) 1

(C) 3 (D) $k > 1$

10. $\int_0^a \frac{dx}{a^2 - x^2}$ where $a > 0$ is

(A) ℓn a (B) $\frac{1}{2}[\ell n a + 1]$

(C) $\frac{1}{2}[\ell n a - 1]$ (D) None of these

11. The minimum value of $\left|x^2 - 5x + 2\right|$ is

(A) –5 (B) 0

(C) –1 (D) –2

12. The maximum value of

$f(x) = \frac{\sin x - \cos x}{\sqrt{2}}$ is

(A) 1 (B) $\sqrt{2}$

(C) $1/\sqrt{2}$ (D) 3

13. The maximum value of $1/x^x$ is

(A) e (B) e^{-e}

(C) $e^{-1/e}$ (D) $e^{1/e}$

14. The function $f(c) = x^5 - 5x^4 + 5x^3 - 1$ has

(A) 1 minima and 2 maxima

(B) 2 minima and 1 maxima

(C) 2 minima and 2 maxima

(D) 1 minima and 1 maxima

15. The greatest and least value of

$\frac{x^3}{3} - \frac{3x^2}{2} + 2x \text{ in } [0,2]$ are

(A) $\frac{5}{6}$ and $\frac{2}{3}$ (B) $\frac{11}{6}$ and $\frac{1}{6}$

(C) $\frac{8}{3}$ and $\frac{2}{3}$ (D) $\frac{13}{3}$ and $\frac{1}{3}$

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Test Paper – 3

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. The $\lim_{x\to1}\frac{(x^{16}-1)}{(x^5-1)}$ is

(A) $\frac{5}{16}$ (B) $\frac{2}{17}$

(C) $\frac{16}{5}$ (D) $\frac{17}{2}$

2. $\lim_{x\to2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ equal to

(A) 8 (B) – 8

(C) 4 (D) – 4

3. $\lim_{x\to0}\frac{a^x-b^x}{x}$ equals to

(A) $\log\left(\frac{a}{b}\right)$ (B) $\log\left(\frac{b}{a}\right)$

(C) $\log\left(\frac{x}{a}\right)$ (D) $\log\left(\frac{b}{x}\right)$

4. The function $f(x) = \frac{|x|}{x}$ at x = 0 has

(A) discontinuity of second kind

(B) discontinuity of first kind

(C) mixed discontinuity

(D) continuous

5. The function

$f(x) = x^2 \sin\frac{1}{x}$, if $x \neq 0$

$= 0$, if $x = 0$ is

(A) differentiable at x = 0 and f ‘(0) = 0

(B) not differentiable at x = 0 since $\frac{1}{x} \to \infty$, $\cos x \to 0$

(C) differentiable at x = 0 and the derivative is continuous at x = 0

(D) not differentiable at any x since it is not continuous for any x

Q6 to Q15 carry two marks each

6. The maximum point on curve $x = e^x y$ is

(A) (1, e) (B) (1, 1/e)

(C) (e, 1) (D) (1/e, 1)

7. If f(x, y) is such that $f_x = e^x \cos y$ and $f_y = e^x \sin y$ then which of following is true

(A) $f(x, y) = e^{x+y}\sin(x + y)$

(B) $f(x, y) = e^x \sin$

(C) f(x, y) does not exist

(D) None of these

8. If $f(x, y) = \ln(x^2 + y^2)$ then

(A) $f_{xx} + f_{yy} = 0$ (B) $f_{xx} - f_{yy} = 0$

(C) $f_{xx} + 2f_{yy} = 0$ (D) $2f_{xx} + f_{yyt} = 0$

9. If $z = x^4 + 3xy - y^2$ and $y = \sin x$ then $\frac{dz}{dx}$ is

(A) $(4x^3 + 3\sin x) + (3x - 2\sin x)\cos x$

(B) $(2x^2 - 3\sin x) + (3x - 2\cos x)\sin x$

(C) $(4x^3 - 3\sin x) - (3x + 2\sin x)\cos x$

(D) none of these

10. A point P is moving along curve of intersection of paraboloid $\frac{x^2}{16} - \frac{y^2}{9} = z$ and cylinder $x^2 + y^2 = 5$. If x is increasing at the rate of 5cm/s how fast is z changing when x = 2m and y = 1 cm ?

(A) $\frac{25}{36}$ (B) $\frac{125}{36}$

(C) $\frac{144}{25}$ (D) $\frac{18}{25}$

11. The general solution of $\frac{\partial^2 f}{\partial x^2} = 0$ is

(A) $f(x, y) = x\,g(y) + h(y)$

(B) $f(x, y) = y\,g(x) + h(x)$

(C) $f(x, y) = x\,g(y) + h(x)$

(D) $f(x, y) = g(x) + h(y)$

12. If $z = xe^{y/x}$ then

(A) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

(B) $x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial z} = z$

(C) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$

(D) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$

13. Find the general solution of $\frac{\partial^2 f}{\partial x \partial y} = 1$ is

(A) $f(x, y) = A(x) + B(y) + xy$

(B) $f(x, y) = yA(x) + xB(y)$

(C) $f(x, y) = A(x) + B(y)$

(D) none of these

14. The equation of tangent plane sphere phase $x^2 + y^2 + z^2 = 1$ at point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ is

(A) $x + y + z = 2$

(B) $x + y + \sqrt{2}\,z = 2$

(C) $2x - y + 32 = 5$

(D) $2x + y = 2$

15. The altitude h of right circular cone is decreasing at rate of 3 mm/s while radius is increasing at 2mm/s. How fast is volume v changing when altitude and radius are 100mm and 50mm respectively ?

(A) $\frac{2200\pi}{3}$ (B) $\frac{2500\pi}{3}$

(C) $\frac{12500\pi}{3}$ (D) $\frac{1600\pi}{3}$

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Test Paper – 4

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. Maxima and minima of a continuous function occur
(A) simultaneously
(B) once
(C) alternately
(D) rarely

2. Let f(x) = | x | than
(A) $f'(0) = 0$
(B) f(x) is maximum at x = 0
(C) f(x) is minimum of x = 0
(D) none of these

3. Find the maximum speed with which a snail moves if the equation of distance covered (s) in terms of time (t) is $s = -3t^3 + \frac{t^2}{2} + 15$ units ?
(A) $\frac{1}{9}$ speed units
(B) $\frac{1}{18}$ speed units
(C) $\frac{1}{27}$ speed units
(D) None of these

4. The function $g(x) = \frac{f(x)}{x}$, $x \neq 0$ has an extreme value then
(A) $f'(x) = f(x)$ (B) $g'(x) = f(x)$
(C) $f(x) = 0$ (D) $g(x) = f'(x)$

5. Which of following is false ?
(A) If $f'(a) = 0$ and $f''(a) > 0$ then f (a) is minimum value
(B) If $f'(a) = 0$ and $f''(a) \neq 0$ then f(a) is extreme value
(C) a maximum value is greater than greatest values
(D) f(a) is minimum value if f(a) is least value of f(x) in immediate neighborhood of point [a, f(a)]

Q6 to Q15 carry two marks each

6. The area included between curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$ is
(A) 10 (B) 45/2
(C) 15 (D) 30

7. The area included between curve $r = a(\sec\theta + \cos\theta)$ and its asymptote $r = a\sec\theta$ is
(A) πa^2 (B) $\frac{5\pi a^2}{4}$
(C) $\frac{2\pi a^2}{3}$ (D) $2\pi a^2$

8. The mass distributed over the area bounded by curve $16y^2 = x^3$ and line $2y = x$ assuming that density of a point of area varies as distance of point from x axis. Take k as constant of proportionality
(A) k (B) 1/3k
(C) 2/3k (D) 2k

9. The density at any point in circular lamina of radius a varies as its distance from a fixed point on its circumference. Take k as proportional constant, the mass of lamina is

(A) ka^2 (B) $\frac{1}{9}ka^3$

(C) $\frac{8}{9}ka^3$ (D) $\frac{32}{9}ka^3$

10. The volume of sphere $x^2 + y^2 + z^2 = a^2$ cut off by plane z = 0 and cylinder $x^2 + y^2 = ax$ is

(A) $\frac{\pi a^3}{3}$ (B) $\frac{2\pi a^3}{3}$

(C) πa^3 (D) $2\pi a^3$

11. The volume cut off from paraboloid $x^2 + \frac{1}{4}y^2 + z = 1$ by plane z = 0 is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\frac{3\pi}{4}$ (D) π

12. The volume of solid obtained by revolution of loop of curve $y^2 = x^2 \frac{a+x}{a-x}$ about x axis is

(A) $2\pi a^3$

(B) $2\pi a^3\left[\log 2 - \frac{2}{3}\right]$

(C) $\pi a^3\left[\log 2 + \frac{2}{3}\right]$

(D) πa^3

13. The volume of solid formed by revolution of the curve $(a - x)y^2 = a^2x$ about its asymptote

(A) $\frac{\pi^2 a^3}{2}$ (B) $\pi^2 a^3$

(C) $\frac{3\pi^2 a^3}{2}$ (D) $4\pi^2 a^3$

14. The surface of reel formed by revolution of cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ round a tangent at its vertex is

(A) $\frac{\pi a^2}{3}$ (B) $\frac{2\pi a^2}{3}$

(C) $\frac{16\pi a^2}{3}$ (D) $\frac{32\pi a^2}{3}$

15. The volume generated by revolving area below the x–axis between curve $\frac{y+8}{x} = x - 2$ and x axis about line $x + 5 = 0$ is

(A) 18π (B) 32π

(C) 432π (D) 216π

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Test Paper – 5

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. The minimum value of $x + \frac{1}{x}$, $x > 0$

(A) 1 (B) 3/2
(C) 5 (D) 2

2. The triangle of maximum area inscribed in a circle of radius r is

(A) a right angled Δ with hypotenuse 2r
(B) an equilateral Δ
(C) an isoceles Δ of height r
(D) does not exist

3. The maximum value of $\frac{\log x}{x}$ in $(0, \infty)$ is

(A) e (B) 1/e
(C) 1 (D) none of these

4. A manufacturer sells 'x' items at a price of Rs. $\left(3 - \frac{x}{1000}\right)$ per item and it costs Rs. $\left(\frac{x}{2} + 200\right)$ to produce them. Find the production level for maximum profit.

(A) 1250
(B) 1500
(C) 1750
(D) Data insufficient

5. If z is defined as function of x and y by $xy - yz + xz = 0$ then $\frac{\partial z}{\partial x}$ is

(A) $\frac{y+z}{y-x}$ (B) $\frac{y-z}{y+x}$
(C) $\frac{y+x}{y-z}$ (D) $\frac{x+z}{x-y}$

Q6 to Q15 carry two marks each

6. If $f(x) = e^x$ and $g(x) = e^{-x}$ then value of c in interval [a, b] by Cauchy's mean value theorem

(A) $\frac{a+b}{2}$ (B) $\sqrt{ab}$
(C) $\frac{2ab}{a+b}$ (D) none of these

7. If f and F be both continuous in [a, b] and are derivable in (a, b) and if $f'(x) = F'(x)$ for all x in [a, b] then f(x) and F (x) differ

(A) by 1 in [a, b]
(B) by x in [a, b]
(C) by constant in [a, b]
(D) none of these

8. If $f(x) = 4x^2$ then the value of c in [–1, 3] for which $f'(c) = \frac{f(3) - f(-1)}{4}$ is

(A) 0 (B) 1
(C) 2 (D) 3

9. $I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^4 \cos\theta\, dr\, d\theta$

(A) $\frac{5}{16}\pi a^5$ (B) $\frac{21}{16}\pi a^5$

(C) $\frac{3}{16}\pi a^5$ (D) $\frac{1}{4}\pi a^4$

10. $I = \int_{-2}^{4}\int_{y^2/4}^{\frac{y}{2}+2} x\, dx\, dy$

(A) $\frac{24}{5}$ (B) $\frac{36}{5}$

(C) $\frac{72}{5}$ (D) $\frac{108}{5}$

11. $I = \int_0^{b}\int_0^{b} y(2b+y)\, dx\, dy$

(A) $\frac{b^4}{3}$ (B) $\frac{2b^4}{3}$

(C) b^4 (D) $\frac{4b^4}{3}$

12. $I = 2\int_0^{\pi/4}\int_0^{2\sin\theta} r^3\, dr\, d\theta$

(A) $\frac{3\pi}{4}$ (B) $\frac{3\pi}{4} - 2$

(C) $\frac{3\pi}{4} + 2$ (D) $\frac{5\pi}{2}$

13. $I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^4 \sin^3\theta\, dr\, d\theta$

(A) $\frac{2}{35}\pi a^5$ (B) $\frac{2^4}{35}\pi a^5$

(C) $\frac{2^6 \pi a^5}{35}$ (D) $\frac{2^8 \pi a^5}{35}$

14. $I = \int_0^{a}\int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\, dx\, dy$

(A) $\frac{a^2}{2}\log a$ (B) $\frac{a^2}{8}(\log a + 1)$

(C) $8a\log a$ (D) $\frac{a^2}{8}[2\log a + 1]$

15. $I = \int_0^{a}\int_{x^2/a}^{2a-x} xy\, dx\, dy$

(A) $\frac{3}{8}a^4$ (B) $\frac{5}{8}a^4$

(C) $\frac{7}{8}a^4$ (D) a^4

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Test Paper – 6

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. $\lim\limits_{x \to 8} \dfrac{\sqrt{x} - 2\sqrt{2}}{x - 8}$ is

(A) $\dfrac{0}{0}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{1}{4\sqrt{2}}$ (D) $\dfrac{1}{\sqrt{2}}$

2. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$ is

(A) 1/6 (B) 1/3

(C) –1/3 (D) none of these

3. $\lim\limits_{x \to a^+} (x - a)\cos\dfrac{1}{x - a}$ is

(A) 0 (B) a

(C) cos a (D) does not exist

4. The function $f(x) = \dfrac{1 - \cos x}{x^2}$ $(x \neq 0)$ be made continuous at x = 0 by defining f(0) to be

(A) 1 (B) 1/2

(C) 0 (D) e

5. The differentiability of a function continuity is

(A) sufficient

(B) necessary

(C) sufficient and necessary

(D) none of these

Q6 to Q15 carry two marks each

6. If $I = \iint\limits_R y^2 dA$ where R is region bounded by y = 2x, y = 5x and x = 1. Then I is equal to

(A) $\dfrac{39}{2}$ (B) $\dfrac{39}{4}$

(C) $\dfrac{39}{8}$ (D) $\dfrac{39}{16}$

7. The value of $I = \iint\limits_R \dfrac{1}{\sqrt{2y - y^2}} dA$ where R is region in the first quadrant bounded by $x^2 = 4 - 2y$ is

(A) 0 (B) 2

(C) 4 (D) –1

8. The volume V under plane z = 3x + 4y and over the rectangle

$R : 1 \leq x \leq 2,\ 0 \leq y \leq 3$ is

(A) 25 (B) $\dfrac{49}{2}$

(C) $\dfrac{63}{2}$ (D) 16

9. The volume V in first octant bounded by $z = y^2$, x = 2 and y = 4 is

(A) $\dfrac{64}{3}$ (B) $\dfrac{128}{3}$

(C) $\dfrac{16}{3}$ (D) $\dfrac{32}{3}$

10. The area of region R bounded by $xy = 1$ and $2x + y = 3$ is

(A) $\ell n\ 2$ (B) $2 + \ell n\ 2$

(C) $\frac{3}{4} - \ell n 2$ (D) None of these

11. The volume in first octant bounded by $2x + 2y - z + 1 = 0$ $y = x$ and $x = 2$ is

(A) 10 (B) 28

(C) 12 (D) 16

12. The area of region enclosed by cardioide $r = 1 + \cos\theta$ is

(A) $\frac{\pi}{2}$ (B) $\frac{3\pi}{2}$

(C) π (D) 2π

13. $I = \int_{-\pi/4}^{\pi/4} d\theta \int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr}{\sqrt{a^2 + r^2}}$

(A) $2a\left[1 + \frac{\pi}{4}\right]$ (B) $2a\left[2 - \frac{\pi}{4}\right]$

(C) $2a\left[1 - \frac{\pi}{4}\right]$ (D) $2a\left[2 + \frac{\pi}{4}\right]$

14. $I = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2-x-y} dx\,dy\,dz$

(A) $\frac{11}{30}$ (B) $\frac{13}{30}$

(C) $\frac{1}{8}$ (D) 16

15. $I = \int_1^2 \int_{y-1}^{1} \left(x^2 - y^2\right) dx\,dy$

(A) $\frac{1}{3}$ (B) $\frac{-1}{3}$

(C) $\frac{-2}{3}$ (D) $\frac{2}{3}$

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Chapter - 3 : Probability and Statistics

3.1 Basic Terms

Random Experiment

Consider an action which is repeated under essentially identical conditions. If it results in any one of the several possible outcomes, but it is not possible to predict which outcome will appear. Such an action is called as a Random Experiment. One performance of such an experiment is called as a **Trial.**

Sample Space

The set of all possible outcomes of a random experiment is called as the sample space. All the elements of the sample space together are called as 'exhaustive cases'. The number of elements of the sample space i.e. the number of exhaustive cases is denoted by n(S) or N or n.

Event

Any subset of the sample space is called as an 'Event' and is denoted by some capital letter like A, B, C or A_1, A_2, A_3,.. or B_1, B_2, ... etc.

Favourable cases

The cases which ensure the happening of an event A, are called as the cases favourable to the event A. The number of cases favourable to event A is denoted by n(A) or N_A or n_A.

Mutually Exclusive Events or Disjoint Events

Two events A and B are said to be mutually exclusive or disjoint if $A \cap B = \phi$ i.e. if there is no element common to A & B.

Equally Likely Cases

Cases are said to be equally likely if they all have the same chance of occurrence i.e. no case is preferred to any other case.

Permutation

A permutation is an arrangement of all or part of a set of objects. The number of permutations of n distinct objects taken r at a time is

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Note: **The number of permutations of n distinct objects is n! i.e., $^nP_n = n!$**

The number of permutations of n distinct objects arranged in a circle is $(n-1)!$

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind … n_k of a k^{th} kind is $\dfrac{n!}{n_1!\ n_2!\ldots\ldots n_k!}$

Combination

A combination is selection of all or part of a set of objects. The number of combinations of n distinct objects taken r at a time is

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Note: **In a permutation, the order of arrangement of the objects is important. Thus abc is a different permutation from bca.**

In a combination, the order in which objects are selected does not matter. Thus abc and bca are the same combination.

3.2 Definition of Probability

Consider a random experiment which results in a sample space containing n(S) cases which are exhaustive, mutually exclusive and equally likely. Suppose, out of n(S) cases, n(A) cases are favourable to an event A. Then the probability of event A is denoted by P(A) and is defined as follows.

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{number of cases favourable to event A}}{\text{number of cases in the sample space S}}$$

3.3 Complement of an event

The complement of an event A is denoted by $\bar{A}$ and it contains all the elements of the sample space which do not belong to A.

For example: Random experiment: an unbiased die is rolled.

S = {1, 2, 3, 4, 5, 6}

(i) Let A: number on the die is a perfect square

$\therefore$ A = {1, 4} $\therefore$ $\bar{A}$ = {2, 3, 5, 6}

(ii) Let B: number on the die is a prime number

$\therefore$ B = {2, 3, 5} $\therefore \bar{B}$ = {1, 4, 6}

Note: **$P(A) + P(\bar{A}) = 1$ i.e. $P(A) = 1 - P(\bar{A})$**

For any events A and B, $P(A) = P(A \cap B) + P(A \cap \bar{B})$

3.4 Independent Events

Two events A & B are said to be independent if

$$P(A \cap B) = P(A).P(B)$$

Note: **If A & B are independent then**

A & $\bar{B}$ are independent

$\bar{A}$ & B are independent

$\bar{A}$ & $\bar{B}$ are independent

3.5 Theorems of Probability

Addition Theorem

If A and B are any two events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note:

1. **$A \cup B$: either A or B or both i.e. at least one of A & B**

   $\bar{A} \cap \bar{B}$: neither A nor B i.e. none of A & B

   $A \cup B$ & $\bar{A} \cap \bar{B}$ are complement to each other

   $\therefore P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

2. **If A & B are mutually exclusive, $P(A \cap B) = 0$**

   $\therefore P(A \cup B) = P(A) + P(B)$

3. $$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

Multiplication Theorem

If A & B are any two events then

$$P(A \cap B) = P(A).P(B/A) = P(B).P(A/B)$$

1. **Conditional probability** of occurrence of event B given that event A has already occurred.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

2. **Conditional probability** of occurrence of event A given that event B has already occurred

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Theorem

Suppose that a sample space S is a union of mutually disjoint events B_1, B_2, B_3, ..., B_n, suppose A is an event in S, and suppose A and all the B_i's have nonzero probabilities. If k is an integer with $1 \le k \le n$, then

$$P(B_k / A) = \frac{P(A / B_k)P(B_k)}{P(A / B_1)P(B_1) + P(A / B_2)P(B_2) + ... + P(A / B_n)P(B_n)}$$

Solved Example 1 :

A single die is tossed. Find the probability of a 2 or 5 turning up.

Solution :

The sample space is S = {1, 2, 3, 4, 5, 6}

$$P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$$

The event that either 2 or 5 turns up in indicated by $2 \cup 5$. Thus

$$P(2 \cup 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Solved Example 2 :

A coin is tossed twice. What is the probability that at least one head occurs ?

Solution :

The sample space is

S = {HH, HT, TH, TT}

Probability of each outcomes = 1/4

Probability of atleast one head occurring is

$$P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Solved Example 3 :

A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die. Find P(E).

Solution :

$S = \{1, 2, 3, 4, 5, 6\}$

We assign a probability of w to each odd number and a probability of 2w to each even number. Since the sum of the probabilities must be 1, we have 9w = 1 or w = 1/9.

Hence probabilities of 1/9 and 2/9 are assigned to each odd and even number respectively.

$\therefore \quad E = \{1, 2, 3\}$

and $\quad P(E) = \frac{1}{9}+\frac{2}{9}+\frac{1}{9}=\frac{4}{9}$

Solved Example 4 :

In the above example let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

Solution :

$A = \{2, 4, 6\}$ and

$B = \{3, 6\}$

We have,

$A \cup B = \{2, 3, 4, 6\}$ and

$A \cap B = \{6\}$

By assigning a probability of 1/9 to each odd number and 2/9 to each even number

$$P(A \cup B) = \frac{2}{9}+\frac{1}{9}+\frac{2}{9}+\frac{2}{9}=\frac{7}{9}$$

and $\quad P(A \cap B) = \frac{2}{9}$

Solved Example 5 :

A mixture of candies 6 mints, 4 toffees and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting (a) a mint, or

(b) a toffee or a chocolate.

Solution :

(a) Since 6 of the 13 candies are mints, the probability of event M, selecting mint at random, is

$$P(M) = \frac{6}{13}$$

(b) Since 7 of the 13 candies are toffees or chocolates it follows that

$$P(T \cup C) = \frac{7}{13}$$

Solved Example 6 :

In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution :

The number of ways of being dealt 2 aces from 4 is

$$^4C_2 = \frac{4!}{2!2!} = 6$$

The number of ways of being dealt 3 jacks from 4 is

$$^4C_3 = \frac{4!}{3!1!} = 4$$

$\therefore$ Total number of ways n = 6.4 = 24 hands with 2 aces and 3 jacks.

The total number of 5–card poker hands all of which are equally likely, is

$$N = {}^{52}C_5 = \frac{52!}{5!47!} = 2,598,960$$

$\therefore$ The probability of event C of getting 2 aces and 3 jacks in a 5–card poker hand is

$$P(C) = \frac{24}{2,598,960} = 0.9 \times 10^{-5}$$

Solved Example 7 :

The probability that Paula passes mathematics is 2/3 and the probability that she passes English is 4/9. If the probability of passing both courses is 1/4, what is the probability that Paula will pass at least one of these courses ?

Solution :

If M is the event "passing Mathematics" and E the event "passing English" then

$$P(M \cup E) = P(M) + P(E) - P(M \cap E)$$

$$= \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

Solved Example 8 :

What is the probability of getting a total of 7 or 11 when a pair of dice are tossed ?

Solution :

Let A be the event that 7 occurs and B the event that 11 comes up. Now a total of 7 occurs for 6 of the 36 sample points and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have P(A) = 1/6 and P(B) = 1/18.

The events A and B are mutually exclusive since a total of 7 and 11 cannot both occur on the same toss.

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$$

Solved Example 9 :

If the probabilities are respectively 0.09, 0.15, 0.21 and 0.23 that a person purchasing a new automobile will choose the colour green, white, red or blue. What is the probability that a given buyer will purchase a new automobile that comes in one of those colours ?

Solution :

Let G, W, R and B be the events that a buyer selects, respectively, a green, white, red or blue automobile. Since these four events are mutually exclusive the probability is

$$P(G \cup W \cup R \cup B)$$
$$= P(G) + P(W) + P(R) + P(B)$$
$$= 0.09 + 0.15 + 0.21 + 0.23 = 0.68$$

Solved Example 10 :

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7 or 8 more cars on any given workday are respectively 0.12, 0.19, 0.28, 0.24, 0.10 and 0.07. What is the probability that he will service at least 5 cars on next day at work?

Solution :

Let E be the event that at least 5 cars are serviced. Now

$P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced.

Since $P(E') = 0.12 + 0.19 = 0.31$

$\therefore \quad P(E) = 1 - 0.31 = 0.69$

Solved Example 11 :

In a garden 40% of the flowers are roses and the rest are carnations. If 25% of the roses and 10% of the carnations are red, the probability that a red flower selected at random is a rose.

(A) 5/6 (B) 1/4

(C) 4/5 (D) 5/8

Solution :

Suppose there are 100 flowers

Number of roses = 40; Number of carnations = 60

25% of 40 = 10 roses are red and

10% of 60 = 6 carnations are red

Let A be the event that the flower is red and B the event that the flower is a rose.

$\therefore$ $A \cap B$ is the event that the flower is a red rose.

$n(A) = 16 \qquad \therefore P(A) = \dfrac{16}{100}$

$n(A \cap B) = 10 \qquad \therefore P(A \cap B) = \dfrac{10}{100}$

P(B/A) = probability that a selected flower is a rose red is colour

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{10/100}{16/100} = \frac{5}{8}$$

3.6 Random Variables

A random variable is a function that associates a real number with each element in the sample space. It is a set of possible values from a random experiment.

For example: Suppose that a coin is tossed twice so that the sample space is S = {HH, HT, TH, TT}. Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in the table.

SAMPLE POINT	HH	HT	TH	TT
X	2	1	1	0

Discrete Random Variable

A random variable is discrete if it can take only discrete or separate values. Value of a discrete random variable is often obtained by counting.

For example: Number of heads appearing when three coins are tossed.

Continuous Random Variable

A random variable is continuous if can take any value in a given range. Value of a continuous random variable is often obtained by measuring.

For example: Time required to travel certain distance.

3.7 Probability Distribution Function

In general sense, probability distribution function may refer to:

- Probability mass function
- Probability density function
- Cumulative distribution function

Probability mass function

It is a function that gives probability that a discrete random variable is exactly equal to some value.

Probability density function

It is used to specify the probability of a continuous random variable falling within a particular range of values and the probability is given by the integral of the probability density function over that range.

Cumulative distribution function

It is a probability that random variable X will take a value less than or equal to x.
i.e., $F(x) = P(X \leq x)$

3.8 Expectation(Mean), Variance and Standard Deviation

Expected value for a discrete random variable is given by

$$E(X) = \sum_{i=1}^{n} x_i \, P(x_i)$$

Expected value for a continuous random variable is given by

$$E(X) = \int_{a}^{b} x \, f(x) \, dx$$

Variance of random variable is given by

$$\mathbf{Var(X) = E((X - \mu)^2) = E(X^2) - E(X)^2 = E(X^2) - \mu^2}$$

Standard deviation is square root of variance

$$\sigma = \sqrt{\mathbf{Var(X)}}$$

Note:	**1.**	**Expected value μ = E(X) is a measure of central tendency.**
	2.	**Standard deviation σ is a measure of spread.**

Solved Example 12 :

Thirteen cards are drawn simultaneously from a deck of 52 cards. If aces count 1, face cards 10 and other count by their denomination, find the expectation of the total score on 13 cards.

Solution :

Let x denote the number corresponding to the i^{th} card. Then x takes the value, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10 each having the same probability $\frac{1}{13}$ so that

$$p(x) = \frac{1}{13}(1+2+...+10+10+10+10)$$

$$= \frac{85}{13}$$

Hence the required expectation is 85.

3.9 Standard Distributions

Binomial Distribution

Probability of exactly *k* successes in *n* trials is given by

$$\mathbf{P(X=k) = {}^nC_k\, p^k\, (1 - p)^{n-k}}$$

where p is the probability of success in each trial

Here the trial are Bernouilli trials, where each trial can have exactly two outcomes viz. success and failure.

If p is the probability of success

then q = 1 – p is the probability of failure.

Then

$$\mathbf{P(X=k) = {}^nC_k\, p^k\, q^{n-k}}$$

For Binomial Distribution,

Mean = np

Variance = npq = np(1 − p)

Variance < Mean

Cumulative distribution function is, $F(x) = \sum_{i=0}^{k} {}^{n}C_i \, p^i \, q^{n-i}$

Poisson Distribution

Probability of k events occurring in an interval is given by

$$\mathbf{P(k\ events) = \frac{\lambda^k e^{-\lambda}}{k!}}$$

where λ is the average number of events per interval.

For a Poisson Distribution,

Mean = λ

Variance = λ

Variance = Mean

Cumulative distribution function is, $F(x) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$

Normal Distribution

Probability density function of normal distribution is

$$\mathbf{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma^2}}}$$

where μ is the mean and σ^2 is the variance

Graph of f(x) is symmetric about x = μ and is a bell shaped curve.

Cumulative distribution function is, $F(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

Note: **Standard Normal Distribution**

Special case with $\mu = 0$ and $\sigma = 1$

Probability density function is $f(x) = \dfrac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$

This graph is symmetric about x = 0 where it attains its maximum value $\dfrac{1}{\sqrt{2\pi}}$

Uniform Distribution

Probability density function for uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}$$

For Uniform Distribution,

Mean = $\dfrac{a+b}{2}$

Variance = $\dfrac{(b-a)^2}{12}$

Cumulative distribution function is, $F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$

Exponential Distribution

Probability density function for uniform distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

For exponential distribution

Mean = $\dfrac{1}{\lambda}$

Variance = $\dfrac{1}{\lambda^2}$

Cumulative distribution function is, $F(x) = 1 - e^{-\lambda x}$

Solved Example 13 :

The probability of getting exactly 2 heads in 6 tosses of a fair coin is

Solution :

$$P(X = 2) = {}^6C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{6!}{2!4!}\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

Solved Example 14 :

The probability that a certain kind of component will survive a given shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive.

Solution :

Assuming the tests are independent and p = ¾ for each of the 4 tests we get

$${}^4C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{4!}{2!2!}\frac{3^2}{4^4} = \frac{27}{128}$$

Solved Example 15 :

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are to have contracted this disease, what is the probability that

a) at least 10 survive

b) from 3 to 8 survive

c) exactly 5 survive

Solution :

Let X be the number of people that survive.

a) $P(X \geq 10) = 1 - P(X < 10)$

$$= 1 - \sum_{i=0}^{9} {}^{15}C_i\,(0.4)^i\,(0.6)^{15-i}$$

$$= 1 - 0.9662 = 0.0338$$

b) $P(3 \leq X \leq 8) = \sum_{i=3}^{8} {}^{15}C_i\,(0.4)^i\,(0.6)^{15-i}$

$= 0.8779$

c) $P(X = 5) = {}^{15}C_5\,(0.4)^5\,(0.6)^{15-5}$

$= 0.1859$

Solved Example 16 :

The probability is 0.02 that an item produced by a factory is defective. A shipment of 10,000 items are sent to its warehouse. Find the expected number E of defective items and the standard deviation.

Solution :

$$\mu = np = 10000 \times 0.02 = 200$$

$$\sigma = \sqrt{npq} = \sqrt{10000 \times 0.02 \times 0.98}$$

$$= \sqrt{196} = 14$$

Solved Example 17 :

Ten percent of the tools produces in a certain manufacturing turn out to be defective. Find the probability that in a sample of 10 tools chosen at random exactly two will be defective.

Solution :

We have $\lambda = np = 10\,(0 - 1) = 1$

Then according to the Poisson distribution

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

or $\quad P(X = 2) = \dfrac{(1)^2\, e^{-1}}{2!} = 0.1839$ or 0.18

In general the approximation is good if

$P \leq 0.1$ and $\lambda = np \leq 5$

Solved Example 18 :

If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 2000 individuals

a) exactly 3

b) more than 2, individuals will suffer a bad reaction.

Solution :

Let X denote the number of individuals suffering a bad reaction. X is poisson distributed.

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where $\lambda = np = 2000\ (0.001) = 2$

a) $P(X = 3) = \dfrac{2^3 e^{-2}}{3!} = 0.180$

b) $P(X > 2)$

$= 1 - \left[P(X=0) + P(X=1) + P(X=2)\right]$

$= 1 - \left[\dfrac{2^0 e^{-2}}{0!} + \dfrac{2^1 e^{-2}}{1!} + \dfrac{2^2 e^{-2}}{2!}\right]$

$= 1 - 5e^{-2} = 0.323$

3.10 Mean, Median, Mode and Standard Deviation

Mean

Mean of a data generally refers to arithmetic mean of the data. However, there are two more types of mean which are geometric mean and harmonic mean. The different types of mean for set of values and grouped data are given in the following table.

	Set of Values	**Grouped Data**
Arithmetic Mean	$\dfrac{\sum x_i}{n}$	$\dfrac{\sum f_i x_i}{\sum f_i}$
Geometric Mean	$(x_1 \cdot x_2 \cdot x_3 \ldots\ldots x_n)^{1/n}$	$\text{Antilog}\left(\dfrac{\sum f_i \log x_i}{\sum f_i}\right)$
Harmonic Mean	$\dfrac{n}{\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3} \cdots \dfrac{1}{x_n}\right)}$	$\dfrac{\sum f_i}{\sum f_i / x_i}$

Median

For an ordered set of values, median is the middle value. If the number of values is even, median is taken as mean of the middle two values.

For a grouped data, median is given by

$$\text{Median} = L + \frac{\left(\frac{N}{2} - C\right)}{f} h$$

where, L = Lower boundary of the median class

$N = \sum f_i$

C = Cumulative frequency upto the class before the median class

h = Width of median class

f = Frequency of the median class

Mode

For a set of values, mode is the most frequently occurring value. For a grouped data, mode is given by

$$\text{Mode} = L + \frac{(f_i - f_{i-1})}{(2f_i - f_{i-1} - f_{i+1})} h$$

where, L = Lower boundary of the modal class

f_i = Frequency of the modal class (Highest frequency)

f_{i-1} = Frequency of the class before the modal class

f_{i+1} = Frequency of the class after the modal class

h = Width of the modal class

<u>Note:</u> Relation between Mean, Median and Mode :

Mean − Mode = 3(Mean − Median)

Standard Deviation

Standard deviation is given by,

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} \qquad \textbf{For a Set of Values}$$

$$\sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} \qquad \textbf{For a Grouped Data}$$

3.11 Correlation and Regression Analysis

Correlation

Correlation analysis deals with finding linear association between two variables i.e. if the two variables are linearly related to each other.

Correlation coefficient between variables x and y can be found as follows.

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{\left[n\sum x_i^2 - (\sum x_i)^2\right]\left[n\sum y_i^2 - (\sum y_i)^2\right]}}$$

where, $\bar{x}$ **= mean of x**

$\bar{y}$ **= mean of y**

Regression

Regression analysis involves identifying the relationship between a dependent variable and one or more independent variables.

If X and Y are the variables then equation for regression line of Y on X is given by

$$Y = a + bX$$

where, $b = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$

and a can be found by using

$$\bar{y} = a + b\bar{x}$$

Note: **Correlation coefficient measures linear association between two variables.**

r can vary between –1 to 1 i.e. $-1 \le r \le 1$.

If there is no linear association between the two variables, r will be zero.

List of Formulae

- **Permutation and Combination**

$$^nP_r = \frac{n!}{(n-r)!}$$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

- **Probability**

$$P(A) = \frac{n(A)}{n(S)}$$

- $P(A) + P(\bar{A}) = 1$
- $P(A) = P(A \cap B) + P(A \cap \bar{B})$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$
- Two events A & B are said to be independent if
  $P(A \cap B) = P(A).P(B)$
- $P(A \cap B) = P(A).P(B/A) = P(B).P(A/B)$
- **Conditional probability** of occurrence of event B given that event A has already occurred

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

- **Bayes' Theorem**

$$P(B_k/A) = \frac{P(B_k)P(A/B_k)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$

- **Expectation(Mean), Variance and Standard Deviation**

$$E(X) = \sum_{i=1}^{n} x_i P(x_i)$$

…. For Discrete Random Variable

$$E(X) = \int_a^b x\, f(x)\, dx$$

….. For Continuous Random Variable

$$Var(X) = E((X-\mu)^2) = E(X^2) - E(X)^2$$
$$= E(X^2) - \mu^2$$

$$\sigma = \sqrt{Var(X)}$$

- **Binomial Distribution**

$P(X=k) = {}^nC_k\, p^k\, (1-p)^{n-k}$

Mean = np

Variance = npq = np(1 – p)

Variance < Mean

Cumulative distribution function is,

$$F(x) = \sum_{i=0}^{k} {}^nC_i\, p^i\, q^{n-i}$$

- **Poisson Distribution**

$$P(k \text{ events}) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Mean = λ

Variance = λ

Variance = Mean

Cumulative distribution function is,

$$F(x) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

- **Normal Distribution**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma^2}}$$

Mean = μ

Variance = σ^2

Cumulative distribution function is,

$$F(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

- **Uniform Distribution**

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$$\text{Mean} = \frac{a+b}{2}$$

$$\text{Variance} = \frac{(b-a)^2}{12}$$

Cumulative distribution function is,

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

- **Exponential Distribution**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\text{Mean} = \frac{1}{\lambda}$$

$$\text{Variance} = \frac{1}{\lambda^2}$$

Cumulative distribution function is,

$$F(x) = 1 - e^{-\lambda x}$$

- **Mean**

Arithmetic Mean

$= \dfrac{\sum x_i}{n}$ …. for set of values

$= \dfrac{\sum f_i x_i}{\sum f_i}$ …. for grouped data

Geometric Mean

$= (x_1.x_2.x_3.\dots.x_n)^{1/n}$ …. for set of values

$= \text{Antilog}\left(\dfrac{\sum f_i \log x_i}{\sum f_i}\right)$

…. for grouped data

Harmonic Mean

$$= \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \cdots \frac{1}{x_n}\right)}$$

…. for set of values

$= \dfrac{\sum f_i}{\sum f_i / x_i}$ …. for grouped data

- **Median**

For an ordered set of values, median is the middle value. If the number of values is even, median is taken as mean of the middle two values.

For a grouped data,

$$\text{Median} = L + \frac{\left(\frac{N}{2} - C\right)}{f} h$$

where,

L = Lower boundary of the median class

$N = \sum f_i$

C = Cumulative frequency upto the class before the median class

h = Width of median class

f = Frequency of the median class

- **Mode**

  For a set of values, mode is the most frequently occurring value.

  For a grouped data,

  $$\text{Mode} = L + \frac{(f_i - f_{i-1})}{(2f_i - f_{i-1} - f_{i+1})} h$$

  where,

  L = Lower boundary of the modal class

  f_i = Frequency of the modal class (Highest frequency)

  f_{i-1} = Frequency of the class before the modal class

  f_{i+1} = Frequency of the class after the modal class

  h = Width of the modal class

- **Relation between Mean, Median and Mode :**

  Mean – Mode = 3(Mean – Median)

- **Standard Deviation**

  $$\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n} \quad \text{For a Set of Values}$$

  $$\sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} \quad \text{For a Grouped Data}$$

❑ ❑ ❑ ❑ ❑

Assignment – 1

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. A sample of 15 data is as follows: 17, 18, 17, 17, 13, 18, 5, 5, 6, 7, 8, 9, 20, 17, 3. The mode of the data is **[ME – 2017]**

(A) 4 (B) 13
(C) 17 (D) 20

2. In a housing society, half of the families have a single child per family, while the remaining half have two children per family. The probability that a child picked at random, has a sibling is _______. **[EC–2014]**

3. An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the tenth toss is **[EC–2014]**

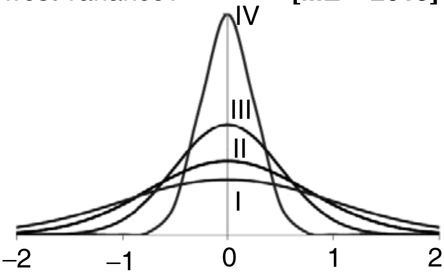
(A) 0.067 (B) 0.073
(C) 0.082 (D) 0.091

4. The probability density function of evaporation E on any day during a year in a watershed is given by

$$f(E) = \begin{cases} \dfrac{1}{5} & 0 \le E \le 5\,\text{mm / day} \\ 0 & \text{otherwise} \end{cases}$$

The probability that E lies in between 2 and 4 mm/day in a day in the watershed is (in decimal) _________ **[CE–2014]**

5. Among the four normal distributions with probability density functions as shown below, which one has the lowest variance? **[ME – 2015]**



(A) I (B) II
(C) III (D) IV

6. A random variable X has probability density function f(x) as given below:

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value E[X] = 2/3, then Pr[X < 0.5] is _______. **[EE–2015]**

7. Consider a Poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by **[ME–2016]**

(A) $\sqrt{\mu}$ (B) μ^2
(C) μ (D) $1/\mu$

8. The probability of getting a "head" in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a "head" is obtained. If the tosses are independent, then the probability of getting "head" for the first time in the fifth toss is ________ **[EC–2015]**

9. X and Y are two random independent events. It is known that P(X) = 0.40 and $P(X \cup Y^C) = 0.7$. Which one of the following is the value of $P(X \cup Y)$? **[CE–2016]**

(A) 0.7 (B) 0.5
(C) 0.4 (D) 0.3

10. Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is ________. **[CS–2016]**

Q 11 to Q 20 carry two marks each

11. In the following table, x is a discrete random variable and p(x) is the probability density. The standard deviation of x is **[ME–2014]**

x	1	2	3
P(x)	0.3	0.6	0.1

(A) 0.18 (B) 0.36
(C) 0.54 (D) 0.6

12. The number of accidents occurring in a plant in a month follows Poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is **[ME–2014]**

(A) 0.029 (B) 0.034
(C) 0.039 (D) 0.044

13. Parcels from sender S to receiver R pass sequentially through two post-offices. Each post-office has a probability $\frac{1}{5}$ of losing an incoming parcel, independently of all other parcels. Given that a parcel is lost, the probability that it was lost by the second post-office is ____. **[EC–2014]**

14. A traffic office imposes on an average 5 number of penalties daily on traffic violators. Assume that the number of penalties on different days is independent and follows a Poisson distribution. The probability that there will be less than 4 penalties in a day is ________. **[CE–2014]**

15. The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks in it is 5%. GIVEN that a student passes the examination, the probability that the student gets above 90% marks is **[ME–2015]**

(A) $\frac{1}{18}$ (B) $\frac{1}{4}$

(C) $\frac{2}{9}$ (D) $\frac{5}{18}$

16. The probability density function of a random variable, x is

$f(x) = \frac{x}{4}(4 - x^2)$ for $0 \le x \le 2$

$= 0$ otherwise

The mean, μ_x of the random variable is ___________. **[CE–2015]**

17. Suppose a fair six-sided die is rolled once. If the value on the die is 1, 2, or 3, the die is rolled a second time. What is the probability that the sum total of values that turn up is at least 6?

