Continuous Math

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Contents

***Abstract*—This book provides a computational approach to continuous mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

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svn co https://github.com/gadepall/school/trunk/
    ncert/continuous/codes
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1 Curves

1.1 Examples

1. Find the value of each of the following polynomials at the indicated value of variables:
   a) $q(y) = 3y^3 - 4y + 11$ at $y = 2$.
   b) $p(t) = 4t^4 + 5t^3 - t2 + 6$ at $t = a$.
2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(t) = 2 + t + 2t^2 - t^3$
   b) $p(x) = x^3$
3. Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.
4. Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.
5. Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.
6. Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.
7. Factorize $x^3 - 23x^2 + 142x - 120$.
8. Verify that $3, -1, \frac{1}{3}$, are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.
9. Show that the function f given by
$$f(x) = \begin{cases} x^3 + 3 & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{1.1.9.1}$$
is not continuous at $x = 0$.
10. Discuss the continuity of the function f defined by $f(x) = x^2 + x + 1$.
11. Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}, x \neq 0$.
12. Show that every polynomial function is continuous.
13. Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.
14. Discuss the continuity of sine function.
15. Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.
16. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$
17. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$
18. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
19. Find the equation of the tangent to the curve $\begin{pmatrix} a\sin^3 t \\ b\cos^3 t \end{pmatrix}$ at $t = \frac{\pi}{2}$.

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20. Find the equation of tangents to the curve $y = \cos(x+y), -2\pi \le x \le 2\pi$ that are parallel to the line $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 0$.
21. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.
22. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.
23. Show that the function f given by
$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R} \tag{1.1.23.1}$$
is increasing on $\mathbf{R}$.
24. Prove that the function given by $f(x) = \cos x$ is
 a) decreasing in $(0, \pi)$.
 b) increasing in $(\pi, 2\pi)$ and
25. Find the intervals in which the function
$$f(x) = 4x^3 - 6x^2 - 72x + 30 \tag{1.1.25.1}$$
is
 a) increasing
 b) decreasing.
26. Find the intervals in which the function given by
$$f(x) = \sin x, x \in \left[0, \frac{\pi}{2}\right] \tag{1.1.26.1}$$
is
 a) increasing
 b) decreasing.
27. Find the intervals in which the function given by
$$f(x) = \sin x + \cos x, x \in [0, 2\pi] \tag{1.1.27.1}$$
is increasing or decreasing.
28. Find all points of local maxima and local minima of the function f given by
$$f(x) = x^3 - 3x + 3 \tag{1.1.28.1}$$
29. Find all points of local maxima and local minima of the function f given by
$$f(x) = 2x^3 - 6x^2 + 6x + 5 \tag{1.1.29.1}$$
30. Find the local maxima and minima of the function f given by
$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12 \tag{1.1.30.1}$$
31. Find the absolute maximum and minimum values of a function f given by
$$f(x) = 2x^3 - 15x^2 + 36x + 1, \quad x \in [1, 5]. \tag{1.1.31.1}$$
32. Find the absolute maximum and minimum values of a function f given by
$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, \quad x \in [1, 1]. \tag{1.1.32.1}$$
33. A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by
$$x = t^2\left(2 - \frac{t}{3}\right). \tag{1.1.33.1}$$
Find the time taken by it to reach Q and also find the distance between P and Q.
34. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.
35. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.
36. Find intervals in which the function given by
$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5} + 11 \tag{1.1.36.1}$$
is
 a) decreasing
 b) increasing
37. Show that the function f given by
$$f(x) = \tan^{-1}(\sin x + \cos x), \quad x > 0 \tag{1.1.37.1}$$
is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.
38. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.
39. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the

volume of the largest such box.

40. A manufacturer can sell x items at a price of $|\left(5-\frac{x}{500}\right)$ each. The cost price of x items is $|\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.
41. Find the limits
    a) $\lim_{x\to 1} x^3-x^2+1$
    b) $\lim_{x\to 1} x(x+1)$
    c) $\lim_{x\to 1} 1+x+x^2+\cdots+x^{1}0$
42. Find the limits
    a) $\lim_{x\to 1} \frac{x^2+1}{x+100}$
    b) $\lim_{x\to 2} \frac{x^3-4x^2+4x}{x^2-4}$
    c) $\lim_{x\to 1} \frac{x^2-4}{x^3-4x^2+4x}$
    d) $\lim_{x\to 1} \frac{x^3-2x^2}{x^2-5x+6}$
    e) $\lim_{x\to 1} \left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$
43. Evaluate
    a) $\lim_{x\to 1} \frac{x^{1}5-1}{x^{1}0-1}$
    b) $\lim_{x\to 2} \frac{\sqrt{1+x}}{x}$
44. Evaluate

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 2x} \tag{1.1.44.1}$$

45. Evaluate

$$\int_{-1}^{\frac{3}{2}} |x\sin(\pi x)|\,dx \tag{1.1.45.1}$$

46. Evaluate

$$\int_0^{\pi} \frac{x}{a^2\cos^2 x+b^2\sin^2 x}\,dx \tag{1.1.46.1}$$

47. Evaluate

$$\int_0^2 e^x\,dx \tag{1.1.47.1}$$

as a limit of a sum.

48. Evaluate the following integrals:
    a) $\int_2^3 x^2\,dx$
    b) $\int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2}\,dx$
    c) $\int_1^2 \frac{x}{(x+1)(x+2)}\,dx$
    d) $\int_0^{\frac{\pi}{4}} \sin^3 2t\cos 2t\,dt$
49. Evaluate

$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx \tag{1.1.49.1}$$

50. Evaluate

$$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx \tag{1.1.50.1}$$

51. Evaluate

$$\int_{-1}^{2} \left|x^3-x\right|\,dx \tag{1.1.51.1}$$

52. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx \tag{1.1.52.1}$$

53. Evaluate

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \tag{1.1.53.1}$$

54. Evaluate

$$\int_{-1}^{1} \sin^5 x\cos^4 x\,dx \tag{1.1.54.1}$$

55. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4+\cos^4 x}\,dx \tag{1.1.55.1}$$

56. Evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}\,dx \tag{1.1.56.1}$$

57. Evaluate

$$\int_0^{\frac{\pi}{2}} \log\sin x\,dx \tag{1.1.57.1}$$

58. Solve the differential equation

$$y_1=-4xy^2,\quad y(0)=1 \tag{1.1.58.1}$$

59. Find the equation of the curve passing through the point $\begin{pmatrix}1\\1\end{pmatrix}$, whose differential equation is $xdy=\left(2x^2+1\right)dx\,(x\neq 0)$.
60. Find the equaion of a curve passing through the point $\begin{pmatrix}-2\\3\end{pmatrix}$ given that the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is $\frac{2x}{y^2}$.
61. In a bank, principal increases continuously at the rate of 5% per year. In how many years will |100 double itself?

62. Solve
$$2ye^{\frac{x}{y}}\,dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0, \quad y(0) = 1 \tag{1.1.62.1}$$
63. Solve
$$y_1 + y\cot x = 2x + x^2 \cot x\,(x \neq 0), \quad y\left(\frac{\pi}{2}\right) = 0 \tag{1.1.63.1}$$
64. Find the equation of a curve passing through the point $\begin{pmatrix}0\\1\end{pmatrix}$. If the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.
65. Solve
$$\log y_1 = 3x + 4y, \quad y(0) = 0 \tag{1.1.65.1}$$

