

Math Document Template

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Abstract—This is a document explaining for a question on the concept of linear algebra.

Download all python codes from

```
svn co https://github.com/Ashuwin/summer_20/
trunk/linear_algebra/codes
```

and latex-tikz codes from

```
svn co https://github.com/Ashuwin/summer_20/
trunk/linear_algebra/figs
```

1 TRIANGLE

1.1 Problem

- Find the area of triangle whose vertices are

- $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
- $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

1.2 Solution

- The area of triangle ABC :

Solution: The area of triangle ABC using cross product is obtained as:

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\|$$

and it is found in the following python code:

```
codes/triangle/tri_area_ABC.py
```

Area of $\triangle ABC = 10.5 \text{ units}^2$

$\triangle ABC$ in Fig.?? is generated using the following python code

```
codes/triangle/triangle1.py
```

- The area of triangle PQR :

Solution: The area of triangle PQR using Heron's formula is obtained as:

$$\frac{1}{2} \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\|$$

and it is found in the following python code:

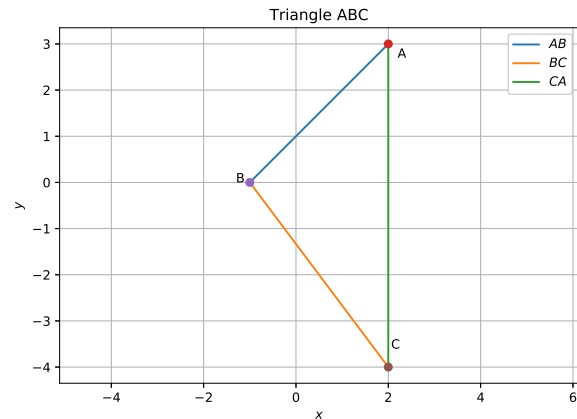


Fig. 1.2.1: Triangle ABC using python

```
codes/triangle/tri_area_PQR.py
```

Area of $\triangle PQR = 32 \text{ units}^2$ $\triangle PQR$ in Fig.?? is generated using the following python code

```
codes/triangle/triangle2.py
```

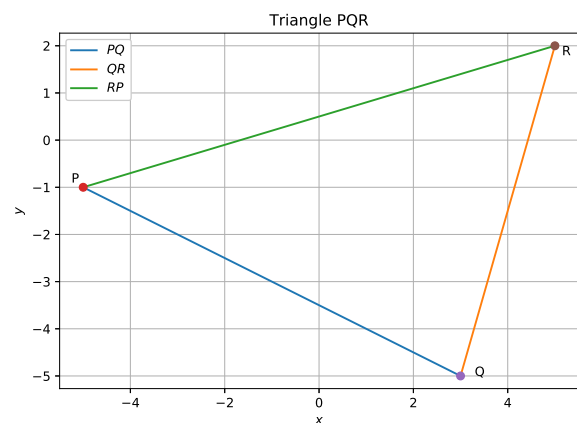


Fig. 1.2.2: Triangle PQR using python

2 QUADRILATERAL

2.1 Problem

- Find the area of the quadrilateral whose vertices are, taken in order, are

$$\begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

3 LINE EXERCISES

2.2 Solution

1. The area of triangle ABC :

Solution: The area of triangle ABC using cross product is obtained as:

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\|$$

and it is found in the following python code:

```
codes/tri_area_ABC.py
```

Area of $\triangle ABC = 22.5 \text{ units}^2$

2. The area of triangle ACD :

Solution: The area of triangle ACD using Heron's formula is obtained as:

$$\frac{1}{2} \|(\mathbf{C} - \mathbf{A}) \times (\mathbf{D} - \mathbf{A})\|$$

and it is found in the following python code:

```
codes/tri_area_ACD.py
```

Area of $\triangle ACD = 15.5 \text{ units}^2$

3. The area of quadrilateral $ABCD$:

Solution: Area of Quadrilateral $ABCD = \text{Area of } \triangle ABC + \text{Area of } \triangle ACD = 38 \text{ units}^2$

