Geometry through Trigonometry

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Contents

***Abstract*—The focus of this text is on investigating the properties of triangles and circles through trigonometry. Several proofs in geometry require a lot of constructions, which may not be easy for all the students in a class. This book provides an alternative approach to geometry which may be useful to those having difficulties with traditional methods. In the process, almost all the basic concepts of trigonometry are covered. Instead of teaching geometry and trigonometry as separate subjects in high school, this textbook shows how to develop a holistic approach for teaching math.**

1 The Right Angled Triangle

Definition 1.1. A right angled triangle looks like Fig. 1.0. with angles $\angle A, \angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle C$ which is defined to be 90°.

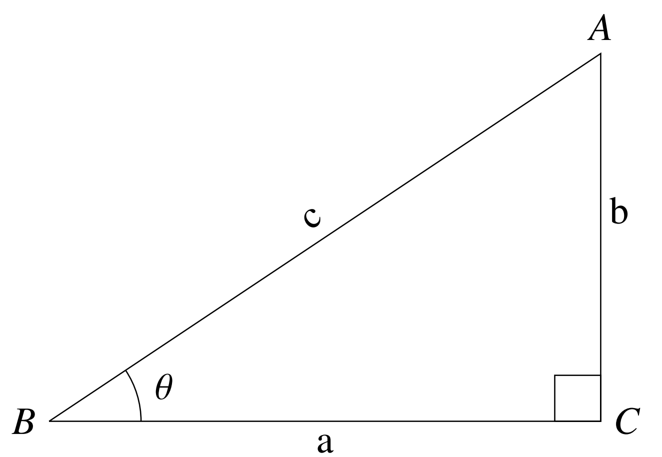

Fig. 1.0: Right Angled Triangle

Definition 1.2. For simplicity, let the greek letter $\theta = \angle B$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \frac{a}{c} & \cos\theta &= \frac{b}{c} \\ \tan\theta &= \frac{b}{a} & \cot\theta &= \frac{1}{\tan\theta} \\ \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} \end{aligned} \tag{1.1}$$

1.1 Sum of Angles

Definition 1.3. In Fig. 1.0, the sum of all the angles on the top or bottom side of the straight line XY is 180°.

Definition 1.4. In Fig. 1.0, the straight line making an angle of 90° to the side AC is said to be parallel to the side BC. Note there is an angle at A that is equal to θ. This is one property of parallel lines. Thus, $\angle YAZ = 90^{\circ}$.

Problem 1.1. Show that $\angle VAZ = 90^{\circ} - \theta$

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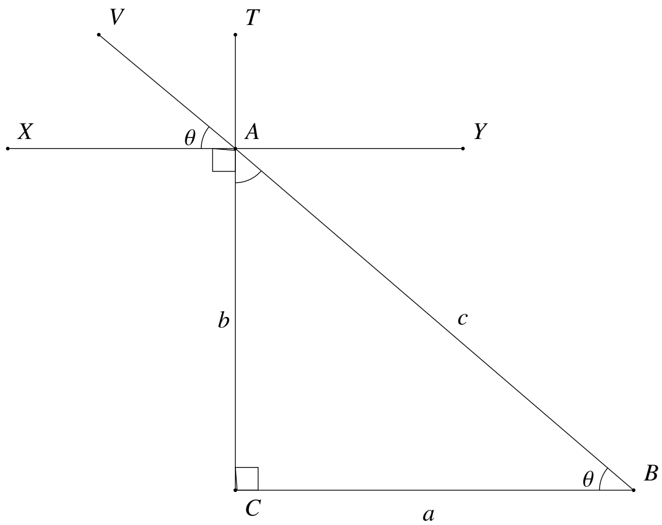

Fig. 1.0: Sum of angles of a triangle

Proof. Considering the line XAZ,

$$\theta + 90^\circ + \angle VAZ = 180^\circ \tag{1.2}$$

$$\Rightarrow \angle VAZ = 90^\circ - \theta \tag{1.3}$$

Problem 1.2. Show that $\angle BAC = 90^\circ - \theta$.

Proof. Consider the line VAB and and use the approach in the previous problem. Note that this implies that $\angle VAZ = \angle BAC$. Such angles are known as vertically opposite angles.

Problem 1.3. Sum of the angles of a triangle is equal to 180°

1.2 Budhayana Theorem

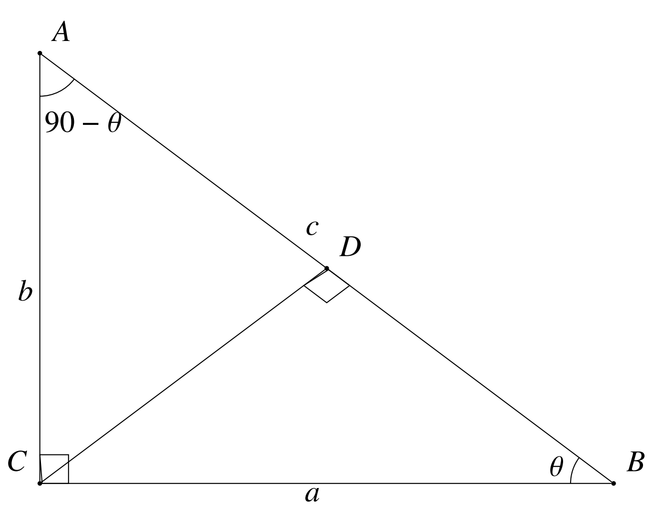

Fig. 1.3: Budhayana Theorem

Problem 1.4. Using Fig. 1.0, show that

$$\cos\theta = \sin\left(90^\circ - \theta\right) \tag{1.4}$$

Proof. From Problem 1.2 and (1.1)

$$\cos\left(90^\circ - \theta\right) = \frac{b}{c} = \sin\theta \tag{1.5}$$

Problem 1.5. Using Fig. 1.3, show that

$$c = a\cos\theta + b\sin\theta \tag{1.6}$$

Proof. We observe that

$$BD = a\cos\theta \tag{1.7}$$

$$AD = b\cos(90-\theta) = b\sin\theta \quad (\text{From} \quad (1.2)) \tag{1.8}$$

Thus,

$$BD + AD = c = a\cos\theta + b\sin\theta \tag{1.9}$$

Problem 1.6. From (1.6), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.10}$$

Proof. Dividing both sides of (1.6) by c,

$$1 = \frac{a}{c}\cos\theta + \frac{b}{c}\sin\theta \tag{1.11}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad (\text{from} \quad (1.1)) \tag{1.12}$$

Problem 1.7. Using (1.6), show that

$$c^2 = a^2 + b^2 \tag{1.13}$$

(1.13) is known as the Budhayana theorem. It is also known as the Pythagoras theorem.

