JEE Advanced:2019

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Contents

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***Abstract*—This manual has exercises based on problems in JEE advanced.**

1 Complex Numbers: Optimization

1.1 Consider the optimization problem

$$\max_{z} \frac{1}{|z-1|} \tag{1.1}$$
$$s.t. \quad \left|z-2+\jmath\right| \geq \sqrt{5} \tag{1.2}$$

Show that it can be reframed as

$$\min_{\mathbf{x}} \|\mathbf{x}-\mathbf{c}_1\|^2 \tag{1.3}$$
$$s.t. \quad \|\mathbf{x}-\mathbf{c}_2\|^2 \geq 5 \tag{1.4}$$

where

$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.5}$$

1.2 Explain the optimization problem with a figure.
Solution: Fig. 1.1 explains (1.3) where z_0 is the set of points comprising of the intersection of the smallest circle Γ : with the largest circle Ω : $r_2 \geq \sqrt{5}$ with radii r_1 and $r_2 \geq \sqrt{5}$ respectively.

1.3 Obtain the Lagrangian.
Solution: The Lagrangian is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x}-\mathbf{c}_1\|^2 - \lambda\left\{\|\mathbf{x}-\mathbf{c}_2\|^2 - r_2^2\right\} \tag{1.6}$$

1.4 Use the KKT conditions to obtain the minima.
Solution: From the KKT conditions,

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \tag{1.7}$$
$$\implies \mathbf{x}-\mathbf{c}_1 - \lambda(\mathbf{x}-\mathbf{c}_2) = 0 \tag{1.8}$$
$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1-\lambda} \tag{1.9}$$

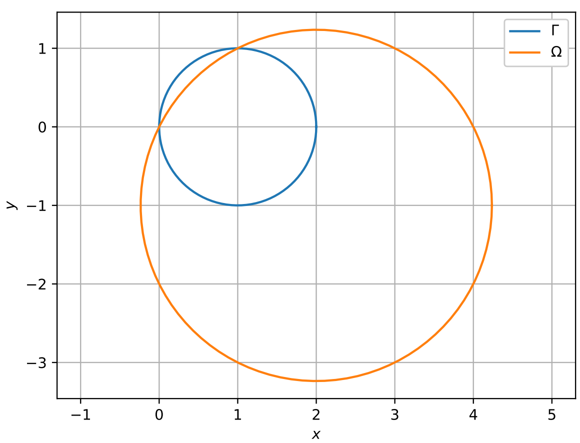

Fig. 1.1

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{1.10}$$

$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \tag{1.11}$$

Substituting from (1.9) in (1.11),

$$\left\| \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} - \mathbf{c}_2 \right\|^2 - r_2^2 = 0 \tag{1.12}$$

$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \tag{1.13}$$

$$= 1 \pm \sqrt{\frac{2}{5}} \tag{1.14}$$

Fig. 1.2 plots Γ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \tag{1.15}$$

1.5 If the maximum value is obtained at z_0, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{1.16}$$

Solution: From (1.9),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} \tag{1.17}$$

$$\implies z_0 = \frac{1}{1 - \lambda} \left(1 - 2\lambda + \jmath\lambda\right) \tag{1.18}$$

$$\text{or,} \ \arg \frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\left\{z_0\right\}}{\jmath\left(\Im\left\{z_0\right\} + 1\right)} \tag{1.19}$$

$$= \frac{2\left(1 - \lambda\right) - \left(1 - 2\lambda\right)}{\jmath} \tag{1.20}$$

$$= -\jmath \tag{1.21}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

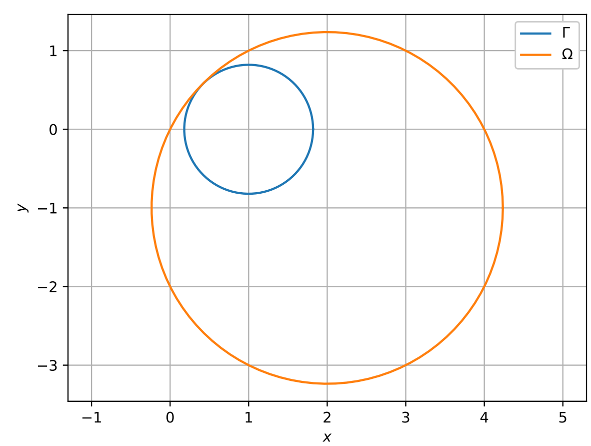

Fig. 1.2

1.6 Show that the set

$$D = \left\{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\right\}, r_2 > 0 \tag{1.22}$$

is nonconvex.
Solution: Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \tag{1.23}$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \tag{1.24}$$

$$\implies \mathbf{x}_2 \in D. \tag{1.25}$$

Suppose

$$\mathbf{x} = \theta \mathbf{x}_1 + \left(1 - \theta\right) \mathbf{x}_2 \tag{1.26}$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \tag{1.27}$$
$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \tag{1.28}$$
$$\text{or, } \mathbf{x} \notin D \tag{1.29}$$

Thus, by definition, D is not a convex set.