4. Quadrilateral $ABCD$ in Fig.?? is generated using the following python code

```
codes/quadrilateral/quadr.py
```

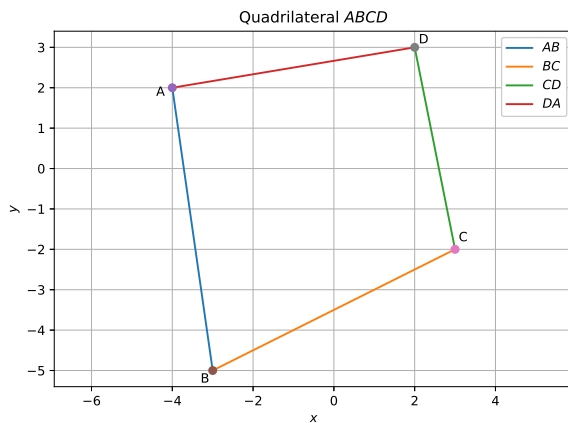


Fig. 2.2.4: Quadrilateral $ABCD$ using python

3.1 Points and Vectors

3.1.1 Problem: Find the point on the x-axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}$$

3.1.2 Solution:

1. From the given information,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 = \left\| \mathbf{x} - \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2$$

$$\|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 2 & -5 \end{pmatrix} \mathbf{x} = \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} -2 & 9 \end{pmatrix} \mathbf{x}$$

which can be simplified to obtain

$$\begin{pmatrix} 8 & -28 \end{pmatrix} \mathbf{x} = -56$$

Choose $\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix} 8 & -28 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = -56$$

$$\Rightarrow x = -7$$

$$\Rightarrow \text{The point is } \begin{pmatrix} -7 \\ 0 \end{pmatrix}$$

3.2 Points on a line

3.2.1 Problem: If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of \mathbf{P} such that $AP = \frac{3}{7}AB$ and \mathbf{P} lies on the line segment AB

3.2.2 Solution:

$$1. \mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Then \mathbf{P} that divides \mathbf{A}, \mathbf{B} in the ratio $k:1$ is

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (3.2.2.1.1)$$

For the given problem, $k = \frac{3}{4}$

Using the equation ??, the desired point is

$$\mathbf{P} = \frac{\frac{3}{4} \begin{pmatrix} 2 \\ -4 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ -2 \end{pmatrix}}{\frac{3}{4} + 1} \quad (3.2.2.1.2)$$

$$\mathbf{P} = \begin{pmatrix} -2/7 \\ -20/7 \end{pmatrix} \quad (3.2.2.1.3)$$

The following python code plots the Fig.??

```
codes/point_line/int_sec.py
```

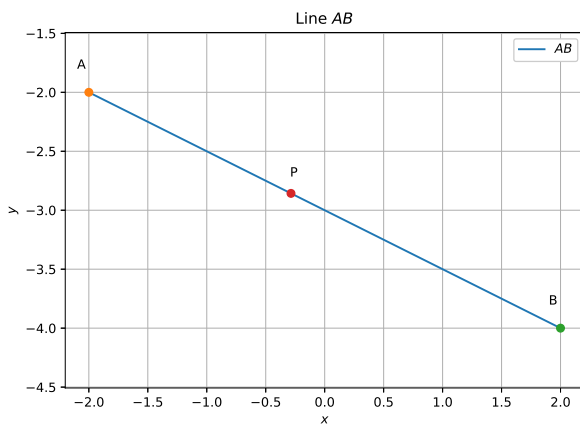


Fig. 3.2.2.1: Line AB using python

3.3 Lines and Planes

3.3.1 Problem: Write four solutions for each of the following equations

a) $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 7$

b) $\begin{pmatrix} \pi & 1 \end{pmatrix} \mathbf{x} = 9$

c) $\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = 0$

3.3.2 Solution:

1. \mathbf{x} are randomly chosen and substituted in the equation and solutions are found.

a)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 7 \quad (3.3.2.1.1)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation ??, $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 7$
 $\Rightarrow a = \frac{7}{2}$

Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 7$$

$\Rightarrow b = 7$
 Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 7$$

$\Rightarrow c = 3$
 Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 7$$

$\Rightarrow d = 5$

b)

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \mathbf{x} = 9 \quad (3.3.2.1.2)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation ??, $\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 9$

$$\Rightarrow a = \frac{9}{\pi}$$

Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 9$$

$\Rightarrow b = 9$
 Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 9$$

$$\Rightarrow c = \frac{8}{\pi}$$

Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} \pi & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 9$$

$$\Rightarrow d = 9 - \pi$$

c)

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.3.2.1.3)$$

Solution: Let $\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ Substituting in equation ??, $\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = 0$

$$\Rightarrow a = 0$$

Let $\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = 0$$

$$\Rightarrow b = 0$$

Let $\mathbf{x} = \begin{pmatrix} c \\ 1 \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow c = 4$$

Let $\mathbf{x} = \begin{pmatrix} 1 \\ d \end{pmatrix}$ Substituting in equation ??,

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ d \end{pmatrix} = 0$$

$$\Rightarrow d = \frac{1}{4}$$

3.4 Motion in a plane

3.4.1 Problem:

1. A man can swim with a speed of 4.0km/h in still water. How long does he take to cross a river 1.0km wide if the river flows steadily at 3.0km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?