[EC–2015]

(A) 10/21 (B) 5/12

(C) 2/3 (D) 1/6

18. The variance of the random variable X with probability density function $f(x) = \frac{1}{2}|x|e^{-|x|}$ is _____. **[EC–2015]**

19. Two random variables X and Y are distributed according to

$F_{X,Y}(x, y)$

$$= \begin{cases} (x+y), & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The probability $P(X + Y \le 1)$ is ______.

[EC–2016]

20. An urn contains 5 red and 7 green balls. A ball is drawn at random and its colour is noted. The ball is placed back into the urn along with another ball of the same colour. The probability of getting a red ball in the next draw is

[IN–2016]

(A) $\frac{65}{156}$ (B) $\frac{67}{156}$

(C) $\frac{79}{156}$ (D) $\frac{89}{156}$

❑ ❑ ❑ ❑ ❑

Assignment – 2

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. A group consists of equal number of men and women. Of this group 20% of the men and 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is_______ **[ME–2014]**

2. Vehicle arriving at an intersection from one of the approach roads follow the Poisson distribution. The mean rate of arrival is 900 vehicles per hour. If a gap is defined as the time difference between two successive vehicle arrivals (with vehicle assumed to be points), the probability (up to four decimal places) that the gap is greater than 8 second is _______. **[CE–2017]**

3. Let X be a zero mean unit variance Gaussian random variable. E[|X|] is equal to _______. **[EC–2014]**

4. A fair (unbiased) coin was tossed four times in succession and resulted in the following outcomes: (i) Head, (ii) Head, (iii) Head, (iv) Head. The probability of obtaining a 'Tail' when the coin is tossed again is **[CE–2014]**

(A) 0 (B) $\frac{1}{2}$

(C) $\frac{4}{5}$ (D) $\frac{1}{5}$

5. Three vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specifications are 0.8, 0.7 and 0.5. Each vendor supplies one component. The probability that out of total three components supplied by the vendors, at least one will meet the design specification is ________. **[ME–2015]**

6. Consider the following probability mass function (p.m.f.) of a random variable X:

$$p(x, q) = \begin{cases} q & \text{if } X = 0 \\ 1-q & \text{if } X = 1 \\ 0 & \text{otherwise} \end{cases}$$

If q = 0.4, the variance of X is ______.

[CE–2015]

7. The area (in percentage) under standard normal distribution curve of random variable Z within limits from –3 to +3 is __________ **[ME–2016]**

8. Types II error in hypothesis testing is

(A) acceptance of the null hypothesis when it is false and should be rejected

(B) rejection of the null hypothesis when it is true and should be accepted

(C) rejected of the null hypothesis when it is false and should be rejected

(D) acceptance of the null hypothesis when it is true and should be accepted **[CE–2016]**

9. A voltage V_1 is measured 100 times and its average and standard deviation are 100 V and 1.5 V respectively. A second voltage V_2, which is independent of V_1, is measured 200 times and its average and standard deviation are 150 V and 2 V respectively. V_3 is computed as: $V_3 = V_1 + V_2$. Then the standard deviation of V_3 in volt is ________.

[IN–2016]

10. Suppose p is the number of cars per minute passing through a certain road junction between 5 PM and 6 PM, and p has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval? **[CS–2013]**

(A) $8/(2e^3)$ (B) $9/(2e^3)$

(C) $17/(2e^3)$ (D) $26/(2e^3)$

Q 11 to Q 20 carry two marks each

11. Consider an unbiased cubic dice with opposite faces coloured identically and each face coloured red, blue or green such that each colour appears only two times on the dice. If the dice is thrown thrice, the probability of obtaining red colour on top face of the dice at least twice is ____. **[ME–2014]**

12. Let X be a real-valued random variable with E[X] and E[X^2] denoting the mean values of X and X^2, respectively. The relation which always holds true is

(A) $(E[X])^2 > E[X^2]$ **[EC–2014]**

(B) $E[X^2] \geq (E\,[X])^2$

(C) $E[X^2] = (E[X])^2$

(D) $E[X^2] > (E[X])^2$

13. A fair coin is tossed n times. The probability that the difference between the number of heads and tails is (n – 3) is **[EE–2014]**

(A) 2^{-n} (B) 0

(C) ${}^{n}C_{n-3}\, 2^{-n}$ (D) 2^{-n+3}

14. An observer counts 240 veh/h at a specific highway location. Assume that the vehicle arrival at the location is Poisson distributed, the probability of having one vehicle arriving over a 30–second time interval is _________.

[CE–2014]

15. Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is **[EE–2015]**

(A) 5/11 (B) 1/2

(C) 7/13 (D) 6/11

16. The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors randomly picked up, 3 are defective is **[IN–2015]**

(A) 0.001 (B) 0.057

(C) 0.107 (D) 0.3

17. Let $X \in \{0, 1\}$ and $Y \in (0, 1)$ be two independent binary random variables. If $P(X = 0) = p$ and $P(Y = 0) = q$, the $P(X + Y \geq 1)$ is equal to **[EC–2015]**

(A) $pq + (1 - p)(1 - q)$

(B) pq

(C) $p(1 - q)$

(D) $1 - pq$

18. The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is _______. **[ME–2016]**

19. If a random variable X has a Poisson distribution with mean 5, then the expectation $E[(X + 2)^2]$ equals _________. **[CS–2017]**

20. Consider the following experiment.

Step 1. Flip a fair coin twice.

Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.

Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.

Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is (up to two decimal places_______. **[CS–2016]**

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Assignment – 3

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 10 carry one mark each

1. A nationalized bank has found that the daily balance available in its savings accounts follows a normal distribution with a mean of Rs. 500 and a standard deviation of Rs. 50. The percentage of savings account holders, who maintain an average daily balance more than Rs. 500 is________ **[ME–2014]**

2. Let X be a random variable which is uniformly chosen from the set of positive odd numbers less than 100. The expectation, E[X], is _________. **[EC–2014]**

3. Ram and Ramesh appeared in an interview for two vacancies in the same department. The probability of Ram's selection is 1/6 and that of Ramesh is 1/8. What is the probability that only one of them will be selected? **[EC–2015]**

(A) 47/48 (B) 1/4
(C) 13/48 (D) 35/48

4. If {x} is a continuous, real valued random variable defined over the interval $(-\infty, +\infty)$ and its occurrence is defined by the density function given as:

$$f(x) = \frac{1}{\sqrt{2\pi} * b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$ where 'a' and 'b' are the statistical attributes of the random variable {x}. The value of the integral $\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi} * b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$ is **[CE–2014]**

(A) 1 (B) 0.5
(C) π (D) $\frac{\pi}{2}$

5. If P(X) = 1/4, P(Y) = 1/3, and $P(X \cap Y) = 1/12$, the value of P(Y/X) is **[ME–2015]**

(A) $\frac{1}{4}$ (B) $\frac{4}{25}$
(C) $\frac{1}{3}$ (D) $\frac{29}{50}$

6. Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\bar{A}$ and $\bar{B}$ be their complements. Which one of the following statements is FALSE?

[EC–2015]

(A) $P(A \cap B) = P(A)\,P(B)$

(B) $P(A|B) = P(A)$

(C) $P(A \cup B) = P(A) + P(B)$

(D) $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

7. The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is _____.

[EC–2016]

8. The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is ______ *(Note: answer with one decimal accuracy)* **[CE–2016]**

9. A probability density function on the interval [a, 1] is given by $1/x^2$ and outside this interval the value of the function is zero. The value of a is ______. **[CS–2016]**

10. Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $P(3V \geq 2U)$ is **[ME–2013]**

(A) 4/9 (B) 1/2

(C) 2/3 (D) 5/9

Q 11 to Q 20 carry two marks each

11. A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3 and 1/6, respectively. The mean value and the variance of the number of defective pieces produced by the machine in a day, respectively, are

[ME–2014]

(A) 1 and 1/3 (B) 1/3 and 1

(C) 1 and 4/3 (D) 1/3 and 4/3

12. A box contains 4 red balls and 6 black balls. Three balls are selected randomly from the box one after another without replacement. The probability that the selected set contains one red ball and two black balls is **[EC–2014]**

(A) $\frac{1}{20}$ (B) $\frac{1}{12}$

(C) $\frac{3}{10}$ (D) $\frac{1}{2}$

13. Let X be a random variable with probability density function

$$f(x) = \begin{cases} 0.2, & \text{for } |x| \leq 1 \\ 0.1, & \text{for } 1 < |x| \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

The probability $P(0.5 < x < 5)$ is ________. **[EE–2014]**

14. The probability of obtaining at least two "SIX" in throwing a fair dice 4 times is **[ME–2015]**

(A) 425/432 (B) 19/144
(C) 13/144 (D) 125/432

15. Two coins R and S are tossed. The 4 joint events H_RH_S, T_RT_S, H_RT_S, T_RH_S have probabilities 0.28, 0.18, 0.30, 0.24, respectively, where H represents head and T represents tail. Which one of the following is TRUE? **[EE–2015]**

(A) The coin tosses are independent
(B) R is fair, S is not
(C) S is fair, R is not
(D) The coin tosses are dependent

16. The probability density function of a random variable X is $p_X(x) = e^{-x}$ for $x \geq 0$ and 0 otherwise. The expected value of the function $g_X(x) = e^{3x/4}$ is ___________. **[IN–2015]**

17. A fair die with faces {1, 2, 3, 4, 5, 6} is thrown repeatedly till '3' is observed for the first time. Let X denote the number of times the die is thrown. The expected value of X is __________. **[EC–2015]**

18. Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is **[ME–2016]**

(A) $\frac{16}{5525}$ (B) $\frac{64}{2197}$
(C) $\frac{3}{13}$ (D) $\frac{8}{16575}$

19. Probability density function of a random variable X is given below.

$$f(x) = \begin{cases} 0.25 & \text{if } 1 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$P(X \leq 4)$ is **[CE–2016]**

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$
(C) $\frac{1}{4}$ (D) $\frac{1}{8}$

20. Consider the random process

$$X(t) = U + Vt$$

where U is a zero-mean Gaussian random variable and V is a random variable uniformly distributed between 0 and 2. Assume that U and V are statistically independent. The mean value of the random process at t = 2 is ________. **[EC–2017]**

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Assignment – 4

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 6 carry one mark each

1. Probability of getting an even number in a single throw with a die is

(A) 1/2 (B) 2/3
(C) 1/4 (D) 1/3

2. Probability of getting tail in a throw of a coin is

(A) 1 (B) 1/3
(C) 1/4 (D) 1/2

3. A bag contains 6 white balls, 9 black balls. The probability of drawing a black ball is

(A) 2/5 (B) 3/5
(C) 1/5 (D) 4/5

4. Probability of a card drawn at random from an ordinary pack of cards to be club card is

(A) 3/4 (B) 2/4
(C) 1/4 (D) 1/5

5. Probability of throwing a number greater than 3 with an ordinary die is

(A) 1/2 (B) 1/3
(C) 1/4 (D) 2/3

6. Probability of getting a total of more than 10 in a single throw with 2 dice

(A) 1/6 (B) 1/8
(C) 1/12 (D) 2/3

Q 7 to Q 18 carry two marks each

7. Probability than a leap year selected at random will have 53 Sundays.

(A) 2/5 (B) 1/8
(C) 2/7 (D) 1/7

8. A card is drawn from an ordinary pack of playing cards and a person bets that it is a spade or an ace. Then the odds against his winning this bet is

(A) 9 to 4 (B) 8 to 5
(C) 7 to 6 (D) none of above

9. In a horse race the odds in favour of four horses H_1, H_2, H_3, H_4 are 1 : 3, 1 : 4, 1 : 5; 1 : 6 respectively not more than one wins at a time. Then the chance that one of them wins is

(A) 320/419 (B) 319/420
(C) 419/520 (D) 520/519

10. In a garden 40% of the flowers are roses and the rest are carnations. If 25% of the roses and 10% of the carnations are red, the probability that a red flower selected at random is a rose.

(A) 5/6 (B) 1/4
(C) 4/5 (D) 5/8

11. Find the chance of throwing more than 15 in one throw with 3 dice

(A) 1/54
(B) 17/216
(C) 5/108
(D) cannot be determined

12. In a race where 12 horses are running, the chance that horse A will win is 1/6, that B will win is 1/10 and that C will win is 1/8. Assuming that a dead heat is impossible the chance that one of them will win

(A) 1/390 (B) 47/120
(C) 3/20 (D) 1/54

13. There are two bags, one of which contains 5 red and 7 white balls and the other 3 red and 12 white balls. A ball is to be drawn from one or other of the two bags, find the chance of drawing a red ball.

(A) 37/120 (B) 1/10
(C) 1/13 (D) 1/96

14. What is the probability of a particular person getting 9 cards of the same suit in one hand at a game of bridge where 13 cards are dealt to a person ?

(A) $\dfrac{{}^{13}C_9 \times 4}{{}^{39}C_4}$

(B) $\dfrac{{}^{13}C_9 \times {}^{13}C_9 \times {}^{13}C_2 \times 4}{{}^{52}C_9}$

(C) $\dfrac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}$

(D) $\dfrac{{}^{13}C_9 \times 4}{{}^{52}C_{13}}$

15. A man and his wife appear for an interview for two posts. The probability of the husband's selection is 1/7 and that of the wife's selection is 1/5. The probability that only one of them will be selected is

(A) 2/7 (B) 4/5
(C) 4/35 (D) 6/35

16. There are two bags. One bag contains 4 white and 2 black balls. Second bag contains 5 white and 4 black balls. Two balls are transferred from first bag to second bag. Then one ball is taken from the second bag. The probability that it is white is

(A) 42/165 (B) 95/165
(C) 5/165 (D) 48/165

17. In a single throw of two dice find the probability that neither a doublet (same number on the both dice) nor a total of 9 will appear.

(A) 5/1 (B) 1/9
(C) 13/18 (D) 1/4

18. A speaks truth in 75% and B in 80% of the cases. In what percentage of cases are they likely to contradict each other narrating the same incident ?

(A) 75% (B) 80%
(C) 35% (D) 100%

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Assignment – 5

Duration : 45 Min. **Max. Marks : 30**

Q 1 to Q 6 carry one mark each

1. A die is rolled. The probability of getting a number 1 or 6 on the upper face is

(A) 1/3 (B) 1/2
(C) 2/3 (D) 1/4

2. The probability of the horse A winning the race is 1/5 and the probability of the horse B winning the same race is 1/6, then the probability of one of the horse to win the race is

(A) 11/29 (B) 11/28
(C) 11/30 (D) 11/31

3. A card is drawn from a pack of 52 cards and then a second is drawn. Probability that both the cards drawn are queen is

(A) 1/219 (B) 1/13
(C) 1/17 (D) 1/221

4. A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. The probability that both the balls drawn are black is

(A) 3/28 (B) 3/8
(C) 2/7 (D) 3/29

5. 4 coins are tossed. Then the probability that at least one head turns up is

(A) 15/26 (B) 1/16
(C) 14/16 (D) 15/16

6. In a throw of 3 dice the probability that at least one die shows up 1 is

(A) 5/6 (B) 1/6
(C) 91/216 (D) 90/215

Q 7 to Q 18 carry two marks each

7. Match the following :

List – I

(a) $P(\phi)$, ϕ is the empty set
(b) $P(A/B)\cdot P(B)$
(c) $P(\bar{A})$
(d) $P(\bar{A} \cap \bar{B})$
(e) $P(A \cup B)$

List – II

(i) $1 - P(A)$
(ii) $P(A \cap B)$
(iii) $1 - P(A \cup B)$
(iv) 0
(v) $P(A) + P(B) - P(A \cap B)$

(A) a – iv, b – ii, c – i, d – iii, e – v
(B) a – iii, b – ii, c – i, d – iv, e – v
(C) a – ii, b – iii, c – i, d – iv, e – v
(D) a – i, b – ii, c – iii, d – iv, e – v

8. Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentages of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and found defective. Find the probability that the item was produced by machine C.

(A) 2/25 (B) 1/25
(C) 4/25 (D) 3/25

9. A fair die is tossed 180 times. The expected number of sixes is

(A) 40 (B) 25
(C) 180 (D) 30

10. In a population having 50% rice consumers, what is the probability that three or less out of 10 are rice consumers?

(A) 17% (B) 10%
(C) 40% (D) 50%

11. The chances of a person being alive who is now 35 years old, till he is 75 are 8 : 6 and of another person being alive now 40 years old till he is 80 are 4 : 5. The probability that at least one of these persons would die before completing 40 years hence is

(A) 8/14 (B) 16/63
(C) 4/9 (D) 47/63

12. If 10 coins are tossed 100 times, how many times would you expect 7 coins to fall head upward ?

(A) 12 (B) 11
(C) 10 (D) 9

13. If the probability of a defective bolt is 0.1, the mean and standard deviation for the distribution of defective bolts in a total of 500 are

(A) 50, 6.7 (B) 40, 7.3
(C) 20, 3.1 (D) none of these

14. Two cards are drawn with replacement from a well shuffled deck of 52 cards. The mean and standard deviation for the number of aces are

X	0	1	2
P(x)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Probability distribution table

(A) $\frac{2}{13}$,0.377 (B) $\frac{1}{13}$,0.277
(C) $\frac{3}{13}$,0.477 (D) none of these

15. In a Binomial distribution, the mean and standard deviation are 12 and 2 respectively. The values of n and p are respectively.

(A) 12, 2 (B) 9, $\frac{1}{3}$
(C) 18, $\frac{2}{3}$ (D) $\frac{1}{3}$, 18

16. For a biased die the probabilities for different faces to turn up are given below.

Face	Probability
1	0.10
2	0.32
3	0.21
4	0.15
5	0.05
6	0.17

The die is tossed and you are told that either face 1 or face 2 has turned up. The probability that it is face 1 is

(A) 2/11 (B) 3/21

(C) 11/13 (D) 5/21

17. There are three events A, B and C, one of which must occur and only one can happen, the odds are 8 to 3 against A, 5 to 2 against B; find the probability of C.

(A) 41/57 (B) 3/11

(C) 34/77 (D) 2/7

18. Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability that she studies computer science on Wednesday ?

(A) 0.24 (B) 0.36

(C) 0.4 (D) 0.6

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Assignment – 6

Duration : 45 Min. **Max. Marks : 30**

Q1 to Q6 carry one mark each

1. The probability that the birthday of a child is Saturday or Sunday is

(A) 1 / 2 (B) 2 / 7
(C) 1 / 3 (D) 1 / 7

2. The probability of getting a multiple of 2 in the throw of a die

(A) 1 / 3 (B) 1 / 4
(C) 1 / 6 (D) 1 / 2

3. Probability that the sum of the score is odd in a throw of two dice is

(A) 1 / 4 (B) 1 / 5
(C) 1 / 2 (D) 1 / 3

4. A committee of 5 students is to be chosen from 6 boys and 4 girls. The probability that the committee contains exactly 2 girls is

(A) 9 / 20 (B) 1 / 6
(C) 10 / 21 (D) 1 / 9

5. A bag contains 7 red, 5 blue, 4 white and 4 black balls. Then the probability that a ball drawn at random is red or white is

(A) 11 / 20 (B) 1 / 7
(C) 1 / 11 (D) 3 / 20

6. Two cards are drawn at random from a pack of 52 cards. The probability that one may be a jack and other an Ace is

(A) 1 / 16 (B) 7 / 663
(C) 8 / 663 (D) none of these

Q7 to Q18 carry two marks each

7. The probability that a man aged 50 years will die within a year is 0.01125. The probability that out of 12 such men at least 11 will reach their fifty first birthday.

(A) 1 (B) 0
(C) 0.9923 (D) 0.8823

8. The probability that a certain beginner at golf gets a good shot if he uses the correct club is 1/3 and the probability of a good shot with an incorrect club is 1/4. In his bag are 5 different clubs, only one of which is correct for the shot in question. If he chooses a club at random and takes a stroke, the probability that he gets a good shot is

(A) 2 / 3 (B) 4 / 15
(C) 5 / 9 (D) 7 / 36

9. From a pack of cards two are drawn, the first being replaced before the second is drawn. The probability that the first is a diamond and the second is a king is

(A) 1 / 52 (B) 4 / 13
(C) 57 / 64 (D) 11 / 52

10. One of the two mutually exclusive events must occur, if the chance of one is 2/3 of the other, then odds in favour of the other are

(A) 1 : 1 (B) 1 : 2
(C) 2 : 3 (D) 3 : 2

11. Given $P(A \cap \bar{B}) = \frac{1}{3}$ and $P(A \cup B) = \frac{2}{3}$ then P(B) is

(A) 2 / 3 (B) 1 / 5
(C) 1 / 3 (D) 4 / 5

12. There are 64 beds in a garden and 3 seeds of a particular type of flower are sown in each bed. The probability of a flower being white is 1/4. The number of beds with 3, 2, 1, and 0 white flowers is respectively

(A) 1, 9, 27, 27 (B) 27, 9, 1, 27
(C) 27, 9, 1, 1 (D) 27, 9, 9, 1

13. A sample space has two events A and B such that probabilities $P(A \cap B) = 1/2$, $P(\bar{A}) = 1/3$, $P(\bar{B}) = 1/3$. What is $P(A \cup B)$?

(A) 11/12 (B) 10/12
(C) 9/12 (D) 8/12

14. A man takes a step forward with probability 0.4 and backward with probability 0.6. The probability that at the end of eleven steps he is one step away from the starting point.

(A) 0.3678 (B) 0.25
(C) ${}^{11}c_2(0.24)^3$ (D) none of these

15. Find the probability of drawing one rupee coin from a purse with two compartments one of which contains 3 fifty paisa coins and 2 one–rupee coins and the other contains 2 fifty paisa coins and 3 one rupee coins.

(A) 1 / 2 (B) 2 / 5
(C) 3 / 5 (D) none of these

16. If A and B are two events such that $P(A \cup B) = 5/6$. $P(A \cap B) = 1/3$ $P(\bar{B}) = 1/2$ then the events A and B are

(A) dependent
(B) independent
(C) mutually exclusive
(D) none of these

17. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that value of the determinant chosen is positive is

(A) 2 / 17 (B) 1 / 17
(C) 1 / 16 (D) 3 / 16

18. One hundred identical coins each with probability p of showing up heads are tossed. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of the heads showing in 51 coins, then the value of p is

(A) 1 / 2 (B) 49 / 101
(C) 50 / 101 (D) 51 / 101

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Test Paper – 1

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. In a shooting competition, the probability of hitting the target by A is 2/5, by B is 2/3 and by C is 3/5. If all of them fire independently at the same target, calculate the probability that only one of them will hit the target.

(A) 2/5 (B) 2/3
(C) 3/5 (D) 1/3

2. A box contains 6 white balls and 3 black balls and another box contains 4 white balls and 5 black balls. The probability that a ball selected from one of the box again selected at random is a white ball.

(A) 6/18 (B) 4/18
(C) 1/2 (D) 5/9

3. Two sisters A and B appeared for an Audition. The probability of selection of A is 1/5 and that of B is 2/7. The probability that both of them are selected is

(A) 2/35 (B) 1/5
(C) 2/7 (D) none of these

4. Two balls are to be drawn from a bag containing 5 red and 7 white balls. Find the chance that they will both be white

(A) 5/108 (B) 5/7
(C) 7/22 (D) 2/35

5. From a bag containing 4 white and 5 black balls a man draws 3 at random. Then the probability that all 3 are black is

(A) 5/42 (B) 1/42
(C) 1/20 (D) 1/15

Q6 to Q15 carry two marks each

6. Urn A contains 6 red and 4 black balls and urn B contains 4 red and 6 black balls, one ball is drawn at random from urn A and placed in urn B. Then one ball drawn at random from urn B and placed in urn A. If one ball is now drawn from urn A, the probability that it is found to be red is

(A) 3 / 11 (B) 1 / 17
(C) 32 / 55 (D) 31 / 56

7. Two persons each make a single throw with a dice. The probability they get equal value is P_1. Four persons each make a single throw and probability of three being equal is P_2. Then

(A) $P_1 = P_2$ (B) $P_1 < P_2$
(C) $P_1 > P_2$ (D) none of these

8. A bag has 13 red, 14 green and 15 black balls. The probability of getting exactly 2 blacks on pulling out 4 balls is P_1. Now the number of each colour ball is doubled and 8 balls are pulled out.

The probability of getting exactly 4 blacks is P_2. Then

(A) $P_1 = P_2$ (B) $P_1 > P_2$

(C) $P_1 < P_2$ (D) none of these

9. If A and B are arbitrary events, then

(A) $P(A \cap B) \geq P(A) + P(B) - 1$

(B) $P(A \cap B) \leq P(A) + P(B) - 1$

(C) $P(A \cap B) = P(A) + P(B) - 1$

(D) none of above

10. One mapping is selected at random from all the mappings from the set $S = \{1, 2, 3, ...n\}$ into itself. The probability that the selected mapping is one to one is

(A) $1/n^n$ (B) $1/n!$

(C) $n/n!$ (D) none of these

11. For two events A and B $P(A \cap B)$ is

(i) not less than $P(A) + P(B) - 1$ (ii) not greater than $P(A) + P(B)$

(iii) equal to $P(A) + P(B) - P(A \cup B)$

(iv) equal to $P(A) + P(B) + P(A \cup B)$

(A) i, ii, iii (B) i, iv

(C) i, iii (D) i, ii, iii, iv

12. Two persons A and B have respectively n + 1 and n coins, which they toss simultaneously. Then the probability that A will have more heads than B is

(A) 1 / 2 (B) > 1 / 2

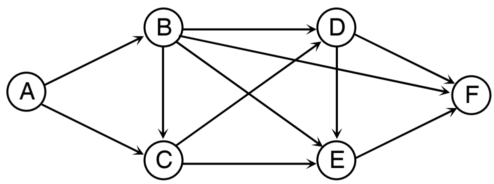
(C) < 1 / 2 (D) none of these

13. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are p, q and 1/2 respectively. If the probability that the student is successful is 1/2. Then

(A) $p = q = 1$ (B) $p = q = 1/2$

(C) $p = 1,\ q = 0$ (D) $p = 1,\ q = 1/2$

14. The figure below shows the network connecting cities A, B, C, D, E and F. The arrows indicate permissible directions of travel. If Deepak from city A wants to visit city F, what is the probability he will pass through city C



(A) 3/5 (B) 4/5

(C) 1/2 (D) 3/4

15. Mean of 200 observations was found to be 90. However, when the observations were being made, two observations were wrongly taken as 15 and 80 instead of 40 and 87. Then the correct mean is

(A) 90 (B) 91.16

(C) 90.16 (D) 91.6

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Test Paper – 2

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. What is the chance of throwing a number greater than 4 with an ordinary die whose faces are numbered from 1 to 6 ?

(A) 1/6 (B) 1/3
(C) 1/4 (D) none of these

2. Find the chance of throwing at least one ace in a simple throw with two dice numbered 1 to 6.

(A) 1/11 (B) 1/36
(C) 11/18 (D) 11/36

3. What is the probability that a digit selected at random from the logarithmic table is 1 ?

(A) 1/2 (B) 1/5
(C) 1/3 (D) 1/10

4. A problem in mathematics is given to three students Dayanand, Ramesh and Naresh whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. The probability that the problem is solved is

(A) 1/4 (B) 1/2
(C) 3/4 (D) 1/3

5. A bag contains 3 black and 5 white balls. One ball is drawn from the bag. What is the probability that the ball is not black ?

(A) 3/8
(B) 5/8
(C) 2/7
(D) none of the these

Q6 to Q15 carry two marks each

6. A single letter is selected at random from the word "PROBABILITY". The probability that it is a vowel is

(A) 1/3 (B) 2 / 21
(C) 0 (D) 2 / 9

7. A and B throw a coin alternately till one of them gets a 'head' and wins the game. Their respective probabilities of winning are

(A) 1/3, 2/3 (B) 1/2, 1/2
(C) 1/4, 3/4 (D) 2/3, 1/3

8. Two dice are thrown the probability of getting an odd number on the one and a multiple of 3 on the other is

(A) 11/36 (B) 2/17
(C) 10/36 (D) 12/36

9. In a certain college, 4% of the men and 1% of the women are taller than 1.8m. Further 60% of the students are woman. If a student is selected at random and is taller than 1.8m, the probability that the student is women is

(A) 3/11 (B) 2/11

(C) 4/11 (D) 1/11

10. A and B throws two dice; if A throws 9 then B's chance of throwing a higher number is $\frac{1}{6}$, state whether the above statement is true or false.

(A) True

(B) False

(C) Can not say

(D) Data insufficient

11. A pack of cards contains 4 aces, 4 queens and 4 jacks. Two cards are drawn at random. The probability that at least one of them is an ace is

(A) $\frac{19}{33}$ (B) $\frac{3}{16}$

(C) $\frac{1}{6}$ (D) $\frac{1}{9}$

12. x is a random variable with mean μ and S. D. $\sigma > 0$

$$x' = \frac{x-\mu}{\sigma}\text{; then var}(x') = \_\_\_\_\_\_\_\_$$

(A) 0 (B) 2

(C) can't say (D) 1

13. If A and B are two events such that P(A) = 0 and P(B) $\neq$ 1, then $P(\bar{A}/\bar{B})$ is equal to

(A) $1 - P(A/\bar{B})$ (B) $1 - P(\bar{A}/B)$

(C) $1 - \frac{P(A/B)}{P(\bar{B})}$ (D) $\frac{P(\bar{A})}{P(\bar{B})}$

14. Ten points are marked on a straight line and 11 points are marked on another straight line. If any three points are chosen, what is the probability that it forms a triangle?

(A) $\frac{11}{14}$ (B) $\frac{^{21}C_3}{\left(^{10}C_3 + {}^{11}C_3\right)}$

(C) $\frac{6}{7}$ (D) $\frac{^{10}C_3 + {}^{11}C_3}{1045}$

15. Three machines A, B and C produce identical items of their respective output 5%, 4% and 3% items are faulty. If A produces 25% of the total output, B produces 30% and C the remainder and an item selected at random is found to be faulty, what is the probability that it was produced by machine C?

(A) 0.155 (B) 0.255

(C) 0.355 (D) 0.455

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Test Paper – 3

Duration : 30 Min. **Max. Marks : 25**

Q1 to Q5 carry one mark each

1. An urn contains 25 balls numbered 1 through 25. Two balls are drawn from the urn with replacement. The probability of getting at least one odd is

(A) 144/625 (B) 312/625
(C) 12/25 (D) 481/625

2. Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. The chance that exactly 2 of them will be children is

(A) 10/21 (B) 5/8
(C) 13/32 (D) 3/32

3. Let A and B be events with P(A) = $\frac{1}{2}$, P(B) = $\frac{1}{3}$ and P(A $\cap$ B) = $\frac{1}{4}$. Then P(A/B) is

(A) 1/4 (B) 1/3
(C) 1/2 (D) 3/4

4. How many different words can be formed from the letters of the word Ganeshpuri when all the letters taken

(A) 2! (B) 10!
(C) 9! (D) 5!

5. A bag contains 8 green and 10 white balls. Two balls are drawn. What is the probability that one is green and the other is white ?

(A) $\frac{80}{153}$ (B) $\frac{10}{17}$
(C) $\frac{8}{17}$ (D) $\frac{81}{157}$

Q6 to Q15 carry two marks each

6. Fifteen coupons are numbered 1, 2, 3, …….., 15. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on selected coupon is 9, is

(A) $\left(\frac{9}{16}\right)^6$ (B) $\left(\frac{7}{15}\right)^8$
(C) $\left(\frac{2}{5}\right)^7$ (D) none of these

7. A bag contains 2 white and 3 black balls. A ball is drawn 5 times, each being replaced before another is drawn. Find the probability that exactly 4 of the balls drawn are white.

(A) $\frac{2}{5}$ (B) $\frac{48}{625}$
(C) $\frac{31}{725}$ (D) $\frac{3}{5}$

8. From a pack of 52 cards two are drawn at random. The probability that one is a king and the other a queen is

(A) 3/ 8 (B) 8 / 663
(C) 7/ 663 (D) 9 / 663

9. In three throws of two dice, the probability of throwing doublets not more than two times is

(A) 1/216 (B) 1/6
(C) 4/216 (D) 215 / 216

10. If the sum of the mean and the variance of a Binomial distribution for 18 trials is 10, find the distribution.

(A) $(p + q)^n$ (B) $\left(\frac{1}{3}+\frac{2}{3}\right)^{18}$

(C) $\left(\frac{1}{2}+\frac{1}{2}\right)^{18}$ (D) $\left(\frac{1}{3}+\frac{2}{3}\right)^{11}$

11. If the sum of the mean and the variance of a Binomial distribution for 5 trials is 1.8 find the distribution.

(A) $\left(\frac{1}{3}+\frac{2}{3}\right)^{5}$ (B) $\left(\frac{1}{2}+\frac{1}{2}\right)^{5}$

(C) $\left(\frac{6}{7}+\frac{1}{7}\right)^{9}$ (D) $\left(\frac{4}{5}+\frac{1}{5}\right)^{5}$

12. For a Binomial distribution, the mean is 6 and the standard deviation is $\sqrt{2}$. The Binomial distribution is

(A) $\left(\frac{1}{3}+\frac{2}{3}\right)^{9}$ (B) $\left(\frac{1}{2}+\frac{1}{2}\right)^{9}$

(C) $\left(\frac{6}{7}+\frac{1}{7}\right)^{9}$ (D) $\left(\frac{4}{5}+\frac{1}{5}\right)^{9}$

13. If 3 squares are chosen at random on a chess board the probability that they should be in a diagonal line is

(A) 5/744 (B) 3/744
(C) 7/744 (D) 9/744

14. A man has 9 friends (4 men and 5 women). In how many ways can he invite them, if there have to be exactly 3 women in the invitees ?

(A) 320 (B) 160
(C) 80 (D) 200

15. Given two events A and B and P(A) = 1/4, P(B/A) = 1/2, P(A/B) = 1/4. Which of the following is true?

(A) $P(A/\bar{B}) = 1/2$

(B) A is subevent of B

(C) $P(A/B) + P(A/\bar{B}) = 1$

(D) None of these

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Solutions – Linear Algebra

Answer Key on Assignment – 1

1.	(A)	2.	(D)	3.	0.99 to 1.01	4.	(A)
5.	(A)	6.	(B)	7.	(A)	8.	2
9.	6	10.	(C)	11.	(B)	12.	2.9 to 3.1
13.	(A)	14.	(B)	15.	(D)	16.	(A)
17.	–6 to –6	18.	(C)	19.	(B)	20.	(D)

Answer Key on Assignment – 2

1.	(A)	2.	(D)	3.	199 to 201	4.	23 to 23
5.	88 to 88	6.	2	7.	4.49 to 4.51	8.	16.5 to 17.5
9.	(D)	10.	(C)	11.	48.9 to 49.1	12.	2.0 to 2.0
13.	(D)	14.	(D)	15.	2 to 2	16.	(B)
17.	(B)	18.	(D)	19.	(A)	20.	0 to 0

Answer Key on Assignment – 3

1.	(D)	2.	(D)	3.	(B)	4.	(A)
5.	(A)	6.	(C)	7.	2.0 to 2.0	8.	(C)
9.	(A)	10.	0.95 to 1.05	11.	(B)	12.	(A)
13.	5.0 to 5.0	14.	0	15.	2.9 to 3.1	16.	(B)
17.	(A)	18.	(B)	19.	(A)	20.	(A)

Answer Key on Assignment – 4

1.	(C)	2.	(B)	3.	(A)	4.	(B)
5.	(D)	6.	(A)	7.	(A)	8.	(C)
9.	(A)	10.	(A)	11.	(B)	12.	(D)
13.	(B)	14.	(C)	15.	(A)	16.	(D)
17.	(B)	18.	(B)				

Answer Key on Assignment – 5

1.	(B)	2.	(C)	3.	(A)	4.	(B)
5.	(B)	6.	(A)	7.	(B)	8.	(A)
9.	(B)	10.	(A)	11.	(C)	12.	(B)
13.	(C)	14.	(A)	15.	(B)	16.	(C)
17.	(D)	18	(D)				

Answer Key on Assignment – 6

1.	(B)	2.	(C)	3.	(C)	4.	(C)
5.	(B)	6.	(C)	7.	(B)	8.	(B)
9.	(B)	10.	(D)	11.	(C)	12.	(C)
13.	(C)	14.	(A)	15.	(C)	16.	(A)
17.	(C)	18.	(B)				

Answer Key on Assignment – 7

1.	(C)	2.	(D)	3.	(B)	4.	(B)
5.	(D)	6.	(A)	7.	(A)	8.	(C)
9.	(A)	10.	(A)	11.	(B)	12.	(C)
13.	(B)	14.	(A)	15.	(B)	16.	(A)
17.	(A)	18.	(B)				

Answer Key on Assignment – 8

1.	(C)	2.	(C)	3.	(A)	4.	(C)
5.	(C)	6.	(C)	7.	(A)	8.	(A)
9.	(B)	10.	(C)	11.	(C)	12.	(C)
13.	(C)	14.	(C)	15.	(C)	16.	(B)
17.	(B)	18.	(A)				

Answer Key on Assignment – 9

1.	(B)	2.	(B)	3.	(C)	4.	(A)
5.	(C)	6.	(B)	7.	(C)	8.	(A)
9.	(C)	10.	(B)	11.	(C)	12.	(C)
13.	(C)	14.	(C)	15.	(B)	16.	(C)
17.	(D)	18.	(D)				

Answer Key on Assignment – 10

1.	(D)	2.	(B)	3.	(B)	4.	(C)
5.	(B)	6.	(C)	7.	(A)	8.	(A)
9.	(A)	10.	(B)	11.	(C)	12.	(B)
13.	(C)	14.	(C)	15.	(D)	16.	(C)
17.	(D)	18.	(C)				

Answer Key on Assignment – 11

1.	(C)	2.	(C)	3.	(C)	4.	(D)
5.	(C)	6.	(C)	7.	(C)	8.	(C)
9.	(A)	10.	(B)	11.	(D)	12.	(B)
13.	(C)	14.	(D)	15.	(B)	16.	(C)
17.	(D)	18.	(D)				

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Model Solution on Assignment – 1

1. (A)

$$\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix} = (2)^3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= 8 \times (-12) = -96$$

2. (D)

We know that the Eigen vectors corresponding to distinct Eigen values of real symmetric matrix are orthogonal.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 = 0$$

3. 0.99 to 1.01

Given, $A^2 = I$

We know that above equation is satisfied by identity matrix.

i.e., $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Eigen value of identity matrix is 1.

4. (A)

5. (A)

Area of triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{1[2-3]-2[0-3]+4[0-2]\}$$

$$= \frac{1}{2}\{-1+6-8\} = \frac{3}{2}$$

6. (B)

7. (A)

8. 2

$x - 2y + 3z = -1$

$x - 3y + 4z = 1$

$-2x + 4y - 6z = k$

Arugmented matrix (A/B) is given by

$$(A/B) = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & -3 & 4 & 1 \\ -2 & 4 & -6 & k \end{bmatrix}$$

Now applying row reduction technique –

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

We get

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & k-2 \end{bmatrix}$$

For infinite solution

$\rho(A/B) = \rho(A) = r$

$\therefore\ k - 2 = 0$

$\therefore\ k = 2$

9. 6

The given matrix is

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

Inorder to find eigen values of above matrix.

$|A - \lambda I| = 0$

$\therefore \quad \begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\therefore \quad (4 - \lambda)(1 - \lambda) - 10 = 0$

$\therefore \quad 4 - 5\lambda + \lambda^2 - 10 = 0$

$\therefore \quad \lambda^2 - 5\lambda - 6 = 0$

$\therefore \quad \lambda^2 - 6\lambda + 1\lambda - 6 = 0$

$\therefore \quad \lambda(\lambda - 6) + 1(\lambda - 6) = 0$

$\therefore \quad \lambda + 1 = 0$ or λ

$\therefore \quad \lambda = -1$ or $\lambda = 6$

$\therefore$ Larger eigen value of matrix is 6.

10. (C)

11. (B)

12. 2.9 to 3.1

Characteristic equation is given by

$|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} 0-\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$\therefore \quad \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\therefore \quad \lambda = -1, -2, -3$ are the eigen values of A

$\therefore \quad \lambda_{max} = -1$ and $\lambda_{min} = -3$

$$\therefore \quad \left|\frac{\lambda_{max}}{\lambda_{min}}\right| = \left|\frac{-1}{-3}\right| = \frac{1}{3}$$

13. (A)

$$A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$$

14. (B)

$$\text{Let} \quad A = \begin{bmatrix} 10 & 5+J & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

Given that all eigen values of A are real.

