# Math Document Template

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***Abstract*—This is a document explaining for a question on the concept of linear algebra.**

Download all python codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/codes
```

and latex-tikz codes from

```
svn co https://github.com/Ashuwin/summer_20/
    trunk/linear_algebra/figs
```

## 1 Triangle

### 1.1 Problem

1. Find the area of triangle whose vertices are
   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
   b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

### 1.2 Solution

1. The area of triangle $ABC$:
   **Solution:** The area of triangle $ABC$ using cross product is obtained as:
   $$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\|$$
   and it is found in the following python code:
   ```
   codes/triangle/tri_area_ABC.py
   ```
   Area of $\triangle ABC$ = $10.5 units^2$
   $\triangle ABC$ in Fig.1.2.1 is generated using the following python code
   ```
   codes/triangle/triangle1.py
   ```

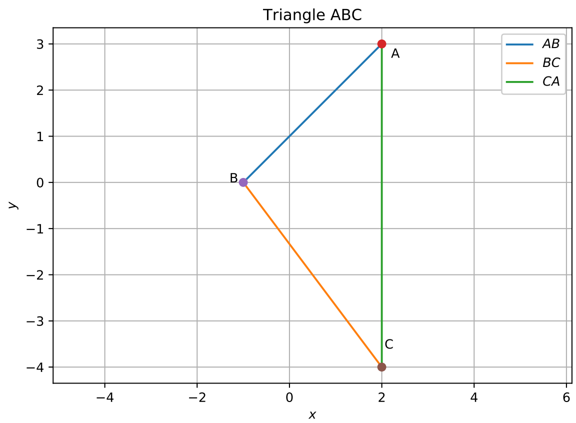


Fig. 1.2.1: Triangle $ABC$ using python

2. The area of triangle $PQR$:
   **Solution:** The area of triangle $PQR$ using Heron's formula is obtained as:
   $$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{R}-\mathbf{P})\right\|$$
   and it is found in the following python code:
   ```
   codes/triangle/tri_area_PQR.py
   ```
   Area of $\triangle PQR$ = $32 units^2$ $\triangle PQR$ in Fig.1.2.2 is generated using the following python code
   ```
   codes/triangle/triangle2.py
   ```

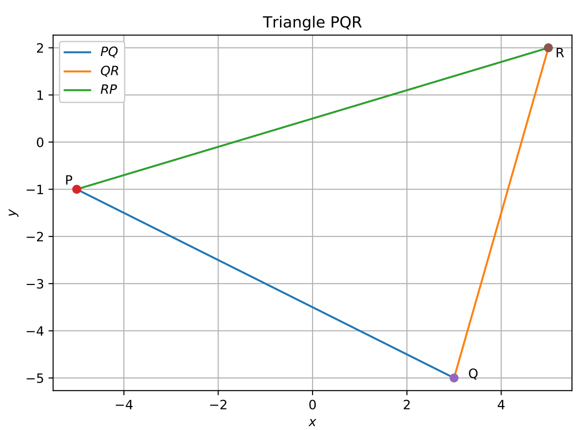


Fig. 1.2.2: Triangle $PQR$ using python

## 2 Quadrilateral

### 2.1 Problem

1. Find the area of the quadrilateral whose vertices are, taken in order, are

$$\begin{pmatrix}-4\\2\end{pmatrix},\begin{pmatrix}-3\\-5\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}$$

### 2.2 *Solution*

1. The area of triangle $ABC$:
   **Solution:** The area of triangle $ABC$ using cross product is obtained as:
   $$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\|$$
   and it is found in the following python code:
   ```
   codes/tri_area_ABC.py
   ```
   Area of $\triangle ABC = 22.5 units^2$
2. The area of triangle $ACD$:
   **Solution:** The area of triangle $ACD$ using Heron's formula is obtained as:
   $$\frac{1}{2}\left\|(\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{A})\right\|$$