3.4.2 Solution:

1. Let the man be at point $\mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The speed of the man is $\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

The speed of the river is $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Since the swimmer dive the river normal to the flow of river, therefore time taken by swimmer to cross the river,

$$t = \frac{d}{\|\mathbf{u}\|} = \frac{1\text{km}}{4\text{km/h}} = 15\text{mins}$$

Distance covered down the river = $t \times \|\mathbf{v}\|$

$$x = \frac{1}{4}\text{hr} \times 3\text{km/h} = 750\text{m}$$

The code for diagrammatic representation(??) of the solution is

codes/line/motion_plane/man_river.py

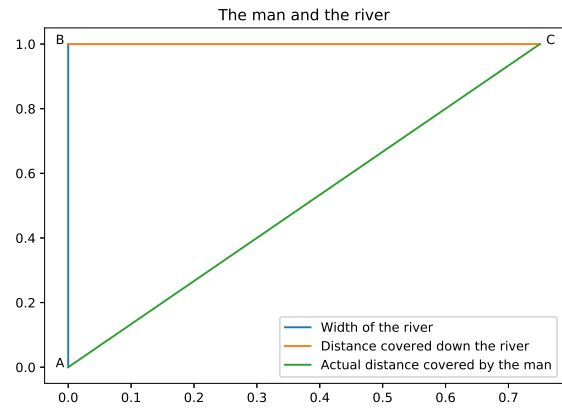


Fig. 3.4.2.1

$$\text{c) } \begin{pmatrix} x + y + z \\ x + y \\ y + z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

3.5.2 Solution:

1. This problem is solved by comparing the respective elements in both the matrices

a)

$$x = 1, y = 4, z = 3$$

b) $5 + z = 5$

$$\Rightarrow z = 0$$

$$x + y = 6 \text{ and } xy = 8$$

$$\Rightarrow x = 4, y = 2 \text{ or } x = 2, y = 4$$

$$x = 4, y = 2, z = 0$$

or

$$x = 2, y = 4, z = 0$$

c) $x + y + z = 9$ and $x + y = 5$

$$\Rightarrow z = 4$$

$$x + y + z = 9 \text{ and } y + z = 7$$

$$\Rightarrow x = 2$$

$$x + y + z = 9$$

$$2 + y + 4 = 9$$

$$\Rightarrow y = 3$$

$$x = 2, y = 3, z = 4$$

3.5 Matrix

3.5.1 Problem:

1. Find the values of x, y and z from the following equations:

$$\text{a) } \begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$$

$$\text{b) } \begin{pmatrix} x + y & 2 \\ 5 + z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$$

3.6 Determinants

3.6.1 Problem:

1. If $A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$.

3.6.2 Solution:

1. The determinant of a 3×3 matrix is given by:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \text{Det} &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ & a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ \Rightarrow \text{Det} &= 0 \end{aligned}$$

3.7 Linear Inequalities

3.7.1 Problem:

1. Solve $3x + 2y > 6$ graphically

3.7.2 Solution:

1. Let $3x + 2y = 6$ intersects the x-axis and y-axis at **A** and **B** respectively.

a) Let **A** = $\begin{pmatrix} x \\ 0 \end{pmatrix}$

$$3x = 6$$

$$\Rightarrow x = 2$$

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

b) Let **B** = $\begin{pmatrix} 0 \\ y \end{pmatrix}$

$$2y = 6$$

$$\Rightarrow y = 3$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

c) Origin = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not satisfy the equation $3x + 2y < 6$.

\Rightarrow The solution is the right side of the line $3x + 2y = 6$

2. The following python code is the diagrammatic representation of the solution in Fig.??

```
codes/linear_inequalities/linear_inequalities.py
```

4 CIRCLE

4.1 Problem

1. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$.