1.2 Exercises

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by
   a) $x + 1$
   b) $x - \frac{1}{2}$
   c) x
   d) $x + \pi$
   e) $5 + 2x$
2. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.
3. Determine which of the following polynomials has $(x + 1)$ as a factor:
   a) $x^3 + x^2 + x + 1$
   b) $x^4 + x^3 + x^2 + x + 1$
   c) $x^4 + 3x^3 + 3x^2 + x + 1$
   d) $x^3 - x^2 - \left(2 + \sqrt{2}\right) + \sqrt{2}$.
4. Determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $p(x) = x^4 - 4x^2 + x + 6, g(x) = x - 3$
5. Factorise :
   a) $x^3–2x^2–x + 2$
   b) $x^3–3x^2–9x–5$
   c) $x^3 + 13x^2 + 32x + 20$
   d) $2y^3 + y^2–2y–1$
6. Find the roots of the following equations:
   a) $x - \frac{1}{x} = 3, x \neq= 0$
   b) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq= -4, 7$
7. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
8. Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 2.
9. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
10. Find the slope of the normal to the curve $\mathbf{x} = a\begin{pmatrix}\cos^3\theta\\ \sin^3\theta\end{pmatrix}$ at $\theta = \frac{\pi}{4}$.
11. Find the slope of the normal to the curve $\mathbf{x} = \begin{pmatrix}1 - a\sin\theta\\ b\cos^2\theta\end{pmatrix}$ at $\theta = \frac{\pi}{2}$.
12. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to th x-axis.
13. Find the point on the curve $y = x^3–11x + 5$ at which the tangent is $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 11$.
14. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$.
15. Find the equations of the tangent and normal to the given curves at the indicated points:
    a) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}0\\5\end{pmatrix}$.
    b) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}1\\3\end{pmatrix}$.
    c) $y = x^3$ at $\begin{pmatrix}1\\1\end{pmatrix}$.
16. Show that the tangents to the curve $y = 7x^3+11$ at the points where $x = 2$ and $x = -2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3–2x^5$ find all the points at which the tangent passes through the origin.
19. Find the equation of the normal at the point $\begin{pmatrix}am^2\\am^3\end{pmatrix}$ for the curve $ay^2 = x^3$
20. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $\begin{pmatrix}1 & 14\end{pmatrix} + 4 = 0$.
21. Find the slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$. Show that the normal at any point θ to the curve $\mathbf{x} = \begin{pmatrix}a\cos\theta + a\theta\sin\theta\\ a\sin\theta–a\theta\cos\theta\end{pmatrix}$ is at a constant distance from the origin.
22. Find the slope of the tangent to the curve $\mathbf{x} = \begin{pmatrix}t^2 + 3t - 8\\ 2t^2 - 2t - 5\end{pmatrix}$ at the point $\begin{pmatrix}2\\-1\end{pmatrix}$.
23. Find the points on the curve $9y^2 = x^3$, where

the normal to the curve makes equal intercepts with the axes.

24. Find the area under $y = x^4, x = 1, x = 5$ and x-axis.
25. Find the area bounded by the curve $y = x^3, x = -2, x = 1$ and the x-axis.
26. Find the area bounded by the curve $y = x|x|, x = -1, x = 1$ and the x-axis.
27. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$.
28. Show that the function given by $f(x) = 3x+17$ is increasing on $\mathbf{R}$.
29. Show that the function given by $f(x) = e^{2x}$ is increasing on $\mathbf{R}$.
30. Show that the function given by

$$f(x) = \sin x \tag{1.2.30.1}$$

is

a) increasing in $\left(0, \frac{\pi}{2}\right)$
b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

31. Find the intervals in which the function given by

$$f(x) = 2x^3 - 3x^2 - 36x + 7 \tag{1.2.31.1}$$

is

a) increasing
b) decreasing.

32. Find the intervals in which the following functions are strictly increasing or decreasing

a) $(x+1)^3(x-3)^3$
b) $-2x^3 - 9x^2 - 12x + 1$

33. Show that

$$y = \log(1+x) - \frac{2x}{2+x}, x > -1, \tag{1.2.33.1}$$

is an increasing function of x throughout its domain.

34. Find the values of x for which $y = x(x-2)^2$ is an increasing function.
35. Prove that

$$y = \frac{4\sin\theta}{2+\cos\theta} - \theta \tag{1.2.35.1}$$

is an incresing function of θ in $\left[0, \frac{\pi}{2}\right]$.

36. Prove that the logarithmic function is increasing on $(0, \infty)$.
37. Which of the following functions are decreasing on $\left[0, \frac{\pi}{2}\right]$?

a) $\cos x$
b) $\cos 2x$
c) $\cos 3x$
d) $\tan x$

38. Find the intervals on which

$$f(x) = x^{100} + \sin x - 1 \tag{1.2.38.1}$$

is decreasing.

39. Let I be any interval disjoint from $[1, -1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.
40. Prove that the function f given by $f(x) = \log\sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.
41. Prove that the function f given by $f(x) = \log|\cos x|$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.
42. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in $\mathbf{R}$.
43. Find the interval(s) in which $f(x) = x^2e^{-x}$ is increasing.
44. Find the maximum and minimum values, if any, of $g(x) = x^3 + 1$.
45. Find the maximum and minimum values, if any of the following functions given by

a) $h(x) = \sin(2x) + 5$
b) $f(x) = |\sin(4x) + 3|$

46. Find the local maximum and minima, if any, of the following functions. Find also the local maximum and local minimum values, as the case may be

a) $g(x) = x^3 - 3x$
b) $h(x) = \sin x + \cos x, x \in \left(0, \frac{\pi}{2}\right)$
c) $f(x) = \sin x - \cos x, x \in (0, 2\pi)$
d) $f(x) = x^3 - 6x^2 + 9x + 15$
e) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
f) $g(x) = \frac{1}{x^2+2}$
g) $f(x) = x\sqrt{1-x}, 0 < x < 1$.

47. Prove that the following functions do not have maxima or minima:

a) $f(x) = e^x$
b) $g(x) = \log x$
c) $h(x) = x^3 + x^2 + x + 1$

48. Find the absolute maximum and absoute minimum value of the following functions in the given intervals

a) $f(x) = x^3, x \in (-2, 2)$
b) $f(x) = \sin x + \cos x, x \in (0, \pi)$.

49. Find both the maximum value and the minimum value of
$$3x^4 - 8x^3 + 12x^2 - 48x + 25, x \in [0,3]. \tag{1.2.49.1}$$
50. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?
51. What is the maximum value of the function $\sin x + \cos x$?
52. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, 1]$.
53. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Find the value of a.
54. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.
55. For all real values of x, the minimum value of
$$\frac{1 - x + x^2}{1 + x + x^2}. \tag{1.2.55.1}$$
56. Find the maximum value of
$$[x(x-1)]^{\frac{1}{3}}. \tag{1.2.56.1}$$
57. Using differentials, find the approximate value of each of the following
    a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$
    b) $(33)^{-\frac{1}{5}}$
58. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = 3$.
59. Find the intervals in which the function f given by
$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x} \tag{1.2.59.1}$$
is
    a) increasing
    b) decreasing
60. Find the interals in which the function f given by
$$f(x) = x^3 + \frac{1}{x^3}, \quad x \neq 0 \tag{1.2.60.1}$$
is
    a) increasing
    b) decreasing
61. Find the absolute maximum and minimum values of the function f given by
$$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi] \tag{1.2.61.1}$$
62. Find the points at which the function f given by
$$f(x) = (x-2)^4 (x+1)^3 \tag{1.2.62.1}$$
has
    a) local maxima
    b) local minima
    c) point of inflexion
63. Examine the following functions for continuity.
    a) $f(x) = \frac{1}{x-5}$
    b) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$
64. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.
    a) $f(x) = \begin{cases} x^3 - 3, & x \leq 2, \\ x^2 + 1, & x > 2 \end{cases}$
    b) $f(x) = \begin{cases} x^10 - 1, & x \leq 1, \\ x^2, & x > 1 \end{cases}$
65. Discuss the continuity of the following functions:
    a) $f(x) = \sin x + \cos x$
    b) $f(x) = \sin x - \cos x$
    c) $f(x) = \sin x \cos x$
66. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.
67. Find all points of discontinuity of f, where
$$f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0, \\ x + 1, & x \geq 0 \end{cases} \tag{1.2.67.1}$$
68. Determine if
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases} \tag{1.2.68.1}$$
is a continuous function.
69. Examine the continuity of
$$f(x) = \begin{cases} \sin x - \cos x, & x \neq 0, \\ -1, & x = 0 \end{cases} \tag{1.2.69.1}$$
70. Find values of k so that the following functions are continuous at the points indicated
    a) $\begin{cases} \frac{k\cos x}{\pi - 2x} & x \neq \frac{\pi}{2}, \\ 3, & x = \frac{\pi}{2} \end{cases}, \quad x = \frac{\pi}{2}$

b) $\begin{cases} kx+1 & x \le \pi, \\ \cos x, & x > \pi, \end{cases} \quad x = \pi$