   and it is found in the following python code:
   ```
   codes/tri_area_ACD.py
   ```
   Area of $\triangle ACD = 15.5 units^2$
3. The area of quadrilateral $ABCD$:
   **Solution:** Area of Quadrilateral $ABCD$ = Area of $\triangle ABC$ + Area of $\triangle ACD = 38 units^2$
4. Quadrilateral $ABCD$ in Fig.2.2.4 is generated using the following python code
   ```
   codes/quadrilateral/quad.py
   ```

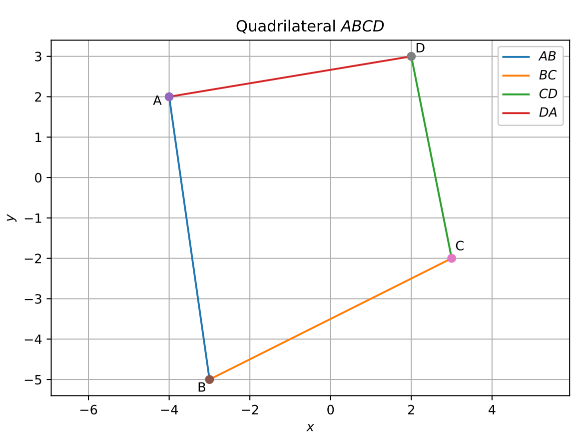


Fig. 2.2.4: Quadrilateral $ABCD$ using python

## 3 Line exercises

### 3.1 *Points and Vectors*

#### 3.1.1 *Problem:*
Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix},\begin{pmatrix}-2\\9\end{pmatrix}$$

#### 3.1.2 *Solution:*

1. From the given information,
   $$\left\|\mathbf{x}-\begin{pmatrix}2\\-5\end{pmatrix}\right\|^2=\left\|\mathbf{x}-\begin{pmatrix}-2\\9\end{pmatrix}\right\|^2$$
   $$\|\mathbf{x}\|^2+\left\|\begin{pmatrix}2\\-5\end{pmatrix}\right\|^2-2\begin{pmatrix}2 & -5\end{pmatrix}\mathbf{x}=\|\mathbf{x}\|^2+\left\|\begin{pmatrix}-2\\9\end{pmatrix}\right\|^2-2\begin{pmatrix}-2 & 9\end{pmatrix}\mathbf{x}$$
   which can be simplified to obtain
   $$\begin{pmatrix}8 & -28\end{pmatrix}\mathbf{x}=-56$$
   Choose $\mathbf{x}=\begin{pmatrix}x\\0\end{pmatrix}$ as the point lies on the x-axis
   $$\begin{pmatrix}8 & -28\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix}=-56$$
   $$\implies x=-7$$
   $$\implies \text{The point is }\begin{pmatrix}-7\\0\end{pmatrix}$$

### 3.2 *Points on a line*

#### 3.2.1 *Problem:*
If $\mathbf{A}=\begin{pmatrix}-2\\-2\end{pmatrix},\mathbf{B}=\begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP=\frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment $AB$

#### 3.2.2 *Solution:*

1. $\mathbf{A}=\begin{pmatrix}-2\\-2\end{pmatrix},\mathbf{B}=\begin{pmatrix}2\\-4\end{pmatrix}$
   Then $\mathbf{P}$ that divides $\mathbf{A},\mathbf{B}$ in the ratio k:1 is
   $$\mathbf{P}=\frac{k\mathbf{B}+\mathbf{A}}{k+1} \tag{3.2.2.1.1}$$
   For the given problem,$k=\frac{3}{4}$
   Using the equation 3.2.2.1.1, the desired point

is

$$\mathbf{P} = \frac{\frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix} + 1\begin{pmatrix}-2\\-2\end{pmatrix}}{\frac{3}{4}+1} \tag{3.2.2.1.2}$$

$$\mathbf{P} = \begin{pmatrix}-2/7\\-20/7\end{pmatrix} \tag{3.2.2.1.3}$$

The following python code plots the Fig.**??**