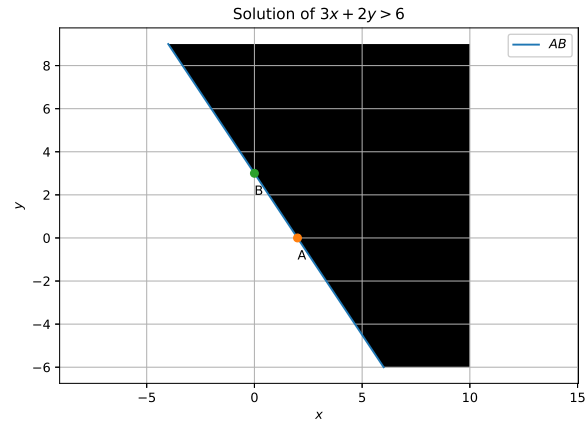


Fig. 3.7.2.2

4.2 Solution

1. The general equation of circle,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (4.2.1.1)$$

whose centre is $\begin{pmatrix} -g \\ -f \end{pmatrix}$ The vector form is,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2g & 2f \end{pmatrix} \mathbf{x} + c = 0$$

2. Point **A** = $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on the circle. So, point A satisfies the equation ??

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2g & 2f \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c = 0$$

$$8g + 2f + c + 17 = 0 \quad (4.2.2.1)$$

3. Point **B** = $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ lies on the circle. So, point B satisfies the equation ??

$$\begin{pmatrix} 6 \\ 5 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 2g & 2f \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} + c = 0$$

$$12g + 10f + c + 61 = 0 \quad (4.2.3.1)$$

4. Centre $\begin{pmatrix} -g \\ -f \end{pmatrix}$ lies on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} -g \\ -f \end{pmatrix} = 16$$

$$4g + f + 16 = 0 \quad (4.2.4.1)$$

5. Solving equations ??, ?? and ??

$$g = -3, f = -4, c = 15$$

$$\Rightarrow \text{Equation of the circle is } x^2 + y^2 - 6x -$$

$$8y + 15 = 0$$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6 & -8 \end{pmatrix} \mathbf{x} + 15 = 0$$

6. The circle in Fig.?? is generated using the following python code

```
codes/circle/circle.py
```

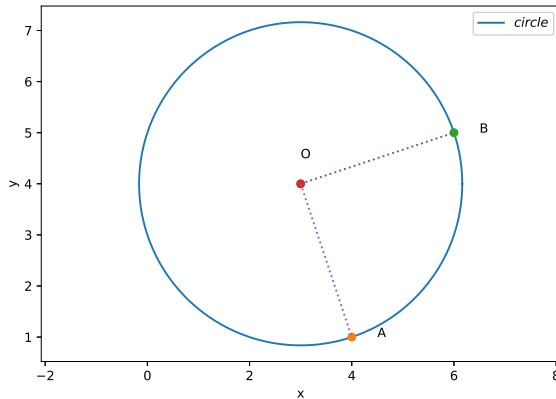


Fig. 4.2.6: Circle generated using python

5 CONICS

5.1 Problem

1. Find the roots of the quadratic equations:

- $x^2 - 3x - 10 = 0$
- $2x^2 + x - 6 = 0$
- $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
- $2x^2 - x + \frac{1}{8} = 0$
- $100x^2 - 20x + 1 = 0$

5.2 Solution

1. For a general polynomial equation of degree 2

$$p(x, y) \Rightarrow Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (5.2.1.1)$$

- a) $x^2 - 3x - 10 = 0$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0$$

The values of \mathbf{x} are found in the following python code

```
codes/conics/conics_1.py
```

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

which can be verified from the Fig.?.?. The following python code generates the fig.??

```
codes/conics/conics_1.py
```

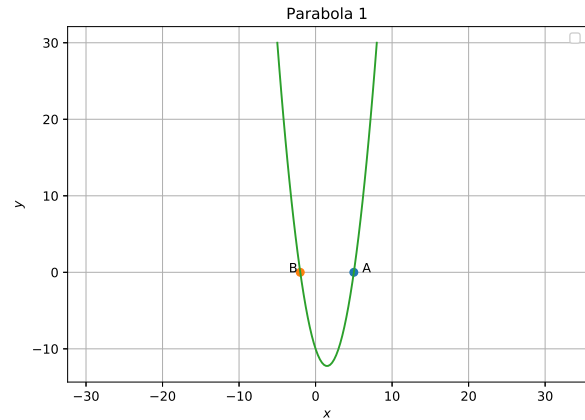


Fig. 5.2.1: Parabola 1

- b) $2x^2 + x - 6 = 0$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 6 = 0$$

The values of \mathbf{x} are found in the following python code

```
codes/conics/conics_2.py
```

$$\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

which can be verified from the Fig.?.?. The following python code generates the fig.??

```
codes/conics/conics_2.py
```

- c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 5\sqrt{2} = 0$$

The values of \mathbf{x} are found in the following python code

```
codes/conics/conics_3.py
```