Proof. From (1.6),

$$c = a\frac{a}{c} + b\frac{b}{c} \quad (\text{from} \quad (1.1)) \tag{1.14}$$

$$\Rightarrow c^2 = a^2 + b^2 \tag{1.15}$$

2 Medians of a Triangle

2.1 Area of a Triangle

Definition 2.1. The area of the rectangle $ACBD$ shown in Fig. 2.0 is defined as ab. Note that all the angles in the rectangles are 90°

Definition 2.2. The area of the two triangles constituting the rectangle is the same.

Definition 2.3. The area of the rectangle is the sum of the areas of the two triangles inside.

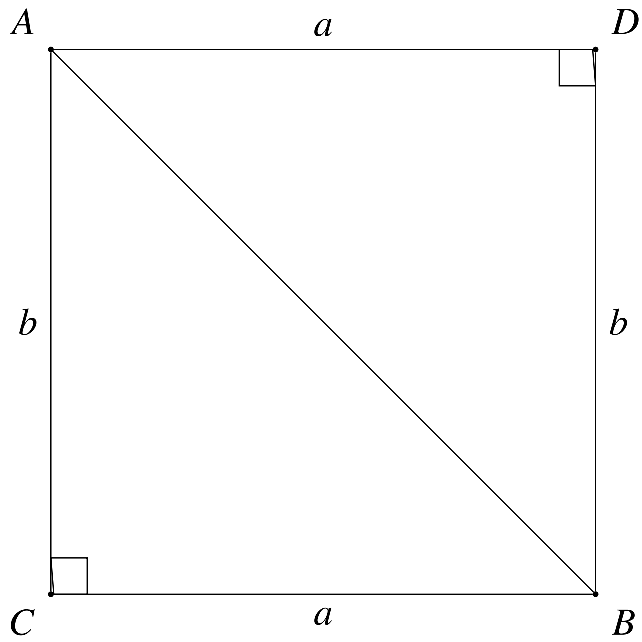

Fig. 2.0: Area of a Right Triangle

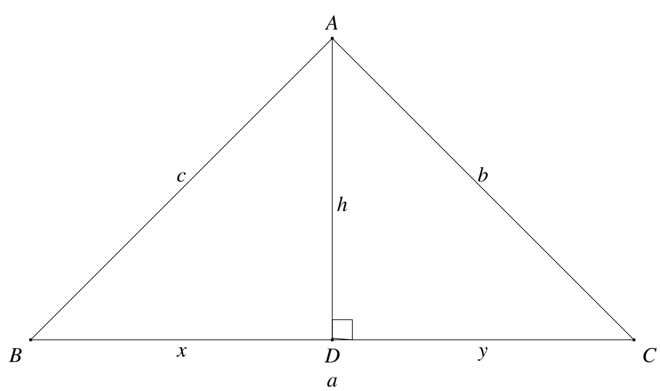

Fig. 2.1: Area of a Triangle

Problem 2.1. Show that the area of ΔABC is $\frac{ab}{2}$

Proof. From(2.3),

$$ar\,(ABCD) = ar\,(ACB) + ar\,(ADB) \tag{2.1}$$

Also from (2.2),

$$ar\,(ACB) = ar\,(ADB) \tag{2.2}$$

From (2.1) and (2.2),

$$2ar\,(ACB) = ar\,(ABCD) = ab\,(\text{from} \quad (2.0)) \tag{2.3}$$

$$\Rightarrow ar\,(ACB) = \frac{ab}{2} \tag{2.4}$$

Problem 2.2. Show that the area of ΔABC in Fig. 2.1 is $\frac{1}{2}ah$.

Proof. In Fig. 2.1,

$$ar\,(\Delta ADC) = \frac{1}{2}hy \tag{2.5}$$

$$ar\,(\Delta ADB) = \frac{1}{2}hx \tag{2.6}$$

Thus,

$$ar\,(\Delta ABC) = ar\,(\Delta ADC) + ar\,(\Delta ADB) \tag{2.7}$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h\,(x+y) \tag{2.8}$$

$$= \frac{1}{2}ah \tag{2.9}$$

Problem 2.3. Show that the area of ΔABC in Fig. 2.1 is $\frac{1}{2}ab\sin C$.

Proof. We have

$$ar\,(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C)\,. \tag{2.10}$$

Problem 2.4. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.11}$$

Proof. Fig. 2.1 can be suitably modified to obtain

$$ar\,(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{s}bc\sin A = \frac{1}{2}ca\sin B \tag{2.12}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.13}$$

This is known as the sine formula.

2.2 Median

Definition 2.4. The line AD in Fig. 2.4 that divides the side a in two equal halfs is known as the median.

Problem 2.5. Show that the median AD in Fig. 2.4 divides ΔABC into triangles ADB and ADC that have equal area.

Proof. We have

$$ar\,(\Delta ADB) = \frac{1}{2}\frac{a}{2}c\sin B = \frac{1}{4}ac\sin B \tag{2.14}$$

$$ar\,(\Delta ADC) = \frac{1}{2}\frac{a}{2}b\sin C = \frac{1}{4}ab\sin C \tag{2.15}$$

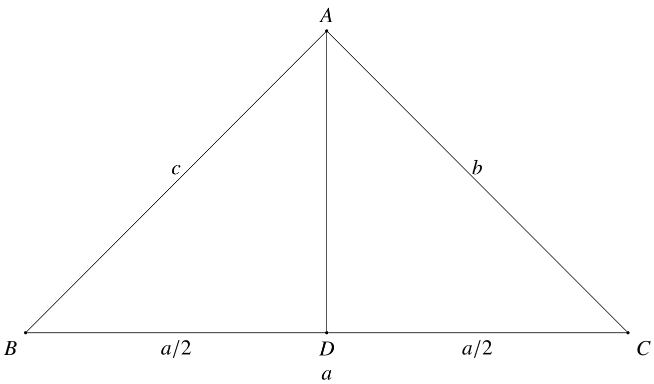

Fig. 2.4: Median of a Triangle

Using the sine formula, $b \sin C = c \sin B$,

$$ar(\Delta ADB) = ar(\Delta ADC) \tag{2.16}$$

Problem 2.6. BE and CF are the medians in Fig. 2.6. Show that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.17}$$