```
codes/point_line/int_sec.py
```

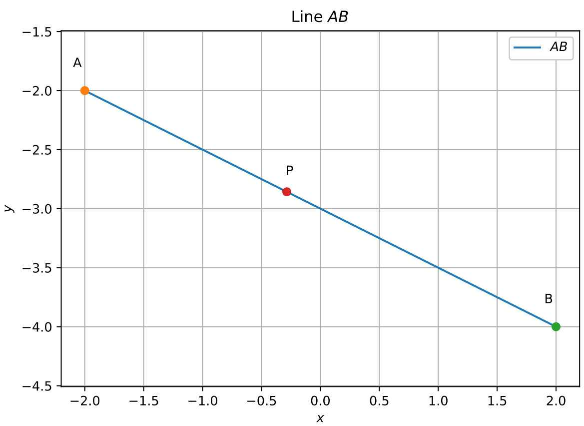


Fig. 3.2.2.1: Line $AB$ using python

### *3.3 Lines and Planes*

*3.3.1 Problem:* Write four solutions for each of the following equations

a)$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$

b)$\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$

c)$\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$

*3.3.2 Solution:*

1. $\mathbf{x}$ are randomly chosen and substituted in the equation and solutions are found.

a)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.3.2.1.1}$$

**Solution:** Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.3.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 7$

$\implies a = \frac{7}{2}$

Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.3.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 7$

$\implies b = 7$

Let $\mathbf{x} = \begin{pmatrix}c\\1\end{pmatrix}$ Substituting in equation 3.3.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 7$

$\implies c = 3$

Let $\mathbf{x} = \begin{pmatrix}1\\d\end{pmatrix}$ Substituting in equation 3.3.2.1.1, $\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 7$

$\implies d = 5$

b)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9 \tag{3.3.2.1.2}$$

**Solution:** Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.3.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 9$

$\implies a = \frac{9}{\pi}$

Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.3.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 9$

$\implies b = 9$

Let $\mathbf{x} = \begin{pmatrix}c\\1\end{pmatrix}$ Substituting in equation 3.3.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 9$

$\implies c = \frac{8}{\pi}$

Let $\mathbf{x} = \begin{pmatrix}1\\d\end{pmatrix}$ Substituting in equation 3.3.2.1.2, $\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 9$

$\implies d = 9 - \pi$

c)

$$\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0 \tag{3.3.2.1.3}$$

**Solution:** Let $\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}$ Substituting in equation 3.3.2.1.3, $\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 0$

$\implies a = 0$

Let $\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}$ Substituting in equation 3.3.2.1.3, $\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 0$

$\implies b = 0$

Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation 3.3.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = 0$
$\implies c = 4$
Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation 3.3.2.1.3, $\begin{pmatrix} 1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ d \end{pmatrix} = 0$
$\implies d = \frac{1}{4}$

### 3.4 Motion in a plane

#### 3.4.1 Problem:

1. A man can swim with a speed of $4.0km/h$ in still water. How long does he take to cross a river $1.0km$ wide if the river flows steadily at $3.0km/h$ and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?

#### 3.4.2 Solution:

1. Let the man be at point $\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   The speed of the man is $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

   The speed of the river is $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

   Since the swimmer dive the river normal to the flow of river, therefore time taken by swimmer to cross the river,

$$t = \frac{d}{\|\mathbf{u}\|} = \frac{1km}{4km/h} = 15mins$$

   Distance covered down the river $= t \times \|\mathbf{v}\|$

$$x = \frac{1}{4}hr \times 3km/h = 750m$$

   The code for diagrammatic representation(3.4.2.1) of the solution is