$\Rightarrow$ A is Hermitian

$\Rightarrow A^\theta = A$ i.e. $\left(\overline{A}\right)^T = A$

$$\begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix} = \begin{bmatrix} 10 & 5+J & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$\Rightarrow x = 5 - j$

15. (D)

The characteristic equation is λ^2 – (sum of diagonal elements) $\lambda + |A| = 0$

$\therefore \lambda^2 - (2 + p)\lambda + (2p - 1) = 0$

$$\therefore \lambda = \frac{(2+p) \pm \sqrt{(4+4p+p^2) - 4(2p-1)}}{2}$$

$$= \frac{(2+p) \pm \sqrt{(8-4p+p^2)}}{2}$$

$$\therefore \quad \frac{(2+p) + \sqrt{(8-4p+p^2)}}{(2+p) - \sqrt{(8-4p+p^2)}} = \frac{3}{1}$$

$$\therefore \frac{2\sqrt{(8-4p+p^2)}}{2(2+p)} = \frac{2}{4}$$

$$\therefore 2\sqrt{8-4p+p^2} = (2 + p)$$

$\therefore$ $4(8 - 4p + p^2) = (4 + 4p + p^2)$

$\therefore$ $3p^2 - 20p + 28 = 0$

$\therefore$ $(3p - 14)(p - 2) = 0$

$\therefore$ $p = 2, 14/3$

16. (A)

17. –6 to –6

18. (C)

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$r < n$. Hence, infinite number of solutions.

19. (B)

$|A - \lambda I| = 0$ …characteristic equation

$$\Rightarrow \left| \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix} = 0$$

$\lambda(5 + \lambda) + 6 = 0$

$\lambda^2 + 5\lambda + 6 = 0$

$\therefore$ By Cayley Hamilton's Theorem,

$A^2 + 5A + 6I = 0$

$\therefore$ $A^2 = -5A - 6I$

Multiply by 'A', We get,

$A^3 = -5A^2 - 6A$

$= +5[5A + 6I] - 6A$

$A^3 = 19A + 30I$

20. (D)

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|P| = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - 0\right) - 0 + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$P.P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ P is an orthogonal matrix

(A) Is correct

$\Rightarrow$ Inverse of P is its transpose only

$\therefore$ (B) and (C) both are correct

$\therefore$ (D) is incorrect

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Model Solution on Assignment – 2

1. (A)

Given that $\frac{dx}{dt} = 3x - 5y$

$\frac{dy}{dt} = 4x + 8y$

Matrix form $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

2. (D)

3. 199 to 201

4. 23 to 23

Given Matrices,

$J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$K^T = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$

$\therefore\ K^TJK = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3+4-1 & 2+8-2 & 1+4-6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 6 + 16 + 1$

$K^TJK = 23$

5. 88 to 88

$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

$\Delta = -1\begin{vmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix}$

$-3\begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 1 \end{vmatrix}$

$= -1[1(-1) - 3(4 - 3)] + 2[1(6)]$

$- 3[1(3) + 3(-9)]$

$= -1[- 1 - 3] + 12 - 3[3 - 27]$

$= 4 + 12 - 3 [-24]$

$= 4 + 12 + 72 = 88.$

6. 2

Let $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 2, 5$

7. 4.49 to 4.51

$$[A : B] = \begin{bmatrix} 2 & 3 & : & 5 \\ 3 & p & : & 10 \end{bmatrix}$$

$2C_2 - 3C_1$

$$\begin{bmatrix} 2 & 3 & : & 5 \\ 0 & 2p-9 & : & 5 \end{bmatrix}$$

There is no solution if

rank $(A : B) \neq$ rank(A).

Now, rank $(A : B) = 2$. So rank$(A) = 1$ only if $2p - 9 = 0$ i.e. $p = 9/2 = 4.5$

8. 16.5 to 17.5

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ P & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 \\ P+7 \\ 36 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{bmatrix}$$

$\Rightarrow \quad \lambda = 12 \quad ....(1)$

$2\lambda = P + 7 \quad ....(2)$

and $\quad 3\lambda = 36$

i.e., $\quad \lambda = 12$

$\therefore$ Equation (2) gives $P + 7 = 24$

$\Rightarrow \quad P = 17$

9. (D)

$2x + 5y = 2 \quad ....(1)$

$-4x + 3y = -30 \quad ....(2)$

$2 \times (1) + (2)$ gives $13y = -26$

$\therefore y = -2$

$\therefore \quad 2x - 10 = 2$

$\therefore \quad 2x = 12$

$\therefore \quad x = 6$

$\therefore (6, -2)$

10. (C)

Given : $M^4 = I$

$\therefore M^{4k} = (M^4)^k = I^k = I$

(A) $M^{4k+1} = M$

(B) $M^{4k+2} = M^2$

(C) $M^{4k+3} = M^{4k} M^4 M^{-1} = M^{-1}$

11. 48.9 to 49.1

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore a + d = 14$ and $bc \geq 0$

Now $|A|$ is maximum when ad is maximum and $bc = 0$ i.e.

when $a = d = 7$

$\therefore$ Maximum value of $|A| = 7 \times 7 = 49$

12. 2.0 to 2.0

The given matrix is

$$A = \begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

The matrix is of order 3×4

Hence $\rho(A) \leq 3$

$$\begin{vmatrix} 6 & 0 & 4 \\ -2 & 14 & 8 \\ 14 & -14 & 0 \end{vmatrix} = 6(0 + 112)$$

$$+ 4(28 - 196)$$

$$= 672 - 672 = 0$$

and $\begin{vmatrix} 0 & 4 & 4 \\ 14 & 8 & 18 \\ -14 & 0 & -10 \end{vmatrix}$

$= -4(-140 + 252) + 4(112)$

$= -4(112) + 4\,(112) = 0$

Hence $\rho(A) \le 2$.

Now $\begin{vmatrix} 6 & 0 \\ -2 & 14 \end{vmatrix} = 84 \ne 0$

Hence $\rho(A) = 2$

$\therefore$ Rank of given matrix is 2

13. (D)

The characteristic equation is

$$\begin{vmatrix} (3-\lambda) & -2 & 2 \\ 4 & (-4-\lambda) & 6 \\ 2 & -3 & (5-\lambda) \end{vmatrix} = 0;$$

$C_2 = C_2 + C_3$ gives

$$\begin{vmatrix} (3-\lambda) & 0 & 2 \\ 4 & (2-\lambda) & 6 \\ 2 & (2-\lambda) & (5-\lambda) \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)\begin{vmatrix} (3-\lambda) & 0 & 2 \\ 4 & 1 & 6 \\ 2 & 1 & (5-\lambda) \end{vmatrix} = 0;$$

$R_3 = R_3 - R_2$ gives

$$(2-\lambda)\begin{vmatrix} (3-\lambda) & 0 & 2 \\ 4 & 1 & 6 \\ -2 & 0 & (-1-\lambda) \end{vmatrix} = 0$$

$\therefore \quad (2 - \lambda)\,\{(3 - \lambda)[(-1 - \lambda) - 0] - 0$

$+ 2[0 - (-2)]\} = 0$

$\therefore \quad (2 - \lambda)\,\{-3 - 2\lambda + \lambda^2 + 4\} = 0$

$\therefore \quad (2 - \lambda)\,(\lambda^2 - 2\lambda + 1) = 0$

$\therefore \quad (2 - \lambda)\,(\lambda - 1)^2 = 0$

$\therefore$ eigen values are 1 and 2

$\therefore$ smallest is 1 and largest is 2.

14. (D)

Sum of Eigen values = 1 + a

$\therefore \quad 6 = 1 + a$

$\therefore \quad a = 5$

and product of Eigen values = det(A)

$-7 = a - 4b$

$\therefore \quad -7 = 5 - 4b$

$\therefore \quad -7 - 5 = -4b$

$\therefore \quad b = 3$

15. 2 to 2

It is an upper triangular matrix. Hence eigen values are the diagonal elements i.e. 2, 2, 3. Hence there are 2 linearly independent eigen vectors.

16. (B)

17. (B)

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 4 & 6 & : & 20 \\ 1 & 4 & \lambda & : & \mu \end{bmatrix}$$

$R_3 \longleftrightarrow R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 4 & 6 & : & 20 \\ 0 & 0 & \lambda-6 & : & \mu-20 \end{bmatrix}$$

For $\lambda = 6$, $\mu \ne 20$ System is inconsistent

$\therefore$ It has no solution.

18. (D)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^5.A^5 = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^{10}.A^5 = \begin{bmatrix} 128 & 128 \\ 128 & -128 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$$

$$A^{15}.A^4 = \begin{bmatrix} 512 & 512 \\ 512 & -512 \end{bmatrix}$$

$$\therefore \quad A^{19} = \begin{bmatrix} 512 & 512 \\ 512 & -512 \end{bmatrix}$$

$$\begin{vmatrix} 512-\lambda & 512 \\ 512 & -(512+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad -\left((512)^2 - \lambda^2\right) - (512)^2 = 0$$

$$\Rightarrow \quad \lambda^2 = 2 \times (512)^2$$

$$\Rightarrow \quad \lambda = \pm \sqrt{2}\,(512)$$

If eigen values of A are $\lambda_1, \lambda_2, \ldots$, then eigen values of A^k are

$\lambda_1^k, \lambda_2^k, \ldots (k > 0)$

19. (A)

If matrix B is obtained from matrix A by replacing the l^{th} row by itself plus k times the m^{th} row, for $l \neq m$ then det(B) = det(A). With this property given matrix is equal to the matrices given in options (B), (C) and (D).

20. 0 to 0

$$A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$$

Eigenvectors are

$$X_1 = \begin{pmatrix} 70 \\ \lambda_1 - 50 \end{pmatrix}; \quad X_2 = \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$$

where λ_1, λ_2 = Eigenvalues of A

$$X_1^T X_2 \begin{pmatrix} 70 & \lambda_1 - 50 \end{pmatrix} \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$$

$= 70(\lambda_2 - 80) + (\lambda_1 - 50)\, 70$

$= 70\lambda_2 - 5600 + 70\lambda_1 - 3500$

$= 70(\lambda_1 + \lambda_2) - 9100$

$= 70(130) - 9100$

$= 9100 - 9100 = 0$

$$\begin{pmatrix} \text{sum of eigenvalues} = \lambda_1 + \lambda_2 \\ \text{Trace} = 50 + 80 = 130 \end{pmatrix}$$

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Model Solution on Assignment – 3

1. (D)

Eigen values of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ are 4, –3

$\therefore$ The eigen vector corresponding to eigen vector λ is $Ax = \lambda x$ (verify the options)

$\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ is eigen vector corresponding to eigen value $\lambda = 3$

2. (D)

MN is not always equal to NM.

3. (B)

$$[A:B] = \begin{bmatrix} 1 & 2 & 2 & : & b_1 \\ 5 & 1 & 3 & : & b_2 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & 2 & 2 & : & b_1 \\ 0 & -9 & -7 & : & b_2 - 5b_1 \end{bmatrix}$$

$\therefore$ $\rho(A) = \rho(A/B) <$ number of unknowns, for all values of b_1 and b_2.

$\therefore$ The equations have infinitely many solutions, for any given b_1 and b_2.

4. (A)

$$\text{Given, Matrix } [M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$$

Sum of the eigenvalues

= Trace of matrix

= 215 + 150 + 550 = 915

5. (A)

6. (C)

Characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & 5 \\ 0 & 5-\lambda & 6 \\ 0 & -6 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1 - \lambda)\,[(5 - \lambda)^2 + 36] = 0$

$\lambda = 1;\ \lambda^2 - 10\lambda + 61 = 0$

$$\Rightarrow \lambda = \frac{10 \pm \sqrt{100 - 224}}{2}$$

$$= \frac{10 \pm 12j}{2} = 5 \pm j6$$

7. 2.0 to 2.0

$$P + Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 8 & 9 & 10 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$8R_3 - R_1 \sim \begin{bmatrix} 8 & -9 & 10 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$R_3 - R_2 \sim \begin{bmatrix} 8 & -9 & 10 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank is 2

8. (C)

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

Now $\quad A^{-1} = \dfrac{1}{|A|}(\text{Adj}(A))$

$$A = \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$\therefore \quad |A^T.A^{-1}| = 1$

9. (A)

Given systems

$3x_1 + 2x_2 = c_1$

$4x_1 + x_2 = c_2$

Matrix Form is $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

AX = B

Characteristic equations of above system is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

By expanding $\lambda^2 - 4\lambda - 5 = 0$

10. 0.95 to 1.05

Matrix A has zero as an eigen value if

$|A| = 0$

$\therefore 3[(-63 + 7x) + 52] - 2[(-81 + 9x) + 78] + 4[-36 + 42] = 0$

$\therefore 3(7x - 11) - 2(9x - 3) + (4 \times 6) = 0$

$\therefore 21x - 33 - 18x + 6 + 24 = 0$

$\therefore 3x = 3 \;\; \therefore x = 1$

11. (B)

12. (A)

$|P| = (16 + 9) - (1) = 24$

$$P^{-1} = \frac{1}{|P|}\text{adj}(P) = \frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$$

13. 5.0 to 5.0

Since one eigenvalue of M is 2

$\therefore 2^3 - 4(2)^2 + a(2) + 30 = 0$

$\Rightarrow a = -11$

$\therefore$ Characteristic polynomial is

$\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0$

$(\lambda - 2)(\lambda - 5)(\lambda + 3) = 0$

$\therefore \lambda = 2, 5, -3$

Largest absolute value of 'λ' is 5

14. 0

The given matrix is

$$A = \begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$$

Now according to question

(i) $R_2 \to R_2 + R_3$ (Adding third row to the second row)

$$\therefore \quad B = \begin{bmatrix} 3 & 4 & 45 \\ 20 & 11 & 300 \\ 13 & 2 & 195 \end{bmatrix}$$

(ii) $C_1 \to C_1 - C_3$ (Subtracting third column from first column)

$$\therefore \quad C = \begin{bmatrix} -42 & 4 & 45 \\ -280 & 11 & 300 \\ -182 & 2 & 195 \end{bmatrix}$$

Now $|C| = -42(2145 - 600) -$

$4(-54600 + 54600) + 45(-560 + 2002)$

$|C| = -42(1545) - 4(0)$

$+ 45(1442)$

$= -64890 + 64890 = 0$

15. 2.9 to 3.1

sum of diagonal elements

= sum of eigen values

$\therefore$ a + b + 7 = 14

$\therefore$ a + b = 7

$(a - b)^2 = (a + b)^2 - 4ab = 49 - 40 = 9$

$\therefore$ |a – b| = 3

16. (B)

$$[A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 2 & 3 & 3 & : & b \\ 5 & 9 & -6 & : & c \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\therefore [A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 5 & 9 & -6 & : & c \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$

$$\therefore [A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 0 & -1 & 9 & : & c-5a \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\therefore [A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 0 & 0 & 0 & : & c-b-3a \end{bmatrix}$$

$\therefore$ Equation are consistent if

c – b – 3a = 0 i.e. 3a + b – c = 0

17. (A)

Let $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$

Characteristic equations is $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3, 3$

Eigen value 3 has multiplicity 2.

Eigen vectors corresponding to $\lambda = 3$ is $(A - 3I)X = 0$

$$\begin{pmatrix} 5-3 & -1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R^2 \to R_2 - 2R_1 \Rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

e(A) = 1

Number of linearly independent eigenvectors corresponding to eigenvalue $\lambda = 3$ is n – r = 2 – 1 = 1

where n = no. of unknowns,

r = rank of $(A - \lambda I)$

$\therefore$ One linearly independent eigenvector exists corresponding to $\lambda = 3$

18. (B)

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5 - \lambda)(3 - \lambda) = 3 = 0$

$\Rightarrow \lambda^2 - 8\lambda + 15 - 3 = 0$

$\Rightarrow \lambda^2 - 8\lambda + 12 = 0 \Rightarrow \lambda = 2, \lambda = 6$

$(A - 2I) \times X = 0$

At, $\lambda = 2$

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + x_2 = 0$

$\therefore\ x_1 = -x_2$

Hence the required vector is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

19. (A)

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix} \xrightarrow{C_1+C_3} \begin{bmatrix} 5 & 5 & 2 \\ 12 & 12 & 7 \\ 7 & 7 & 5 \end{bmatrix}$$

$\Rightarrow$ determinant = 0,

So the matrix is singular

Therefore atleast one of the Eigen value is '0'

As the choices are non negative, the minimum Eigen value is '0'

20. (A)

Characteristic equation is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix}$$

$\Rightarrow\ -\lambda(4\lambda + \lambda^2 + 3) = 0$

$\Rightarrow \lambda(\lambda + 1)\ (\lambda + 3) = 0$

$\Rightarrow\ \lambda = 0, -1, -3$ are the eigenvalues.

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Model Solution on Assignment – 4

1. (C)

2. (B)

If $[A]_{3 \times 4}$ & $[B]_{4 \times 5}$

AB will exist, BA does not exist

3. (A)

Every square, real symmetric matrix is Hermitian and therefore all its eigenvalues are real.

4. (B)

5. (D)

$[A]_{x \times (x+5)}\ [B]_{y \times (11-y)}$

AB exists $\Rightarrow$ $x + 5 = y$ .....(1)

BA exists $\Rightarrow$ $x = 11 - y$ .....(2)

Solving (1) and (2) gives $x = 3$, $y = 8$

6. (A)

A is singular matrix means $|A| = 0$

$$\therefore \quad \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$R_2 \to R_2 + R_3$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$\Rightarrow$ $x(3 - x)(x - 3) = 0$

$\therefore$ $x = 0, 3$

7. (A)

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$c_1 \to c_1 + c_2 + c_3$, we get

$$\Delta = \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} = 0$$

as $1 + \omega + \omega^2 = 0$

8. (C)

For a matrix rank is equal to no. of independent vectors.

9. (A)

We find relation between than as

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$

$\lambda_1 (3, 2, 7) + \lambda_2 (2, 4, 1)$

$+ \lambda_3 (1, -2, 6) = 0$

which gives $3\lambda_1 + 2\lambda_2 + \lambda_3 = 0$

$2\lambda_1 + 4\lambda_2 - 2\lambda_3 = 0$

$7\lambda_1 + \lambda_2 + 6\lambda_3 = 0$

which gives $\lambda_1 = 1$, $\lambda_2 = -1, \lambda_3 = -1$

$\therefore$ $x_1 - x_2 - x_3 = 0$ is linear equation

$\therefore$ They are dependent

10. (A)

11. (B)

By performing

$$ABC = [x\ y\ z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x\ y\ z]\begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$$

12. (D)

As $|A| = 0$

So inverse does not exist.

13. (B)

Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 7 & 4 & 8 \\ -7 & 0 & -4 \end{bmatrix}$

$R_2 \to R_2 + R_3$

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 4 & 4 \\ -7 & 0 & -4 \end{bmatrix}$$

$\therefore |A| = 0$ ….. rank < 3

Now minor $\begin{bmatrix} 0 & 2 \\ 7 & 4 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} 0 & 2 \\ 7 & 4 \end{vmatrix} = -14 \neq 0$

$\therefore$ rank = 2

14. (C)

Eigen values are obtained by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

which gives $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 5$

15. (A)

Given equations are AX = B

It will be consistent if

Rank A = Rank [A : B]

We have

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} = B$$

Augmented matrix

$$[A\ B] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 3 & 1 & -2 & : & -2 \\ 2 & 4 & 7 & : & 7 \end{bmatrix}$$

We reduce [A B] to Echelon form by applying successively.

$R_2 \to R_2 - 3R_1,\quad R_3 \to R_3 - 2R_1$

transformations

$$\therefore\ [A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 2 & 5 & : & 13 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 0 & 0 & : & 20 \end{bmatrix}$$

$\Rightarrow$ Rank [A : B] = 3

(No. of non-zero rows in Echelon form.)

$$\therefore\ A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank A = 2

Since Rank A $\neq$ Rank [A : B]

$\therefore$ Inconsistent equations.

16. (D)

17. (B)

18. (B)

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Model Solution on Assignment – 5

1. (B)

As we know $A^{-1} A = I = A A^{-1}$

$\therefore$ B must be inverse of A

2. (C)

3. (A)

4. (B)

5. (B)

6. (A)

$$PQ = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}\begin{bmatrix} r & s \\ -s & r \end{bmatrix}$$

$$= \begin{bmatrix} pr-qs & ps+qr \\ -qr-ps & -qs+pr \end{bmatrix}$$

7. (B)

Matrix is involutory if it satisfies condition that $A^2 = I$ and given matrix satisfies this condition.

8. (A)

$A^{-1} = \dfrac{\text{adj } A}{|A|}$ we calculate inverse using this formula.

9. (B)

10. (A)

By writing augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \\ 1 & -1 & 1 & : & -1 \end{bmatrix}$$

performing $R_2 \to R_2 - 3R_1$,

$R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

and subsequently.

$R_2 \to R_2 - R_3$, $R_3 \to R_3 - 6R_2$,

$R_4 \to R_4 - 3R_2$,

$R_3 \to \dfrac{1}{5}R_3$, $R_4 \to \dfrac{1}{2}R_4$ &

$R_4 \to R_4 - R_3$.

Consequently we get

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -1 & 0 & : & -4 \\ 0 & 0 & 1 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

which is Echelon form with rank

Number of non–zero rows = 3

Also $\quad AI = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Here rank = 3

As rank [A : B] = rank A

$\therefore$ It has solution, which is unique

These equations are equivalent to

$$x + 2y - z = 3$$
$$-y = -4$$
$$z = 4$$

which gives $x = -1$, $y = 4$, $z = 4$

11. (C)

Eigen values of given matrix = 2, 6

For $\lambda = 6$,

eigen vector = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$

12. (B)

For triangular matrix eigen values are same as diagonal elements

$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 2.$

13. (C)

Explanation: Given matrix is, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now $\begin{vmatrix} (1-\lambda) & 1 & 1 \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = 0$

or $\begin{vmatrix} (3-\lambda) & (3-\lambda) & (3-\lambda) \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = 0$

or $(3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{vmatrix} = 0$

or $(3-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^2 (3 - \lambda) = 0$

Hence eigenvalues are 0, 0, 3.

14. (A)

Reducing to Echelon form,

$R_4 \to R_4 - R_1$

We have number of non–zero rows = 3

$\therefore$ rank = 3

15. (B)

Augmented matrix

$[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 3 & -2 & 1 & : & 2 \\ 4 & 2 & 1 & : & 7 \end{bmatrix}$

By performing $R_2 \to R_2 - 3R_1$,

$R_3 \to R_3 - 4R_1$ and $-\frac{1}{8}R_2$

$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 1 & : & 2 \\ 0 & -6 & -11 & : & -17 \end{bmatrix}$

Performing $R_3 \to R_3 + 6R_2$

$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 1 & : & 2 \\ 0 & 0 & -5 & : & -5 \end{bmatrix}$

we get $x = y = z = 1$

16. (C)

17. (D)

18. (D)

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Model Solution on Assignment – 6

1. (B)

Characteristic equation is

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 10 = 0$$

$$4 - 5\lambda + \lambda^2 - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

$\therefore \quad \lambda_1 = 6, \quad \lambda_2 = -1$

2. (C)

3. (C)

4. (C)

5. (B)

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}, |A| = -1$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}}{-1} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$$

6. (C)

As $\quad A A^{-1} = I$

$$\therefore \quad \begin{bmatrix} 2x & 0 \\ 0 & 2x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad 2x = 1$

$x = 1/2$

7. (B)

8. (B)

Characteristic Equation:

$|A - \lambda I| = 0$

$$\therefore \quad \begin{vmatrix} (1-\lambda) & 1 & 0 \\ 0 & (1-\lambda) & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)^3 = 0$$

$$1 - 3\lambda + 3\lambda^2 - \lambda^3 = 0$$

$$\therefore \quad I - 3A + 3A^2 - A^3 = 0$$

…. Cayley-Hamilton Theorem

$$\therefore \quad I - 3A + 3A.A - A^2A = 0$$

Post multiplying by A^{-1},

$$I.A^{-1} - 3AA^{-1} + 3AAA^{-1} - A^2AA^{-1} = 0A^{-1}$$

$$\therefore \quad A^{-1} - 3I + 3AI - A^2I = 0$$

$$\therefore \quad A^{-1} - 3I + 3A - A^2 = 0$$

$$\therefore \quad A^{-1} = A^2 - 3A + 3I$$

9. (B)

For given matrix of rank = 1

$\lambda = -1 \Rightarrow$ rank = 1

10. (D)

As $|A| \neq 0$ for unit matrix

$\therefore$ rank equal to number and order

11. (C)

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 15 & -6 & 5 \\ \lambda & -2 & -2 \end{bmatrix}$$

For non–trivial solution P(A) < 3

$\therefore\ |A| = 0 \Rightarrow -6 + \lambda(0 + 1) = 0$

$\lambda = 6$

12. (C)

Solving characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 6 & -5-\lambda \end{vmatrix} = 0$$

Gives $\lambda = -1 \pm 2\sqrt{7}$

13. (C)

As determinant of A i.e. $|A| \neq 0$

14. (A)

The sum of eigen values of a matrix is the sum of diagonal elements

$\lambda_1 + \lambda_2 = 12.$

Only option (A) has sum of diagonal elements of 12.

15. (C)

$|A^T A| = |I| = |A^T| \,.\, |A| = 1$

$\Rightarrow\ |A| \,.\, |A| = 1$

$\Rightarrow\ (|A|)^2 = 1$

$\Rightarrow\ |A| = \pm 1$

16. (A)

$A.A^{-1} = I$

$$\begin{bmatrix} 2 & \frac{-1}{10} \\ 0 & 3 \end{bmatrix} . \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2a - \frac{b}{10} \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

comparing the corresponding terms, we get

$b = \frac{1}{3},\ 2a - \frac{b}{10} = 0 \quad \therefore\ 2a = \frac{b}{10}$

$\therefore\quad a = \frac{b}{20} \qquad \therefore\ a = \frac{1}{60}$

$\therefore\quad a + b = \frac{1}{60} + \frac{1}{3} = \frac{1+20}{60} = \frac{21}{60} = \frac{7}{20}$

17. (C)

18. (B)

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Model Solution on Assignment – 7

1. (C)

As per theorem for inverse matrix, that eigen values of inverse matrix are inverse of eigen values of the matrix

i.e. if λ is eigen value of A, then $\frac{1}{\lambda}$ is eigen value of A^{-1}

2. (D)

For non zero matrix rank must be greater than or equal to 1, but not equal to zero.

3. (B)

As determinant $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\therefore$ rank equal to no. of rows = 2

4. (B)

For given equations

$$\begin{bmatrix} 1 & 2 \\ 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$R_2 \longleftrightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & \lambda - 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

i.e. $(\lambda - 6)\, y = 0$

i.e. $\lambda - 6 = 0$

$\lambda = 6$ will give no solution to have unique solution $\lambda \neq 6$

5. (D)

As $A^{-1} = \frac{\text{Adj A}}{|A|}$

6. (A)

$$A = \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \longleftrightarrow R_1 - (R_2 + R_3)$

$$\begin{vmatrix} 1-\lambda & -1+\lambda & -1+\lambda \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1 & -1 & -1 \\ 4 & 1-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\,[(1-\lambda)(4-\lambda) + 2] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda - 2)(\lambda - 3) = 0$$

Roots of equation are 1, 2 and 3.

As the eigen values of the matrix A are all distinct hence A is similar to diagonal matrix.

7. (A)

By using formula $A^{-1} = \frac{\text{Adj A}}{|A|}$

We find $A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$

8. (C)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & \alpha \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 = 0$$

$\therefore \quad \lambda = 0$

Hence if $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector then,

$AX = \lambda X$

$$\therefore \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \qquad \alpha z = 0$$

$\therefore$ The vectors must be $\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}$

As the vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

9. (A)

Hint : After taking the common from each column, we get matrix having all values = 1 and we get after reducing all rows and columns to zero the matrix of rank 1 only

10. (A)

For characteristic equation

$$A = \begin{vmatrix} -\lambda & h & g \\ h & -\lambda & f \\ g & f & -\lambda \end{vmatrix}$$

$$= \lambda^3 - \lambda(f^2 + g^2 + h^2) - 2fgh = 0$$

11. (B)

By solving, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

We get $\lambda = 1$ and $\lambda = 6$

12. (C)

Hint : By reducing the given matrix to echelon form, we come to know that, the rank of given matrix is 3.

13. (B)

By solving,

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix},$$

We get the values only one for each parameter, so it has unique solution.

14. (A)

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

i.e. $X^T AX$ gives,

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$\therefore$ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then for

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$a_{11} = a \qquad a_{22} = b \qquad a_{33} = c$

$a_{12} = a_{21} = h = \frac{1}{2} \times$ (co-efficient of xy)

$a_{23} = a_{32} = f = \frac{1}{2} \times$ (co-efficient of yz)

$a_{31} = a_{13} = g = \frac{1}{2} \times$ (co-efficient of zx)

15. (B)

Hint : The property of the unitary matrix

16. (A)

17. (A)

18. (B)

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Model Solution on Assignment – 8

1. (C)

Here P diagonals matrix B by pre & post multiplication such that

$B = P^{-1}AP$

2. (C)

Hint : As per the properties of the matrices.

3. (A)

$$\Delta = \begin{vmatrix} 1 & bc+ca+ab & a(b+c) \\ 1 & bc+ca+ab & b(c+a) \\ 1 & bc+ca+ab & c(a+b) \end{vmatrix}$$

$$= (ab + bc + ca)\begin{vmatrix} 1 & 1 & a(b+c) \\ 1 & 1 & b(c+a) \\ 1 & 1 & c(a+b) \end{vmatrix}$$

$$= (ab + bc + ca)\ 0 = 0$$

4. (C)

As per the property of multiplication of matrix.

5. (C)

6. (C)

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \end{bmatrix}\begin{bmatrix} 8 & 4 & 5 \\ -1 & 0 & -2 \\ 3 & 4 & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 17 & 29 & 32 \\ 4 & 12 & 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 17 & 24 & 32 \\ 4 & 12 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8x+3y & 6z & 3z \\ 4 & 12 & 26x-5y \end{bmatrix}$$

$\Rightarrow$ x = 1, y = 3, z = 4

7. (A)

As the rows are similar if common is taken from each row

8. (A)

Hint : $AX = \lambda X$

$A(AX) = A(\lambda X)$

$\therefore \quad A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

$A^2X = \lambda^2 X$

9. (B)

Given vectors are

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

Which gives

$$\lambda_1 x_1 + \lambda_2 x_2 + 2\lambda_3 x_3 = 0$$
$$\lambda_1 x_1 + 2\lambda_2 x_2 + 3\lambda_3 x_3 = 0$$
$$\lambda_1 x_1 + 3\lambda_2 x_2 + 4\lambda_3 x_3 = 0$$
$$3\lambda_1 x_1 + 4\lambda_2 x_2 + 9\lambda_3 x_3 = 0$$

which on solving gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$

$\therefore$ They are not linearly dependent

10. (C)

$$\text{Now} \begin{bmatrix} 1 & 2 & 1 & : & 6 \\ 2 & 1 & 2 & : & 6 \\ 1 & 1 & 1 & : & 5 \end{bmatrix}$$

Now, $R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & : & 6 \\ 2 & 1 & 2 & : & 6 \\ 0 & -1 & 0 & : & -1 \end{bmatrix}$$

$R_2 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & : & 6 \\ 0 & -3 & 0 & : & -6 \\ 0 & -1 & 0 & : & -1 \end{bmatrix}$$

$R_2 + 3R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & : & 6 \\ 0 & 0 & 0 & : & -3 \\ 0 & -1 & 0 & : & -1 \end{bmatrix}$$

Now the left part of dashed line have rank 2 and right part have all elements present so, it does not have solutions.

11. (C)

Hint : If the matrix is in the form of upper / lower triangular matrix , then the diagonal elements are the eigen values of the element.

12. (C)

$\begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$ have eigen values as

$$\begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix} = 0$$

$\therefore (\lambda - 2)(\lambda + 2) - (1 + 2i)(1 - 2i) = 0$

$\lambda^4 - 4 - (1 + 4) = 0$

$\lambda^2 - 9 = 0$

$\therefore$ The eigen values are $\lambda = 3, \lambda = -3$

13. (C)

$$\begin{vmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{vmatrix} = 0$$

Rank < 4

Now minor of order 3

$$\begin{vmatrix} 4 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{vmatrix} = 1(-4-2) = -6 \neq 0$$

Rank = 3

14. (C)

We have

$$\Delta = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$$

$= \Delta_1 + \Delta_2$ (say)

Multiplying R_1, R_2, R_3, R_4 of Δ_2 by a, b, c, d

$$\text{Now } \Delta_2 = \frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = -\Delta_1$$

$\therefore \quad \Delta = \Delta_1 + (-\Delta_1) = 0$

15. (C)

For triangular matrix eigen values are same as major principal diagonal element i.e. 18, 7, 8.

16. (B)

17. (B)

18. (A)

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Model Solution on Assignment – 9

1. (B)

As per the definition of unitary square matrix.

2. (B)

As per the definition of Hermitian matrix

3. (C)

4. (A)

5. (C)

$$\Rightarrow \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

& Adjoint of $A = \begin{bmatrix} \cos\alpha & +\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^T$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\therefore$ $A^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

6. (B)

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix} \;\&$$

$$A' = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & -2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$A + A'$

$$= \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & -2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 8 & 7 \\ -2 & 2 & 8 & -3 \\ 8 & 8 & 14 & -1 \\ 7 & -3 & -1 & 0 \end{bmatrix}$$

7. (C)

$$A = \begin{vmatrix} a+ic & -b+id \\ b+id & a-ic \end{vmatrix} = 1$$

$$\Rightarrow a^2 + b^2 + c^2 + d^2 = 1$$

8. (A)

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

i.e. $X^T AX$ gives,

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$\therefore$ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then for

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$a_{11} = a \qquad a_{22} = b \qquad a_{33} = c$

$a_{12} = a_{21} = h = \frac{1}{2} \times$ (co-efficient of xy)

$a_{23} = a_{32} = f = \frac{1}{2} \times$ (co-efficient of yz)

$a_{31} = a_{13} = g = \frac{1}{2} \times$ (co-efficient of zx)

Extension of this method

9. (C)

As the number of equations & unknowns are same and they are equal to zero, so their values are also zero.

Or $|A| \neq 0$

$\therefore$ the system will have trivial solution.

10. (B)

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

i.e. $X^T AX$ gives,

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$$\therefore \quad \text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then for

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$a_{11} = a \qquad a_{22} = b \qquad a_{33} = c$

$a_{12} = a_{21} = h = \frac{1}{2} \times$ (co-efficient of xy)

$a_{23} = a_{32} = f = \frac{1}{2} \times$ (co-efficient of yz)

$a_{31} = a_{13} = g = \frac{1}{2} \times$ (co-efficient of zx)

11. (C)

$|A|_{3\times 3} = 5$

$\therefore$ Rank of A is 3.

12. (C)

Hint : Eigen values are the values of diagonal for diagonal matrix. So reduce given matrix to diagonal matrix. Or sum of the eigen values of a matrix = trace of the matrix

13. (C)

Characteristic Equation:

$$|A - \lambda I| = 0$$

$$\therefore \quad \begin{vmatrix} (3-\lambda) & 1 \\ -1 & (2-\lambda) \end{vmatrix} = 0$$

$$\therefore \quad (3-\lambda)(2-\lambda) - (-1) = 0$$

$$\therefore \quad 6 - 5\lambda + \lambda^2 + 1 = 0$$

$$\therefore \quad \lambda^2 - 5\lambda + 7 = 0$$

Cayley-Hamilton Theorem,

$A^2 - 5A + 7I = 0$

14. (C)

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \begin{matrix}\cos\alpha\sin\alpha \\ +\sin\alpha\cos\alpha\end{matrix} \\ \begin{matrix}-\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha\end{matrix} & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

As $\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$

& $2\sin\alpha\cos\alpha = \sin 2\alpha$

Similarly , we get

$$A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

15. (B)

As $(ABC)^T = C^T B^T A^T$

$\therefore$ (B) is wrong.

16. (C)

Let $\Delta = \begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

Here, T_p, T_q and T_r are p^{th} q^{th} and r^{th} term of an AP.

Let the first term of the AP be a and common difference of the AP be d.

Then

$T_p = a + (p-1)d$

$T_q = a + (q-1)d$

$T_r = a + (r-1)d$

$$\Delta = \begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$R_2 \leftrightarrow R_2 - R_3$

$$\Delta = \begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ p-1 & q-1 & r-1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{d}\begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ (p-1)d & (q-1)d & (r-1)d \\ 1 & 1 & 1 \end{vmatrix}$$

.....$(\because d \neq 0)$

$R_1 \leftrightarrow R_1 - R_2$

$$\therefore \quad \Delta = \frac{1}{d}\begin{vmatrix} a & a & a \\ (p-1)d & (q-1)d & (r-1)d \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{a}{d}\begin{vmatrix} 1 & 1 & 1 \\ (p-1)d & (q-1)d & (r-1)d \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore \quad \Delta = 0$

17. (D)

18. (D)

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Model Solution on Assignment – 10

1. (D)

As $R_2 = (-2) \times R_1$

$\therefore \quad |A|_{3\times 3} = 0$

$\therefore \quad \text{rank} \neq 0$

$$|A|_{2\times 2} = \begin{vmatrix} 3 & 2 \\ -6 & -4 \end{vmatrix} = -12 + 12 = 0$$

$$\begin{vmatrix} -4 & 18 \\ 8 & -36 \end{vmatrix} = 144 - 144 = 0$$

$|A|_{2\times 2} = 0$

$\therefore \quad |A|_{1\times 1} = |3| = 3$

$|A|_{1\times 1} \neq 0$

$\therefore \quad \text{rank} = 1$

Or $\quad R_2 = -2R_1$

$\quad R_3 = 4R_1$

$\therefore \quad \text{rank} = 1$

2. (B)

This is like a property of triangular matrix.

3. (B)

4. (C)

The diagonal matrix is

$$= [x_1 \ldots\ldots x_n] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ : \\ x_n \end{bmatrix}$$

$$= \lambda x_1^2 + \lambda_2 x_2^2 + \ldots\ldots + \lambda_n x_n^2$$

5. (B)

Q is obtained by interchanging C_1 and C_2

$|Q| = -|P| = -8$

6. (C)

By performing $R_4 \to R_4 - R_3 - R_2 - R_1$ & $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$. Sequentially we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $|A|_{4\times 4} = 0$ … As E transform does not change rank of matrix

$\therefore \quad \text{rank} \neq 4$

$$\text{Now} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & -4 & -8 \end{vmatrix} = -12 \quad \text{i.e.} \neq 0$$

$\therefore$ Rank = 3

7. (A)

A matrix is said to be orthogonal if

$A^T A = I$

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal

8. (A)

For triangular matrix the eigen values are same as principal diagonal elements.

9. (A)

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

i.e. $X^T AX$ gives,

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$\therefore$ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then for

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$a_{11} = a \qquad a_{22} = b \qquad a_{33} = c$

$a_{12} = a_{21} = h = \dfrac{1}{2} \times$ (co-efficient of xy)

$a_{23} = a_{32} = f = \dfrac{1}{2} \times$ (co-efficient of yz)

$a_{31} = a_{13} = g = \dfrac{1}{2} \times$ (co-efficient of zx)

10. (B)

Characteristic equation of A is

$| A - \lambda I | = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

by $C_3 \to C_3 + C_2$

$$\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 - R_3$

$$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$(2 - \lambda) . [(6 - \lambda) . (4 - \lambda) - 8] = 0$

$[(2 - \lambda) (\lambda^2 - 10\lambda + 16)] = 0$

$(2 - \lambda) (\lambda - 2) (\lambda - 8) = 0$

$\therefore \lambda = 2, 2, 8$

Or sum of the eigen values of a matrix

= trace of the matrix

11. (C)

As $R_4 = 3R_1$

$\therefore | A |_{4 \times 4} = 0$

Now $\begin{vmatrix} 1 & 4 & 8 \\ 0 & 0 & 3 \\ 4 & 2 & 3 \end{vmatrix}$

$= 1(0 - 6) - 4(0 - 12) + 8(0)$

$= -6 + 48 = 42$ i.e. $\neq 0$

$\therefore$ rank = 3

12. (B)

$| A | = -37$

$$\text{Adj. } A = \begin{bmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{37}\begin{bmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{bmatrix}$$

13. (C)

As A is not symmetric matrix.

14. (C)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 3 & 2 & 1 \end{vmatrix}$$

$= 1(5 - 16) - 2(4 - 24) + 3(8 - 15)$

$= -11 + 40 - 21$

$= 8$ i.e. $\neq 0$

$\therefore$ rank $= 3$

15. (D)

As $R_3 = R_1$

$\therefore$ Value of determinant = 0

16. (C)

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$R_1 \leftrightarrow R_1 + (R_2 + R_3)$

$$\begin{vmatrix} \begin{matrix} a-x \\ +c+b \end{matrix} & \begin{matrix} c+b \\ -x+a \end{matrix} & \begin{matrix} b+a \\ +c-x \end{matrix} \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$$\begin{vmatrix} \begin{matrix} a+b \\ +c-x \end{matrix} & \begin{matrix} a+b \\ +c-x \end{matrix} & \begin{matrix} a+b \\ +c-x \end{matrix} \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$\because a + b + c = 0$

$$\begin{vmatrix} -x & -x & -x \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$$x\begin{vmatrix} -1 & -1 & -1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

$\therefore$ $x = 0$ is one root of the given equation.

17. (D)

$$\begin{vmatrix} a-x & b & c \\ 0 & b-x & a \\ 0 & 0 & c-x \end{vmatrix} = 0$$

$$\therefore (a-x)\begin{vmatrix} b-x & a \\ 0 & c-x \end{vmatrix} - 0\begin{vmatrix} b & c \\ 0 & c-x \end{vmatrix}$$

$$+0\begin{vmatrix} b & c \\ b-x & a \end{vmatrix} = 0$$

$\therefore$ $(a-x)\left[(b-x)(c-x) - 0\right] - 0 + 0 = 0$

$\therefore$ $(a - x)(b - x)(c - x) = 0$

$\therefore$ $x = a, b, c$ are the roots.