Proof. Since BE and CF are the medians,

$$ar(\Delta BFC) = \frac{1}{2} ar(\Delta ABC) \tag{2.18}$$
$$ar(\Delta BEC) = \frac{1}{2} ar(\Delta ABC) \tag{2.19}$$

From the above, we infer that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.20}$$

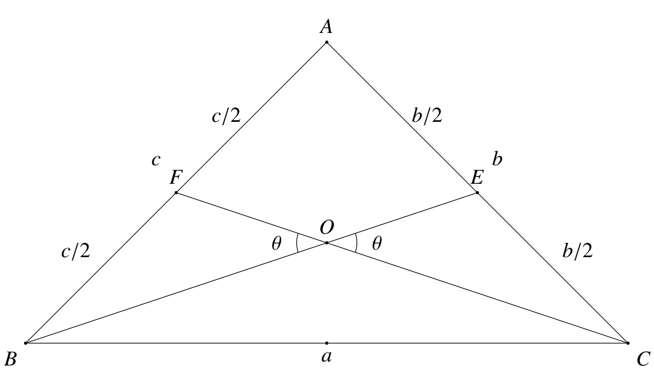

Fig. 2.6: O is the Intersection of Two Medians

Problem 2.7. The medians BE and CF in Fig. 2.6 meet at point O. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} \tag{2.21}$$

Proof. From Problem 2.6,

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.22}$$

$$\Rightarrow ar(\Delta BOF) + ar(\Delta BOC) = ar(\Delta BOC) + ar(\Delta COE) \tag{2.23}$$

resulting in

$$ar(\Delta BOF) = ar(\Delta COE) \tag{2.24}$$

Using the sine formula for area of a triangle, the above equation can be expressed as

$$\frac{1}{2} OB\, OF \sin\theta = \frac{1}{2} OC\, OE \sin\theta \tag{2.25}$$
$$\Rightarrow \frac{OB}{OE} = \frac{OC}{OF} \tag{2.26}$$

Definition 2.5. We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 2.7,

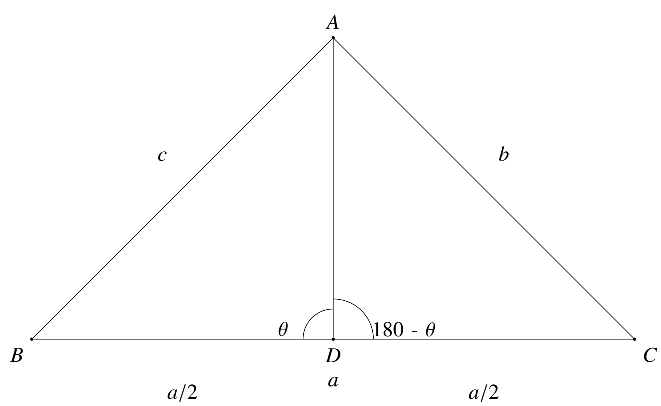

Fig. 2.7: $\sin\theta = \sin\left(180^\circ - \theta\right)$

$$\frac{1}{2}\frac{a}{2} AD \sin\theta = \frac{1}{2}\frac{a}{2} AD \sin\left(180^\circ - \theta\right) \tag{2.27}$$
$$\Rightarrow \sin\theta = \sin\left(180^\circ - \theta\right). \tag{2.28}$$

Note that our geometric definition of $\sin\theta$ holds only for $\theta < 90^\circ$. (2.28) allows us to extend this definition for $\angle ADC > 90^\circ$.

Problem 2.8. In Fig. 2.8, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{2.29}$$

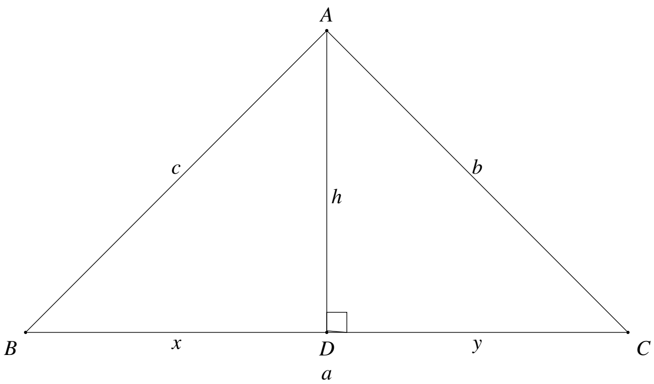

Fig. 2.8: The cosine formula

Proof. From the figure, the first of the following equations

$$a = b\cos C + c\cos B \tag{2.30}$$
$$b = c\cos A + a\cos C \tag{2.31}$$
$$c = b\cos A + a\cos B \tag{2.32}$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.33}$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{2.34}$$

Problem 2.9. In Fig. 2.9, show that $EF = \frac{a}{2}$.

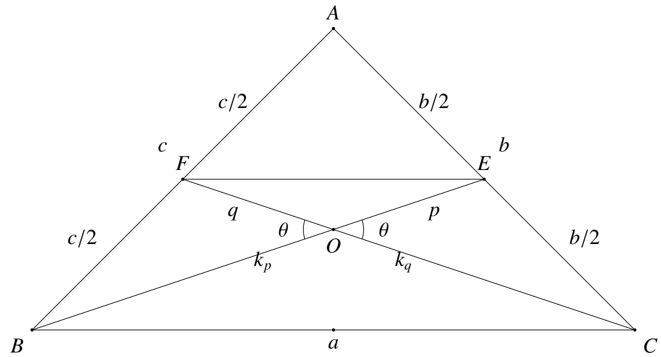

Fig. 2.9: Similar Triangles

Proof. Using the cosine formula for ΔAEF,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \tag{2.35}$$
$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \tag{2.36}$$
$$= \frac{a^2}{4} \tag{2.37}$$
$$\Rightarrow EF = \frac{a}{2} \tag{2.38}$$

Definition 2.6. The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.

Problem 2.10. Show that similar triangles have the same angles.

Proof. Use cosine formula and the proof is trivial.

Problem 2.11. Show that in Fig. 2.9, $EF \| BC$.

Proof. Since $\Delta AEF \sim \Delta ABC$, $\angle AEF = \angle ACB$. Hence the line $EF \| BC$

Problem 2.12. Show that $\triangle OEF \sim \triangle OEC$.

Problem 2.13. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \tag{2.39}$$

Problem 2.14. In Fig. 2.14,BE is a median of ΔABC and $\frac{OB}{OE} = 2$. Show that AD is also a median.

