# 3.1 Boundary-Layer Flow

## Question 1

We aim to analyse the Falkner Skan Equation, which is

$$m(f')^2 - \frac{1}{2}(m+1)ff'' = m + f''' \tag{1}$$

where $f = f(\eta)$ and boundary conditions

$$f' = f = 0 \text{ on } \eta = 0,\ f' \to \text{sgn } A \text{ as } \eta \to \infty \tag{2}$$

In particular, we aim to analyse the different possible terminal behaviours as $\eta \to \infty$ of the solution to (1).

For most of this project, we concentrate on the case $A > 0$.

**A Very Important Interlude**

We shall do a rather important analysis that we will reference throughout this project.

Consider the case of $m \neq -1$. Let $\xi = \alpha\eta$ for some constant $\alpha > 0$. Then (1), under this change of variable becomes

$$\alpha^2 ff'' + \frac{2}{m+1}\alpha^3 f''' + \beta(1 - \alpha^2(f')^2) = 0$$

, where $\beta = \frac{2m}{m+1}$. Let $g = \alpha f$ such that

$$gg'' + \frac{2}{m+1}\alpha^2 g''' + \beta(1 - (g')^2) = 0$$

Further, the boundary conditions are accordingly changed such that $g = g' = 0$ at $\xi = 0$, and $g' \to 1$ as $\xi \to \infty$.

We shall do a clever substitution! First consider the case $m > -1$. Let $\alpha = \sqrt{\frac{1+m}{2}}$. Then, from above we get that

$$gg'' + g''' + \beta(1 - (g')^2) = 0 \tag{3}$$

Now consider the case of a perturbed solution, ie for small $\epsilon$

$$g = g_0 + \epsilon g_1 + O(\epsilon^2)$$

, where $g_0$ is a solution to (3).

Therefore using this, expand (3) up to linear order in $\epsilon$. Further use the fact that as $\xi \to \infty$, $g_0 \to 1$. This implies that $y = g_1'$ (this is different from the $y$ in project brief, and is used here solely for convenience) asymptotically (as $\xi \to \infty$) satisfies the following ODE:

$$y'' + \xi y' - 2\beta y = 0$$

Now, further use the substitution $y = we^{\frac{-\xi^2}{4}}$. This leads the following ODE

$$w'' - \left(\frac{1}{4}\xi^2 + (2\beta + \frac{1}{2})\right) = 0$$

This is an ODE solved by Parabolic Cylinder Functions! (as found in Appendix A of [3]).

Using the asymptotics of Parabolic Cylinder Functions that solve the above ODE, we can deduce that, asymptotically, as $\xi \to \infty$, we must have

$$g' \sim 1 + a_0 e^{\frac{-\xi^2}{2}} \xi^{-(1+2\beta)} + a_1 \xi^{2\beta} + \cdots \tag{4}$$

Here, using details in Appendix A of [3], we note that if $\beta \geq 0$, we have $a_1 = 0$ (to satisfy the boundary condition), and if $\beta < 0$, there are no restrictions on $a_0$ and $a_1$.

Using above notation, but extending the analysis for $m < -1$, we note that instead of equation 3, we have (with $\alpha^2 = \frac{-(1+m)}{2}$)

$$-gg'' + g''' - \beta(1 - (g')^2) = g''' - gg'' + \gamma(1 - (g')^2) = 0 \tag{5}$$

with same boundary conditions, and $\gamma = -\beta$. Notice, in this case $\beta > 0$, ie $\gamma < 0$. Using analogous analysis as above, we notice that the asymptotic solution is, for any value of $\gamma$ is (ignoring the boundary condition temporarily):

$$g' \sim 1 + a_0 \xi^{-2\gamma} + a_1 \xi^{2\gamma-1} e^{\frac{\xi^2}{2}}$$

However, as $m < -1$, so $\gamma < 0$. But then, there are no values of $a_0, a_1$ we can choose such that the above asymptotic solution satisfies the boundary condition as $\xi \to \infty$. So there exists no solutions to (1) with boundary conditions (2) for $m < -1$. The case of $m < -1$ shall, thus, be avoided throughout this report.

Finally to complete our analysis, we notice that if $m = -1$, we are concerned with the simple ODE

$$y''' - (1 - (y')^2) = 0. \tag{6}$$

This leads to

$$f' \sim 1 + a_0 e^{-\lambda_1 \eta} + a_1 e^{-\lambda_2 \eta} + \cdots$$

where $\lambda_1, \lambda_2$ are appropriate constants, which in this case are complex. That is, asymptotically periodic solutions!

**(i) Algebraic Convergence**

Algebraic convergence means that there is some constant $B \neq 0$ and $k > 0$ such that

$$f' = 1 + B\eta^{-k} + \cdots \text{ as } \eta \to \infty$$

Clearly this is a possible behaviour exhibited by a solution to (1) with boundary conditions (2).

**(a) Which, if any, of the constants can be determined without using the conditions at $\eta = 0$?**

In this case, there are few possibilities for the asymptotic series solution of $f$ as $\eta \to \infty$. Upto leading order:

$$f' \sim 1 + B\eta^{-k} \cdots$$

$$f'' \sim -kB\eta^{-(k+1)} + \cdots$$

$$f'' \sim k(k+1)B\eta^{-(k+2)} + \cdots$$

Then, upto leading order, (1) implies

$$m + \eta^{-k}\left(2Bm + \frac{1}{2}(m+1)kB\right) + O(\eta^{-(k+1)}) = m + O(\eta^{-(k+2)})$$

This implies that

$$B(4m + (m+1)k) = 0$$

As $B \neq 0$ and $k > 0$, we get that for $-1 < m < 0$ with

$$k = -\frac{4m}{m+1}$$

More importantly, this analysis helps us realise that we can, without using the condition at $\eta = 0$, evaluate the value of $k$. However, this analysis also shines a light on the fact that although we can use our asymptotic series solution for $f'$ to get recurrence relation between coefficients of the asymptotic series, we can't explicitly evaluate $B$ as above using just the information given (and not using the boundary condition at $\eta = 0$) for any arbitrary value of $m$ (as that is hard).

However, once we know the value of $m$ and thus $k$, we may be able use these recurrence relations to obtain $B$ (however, only when $m \neq -1$. But as we will see soon below, the case of $m = -1$ is useless for algebraic convergence).

Note that the value of $k$ above agrees with the value of $k$ we would obtain if we follow the analysis in the `Very Important Interlude` section (!). We conclude that we can determine the constants $k$ and $B$ without using the boundary condition at $\eta = 0$.

**(b) What restrictions (if any) must be placed on the value of m for this type of behaviour to be possible?**

By our analysis in the Q1(i)(a), and noting that $k > 0$, we notice that as $k = \frac{-4m}{m+1}$, algebraic convergence is possible only when $-1 < m < 0$.

This analysis is in fact supported by the important interlude earlier, as for $m = -1$, we have asymptotically periodic solution which do not exhibit algebraic convergence. And for $m > -1$, we note that the only way to have algebraic convergence is when $a_1 \neq 0$, which is possible when $\beta < 0$. So algebraic convergence possible if and only if $\beta < 0$, which happens if and only if $-1 < m < 0$, thus, supporting our analysis above!

**(ii) Exponential Convergence**

We have constants $\eta_0, k', k''$ and $k \neq 0$, such that if $\xi = \eta - \eta_0$,

$$f \sim \eta + e^{-\sigma(\xi)} + \cdots \text{ as } \eta \to \infty$$

where $\sigma(\xi)$ such that

$$\sigma'(\xi) = k\xi + k' + k''\xi^{-1} + O(\xi^{-2}).$$

This definition of $\sigma$ gives us that, asymptotically,

$$\sigma(\xi) \sim \frac{k}{2}\xi^2 + k'\xi + k'' \log \xi + C + O(\xi^{-1})$$

where $C$ is a constant.

This implies that (if we let $D = e^{-C} > 0$, another constant):

$$\begin{aligned} f' &\sim 1 - \sigma'(\xi)e^{-\sigma(\xi)} + \cdots \\ &\sim 1 - D(k\xi + k' + k''\xi^{-1} + O(\xi^{-2}))\xi^{-k''} e^{(-\frac{k}{2}\xi^2 - k'\xi)} + \cdots \\ &\xrightarrow{\eta \to \infty} 1 \end{aligned}$$

So it is a possible solution to (1) with the boundary conditions (2).

**(a) Which, if any, of the constants can be determined without using the conditions at $\eta = 0$?**

The above analysis suggests that asymptotically,

$$f \sim \eta + \left(D\eta^{-k''} + \cdots\right) e^{(-\frac{k}{2}\xi^2 - k'\xi)}$$

Then, up to leading order:

$$\begin{aligned} f' &\sim 1 - (Dk\xi^{-k''+1} + \cdots)e^{(-\frac{k}{2}\xi^2 - k'\xi)} \\ f'' &\sim (Dk^2\xi^{-k''+2} + \cdots)e^{(-\frac{k}{2}\xi^2 - k'\xi)} \\ f''' &\sim -(Dk^3\xi^{-k''+3} + \cdots)e^{(-\frac{k}{2}\xi^2 - k'\xi)} \end{aligned}$$

Using this, and substituting into (1), we can solve for the coefficients in our asymptotic series, and thus get the values $k, k', k''$. However, we can't obtain the value of $\eta_0$ without using the boundary condition at $\eta = 0$.

A keen reader might notice and try comparing the asymptotic solution to the asymptotic solution in the `Very Important Interlude` section, and note that this implies $k' = 0$ and $k = \alpha^2$, and $k'' = 2\beta$, but we still can't determine the value of $\eta_0$ using only the boundary condition at $\eta = \infty$.

**(b) What restrictions (if any) must be placed on the value of m for this type of behaviour to be possible?**

Using the analysis in the `Very Important Interlude` section (!), we notice that if $m = -1$, we have periodic solutions, which do not exhibit exponential convergence. For $m > -1$, we notice that exponentially convergent solution is possible regardless of the value of $\beta$ (ie in both cases $\beta \geq 0$ and $\beta < 0$). This implies that exponential convergent solution is always (!) possible.