71. Show that the function defined by $f(x) = cos(x^2)$ is a continuous function.
72. Show that the function defined by $f(x) = |cos x|$ is a continuous function.
73. Examine that $sin\,|x|$ is a continuous function.
74. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.
75. Evaluate the following limits
    a) $\lim_{x\to 4} \frac{4x+3}{x-2}$
    b) $\lim_{x\to -1} \frac{x^10+x^5+1}{x-1}$
    c) $\lim_{x\to 0} \frac{(x+1)^5-1}{x}$
    d) $\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4}$
    e) $\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3}$
    f) $\lim_{x\to 0} \frac{ax+b}{cx+1}$
    g) $\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$
    h) $\lim_{x\to 1} \frac{ax^2+bx+3}{cx^2+bx+a}, \quad a+b+c \neq 0$
    i) $\lim_{x\to 2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$
    j) $\lim_{x\to 0} \frac{\sin ax}{bx}$
    k) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}, \quad a, b \neq 0$
    l) $\lim_{x\to \pi} \frac{\sin(\pi-x)}{\pi(\pi-x)}, \quad a, b \neq 0$
    m) $\lim_{x\to 0} \frac{\cos x}{\pi-x}$
    n) $\lim_{x\to 0} \frac{\cos 2x-1}{\cos x-1}$
    o) $\lim_{x\to 0} \frac{ax+x\cos x}{b\sin x}$
    p) $\lim_{x\to 0} x\sec x$
    q) $\lim_{x\to 0} \frac{\sin ax+bx}{ax+\sin bx}, \quad a, b, a+b \neq 0$
    r) $\lim_{x\to 0} \csc - \cot x$
    s) $\lim_{x\to \frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}}$
76. Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$where
$$f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1), & x > 0 \end{cases} \tag{1.2.76.1}$$
77. Let $a_1, a_2, \ldots a_n$ be fixed real numbers and define a function
$$f(x) = (x-a_1)(x-a_2)\ldots(x-a_n) \tag{1.2.77.1}$$
What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x\to a} f(x)$.
78. If
$$\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi, \tag{1.2.78.1}$$
evaluate $\lim_{x\to 1} f(x)$.
79. If
$$f(x) = \begin{cases} mx^2+n & x < 0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x > 1, \end{cases} \tag{1.2.79.1}$$
for what integers m and n does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist ?
80. Integrate the following as limit of sums:
    (i) $\int_{-1}^{1} e^x\,dx$
    (ii) $\int_{-1}^{1} \left(x - e^{2x}\right) dx$
81. Evaluate the following definite integrals
    (i) $\int_2^3 \frac{1}{x}\,dx$
    (ii) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$
    (iii) $\int_0^{\frac{\pi}{4}} \sin 2x\,dx$
    (iv) $\int_0^{\frac{\pi}{2}} \cos 2x\,dx$
    (v) $\int_4^5 e^x\,dx$
    (vi) $\int_0^{\frac{\pi}{4}} \tan 2x\,dx$
    (vii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc 2x\,dx$
    (viii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,dx$
    (ix) $\int_0^1 \frac{dx}{1+x^2}\,dx$
    (x) $\int_2^3 \frac{dx}{x^2-1}\,dx$
    (xi) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
    (xii) $\int_2^3 \frac{x}{1+x^2}\,dx$
    (xiii) $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$
    (xiv) $\int_0^1 xe^{x^2}\,dx$
    (xv) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$
    (xvi) $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$
    (xvii) $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}\right) dx$
    (xviii) $\int_0^2 \frac{6x+3}{x^2+4}\,dx$
    (xix) $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4}\right) dx$
82. Find $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}\,dx$
83. Find $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}\,dx$
84. Evaluate the following definite integrals
    (i) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
    (ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$
    (iii) $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}\,dx$
    (iv) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x}\,dx$
    (v) $\int_0^1 x(1-x)^n\,dx$
    (vi) $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

(vii) $\int_0^1 x\sqrt{2-x}\,dx$
(viii) $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$
(ix) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$
(x) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$
(xi) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$
(xii) $\int_0^{2\pi} \cos^5 x\,dx$
(xiii) $\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}\,dx$
(xiv) $\int_0^{\pi} \log(1+\cos x)\,dx$
(xv) $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

85. Find the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1\right)dx \quad (1.2.85.1)$$

86. Find the value of

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \quad (1.2.86.1)$$

87. Evaluate the following definite integrals
(i) $\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1-\sin x}{1-\cos x}\right)dx$
(ii) $\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$
(iii) $\int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}\,dx$
(iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$
(v) $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}\,dx$
(vi) $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$
(vii) $\int_{\frac{\pi}{2}}^{\pi} \sin 2x\tan^{-1}(\sin x)\,dx$
(viii) $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx$

88. Prove that
(i) $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$
(ii) $\int_0^1 e^x\,dx = 1$
(iii) $\int_{-1}^1 x^{17}\cos^4\,dx = 0$
(iv) $\int_{\frac{\pi}{2}}^{\pi} \sin^3 x\,dx = \frac{2}{3}$
(v) $\int_{\frac{\pi}{4}}^{\pi} 2\tan^3 x\,dx = 1-\log 2$
(vi) $\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2}-1$

89. Evaluate $\int_0^1 e^{2-3x}\,dx$ as a limit of a sum.

90. Find the value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx = \frac{\pi}{2}-1$

91. Solve
(i) $\left(x^3+x^2+x+1\right)y_1 = 2x^2+x \quad y(0)=1$
(ii) $\left(x\left(x^2-1\right)\right)y_1 = 2x^2+x \quad y(2)=0$
(iii) $\cos(y_1) = y\tan x;\ y(0)=1$

92. Find the equation of a curve passing through the origin and whose differential equation is $y_1 = e^x\sin x$

93. For the differential equation $xyy_1 = (x+2)(y+2)$, find the solution curve passing through the point $\begin{pmatrix}1\\-1\end{pmatrix}$

94. Find the equation of a curve passing through the point $(0,-2)$ given that at any point (x,y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

95. At any point (x,y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4,-3)$. Find the equation of the curve given that it passes through $(-2,1)$.

96. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

97. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹100 double itself in 10 years.

98. In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹1000 is deposited with this bank, how much will it worth after 10 years.

99. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

100. Solve
(i) $(x+y)\,dy + (x-y)\,dx = 0, y(1)=1$
(ii) $x^2\,dy + \left(xy+y^2\right)dx = 0, y(1)=1$
(iii) $\left[x\sin^2\left(\frac{y}{x}-y\right)\right]dx + x\,dy = 0, y(1) = \frac{\pi}{4}$
(iv) $y_1 - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0, y(1)=0$
(v) $2xy + y^2 - 2x^2y_1 = 0, y(1)=2$

101. Solve
(i) $y_1 + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$
(ii) $\left(1+x^2\right)y_1 + 2xy = \frac{1}{1+x^2}, y(0)=1$
(iii) $y_1 - 3y\cot x = \sin 2x, y\left(\frac{\pi}{2}\right) = 2$

102. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is equal to the sum of the coordinates of the point.

103. Find the equation of a curve passing through

the point $\begin{pmatrix}0\\2\end{pmatrix}$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

104. Find the equation of the curve passing through the point $\begin{pmatrix}0\\\frac{\pi}{4}\end{pmatrix}$ whose differential equation is $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$
105. Solve
$$\left(1 + e^{2x}\right) dy + \left(1 + y^2\right) e^x\, dx = 0, \quad y(0) = 1 \tag{1.2.105.1}$$
106. Solve
$$(x - y)(dx - dy) = dx - dy, \quad y(0) = -1 \tag{1.2.106.1}$$
107. Solve
$$y_1 + y \cot x = 4x \csc x \quad y\left(\frac{\pi}{2}\right) = 0 \tag{1.2.107.1}$$
108. Solve
$$(x + 1)\, y_1 = 2e^{-y} - 1 \quad y(0) = 0 \tag{1.2.108.1}$$
109. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