Proof. Since BE is a median, the areas of triangles $\triangle BEC$ and $\triangle BEA$ are equal. Hence,

$$\frac{1}{2}BE.BC\sin\theta_1 = \frac{1}{2}BE.AB\sin\theta_2 \tag{2.40}$$
$$\implies a\sin\theta_1 = c\sin\theta_2 \tag{2.41}$$

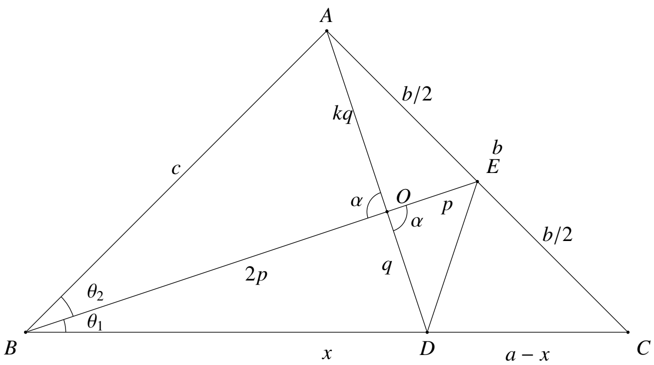

Fig. 2.14: Medians meet at a point

Using the sine formula in $\triangle OBD$ and $\triangle BOA$,

$$\frac{q}{\sin\theta_1} = \frac{x}{\sin\alpha} \tag{2.42}$$
$$\frac{c}{\sin\alpha} = \frac{kq}{\sin\theta_2} \tag{2.43}$$

Mutiplying (2.41),(2.42) and (2.43), we obtain

$$x = \frac{a}{k} \tag{2.44}$$

after simplification. Let the area of $\triangle ABC = \Delta$. Then, in terms of area,

$$\triangle ABD = \triangle OBD + \triangle OBA \tag{2.45}$$
$$\implies \frac{1}{2}(c)(x)\sin B = \frac{1}{2}(2p)(q)\sin\alpha \tag{2.46}$$
$$+\frac{1}{2}(2p)(kq)\sin\alpha \tag{2.47}$$
$$\implies k(k+1)\,pq\sin\alpha = \frac{1}{2}ca\sin B = \Delta \tag{2.48}$$

Similarly,

$$\triangle ABE = \triangle AOB + \triangle AOE \tag{2.49}$$
$$\implies \frac{\Delta}{2} = \frac{1}{2}(2p)(kq)\sin\alpha \tag{2.50}$$
$$+\frac{1}{2}(p)(kq)\sin\alpha \tag{2.51}$$
$$\implies 3kpq\sin\alpha = \Delta \tag{2.52}$$

after simplification. From (2.48) and (2.52),

$$k(k+1)\,pq\sin\alpha = 3kpq\sin\alpha \tag{2.53}$$
$$\implies k = 2. \tag{2.54}$$

Thus,

$$BD = DC \tag{2.55}$$
$$\frac{OA}{OD} = 2. \tag{2.56}$$

Thus, AD is a median.
Conculusion: The medians of a triangle meet at a point.

3 Angle and Perpendicular Bisectors

3.1 Angle Bisectors

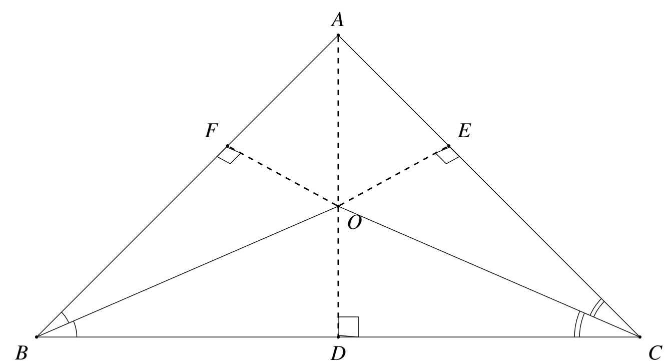

Fig. 3.0: Angle bisectors meet at a point

Definition 3.1. In Fig. 3.0, OB divides the $\angle B$ into half, i.e.

$$\angle OBC = \angle OBA \tag{3.1}$$

OB is known as an angle bisector.

OB and OC are angle bisectors of angles B and C. OA is joined and OD, OF and OE are perpendiculars to sides a, b and c.

Problem 3.1. Show that $OD = OE = OF$.

Proof. In Δs ODC and OEC,

$$OD = OC\sin\frac{C}{2} \tag{3.2}$$
$$OE = OC\sin\frac{C}{2} \tag{3.3}$$
$$\Rightarrow OD = OE. \tag{3.4}$$

Similarly,

$$OD = OF. \tag{3.5}$$

Problem 3.2. Show that OA is the angle bisector of $\angle A$

Proof. In Δs OFA and OEA,

$$OF = OE \tag{3.6}$$
$$\Rightarrow OA \sin OAF = OA \sin OAE \tag{3.7}$$
$$\Rightarrow \sin OAF = \sin OAE \tag{3.8}$$
$$\Rightarrow \angle OAF = \angle OAE \tag{3.9}$$

which proves that OA bisects $\angle A$. *Conclusion:* The angle bisectors of a triangle meet at a point.

3.2 Congruent Triangles

Problem 3.3. Show that in Δs ODC and OEC, corresponding sides and angles are equal.

Definition 3.2. Note that Δs ODC and OEC are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.

Problem 3.4. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.

Problem 3.5. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.

Problem 3.6. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.

Problem 3.7. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

3.3 Perpendicular Bisectors

Definition 3.3. In Fig. 3.8, OD $\perp$ BC and $BD = DC$. OD is defined as the perpendicular bisector of BC.

Problem 3.8. In Fig. 3.8, show that $OA = OB = OC$.

Proof. In Δs ODB and ODC, using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \tag{3.10}$$

Since $BD = DC = \frac{a}{2}$, $OB = OC$. Similarly, it can be shown that $OA = OC$. Thus, $OA = OB = OC$.

Definition 3.4. In ΔAOB, $OA = OB$. Such a triangle is known as an isoceles triangle.

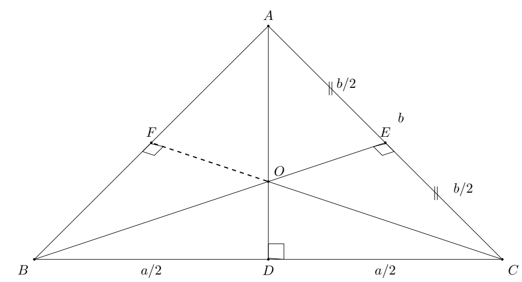

Fig. 3.8: Perpendicular bisectors meet at a point

Problem 3.9. Show that $AF = BF$.