2 Matrices: Cayley-Hamilton Theorem

2.1 Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{2.1}$$

where α, β are real functions of θ and $\mathbf{I}$ is the identity matrix. Find the characteristic equation of $\mathbf{M}$.
Solution: (2.1) can be expressed as

$$\mathbf{M}^2 - \alpha\mathbf{M} - \beta\mathbf{I} = 0 \tag{2.2}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{2.3}$$

2.2 Find α and β.
Solution: Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta + \cos^4\theta \tag{2.4}$$
$$\beta = -\sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{2.5}$$

2.3 If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{2.6}$$
$$\beta^* = \min_\theta \beta(\theta), \tag{2.7}$$

find $\alpha^* + \beta^*$.
Solution:

$$\because \alpha = \sin^4\theta + \cos^4\theta = 1 - \frac{\sin^2 2\theta}{2}, \tag{2.8}$$
$$\alpha^* = \frac{1}{2}, \tag{2.9}$$

Similarly,

$$-\beta = \sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{2.10}$$
$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \tag{2.11}$$
$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2}\right)^2 + \frac{7}{4} \tag{2.12}$$

Thus,

$$\beta^* = -\frac{37}{16} \tag{2.13}$$
$$\implies \alpha^* + \beta^* = -\frac{29}{16} \tag{2.14}$$

3 Vector Algebra

3.1 The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.1}$$

intersects the circle

$$\Omega : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{3.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{R} = -\frac{3}{5} \tag{3.3}$$

Find m.
Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.4}$$

The intersection of (3.1) and (3.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{3.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T(\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \tag{3.6}$$

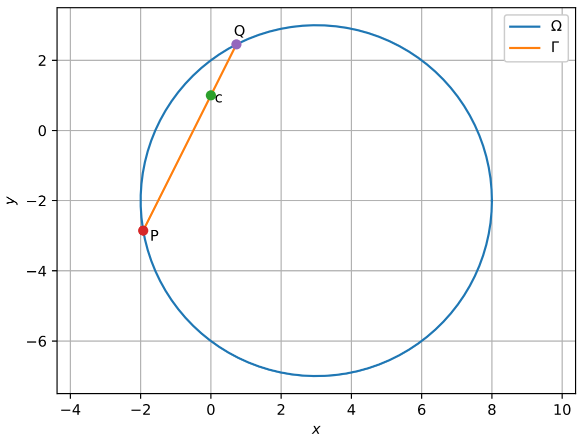

Fig. 3.1

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1 \mathbf{m} \tag{3.7}$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2 \mathbf{m} \tag{3.8}$$

$$\implies \frac{\mathbf{P}+\mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2}\mathbf{m} \tag{3.9}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{m} \tag{3.10}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} - \frac{\mathbf{m}^T \left(\mathbf{c} - \mathbf{O}\right)}{\|\mathbf{m}\|^2} \tag{3.11}$$

using the sum of roots in (3.6). From (3.3) and (3.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5}\left(1 + m^2\right) \tag{3.12}$$

$$\implies m^2 - 5m + 6 = 0 \tag{3.13}$$

$$\implies m = 2 \text{ or } 3 \tag{3.14}$$

From (3.6),

$$\lambda = \frac{-\mathbf{m}^T \left(\mathbf{c} - \mathbf{O}\right)}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{\left(\mathbf{m}^T \left(\mathbf{c} - \mathbf{O}\right)\right)^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{3.15}$$

Fig. 3.1 summarizes the solution for $m = 2$.

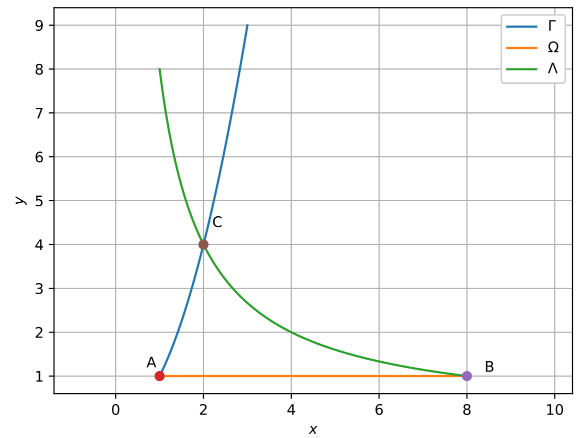

Fig. 4.1

4 Calculus: Integration

4.1 Sketch the region

$$\begin{pmatrix} x \\ y \end{pmatrix} : xy \le 8, 1 \le y \le x^2 \tag{4.1}$$

4.2 Find the area of the region.
Solution: The intersection of $y = 1, y = x^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.2}$$

The intersection of $y = 1, xy = 8$ is

$$\mathbf{B} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{4.3}$$

The intersection of $y = x^2, xy = 8$ is

$$\mathbf{C} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{4.4}$$

The desired region is enclosed by the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ Thus, the area is obtained as

$$\int_1^2 x^2 \, dx + \int_2^8 \frac{8}{x} \, dx = \left[\frac{x^3}{3}\right]_1^2 + 8\left[\ln x\right]_2^8 - 7 \tag{4.5}$$

$$= 16 \ln 2 - \frac{14}{3} \tag{4.6}$$

5 Signal Processing: Z Transform

5.1 Let

$$a(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n) \tag{5.1}$$

$$b(n) = a(n-1) + a(n+1) - \delta(n) \tag{5.2}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{5.3}$$

and

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \ge 0 \end{cases} \tag{5.4}$$

$$\delta(n) = \begin{cases} 0, & n \ne 0 \\ 1, & n = 0 \end{cases} \tag{5.5}$$

5.2 Verify your results through a C program.

5.3 Show that the Z transform of $u(n)$

$$U(z) \triangleq \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{5.6}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{5.7}$$