```
codes/line/motion_plane/man_river.py
```

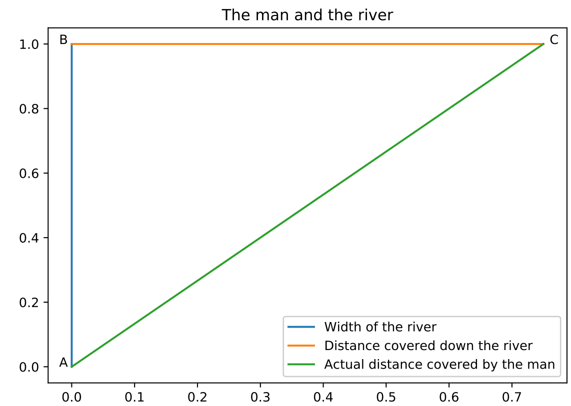


Fig. 3.4.2.1

### 3.5 Matrix

#### 3.5.1 Problem:

1. Find the values of x,y and z from the following equations:

   a) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$

   b) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$

   c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

#### 3.5.2 Solution:

1. This problem is solved by comparing the respective elements in both the matrices

   a)

$$x = 1, y = 4, z = 3$$

   b) $5 + z = 5$
   $\implies z = 0$
   $x + y = 6$ and $xy = 8$
   $\implies x = 4, y = 2$ or $x = 2, y = 4$

$$x = 4, y = 2, z = 0$$

   or

$$x = 2, y = 4, z = 0$$

   c) $x + y + z = 9$ and $x + y = 5$
   $\implies z = 4$
   $x + y + z = 9$ and $y + z = 7$
   $\implies x = 2$
   $x + y + z = 9$
   $2 + y + 4 = 9$
   $\implies y = 3$

$$x = 2, y = 3, z = 4$$

### 3.6 Determinants

#### 3.6.1 Problem:

1. If $A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$.

#### 3.6.2 Solution:

1. The determinant of a $3 \times 3$ matrix is given by:
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$Det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$
$\implies Det = 0$

### 3.7 *Linear Inequalities*

#### 3.7.1 *Problem:*

1. Solve $3x + 2y > 6$ graphically

#### 3.7.2 *Solution:*

1. Let $3x + 2y = 6$ intersects the x-axis and y-axis at $\mathbf{A}$ and $\mathbf{B}$ respectively.
   a) Let $\mathbf{A} = \begin{pmatrix} x \\ 0 \end{pmatrix}$
   $3x = 6$
   $\implies x = 2$
   $\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
   b) Let $\mathbf{B} = \begin{pmatrix} 0 \\ y \end{pmatrix}$
   $2y = 6$
   $\implies y = 3$
   $\mathbf{B} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
   c) Origin $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not satisfy the equation $3x + 2y < 6$.
   $\implies$ The solution is the right side of the line $3x + 2y = 6$

2. The following python code is the diagrammatic representation of the solution in Fig.3.7.2.2

```
codes/linear_inequalities/linear_inequalities.
    py
```

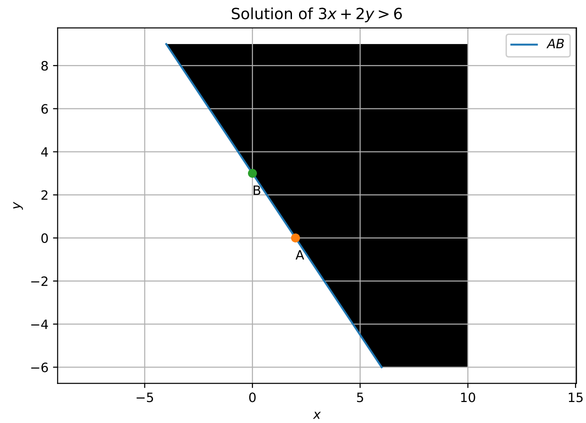


Fig. 3.7.2.2

## 4 Circle

### 4.1 *Problem*

1. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$.

### 4.2 *Solution*

1. The general equation of circle,
$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{4.2.1.1}$$
whose centre is $\begin{pmatrix} -g \\ -f \end{pmatrix}$ The vector form is,
$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2g & 2f \end{pmatrix} \mathbf{x} + c = 0$$
2. Point $\mathbf{A} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on the circle. So, point $A$ satisfies the equation 4.2.1.1
$$\begin{pmatrix} 4 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2g & 2f \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c = 0$$
$$8g + 2f + c + 17 = 0 \tag{4.2.2.1}$$
3. Point $\mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ lies on the circle. So, point $B$ satisfies the equation 4.2.1.1
$$\begin{pmatrix} 6 \\ 5 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 2g & 2f \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} + c = 0$$
$$12g + 10f + c + 61 = 0 \tag{4.2.3.1}$$
4. Centre $\begin{pmatrix} -g \\ -f \end{pmatrix}$ lies on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$
$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} -g \\ -f \end{pmatrix} = 16$$
$$4g + f + 16 = 0 \tag{4.2.4.1}$$
5. Solving equations 4.2.2.1, 4.2.3.1 and 4.2.4.1
$$g = -3, f = -4, c = 15$$
$\implies$ Equation of the circle is $x^2 + y^2 - 6x -$

$8y + 15 = 0$
The vector form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6 & -8 \end{pmatrix} \mathbf{x} + 15 = 0$$

6. The circle in Fig.4.2.6 is generated using the following python code