Proof. Trivial using Budhayana's theorem. This shows that OF is a perpendicular bisector of AB. *Conclusion:* The perpendicular bisectors of a triangle meet at a point.

3.4 Perpendiculars from Vertex to Opposite Side

In Fig. 3.10, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F.

Problem 3.10. Show that

$$OE = c \cos A \cot C \tag{3.11}$$

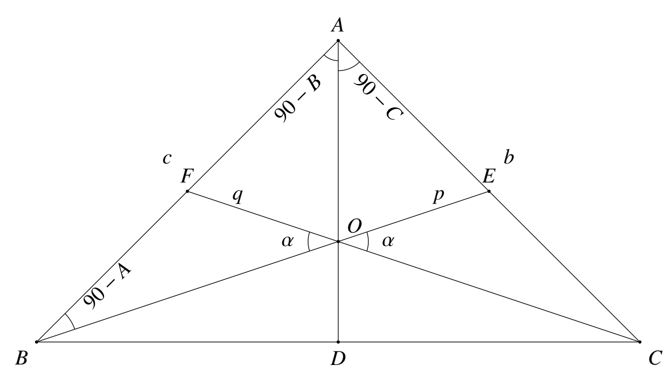

Fig. 3.10: Perpendiculars from vertex to opposite side meet at a point

Proof. In Δ s AEB and AEO,

$$AE = c\cos A \tag{3.12}$$

$$OE = AE\tan\left(90^\circ - C\right)(\because ADC \text{ is right angled}) \tag{3.13}$$

$$= AE\cot C \tag{3.14}$$

From both the above, we get the desired result.

Problem 3.11. Show that $\alpha = A$.

Proof. In ΔOEC,

$$CE = a\cos C\,(\because BEC \text{ is right angled}) \tag{3.15}$$

Hence,

$$\begin{aligned}\tan\alpha &= \frac{CE}{OE} \\ &= \frac{a\cos C}{c\cos A\cot C} \\ &= \frac{a\cos C\sin C}{c\cos A\cos C} \\ &= \frac{a\sin C}{c\cos A} \\ &= \frac{c\sin A}{c\cos A}\left(\because \frac{a}{\sin A} = \frac{c}{\sin C}\right) \\ &= \tan A\end{aligned} \tag{3.16}$$

$$\Rightarrow \alpha = A$$

Problem 3.12. Show that $CF \perp AB$

Proof. Consider triangle OFB and the result of the previous problem. $\because$ the sum of the angles of a triangle is 180°, $\angle CFB = 90^\circ$. *Conclusion: The perperdiculars from the vertex of a triangle to the opposite side meet at a point.*

4 Circle

4.1 Chord of a Circle

Definition 4.1. Fig. 4.0 represents a circle. The points in the circle are at a distance r from the centre O. r is known as the radius.

4.2 Chords of a circle

Definition 4.2. In Fig. 4.0, A and B are points on the circle. The line AB is known as a chord of the circle.

Problem 4.1. In Fig. 4.1 Show that $\angle AOB = 2\angle ACB$.

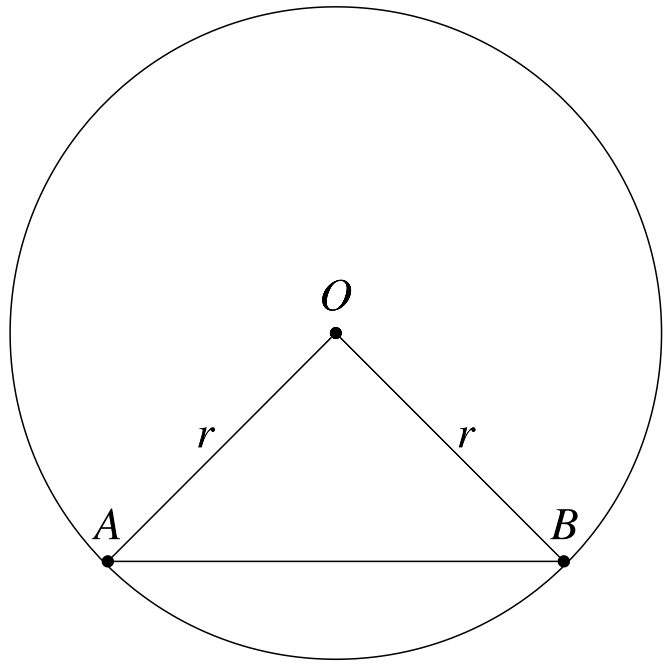

Fig. 4.0: Circle Definitions

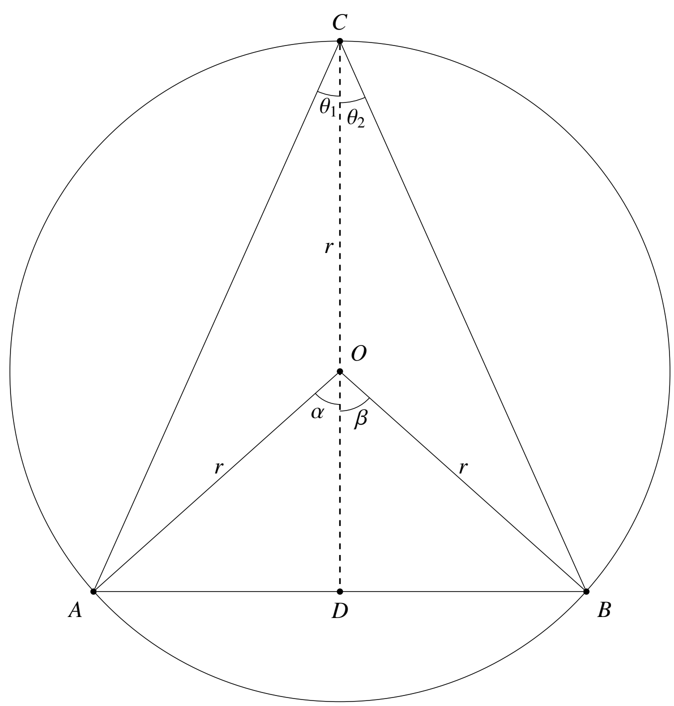

Fig. 4.1: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Proof. In Fig. 4.1, the triangeles OPA and OPB are isosceles. Hence,

$$\angle OCA = \angle OAC = \theta_1 \tag{4.1}$$

$$\angle OCB = \angle OBC = \theta_2 \tag{4.2}$$

Also, α and β are exterior angles corresponding to the triangle AOC and BOC respectively. Hence

$$\alpha = 2\theta_1 \tag{4.3}$$
$$\beta = 2\theta_2 \tag{4.4}$$

Thus,

$$\begin{aligned} \angle AOB &= \alpha + \beta && (4.5)\\ &= 2(\theta_1 + \theta_2) && (4.6)\\ &= 2\angle ACB && (4.7) \end{aligned}$$

Definition 4.3. The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 4.2, AB is the diameter and passes through the centre O

Problem 4.2. In Fig. 4.2, show that $\angle APB = 90^\circ$.