### (iii) Algebraic Divergence

In the algebraic divergence case, for constants $B \neq 0$ and $k > 0$:

$$f \sim B\eta^{1+k} + \cdots \text{ as } \eta \to \infty.$$

It is trivially clear that if a solution of Falkner-Skan Equation exhibits this behaviour, then it can't satisfy one of the boundary conditions in (2), particularly the one as $\eta \to \infty$

**(a) Which, if any, of the constants can be determined without using the conditions at $\eta = 0$?**

One might be interested in Falkner-Skan Equation, without the boundary condition at $\eta = \infty$. Then we might have the following asymptotic expression:

$$f \sim \sum_{n=-1}^{\infty} a_n \eta^{-n+k}$$

where $a_{-1} = B$. Then

$$f' \sim \sum_{n=0}^{\infty} a_{n-1}(k-n+1)\eta^{k-n}$$

$$f' \sim \sum_{n=1}^{\infty} a_{n-2}\eta^{k-n}(k-n+2)(k-n+1)$$

$$f' \sim \sum_{n=2}^{\infty} a_{n-3}\eta^{k-n}(k-n+3)(k-n+2)(k-n+1)$$

This implies that the LHS in (1) then, up to leading order:

$$\text{LHS} \sim B^2(k+1)\left(m(k+1) - \frac{1}{2}(m+1)k\right)\eta^{2k} + O(\eta^{2k-1})$$

Similarly, for RHS in (1), we obtain

$$\text{RHS} \sim m + B(k-1)k(k+1)\eta^{k-2} + \cdots$$

Since $k > 0$, we have that the leading term on the RHS is of constant order or of order $k-2$(!). Furter, leading term on the LHS is of order $2k$. By asymptotic matching, thus, we require $k - 2 = 2k$, ie $k = -2$, which is not possible, or $k = 0$, which is again not possible as $k > 0$. Thus the leading term on the LHS must have coefficient 0. This implies that (as $B \neq 0$ and $k > 0$):

$$m(k+1) - \frac{1}{2}(m+1)k = 0.$$

This thus implies that $\text{sgn}(m) = \text{sgn}(m+1)$, and so we must have $m > 0$ (as we are ignoring the case of $m < -1$ throughout the project). In this case, we also obtain that $\frac{2m}{m+1} = \frac{k}{k+1} < 1$, thus $m < 1$. In this case, then:

$$k = \frac{2m}{1-m}.$$

From this analysis, it is clear that we can obtain the value of the constant $k$, but not $B$ without the boundary condition at $\eta = 0$.

**(b) What restrictions (if any) must be placed on the value of m for this type of behaviour to be possible?**

By our analysis above, we have that $k = \frac{2m}{1-m}$, with $m > 0$ and $m < 1$. So for this type of behaviour to be possible, we require that $0 < m < 1$

**(iv) Exponential Divergence**

In the exponential divergence case, for constants $B \neq 0$ and $k > 0$:

$$f \sim Be^{k\eta} + \cdots \text{ as } \eta \to \infty.$$

It is trivially clear that if a solution of Falkner-Skan Equation exhibits this behaviour, then it can't satisfy one of the boundary conditions in (2), particularly the one as $\eta \to \infty$

**(a) Which, if any, of the constants can be determined without using the conditions at $\eta = 0$?**

We might guess the following asymptotic expression for $f$ as $\eta \to \infty$

$$f \sim Be^{k\eta} + \cdots$$

Then, upon substituting the above asymptotic expansion into (1) and analysing the solution as $\eta \to \infty$:

$$\left(B^2k^2\left(m - \frac{1}{2}(m+1)\right)\right)e^{2k\eta} + \cdots = \left(Bk^3\right)e^{k\eta} + \cdots$$

Simplifying the left hand side, we obtain

$$\left(B^2k^2\frac{1}{2}(m-1)\right)e^{2k\eta} + \cdots = \left(Bk^3\right)e^{k\eta} + \cdots$$

It quickly becomes apparent that we can not explicitly determine $B$ and $k$ in such a case without the boundary condition at $\eta = 0$. Furthermore, this asymptotic expression makes sense if and only if the coefficient of $e^{2k\eta}$ on the LHS is 0 (as $B \neq 0$, $k > 0$, so nothing else is possible). But this is possible if and only if $m = 1$.

**(b) What restrictions (if any) must be placed on the value of m for this type of behaviour to be possible?**

From the above analysis, we deduce that the only restriction on $m$ for this type of behaviour to be possible is that $m = 1$.

**(v) A Finite Distance Singularity**

In this case, as $\eta \to \eta_0$, we have that

$$f \sim B(\eta_0 - \eta)^{-1} + \cdots$$

, where $B \neq 0$ and $\eta_0$ are constants.

**(a) Which, if any, of the constants can be determined without using the conditions at $\eta = 0$?**

This suggests an asymptotic expansion, as $\eta \to \eta_0$:

$$f \sim B(\eta_0 - \eta)^{-1} + C + O(\eta_0 - \eta)$$
$$f' \sim B(\eta_0 - \eta)^{-2} + \cdots$$
$$f'' \sim 2B(\eta_0 - \eta)^{-3} + \cdots$$
$$f''' \sim 6B(\eta_0 - \eta)^{-4} + \cdots$$

Then substituting into (1), and analysing the behaviour of the solution as $\eta \to \eta_0$, we obtain, up to leading order, that:

$$-B^2(\eta_0 - \eta)^{-4} + \cdots = 6B(\eta - \eta_0)^{-4} + \cdots$$

This suggests that we must have

$$B^2 + 6B = 0$$

Since $B \neq 0$, this implies $B = -6$. So we found the value of the constant $B$ without using the boundary condition at $\eta = 0$!

**(b) What restrictions (if any) must be placed on the value of m for this type of behaviour to be possible?**

There are no restrictions places on value of $m$ for this type of behaviour to be possible by our analysis above.

## Programming Task

We aim to write a program to integrate the Falkner-Skan Equation (1) subject to

$$f(0) = f'(0) = 0,\ f''(0) = S$$

where S is a known constant. We use the black-box numerical integrator `odeint` in the `scipy` library. The program source code can be found in the program listings in the appendix. We use relative and absolute tolerance of $1e-8$ throughout the entire project (unless mentioned otherwise).

## Question 2

Consider $m = 0, S = 1$. We observe that $f'$ converges to a constant as $\eta \to \infty$. Let this constant be $f_0'$. One observes that this convergence is exponential convergence, as demonstrated by the roughly straight line on log-linear plot below for $\log(f_0' - f'(\eta))$ vs $\eta$. The plot of $f'(\eta)$ vs $\eta$ below further demonstrates this exponential convergence.

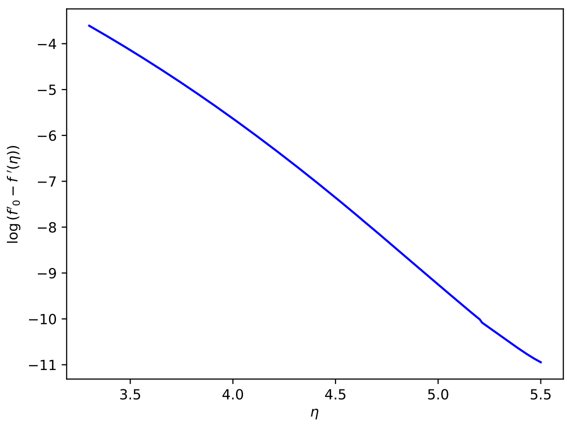


Figure 1: Log-Linear plot of $\log(f_0' - f'(\eta))$ vs $\eta$. The (almost) straight line demonstrates exponential convergence.

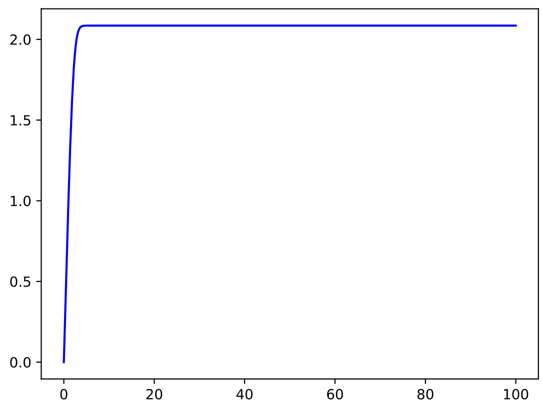


Figure 2: Plot of $f(\eta)$ vs $\eta$

Further, we note some of the tabulated values of $f'(\eta)$, when we integrated from $\eta = 0$ to $\eta = 20$. We determine that

$$f_0' = 2.0854$$

, to at least 5 significant figures. The table illustrates evidence of this accuracy being achieved, as this value of $f_0$ is sustained for multiple values of $\eta$ near the end (and to a higher accuracy than we claim here). This is further supported by the fact that we set our relative and absolute tolerances to $1e-8$, which guarantees **at least** four significant figures of accuracy. To further aid in our evidence, we integrated till $\eta = 30$ to observe that $f_0' = 2.0854$ to 5 significant figures still.

| $\eta$ | $f'(\eta)$ |
|---|---|
| 0 | 0 |
| 0.04 | 0.03999968873457803 |
| ... | ... |
| 19.88 | 2.0854000298138207 |
| 19.92 | 2.0854000298129023 |
| 19.96 | 2.085400029811984 |
| 20.0 | 2.0854000298110655 |

Table 1: Some of the calculated numerical values of $\eta$ and $f'(\eta)$, for a run from $\eta = 0$ to $\eta = 20$

We notice that without further numerical computation, we can deduce a solution to the BVP (1) and (2) for $m = 0$. With $m = 0$, we notice that (1) becomes

$$f''' + \frac{1}{2} f f'' = 0$$

Suppose $f$ is the solution we numerically computed above. Suppose further that $f$ satisfies the above ODE and the boundary conditions $f(0) = f'(0) = 0$, and $f' \to f_0$ as $\eta \to \infty$. Let $g(\eta) = af(b\eta)$ for constants $a, b$ we need to find, such that $g$ satisfies the same ODE as $f$, but further $g(0) = g'(0) = 0$ and $g' \to 1$ as $\eta \to 1$.