2 Trigonometry

2.1 Examples

1. Convert $40°20'$ into radian measure.
2. Convert 6 radians into radian measure.
3. Find the radius of the circle in which a central angle of 60^o intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).
4. The minute hand of watch is 1.5 cm long. How far does its tip move in 40 minutes? ($\pi = 3.14$)
5. If the arcs of the same lengths in two circles subtend angles 65^o and 110^o at the centre, find the ratio of their radii.
6. If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric function.
7. If $\cot x = -\frac{5}{12}$, x lies in the second quadrant, find the values of other five trigonometric function.
8. Find the value of $\sin \frac{31\pi}{3}$.
9. Find the value of $\cos(-1710^o)$.
10. Prove that $3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$.
11. Find the value of $\sin 15^o$.
12. Find the value of $\tan\frac{13\pi}{12}$.
13. Prove that $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}$
14. Show that
$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x.$$
15. Prove that
$$\cos(\tfrac{\pi}{4} + x) + \cos(\tfrac{\pi}{4} - x) = \sqrt{2} \cos x$$
16. Prove that $\frac{\cos 7x + \cos 5x}{\cos 7x - \cos 5x} = \cot x$
17. Prove that $\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$
18. Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.
19. Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.
20. Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.
21. Solve $\cos x = \frac{1}{2}$.
22. Solve $\tan 2x = -\cot(x + \frac{\pi}{3})$.
23. Solve $\sin 2x - \sin 4x + \sin 6x = 0$.
24. Solve $2\cos^2 x + 3\sin x = 0$
25. If $\sin x = \frac{3}{5}, \cos y = -\frac{12}{13}$, where x and y both lies in second quadrant, find the value of $\sin(x + y)$.
26. Prove that $\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}$
.
27. Find the value of $\tan\frac{\pi}{8}$.

28. If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$
29. Prove that
$\cos^2 x + \cos^2(x + \frac{\pi}{3}) + \cos^2(x - \frac{\pi}{3}) = \frac{3}{2}$

2.2 Exercises

1. Find the radian measures corresponding to the following meausres:
(i) $25°$
(ii) $-47°30'$
(iii) $240°$
(iv) $520°$
2. Find the degree measures corresponding to the following radian measures(use π=3.14)
(i) $\frac{11}{16}$
(ii) -4
(iii) $\frac{5\pi}{3}$
(iv) $\frac{7\pi}{6}$
3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?
4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm?
5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.
6. If in two circles, arcs of the same length subtend angles $60°$ and $75°$ at the centre, find the ratio of their radii?
7. Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length
(i) 10 cm
(ii) 15 cm
(iii) 21 cm
8. Find the values of other five trigonometric functions
1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.
9. Find the values of the trigonometric functions
1. $\sin 765^o$
2. $cosec(-1410^o)$
3. $\tan\frac{19\pi}{3}$
4. $\sin\frac{-11\pi}{3}$
5. $\cot\frac{-15\pi}{4}$
10. Prove that
1. $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$
2. $2\sin^2\frac{\pi}{6} + cosec^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = -\frac{3}{2}$
3. $\cot^2\frac{\pi}{6} + cosec^2\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6}$=6
4. $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3}$=10
11. Find the value of
(i) $\sin 75^o$
(ii) $\tan 15^o$
12. Prove that
$\cos(\frac{\pi}{4} - x)\cos(\frac{\pi}{4} - y) - \sin(\frac{\pi}{4} - x)\sin(\frac{\pi}{4} - y) = \sin(x + y)$
13. Prove that
$\frac{\tan(\frac{\pi}{4}+x)}{\tan(\frac{\pi}{4}-x)} = (\frac{1+\tan x}{1-\tan x})^2$
14. Prove that
$\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos(\frac{\pi}{2}+x)} = \cot^2 x$
15. Prove that
$\cos(\frac{3\pi}{2}+x)\cos(2\pi+x)[\cot(\frac{3\pi}{2}-x)+\cot(2\pi+x)] = 1$
16. Prove that
$\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x = \cos x$
17. Prove that
$\cos(\frac{3\pi}{4} + x) - \cos(\frac{3\pi}{4} - x) = -\sqrt{2}\sin x$
18. Prove that
$\sin^2 6x - \sin^2 4x = \sin 2x\sin 10x$
19. Prove that
$\cos^2 2x - \cos^2 6x = \sin 4x\sin 8x$

20. Prove that

$$\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$$

21. Prove that

$$\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$$

22. Prove that

$$\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$$

23. Prove that

$$\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$$

24. Prove that

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan(\frac{x-y}{2})$$

25. Prove that

$$\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$$

26. Prove that

$$\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$$

27. Prove that

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$$

28. Prove that

$$\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$$

29. Prove that

$$\tan 4x = \frac{4\tan x(1 - \tan^2 x)}{1 - 6\tan^2 x + \tan^4 x}$$

30. Prove that

$$\cos 4x = 1 - 8\sin^2 x \cos^2 x$$

31. Prove that

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

32. Find the principle and general solutions of the following equations:
    1. $\tan x = \sqrt{3}$
    2. $\sec x = 2$
    3. $\cot x = -\sqrt{3}$
    4. $cosec x = -2$

33. Find the general solution for each of the following equations:
    1. $\cos 4x = \cos 2x$
    2. $\cos 3x + \cos x - \cos 2x = 0$
    3. $\sin 2x + \cos x = 0$
    4. $\sec^2 2x = 1 - \tan 2x$
    5. $\sin x + \sin 3x + \sin 5x = 0$

34. Prove that
    1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$
    2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$
    3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2(\frac{x+y}{2})$
    4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2(\frac{x-y}{2})$
    5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$
    6. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$
    7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\frac{x}{2\cos\frac{3x}{2}}$

35. Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following:
    1. $\tan x = -\frac{4}{3}$, x in second quadrant.
    2. $\sin x = \frac{1}{4}$, x in second quadrant.
    3. $\cos x = -\frac{1}{3}$, x in third quadrant.

3 Calculus

3.1 Examples

1. Find the derivative of the function given by $f(x) = \sin\left(x^2\right)$.
2. Find the derivative of $\tan(2x + 3)$.
3. Find $\frac{dy}{dx}$ if $y + \sin y = \cos x$.
4. Find the derivative of $f(x) = \sin^{-1} x$ assuming it exists.
5. Find the derivative of $f(x) = \tan^{-1} x$ assuming it exists.
6. Differentiate the following with respect to x.
   a) e^x
   b) $\sin(\log x), x > 0$
   c) $\cos^{-1}(e^x)$
   d) $e^{\cos x}$.

7. Differentiate
$$\sqrt{\frac{(x-3)\left(x^2+4\right)}{3x^2+4x+5}} \tag{3.1.7.1}$$
8. Differentiate a^x w.r.t. x, where a is a positive constant.
9. Differentiate $x^{\sin x}, x > 0$ w.r.t. x.
10. Find $\frac{dy}{dx}$, if $Y^x + x^y + x^x = a^b$.
11. Find $\frac{dy}{dx}$, if $x = a\cos\theta, y = a\sin\theta$.
12. Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$.
13. Find $\frac{dy}{dx}$, if $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$.
14. Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
15. Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.
16. If $y = A\sin x + B\cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.
17. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.
18. If $y = \sin^{-1} x$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.
19. Differentiate the following with respect to x.
    a) $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$
    b) $e^{\sec^2 x} + 3\cos^{-1} x$
    c) $\log_7(\log x)$
    d) $\cos^{-1}(\sin x)$
    e) $\tan^{-1}\left(\frac{1}{1+\cos x}\right)$
    f) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$
20. Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $x \in (0, \pi)$.
21. For a positive constant a, find $\frac{dy}{dx}$, where
$$y = a^{t+\frac{1}{t}}, x = \left(t + \frac{1}{t}\right)^a \tag{3.1.21.1}$$
22. Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.
23. Find the derivative of $\sin x$ at $x = 0$.
24. Find the derivative of $f(x) = \frac{1}{x}$.
25. Find the derivative of $f(x) = 1 + x + x^2 + x^3 + \dots + x^50$ at $x = 1$.
26. Find the derivative of $f(x) = \frac{x+1}{x}$.
27. Find the derivative of $\sin x$.
28. Find the derivative of $\tan x$.
29. Find the derivative of $f(x) = \sin^2 x$.
30. Find the derivative of f from the first principle, where f is given by
    a) $f(x) = \frac{2x+3}{x-2}$
    b) $f(x) = x + \frac{1}{x}$
31. Find the derivative of f from the first principle, where $f(x)$ is
    a) $\sin x + \cos x$
    b) $x\sin x$
32. Compute the derivative of
    a) $f(x) = \sin 2x$
    b) $g(x) = \cot x$
33. Find the derivative of
    a) $\frac{x^5 - \cos x}{\sin x}$
    b) $\frac{x + \cos x}{\sin x}$
34. Write an an anti-derivative for each of the following functions using the method of inspection:
    a) $\cos 2x$
    b) $3x^2 + 4x^3$
    c) $\frac{1}{x}, x \neq 0$
35. Find the following integrals:
    a) $\int \frac{x^3-1}{x^2}\,dx$
    b) $\int \left(x^{\frac{2}{3}} + 1\right)x^2\,dx$
    c) $\int \left(x^{\frac{2}{3}} + 2e^x - \frac{1}{x}\right)x^2\,dx$
36. Find the following integrals:
    a) $\int (\sin x + \cos x)\,dx$
    b) $\int \csc x(\csc + \cot x)\,dx$
    c) $\int \frac{1-\sin x}{\cos^2 x}\,dx$
37. Find an anti-derivative F of f defined by $f(x) = 4x^3 - 6$, where $F(0) = 3$.
38. Integrate the following functions w.r.t x:
    a) $\sin mx$
    b) $2x\sin\left(x^2+1\right)$
    c) $\frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}$
    d) $\frac{\sin\left(\tan^{-1} x\right)}{1+x^2}$
39. Find the following integrals:
    a) $\int \sin^3 x\cos^2 x\,dx$
    b) $\int \frac{\sin x}{\sin(x+a)}\,dx$
    c) $\int \frac{1}{1+\tan x}\,dx$
40. Find
    a) $\int \cos^2 x\,dx$
    b) $\int \sin 2x\cos 3x\,dx$
    c) $\int \sin^3 x\,dx$
41. Find the following integrals
    a) $\int \frac{dx}{x^2-16}$
    b) $\int \frac{dx}{\sqrt{2x-x^2}}$
42. Find the following integrals
    a) $\int \frac{dx}{x^2-6x+13}$
    b) $\int \frac{dx}{3x^2+13x-10}$
    c) $\int \frac{dx}{\sqrt{5x^2-2x}}$
43. Find the following integrals
    a) $\int \frac{x+2}{2x^2+6x+5}\,dx$
    b) $\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx$

