

Problems in Linear Algebra

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1 INTRODUCTION

1.1 Definitions

1. The *inner product* of \mathbf{P} and \mathbf{Q} is defined as

$$\mathbf{P}^T \mathbf{Q} = p_1 q_1 + p_2 q_2 \quad (1.1.1)$$

2. The *norm* of a vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (1.1.2)$$

is defined as

$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \quad (1.1.2)$$

3. The *length* of PQ is defined as

$$\|\mathbf{P} - \mathbf{Q}\| \quad (1.1.3)$$

4. The *direction vector* of the line PQ is defined as

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \quad (1.1.4)$$

5. *Orthogonality*

$$PQ \perp RS \iff (\mathbf{P} - \mathbf{Q})^T (\mathbf{R} - \mathbf{S}) = 0 \quad (1.1.5)$$

6. The point dividing PQ in the ratio $k : 1$ is

$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k + 1} \quad (1.1.6)$$

7. The *area* of $\triangle PQR$ is the *determinant*

$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \quad (1.1.7)$$

1.2 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.2.1)$$

2. Find the length of PQ for

a) $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$;

b) $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$;

c) $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$.

3. Using direction vectors, show that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are the vertices of a parallelogram.

4. Using Baudhayana's theorem, show that the points $\begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ are the vertices of a right-angled triangle. Repeat using orthogonality.

5. Plot the points $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and prove that they are the vertices of a rectangle.

6. Show that $\mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are the vertices of an isosceles triangle.

7. In the last question, find the distance of the vertex \mathbf{A} of the triangle from the middle point of the base BC .

8. Prove that the points $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are the vertices of a square.

9. Prove that the points $\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are the vertices of a parallelogram. Find $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$, the mid points of AB, BC, CD, AD respectively. Show that EG and FH bisect each other.

10. Prove that the points $\begin{pmatrix} 21 \\ -2 \end{pmatrix}, \begin{pmatrix} 15 \\ 10 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -12 \end{pmatrix}$ are the vertices of a rectangle, and find

the coordinates of its centre.

11. Find the lengths of the medians of the triangle whose vertices are at the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.
12. Find the coordinates of the points that divide the line joining the points $\begin{pmatrix} -35 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -10 \end{pmatrix}$ into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 25 \\ 10 \end{pmatrix}$.
14. Prove that the middle point of the line joining the points $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ is a point of trisection of the line joining the points $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$.
15. The points $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$.
16. The point $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is the intersection of the diagonals of a parallelogram two of whose vertices are at the points $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$. Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.
18. Find the coordinates of points which divide the join of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ externally in the ratio 2 : 3, and also externally in the ratio 3 : 2.
19. Prove the centroid of $\triangle ABC$ is

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.2.19)$$

1.3 Loci

1. A point moves so that its distance from the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is double its distance from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the equation of its locus.
2. Find the equation of the perpendicular bisector of the line joining the points $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

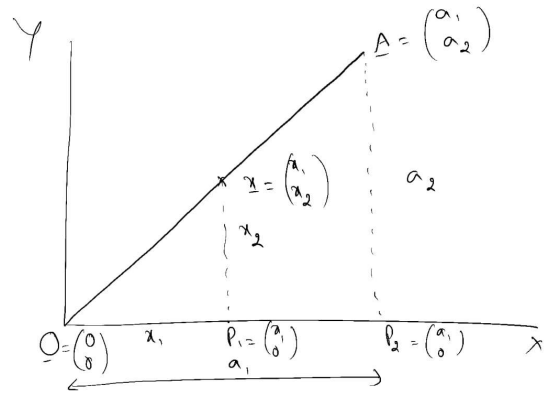


Fig. 2.1.1

3. Find the equation of the circle of radius 5 with centre at $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.
4. A point moves so that its distance from the y-axis is equal to the distance from the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the equation of its locus.
5. A point moves so that the sum of the squares of its distance from the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is constant. Find the equation of the locus.
6. A point moves so that its distance from the axis of x is twice its distance from the point $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the equation of the locus.
7. A point moves in such a way that with the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ it forms a triangle of area 8.5. Show that its locus has an equation

$$\{(1 \ 5)\mathbf{x}\}\{(1 \ 5)\mathbf{x} - 34\} = 0 \quad (1.3.7)$$

2 THE STRAIGHT LINE

2.1 Definitions

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 2.1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \quad (2.1.1.1)$$

$$\Rightarrow x_2 = mx_1 \quad (2.1.1.2)$$

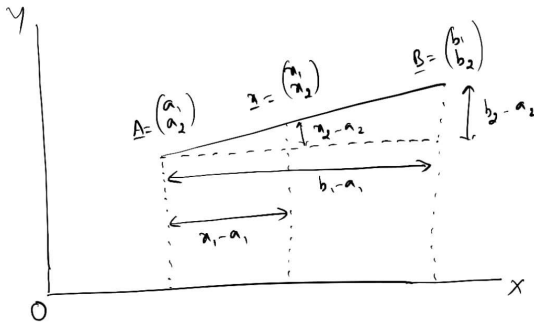


Fig. 2.1.2

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = x_1 \mathbf{m} \quad (2.1.1.3)$$

In general, the above equation is written as

$$\mathbf{x} = \lambda \mathbf{m}, \quad (2.1.1.4)$$

where \mathbf{m} is the direction vector of the line.

2. Find the equation of AB in Fig. 2.1.2

Solution: From Fig. 2.1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \quad (2.1.2.1)$$

$$\Rightarrow x_2 = mx_1 + a_2 - ma_1 \quad (2.1.2.2)$$

From (2.1.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \quad (2.1.2.3)$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (2.1.2.4)$$

$$= \mathbf{A} + \lambda \mathbf{m} \quad (2.1.2.5)$$

3. *Translation:* If the line shifts from the origin by \mathbf{A} , (2.1.2.5) is obtained from (2.1.1.4) by adding \mathbf{A} .