18. (C)

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Model Solution on Assignment – 11

1. (C)

2. (C)

3. (C)

4. (D)

5. (C)

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

$\therefore$ $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ is not invertible.

6. (C)

7. (C)

$A = (a_{ij})$ is 3×2 matrix

Elements are given as

$a_{ij} = 2i - j \qquad i > j$

$\quad = 2j - I \qquad i \le j$

Elements of A will be a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}

Elements having $i > j = a_{21}, a_{31}, a_{32}$

$\therefore$ $a_{21} = 2(2) - (1) = 3$

$a_{31} = 2(3) - (1) = 5$

$a_{32} = 2(3) - 2 = 4$

Elements having $i \le j$; a_{11}, a_{12}, a_{22}

$\therefore$ $a_{11} = 2(1) - (1) = 1$

$a_{12} = 2(2) - (1) = 3$

$a_{22} = 2(2) - (2) = 2$

$\therefore$ Matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

8. (C)

$$X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \qquad \text{....(i)}$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{....(ii)}$$

Addition of (i) and (ii) gives

$$2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow \quad Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\therefore \quad Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

9. (A)

Check with options

10. (B)

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$|A| = 1 \ne 0$$

$\therefore$ A^{-1} exists

Cofactors :

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix} = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}0 & 2\\0 & 1\end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}2 & -3\\0 & 1\end{vmatrix} = -2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & -3\\0 & 1\end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & 2\\0 & 0\end{vmatrix} = 0$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}2 & -3\\1 & 2\end{vmatrix} = 7$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & -3\\0 & 2\end{vmatrix} = -2$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix} = 1$$

$$\therefore \quad \text{Cofactor Matrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\7 & -2 & 1\end{bmatrix}$$

$$\text{Adj A} = [\text{Cofactor matrix}]^T$$

$$= \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\7 & -2 & 1\end{bmatrix}^T$$

$$\text{Adj A} = \begin{bmatrix}1 & -2 & 7\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adjA}}{|A|}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -2 & 7\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix} \text{.....}(\because |A| = 1)$$

$$\therefore \quad A^{-1} = \begin{bmatrix}1 & -2 & 7\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$$

11. (D)

12. (B)

13. (C)

a_1, a_2, ……. are in GP

$$\therefore \quad \frac{a_{m+1}}{a_m} = \frac{a_{m+2}}{a_{m+1}} = \frac{a_{m+3}}{a_{m+2}} \cdots\cdots k\,(\text{say})$$

$a_{m+1} = a_m k$

$a_{m+2} = a_{m+1}k = a_m k^2$

$a_{m+3} = a_{m+2}k = a_m k^3$

and so on

$$\Delta = \begin{vmatrix}\log a_m & \log a_{m+1} & \log a_{m+2}\\ \log a_{m+3} & \log a_{m+4} & \log a_{m+5}\\ \log a_{m+6} & \log a_{m+7} & \log a_{m+8}\end{vmatrix}$$

$$= \begin{vmatrix}\log a_m & \log a_m + \log k & \log a_m + 2\log k\\ \log a_m + 3\log k & \log a_m + 4\log k & \log a_m + 5\log k\\ \log a_m + 6\log k & \log a_m + 7\log k & \log a_m + 8\log k\end{vmatrix}$$

$R_2 \longleftrightarrow R_2 - R_1$; $R_3 \longleftrightarrow R_3 - R_1$

$$\Delta = \begin{vmatrix} \log a_m & \log a_m + \log k & \log a_m + 2\log k \\ 3\log k & 3\log k & 3\log k \\ 6\log k & 6\log k & 6\log k \end{vmatrix}$$

$$= 3 \times 6 \times \begin{vmatrix} \log a_m & \log a_m + \log k & \log a_m + 2\log k \\ \log k & \log k & \log k \\ \log k & \log k & \log k \end{vmatrix}$$

$\therefore\ \Delta = 0$

14. (D)

Given determinant, Δ

$$= \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$

.... Since $\cos(-\theta) = \cos\theta$

$= 1(1 - \cos^2(\beta-\gamma))$

$-\cos(\alpha-\beta)\begin{Bmatrix} \cos(\alpha-\beta) \\ -\cos(\gamma-\alpha)\cos(\beta-\gamma) \end{Bmatrix}$

$+ \cos(\gamma-\alpha)\ \begin{Bmatrix} \cos(\alpha-\beta)\cos(\beta-\gamma) \\ -\cos(\gamma-\alpha) \end{Bmatrix}$

$= 1 - \cos^2(\beta-\gamma) - \cos^2(\alpha-\beta)$

$+ \cos(\alpha-\beta)\cos(\gamma-\alpha)\cos(\beta-\gamma)$

$+ \cos(\gamma-\alpha)\cos(\alpha-\beta)\cos(\beta-\gamma)$

$- \cos^2(\gamma-\alpha)$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma) - \cos^2(\gamma-\alpha)$

$+ 2\{\cos(\alpha-\beta)\cos(\beta-\gamma)\}\cos(\gamma-\alpha)$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma)$

$- \cos^2(\gamma-\alpha) + \cos\{(\alpha-\beta) - (\beta-\gamma)\}$

$+ \cos\{(\alpha-\beta) + (\beta-\gamma)\}\cos(\gamma-\alpha)$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma) - \cos^2(\gamma-\alpha)$

$+ \cos(\gamma-\alpha)\cos(\alpha-2\beta+\gamma)$

$+ \cos(\gamma-\alpha)\cos(\alpha-\gamma)$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma)$

$$+ \frac{\begin{Bmatrix} \cos[(\gamma-\alpha)-(\alpha-2\beta+\gamma)] \\ +\cos[(\gamma-\alpha)+(\alpha-2\beta+\gamma)] \end{Bmatrix}}{2}$$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma)$

$$+ \frac{\cos(-2\alpha+2\beta)+\cos(2\gamma-2\beta)}{2}$$

$= 1 - \cos^2(\alpha-\beta) - \cos^2(\beta-\gamma)$

$$+ \frac{\cos 2(\alpha-\beta)+\cos 2(\beta-\gamma)}{2}$$

$$= \frac{\begin{Bmatrix} 2 - 2\cos^2(\alpha-\beta) - 2\cos^2(\beta-\gamma) \\ +2\cos^2(\alpha-\beta) - 1 \\ +2\cos^2(\beta-\gamma) - 1 \end{Bmatrix}}{2}$$

$\therefore\ \Delta = 0$

15. (B)

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$$

Characteristic Equation of A is

$|A - \lambda I| = 0$

$$\therefore \quad \begin{vmatrix} 0-\lambda & 2 & 4 \\ 1 & 1-\lambda & -2 \\ -2 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\text{i.e.} \quad \begin{vmatrix} -\lambda & 2 & 4 \\ 1 & 1-\lambda & -2 \\ -2 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\therefore -\lambda \begin{vmatrix} 1-\lambda & -2 \\ 0 & 5-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & -2 \\ -2 & 5-\lambda \end{vmatrix}$$

$$+4 \begin{vmatrix} 1 & 1-\lambda \\ -2 & 0 \end{vmatrix} = 0$$

$$\therefore -\lambda\left[(1-\lambda)(5-\lambda)-0\right]-2\left[5-\lambda-4\right]$$

$$+4\left[0+2(1-\lambda)\right]=0$$

$$\therefore -\lambda\left[5-6\lambda+\lambda^2\right]-2\left[1-\lambda\right]$$

$$+4\left[2-2\lambda\right]=0$$

$$\therefore \quad -5\lambda+6\lambda^2-\lambda^3-2+2\lambda+8-8\lambda=0$$

$$\therefore \quad -\lambda^3+6\lambda^2-1\lambda+6=0$$

i.e. $\lambda^3 - 6\lambda^2 + 1\lambda - 6 = 0$

16. (C)

Characteristic equation: $|A - \lambda I| = 0$

$$\therefore \quad \begin{vmatrix} (3-\lambda) & 10 & 5 \\ -2 & (-3-\lambda) & -4 \\ 3 & 5 & (7-\lambda) \end{vmatrix} = 0$$

gives $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\Rightarrow \lambda = 2, 2, 3$

For $\lambda = 2$

$$\begin{bmatrix} (3-2) & 10 & 5 \\ -2 & (-3-2) & -4 \\ 3 & 5 & (7-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$ satisfies the above equation.

17. (D)

$$\Delta_1 = \begin{vmatrix} 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \\ 4 & 4^2 & 4^3 \end{vmatrix}$$

$$= 2\times3\times4\times \begin{vmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{vmatrix}$$

$$= 24 \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix}$$

$\Delta_1 = 24\Delta$

18. (D)

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Answer Key on Test Paper – 1

1.	(B)	2.	(C)	3.	(C)	4.	(B)
5.	(C)	6.	(D)	7.	(C)	8.	(B)
9.	(A)	10.	(C)	11.	(D)	12.	(A)
13.	(B)	14.	(C)	15.	(C)		

Answer Key on Test Paper – 2

1.	(B)	2.	(B)	3.	(A)	4.	(A)
5.	(B)	6.	(B)	7.	(D)	8.	(B)
9.	(B)	10.	(D)	11.	(C)	12.	(B)
13.	(C)	14.	(A)	15.	(A)		

Answer Key on Test Paper – 3

1.	(C)	2.	(C)	3.	(D)	4.	(A)
5.	(C)	6.	(A)	7.	(B)	8.	(D)
9.	(D)	10.	(C)	11.	(C)	12.	(B)
13.	(B)	14.	(C)	15.	(A)		

Answer Key on Test Paper – 4

1.	(B)	2.	(C)	3.	(C)	4.	(C)
5.	(B)	6.	(A)	7.	(A)	8.	(C)
9.	(B)	10.	(B)	11.	(A)	12.	(A)
13.	(B)	14.	(A)	15.	(B)		

Answer Key on Test Paper – 5

1.	(A)	2.	(B)	3.	(B)	4.	(B)
5.	(C)	6.	(C)	7.	(B)	8.	(A)
9.	(B)	10.	(D)	11.	(B)	12.	(A)
13.	(B)	14.	(C)	15.	(D)		

Answer Key on Test Paper – 6

1.	(B)	2.	(B)	3.	(B)	4.	(A)
5.	(A)	6.	(C)	7.	(C)	8.	(B)
9.	(C)	10.	(D)	11.	(A)	12.	(B)
13.	(B)	14.	(B)	15.	(C)		

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Model Solution on Test Paper – 1

1. (B)

2. (C)

$x + 3 = 0 \quad \therefore\ x = -3$

similarly y, z & a can be calculated.

3. (C)

4. (B)

5. (C)

Characteristic equation is

$$\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

$$16 - 8\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 8\lambda + 15 = 0$$

$$(\lambda - 5)(\lambda - 3) = 0$$

$\therefore\ \lambda_1 = 3, \quad \lambda_3 = 5$

Or following property can be used

Sum of the eigen values = trace

6. (D)

7. (C)

Relationship obtained by using

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$

$\therefore\ \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = 0$

gives 4 equations

i.e. $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$

$$3\lambda_1 - 5\lambda_2 - \lambda_3 = 0$$

$$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$$

$$2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

By solving these equations we get

$\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = -2$

$\therefore$ Equation is $x_1 + x_2 - 2x_3 = 0$

8. (B)

Suppose these points are collinear on line $ax + by + c = 0$

Then $ax_1 + by_1 + c = 0$ ……(i)

$ax_2 + by_2 + c = 0$ …….(ii)

$ax_3 + by_3 + c = 0$ ……(iii)

Eliminate a, b, c, between (i), (ii) and (iii)

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Thus rank of matrix $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is less than 3

Conversely if rank of matrix A is less than 3, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\therefore$ Area of triangle with vertices (x_1, y_1) (x_2, y_2) (x_3, y_3) is equal to 0

$\therefore$ They are collinear points.

9. (A)

We have equation as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$AX = B$$

Now augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 5 & 7 & : & 52 \\ 2 & 1 & -1 & : & 0 \end{bmatrix}$$

We reduce it in Echelon form

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 3 & 5 & : & 34 \\ 0 & -1 & -3 & : & -18 \end{bmatrix}$$

$R_2 \to \dfrac{R2}{3}$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 5/3 & : & 34/3 \\ 0 & -1 & -3 & : & -18 \end{bmatrix}$$

$R_3 \to \quad R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 5/3 & : & 34/3 \\ 0 & 0 & -4/3 & : & -20/3 \end{bmatrix}$$

$\therefore$ Rank A = Rank [A : B] = 3

$\therefore$ Equations are consistent.

10. (C)

By taking product

$$AB = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} a^2 & ab & \bar{a}c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

11. (D)

$f(x) = x^2 - 5x + 6$

$f(A) = A^2 - 5A + 6\,I$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$+6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which gives

$$f(A) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

12. (A)

A matrix is said to be idempotent if

$A^2 = A$

Here, $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

gives $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

which is same as A.

$\therefore$ It is idempotent

13. (B)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad |A| = 1$$

$$\therefore \quad A^{-1} = \frac{\text{adj}A}{|A|}$$

$$\text{Adj } A = [\text{cofactor matrix } A]^T$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. (C)

We have $(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1 - \lambda) [(1 - \lambda)^2 - 1] - 1[(1 - \lambda) - 1]$

$+ 1[1 - (1 - \lambda)] = 0$

$\therefore \quad (1 - \lambda) [(1 - \lambda)^2 - 1] - 1[(1 - \lambda) - 1]$

$+ 1[1 - (1 - \lambda)] = 0$

$\therefore \quad (1 - \lambda) (\lambda^2 - 2\lambda) + 2\lambda = 0$

$\therefore \quad \lambda^2 (3 - \lambda) = 0$

$\therefore \quad \lambda = 0, 0, 3$

15. (C)

We have $C_1 \to \frac{1}{8}C_1$

$$\begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix}$$

Now $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$ which is in Echelon

form so rank is equal to no. of nonzero rows

Rank = 3

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Model Solution on Test Paper – 2

1. (B)

2. (B)

3. (A)

4. (A)

$$2A + 3B = 2\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & 5 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & -2 \\ 2 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 12 \\ 8 & 8 & 28 \end{bmatrix}$$

5. (B)

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Performing $R_2 \to R_2 - R_1$,

$R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix} = 0$$

$\therefore$ $(b - a)(ab - bc)$

$- (c - a)(ca - bc) = 0$

$(b - a).\ b(a - c)$

$- (a - c)\ c(a - b) = 0$

$(a - c)(b - a)(b - c) = 0$

$\therefore$ $(a - b)$ is a factor of Δ

6. (B)

The given system of equation is equivalent to the single matrix equation

$$AX = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} = B$$

We reduce this matrix into triangular matrix by sequentially performing

$R_2 \to R_2 - 3R_1$,

$R_3 \to R_3 - 3R_2$, we get

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$$

We have $0x + 0y + 0z = -91$, this shows systems is not consistent

7. (D)

Using

$\cos\theta\cos\phi - \sin\theta\sin\phi = \cos(\theta + \phi)$

$\cos\theta\sin\phi + \sin\theta\cos\phi = \sin(\theta + \phi)$

$\cos\theta\cos\phi - \sin\theta\sin\phi = \cos(\theta + \phi)$

we get $\begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

8. (B)

As $A^3 = A\ .\ A\ .\ A = 0$

where A is given matrix

$\therefore$ It is nilpotent matrix

9. (B)

$R_3 \rightarrow R_3 + R_1$, $R_2 \rightarrow R_2 + 2R_1$ gives

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix}$$

The determinant of above matrix is 0

Checking for non-zero minor of order 2

$$\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = 2 - 12 = -10 \neq 0$$

$\therefore$ rank = 2

10. (D)

11. (C)

As it is not a singular matrix

12. (B)

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ $I + A + A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 9 & 17 \\ 0 & 4 & -5 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 12 & 22 \\ 0 & 7 & -6 \\ 0 & 0 & 13 \end{bmatrix}$$

Now for triangular matrix eigen values are same as principal diagonal element.

$\therefore$ $\lambda_1 = 3$, $\lambda_2 = 7$, $\lambda_3 = 13$ are eigen values.

13. (C)

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Here AB = BA if a = b

14. (A)

$R_4 = 3R_1$

Hence rank < 4 and

$$\begin{vmatrix} 1 & 4 & 8 \\ 0 & 0 & 3 \\ 4 & 2 & 3 \end{vmatrix} = 48 \neq 0$$

$\therefore$ Rank = 3

15. (A)

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Model Solution on Test Paper – 3

1. (C)

As for triangular matrix eigen values are principal diagonal elements

2. (C)

As per definition of rank of matrix

3. (D)

As columns of A $\neq$ rows of B

So product is not possible

4. (A)

5. (C)

As these are properties of skew symmetric matrix.

6. (A)

e.g. A = $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ for 2×2 matrix

Rank of A = 1

7. (B)

8. (D)

As per property of idempotent matrix if A & B are idempotent implies AB = BA.

9. (D)

The determinant of 3×3 element of given matrix is not equal to zero or when reduced to echelon form, it gives 3 rowed matrix.

10. (C)

The given equation in Matrix Form:

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$AX = B$

$$[A : B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$R_3 \longleftrightarrow R_3 - R_1$

$$\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 0 & 0 & (\lambda-5) & : & (\mu-9) \end{bmatrix}$$

If rank(A) $\neq$ rank (A : B), system has no solution.

$\Rightarrow \lambda = 5, \mu \neq 9$

11. (C)

Just by the definition of the rank of matrix we find that the rank of given matrix A is 3.

12. (B)

Characteristic equation of A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (1 - \lambda) [(2 - \lambda) (3 - \lambda) - 0]$

$+ 2 \times 2 (2 - \lambda) = 0$

$\therefore \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

13. (B)

By solving the above equations as follows

$$\begin{vmatrix} 2 & 1 & 2 & : & 1 \\ 1 & 1 & 0 & : & 0 \\ 1 & -k & 6 & : & 3 \end{vmatrix}$$

$\Rightarrow$ That is, after reducing to Echelon form, if we want unique solution, then $K \neq 2$

14. (C)

Hint : By just the definition

i.e. transpose of all elements' cofactor matrix.

OR [cofactor Matrix $]^T$

15. (A)

$R_3 \to R_3 - (R_1 + R_2)$, we get

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

By $R_2 \to R_2 - R_1$ & $R_1 \to R_1 + R_3$

$$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

$$(x+1)(x+2)\begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$(x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Now,

$(x+1)(x+2)[-(3x+8-5)] = 0$

$-3(x+1)(x+2)(x+1) = 0$

$\therefore \quad x = -1, -1, -2$

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Model Solution on Test Paper – 4

1. (B)
As per the property of the skew Hermitian matrix

2. (C)
As per the property of the eigen values for orthogonal matrices. i.e. for orthogonal matrix, the eigen value is reciprocal of the original eigen value.

3. (C)
Hint : The transpose of the matrix does not change the rank of the matrix.

4. (C)
As per the defitnition of nilpotent matrix for index K.

5. (B)

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

i.e. X^T AX gives,

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$$\therefore \quad \text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then for

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$

$a_{11} = a \qquad a_{22} = b \qquad a_{33} = c$

$$a_{12} = a_{21} = h = \frac{1}{2} \times (\text{co-efficient of } xy)$$

$$a_{23} = a_{32} = f = \frac{1}{2} \times (\text{co-efficient of } yz)$$

$$a_{31} = a_{13} = g = \frac{1}{2} \times (\text{co-efficient of } zx)$$

Extension of this method

6. (A)
When reduced to Echelon form, non-zero rows obtained is 1.
Or $R_2 = -6R_1$ & $R_3 = -5R_1$

7. (C)

8. (C)

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Characteristic equation of matrix

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \qquad \lambda^3 - 6\lambda^2 + 9\lambda + 4 = 0$$

9. (B)

10. (B)

11. (A)
Inverse of matrix A is,

$$A^{-1} = \frac{1}{|A|} \times \text{Adj } A$$

12. (A)

0 is an eigen value of A $\Rightarrow$ $\lambda = 0$

satisfies the equation $|A - \lambda I| = 0$

$\Rightarrow$ $|A| = 0 \Rightarrow$ A is singular

Conversely $\Rightarrow |A| = 0$

$\lambda = 0$ satisfies the equation

$|A - \lambda I| = 0$

$\Rightarrow$ 0 is eigen value of A

13. (B)

$R_2 = 3R_1$

As one of the row is repeated, so the no. of rows for considering rank reduces to 2. Now, for given row matrix, determinant is non zero and hence the matrix is 2.

14. (A)

15. (B)

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Model Solution on Test Paper – 5

1. (A)

For symmetric matrix $A = A^T$

2. (B)

We have, $a + 3 = 2a + 1$

$\therefore \quad a = 2$

Similarly, $b = 1, 2$

3. (B)

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore A^2 = 0 \quad \Rightarrow \quad K = 2$

$\therefore$ Index = 2

4. (B)

It is the definition of eigen values.

5. (C)

For diagonal matrix, eigen values are different.

6. (C)

7. (B)

$C_2 = 2C_1$

$\therefore \quad |A|_{3\times 3} = 0$

$\therefore$ rank < 3

minor of order 2

$$\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 12 - 4 = 8 \neq 0$$

$\therefore$ rank = 2

8. (A)

For triangular matrix eigen values are same as major principal diagonal elements.

9. (B)

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$\lambda_1(1, 2, 4) + \lambda_2(2, -1, 3) + \lambda_3(0, 1, 2)$

$+ \lambda_4(-3, 7, 2) = 0$

$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$

$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$

$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$

By solving these equations we get

$\lambda_1 = 9, \ \lambda_2 = -12, \ \lambda_3 = 5, \ \lambda_4 = -5$

$\therefore$ relation is $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$

10. (D)

11. (B)

$|A| = 1$

$$\text{adj } A = \begin{bmatrix} a - ib & c - id \\ -c - id & a + ib \end{bmatrix}^T$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \begin{bmatrix} a - ib & -c - id \\ c - id & a + ib \end{bmatrix}$$

12. (A)

Idempotent matrix is one which satisfies $A^2 = A$. But given matrix not satisfy this relation

$\therefore$ It is not a idempotent matrix

13. (B)

$|A|_{3 \times 3} = 0$

$\therefore$ Rank < 3

Minor of order 2, $\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -2$

i.e. $\neq 0$

$\therefore$ rank = 2

14. (C)

determinant

$$-1\begin{vmatrix} 6 & 0 & 0 \\ -8 & 2 & 0 \\ 1 & 4 & 4 \end{vmatrix} = -1[6\ [8-0]\ ] = -48$$

15. (D)

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Model Solution on Test Paper – 6

1. (B)

$|A| = a[b(d \times \ell)] = abd\ell$

2. (B)

As $[A]_{3\times3}$ $[B]_{3\times2}$

$\therefore$ $[B]_{3\times2}$ $[A]_{3\times3}$

BA is not possible as

columns of B $\neq$ rows of A

3. (B)

$\bar{A}$ is obtained by substituting i by –i

4. (A)

$$|A|_{3\times3} = \begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{vmatrix} = 0$$

$$|A|_{2\times2} = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0$$

Similarly other minors of order 2 are 0

$\therefore$ $|A|_{1\times1} = |3| \neq 0$

$\therefore$ rank = 1

5. (A)

Characteristic equation are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & -3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-3-\lambda) - 9 = 0$$

$$-15 + \lambda^2 + 2\lambda - 9 = 0$$

$$\lambda^2 - 2\lambda - 24 = 0$$

$\therefore$ $\lambda = 6, -4$

6. (C)

As per theorem,

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Now here $a = b = c = d = -4$

$\therefore$ $= (-4)^4 \times (0) = 0$

7. (C)

By $R_4 \to R_4 - R_3 - R_2 - R_1$, $R_1 \to R_2$,

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

Subsequently, $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We see that $\begin{vmatrix} 1 & -1 & -2 \\ 0 & 5 & 3 \\ 0 & 4 & 9 \end{vmatrix} = 33 \neq 0$

$\therefore$ Rank = 3

8. (B)

Hint : Adjoint A = [Cofactor matrix A]^T

9. (C)

Characteristic roots are obtained by

$|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

$-\lambda(\lambda^2 - 1) - 1(-\lambda + 2) + 2(-1 + 2\lambda) = 0$

$-\lambda^3 + 6\lambda - 4 = 0$

$\therefore \quad \lambda^3 + 0\lambda^2 - 6\lambda + 4 = 0$

By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & 0 & -6 & 4 \\ & 0 & 2 & 4 & -4 \\ \hline & 1 & 2 & -2 & \boxed{0} \end{array}$$

$\therefore \quad \lambda = 2$

$\lambda^2 + 2\lambda - 2 = 0$

$\Rightarrow \quad \lambda = -1 \pm \sqrt{3}$

$\therefore$ roots are $2, -1 \pm \sqrt{3}$

10. (D)

As for triangular matrix eigen values are same as principal diagonal element.

11. (A)

12. (B)

As $C_4 = 2C_1$

$\therefore |A|_{4\times4} = 0$

$\therefore$ rank < 4

Minor of order 3 $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{vmatrix}$

$= 1(9 - 4) - 2(3 - 2) + 1(4 - 6)$

$= 1 \neq 0$

$\therefore$ rank = 3

13. (B)

14. (B)

$$AB = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -9 & 10 \\ 3 & 4 & 6 \\ 4 & 2 & 4 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ -4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ -9 & 4 & 2 \\ -10 & 6 & 4 \end{bmatrix}$$

15. (C)

The given equations in matrix form

$$\begin{bmatrix} 1 & a & a \\ b & 1 & b \\ c & c & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For non trivial solution,

$|A| = 0$

$$\therefore \begin{vmatrix} 1 & a & a \\ b & 1 & b \\ c & c & 1 \end{vmatrix} = 0$$

$\therefore 1(1 - bc) - a(b - bc) + a(bc - c) = 0$

$\therefore \quad 1 - bc - ab + abc + abc - ac = 0$

$\therefore \quad ab + bc + ac - abc = 1 + abc \quad \text{....(i)}$

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}$$

$$= \frac{\left\{\begin{matrix} a(1-b)(1-c) + b(1-a)(1-c) \\ + c(1-a)(1-b) \end{matrix}\right\}}{(1-a)(1-b)(1-c)}$$

$$= \frac{\begin{Bmatrix} a(1-b-c+bc) \\ +b(1-a-c+ac) \\ +c(1-a-b+ab) \end{Bmatrix}}{(1-a)(1-b-c+bc)}$$

$$= \frac{\begin{Bmatrix} a-ab-ac+abc+b-ab-bc \\ +abc+c-ac-bc+abc \end{Bmatrix}}{1-b-c+bc-a+ab+ac-abc}$$

$$= \frac{\begin{Bmatrix} a+b+c+abc \\ -2(ab+bc+ac-abc) \end{Bmatrix}}{1-a-b-c+(ab+bc+ac-abc)}$$

$$= \frac{a+b+c+abc-2(1+abc)}{1-a-b-c+(1+abc)}$$

....From (i)

$$= \frac{a+b+c+abc-2-2abc}{1-a-b-c+1+abc}$$

$$= \frac{-2+a+b+c-abc}{2-a-b-c+abc}$$

$$= -1$$

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Solutions – Calculus

Answer Key on Assignment – 1

1.	(A)	2.	(D)	3.	(C)	4.	(D)
5.	(C)	6.	(C)	7.	(C)	8.	2
9.	(C)	10.	(D)	11.	–100.01 to –99.99	12.	5.9 to 6.1
13.	1.85 to 1.87	14.	–1	15.	4.66 to 4.76	16	85.0 to 85.5
17.	(B)	18.	(C)	19.	(C)	20.	(C)

Answer Key on Assignment – 2

1.	(B)	2.	(B)	3.	–0.01 to 0.01	4.	(A)
5.	(A)	6.	(D)	7.	(B)	8.	–5.1 to –4.9
9.	(B)	10.	2.0 to 2.0	11.	(B)	12.	(B)
13.	1.9 to 2.1	14.	(C)	15.	(B)	16	–13 to –13
17.	(D)	18.	(B)	19.	(A)	20.	0.99 to 1.01

Answer Key on Assignment – 3

1.	40 to 40	2.	(A)	3.	(C)	4.	(C)
5.	–0.35 to –0.30	6.	(C)	7.	(A)	8.	(C)
9.	(B)	10.	0.0 to 0.0	11.	(A)	12.	(A)
13.	0.95 to 1.05	14.	18 to 22	15.	(D)	16	(A)
17.	(A)	18.	(D)	19.	(C)	20.	(A)

Answer Key on Assignment – 4

1.	(B)	2.	(D)	3.	(B)	4.	(D)
5.	(D)	6.	(B)	7.	(A)	8.	(A)
9.	(C)	10.	(D)	11.	(D)	12.	(C)
13.	(C)	14.	(C)	15.	(B)	16	(A)
17.	(D)	18.	(B)				

Answer Key on Assignment – 5

1.	(D)	2.	(A)	3.	(C)	4.	(B)
5.	(C)	6.	(A)	7.	(A)	8.	(B)
9.	(A)	10.	(A)	11.	(D)	12.	(B)
13.	(B)	14.	(D)	15.	(B)	16.	(B)
17.	(B)	18	(D)				

Answer Key on Assignment – 6

1.	(A)	2.	(A)	3.	(B)	4.	(D)
5.	(B)	6.	(A)	7.	(A)	8.	(B)
9.	(C)	10.	(A)	11.	(B)	12.	(C)
13.	(A)	14.	(A)	15.	(B)	16.	(B)
17.	(C)	18.	(C)				

Answer Key on Assignment – 7

1.	(B)	2.	(C)	3.	(A)	4.	(B)
5.	(A)	6.	(D)	7.	(A)	8.	(B)
9.	(B)	10.	(C)	11.	(A)	12.	(B)
13.	(A)	14.	(D)	15.	(B)	16.	(A)
17.	(B)	18.	(A)				

Answer Key on Assignment – 8

1.	(C)	2.	(C)	3.	(D)	4.	(D)
5.	(D)	6.	(D)	7.	(B)	8.	(C)
9.	(C)	10.	(D)	11.	(A)	12.	(A)
13.	(B)	14.	(A)	15.	(C)	16.	(B)
17.	(C)	18.	(B)				

Answer Key on Assignment – 9

1.	(B)	2.	(B)	3.	(A)	4.	(C)
5.	(B)	6.	(D)	7.	(B)	8.	(A)
9.	(A)	10.	(C)	11.	(A)	12.	(B)
13.	(D)	14.	(C)	15.	(A)	16.	(D)
17.	(B)	18.	(B)				

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Model Solution on Assignment – 1

1. (A)

$$\lim_{x\to 0}\frac{x-\sin x}{1-\cos x} = \left(\frac{0}{0}\right)$$

Applying L. Hospital Rule,

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x} = \left(\frac{0}{0}\right)$$

Once again, L. Hospital rule

$$\lim_{x\to 0}\frac{\sin x}{\cos x} = \left(\frac{0}{1}\right) = 0$$

2. (D)

We know that f(x) is continuous at x = a,

if $\lim\limits_{x\to a} f(x)$ exists and equal to f(a).

3. (C)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Let $\frac{1}{x} = h$

as $x \to \infty$, $h \to 0$

$$\therefore \quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$= e$$

4. (D)

We have $e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + ...$

$$= \sum_{n=0}^{\infty}\frac{a^n}{n!}$$

Put a = 1

$$\therefore \quad \sum_{n=0}^{\infty}\frac{1}{n!} = e^1 = e$$

5. (C)

$$\lim_{x\to 0}\frac{1-\cos(x^2)}{2x^4} = \lim_{x\to 0}\frac{2\sin^2(x^2/2)}{2x^4}$$

$$= \lim_{x\to 0}\frac{\sin^2(x^2/2)}{(x^2/2)^2}\times\frac{1}{4}$$

$$= \frac{1}{4}\lim_{x\to 0}\left[\frac{\sin(x^2/2)}{(x^2/2)}\right]^2$$

$$= \frac{1}{4}\times(1)^2 = \frac{1}{4}$$

Alternate Method

$$\lim_{x\to 0}\frac{1-\cos(x^2)}{2x^4} = \frac{0}{0}$$

Using L Hospital Rule

$$\lim_{x\to 0}\frac{(\sin x^2)2x}{-8x^3} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(\cos x^2)2x2x + (\sin x^2)2}{24x^2}$$

$$\lim_{x\to 0}\frac{(\cos x^2)4x^2 + 2\sin x^2}{24x^2}$$

$$= \lim_{x\to 0}\frac{\left\{\begin{matrix}(-\sin x^2)4x^2 + \cos x^2(8x)\\ +2(\cos x^2)2x\end{matrix}\right\}}{48x}$$

$$= \lim_{x\to 0}\frac{\left\{\begin{matrix}(-\cos x^2)2x\left(4x^2\right)\\ +(-\sin x^2)(8x)\\ +(-\sin x^2)(2x)(8x) + 12\cos x^2\\ +(-\sin x^2)(2x)4x\end{matrix}\right\}}{48}$$

$$= \frac{12}{48} = \frac{1}{4}$$

6. (C)

Given: f(x) does not have a root in [a, b]. That is from x = a to x = b, then curve f(x) does not cross the X-axis. Hence the curve is entirely above X-axis or entirely below it.

$\therefore$ f(a) > 0 and f(b) > 0 OR f(a) < 0 and f(b) < 0

$\therefore$ f(a) . f(b) > 0

7. (C)

The ranges are given by 0 < y < a and 0 < x < y i.e. 0 < x < y < a.

This can also be written as 0 < x < a and x < y < a

8. 2

$$\sum_{n=0}^{\infty} n.\left(\frac{1}{2}\right)^n = ?$$

This can be solved using Z-transform and also using algebra as arithmetic-geometric series. We will use algebra.

$$\sum_{n=0}^{\infty} n.\left(\frac{1}{2}\right)^n = 0\times 1 + 1\times\frac{1}{2} + 2\times\left(\frac{1}{2}\right)^2 + ...$$

[a + (a + d)r + (a + 2d)r^2 + …..infinite terms]

$$= \frac{a}{1-r} + \frac{rd}{(1-r)^2} \qquad (-1 < r < 1)$$

For above series, a = 0, d = 1, r = $\frac{1}{2}$.

$$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n.n = \frac{0}{1-\frac{1}{2}} + \frac{(1/2)(1)}{\left(1-\frac{1}{2}\right)^2} = 2$$

9. (C)

$$\lim_{x\to 0}\frac{\log_e(1+4x)}{e^{3x}-1} = \lim_{x\to 0}\frac{\left(\frac{4}{1+4x}\right)}{3e^{3x}} = \frac{4}{3}$$

10. (D)

$$\lim_{x\to 0}\frac{x^3-\sin x}{x}$$

$$= (\lim_{x\to 0} x^2) - \left(\lim_{x\to 0}\frac{\sin x}{x}\right)$$

$$= 0 - 1 = -1$$

11. –100.01 to –99.99

$$f(x) = \frac{1}{3}x(x^2-3) = \frac{x^3}{3} - x$$

$$f'(x) = \frac{3x^2}{3} - 1 = x^2 - 1$$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

$f''(x) = 2x$

$f''(1) = 2 > 0$

$\Rightarrow$ at x = 1, f(x) has local minimum.

$f''(-1) = -2 < 0$

$\Rightarrow$ at x = –1, f(x) has local maximum

For x = 1, local minimum value

$$f(1) = \frac{1}{3} - 2 = \frac{-2}{3}$$

Finding f(–100) = –333433.33

f(100) = 333233.33

($\because$ x = 100, –100 are end points of interval)

$\therefore$ Minimum occurs at x = –100

12. 5.9 to 6.1

$f(x) = 2x^3 - 9x^2 + 12x - 3$

$\therefore \quad f'(x) = 6x^2 - 18x + 12$

$\therefore \quad f''(x) = 12x - 18$

$f'(x) = 0$ gives $6(x^2 - 3x + 2) = 0$

$\therefore$ x = 1, x = 2

$f''(1) = 12 - 18 = -6$

and $f''(2) = 24 - 18 = 6$

$\therefore$ There is local maxima at x = 1 and local minima at x = 2

Now, f(0) = –3,

f(1) = 2 – 9 + 12 – 3 = 2 and

f(3) = 54 – 81 + 36 – 3 = 6

13. 1.85 to 1.87

The length of the curve

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{2}{\pi}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{\sin^2 t + \cos^2 t + \frac{4}{\pi^2}}\, dt$$

$$= \int_0^{\pi/2} \sqrt{1 + \frac{4}{\pi^2}}\, dt$$

$$= \sqrt{1 + \frac{4}{\pi^2}}\,.\,t\Big|_0^{\pi/2}$$

$$= \sqrt{1 + \frac{4}{\pi 2}}\ x\ \frac{\pi}{2} = 1.8622$$

14. –1

$$I = \int_{2/\pi}^{2/\pi} \left(\frac{1}{x^2}\right) \cos\left(\frac{1}{x}\right) dx$$

Put $\left(\frac{1}{x}\right) = y$

$$\therefore\ -\left(\frac{1}{x^2}\right) dx = dy$$

Further, if x = $1/\pi$ then y = π and if x = $2/\pi$ then y = $\pi/2$

$$\therefore\ I = \int_{\pi}^{\pi/2} \cos y(-dy) = \int_{\pi/2}^{\pi} \cos y\, dy$$

$$= \sin(\pi) - \sin(\pi/2) = 0 - 1 = -1$$

15. 4.66 to 4.76

$$\text{Volume} = \int_{\rho=3}^{5} \int_{\varphi=\pi/8}^{\pi/4} \int_{z=3}^{4.5} \rho\, dz\, d\varphi\, d\rho$$

$$= \int_{\rho=3}^{5} \int_{\varphi=\pi/8}^{\pi/4} \rho(1.5)\, d\varphi\, d\rho$$

$$= \int_{\rho=3}^{5} \rho\, 1.5 \times (\pi/8)\, d\rho$$

$= 1.5 \times (\pi/8) \times 8$

$= 1.5\pi = 4.7124$

16. 85.0 to 85.5

17. (B)

$y = 2x - 0.1x^2$

$$\frac{dy}{dx} = 2 - 0.2x$$

$$\frac{d^2y}{dx^2} < 0$$

$\therefore$ y maximizes at $2 - 0.2x = 0$

$\Rightarrow$ x = 10

$\therefore$ y = 20 – 10 = 10 m

18. (C)

Given function is $f(x) = |x|$

$|x|$ is continuous at $x = 0$ but not differentiable

19. (C)

To find maxima

We differentiate $f(x)$ w.r.t. 'x' & equate to zero

$$\therefore \quad \frac{df(x)}{dx} = 3x^2 - 18x + 24 = 0$$

$$\Rightarrow \quad x^2 - 6x + 8 = 0$$

$$(x-4)(x-2)$$

$$x = 2 \text{ or } x = 4$$

Now,

$$\left.\frac{d^2f(x)}{dx^2}\right|_{x=2} = 6x - 18 = -6 < 0 \text{ (maxima)}$$

$$\left.\frac{d^2f(x)}{dx^2}\right|_{x=4} = 6 > 0 \text{ (minimum)}$$

$$\therefore \ f(x)\big|_{x=2} = (2)^3 - 9(2)^2 + 24(2) + 5$$

$$= 8 - 36 + 48 + 5 = 25$$

But since interval [1, 6] i.e. inclusive, we have to find

$f(1) = 1 - 9 + 24 + 5 = 21$

$f(6) = (6)^3 - 9(6)^2 + 24(6) + 5$

$= 216 - 324 + 144 + 5 = 41$

Thus, in the interval maximum value is 41.