5.4 Show that

$$A(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \tag{5.8}$$

5.5 Let

$$y(n) = a(n) * u(n) \triangleq \sum_{k=-\infty}^{\infty} a(k) u(n-k) \tag{5.9}$$

Show that

$$y(n) = \sum_{k=0}^{n} a(k) \tag{5.10}$$

5.6 Show that

$$Y(z) = A(z) U(z) \tag{5.11}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{5.12}$$

5.7 Show that

$$w(n) = [a(n+2) - 1]\, u(n-1) \tag{5.13}$$

$$= a(n+2) - u(n+1) + 2\delta(n) \tag{5.14}$$

5.8 Is $W(z) = Y(z)$?

5.9 Verify if

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} = \frac{10}{89} \tag{5.15}$$

5.10 Verify if

$$\sum_{n=1}^{\infty} \frac{b(n)}{10^n} = \frac{8}{89} \tag{5.16}$$

6 Matrices: Adjugate

Let

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}, \quad \operatorname{adj}(\mathbf{M}) = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \tag{6.1}$$

6.1 Show that $a + b = 3$

Solution:

$$\because \mathbf{M} \operatorname{adj}(\mathbf{M}) = \det(\mathbf{M})\,\mathbf{I}, \tag{6.2}$$

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} = 0 \tag{6.3}$$

$$\begin{pmatrix} 3 & b & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = 0 \tag{6.4}$$

resulting in

$$a = 2, b = 1 \tag{6.5}$$

Hence, $a + b = 3$.

6.2 Verify if

$$\left(\operatorname{adj}(\mathbf{M})\right)^{-1} + \operatorname{adj}\left(\mathbf{M}^{-1}\right) = -\mathbf{M} \tag{6.6}$$

Solution: From (6.2)

$$\left(\operatorname{adj}(\mathbf{M})\right)^{-1} = \frac{\mathbf{M}}{\det(\mathbf{M})} \tag{6.7}$$

and

$$\left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) = \frac{\mathbf{M}^{-1}}{\det\left(\mathbf{M}^{-1}\right)} \tag{6.8}$$

$$= \mathbf{M}^{-1} \det(\mathbf{M}) \tag{6.9}$$

Thus,

$$\left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) + \operatorname{adj}\left(\mathbf{M}^{-1}\right)$$
$$= \mathbf{M}^{-1} \det(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})}$$
$$= \operatorname{adj}(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \tag{6.10}$$

From (6.2)

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = \det(\mathbf{M}) \tag{6.11}$$

$$\implies \det(\mathbf{M}) = 8 - 5a = -2 \tag{6.12}$$

If

$$\left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) + \operatorname{adj}\left(\mathbf{M}^{-1}\right) = -\mathbf{M},$$
$$\operatorname{adj}(\mathbf{M}) - \frac{\mathbf{M}}{2} = -\mathbf{M}$$
$$\implies \mathbf{M} = -\operatorname{adj}(\mathbf{M})$$

which is incorrect.

6.3 Verify if

$$\det\left(\operatorname{adj}\left(\mathbf{M}^2\right)\right) = 81 \tag{6.13}$$

Solution:

$$\operatorname{adj}\left(\mathbf{M}^2\right) = \mathbf{M}^{-2}\det(\mathbf{M})^2 \tag{6.14}$$
$$= 4\mathbf{M}^{-2} \tag{6.15}$$
$$\implies \det\left(\operatorname{adj}\left(\mathbf{M}^2\right)\right) = 4^3\det(\mathbf{M})^{-2} \tag{6.16}$$
$$= 16 \neq 81 \tag{6.17}$$

6.4 If

$$\mathbf{M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.18}$$

show that

$$\alpha - \beta + \gamma = 3 \tag{6.19}$$

Solution:

$$\mathbf{M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.20}$$
$$\implies \operatorname{adj}(\mathbf{M})\,\mathbf{M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \operatorname{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.21}$$

which can be expressed as

$$\det(\mathbf{M})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \operatorname{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.22}$$
$$\text{or, } \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = -\frac{1}{2}\operatorname{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.23}$$

Thus,

$$\alpha - \beta + \gamma = \begin{pmatrix}1 & -1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \tag{6.24}$$
$$= -\frac{1}{2}\begin{pmatrix}1 & -1 & 1\end{pmatrix}\operatorname{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{6.25}$$
$$= \begin{pmatrix}7 & -5 & 2\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = 3 \tag{6.26}$$

7 Probability

Table 4 lists the number of red (R) and green (G) balls in bags B_1, B_2 and B_3. Also listed are the probabilities of each bag.