4. Find the length of \mathbf{A} in Fig. 2.1.1

Solution: Using Baudhayana's theorem, the length of the vector \mathbf{A} is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \quad (2.1.4.1)$$

Also, from (2.1.1.4),

$$\|\mathbf{A}\| = \lambda \sqrt{1 + m^2} \quad (2.1.4.2)$$

Note that λ is the variable that determines the length of \mathbf{A} , since m is constant for all points

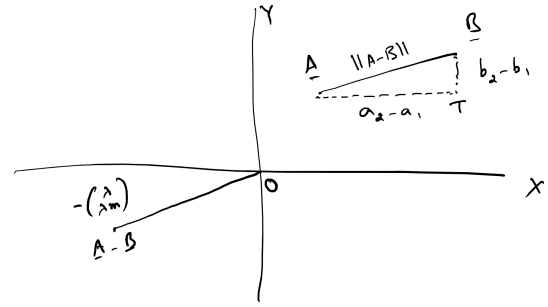


Fig. 2.1.5

on the line.

5. Find $\mathbf{A} - \mathbf{B}$.

Solution: See Fig. 2.1.5. From (2.1.2.5), for some λ ,

$$\mathbf{B} = \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (2.1.5.1)$$

$$\Rightarrow \mathbf{A} - \mathbf{B} = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (2.1.5.2)$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 2.1.5.

6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

7. Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (2.1.7.1)$$

8. The *normal* to the vector \mathbf{m} is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \quad (2.1.8.1)$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (2.1.8.2)$$

9. From (2.1.7.1), the equation of a line can also be expressed as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A} + \lambda \mathbf{n}^T (\mathbf{B} - \mathbf{A}) \quad (2.1.9.1)$$

$$\Rightarrow \mathbf{n}^T \mathbf{x} = c \quad (2.1.9.2)$$

10. The unit vectors on the x and y axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.1.10.1)$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.1.10.2)$$

11. If a be the *intercept* of the line

$$\mathbf{n}^T \mathbf{x} = c \quad (2.1.11.1)$$

on the x -axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line.
Thus,

$$\mathbf{n}^T \begin{pmatrix} a \\ 0 \end{pmatrix} = c \quad (2.1.11.2)$$

$$\Rightarrow a = \frac{c}{\mathbf{n}^T \mathbf{e}_1} \quad (2.1.11.3)$$

2.2 Intercepts

- Find the intercepts made on the axes by the straight lines whose equations are

$$\begin{array}{ll} \text{a) } (2 \ 3)\mathbf{x} = 2 & \text{d) } \left(\frac{1}{a+b} \ \frac{1}{a-b}\right)\mathbf{x} = \frac{1}{a^2-b^2} \\ \text{b) } (1 \ -3)\mathbf{x} = -5 & \text{e) } (1 \ -m)\mathbf{x} = -c \\ \text{c) } (1 \ -1)\mathbf{x} = 0 & \end{array}$$

- Write down the equations of straight lines which make the following pairs of intercepts on the axes:

$$\begin{array}{ll} \text{a) } 3, -4 & \text{c) } \frac{1}{a}, \frac{1}{b} \\ \text{b) } -5, 6 & \text{d) } 2a, -2a \end{array}$$

- A straight line passes through a fixed point $\begin{pmatrix} h \\ k \end{pmatrix}$ and cuts the axes in **A**, **B**. Parallels to the axes through **A** and **B** intersect in **P**. Find the equation of the locus of **P**.

2.3 Line Equation

- Find the equations of two straight lines at a distance 3 from the origin and making an angle of 120° with OX .
- Find the equation of a straight line making an angle of 60° with OX and passing through the point $(2, -2)$. Transform the equation to the form

$$x \cos \alpha + y \sin \alpha = p$$

- Find the equation of the straight line that passes through the points $(2, 3)$ and $(3, 2)$. What is its inclination to OX ?
- Find the equation of the straight line through the point $(5, 7)$ that makes equal intercepts on the axes.
- Find the equations of the sides of a triangle whose corners are $(2, 4)$, $(-4, 1)$ and $(2, -3)$.
- For the same triangle find the equations of the lines joining the corners to the middle points of the opposite sides.

- Find the equation of a straight line passing through the point $(2, -3)$ parallel to the line $4x - y + 7 = 0$.
- Find the intercepts on the axes made by a straight line which passes through the point $(3, -1)$ and makes an angle of 30° with OX .
- Find the equation of the straight line through the points $(3, -4)$, $(2, 3)$ and of the parallel line through $(5, 2)$.
- What is the distance from the origin of the line $4x - y - 7$? Write down the equation of a parallel line at double the distance.
- Find the equation of the straight line through the point $(3, -4)$ parallel to the line joining the origin to the point $(2, -1)$.
- Write down the equation of the straight line which makes intercepts 2 and -7 on the axes, and of the parallel line through the point $(3, -1)$.
- Find the equations of the straight line joining the points $(3, 4)$, $(-2, 1)$ and of the parallel line through the origin.
- ABC is a triangle and A , B and C are the points $(2, 3)$, $(5, -1)$ and $(-4, 2)$. Find the equation of the straight line through A parallel to BC .
- Find the equation of a line parallel to $2x + 5y = 11$ passing through the middle point of the join of the points $(-7, 3)$, $(5, -11)$.
- The base of a triangle passes through a fixed point (f, g) and the sides are bisected at right angles by the axes. Prove that the locus of the vertex is the line

$$gx + fy = 0$$