Linear Algebra through Coordinate Geometry

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Contents

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***Abstract*—This book provides a computational approach to linear algebra and matrices by solving problems in 2D and 3D coordinate geometry from IIT-JEE. An introduction to convex optimization is also provided in the process. Links to sample Python codes are available in the text. The book provides sufficient math basics for Machine Learning and is also recommended for high school students who wish to explore topics in Artificial Intelligence.**

Download python codes using

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svn co https://github.com/gadepall/school/trunk/
    linalg/book/codes
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1 The Straight Line

1.1 Point

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$
3. The *length* of PQ is defined as
$$\|\mathbf{P} - \mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$
5. The point dividing PQ in the ratio $k : 1$ is
$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k + 1} \tag{1.1.5}$$

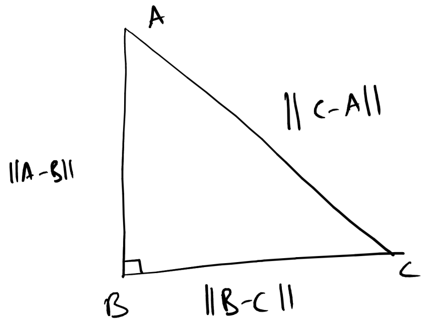

Fig. 1.1.7

6. The *area* of $\triangle PQR$ is the *determinant*

$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$

7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that

$$\left(\mathbf{A} - \mathbf{B}\right)^T \left(\mathbf{B} - \mathbf{C}\right) = 0 \tag{1.1.7}$$

Solution: Using Baudhayana's theorem,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{1.1.7}$$

$$\implies \left(\mathbf{A} - \mathbf{B}\right)^T \left(\mathbf{A} - \mathbf{B}\right) + \left(\mathbf{B} - \mathbf{C}\right)^T \left(\mathbf{B} - \mathbf{C}\right) = \left(\mathbf{C} - \mathbf{A}\right)^T \left(\mathbf{C} - \mathbf{A}\right)$$

$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$

which can be simplified to obtain (1.1.7).

8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show that

$$\left(\mathbf{x} - \mathbf{A}\right)^T \left(\mathbf{B} - \mathbf{C}\right) = 0 \tag{1.1.8}$$

9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$\left(\mathbf{x} - \mathbf{y}\right)^T \left(\mathbf{B} - \mathbf{C}\right) = 0 \tag{1.1.9}$$

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1.2 Line

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.2.1.1}$$

$$\implies x_2 = mx_1 \tag{1.2.1.2}$$

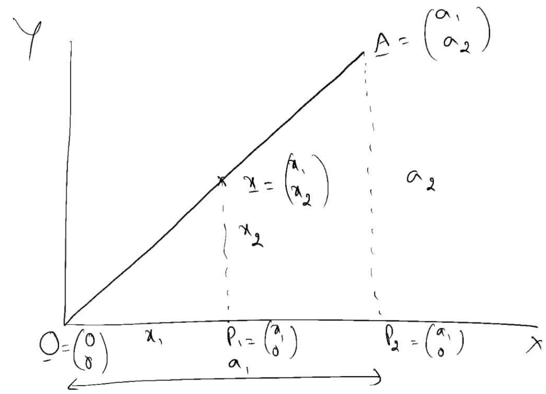

Fig. 1.2.1

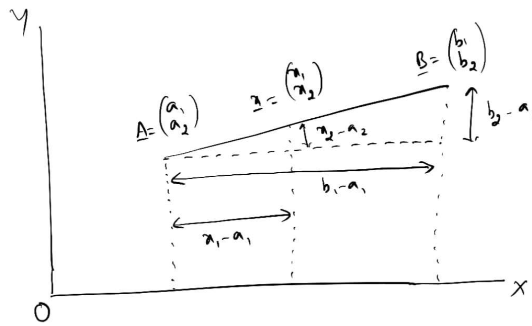

Fig. 1.2.2

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = x_1\mathbf{m} \tag{1.2.1.3}$$

In general, the above equation is written as

$$\mathbf{x} = \lambda\mathbf{m}, \tag{1.2.1.4}$$

where $\mathbf{m}$ is the direction vector of the line.

2. Find the equation of AB in Fig. 1.2.2

Solution: From Fig. 1.2.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.2.2.1}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.2.2.2}$$

From (1.2.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.2.2.3}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2.4}$$

$$= \mathbf{A} + \lambda\mathbf{m} \tag{1.2.2.5}$$

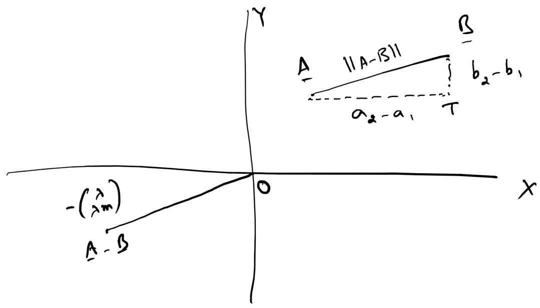

Fig. 1.2.5

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (1.2.2.5) is obtained from (1.2.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 1.2.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as
$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \quad (1.2.4.1)$$
Also, from (1.2.1.4),
$$\|\mathbf{A}\| = \lambda\sqrt{1+m^2} \quad (1.2.4.2)$$
Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.2.5. From (1.2.2.5), for some λ,
$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \quad (1.2.5.1)$$
$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix}1\\m\end{pmatrix}, \quad (1.2.5.2)$$
$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.
6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is
$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \quad (1.2.7.1)$$
8. The *normal* to the vector $\mathbf{m}$ is defined as
$$\mathbf{n}^T\mathbf{m} = 0 \quad (1.2.8.1)$$
$$\mathbf{n} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{m} \quad (1.2.8.2)$$
9. From (1.2.7.1), the equation of a line can also be expressed as
$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) \quad (1.2.9.1)$$
$$\implies \mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} = c \quad (1.2.9.2)$$
10. The unit vectors on the x and y axis are defined as
$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad (1.2.10.1)$$
$$\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} \quad (1.2.10.2)$$
11. If a be the *intercept* of the line
$$\mathbf{n}^T\mathbf{x} = c \quad (1.2.11.1)$$
on the $x-$axis, then $\begin{pmatrix}a\\0\end{pmatrix}$ is a point on the line. Thus,
$$\mathbf{n}^T\begin{pmatrix}a\\0\end{pmatrix} = c \quad (1.2.11.2)$$
$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \quad (1.2.11.3)$$
12. The *rotation matrix* is defined as
$$\mathbf{Q} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \quad (1.2.12)$$
where θ is anti-clockwise.
13.
$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \quad (1.2.13)$$
where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix*.
14. Find the equation of line L in Fig. 1.2.14.
Solution: The equation of the $x-$axis is
$$\mathbf{x} = \lambda\mathbf{e}_1 \quad (1.2.14.1)$$
Translation by p units along the $y-$axis results in
$$L_0: \quad \mathbf{x} = \lambda\mathbf{e}_1 + p\mathbf{e}_2 \quad (1.2.14.2)$$
Rotation by $90° - \alpha$ in the anti-clockwise direction yields
$$L: \quad \mathbf{x} = \mathbf{Q}\{\lambda\mathbf{e}_1 + p\mathbf{e}_2\} \quad (1.2.14.3)$$
$$= \lambda\mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \quad (1.2.14.4)$$

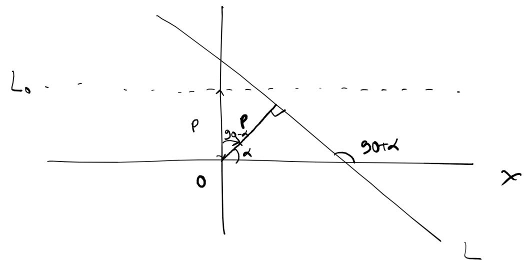

Fig. 1.2.14

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \quad (1.2.14.5)$$

$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \quad (1.2.14.6)$$

From (1.2.14.4),

$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$
$$= \lambda\mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \quad (1.2.14.7)$$

resulting in

$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \quad (1.2.14.8)$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T\mathbf{x} = c \quad (1.2.15.1)$$

is

$$p = \frac{c}{\|n\|} \quad (1.2.15.2)$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \quad (1.2.16.1)$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \quad (1.2.16.2)$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \quad (1.2.16.3)$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \quad (1.2.16.4)$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \quad (1.2.17.1)$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \quad (1.2.18.1)$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \quad (1.2.18.2)$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \quad (1.2.18.3)$$

where

$$\mathbf{n}^T\mathbf{A} = c, \mathbf{n}^T\mathbf{m} = 0 \quad (1.2.18.4)$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \quad (1.2.18.5)$$
$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \quad (1.2.18.6)$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.2.19.1)$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (1.2.19.2)$$

where

$$\mathbf{V} = \mathbf{V}^T \quad (1.2.19.3)$$
$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (1.2.19.4)$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.2.20.1)$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \quad (1.2.20.2)$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \quad (1.2.21.1)$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k + 1} = \mathbf{B}. \quad (1.2.21.2)$$

Solution: From (1.2.2.5),

$$\begin{aligned} \mathbf{B} &= \mathbf{A} + \lambda_1 \mathbf{m} \\ \mathbf{B} &= \mathbf{C} - \lambda_2 \mathbf{m} \end{aligned} \tag{1.2.21.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{1.2.21.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{1.2.21.5}$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A} - \mathbf{B} = k\left(\mathbf{B} - \mathbf{C}\right) \tag{1.2.21.6}$$

resulting in (1.2.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{1.2.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{1.2.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{1.2.23.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{1.2.23.3}$$

24. In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of ΔABC.
25. Find the intersection of AP and BQ.
Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From (1.2.8.1) and (1.2.9.2), the equation of AP and BQ are

$$\left(\mathbf{B} - \mathbf{C}\right)^T\left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2.25.1}$$

$$\left(\mathbf{C} - \mathbf{A}\right)^T\left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{1.2.25.2}$$

which can be solved to obtain the intersection point using (1.2.16.3).

26. Show that the equation of the angle bisectors

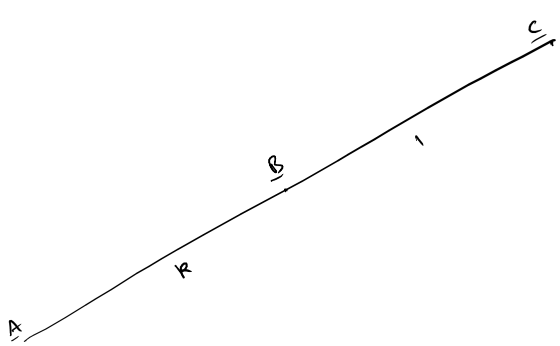

Fig. 1.2.21

of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.26.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.26.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{1.2.26.3}$$

27. Find the equation of a line passing through the intersection of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.27.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.27.2}$$

and passing through the point $\mathbf{p}$.
Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T}\mathbf{c} \tag{1.2.27.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{1.2.27.4}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{1.2.27.5}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda\left(\mathbf{N}^{-T}\mathbf{c} - \mathbf{p}\right) \tag{1.2.27.6}$$

$$\implies \mathbf{N}^T\mathbf{x} = \mathbf{N}^T\mathbf{p} + \lambda\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right) \tag{1.2.27.7}$$

resulting in

$$\begin{aligned} &\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{x} \\ &\quad = \left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{p} \end{aligned} \tag{1.2.27.8}$$

28. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.28.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.2.28.2}$$

$$\mathbf{n}^T\mathbf{Q} = c \tag{1.2.28.3}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{1.2.28.4}$$

where $\mathbf{m}$ is the direction vector of L. From (1.2.28.2) and (1.2.28.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{1.2.28.5}$$

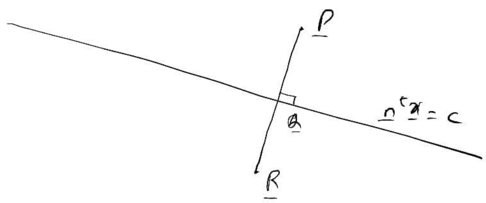

Fig. 1.2.28

From (1.2.28.5) and (1.2.28.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \quad (1.2.28.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \quad (1.2.28.7)$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \quad (1.2.28.8)$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \quad (1.2.28.9)$$

(1.2.28.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \quad (1.2.28.10)$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \quad (1.2.28.11)$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \quad (1.2.28.12)$$

29. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (1.2.29.1)$$

1.3 *Example*

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix} \quad (1.3.1.1)$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \quad (1.3.1.2)$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \quad (1.3.1.3)$$

Find the area of $\triangle ABC$.

Solution: The centroid $\mathbf{O}$ is the solution of (1.3.1.2),(1.3.1.3) and is obtained as the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} \quad (1.3.1.4)$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix}1 & 1 & 5\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 5\\0 & 1 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 1\end{pmatrix} \quad (1.3.1.5)$$

Thus,

$$\mathbf{O} = \begin{pmatrix}4\\1\end{pmatrix} \quad (1.3.1.6)$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \quad (1.3.1.7)$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix}11\\1\end{pmatrix} \quad (1.3.1.8)$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}11\\1\end{pmatrix} \quad (1.3.1.9)$$

From (1.3.1.3) and (1.3.1.9),

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} = 5 \quad (1.3.1.10)$$

$$\implies 5 + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 12 \quad (1.3.1.11)$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 7 \quad (1.3.1.12)$$

From (1.3.1.12) and (1.3.1.3), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}7\\4\end{pmatrix} \quad (1.3.1.13)$$

using the augmented matrix as

$$\begin{pmatrix}1 & 1 & 7\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 7\\0 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 3\end{pmatrix} \quad (1.3.1.14)$$

$$\implies \mathbf{C} = \begin{pmatrix}4\\3\end{pmatrix} \quad (1.3.1.15)$$

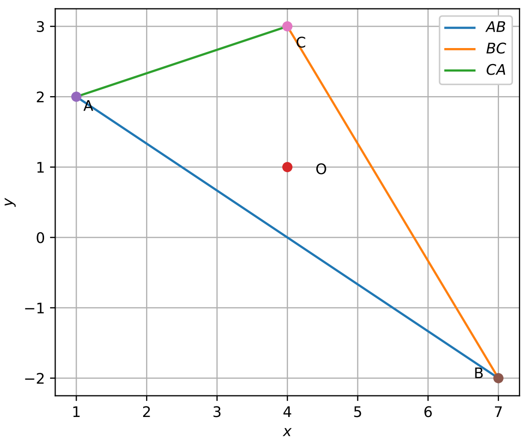

Fig. 1.3.2

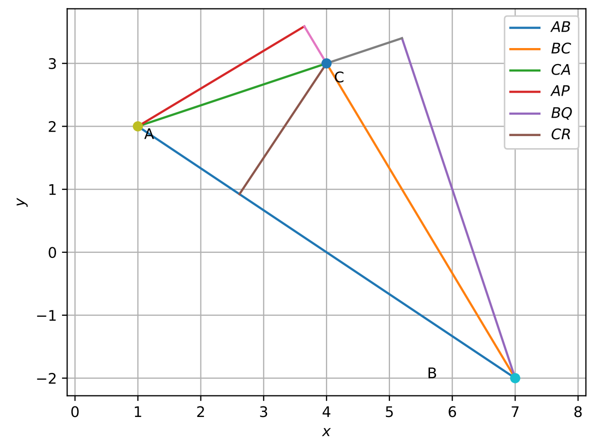

Fig. 1.4.4

From (1.3.1.8),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.3.1.16}$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \tag{1.3.1.17}$$

2. Summarize all the above computations through a Python script and plot $\triangle ABC$.
Solution:

```
codes/2d/triang.py
```

1.4 Programming

1. Find the *orthocentre* of $\triangle ABC$.
Solution: The following code finds the required point using (1.2.25.1) and (1.2.25.2) .

```
codes/2d/orthocentre.py
```

2. Find $\mathbf{P}$, the foot of the altitude from $\mathbf{A}$ upon BC.
Solution:

```
codes/2d/alt_foot.py
```

3. Find $\mathbf{Q}$ and $\mathbf{R}$.
4. Draw AP, BQ and CR and verify that they meet at a point $\mathbf{H}$.
Solution: The following code plots the altitudes in Fig. 1.4.4

```
codes/2d/alt_draw.py
```

5. Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.
6. Find the equations of AD, BE and CF.
7. Find the point of intersection of AD and CF.
8. Verify that $\mathbf{O}$ is the point of intersection of BE, CF as well.
9. Graphically show that the medians of ΔABC meet at the centroid.

1.5 Solved Problems

1. A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.5.1}$$
$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.5.1}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.5.1}$$

Then (1.5.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5.1}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.5.1) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.5.1}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.5.1}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.5.1}$$

$$\mathbf{A} = -k\mathbf{B} \tag{1.5.1}$$

Substituting in (1.5.1), and simplifying,

$$\mathbf{n}^T\mathbf{B} = \frac{10}{-k} \tag{1.5.1}$$

$$\mathbf{n}^T\mathbf{B} = \frac{-5}{2} \tag{1.5.1}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.5.1}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.5.2}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{1.5.2}$$

by $d = 2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (1.5.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.2}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \tag{1.5.2}$$

However,

$$PQ = d \tag{1.5.2}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = d \tag{1.5.2}$$

$$\implies \lambda = \pm\frac{d}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{1.5.2}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T\mathbf{m}} = \sqrt{2} \tag{1.5.2}$$

from (1.5.2). Since $\mathbf{Q}$ lies in the third quadrant, from (1.5.2) and (1.5.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{1.5.2}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{Q}) = 0 \tag{1.5.2}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - 2\sqrt{6} \tag{1.5.2}$$

3. Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.5.3}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{1.5.3}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{1.5.3}$$

find its vertices.
Solution: From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{1.5.3}$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1.5.3}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2}$$

$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.5.3}$$

$$\because AD \parallel BC,$$

$$BC : \begin{pmatrix} 7 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$

$$\implies \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} = -15 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \implies \mathbf{B} = -\frac{1}{3}\begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.5.3}$$

$$\because AB \parallel CD,$$
$$CD : \begin{pmatrix} 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \implies \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \tag{1.5.3}$$

4. Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{1.5.4}$$

has area 28. Find the orthocentre of this triangle.
Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \tag{1.5.4}$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.5.4}$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \tag{1.5.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \tag{1.5.4}$$

The orthocentre is obtained by solving (1.5.4) and (1.5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \tag{1.5.4}$$

which can be expressed using (1.5.4), (1.5.4), (1.5.4) and (1.5.4) as

$$\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix}$$
$$= k \begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix} \tag{1.5.4}$$

From (1.5.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$
$$\implies \begin{vmatrix} k & 5 - k & -2k \\ -3k & 4k & 2 + 3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \tag{1.5.4}$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \tag{1.5.4}$$
$$\implies 5k^2 + 13k - 46 = 0 \tag{1.5.4}$$
$$\text{or, } k = 2, -\frac{23}{5} \tag{1.5.4}$$

Substituting the above in (1.5.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.5.5}$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0} \tag{1.5.5}$$

If AB be the line in (1.5.5), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.5.5}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.5.5}$$

since CF passes through the origin and $CF \perp AB$. From (1.5.5) and (1.5.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.5.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

From (1.5.5),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

after substituting from (1.5.5). Thus,

$$CF = \|\mathbf{C} - \mathbf{F}\| = 3\sqrt{2} \tag{1.5.5}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = 2\sqrt{6} \tag{1.5.5}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 6\sqrt{3} \tag{1.5.5}$$

6. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.
Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and y-axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.6}$$

Multiplying (1.5.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.5.6}$$

$$\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 4\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= 4\begin{pmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix} \tag{1.5.6}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})$$
$$= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3} - 1\right) \tag{1.5.6}$$

for $\theta = 30^{\circ}$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.

1.6 JEE Exercises

1. Find the area enclosed within the curve $|x| + |y| = 1$.
2. Find the equation of the line about which y=10^x is the reflection of y=$\log_{10} x$.
3. If $3a + 2b + 4c = 0$, find the intersection of the set of lines
$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + c = 0. \tag{1.6.3.1}$$
4. Given the points $A = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$, find the equation of the locus of the point $P = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $|AP - BP|$=6.
5. If a,b and c are in A.P, show that the straight line
$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + c = 0 \tag{1.6.5.1}$$
will always pass through a fixed point and find its coordinates.
6. Find the quadrant in which the orthocentre of the triangle formed by the lines
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 1 \tag{1.6.6.1}$$
$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 6 \tag{1.6.6.2}$$
$$\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.6.6.3}$$
lies.
7. Let the algebraic sum of the perpendicular distances from the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to a variable straight line be zero. Show that the line passes through a fixed point and find its coordinates.
8. The vertices of a triangle are $\mathbf{A} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find the equation of the bisector of $\angle ABC$.
9. Verify if the straight line
$$\begin{pmatrix} 5 & 4 \end{pmatrix}\mathbf{x} = 0 \tag{1.6.9.1}$$
passes through the point of intersection of the straight lines
$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} - 10 = 0 \tag{1.6.9.2}$$
and
$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} + 5 = 0 \tag{1.6.9.3}$$

10. Do the lines
$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 19 = 0 \tag{1.6.10.1}$$
and
$$\begin{pmatrix} 9 & 6 \end{pmatrix} \mathbf{x} - 17 = 0 \tag{1.6.10.2}$$
cut the coordinate axes in concyclic points?
11. The points $\begin{pmatrix} -a \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} a^2 \\ ab \end{pmatrix}$ are:
    a) Collinear
    b) Vertices of a parallelogram
    c) Vertices of a rectangle
    d) None of these
12. The point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ undergoes the following three transformations successively (i) Reflection about the line
$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.12.1}$$
(ii)Translation through a distance 2 units along the positive direction of x-axis (iii) Rotation through an angle $\frac{\pi}{4}$ about the origin in counter clockwise direction. Find the final position of the point.
13. The straight lines
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.13.1}$$
$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{1.6.13.2}$$
$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{1.6.13.3}$$
form a triangle which is
    a) isosceles
    b) equilateral
    c) right angled
    d) none of these
14. If $\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ are three given points, then the locus of the point $\mathbf{S}$ satisfying the relation $SQ^2 + SR^2 = 2SP^2$, is
    a) a straight line parallel to X-axis
    b) a circle passing through the origin
    c) a circle with centre at the origin
    d) a straight line parallel to Y-axis.
15. Line L has intercepts a and b on the coordinate axes.When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q, then
    a) $a^2 + b^2 = p^2 + q^2$
    b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
    c) $a^2 + p^2 = b^2 + q^2$
    d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$
16. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
    a) Square
    b) Circle
    c) Straight line
    d) Two intersecting lines
17. The locus of a variable point whose distances from $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ is $\frac{2}{3}$ times its distance from the line $x = -\frac{9}{2}$ is
    a) Ellipse
    b) Parabola
    c) Hyperbola
    d) None of these
18. The equations of a pair of opposite sides of a parallelogram are
$$x^2 - 5x + 6 = 0 \tag{1.6.18.1}$$
$$y^2 - 6y + 5 = 0 \tag{1.6.18.2}$$
Find the equations of its diagonals.
19. Find the orthocentre of the triangle formed by the lines
$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.19.1}$$
and
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1.6.19.2}$$
20. Let PQR be an isosceles triangle, right angled at $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. If the equation of the line QR is
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3, \tag{1.6.20.1}$$
then find the equation representing the pair of lines PQ and PR is
21. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P with the same common ratio, then the points$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$
    a) lie on a straight line
    b) lie on a ellipse
    c) lie on a circle
    d) are the vertices of a triangle
22. Let PS be the median of the triangle with

vertices $\mathbf{P} = \begin{pmatrix}2\\2\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-1\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}7\\3\end{pmatrix}$. Find the equation of the line passing through $\begin{pmatrix}1\\-1\end{pmatrix}$ and parallel to PS.