Plugging the definition of $g$ into the ODE satisfies by $f$, we obtain that $g$ satisfies

$$g''' + \frac{b}{2a} g g'' = 0.$$

Since we want $g$ to satisfy the same ODE as $f$, we require $b = a$. Furthermore, the required boundary conditions at 0 are clearly satisfied by $g$, as they are satisfied by $f$. For the boundary condition as $\eta \to \infty$, we note that $g'(\eta) = a^2 f'(a\eta)$, and so taking limits for both sides, we notice that we require

$$1 = a^2 f_0'$$

. Since we have defined $f$ for non-negative $\eta$, we require $a > 0$, and so

$$a = \frac{1}{\sqrt{f_0'}}$$

, and so

$$g(\eta) = \frac{1}{\sqrt{f_0'}} f\left(\frac{\eta}{\sqrt{f_0'}}\right)$$

satisfies the BVP (1) and (2).

Furthermore,

$$g''(\eta) = \frac{1}{\sqrt{(f_0')^3}} f''\left(\frac{\eta}{\sqrt{f_0'}}\right)$$

, thus

$$g''(0) = \frac{S}{(f_0')^{3/2}} = \frac{1}{(f_0')^{3/2}}$$

## Question 3

Now we intend to use the shooting method for $0 \leq m \leq 1$. Trying for multiple values, we notice that for each $m$ in this range, there exists a unique value of $S$, call it $S_m$, for which $f' \to 1$ as $\eta \to \infty$. We plot graphs of $f'$ for various values of $S$ for $m = \frac{2}{5}$ and $m = 1$ below (both less than and greater than $S_m$).

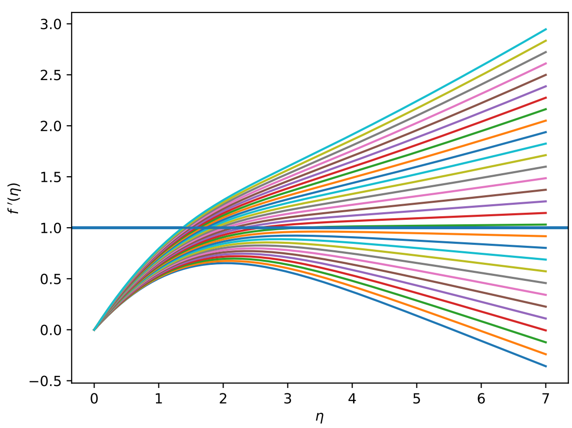


Figure 3: Plots for $S \in [0.7, 1]$

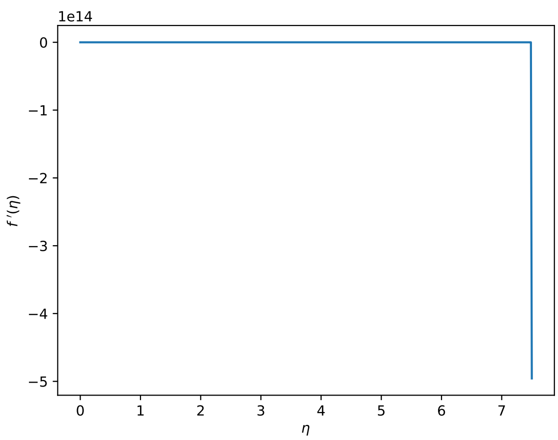


Figure 4: Plot for solution with finite distance singularity, $S = 0.4$

Plots of $f'(\eta)$ vs $\eta$ for various values of $S$, $m = \frac{2}{5}$

We can observe variety of terminal behaviour in the above figures 3 and 4. For the case of $m = \frac{2}{5}$, we observe that the solution at $S = S_m$ exhibits exponential convergence. This is further demonstrated by the plot for the same, which can be found below. Furthermore, we notice there are some solutions in this case, for different values of S, that exhibit algebraic divergence (which we observe in the plot above). This is further validated by choosing certain values of S, and looking at the Log-Log plots (the straight line would suggest algebraic divergence). These plots for specific values can be found below. Furthermore, another possible terminal behaviour is a finite-distance singularity, as can be seen in a plot above. A keen reader might also notice that these possible terminal behaviours agree with our earlier analysis in Question 1.