```
codes/circle/circle.py
```

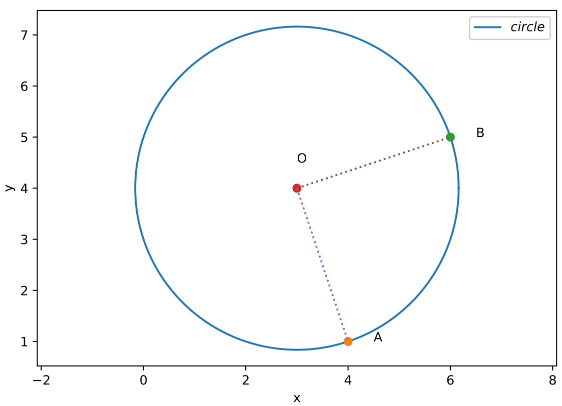


Fig. 4.2.6: Circle generated using python

# 5 Conics

## 5.1 Problem

1. Find the roots of the quadratic equations:
   a) $x^2 - 3x - 10 = 0$
   b) $2x^2 + x - 6 = 0$
   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   d) $2x^2 - x + \frac{1}{8} = 0$
   e) $100x^2 - 20x + 1 = 0$

## 5.2 Solution

1. For a general polynomial equation of degree 2 , $p(x, y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
   The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.2.1.1)$$

   a) $x^2 - 3x - 10 = 0$
   The vector form is
   $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0$
   The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/conics_1.py
```

   $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_1.py
```

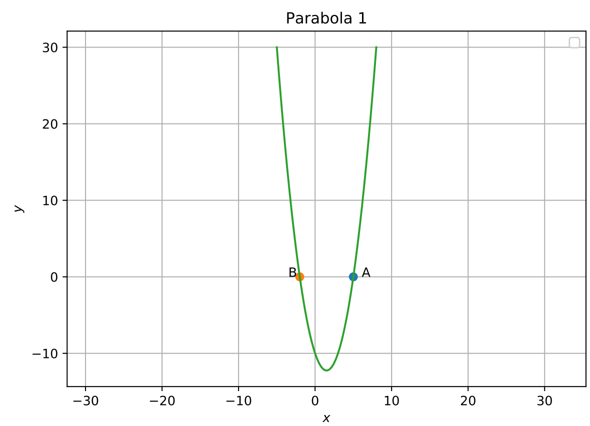


Fig. 5.2.1: Parabola 1

   b) $2x^2 + x - 6 = 0$
   The vector form is
   $\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 6 = 0$ The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/conics_2.py
```

   $\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
   which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_2.py
```

   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   The vector form is
   $\mathbf{x}^T \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 5\sqrt{2} = 0$
   The values of $\mathbf{x}$ are found in the following python code

```
codes/conics/conics_3.py
```

   $\mathbf{x} = \begin{pmatrix} -1.414 \\ 0 \end{pmatrix}, \begin{pmatrix} -3.535 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