23. Find the incentre of the triangle with vertices $\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.
24. The number of integer values of m, for which the x coordinate of the point of intersection of the lines $\begin{pmatrix}3\\4\end{pmatrix}\mathbf{x} = 9$ and $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x} - 1 = 0$ is also an integer, is
    a) 2
    b) 0
    c) 4
    d) 1
25. Find the area of the parallelogram formed by the lines $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x}$=0, $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x}$+1=0 ,$\begin{pmatrix}-n\\1\end{pmatrix}\mathbf{x}$=0 and $\begin{pmatrix}-n\\1\end{pmatrix}\mathbf{x}$+ 1 =0.
26. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $\mathbf{P} = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}\cos(\alpha-\theta)\\\sin(\alpha-\theta)\end{pmatrix}$ then the $\mathbf{Q}$ is obtained from $\mathbf{P}$ by
    a) clockwise rotation around origin through an angleα
    b) anticlockwise rotation around origin through an angleα
    c) reflection in the line through origin with slope $\tan\alpha$
    d) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$
27. Let $\mathbf{P} = \begin{pmatrix}-1\\0\end{pmatrix} \mathbf{Q} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}3\\3\sqrt{3}\end{pmatrix}$ be three points. Then find the equation of the bisector of the angle PQR.
28. A straight line through the origin $\mathbf{O}$ meets the parallel lines
$$\begin{pmatrix}4 & 2\end{pmatrix}\mathbf{x} = 9 \tag{1.6.28.1}$$
and
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} + 6 = 0 \tag{1.6.28.2}$$
at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. Find the ratio in which $\mathbf{O}$ divides the segment PQ.
29. The number of integral points (integral points means both the coordinate should be integer) exactly in the interior of the triangle with the vertices $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\21\end{pmatrix}$ and $\begin{pmatrix}21\\0\end{pmatrix}$ is
    a) 133
    b) 190
    c) 233
    d) 105
30. Find the orthocentre of a triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}4\\0\end{pmatrix}$
31. Find the area of the triangle formed by the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \tag{1.6.31.1}$$
and angle bisectors of the pair of straight lines
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} = 1. \tag{1.6.31.2}$$
32. Let $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{P} = \begin{pmatrix}3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\0\end{pmatrix}$ be the vertices of the triangle OPQ. The point $\mathbf{R}$ inside the triangle OPQ is such that the triangles OPR,PQR, OQR are of equal area. Find the coordinates of $\mathbf{R}$.
33. A straight line L through the point $\begin{pmatrix}3\\-2\end{pmatrix}$ is inclined at an angle of 60° to the line
$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1. \tag{1.6.33.1}$$
If L also intersects the x-axis, then find the equation of L.
34. Three lines
$$\begin{pmatrix}p & q\end{pmatrix}\mathbf{x} + r = 0, \tag{1.6.34.1}$$
$$\begin{pmatrix}q & r\end{pmatrix}\mathbf{x} + p = 0 \tag{1.6.34.2}$$
and
$$\begin{pmatrix}r & p\end{pmatrix}\mathbf{x} + q = 0 \tag{1.6.34.3}$$
are concurrent if
    a) $p + q + r = 0$
    b) $p^2 + q^2 + r^2 = qr + rp + pq$
    c) $p^3 + q^3 + r^3 = 3pqr$
    d) none of these
35. The points $\begin{pmatrix}0\\\frac{8}{3}\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}82\\30\end{pmatrix}$ are vertices of
    a) an obtuse angled triangle
    b) an acute angled triangle

c) a right angled triangle
d) none of these

36. All points lying inside the triangle formed by the points $\begin{pmatrix}1\\3\end{pmatrix}$, $\begin{pmatrix}5\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$ satisfy
a) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} \geq 0$
b) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 13 \geq 0$
c) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} - 12 \leq 0$
d) $\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} \geq 0$
e) none of these

37. A vector $\mathbf{a} = \begin{pmatrix}2p\\1\end{pmatrix}$ with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counter clockwise sense. If, with respect to the new system, $\mathbf{a} = \begin{pmatrix}p+1\\1\end{pmatrix}$, then
a) p=0
b) p=1 or p=$-\frac{1}{3}$
c) p=-1 or p=$\frac{1}{3}$
d) p=1 or p=-1
e) none of these

38. If $\mathbf{P} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}4\\6\end{pmatrix}, \mathbf{R} = \begin{pmatrix}5\\7\end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix}a\\b\end{pmatrix}$ are the vertices of a parallelogram PQRS, find a and b.

39. The diagonals of a parallelogram PQRS are along the lines
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 4 \tag{1.6.39.1}$$
and
$$\begin{pmatrix}6 & -2\end{pmatrix}\mathbf{x} = 7 \tag{1.6.39.2}$$
. Then PQRS must be a
a) rectangle
b) square
c) cyclic quadrilateral
d) rhombus

40. If the vertices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ of a triangle PQR are rational points, which of the following points of the triangle PQR is (are) always rational point(s)?
a) centroid
b) incentre
c) circumcentre
d) orthocentre

41. Let L_1 be a straight line passing through the origin and L_2 be the straight line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1. \tag{1.6.41.1}$$
If the intercepts made by the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-1 & 3\end{pmatrix}\mathbf{x} = 0 \tag{1.6.41.2}$$
on L_1 and L_2 are equal, then which of the following equations can represent L_1?
a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 0$
d) $\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = 0$

42. For $a > b > c > 0$, the distance between $\begin{pmatrix}1\\1\end{pmatrix}$ and the point of intersection of the lines $\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} + c = 0$ and $\begin{pmatrix}b & a\end{pmatrix}\mathbf{x} + c = 0$ is less than $2\sqrt{2}$. Then
a) a+b-c>0
b) a-b+c<0
c) a-b+c>0
d) a+b-c<0

43. A straight line segment of length l, moves with its ends on two mutually perpendicular lines. Find the locus of the points which divides the line segment in the ratio 1:2.

44. The area of triangle is 5. Two of its vertices are $\mathbf{A} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-2\end{pmatrix}$. The third vertex $\mathbf{C}$ lies on $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 3$. Find $\mathbf{C}$.

45. One side of a rectangle lies along the line
$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} + 5 = 0. \tag{1.6.45.1}$$
Two of its vertices are $\begin{pmatrix}-3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$. Find the equations of the other three sides.

46. Two vertices of a triangle are $\begin{pmatrix}5\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$. If the orthocentre of the triangle is the origin, find the coordinates of the third point.

47. Find the equation of the line which bisects the obtuse angle between the lines
$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.6.47.1}$$
and
$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} - 2 = 0. \tag{1.6.47.2}$$

48. A straight line L is perpendicular to the line
$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = 1. \tag{1.6.48.1}$$
The area of the triangle formed by the line L and the coordinate axes is 5. Find the equation of the line L.
49. The end $\mathbf{A}, \mathbf{B}$ of a straight line segment of constant length c slide upon the fixed rectangular axis OX,OY respectively. If the rectangle OAPB be completed, then show that the locus of the foot of the perpendicular drawn from $\mathbf{P}$ to AB is $x^{2/3} + y^{2/3} = c^{2/3}$.
50. The vertices of a triangle are $\begin{pmatrix}at_1t_2\\a(t_1+t_2)\end{pmatrix}, \begin{pmatrix}at_2t_3\\a(t_2+t_3)\end{pmatrix}, \begin{pmatrix}at_3t_1\\a(t_3+t_1)\end{pmatrix}$. Find the orthocentre of the triangle.
51. The coordinates of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $\begin{pmatrix}6\\3\end{pmatrix}, \begin{pmatrix}-3\\5\end{pmatrix}, \begin{pmatrix}4\\-2\end{pmatrix}$ respectively, and $\mathbf{P}$ is any point $\mathbf{x}$. Show that the ratio of the area of the triangles $\triangle PBC$ and $\triangle ABC$ is $\frac{\left|\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}-2\right|}{7}$
52. Two equal sides of an isosceles triangles are given by the equations
$$\begin{pmatrix}7 & -1\end{pmatrix}\mathbf{x} + 3 = 0 \tag{1.6.52.1}$$
and
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} - 3 = 0 \tag{1.6.52.2}$$
and its third side passes through the point $\begin{pmatrix}1\\10\end{pmatrix}$. Determine the equation of third side.
53. One of the diameters of the circle circumscribing the rectangle ABCD is
$$\begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} = 7. \tag{1.6.53.1}$$
If A and B are the points $\begin{pmatrix}-3\\4\end{pmatrix}$ and $\begin{pmatrix}5\\4\end{pmatrix}$ respectively, then find the area of the rectangle.
54. Two sides of a rhombus ABCD are parallel to the lines
$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.6.54.1}$$
and
$$\begin{pmatrix}-7 & 1\end{pmatrix}\mathbf{x} = 3 \tag{1.6.54.2}$$
. If the diagonals of the rhombus intersect at the point $\begin{pmatrix}1\\2\end{pmatrix}$ and vertex $\mathbf{A}$ is on the y axis. Find possible coordinates of $\mathbf{A}$.
55. Lines
$$L_1 \equiv \begin{pmatrix}a & b\end{pmatrix}\mathbf{x} + c = 0 \tag{1.6.55.1}$$
$$L_2 \equiv \begin{pmatrix}l & m\end{pmatrix}\mathbf{x} + n = 0 \tag{1.6.55.2}$$
intersect at the point $\mathbf{P}$ and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through $\mathbf{P}$ and makes the same angle θ with L_1.
56. Let ABC be a traingle with AB=AC. If $\mathbf{D}$ is the mid point of BC, $\mathbf{E}$ is the foot of the perpendicular drawn from $\mathbf{D}$ to AC and $\mathbf{F}$ the mid-point of DE, Prove that AF perpendicular to BE.
57. Straight lines
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{1.6.57.1}$$
and
$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 15 \tag{1.6.57.2}$$
intersect at the point $\mathbf{A}$. Points $\mathbf{B}$ and $\mathbf{C}$ are chosen on these two lines such that AB=AC. Determine the possible equations of the lines BC passing through the point$\begin{pmatrix}1\\2\end{pmatrix}$
58. A line cuts the x-axis at $\mathbf{A} = \begin{pmatrix}7\\0\end{pmatrix}$ and the y-axis at $\mathbf{B} = \begin{pmatrix}0\\-5\end{pmatrix}$. A variable line PQ is draw perpendicular to AB cutting the x-axis in $\mathbf{P}$ and the y-axis in $\mathbf{Q}$. If AQ and BP intersect at $\mathbf{R}$, find the locus of $\mathbf{R}$.
59. Find the equation of the line passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ and making an intercept of length 2 units between the lines $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$=3 and $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$=5.
60. Show that all chords of the curve
$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-2 & 4\end{pmatrix}\mathbf{x} = 0 \tag{1.6.60.1}$$
which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point.
61. Determine all values of α for which the point

$\begin{pmatrix}\alpha \\ \alpha^2\end{pmatrix}$ lies inside the triangle formed by the lines

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 1 = 0 \tag{1.6.61.1}$$
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} - 3 = 0 \tag{1.6.61.2}$$
$$\begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} - 1 = 0 \tag{1.6.61.3}$$

62. The tangent at a point $\mathbf{P}_1$ (other than the origin) on the curve

$$y = x^3 \tag{1.6.62.1}$$

meets the curve again at $\mathbf{P}_2$. The tangent at $\mathbf{P}_2$ meets the curve at $\mathbf{P}_3$ and so on. Show that the abscissae of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \ldots \mathbf{P}_n$, form a G.P. Also find the ratio $\frac{\text{area}(\triangle P_1,P_2,P_3)}{\text{area}(\triangle P_2,P_3,P_4)}$.

63. A line through $\mathbf{A} = \begin{pmatrix}-5 \\ -4\end{pmatrix}$ meets the line $\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} + 2 = 0, \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$ +4=0 and $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x}$ -5=0 at the points $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2, \tag{1.6.63.1}$$

find the equation of the line.

64. A rectangle PQRS has its side PQ parallel to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.64.1}$$

and vertices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{S}$ on the lines

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = a \tag{1.6.64.2}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = b \tag{1.6.64.3}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -b \tag{1.6.64.4}$$

respectively. Find the locus of vertex $\mathbf{R}$.

65. Using coordinate geometry, prove that the three altitudes of any triangle are concurrent.

66. For points $\mathbf{P} = \begin{pmatrix}x_1 \\ y_1\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}x_2 \\ y_2\end{pmatrix}$ of the coordinate plane, a new distance $d(\mathbf{P}, \mathbf{Q}) = |x_1 - x_2| + |y_1 - y_2|$. Let $\mathbf{O} = \begin{pmatrix}0 \\ 0\end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix}3 \\ 2\end{pmatrix}$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from $\mathbf{O}$ and $\mathbf{A}$ consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.

67. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to the sides QR,RP,PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ to BC,CA,AB respectively are also concurrent.

68. Let a,b,c be real numbers with $a^2 + b^2 + c^2 = 1$. show that the equation

$$\begin{vmatrix}\begin{pmatrix}a & -b\end{pmatrix}\mathbf{x} - c & \begin{pmatrix}b & a\end{pmatrix}\mathbf{x} & \begin{pmatrix}c & 0\end{pmatrix}\mathbf{x} + a \\ \begin{pmatrix}b & a\end{pmatrix}\mathbf{x} & \begin{pmatrix}-a & b\end{pmatrix}\mathbf{x} - c & \begin{pmatrix}0 & c\end{pmatrix}\mathbf{x} + b \\ \begin{pmatrix}c & 0\end{pmatrix}\mathbf{x} + a & \begin{pmatrix}0 & c\end{pmatrix}\mathbf{x} + b & \begin{pmatrix}-a & -b\end{pmatrix}\mathbf{x} + c\end{vmatrix} = 0 \tag{1.6.68.1}$$

represents a straight line.

69. A straight line L through the origin meets the line and

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \tag{1.6.69.1}$$

at $\mathbf{P}$ and $\mathbf{Q}$ respectively. Through $\mathbf{P}$, straight lines L_1 and L_2 are drawn parallel to

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 5 \tag{1.6.69.2}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 5 \tag{1.6.69.3}$$

respectively. Lines L_1 and L_2 intersect at that the locus of $\mathbf{R}$, as L varies, is a straight line.

70. A straight line L with negative slope passes through point $\begin{pmatrix}8 \\ 2\end{pmatrix}$ and cuts the positive coordinates are $\mathbf{P}$ and $\mathbf{Q}$. Find the absolute minimum value of OP varies, where $\mathbf{O}$ is origin.

71. The area of the triangle formed by the intersection of a line parallel to the x-axis and passing through $\mathbf{P} = \begin{pmatrix}h \\ k\end{pmatrix}$ with the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.71.1}$$
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.6.71.2}$$

is $4h^2$. Find the locus of the point.

72. Lines

$$L_1 : \begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.72.1}$$
$$L_2 : \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.72.2}$$

intersect the line

$$L_3 : \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + 2 = 0 \tag{1.6.72.3}$$

at $\mathbf{P}$ and $\mathbf{Q}$ respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at $\mathbf{R}$.

Statement-1. The ratio PR:RQ equals $2\sqrt{2} : \sqrt{5}$

Statement-2. In any triangle, the bisector of an angle divides the triangle into two similar triangles.

a) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement-1
b) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement-1
c) Statement-1 is True,Statement False
d) Statement-1 is False,Statement True

73. For a point $\mathbf{P}$ in the plane, let $d_1(\mathbf{P})$ and $d_2(\mathbf{P})$ be the distance of the point $\mathbf{P}$ from the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.73.1}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.73.2}$$

respectively. Find the area of the region R consisting of all points $\mathbf{P}$ lying in the first quadrant of the plane and satisfying

$$2 \le d_1(\mathbf{P}) + d_2(\mathbf{P}) \le 4. \tag{1.6.73.3}$$

74. A triangle with vertices $\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is

a) isosceles and right angled
b) isosceles but not right angled
c) right angled but not isosceles
d) neither right angled nor isosceles

75. Find the locus of mid points of the portion between the axis of

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \mathbf{x} = 0 \tag{1.6.75.1}$$

where p is constant

76. If the pair of the lines

$$\mathbf{x}^T \begin{pmatrix} a & 2h \\ 0 & b \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} g & f \end{pmatrix} \mathbf{x} + c = 0 \tag{1.6.76.1}$$

intersects the y-axis then

a) $2fgh = bg^2$+ch^2
b) $bg^2 \neq ch^2$
c) $abc = 2fgh$
d) none of these

77. A pair of lines represented by

$$\mathbf{x}^T \begin{pmatrix} 3a & 5 \\ 0 & (a^2 - 2) \end{pmatrix} \mathbf{x} = 0 \tag{1.6.77.1}$$

are perpendicular to each other for

a) two values of a
b) $\forall$ a
c) for one value of a
d) for no values of a

78. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle $\alpha(0<\alpha<\pi/4)$ with the positive direction of the x-axis. Find the equation of its diagonal not passing through the origin.

79. If the pair of straight lines

$$\mathbf{x}^T \begin{pmatrix} 1 & -2p \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.79.1}$$

and

$$\mathbf{x}^T \begin{pmatrix} 1 & -2q \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.79.2}$$

be such that each pair bisects the angle between the other pair, then find the relation between p and q.

80. Find the locus of the centroid of the triangle whose vertices are $\begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}, \begin{pmatrix} b\sin t \\ -b\cos t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

81. If x_1,x_2,x_3 and y_1,y_2,y_3 are both in G.P. with the same common ratio, then the common points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

a) are verices of a triangle
b) lie on a straight line
c) lie on a ellipse
d) lie on a circle

82. If the equation of the locus of a point equidistant from the points $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is

$$\begin{pmatrix} a_1 - b_2 & a_1 - b_2 \end{pmatrix} \mathbf{x} + c = 0 \tag{1.6.82.1}$$

then find the value of c.

83. Let $\mathbf{A} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ be the vertices of a triangle ABC. If the centroid of the triangle moves on the line

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{1.6.83.1}$$

then find the locus of the vertex $\mathbf{C}$.

84. Find the equation of the straight line passing through the point $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and making intercepts on the coordinate axis whose sum is -1.

85. If the sum of the slopes of the lines given by $\mathbf{x}^T\begin{pmatrix}1 & -2c\\0 & -1\end{pmatrix}\mathbf{x} = 0$ is 4 times their product, then find c.
86. If one of the lines given by

$$\mathbf{x}^T\begin{pmatrix}6 & -1\\0 & 4c\end{pmatrix}\mathbf{x} = 0 \tag{1.6.86.1}$$

is

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 0 \tag{1.6.86.2}$$

then find c.
87. The line parallel to the x-axis and passing through the intersection of the lines

$$\begin{pmatrix}a & 2b\end{pmatrix}\mathbf{x} + 3b = 0 \tag{1.6.87.1}$$
$$\begin{pmatrix}b & -2a\end{pmatrix}\mathbf{x} - 3a = 0, \tag{1.6.87.2}$$

where $\begin{pmatrix}a\\b\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$ is
a) below the x-axis at a distance of 3/2 from it
b) below the x-axis at a distance of 2/3 from it
c) above the x-axis at a distance of 3/2 from it
d) above the x-axis at a distance of 2/3 from it
88. If a vertex of a triangle is $\begin{pmatrix}1\\1\end{pmatrix}$ and the mid points of two sides through this vertex are $\begin{pmatrix}-1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$, then find the centroid of the triangle.
89. A straight line through the point $\mathbf{A} = \begin{pmatrix}3\\4\end{pmatrix}$ is such that its intercepts between the axes is bisected at $\mathbf{A}$. Find its equation.
90. If $\begin{pmatrix}a\\a^2\end{pmatrix}$ falls inside the angle made by the lines

$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 0, \tag{1.6.90.1}$$
$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.90.2}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} > 0, \tag{1.6.90.3}$$

then find the range of a.
91. Let $\mathbf{A} = \begin{pmatrix}h\\k\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}2\\1\end{pmatrix}$ be the vertices of a right angled triangle with AC has its hypotenuse. If the area of the triangle is 1 sq.unit, then find the range of k.
92. Let $\mathbf{P} = \begin{pmatrix}-1\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}3\\\sqrt{3}\end{pmatrix}$ be three points. Find the equation of the bisector of the angle PQR.
93. If one of the lines of

$$\mathbf{x}^T\begin{pmatrix}-m & (1-m^2)\\0 & m\end{pmatrix}\mathbf{x} = 0 \tag{1.6.93.1}$$

is a bisector of the angle between the lines

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\mathbf{x} = 0, \tag{1.6.93.2}$$

then find m.
94. The perpendicular bisector of the line segment joining $\mathbf{P} = \begin{pmatrix}1\\4\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}k\\3\end{pmatrix}$ has y-intercept -4. Then a possible value of k is
a) 1
b) 2
c) -2
d) -4
95. Find the shortest distance between the line

$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{1.6.95.1}$$

and the curve

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1 & 0\end{pmatrix} = 0 \tag{1.6.95.2}$$

.
96. The lines

$$\begin{pmatrix}(p(p^2+1) & -1\end{pmatrix}\mathbf{x} + q = 0 \tag{1.6.96.1}$$
$$\begin{pmatrix}(p^2+1)^2 & (p^2+1)\end{pmatrix}\mathbf{x} + 2q = 0 \tag{1.6.96.2}$$

are perpendicular to a common line for:
a) exactly one values of p
b) exactly two values of p
c) more than two values of p
d) no value of p
97. Three distinct points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are given in the two dimensional coordinates plane such that the ratio of the distance of any one of them from the point $\begin{pmatrix}1\\0\end{pmatrix}$ to the distance from the point $\begin{pmatrix}-1\\0\end{pmatrix}$ is equal to $\frac{1}{3}$. Find the circumcentre of the triangle ABC.
98. The line L given by

$$\begin{pmatrix}1/5 & 1/b\end{pmatrix}\mathbf{x} = 1 \tag{1.6.98.1}$$

passes through the point $\begin{pmatrix}13\\32\end{pmatrix}$. The line K is parallel to L and has the equation

$$\begin{pmatrix}\frac{1}{c} & \frac{1}{3}\end{pmatrix}\mathbf{x} = 1 \tag{1.6.98.2}$$

. Find the distance between L and K.

99. If the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = k \tag{1.6.99.1}$$

passes through the point which divides the line segment joining the points $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix}$ in the ratio 3:2, then find k.

100. A ray of light along

$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = \sqrt{3} \tag{1.6.100.1}$$

gets reflected upon reaching the x-axis, then find the equation of the reflected ray.

101. Find the x coordinate of the incentre of the triangle that has the coordinates of the mid points of its sides as $\begin{pmatrix}0\\1\end{pmatrix}$,$\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$

102. Let PS be the median of the triangle with vertices $\mathbf{P} = \begin{pmatrix}2\\2\end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix}6\\-1\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}7\\3\end{pmatrix}$. Find the equation of the line passing through $\begin{pmatrix}1\\-1\end{pmatrix}$ and parallel to PS.

103. Let a,b,c and d be non zero numbers. If the point of intersection of the lines $\begin{pmatrix}4a & 2a\end{pmatrix}\mathbf{x} + c = 0$ and $\begin{pmatrix}5b & 2b\end{pmatrix}\mathbf{x} + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then
   a) 3bc-2ad=0
   b) 3bc+2ad=0
   c) 2bc-3ad=0
   d) 2bc+3ad=0

104. The number of points, having both coordinates as integers, that lie in the interior of the triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix}$,$\begin{pmatrix}0\\41\end{pmatrix}$ and$\begin{pmatrix}41\\0\end{pmatrix}$. is:
   a) 820
   b) 780
   c) 901
   d) 861

105. Two sides of a rhombus are along the lines, $\begin{pmatrix}1 & -1\end{pmatrix}$ $\mathbf{x}$+1=0 and $\begin{pmatrix}7 & -1\end{pmatrix}\mathbf{x}$-5=0. If its diagonals intersect at $\begin{pmatrix}-1\\-2\end{pmatrix}$ then which one of the following is a vertex of the rhombus?
   a) $\begin{pmatrix}\frac{1}{3}\\\frac{8}{3}\end{pmatrix}$
   b) $\begin{pmatrix}\frac{10}{3}\\\frac{7}{3}\end{pmatrix}$
   c) $\begin{pmatrix}-3\\-9\end{pmatrix}$
   d) $\begin{pmatrix}-3\\-8\end{pmatrix}$

106. A straight the through a fixed point $\begin{pmatrix}2\\3\end{pmatrix}$ intersects the coordinate axes at distinct points $\mathbf{P}$ and $\mathbf{Q}$. If $\mathbf{O}$ is the origin and the rectangle OPRQ is completed, then find the locus of $\mathbf{R}$.

107. Consider the set of all lines $\begin{pmatrix}p & q\end{pmatrix}$ $\mathbf{x}$+r=0 such that $3p + 2q + 4r = 0$. Which one of the following statements is true?
   a) The lines are concurrent at the point $\begin{pmatrix}\frac{3}{4}\\\frac{1}{2}\end{pmatrix}$
   b) Each line passes through the origin
   c) The lines are all parallel
   d) The lines are not concurrent

108. Find the slope of a line passing through $\mathbf{P} = \begin{pmatrix}2\\3\end{pmatrix}$ and intersecting the line $\begin{pmatrix}1 & 1\end{pmatrix}$ $\mathbf{x}$=7 at a distance of 4 units from $\mathbf{P}$.

2 The Circle

2.1 *Definitions*

1. The equation of a circle is

$$\|\mathbf{x} - \mathbf{c}\| = r \tag{2.1.1.1}$$

where $\mathbf{c}$ is the centre and r is the radius.