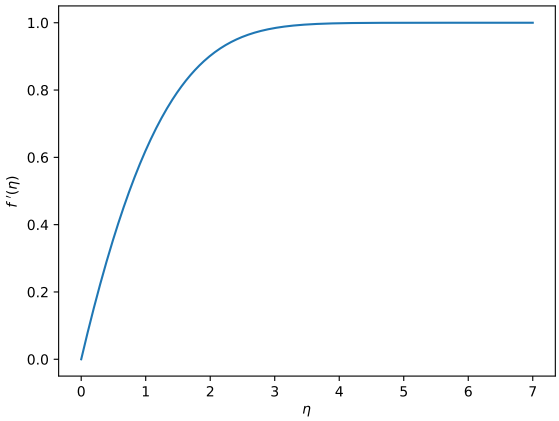


Figure 5: Exponential Convergence, $S = S_{\frac{2}{5}}$

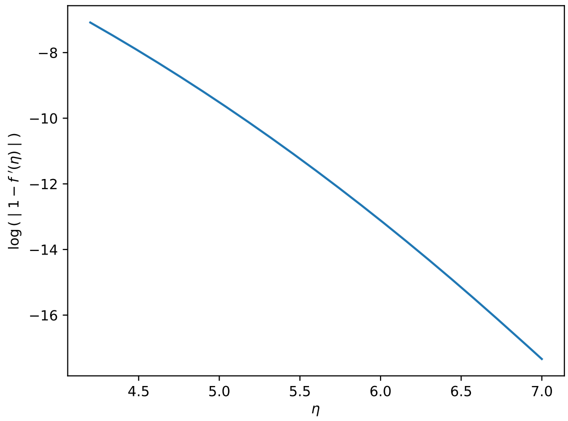


Figure 6: Log-Linear plot to exhibit exponential convergence

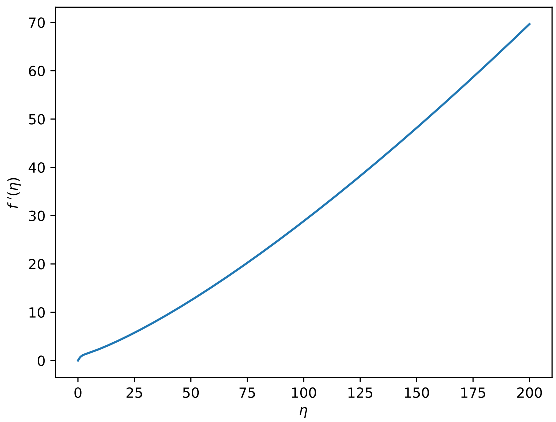


Figure 7: Algebraic Divergence, $S = 0.9$

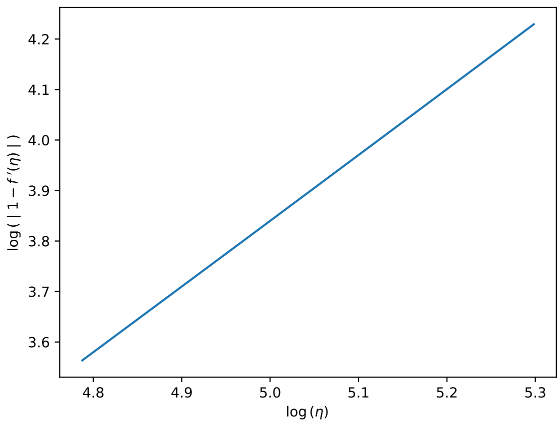


Figure 8: Log-Log plot to exhibit algebraic divergence

Plots to exhibit various behaviours for the case $m = \frac{2}{5}$

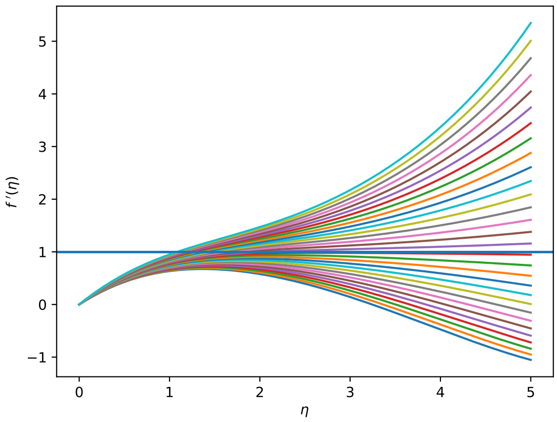


Figure 9: Plots for $S \in [1.1, 1.4]$

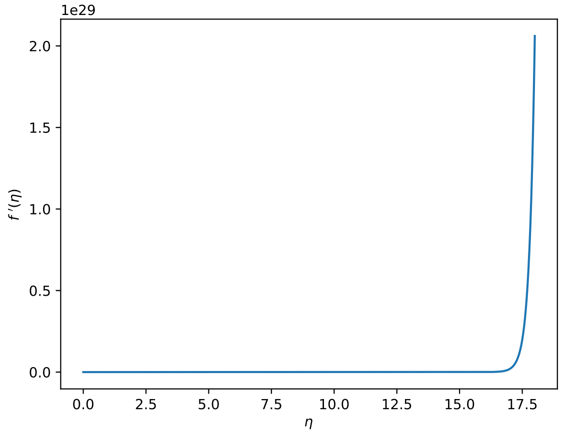


Figure 10: Plot for a solution with a finite distance singularity, $S = 0.3$,

Plots of $f'(\eta)$ vs $\eta$ for various values of $S$, $m = 1$

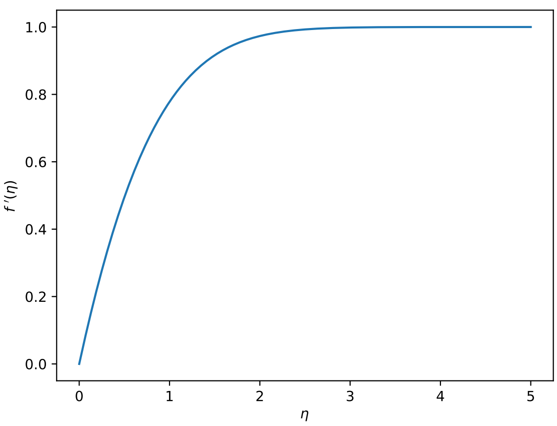


Figure 11: Exponential Convergence, $S = S_1$

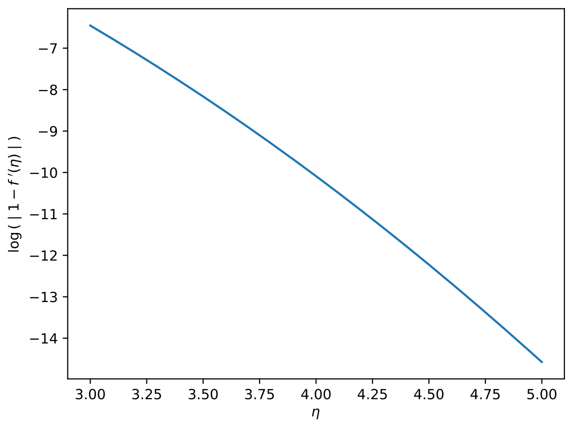


Figure 12: Log-Linear plot to exhibit exponential convergence

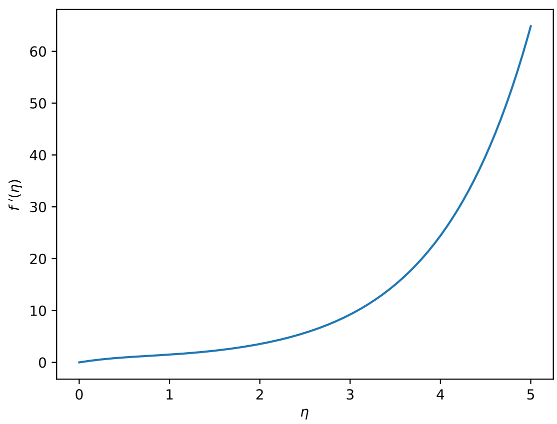


Figure 13: Exponential Divergence, $S = 1.4$

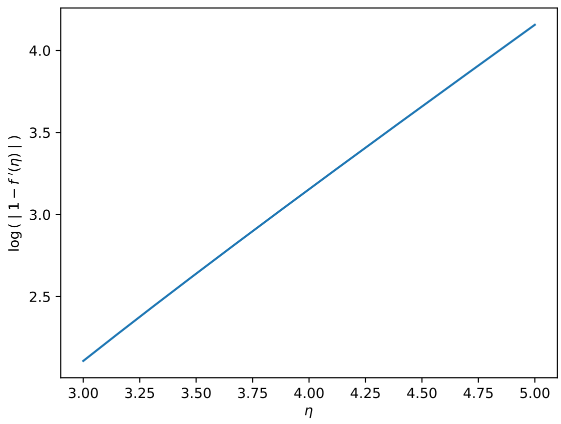


Figure 14: Log-Linear plot to exhibit exponential divergence

Plots to exhibit various behaviours for the case $m = 1$

For the case of $m = 1$, we notice that solutions exhibit either exponential convergence (at $S = S_m$), exponential divergence or a finite-distance singularity. To demonstrate exponential convergence/divergence, we use Log-Linear plots, and observe that we have straight lines, again. These plots can be found above.

We further wrote a program to calculate the value of $S_m$, which can be found in the program listings in the appendix. We use a black-box root-finder, `fsolve`, found in the `scipy` library. Using this, we tabulate various values of $S_m$ for $0 \le m \le 1$ below.

| $m$ | $S_m$ |
|---|---|
| 0.0 | 0.33205732973880314 |
| 0.1 | 0.4965715376642754 |
| 0.2 | 0.6213238198636344 |
| ... | ... |
| 0.8 | 1.1113618974795623 |
| 0.9 | 1.17353070910866 |
| 1.0 | 1.232587680335879 |

Table 2: Some of the calculated values for $S_m$ for $0 \le m \le 1$

Furthermore, we plot $m$ vs $S_m$ for $0 \le m \le 1$ below.

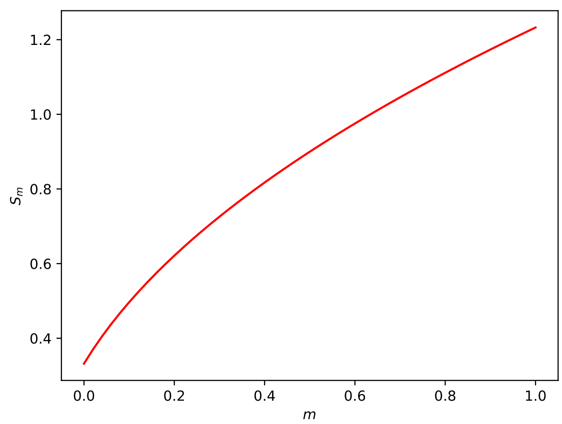


Figure 15: $m$ vs $S_m$ for $0 \le m \le 1$

To plot, we choose $\eta$ up to 20 (as it is large enough to exhibit long-term behaviour well to at least 4 significant figures in the case of exponential convergence (as we had observed in the case of $m = 0, S = 1$ earlier)), and tolerances of $1e - 8$ (in both, the numerical integrator and the black-box root finder). We are satisfied with the accuracy above (which we claim is accurate to at least 4 significant figures) due to our choice of tolerance, particularly for `fsolve`, and our choice of the upper limit for the integration. Further evidence can be found below where we have tabulated the value of $S_m$ for various different initial values of $S$, $S_{\text{initial}}$, in the case of $m = \frac{2}{5}$ as an example, and observe that the variance suggests accuracy of at least 4 significant figures.

| $S_{\text{initial}}$ | $S_m$ |
|---|---|
| 0.8 | 0.8172703247234739 |
| 0.9 | 0.8172703246852807 |
| 1.0 | 0.8172703246842572 |
| 1.2 | 0.8172703246858448 |
| 1.4 | 0.8172703246855043 |
| 2.0 | 0.817270324691836 |

Table 3: Tabulated values of $S_m$ for various values of $m$, $0 \leq m \leq 1$

If we increase $m$, we observe from the plot above that $S_m = f''(0)$ increases. We further recall that in our problem, $A > 0$ and for $x > 0$,

$$U_e(x) = Ax^m.$$

So as $m$ increase, $U_e(x)$ increases more rapidly as $x \to \infty$. So throughout the boundary layer, as $x \to \infty$, we require the tangential velocity to increase rapidly to match the external flow, and this increase has to be increasingly rapid as $m$ increases.

Furthermore, the tangential stress exerted by the fluid on the boundary $y = 0$ is such that it satisfies:

$$\tau_0 \propto f''(0)x^{\frac{3m-1}{2}}$$

For $m \geq \frac{1}{3}$, as $m$ increases, $\tau_0$ increases more rapidly as $x \to \infty$. But this makes physical sense, as we observed above that the tangential velocity in the boundary layer needs to increase rapidly to match the external flow, and this increase has to be increasingly rapid as $m$ increases. So there is greater angle of steepness or greater velocity gradient at the rigid boundary as $m$ increases (and also as $x$ increases), thus greater tangential stress at the rigid boundary at $y = 0$!

For $0 \leq m < \frac{1}{3}$, we note that as $x \to \infty$,

$$\left|U_e(x)\right|^3 \sim x^{3m} < x^1$$

But this implies that $\tau_0$ decays as $x \to \infty$ in this case, but still as $m$ increases, $\tau_0$ decreases less rapidly as $x \to \infty$. This makes physical sense, though, as for $m < 1/3$, we have flow past a wedge with semi-angle of $\theta < \frac{\pi}{4}$, and so we have a singularity for tangential stress at $x = 0$. Thus it makes sense to expect $\tau_0$ to decrease as $x \to \infty$. Also, our earlier analysis (and physical interpretation) of rate of decay decreasing/rate of growth increasing for $\tau_0$ as $m$ increases extends to this case all the same.

## Question 4

We are now concerned with the cases of $-1 < m < 0$. We illustrate the different terminal behaviours for these values of $m$. The possible different terminal behaviours for these values of $m$ are : exponential convergence, algebraic convergence, and a solution with finite distance singularity. These are illustrated below for various values of $m$ and $S$. These possible terminal behaviours, in fact, agree with our predictions in Question 1 as the only possible terminal behaviours to solutions of the Falkner-Skan equation.

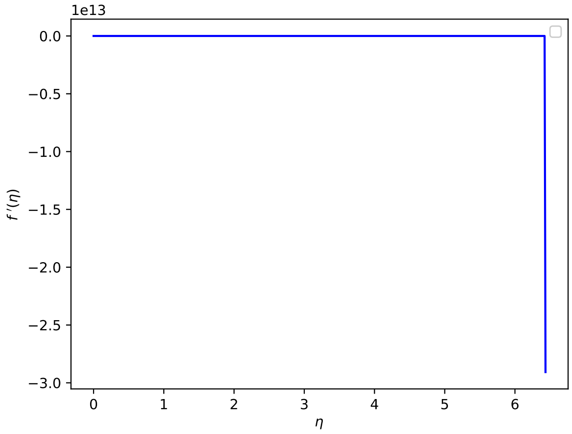


Figure 16: Plots for a solution with finite distance singularity, $m = -0.2$, $S = -0.5$