Bag	**R**	**G**	**Probability**
B_1	5	5	$\Pr(B_1) = \frac{3}{10}$
B_2	3	5	$\Pr(B_2) = \frac{3}{10}$
B_3	5	3	$\Pr(B_3) = \frac{4}{10}$

TABLE 4

7.1 Show that

$$\Pr(G|B_3) = \frac{3}{8} \tag{7.1}$$

7.2 Show that

$$\Pr(G) = \frac{39}{80} \tag{7.2}$$

Solution:

$$\because \Pr(G|B_1) = \frac{1}{2}, \Pr(G|B_2) = \frac{5}{8}, \Pr(G|B_3) = \frac{3}{8},$$
$$\Pr(G) = \sum_{i=1}^{3}\Pr(G|B_i)\Pr(B_i) \tag{7.3}$$
$$= \frac{1}{2}\times\frac{3}{10} + \frac{5}{8}\times\frac{3}{10} + \frac{3}{8}\times\frac{4}{10} \tag{7.4}$$
$$= \frac{39}{80} \tag{7.5}$$

7.3 Is

$$\Pr(B_3|G) = \frac{5}{13}\,? \tag{7.6}$$

Solution:

$$\Pr(B_3|G) = \frac{\Pr(G|B_3)\Pr(B_3)}{\Pr(G)} \tag{7.7}$$

$$= \frac{\frac{3}{8} \times \frac{4}{10}}{\frac{39}{80}} = \frac{4}{13} \neq \frac{5}{13} \tag{7.8}$$

7.4 Is

$$\Pr(B_3 \cap G) = \frac{3}{10}\,? \tag{7.9}$$

Solution:

$$\Pr(B_3 \cap G) = \Pr(G|B_3)\Pr(B_3) \tag{7.10}$$

$$\tag{7.11}$$

$$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20} \neq \frac{3}{10} \tag{7.12}$$

8 Trigonometry

8.1 In $\triangle PQR$, which is not right angled, let

$$PQ = r, QR = p, RP = q \tag{8.1}$$

The median RS and the altitude PE intersect at **O**. $p = \sqrt{3}, q = 1$ and the radius of the circumcircle of $\triangle PQR = k = 1$.

8.2 Find RS

Solution: Using the sine formula,

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2k \tag{8.2}$$

$$\implies \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2} \tag{8.3}$$

If $\angle R \neq \frac{\pi}{2}$, the only possible solution is

$$\angle P = \frac{2\pi}{3}, \angle Q = \frac{\pi}{6}, \angle R = \frac{\pi}{6} \tag{8.4}$$

$\because \angle Q = \angle R, q = r = 1$. The given information is shown in Fig. 8.1 Using the cosine formula,

$$RS = \sqrt{q^2 + \left(\frac{r}{2}\right)^2 - qr\cos P} \tag{8.5}$$

$$= \sqrt{1 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{2}} \tag{8.6}$$

8.3 Find OE.

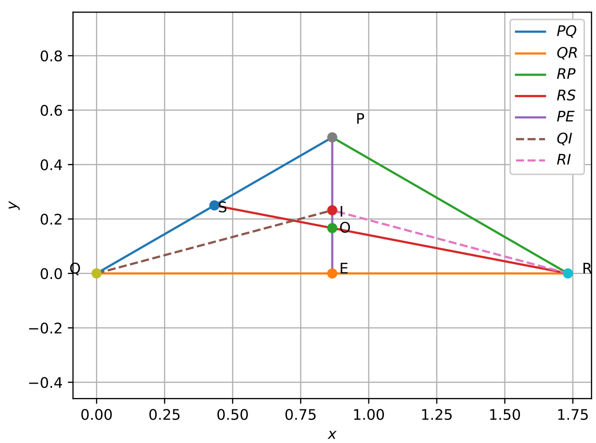

Fig. 8.1

Solution: Using Baudhayana's theorem,

$$OE = \sqrt{OR^2 - ER^2} \tag{8.7}$$

$$= \sqrt{\left(\frac{2RS}{3}\right)^2 - \left(\frac{p}{2}\right)^2} \tag{8.8}$$

$$= \sqrt{\frac{7}{9} - \frac{3}{4}} = \frac{1}{6} \tag{8.9}$$

8.4 Find the area of $\triangle SOE$

Solution: $\because$ PE and RS are medians,

$$\text{ar}(\triangle SOE) = \frac{1}{4}\text{ar}(\triangle POR), \tag{8.10}$$

$$\text{ar}(\triangle POR) = \frac{2}{3}\text{ar}(\triangle PER), \tag{8.11}$$

$$\text{ar}(\triangle PER) = \frac{1}{2}\text{ar}(\triangle PQR), \tag{8.12}$$

$$\implies \text{ar}(\triangle SOE) = \frac{1}{12}\text{ar}(\triangle PQR) = \frac{\sqrt{3}}{24} \tag{8.13}$$

8.5 Find the radius of the incircle of $\triangle PQR$.

Solution: I is the incentre in Fig. 8.1. The radius of the incircle is

$$\frac{p}{2\cos\frac{Q}{2}} = \frac{p}{\sqrt{2(1+\cos Q)}} \tag{8.14}$$

$$= \sqrt{\frac{3}{1+\sqrt{3}}} \tag{8.15}$$

8.6 Repeat all the above exercises using vector algebra and plot Fig. 8.1.

9 Linear Agebra: Coordinate Geometry

Let the ellipse $E_1, n = 1, 2, \ldots$ have the equation

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \tag{9.1}$$

where

$$\mathbf{D} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \tag{9.2}$$

9.1 Let the largest rectangle inside E_1 with sides parallel to the axebe be R_1. Show that the coordinates of the R_1 have the form

$$\begin{pmatrix} \pm p_1 \\ \pm p_2 \end{pmatrix} \tag{9.3}$$

Solution: Let R_1 be the rectangle $PQRS$, where $PQ \parallel RS \parallel x-axis, QR \parallel PS \parallel y-axis$. Their corresponding equations are