20. (C)

Given $\quad x = \cos\left(\frac{\pi u}{2}\right)$,

$$y = \sin\left(\frac{\pi u}{2}\right) \ 0 \le u \le 1$$

$$\frac{dx}{du} = \frac{-\pi}{2}\sin\left(\frac{\pi u}{2}\right)$$

$$\frac{dy}{du} = \frac{\pi}{2}\cos\frac{\pi u}{2}$$

We know that surface area when the curve revolved about X-axis of a parametric curve is

$$= 2\pi\int_0^1 y\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

$$= 2\pi\int_0^1 \sin\left(\frac{\pi u}{2}\right)$$

$$\sqrt{\left(\frac{-\pi}{2}\sin\left(\frac{\pi u}{2}\right)\right)^2 + \left(\frac{\pi}{2}\cos\left(\frac{\pi u}{2}\right)\right)^2}\, du$$

$$= 2\pi\int_0^1 \sin\frac{\pi u}{2}\sqrt{\frac{\pi^2}{4}}\, du$$

$$= 2\pi \times \frac{\pi}{2}\int_0^1 \sin\frac{\pi u}{2}\, dx$$

$$= \pi^2\left(\frac{-1\cos\frac{\pi u}{2}}{\frac{\pi}{2}}\right)_0^1$$

$$= -\pi^2 \times \frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= 2\pi[\cos 0 - 1] = 2\pi$$

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Model Solution on Assignment – 2

1. (B)

$\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\sin 4x}$ is $\left(\dfrac{0}{0}\right)$

So, Applying L-Hospital Rule,

$\lim\limits_{x\to 0}\dfrac{2e^{2x}}{4\cos 4x} = \dfrac{2}{4} = 0.5$

2. (B)

Let I = $\dfrac{\int_0^2 (x-1)^2\sin(x-1)}{(x-1)^2+\cos(x-1)}dx$

We know that,

$\int_0^2 \dfrac{(2-x-1)^2\sin(2-x-1)}{(2-x-1)^2+\cos(2-x-1)}dx$

$= \int_0^2 \dfrac{(1-x)^2\sin(1-x)}{(1-x)^2+\cos(1-x)}dx$

$= -\int_0^2 \dfrac{(x-1)^2\sin(x-1)}{(x-1)^2+\cos(x-1)}dx = -I$

$\Rightarrow \quad I + I = 0$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

3. –0.01 to 0.01

$f(x) = \ell n(1+x) - x$

$\therefore \quad f'(x) = \dfrac{1}{1+x} - 1$ and $f''(x)$

$= -\dfrac{1}{(1+x)^2} < 0$

$\therefore$ Maxima $f'(x) = 0$ gives $\dfrac{1}{1+x} - 1 = 0$

$\therefore \quad \dfrac{1}{1+x} = 1$

$\therefore\ 1 + x = 1$

$\therefore\ x = 0$

$\therefore$ There is a maxima at x = 0

$\therefore$ Maximum value = f(0) = $\ell n(1) - 0 = 0$

4. (A)

In order to find maximum value of the function we have

$f'(x) = 0$

$\therefore \quad -x\, e^{-x} + e^{-x} = 0$

$\therefore \quad e^{-x}(1 - x) = 0$

$\Rightarrow \quad x = 1$

And $\quad f'(x) < 0$ at x = 1

$\therefore$ Maximum value is $f(1) = (1)\, e^{-1} = e^{-1}$

5. (A)

For negative value of x, f(x) will be positive

For positive values of x, f(x) will be positive

$\therefore$ minimum value of f(x) will occur at x = 0

6. (D)

7. (B)

The given function is

$F(x) = 1 - x^2 + x^3,$

where $x \in [-1, 1]$

$\Rightarrow \quad F'(x) = -2x + 3x^2$

By mean value theorem

$F'(x) = \dfrac{F(1) - F(-1)}{1-(-1)}$

Now $F(1) = 1 - (1)^2 + (1)^3 = 1$

and $F(-1) = 1 - (-1)^2 + (-1)^3$

$= 1 - 1 - 1 = -1$

$\therefore \quad F'(x) = \dfrac{1-(-1)}{1-(-1)} = \dfrac{2}{2} = 1$

$\therefore \quad F'(x) = 1$

$\therefore \quad -2x + 3x^2 = 1$

$\therefore \quad 3x^2 - 2x - 1 = 0$

$\therefore \quad 3x^2 - 3x + x - 1 = 0$

$\therefore \quad 3x(x - 1) + 1(x - 1) = 0$

$\therefore \quad (3x + 1) = 0$

or $\quad x - 1 = 0$

$\therefore \quad x = -\dfrac{1}{3}$

or $\quad x = 1$

Now $-\dfrac{1}{3}$ lies between $(-1, 1)$

$\therefore \quad x = -\dfrac{1}{3}$

8. –5.1 to –4.9

$f(x) = 2x^3 - 3x^2$ where $x \in [-1, 2]$

$\therefore f'(x) = 6x^2 - 6x$ and $f''(x) = 12x - 6$

$\therefore f'(x) = 0$ gives $6x^2 - 6x = 0$

$\therefore 6x(x - 1) = 0 \ \therefore x = 0, 1$

$f''(0) = -6 < 0$

$\therefore$ There is a maxima at $x = 0$

$f''(1) = 12 - 6 = 6 > 0$

$\therefore$ There is a minima at $x = 1$

$f(1) = 2 - 3 = -1$; $f(-1) = -2 - 3 = -5$;

$f(2) = 16 - 12 = 4$

$\therefore$ Global minimum value of $f(x) = -5$

9. (B)

10. 2.0 to 2.0

$I = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-x)}}\,dx$;

Use: $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore I = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-(1-x))}}\,dx$

$= \displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$

$= \left[\dfrac{x^{1/2}}{1/2}\right]_0^1 = 2$

11. (B)

$\displaystyle\int_0^2\int_0^x e^{x+y}\,dy\,dx$

$= \displaystyle\int_0^2 e^x \left(e^x \int_0^x e^y\,dy\right) dx$

$= \displaystyle\int_0^2 e^x \left(e^y\right)_0^x dx = \int_0^2 e^x\,(e^x - 1)\,dx$

$= \displaystyle\int_0^2 (e^{2x} - e^x)\,dx = \left(\frac{e^{2x}}{2} - e^x\right)_0^2$

$= \dfrac{e^4}{2} - e^2 - \dfrac{1}{2} + 1 = \dfrac{e^4}{2} - e^2 + \dfrac{1}{2}$

$= \dfrac{1}{2}(e^4 - 2e^2 + 1)$

$= \dfrac{1}{2}(e^2 - 1)^2$

12. (B)

$f(x) = x^3 - 3x^2 - 24x + 100$

where $x \in [-3, 3]$

$\therefore$ $f'(x) = 3x^2 - 6x - 24$ and

$f''(x) = 6x - 6$

$f'(x) = 0$ gives $3(x^2 - 2x - 8) = 0$

$\therefore$ $(x - 4)(x + 2) = 0$ $\therefore$ $x = -2, 4$

As $x = 4 > 3$, consider $x = -2 \in [-3, 3]$

$f''(-2) = -12 - 6 < 0$

$\therefore$ There is a maxima at $x = -2$

$f(-3) = -27 - 27 + 72 + 100 = 118$

$f(3) = 27 - 27 - 72 + 100 = 28$

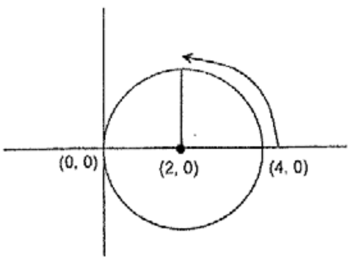
$\therefore$ Minimum value of $f(x) = 28$

13. 1.9 to 2.1

$\frac{1}{4}$ x circumference

$\Rightarrow$ $\frac{1}{4} \times \pi \times 4$

time $= \frac{\pi}{1.57} = 2$ sec



14. (C)

$$\lim_{x\to\infty} \sqrt{x^2 + x - 1} - x$$

$$= \lim_{x\to\infty} \left(\sqrt{x^2 + x - 1} - x\right) \frac{\left(\sqrt{x^2 + x - 1} + x\right)}{\left(\sqrt{x^2 + x - 1} + x\right)}$$

$$= \lim_{x\to\infty} \frac{\left(x^2 + x - 1 - x^2\right)}{\sqrt{x^2 + x - 1} + x}$$

$$= \lim_{x\to\infty} \frac{x\left(1 - \frac{1}{x}\right)}{x\left(\sqrt{1 + \frac{1}{x} - \frac{1}{x^2}} + 1\right)} = \frac{1}{2}$$

15. (B)

16. –13 to –13

$f(x) = 2x^3 - x^4 - 10$ where $x \in [-1, 1]$

$\therefore$ $f'(x) = 6x^2 - 4x^3$ and

$f''(x) = 12x - 12x^2$

$f'(x) = 0$ gives $2x^2(3 - 2x) = 0$

$\therefore$ $x = 0. 1.5$

As $x = 1.5 > 1$, consider $x = 0 \in [-1, 1]$

$f''(0) = 0$ $\therefore$ There is a neither maxima nor minima at $x = 0$

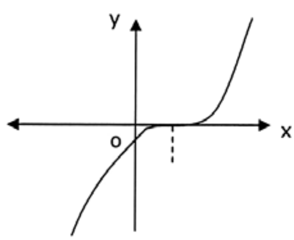
$f(-1) = -2 - 1 - 10 = -13$

$f(1) = 2 - 1 - 10 = -9$

$\therefore$ Minimum value of $f(x) = -13$

17. (D)

The function $f(x) = x^3 + 1$ has a point of inflection at $x = 0$, since in the graph sign of the curvature (i.e., the concavity) is changed.



18. (B)

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$=2\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x^2}{2}\right)\times 4}$$

$$=2\times\frac{1}{4}=\frac{1}{2}$$

19. (A)

$$\lim_{x\to 3+} f(x)=\lim_{x\to 3+}(x-1)=2=f(3)$$

$$\lim_{x\to 3-} f(x)=\lim_{x\to 3-}\left(\frac{x+3}{3}\right)=2=f(3)$$

$\therefore$ f(x) is continuous at x = 3

20. 0.99 to 1.01

$$\iint_R xy^2\,dxdy$$

$$=\iint_{R_1} xy^2dxdy+\iint_{R} xy^2dxdy$$

$$=\int_{x=1}^{5}\int_{y=0}^{2} xy^2dxdy+\int_{x=1}^{5}\int_{y=2}^{2x} xy^2dx$$

$$=\left(\frac{x^2}{2}\right)_1^5\left(\frac{y^3}{3}\right)_0^2+\int_1^5 x\left(\frac{y^3}{3}\right)_2^{2x}dx$$

$$=(12)\left(\frac{8}{3}\right)+\frac{1}{3}\int_1^5 x(8x^3-8)\,dx$$

$$=32+\frac{1}{3}\left[8\left(\frac{x^5}{5}\right)_1^5-8\left(\frac{x^2}{2}\right)_1^5\right]$$

$$=32+\frac{1}{3}\left(\frac{24992}{5}\right)-32$$

$$=\frac{24992}{15}$$

$$\therefore\ C\iint_R xy^2dxdy=\frac{2}{5}(24992)\times 10^{-4}$$

$$=0.99968\approx 1$$

OR

$$\iint_R xy^2\,dxdy=\int_{x=1}^{5}\left(\int_{y=0}^{2x} xy^2dy\right)dx$$

$$=\int_1^5 x\left(\frac{y^3}{3}\right)_0^{2x}dx$$

$$=\frac{8}{3}\int_1^5 x(x^3)\,dx$$

$$=\frac{8}{3}\left(\frac{x^5}{5}\right)_1^5$$

$$=\frac{8}{15}(3124)=\frac{24992}{15}$$

$$\therefore\ C\iint_R xy^2dxdy=\frac{24992}{15}\times 10^{-4}\times 6$$

$$=\frac{2}{5}\times 2.4992$$

$$=0.9968\approx 1$$

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Model Solution on Assignment – 3

1. 40 to 40

$$\frac{\partial f}{\partial x} = ((y^2 + z^2)(2x))$$

at x = 2, y = 1, z = 3 = (1 + 9) (4) = 40

2. (A)

$f(t) = e^{-t} - 2e^{-2t}$ where $0 \le t < \infty$

$\therefore\ f'(t) = -e^{-t} - 2e^{-2t}(-2) = 4e^{-2t} - e^{-t}$

and $f''(t) = e^{-t} - 8e^{-2t}$

$f'(t) = 0$ gives $4e^{-2t} - e^{-t} = 0$

$\therefore\ e^{-t}(4e^{-t} - 1) = 0$

$\therefore\ e^{-t} = 0$ or $e^{-t} = 1/4$

$\therefore\ t = \infty$ or $e^{t} = 4$ $\therefore\ t = \infty,\ t = \ell n(4)$

$f''(\infty) = 0$

$\therefore$ Neither maxima or minima at $t = \infty$

$$f''(\ell n4) = \frac{1}{4} - 8\times\frac{1}{16} = -\frac{1}{4} < 0$$

$\therefore$ Maxima at $t = \ell n(4)$

3. (C)

$z = xy\,\ell n(xy) = xy\,\ell n(x) + xy\,\ell n(y)$

$$\therefore\ \frac{\partial z}{\partial x} = y[1 + \ell n(x)] + y\,\ell n(y)$$

$$= y + y\,\ell n(xy)$$

$$\text{and } \frac{\partial z}{\partial y} = x\,\ell n(x) + x[1 + \ell n(y)]$$

$$= x + x\,\ell n(xy)$$

$$\therefore\ x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$$

4. (C)

Given, $\lim\limits_{x\to\infty}\left(\dfrac{x + \sin x}{x}\right)$

Put $x = \dfrac{1}{h}$ and $x \to \infty$ or $h \to 0$

$$\text{Now, } \lim_{h\to0}\left(\frac{\frac{1}{h} + \sin\frac{1}{h}}{\frac{1}{h}}\right) = \lim_{h\to0}\left(\frac{1 + h\sin\frac{1}{h}}{1}\right)$$

$$= \frac{1+0}{1} = 1$$

5. –0.35 to –0.30

$$\lim_{x\to0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right)\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to0}\left(\frac{-\cos x}{2\cos x + \cos x - x\sin x}\right)$$

(L – Hospital Rule)

$$= \frac{-1}{3}$$

6. (C)

$$I = \int_0^{\pi/2}\left(\frac{\cos x + i\sin x}{\cos x + i\sin x}\right)dx$$

$$= \int_0^{\pi/2}\left(\frac{e^{ix}}{e^{-ix}}\right)dx = \int_0^{\pi/2} e^{2ix}\,dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2} = \frac{1}{2i}[e^{i\pi} - e^0]$$

$$= \frac{1}{2i}[\cos(\pi) - i\sin(\pi) - 1]$$

$$= \frac{i}{2i^2}[-1 - 0 - 1] = i$$

7. (A)

The partial derivative of x^2y^2 with respect to y is $0 + 2y \Rightarrow 2y$.

The partial derivate of $6y + 4x$ with respect x is $0 + 4 = 4$.

Given that both are equal.

$\Rightarrow \quad 2y = 4$

$\Rightarrow \quad y = 2$

8. (C)

f(x) is discontinuous when

denominator = 0

$\therefore (x^2 + 3x - 4) = 0$

$\therefore (x + 4)(x - 1) = 0$

$\therefore x = -4, 1$

9. (B)

$f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x = 3x(x - 2)$

$\therefore$ Between -1 & 0, $f'(x) > 0$ i.e. f(x) is increasing

Between 0 & 2, $f'(x) < 0$ i.e. f(x) is decreasing

above 2, $f'(x) > 0$ i.e. f(x) is increasing

10. 0.0 to 0.0

$f(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x$ where $x \in [1, 2]$

$\therefore f'(x) = 3x^2 - 6x + 2$ and $f''(x) = 6x - 6$

$\therefore f'(x) = 0$ gives $3x^2 - 6x + 2 = 0$

$$\therefore x = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$= 1 \pm \frac{\sqrt{12}}{6} = 1 \pm 0.5774$$

$= 0.4226, 1.5774$

Now, $x = 0.4226 < 1$.

Consider $x = 1.5774$

$f''(1.5774) = 6(1.5774 - 1) > 0$

$\therefore$ Minima at $x = 1.5774$

$f(1) = 0$ and $f(2) = 0$

$\therefore$ Maximum value is $f(x) = 0$

11. (A)

3 sin x + 2 cos x

replacing sin x and cos x by its series,

$$= 3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\ldots\right]$$

$$+ 2\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots\right]$$

$$= 3x - \frac{3x^3}{3!} + \frac{3x^5}{5!} \ldots\ldots$$

$$+ 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} \ldots\ldots$$

$$= 2 + 3x - x^2 - \frac{x^3}{2} + \ldots\ldots$$

12. (A)

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha}\left[\frac{0}{0}\text{ form}\right]$$

Hence applying L Hospital rule

Let $\quad y = x^\alpha$

$\therefore \quad \log y = \alpha \log x$

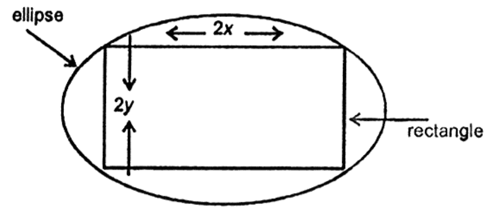
$$\frac{1}{y}\frac{dy}{dx} = \frac{\alpha}{x} + \log x \frac{d\alpha}{dx}$$

$$\therefore \quad \frac{dy}{dx} = x^\alpha\left[\frac{\alpha}{x} + \log x \frac{d\alpha}{dx}\right]$$

Now applying L Hospital rule

$$\lim_{\alpha\to 0}\frac{x^{\alpha}\left[\frac{\alpha}{x}+\log x\frac{d\alpha}{dx}\right]}{\frac{d\alpha}{dx}} = \log x$$

13. 0.95 to 1.05



Here rectangle is inscribed in ellipse.

Let the length and breadth of rectangle be 2x and 2y respectively.

Now area of rectangle = (2x) × (2y)

$= 4xy$

Let $F = (\text{Area of rectangle})^2$

$= (4xy)^2$

$= 16\, x^2\, y^2$

$= 16x^2\left(\frac{1-x^2}{4}\right)$

$\left[\because y^2 = \frac{1-x^2}{4} \text{ from equation of ellipse}\right]$

$= 4x^2\,(1 - x^2)$

Now $F'(x) = 0$

$\therefore$ $x(1 - 2x^2) = 0$

$\therefore$ $x = 0$

or $1 - 2x^2 = 0$

$\Rightarrow$ $2x^2 = 1$

$\Rightarrow$ $x^2 = \frac{1}{2}$

$\Rightarrow$ $x = \frac{1}{\sqrt{2}}$

For $x = 0$,

$y^2 = \frac{1}{4}$

$\Rightarrow$ $y = \pm\frac{1}{2}$

and for $x = \frac{1}{\sqrt{2}}$

$y^2 = \frac{1-\frac{1}{2}}{4} = \frac{1}{8}$

$\Rightarrow$ $y = \frac{1}{\sqrt{8}}$

Now $F''(x) = 8 - 48\, x^2$

$\therefore$ $F''(x) < 0$ for

$x = \frac{1}{\sqrt{2}}$

$\Rightarrow$ F(x) is maximum at $x = \frac{1}{\sqrt{2}}$

The maximum area of rectangle inscribed in ellipse is

$$4xy = 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{8}}\right)$$

$$= \frac{4}{4} = 1$$

14. 18 to 22

$I = \frac{1}{2\pi}\iint_D (x+y+10)\,dx\,dy$ where D denotes $x^2 + y^2 \le 4$

Use polar co-ordinates. Put $x = r\cos\theta$, $y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$

$$I = \frac{1}{2\pi}\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r\cos\theta + r\sin\theta + 10)\, r\, dr\, d\theta$$

$$= \frac{1}{2\pi}\int_{r=0}^{2}\left[r(r\sin\theta - r\cos\theta + 10\theta\right]_0^{2\pi} dr$$

$$= \frac{1}{2\pi}\int_{r=0}^{2}(10r \times 2\pi)\, dr$$

$$= 10\left[\frac{r^2}{2}\right]_0^2 = 10 \times 2 = 20$$

15. (D)

16. (A)

$P = 50q - 5q^2$

$$\frac{dp}{dq} = 50 - 10q; \frac{d^2p}{dq^2} < 0$$

$\therefore$ p is maximum at $50 - 10q = 0$ or,

$q = 5$

Else check with options

17. (A)

The given curves are $y = x$ and $y = x^2$

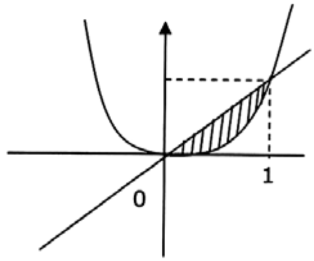
solving (1) and (2) we have

$x = 0$, $x = 1$

$$\text{Area} = \int_0^1 (x - x^2)\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ sq units.}$$



18. (D)

19. (C)

$$\int_1^e \sqrt{x}\ \ln(x)\, dx$$

$$= \left[\ln(x) \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^e - \int\left[\frac{1}{x} \times \frac{x^{\frac{3}{2}}}{\frac{2}{3}}\right] dx$$

$$= \left[\ln(x) \times x^{\frac{3}{2}} \times \frac{2}{3} - \frac{4}{9} \times x^{\frac{3}{2}}\right]_1^e$$

$$= \frac{2}{9}\sqrt{e^3} + \frac{4}{9}$$

20. (A)
Given

$$I = \int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}}\, dx$$

$$= \left.\frac{(\sin^{-1} x)^3}{3}\right|_0^1 \left(\because \int f^n(x) f'(x)\, dx = \frac{f^{n+1}(x)}{n+1}\right)$$

$$= \frac{1}{3}\left[(\sin^{-1})^3 - \sin^{-1} 0\right]$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - 0\right] = \frac{\pi^3}{24}$$

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Model Solution on Assignment – 4

1. (B)

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\left(\frac{\sin(x/2)}{(x/2)}\right)^2 \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\left(\therefore \lim_{x\to 0}\frac{\sin\theta}{\theta} = 1\right)$$

2. (D)

Applying L' Hospitals rule, we have

$$\lim_{\theta\to 0}\frac{\cos\theta}{1} = \cos 0 = 1$$

3. (B)

$$L = \lim_{x\to 8}\frac{x^{1/3}-2}{(x-8)}$$

Let $x = a^3 \Rightarrow a \to 2$

$$\therefore L = \lim_{a\to 2}\frac{(a-2)}{a^3-8}$$

Now by partial fractions,

$(a^3 - 8) = (a - 2)\,(a^2 + 2a + 4)$

$$\Rightarrow \quad L = \lim_{a\to 2}\frac{1}{a^2+2a+4} = \frac{1}{12}$$

Alternate Method

$$\text{Since } \lim_{x\to a}\frac{x^n-a^n}{x-a} = n\,a^{n-1}$$

$$L = \lim_{x\to 8}\frac{x^{1/3}-2}{(x-8)}$$

$$= \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$$

4. (D)

$$\lim_{x\to\infty}\frac{ax+b}{cx} = \lim_{x\to\infty}\frac{a}{c}+\frac{b}{x}$$

$$= \frac{a}{c} + 0 = \frac{a}{c}$$

5. (D)

$$\operatorname*{Lim}_{x\to 0} f(x) = \operatorname*{Lim}_{x\to 0}\frac{e^x-1}{x} = 1$$

But f (0) = $\frac{0}{0}$, which is not defined.

Since $\operatorname*{Lim}_{x\to 0} f(x) \neq f(0)$ therefore f (x) has removable discontinuity at x = 0.

6. (B)

$$\lim_{x\to 0}\frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{\left\{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right) - \left(1+x+\frac{x^2}{2}\right)\right\}}{x^3}$$

$$= \lim_{x\to 0}\frac{1}{6}+\frac{x}{4!}+\frac{x^2}{5!}+\ldots = \frac{1}{6}$$

Else, use L'Hospital's rule

7. (A)

8. (A)

9. (C)

10. (D)

$f(x) = x^3 - 6x^2 + 24x + 4$

$f'(x) = 3x^2 - 12x + 24$

$f''(x) = 6x - 12$

For maxima and minima, $f'(x) = 0$

or $3x^2 - 12x + 24 = 0$

or $x = 2 \pm 2i$

$\therefore$ f (x) has no maximum and no minimum at any real value of (x).

11. (D)

Two parts are 30 and 30 – x

$f(x) = x^3 .(30 - x)$

$f(x) = 30x^3 - x^4$

$f'(x) = 90x^2 - 4x^3$

$f''(x) = 180x - 12x^2$

For maximum and Minimum $f'(x) = 0$

or $90x^2 - 40x^3 = 0$

or $x = 0, x = \frac{90}{4} = \frac{45}{2}$

But, x = 0 is impossible, hence consider only $x = \frac{45}{2}$

$$f''\left(\frac{45}{2}\right) = 180 \times \frac{45}{2} - 12\left(\frac{45}{2}\right)^2$$

$$f''\left(\frac{45}{2}\right) < 0$$

$\therefore$ f (x) is maximum at $x = \frac{45}{2} = 22.5$

$\therefore$ Two parts of 30 are 22.5 and 7.5

12. (C)

$f(x) = (x-2)^3$

$f'(x) = 3(x-2)^2$

$f''(x) = 0$ for maximum and minimum.

or $x = 2$

When $x > 2$, $f'(x) = 3(x-2)^2 > 0$

When $x < 2$, $f'(x) > 0$

Hence neither maximum nor minimum at x = 2.

13. (C)

$y = \sin x (1 + \cos x)$

$$\frac{dy}{dx} = \cos x - \sin^2 x + \cos^2 x$$

$$= \cos x + \cos 2x$$

$$\frac{d^2y}{dx^2} = -(\sin x + 2 \sin 2x)$$

For maximum and minimum, $\frac{dy}{dx} = 0$

$\cos x + \cos 2x = 0$

or $2\cos^2 x + \cos x - 1 = 0$

or $\cos x = \frac{1}{2}, \cos x = -1$

$\therefore x = \frac{\pi}{3}$ or $x = \pi$

But $\frac{\pi}{3} \in (0, \pi)$ and $\pi \notin (0, \pi)$

$$\frac{d^2y}{dx^2} < 0 \text{ at } x = \frac{\pi}{3}$$

14. (C)

$f(x) = 4x^3 - 8x^2 + 1$ in $[-1, 1]$

$f'(x) = 12x^2 - 16x$

Equating $f'(x)$ to 0

$\therefore \quad 12x^2 - 16x = 0$

$3x^2 - 4x = 0$

$x(3x - a) = 0$

$\therefore \quad x = 0, \quad x = \frac{4}{3}$

Now, $\quad x = \frac{4}{3}[-1, 1]$

So $x = 0$

$f''(x) = 6x - 4$

$\therefore \quad f''(0) = -4 < 0$

$\therefore$ f(x) in not minimum at x = 0.

Checking at endpoints :

$f(-1) = 4(-1)^3 - 8(-1)^2 + 1$

$= -4 - 8 + 1 = -11$

$f(1) = 4(1)^3 - 8(1)^2 + 1$

$= 4 - 8 + 1 = -3$

$\therefore$ f(x) has minimum at x = –1 in [–1, 1]

15. (B)

$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow -4\pi = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{-1}{r^2} = \frac{-1}{100} = -0.01$

Since diameter = 2 $\times$ radius

Hence diameter is decreasing at the rate of 0.02 cm /hr.

16. (A)

$\frac{dV}{dt} \propto -4\pi r^2$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$

$\therefore \quad 4\pi r^2 \frac{dr}{dt} \propto -4\pi r^2$

i.e. $4\pi r^2 \frac{dr}{dt} = -k\, 4\pi r^2$

$\therefore \quad \frac{dr}{dt} = -k$

17. (D)

$f'(x) = 4x^3 - 6x^2 - 2x - 4$

$= 2(x - 2)(2x^2 + x + 1)$

Thus only critical number is x = 2

So $f(0) = 3$

$f(2) = -9$

$f(4) = 99$

18. (B)

$f'(x) = \frac{(x+2)(3x^2) - x^3}{(x+2)^3}$

$= \frac{2x^2(x+3)}{(x+2)^2} = 0$

x = 0 and x = –3 are critical.

However x = –3 does not lie in interval so we list 0 and end points –1 and 1.

f(0) = 0, f (–1) = –1, f (1) = 1/3

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Model Solution on Assignment – 5

1. (D)

$$\lim_{x\to\infty} (\sqrt{x^2+1} - \sqrt{x+1})$$

$$= \lim_{x\to\infty}(\sqrt{x^2+1} - \sqrt{x+1})\frac{(\sqrt{x^2+1}+\sqrt{x+1})}{(\sqrt{x^2+1}+\sqrt{x+1})}$$

$$= \lim_{x\to\infty} \frac{(x^2+1)-(x+1)}{(\sqrt{x^2+1}+\sqrt{x+1})}$$

$$= \lim_{x\to\infty} \frac{x(x-1)}{x\left[\sqrt{1+\frac{1}{x^2}} + \frac{1}{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}\right]}$$

$$= \lim_{x\to\infty} \frac{(x-1)}{\left[\sqrt{1+\frac{1}{x^2}} + \frac{1}{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}\right]}$$

$$= \frac{\infty}{0+1} = \infty$$

2. (A)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{ax+b}{x+1}$$

$$= \frac{\lim_{x\to 0} ax+b}{\lim_{x\to 0} x+1} = b \quad ....(1)$$

But $\lim_{x\to 0} f(x) = 2$ (Given)

$\therefore \quad b = 2 \quad$ (from eq. (1))

Also, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{ax+b}{x+1}$

$$= \lim_{x\to\infty} \frac{a+(b/x)}{1+(1/x)}$$

$$= a \quad ....(2)$$

But $\lim_{x\to\infty} f(x) = 1$ (Given)

$\therefore \quad a = 1$ (From eq. (2))

$$\therefore \quad f(x) = \frac{x+2}{x+1}$$

$$f(-2) = \frac{-2+2}{-2+1} = 0$$

3. (C)

Here

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+ .... + \frac{n^2}{n^3}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1^2+2^2+3^2+.....+n^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right]$$

$$\left[\because 1^2+2^2+3^2+....+n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2n^2+3n+1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{6}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$= \frac{2}{6} = \frac{1}{3}$$

4. (B)

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = \left(\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}\right)^{-2} = e^{-2}$$

5. (C)

Put $x - \frac{\pi}{2} = t$, so that, $x = \frac{\pi}{2} + t$

or $2x = \pi + 2t$

when $x \to \frac{\pi}{2}$ then $t \to 0$

Now $\lim\limits_{x \to (\pi/2)} \frac{1 + \cos 2x}{(\pi - 2x)^2}$

$= \lim\limits_{t \to 0} \frac{1 + \cos(\pi + 2t)}{(-2t)^2}$

$= \lim\limits_{t \to 0} \frac{1 - \cos 2t}{4t^2}$

$= \lim\limits_{t \to 0} \frac{2\sin^2 t}{4t^2}$

$= \lim\limits_{t \to 0} \left(\frac{\sin t}{t}\right)^2 . \frac{1}{2} = \frac{1}{2}$

6. (A)

Put $\frac{\pi}{2} - \theta = y$, as $\theta \to \frac{\pi}{2}$, $y \to 0$

$\therefore \lim\limits_{\theta \to (\pi/2)} \frac{\cot\theta}{(\pi/2) - \theta}$

$= \lim\limits_{\theta \to (\pi/2)} \frac{\tan((\pi/2) - \theta)}{(\pi/2) - \theta}$

$= \lim\limits_{y \to 0} \frac{\tan y}{y}$

$= 1$

7. (A)

$\frac{dy}{dx} = 7 - 3x^2$ hence rate of change of slope is $\frac{d}{dt}\left(\frac{dy}{dx}\right) = -6x\frac{dx}{dt}$

$= -6 \times 3 \times 4 = -72$

8. (B)

$3\frac{dy}{dt} = 2x\frac{dx}{dt}$

When $x = 3$, $\frac{dx}{dt} = 1 \text{foot/min}$

$\Rightarrow 3\frac{dy}{dt} = 6 \Rightarrow \frac{dy}{dt} = 2 \text{ft/min}$

9. (A)

$A = 4\pi r^2$. $\qquad v = \frac{4}{3}\pi r^3$

$\frac{dA}{dt} = 8\pi r\frac{dr}{dt} \qquad \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$

$10 = 4\pi r^2\frac{dr}{dt} = \frac{1}{2} \times r \times \frac{dA}{dt}$

When $r = 5$,

$10 = \frac{1}{2} \times 5 \times \frac{dA}{dt}$

$\Rightarrow \frac{dA}{dt} = 4 \text{cm}^2/\text{h}$

10. (A)

$b^2 = r^2 + (h/2)^2$

$v = \pi r^2 h = \pi(b^2 - h^2/4)h$

$\frac{dv}{dh} = \pi\left(b^2 - 3h^2/h\right) = 0 \Rightarrow h = 2b/\sqrt{3}$

$\frac{d^2v}{dh^2} = -\frac{3\pi h}{2} \Rightarrow \left.\frac{d^2v}{dh^2}\right]_{2b/\sqrt{3}}$

$= \frac{-3\pi}{2} \times \frac{2b}{\sqrt{3}} b - ve$

Hence there is relative maxima at

$h = 2b/\sqrt{3}$

$\therefore r = b\sqrt{2/3}$

11. (D)

Let x be thickness of ice

$v = 300\,[\,\pi\,(10 + x)^2 - 100\pi]$

$$\frac{dv}{dt} = 300\pi\left[2(10 + x)\frac{dx}{dt}\right]$$

$$\frac{dv}{dt} = -2\pi$$

$$\Rightarrow \frac{dx}{dt} = \frac{-1}{300(10+x)}$$

$$= \frac{-1}{300(10+2)} = \frac{-1}{3600}$$

12. (B)

Let length be 2x and breadth be 2y

$x^2 + y^2 = 1 \Rightarrow dy/dx = -x/y$

$A = (2x)\,(2y) = 4xy$

$$\Rightarrow \frac{dA}{dx} = \frac{4}{y}\left(-x^2 + y^2\right) = 0$$

$$\Rightarrow x = y = 1/\sqrt{2}$$

$$\frac{d^2A}{dx^2} = -\frac{4x}{y^3}$$

$$\Rightarrow \left.\frac{d^2A}{dx^2}\right|_{1/\sqrt{2},\,-1/\sqrt{2}} = \frac{-4\times 1/\sqrt{2}}{\left(1/\sqrt{2}\right)^3} = -8$$

$$\therefore \left.\frac{d^2A}{dx^2}\right|_{1/\sqrt{2},\,-1/\sqrt{2}} = -16$$

$$\therefore \text{Area} = 4xy = 4\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = 2$$

13. (B)

Given : $y = 3x^4 - 16x^3 + 24x^2 + 37$

For maximum value or minimum

$$\frac{dy}{dx} = 12x^3 - 48x^2 + 48x = 0$$

$$\Rightarrow \quad x\,(x^2 - 4x + 4) = 0$$

$$\Rightarrow \quad x\,(x-2)^2 = 0$$

$$\Rightarrow \quad x = 0, 2, 2$$

$$\frac{d^2y}{dx^2} = 36x^2 - 96x + 48$$

At x = 0,

$$\frac{d^2y}{dx^2} = +48$$

At x = 2,

$$\frac{d^2y}{dx^2} = 36\times 4 - 96\times 2 + 48 = 0$$

so only 1 local minima.

either x = 0 or complex root

so only value = 0

14. (D)

$$\int_0^{\pi/4}\frac{(1-\tan x)}{(1+\tan x)}dx$$

$$I = \int_0^{\pi/4}\frac{\cos x - \sin x}{\cos x + \sin x}dx$$

Let t = cos x + sin x

$\therefore$ dt = (–sin x + cos x) dx

$$\therefore \quad \frac{dt}{t} = \frac{\cos x - \sin x}{\cos x + \sin x}dx$$

$$\therefore \quad I = \int_0^{\pi/4}\frac{dt}{t} = [\ln t]_0^{\pi/4}$$

$$= \ln(\cos x + \sin x)_0^{\pi/4}$$

$$= \ln(\cos\pi/4 + \sin\pi/4)$$

$$= \ln\left(\sqrt{2}\right) = \ln 2^{1/2} = \frac{1}{2}\ln 2$$

15. (B)

If 2 points on line are $A_1(0, 1)$ and $A_2(-1, 0)$

$\therefore$ Equation of line is $\dfrac{y-y_1}{x-x_1} = \dfrac{y_1-y_2}{x_1-x_2}$

$$\frac{y-1}{x-0} = \frac{1-0}{0-(-1)}$$

$$\frac{y-1}{x} = 1$$

$$y = x + 1$$

$$I = \int_1^2 y\,dx = \int_1^2 (x+1)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^2 = (2+2) - \left(\frac{1}{2} + 1\right)$$

$$= 4 - \frac{3}{2} = \frac{5}{2} = 2.5$$

16. (B)

$$\int_0^1 e^x dx = x\int_0^1 e^x dx - \int_0^1 (1)\int_0^1 e^x dx$$

$$\left(xe^x\right)_0^1 - \left(e^x\right)_0^1 = (e-0) - (e-1) = 1$$

17. (B)

18. (D)

Let $I = \int_0^\infty\int_0^\infty e^{-x^2} e^{-y^2} dxdy$

$\therefore \quad I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy$

Put $x = r\cos\theta$

$y = r\sin\theta$

Then $J = r$

Now $0 \le x \le \infty$ and $0 \le y \le \infty$

$\Rightarrow$ region of integration is the first quadrant.

$\therefore \quad 0 \le r \le \infty, \quad 0 \le \theta \le \pi/2$

$\therefore \quad I = \int_0^{\pi/2}\int_0^\infty e^{-r^2} \cdot r\,dr\,d\theta$

$$= \int_0^{\pi/2} d\theta \int_0^\infty e^{-r} r dr = \frac{\pi}{2}\int_0^\infty e^{-r^2} dr \cdot \frac{2r}{2}$$

$$= \frac{\pi}{4}\int_0^\infty e^{-r^2} \cdot 2r\,dr = \frac{\pi}{4}\left[-e^{-r^2}\right]_0^\infty$$

$$= \frac{\pi}{4}\left[-0-(-1)\right]$$

$\therefore \quad I = \dfrac{\pi}{4}$

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Model Solution on Assignment – 6

1. (A)

Here f (0) = 0

Right Hand Limit

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{|x|}{x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{x}{x}\right) = 1$$

$$\left[\begin{array}{ll}\because |x| = x & \text{if } x \geq 0 \\ \quad\;\; = -x & \text{if } x < 0\end{array}\right]$$

Left Hand Limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{|x|}{x}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{-x}{x}\right) = -1$$

$\therefore$ R.H.L. $\neq$ L.H.L.

The function f (x) is discontinuous at x = 0 and the discontinuity at x = 0 is of first kind because R.H.L. and L.H.L. both exist finitely but are unequal.

2. (A)

Here f (1) = – 1

Also, Right Hand Limit

$$\operatorname{Lim}_{x\to 1^+} f(x) = -1$$

Left Hand Limit

$$\operatorname{Lim}_{x\to 1^-} f(x) = a - 2$$

Now, L.H.L. = R.H.L.

$\Rightarrow$ a – 2 = – 1 $\Rightarrow$ a = 1

3. (B)

Here f (x) = 1

Also, Left Hand Limit

$$\operatorname{Lim}_{f(x)\to 3^-} 1 = 1$$

Right Hand Limit

$$\operatorname{Lim}_{x\to 3^+} (ax+b) = 3a + b$$

Also, f (3) = 1

Since f (x) is continuous, therefore

3a + b = 1 ….(1)

Also, f (s) = 7

and $\operatorname{Lim}_{f(x)\to 5^-} f(x) = \operatorname{Lim}_{x\to 5^-} (ax + b)$

= 5a + b

Since f (5) is continuous at x = 5, therefore

5a + b = 7 ….(2)

Solving eq. (1) and eq. (2), we get

a = 3, b = – 8.

4. (D)

The given function is of straight line hence no maximum or minimum.

5. (B)

y = log x

$$\frac{dy}{dx} = \frac{1}{x}$$

Since log x is defined only for +ve values of x.

So, $\frac{1}{x} > 0 \Rightarrow \frac{dy}{dx} > 0 \Rightarrow$ y = log x is increasing.

6. (A)

$y = \log(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$= \frac{\cos x}{\sin x} = \cot x$$

For the interval $\left[0, \frac{\pi}{2}\right]$, cot x is +ve

$$\Rightarrow \frac{dy}{dx} > 0 \text{ on } \left[0, \frac{\pi}{2}\right].$$

$\Rightarrow$ log (sin x) is increasing on $\left[0, \frac{\pi}{2}\right]$

Again cot x is negative on the interval $\left]\frac{\pi}{2}, \pi\right[$

$$\therefore \frac{dy}{dx} < 0$$

$\Rightarrow$ log (sin x) is decreasing.

7. (A)

The series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ...$ is a G.P

$$S_n = \frac{1\left[1-(-1/2)^n\right]}{1-\left(\frac{-1}{2}\right)} = \frac{2}{3}\left[1-\left(\frac{-1}{2}\right)^n\right]$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{2}{3}\left[1-\left(-\frac{1}{2}\right)^n\right]$$

$$= \frac{2}{3}[1-0] = \frac{2}{3}$$

Hence the series is convergent

8. (B)

The series 1 + 2 + 3 + ….. is an A.P

$$S_n = \frac{1}{2}n(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}n(n+1) = \infty$$

hence the series is divergent.

9. (C)

The series 1 – 2 + 3 – 4 + – 6 + . ……

is an alternatory series

S_n = (1 – 2) + (3 – 4) + (5 – 6) +. . . .+ [(n – 1) – n]

$$= (-1) + (-1) + ... + (-1) = \frac{-1}{2}n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\frac{1}{2}n\right) = -\infty$$

Series can also be written as

S_n = 1 – (2 – 3) – (4 – 5) – (6 –7)……..

– ((n–1) – n))

$$= 1 + 1 + 1 + 1 ....+ 1 = \frac{n+1}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{2}(n+1) = \infty$$

Since limit does not exist because the sum of infinite forms of series is not unique

10. (A)

The series is alternatory series

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4}.......$$

$$\lim_{n\to\infty} 1_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

here the series is convergent

11. (B)

Here $U_n = \dfrac{n}{1+2^{-n}}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{1+2^{-n}} = \infty \neq 0$$

hence the series is divergent.

12. (C)

The infinite series

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \text{.....is}$$

i) convergent of $p > 1$

ii) divergent of $p \leq 1$

13. (A)

The series $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27}$......is a G.P.

Where a = 1 ,

r = $-\dfrac{1}{3}$ Which lie between -1 and 1

Hence the given series is convergent.

14. (A)

$U_n = \sqrt{n^3+1} - \sqrt{n^3}$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

Take $V_n = \dfrac{1}{n^{3/2}}$

$$\lim_{A\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \left[\frac{\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}}{\frac{1}{n^{3/2}}} \right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{\sqrt{1+\frac{1}{n^{3/2}}} + \sqrt{1}} \right)$$

$$= \frac{1}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$$

Hence it is convergent.

15. (B)

here $U_n = \text{Sin } \dfrac{1}{n}$

$$= \frac{1}{n} \frac{1}{\lfloor 3} \left(\frac{1}{n}\right)^3 + \ldots = \frac{1}{n} - \frac{1}{6n^3} + \text{.....}$$

Take $V_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{V_n}{V_n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{6n^3} + \ldots \right) \times \frac{n}{1}$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{6n^2} + \ldots \right) = 1$$

series ΣV_n n a divergent. Hence Σ U_n is also divergent.

16. (B)

$U_n = \dfrac{n+1}{n^2}$, Take $V_n = \dfrac{n}{n^2} = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{V_n}{V_n} = \lim_{n\to\infty} \left(\frac{\left(\frac{n+1}{n^2}\right)}{\frac{1}{n}} \right) = 1$$

By comparison test ΣU_n & Σ V_n behave alike.

Now ΣV_n = $\Sigma \dfrac{1}{n}$ is a divergent series

hence ΣU_n is also divergent.

17. (C)

Since $f(x) = 1/x$ is not continuous in $[-3, 3]$ $[-4, 2]$ or $[-1, 1]$ the point of discontinuity is 0. only in $[2, 3]$ the function is continuous and differentiable.

18. (C)

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(b) - f(a) = (b - a)\ f'(c)$

$b^3 - a^3 = (b - a) \times 3c^2$

$$\Rightarrow 3c^2 = \frac{(b-a)\left(a^2+ab+b^2\right)}{b-a}$$

$$\Rightarrow c = \sqrt{\frac{a^2+b^2+ab}{3}}$$

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Model Solution on Assignment – 7

1. (B)

$f(x) = e^{(1/x)}$

$f'(x) = e^{(1/x)}\left(\frac{-1}{x^2}\right)$

Since $e^{(1/x)} > 0$ and $x^2 > 0$,

for all $x \neq 0$

$\therefore$ $f'(x) < 0$, for all $x \neq 0$, $x > 0$

$\therefore$ f (x) is decreasing in the interval $[0, \infty]$

Hence $f(x) = e^{1/x}$, $x > 0$ is a decreasing.