2. By expanding (2.1.1.1), the equation of a circle can also be expressed as

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.1.2.1}$$
$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} - r^2 = 0 \tag{2.1.2.2}$$

3. Find the equation of the *circumcircle* of $\triangle ABC$ in Fig. 2.1.4.
Solution: Let $\mathbf{O}$ be the centre and R the radius. From (2.1.2.2),

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = R^2 \tag{2.1.3.1}$$
$$\implies \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \tag{2.1.3.2}$$

which can be simplified to obtain

$$(\mathbf{A}-\mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad \text{and} \tag{2.1.3.3}$$

$$(\mathbf{A}-\mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{C}\|^2}{2} \tag{2.1.3.4}$$

Solving the two yields $\mathbf{O}$, which can then be used to obtain R.

4. Given $OD \perp BC$ as in Fig. 2.1.4. Show that

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{2.1.4.1}$$

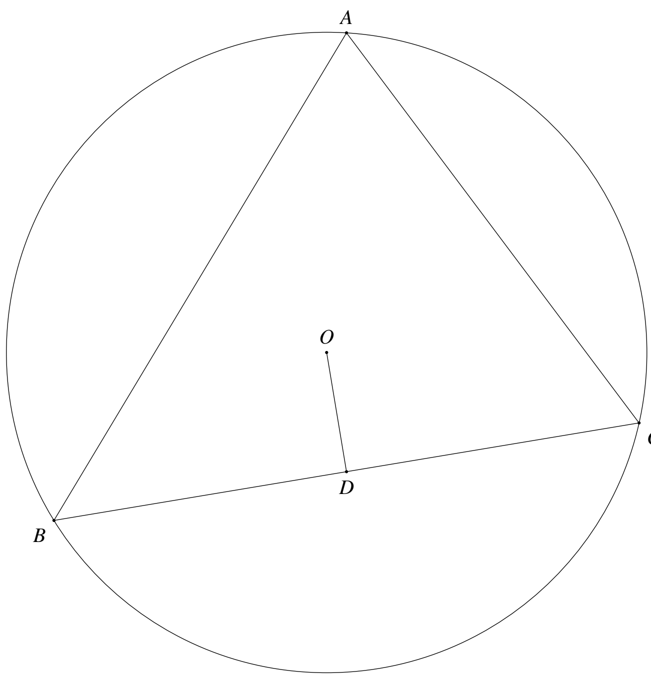

Fig. 2.1.4: Circumcircle.

Solution: From (2.1.1.1)

$$\|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = R^2 \tag{2.1.4.2}$$

$$\implies (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = (\mathbf{C}-\mathbf{O})^T(\mathbf{C}-\mathbf{O}) \tag{2.1.4.3}$$

$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{B}+\mathbf{C}}{2} - \mathbf{O}\right) = 0 \tag{2.1.4.4}$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{O}) = 0 \tag{2.1.4.5}$$

Since D and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on BC, using (1.2.7.1),

$$\frac{\mathbf{B}+\mathbf{C}}{2} = \mathbf{B} + \lambda_1(\mathbf{B}-\mathbf{C}) \tag{2.1.4.6}$$

$$\mathbf{D} = \mathbf{B} + \lambda_2(\mathbf{B}-\mathbf{C}) \tag{2.1.4.7}$$

Multiplying (2.1.4.6) and (2.1.4.7) with $(\mathbf{B}-\mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{2.1.4.8}$$

5. Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.
6. The *incircle* with centre $\mathbf{I}$ and radius r in Fig.2.1.6 is inside $\triangle ABC$ and touches AB, BC and CA at $\mathbf{W}, \mathbf{U}$ and $\mathbf{V}$ respectively. AB, BC and CA are known as *tangents* to the circle.

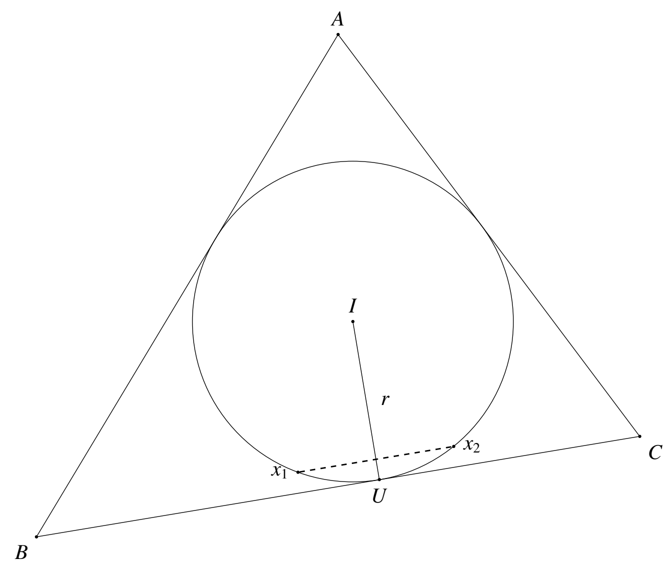

Fig. 2.1.6: Tangent and incircle.

7. Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \tag{2.1.7.1}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{2.1.7.2}$$

$$\implies (\mathbf{B} - \mathbf{C})^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{2.1.7.3}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the above equation becomes

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{U}-\mathbf{I}) = 0 \implies OD \perp BC \tag{2.1.7.4}$$

8. Give an alternative proof for the above.

Solution: Let

$$\mathbf{B} = \mathbf{0} \tag{2.1.8.1}$$
$$\mathbf{U} = \lambda\mathbf{m} \tag{2.1.8.2}$$

Then

$$\|\mathbf{U} - \mathbf{I}\|^2 = r^2 \tag{2.1.8.3}$$
$$\implies \lambda^2 \|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{I} + \|\mathbf{I}\|^2 = r^2 \tag{2.1.8.4}$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T\mathbf{I}}{\|\mathbf{m}\|^2} \tag{2.1.8.5}$$

Thus,

$$(\mathbf{U} - \mathbf{B})^T (\mathbf{U} - \mathbf{I}) = (\lambda\mathbf{m})^T (\lambda\mathbf{m} - \mathbf{I}) \tag{2.1.8.6}$$
$$= \lambda^2 \|\mathbf{m}\|^2 - \lambda\mathbf{m}^T\mathbf{I} \tag{2.1.8.7}$$
$$= \mathbf{O} \text{ (from 2.1.8.5).} \tag{2.1.8.8}$$
$$\implies OD \perp BC \tag{2.1.8.9}$$

9. Find the equation of the tangent at $\mathbf{U}$.
Solution: The equation of the tangent is given by

$$(\mathbf{I} - \mathbf{U})^T (\mathbf{x} - \mathbf{U}) = 0 \tag{2.1.9.1}$$

10. The direction vector of *normal to the circle* in (2.1.2.2) at point $\mathbf{U}$ is

$$\mathbf{n} = \mathbf{U} - \mathbf{I} \tag{2.1.10.1}$$

11. Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \tag{2.1.11.1}$$
$$\implies \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \tag{2.1.11.2}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{2.1.11.3}$$
$$\implies \mathbf{I} - \mathbf{U} = \lambda\mathbf{n} \tag{2.1.11.4}$$
$$\text{or} \quad r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \, \|\mathbf{n}\| \tag{2.1.11.5}$$

From (2.1.11.2) and (2.1.11.3)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{2.1.11.6}$$
$$\implies \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda \|\mathbf{n}\|^2 \tag{2.1.11.7}$$
$$\implies r = |\lambda| \, \|\mathbf{n}\| = \frac{\left|\mathbf{n}^T (\mathbf{I} - \mathbf{B})\right|}{\|\mathbf{n}\|} \tag{2.1.11.8}$$

from (2.1.11.5). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{2.1.11.9}$$
$$r = \left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| \tag{2.1.11.10}$$

12. Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (2.1.11.10),

$$\left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| = \left|\mathbf{n}_2^T (\mathbf{I} - \mathbf{C})\right| = \left|\mathbf{n}_3^T (\mathbf{I} - \mathbf{A})\right| \tag{2.1.12.1}$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (2.1.12.1) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \tag{2.1.12.2}$$
$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2\mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \tag{2.1.12.3}$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3 \end{pmatrix}^T \mathbf{I} = \begin{pmatrix} \mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A} \end{pmatrix} \tag{2.1.12.4}$$

13. Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$
14. Let $a = BC, b = CA, c = AB, x = BU = BW, y = CU = CV, z = AV = AW$. Find $\mathbf{U}$.
Solution: It is easy to verify that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.1.14.1}$$

which can be used to obtain x and y. Using the section formula,

$$\mathbf{U} = \frac{x\mathbf{B} + \mathbf{A}}{x + y} \tag{2.1.14.2}$$

15. Show that the angle in a semi-circle is a right angle.

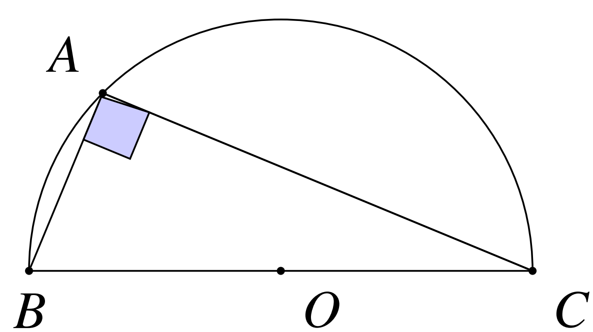

Fig. 2.1.15: Angle in a semi-circle.

Solution: Let

$$\mathbf{O} = 0 \quad (2.1.15.1)$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \quad (2.1.15.2)$$
$$\|\mathbf{B} - \mathbf{C}\|^2 = (2r)^2 \quad (2.1.15.3)$$
$$\mathbf{B} + \mathbf{C} = 0 \quad (2.1.15.4)$$

where r is the radius of the circle. Thus,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T(\mathbf{B} + \mathbf{C}) \quad (2.1.15.5)$$

From (2.1.15.4) and (2.1.15.2),

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 4r^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (2.1.15.6)$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

16. Show that $PA.PB = PC^2$, where PC is the tangent to the circle in Fig. 2.1.16.

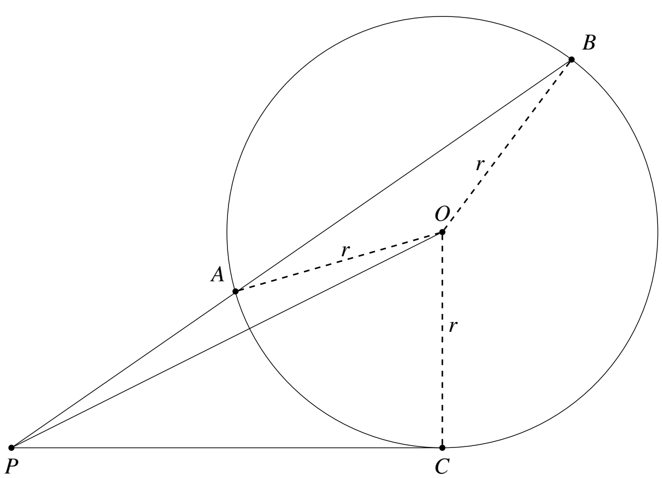

Fig. 2.1.16: $PA.PB = PC^2$.

Solution: Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda\|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda\mathbf{A}) \quad (2.1.16.1)$$
$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \quad (2.1.16.2)$$
$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \quad (2.1.16.3)$$

$\because$

$$\|\mathbf{B} - \mathbf{O}\|^2 - \|\mathbf{A} - \mathbf{O}\|^2 = 0, \quad (2.1.16.4)$$
$$\left(\lambda^2 - 1\right)\|\mathbf{A}\|^2 - 2(\lambda - 1)\mathbf{A}^T\mathbf{O} = 0 \quad (2.1.16.5)$$
$$\implies PA.PB = \lambda\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \quad (2.1.16.6)$$

after substituting from (2.1.16.1) and simplifying. From (2.1.16.3),

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad (2.1.16.7)$$
$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 = PC^2 \quad (2.1.16.8)$$

From (2.1.16.6) and (2.1.16.8),

$$PA.PB = PC^2 \quad (2.1.16.9)$$

17. In Fig. 2.1.17 show that $PA.PB = PC.PD$.

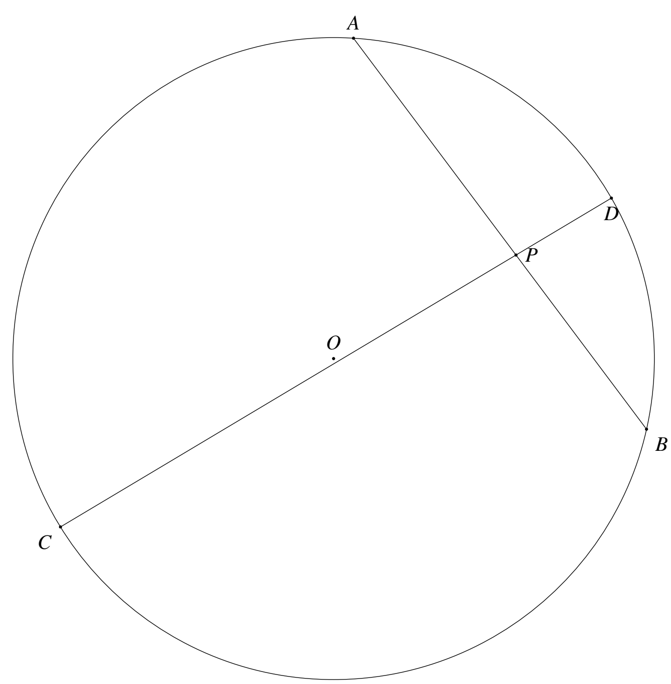

Fig. 2.1.17: Chords of a circle

Solution: Let $\mathbf{P} = \mathbf{0}$. We then have the following equations

$$\mathbf{B} = k_1\mathbf{A}, k_1 = \frac{PB}{PA} \quad \mathbf{D} = k_2\mathbf{C}, k_2 = \frac{PD}{PC} \tag{2.1.17.1}$$

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = \|\mathbf{D}-\mathbf{O}\|^2 = r^2 \tag{2.1.17.2}$$

where r is the radius of the circle and $\mathbf{O}$ is the centre. From (2.1.17.2),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 \tag{2.1.17.3}$$

$$\implies \|\mathbf{A}-\mathbf{O}\|^2 = \|k\mathbf{A}-\mathbf{O}\|^2 \quad \text{(from (2.1.17.1))} \tag{2.1.17.4}$$

which can be simplified to obtain

$$k_1\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{2.1.17.5}$$

Similarly,

$$k_2\|\mathbf{C}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.6}$$

From (2.1.17.2), we also obtain

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 \tag{2.1.17.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.8}$$

after simplification. Using this result in (2.1.17.5) and (2.1.17.6),

$$k_1\|\mathbf{A}\|^2 = k_2\|\mathbf{C}\|^2 \tag{2.1.17.9}$$

$$\implies \|\mathbf{A}\|\,\|\mathbf{B}\| = \|\mathbf{C}\|\,\|\mathbf{D}\| \tag{2.1.17.10}$$

which completes the proof.

18. (Pole and Polar:) The polar of a point $\mathbf{x}$ with respect to the curve

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.18.1}$$

is the line

$$\mathbf{n}^T\mathbf{x} = c \tag{2.1.18.2}$$

where

$$\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ 1\end{pmatrix} \tag{2.1.18.3}$$

The pole of the line in (2.1.18.2) is obtained as $\frac{1}{x_3}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$, where

$$\begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} \tag{2.1.18.4}$$

19. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be conjugate points for (2.1.18.1) if $\mathbf{x}_2$ lies on the polar of $\mathbf{x}_1$ and vice-versa. A similar definition holds for conjugate lines as well.
20. Let $\mathbf{p}$ be a point of intersection of two circles with centres $\mathbf{c}_1$ and $\mathbf{c}_2$. The circles are said to be orthogonal if their tangents at $\mathbf{p}$ are perpendicular to each other. Show that if r_1 and r_2 are their respective radii,

$$\|\mathbf{c_1} - \mathbf{c}_2\|^2 = r_1^2 + r_2^2 \tag{2.1.20.1}$$

21. Show that the length of the tangent from a point $\mathbf{p}$ to the circle

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{2.1.21.1}$$

is

$$\mathbf{p}^T\mathbf{p} - 2\mathbf{c}^T\mathbf{p} + f \tag{2.1.21.2}$$

This length is also known as the *power* of the point $\mathbf{p}$ with respect to the circle.

22. The *radical axis* of the circles

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.22.1}$$

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.2}$$

is the locus of the points from which lengths of the tangents to the circles are equal. From (2.1.25), this locus is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.3}$$

$$\implies 2\left(\mathbf{c}_1 - \mathbf{c}_2\right)^T\mathbf{x} + f_2 - f_1 = 0 \tag{2.1.22.4}$$

23. Show that the radical axis of the circles is perpendicular to the line joining their centres.
24. *Coaxal circles* have the same radical axis.
25. Obtain a family of coaxal circles from and find their *limit points*.
Solution: The family of circles is obtained as

$$\mathbf{x}^T\mathbf{x} - 2\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\mathbf{x} + f_1 + \lambda f_2 = 0 \tag{2.1.25.1}$$

The limit points are the centres of those cir-

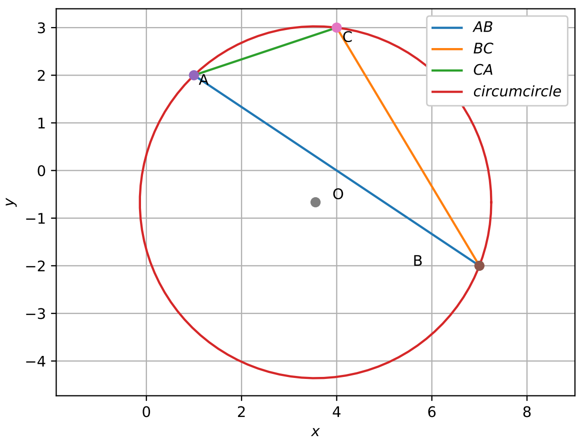

Fig. 2.2.2

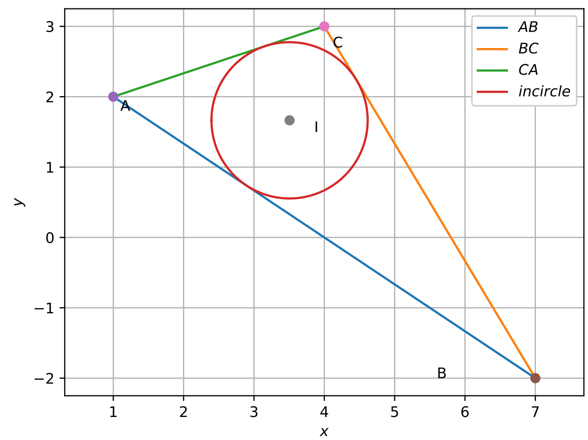

Fig. 2.2.5

cles whose radii are 0. From (2.1.25.1) and (2.1.2.2), this results in

$$f_1 + \lambda f_2 = (\mathbf{c}_1 + \lambda \mathbf{c}_2)^T (\mathbf{c}_1 + \lambda \mathbf{c}_2) \tag{2.1.25.2}$$

$$\implies \lambda^2 \|\mathbf{c}_2\|^2 + \lambda \left(2\mathbf{c}_1^T \mathbf{c}_2 - f_2\right) + \|\mathbf{c}_1\|^2 - f_1 = 0 \tag{2.1.25.3}$$

Solving for λ, the limit points are given by

$$\mathbf{c}_1 + \lambda \mathbf{c}_2 \tag{2.1.25.4}$$

2.2 Programming

1. Find the circumcentre $\mathbf{O}$ and radius R of $\triangle ABC$ in Fig. 1.3.2
Solution: $\mathbf{O}$ can be obtained from The following code computes $\mathbf{O}$ using (2.1.3.3) and (2.1.3.4)

```
codes/2d/circumcentre.py
```

2. Plot the circumcircle of $\triangle ABC$.
Solution: The following code plots Fig. 2.2.2

```
codes/2d/circumcircle.py
```

3. Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.
4. Compute $\mathbf{I}$ and r.
Solution: The following code uses (2.1.12.4) and (2.1.11.10) to compute $\mathbf{I}$ and r respectively.

```
codes/2d/incentre.py
```

5. Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 2.2.5

```
codes/2d/incircle.py
```

2.3 Example

1. Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} - 1 = 0 \tag{2.3.1.1}$$

Solution: let $\mathbf{c}$ be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.3.1.2}$$

$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \tag{2.3.1.3}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} = r^2 - \mathbf{c}^T\mathbf{c} \tag{2.3.1.4}$$

Comparing with (2.3.1.1),

$$\mathbf{c} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.3.1.5}$$

$$r^2 - \mathbf{c}^T\mathbf{c} = 1 \implies r = \sqrt{2} \tag{2.3.1.6}$$

2. Find the tangent to the circle C_1 at the point $\begin{pmatrix}2\\1\end{pmatrix}$.
Solution: From (3.1.6), the tangent T is given by

$$\left[\begin{pmatrix}2 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\end{pmatrix}\right]\mathbf{x} - \begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1 \tag{2.3.2.1}$$

$$\implies T : \mathbf{n}^T\mathbf{x} = 3 \tag{2.3.2.2}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3.2.3}$$

3. The tangent T in (2.3.2.2) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{2.3.3.1}$$

Find $\mathbf{A} + \mathbf{B}$.
Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \tag{2.3.3.2}$$
$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \tag{2.3.3.3}$$

Subtracting (2.3.3.3) from (2.3.3.2),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.4}$$
$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$
$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.5}$$

$\because \mathbf{A}, \mathbf{B}$ lie on T, from (2.3.2.2),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{2.3.3.6}$$
$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \tag{2.3.3.7}$$

From (2.3.3.5) and (2.3.3.7)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{2.3.3.8}$$
$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{2.3.3.9}$$
$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{2.3.3.10}$$
$$\implies k = 2 \tag{2.3.3.11}$$

using (2.3.3.6). Substituting in (2.3.3.8)

$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \tag{2.3.3.12}$$

4. If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.
Solution: From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.4.1}$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \tag{2.3.4.2}$$
$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \tag{2.3.4.3}$$

using (2.3.3.12) and simplifying.

5. Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \tag{2.3.5.1}$$

Solution:

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.5.2}$$
$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \tag{2.3.5.3}$$

From (2.3.5.3),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T \, [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \tag{2.3.5.4}$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \tag{2.3.5.5}$$

Upon substituting from (2.3.3.3) and (2.3.3.2) and simplifying, (2.3.5.1) is obtained.

6. Find r.
Solution: (2.3.5.1) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{2.3.6.1}$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \tag{2.3.6.2}$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \tag{2.3.6.3}$$
$$\implies r = \sqrt{6}. \tag{2.3.6.4}$$

7. Summarize all the above computations through a Python script and plot the tangent and circle.
Solution: The following code generates Fig. 2.3.7.

```
wget
codes/2d/circ.py
```

2.4 Lagrange Multipliers

1. Find

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 = r^2 \tag{2.4.1.1}$$
$$\text{s.t.} \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{2.4.1.2}$$

by plotting the circles $f(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.
Solution: The following code plots Fig. 2.4.1

```
codes/optimization/2.1.py
```

2. Show that

$$\min r = \frac{5}{\sqrt{2}} \tag{2.4.2.1}$$

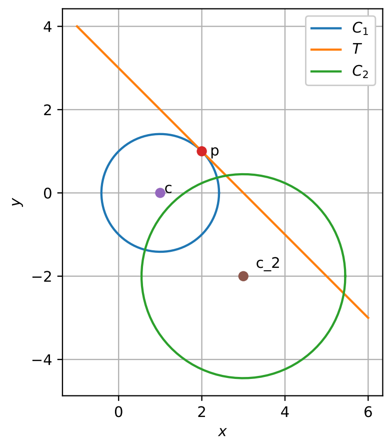

Fig. 2.3.7

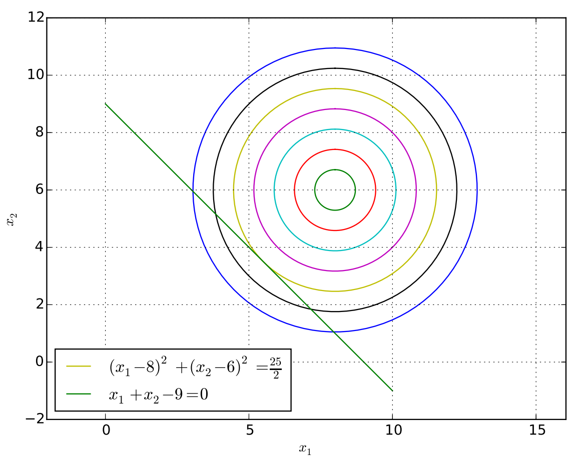

Fig. 2.4.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$

.

3. Show that

$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.4.3.1}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{2.4.3.2}$$

4. Show that

$$\nabla f(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\} \tag{2.4.4.1}$$

is the direction vector of the normal at $\mathbf{x}$.