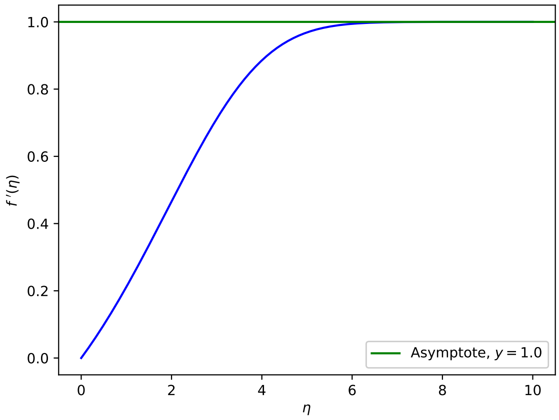


Figure 17: Plots of $f'(\eta)$ vs $\eta$

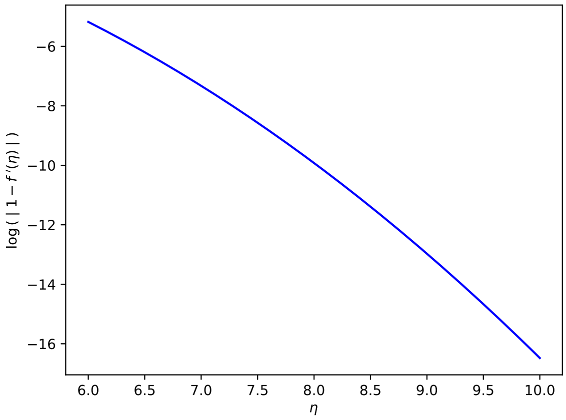


Figure 18: Log-Linear Plot

Plots to demonstrate a solution exhibiting exponential convergence, $m = -0.06$, $S = 0.18256167517446925$

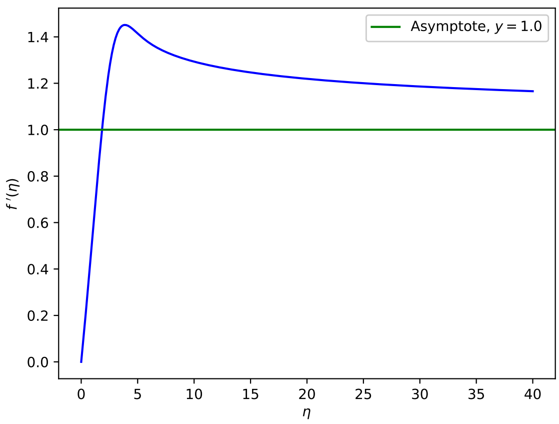


Figure 19: Plots of $f'(\eta)$ vs $\eta$

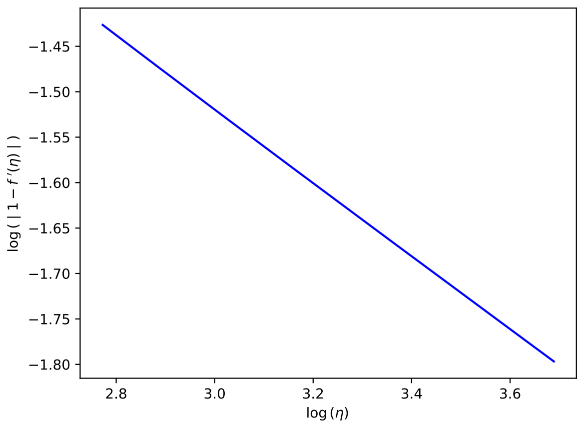


Figure 20: Log-Log Plot

Plots to demonstrate a solution exhibiting algebraic convergence, $m = -0.1$, $S = 0.5$

We observe that there exists a value $m_c$ such that for $m_c < m < 0$, there are two branches of exponentially converging solutions, one of which is a continuation of that already found for $0 \leq m \leq 1$ in Question 3. To determine $m_c$ up to 4 significant figures, we use 0.000001 increments and find (for the first branch) the first value of $m$ such that $S_m = 0$ numerically. The fact that at $m = m_c$, we expect $S_m = 0$ arises from the numerical observations we make when finding values of $S_m$ for $-1 < m < 0$, and noting that (as will be further explored later), at $m = m_c$, we would expect $f''(0) = 0$ due to creation of a seperation profile. This is observed in the table below (by noting that for $m < m_c$, we might expect $S_m$ to start increasing or become negative). This method suggests that, to 4 significant figures,

$$m_c = 0.09043$$

In fact, our evidence suggests that $m_c \approx 0.090429$.

| $m$ | $S_m$ |
|---|---|
| $\dots$ | $\dots$ |
| $-0.090434$ | $-6.683769926367714e-06$ |
| $-0.090433$ | $-6.026697883036959e-05$ |
| $-0.090432$ | $0.00017843349603111026$ |
| $-0.090431$ | $4.732127930300189e-054$ |
| $-0.09043$ | $4.74407843077477e-05$ |
| $-0.090429$ | $3.724959726594514e-05$ |
| $-0.090428$ | $0.0006724218238122436$ |
| $-0.090427$ | $0.0011135123308174697$ |
| $-0.090426$ | $0.0014239185245522978$ |
| $\dots$ | $\dots$ |

Table 4: Tabulated values of $S_m$ for the first branch

| $m$ | $S_m$ |
|---|---|
| $0.0$ | $0.3320573297388055$ |
| $\dots$ | $\dots$ |
| $-0.02$ | $0.289639108911829$ |
| $\dots$ | $\dots$ |
| $-0.04$ | $0.24109925277685265$ |
| $\dots$ | $\dots$ |
| $-0.06$ | $0.18256167517447275$ |
| $\dots$ | $\dots$ |
| $-0.09043$ | $4.74407843077477 \times 10^{-5}$ |

Table 5: Tabulated values of $m$ and $S_m$

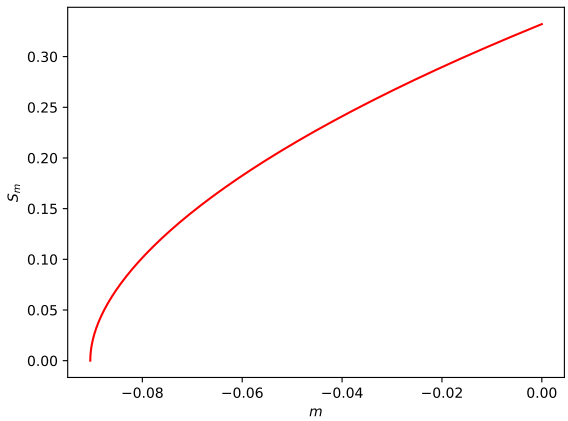


Figure 21: Plot of $S_m$ vs $m$

Data for the first branch, $m_c < m < 0$

| $m$ | $S_m$ |
|---|---|
| $-1 \times 10^{-5}$ | $-4.3864862126573245 \times 10^{-06}$ |
| $\dots$ | $\dots$ |
| $-0.0001$ | $-0.0016074694310685572$ |
| $\dots$ | $\dots$ |
| $-0.02$ | $-0.09766995977426081$ |
| $\dots$ | $\dots$ |
| $-0.04$ | $-0.13401870657434034$ |
| $\dots$ | $\dots$ |
| $-0.06$ | $-0.14215318900642832$ |
| $\dots$ | $\dots$ |
| $-0.09043$ | $-6.816825847537136 \times 10^{-6}$ |

Table 6: Tabulated values of $m$ and $S_m$

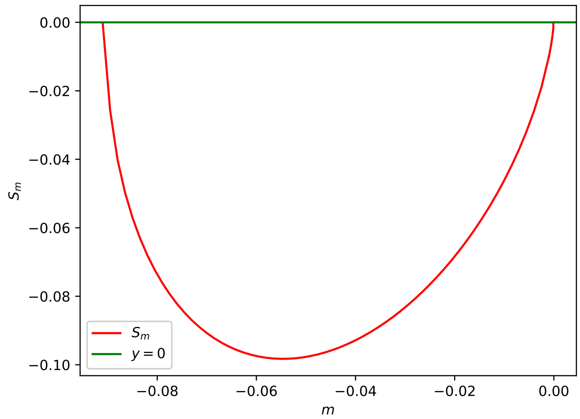


Figure 22: Plot of $S_m$ vs $m$

Data for the second branch, $m_c < m < 0$

To tabulate values of $S_m$ in the second branch, we notice that we would require that $S_m \leq 0$ in this branch. Using this condition, we modify our program slightly (as can be seen in the program listings in the appendix) to obtain $S_m$ values in the second branch. We also note that due to the values becoming too small as $m \to 0$, we decrease our steps in $m$ near 0, and use a different initial value of $S$. We can clearly extrapolate and notice that $S_m \to 0$ as $m \uparrow 0$.

We further notice that both these branches come together at $m = m_c$, where on both branches $S_m = 0$.

Further, the figures below displays plots of $f'$ vs $\eta$ for a particular value of $m$, particularly $m = -0.06$. It shows two different plots, to display the two different branches of solutions.

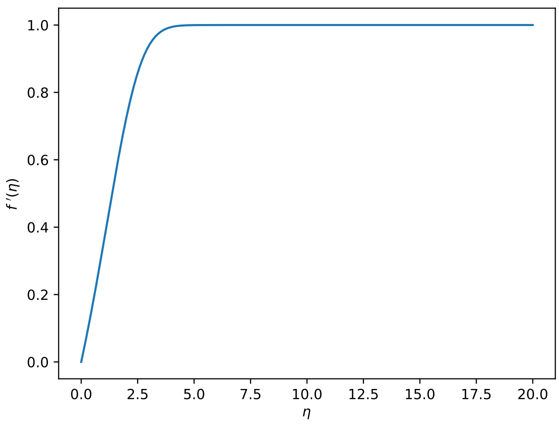


Figure 23: First branch solution, $m = -0.05$, $S_m = 0.21348374500963493$

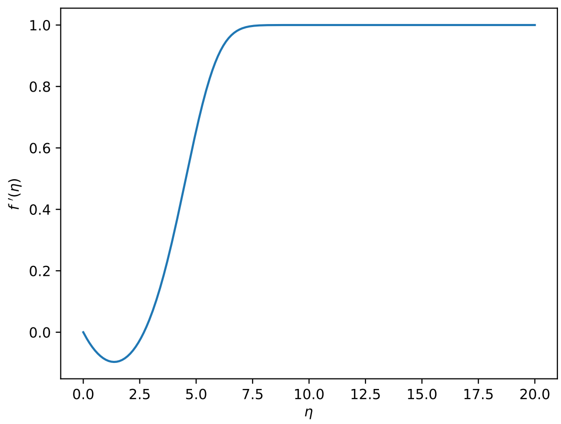


Figure 24: Second branch solution, $m = -0.05$, $S_m = -0.09770471439834048$

Plots to demonstrate two different solution exhibiting exponential convergence, $m = -0.05$

We notice that for different values of in the second branch, as $m \uparrow 0$, we notice that the point where the solution becomes positive, call it $x_0$, increases as $m \uparrow 0$. Further, the negative part of the solution seems to flatten as $m \uparrow 0$. This trend can be observed, for instance, by choosing values of $m$ that are closer and closer to 0, . An example of this trend can be observed by comparing the plot for the second branch solution for

$m = -0.06$ and the plot for the second branch solution for $m = -0.0005$ (which can be found below). These observations suggest that as $m \uparrow 0$, the region of reversed flow increases from 0 at $m = m_c$ to infinity.