2. (C)

$f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$; $f''(x) = \frac{2}{x^3}$

for maximum and minimum, $f'(x) = 0$

or $1 - \frac{1}{x^2} = 0$

or $x = \pm 1$

$f''(-1) = -2 < 0$

$\therefore$ f (x) is maximum at $x = -1$.

3. (A)

$f(x) = -x^2 + 2x + 1$

$f'(x) = -2x + 2$; $f''(x) = -2$

$f'(x) = 0$

[for maximum and minimum]

$-2x + 2 = 0$

$x = 1$

$f''(x) = -2 < 0$

$\therefore$ f (x) is maximum at $x = 1$

4. (B)

$$f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$$

$$f'(\theta) = -5\sin\theta - 3\sin\left(\theta + \frac{\pi}{3}\right)$$

$$f''(\theta) = -5\cos\theta - 3\cos\left(\theta + \frac{\pi}{3}\right)$$

Equating $f'(\theta)$ to 0

$$-5\sin\theta - 3\sin\left(\theta + \frac{\pi}{3}\right) = 0$$

$$\therefore \quad -5\sin\theta + 3\sin\left(\theta + \frac{\pi}{3}\right) = 0$$

$$\therefore \quad 5\sin\theta + 3\left[\sin\theta \cdot \frac{1}{2} + \cos\theta \cdot \frac{\sqrt{3}}{2}\right] = 0$$

$$\therefore \quad 5\sin\theta + \frac{3}{2}\sin\theta + 3\frac{\sqrt{3}}{2}\cos\theta = 0$$

$$\therefore \quad \frac{13}{2}\sin\theta + \frac{3\sqrt{3}}{2}\cos\theta = 0$$

$$\therefore \quad \tan\theta = -\frac{3\sqrt{3}}{13}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{-3\sqrt{3}}{13}\right)$$

$$\theta = -21.79^\circ$$

$f''(-21.79^\circ) < 0$

$\Rightarrow$ maximum at $\theta = -21.79^\circ$

$f(-21.79^\circ) = 5\cos(-21.79^\circ) + 3\cos(-21.79^\circ + 60^\circ)$

$f(-21.79^\circ) = 10$

5. (A)

$f(x) = x^3 - 12x^2 + 45x + 11$

$f'(x) = 3x^2 - 24x + 45$

$f''(x) = 6x - 24$

$f'(x) = 0$ or $3x^2 - 24x + 45 = 0$

$x = 3, x = 5$

$f''(5) = (6 \times 5) - 24$

$= 6 > 0$

$\therefore$ f (x) is minimum at x = 5.

6. (D)

7. (A)

Using lagrange's mean value theorem

$f'(c) = \frac{f(x) - f(0)}{x - 0}$ or $e^c = \frac{e^x - 1}{x}$ ....(i)

Now 0 <<< x $\Rightarrow e^0 < e^c < e^x$ ....(ii)

From equation (i) and (ii)

$1 < \frac{e^x - 1}{x} < e^x$

$\Rightarrow x < e^x - 1 < xe^x$

$\Rightarrow 1 + x < e^x < 1 + xe^x$

8. (B)

$x^{1/x}$ is decreasing function then

$f'(x) = x^{1/x}\left(\frac{1 - \log_e x}{x^2}\right)$

f(x) is decreasing if $f'(x) < 0$

i.e. $x^{1/x}\left(\frac{1 - \log x}{x^2}\right) < 0$

$\Rightarrow 1 - \log_e x < 0 \Rightarrow \log_e x > 1$

$\Rightarrow x > e$

9. (B)

let $f(x) = e^x$. Then $f^n(x) = e^x$.

Hence $f^n(o) = 1$ for all $n \geq 0$

$\therefore$ Maclaurin series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

10. (C)

Let f(x) = sin x.

Then $f(\pi/4) = \sqrt{2}/2 . f'(\pi/4)$

$= \sqrt{2}/2\, f''(\pi/4) = \frac{-\sqrt{2}}{2}$

$f'''(\pi/4) = \frac{(-\sqrt{2})}{2}$.Thus

Taylor series for sin x about $\pi/4$ is

$$\frac{\sqrt{2}}{2}\left[1 + \frac{x - \pi/4}{1!} - \frac{(x - \pi/4)^2}{2!} - \frac{(x - \pi/4)^3}{3!} + ...\right]$$

11. (A)

Let f(x) = ln (1–x). Then f(0) = 0.

$f'(0) = -1,\ f''(0) = -1,\ f'''(0) = -1 \times 2$

$f^{(4)} = -1.2.3$ and $f^n(0) = -(n-1)!$

Thus for $n \geq 1$ $f^n(0)/n! = -1/n$ and

Maclaurin series is $-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}...$

12. (B)

Let $f(x) = \frac{1}{x}$ Then

$f'(x) = \frac{-1}{x^2}, f''(x) = \frac{2}{x^3}, f'''(x) = \frac{-2.3}{x^4}$

$f^4(x) = \frac{2 \times 3 \times 4}{x^5}$ and in general

$f^n(x) = (-1)^n \frac{n!}{x^{n+1}}$ so $f^n(1) = (-1)^n n!$

Thus Taylor series is

$$\sum_{n=0}^{x}\frac{f^n(1)}{n!}(x-1)^n=\sum_{n=0}^{x}(-1)^n(x-1)^n$$

13. (A)

Let f(x) sec x. Then

$f'(x) = \text{Sec}\, x \tan x, f''(x)$

$= \sec x\left(1+2\tan^2 x\right)$

$f'''(x) = \sec x \tan x(5+6\tan^2 x)$

$\therefore \quad f(0) = 1,\ f'(0)=0,\ f''(0)=1$

$f'''(0)=0, f^4 = 5$

The Maclaurin series is

$$1+\frac{1}{2}x^2+\frac{5}{24}x^4+....$$

14. (D)

In general,

$a_n = \dfrac{f^n(0)}{n!}$.so $f^{33}(0)=33!a_{33} = 33!2^{33}$

15. (B)

$$\tan^{-1}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Since $a_n = \dfrac{f^n(0)}{n!}, f^{99}(0)=99!a_{99}$

But $a_{99} = (-1)^{49}\ \dfrac{1}{99}=-\dfrac{1}{99}$

Thus $f^{99}(0) = -(99!)/99 = -98!$

16. (A)

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-.....$$

$$+(-1)^{n+1}\frac{x^n}{n}+...\text{for } |x|<1$$

Hence

$$\ln(1+x^2) = x^2-\frac{x^4}{2}+\frac{x^6}{3}+.....$$

$$+(-1)^{n+1}\frac{x^{2n}}{n}.....$$

Thus term is $(-1)^{n+1}\dfrac{x^{2n}}{n}$

17. (B)

$f(x) = \dfrac{1-x}{1-x^3}$

from $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$

& so $f(x)=\dfrac{1}{1-x^3}-x\dfrac{1}{1-x^3}$

$$=\sum_{n=0}^{\infty}x^{3n}-\sum_{n=0}^{\infty}x^{3n+1}$$

From Maclaurin expansion,

$\dfrac{f^{36}(0)}{36!}=a_{36} = +1$

Hence $f^{36}(0) = 36!$

18. (A)

Let f(x) = tan x . then

$f'(x)=\sec^2 x, f''(x)=2\tan x\sec^2 x$

$f'''(x)=2(\sec^4 x+2\tan^2 x+\tan^4 x)$

So f(0) = 0,

$f'(0)=1, f''(0)=0, f'''(0)=2$

Thus Maclaurin series is

$$x+\frac{1}{3}x^3+\frac{2}{15}x^5+...$$

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Model Solution on Assignment – 8

1. (C)

$f'(x) = 1 + k\,x^{-2}$

For x = –2 to be critical

$f'(2) = 1 + k(-2)^{-2} = 0$

$\Rightarrow k = -4$

Hence

$f'(x) = 1 - \dfrac{4}{x^2}, f''(x) = \dfrac{8}{x^3}$

$f''(-2) = -1$

Hence relative maximum at x = –2

2. (C)

Let x and y be dimension

$2x + 2y = 16 \Rightarrow A = x(8 - x) = 8x - x^2$

$\dfrac{dA}{dx} = 8 - 2x, \dfrac{d^2A}{dx^2} = -2$

Critical number is $x = 4 \Rightarrow \left.\dfrac{d^2A}{dx^2}\right]_4 = -2$

here x = 4 and y = 4

3. (D)

$\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$

Put x – 2 = h

as $x \to 2, \quad h \to 0$

$\therefore \quad \lim\limits_{x\to 2}\dfrac{|x-2|}{x-2} = \lim\limits_{h\to 0}\dfrac{|h|}{h}$

Left Hand Limit :

$\lim\limits_{h\to 0^-}\dfrac{|h|}{h} = \lim\limits_{h\to 0}\dfrac{-h}{h}$

$\ldots\left(\because |h| = -h \text{ for } h < 0\right)$

$= -1$

Right Hand Limit :

$\lim\limits_{h\to 0^+}\dfrac{|h|}{h} = \lim\limits_{h\to 0}\dfrac{h}{h} \quad \ldots\left(\because |h| = h \text{ for } h > 0\right)$

$= 1$

$\therefore \quad \lim\limits_{h\to 0^-}\dfrac{|h|}{h} \neq \lim\limits_{h\to 0^+}\dfrac{|h|}{h}$

$\therefore \quad \lim\limits_{h\to 0}\dfrac{|h|}{h}$ does not exist.

4. (D)

$\lim\limits_{x\to 1}\dfrac{f(x)-2}{f(x)+2} = 0 \Rightarrow \lim\limits_{x\to 1}f(x) = 2$

$\Rightarrow f(x) = 2x$

hence $\lim\limits_{x\to 1}f(x) = 2$

5. (D)

6. (D)

Let $\quad A = \lim\limits_{n\to 0}\dfrac{1}{e^{n\log n}}$

$\Rightarrow \log_e A = \lim\limits_{n\to 0}\dfrac{1}{n\log n}\log e^e$

$= \lim\limits_{n\to 0}\dfrac{1/n}{\log n} = \lim\limits_{n\to 0}\dfrac{-1/n^2}{1/n}$

$= \lim\limits_{n\to 0}\dfrac{-1}{n} = -\infty$

7. (B)

Let f(x) = $\ell n\, x$. then

$f'(x) = \dfrac{1}{x}, f''(x) = -\dfrac{1}{x^2}, f'''(x)$

$= \dfrac{2}{x^3} f^4(x) = \dfrac{-2.3}{x^4}$ & in general

$f^n(x) = (-1)^{n+1}\dfrac{(n-1)!}{x^n}$

So $f(2) = \ell n 2$,

$f^n(2) = (-1)^{n+1}\dfrac{(n-1)!}{2^n}$

Thus Taylor series is $\sum_{n=0}^{\infty}\dfrac{f^n(2)}{n!}(x-2)^n$

$= \ell n 2 + \left(\dfrac{x-2}{2}\right) - \dfrac{1}{8}(x-2)^2 + ...$

8. (C)

$2^x = e^{x\ell n 2}$

Now, $e^x = \sum_{n=0}^{\infty}\dfrac{x^n}{n!}$.

Therefore $2^x = e^{x\ell n 2} = \sum_{n=0}^{\infty}\dfrac{(\ell n 2)^n}{n!}x^n$

9. (C)

$e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$

Hence

$f(x) = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^{2n}, \dfrac{f^{100}(0)}{100} = G_{100} = \dfrac{1}{(50)!}$

Hence $f^{100}(0) = \dfrac{100!}{50!}$

$f^n(0) = \dfrac{n!}{(n/2)!}$ for $f(x) = e^{x^2}$

10. (D)

The Maclaurin series for f(x) is the polynomial $2 + x + \dfrac{4}{2!}x^2 + \dfrac{12}{3!}x^3$

Thus $f(x) = 2 + x + 2x^2 + 2x^3$

11. (A)

$f'(x) = 4x + 4, f''(x)$

$= 4 \;\&\; f^n(x) = 0 \text{ for } n > 2$

Thus f(1) = 3,

$f'(1) = 8, f''(1) = 4 \;\&\; f(x)$

$= 3 + 8(x-1) + 2(x-1)^2$

12. (A)

13. (B)

$I = \int_0^{\pi}\log\left[1+\cos(\pi-\theta)\right]d\theta$

$2I = \int_0^{\pi}\log(1+\cos\theta)d\theta$

$+\int_0^{\pi}\log(1-\cos\theta)d\theta$

$= \int_0^{\pi}\log\left(1-\cos^2\theta\right)d\theta$

$= 2\int_0^{\pi}\log\sin\theta\, d\theta$

$= 4\left(-\dfrac{\pi}{2}\log 2\right)$

$I = -\pi\log 2$

14. (A)

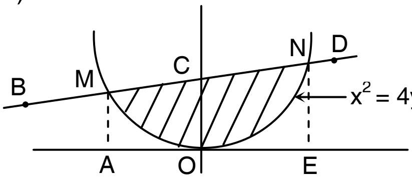
$I = \int_0^{\pi/2}\sin 2\left(\dfrac{\pi}{2}-x\right)\log\tan\left(\dfrac{\pi}{2}-x\right)dx$

$2I = \int_0^{\pi/2}\left\{\begin{matrix}\sin 2x\log(\tan x)\\ +\sin 2x\log\cot x\end{matrix}\right\}dx$

$= \int_0^{\pi/2}\sin 2x\log(\tan x\cot x)dx$

$= \int_0^{\pi/2}\sin 2x\log(1)dx = 0$

15. (C)



Required area

$$BOD = \int_{-1}^{2}\left(\frac{x+2}{4}\frac{-x^2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{8}$$

16. (B)

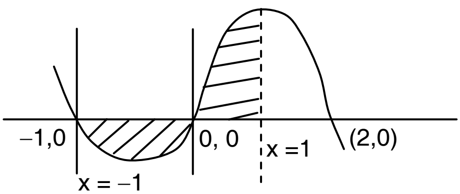
The given curve passes through the point (2, 3) and (4, $\frac{3}{2}$)

$$\text{Required area} = \int_2^4\left(1+\frac{8}{x^2}\right)dx$$

$$= \left[x-\frac{8}{x}\right]_2^4 = 4$$

17. (C)

Required area



$$= \int_0^1\left(2x+x^2-x^3\right)dx$$

$$- \int_{-1}^0\left(2x+x^2-x^3\right)dx$$

$$= \left|x^2+\frac{x^3}{3}-\frac{x^4}{4}\right|_0^1 - \left|x^2+\frac{x^3}{3}-\frac{x^4}{4}\right|_{-1}^0 = \frac{3}{2}$$

18. (B)

$$I = \int\frac{x-\sin x}{1-\cos x}dx$$

$$= \int\frac{x}{1-\cos x}dx - \int\frac{\sin x}{1-\cos x}dx$$

$$= \int\frac{x}{2\sin^2 x/2}dx - \int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2 x/2}$$

$$= \int\frac{x}{2}\operatorname{cosec}^2\frac{x}{2}dx - \int\cot\frac{x}{2}dx$$

$$= -x\cot\frac{x}{2}+C$$

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Model Solution on Assignment – 9

1. (B)

let $A = \lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{2x} = \lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-2x}$

$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2x}$

$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-2}$

$= [e]^{-2} = \frac{1}{e^2}$

2. (B)

$A = \lim_{x\to\infty}\left(\frac{1}{x}\right)^{\tan x}$

$\Rightarrow \log_e A = \lim_{x\to 0}\tan x \log 1/x$

$= \lim_{x\to 0}\frac{\log 1/x}{\cot x}$

$= \lim_{x\to\infty}\frac{x\left(-1/x^2\right)}{-\operatorname{cosec}^2 x}$

[By L′Hospital rule]

$= \lim_{x\to 0}\frac{\sin^2 x}{x}$

$= \lim_{x\to 0}\frac{2\sin x\cos x}{1} = 0$

[By L′Hospital rule]

$\log_e A = 0 \Rightarrow A = e^0 = 1$

3. (A)

$A = \lim_{x\to 0}(\operatorname{cosec} x)^x$

$\log_e A = \lim_{x\to 0} x\log(\operatorname{cosec} x)$

$= \lim_{x\to 0}\frac{\log(\operatorname{cosec} x)}{\frac{1}{x}}$

[By L′Hospital rule]

$= \lim_{x\to 0}\frac{\frac{1}{\operatorname{cosec} x}(-\operatorname{cosec} x\cot x)}{\frac{-1}{x^2}}$

$= \lim_{x\to 0}\frac{x^2}{\tan x} = \lim_{x\to 0} 2x/\sec^2 x$

[By L′Hospital rule]

$= \lim_{x\to 0} 2x\cos^2 x$

$\log_e A = 0 \Rightarrow A = e^0 = 1$

4. (C)

$\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{1}{n}\right] = \lim_{n\to\infty}\frac{n(n+1)/2}{n^2}$

$= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)$

$= \frac{1}{2}(1+0) = 1/2$

5. (B)

L. $-\lim_{x\to 2} f(x) = \lim_{h\to 0} 2a(2-h) - b = 4a - b$

R. $\lim_{x\to 2} f(x) = \lim_{h\to 0} a(2+h)^2 + b = 4a + b$

L. $\lim_{x\to 2} f(x) = R\lim_{x\to 2} f(2)$

$4a - b = 2$ and $4a + b = 2$

$\Rightarrow a = 1/2,\ b = 0$

6. (D)

$$\lim_{n\to\infty}\left(\frac{1+2+....+n}{n}\right)=\lim_{n\to\infty}\frac{n(n+1)}{2n}$$

$$=\frac{n+1}{2}=\infty$$

7. (B)

$$I=\int_0^{\pi/2}\cos^4 x\sin^3 x\,dx$$

$$=\int_0^{\pi/2}\cos^4 x\sin^2 x\sin x\,dx$$

$$=\int_0^{\pi/2}\cos^4 x\left(1-\cos^2 x\right)\sin x\,dx$$

let $\cos x = t$

$$I=-\int_1^0 t^4\left(1-t^2\right)dt$$

$$=\left[\frac{t^5}{5}-\frac{t^7}{7}\right]_0^1=\frac{2}{35}$$

8. (A)

Put $x = \sin\theta$

$$I=\int\frac{1+\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$$=\int\left(1+\sin^2\theta\right)d\theta$$

$$=\theta+\frac{\theta}{2}-\frac{1}{2}\frac{\sin2\theta}{2}+C$$

$$=\frac{3}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+C$$

9. (A)

$$I=\frac{1}{\sqrt2}\int\frac{dx}{\frac{1}{\sqrt2}\cos+\frac{1}{\sqrt2}\sin x}$$

$$=\frac{1}{\sqrt2}\int\frac{dx}{\sin(x+\pi/4)}$$

$$=\frac{1}{\sqrt2}\int\operatorname{cosec}(x+\pi/4)dx$$

$$=\frac{1}{\sqrt2}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+C$$

10. (C)

$$I=\int\frac{1}{1+3\sin^2 x}=\int\frac{\sec^2 x\,dx}{\sec^2 x+3\tan^2 x}$$

(Divide numerator and denominator by $\cos^2 x$)

$$=\int\frac{\sec^2 x dx}{1+\tan^2 x+3\tan^2 x}$$

$$=\frac{1}{4}\int\frac{\sec^2 x\,dx}{\left(\frac{1}{2}\right)^2+\tan^2 x}$$

Put $\tan x = t$

$$=\frac{1}{4}\int\frac{dt}{(1/2)^2+t^2}$$

$$=\frac{\tan^{-1}2t}{2}+C$$

$$=\frac{\tan^{-1}(2\tan x)}{2}+C$$

11. (A)

$$I=\int\frac{\sec^4\theta}{\tan^4+1}d\theta$$

$$=\int\frac{\left(1+\tan^2\theta\right)\sec^2\theta}{1+\tan^4\theta}d\theta$$

Put $\tan\theta = x$

$$I=\int\frac{\left(1+x^2\right)dx}{1+x^4}=\int\frac{\left(1+1/x^2\right)dx}{\left(x-\frac{1}{x}\right)^2+2}$$

Put $x-\frac{1}{x}=t\Rightarrow\left(x+\frac{1}{x^2}\right)dx=dt$

$$I = \int \frac{dt}{t^2 + \left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{\sqrt{2}} = \tan\left(\frac{x - 1/x}{\sqrt{2}}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan^2\theta - 1}{\sqrt{2}\tan\theta}\right) + C$$

12. (B)

$$I = \int \frac{\sin x dx}{3\sin x - 4\sin^3 x}$$

$$= \int \frac{dx}{3 - 4\sin^2 x}$$

$$= \int \frac{dx}{3\cos^2 x - \sin^2 x}$$

$$= \int \frac{\sec^2 x dx}{3 - \tan^2 x}$$

Put tan x = t

$$I = \int \frac{dt}{3 - t^2} = \frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3} + t}{\sqrt{3} - t}\right) + C$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right) + C$$

13. (D)

Integrating by parts

$$I = \int x^2 \sin x dx$$

$$= -x^2 \cos x + \int 2x \cos x \, dx$$

$$= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C\Big|_0^{\pi/2}$$

$$= \pi - 2$$

14. (C)

For $x \in (0, \pi/2)$, $0 < \sin x < x$

$$0 < \frac{\sin x}{x} < 1 \quad \Rightarrow 0 < \frac{n\sin x}{x} < n \quad \text{...(1)}$$

$$\therefore \quad \left[\frac{n\sin x}{x}\right] = n - 1 \quad \text{...(2)}$$

Divide by x

$$1 > \frac{\sin x}{x} > 0 \quad \text{...(3)}$$

$$\therefore \quad 0 < \frac{n\sin x}{x} < n$$

$$\therefore \quad \left[\frac{n\sin x}{x}\right] = n - 1 \quad \text{...(4)}$$

(2) & (4)

$$\lim_{x\to 0}\left[\frac{n\sin x}{x}\right] = n - 1$$

15. (A)

$$I = \int_C \left[t\sin(2\pi t)\right]dx + \left(-t^3\right)dy + t^2 dz$$

$$\frac{dx}{dt} = 2\pi\cos(2\pi t)$$

$$\Rightarrow \quad dx = 2\pi\cos(2\pi t)dt$$

Since dy = dt and dz = 2tdt

$$\therefore I = \int_{-1}^{1}\left\{\begin{matrix}(t\sin 2\pi t)\cdot 2\pi\cos 2\pi t \\ -t^3 + 2t^3\end{matrix}\right\}dt$$

$$= 2\pi\int_{-1}^{1} t\sin(4\pi t)dt + \int_{-1}^{1} t^3 dt$$

$$= -1$$

16. (D)

$$I_n = \int_0^\infty \underset{u}{\underline{e^{-x}\log x}}\ \underset{\frac{dv}{dx}}{x^{n-1}}\, dx$$

$$I_n = \left[e^{-x}\log x \frac{x^n}{n}\right]_0^\infty - \int_0^\infty \left(\frac{e^{-x}}{x} - e^{-x}\log x\right)\frac{x^n}{n}dx$$

$$= 0 + \frac{1}{n}\ I_{n+1} - \frac{(n-1)!}{n}$$

$$\because \int_0^\infty e^{-x}x^{n-1}dx = (n-1)!$$

$$\therefore\ I_n = \frac{1}{n}\ I_{n+1} - \frac{(n-1)!}{n} \qquad ....(1)$$

Replace

$$n \to n+1 : (n+1)\ I_{n+1} - \frac{(n-1)!}{n} \qquad ....(2)$$

$$(1)\ \&\ (2) : I_{n+2} - (2n+1)I_{n+1} + n^2 I_n = 0$$

17. (B)

$f(0) = 0$

$\lim\limits_{x\to 0} f(h) = 0$

$\therefore$ f(x) is continuous at x = 0

$$f'(0^+) = \lim_{x\to 0^+} \frac{\sqrt{1-\sqrt{1-x^2}} - 0}{x}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{1\sqrt{1-x^2}}}{x\sqrt{1+\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^2}}{x\sqrt{1+\sqrt{1-x^2}}}$$

$$= \lim_{x\to 0^+} \frac{x}{x\sqrt{1+\sqrt{1-x^2}}},$$

because $\sqrt{x^2} \begin{array}{l} = x \text{ if } x \ge 0 \\ = -x \text{ if } x < 0 \end{array}$

$$= \frac{1}{\sqrt{2}}$$

But $f'(0^-) = -\dfrac{1}{\sqrt{2}}$

$\therefore$ F is not differentiable of x = 0.

18. (B)

$$\frac{\partial z}{\partial x} = 5y - 8x - 2, \quad \frac{\partial z}{\partial y} = 5x + 2y - 1$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0 \Rightarrow x = \frac{1}{41}, \ y = \frac{18}{41}$$

Here r = –8, s = 5, t = 2

$\therefore$ $rt - s^2 < 0$

$\therefore$ $\left(\dfrac{1}{41}, \dfrac{18}{41}\right)$ is a saddle point.

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Answer Key on Test Paper – 1

1.	(B)	2.	(D)	3.	(A)	4.	(D)
5.	(A)	6.	(A)	7.	(B)	8.	(D)
9.	(B)	10.	(C)	11.	(B)	12.	(D)
13.	(B)	14.	(B)	15.	(B)		

Answer Key on Test Paper – 2

1.	(D)	2.	(B)	3.	(A)	4.	(D)
5.	(A)	6.	(D)	7.	(C)	8.	(D)
9.	(D)	10.	(D)	11.	(B)	12.	(A)
13.	(D)	14.	(D)	15.	(A)		

Answer Key on Test Paper – 3

1.	(C)	2.	(D)	3.	(A)	4.	(B)
5.	(A)	6.	(B)	7.	(C)	8.	(A)
9.	(A)	10.	(B)	11.	(A)	12.	(A)
13.	(A)	14.	(B)	15.	(C)		

Answer Key on Test Paper – 4

1.	(C)	2.	(C)	3.	(D)	4.	(D)
5.	(C)	6.	(B)	7.	(B)	8.	(C)
9.	(D)	10.	(A)	11.	(D)	12.	(B)
13.	(A)	14.	(D)	15.	(C)		

Answer Key on Test Paper – 5

1.	(D)	2.	(B)	3.	(B)	4.	(A)
5.	(A)	6.	(A)	7.	(B)	8.	(B)
9.	(B)	10.	(C)	11.	(D)	12.	(B)
13.	(C)	14.	(D)	15.	(A)		

Answer Key on Test Paper – 6

1.	(C)	2.	(A)	3.	(A)	4.	(B)
5.	(D)	6.	(B)	7.	(C)	8.	(C)
9.	(B)	10.	(C)	11.	(A)	12.	(B)
13.	(C)	14.	(A)	15.	(C)		

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Model Solution on Test Paper – 1

1. (B)

$$\lim_{x\to 0}\frac{3\sin x-4\sin^3 x-\sin x}{\sin x}$$

$$=\lim_{x\to 0}(2-4\sin^2 x)=2$$

2. (D)

$\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist because

$\sin\frac{1}{x}$ oscillates.

3. (A)

Now, $\lim\limits_{x\to 0}\frac{\sin 3x}{5x}=\lim\limits_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3x}{5x}$

$$=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3}{5}$$

$$=1\times\frac{3}{5}=\frac{3}{5}$$

4. (D)

Left Hand Limit,

$$\lim_{x\to 2^-}[x]=\lim_{h\to 0^-}[2+h]$$

$$=\lim_{h\to 0^-}1$$

$$=1$$

Right Hand Limit,

$$\lim_{x\to 2^+}[x]=\lim_{h\to 0^+}[2+h]$$

$$=\lim_{h\to 0^+}2$$

$$=2$$

$\therefore$ L.H.L. $\lim\limits_{x\to 2^-}[x]\neq$ R.H.L. $\lim\limits_{x\to 2^+}[x]$

5. (A)

Put $1+x=y$, so that $x\to 0\Rightarrow y\to 1$

$$\frac{(1+x)^n-1}{x}=\frac{y^n-1}{y-1}$$

$$\therefore\quad \lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{y\to 1}\frac{y^n-1}{y-1}$$

$$=n(1)^{n-1}$$

$$=n$$

6. (A)

7. (B)

8. (D)

$$\int_2^\infty\frac{1}{x\ell n^2 x}=\int_2^\infty(\ell nx)^{-2}\cdot\frac{1}{x}dx$$

$$=\left[-(\ell nx)^{-1}\right]_2^\infty$$

$$=-(\ell n\infty)^{-1}+(\ell n2)^{-1}=\frac{1}{\ell n2}$$

9. (B)

There is discontinuity at $x=2$. Thus

$$\int_1^5\frac{dx}{\sqrt[3]{x-2}}=\int_1^2\frac{dx}{\sqrt[3]{x-2}}+\int_2^5\frac{dx}{\sqrt[3]{x-2}}$$

$$=\left[\frac{3}{2}(x-2)^{2/3}\right]_1^2+\left[\frac{3}{2}(x-2)^{2/3}\right]_2^5$$

$$=\frac{3}{2}\left(\sqrt[3]{9}-1\right)$$

10. (C)

$$A = \int_0^1 \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx$$

$$= \left[-\left(1-x^2\right)^{1/2}\right]_0^1 = 1$$

11. (B)

By Stoke's formula

$$V = \pi\int_0^\infty \left(e^{-2x}\right)^2 dx = \pi\int_0^\infty e^{-4x}dx$$

$$= \pi \lim_{v\to+\infty}\int_0^v e^{-4x}dx = \pi \lim_{v\to+\infty}\left(-\frac{1}{4}e^{-4x}\right)\Bigg]_0^v$$

$$= -\frac{\pi}{4}\lim_{v\to+\infty}\left(e^{-4v}-1\right) = \frac{\pi}{4}$$

12. (D)

There is discontinuity at x = 1. Then

$$\int_0^9 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^9 \frac{dx}{(x-1)^{2/3}}$$

$$= 3(x-1)^{1/3}\Big]_0^1 + 3(x-1)^{1/3}\Big]_1^9$$

$$= 3 + 6 = 9$$

13. (B)

$$A = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[\ell n|x| - \ell n|x+1|\right]_1^\infty$$

$$= \ell n\ 2$$

14. (B)

There is a discontinuity at x = 2

So $\displaystyle\int_1^3 \frac{dx}{x-2} = \int_1^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2}$

$$= \left(\ell n|x-2|\right)_1^2 + \left(\ell n|x-2|\right)_2^3$$

= neither limit exists.

Therefore integral diverges.

15. (B)

By successive application of L'Hospital rule, $\displaystyle\lim_{x\to+\infty}\frac{(\ell nx)^p}{x} = 0$.

Hence $\dfrac{(\ell nx)^p}{x} < 1$ for sufficiently large x. Thus for some x_0.

If $x \ge x_0$ $(\ell nx)^p < x$, $\dfrac{1}{(\ell nx)^p} > \dfrac{1}{x}$.

So $\displaystyle\int_e^\infty \frac{dx}{(\ell nnx)^p} > \int_e^\infty \frac{dx}{\ell nnx} \to +\infty$.

Hence the integral is divergent.

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Model Solution on Test Paper – 2

1. (D)

Left Hand Limit,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x = 4$

Right Hand Limit,

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4$

Thus, $\lim_{x\to 2} f(x) = 4$

But,f (2) = $2 \neq \lim_{x\to 2} f(x)$

Thus the given function is discontinuous at x = 2.

But if we change f (x) = 4 at x = 2, then the given function f becomes continuous at x = 1.

Hence 2 is the removable discontinuity.

2. (B)

Left Hand Limit,

$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} (2 - h - 1)$

$= 1$ ….(1)

Right Hand Limit,

$\lim_{h\to 0} f(2+h) = \lim_{h\to 0}\left([2(2+h)] - 3\right)$

$= 1$ ….(2)

From eq. (1) and eq. (2),

L.H.L. = R.H.L.

$\lim_{x\to 2} f(x)$ exists.

f (2) = value of f (x)

= 2 . 2 – 3 = 1

$\lim_{x\to 2} f(x) = f(2)$

$\Rightarrow$ f (x) is continuous at x = 2.

3. (A)

Left Hand Limit,

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0}(1-h) - [1-h]$

$= \lim_{h\to 0} (1 - h) - 0 = 1$

Right Hand Limit,

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} (1+h) - [1+h]$

$= \lim_{h\to 0} (1+h) - 1$

$= \lim_{h\to 0} h = 0$

L.H.L. $\lim_{x\to 1} f(x) \neq$ R.H.L. $\lim_{x\to 1} f(x)$

f (x) = x – [x] is discontinuous at x = 1.

4. (D)

At x = a (say), where ‘a’ is any rational number.

Given, f (x) = 1, when x is rational.

Left Hand Limit,

$\lim_{x\to a} f(x) = \lim_{h\to a} f(a+h)$

$= \lim_{h\to 0} (-1)$

(because a – h is an irrational number)

Right Hand Limit,

$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a + h)$

$= \lim_{h\to 0} (-1) = -1$

But, f (a) = 1

L.H.L. $\underset{x\to a}{\text{Lim}}\, f(x) = $ R.H.L. $\underset{x\to a}{\text{Lim}}\, f(x) \neq f(a)$

$\therefore$ f (x) is discontinuous at x = a, where 'a' is a rational number.

Similarly we can prove that at x = b (an irrational number) f (x) is discontinuous.

$\therefore$ f (x) is discontinuous at every real point.

5. (A)

Left Hand Limit,

$$\underset{h\to 0}{\text{Lim}}\, f(4-h) = \underset{h\to 0}{\text{Lim}}\, 4(4-h)+3$$

$$= 19 \qquad ....(1)$$

Right Hand Limit,

$$\underset{h\to 0}{\text{Lim}}\, f(4+h) = \underset{h\to 0}{\text{Lim}}\, 4(4+h)+3$$

$$= 19 \qquad ....(2)$$

From eq. (1) and eq. (2),

$$\Rightarrow \underset{x\to 4}{\text{Lim}}\, f(x) = 19$$

Also, $\quad f(h) = 3 . 4 + 7$

$$= 19$$

$$\underset{x\to 4}{\text{Lim}}\, f(x) = f(4) = 19$$

$\therefore$ f (x) is continuous at x = 4.

6. (D)

$$I = \int_0^{\pi/2} \cot x\,dx = \ln(\sin x)\Big]_0^{\pi/2} = +\infty$$

7. (C)

$$A = \int_0^{\infty} \frac{dx}{x^2+6x+10} = \int_0^{\infty} \frac{dx}{(x+3)^2+1}$$

$$= \tan^{-1}(x+3)\Big]_0^{\infty}$$

$$= \tan^{-1}(\infty+3) - \tan^{-1}(0+3)$$

$$= \frac{\pi}{2} - \tan^{-1} 3$$

8. (D)

$$A = \int_{a+1}^{\infty} \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|_{a+1}^{\infty}$$

$$= \frac{1}{2a}\ell n|2a+1|$$

9. (D)

$$\int_0^1 \frac{1}{x^k}dx = \lim_{v\to 0+}\int_v^1 \frac{1}{x^k}dx$$

$$= \lim_{v\to 0+} \frac{1}{-k+1}x^{-k+1}\Big]_0^1$$

$$= \lim_{v\to 0+} \frac{1}{1-k}\left(1-\frac{1}{v^{k-1}}\right).$$

If k > 1 this limit is $+\infty$ whereas if k < 1 the limit is $\frac{1}{(1-k)}$

10. (D)

$$\int_0^a \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\frac{a+x}{a-x}\Big|_0^a$$

$$= -\frac{1}{2a}\ell n|a-x|$$

$$+\frac{1}{2a}\ell n(a+x)\Big]_0^a$$

$$= +\infty$$

Thus the integral diverges.

11. (B)

The value of mod function cannot be less than 0

$f(x) = | x^2 - 5x + 2 | = 0$

12. (A)

let $f(x) = \dfrac{(\sin x - \cos x)}{\sqrt{2}} = \sin(x - \pi/4)$

since maximum value of $\sin\theta$ is 1

$\therefore$ maximum is 1

13. (D)

let $y = x^{-x}$

$\log y = -x \log x$

$dy/dx = -1/x^x (1 + \log x)$

$d^2y/dx^2 = -[\,y/x + (1 + \log x)$

$\{-y(1 + \log x)\}]$

for maxima and minima $dy/dx = 0$

$\Rightarrow -1/x^x (1 + \log x) = 0 \Rightarrow x = 1/e$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{e}} = \frac{1}{\left(\frac{1}{e}\right)^{1/e}}\left[0^2 - \frac{1}{\frac{1}{e}}\right]$$

$$= e^{1/e}(-e) < 0$$

Hence maximum at $x = 1/e$

Maximum value $= \dfrac{1}{\left(\frac{1}{e}\right)^{1/e}} = e^{1/e}$

14. (D)

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$f''(x) = 20x^3 - 60x^2 + 30x$

$f'''(x) = 60x^2 - 120x + 30$

for maxima or minima $f'(x) = 0$

$x^4 - 4x^3 + 3x^2 = 0 \Rightarrow x = 0, 1, 3$

$f''(0) = 0,\ f'''(0) = 30 \neq 0$

$f''(1) = -10 < 0$

$f''\,3 = 90 > 0$

Hence f (x) is neither maximum and minimum at x = 0 f (x) is maximum at x = 1 and minimum at x = 3

Hence 1 maxima and 1 minima

15. (A)

$f(x) = x^3/3 - 3x^2/2 + 2x$

$f'(x) = x^2 - 3x + 2$

$f''(x) = 2x - 3$

For maxima minima

$f'(x) = 0\ x^3 - 3x + 2 = 0$

$\Rightarrow (x - 1)(x - 2) = 0$

$\Rightarrow x = 1, 2$

$f''(1) = 2 - 3 = -1 < 0$

$f''(2) = 2 \times 2 - 3 = 1 > 0$

$f(1) = 1/3 - 3/2 + 2 = 5/6$

$f(2) = 8/3 - 6 + 4 = 4/6 = 2/3$

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Model Solution on Test Paper – 3

1. (C)

$$\lim_{x\to 1}\frac{x^{16}-1}{x^5-1}$$

$$= \lim_{x\to 1}\left(\frac{x^{16}-1}{x-1}\div\frac{x^5-1}{x-1}\right)$$

$$= \lim_{x\to 1}\frac{x^{16}-1}{x-1}\div\lim_{x\to 1}\frac{x^5-1}{x-1}$$

$$= 16 \div 5$$

$$= \frac{16}{5} \qquad \left[\because \lim_{x\to 0}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

2. (D)

$$\lim_{x\to 0}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$$

$$= \lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\,\frac{\sqrt{x+2}+\sqrt{3x-2}}{\sqrt{x+2}+\sqrt{3x-2}}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)\,(\sqrt{x+2}+\sqrt{3x-2})}{(x+2)-(3x-2)}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)\,(\sqrt{x+2}+\sqrt{3x-2})}{-2\,(x-2)}$$

$$= \lim_{x\to 2}\frac{(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2}$$

$$= \frac{4\,(\sqrt{4}+\sqrt{4})}{-2} = -8$$

3. (A)

Now, $$\frac{a^x-b^x}{x} = \frac{(a^x-1)-(b^x-1)}{x}$$

$$= \frac{a^x-1}{x}-\frac{b^x-1}{x}$$

$$\therefore \lim_{x\to 0}\frac{a^x-b^x}{x}$$

$$= \lim_{x\to 0}\frac{a^x-1}{x}-\lim_{x\to 0}\frac{b^x-1}{x}$$

$$= \log a - \log b = \log\left(\frac{a}{b}\right)$$

4. (B)

Left Hand Limit,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{|0-h|}{0-h}$$

$$= \lim_{h\to 0}\frac{h}{-h} = -1$$

Right Hand Limit,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{|0+h|}{0+h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$\therefore$ L.H.L. $\lim_{x\to 0} f(x) \neq$ R.H.L. $\lim_{x\to 0} f(x)$

Hence f (x) is discontinuous function of first kind at x = 0.

5. (A)

Left Hand Limit,

$$f'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

Right Hand Limit,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}$$

$= \lim_{h\to 0} h \sin\frac{1}{h} = 0$

$\therefore$ L.H.L. f '(0) = R.H.L. f '(0) = 0

6. (B)

$y = \frac{x}{e^x}$

log y = log x – x

$\frac{1}{y}dy/dx = \frac{1}{x} - 1 \Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x} - 1\right] = 0$

$\Rightarrow x = 1$

$\frac{1}{y}\frac{d^2y}{dx^2}\frac{-1}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{-1}{x^2}$

$\frac{1}{e}\frac{d^2y}{dx^2}\bigg]_{x=1} - 0 = -1 \Rightarrow \frac{d^2y}{dx^2}\bigg]_{x=1} = -e < 0$

Hence maxima at x = 1

Maximum value $\Rightarrow$ $y_{max} = \frac{1}{e}$

7. (C)

Assume that there is such a function. Then f_{xy} and f_{yx} will be continuous everywhere. Hence $f_{xy} = f_{yx}$ Thus – e^x sin y = e^x sin y = 0 for all y which is false

8. (A)

$fx = \frac{2x}{x^2 + y^2}$,

$fxx = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2(y^2 - x^2)}{(x^2+y^2)^2}$

$fy = \frac{2y}{x^2 + y^2}$,

$fyy = \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2} = \frac{2(x^2 - y^2)}{(x^2+y^2)^2}$

Hence fxx + fyy = 0

9. (A)

By chain rule

$\frac{dz}{dx} = f_x\frac{dx}{dx} + f_y\frac{dy}{dx}$

$= (4x^3 + 3y) + (3x - 2y)\cos x$

10. (B)

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= \frac{x}{8}\frac{dx}{dt} - \frac{2y}{9}\frac{dy}{dt}$ since $\frac{dx}{dt} = 5$

and

$5x + y\frac{dy}{dt} = 0$

so when $x = 2$ and $y = 1$,

$dy/dt = -10$ and

$\frac{dz}{dt} = \frac{2}{8}(5) - \frac{2}{9}(-10)\ \frac{125}{36}$ cm / s

11. (A)

Let k (x, y) = $\frac{\partial f}{\partial x}$, then $\frac{\partial k}{\partial x} = 0$

then k (x, y) = g(y).