44. Find
$$\int \frac{dx}{(x+1)(x+2)} \tag{3.1.44.1}$$
45. Find
$$\int \frac{x^2+1}{x^2-5x+6}\,dx \tag{3.1.45.1}$$
46. Find
$$\int \frac{3x-2}{(x+1)^2(x+3)}\,dx \tag{3.1.46.1}$$
47. Find
$$\int \frac{x^2}{(x^2+1)^2(x^2+4)}\,dx \tag{3.1.47.1}$$
48. Find
$$\int \frac{(3\sin\phi-2)\cos\phi}{5-\cos^2\phi-4\sin\phi}\,dx \tag{3.1.48.1}$$
49. Find
$$\int \frac{x^2+x+1}{(x+2)(x^2+1)}\,dx \tag{3.1.49.1}$$
50. Find
$$\int x\cos x\,dx \tag{3.1.50.1}$$
51. Find
$$\int \log x\,dx \tag{3.1.51.1}$$
52. Find
$$\int xe^x\,dx \tag{3.1.52.1}$$
53. Find
$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx \tag{3.1.53.1}$$
54. Find
$$\int e^x\sin x\,dx \tag{3.1.54.1}$$
55. Find
   a) $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$
   b) $\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx$
56. Find
$$\int \sqrt{x^2+2x+5}\,dx \tag{3.1.56.1}$$
57. Find
$$\int \sqrt{3-2x-x^2}\,dx \tag{3.1.57.1}$$
58. Find
$$\int \cos 6x\sqrt{1+\sin 6x}\,dx \tag{3.1.58.1}$$
59. Find
$$\int \frac{\left(x^4-x\right)^{\frac{1}{4}}}{x^5}\,dx \tag{3.1.59.1}$$
60. Find
$$\int \frac{x^4}{(x-1)(x^2+1)}\,dx \tag{3.1.60.1}$$
61. Find
$$\int [\log(\log x)] + \frac{1}{(\log x)^2}\,dx \tag{3.1.61.1}$$
62. Find
$$\int \left[\sqrt{\cot x}+\sqrt{\tan x}\right]dx \tag{3.1.62.1}$$
63. Find
$$\int \frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4(2x)}}\,dx \tag{3.1.63.1}$$
64. Verify that $y = e^{-3x}$ is a solution of the differential equation
$$y_2+y_1-6y=0 \tag{3.1.64.1}$$
65. Verify that $y = a\cos x + b\sin x$ is a solution of the differential equation
$$y_2+y=0 \tag{3.1.65.1}$$
66. Form the differential equation representing the family of curves $y = a\sin(x+b)$, where a, b are arbitrary constants.
67. Find the general solution of the differential equation
$$y_1 = \frac{x+1}{2-y} \tag{3.1.67.1}$$
68. Find the general solution of the differential equation
$$y_1 = \frac{1+y^2}{1+x^2} \tag{3.1.68.1}$$
69. Show that the differential equation $(x-y)\,y_1 =$

$x + 2y$ is homogeneous and solve it.

70. Solve $x\cos\left(\frac{x}{y}\right)y_1 = y\cos\left(\frac{y}{x}\right) + x$.
71. Show that the family of curves for which the slope of the tangent at any point $\begin{pmatrix} x \\ y \end{pmatrix}$ on it is $\frac{x^2+y^2}{xy}$, is given by $x^2 - y^2 = c$.
72. Solve
$$y_1 - y = \cos x \tag{3.1.72.1}$$
73. Solve
$$xy_1 + 2y = x^2 \tag{3.1.73.1}$$
74. Solve
$$y\,dx - \left(x + 2y^2\right)dy = 0 \tag{3.1.74.1}$$
75. Solve
$$y\,dx - \left(x + 2y^2\right)dy = 0 \tag{3.1.75.1}$$
76. Verify that $y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation
$$y_2 - 2ay_1 + \left(a^2 + b^2\right)y = 0 \tag{3.1.76.1}$$
77. Solve
$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right) \tag{3.1.77.1}$$
78. Solve the differntial equation
$$\left(\tan^{-1}x - x\right)dy = \left(1 + y^2\right)dx \tag{3.1.78.1}$$

3.2 Exercises

1. Differentiate the following functions with respect to x
   a) $\sin\left(x^2 + 5\right)$
   b) $\cos(\sin x)$
   c) $\sin(ax + b)$
   d) $\sec\left(\tan\sqrt{x}\right)$
   e) $\frac{\sin(ax+b)}{\cos(cx+d)}$
   f) $\cos x^3 \sin^2\left(x^5\right)$
   g) $2\sqrt{\cot\left(x^2\right)}$
   h) $\cos\left(\sqrt{x}\right)$
2. Find $\frac{dy}{dx}$ in the following:
   a) $2x + 3y = \sin x$
   b) $2x + 3y = \sin y$
   c) $ax + by^2 = \cos y$
   d) $xy + y^2 = \tan x + y$
   e) $x^3 + x^2y + xy^2 + y^3 = 81$
   f) $\sin^2 y + \cos xy = \kappa$
   g) $\sin^2 x + \cos^2 y = 1$
   h) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
   i) $y = \tan^{-1}\left(\frac{3x-x^2}{1-3x^2}\right), x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
   j) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
   k) $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) < x < 1$
   l) $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$
   m) $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
   n) $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$
3. Differentiate the following w.r.t. x:
   a) $\frac{e^x}{\sin x}$
   b) $e^{\sin^{-1}x}$
   c) e^{x^3}
   d) $\sin\left(\tan^{-1}e^{-x}\right)$
   e) $\log(\cos e^x)$
   f) $e^x + e^{x^2} + \cdots + e^{x^5}$
   g) $\sqrt{e^{\sqrt{x}}}, x > 0$
   h) $\log(\log x), x > 1$
   i) $\frac{\cos x}{\log x}, x > 0$
   j) $\cos(\log x + e^x), x > 0$
4. Differentiate the following w.r.t. x
   a) $\cos x\cos 2x\cos 3x$
   b) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
   c) $(\log x)^{\cos x}$
   d) $x^x - 2^{\sin x}$
   e) $(x+3)^2(x+4)^3(x+5)^4$
   f) $\left(x + \frac{1}{x}\right)^x + x^{1+\frac{1}{x}}$
   g) $(\log x)^x + (\sin x)^{\cos x}$
   h) $(\sin x)^x + \sin^{-1}\sqrt{x}$
   i) $x^{\sin x} + (\sin x)^{\cos x}$
   j) $x^{\cos x} + \frac{x^2+1}{x^2-1}$
   k) $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$
   l) $x^y + y^x = 1$
   m) $y^x = x^y$
   n) $(\cos x)^y = (\cos y)^x$
   o) $xy = e^{x-y}$
5. Find the derivative of the function given by $f(x) = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'1$.
6. Differentiate $\left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$ in three ways mentioned below:
   a) by using product rule
   b) by expanding the product to obtain a single polynomial
   c) by logarithmic differentiation.