$$PQ : \mathbf{x} = \mathbf{P} + \lambda_1\mathbf{m}_1 \tag{9.4}$$

$$PS : \mathbf{x} = \mathbf{P} + \lambda_2\mathbf{m}_2 \tag{9.5}$$

$$QR : \mathbf{x} = \mathbf{Q} + \lambda_3\mathbf{m}_2 \tag{9.6}$$

where

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.7}$$

The intersection of PQ with E_1 is

$$[\mathbf{P} + \lambda_1\mathbf{m}_1]^T \mathbf{D} [\mathbf{P} + \lambda_1\mathbf{m}_1] = 1$$
$$\implies \lambda_1^2 \|\mathbf{m}_1\|^2 + 2\lambda_1\mathbf{m}_1^T\mathbf{P} + \mathbf{P}^T\mathbf{D}\mathbf{P} - 1 = 0$$

$$\because \mathbf{P} \in E_1, \|\mathbf{m}_1\|^2 = 1, \mathbf{P}^T\mathbf{D}\mathbf{P} - 1 = 0$$
$$\implies \lambda_1 = 0, -2\mathbf{m}_1^T\mathbf{P} \tag{9.8}$$

Thus,

$$\mathbf{Q} = \mathbf{P} - 2\mathbf{m}_1^T\mathbf{P}\mathbf{m}_1$$
$$= \begin{pmatrix} -p_1 \\ p_2 \end{pmatrix} \tag{9.9}$$

Similarly,

$$\mathbf{S} = \mathbf{P} - 2\mathbf{m}_2^T\mathbf{P}\mathbf{m}_2$$
$$= \begin{pmatrix} p_1 \\ -p_2 \end{pmatrix} \tag{9.10}$$

and

$$\mathbf{R} = \mathbf{Q} - 2\mathbf{m}_2^T\mathbf{Q}\mathbf{m}_2$$
$$= \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix} \tag{9.11}$$

9.2 Find an expression for the square of the area of R_1.
Solution:

$$\because \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1,$$
$$p_2 = b\sqrt{1 - \frac{p_1^2}{a^2}}. \tag{9.12}$$

Hence the desired expression is

$$F = (PQ \times QR)^2 = 16p_1^2p_2^2 = 16p_1^2b^2\left(1 - \frac{p_1^2}{a^2}\right). \tag{9.13}$$

9.3 Find p_1 that maximises F.
Solution: (9.13) can be expressed as

$$F = a^2b^2\left(16a^2p_1^2 - 16p_1^4\right)$$
$$= a^2b^2\left\{a^4 - \left(a^2 - 4p_1^2\right)^2\right\} \tag{9.14}$$

Thus, F is maximum when

$$\left(a^2 - 4p_1^2\right)^2 = 0$$
$$\implies p_1 = \pm\frac{a}{2} \tag{9.15}$$

9.4 Verify the above result graphically.

9.5 Find p_2.
Solution: From (9.12)

$$p_2 = \pm\frac{\sqrt{3}}{2}b \tag{9.16}$$

9.6 Find E_2, the largest ellipse within R_1.
Solution: From (9.15) and (9.16), the semi-major/minor axes of E_2 are

$$E_2 : \left(\frac{a}{2}, \frac{\sqrt{3}}{2}b\right) \tag{9.17}$$

9.7 find E_n and R_n,
Solution: From (9.15) and (9.16), the vertices of R_n and semi-major/minor axes of E_n are

$$R_n : \left\{\pm\frac{a}{2^n}, \pm\left(\frac{\sqrt{3}}{2}\right)^n b\right\}$$
$$E_n : \left\{\frac{a}{2^{n-1}}, \left(\frac{\sqrt{3}}{2}\right)^{n-1} b\right\} \tag{9.18}$$

In the following questions, $a = 3, b = 2$. Use a computer program.

9.8 Is the eccentricity $e_18 = e_19$?

9.9 Verify if

$$\sum_{n=1}^{N} (\text{Area of } R_n) < 24, \tag{9.19}$$

for each positive integer N.

9.10 Is the length of the latus rectum of $E_9 = \frac{1}{6}$?

9.11 Is the distance of a focus from the centre in $E_9 = \frac{\sqrt{5}}{32}$?

10 Calculus: Differentiation

Let

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x-2)\ln(x-2) - x + \frac{10}{3} & x \ge 3 \end{cases} \tag{10.1}$$

10.1 Is f increasing in $(-\infty, 0)$?
Solution:

$$f'(x) = 5x^4 + 20x^3 + 30x^2 + 20x + 3 \quad x < 0$$
$$\implies f'(-1) = 5 - 20 + 30 - 20 + 3 = -2 < 0 \tag{10.2}$$

Hence $f'(x)$ is non-increasing.

10.2 Does f' have a local maximum at $x = 1$?
Solution:

$$f'(x) = \begin{cases} 2x - 1 > 0, & \frac{1}{2} < x < 1, \\ 2(x-2)^2 - 1 < 0 & 1 \le x < 3 \end{cases} \tag{10.3}$$

Hence, f is increasing in $\left(\frac{1}{2}, 1\right)$ and decreasing between $(1, 3) \implies f$ has a local maximum at $x = 1$

10.3 Show that f' is differentiable at $x = 1$.
Solution: Since

$$f'(1-) = f'(1) = 1, \tag{10.4}$$

f is differentiable at $x = 1$.