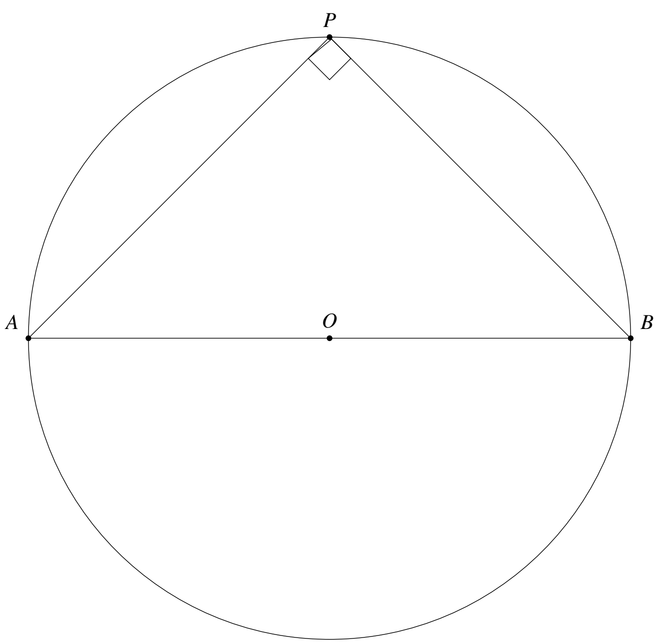

Fig. 4.2: Diameter of a circle.

Problem 4.3. In Fig. 4.3, show that

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \tag{4.8}$$

Proof. Use Problem 4.1.

Problem 4.4. In Fig. 4.3, show that the triangles PAB and PBD are similar

Proof. Trivial using previous problem

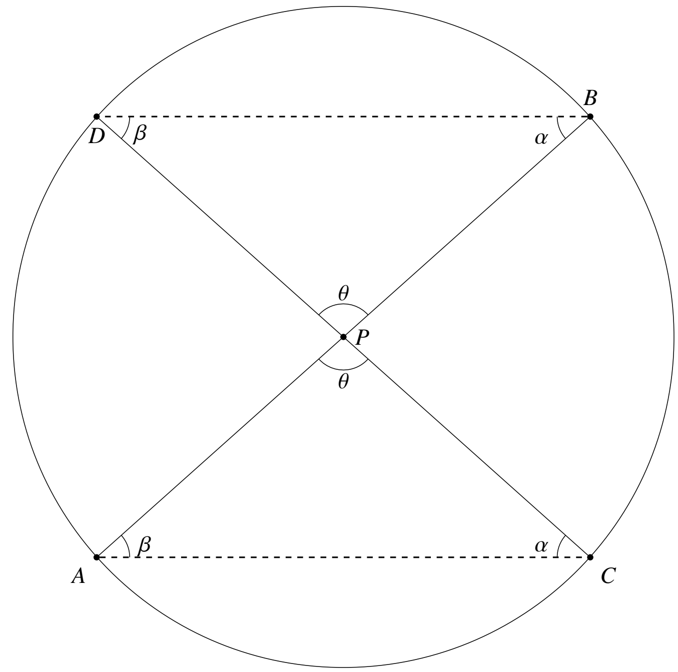

Fig. 4.3: $PA.PB = PC.PD$

Problem 4.5. In Fig. 4.3, show that

$$PA.PB = PC.PD \tag{4.9}$$

Proof. Since triangles PAC and PBD are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{4.10}$$
$$\Rightarrow PA.PB = PC.PD \tag{4.11}$$

Definition 4.4. The line PX in Fig. 4.6 touches the circle at exactly one point P. It is known as the tangent to the circle.

Problem 4.6. Show that $OP \perp PX$.

Proof. Without loss of generality, let $0 \le \theta \le 90^\circ$. Using the cosine formula in $\triangle OPP_n$,

$$(r + d_n)^2 > r^2, \tag{4.12}$$

$$(r + d_n)^2 = r^2 + x_n^2 - 2rx_n \cos\theta > r^2 \tag{4.13}$$
$$\implies 0 < \cos\theta < \frac{x_n}{2r}, \tag{4.14}$$

where x_n can be made as small as we choose. Thus,

$$\cos\theta = 0 \implies \theta = 90^\circ. \tag{4.15}$$

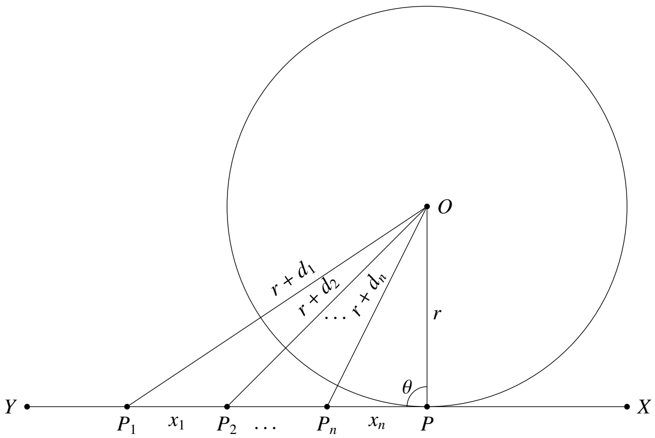

Fig. 4.6: Tangent to a Circle.

Problem 4.7. In Fig. 4.7 show that

$$\angle PCA = \angle PBC \tag{4.16}$$

O is the centre of the circle and PC is the tangent.

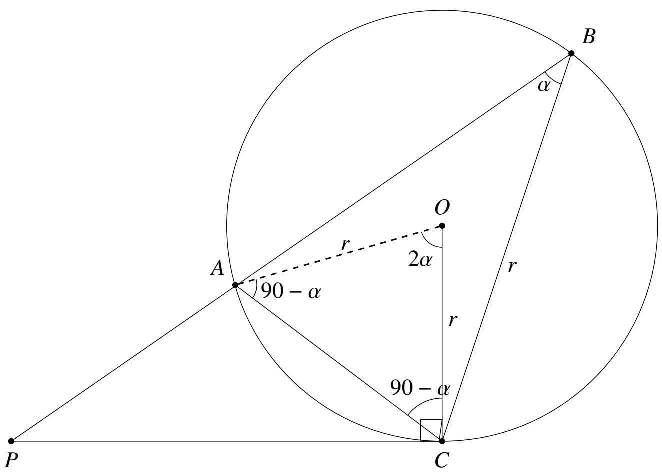

Fig. 4.7: $PA.PB = PC^2$.