Hence $\frac{\partial f}{\partial x}$ = g(y) then

f(x,y) = xg(y) + h(y) for a suitable function h

12. (A)

If z = x f(y/x) then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$$

$$= x\left[xf'(y/x)\left(-y/x^2\right)+f(y/x)\right]$$

$$+xyf'(y/x)1/x$$

$$= xf(y/x) = z$$

13. (A)

let c (x, y) = f (x, y) –xy,

then $\frac{\partial^2 c}{\partial x\partial y} = 1-1 = 0$ then

c (x, y) = A (x) + B (y)

$\therefore$ f (x, y) = A (x) + B(y) + xy

14. (B)

A normal vector to tangent plane will be (2x, 2y, 2z) = $\left(1, 1, \sqrt{2}\right)$

Hence equation of tangent plane is

$$\left(x\frac{-1}{2}\right)+\left(y-\frac{1}{2}\right)+\sqrt{2}\left[z-\frac{1}{\sqrt{2}}\right] = 0$$

Or x + y + $\sqrt{2}z = 2$

15. (C)

$$V = \frac{1}{3}\pi r^2 h\frac{dh}{dt}$$

$$= \frac{\partial V}{\partial r}\frac{dr}{dt}+\frac{\partial v}{\partial h}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi rh(2)+\frac{1}{3}\pi r^2(-3)$$

$$= \pi\left[\frac{4}{3}(5000)-2500\right]$$

$$= \frac{12500\pi}{3}mm^3/s$$

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Model Solution on Test Paper – 4

1. (C)

2. (C)

3. (D)

$$s = -3t^3 + \frac{t^2}{2} + 15$$

$$\frac{ds}{dt} = \text{speed} = -9t^2 + t$$

$$\frac{d^2s}{dt^2} = -18t + 1 \qquad \frac{d^2s}{dt^2} = 0 \Rightarrow t = \frac{1}{18}$$

again; $\frac{d^3s}{dt^3} = -18 < 0,$ hence speed has maxima.

$$\text{Maximum speed} = \left.\frac{ds}{dt}\right|_{t=\frac{1}{18}}$$

$$= -9\left(\frac{1}{18}\right)^2 + \frac{1}{18}$$

$$= -\frac{1}{36} + \frac{1}{18}$$

$$= \frac{2-1}{36}$$

$$= \frac{1}{36} \text{ speed units.}$$

4. (D)

5. (C)

6. (B)

$$A = \int_{-1}^{2}\int_{3x^2-x-3}^{-2x^2+4x+7} dx\,dy$$

$$= 5\int_{-1}^{2} dx, \left(-x^2 + x + 2\right)$$

$$= 5\left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = \frac{45}{2}$$

7. (B)

$$A = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[r^2\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}\left\{\left(\sec\theta + \cos\theta\right)^2 - \sec^2\theta\right\} d\theta$$

$$= a^2\int_0^{\pi/2}\left[2 + \cos^2\theta\right] d\theta$$

$$= \frac{5\pi a^2}{4}$$

8. (C)

$$\text{Mass} = k\int_0^4 dx \int_{\frac{x^{3/2}}{4}}^{x/2} y\,dy$$

$$= k\int_0^4\left[\frac{y^2}{2}\right]_{\frac{x^{3/2}}{4}}^{x/2} dx$$

$$= \frac{k}{2}\left[\frac{x^3}{12} - \frac{x^4}{64}\right]_0^4 = \frac{2}{3}k$$

9. (D)

Equation of circle $r = 2a\cos\theta$

$$\text{Mass} = k\int_{-\pi/2}^{\pi/2} d\theta \int_0^{2a\cos\theta} r^2\,dr$$

$$= \frac{k}{3}\int_{-\pi/2}^{\pi/2} d\theta\left[r^3\right]_0^{2a\cos\theta}$$

$$= 2\frac{8a^3k}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$$

$$= \frac{32}{9}ka^3$$

10. (A)

$$v = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\, r\,dr$$

Put $t^2 = a^2 - r^2$

$$= -\int_a^{a\sin\theta} t^2 dt = -\left.\frac{t^3}{3}\right]_a^{a\sin\theta}$$

$$= \frac{1}{3}a^3\left[1 - \sin^3\theta\right]$$

11. (D)

$$v = \int_{-1}^{1} dx \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \left(1 - x^2 - y^2/4\right) dy$$

$$= \int_{-1}^{1}\left\{\left(1-x^2\right)y - \frac{y^3}{12}\right\}_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} \frac{8}{3}\left(1-x^2\right)^{3/2} dx = \pi$$

12. (B)

$$v = \pi\int_{-a}^{0} y^2 dx = \pi\int_{-a}^{0} x^2 \frac{a+x}{a-x} dx$$

put $z = a - x$ so

$$v = \pi \int_0^{2a} \frac{(a-z)^2(2a-z)}{z} dz$$

$$= \pi\int_0^{2a}\left[\frac{2a^3}{z} - 5a^2 + 4az - z^2\right] dz$$

$= 2\pi a^3$ [log2 – 2/3]

13. (A)

$$V = 2\int_0^{\infty} \pi(a-x)^2 dy$$

$$= 2\pi\int_0^{a}(a-x)^2 \frac{dy}{dx} dx$$

$$\frac{dy}{dx} = \frac{a^2}{2\sqrt{x}\,(a-x)^{3/2}}$$

$$\left(\text{from curve } y^2 = \frac{a^2x}{a-x}\right)$$

$$V = \pi a^2 \int_0^4 \frac{\sqrt{a-x}}{\sqrt{x}} dx$$

Put $x = a\sin^2\theta$

$$V = 2\pi a^3 \int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi^2 a^3}{2}$$

14. (D)

$$S = 2\pi\int y\,ds = 2\pi\int_{-\pi}^{\pi} y\frac{ds}{d\theta} d\theta$$

$$= 4\pi\int_0^{\pi} y\frac{ds}{d\theta} d\theta$$

$$\frac{ds}{d\theta} = \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} = 2a\cos\theta/2$$

$$S = 4\pi\int_0^{\pi} 2a\sin^2\frac{\theta}{2}\, 2a\cos\frac{\theta}{2} d\theta$$

$$= \frac{32\pi a^2}{3}.$$

15. (C)

$$V = 2\pi\int_{-2}^{4}\int_{x(x-2)-8}^{0} (5+x)\,dx\,dy$$

$$= 2\pi \int_{-2}^{4} (5+x)\left[8 - x(x-2)\right] dx$$

$$= 2\pi \int_{-2}^{4}\left[40 + 18x - 3x^2 - x^3\right] dx$$

$= 2\pi \times 216 = 432\,\pi$

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Model Solution on Test Paper – 5

1. (D)

Let f(x) = x + $\frac{1}{x}$ $\Rightarrow$ f′ (x) = $1-\frac{1}{x^2}$

$\Rightarrow f''(x) = 2/x^3$

For maxima and minima f′ (x) = 0

$\Rightarrow 1-\frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

$f''(x) = \frac{2}{1^3} = 2 > 0$

f(x) is minimum at x = 1

minimum value = f (1) = 1 + $\frac{1}{1}$ = 2

2. (B)

Let θ be semi vertical angle of

inscribed triangle

Let S be the area

Then s = $\frac{1}{2}\times 2x \sin 2\theta\,(r + r\cos 2\theta)$

$\frac{dS}{d\theta} = 2r^2 (\cos 4\theta + \cos 23\theta) = 0$

$\Rightarrow \theta = \pi/6$

S is maximum at $\theta = \pi / 6$

vertical angle = $2\theta = \pi/3$

3. (B)

f′ (x) = $\frac{1-\log x}{x^2}$

$\Rightarrow 1 - \log x = 0 \Rightarrow x = e$

f″ (x) = $\frac{2\log x - 3}{x^3}$

$\Rightarrow$ f″ (e) = $\frac{2-3}{e^3} = \frac{-1}{e^3} < 0$

f(x) is maximum at x = e

maximum value f (e) = $\frac{\log e}{e} = \frac{1}{e}$

4. (A)

Total SP = $x\left(3-\frac{x}{1000}\right) = 3x - \frac{x^2}{1000}$

Total CP = $\frac{x}{2} + 200$

$\therefore$ Profit, $f(x) = SP - CP$

$$= \frac{5}{2}x - \frac{x^2}{1000} - 200$$

$\therefore\ f'(x) = \frac{5}{2} - \frac{2x}{1000}$

$\therefore\ f''(x) = -\frac{2}{1000} < 0$

$\frac{5}{2} - \frac{2x}{1000} = 0 => x = 1250$

$\therefore f''(1250) = -\frac{1}{500}$ negative

So, profit is maximum when x = 1250

5. (A)

By implicit differentiation w.r.t. x

$y - y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial x} + z = 0$

$\Rightarrow \frac{\partial z}{\partial x}(x-y) = -(y+z)\frac{\partial z}{\partial x} = \frac{y+z}{y-x}$

6. (A)

By Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Choosing f(x) = e^x, g(x) = c^{-x}

$$\therefore \frac{e^b-e^a}{e^{-b}-e^{-a}}=\frac{e^c}{-e^{-c}} \Rightarrow c=\frac{a+b}{2}$$

7. (B)

$$f'(x) = F'(x)$$

let $\phi(x) = f(x) - F(x)$

$$\phi'(x) = f'(x) - F'(x) = 0$$

$$\phi(x) = \text{constant}$$

8. (B)

9. (B)

$$\iint r^4 \cos\theta \, dr\, d\theta$$

$$= \int_0^{\pi} \cos \int_0^{a(1+\cos\theta)} r^4 dr d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi}(1+\cos\theta)^5 \cos\theta \, d\theta$$

$$= \frac{32a^5}{5}\int_0^{\pi}\cos^{10}\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= \frac{64}{5}a^5\int_0^{\pi/2}\left[2\cos^{12}\phi-\cos^{10}\phi\right]d\phi$$

$$\ldots\left(\frac{\theta}{2}=\phi\right)$$

$$= \frac{21}{16}\pi a^5 \text{ (by reduction formula)}$$

10. (C)

$$I = \iint x\,dxdy = \int_{-2}^{4} dy \int_{y^2/4}^{\frac{y}{2}+2} x dx$$

$$= \int_{-2}^{4} dy \left[\frac{x^2}{2}\right]_{y^2/4}^{y/2+2}$$

$$= \frac{1}{2}\left[\frac{y^3}{12}+y^2+4y-\frac{y^5}{80}\right]_{-2}^{4} = \frac{72}{5}$$

11. (D)

$$I = \int_0^b \int_0^b y(2b+y) dx\, dy$$

$$= \int_0^b \left[by^2+\frac{y^3}{3}\right]_0^b dx$$

$$= \int_0^b \left(b^3+\frac{b^3}{3}\right)dx = \frac{4b^4}{3}$$

12. (B)

$$I = 2\int_0^{\pi/4}\int_0^{2\sin\theta} r^3 dr d\theta$$

$$= 8\int_0^{\pi/4}\sin^4\theta\, d\theta$$

$$= 8\int_0^{\pi/4}\frac{(1-\cos 2\theta)^2}{4}d\theta$$

$$= \int_0^{\pi/2}(1-\cos\phi)^2 d\phi \qquad ....[\phi = 2\theta]$$

$$= \left[\frac{\pi}{2}-2+\frac{1}{2}\times\frac{\pi}{2}\right] = \frac{3\pi}{4}-2$$

13. (C)

$$I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^4 \sin^3\theta \, dr\, d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{\pi}(1+\cos\theta)^5 \sin^3\theta \, d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{2}\left(2z^6 - z^7\right)dz$$

$$\ldots.(z = 1 + \cos\theta)$$

$$= \frac{2^6}{35}\pi a^5$$

14. (D)

$$I = \int_0^a dx \frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a}$$

$$= \frac{1}{2}\int_0^a dx \frac{x\log(x+a)}{(x-a)^2}\left[a^2 - 2ax + x^2\right]$$

Integrating with parts, we get

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{a^2}{8}\left[2\log a + 1\right]$$

15. (A)

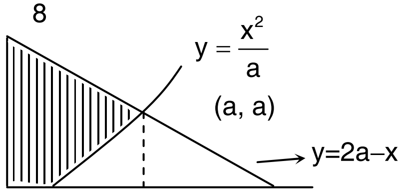
$$\int_0^a dx \int_{x^2/a}^{2a-x} xy\, dy$$

$$= \frac{1}{2}\int_0^a x\, dx\left[(2a-x)^2 - \frac{x^4}{a^2}\right]$$

$$= \frac{1}{2}\int_0^a \left[4a^2x - 4ax^2 + x^3 - \frac{x^5}{a^2}\right]dx$$

$$= \frac{1}{2}\left[2a^4 - \frac{4}{3}a^4 + \frac{1}{4}a^4 - \frac{a^4}{6}\right]$$

$$= \frac{3a^4}{8}$$



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Model Solution on Test Paper – 6

1. (C)

2. (A)

3. (A)

$$\lim_{x\to a}(x-a)\cos\frac{1}{x-a}$$

$$=\lim_{h\to 0} h\cos\frac{1}{h}[x-a=h]$$

= 0 × some finite a + y = 0

4. (B)

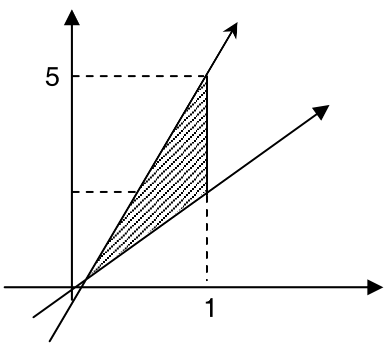
5. (D)

6. (B)

$$I=\int_0^1\int_{2x}^{5x} y^2 dydx$$

$$=\int_0^1 \frac{1}{3}y^3\Big]_{2x}^{5x} dx=\frac{1}{3}\int_0^1\left(125x^3-8x^3\right)dx$$

$$=\frac{39}{4}x^4\Big]_0^1=\frac{39}{4}$$



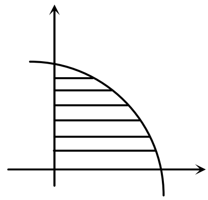
7. (C)

$$I=\int_0^2\int_0^{\sqrt{4-2y}}\frac{1}{\sqrt{2y-y^2}}dxdy$$

$$=\int_0^2\left[\frac{x}{\sqrt{2y-y^2}}\right]_0^{\sqrt{4-2y}} dy$$

$$=\int_0^2\frac{\sqrt{2}}{\sqrt{y}}\frac{\sqrt{2-y}}{\sqrt{2-y}}dy$$

$$=\sqrt{2}\cdot 2y^{1/2}\Big]_0^2=4$$



8. (C)

$$V=\iint_R(3x+4y)dA=\int_0^3\int_1^2(3x+4y)dxdy$$

$$=\int_0^3\left(\frac{3}{2}x^2+4yx\right)_1^2 dy$$

$$=\int_0^3\left(\frac{9}{2}+4y\right)dy=\left(\frac{9}{2}y+2y^2\right)_0^3=\frac{63}{2}$$

9. (B)

$$V=\iint_R y^2 dA=\int_0^2\int_0^4 y^2 dydx$$

$$=\int_0^2\left[\frac{1}{3}y^3\right]_0^4 dx=\int_0^2\frac{64}{3}dx$$

$$=\frac{64}{3}\cdot 2=\frac{128}{3}$$

10. (C)

$$A=\iint_R 1dA=\int_{1/2}^1\int_{1/x}^{3-2x} dydx$$

$$=\int_{1/2}^1\left(3-2x-\frac{1}{x}\right)dx$$

$$= \left[3x - x^2 - \ell n x\right]_{1/2}^{1}$$

$$= (3-1-0) - \left(\frac{3}{2} - \frac{1}{4} - \ell n \frac{1}{2}\right)$$

$$= 2 - \left(\frac{5}{4} + \ell n 2\right)$$

$$= \frac{3}{4} - \ell n 2$$

11. (A)

$$V = \iint_R (2x + 2y + 1)\, dA$$

$$= \int_0^2 \int_0^x (2x + 2y + 1)\, dy dx$$

$$= \int_0^2 \left[2xy + y^2 + y\right]_0^x dx$$

$$= \left(x^3 + \frac{1}{2}x^2\right)_0^2$$

$$= 8 + 2 = 10$$

12. (B)

$$A = \iint_R dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2\right]_0^{1+\cos\theta} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2} \times \left(\frac{3}{2} 2\pi\right) = \frac{3\pi}{2}$$

13. (C)

$$I = \int_{-\pi/4}^{\pi/4} d\theta \left[\sqrt{\alpha^2 + r^2}\,\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= \int_{-\pi/4}^{\pi/4} d\theta \left\{a\sqrt{1 + \cos 2\theta} - a\right\}$$

$$= a \int_{-\pi/4}^{\pi/4} \left[\sqrt{2}\cos\theta - 1\right] d\theta$$

$$= a \left[\sqrt{2}\sin\theta - \theta\right]_{-\pi/4}^{\pi/4}$$

$$= a[2 - \pi/2] = 2a\left[1 - \frac{\pi}{4}\right]$$

14. (A)

$$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{2-x-y} dz$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy\, [z]_0^{2-x-y}$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (2 - x - y)\, dy$$

$$= \int_0^1 \left[(2-x)\sqrt{x} - \frac{x}{2} - (2-x)x^2 + \frac{x^4}{2}\right] dx$$

$$= \frac{11}{30}$$

15. (C)

$$I = \int_1^2 dy \int_{y-1}^{1} \left(x^2 - y^2\right) dx$$

$$= \int_1^2 dy \left[\frac{x^3}{3} - y^2 x\right]_{y-1}^{y}$$

$$= \int_1^2 \left(\frac{1}{3} - y^2 - \frac{(y-1)^3}{3} + y^2(y-1)\right) dy$$

$$= \left[\frac{y}{3} \frac{-2y^3}{3} - \frac{(y-1)^4}{12} + \frac{y^4}{4}\right]_1^2 = \frac{-2}{3}$$

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Solutions – Probability and Statistics

Answer Key on Assignment – 1

1.	(C)	2.	0.65 to 0.68	3.	(C)
4.	0.4 to 0.4	5.	(D)	6.	0.25
7.	(A)	8.	0.07 to 0.08	9.	(A)
10.	0.55 to 0.55	11.	(D)	12.	(B)
13.	0.43 to 0.45	14.	0.26 to 0.27	15.	(B)
16.	1.06 to 1.07	17.	(B)	18.	6
19.	0.32 to 0.34	20.	(A)		

Answer Key on Assignment – 2

1.	0.64 to 0.66	2.	0.1276 to 0.1372	3.	0.79 to 0.81
4.	(B)	5.	0.96 to 0.98	6.	0.23 to 0.25
7.	99.6 to 99.8	8.	(A)	9.	2.5 to 2.5
10.	(C)	11.	0.25 to 0.27	12.	(B)
13.	(B)	14.	0.25 to 0.28	15.	(D)
16.	(B)	17.	(D)	18.	0.39 to 0.43
19.	54.0 to 54.0	20.	0.33 to 0.34		

Answer Key on Assignment – 3

1.	49 to 51	2.	49.9 to 50.1	3.	(B)
4.	(B)	5.	(C)	6.	(C)
7.	0.9 to 1.1	8.	54.49 to 54.51	9.	0.5 to 0.5
10.	(B)	11.	(A)	12.	(D)
13.	0.35 to 0.45	14.	(B)	15.	(D)
16.	4	17.	6	18.	(A)
19.	(A)	20.	2.0 to 2.0		

Answer Key on Assignment – 4

1.	(A)	2.	(D)	3.	(B)
4.	(C)	5.	(A)	6.	(C)
7.	(C)	8.	(A)	9.	(B)
10.	(D)	11.	(C)	12.	(B)
13.	(A)	14.	(C)	15.	(A)
16.	(B)	17.	(C)	18.	(C)

Answer Key on Assignment – 5

1.	(A)	2.	(C)	3.	(D)
4.	(A)	5.	(D)	6.	(C)
7.	(A)	8.	(C)	9.	(D)
10.	(A)	11.	(D)	12.	(A)
13.	(A)	14.	(A)	15.	(C)
16.	(D)	17.	(C)	18.	(C)

Answer Key on Assignment – 6

1.	(B)	2.	(D)	3.	(C)
4.	(C)	5.	(A)	6.	(C)
7.	(C)	8.	(B)	9.	(A)
10.	(D)	11.	(C)	12.	(A)
13.	(B)	14.	(A)	15.	(A)
16.	(B)	17.	(D)	18.	(D)

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Model Solution on Assignment – 1

1. (C)

We know that mode is the value of the data which occurred most of

$\therefore$ 17 is mode.

2. 0.65 to 0.68

If there are N families, then $\frac{N}{2}$ have single child per family and $\frac{N}{2}$ have two children per family.

N

Single child per family $\frac{N}{2}$ — $\frac{N}{2}$ Two children per family

$\therefore$ Total children = $\frac{N}{2}$ $\quad 2\times\frac{N}{2} = N$

$\therefore$ Total number of children $= \frac{N}{2}+N$

$= \frac{3}{2}N$

If a child picked at random should have a sibling, then that child should come from the family which has 2 child. There are N such children.

$\therefore$ Required probability $= \dfrac{N}{\frac{3}{2}N} = \dfrac{2}{3}$

$= 0.67$

Alternative

Suppose there are total 10 families.

10

One child per family ← 5 — 5 → Two child per family

Number of children in these 5 families is 5 — Number of children in these 5 families is 10

$\therefore$ Total children = 15

Required probability = $\frac{10}{15} = \frac{2}{3} = 0.67$

3. (C)

P(Head) = $\frac{1}{2}$ $\quad\Rightarrow\quad$ P(Tail) = $\frac{1}{2}$

P(Fourth Head in tenth toss)

$$= {}^9C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^6\times\frac{1}{2} = 0.082$$

4. 0.4 to 0.4

Given, probability density function of evaporation E is

$$f(E) = \begin{cases} \frac{1}{5}, & 0 \le E \le 5\ \text{mm / day} \\ 0, & \text{otherwise} \end{cases}$$

As, the probability has to be find between 2 and 4 mm/day, therefore

$$P(2 < E < 4) = \int_2^4 f(E)\,dE = \int_2^4 \frac{1}{5}\,dE$$

$$= \frac{1}{5}[E]_2^4 = \frac{4-2}{5} = \frac{2}{5}$$

$\therefore\quad P(2 < E < 4) = 0.4$

5. (D)

We have Probability distribution function of Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{x}} e^{\frac{(x-\mu)^2}{2\sigma^2}} \qquad \dots (1)$$

Variance = σ^2 is lowest

$\Rightarrow$ σ also lowest

$\Rightarrow$ If σ decreases $\Rightarrow$ f(x) increases

($\because$ from (1))

$\Rightarrow$ Curve will have highest peak

6. 0.25

As f(x) is a pdf,

$$\int_0^1 f(x)\,dx = 1$$

$$\therefore \int_0^1 (a + bx)\,dx = 1$$

$$\therefore \left[ax + b\frac{x^2}{2}\right]_0^1 = 1$$

$$\therefore a + \frac{b}{2} = 1$$

$\therefore$ 2a + b = 2 ....(1)

Also, $E(X) = \frac{2}{3}$

$$\therefore \int_0^1 x\, f(x)\,dx = \frac{2}{3}$$

$$\therefore \int_0^1 x\,(a + bx)\,dx = \frac{2}{3}$$

$$\therefore \left[a\frac{x^2}{2} + b\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\therefore \frac{a}{2} + \frac{b}{3} = \frac{2}{3}$$

$\therefore$ 3a + 2b = 4 ....(2)

Solving (1) & (2) we get, a = 0, b = 2

$$\therefore P(X < 0.5) = \int_0^{0.5} f(x)\,dx = \int_0^{0.5} 2x\,dx$$

$$= \left[2 \times \frac{x^2}{2}\right]_0^{0.5} = (0.5)^2 = 0.25$$

7. (A)

8. 0.07 to 0.08

P(Head) = 0.3 $\Rightarrow$ P(Tail) = 0.7

P (getting Head first time in fifth toss)

$= 0.7 \times 0.7 \times 0.7 \times 0.7 \times 0.3$

$= 0.072$

9. (A)

P(X) = 0.40

$P(X \cup Y^c) = 0.7$

$\because$ X and Y are independent events

$P(X \cap Y) = P(X) . P(Y)$ ....(1)

$\therefore$ X and Y^c are also independent

$P(X \cap Y^c) = P(X) . P(Y^c)$ ....(2)

Now,

$P(X \cup Y^c) = P(X) + P(Y^c) - P(X \cap Y^c)$

$P(X \cup Y^c) = P(X) + P(Y^c) - P(X)\, P(Y^c)$

$\therefore 0.7 = 0.4 + P(Y^c) - 0.4\, P(Y^c)$

$\therefore 0.7 - 0.4 = P(Y^c)\, [1 - 0.4]$

$\therefore 0.3 = 0.6\, P(Y^c)$

$$P(Y^c) = \frac{0.3}{0.6} = 0.5$$

$\because P(Y^c) = 1 - P(Y)$

$\therefore$ 0.5 = 1 – P(Y)

$\therefore$ P(Y) = 1 – 0.5 = 0.5

Then,

$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= P(X) + P(Y) - P(X)\,P(Y)$

$= 0.4 + 0.5 - 0.4 \times 0.5$

$\therefore P(X \cup Y) = 0.7$

10. 0.55 to 0.55

There are equal number of bulbs of two different types.

Probability that the bulb selected is of

Type 1 $P(T_1) = \frac{1}{2} = 0.5$

Probability that the bulb selected is of

type 2 $P\left(T_2\right) = \frac{1}{2} = 0.5$

Let E be the event that a bulb selected at random lasts more than 100 hours.

$P(E/T_1) = 0.7$

$P(E/T_2) = 0.4$

By total probability

$P(E) = P(T_1)\,P(E/T_1) + P(T_2)\,P(E/T_2)$

$= 0.5 \times 0.7 + 0.5 \times 0.4$

$\therefore P(E) = 0.55$

11. (D)

Given

x	1	2	3
p(x)	0.3	0.6	0.1

mean (μ) = exp(x)

$= 1 \times 0.3 + 2 \times 0.6 + 3 \times 0.1$

$= 0.3 + 1.2 + 0.3 = 1.8$

$E(x^2) = \sum x^2\,P(x)$

$= 1 \times 0.3 + 4 \times 0.6 + 9 \times 0.1$

$= 0.3 + 2.4 + 0.9 = 3.6$

Variance $v(x) = E(x^2) - \mu^2$

$= 3.6 - (1.8)^2$

S.D(σ) = $+\sqrt{v(x)}$

$= +\sqrt{3.6 - (1.8)^2} = \sqrt{0.36}$

$= 0.6$

12. (B)

Given $\lambda = 5.2$

Let x be random variable which follows Poission's distribution

$P(x < 2) = P(x = 0) + P(x = 1)$

$= \frac{e^{-\lambda}\,\lambda^0}{0!} + \frac{e^{-\lambda}}{1!}\lambda^1$

$= e^{-5.2}\,(6.2)$

$= 0.0055 \times 6.2 = 0.034$

13. 0.43 to 0.45

Parcels pass sequentially through two post offices.

Probability that parcel reaches first post office

P(01) = 1

Probability that parcel is lost by first post office

P(L/01) = $\frac{1}{5}$

Parcel will reach second post office if it is not lost at first post office.

Probability that parcel reaches second post office

$P(02) = \frac{4}{5}$

Probability that parcel is lost by second post office

$P(L/02) = \frac{1}{5}$

By Total Probability,

Probability that parcel is lost

$P(L) = P(01)\,P(L/01) + P(02)\,P(L/02)$

$$= 1\times\frac{1}{5}+\frac{4}{5}\times\frac{1}{5} = \frac{9}{25}$$

By Baye's Theorem, knowing that the parcel is lost, probability that it was lost by second post office

$$P(02/L) = \frac{P(02)\,P(L/02)}{P(L)}=\frac{4/25}{9/25}=\frac{4}{9}$$

$P(02/L) = 0.444$

14. 0.26 to 0.27

Given, mean $\lambda = 5$

The probability that there will be less than 4 penalties in a day is

$P(y < 4) = P(y = 0) + P(y = 1)$
$+ P(y = 2) + P(y = 3)$

$$= \frac{e^{-5}5^0}{0!}+\frac{e^{-5}5^1}{1!}+\frac{e^{-5}5^2}{2!}+\frac{e^{-5}5^3}{3!}$$

$$= e^{-5}\left[\frac{5^0}{0!}+\frac{5^1}{1!}+\frac{5^2}{2!}+\frac{5^3}{3!}\right]$$

$= 0.006737\,[1 + 5 + 12.5 + 20.833]$

$= 0.006737 \times 39.33 = 0.2649$

15. (B)

Given that the student is passing exam, i.e., if only 20 students (out of 100) are considered, of the 5 students get more than 90%

$\therefore$ 5 out of 20 is the probability
Or 1/4 is the answer.

Alternative Method:

Let the student pass the examination be A and student pass the examination and got about 90% marks be B

Now $P(A) = 20\%$ and $P(A \cap B) = 5\%$

$$\therefore\quad P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{5\%}{20\%}=\frac{1}{4}$$

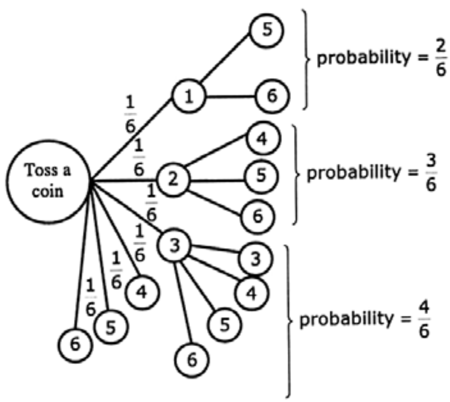
16. 1.06 to 1.07

$$\text{Mean} = \int_0^2 x\, f(x)\, dx$$

$$= \frac{1}{4}\left[4\frac{x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{4}\left(\frac{32}{3}-\frac{32}{5}\right) = 1.0666$$

17. (B)



$\therefore$ Required probability

$$= \frac{1}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{3}{6}+\frac{1}{6}\times\frac{4}{6}+\frac{1}{6}$$

$$= \frac{15}{36}=\frac{5}{12}$$

18. 6

$\text{Var}(x) = E(x^2) - (E(x))^2$

$E(x) = \int_{-\infty}^{\infty} x.\frac{1}{2}|x|e^{-|x|}\,dx$

$= \frac{-1}{2}\int_{-\infty}^{0} x^2 e^x dx + \frac{1}{2}\int_{0}^{\infty} x^2.e^{-x} dx$

$= 0$

$E(x)^2 = \int_{-\infty}^{\infty} x^2.\frac{1}{2}|x|e^{-|x|}\,dx$

$= 2\left[\frac{1}{2}\int_{0}^{\infty} x^2.x.e^{-x}\,dx\right]$

$= \int_{0}^{\infty} x^3.e^{-x}\,dx$

$= x^3.\frac{e^{-x}}{-1}\Big|_0^{\infty} - (3x^2)(e^{-x})\Big|_0^{\infty}$

$+ 6x(-e^{-x})\Big|_0^{\infty} - 6e^{-x}\Big|_0^{\infty}$

$= 6$

$\Rightarrow \quad \text{Var}(x) = 6 - 0 = 6$

19. 0.32 to 0.34

20. (A)

5 red and 7 green balls

Total 12 balls.

A ball is drawn and placed back along with another ball of same colour, this can be done in two ways:

I) Red ball in first draw

P(red in first draw) = $\frac{5}{12}$

When the red ball is placed back with another red ball, there will be total 13 balls with 6 balls of red colour.

P(red/red in first draw) = $\frac{6}{13}$

II) Green Ball in first draw

P(green in first draw) = $\frac{7}{12}$

When the green ball is placed back with another green ball, there will be total 13 balls with 5 balls of red colour.

P(red/green in first draw) = $\frac{5}{13}$

$\therefore \quad P(\text{red}) = \frac{5}{12}\times\frac{6}{13} + \frac{7}{12}\times\frac{5}{13}$

$= \frac{30}{156} + \frac{35}{156}$

$\therefore \quad P(\text{red}) = \frac{65}{156}$

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Model Solution on Assignment – 2

1. 0.64 to 0.66

Let $M \rightarrow$ men

$W \rightarrow$ women

$E \rightarrow$ employed

$U \rightarrow$ unemployed

Given $P(M) = 0.5$

$P(W) = 0.5$

$P(U/M) = 0.20$

$P(U/W) = 0.50$

By Total probability,

$P(U) = P(M)P(U/M) + P(W)P(U/M)$

$= 0.5 \times 0.20 + 0.5 \times 0.50 = 0.35$

Required probability = P(E)

$= 1 - P(U)$

$= 1 - 0.35 = 0.65$

2. 0.1276 to 0.1372

By Poission's distribution

$p(h \geq 8) = e^{-8\lambda}$

$\lambda = \dfrac{900}{3600} = 0.25$

$p(h \geq 8) = e^{-8 \times 0.25} = 0.1354$

3. 0.79 to 0.81

$$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= 2 \times \int_0^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

….(even function)

$$= 2 \times \int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Put $\frac{1}{2}x^2 = y \quad \therefore \quad x\,dx = dy$

$$\therefore E(|X|) = 2 \times \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y}\, dy$$

$$= 2 \times \frac{1}{\sqrt{2\pi}} \Gamma(1)$$

$= 0.7979$

4. (B)

The probability of getting 'tail' when the coin is tossed again is $\frac{1}{2}$

5. 0.96 to 0.98

P(at least one will meet specification)

= 1 – P(none will meet specification)

= 1 – (1 – 0.8) x (1 – 0.7) x (1 – 0.5)

= 1 – 0.2 x 0.3 x 0.5

= 1 – 0.03 = 0.97

6. 0.23 to 0.25

$E(X) = (0 \times q) + (1 \times p) = p$

$E(X^2) = (0^2 \times q) + (1^2 \times p) = p$

$V(X) = E(X^2) - [E(X)]^2$

$= p - p^2 = p(1 - p) = pq$

$= 0.6 \times 0.4 = 0.24$

7. 99.6 to 99.8

For SN distribution, percentage of area from –3 to 3 is 99.74

8. (A)

9. 2.5 to 2.5

10. (C)

$P(p < 3) = P(p = 0) + P(p = 1) + P(p = 2)$

$$= \frac{e^{-\mu}\mu^0}{0!} + \frac{e^{-\mu}\mu^1}{1!} + \frac{e^{-\mu}\mu^2}{2!}$$

(where μ = 3)

$$= e^{-3} + e^{-3} \times 3 + \frac{e^{-3} \times 9}{2}$$

$$= e^{-3}\left(1 + 3 + \frac{9}{2}\right) - \frac{17}{2e^3}$$

11. 0.25 to 0.27

$p = \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

Using Binomial distribution

$p(x \geq 2)$

$$= 3_{C_2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 + 3_{C_3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0$$

$$= \frac{6}{27} + \frac{1}{27} = \frac{7}{27}$$

12. (B)

Variance = $E[X^2] - [E[X]]^2$

We know that, variance ≥ 0

$\therefore \quad E[X^2] - [E[X]]^2 \geq 0$

$E[X^2] \geq [E[X]]^2$

13. (B)

Let X = difference between the number of heads and tails.

Taken $x = 2$

$\Rightarrow$ S = {HH, HT, TH, TT}

and $X = -2, 0, 2$

Here, $n - 3 = -1$ is not possible

Taken $n = 3$

$\Rightarrow$ S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

and $X = -3, -1, 1, 2$

Here $n - 3 = 0$ is not possible

Similarly, if a coin is tossed n times then the difference between heads and tails is $n - 3$ is not possible.

$\therefore$ Required probability is 0.

14. 0.25 to 0.28

Given: Poisson distribution,

$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$; x = 0, 1, 2, ...

λ = 240 per hour = 4 per minute = 2 per 30 seconds

$$P(X = 1) = \frac{e^{-\lambda}\lambda^1}{1!} = 2 \times e^{-2} = 0.2706$$

15. (D)

Player A Starts the game

Player A can with the game in the following manner :

A wins by getting 6 in the first try)

A loses, B loses, A wins

A loses, B loses, A loses, B loses, A wins and so on,

P(A winning)

$$= \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + ...$$

$$= \frac{1}{6}\left[1 + \frac{25}{36} + \frac{25}{36} \times \frac{25}{36} + ...\right]$$

Which forms a G.P.

$\therefore$ P(A winning)

$$= \frac{1}{6}\times\left[\frac{1}{1-\frac{25}{36}}\right]....\left(\because S_n = \frac{9}{1-r}\right)$$

$$= \frac{1}{6}\times\frac{1}{\frac{11}{36}}$$

$\therefore$ P(A winning) = $\frac{6}{11}$

16. (B)

Use Binomial distribution:

$p(x) = {}^nC_x\, p^x\, q^{\,n-x}$

Here n = 10, p = 0.1, q = 0.9

P(3 out of 10 are defective)

$= {}^{10}C_3(0.1)^3(0.9)^7 = 0.0574$

17. (D)

$P\{x = 0\} = P \Rightarrow P\{x = 1\} = 1 - p$

$P\{y = 0\} = q \Rightarrow P\{y = 1\} = 1 - q$

Let Z = X + Y

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	2

From above table,

$P\{X + Y + Z\} \Rightarrow P < Z \geq B$

$P\{Z \geq 1\} = P\{X = 0 \text{ and } Y = 1\}$

$+ P\{X = 1 \text{ and } Y = 0\}$

$+ P\{X = 1 \text{ and } Y = 1\}$

$= 1 - P\{X = 0 \text{ and } Y = 0\}$

$= 1 - pq$

18. 0.39 to 0.43

Use Binomial distribution:

$p(x) = {}^nC_x\, p^x\, q^{\,n-x}$

Here n = 5, p = 0.1, q = 0.9

P(packet is replaced)

$= P(X \geq 1) = 1 - p(0)$

$= 1 - q^n = 1 - (0.9)^5$

$= 0.40951$

19. 54.0 to 54.0

E(X) = 5

$\Rightarrow$ $E(X^2) = 30$, where $X \sim P(\lambda)$, $\lambda = 5$

$\therefore E[(X + 2)^2] = E(X^2) + 4E(X) + 4$

$= 30 + 20 + 4 = 54$

$(\because V(X) = E(X^2) - (E(X))^2)$

20. 0.33 to 0.34

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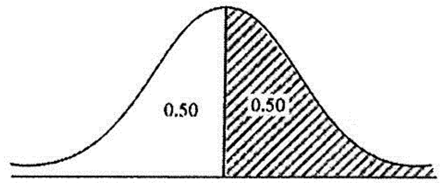
Model Solution on Assignment – 3

1. 49 to 51

Given $M = 500$

$\sigma = 50$

$P(X > 500) = ?$



where X follows normal distribution

We know that standard normal variable

$$z = \frac{x - M}{\sigma}$$

$$z = \frac{500 - 500}{50} = 0$$

$\therefore$ $(X > 500) = P(z > 0) = 0.50$ (see figure)

2. 49.9 to 50.1

Let X be a positive odd number less than 100

$\therefore$ X = 1, 3, 5, ..., 99 each with probability $\frac{1}{50}$

$$\therefore\ E(X) = \frac{1}{50}(1 + 3 + 5 + ... + 99)$$

$$= \frac{1}{50} \times 100 \times 25 = 50$$

3. (B)

$P(\text{Ram}) = \frac{1}{6}$; $P(\text{Ramesh}) = \frac{1}{8}$

$P(\text{only one}) = P(\text{Ram}) \times P(\text{not Ramesh}) + P(\text{Ramesh}) \times P(\text{not Ram})$

$$= \frac{1}{6} \times \frac{7}{8} + \frac{1}{8} \times \frac{5}{6}$$

$$= \frac{12}{48} = \frac{1}{4}$$

4. (B)

5. (C)

$$P(Y/X) = \frac{P(X \cap Y)}{P(X)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

6. (C)

Here A and B are two independent events

$\therefore$ $P(A \cap B) = P(A)\,P(B)$

and $P(A/B) = P(A)$

and $P(B/A) = P(B)$

Also if A and B are independent event then $\bar{A}$ and $\bar{B}$ are also independent event.

$\therefore$ $P\left(\bar{A} \cap \bar{B}\right) = P(\bar{A})\,P(\bar{B})$

Hence option (A), (B) and (D) are correct.

Now,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\,P(B)$

$\therefore$ Option (C) is false.

7. 0.9 to 1.1

8. 54.49 to 54.51

Observed speeds (in km/hr) : 66, 62, 45, 79, 32, 51, 56, 60, 53 and 49.

Number of observations = 10

$\therefore$ Median will be the mean of middle two values in the ordered set.