Do they all give the same answer?

7. Without eliminating the parameter, find $\frac{dy}{dx}$ in the following
   a) $x = 2at^2, y = at^4$
   b) $x = a\cos\theta, y = b\cos\theta$
   c) $x = \sin t, y = \cos t$
   d) $x = 4t, y = \frac{4}{t}$
   e) $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$
   f) $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$
   g) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
   h) $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$
   i) $x = a\sec\theta, y = b\tan\theta$
   j) $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$
   k) If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$
   show that $\frac{dy}{dx} = -\frac{y}{x}$.
8. Find the second order derivatives of the following functions
   a) $x^2 + 3x + 2$
   b) x^{20}
   c) $x\cos x$
   d) $\log x$
   e) $x^3\log x$
   f) $x^x\sin 5x$
   g) $e^{6x}\cos 3x$
   h) $\tan^{-1} x$
   i) $\log(\log x)$
   j) $\sin(\log x)$
9. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$
10. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y.
11. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$
12. If $y = Ae^{mx} + Be^{nx}$, show that $y_2 - (m+n)y_1 + mny = 0$.
13. If $y = 500e^{7x} + 600e^{-7x}$, show that $y_2 = 49y$
14. If $e^y(x+1) = 1$, show that $y_2 = y_1^2$
15. If $y = \left(\tan^{-1} x\right)^2$, show that $\left(x^2+1\right)y_2 + 2x\left(x^2+1\right)y_1 = 2$
16. If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.
17. Verify mean value theorem, if $f(x) = x^3 - 5x^2 - 3x, x \in [a, b]$ where $a = 1, b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$
18. Differentiate the following functions w.r.t x
   a) $\left(3x^2 - 9x + 5\right)^9$
   b) $\sin^3 x + \cos^6 x$
   c) $(5x)^{3\cos 2x}$
   d) $\sin^{-1}\left(x\sqrt{x}\right), \quad 0 \le x \le 1$
   e) $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$
   f) $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$
   g) $(\log x)^{\log x}, x > 1$
   h) $\cos(a\cos x + b\sin x)$, for some constant a and b.
   i) $(\sin x - \cos x)^{\sin x - \cos x}, \quad \frac{\pi}{4}, < x < \frac{3\pi}{4}$
   j) $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.
   k) $x_+^{x^2-3}(x-3)^{x^2}$, for $x > 3$.
19. Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < x < \frac{\pi}{2}$.
20. Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1 - x^2}, \quad 0 < x < 1$
21. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that
$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} \tag{3.2.21.1}$$
22. If $\cos y = x\cos(a + y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$
23. if $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find y_2
24. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.
25. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.
26. Using the fact that
$$\sin(x + y) = \sin x\cos y + \cos x\sin y, \tag{3.2.26.1}$$
show that
$$\cos(x + y) = \cos x\cos y - \sin x\sin y \tag{3.2.26.2}$$
27. If
$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}, \tag{3.2.27.1}$$
prove that
$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} \tag{3.2.27.2}$$
28. If $y = e^{a\cos^{-1} x}, -1 \le x \le 1$, show that
$$\left(1 - x^2\right)y_2 - xy_1 - ay^2 = 0. \tag{3.2.28.1}$$

29. Find the derivative of the following functions from the first principle:
   a) $x^3 - 27$
   b) $\frac{1}{x^2}$
   c) $\frac{x+1}{x-1}$
30. For the function
$$f(x) = \frac{x^100}{100} + \frac{x^99}{99} + \cdots + \frac{x^2}{2} + x + 1. \tag{3.2.30.1}$$
prove that $f^{'}(1) = 100f^{'}(0)$.
31. Find the derivative of
$$x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n \tag{3.2.31.1}$$
for some fixed real number a.
32. For some constans a and b, find the derivative of
   a) $\left(ax^2 + b\right)^2$
   b) $\frac{x-a}{x-b}$
33. Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.
34. Find the derivative of
   a) $2x - \frac{3}{4}$
   b) $\left(5x^3 + 3x - 1\right)(x - 1)$
   c) $x^{-3}\left(3 - 4x^{-5}\right)$.
   d) $x^5\left(x - 6x^{-9}\right)$
   e) $x^{-4}\left(3 - 4x^{-5}\right)$
   f) $\frac{2}{x+1} - \frac{x^2}{3x-1}$
35. Find the derivative of $\cos x$ from the first principle.
36. Find the derivative of the following functions:
   a) $\sin x \cos x$
   b) $\sec x$
   c) $5 \sec x + 4 \cos x$.
   d) $\csc x$
   e) $3 \cot x + 5 \csc x$
   f) $5 \sin x - 6 \cos x + 7$
   g) $2 \tan x - 7 \sec x$
37. Find the derivative of the following functions:
   (i) $(-x)^{-1}$
   (ii) $\sin(x + 1)$
   (iii) $\cos\left(x - \frac{\pi}{8}\right)$
   (iv) $\frac{ax+b}{cx+d}$
   (v) $(px + q)\left(\frac{r}{x} + s\right)$
   (vi) $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$
   (vii) $\frac{1}{ax^2+bx+c}$
   (viii) $\frac{ax+b}{px^2+qx+r}$
   (ix) $\frac{px^2+qx+r}{ax+b}$
   (x) $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$
   (xi) $4\sqrt{x} - 2$
   (xii) $(ax + b)^n$
   (xiii) $(ax + b)^n (cx + d)^m$
   (xiv) $\sin(x + a)$
   (xv) $\csc x \cot x$
   (xvi) $\frac{\cos x}{1+\sin x}$
   (xvii) $\frac{\sin x+\cos x}{\sin x-\cos x}$
   (xviii) $\frac{\sec x-1}{\sec x+1}$
   (xix) $\sin^n x$
   (xx) $\frac{a+b \sin x}{c+d \cos x}$
   (xxi) $\frac{\sin(x+a)}{\cos x}$
   (xxii) $x^4 (5 \sin x - 3 \cos x)$
   (xxiii) $\left(x^2 + 1\right) \cos x$
   (xxiv) $\left(ax^2 + \sin x\right)(p + q \cos x)$
   (xxv) $(x - \tan x)(x + \cos x)$
   (xxvi) $\frac{4x+5 \sin x}{3x+7 \cos x}$
   (xxvii) $\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$
   (xxviii) $(x)(1 + \tan x)$
   (xxix) $(x + \sec x)(x - \tan x)$
   (xxx) $\frac{x}{\sin^n x}$
38. Find anti-derivative of each of the following functions
   a) $\sin 2x$
   b) $\cos 2x$
   c) e^{2x}
   d) $(ax + b)^2$
   e) $\sin 2x - 4e^{2x}$
39. Find the following integrals:
   a) $\int 4e^{3x} + 1, dx$
   b) $\int x^2\left(1 - \frac{1}{x^2}\right), dx$
   c) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$
   d) $\int \left(ax^2 + bx + c\right), dx$
   e) $\int \left(2x^2 + e^x\right), dx$
   f) $\int \frac{x^3+5x^2-4}{x^2}, dx$
   g) $\int \frac{x^3-x^2+x-1}{x-1}, dx$
   h) $\int (1 - x)\sqrt{x}, dx$
   i) $\int \sqrt{x}\left(3x^2 + 2x + 3\right), dx$
   j) $\int (2x - 3 \cos x + e^x), dx$
   k) $\int \left(2x^2 - 3 \sin x + 5\sqrt{x}\right), dx$
   l) $\int \sec x (\sec x + \tan x), dx$
   m) $\int \frac{\sec^2 x}{\csc^2 x}, dx$

40. Find anti-derivative of

$$\sqrt{x}+\frac{1}{\sqrt{x}} \quad (3.2.40.1)$$

41. If

$$\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}, \quad f(2)=0 \quad (3.2.41.1)$$

Find $f(x)$.