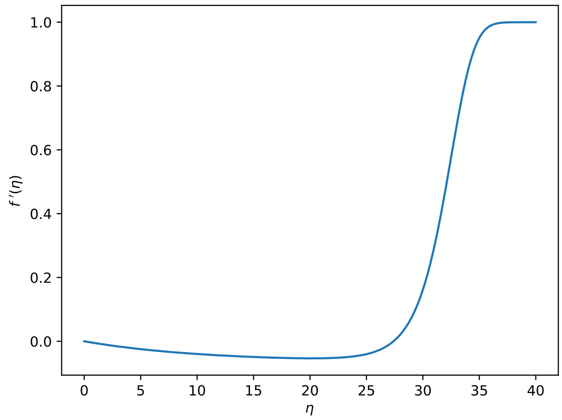


Figure 25: Second branch solution, $m = -0.0005$, $S_m = -0.006089475441101154$

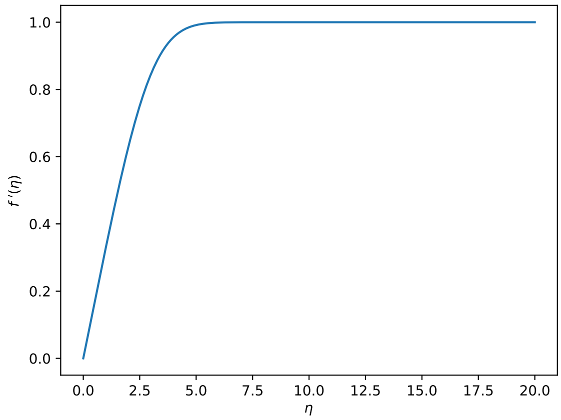


Figure 26: First branch solution, $m = 0$, $S_m = 0.3320573297388055$

We also notice that the solution for the first branch, as $m \uparrow 0$, approaches the solution for $m = 0$ we have found in Question 3. The appropriate plot can be found above.

We now aim to discuss the physical significance of these solutions. The first branch solutions are clearly the solutions with no back flow, and what we would expect if we naturally extended the solutions in Question 3 to negative values of $m$. However, we observe that if we further try to extend these solutions to $m < m_c$, we notice that the solutions represent an overshoot. This is illustrated in an example below. These supervelocity solutions are not physical viable.

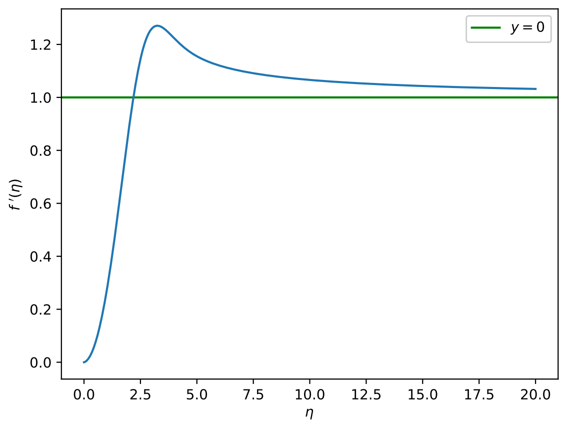


Figure 27: Overshoot solution, $m = -0.2$, $S = -0.006089475441101154$

Furthermore, for the second branch solutions, we notice that these solutions have reversed flow/ back flow (indicated by sections with negative values of $f'$). At $m = m_c$, we notice existence of a separation point. At this point, we have that the tangential velocity and the shear stress at the rigid boundary reach 0. Due to this, we have separation of the boundary layer from the surface (!) The solutions with reverse flow are probably unstable and impossible to set up in an experiment. Furthermore, something worth noting is that the boundary layer approximations fail in back flow situations. Also, as was observed in passing in the Part II Fluid Dynamics lectures, these solutions are in fact acceptable for slightly decelerating external flows!If the external flow is slowing down, we have advection away from the rigid surface due to mass conservation. As advection and diffusion are now **both** away from the rigid surface, the vorticity can no longer be confined to be near the rigid surface, thus leading to breakdown of the boundary layer structure. This leads to the boundary layer separation!

In the interval $-1 < m < m_c$, we can observe other branches of exponentially converging solutions. These branches are found by extending the method we used to find the second branch to other values of $m$.

First, we discuss a branch for $-0.40298507462686567 < m < -0.3333333333333333$. The plot for $S_m$ vs $m$ in this branch is plotted below.

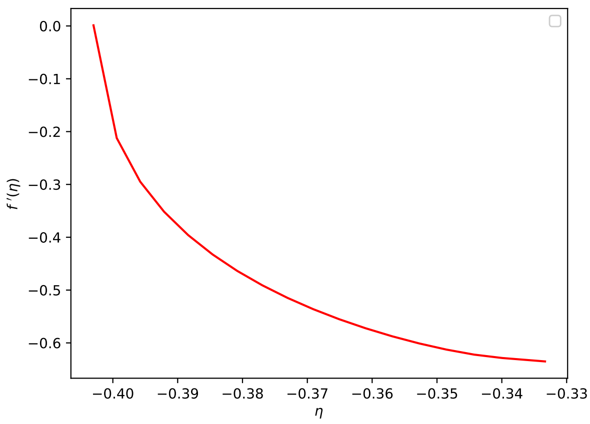


Figure 28: Plot of $S_m$ vs $m$ for the third branch, $-0.40298507462686567 < m < -0.3333333333333333$

We further present few plots of $f'(\eta)$ vs $\eta$ for solutions on this third branch.

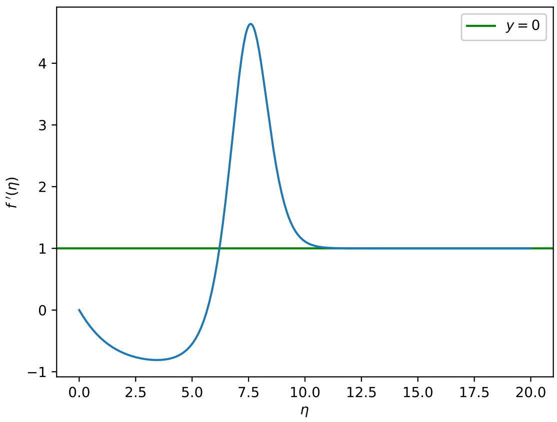


Figure 29: $m = -0.3399$, $S_m = -0.62861$

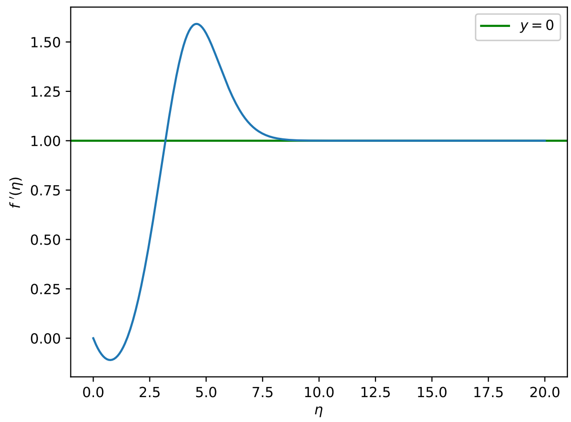


Figure 30: $m = -0.39577$, $S_m = -0.29493$

Few plot of $f(\eta)$ vs $\eta$ of third branch solutions

Next, we discuss another branch of solutions, found in $-0.497487437185929640 < m < -0.5475113122171945$. The plot for $S_m$ vs $m$ in this branch is plotted below.

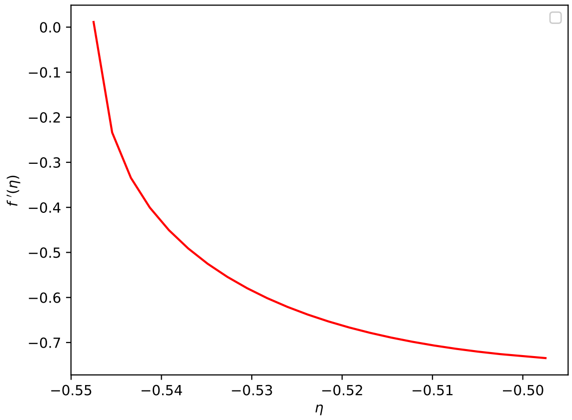


Figure 31: Plot of $S_m$ vs $m$ for the fourth branch, $-0.49748743718592964 < m < -0.5475113122171945$

We further present few plots of $f'(\eta)$ vs $\eta$ for solutions on this fourth branch.