Proof. Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

Problem 4.8. In Fig. 4.7, show that the triangles PAC and PBC are similar.

Proof. From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

Problem 4.9. Show that $PA.PB = PC^2$

Proof. Since $\Delta PAC \sim \Delta PBC$, their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{4.17}$$

$$\Rightarrow PA.PB = PC^2 \tag{4.18}$$

Problem 4.10. Given that $PA.PB = PC^2$, show that PC is a tangent to the circle.

Problem 4.11. In Fig. 4.11, show that

$$PA.PB = PC.PD \tag{4.19}$$

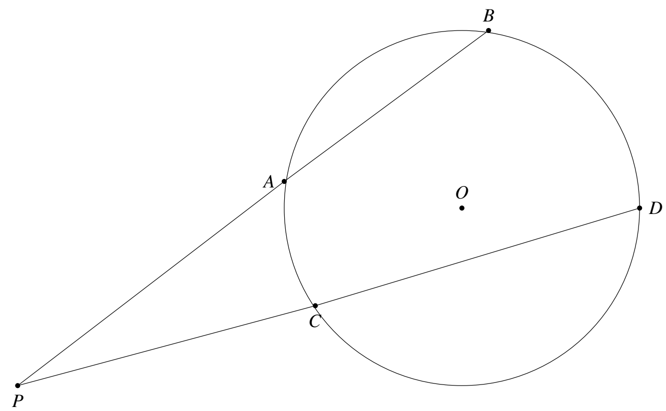

Fig. 4.11: $PA.PB = PC^2$.

Proof. Draw a tangent and use the previous problem.

5 Area of a Circle

5.1 The Regular Polygon

Definition 5.1. In Fig. 5.0, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.

Definition 5.2. The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{360^\circ}{n}$.

Problem 5.1. Show that the area of a regular polygon is given by

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} \tag{5.1}$$

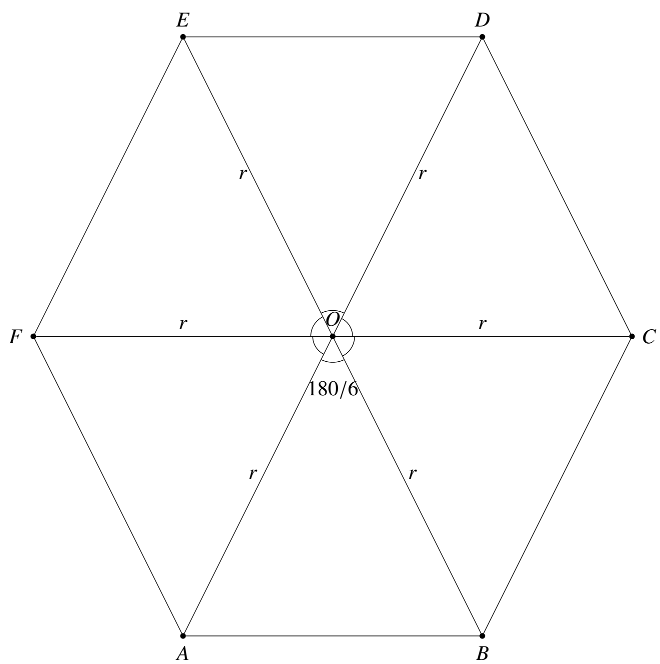

Fig. 5.0: Polygon Definition

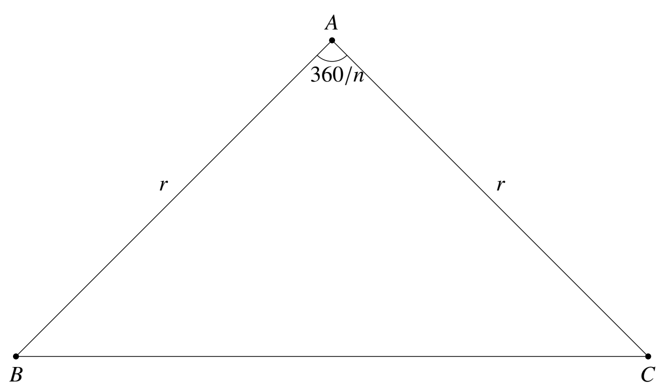

Fig. 5.1: Polygon Area

Proof. The triangle that forms the polygon of n sides is given in Fig. 5.1. Thus,

$$\begin{aligned} ar\,(polygon) &= nar\,(\Delta ABC) \\ &= \frac{n}{2} r^2 \sin \frac{360^\circ}{n} \end{aligned} \tag{5.2}$$

Problem 5.2. Using Fig. 5.2, show that

$$\frac{n}{2} r^2 \sin \frac{360^\circ}{n} < \text{ area of circle } < nr^2 \tan \frac{180^\circ}{n} \tag{5.3}$$

The portion of the circle visible in Fig. 5.2 is defined to be a sector of the circle.

Proof. Note that the circle is squeezed between the

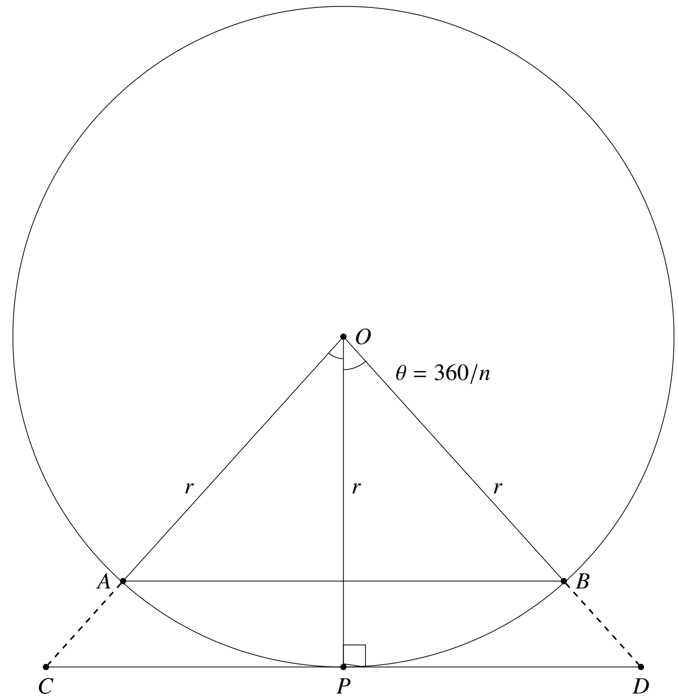

Fig. 5.2: Circle Area in between Area of Two Polygons

inner and outer regular polygons. As we can see from Fig. 5.2, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$ar\,(\Delta OAB) = \frac{1}{2} r^2 \sin \frac{360^\circ}{n} \tag{5.4}$$

$$ar\,(\Delta OPQ) = 2 \times \frac{1}{2} \times r \tan \frac{360/n}{2} \times r \tag{5.5}$$

$$= r^2 \tan \frac{180^\circ}{n}, \tag{5.6}$$

we obtain (5.3).