5. From Fig. 2.4.1, show that

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{2.4.5.1}$$

where $\mathbf{p}$ is the point of contact.

6. Use (2.4.5.1) and $\mathbf{g}(\mathbf{p}) = 0$ from (2.4.1.2) to obtain $\mathbf{p}$.
7. Define

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{2.4.7.1}$$

and show that $\mathbf{p}$ can also be obtained by solving the equations

$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{2.4.7.2}$$

What is the sign of λ? L is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

Solution:

```
codes/optimization/2.3.py
```

2.5 Inequality Constraints

1. Modify the code in problem 2.4.1 to find a graphical solution for minimising

$$f(\mathbf{x}) \tag{2.5.1.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{2.5.1.2}$$

Solution: This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. 2.5.1 . It is clear that this radius is 0.

```
codes/optimization/2.4.py
```

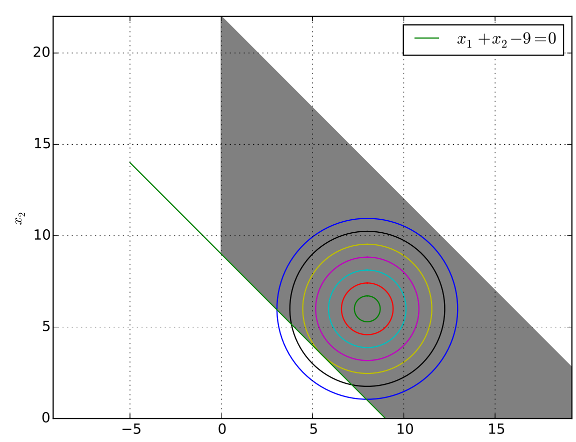

Fig. 2.5.1: Smallest circle in the shaded region is a point.

2. Now use the method of Lagrange multipliers to solve problem 2.5.1 and compare with the graphical solution. Comment.
Solution: Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem 2.5.1, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.
3. Repeat problem 2.5.2 by keeping $\lambda = 0$. Comment.
Solution: Keeping $\lambda = 0$ results in $\mathbf{x} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, which is the correct solution. The minimum value of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.
4. Find a graphical solution for minimising

$$f(\mathbf{x}) \tag{2.5.4.1}$$

with constraint

$$g(\mathbf{x}) \leq 0 \tag{2.5.4.2}$$

Summarize your observations.
Solution: In Fig. 2.5.4, the shaded region represents the constraint. Thus, the solution is the same as the one in problem 2.5.1. This implies that the method of Lagrange multipliers can be used to solve the optimization problem with this inequality constraint as well. Table 2.5.4 summarizes the conditions for this based on the observations so far.

```
codes/optimization/2.7.py
```

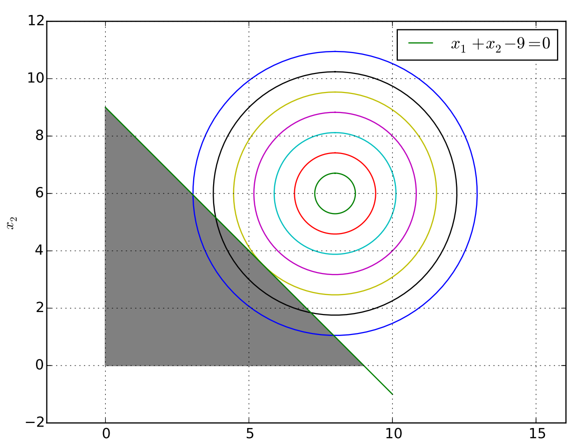

Fig. 2.5.4: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

TABLE 2.5.4: Summary of conditions.

Cost	Constraint	λ
$f(\mathbf{x})$	$g(\mathbf{x}) = 0$	< 0
	$g(\mathbf{x}) \geq 0$	0
	$g(\mathbf{x}) \leq 0$	< 0

5. Find a graphical solution for

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 \tag{2.5.5.1}$$

with constraint

$$g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 = 0 \tag{2.5.5.2}$$

Solution:

```
codes/optimization/2.8.py
```

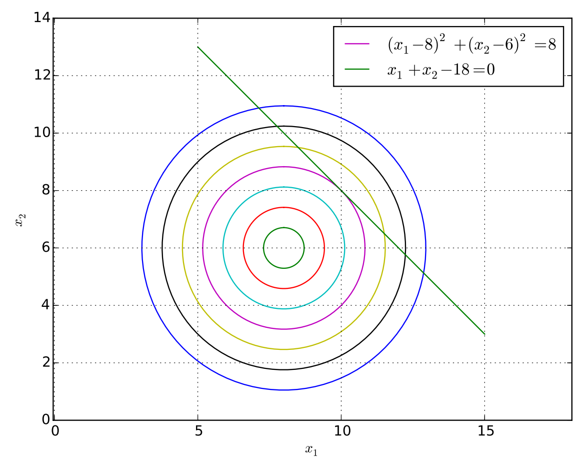

Fig. 2.5.5: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

6. Repeat problem 2.5.5 using the method of Lagrange mutipliers. What is the sign of λ?
Solution: Using the following python script, λ is positive and the minimum value of f is 8.

```
codes/optimization/2.9.py
```

7. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{2.5.7.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{2.5.7.2}$$

Solution: Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem 2.5.5.

8. Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \tag{2.5.8.1}$$

Solution: Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 9 \leq 0, \lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \tag{2.5.8.2}$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \tag{2.5.8.3}$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \tag{2.5.8.4}$$

else, $\lambda = 0$.

9. Solve

$$\min_{\mathbf{x}} \quad x_1 + x_2 \tag{2.5.9.1}$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \tag{2.5.9.2}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solution:

Graphical solution:

```
codes/optimization/2.15.py
```

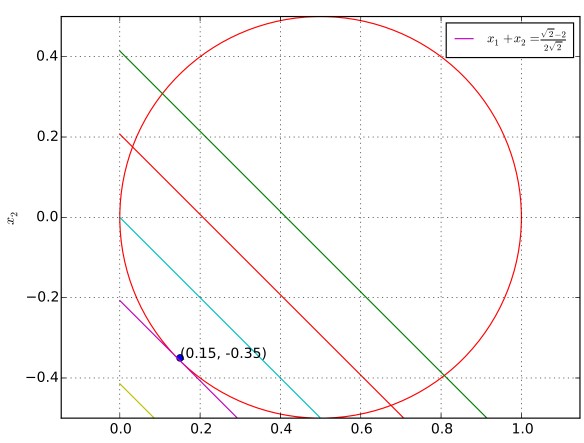

Fig. 2.5.9: Optimal solution is the lower tangent to the circle

2.6 Solved Problems

1. A circle passes through the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix} -1 & 4 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.6.1.1}$$

find its radius.

Solution: Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.6.1.2}$$

The direction vector of AB is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{2.6.1.3}$$

$$\begin{aligned} \because OC &\perp AB, \\ OC : \mathbf{m}^T(\mathbf{x} - \mathbf{C}) &= 0 \\ \implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} &= 7 \end{aligned} \tag{2.6.1.4}$$

Thus, $\mathbf{O}$ is the intersection of (2.6.1.1) and (2.6.1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{2.6.1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.6.1.6}$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \tag{2.6.1.7}$$

2. If a circle C_1, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix}\mathbf{x} = 4 \tag{2.6.2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.

Solution: From (2.6.2.1), the centre of C_2 is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.6.2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T\mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.6.2.3}$$

Since the radius of C_1 is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$
$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$
$$\implies \mathbf{O}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.6.2.4}$$

The intercepts of C_1 on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda\mathbf{m} \tag{2.6.2.5}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6.2.6}$$

Susbtituting in the equation for C_1,

$$\|\lambda\mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \tag{2.6.2.7}$$

which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$
$$\implies \lambda^2 - 10\lambda + 20 = 0 \tag{2.6.2.8}$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \tag{2.6.2.9}$$

after substituting from (2.6.2.6) and (2.6.2.4).

3. A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{2.6.3.1}$$

cuts the circle

$$C : \mathbf{x}^T\mathbf{x} = 9 \tag{2.6.3.2}$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$. Draw PAB for any two points $\mathbf{A}, \mathbf{B}$ on the circle.
Solution: Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line PAB can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{2.6.3.3}$$

for $\|\mathbf{m}\| = 1$. The intersection of L and C yields

$$(\mathbf{P} + \lambda\mathbf{m})^T (\mathbf{P} + \lambda\mathbf{m}) = 9$$
$$\implies \lambda^2 + 2\lambda\mathbf{m}^T\mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \tag{2.6.3.4}$$

The product of the roots in (2.6.3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \tag{2.6.3.5}$$

4. Find the equation of the circle C_2, which is the mirror image of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{2.6.4.1}$$

in the line

$$L : \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3. \tag{2.6.4.2}$$

Solution: From (2.6.4.1), circle C_1 has centre at

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6.4.3}$$

and radius

$$r_1 = \mathbf{O}_1^T\mathbf{O}_1 = 1 \tag{2.6.4.4}$$

The centre of C_2 is the reflection of $\mathbf{O}_1$ about L and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{O}_1 + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.6.4.5}$$

where the relevant parameters are obtained from (2.6.4.2) as

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 & -1 \end{pmatrix}, c = 3. \tag{2.6.4.6}$$

Substituting the above in (2.6.4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4}\mathbf{O}_1 + c\frac{\mathbf{n}}{2}$$
$$\implies \mathbf{O}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.6.4.7}$$

Thus

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = 1 \tag{2.6.4.8}$$

5. One of the diameters of the circle, given by

$$C : \mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} -2 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{2.6.5.1}$$

is a chord of a circle S, whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \tag{2.6.5.2}$$

Find the radius of S.

Solution: From (2.6.5.1), the centre of C is

$$\mathbf{O}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.6.5.3}$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T\mathbf{O}_1 - 12} = 5 \tag{2.6.5.4}$$

From (2.6.5.3) and (2.6.5.2),

$$O_1O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$
$$\implies r_2 = \sqrt{O_1O_2^2 - r_1^2} = 5 \tag{2.6.5.5}$$

6. A circle C passes through

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \tag{2.6.6.1}$$

and touches the y-axis at

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{2.6.6.2}$$

Which one of the following equations can represent a diameter of this circle?

(i) $\begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} = 6$ (iii) $\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 3$

(ii) $\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} + 10 = 0$(iv) $\begin{pmatrix} 5 & 2 \end{pmatrix}\mathbf{x} + 4 = 0$

Solution: Let $\mathbf{O}$ be the centre of C. Then the equation of the normal, OQ is

$$\begin{pmatrix} 0 & 1 \end{pmatrix}(\mathbf{O} - \mathbf{Q}) = 0$$
$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{O} = 2 \tag{2.6.6.3}$$

Also,

$$\|\mathbf{O} - \mathbf{P}\|^2 = \|\mathbf{O} - \mathbf{Q}\|^2$$
$$\implies 2(\mathbf{P} - \mathbf{Q})^T\mathbf{O} = \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2$$
$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{O} = -4 \tag{2.6.6.4}$$

(2.6.6.3) and (2.6.6.4) result in the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\mathbf{O} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{2.6.6.5}$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{O} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{2.6.6.6}$$

Hence, option ii) is correct.

7. Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{2.6.7.1}$$

whose centre $\mathbf{O}$ is the point of intersection of the straight lines

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{2.6.7.2}$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 1 \tag{2.6.7.3}$$

Solution: From (2.6.7.2) and (2.6.7.3), we obtain the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{O} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.6.7.4}$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 1 \end{pmatrix} \implies \mathbf{O} = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{2.6.7.5}$$

Thus, the equation of the desired tangent is

$$(\mathbf{O} - \mathbf{P})^T(\mathbf{x} - \mathbf{P}) = 0$$
$$\implies \begin{pmatrix} 1 & 4 \end{pmatrix}\mathbf{x} = -3 \tag{2.6.7.6}$$

8. The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.1}$$

intersects the circle

$$\Omega : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{2.6.8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{R} = -\frac{3}{5} \tag{2.6.8.3}$$

Find m.
Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.4}$$

The intersection of (2.6.8.1) and (2.6.8.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{2.6.8.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^T (\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \quad (2.6.8.6)$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1 \mathbf{m} \quad (2.6.8.7)$$
$$\mathbf{Q} = \mathbf{c} + \lambda_2 \mathbf{m} \quad (2.6.8.8)$$
$$\implies \frac{\mathbf{P} + \mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \mathbf{m} \quad (2.6.8.9)$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P} + \mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{m} \quad (2.6.8.10)$$
$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} - \frac{\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \quad (2.6.8.11)$$

using the sum of roots in (2.6.8.6). From (2.6.8.3) and (2.6.8.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5} \left(1 + m^2\right) \quad (2.6.8.12)$$
$$\implies m^2 - 5m + 6 = 0 \quad (2.6.8.13)$$
$$\implies m = 2 \text{ or } 3 \quad (2.6.8.14)$$

From (2.6.8.6),

$$\lambda = \frac{-\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{(\mathbf{m}^T (\mathbf{c} - \mathbf{O}))^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \quad (2.6.8.15)$$

Fig. 2.6.8 summarizes the solution for $m = 2$.

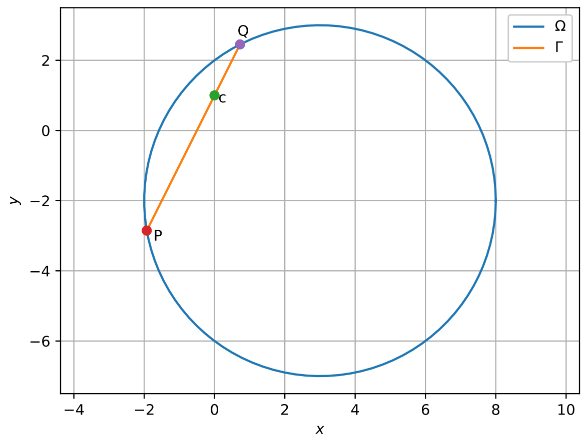

Fig. 2.6.8

2.7 JEE Exercises

1. If A and B are points in the plane such that $\frac{PA}{PB} = k$(constant) for all $\mathbf{P}$ on a given circle, then the value of k cannot be equal to?
2. Find the points of intersection of the line
$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} - 10 = 0 \quad (2.7.2.1)$$
and the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 4 \end{pmatrix} \mathbf{x} - 20 = 0. \quad (2.7.2.2)$$
are...................and................
3. The lines
$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 4 = 0 \quad (2.7.3.1)$$
$$\begin{pmatrix} 6 & -8 \end{pmatrix} \mathbf{x} - 7 = 0 \quad (2.7.3.2)$$
are tangents to the same circle. The radius of the circle is.............
4. Let
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -4 & -2 \end{pmatrix} \mathbf{x} - 11 = 0 \quad (2.7.4.1)$$
be circle. A pair of tangents $\begin{pmatrix} 4 & 5 \end{pmatrix}$ with a pair radii from quadrilateral of area..............
5. From the origin chords are drawn to the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (2.7.5.1)$$
The equation of the locus of the mid-points of these chords is............
6. The equation of the line passing through the points of intersection of the circles
$$3(\mathbf{x}^T \mathbf{x}) + \begin{pmatrix} -2 & 12 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (2.7.6.1)$$
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 6 & 2 \end{pmatrix} \mathbf{x} - 15 = 0 \quad (2.7.6.2)$$
is................
7. From the point A$\begin{pmatrix} 0 & 3 \end{pmatrix}$ on the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} + 9 = 0 \quad (2.7.7.1)$$
a chord AB is drawn and extended to a point $\mathbf{M}$ such that AM = 2AB. The equation of the locus of $\mathbf{M}$ is.............
8. The area of the triangle is formed by tangents

from the point $\begin{pmatrix} 4 & 3 \end{pmatrix}$ to the circle

$$\mathbf{x}^T\mathbf{x} = 9 \tag{2.7.8.1}$$

and the line joining their points of contact is

9. If the circle

$$C_1 : \mathbf{x}^T\mathbf{x} = 16 \tag{2.7.9.1}$$

intersects another circle C_2 of radius 5 in such a manner that common chord is of maximum length and has a slope equal to $\frac{3}{4}$, then the coordinates of the centre C_2 are...........

10. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle

$$\mathbf{x}^T\mathbf{x} = 4 \tag{2.7.10.1}$$

at $(1, \sqrt{3})$is?

11. If a circle passes through the points of intersection of the coordinate axes with the lines

$$\begin{pmatrix} \lambda & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{2.7.11.1}$$

$$\begin{pmatrix} 1 & -2 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{2.7.11.2}$$

then the value of $\lambda =$ ....

12. The equation of the locus of the midpoints of the circle

$$4(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix} -12 & 4 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{2.7.12.1}$$

that subtend of a angle of $\frac{2\pi}{3}$ at its centre is.............

13. The intercept on the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0 \tag{2.7.13.1}$$

by the circle

$$C1 : \mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{2.7.13.2}$$

is AB. Equation of the circle with AB as a diameter is..............

14. For each natural number k, let C_k denote the circle with radius k centimetres and center at origin. On the circle C_k, α-particle moves k centimetres in the counter clockwise direction. After completing its motion on C_k the particle moves to C_{k+1} in the radial direction. The motion of the particle is continues in the manner. The particle starts at $\begin{pmatrix} 1 & 0 \end{pmatrix}$. If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n=...............

15. The chords of contact of the pair of tangents drawn from each point on the line

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 4 \tag{2.7.15.1}$$

to circle

$$\mathbf{x}^T\mathbf{x} = 1 \tag{2.7.15.2}$$

pass through the point?

16. A square is inscribed in the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 4 \end{pmatrix}\mathbf{x} + 3 = 0. \tag{2.7.16.1}$$

Its sides are parallel to the coordinate axes. The one vertex of the square is

a) $(1 + \sqrt{2}, -2)$
b) $(1 - \sqrt{2}, -2)$
c) $(1, -2 + \sqrt{2})$
d) none of these

17. Two circles

$$\mathbf{x}^T\mathbf{x} = 6 and \tag{2.7.17.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -6 & 8 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{2.7.17.2}$$

are given. Then the equation of the circle through their points of intersection and the point $\begin{pmatrix} 1 & 1 \end{pmatrix}$ is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -6 & 0 \end{pmatrix}\mathbf{x} + 4 = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix}\mathbf{x} + 1 = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 0 & -4 \end{pmatrix}\mathbf{x} + 2 = 0$
d) none of these

18. The centre of the circle passing through the point $\begin{pmatrix} 0 & 1 \end{pmatrix}$ and touching the curve

$$y = \mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} \tag{2.7.18.1}$$

at $\begin{pmatrix} 2 & 4 \end{pmatrix}$ is

a) $(\frac{-16}{5}, \frac{27}{10})$
b) $(\frac{-16}{7}, \frac{53}{10})$
c) $(\frac{-16}{5}, \frac{53}{10})$
d) none of these

19. The equation of the circle passing through the point $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and the points of intersection of

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 13 & -3 \end{pmatrix}\mathbf{x} = 0 \tag{2.7.19.1}$$

$$2(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix} 4 & -7 \end{pmatrix}\mathbf{x} - 25 = 0 \tag{2.7.19.2}$$

is

a) $4(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix} -30 & -10 \end{pmatrix}\mathbf{x} - 25 = 0$

b) $4(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix}30 & -13\end{pmatrix}\mathbf{x} - 25 = 0$
c) $4(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix}-17 & -10\end{pmatrix}\mathbf{x} + 25 = 0$
d) none of these

20. The locus of the mid point of a chord of the circle

$$\mathbf{x}^T\mathbf{x} = 4 \tag{2.7.20.1}$$

which subtends a right angle at the origin is
a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$
b) $\mathbf{x}^T\mathbf{x} = 1$
c) $\mathbf{x}^T\mathbf{x} = 2$
d) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1$

21. If a circle passes through the point $\begin{pmatrix}a & b\end{pmatrix}$ and cuts the circle

$$\mathbf{x}^T\mathbf{x} = k^2 \tag{2.7.21.1}$$

orthogonally, then the equation of the locus of its centre is
a) $\begin{pmatrix}2a & 2b\end{pmatrix}\mathbf{x} - (a^2 + b^2 + k^2) = 0$
b) $\begin{pmatrix}2a & 2b\end{pmatrix}\mathbf{x} - (a^2 - b^2 + k^2) = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-3a, -4b\end{pmatrix}\mathbf{x} + (a^2 + b^2 - k^2) = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-3a, -4b\end{pmatrix}\mathbf{x} + (a^2 - b^2 - k^2) = 0$

22. If the two circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & -2\end{pmatrix}\mathbf{x} + 2 = r^2 \tag{2.7.22.1}$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-8 & 2\end{pmatrix}\mathbf{x} + 8 = 0 \tag{2.7.22.2}$$

intersect in two distinct points, then
a) $2 < r < 8$
b) $r < 2$
c) r = 2
d) $r > 2$

23. The lines

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 5 \tag{2.7.23.1}$$
$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 7 \tag{2.7.23.2}$$

are diameters of a circle of area 154 sq.units. Then the equation of this circle is
a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 62$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 47$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & 2\end{pmatrix}\mathbf{x} = 47$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & 2\end{pmatrix}\mathbf{x} = 62$

24. Find the centre of a circle passing through the points $\begin{pmatrix}0 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 0\end{pmatrix}$ and touching the circle

$$\mathbf{x}^T\mathbf{x} = 9. \tag{2.7.24.1}$$

a) $(\frac{3}{2}, \frac{1}{2})$
b) $(\frac{1}{2}, \frac{3}{2})$
c) $(\frac{1}{2}, \frac{1}{2})$
d) $(\frac{1}{2}, (-2)^{\frac{1}{2}})$

25. The locus of the centre of a circle, which touches externally the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-6 & -6\end{pmatrix}\mathbf{x} + 14 = 0 \tag{2.7.25.1}$$

and also touches the y-axis, is given by the equation:
a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-6 & 10\end{pmatrix}\mathbf{x} + 14 = 0$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-10 & -6\end{pmatrix}\mathbf{x} + 14 = 0$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-6 & -10\end{pmatrix}\mathbf{x} + 14 = 0$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-10 & -6\end{pmatrix}\mathbf{x} + 14 = 0$

26. The circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-10 & 0\end{pmatrix}\mathbf{x} + 16 = 0 \tag{2.7.26.1}$$
$$\mathbf{x}^T\mathbf{x} = r^2 \tag{2.7.26.2}$$

intersect each other in two distinct points if
a) $r < 2$
b) $r > 8$
c) $2 < r < 8$
d) $2 \le r \le 8$

27. The angle between a pair of tangents drawn from a point $\mathbf{P}$ to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} + 9\sin^2(\alpha) + 13\cos^2(\alpha) = 0 \tag{2.7.27.1}$$

is 2π. The equation of the locus of the point $\mathbf{P}$ is
a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} + 4 = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} - 9 = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} - 4 = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} + 9 = 0$

28. If two distinct chords drawn from the point (p, q) on the circle

$$\mathbf{x}^T\mathbf{x} = \begin{pmatrix}p & q\end{pmatrix}\mathbf{x} \tag{2.7.28.1}$$

(where pq $\neq$ 0) are bisected by the x-axis, then

a) $p^2 = q^2$
b) $p^2 = 8q^2$
c) $p^2 < 8q^2$
d) $p^2 > 8q^2$

29. The triangle PQR is inscribed in the circle

$$\mathbf{x}^T\mathbf{x} = 25. \tag{2.7.29.1}$$

If $\mathbf{Q}$ and $\mathbf{R}$ have co-ordinates $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-4\\3\end{pmatrix}$ respectively, then $\angle QPR$ is equal to

a) $\frac{\pi}{2}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{4}$
d) $\frac{\pi}{6}$

30. If the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 2k\end{pmatrix}\mathbf{x} + 6 = 0 \tag{2.7.30.1}$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 2k\end{pmatrix}\mathbf{x} + k = 2 \tag{2.7.30.2}$$

intersect orthogonally, then find k.