42. Integrate the following functions:

(i) $\frac{2x}{1+x^2}$
(ii) $\frac{\left(\log x\right)^2}{x}$
(iii) $\frac{1}{x+x\log x}$
(iv) $\sin x\sin\left(\cos x\right)$
(v) $\sin\left(ax+b\right)\cos ax+b$
(vi) $\sqrt{ax+b}$
(vii) $x\sqrt{x+2}$
(viii) $x\sqrt{1+2x^2}$
(ix) $(4x+2)\sqrt{x^2+x+1}$
(x) $\frac{1}{x-\sqrt{x}}$
(xi) $\frac{x}{\sqrt{x+4}}, \quad x>0$
(xii) $\left(x^3-1\right)^{\frac{1}{3}}x^5$
(xiii) $\frac{x^2}{\left(2+3x^3\right)^3}$
(xiv) $\frac{1}{x\left(\log x\right)^m} \quad x>0, m\neq 1$
(xv) $\frac{x}{9-4x^2}$
(xvi) e^{2x+3}
(xvii) $\frac{x}{e^{x^2}}$
(xviii) $e^{\frac{\tan^{-1}x}{1+x^2}}$
(xix) $\frac{e^{2x}-1}{e^2x+1}$
(xx) $\frac{e^{2x}-e^{-2x}}{e^2x+e^{-2x}}$
(xxi) $\tan^2\left(2x-3\right)$
(xxii) $\sec^2\left(7-4x\right)$
(xxiii) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$
(xxiv) $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$
(xxv) $\frac{1}{\cos^2x(1-\tan x)^2}$
(xxvi) $\frac{\cos\sqrt{x}}{\sqrt{x}}$
(xxvii) $\sqrt{\sin 2x}\cos 2x$
(xxviii) $\frac{\cos x}{\sqrt{1+\sin x}}$
(xxix) $\cot x\log x\sin x$
(xxx) $\frac{\sin x}{1+\cos x}$
(xxxi) $\frac{\sin x}{(1+\cos x)^2}$
(xxxii) $\frac{1}{1+\cot x}$
(xxxiii) $\frac{1}{1-\tan x}$
(xxxiv) $\frac{\sqrt{\tan x}}{\sin x\cos x}$
(xxxv) $\frac{\left(1+\log x\right)^2}{x}$
(xxxvi) $\frac{(x+1)\left(x+\log x\right)^2}{x}$
(xxxvii) $\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

43. Find $\int\frac{10x^9+10^x\ln 10}{x^10+10^x}, dx$

44. Find $\int\frac{dx}{\sin^2x\cos^2x}, dx$

45. Find the integrals of the following functions:

(i) $\sin^2\left(2x+5\right)$
(ii) $\sin 3x\cos 4x$
(iii) $\cos 2x\cos 4x\cos 6x$
(iv) $\sin^3\left(2x+1\right)$
(v) $\sin^3x\cos^3x$
(vi) $\sin x\sin 2x\sin 3x$
(vii) $\sin 4x\sin 8x$
(viii) $\frac{1-\cos x}{1+\cos x}$
(ix) $\frac{\cos x}{1+\cos x}$
(x) $\sin^4x$
(xi) $\cos^4x$
(xii) $\frac{\sin^2x}{1+\cos x}$
(xiii) $\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$
(xiv) $\frac{\cos x-\sin x}{1+\sin 2x}$
(xv) $\tan^3 2x\sec 2x$
(xvi) $\tan^4x$
(xvii) $\frac{\sin^3x+\cos^3x}{\sin^2x\cos^2x}$
(xviii) $\frac{\cos 2x+2\sin^2x}{\cos^2x}$
(xix) $\frac{\cos 2x}{(\cos x+\sin x)^2}$
(xx) $\sin^{-1}\left(\cos x\right)$
(xxi) $\frac{1}{\cos(x-a)\cos(x-b)}$

46. Find $\frac{\sin^2x-\cos^2x}{\sin^2x\cos^2x}$

47. Find $\frac{e^x(1+x)}{\cos^2(e^x)}$

48. Integrate the following functions:

(i) $\frac{3x^2}{x^6+1}$
(ii) $\frac{1}{\sqrt{1+4x^2}}$
(iii) $\frac{1}{\sqrt{1+(2-x)^2}}$
(iv) $\frac{1}{\sqrt{9-25x^2}}$
(v) $\frac{3x}{1+2x^4}$
(vi) $\frac{x^2}{1-x^6}$
(vii) $\frac{x-1}{\sqrt{x^2-1}}$
(viii) $\frac{x^2}{\sqrt{x^6+a^6}}$
(ix) $\frac{\sec^2x}{\sqrt{\tan^2x+4}}$
(x) $\frac{1}{\sqrt{x^2+2x+2}}$
(xi) $\frac{1}{9x^2+6x+5}$
(xii) $\frac{1}{\sqrt{7-6x-x^2}}$
(xiii) $\frac{1}{\sqrt{(x-1)(x-2)}}$
(xiv) $\frac{1}{\sqrt{8+3x-x^2}}$
(xv) $\frac{1}{\sqrt{(x-a)(x-b)}}$

(xvi) $\frac{4x+1}{\sqrt{2x^2+x-3}}$
(xvii) $\frac{x+2}{\sqrt{x^2-1}}$
(xviii) $\frac{5x-2}{1+2x+3x^2}$
(xix) $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$
(xx) $\frac{x+2}{\sqrt{4x-x^2}}$
(xxi) $\frac{x+2}{\sqrt{x^2+2x+3}}$
(xxii) $\frac{x+3}{3x^2-2x-5}$
(xxiii) $\frac{5x+3}{\sqrt{x^2+4x+10}}$

49. Find $\int \frac{dx}{x^2+2x+2}\, dx$
50. Find $\int \frac{dx}{\sqrt{9x-4x^2}}\, dx$
51. Integrate the following:
   (i) $\frac{x}{(x+1)(x+2)}$
   (ii) $\frac{1}{x^2-9}$
   (iii) $\frac{3x-1}{(x-1)(x-2)(x-3)}$
   (iv) $\frac{x}{(x-1)(x-2)(x-3)}$
   (v) $\frac{x}{x^2+3x+2}$
   (vi) $\frac{1-x^2}{x(1-2x)}$
   (vii) $\frac{x}{\left(x^2+1\right)(x-1)}$
   (viii) $\frac{x}{(x+2)(x-1)^2}$
   (ix) $\frac{3x+5}{x^3-x^2-x+1}$
   (x) $\frac{2x-3}{\left(x^2-1\right)(2x+3)}$
   (xi) $\frac{5x}{\left(x^2-4\right)(x+1)}$
   (xii) $\frac{x^3+x+1}{x^2-1}$
   (xiii) $\frac{2}{(1-x)\left(1+x^2\right)}$
   (xiv) $\frac{3x-1}{(x+2)^2}$
   (xv) $\frac{1}{x^4-1}$
   (xvi) $\frac{x}{x^n+1}$
   (xvii) $\frac{\cos x}{(1 \sin x)(2-\sin x)}$
   (xviii) $\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}$
   (xix) $\frac{1}{e^x-1}$
52. Find $\int \frac{x\,dx}{(x-1)(x-2)}$
53. Find $\int \frac{dx}{x\left(x^2+1\right)}$
54. Integrate the following functions:
   (i) $x \sin x$
   (ii) $x \sin 3x$
   (iii) $x^2 e^x$
   (iv) $x \log x$
   (v) $x \log 2x$
   (vi) $x^2 \log x$
   (vii) $x \sin^{-1} x$
   (viii) $x \tan^{-1} x$
   (ix) $x \cos^{-1} x$
   (x) $\left(\sin^{-1} x\right)^2$
   (xi) $\frac{\cos^{-1} x}{\sqrt{1-x^2}}$
   (xii) $x \sec^2 x$
   (xiii) $\tan^{-1} x$
   (xiv) $x (\log x)^2$
   (xv) $\left(x^2+1\right) \log x$
   (xvi) $e^x (\sin x + \cos x)$
   (xvii) $\frac{xe^x}{(1+x)^2}$
   (xviii) $e^x \left(\frac{1+\sin x}{1+\cos x}\right)$
   (xix) $e^x \left(\frac{1}{x} - \frac{1}{x^2}\right)$
   (xx) $\frac{(x-3)e^x}{(x-1)^3}$
   (xxi) $e^2 x \sin x$
   (xxii) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$
55. Find $\int x^2 e^{x^3}\, dx$
56. Find $\int e^x \sec x (1 + \tan x)\, dx$
57. Integrate the following functions:
   (i) $\sqrt{4-x^2}$
   (ii) $\sqrt{1-4x^2}$
   (iii) $\sqrt{x^2+4x+6}$
   (iv) $\sqrt{x^2+4x+1}$
   (v) $\sqrt{1-4x-x^2}$
   (vi) $\sqrt{x^2+4x-5}$
   (vii) $\sqrt{1+3x-x^2}$
   (viii) $\sqrt{x^2+3x}$
   (ix) $\sqrt{1+\frac{x^2}{9}}$
58. Integrate $\int \sqrt{1+x^2}\, dx$
59. Integrate $\int \sqrt{x^2-8x+7}\, dx$
60. Show that