$\mathbf{x} = \begin{pmatrix} -1.414 \\ 0 \end{pmatrix}, \begin{pmatrix} -3.535 \\ 0 \end{pmatrix}$ which can be verified from the Fig.?.?. The following python code generates the fig.??

```
codes/conics/conics_3.py
```

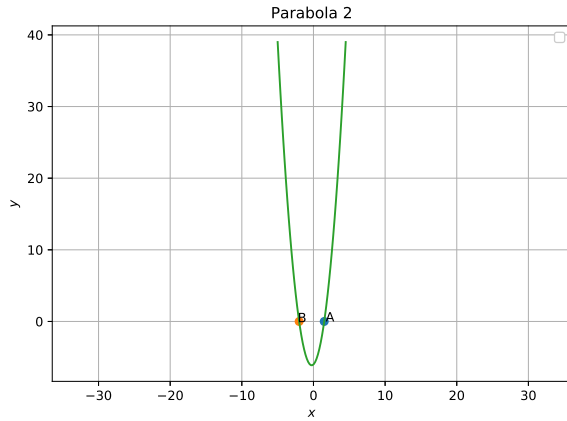


Fig. 5.2.1: Parabola 2

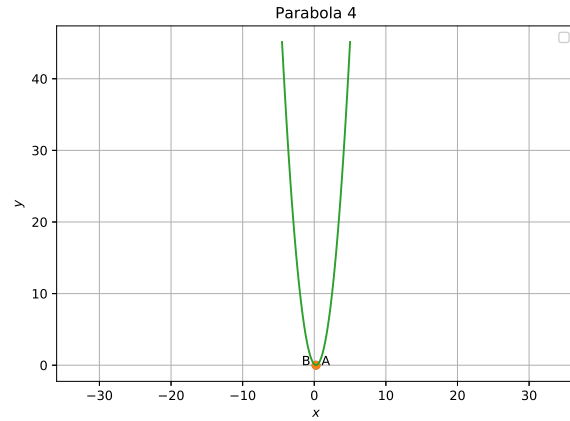


Fig. 5.2.1: Parabola 4

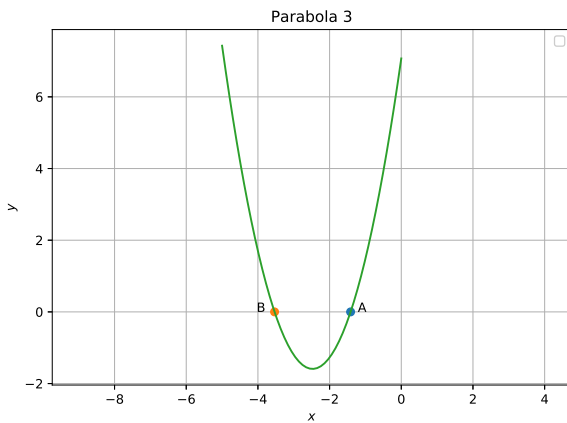


Fig. 5.2.1: Parabola 3

d) $2x^2 - x + \frac{1}{8} = 0$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + \frac{1}{8} = 0$$

The values of \mathbf{x} are found in the following python code

```
codes/conics/conics_4.py
```

$$\mathbf{x} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$$

which can be verified from the Fig.???. The following python code generates the fig.??

```
codes/conics/conics_4.py
```

e) $100x^2 - 20x + 1 = 0$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0$$

The values of \mathbf{x} are found in the following python code

```
codes/conics/conics_5.py
```

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.???. The following python code generates the fig.??

```
codes/conics/conics_5.py
```

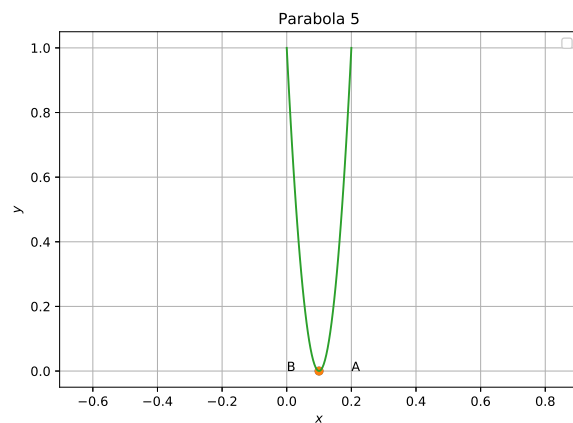


Fig. 5.2.1: Parabola 5