31. Let AB be chord of the circle

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{2.7.31.1}$$

subtending a right angle at at the centre. Then the locus of the centroid of the triangle PAB as $\mathbf{P}$ moves on the circle is

a) a parabola
b) a circle
c) a ellipse
d) a pair of straight lines

32. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point on the circumference of the circle, then 2r equals

a) $\sqrt{PQ.RS}$
b) $\frac{(PQ+RS)}{2}$
c) $\frac{2PQ.RS}{(PQ+RS)}$
d) $\frac{\sqrt{(PQ^2+RS^2)}}{2}$

33. If the tangent at the point $\mathbf{P}$ on the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}6 & 6\end{pmatrix}\mathbf{x} - 2 = 0 \tag{2.7.33.1}$$

meets a straight line

$$\begin{pmatrix}5 & -2\end{pmatrix}\mathbf{x} + 6 = 0 \tag{2.7.33.2}$$

at a point $\mathbf{Q}$ on the y-axis then the length of PQ is

a) 4
b) $2\sqrt{5}$
c) 5
d) $3\sqrt{5}$

34. Find the centre of the circle inscribed in square formed by the lines

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-8 & 0\end{pmatrix}\mathbf{x} + 12 = 0 \tag{2.7.34.1}$$
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -14\end{pmatrix}\mathbf{x} + 45 = 0 \tag{2.7.34.2}$$

35. If one of the diameter of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & -6\end{pmatrix}\mathbf{x} + 6 = 0 \tag{2.7.35.1}$$

is a chord to the circle with centre $\begin{pmatrix}2\\1\end{pmatrix}$. then the radius of the circle is

a) $\sqrt{3}$
b) $\sqrt{2}$
c) 3
d) 2

36. A circle is given by

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & -2\end{pmatrix}\mathbf{x} = 0, \tag{2.7.36.1}$$

another circle C touches it externally and also the x-axis, then the locus of its centre is

a) $\{(x,y) : \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -4\end{pmatrix}\mathbf{x} = 0\} \cup \{(x,y) : y \le 0\}$
b) $\{(x,y) : \mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & -2\end{pmatrix}\mathbf{x} = 3\} \cup \{(x,y) : y \le 0\}$
c) $\{(x,y) : \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} = 0\} \cup \{(0,y) : y \le 0\}$
d) $\{(x,y) : \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0, -4\end{pmatrix}\mathbf{x} = 0\} \cup \{(0,y) : y \le 0\}$

37. Tangents drawn from the point $\mathbf{P} = \begin{pmatrix}1\\8\end{pmatrix}$ to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-6 & -4\end{pmatrix}\mathbf{x} - 11 = 0 \tag{2.7.37.1}$$

touch the circle at the points $\mathbf{A}$ and $\mathbf{B}$. The equation of the circumcircle of the triangle PAB is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} + 19 = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -10\end{pmatrix}\mathbf{x} + 19 = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & 6\end{pmatrix}\mathbf{x} - 29 = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-6 & -4\end{pmatrix}\mathbf{x} + 19 = 0$

38. The circle passing through the point $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and touching the y-axis at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ also passes through the point.
 a) $\begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$
 b) $\begin{pmatrix} -\frac{5}{2} \\ 2 \end{pmatrix}$
 c) $\begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$
 d) $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
39. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line
$$\begin{pmatrix} 4 & -5 \end{pmatrix} \mathbf{x} = 20 \tag{2.7.39.1}$$
 to the circle
$$\mathbf{x}^T \mathbf{x} = 9 \tag{2.7.39.2}$$
 is
 a) $20(\mathbf{x}^T \mathbf{x}) + \begin{pmatrix} -36 & 45 \end{pmatrix} \mathbf{x} = 0$
 b) $20(\mathbf{x}^T \mathbf{x}) + \begin{pmatrix} 36 & -45 \end{pmatrix} \mathbf{x} = 0$
 c) $36(\mathbf{x}^T \mathbf{x}) + \begin{pmatrix} -20 & 45 \end{pmatrix} \mathbf{x} = 0$
 d) $36(\mathbf{x}^T \mathbf{x}) + \begin{pmatrix} 20 & -45 \end{pmatrix} \mathbf{x} = 0$
40. A line
$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1 \tag{2.7.40.1}$$
 intersects the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -6 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{2.7.40.2}$$
 at the points $\mathbf{P}$ and $\mathbf{Q}$. If the mid point of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following option is correct?
 a) $2 \leq m < 4$
 b) $-3 \leq m < -1$
 c) $4 \leq m < 6$
 d) $6 \leq m < 8$
41. The equations of the tangents drawn from the origin to the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2r & -2h \end{pmatrix} \mathbf{x} + h^2 = 0 \tag{2.7.41.1}$$
 are
 a) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}=0$
 b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}=0$
 c) $\begin{pmatrix} (h^2 - r^2) & -2rh \end{pmatrix} \mathbf{x} = 0$
 d) $\begin{pmatrix} (h^2 - r^2) & 2rh \end{pmatrix} \mathbf{x} = 0$
42. The number of common tangents to the circles
$$\mathbf{x}^T \mathbf{x} = 4 \tag{2.7.42.1}$$
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -6 & -8 \end{pmatrix} \mathbf{x} = 24 \tag{2.7.42.2}$$
 is
 a) 0
 b) 1
 c) 3
 d) 4
43. If the circle
$$\mathbf{x}^T \mathbf{x} = a^2 \tag{2.7.43.1}$$
 intersects the hyperbola
$$\mathbf{x}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} = c^2 \tag{2.7.43.2}$$
 in four points $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$ then
 a) $x_1 + x_2 + x_3 + x_4 = 0$
 b) $y_1 + y_2 + y_3 + y_4 = 0$
 c) $x_1 x_2 x_3 x_4 = c^4$
 d) $y_1 y_2 y_3 y_4 = c^4$
44. Circles touching the x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis are
 a) $\mathbf{x}^T \mathbf{x} + (-6, 8)\mathbf{x} + 9 = 0$
 b) $\mathbf{x}^T \mathbf{x} + (-6, 7)\mathbf{x} + 9 = 0$
 c) $\mathbf{x}^T \mathbf{x} + (-6, -8)\mathbf{x} + 9 = 0$
 d) $\mathbf{x}^T \mathbf{x} + (-6, -7)\mathbf{x} + 9 = 0$
45. A circle $\mathbf{S}$ passes through the points $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and is orthongonal to the circles
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = 15 \tag{2.7.45.1}$$
$$\mathbf{x}^T \mathbf{x} = 1. \tag{2.7.45.2}$$
 Then
 a) radius of S is 8
 b) radius of S is 7
 c) radius of S is $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$
 d) radius of S is $\begin{pmatrix} -8 \\ 1 \end{pmatrix}$

46. Let RS be the diameter of the circle
$$\mathbf{x}^T\mathbf{x} = 1 \tag{2.7.46.1}$$
where $\mathbf{S}$ is the point $\begin{pmatrix}1\\0\end{pmatrix}$. Let $\mathbf{P}$ be a variable point on the circle and tangents to the circle at $\mathbf{S}$ and $\mathbf{P}$ meet at the point $\mathbf{Q}$.The normal to the circle at $\mathbf{P}$ intersects a line drawn through $\mathbf{Q}$ parallel to RS at point $\mathbf{E}$. Then the locus of $\mathbf{E}$ passes through the points
    a) $\begin{pmatrix}\frac{1}{3}\\\frac{1}{\sqrt{3}}\end{pmatrix}$
    b) $\begin{pmatrix}\frac{1}{4}\\\frac{1}{2}\end{pmatrix}$
    c) $\begin{pmatrix}\frac{1}{3}\\-\frac{1}{\sqrt{3}}\end{pmatrix}$
    d) $\begin{pmatrix}\frac{1}{4}\\-\frac{1}{2}\end{pmatrix}$
47. Find the equation of the circle whose radius is 5 and which touches the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & -4\end{pmatrix}\mathbf{x} - 20 = 0 \tag{2.7.47.1}$$
at the point $\begin{pmatrix}5\\5\end{pmatrix}$.
48. Let $\mathbf{A}$ be the centre of the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & -4\end{pmatrix}\mathbf{x} - 20 = 0. \tag{2.7.48.1}$$
Suppose that the tangents at the points of $\mathbf{B} = \begin{pmatrix}1\\7\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}4\\-2\end{pmatrix}$ on the circle meet at the point $\mathbf{C}$. Find the area of the quadrilateral ABCD?
49. Find the equations of the circles passing through $\begin{pmatrix}-4\\3\end{pmatrix}$ and the touching the lines
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2 \tag{2.7.49.1}$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \tag{2.7.49.2}$$
50. Through a fixed point $\begin{pmatrix}h\\k\end{pmatrix}$ secants are drawn through the circles
$$\mathbf{x}^T\mathbf{x} = r^2. \tag{2.7.50.1}$$
Show that the locus of the mid-point of the secants intercepted by the circle is
$$\mathbf{x}^T\mathbf{x} = \begin{pmatrix}h & k\end{pmatrix}\mathbf{x} \tag{2.7.50.2}$$
51. The abscissa of the two points $\mathbf{A}$ and $\mathbf{B}$ are the roots of the equation
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}2a & 0\end{pmatrix}\mathbf{x} - b^2 = 0 \tag{2.7.51.1}$$
and their ordinates are the roots of the equation the two points $\mathbf{A}$ and $\mathbf{B}$ are the roots of the equation
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}2p & 0\end{pmatrix}\mathbf{x} - q^2 = 0. \tag{2.7.51.2}$$
Find the equation and the radius of the circle with AB as diameter?
52. Lines
$$\begin{pmatrix}5 & 12\end{pmatrix}\mathbf{x} - 10 = 0 \tag{2.7.52.1}$$
$$\begin{pmatrix}5 & 12\end{pmatrix}\mathbf{x} - 40 = 0 \tag{2.7.52.2}$$
touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, Find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines.
53. Let a given line L_1 intersects the x and y axes at $\mathbf{P}$ and $\mathbf{Q}$ respectively. Let another line L_2, perpendicular to L_1, cut the x and y axes at $\mathbf{R}$ and $\mathbf{S}$ respectively. Show that locus of the point of intersection of the lines PS and QR is circle passing through the origin?
54. The circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 4 = 0 \tag{2.7.54.1}$$
is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcentre of the triangle is
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} - x^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\mathbf{x} + k(\mathbf{x}^T\mathbf{x})^{\frac{1}{2}} = 0 \tag{2.7.54.2}$$
Find the k?
55. If $(m_i, 1/m_i)$,$(m_i > 0)$, i=1,2,3,4, are 4 distinct points on circle, then show that $m_1m_2m_3m_4 = 1$
56. A circle touches the line
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.56.1}$$
at a point $\mathbf{P}$ such that OP=4 $\sqrt{2}$,where $\mathbf{O}$ is the origin. The circle contains the point$\begin{pmatrix}-10\\2\end{pmatrix}$ in its interior and length of its chord on the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.56.2}$$

is $6\sqrt{2}$. Determine the equation of the circle?

57. Two circles each of radius 5 units, touch each other at $\begin{pmatrix}1\\2\end{pmatrix}$. If the equation of their common tangent is

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 10 \tag{2.7.57.1}$$

find the equation of the circles?

58. Let a circle be given by

$$2(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix}-2a & -b\end{pmatrix}\mathbf{x} = 0 \tag{2.7.58.1}$$

$(a \neq 0, b \neq 0)$ Find the condition on a and b if two chords, each bisected by the x-axis can be drawn to the circle form $\begin{pmatrix}a\\ \frac{b}{2}\end{pmatrix}$

59. Consider a family of circles passing through fixed point $\mathbf{A} = \begin{pmatrix}3\\7\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$. Such that the chords in which the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -6\end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.7.59.1}$$

cuts the members of the family are concurrent at a point.Find the coordinate of this point?

60. Find the coordinates the point at which the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{2.7.60.1}$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-12 & 8\end{pmatrix}\mathbf{x} + 36 = 0 \tag{2.7.60.2}$$

touch each other. Also find the equations common tangents touching the circles in the distinct points.

61. Find the intervals of values of a for which the line y+x=0 bisects two chords drawn from a point $\begin{pmatrix}1 + (\sqrt{2}a)/2\\ (1 - (\sqrt{2}a)/2\end{pmatrix}$ to the circle

$$2(\mathbf{x}^T\mathbf{x}) + \begin{pmatrix}-(1 + \sqrt{2}a) & -(1 - \sqrt{2}a)\end{pmatrix}\mathbf{x} = 0 \tag{2.7.61.1}$$

62. A circle passes through three points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ with the line segment AC as its diameter. A line passing through $\mathbf{A}$ intersects the chord BC at a point $\mathbf{D}$ inside the circle. If angles DAB and CAB are α and β respectively and the distance between the point $\mathbf{A}$ and the mid point of the line segments DC is d, prove that the area of the circle is

$$\frac{\pi d^2 cos^2\alpha}{cos^2\alpha + cos^2\beta + 2cos\alpha cos\beta cos(\beta - \alpha)} \tag{2.7.62.1}$$

63. Let C be any circle with centre $\begin{pmatrix}0\\ \sqrt{2}\end{pmatrix}$. Prove that at the most two rational points can be there on C.(A rational point is a point both of whose coordinates are rational numbers).

64. C_1 and C_2 are two concentric circles, the radius C_2 being twice that of C_1. From a point $\mathbf{P}$ on C_2, tangent PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1?

65. Let T_1, T_2 be two tangents drawn from $\begin{pmatrix}-2\\0\end{pmatrix}$ on to the circle C:

$$\mathbf{x}^T\mathbf{x} = 1 \tag{2.7.65.1}$$

Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equation of all possible common tangents to these circles, when take two at a time.

66. Let

$$\mathbf{x}^T\begin{pmatrix}2 & -3\\0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.66.1}$$

be a equation of a pair of tangents drwan from the origin O to a circle of radius 3 with centre is the first coordinate. If $\mathbf{A}$ is the one of the points of contact, find the length of OA?

67. Let C_1 and C_2 be two circles with C_2 lying inside the C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of centre of C?

68. For the circle

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{2.7.68.1}$$

find the value of r for which the area enclosed by the tangents drawn from the point $\mathbf{P} = \begin{pmatrix}6\\8\end{pmatrix}$ to the circle and the chord of contact is maximum?

69. Find the equation of the circle touching the line

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + 1 = 0 \tag{2.7.69.1}$$

at $\begin{pmatrix}1\\-1\end{pmatrix}$ and cutting orthogonally the circle having line segment joining $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ as diameter.

70. Circles with radii 3,4 and 5 touch each other externally. If **P** is the point of intersection of tangents to these circles at their points of contact, find the distance of **P** from the points of contact?

PASSAGE - 1

ABCD is square of side length 2 units. C_1 is the circle touching all the sides of the square ABCD and C_2 is the circumcircle of square ABCD.L is a fixed line in the same plane and **R** is a fixed point.

71. If **P** is any point of C_1 and **Q** is another point on C_2 then

$$\frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2} \tag{2.7.71.1}$$

is equal to

a) 0.75
b) 1.25
c) 1
d) 0.5

72. If a circle is such that it touches the line L and the circle C_1 externally,such that both the circles are on the same side of the line, then the locus of centre of circle is

a) ellipse
b) hyperbola
c) parabola
d) pair of straight line

73. A line L' through **A** is drawn parallel to BD. Point **S** moves such that its distance from the line BD and the vertex **A** arc equal. If locus of **S** cuts L' at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

a) 1/2 sq.units
b) 2/3 sq.units
c) 1 sq.units
d) 2 sq.units

PASSAGE-2

A circle C of radius 1 is inscribed in an equilateral triangle PQR,The points of contact of C with the sides PQ,QR,RP are $\mathbf{D}, \mathbf{E}, \mathbf{F}$ respectively.The line PQ is given by the equation $\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} - 6 = 0$ and point **D** is $\begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix}$ Further,it is given that the origin and the center of C are on the same side of the line PQ.

74. The equation of the circle C is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4\sqrt{3} & -2 \end{pmatrix} \mathbf{x} + 13 = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4\sqrt{3} & 1 \end{pmatrix} \mathbf{x} + 13/4 = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2\sqrt{3} & 2 \end{pmatrix} \mathbf{x} + 3 = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2\sqrt{3} & -2 \end{pmatrix} \mathbf{x} + 3 = 0$

75. Equation of the sides QR,RP are

a) $(2/\sqrt{3}, -1)\mathbf{x} + 1 = 0, (-2/\sqrt{3}, -1)\mathbf{x} - 1 = 0$
b) $(1/\sqrt{3}, -1)\mathbf{x} = 0, (0, 1)\mathbf{x} = 0$
c) $(\sqrt{3}/2, -1)\mathbf{x} + 1 = 0, (\sqrt{3}/2, -1)\mathbf{x} - 1 = 0$
d) $(\sqrt{3}, -1)\mathbf{x} = 0, (0, 1)\mathbf{x} = 0$

PASSAGE-3

A tangent PT is drawn to the circle

$$\mathbf{x}^T\mathbf{x} = 4 \tag{2.7.75.1}$$

at the point $\mathbf{P} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. A straight line L, perpendicular to PT is a tangent to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{2.7.75.2}$$

76. A possible equation of L is

a) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 1$
b) $\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 1$
c) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -1$
d) $\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 5$

77. A common tangent of the two circles is

a) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4$
b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$
c) $\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 4$
d) $\begin{pmatrix} 1 & 2\sqrt{2} \end{pmatrix} \mathbf{x} = 6$

PASSAGE-4

Let S be the circle in the xy-plane defined by the equation

$$\mathbf{x}^T\mathbf{x} = 4 \tag{2.7.77.1}$$

78. Let $E_1\ E_2$ and F_1F_2 be the chords of S passing through the point $\mathbf{P_0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and Parallel to the x-axis and the y-axis respectively.Let G_1G_2 be the chord of S passing through P_0 and having slope -1.Let the tangent to S at F_1 and F_2 meet at F_3 and the tangent to S at G_1 and G_2 meet at G_3. Then,the points E_3, F_3 and G_3 lie on the

curve

a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-8 & -8\end{pmatrix}\mathbf{x} + 16 = 0$
c) $\mathbf{x}^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} = -12$
d) $\mathbf{x}^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\mathbf{x} = 4$

79. Let **P** be a point on the Circle S with both coordinates being positive.Let the tangent to S at **P** intersect the coordinate axes at the points **M** and **N**. Then,the mid-Point of the line segment MN must lie on the curve

a) $\mathbf{x}^T\mathbf{x} = 3x^T\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}\mathbf{x}$
b) $x^{2/3} + y^{2/3} = 2^{4/3}$
c) $\mathbf{x}^T\mathbf{x} = \mathbf{x}^T\begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\mathbf{x}$
d) $\mathbf{x}^T\mathbf{x} = (x^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\mathbf{x})2$

Assertion and Reason Type Questions

80. Tangent are drawn from the point $\mathbf{P} = \begin{pmatrix}17\\7\end{pmatrix}$ to the circle

$$\mathbf{x}^T\mathbf{x} = 169 \tag{2.7.80.1}$$

Statement-1: the tangents are mutually perpendicular
Statement-2: The locus of the point from which mutually perpendicular tangents can be draw to the given circle

$$\mathbf{x}^T\mathbf{x} = 338 \tag{2.7.80.2}$$

a) Statement-1 is True,Statement-2 is True;Statement-2 is correct explanation for Statement-1
b) Statement-1 is True,Statement-2 is False;Statement-2 is NOT correct explanation for Statement-1
c) Statement-1 is True,Statement-2 is False
d) Statement-1 is False,Statement-2 is True

81. Consider

$$L_1 : \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + p - 3 = 0 \tag{2.7.81.1}$$

$$L_2 : \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + p + 3 = 0 \tag{2.7.81.2}$$

where P is a real number, and

$$C : \mathbf{x}^T\mathbf{x} + \begin{pmatrix}6 & -10\end{pmatrix}\mathbf{x} + 30 = 0 \tag{2.7.81.3}$$

Statement-1: If line L_1 is a chord of circle C,Then line L_2 is not always a diameter of circle C
Statement-2: If line L_1 is a diameter of circle C,Then line L_2 is not a chord of circle C

a) Statement-1 is True, Statement-2 is True; Statement-2 is correct explanation for Statement-1
b) Statement-1 is True, Statement-2 is False; Statement-2 is NOT correct explanation for Statement-1
c) Statement-1 is True, Statement-2 is False
d) Statement-1 is False, Statement-2 is True

82. The Center of two Circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centers of C_1 and C_2 and C be the circle touching circle C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of circle C is

83. The straight line

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 1 \tag{2.7.83.1}$$

divides the circular region

$$\mathbf{x}^T\mathbf{x} \le 6 \tag{2.7.83.2}$$

into two parts. If

$$S = \{(2, \frac{3}{4}), (\frac{5}{2}, \frac{3}{4}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{8}, \frac{1}{4})\} \tag{2.7.83.3}$$

then number of points(s) in S lying inside the smaller part is

84. For how many values of p, the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} - p = 0 \tag{2.7.84.1}$$

and the coordinate axes have exactly three common points?

85. Let point B be the reflection of the point $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}$ with respect to the line

$$\begin{pmatrix}8 & -6\end{pmatrix}\mathbf{x} - 23 = 0 \tag{2.7.85.1}$$

Let T_a and T_b be circles of radii 2 and 1 with centers A and B respectively. Let T be a common tangent to a circle T_a and T_b such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is

86. If the chord

$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1 \quad (2.7.86.1)$$

of the circle

$$\mathbf{x}^T \mathbf{x} = 1 \quad (2.7.86.2)$$

subtends an angle of measure 45° at the major segment of the circle then value of m is

a) $2 \pm \sqrt{2}$
b) $-2 \pm \sqrt{2}$
c) $-1 \pm \sqrt{2}$
d) none of these

87. The centers of set points of a circles,each of radius 3,lie on the circle

$$\mathbf{x}^T \mathbf{x} = 25 \quad (2.7.87.1)$$

the locus of any point in the set is

a) $4 \leq \mathbf{x}^T \mathbf{x} \leq 64$
b) $\mathbf{x}^T \mathbf{x} \leq 25$
c) $\mathbf{x}^T \mathbf{x} \geq 25$
d) $3 \leq \mathbf{x}^T \mathbf{x} \leq 9$

88. The center of the circle passing through (0,0) and (1,0) and touching the circle

$$\mathbf{x}^T \mathbf{x} = 9 \quad (2.7.88.1)$$

is

a) $(1/2, 1/2)$
b) $(1/2, -\sqrt{2})$
c) $(3/2, 1/2)$
d) $(1/2, 3/2)$

89. The equation of a circle with origin as a center passing through equilateral triangle whose median is of length 3a is

a) $\mathbf{x}^T.\mathbf{x} = 9a^2$
b) $\mathbf{x}^T.\mathbf{x} = 16a^2$
c) $\mathbf{x}^T.\mathbf{x} = 4a^2$
d) $\mathbf{x}^T.\mathbf{x} = a^2$

90. If the two circles

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & -6 \end{pmatrix} \mathbf{x} + 1 = r^2 \quad (2.7.90.1)$$

and

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -8 & 2 \end{pmatrix} \mathbf{x} + 8 = 0 \quad (2.7.90.2)$$

intersecting in two distinct point, then

a) $r > 2$
b) $2 < r < 8$
c) $r < 2$
d) r=2

91. The lines

$$\mathbf{x} + \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 05 \quad (2.7.91.1)$$

and

$$\mathbf{x} + \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (2.7.91.2)$$

are diameters of a circle having area as 154 sq.units. then the equation of the circle is

a) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix} \mathbf{x} = 62$
b) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 62$
c) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 47$
d) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix} \mathbf{x} = 47$

92. If a circle passes through a point (a,b) and cuts the circle

$$\mathbf{x}^T \mathbf{x} = 4 \quad (2.7.92.1)$$

orthogonally then the locus of its center is

a) $\begin{pmatrix} 2a & -2b \end{pmatrix} \mathbf{x} - (a^2 + b^2 + 4) = 0$
b) $\begin{pmatrix} 2a & 2b \end{pmatrix} \mathbf{x} - (a^2 + b^2 + 4) = 0$
c) $\begin{pmatrix} 2a & -2b \end{pmatrix} \mathbf{x} + (a^2 + b^2 + 4) = 0$
d) $\begin{pmatrix} 2a & 2b \end{pmatrix} \mathbf{x} + (a^2 + b^2 + 4) = 0$

93. A variable circle passes through the fixed point A(p,q) and touches x-axis.the locus of the other end of the diameter through A is

a) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4p & -4q \end{pmatrix} \mathbf{x} + q^2 = 0$
b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2q & 2p \end{pmatrix} \mathbf{x} + q^2 = 0$
c) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4q & -2p \end{pmatrix} \mathbf{x} + q^2 = 0$
d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2p & -4q \end{pmatrix} \mathbf{x} + q^2 = 0$

94. If the lines

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (2.7.94.1)$$

and

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (2.7.94.2)$$

is lie along diameter of a circle of circumference 10π,then the equation of the circle is

a) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} - 23 = 0$
b) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix} \mathbf{x} - 23 = 0$
c) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} - 23 = 0$

d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix}\mathbf{x} - 23 = 0$

95. Intercept on the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0 \tag{2.7.95.1}$$

by the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{2.7.95.2}$$

is AB.Equation of the circle on AB as a diameter is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -1 & -1 \end{pmatrix}\mathbf{x} = 0$