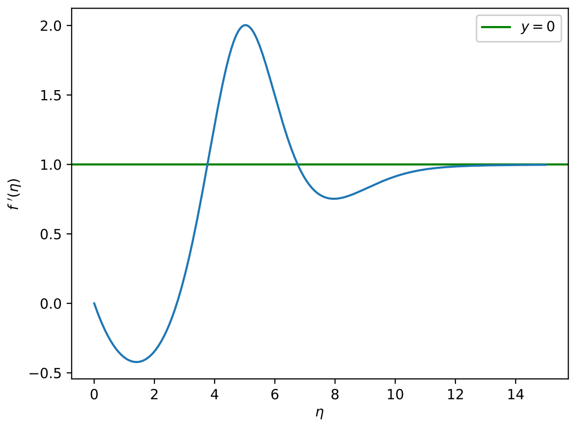


Figure 32: $m = -0.50249$, $S_m = -0.628618$

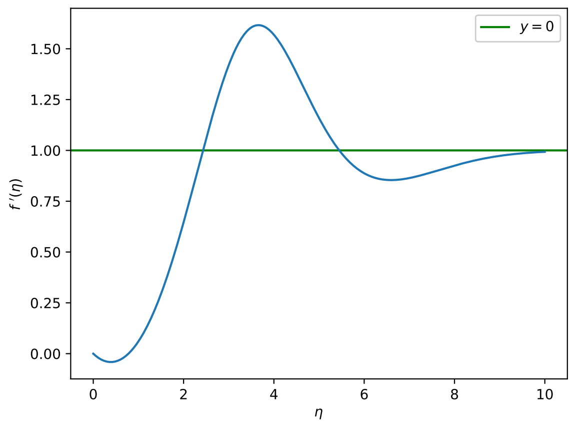


Figure 33: $m = -0.545454$, $S_m = -0.21223$

Few plot of $f(\eta)$ vs $\eta$ of fourth branch solutions

We notice that in these further branches of solutions for $-1 < m < m_c$, we notice that the solutions in both third and fourth branch exhibit back flow/reversed flow. They also exhibit the behaviour we observed for the reversed flow solutions in the second branch in the case $m_c < m < 0$, where as we decrease $m$ from the bottom of the band of the branch to the top of the band of the branch, the region of reversed flow (ie where $f'(\eta)$ is negative) increases in width. Furthermore, at the ends of the bands of these branches, we notice that $S_m = 0$, thus suggesting that these are further points of boundary layer separation!

We also notice that solutions in both these branches exhibit overshoot, ie supervelocity solutions! This is bad, as these imply that the solutions here are not physical. In fact, we hypothesise from experimental considerations (evidenced above by the two branches) and referencing Libby and Liu [1] and Rosenhead [3] that all branches of solutions for $-1 < m < m_c$ exhibit overshoot. In Libby and Lib [1], it is further noted that these overshoot solutions maybe physical plausible as they might be generated by streamwise blowing through an upstream slot. Another thing to notice is that the overshoot peaks for solutions on these third and fourth branches seem to decrease as we go from the top band value of $m$ (for this branch) to the bottom band value of $m$.

Another interesting observation we notice is that all solutions in third branch of solutions have exactly one root to the equation $f'(\eta) - 1 = 0$. Similarly, all solutions in the fourth branch of solutions have exactly two roots to the equation $f'(\eta) - 1 = 0$. This observation is once again noted in Libby and Liu [1], and further elaborated on in Oskam and Veldman [2], and so our observations agree with previously confirmed results.

## Question 5

Now we intend to numerically investigate the case of $m = -1$. Furthermore, we analyse cases where we can have both signs of $A$ (and not solely $A > 0$).

We amend our programs earlier to solve for this case, particularly one that finds $S_m$ for this case, at $m = -1$. We notice that all numerical solutions in the case of $m = -1$ are periodic(!) (regardless of the sign of $A$, as that only impacts the final value). An example of a periodic solution is below, where we plot the solution for $m = -1, S = 0.5$.

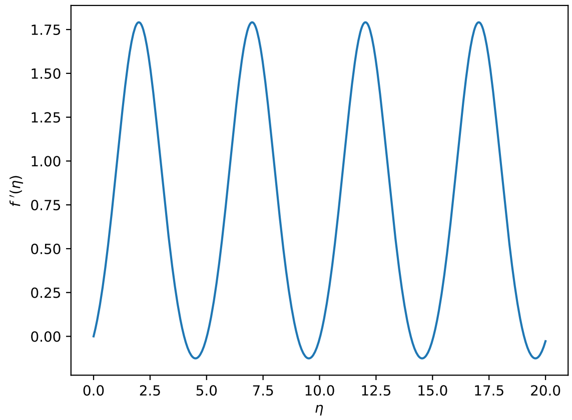


Figure 34: $m = -1$, $S = 0.5$

We first investigate the case sgn$A = -1$. This is the case of a converging channel. We notice that if we restrict $\eta$ to $[0, 10]$, and then use our program to find the value of $S_m$, we get that $S_m = 1.1546314842522847 = S'$. The plot for this value of $S$ is given below. As one might notice, on the restricted domain, it does look (numerically) that the solution is converging to $-1$ exponentially. However, this solution is periodic, still. However, this is fine, as for numerical purposes, integrating to $[0, 10]$ suffices to find the value of $S_m$ as $\eta = 10$ is sufficiently large (as $S \neq S_m$ exactly ever, we have solutions that will be periodic). However, there is another value of $S_m$ that seems to work, and that is negative, $S_m = -1.1546386833847975 = S''$. For this value of $S_m$, we integrate on $[0, 15]$ instead.

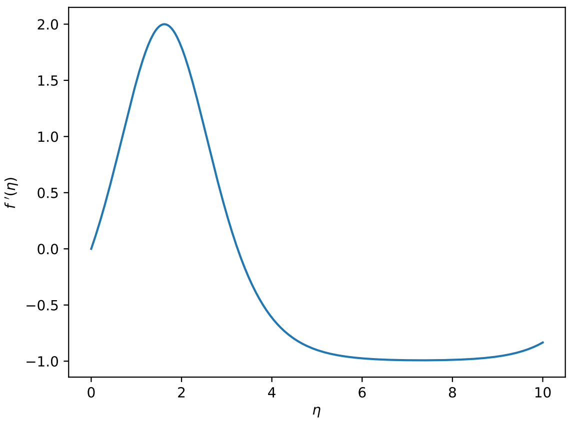


Figure 35: $S_m = 1.1546314842522847$

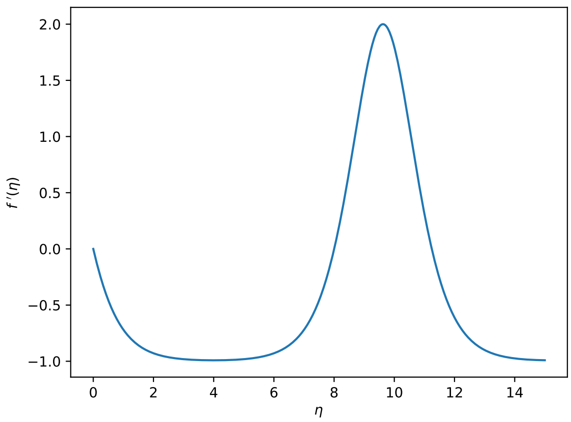


Figure 36: $S_m = -1.1546386833847975$

Plots of $f(\eta)$ vs $\eta$, $m = -1$, sgn$A = -1$

We further notice numerically that for $S < S''$ and $S > S'$, we observe that solutions have a singularity at finite distance (!). This is demonstrated below for $S = \pm 1.155$. This agrees with our analysis of possible terminal behaviours in Question 1!

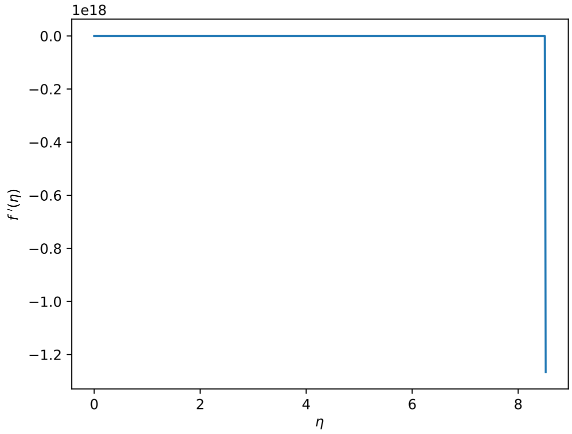


Figure 37: $S = -1.155$

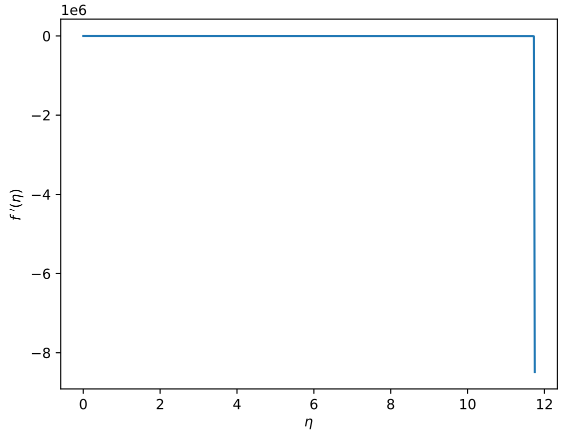


Figure 38: $S_m = -1.155$

Plots of $f(\eta)$ vs $\eta$ with $m = -1$, to demonstrate finite distance singularity

We further notice that for $S'' < S < S'$, we have solutions that are periodic (as has been plotted earlier in this question). We notice that the existence of solution for $A < 0$ agrees with theoretical results, as if we solve (1) and (2), in this case $m = -1, A < 0$, we notice that there exists a solution such that

$$f'(\eta) = 2 - 3\tanh^2\left(\frac{\eta}{\sqrt{2}} \pm \tanh^{-1}\sqrt{\frac{2}{3}}\right)$$

which results in

$$f''(0) = S_{-1} = \mp\sqrt{\frac{4}{3}} \approx 1.1547$$

Then our our numerical values of $S_{-1}$ agree with this theoretical value above up to 4 significant figures!

Physically, $m = -1$ is the situation of a converging channel. In this case, the solution with $S = S''$, gives a velocity profile that is monotonic in the boundary layer, and looks more (intuitively) physically possible. The case of $S = S'$ corresponds to the case of reversed flow in the boundary layer.

We shall now investigate the case of $A > 0$. Investigating numerically, we continue to note that we will have regular solutions if and only if $S'' < S < S'$. We observe that all solutions are periodic, and have oscillating amplitudes. We notice that there are no possible solutions for $S_m$ in this case. And this makes sense, as all solutions (for different values of $S$) seem to overshoot $y = 1$, and don't tend to show any terminal behaviour remotely close to converging to 1 as $\eta \to \infty$ (regardless of our domain of integration). In fact we notice that in the case of $A < 0$, we could conclude convergence due to terminal behaviour on sufficiently large domain of integration that suggested exponential convergence. Particularly, in the case of $A < 0$, we notice that the convergence terminal behaviour happens at the 'trough' of the solution. However in the case of $A > 0$, as can be seen through few examples below, the convergence must take place near the crest, and the crest always overshoots 1 quite comfortably!