Problem 5.3. Using Fig. 5.3, show that

$$\sin\theta_1 = \sin(\theta_1 + \theta_2)\cos\theta_2 - \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{5.7}$$

Proof. The following equations can be obtained from the figure using the forumula for the area of a

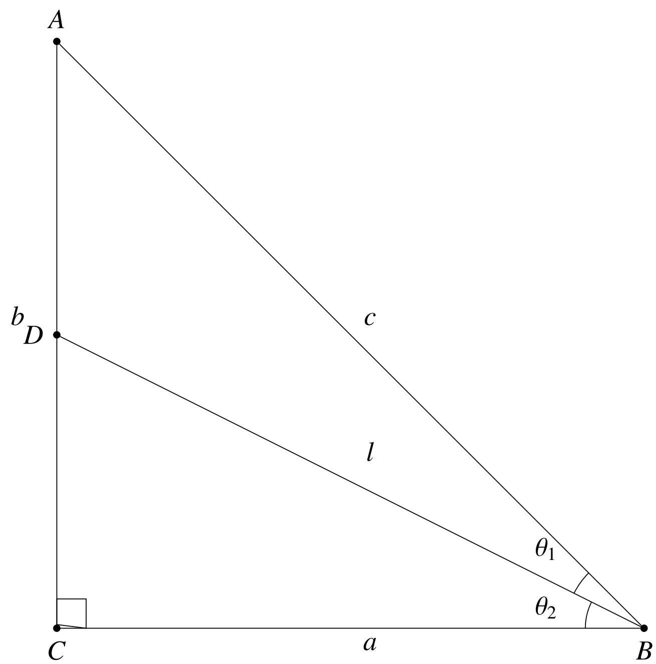

Fig. 5.3: $\sin 2\theta = 2\sin\theta\cos\theta$

triangle

$$ar\,(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1+\theta_2) \tag{5.8}$$
$$= ar\,(\Delta BDC) + ar\,(\Delta ADB) \tag{5.9}$$
$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{5.10}$$
$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{5.11}$$

($\because\, l = a\sec\theta_2$). From the above,

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{5.12}$$
$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \tag{5.13}$$

Multiplying both sides by $\cos\theta_2$,

$$\Rightarrow \sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{5.14}$$

resulting in

$$\Rightarrow \sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{5.15}$$

Problem 5.4. Prove the following identities

1)

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{5.16}$$

2)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{5.17}$$

Proof. In (5.7), let

$$\begin{aligned}\theta_1+\theta_2 &= \alpha\\ \theta_2 &= \beta\end{aligned} \tag{5.18}$$

This gives (5.16). In (5.16), replace α by $90^\circ - \alpha$. This results in

$$\sin\left(90^\circ - \alpha - \beta\right)$$
$$= \sin\left(90^\circ - \alpha\right)\cos\beta - \cos\left(90^\circ - \alpha\right)\sin\beta$$
$$\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{5.19}$$

Problem 5.5. Using (5.7) and (5.17), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{5.20}$$
$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{5.21}$$

Proof. From (5.7),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{5.22}$$

Using (5.17) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{5.23}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{5.24}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{5.25}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{5.26}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{5.27}$$

after factoring out $\cos\theta_2$. Using a similar approach, (5.21) can also be proved.

Problem 5.6. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{5.28}$$

Problem 5.7. Show that

$$\cos^2 \frac{180^\circ}{n} < \frac{\text{area of circle}}{nr^2 \tan \frac{180^\circ}{n}} < 1 \tag{5.29}$$

Proof. From (5.3) and (5.28),

$$\begin{aligned}
\frac{n}{2} r^2 \sin \frac{360^\circ}{n} &< \text{ area of circle} \\
&< nr^2 \tan \frac{180^\circ}{n} \\
\Rightarrow nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} &< \text{ area of circle} \\
&< nr^2 \tan \frac{180^\circ}{n}
\end{aligned} \tag{5.30}$$

Problem 5.8. Show that if

$$\theta_1 < \theta_2, \sin \theta_1 < \sin \theta_2. \tag{5.31}$$

Proof. Trivial using the definition and choosing angles θ_1 and θ_2 appropriately in a right angled triangle.

Problem 5.9. Show that if

$$\theta_1 < \theta_2, \cos \theta_1 > \cos \theta_2. \tag{5.32}$$

Problem 5.10. Show that

$$\sin 0^\circ = 0 \tag{5.33}$$

Proof. Follows from (5.32).

Problem 5.11. Show that

$$\cos 0^\circ = 1 \tag{5.34}$$

Problem 5.12. Show that for large values of n

$$\cos^2 \frac{180^\circ}{n} = 1 \tag{5.35}$$

Proof. Follows from previous problem.

Definition 5.3. The previous result can be expressed as

$$\lim_{n \to \infty} \cos^2 \frac{180^\circ}{n} = 1 \tag{5.36}$$

Problem 5.13. Show that

$$\text{area of circle } = r^2 \lim_{n \to \infty} n \tan \frac{180^\circ}{n} \tag{5.37}$$

Definition 5.4.

$$\pi = \lim_{n \to \infty} n \tan \frac{180^\circ}{n} \tag{5.38}$$

Thus, the area of a circle is πr^2.

Definition 5.5. The radian is a unit of angle defined by

$$1 \text{ radian} = \frac{360^\circ}{2\pi} \tag{5.39}$$

Problem 5.14. Show that the circumference of a circle is $2\pi r$.

Problem 5.15. Show that the area of a sector with angle θ in radians is $\frac{1}{2} r^2 \theta$.

Problem 5.16. Show that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \tag{5.40}$$