$$\int_0^a f(x)g(x)\, dx = 2\int_0^a f(x)\, dx \qquad (3.2.60.1)$$

if

$$f(x) = f(a-x)g(x) + g(a-x) \qquad = 4 \qquad (3.2.60.2)$$

61. Integrate the following functions:
   (i) $\frac{1}{x-x^3}$
   (ii) $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$
   (iii) $\frac{1}{x\sqrt{ax-x^2}}$
   (iv) $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$
   (v) $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$
   (vi) $\frac{5x}{(x+1)\left(x^2+9\right)}$
   (vii) $\frac{\sin x}{\sin(x-a)}$
   (viii) $\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$
   (ix) $\frac{\cos x}{\sqrt{4-\sin^2 x}}$
   (x) $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

(xi) $\frac{1}{\cos(x+a)\cos(x+b)}$
(xii) $\frac{x^3}{\sqrt{1-x^8}}$
(xiii) $\frac{e^x}{\sqrt{2+e^x}}$
(xiv) $\frac{1}{(x^2+1)(x^2+4)}$
(xv) $\cos^3 xe^{\log \sin x}$
(xvi) $e^{3\log x}\left(x^4+1\right)^{-1}$
(xvii) $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$
(xviii) $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x \in [0,1]$
(xix) $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$
(xx) $\frac{2+\sin 2x}{1+\cos 2x}e^x$
(xxi) $\frac{x^2+x+1}{(x+1)^2(x+2)}$
(xxii) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$
(xxiii) $\frac{\sqrt{x^2+1}\log\left(x^2+1\right)-2\log x}{x^4}$

62. Find $\int \frac{dx}{e^x+e^{-x}}\,dx$
63. Find $\int \frac{\cos 2x}{\sin x+\cos x}\,dx$
64. Verify that the given functions is a solution of the corresponding differential equation:
(i) $y = e^x + 1; y_2 - y_1 = 0$
(ii) $y = x^2 + 2x + C; y_1 - 2x - 2 = 0$
(iii) $y = \cos x + C; y_1 + \sin x = 0$
(iv) $y = \sqrt{1+x^2}; y_1 = \frac{xy}{1+x^2}$
(v) $y = Ax; xy_1 = y, x \neq 0$
(vi) $y = x\sin x;$
$xy_1 = y + x\sqrt{x^2-y^2}, (x \neq 0, x > y \text{ or } x < -y)$
(vii) $xy = \log y + C; y_1 = \frac{y^2}{1-xy}, (xy \neq 1)$
(viii) $y - \cos y = x; y^2y_1 + y^2 + 1 = 0$
(ix) $y = \sqrt{a^2-x^2}, x \in (a,-a); x+yy_1 = 0, (y \neq 0)$
65. Form the differential equation representing the following family of curves where a, b are arbitrary constants.
(i) $y = ae^{3x} + be^{-2x}$
(ii) $y = e^{2x}(a+bx)$
(iii) $y = e^x(a\cos x + b\sin x)$
66. Find the general solution for each of the following differntial equations
(i) $y_1 = \frac{1-\cos x}{1+\cos x}$
(ii) $y_1 = \sqrt{4-y^2} \quad (|y| < 2)$
(iii) $y_1 + y = 1 \quad (y \neq 1)$
(iv) $\sec^2 x\tan y\,dx + \sec^2 y\tan x\,dy = 0$
(v) $(e^x+e^{-x})\,dy - (e^y+e^{-y})\,dx = 0$
(vi) $y_1 = \left(1+x^2\right)\left(1+y^2\right)$
(vii) $y\log y\,dx - x\,dy = 0$
(viii) $x^5y_1 = -y^5$
(ix) $y_1 = \sin^{-1} x$
(x) $e^x\tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$
67. Find the general solution of $y_1 = e^{x+y}$
68. Solve
(i) $\left(x^2+xy\right)dy = \left(x^2+y^2\right)$
(ii) $y_1 = \frac{x+y}{x}$
(iii) $(x-y)\,dy - (x+y)\,dx = 0$
(iv) $\left(x^2-y^2\right)dx + 2xy\,dy = 0$
(v) $x^2y_1 = x^2 - 2y^2 + xy$
(vi) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$
(vii) $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y\,dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x\,dx$
(viii) $xy_1 - y + x\sin\left(\frac{y}{x}\right) = 0$
(ix) $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$
(x) $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$
69. Solve
(i) $y_1 + 2y = \sin x$
(ii) $y_1 + 3y = e^{-2x}$
(iii) $y_1 + \frac{y}{x} = x^2$
(iv) $y_1 + y\sec x = \tan x \quad \left(0 \le x \le \frac{\pi}{2}\right)$
(v) $\cos^2 xy_1 + y = \tan x \quad \left(0 \le x \le \frac{\pi}{2}\right)$
(vi) $xy_1 + 2y = x^2\log x$
(vii) $x\log xy_1 + y = \frac{2}{x}\log x$
(viii) $\left(1+x^2\right)dy + 2xy\,dx = \cot x\,dx$
(ix) $xy_1 + y - x + xy\cot x = 0$
(x) $(x+y)\,y_1 = 1$
(xi) $y\,dx + \left(x-y^2\right)dy = 0$
(xii) $\left(x+3y^2\right)y_1 = y, \quad y > 0$
70. Solve

$$xy_1 - y = 2x^2 \tag{3.2.70.1}$$

71. Solve

$$\left(1-y^2\right)y_1 + xy = ay \quad (-1 < y < 1) \tag{3.2.71.1}$$

72. For each of the exercises below, erify that the given function is a solution of the corresponding diferential equation:
73. Solve
(i) $xy = ae^x + be^{-x} + x^2; \quad xy_2 + 2y_1 - xy + x^2 - 2 = 0$
(ii) $y = e^x(a\cos x + b\sin x); \quad y_2 - 2y_1 + 2y = 0$
(iii) $y = x\sin 3x; \quad y_2 + 9y_1 - 6\cos 3x = 0$
(iv) $x^2 = 2y^2\log y; \quad \left(x^2+y^2\right)y_1 - xy = 0$
74. Prove that $x^2 - y^2 = c\left(x^+y^2\right)^2$ is the general

solution of differential equation

$$\left(x^3 - 3xy^2\right) dx = \left(y^3 - 3x^2y\right) dy, \quad (3.2.74.1)$$

where c is a parameter.

75. Find the general solution of the differential equation

$$y_1 + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \quad (3.2.75.1)$$

76. Show that the general solution of the differential equation

$$y_1 + \frac{y^2+y+1}{x^2+x+1} = 0 \quad (3.2.76.1)$$

is

$$(x+y+1) = A\,(1-x-y-2xy)\,, \quad (3.2.76.2)$$

where A is a parameter.

77. Solve

$$ye^{\frac{x}{y}}\,dx = \left(xe^{\frac{x}{y}} + y^2\right) dy \quad (3.2.77.1)$$

78. Solve

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right] \quad (3.2.78.1)$$

79. Solve

$$\frac{y\,dx - x\,dy}{y} = 0 \quad (3.2.79.1)$$

80. Solve

$$e^x\,dy + (ye^x + 2x)\,dx = 0 \quad (3.2.80.1)$$