96. If the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2a & c \end{pmatrix}\mathbf{x} + a = 0 \tag{2.7.96.1}$$

and

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -3a & d \end{pmatrix}\mathbf{x} - 1 = 0 \tag{2.7.96.2}$$

intersect in two distinct points P and Q then the lines

$$\begin{pmatrix} 5 & -b \end{pmatrix}\mathbf{x} - a = 0 \tag{2.7.96.3}$$

passes through P and Q for

a) exactly one value of a
b) no value of a
c) infinitely many values of a
d) exactly two values of a

97. A circle touches the x-axis and also touches the circle with centre at (0,3) and radius 2.The locus of the centre of the circle is

a) an ellipse
b) a circle
c) a hyperbola
d) a parabola

98. If a circle passes through the point (a,b) and cuts the circle

$$\mathbf{x}^T\mathbf{x} + p^2 = 0 \tag{2.7.98.1}$$

orthogonally, then the equation of the locus of its center is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -3a & -4b \end{pmatrix}\mathbf{x} + (a^2 + b^2 - p^2) = 0$
b) $\begin{pmatrix} 2a & 2b \end{pmatrix}\mathbf{x} - (a^2 - b^2 + p^2) = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2a & -3b \end{pmatrix}\mathbf{x} + (a^2 - b^2 - p^2) = 0$
d) $\begin{pmatrix} 2a & 2b \end{pmatrix}\mathbf{x} - (a^2 + b^2 + p^2) = 0$

99. If the lines

$$2(a+b)\mathbf{x}^T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{x} + \mathbf{x}^T\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\mathbf{x} = 0 \tag{2.7.99.1}$$

lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector then

a) $3a^2 - 10ab + 3b^2 = 0$
b) $3a^2 - 2ab + 3b^2 = 0$
c) $3a^2 + 10ab + 3b^2 = 0$
d) $3a^2 + 2ab + 3b^2 = 0$

100. If the lines

$$\begin{pmatrix} 3 & -4 \end{pmatrix}\mathbf{x} - 7 = 0 \tag{2.7.100.1}$$

and

$$\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{2.7.100.2}$$

are two diameters of a circle of area 49π sq.units,the equation of the circle is

a) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & -2 \end{pmatrix}\mathbf{x} - 47 = 0$
b) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -2 \end{pmatrix}\mathbf{x} - 62 = 0$
c) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix}\mathbf{x} - 62 = 0$
d) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 2 \end{pmatrix}\mathbf{x} - 47 = 0$

101. Let C be the circle with center (0,0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtend angle of $2\pi/3$ at its center is

a) $\mathbf{x}^T\mathbf{x} = 3/2$
b) $\mathbf{x}^T\mathbf{x} = 1$
c) $\mathbf{x}^T\mathbf{x} = 27/4$
d) $\mathbf{x}^T\mathbf{x} = 9/4$

102. Consider a family of circles which are passing through the points (-1,1) and are tangent to x-axis. If (h,k) are the coordinates of the centre of the circles,then the set of values of k given by the interval

a) $-1/2 \le k \le 1/2$
b) $k \le 1/2$
c) $0 \le k \le 1/2$
d) $k \ge 1/2$

103. The point diametrically opposite to the point P(1,0) on the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & 4 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.7.103.1}$$

a) (3,-4)

b) (-3,4)
c) (-3,-4)
d) (3,4)

104. The differential equation of family of circles with fixed radius 5 units and center on the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.7.104.1}$$

is

a) $(x-2)y^2 = 25-(y-2)^2$
b) $(y-2)y^2 = 25-(y-2)^2$
c) $(y-2)^2y^2 = 25-(y-2)^2$
d) $(x-2)^2y^2 = 25-(y-2)^2$

105. If P and Q are the points of intersection of the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 3 & 7 \end{pmatrix} \mathbf{x} + 2p - 5 = 0 \tag{2.7.105.1}$$

and

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} - p^2 = 0 \tag{2.7.105.2}$$

then there is a circle passing through P,Q and (1,1) for

a) all except one value of p
b) all except two values of p
c) exactly one value of p
d) all values of p

106. The circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4 & -8 \end{pmatrix} \mathbf{x} = 5 \tag{2.7.106.1}$$

intersects the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = m \tag{2.7.106.2}$$

at two distinct points if

a) -35<m<15
b) 15<m<65
c) 35<m<85
d) -85<m<-35

107. The two circles

$$\mathbf{x}^T\mathbf{x} - a\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.7.107.1}$$

and

$$\mathbf{x}^T\mathbf{x} = c^2 \tag{2.7.107.2}$$

C>0 touch each other if

a) $|a| = c$
b) a=2c
c) $|a| = 2c$
d) $2|a| = c$

108. The length of the diameter of the circle which touches the x-axis at the point (1,0) and passes through the point (2,3) is

a) 10/3
b) 3/5
c) 6/5
d) 5/3

109. The circle passing through (1,-2)and touching the axis of x at (3,0) also passes through the point

a) (-5,2)
b) (2,-5)
c) (5,-2)
d) (-2,5)

110. Let C be the circle with center (1,1) and radius =1. If T is the circle centred at (0,y),passing through origin and touching the circle C externally, then the radius of T is equal to

a) 1/2
b) 1/4
c) $\sqrt{3}/\sqrt{2}$
d) $\sqrt{3}/2$

111. Locus of the image of the point (2,3) in the line

$$[\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} + 4] + k(1,-2)\mathbf{x} + 3 = 0 \tag{2.7.111.1}$$

k∈ R is a

a) circle of radius $\sqrt{2}$
b) circle of radius $\sqrt{3}$
c) straight line parallel to x-axis
d) straight line parallel to y-axis

112. The number of common tangents to the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -4 & -6 \end{pmatrix} \mathbf{x} - 12 = 0 \tag{2.7.112.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 6 & 18 \end{pmatrix} \mathbf{x} + 26 = 0 \tag{2.7.112.2}$$

is

a) 3
b) 4
c) 1
d) 2

113. The centers of those circles which touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -8 & -8 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{2.7.113.1}$$

externally and also touches the x-axis lie on

a) an parabola

b) a circle
c) a hyperbola
d) a ellipse which is not a circle

114. If one of the diameter of the circle given by the equation

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & 6\end{pmatrix}\mathbf{x} - 12 = 0 \quad (2.7.114.1)$$

is a chord of a circle S, whose centre is at (-3,2), then the radius of S is

a) 5
b) 10
c) $5\sqrt{2}$
d) $5\sqrt{3}$

115. If the tangent to the circle

$$\mathbf{x}^T\mathbf{x} = 1 \quad (2.7.115.1)$$

intersects the coordinates axes at distinct points P and Q, then the locus of the mid-point of PQ is

3 Conics

3.1 Definitions

1. From (1.2.19.2), the equation of a conic section is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (3.1.1.1)$$

for

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} \neq 0 \quad (3.1.1.2)$$

2. Show that

$$\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \quad (3.1.2.1)$$

3. Show that

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T\mathbf{x} \quad (3.1.3.1)$$

4. Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \quad (3.1.4.1)$$

5. Find the *normal* vector to the curve in (1.2.19.2) at point $\mathbf{p}$.
Solution: Differentiating (1.2.19.2) with respect to x_1,

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} + \frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} = 0 \quad (3.1.5.1)$$

$$\implies 2\mathbf{x}^T\mathbf{V}\mathbf{m} + 2\mathbf{u}^T\mathbf{m} = 0 \because \left(\frac{d\mathbf{x}}{dx_1} = \mathbf{m}\right) \quad (3.1.5.2)$$

Substituting $\mathbf{x} = \mathbf{p}$ and simplifying

$$(\mathbf{V}\mathbf{p} + \mathbf{u})^T\mathbf{m} = 0 \quad (3.1.5.3)$$

$$\implies \mathbf{n} = \mathbf{V}\mathbf{p} + \mathbf{u} \quad (3.1.5.4)$$

6. The *tangent* to the curve at $\mathbf{p}$ is given by

$$\mathbf{n}^T(\mathbf{x} - \mathbf{p}) = 0 \quad (3.1.6)$$

This results in

$$\left(\mathbf{p}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{x} + \mathbf{p}^T\mathbf{u} + f = 0 \quad (3.1.6)$$

7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \quad (3.1.7)$$

is known as an *affine* transformation.

8. Classify the various conic sections based on (1.2.19.2).
Solution:

Curve	**Property**
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 3.1.8

3.2 Parabola

1. Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} + 6 = 0 \quad (3.2.1.1)$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix}1\\7\end{pmatrix}, V = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \quad (3.2.1.2)$$

in (3.1.6), the desired equation is

$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}+6=0 \quad (3.2.1.3)$$

resulting in

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -5 \quad (3.2.1.4)$$

2. The line in (3.2.1.4) touches the circle

$$\mathbf{x}^T\mathbf{x} + 4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + c = 0 \quad (3.2.2.1)$$

Find c.
Solution: Comparing (1.2.19.2) and (3.2.2.1),

$$V = I, \quad \mathbf{u} = 2\begin{pmatrix}4\\3\end{pmatrix} \quad (3.2.2.2)$$

Comparing (3.1.6) and (3.2.1.4),

$$\mathbf{p} + 2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \quad (3.2.2.3)$$

$$\implies \mathbf{p} = -\begin{pmatrix}6\\7\end{pmatrix} \quad (3.2.2.4)$$

and

$$c + \mathbf{p}^T\mathbf{u} = 5 \quad (3.2.2.5)$$

$$\implies c = 5 + 2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \quad (3.2.2.6)$$

$$= 95 \quad (3.2.2.7)$$

3. Summarize all the above computations through a Python script and plot the parabola, tangent and circle.
Solution: The following code generates Fig. 3.2.3.

```
wget
codes/2d/parab.py
```

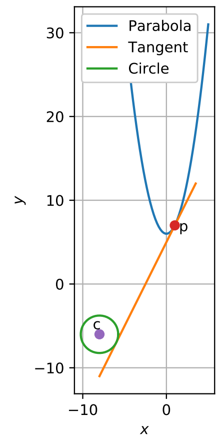

Fig. 3.2.3

3.3 *Affine Transformation*

1. In general, Fig. 3.2.3 was generated using an *affine transformation*.
2. Express

$$y_2 = y_1^2 \quad (3.3.2.1)$$

as a matrix equation.
Solution: (3.3.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{g}^T\mathbf{y} = 0 \quad (3.3.2.2)$$

where

$$\mathbf{D} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{g} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} \quad (3.3.2.3)$$

3. Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0, \quad (3.3.3.1)$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \quad (3.3.3.2)$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \quad (3.3.3.3)$$

(3.3.3.3) is known as an affine transformation. Show that

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P}$$
$$\mathbf{g} = \mathbf{P}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) \quad (3.3.3.4)$$
$$F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0$$

Solution: Substituting (3.3.3.3) in (3.3.3.1),

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + F = 0, \quad (3.3.3.5)$$

which can be expressed as

$$\implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \quad (3.3.3.6)$$

Comparing (3.3.3.6) with (3.3.2.2) (3.3.3.4) is obtained.

4. Show that there exists a $\mathbf{P}$ such that
$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{3.3.4.1}$$
Find $\mathbf{P}$ using
$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.3.4.2}$$
5. Find $\mathbf{c}$ from (3.3.3.4).
Solution:
$$\because \mathbf{g} = \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right), \tag{3.3.5.1}$$
$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{3.3.5.2}$$
$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\left(\mathbf{P}\mathbf{g} - \mathbf{u}\right) = -F - 2\mathbf{u}^T\mathbf{c} \tag{3.3.5.3}$$
resulting in the matrix equation
$$\begin{pmatrix}\mathbf{V}\\ \left(\mathbf{P}\mathbf{g} + \mathbf{u}\right)^T\end{pmatrix}\mathbf{c} = \begin{pmatrix}\mathbf{P}\mathbf{g} - \mathbf{u}\\ -F\end{pmatrix} \tag{3.3.5.4}$$
for computing $\mathbf{c}$.

3.4 *Ellipse: Eigenvalues and Eigenvectors*

1. Express the following equation in the form given in (1.2.19.2)
$$E: \ 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{3.4.1.1}$$
Solution: (3.4.1.1) can be expressed as
$$\mathbf{x}^TV\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{3.4.1.2}$$
where
$$V = \begin{pmatrix}5 & -3\\ -3 & 5\end{pmatrix}, \mathbf{u} = \begin{pmatrix}11\\ -13\end{pmatrix} \tag{3.4.1.3}$$
2. Using the affine transformation in (3.3.3.3), show that (3.4.1.2) can be expressed as
$$\mathbf{y}^TD\mathbf{y} = 1 \tag{3.4.2.1}$$
where
$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.2.2}$$
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.2.3}$$
for
$$\mathbf{P}^\mathbf{T}\mathbf{P} = \mathbf{I} \tag{3.4.2.4}$$
3. Find $\mathbf{c}$
Solution:
$$\mathbf{c} = \begin{pmatrix}1\\ -2\end{pmatrix} \tag{3.4.3.1}$$
4. If
$$D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{3.4.4.1}$$
$$P = \begin{pmatrix}\mathbf{P}_1 & \mathbf{P}_2\end{pmatrix} \tag{3.4.4.2}$$
show that
$$V\mathbf{z} = \lambda\mathbf{z} \tag{3.4.4.3}$$
where $\lambda \in \{\lambda_1, \lambda_2\}, \mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$.
5. Find λ.
Solution: λ is obtained by solving the following equation.
$$\left|\lambda I - V\right| = 0 \tag{3.4.5.1}$$
$$\implies \begin{vmatrix}\lambda - 5 & 3\\ 3 & \lambda - 5\end{vmatrix} = 0 \tag{3.4.5.2}$$
$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{3.4.5.3}$$
$$\implies \lambda = 2, 8 \tag{3.4.5.4}$$
6. Sketch 3.4.2.1.
7. Find $\mathbf{P}_1$ and $\mathbf{P}_2$.
Solution: From (3.4.4.3)
$$V\mathbf{P}_1 = \lambda_1\mathbf{P}_1 \tag{3.4.7.1}$$
$$\implies \left(V - \lambda I\right)\mathbf{y} = 0 \tag{3.4.7.2}$$
$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.7.3}$$
$$\text{or, } \mathbf{P}_1 = k_1\begin{pmatrix}1\\ 1\end{pmatrix} \tag{3.4.7.4}$$
Similarly,
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{P}_2 = 0 \tag{3.4.7.5}$$
$$\text{or, } \mathbf{P}_2 = k_2\begin{pmatrix}1\\ -1\end{pmatrix} \tag{3.4.7.6}$$
8. Find $\mathbf{P}$.
Solution: From (3.4.2.4) and (3.4.4.2),
$$k_1 = \frac{1}{\left\|\begin{pmatrix}1\\ 1\end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.1}$$
$$k_2 = \frac{1}{\left\|\begin{pmatrix}1\\ -1\end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.2}$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{3.4.8.3}$$

9. Find the equation of the major axis for E.
Solution: The major axis for (3.4.2.1) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix}1\\0\end{pmatrix}. \tag{3.4.9.1}$$

Using the affine transformation in (3.3.3.3)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.9.2}$$

$$\implies \mathbf{x} - \mathbf{c} = \lambda_1 \mathbf{P}_1 \tag{3.4.9.3}$$

$$\text{or, } \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} \tag{3.4.9.4}$$

$$= -3 \tag{3.4.9.5}$$

since

$$P\begin{pmatrix}1\\0\end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.9.6}$$

which is the major axis of the ellipse E.

10. Find the minor axis of E.
11. Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{3.4.11.1}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

12. Summarize all the above computations through a Python script and plot the ellipses in (3.4.1.1) and (3.4.2.1).
Solution: The following script plots Fig. 3.4.12 using the principles of an affine transformation.

```
codes/2d/ellipse.py
```

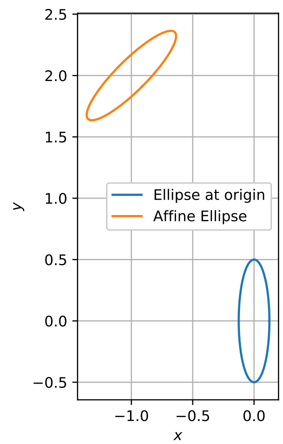

Fig. 3.4.12

3.5 Hyperbola

1. Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \tag{3.5.1.1}$$

where

$$V = \begin{pmatrix}4 & 0\\ 0 & -1\end{pmatrix} \tag{3.5.1.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix}0\\3\end{pmatrix}, \tag{3.5.1.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (3.1.6) as

$$\mathbf{P}^T V \mathbf{x} = 36 \tag{3.5.1.4}$$

$$\mathbf{Q}^T V \mathbf{x} = 36. \tag{3.5.1.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V \mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{3.5.1.6}$$

$$\mathbf{Q}^T V \mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{3.5.1.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix}0 & -3\end{pmatrix}\mathbf{x} = -36 \tag{3.5.1.8}$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = -12 \tag{3.5.1.9}$$

which is the equation of PQ.

2. In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0 \tag{3.5.2.1}$$

using (1.2.9.2) and (3.5.1.9), the equation of TD is

$$\begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{x} - \mathbf{T}) = 0 \tag{3.5.2.2}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.5.2.3}$$

3. Find D.

Solution: From (3.5.1.9) and (3.5.2.3),

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{3.5.3.1}$$

$$\implies \mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{3.5.3.2}$$

4. Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda\mathbf{m} \tag{3.5.4.1}$$

where

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.5.4.2}$$

5. Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda\mathbf{m})^T V (\mathbf{D} + \lambda\mathbf{m}) + F = 0 \tag{3.5.5.1}$$

can be expressed as

$$\lambda^2\mathbf{m}^T V\mathbf{m} + 2\lambda\mathbf{m}^T V\mathbf{D} + \mathbf{D}^T V\mathbf{D} + F = 0 \tag{3.5.5.2}$$

6. Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (3.5.4.1) and (3.5.1.1) (3.5.5.2) is obtained. Substituting from (3.5.4.2), (3.5.1.2) and (3.5.3.2)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 4 \tag{3.5.6.1}$$

$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = 0 \tag{3.5.6.2}$$

$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix}0 & -12\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = -144 \tag{3.5.6.3}$$

Substituting in (3.5.5.2)

$$4\lambda^2 - 144 = 36 \tag{3.5.6.4}$$

$$\implies \lambda = \pm 3\sqrt{5} \tag{3.5.6.5}$$

Substituting in (3.5.4.1),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix}\sqrt{5}\\-4\end{pmatrix} \tag{3.5.6.6}$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix}\sqrt{5}\\4\end{pmatrix} \tag{3.5.6.7}$$

7. Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{3.5.7.1}$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{3.5.7.2}$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \tag{3.5.7.3}$$

8. Repeat the previous exercise using determinants.
9. Summarize all the above computations through a Python script and plot the hyperbola.

3.6 Karush-Kuhn-Tucker (KKT) Conditions

1. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 2\end{pmatrix}\mathbf{x} \tag{3.6.1.1}$$

with constraints

$$g_1(\mathbf{x}) = \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} - 8 = 0 \tag{3.6.1.2}$$

$$g_2(\mathbf{x}) = 15 - \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} \geq 0 \tag{3.6.1.3}$$

Solution: Considering the Lagrangian

$$\nabla L(\mathbf{x}, \lambda, \mu) = 0 \tag{3.6.1.4}$$

resulting in the matrix equation

$$\Rightarrow \begin{pmatrix}8 & 0 & 3 & 2\\0 & 4 & 1 & 4\\3 & 1 & 0 & 0\\2 & 4 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\lambda\\\mu\end{pmatrix} = \begin{pmatrix}0\\0\\8\\15\end{pmatrix} \tag{3.6.1.5}$$

$$\Rightarrow \begin{pmatrix}x_1\\x_2\\\lambda\\\mu\end{pmatrix} = \begin{pmatrix}1.7\\2.9\\-3.12\\-2.12\end{pmatrix} \tag{3.6.1.6}$$

using the following python script. The (incorrect) graphical solution is available in Fig. 3.6.1

```
codes/optimization/2.12.py
```

Note that $\mu < 0$, contradicting the necessary condition in (2.5.8.4).

2. Obtain the correct solution to the previous problem by considering $\mu = 0$.
3. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.3.1}$$

with constraints

$$g_1(\mathbf{x}) = 0 \tag{3.6.3.2}$$

$$g_2(\mathbf{x}) \leq 0 \tag{3.6.3.3}$$

4. Based on whatever you have done so far, list the steps that you would use in general for

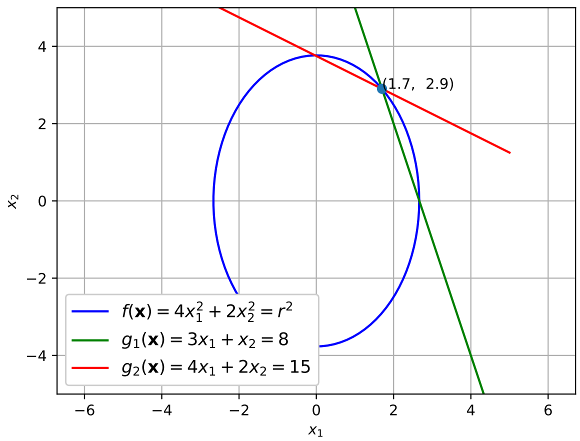

Fig. 3.6.1: Incorrect solution is at intersection of all curves $r = 5.33$

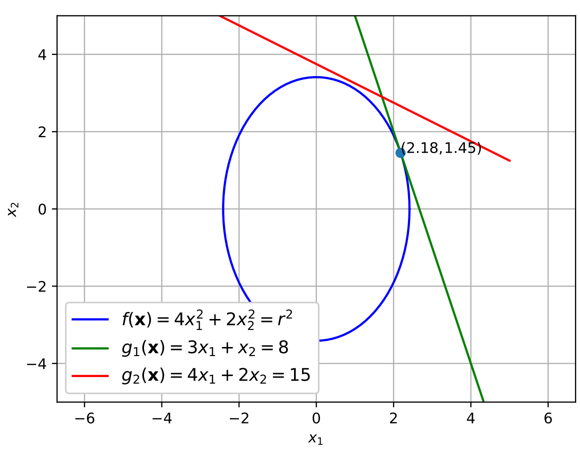

Fig. 3.6.2: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

solving a convex optimization problem like (3.6.1.1) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.
Solution: For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.4.1}$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, .., m \tag{3.6.4.2}$$

$$\text{subject to } g_i(\mathbf{x}) \leq 0, \forall i = 1, .., n \tag{3.6.4.3}$$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \tag{3.6.4.4}$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i g_i(\mathbf{x}), \tag{3.6.4.5}$$

using the KKT conditions

$$\implies \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \tag{3.6.4.6}$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, .., n \tag{3.6.4.7}$$

$$\text{and } \mu_i \geq 0, \forall i = 1, .., n \tag{3.6.4.8}$$

5. Maxmimize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \tag{3.6.5.1}$$

with the constraints

$$x_1^2 + x_2^2 \leq 5 \tag{3.6.5.2}$$

$$x_1 \geq 0, x_2 \geq 0 \tag{3.6.5.3}$$

3.7 Solved Problems

1. Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.
Solution: The equation of a conic is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{3.7.1.1}$$

For the standard parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.7.1.2}$$

$$\mathbf{u} = -2a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.1.3}$$

$$F = 0 \tag{3.7.1.4}$$

The focus

$$\mathbf{F} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.1.5}$$

The Latus rectum is the line passing through $\mathbf{F}$ with direction vector

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.7.1.6)$$

Thus, the equation of the Latus rectum is

$$\mathbf{x} = \mathbf{F} + \lambda\mathbf{m} \quad (3.7.1.7)$$

The intersection of the latus rectum and the parabola is obtained from (3.7.1.4) , (3.7.1.7) and (3.7.1.1) as

$$(\mathbf{F} + \lambda\mathbf{m})^T \mathbf{V} (\mathbf{F} + \lambda\mathbf{m}) + 2\mathbf{u}^T (\mathbf{F} + \lambda\mathbf{m}) = 0 \quad (3.7.1.8)$$

$$\implies \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\lambda^2 + 2(\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m}\lambda + (\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = 0 \quad (3.7.1.9)$$

From (3.7.1.2), (3.7.1.3), (3.7.1.5) and (3.7.1.6),

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = 1 \quad (3.7.1.10)$$

$$(\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m} = 0 \quad (3.7.1.11)$$

$$(\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = -4a^2 \quad (3.7.1.12)$$