10.4 Is f onto?

10.5 Sketch $f(x)$ in Python to verify your answeres.

11 Calculus: Differential Equations

Γ is a curve in the first qudrant and

$$\mathbf{R} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11.1}$$

lies on it. The tangent to Γ at $\mathbf{P}$ intersects the $y-$axis at $\mathbf{Y}_P$. The line segment $PY_P = 1$.

11.1 Find the differential equation of Γ.
Solution: Let

$$\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{Y}_P = \begin{pmatrix} 0 \\ c \end{pmatrix}. \tag{11.2}$$

Then using the equation of a line,

$$\mathbf{Y}_P = \mathbf{P} + \lambda \mathbf{m}, \tag{11.3}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ y' \end{pmatrix}. \tag{11.4}$$

Thus,

$$\begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ y' \end{pmatrix} \tag{11.5}$$
$$\implies \lambda = -x. \tag{11.6}$$

$$\because PY_P = \|\mathbf{P} - \mathbf{Y}_P\| = |\lambda| \, \|\mathbf{m}\| = 1, \tag{11.7}$$
$$x^2\left(1 + (y')^2\right) = 1 \tag{11.8}$$
$$\implies xy' \pm \sqrt{1 - x^2} = 0 \tag{11.9}$$

11.2 Find the equation of Γ.
Solution: From (11.9),

$$dy = \pm \frac{\sqrt{1 - x^2}}{x} \, dx \tag{11.10}$$
$$\implies \int dy = \pm \int \frac{\sqrt{1 - x^2}}{x} \, dx \tag{11.11}$$

Letting

$$z = \sqrt{1 - x^2}, dz = -\frac{x}{\sqrt{1 - x^2}} \, dx$$
$$\implies \int \frac{\sqrt{1 - x^2}}{x} \, dx = -\int \frac{z^2}{1 - z^2} \, dz$$
$$= \int dz - \int \frac{1}{1 - z^2} \, dz$$
$$= z + \frac{1}{2} \ln \frac{1 - z}{1 + z} + C \tag{11.12}$$

Thus,

$$y = \pm\left(\sqrt{1 - x^2} + \frac{1}{2} \ln \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}}\right) \tag{11.13}$$

since $C = 0$ after substituting $x = 0, y = 1$.

11.3 Verify your result through a python sketch.

12 Linear Algebra: Orthogonality

12.1 Let

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \tag{12.1}$$

$$L_2: \quad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{12.2}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.
Solution: Let

$$L_3: \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{12.3}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{12.4}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{12.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{12.6}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1: \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{12.7}$$
$$L_2: \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{12.8}$$
$$L_3: \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{12.9}$$

where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{12.10}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (12.7)-(12.9),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{12.11}$$
$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{12.12}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (12.11)-(12.12) can be expressed as

$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{12.13}$$
$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{12.14}$$
$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{12.15}$$
$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{12.16}$$
$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{12.17}$$
$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{12.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{12.19}$$

$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{12.20}$$

Substituting in (12.11) and (12.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ or} \tag{12.21}$$

$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{12.22}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

13 Convex Optimization

13.1 Show that

$$\min_{a,b,c} \left| a + b\omega + c\omega^2 \right|^2 \tag{13.1}$$

where $\omega^3 = 1, \omega \neq 1$ and a, b, c are distinct nonzero integers can be expressed as

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{13.2}$$

where

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mathbf{A} = 2\mathbf{P}^T \mathbf{P}, \tag{13.3}$$

$$\mathbf{P} = \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}, \theta = \frac{\pi}{3} \tag{13.4}$$

13.2 Show that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \tag{13.5}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ \cos\theta & 1 & -\cos 2\theta \\ -\cos\theta & -\cos 2\theta & 1 \end{pmatrix}, \tag{13.6}$$

resulting in (13.5).

$$\because \cos 2\theta = -\cos\theta = -\frac{1}{2} \tag{13.7}$$

13.3 Show that the characteristic equation of $\mathbf{A}$ is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda \tag{13.8}$$

13.4 Show that the eigenvalues of $\mathbf{A}$ are 0 and 3.
13.5 Verify that $tr(\mathbf{A})$ is the sum of its eigenvalues.
13.6 Verify that $\det(\mathbf{A})$ is the product of its eigenvalues.
13.7 Show that $\mathbf{A}$ is positive definite.
13.8 Show that $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is convex.
13.9 Show that the unconstrained $\mathbf{x}$ that minimizes $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is given by the line

$$\mathbf{x} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{13.9}$$

13.10 Find $\mathbf{y}$ such that

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y} \tag{13.10}$$

where λ is an eigenvalue of $\mathbf{A}$.

13.11 Show that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P} \tag{13.11}$$

where $\mathbf{D}$ is a diagonal matrix comprising of the eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{P}$ are the corresponding eigenvectors.