```
codes/conics/conics_3.py
```

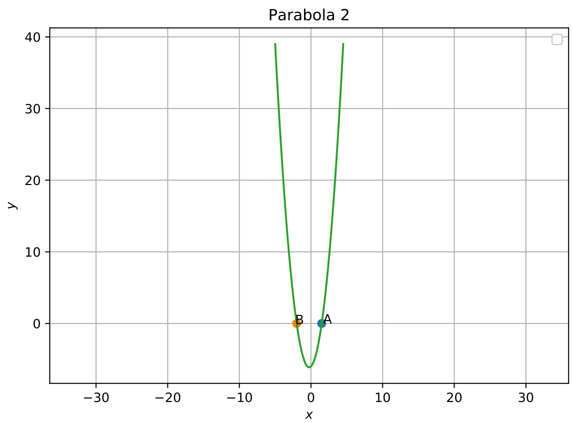


Fig. 5.2.1: Parabola 2

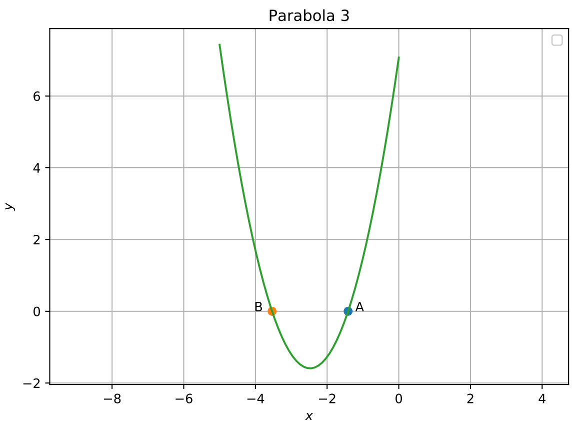


Fig. 5.2.1: Parabola 3

d) $2x^2 - x + \frac{1}{8} = 0$
The vector form is
$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + \frac{1}{8} = 0$
The values of $\mathbf{x}$ are found in the following python code

codes/conics/conics_4.py

$\mathbf{x} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$
which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

codes/conics/conics_4.py

e) $100x^2 - 20x + 1 = 0$
The vector form is
$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0$
The values of $\mathbf{x}$ are found in the following python code

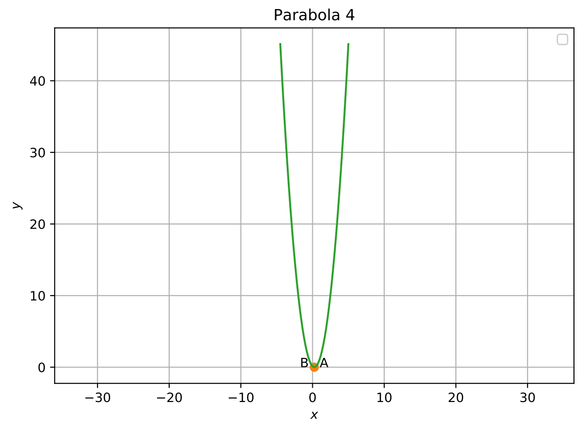


Fig. 5.2.1: Parabola 4

codes/conics/conics_5.py

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.5.2.1. The following python code generates the fig.5.2.1

codes/conics/conics_5.py

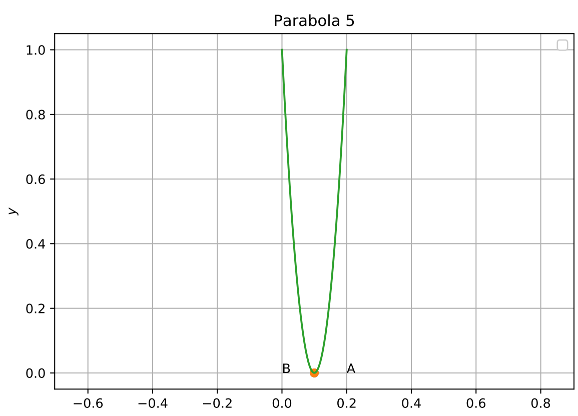


Fig. 5.2.1: Parabola 5