Substituting from (3.7.1.10), (3.7.1.11) and (3.7.1.12) in (3.7.1.9),

$$\lambda^2 - 4a^2 = 0 \quad (3.7.1.13)$$

$$\implies \lambda_1 = 2a, \lambda_2 = -2a \quad (3.7.1.14)$$

Thus, from (3.7.1.6), (3.7.1.7) and (3.7.1.14), the length of the latus rectum is

$$(\lambda_1 - \lambda_2)\|\mathbf{m}\| = 4a \quad (3.7.1.15)$$

From the given information, the two parabolas P_1, P_2 have parameters

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \quad (3.7.1.16)$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a\begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \quad (3.7.1.17)$$

$$4a = 3 \quad (3.7.1.18)$$

Let L be the common tangent for P_1, P_2 with $\mathbf{c}, \mathbf{d}$ being the respective points of contact. The respectivel normal vectors are

$$\mathbf{n}_1 = \mathbf{V}_1\mathbf{c} + \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} \quad (3.7.1.19)$$

$$\mathbf{n}_2 = \mathbf{V}_2\mathbf{d} + \mathbf{u}_2 = d_1\begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \quad (3.7.1.20)$$

From the above equations, since both normals have the same direction vector,

$$\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \implies c_2d_1 = 4a^2 \quad (3.7.1.21)$$

2. Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}\mathbf{x} = 225 \quad (3.7.2.1)$$

upon the tangent to it at the point

$$\frac{1}{2}\begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \quad (3.7.2.2)$$

Solution: For the ellipse in (3.7.2.1),

$$V = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}, \mathbf{u} = 0, F = -1 \quad (3.7.2.3)$$

The equation of the desired tangent is

$$(\mathbf{V}\mathbf{P})^T \mathbf{x} = 1 \quad (3.7.2.4)$$

$$\implies \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\mathbf{x} = 2 \quad (3.7.2.5)$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (3.7.2.6)$$

The product of the perpendiculars is

$$\frac{\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} - 2\right|\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ -4 \end{pmatrix} - 2\right|}{\left\|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\right\|^2} = 9 \quad (3.7.2.7)$$

3. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.
Solution: If a and b be the semi-major and minor-axis respectively, the foci of the ellipse

are

$$\mathbf{F}_1 = ae\begin{pmatrix}1\\0\end{pmatrix}, \mathbf{F}_2 = -ae\begin{pmatrix}1\\0\end{pmatrix} \tag{3.7.3.1}$$

From the given information,

$$e = \frac{3}{5}, 2ae = 6 \tag{3.7.3.2}$$

$$\implies a = 5, b = a\sqrt{1-e^2} = 4 \tag{3.7.3.3}$$

Thus, the vertices of the ellipse are

$$\begin{pmatrix}a\\0\end{pmatrix}, \begin{pmatrix}-a\\0\end{pmatrix}, \begin{pmatrix}0\\b\end{pmatrix}, \begin{pmatrix}0\\-b\end{pmatrix} \tag{3.7.3.4}$$

and the area of the quadrilateral is

$$\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ a & -a & 0\\ 0 & 0 & b\end{vmatrix} + \frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ a & -a & 0\\ 0 & 0 & -b\end{vmatrix} = 2ab = 40 \tag{3.7.3.5}$$

4. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{3.7.4.1}$$

If

$$\mathbf{S} = \begin{pmatrix}5\\0\end{pmatrix} \tag{3.7.4.2}$$

is a focus and

$$\begin{pmatrix}5 & 0\end{pmatrix}\mathbf{x} = 9 \tag{3.7.4.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.
Solution: From (3.7.4.1),

$$(3e-1)(3e-5) = 0 \tag{3.7.4.4}$$

$$\implies e = \frac{5}{3}, \because e > 1 \tag{3.7.4.5}$$

for a hyperbola. Let $\mathbf{x}$ be a point on the hyperbola. From (3.7.4.3), its distance from the directrix is

$$\frac{\left|\begin{pmatrix}5 & 0\end{pmatrix}\mathbf{x} - 9\right|}{5} \tag{3.7.4.6}$$

and from the focus is

$$\left\|\mathbf{x} - \begin{pmatrix}5\\0\end{pmatrix}\right\| \tag{3.7.4.7}$$

From the definition of a hyperbola, the eccentricity is the ratio of these distances and (3.7.4.5) , (3.7.4.6) and (3.7.4.7),

$$\frac{5\left\|\mathbf{x} - \begin{pmatrix}5\\0\end{pmatrix}\right\|}{\left|\begin{pmatrix}5 & 0\end{pmatrix}\mathbf{x} - 9\right|} = \frac{5}{3} \tag{3.7.4.8}$$

$$\implies 9\left\{(x_1-5)^2 + x_2^2\right\} = (5x_1 - 9)^2 \tag{3.7.4.9}$$

$$\text{or, } \mathbf{x}^T\begin{pmatrix}16 & 0\\ 0 & -9\end{pmatrix}\mathbf{x} = 225 \tag{3.7.4.10}$$

which is the equation of the hyperbola. Thus,

$$a^2 = \frac{225}{16}, b^2 = \frac{225}{9} \tag{3.7.4.11}$$

$$\implies a^2 - b^2 = -\frac{175}{16} \tag{3.7.4.12}$$

5. A variable line drawn through the intersection of the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.7.5.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 12 \tag{3.7.5.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.
Solution: The intersection of the lines in (3.7.5.1) is obtained through the matrix equation

$$\begin{pmatrix}4 & 3\\ 3 & 4\end{pmatrix}\mathbf{x} = \begin{pmatrix}12\\12\end{pmatrix} \tag{3.7.5.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix}4 & 3 & 12\\ 3 & 4 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}4 & 3 & 12\\ 0 & 7 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}28 & 0 & 48\\ 0 & 7 & 12\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}7 & 0 & 12\\ 0 & 7 & 12\end{pmatrix} \tag{3.7.5.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix}12\\12\end{pmatrix} \tag{3.7.5.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{R} \tag{3.7.5.6}$$

$$\mathbf{B} = 2\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{R} \tag{3.7.5.7}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x}-\mathbf{C})=0 \tag{3.7.5.8}$$

Since $\mathbf{R}$ lies on AB,

$$\mathbf{n}^T(\mathbf{R}-\mathbf{C})=0 \tag{3.7.5.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A}-\mathbf{B})=0 \tag{3.7.5.10}$$

Substituting from (3.7.5.6) in (3.7.5.10),

$$\mathbf{n}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.11}$$

From (3.7.5.9) and (3.7.5.11),

$$(\mathbf{R}-\mathbf{C})=k\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{R} \tag{3.7.5.12}$$

for some constant k. Multiplying both sides of (3.7.5.12) by

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{3.7.5.13}$$

$$\begin{aligned}\mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(\mathbf{R}-\mathbf{C})&=k\mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{R}\\ &=k\mathbf{R}^T\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{R}=0\end{aligned} \tag{3.7.5.14}$$

$$\because \mathbf{R}^T\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.15}$$

which can be easily verified for any $\mathbf{R}$. from (3.7.5.14),

$$\begin{aligned}\mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(\mathbf{R}-\mathbf{C})&=0\\ \implies \mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{R}-\mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{C}&=0\\ \implies \mathbf{R}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{R}-\mathbf{C}^T\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathbf{R}&=0\end{aligned} \tag{3.7.5.16}$$

which is the desired locus.

3.8 JEE Exercises

1. Find the point of intersection of the tangents at the ends of the latusrectum of the parabola
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x}. \tag{3.8.1.1}$$
2. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $\mathbf{P}=\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}$. Its one directrix is the common tangent, nearer to the point $\mathbf{P}$, to the circle
$$\mathbf{x}^T\mathbf{x}=1 \tag{3.8.2.1}$$
and the hyperbola
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x}=1. \tag{3.8.2.2}$$
Find the equation of the ellipse.
3. The equation
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{1-r} & 0\\ 0 & -\frac{1}{1+r}\end{pmatrix}\mathbf{x}=1, r>1 \tag{3.8.3.1}$$
represents
   a) an ellipse
   b) a hyperbola
   c) a circle
   d) none of these
4. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points $\begin{pmatrix}x_1\\ y_1\end{pmatrix}$ and $\begin{pmatrix}x_2\\ y_2\end{pmatrix}$ in the region, the point $\begin{pmatrix}\frac{x_1+x_2}{2}\\ \frac{y_1+y_2}{2}\end{pmatrix}$ is also in the region. Find the inequality defining this region.
   a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\mathbf{x}\le 1$
   b) $\text{Max}\begin{pmatrix}|x|\\ |y|\end{pmatrix}\le 1$
   c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x}\le 1$
   d) $\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x}\le 0$
5. The equation
$$\mathbf{x}^T\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\mathbf{x}+\begin{pmatrix}-8 & -18\end{pmatrix}\mathbf{x}+35=k \tag{3.8.5.1}$$
represents

a) no locus if $k > 0$
b) an ellipse if$k < 0$
c) a point if k=0
d) a hyperbola if $k > 0$

6. Let E be the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.6.1}$$

and C be the circle

$$\mathbf{x}^T \mathbf{x} = 9. \tag{3.8.6.2}$$

let **P** and **Q** be the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Then

a) Q lies inside C but outside E.
b) Q lies outside both C and E.
c) P lies inside both C and E.
d) P lies inside C but outside E.

7. Consider a circle with its center lying on the focus of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2p & 0 \end{pmatrix} \mathbf{x} \tag{3.8.7.1}$$

such that it touches the directrix of the parabola. Then find the point of intersection.

8. Find the radius of the circle passing through the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1, \tag{3.8.8.1}$$

and having its centre at $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

9. Let $\mathbf{P} = \begin{pmatrix} a \sec\theta \\ b \tan\theta \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} a \sec\phi \\ b \tan\phi \end{pmatrix}$ where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{-1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.9.1}$$

. If $\begin{pmatrix} h \\ k \end{pmatrix}$ is the point of intersection of the normals at **P** and **Q**, then find k.

10. If

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.10.1}$$

is the chord of contact of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.10.2}$$

then find the equation of the corresponding pair of tangents.

11. The curve describes parametrically by

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = t^2 + t + 1 \tag{3.8.11.1}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = t^2 - t + 1 \tag{3.8.11.2}$$

represents

a) a pair of straight lines
b) an ellipse
c) a parabola
d) a hyperbola

12. If

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = k \tag{3.8.12.1}$$

is normal to

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 & 0 \end{pmatrix} \mathbf{x}, \tag{3.8.12.2}$$

then find k.

13. If the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{3.8.13.1}$$

is the directrix of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} k & 0 \end{pmatrix} \mathbf{x} + 8 = 0, \tag{3.8.13.2}$$

then find k.

14. Find the equation of the common tangent touching the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 6 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.8.14.1}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.14.2}$$

15. Find the equation of the directrix of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 4 \end{pmatrix} \mathbf{x} + 2 = 0. \tag{3.8.15.1}$$

16. If $a > 2b > 0$ then the positive value of m for which

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix} \mathbf{x} - b\sqrt{1 + m^2} \tag{3.8.16.1}$$

is the common tangent to

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = b^2 \quad (3.8.16.2)$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2a & 0 \end{pmatrix} \mathbf{x} = a^2 - b^2 \quad (3.8.16.3)$$

is

a) $\frac{2b}{\sqrt{a^2-4b^2}}$
b) $\frac{\sqrt{a^2-4b^2}}{2b}$
c) $\frac{2b}{a-2b}$
d) $\frac{b}{a-2b}$

17. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x} \quad (3.8.17.1)$$

is another parabola with directrix

a) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -a$
b) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{-a}{2}$
c) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0$
d) $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{a}{2}$

18. Find the equation of the common tangent to the curves

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.18.1)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} = -1 \quad (3.8.18.2)$$

19. Find the area of the quadrilateral formed by the tangents at the end points of latusrectum to the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.19.1)$$

20. The focal chord to

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.20.1)$$

is tangent to

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 12 & 0 \end{pmatrix} \mathbf{x} + 36 = 0 \quad (3.8.20.2)$$

then the possible values of the slope of the chord,are

a) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
b) $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$
c) $\begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}$
d) $\begin{pmatrix} 2 \\ -\frac{1}{2} \end{pmatrix}$

21. For hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\cos^2 \alpha} & 0 \\ 0 & -\frac{1}{\sin^2 \alpha} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.21.1)$$

which of the following remains constant with change in $'\alpha'$

a) abscissae of vertices
b) abscissae of foci
c) eccentricity
d) directrix

22. If tangents are drawn to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 2 \quad (3.8.22.1)$$

then the locus of the mid point of the intercept made by the tangents between the coordinate axes is

a) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$
b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$
d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

23. Find the angle between the tangents drawn from the points $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.23.1)$$

24. If the line

$$\begin{pmatrix} 2 & \sqrt{6} \end{pmatrix} \mathbf{x} = 2 \quad (3.8.24.1)$$

touches the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} = 4 \quad (3.8.24.2)$$

then find the point of contact.

25. The minimum area of the triangle is formed by the tangent to the

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.25.1)$$

the coordinate axes is

a) ab sq.units

b) $\frac{a^2+b^2}{2}$sq.units
c) $\frac{(a+b)^2}{2}$sq.units
d) $\frac{a^2+ab+b^2}{3}$sq.units

26. Tangent to the curve

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + 6 \tag{3.8.26.1}$$

at the points $\begin{pmatrix}1\\7\end{pmatrix}$touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}16 & 12\end{pmatrix}\mathbf{x} + c = 0 \tag{3.8.26.2}$$

at a point **Q**.Then the coordinates of **Q** are

a) $\begin{pmatrix}-6\\-11\end{pmatrix}$
b) $\begin{pmatrix}-9\\-13\end{pmatrix}$
c) $\begin{pmatrix}-10\\-15\end{pmatrix}$
d) $\begin{pmatrix}-6\\-7\end{pmatrix}$

27. The axis of a parabola is along the line

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} \tag{3.8.27.1}$$

and the distance of its vertex and focus from the origin are $\sqrt{2}$ and $2\sqrt{2}$ respectively.If vertex and focus both lies in the first quadrant,then find the equation of parabola.

28. A hyperbola, having the transverse axis of length $2\sin\theta$,is confocal with the ellipse

$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 12. \tag{3.8.28.1}$$

Then find its equation.

29. Let a and b be non zero real numbers,then the equation

$$(\mathbf{x}^T\begin{pmatrix}a & 0\\0 & b\end{pmatrix}\mathbf{x} + c)(\mathbf{x}^T\begin{pmatrix}1 & 0\\-5 & 6\end{pmatrix}\mathbf{x}) = 0 \tag{3.8.29.1}$$

represents

a) four straight lines, when c=0 and a,b are of the same sign
b) two straight lines and a circle ,when a=b,and c is of sign opposite to that of a
c) two straight lines and a hyperbola ,when a and b are of the same sign and c is of sign opposite to that of a
d) a circle and an ellipse , when a and b are of the same sign and c is of sign opposite to that of a

30. Consider a branch of the hyperbola

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -2\end{pmatrix}\mathbf{x} + \begin{pmatrix}-2\sqrt{2} & -4\sqrt{2}\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.8.30.1}$$

with the vertex at the point **A**.Let **B** be the one of the end points of its latusrectum.If **C** is the focus of the hyperbola nearer to the point **A**,find the area of the triangle ABC.

31. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 9 \tag{3.8.31.1}$$

meets its auxiliary circle at the point **M** then the area of the triangle with vertices at A, M and the origin O is

a) $\frac{31}{10}$
b) $\frac{29}{10}$
c) $\frac{21}{10}$
d) $\frac{27}{10}$

32. The normal at a point **P** on the ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 16 \tag{3.8.32.1}$$

meets the x-axis at **Q**. If **M** is the mid point of the line segment PQ, then the locus of **M** intersects the latusrectums of the given ellipse at the points

a) $\begin{pmatrix}\pm\frac{3\sqrt{5}}{2}\\\pm\frac{2}{7}\end{pmatrix}$
b) $\begin{pmatrix}\pm\frac{3\sqrt{5}}{2}\\\pm\sqrt{\frac{19}{4}}\end{pmatrix}$
c) $\begin{pmatrix}\pm2\sqrt{3}\\\pm\frac{1}{7}\end{pmatrix}$
d) $\begin{pmatrix}\pm2\sqrt{3}\\\pm\frac{4\sqrt{3}}{7}\end{pmatrix}$

33. The locus of the orthocentre of the triangle formed by the lines

$$\begin{pmatrix}(1+p) & -p\end{pmatrix}\mathbf{x} + p(1+p) = 0 \tag{3.8.33.1}$$

$$\begin{pmatrix}(1+q) & -q\end{pmatrix}\mathbf{x} + q(1+q) = 0 \tag{3.8.33.2}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.8.33.3}$$

, where $p \neq q$ is

a) a hyperbola
b) a parabola
c) an ellipse
d) a straight line

34. Let $\mathbf{P} = \begin{pmatrix}6\\3\end{pmatrix}$ be a points on the hyperbola

$$\mathbf{x}^T \begin{pmatrix}\frac{1}{a^2} & 0\\ 0 & -\frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1. \tag{3.8.34.1}$$

If the normal at the points $\mathbf{P}$ intersects the x-axis at $\begin{pmatrix}9\\0\end{pmatrix}$, then find the eccentricity of the hyperbola.
a) $\sqrt{\frac{5}{2}}$
b) $\sqrt{\frac{3}{2}}$
c) $\sqrt{2}$
d) $\sqrt{3}$

35. Let$\begin{pmatrix}x\\y\end{pmatrix}$be any point on the parabola

$$\mathbf{x}^T \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} \tag{3.8.35.1}$$

. Let $\mathbf{P}$ be the points that divides the lines segment from $\begin{pmatrix}0\\0\end{pmatrix}$ to $\begin{pmatrix}x\\y\end{pmatrix}$ in the ratio 1:3. Then the locus of P is
a) $\mathbf{x}^T \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}$
b) $\mathbf{x}^T \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x}$
c) $\mathbf{x}^T \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}$
d) $\mathbf{x}^T \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x}$

36. The ellipse $\mathbf{E_1}$:

$$\mathbf{x}^T \begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\mathbf{x} = 1. \tag{3.8.36.1}$$

is inscribed in a rectangle R whose sides are parallel to the coordinate axes.Another ellipse $\mathbf{E_2}$ passing through the points $\begin{pmatrix}0\\4\end{pmatrix}$circumscribes the rectangle R. Find the eccentricity of the ellipse $\mathbf{E_2}$.

37. The common tangents to the circle

$$\mathbf{x}^T\mathbf{x} = 2 \tag{3.8.37.1}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} \tag{3.8.37.2}$$

touch the circle at the points $\mathbf{P}, \mathbf{Q}$ and the parabola at the points $\mathbf{R}, \mathbf{S}$. Then find the area of the quadrilateral PQRS.

38. The number of values of c such that the straight line

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} + c \tag{3.8.38.1}$$

touches the curve

$$\mathbf{x}^T \begin{pmatrix}\frac{1}{4} & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.8.38.2}$$

is
a) 0
b) 1
c) 2
d) infinite.

39. If $\mathbf{P} = \begin{pmatrix}x\\y\end{pmatrix}, \mathbf{F_1} = \begin{pmatrix}3\\0\end{pmatrix}, \mathbf{F_2} = \begin{pmatrix}-3\\0\end{pmatrix}$and

$$\mathbf{x}^T \begin{pmatrix}16 & 0\\ 0 & 25\end{pmatrix}\mathbf{x} = 400, \tag{3.8.39.1}$$

then $\mathbf{PF_1} + \mathbf{PF_2}$equals
a) 8
b) 6
c) 10
d) 12

40. On the ellipse

$$\mathbf{x}^T \begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}\mathbf{x} = 1, \tag{3.8.40.1}$$

the points at which the tangents are parallel to the line

$$\begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 9\end{pmatrix}\mathbf{x} \tag{3.8.40.2}$$

are
a) $\begin{pmatrix}\frac{2}{5}\\ \frac{1}{5}\end{pmatrix}$
b) $\begin{pmatrix}-\frac{2}{5}\\ \frac{1}{5}\end{pmatrix}$
c) $\begin{pmatrix}-\frac{2}{5}\\ -\frac{1}{5}\end{pmatrix}$
d) $\begin{pmatrix}\frac{2}{5}\\ -\frac{1}{5})\end{pmatrix}$

41. The equation of the common tangents to the

parabola

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} \tag{3.8.41.1}$$

and

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\4 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.8.41.2}$$

is/are

a) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} + 4{=}0$
b) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} - 4 = 0$
d) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}30 & 0\end{pmatrix}\mathbf{x} + 50 = 0$

42. Let the hyperbola passes through the focus of the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1 \tag{3.8.42.1}$$

The transverse and conjugate axes of this hyperbola coincides with the major and minor axis of the given ellipse also the product of eccentricities of given ellipse and hyperbola is 1, then

a) the equation of the hyperbola is $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
b) the equation of the hyperbola is $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{25}\end{pmatrix}\mathbf{x} = 1$
c) focus of hyperbola is $\begin{pmatrix}5\\0\end{pmatrix}$
d) vertex of hyperbola is $\begin{pmatrix}5\sqrt{3}\\0\end{pmatrix}$

43. Let $\mathbf{P} = \begin{pmatrix}x_1\\y_1\end{pmatrix} and \mathbf{Q} = \begin{pmatrix}x_2\\y_2\end{pmatrix}, y_1 < 0, y_2 < 0$,be the end point of the latus rectum of the ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.8.43.1}$$

.The equation of parabola with latus rectum PQ are

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\sqrt{3}\end{pmatrix}\mathbf{x} = 3 + \sqrt{3}$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -2\sqrt{3}\end{pmatrix}\mathbf{x} = 3 + \sqrt{3}$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\sqrt{3}\end{pmatrix}\mathbf{x} = 3 - \sqrt{3}$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -2\sqrt{3}\end{pmatrix}\mathbf{x} = 3 - \sqrt{3}$

44. In a triangle ABC with fixed base BC,the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the triangle A,B and C,respectively,then

a) b+c=4a
b) b+c=2a
c) locus of point**A** is an ellipse
d) locus of point **A** is a pair of straight lines

45. The tangent PT and the normal PN to the parabola

$$\mathbf{x}^T\begin{pmatrix}-4a & 0\\0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.8.45.1}$$

at a point **P** on it meet its axis at points **T** and **N**, respectively. The locus of the centroid of the triangle PTN is a parabola whose

a) vertex is $\begin{pmatrix}\frac{2a}{3}\\0\end{pmatrix}$
b) directrix is $\begin{pmatrix}1 & 0\end{pmatrix}{=}0$
c) latus rectum is $\frac{2a}{3}$
d) focus is $\begin{pmatrix}a\\0\end{pmatrix}$

46. An ellipse intersects the hyperbola

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & -2\end{pmatrix}\mathbf{x} = 1 \tag{3.8.46.1}$$

orthogonally.The eccentricity of the ellipse is reciprocal of that of the hyperbola.If the axes of the ellipse are along the coordinate axes,then

a) equation of ellipse is$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x} = 2$
b) the foci of ellipse are $\begin{pmatrix}\pm 1\\0\end{pmatrix}$
c) equation of ellipse is $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x} = 4$
d) the foci of ellipse are $\begin{pmatrix}\pm\sqrt{2}\\0\end{pmatrix}$

47. Let**A** and **B** two distinct points on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x}. \tag{3.8.47.1}$$

If the axis of a parabola touches a circle of radius r, having AB as its diameter, then the slop of the line joining A and B can be

a) $-\frac{1}{r}$
b) $\frac{1}{r}$

c) $\frac{2}{r}$
d) $-\frac{2}{r}$

48. Let the eccentricity of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.48.1}$$

. If the hyperbola passes to that of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{3.8.48.2}$$

. If the hyperbola passing through a focus of the ellipse, then

a) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{x} = 1$
b) the focus of the hyperbola is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
d) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} = 3$

49. Let L be a normal to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.49.1}$$

. If **L** passes though the point$\begin{pmatrix} 9 \\ 6 \end{pmatrix}$, then **L** is given by

a) $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} + 3 = 0$
b) $\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 33 = 0$
c) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 15 = 0$
d) $\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + 12 = 0$

50. Tangents are drawn to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1, \tag{3.8.50.1}$$

parallel to the straight line

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.50.2}$$

. The point of contact of the tangents on the hyperbola are

a) $\begin{pmatrix} \frac{9}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
b) $\begin{pmatrix} \frac{9}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
c) $\begin{pmatrix} 3\sqrt{3} \\ -2\sqrt{2} \end{pmatrix}$
d) $\begin{pmatrix} -3\sqrt{3} \\ 2\sqrt{2} \end{pmatrix}$

51. Let **P** and **Q** be distinct points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.51.1}$$

such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are)the coordinates of **P**?

a) $\begin{pmatrix} 4 \\ 2\sqrt{2} \end{pmatrix}$
b) $\begin{pmatrix} 9 \\ 3\sqrt{2} \end{pmatrix}$
c) $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
d) $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

52. Let $\mathbf{E_1}$and $\mathbf{E_2}$be two ellipses whose centers are at the origin. The major axes of $\mathbf{E_1}$ and $\mathbf{E_2}$ lie along the x-axis and the y-axis, respectively. Let S be the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.52.1}$$

. The straight line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.8.52.2}$$

touches the curves S. E_1 and E_2 at **P**,**Q** and **R** respectively. Suppose that PQ=PR=$\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 andE_2, respectively, Then the correct expression(s) is (are)

a) $e_1^2 + e_2^2 = \frac{43}{40}$
b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
c) $\mid e_1^2 - e_2^2 \mid = \frac{5}{8}$
d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

53. Consider a hyperbola H:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.53.1}$$

and a circle S with center $\mathbf{N} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$. Suppose that H and S touches each other at a point $\mathbf{P} =$

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at **P** intersects the x-axis at point **P**. If$\begin{pmatrix} 1 \\ m \end{pmatrix}$ is the centroid of the triangle PMN, then the correct expression is(are)

a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2} for x_1 > 1$
b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{x_1^2-1})}$for $x_1 > 1$
c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}forx_1 > 1$
d) $\frac{dm}{dx_1} = \frac{1}{3}fory_1 > 0$

54. The circle C_1:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.8.54.1}$$

, with centre at O, intersects the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} \tag{3.8.54.2}$$

and centres Q_2Q_3, respectively. If Q_2Q_3 lie on the y-axis, then

a) $Q_2Q_3 = 12$
b) $R_2\ R_3$=4 $\sqrt{6}$
c) area of the triangle O$R_2R_3$is 6 $\sqrt{2}$
d) area of the triangle P$Q_2Q_3$is 4 $\sqrt{2}$

55. Let **P** be the point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.55.1}$$

which is at the shortest distance from the center S of the circle $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -16 \end{pmatrix} \mathbf{x} + 64 = 0$. Let **Q** be the point on the circle dividing the line segment SP internally. Then

a) SP=2 $\sqrt{5}$
b) SQ:QP=($\sqrt{5}$ + 1) : 2
c) the x-intercept of the normal to the parabola at **P** is 6.
d) the slop of the tangent to the circle at **Q** is $\frac{1}{2}$.