Arranging in ascending order : 32, 45, 49, 51, 53, 56, 60, 62, 66, 79

$\therefore$ Median Speed = $\dfrac{53+56}{2}$ = 54.5

9. 0.5 to 0.5

Given: $f(x) = \dfrac{1}{x^2}$; $a < x < 1$

$$\int_a^1 f(x)\,dx = 1$$

$$\therefore \int_a^1 x^{-2}\,dx = 1$$

$$\therefore \left[\frac{x^{-1}}{-1}\right]_a^1 = 1$$

$$\therefore -1 + \frac{1}{a} = 1 \quad \therefore \frac{1}{a} = 2 \quad \therefore a = \frac{1}{2}$$

10. (B)

$p(3V \geq 2U) = p(3V - 2 \geq 0) = p(W \geq 0)$, $W = 3V - 2U$

U, V are independent random variables and $U \sim N\left(0, \dfrac{1}{4}\right)$

$$V \sim N\left(0, \frac{1}{9}\right)$$

$$\therefore W = 3V - 2U \sim N\left(0, 9\times\frac{1}{4} + 4\times\frac{1}{9}\right)$$

$W \sim N(0, 2)$ ie., W has mean $\mu = 0$ and variance, $\sigma^2 = 2$

$$\therefore p(W \geq 0) = p\left(\frac{W-\mu}{\sigma} \geq \frac{0-\mu}{\sigma}\right)$$

$= p(Z \geq 0)$, Z is standard normal variants

$= 0.5 = \dfrac{1}{2}$

11. (A)

Let 'x' be no. of defective pieces.

x	0	1	2
P(x)	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

Mean $(\mu) = E(x) = \Sigma(x)\,P(x)$

$$= \left(0\times\frac{1}{6}\right) + \left(1\times\frac{1}{3}\right) + \left(2\times\frac{1}{6}\right)$$

$$= 0 + \frac{2}{3} + \frac{1}{3} = 1$$

$E(x^2) = \Sigma\, x^2\, P(x)$

$$= \left(0\times\frac{1}{6}\right) + \left(1\times\frac{2}{3}\right) + \left(4\times\frac{1}{6}\right)$$

$$= 0 + \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Variance, $V(x) = E(x^2) - (E(x))^2$

$$= \frac{4}{3} - 1 = \frac{1}{3}$$

12. (D)

Given, $\dfrac{\text{Red}}{4}$ $\quad \dfrac{\text{Black}}{6}$

The selection will be RBB or BBR of BRB

Probability of selecting RBB

$$= \frac{4}{10}\times\frac{6}{9}\times\frac{5}{8}$$

Probability of selecting BBR

$= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$

Probability of selecting BRB

$= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}$

P(Red = 1) = sum of above three probabilities = 0.5

13. 0.35 to 0.45

P(0.5 < x < 5)

$= \int_{0.5}^{5} f(x)dx$

$= \int_{0.5}^{1} 0.2\,dx + \int_{1}^{4} 0.1\,dx + \int_{4}^{5} 0\,dx$

$= 0.2[x]_{0.5}^{1} + 0.1[x]_{1}^{4} + 0$

$= 0.2 \times 0.5 + 0.1 \times 3$

$\therefore \quad P(0.5 < x < 5) = 0.4$

14. (B)

$n = 4;\ p = \frac{1}{6}$

$\Rightarrow \quad q = 1 - \frac{1}{6} = \frac{5}{6}$

$p(x \geq 2)$

$= 1 - p(x < 2)$

$= 1 - [p(x = 0) + p(x = 1)]$

$= 1 - \left[{}^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 + {}^4C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 \right]$

$= \frac{19}{144}$

15. (D)

16. 4

$E(e^{3X/4}) = \int_{0}^{\infty} e^{3x/4} e^{-x}\,dx$

$= \int_{0}^{\infty} e^{-x/4}\,dx = 4$

17. 6

Let S: getting 3 on a die

$\therefore p = P(S) = \frac{1}{6},\ q = 1 - p = \frac{5}{6}$

X = number of times die is thrown to obtain S for the first time

X	1	2	3	4	…
p(X)	p	qp	q^2p	q^3p	…

$E(X) = \Sigma\, x\,(q^{x-1} p) = p\, \Sigma\, x\, q^{x-1}$

$= p \times \frac{1}{(1-q)^2} = \frac{1}{p} = 6$

18. (A)

19. (A)

$P(X \leq 4) = \int_{1}^{4} 0.25\,dx$

$= 0.25 \times (4 - 1) = 0.75$

20. 2.0 to 2.0

Given X(t) = U + Vt

X(2) = U + 2V

E[X(2)] = E[U + 2V]

$= E(U) + 2E[V] = 0 + 2 \times 1 = 2$

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Model Solution on Assignment – 4

1. (A)

The possible cases in the throw of a die are six i.e. 1, 2, 3, 4, 5, 6

Cases of even numbers 2, 4, 6 are three.

$\therefore$ Probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$

2. (D)

When we toss a coin, there are two possible outcomes i.e. Head or Tail.

$\therefore$ Number of possible cases = 2

The outcome of Tail is favourable event

$\therefore$ Probability of getting a tail = 1/2

3. (B)

Total number of equally likely and exhaustive cases = n = 6 + 9 = 15

Number of favourable cases

$= {}^9C_1 = 9$

[$\because$ number of black balls = 9]

Probability of drawing a black ball

$= \frac{9}{15} = \frac{3}{5}$

4. (C)

Number of clubs in a pack = 13

Probability of getting a club $= \frac{13}{52} = \frac{1}{4}$

5. (A)

Sample space = 6

4, 5, 6 are greater than 3

$\therefore$ Probability $= \frac{3}{6} = \frac{1}{2}$

6. (C)

Sample space $= 6^2 = 36$

Favourable cases are (5, 6), (6, 5), (6, 6)

Probability $= \frac{3}{36} = \frac{1}{12}$

7. (C)

There are 366 days in a leap year and it has 52 complete weeks and 2 days extra.

They are

(1) Sunday and Monday

(2) Monday and Tuesday

(3) Tuesday and Wednesday

(4) Wednesday and Thursday

(5) Thursday and Friday

(6) Friday and Saturday

(7) Saturday and Sunday

sample space = 7

Number of favourable cases = 2

$\therefore$ Probability = 2/7

8. (A)

In a pack of 52 cards, 1 card can be drawn in 52 ways

Since there are 13 spades and 3 aces (one ace is present in spade)

∴ Number of favourable case

= 13 + 3 = 16

Sample space = 52

Probability of getting a spade or an ace

$= \frac{16}{52} = \frac{4}{13} = \frac{4}{9+4}$

Odds against winning the bet are

9 to 4.

9. (B)

Events are mutually exclusive

odd in favour of horse $H_1 = 1/3$

$$P(H_1) = \frac{1}{1+3} = \frac{1}{4}$$

Similarly $P(H_2) = \frac{1}{1+4} = \frac{1}{5}$,

$$P(H_3) = \frac{1}{1+5} = \frac{1}{6}$$

$$P(H_4) = \frac{1}{1+6} = \frac{1}{7}$$

$$P(H_1 + H_2 + H_3 + H_4) = P(H_1) + P(H_2) + P(H_3) + P(H_4)$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$= \frac{319}{420}$$

10. (D)

Suppose there are 100 flowers

Number of roses = 40; Number of carnations = 60

25% of 40 = 10 roses are red and

10% of 60 = 6 carnations are red

Let A be the event that the flower is red and B the event that the flower is a rose.

∴ $A \cap B$ is the event that the flower is a red rose.

$n(A) = 16 \quad \therefore P(A) = \frac{16}{100}$

$n(A \cap B) = 10 \quad \therefore P(A \cap B) = \frac{10}{100}$

P(B/A) = probability that a selected flower is a rose red is colour

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{10/100}{16/100} = \frac{5}{8}$$

11. (C)

A throw amounting to 18 must be made up of 6, 6, 6 and this can occur in 1 way. 17 can be made up of 6, 6, 5 which can occur in 3 ways. 16 may be made up of 6, 6, 5 and 6, 5, 5 each of which arrangements can occur in 3 ways.

∴ The number of favourable cases is = 1 + 3 + 3 + 3 = 10 and the total number of cases is 63 = 216

∴ Required chance $= \frac{10}{216} = \frac{5}{108}$

12. (B)

Probability that A wins (P_1) = 1/6

and that B wins (P_2) = 1/10

and that C wins (P_3) = 1/8

As a dead heat is impossible, these are mutually exclusive events, so the chance that one of them will win the race is

$$P_1 + P_2 + P_3 \text{ i.e. } \frac{1}{6} + \frac{1}{10} + \frac{1}{8} = \frac{47}{120}$$

13. (A)

The chance of choosing the first bag is 1/2 and if the first bag be chosen the chance of drawing a red ball from it is 5/12 hence the chance of drawing a red ball from the first bag is

$$\frac{1}{2} \times \frac{5}{12} = \frac{5}{24}$$

Similarly the chance of drawing a red ball from the second bag is $\frac{1}{2} \times \frac{3}{15} = \frac{1}{10}$. Hence, as these events are mutually exclusive, the chance required is $\frac{5}{24} + \frac{1}{10} = \frac{37}{120}$

14. (C)

9 cards of a suit can be selected in ${}^{13}C_9$ ways, 4 cards can be selected from the remaining in ${}^{39}C_4$ ways.

The suit can be selected in 4 ways

$\therefore$ Required probability

$$= \frac{{}^{13}C_9 \times {}^{39}C_4 \times 4}{{}^{52}C_{13}}$$

15. (A)

The probability that husband is not selected = $1 - \frac{1}{7} = \frac{6}{7}$

The probability that wife is not selected

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

Probability that only husband is selected = $\frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$

Probability that only wife is selected

$$= \frac{1}{5} \times \frac{6}{7} = \frac{6}{35}$$

Probability that only one of them is selected = $\frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$

16. (B)

There are 3 mutually exclusive and exhaustive way in which 2 balls are transferred form first bag to second bag.

First Way :

Two white balls are transferred from first bag to second bag so that probability for that is = $\frac{{}^4C_2}{{}^6C_2}$. In the second bag we have 7 white and 4 black balls and the probability of getting a white ball is = $\frac{7}{11}$

$\therefore$ Required probability

$$= \frac{{}^4C_2}{{}^6C_2} \times \frac{7}{11} = \frac{6}{15} \times \frac{7}{11} = \frac{42}{165}$$

Second way :

Two black balls have been transferred from first bag to the second bag so that probability for that is $\frac{{}^2C_2}{{}^6C_2} = \frac{1}{15}$

In the second bag we have five white and 6 black balls and probability of getting a white ball is $\frac{5}{11}$.

$\therefore$ Required probability

$$= \frac{1}{15} \times \frac{5}{11} = \frac{5}{165}$$

Third way :

One black and one white ball have been transferred from first bag to the second so that the probability for this is

$$\frac{{}^4C_1 \times {}^2C_1}{{}^6C_2} = \frac{8}{15}$$

In the second bag we have 6 white and 5 black balls and the probability of drawing a white ball is 6/11.

$\therefore$ Required probability

$$= \frac{8}{15} \times \frac{6}{11} = \frac{48}{165}$$

Since these three cases are mutually exclusive,

$\therefore$ The required probability of drawing a white ball

$$= \frac{42}{165} + \frac{5}{165} + \frac{48}{165} = \frac{95}{165}$$

17. (C)

Total number of cases in a throw of two dice = 36

Number of cases with doublets = 6

$$\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

Number of cases with a total of 9 = 4

$$\{(5,4),(4,5),(3,6),(6,3)\}$$

$\therefore$ Total number of cases in which a doublet or a total of 9 appear = 6 + 4

= 10

P(a doublet or a total of 9) = P(A)

$$= \frac{10}{36} = \frac{5}{18}$$

P(neither a doublet nor a total of 9)

= 1 − P(A)

$$= 1 - \frac{5}{18} = \frac{13}{18}$$

18. (C)

Let P(A), P(B) be the probability of A & B speaking the truths, then

$$P(A) = \frac{75}{100} = \frac{3}{4}, \quad P(B) = \frac{80}{100} = \frac{4}{5}$$

$P(\bar{A}) =$ P (A tells a lie) = 1 − P(A)

$$= 1 - \frac{3}{4} = \frac{1}{4}$$

$P(\bar{B})$ = P(B tells a lie) = 1 − p(B)

$$= 1 - \frac{4}{5} = \frac{1}{5}$$

Now P(A and B will contradict)

$$= P(A)P(\bar{B}) + P(B)P(\bar{A})$$

$$= \frac{3}{4} \times \frac{1}{5} + \frac{4}{5} \times \frac{1}{4} = \frac{7}{20} = 35\%$$

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Model Solution on Assignment – 5

1. (A)

Probability of 1 appearing on upper face = $\frac{1}{6}$

Probability of 6 appearing = $\frac{1}{6}$

$\therefore$ The probability 1 or 6 appear

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

2. (C)

Probability of the winning of the horse A = $\frac{1}{5}$ and the probability of the horse B = $\frac{1}{6}$

$\therefore$ P(A + B) = P(A) + P(B)

$= \frac{1}{5} + \frac{1}{6} = \frac{11}{30}$

3. (D)

First draw : Probability of getting a queen = $\frac{4}{52} = \frac{1}{13}$

Second draw : After drawing the first queen we are left with 51 cards with 3 queens

$\therefore$ Probability of getting a queen in second draw = $\frac{3}{51} = \frac{1}{17}$

$\therefore$ Probability of both the cards are queen = $\frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$

4. (A)

P(A) and P(B/A) denote the probability of drawing a black ball in the first and second attempt

$\therefore$ Probability of drawing a black ball in the first attempt is

$$P(A) = \frac{3}{8} \qquad \left[\frac{3}{5+3}\right]$$

Probability of drawing the second black ball given the first ball drawn is black

$$P(B / A) = \frac{2}{7} \qquad \left[\frac{2}{5+2}\right]$$

Probability that both the balls drawn are black is

$$P(AB) = P(A)\, P(B/A)$$

$$= \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

5. (D)

Probability of getting a head in a toss of a coin $= \frac{1}{2}$;

Probability of getting tail in each case

$= \frac{1}{2}$

Probability of getting a tail in all four cases = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

$\therefore$ Probability of getting at least one head = $1 - \frac{1}{16} = \frac{15}{16}$

6. (C)

P = probability of getting 1 in a throw of a die = 1/6

$\therefore \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Probability of getting at least one die

$$= 1 - q^3 = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$$

7. (A)

8. (C)

Let X = defective items. We seek P (C/X) the probability that an item is produced by machine C, given that the item is defective

By Bayes' theorem,

$$P(C/X) = \frac{P(C)P(X/C)}{\begin{bmatrix} P(A)P(X/A) \\ + P(B)P(X/B) \\ + P(C)P(X/C) \end{bmatrix}}$$

$$= \frac{(0.10)(0.04)}{\begin{bmatrix} (0.60)(0.02) \\ +(0.30)(0.03) \\ +(0.10)(0.04) \end{bmatrix}}$$

$$= \frac{4}{25}$$

9. (D)

Probability of six occurring in one toss = 1/6

Probability of six occurring in 180 toss

$= 180 \times \frac{1}{6} = 30$

10. (A)

Probability that any person selected at random is a rice eater $= \frac{1}{2}$. Probability of a non rice eater $= \frac{1}{2}$. Let A_0, A_1, A_2, A_3 denote the event that none, one, two or three persons respectively are rice eaters out of 10. Then

$$P(A_0) = {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10},$$

$$P(A_1) = {}^{10}C_2 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9$$

$$P(A_2) = {}^{10}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8,$$

$$P(A_3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$$

Now, the required probability

$= P(A_0 + A_1 + A_2 + A_3)$

$= P(A_0) + P(A_1) + P(A_2) + P(A_3)$

$= \frac{1}{2^{10}} [{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3]$

$= \frac{1}{2^{10}} [1 + 10 + 45 + 120]$

$= \frac{176}{1024} = 0.17 = 17\ \%$ (approx)

11. (D)

Probability that the first person lives till he is 75 years $= \frac{8}{14}$

Probability that the second person lives till he is 80 years $= \frac{4}{9}$

Probability of the compound event that both the persons live 40 years hence

$$= \frac{8}{14} \times \frac{4}{9} = \frac{32}{126} = \frac{16}{63}$$

Probability that at least one of them would die without living 40 years hence $= 1 - \frac{16}{63} = \frac{47}{63}$

12. (A)

The probability of getting a head in a single toss of a coin is 1 / 2

$\therefore$ p = 1/2, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Also we are given n = 10, N = 100 and r = 7

The required frequency

$$= N\ {}^{n}C_r\ p^r\ q^{n-r}$$

$$= 100 \times {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3$$

$$= 100.\frac{10!}{7!3!}\left(\frac{1}{2}\right)^{10}$$

$$= 100 \times \frac{10 \times 9 \times 8}{3 \times 2 \times 1}\left(\frac{1}{2}\right)^{10} = \frac{375}{32}$$

$$= 11.7 = 12$$

13. (A)

Let $(q + p)^n$, q + p = 1, be the binomial distribution

P = 0 –1, n = 500

Mean = np = 0.1 × 500 = 50

Now p = 0.1

$\Rightarrow q = 1 - p = 1 - 0.1 = 0.9$

$\therefore$ Variance = npq

$$= 500 \times 0.1 \times 0.9 = 45$$

$\therefore$ Standard deviation $= \sqrt{45} = 6.7$

14. (A)

$$\mu = \Sigma\, p_i\, x_i$$

$$= 0\,.\,\frac{144}{169} + 1.\frac{24}{169} + 2.\frac{1}{169}$$

$$= \frac{26}{169} = \frac{2}{13}$$

$$\sigma^2 = \Sigma\, p_i x_i^2 - (\mu)^2$$

$$= 0.\frac{144}{169} + 1.\frac{24}{169} + 4.\frac{1}{169} - \frac{4}{169}$$

$$= \frac{28}{169} - \frac{4}{169} = \frac{24}{169}$$

$$\therefore \quad \sigma = \sqrt{\frac{24}{169}} = \frac{4.9}{13} = 0.377$$

15. (C)

$$np = 12 \qquad \sqrt{npq} = 2$$

$\therefore$ npq = 4

$$\frac{npq}{np} = \frac{4}{12} = \frac{1}{3} \quad \therefore q = \frac{1}{3}$$

$$\therefore p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

Also np = 12

$$\therefore n \times \frac{2}{3} = 12$$

$$\therefore n = 18 \quad \& \quad p = \frac{2}{3}$$

16. (D)

The probability that face 1 or face 2 turn up = $0.10 + 0.32 = 0.42$

The probability that face 1 turns up

$= 0.10$

$\therefore$ The required probability

$$= \frac{0.10}{0.42} = \frac{10}{42} = \frac{5}{21}$$

17. (C)

Since odds against A are 8 : 3

$$P(A) = \frac{3}{8+3} = \frac{3}{11}$$

$$P(B) = \frac{2}{5+2} = \frac{2}{7}$$

$P(A) + P(B) + P(C) = 1$

{mutually exclusive}

Or $\frac{3}{11} + \frac{2}{7} + P(C) = 1$

$$\therefore \quad P(C) = 1 - \frac{3}{11} - \frac{2}{7} = \frac{34}{77}$$

$\therefore$ odds against C are $77 - 34 = 34$

i.e. 43 : 34

18. (C)

Combinations are CMC, CMM, CCC, CCM.

Probability of

$CMC = 0.6 \times 0.4$

$= 0.24$ – Favourable

$CMM = 0.6 \times 0.6 = 0.36$

$CCC = 0.4 \times 0.4$

$= 0.16$ – Favourable

$CCM = 0.4 \times 0.6 = 0.24$

Total Probability = 1

Favourable = 0.24 + 0.16 = 0.4

So P = 0.4/1

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Model Solution on Assignment – 6

1. (B)

S = {Sun, Mon, Tue, Wed, Thu, Fri, Sat}

n(s) = 7

Let A be the event that the child is born on Sunday or a Saturday

$\therefore$ n(A) = 2

$\therefore$ P(A) = $\frac{n(A)}{n(S)} = \frac{2}{7}$

2. (D)

S = {1, 2, 3, 4, 5, 6 }

n(S) = 6

Let A be the event that the die shows a multiple of 2

$\therefore$ A = {2, 4, 6}

$\therefore$ n(A) = 3

P(A) = $\frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

3. (C)

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

n(S) = 36

Let A be the event that the score on the dice is odd

$$A = \begin{array}{l}\{(1,2),(1,4),(1,6),(2,1),(2,3),\\(2,5),(3,2),(3,4),(3,6),(4,1),\\(4,3),(4,5),(5,2),(5,4),(5,6),\\(6,1),(6,3),(6,5)\}\end{array}$$

n(A) = 18

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$

4. (C)

5 students can be selected from 10 by ${}^{10}C_5$ ways

$\therefore$ n(S) = ${}^{10}C_5 = \frac{10!}{5!5!}$

$= \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} = 252$

Let A be the event that the committee includes exactly 2 girls and 3 boys. The two girls can be selected in ${}^{4}C_2$ ways and the three boys can be selected in ${}^{6}C_3$ ways

n(A) = ${}^{4}C_2 \times {}^{6}C_3 = 6 \times 20 = 120$

P(A) = $\frac{n(A)}{n(S)} = \frac{120}{252} = \frac{10}{21}$

5. (A)

Let A be an event that ball drawn is red or white

$\therefore$ n(A) = ${}^{7}C_1 + {}^{4}C_1 = 7 + 4 = 11$

$\therefore P(A) = \frac{11}{20}$

$[\, n(S) = {}^{20}C_1 = 20 \,]$

6. (C)

Two cards can be selected in $^{52}C_2$ ways

$\therefore$ n(S) = $^{52}C_2 = 26 \times 51$

Let A be the event that 2 cards selected is Jack and an Ace.

Jack can be selected in $^4C_1$ ways and the Ace can also be selected in $^4C_1$ ways.

$\therefore$ n(A) = $^4C_1 \times {^4C_1} = 16$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{16}{26 \times 51} = \frac{8}{663}$$

7. (C)

The required probability

$= {^{12}C_{11}} \times (0.99875)^{11}(0.01125) + {^{12}C_{12}}(0.99875)^{12} = 0.9923$

8. (B)

Required probability

$$= \frac{1}{5}.\frac{1}{3} + \frac{4}{5}.\frac{1}{4} = \frac{4}{15}$$

9. (A)

$$P(A) = \frac{^{13}C_1}{^{52}C_1} \times \frac{^4C_1}{^{51}C_1} = \frac{1}{52}$$

10. (D)

Probability of first event = $\frac{2}{3}$P

Mutually exclusive events

$P + \left(\frac{2}{3}\right)P = 1$ Or $P = \frac{3}{5} = 3 : 5$

Hence odds in favor of the other are 3 : (5 – 3) i.e. 3 : 2

11. (C)

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{1}{3}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = [P(A) - P(A \cap B)] + P(B)$$

$$P(A \cup B) = P(A \cap \bar{B}) + P(B)$$

$$\frac{2}{3} = \frac{1}{3} + P(B)$$

$$\therefore\ P(B) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

12. (A)

Probability P of a white flower = $\frac{1}{4}$

$q = 1 - P = 1 - \frac{1}{4} = \frac{3}{4}$

n = 3 N = 64

$$p(r) = {^3C_r}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{3-r}$$

Number of beds with zero white flowers

$$= 64 \times {^3C_0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3$$

$$= 64 \times \frac{27}{64} = 27$$

Beds with 1 white flower

$$= 64 \times {^3C_1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = 27$$

Beds with 2 white flowers

$$= 64 \times {^3C_2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) = 64 \times \frac{9}{64} = 9$$

Beds with 3 white flowers

$$= 64 \times {^3C_3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = 64 \times \frac{1}{64} = 1$$

13. (B)

$P(A \cap B) = \frac{1}{2}; \ P(\bar{A}) = \frac{1}{3}; \quad P(\bar{B}) = \frac{1}{3}$

Now, $\quad P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(B) = 1 - P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{2}{3} + \frac{2}{3} - \frac{1}{2}$

$= \frac{4}{3} - \frac{1}{2} = \frac{8-3}{6}$

$= \frac{5}{6} = \frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$

14. (A)

To be one step away from the starting point the Man is to take 6 steps forward and 5 steps backward or 5 steps forward and 6 steps backward

$\therefore$ The required probability

$= {}^{11}C_6\,(0.4)^6\,(0.6)^5 + {}^{11}C_5\,(0.4)^5\,(0.6)^6$

$= {}^{11}C_5\,(0.4)^5\,(0.6)^5\,\{0.4 + 0.6\}$

$= {}^{11}C_5\,(0.24)^5$

15. (A)

Required probability

$= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{1}{2}$

16. (B)

$P(B) = 1 - P(\bar{B}) = 1 - \frac{1}{2} = \frac{1}{2}$

and $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $\quad \frac{5}{6} = P(A) + \frac{1}{2} - \frac{1}{3}$

or $\quad P(A) = \frac{2}{3}$

$\therefore \quad P(A)\,P(B) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = P(A \cap B)$

Hence A and B are independent.

17. (D)

n(s) $= 2^4 = 16$ since each of the four places in a determinant of order 2 is to filled by 0 or 1 favourable number of ways is 3.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

Hence $P = \frac{3}{16}$

18. (D)

${}^{100}C_{50}\,p^{50}\,(1-p)^{50} = {}^{100}C_{51}\,p^{51}\,(1-p)^{49}$

or $\quad \frac{1-p}{p} = \frac{100!}{51!\,49!} \times \frac{50!\,50!}{100!} = \frac{50}{51}$

or $\quad 51 - 51p = 50\,p$

giving $\quad P = \frac{51}{101}$

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Answer Key on Test Paper – 1

1.	(D)	2.	(D)	3.	(A)
4.	(C)	5.	(A)	6.	(C)
7.	(C)	8.	(B)	9.	(A)
10.	(D)	11.	(A)	12.	(A)
13.	(C)	14.	(A)	15.	(C)

Answer Key on Test Paper – 2

1.	(B)	2.	(D)	3.	(D)
4.	(C)	5.	(B)	6.	(A)
7.	(D)	8.	(A)	9.	(A)
10.	(A)	11.	(A)	12.	(D)
13.	(A)	14.	(A)	15.	(C)

Answer Key on Test Paper – 3

1.	(D)	2.	(A)	3.	(D)
4.	(B)	5.	(A)	6.	(D)
7.	(B)	8.	(B)	9.	(D)
10.	(B)	11.	(D)	12.	(A)
13.	(C)	14.	(B)	15.	(D)

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Model Solution on Test Paper – 1

1. (D)

$P(A) = \frac{2}{5}$ $\quad$ $P(A') = \frac{3}{5}$

$P(B) = \frac{2}{3}$ $\quad$ $P(B') = \frac{1}{3}$

$P(C) = \frac{3}{5}$ $\quad$ $P(C') = \frac{2}{5}$

Probability that only one of them hits the target

= probability that A hits the target but not B and C + probability that B hits the target but not A and C + probability that C hits the target but not A and B

$$= P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$$

$$= \frac{2}{5} \times \frac{1}{3} \times \frac{2}{5} + \frac{2}{3} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{1}{3}$$

$$= \frac{4}{75} + \frac{12}{75} + \frac{9}{75} = \frac{25}{75} = \frac{1}{3}$$

2. (D)

Probability of selecting any bag is 1/2

Probability of getting a white ball

$$= \frac{1}{2} \times \frac{{}^6C_1}{{}^9C_1} + \frac{1}{2} \times \frac{{}^4C_1}{{}^9C_1} = \frac{5}{9}$$

3. (A)

Let A be the event that A is selected and B be the event that B is selected

$\therefore$ P(A) = 1/5 and P(B) = 2/7

Let C be the event that both are selected

$\therefore$ C be the event that both are selected.

$\therefore$ $C = A \cap B$

$\therefore$ $P(C) = P(A \cap B)$

$\therefore$ $P(C) = P(A) \cdot P(B)$ as A and B are independent events $= \frac{1}{5} \times \frac{2}{7} = \frac{2}{35}$

4. (C)

Two balls are selected from 12 in ${}^{12}C_2$ ways. Two white balls can be selected in ${}^7C_2$ ways.

$\therefore$ Required probability $= \frac{{}^7C_2}{{}^{12}C_2}$

$$= \frac{21}{66} = \frac{7}{22}$$

5. (A)

Three balls can be selected from 9 in ${}^9C_3$ ways

3 black balls can be selected from 5 in ${}^5C_3$ ways

$\therefore$ Required probability $= \frac{{}^5C_3}{{}^9C_3}$

$$= \frac{10}{84} = \frac{5}{42}$$

6. (C)

R stands drawing a red ball and B for drawing black ball. Then required probability is

= RRR + RBR + BRR + BBR

$$= \left(\frac{6}{10}\times\frac{5}{11}\times\frac{6}{10}\right)+\left(\frac{6}{10}\times\frac{6}{11}\times\frac{5}{10}\right)$$

$$+\left(\frac{4}{10}\times\frac{4}{11}\times\frac{7}{10}\right)+\left(\frac{4}{10}\times\frac{7}{11}\times\frac{6}{10}\right)$$

$$= \frac{6400}{1100} = \frac{32}{55}$$

7. (C)

$$P_1 = \frac{6}{36} = \frac{1}{6}$$

[$\because$ out of total of 36 ways both the persons can throw equal values in 6 ways]

To find p_2 the total number of ways $n \neq 6^4$ and the favorable number of ways, M =15 × 8 = 120

Since any two numbers out of 6 can be selected in ${}^6c_2$ i.e. 15 ways and corresponding to each of these ways, there are 8 ways e.g. corresponding to the numbers 1 and 2 the eight ways are (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1, 1) , (2, 2, 2, 1) (2, 2, 1, 2) (2, 1, 2, 2) (1, 2, 2, 2).

Hence $P = \dfrac{120}{64} = \dfrac{5}{54}$

Since $\dfrac{1}{6} > \dfrac{5}{54}$, we have $P_1 > P_2$

8. (B)

$$P_1 = \frac{{}^{15}c_2 \times {}^{27}c_2}{{}^{42}c_4}$$

$$= \frac{15\times14\times27\times26\times1\times2\times3\times4}{1\times2\times1\times2\times42\times41\times40\times39}$$

$$= \frac{27}{82}$$

$$P_2 = \frac{{}^{30}c_4 \times {}^{54}c_4}{{}^{84}c_8}$$

$$= \frac{30.29.28.27.54.53.52.51.8!}{4!4!84.83.82.81.80.79.78.77}$$

$$= \frac{17.29.45.53}{11.79.82.83}$$

$$\frac{P_1}{P_2} = \frac{27}{82}\times\frac{11.79.82.83}{17.29.45.53} = \frac{33.79.83}{29.53.85}$$

$$= \frac{216381}{130645} > 1$$

Hence $P_1 > P_2$

9. (A)

$P(A\cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \le 1$

$\therefore\ P(A) + P(B) - P(A \cap B) \le 1$

$P(A) + P(B) - 1 \le P(A \cap B)$

$P(A \cap B) \ge P(A) + P(B) - 1$

10. (D)

We first find the total number of cases. For each element of set A with n element, the number of possible images is n.

$\therefore$ The total number of cases = n^n.

When next find the number of favourable cases. For the first element we have n choices. For the seconds element we have (n − 1) choices and so on.

∴ The number of favourable cases

$$= n(n-1)(n-2)...2\cdot 1 = n!$$

Hence the required probability

$$= \frac{n!}{n^n} = \frac{(n-1)!}{n^{n-1}}$$

11. (A)

12. (A)

13. (C)

14. (A)

Distinct paths from A to F are given below.

1) ABDF
2) ACEF
3) ABF
4) ABEF
5) ACDF
6) ABCDEF
7) ACDEF
8) ABDEF
9) ABCDF
10) ABCEF

There are 6 way to go to city F through C.

Total no. of paths = 10

∴ Hence, required probability

= 6/10 = 3/5

15. (C)

Mean $\bar{x} = \frac{\sum x}{N}$

$\therefore \quad 90 = \frac{\sum x}{200}$

$\sum x = 200 \times 90 = 18000$

∴ Sum of observations is 18000

Of these, two wrong observations are 15 and 80.

Subtracting 15 and 80 from 18000, we get 17905

Adding the correct observations to 17905,

i.e. 17905 + 40 + 87 = 18032

∴ Correct sum of 200 observation is 18032

∴ Correct mean is $\frac{18032}{200} = 90.16$

❑ ❑ ❑ ❑ ❑ ❑

Model Solution on Test Paper – 2

1. (B)

There are 6 possible ways in which the die can fall and of these two are favourable events required.

$\therefore$ Required chance = $\frac{2}{6} = \frac{1}{3}$

2. (D)

The possible number of cases is

$6 \times 6 = 36$

An ace on one die may be associated with any of the 6 numbers on the other die and the remaining 5 numbers on the first die may be associated with the ace on the second die, thus the number of favorable cases is 11.

$\therefore$ Required chance = $\frac{11}{36}$

3. (D)

Various Digits in the log table are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

i.e. a total of 10 digits are used.

The number of favourable cases for getting 1 out of the 10 all equally likely cases is one

$\therefore$ Probability of getting 1 is 1/10.

4. (C)

The probabilities of Dayanand, Ramesh and Naresh solving the problem are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively

$\therefore$ The probabilities of Dayanand, Ramesh, Naresh not solving the problem are

$1-\frac{1}{2}=\frac{1}{2}; 1-\frac{1}{3}=\frac{2}{3}; 1-\frac{1}{4}=\frac{3}{4}$

respectively.

$\therefore$ The probability that the problem is not solved by any one of them is

$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$

$\therefore$ The probability that the problem will be solved by at least one of them = $1 - \frac{1}{4} = \frac{3}{4}$

5. (B)

The probability of drawing a black ball from the bag = 3/8

The probability that the drawn ball is not black $= 1-\frac{3}{8}=\frac{5}{8}$

6. (A)

In "PROBABILITY" there are

O A I : 3 distinct vowels

P R B L T Y : 6 distinct consonants

Total number of distinct letters = 9,

Required probability $= \frac{3}{9}=\frac{1}{3}$

7. (D)

Let P(A), P($\bar{A}$) be probabilities of A's getting the head and not getting the head respectively, then $P(A) = \frac{1}{2}$

$\therefore P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

Similarly, $P(B) = \frac{1}{2}$ and $P(\bar{B}) = \frac{1}{2}$

Let A start the game. He can win it in the 1^{st} throw 3^{rd} throw 5^{th} throw, 7^{th} throw and so on. Probability of A's winning in 1^{st} throw $= P(A) = \frac{1}{2}$

Probability of A's winning in 3^{rd} throw

$= P(\bar{A})\, P(\bar{B})\, P(A)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^3$

Probability of A's winning in 5^{th} throw

$= P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(\bar{B})\, P(A)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^5$

Probability of A's winning in 7^{th} throw

$= P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(\bar{B})\, P(A)$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^7$

Mutually exclusive cases

$\therefore$ Probability of A's winning the game first is

$$\frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + ... = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$$

$\therefore$ Probability of B winning the game first $= 1 - \frac{2}{3} = \frac{1}{3}$

8. (A)

$n(S) = 36$

The outcomes which have odd number and multiple of 3 are

(1, 3) (1, 6) (3, 1) (3, 3) (3, 5) (3, 6) (5, 3) (5, 6) (6, 1) (6, 3) (6, 5)

favourable cases = 11

Probability $= \frac{11}{36}$

9. (A)

Let A = The event that students are taller than 1–8m

$P(w/A)$ = Probability that a student is a woman given A

$$= \frac{P(w)P(A/w)}{P(w)P(A/w) + P(M)P(A/M)}$$

$$= \frac{(0.60)(0.01)}{(0.60)(0.01) + (0.40)(0.04)} = \frac{3}{11}$$

10. (A)

n = total number of ways $= 6 \times 6 = 36$

The numbers higher than 9 are 10, 11, 12 in the case of two dice

$\therefore$ m = favourable number of ways

$= 3 + 2 + 1 = 6$

Hence $P = \frac{M}{n} = \frac{6}{36} = \frac{1}{6}$

11. (A)

Total number of cards =12

Probability of at least one are

= 1– probability no are

$$= 1 - \frac{{}^{8}C_2}{{}^{12}C_2} = 1 - \frac{14}{33} = \frac{19}{33}$$

12. (D)

Var (x′) = 1

13. (A)

The probability that event $\bar{A}$ occurred, provided that $\bar{B}$ took place will be denoted by $P(\bar{A}\ \bar{B})$. Now the probability that event A occurred, provided that B took place will be denoted by $P(A/\bar{B})$. The event $\bar{A}$ is the non occurrence of the event A.

$\therefore$ We have $P(A/\bar{B}) + P(\bar{A}/\bar{B}) = 1$

Hence $P(\bar{A}/\bar{B}) = 1 - P(A/\bar{B})$

14. (A)

Two points are always collinear so we can say a triangle consists of two collinear points and one point which is not collinear to both of them simultaneously.

So here, we have no. of triangles

= No. of triangles having 2 points on line 1 and one point on line 2 + No. of triangles having 2 points on line 2 and one point on line 1

$= {}^{10}C_2 \times 11 + {}^{11}C_2 = 45 \times 11 + 55 \times 10$

= 1045

Total number of points = 10 + 11 = 21

$\therefore$ No. of 3 points group

$= {}^{21}C_3$

$= \dfrac{21 \times 20 \times 19}{3 \times 2 \times 1} = 7 \times 10 \times 19$

$\therefore$ Required probability

$= \dfrac{1045}{7 \times 10 \times 19} = \dfrac{209}{14 \times 19}$

$= \dfrac{11 \times 19}{14 \times 19} = \dfrac{11}{14}$

15. (C)

Probability that the item is produced by machine A, P(A) = 0.25

Probability that the item produced by A is faulty, P(F/A) = 0.05

Similarly,

P(B) = 0.30

P(F/B) = 0.04

P(C) = 1 − 0.25 − 0.3 = 0.45

P(F/C) = 0.03

By total probability,

Probability that the item selected is faulty

= P(A) P(F/A) + P(B) P(F/B) + P(C) P(F/C)

= 0.25 × 0.05 + 0.30 × 0.04 + 0.45 × 0.03

= 0.038

By Baye's Theorem,

Probability that the faulty item was produced by machine C,

$P(C/F) = \dfrac{P(C)\ F(F/C)}{P(F)} = \dfrac{0.45 \times 0.03}{0.038}$

$\therefore$ P(C/F) = 0.355

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Model Solution on Test Paper – 3

1. (D)

Probability of getting both the balls even numbered

$P(EE) = P(E) \cdot P(E)$

$$= \frac{12}{25} \times \frac{12}{25} = \frac{144}{625}$$

P(at least one odd) = $1 - P(EE)$

$$= 1 - \frac{144}{625}$$

$$= \frac{481}{625}$$

2. (A)

The total number of ways in which four persons can be selected out of 9 persons is ${}^9C_4$.

For favourable cases, we want that 2 out of the four selected should be children. Two children can be selected out of 4 in ${}^4C_2$ ways. The other two are to be selected out of 5 persons (3 men and 2 women). Two persons can be selected out of 5 in ${}^5C_2$ ways.

$\therefore$ The number of favourable cases

${}^4C_2 \times {}^5C_2$

$\therefore$ Required probability

$$= \frac{{}^4C_2 \times {}^5C_2}{{}^9C_4} = \frac{10}{21}$$

3. (D)

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

4. (B)

When all the letters are taken they can be arranged in ${}^{10}p_{10}$ ways or 10 !

5. (A)

Here the balls are drawn without replacement. Let P(W) and P(G) denote the probability of drawing white ball and green ball respectively.

P(W and G) = P(W) P(G) + P(G) P(W)

$$= \frac{10}{18} \times \frac{8}{17} + \frac{8}{18} \times \frac{10}{17} = \frac{80}{153}$$

6. (D)

The largest number on the selected coupon is 9, and hence the selection is to be made from the coupons numbered 1 to 9. Since there are 15 possible cases for selecting a coupon and seven coupons are selected the total number of possible cases = 15^7

The number of favorable cases $= 9^7 - 8^7$. Observe that out of 9^7 cases 8^7 cases do not contain the number 9

The required probability

$$= \frac{\text{Number of favourable cases}}{\text{Total number of cases}}$$

$$= \frac{9^7 - 8^7}{15^7}$$

7. (B)

$P = \frac{2}{5} \quad q = \frac{3}{5} \quad n = 5$

$P(r) = {}^{n}C_r \,.\, q^{n-r} p^r = {}^{5}C_r \left(\frac{3}{5}\right)^{5-r} . \left(\frac{2}{5}\right)^r$

$\therefore$ The required chance

$= P(4)$

$= {}^{5}C_4 \,.\frac{3}{5}.\left(\frac{2}{5}\right)^4$

$= 5 \times \frac{48}{5^5} = \frac{48}{5^4} = \frac{48}{625}$

8. (B)

Total number of ways in which two cards are drawn out of 52 is ${}^{52}C_2$.

A king can be drawn in 4 ways and Queen can be drawn in 4 ways.

Required probability = $\frac{4 \times 4}{{}^{52}C_2} = \frac{8}{663}$

9. (D)

In a single throw of two dice, the probability of throwing a doublet is

$\frac{6}{36} = \frac{1}{6}$

$\therefore$ The probability of throwing three doublets in three throws is

$\frac{1}{6}.\frac{1}{6}.\frac{1}{6} = \frac{1}{216}$

$\therefore$ Hence the required probability

$= 1 - \frac{1}{216} = \frac{215}{216}$

10. (B)

Here $np + npq = 10$, $n = 18$

$18p + 18pq = 10 \Rightarrow 18p(1 + q) = 10$

$\therefore \quad p + q = 1 \Rightarrow p = 1 - q$

$18(1 - q)(1 + q) = 10$

$1 - q^2 = \frac{10}{18}$

$-q^2 = \frac{10}{18} - 1 = \frac{-8}{18}$

$q^2 = \frac{8}{18} = \frac{4}{9}$

$q = \pm\frac{2}{3}$

$\therefore \quad q = \frac{2}{3}$ (–ve sign rejected),

$p = \frac{1}{3}$

Hence Binomial distribution is

$\left(\frac{1}{3} + \frac{2}{3}\right)^{18} \{(p + q)^n\}$

11. (D)

12. (A)

13. (C)

14. (B)

Out of the 5 women, 3 women can be invited in ${}^{5}C_3$ ways. Nothing is mentioned about the number of men that he has to invite. He can invite one, two, three, four or even number of men. Out of 4 men, he can invite them in the said manner in $(2)^4$ ways. Thus, the total number of ways is ${}^5C_3 \times (2^4)$ $= 10 \times 16 = 160$.

15. (D)

$P(A) = 1/4$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{2}$$

$$\therefore\ P(A \cap B) = \frac{1}{8}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4}$$

$$\therefore\ P(B) = \frac{1}{2}$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= \frac{1}{4} - \frac{1}{8}$$

$$P(A \cap \bar{B}) = \frac{1}{8}$$

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{1/8}{1/2} = \frac{1}{4}$$

$\therefore\ P(A/\bar{B}) = \frac{1}{2}$ is false.

If $A \subset B$, $A \cap B = A$

then $P(A \cap B) = P(A)$

but $P(A \cap B) = \frac{1}{8}$ and $P(A) = \frac{1}{4}$

$\therefore$ This is false.

$$P(A/B) + P(A/\bar{B}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$\therefore$ $P(A/B) + P(A/\bar{B}) = 1$ is false

$\therefore$ None of the options (A), (B) or (C) is true.

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