13.12 Find $\mathbf{U}$ such that

$$\mathbf{A} = \mathbf{U}^T\mathbf{D}\mathbf{U}, \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{13.12}$$

13.13 Show that

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 3\mathbf{v}^T\mathbf{v}, \tag{13.13}$$

where

$$\mathbf{v} = \mathbf{U}\mathbf{x} \tag{13.14}$$

13.14 Show that when the entries of $\mathbf{x}$ are unequal and integers, the solution of (13.2) can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{13.15}$$

14 Algebra: Modular Arithmetic

Let $AP(a; d)$ denote an A.P. with $d > 0$

14.1 Express $AP(a; d)$ in modulo arithmetic.
Solution:

$$A \equiv a \pmod{d} \tag{14.1}$$

14.2 Express the intersection of $AP(1; 3)$, $AP(2; 5)$ and $AP(3; 7)$ using modulo arithmetic.
Solution: The desired AP can be expressed as

$$A \equiv 1 \pmod 3 \tag{14.2}$$
$$\equiv 2 \pmod 5 \tag{14.3}$$
$$\equiv 3 \pmod 7 \tag{14.4}$$

14.3 Two numbers are said to be coprime if their greatest common divisor (gcd) is 1. Verify if (3,5), (5,7) and (3,7) are pairwise coprime.
14.4 Does a solution for (14.2) exist?
Solution: The Chinese remainder theorem guarantees that the system in (14.2) has a solution since 3,5,7 are pairwise coprime.
14.5 Simplify

$$(7 \times 5) \pmod 3 \tag{14.5}$$

Solution: (14.5) can be expressed as

$$(7 \times 5) \pmod 3 = 35 \pmod 3$$
$$= 2 \pmod 3 \tag{14.6}$$

14.6 Find x in

$$2x = 1 \pmod 3 \tag{14.7}$$

Solution: By inspection, for $x = 2$,

$$2x = 2 \times 2 = 4 = 3 + 1 = 1 \pmod 3 \tag{14.8}$$

Thus $x = 2$ is a solution of (14.7).

14.7 In general, x in

$$ax = 1 \pmod d \tag{14.9}$$

is defined to be the modular multiplicative inverse of (14.1).

14.8 Show that the multiplicative inverse of

$$(3\times 5) \pmod 7 = y = 1 \tag{14.10}$$

14.9 Show that the multiplicative inverse of

$$(3\times 7) \pmod 5 = z = 1 \tag{14.11}$$

14.10 Find $a+d$.
Solution:

$$(5\times 7\times 1\times x)+(3\times 5\times 3\times y) + (3\times 7\times 2\times z) = 157 \tag{14.12}$$

14.11 Find a and d.
Solution:

$$d = LCM\,(3,5,7) = 105 \tag{14.13}$$

$$\begin{aligned} A &= 157 \pmod{105} \\ &= 52 \pmod{105} \end{aligned}$$

$$\implies a = 52 \tag{14.14}$$

14.12 Given the APs

$$a_1 \pmod{d_1} \tag{14.15}$$

$$a_2 \pmod{d_2} \tag{14.16}$$

$$a_3 \pmod{d_3}, \tag{14.17}$$

such that

$$gcd(d_1,d_2) = gcd(d_2,d_3) = gcd(d_3,d_1) = 1, \tag{14.18}$$

show that their intersection

$$a \pmod d \tag{14.19}$$

is obtained through

$$a+d = (d_1\times d_2\times a_3\times x)+(d_2\times d_3\times a_1\times y) + (d_3\times d_1\times a_2\times z) \tag{14.20}$$

$$d = LCM\,(d_1,d_2,d_3)\,, \tag{14.21}$$

where x,y,z are the modular multiplicative inverses given by

$$x = [(d_1\times d_2) \pmod{d_3}]^{-1} \tag{14.22}$$

$$y = [(d_2\times d_3) \pmod{d_1}]^{-1} \tag{14.23}$$

$$z = [(d_3\times d_1) \pmod{d_2}]^{-1} \tag{14.24}$$

respectively.

14.13 Write a **C** program to find x,y and z.

15 Linear Algebra: Binary Matrices

Let S be the set of all 3×3 matrices whose entries are from $\{0,1\}$ and

$$E_1 = \{\mathbf{A}\in S : \det(\mathbf{A}) = 0\} \tag{15.1}$$

and

$$E_2 = \{\mathbf{A}\in S : \text{ sum of entries of } A \text{ is } 7\} \tag{15.2}$$

15.1 Find $|E_2|$.
Solution:

$$|E_2| = \frac{9!}{7!2!} = 72 \tag{15.3}$$

15.2 Find $|(E_1|E_2)|$.
Solution: E_2 is the set of matrices with rows $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and the following combinations in Table 15.2. $\mathbf{e}_i, i = 1,2,3$ are the standard basis vectors. The equation

$$\mathbf{v}_1 = \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \tag{15.4}$$

has a solution only for the first combination in Table 15.2. Thus, $\det(A) = 0$ only for this combination. Thus

$$|(E_1|E_2)| = 3\times 3 = 9 \tag{15.5}$$

$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\mathbf{e}_i$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$

TABLE 15.2

15.3 Find $\Pr(E_1|E_2)$.
Solution: From (15.3) and (15.5),

$$\Pr(E_1|E_2) = \frac{|(E_1|E_2)|}{|E_2|} = \frac{9}{72} = \frac{1}{8} \tag{15.6}$$

15.4 Verify using a python script.

16 Linear Algebra: Reflection

16.1 Let $\mathbf{B}$ be the reflection of $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ with respect to the line

$$L : \begin{pmatrix} 8 & -6 \end{pmatrix} \mathbf{x} = 23 \tag{16.1}$$

Show that $\mathbf{B} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

16.2 Find the equation of AB.
Solution: The normal vector of L is

$$\mathbf{n} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \tag{16.2}$$

Thus, the equation of AB is

$$AB : \mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{16.3}$$

$$\implies \begin{pmatrix} 6 & 8 \end{pmatrix} \mathbf{x} = 36 \tag{16.4}$$

16.3 Let Γ_A and Γ_B be circles of radii $r_1 = 2, r_2 = 1$ with centres at $\mathbf{A}$ and $\mathbf{B}$ respectively. Let T be the common tangent to both the circles such that they are on the same side of T.