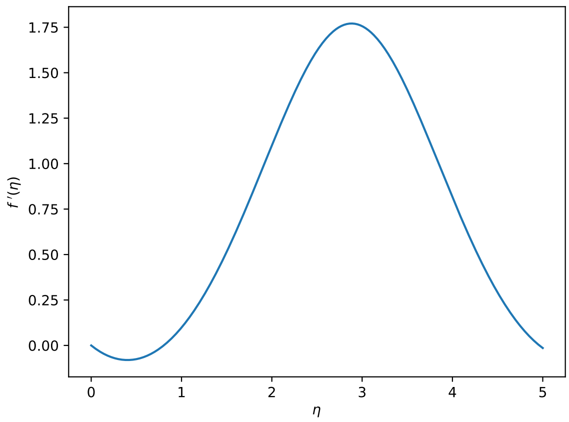


Figure 39: $S = -0.4$

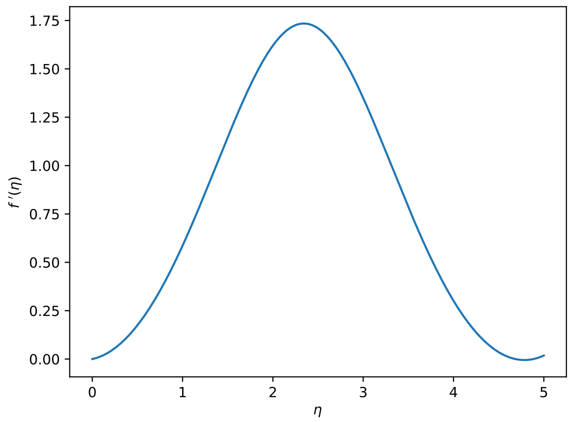


Figure 40: $S_m = 0.1$

Some more plots of $f(\eta)$ vs $\eta$ with $m = -1$

Thus we conclude that the for the case of $A > 0$, no solution exists. This result correlates well with our analysis in Question 1, particularly in the section `A Very Important Interlude`, as asymptotically we get a periodic solution without a decaying amplitude, suggesting lack of a solution. This can further be noticed via solving (1) and (2) and observing that we get a contradiction as we end up concluding that

$$\frac{1}{2}(f''(0))^2 = -\frac{2}{3}$$

Physically, lack of solution makes sense as $A > 0$ is the case of flow in a diverging channel. The typical flow, using the approach in this question, leads to 'solutions' where the radial velocity profile has an increasing number of oscillations as the Reynolds number tends to infinity. So the viscous effects remain important throughout the domain, and not just the boundary layer. This can be further explained by realising that the velocity profile is of the form $\frac{1}{x}$, and so have a very very rapid deceleration of the flow away from the singular point $x = 0$. Regardless, in this situation, the boundary layer flow approximation is not valid, which explains why there is no solution to Falkner-Skan equation with $m = -1, A > 0$!

# References


[1] Paul A Libby and TM Liu. Further solutions of the falkner-skan equation. AIAA Journal, 5(5):1040–1042, 1967.

[2] B Oskam and AEP Veldman. Branching of the falkner-skan solutions for $\lambda < 0$. Journal of Engineering Mathematics, 16(4):295–308, 1982.

[3] Louis Rosenhead. Laminar boundary layers: an account of the development, structure, and stability of laminar boundary layers in incompressible fluids, together with a description of the associated experimental techniques. Clarendon Press, 1963.

# Appendix A: Program Listings

Listing 1: Program for the programming task and Questions 2 and 3

```
#---start of def---
def falkner_skan_setup(f,nu,m):
    dfdnu=[0,0,0]
    dfdnu[0]=f[1]
    dfdnu[1]=f[2]
    dfdnu[2]=m*(f[1]**2)-(1/2)*(m+1)*f[0]*f[2]-m
    return dfdnu
#---end of def---
#---start of def---
def falkner_skan_integrator_type1(m,S,nuspan,reltol,abstol):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup, f_init, nuspan, args=(m,),rtol=reltol,
   atol=abstol)
    return sol
#---end of def---
#---start of def---
def falkner_skan_shooting_method_objective(S,m,nuspan,reltol=1e-8,abstol=1e
   -8):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup, f_init, nuspan, args=(m,),rtol=reltol,
   atol=abstol)
    df= sol[:,1]
    return df[-1]-1
#---end of def---
#---start of def---
def falkner_skan_shooting_method(m,initval):
    Sm, = fsolve(falkner_skan_shooting_method_objective, initval, args=(m,np
   .linspace(0,20,501),),xtol=1e-8)
    return Sm
# np.info(scipy.optimize.fsolve)
#---end of def---
#---start of def---
def falkner_skan_shooting_method_plot(m,initval):
    S_ideal=falkner_skan_shooting_method(m,initval);
    print('S_ideal',S_ideal)
    sol=falkner_skan_integrator_type1(m,S_ideal,np.linspace(0,10,501),1e-8,1
   e-8);
    print(sol[:,1][-1]-1);
    plt.plot(np.linspace(0,10,501), sol[:, 1], 'b', label='theta(t)')
#---end of def---
```

Listing 2: Program for Question 4

```
#---start of def---
def falkner_skan_shooting_method_objective2(S,m,nuspan,reltol=1e-8,abstol=1e-8):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup, f_init, nuspan, args=(m,),rtol=reltol,atol=abstol)
    df= sol[:,1]
    if(S>0):
        return 100
    return df[-1]-1
#---end of def---
#---start of def---
def falkner_skan_shooting_method2(m,initval):
    Sm, = fsolve(falkner_skan_shooting_method_objective2, initval, args=(m,np.linspace(0,40,501),),xtol=1e-8)
    return Sm
# np.info(scipy.optimize.fsolve)
#---end of def---
#---start of def---
def falkner_skan_shooting_method_plot2(m,initval):
    S_ideal=falkner_skan_shooting_method2(m,initval);
    print('S_ideal',S_ideal)
    sol=falkner_skan_integrator_type1(m,S_ideal,np.linspace(0,40,501),1e-8,1e-8);
    print(sol[:,1][-1]-1);
    plt.plot(np.linspace(0,40,501), sol[:, 1], 'b', label='theta(t)')
#---end of def---
#---start of def---
def falkner_skan_setup3(f,nu,beta):
    dfdnu=[0,0,0]
    dfdnu[0]=f[1]
    dfdnu[1]=f[2]
    dfdnu[2]=beta*(f[1]**2-1)-f[0]*f[2]
    return dfdnu
#---end of def---
#---start of def---
def falkner_skan_integrator_type3(beta,S,nuspan,reltol,abstol):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup3, f_init, nuspan, args=(beta,),rtol=reltol,atol=abstol)
    return sol
#---end of def---
#---start of def---
def falkner_skan_shooting_method_objective3(S,beta,nuspan,reltol=1e-8,abstol=1e-8):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup3, f_init, nuspan, args=(beta,),rtol=reltol,atol=abstol)
    df= sol[:,1]
    return df[-1]-1
#---end of def---
#---start of def---
def falkner_skan_shooting_method3(beta,initval):
    Sm, = fsolve(falkner_skan_shooting_method_objective3, initval, args=(beta,np.linspace(0,20,501),),xtol=1e-8)
    return Sm
# np.info(scipy.optimize.fsolve)
#---end of def---
#---start of def---
```

```
def falkner_skan_shooting_method_plot3(beta,initval):
    S_ideal=falkner_skan_shooting_method3(beta,initval);
    print('S_ideal',S_ideal)
    sol=falkner_skan_integrator_type3(beta,S_ideal,np.linspace(0,40,501),1e
    -8,1e-8);
    print(sol[:,1][-1]-1);
    plt.plot(np.linspace(0,40,501), sol[:, 1], 'b', label='theta(t)')
#---end of def---
```

Listing 3: Program for Question 5

```
#---start of def---
def falkner_skan_setup_m0(f,nu):
    dfdnu=[0,0,0]
    dfdnu[0]=f[1]
    dfdnu[1]=f[2]
    dfdnu[2]=-(f[1]**2)+1
    return dfdnu
#---end of def---
#---start of def---
def falkner_skan_integrator_type_m0(S,nuspan,reltol,abstol):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup_m0, f_init, nuspan,rtol=reltol,atol=abstol)
    return sol
#---end of def---
#---start of def---
def falkner_skan_shooting_method_objective_m0(S,nuspan,reltol=1e-8,abstol=1e-8):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup_m0, f_init, nuspan,rtol=reltol,atol=abstol)
    df= sol[:,1]
    return df[-1]+1
#---end of def---
#---start of def---
def falkner_skan_shooting_method_m0(initval,x):
    Sm, = fsolve(falkner_skan_shooting_method_objective_m0, initval, args=(x,),xtol=1e-8)
    return Sm
# np.info(scipy.optimize.fsolve)
#---end of def---
#---start of def---
def falkner_skan_shooting_method_plot_m0(initval):
    S_ideal=falkner_skan_shooting_method_m0(initval);
    print('S_ideal',S_ideal)
    sol=falkner_skan_integrator_type_m0(m,S_ideal,np.linspace(0,10,501),1e-8,1e-8);
    print(sol[:,1][-1]-1);
    plt.plot(np.linspace(0,10,501), sol[:, 1], 'b', label='theta(t)')
#---end of def---
#---start of def---
def falkner_skan_shooting_method_objective_m1(S,nuspan,reltol=1e-8,abstol=1e-8):
    f_init=[0,0,S]
    sol=odeint(falkner_skan_setup_m0, f_init, nuspan,rtol=reltol,atol=abstol)
    df= sol[:,1]
    return df[-1]-1
#---end of def---
#---start of def---
def falkner_skan_shooting_method_m1(initval,x):
    Sm, = fsolve(falkner_skan_shooting_method_objective_m1, initval, args=(x,),xtol=1e-8)
    return Sm
# np.info(scipy.optimize.fsolve)
#---end of def---
```