16.4 Find the point $\mathbf{C}$ where AB meets T.
Solution: Let $\mathbf{D}, \mathbf{E}$ be the points of contact for T with Γ_A and $4\Gamma_B$ respectively. It is obvious that $\triangle ADC \sim \triangle BEC$. Hence,

$$AB = BC \tag{16.5}$$

$$\implies \mathbf{C} = 2\mathbf{B} - A = \begin{pmatrix} 10 \\ -3 \end{pmatrix} \tag{16.6}$$

16.5 Find AC.
Solution:

$$AC = \|\mathbf{A} - \mathbf{C}\| = 10 \tag{16.7}$$

-

16.6 Find $\mathbf{D}$ and $\mathbf{E}$.

17 Calculus: Definite Integral

17.1 If

$$I = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1 + e^{\sin x}\right)(2 - \cos 2x)}, \tag{17.1}$$

find $27I^2$.
Solution: Substituting $-x$ for x,

$$I = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1 + e^{-\sin x}\right)(2 - \cos 2x)}, \tag{17.2}$$

Adding (17.1) and (17.2),

$$2I = \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2 - \cos 2x)} \left[\frac{1}{\left(1 + e^{\sin x}\right)} + \frac{1}{\left(1 + e^{-\sin x}\right)}\right], \tag{17.3}$$

which can be simplified to obtain

$$I = \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2 - \cos 2x)} \frac{\left(1 + e^{\sin x} + 1 + e^{-\sin x}\right)}{\left(1 + e^{\sin x} + e^{-\sin x} + 1\right)}$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2 - \cos 2x)} \tag{17.4}$$

Substituting

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2}, \tag{17.5}$$

in (17.4) and simplifying,

$$I = \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{\left(1 + 3\tan^2 x\right)} dx$$

$$= \frac{1}{\pi\sqrt{3}} \left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{2}{3\sqrt{3}} \tag{17.6}$$

resulting in

$$27I^2 = 4 \tag{17.7}$$

18 Linear Algebra: Area of a Triangle

18.1 Let the lines

$$L_1 : \mathbf{x} = \lambda_1 \mathbf{a} \tag{18.1}$$

$$L_2 : \mathbf{x} = \lambda_2 \mathbf{b} \tag{18.2}$$

$$L_3 : \mathbf{x} = \lambda_3 \mathbf{c} \tag{18.3}$$

intersect the plane

$$P : \mathbf{n}^T \mathbf{x} = c \tag{18.4}$$

at the points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ respectively. Find λ_1, λ_2 and λ_3.

Solution: From the given information, $\mathbf{A} \in P, L_1$. $\because \mathbf{A} = \lambda_1 \mathbf{a}$,

$$\lambda_1 \mathbf{n}^T \mathbf{a} = c \implies \lambda_1 = \frac{c}{\mathbf{n}^T \mathbf{a}} \tag{18.5}$$

Similarly, λ_2 and λ_3 are obtained.

18.2 Find the area Δ of $\triangle ABC$

Solution:

$$\begin{aligned}\Delta &= \left\| \frac{1}{2} (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| \\ &= \frac{1}{2} \|(\lambda_1 \mathbf{a} - \lambda_2 \mathbf{b}) \times (\lambda_1 \mathbf{a} - \lambda_3 \mathbf{c})\| \\ &= \frac{1}{2} \|\lambda_1 \lambda_2 (\mathbf{a} \times \mathbf{b}) + \lambda_2 \lambda_3 (\mathbf{b} \times \mathbf{c}) \\ &\quad + \lambda_1 \lambda_3 (\mathbf{c} \times \mathbf{a})\|\end{aligned} \tag{18.6}$$

18.3 Find $(6\Delta)^2$ given

$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{18.7}$$

$$\mathbf{n} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}, c = 1 \tag{18.8}$$

Solution:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \\ &= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\end{aligned} \tag{18.9}$$

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix} \tag{18.10}$$

$$\mathbf{c} \times \mathbf{a} = \begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} \tag{18.11}$$

Using (18.5),

$$\lambda_1 = 1, \lambda_2 = \frac{1}{2}, \lambda_3 = \frac{1}{3} \tag{18.12}$$

Thus,

$$\Delta = \frac{1}{2}\left\| \frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{6}\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\-1\end{pmatrix} \right\| \tag{18.13}$$

$$= \frac{1}{12}\left\| \begin{pmatrix}1\\1\\1\end{pmatrix} \right\| = \frac{\sqrt{3}}{12} \tag{18.14}$$

Hence,

$$(6\Delta)^2 = \frac{3}{2} \tag{18.15}$$