56. If $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 1 = 0$ is a tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1. \tag{3.8.56.1}$$

then which of the can not be sides of a right angled triangle ?

a) a,4,1
b) a,4,2
c) 2a,8,1
d) 2a,4,1

57. If a chord, which is not a tangent, of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.57.1}$$

has the equation $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = p$, and midpoint $\begin{pmatrix} h \\ k \end{pmatrix}$, then which of the following are possible values of p,h and k?

a) p=-2,h=2,k=-4
b) p=-1,h=1,k=-3
c) p=2,h=3,k=-4
d) p=5,h=4,k=-3

58. Consider two straight lines, each of which is tangents to both the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \tag{3.8.58.1}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.58.2}$$

. Let these lines intersect at a point **Q**. Consider the ellipse whose centre is at the origin $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and whose semi major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is(are) TRUE?

a) For the ellipse,the eccentricity is $\frac{1}{\sqrt{2}}$and the length of the latus rectum is 1
b) For the ellipse,the eccentricity is $\frac{1}{\sqrt{2}}$and the length of the latus rectum is $\frac{1}{2}$
c) the area of the region bounded by the ellipse between the lines $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{\sqrt{2}}$and $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1$ is $\frac{1}{4\sqrt{2}}(\Pi - 2)$
d) the area of the region bounded by the ellipse between the lines $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{\sqrt{2}}$and $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1$ is $\frac{1}{16}(\Pi - 2)$

Subjective Problems

59. Suppose that the normals drawn at the different

points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.59.1}$$

pass through the point $\begin{pmatrix} h \\ k \end{pmatrix}$. Show that $h > 2$.

60. **A** is a point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.60.1}$$

The normal A cuts the parabola again at the point B. if AB subtends a right angle at the vertex of the parabola. find the slop of AB.

61. Three normals are drawn from the point $\begin{pmatrix} c \\ 0 \end{pmatrix}$ to the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.61.1}$$

. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other.

62. Through the vertex O of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x}, \tag{3.8.62.1}$$

chords OQ and OP are drawn at right angles to one other. Show that for all positions of **P**, PQ cuts the axis of the parabola at a fixed point. also find the locus of the middle point of PQ.

63. Show that the locus of point that divides a chord of slop 2 of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.63.1}$$

internally in the ratio 1:2 is a parabola. Find the vertex of this parabola.

64. Let $'d'$ be the perpendicular distance from the centre of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.64.1}$$

to the tangent drawn at a point **P** on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2(1 - \frac{b^2}{d^2})$.

65. Points **A**,**B** and **C** lie on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.65.1}$$

The tangents to the parabola at A,B and C, taken in pairs, intersects at points **P**, **Q** and **R**. Determine the ratio of the areas of the triangles ABC and PQR.

66. From a point **A** common tangents are drawn to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{a^2}{2} \tag{3.8.66.1}$$

and parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.66.2}$$

Find the area of the quadrilateral formed by the common tangents, the chord of the contact of the circle and the chord of the contact of the parabola

67. A tangent to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{3.8.67.1}$$

meets the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{3.8.67.2}$$

at point **P** and **Q**. Prove that the tangents at point **P** and **Q** of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{3.8.67.3}$$

are at right angles.

68. The angle between a pair of tangents drawn from a point **P** to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.68.1}$$

is 45°. Show that the locus of the point **P** is a hyperbola.

69. Consider the family of circles

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = r^2 \tag{3.8.69.1}$$

, $2 < r < 5$. If in the first quadrant, the common tangent to the circle of this family and the ellipse

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{x} = 100 \tag{3.8.69.2}$$

meet the coordinate axes at A and B, then find the equation of the locus of the midpoint of AB.

70. Find co-ordinates of all the points **P** on the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1, \quad (3.8.70.1)$$

for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P.

71. Let ABC be an equilateral triangle inscribed in the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2. \quad (3.8.71.1)$$

Suppose perpendiculars from A,B,C to the major axis of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1, (a > b) \quad (3.8.71.2)$$

meets the ellipse respectively, at **P**, **Q**, **R**. So, that P,Q,R lie on the same side of the major axis as ABC respectively. Prove that the normals to the ellipse drawn at the points **P**,**Q** and **R** are concurrent.

72. Let C_1 and C_2 be respectively, the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} - 1 \quad (3.8.72.1)$$

and

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 1. \quad (3.8.72.2)$$

Let **P** be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflection of **P** and **Q**, respectively. with respect to the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}. \quad (3.8.72.3)$$

Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min \begin{pmatrix} PP_1 \\ QQ_1 \end{pmatrix}$. Hence or otherwise determine points P_0 and Q_0 on parabolas C_1 and C_2 respectively such that $\left(P_0Q_0 \leq PQ\right)$ for all pairs points $\begin{pmatrix} P \\ Q \end{pmatrix}$ with **P** on C_1 and **Q** on C_2.

73. Let**P** be a point on the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1, \quad (3.8.73.1)$$

$0 < b < a$. Let the line parallel to y-axis passing through **P** meet the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2 \quad (3.8.73.2)$$

at the point **Q** such that **P** and **Q** are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point **R** on PQ such that $PR : RQ = r : s$ as **P** varies over ellipse.

74. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding dirctrix.

75. Normals are drawn from the point **P** with slopes m_1, m_2, m_3 to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.75.1)$$

. If locus of P with m_1,m_2=α. is a part of the parabola it self then find α.

76. Tangents is drawn to parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -2 \end{pmatrix} \mathbf{x} + 5 = 0 \quad (3.8.76.1)$$

at a point **Q**. A point **R** is such that it divides QP externally in the ratio $\frac{1}{2} : 2$. Find the locus of point **R**.

77. Tangents are drawn from any point on the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.77.1)$$

to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 9. \quad (3.8.77.2)$$

Find the locus of mid-point of the chord of contact.

78. Find the equation of the common tangents in the 1st quadrant to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 16 \quad (3.8.78.1)$$

and the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.78.2)$$

Also find the length of the intercept of the tangent between the coordinate axes.

Comprehension Based Questions

PASSAGE I

Consider the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.78.3}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.78.4}$$

They intersects at **P** and **Q** in the first and fourth quadrants, respectively. Tangents to the circle at **P** and **Q** intersects the x-axis at **R** and tangents to the parabola at **P** and **Q** intersects the x-axis at **S**.

79. The ratio of the areas of the triangles PQS and PQR is
   a) $1 : \sqrt{2}$
   b) $1 : 2$
   c) $1 : 4$
   d) $1 : 8$
80. The radius of the circumcircle of the triangle PRS is
   a) 5
   b) $3\sqrt{3}$
   c) $3\sqrt{2}$
   d) $2\sqrt{3}$
81. The radius of the incircle of the triangle PQR is
   a) 4
   b) 3
   c) $\frac{8}{3}$
   d) 2

PASSAGE 2 The circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.8.81.1}$$

and hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.81.2}$$

intersect at the points **A** and **B**.

82. Equation of a common tangent with positive slop to the circle as well as to the hyperbola is
   a) $\begin{pmatrix} 2 & -\sqrt{5} \end{pmatrix} \mathbf{x} - 20 = 0$
   b) $\begin{pmatrix} 2 & -\sqrt{5} \end{pmatrix} \mathbf{x} + 4 = 0$
   c) $\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 8 = 0$
   d) $\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} + 4 = 0$
83. Equation of the circle with AB as its diameter is
   a) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -12 & 0 \end{pmatrix} \mathbf{x} + 24 = 0$
   b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 12 & 0 \end{pmatrix} \mathbf{x} + 24 = 0$
   c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 24 & 0 \end{pmatrix} \mathbf{x} - 12 = 0$
   d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -24 & 0 \end{pmatrix} \mathbf{x} - 12 = 0$

PASSAGE 3 Tangents are drawn from the point $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ to the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.83.1}$$

touches the ellipse at points **A** and **B**.

84. The coordinates of A and B are
   a) $\begin{pmatrix} 3 \\ 0 \end{pmatrix} and \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
   b) $\begin{pmatrix} -\frac{8}{5} \\ \frac{2\sqrt{161}}{15} \end{pmatrix} \text{and} \begin{pmatrix} -\frac{9}{5} \\ \frac{8}{5} \end{pmatrix}$
   c) $\begin{pmatrix} -\frac{8}{5} \\ \frac{2\sqrt{161}}{15} \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
   d) $\begin{pmatrix} 3 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} -\frac{9}{5} \\ \frac{8}{5} \end{pmatrix}$
85. The orthocenter of the triangle PAB is
   a) $\begin{pmatrix} 5 \\ \frac{8}{7} \end{pmatrix}$
   b) $\begin{pmatrix} \frac{7}{5} \\ \frac{25}{8} \end{pmatrix}$
   c) $\begin{pmatrix} \frac{11}{5} \\ \frac{8}{5} \end{pmatrix}$
   d) $\begin{pmatrix} \frac{8}{25} \\ \frac{7}{5} \end{pmatrix}$
86. The equation of the locus of a point whose distances from the point **P** and the line AB are equal, is
   a) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ -6 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -54 & -62 \end{pmatrix} \mathbf{x} + 241 = 0$
   b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 6 & 9 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -54 & 62 \end{pmatrix} \mathbf{x} - 241 = 0$
   c) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ -6 & 9 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -54 & -62 \end{pmatrix} \mathbf{x} - 241 = 0$
   d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 27 & 31 \end{pmatrix} \mathbf{x} - 120 = 0$

PASSAGE 4
Let PQ be the focal chord of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.86.1}$$

The tangents to the parabola at **P** and **Q** meet at a point lying on the line $\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} + a, a > 0$.

87. Length of chord PQ is
    a) 7a
    b) 5a
    c) 2a
    d) 3a
88. If the chord PQ subtends an angle θ at the vertex of

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.88.1}$$

, then $\tan\theta$=
    a) $\frac{2}{3}\sqrt{7}$
    b) $-\frac{2}{3}\sqrt{7}$
    c) $\frac{2}{3}\sqrt{5}$
    d) $-\frac{2}{3}\sqrt{5}$

PASSAGE 5 Let a,r,s,t be the non zero real numbers. Let $\mathbf{P} = \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$,$\mathbf{Q}$, $\mathbf{R} = \begin{pmatrix} ar^2 \\ 2ar \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} as^2 \\ 2as \end{pmatrix}$ be distinct points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.88.2}$$

Suppose that PQ is the focal chord and lines QR and PK are parallel, where $\mathbf{K} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}$

89. The value of r is
    a) $-\frac{1}{t}$
    b) $\frac{t^2+1}{t}$
    c) $\frac{1}{t}$
    d) $\frac{t^2-1}{t}$
90. If st=1, then the tangent at **P** and the normal at **S** to the parabola meet at a point whose ordinate is
    a) $\frac{(t^2+1)^2}{2t^3}$
    b) $\frac{a(t^2+1)^2}{2t^3}$
    c) $\frac{a(t^2+1)^2}{t^3}$
    d) $\frac{a(t^2+2)^2}{t^3}$

PASSAGE 6

Let $F_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $F_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{8} \end{pmatrix} \mathbf{x} = 1. \tag{3.8.90.1}$$

Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point **M** in the first quadrant and at point **N** in the fourth quadrant.

91. The orthocentre of the triangle F_1MN is
    a) $\begin{pmatrix} -\frac{9}{10} \\ 0 \end{pmatrix}$
    b) $\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$
    c) $\begin{pmatrix} \frac{9}{10} \\ 0 \end{pmatrix}$
    d) $\begin{pmatrix} \frac{2}{3} \\ \sqrt{6} \end{pmatrix}$
92. If the tangents of the ellipse at **M** and **N** meet at **R** and the normals to the parabola at **M** meets the x-axis at **Q**, then the ratio of the triangle MQR to area of the quadrilateral MF_1NF_2 is
    a) 3:4
    b) 4:5
    c) 5:8
    d) 2:3

Assertion and Reason Type Questions
STATEMENT-I: The curve $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}$=

$$\mathbf{x}^T \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 1 \tag{3.8.92.1}$$

. because
STATEMENT-2:A parabola is symmetric about its axis.
    a) Statement-1 is True,Statement-2 is True;Statement-2 is a correct explanation for Statement-1
    b) Statement-1 is True,Statement-2 is True;Statement-2 is NOT correct explanation for Statement-1
    c) Statement-1 is True,Statement-2 is False
    d) Statement-1 is False,Statement-2 is True.

I Integer Value Correction Type

93. The line $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 1$ is tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.93.1}$$

If this line passes through the point of intersection of the nearest directrix and the x-axis,then the eccentricity of the hyperbola is

94. Consider the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.94.1}$$

. Let $\Delta_1$be the area of the triangle formed by the end points of its latus rectum and the point $\mathbf{P} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix}$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangents at **P** and at the end of the points of the latus rectum.Then $\frac{\Delta_1}{\Delta_2}$ is

95. Let S be the focus of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.95.1}$$

and let PQ be the common chord of the cicle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{3.8.95.2}$$

and the given parabola. The area of the triangle PQS is

96. A vertical line passing through the point $\begin{pmatrix} h \\ 0 \end{pmatrix}$ intersects the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.96.1}$$

. at the point **P** and **Q**. Let the tangents to the ellipse at **P** and **Q** meet at the point **R**. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max \frac{1}{2} < h < 1\Delta(h)$ and $\Delta_2 = \min\frac{1}{2} < h < 1\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$

a) g(x) is continuous but not differentiable at a
b) g(x) is differentiable on R
c) g(x) is continuous but not differentiable at b
d) g(x) is continuous but not differentiable at either (a)or (b) but not both.

97. If the normals of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.97.1}$$

drawn at the end points of its latus rectum are tangents to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6 & 4 \end{pmatrix} \mathbf{x} - 5 = r^2, \tag{3.8.97.2}$$

then the value of r^2 is

98. Let the curve C be the mirror image of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.98.1}$$

with respect to the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 4 = 0$. If **A** and **B** are the points of intersecting of **C** with the line $\begin{pmatrix} 0 & 1 \end{pmatrix} = -5$, then the distance between A and B is

99. Suppose that the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.99.1}$$

are $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} f_2 \\ 0 \end{pmatrix}$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and with foci at $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2f_2 \\ 0 \end{pmatrix}$ respectively. Let T_1 be a tangent to P_1 which passes through $\begin{pmatrix} 2f_2 \\ 0 \end{pmatrix}$ and T_2 be a tangent to P_2 which passes through $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. If m_1 is the slope of the T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

Section-B

JEE Main/AIEEE

100. Two common tangents to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 2a^2 \tag{3.8.100.1}$$

and parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.100.2}$$

are

a) $\begin{pmatrix} 1 & 0 \end{pmatrix} = \pm(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + 2a)$
b) $\begin{pmatrix} 0 & 1 \end{pmatrix} = \pm(\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2a)$
c) $\begin{pmatrix} 1 & 0 \end{pmatrix} = \pm(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + a)$
d) $\begin{pmatrix} 0 & 1 \end{pmatrix} = \pm(\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + a)$

101. The normals at the point $\begin{pmatrix} bt_1^2 \\ 2bt_1 \end{pmatrix}$ on a parabola meets the parabola again in the point $\begin{pmatrix} bt_2^2 \\ 2bt_2 \end{pmatrix}$, then

a) $t_2 = t_1 + \frac{2}{t_1}$
b) $t_2 = -t_1 - \frac{2}{t_1}$

c) $t_2 = -t_1 + \frac{2}{t_1}$
d) $t_2 = t_1 - \frac{2}{t_1}$

102. The foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.102.1}$$

and the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{144} & 0 \\ 0 & -\frac{1}{81} \end{pmatrix} \mathbf{x} = \frac{1}{25} \tag{3.8.102.2}$$

coincide.Then the value of b^2 is
a) 9
b) 1
c) 5
d) 7

103. If $a \neq 0$ and the line

$$\mathbf{x}^T \begin{pmatrix} 2b & 0 \\ 0 & 3c \end{pmatrix} \mathbf{x} + 4d = 0 \tag{3.8.103.1}$$

passes through the point of intersection of the parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.103.2}$$

and

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 4a \end{pmatrix} \mathbf{x} \tag{3.8.103.3}$$

, then
a) $d^2 + (3b - 2c)^2 = 0$
b) $d^2 + (3b + 2c)^2 = 0$
c) $d^2 + (2b - 3c)^2 = 0$
d) $d^2 + (2b + 3c)^2 = 0$

104. The eccentricity of an ellipse, with its centre at the origin, is $\frac{1}{2}$.If one of the directrices is $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4$, then the equation of the ellipse is:
a) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1$
b) $\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 12$
c) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 12$
d) $\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 1$

105. Let $\mathbf{P}$ be the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{Q}$ a point on the locus

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.105.1}$$

the locus of mid point of PQ is
a) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$
b) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$
c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} + 2 = 0$
d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$

106. The locus of a point $\mathbf{P} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ moving under the condition that the line $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \alpha & 0 \end{pmatrix} \mathbf{x} + \beta$ is a tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.106.1}$$

is
a) an ellipse
b) a circle
c) a parabola
d) a hyperbola

107. An ellipse has OB as semi minor axis, F and F' it's foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is
a) $\frac{1}{\sqrt{2}}$
b) $\frac{1}{2}$
c) $\frac{1}{4}$
d) $\frac{1}{\sqrt{3}}$

108. The locus of the vertices of the family of parabolas

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x}^T \begin{pmatrix} \frac{a^3}{3} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{a^2}{2} & 0 \end{pmatrix} \mathbf{x} - 2a \tag{3.8.108.1}$$

is
a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{105}{64}$
b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{3}{4}$
c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{35}{16}$
d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{64}{105}$

109. In an ellipse, the distance between its foci is 6

and minor axis is 8. Then its eccentricity is

a) $\frac{3}{5}$
b) $\frac{1}{2}$
c) $\frac{4}{5}$
d) $\frac{1}{\sqrt{5}}$

110. Angle between the tangents to the curve $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-5 & 0\end{pmatrix}\mathbf{x} + 6$ at the points $\begin{pmatrix}2\\0\end{pmatrix}$ and $\begin{pmatrix}3\\0\end{pmatrix}$ is

a) π
b) $\frac{\pi}{2}$
c) $\frac{\pi}{6}$
d) $\frac{\pi}{4}$

111. For the hyperbola

$$\mathbf{x}^T \begin{pmatrix}\frac{1}{\cos^2\alpha} & 0\\ 0 & -\frac{1}{\sin^2\alpha}\end{pmatrix}\mathbf{x} = 1 \tag{3.8.111.1}$$

, which of the following remains constant when α varies=?

a) abscissae of vertices
b) abscissae of foci
c) eccentricity
d) directrix.

112. The equation of a tangent to the parabola

$$\mathbf{x}^T \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} \tag{3.8.112.1}$$

is

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 2. \tag{3.8.112.2}$$

The point on this line from which the other tangents to the parabola is perpendicular to the given tangent is

a) $\begin{pmatrix}2\\4\end{pmatrix}$
b) $\begin{pmatrix}-2\\0\end{pmatrix}$
c) $\begin{pmatrix}-1\\-1\end{pmatrix}$
d) $\begin{pmatrix}0\\2\end{pmatrix}$

113. The normal to a curve at $\mathbf{P} = \begin{pmatrix}x\\y\end{pmatrix}$ meets the x-axis at G. If the distance G from the origin is twice the abscissa of **P**, then the curve is a

a) circle
b) hyperbola
c) ellipse
d) parabola.

114. A focus of an ellipse is at the origin. The directrix is the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \tag{3.8.114.1}$$

and the eccentricity is $\frac{1}{2}$. Then the length of the semi major axis is

a) $\frac{8}{3}$
b) $\frac{2}{3}$
c) $\frac{4}{3}$
d) $\frac{5}{3}$

115. A parabola has the origin as its focus and the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 2 \tag{3.8.115.1}$$

as directrix. Then the vertex of the parabola is at

a) $\begin{pmatrix}0\\2\end{pmatrix}$
b) $\begin{pmatrix}1\\0\end{pmatrix}$
c) $\begin{pmatrix}0\\1\end{pmatrix}$
d) $\begin{pmatrix}2\\0\end{pmatrix}$

116. The ellipse

$$\mathbf{x}^T \begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.8.116.1}$$

is inscribed in a rectangular aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point $\begin{pmatrix}4\\0\end{pmatrix}$. Then the equation of the ellipse is :

a) $\mathbf{x}^T \begin{pmatrix}1 & 0\\0 & 12\end{pmatrix}\mathbf{x} = 16$
b) $\mathbf{x}^T \begin{pmatrix}4 & 0\\0 & 48\end{pmatrix}\mathbf{x} = 48$
c) $\mathbf{x}^T \begin{pmatrix}4 & 0\\0 & 64\end{pmatrix}\mathbf{x} = 48$
d) $\mathbf{x}^T \begin{pmatrix}1 & 0\\0 & 16\end{pmatrix}\mathbf{x} = 16$

117. If two tangents drawn from a point **P** to the parabola

$$\mathbf{x}^T \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} \tag{3.8.117.1}$$

are at right angles, then the locus of P is

a) $\begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} + 1 = 0$
b) $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -1$
c) $\begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} - 1 = 0$
d) $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1$

118. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and has eccentricity $\sqrt{\frac{2}{5}}$ is

a) $\mathbf{x}^T\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x} - 48 = 0$
b) $\mathbf{x}^T\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\mathbf{x} - 15 = 0$
c) $\mathbf{x}^T\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x} - 32 = 0$
d) $\mathbf{x}^T\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\mathbf{x} - 32 = 0$

119. **Statement-1:**An equation of a common tangent to the parabola

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 16\sqrt{3} & 0 \end{pmatrix}\mathbf{x} \qquad (3.8.119.1)$$

and the ellipse

$$\mathbf{x}^T\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 4 \qquad (3.8.119.2)$$

is

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} + 2\sqrt{3} \qquad (3.8.119.3)$$

Statement-2: If the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix}\mathbf{x} + \frac{4\sqrt{3}}{m}(m \neq 0) \qquad (3.8.119.4)$$

, is a common tangent to the parabola

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 16\sqrt{3} & 0 \end{pmatrix}\mathbf{x} \qquad (3.8.119.5)$$

and the ellipse

$$\mathbf{x}^T\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 4 \qquad (3.8.119.6)$$

, then m satisfies $m^4 + 2m^2 = 24$

a) Statement-1 is false, Statement-2 is true.
b) Statement-1 is true, Statement-2 is true;Statement-2 is correct explanation for Statement-1.
c) Statement-1 is true, Statement-2 is true;Statement-2 is NOT correct explanation for Statement-1.
d) Statement-1 is true, Statement-2 is false.

120. An ellipse is drawn by taking a diameter of the circle

$$\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0 \qquad (3.8.120.1)$$

as its semi-minor axis and a diameter of the circle

$$\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix}\mathbf{x} = 0 \qquad (3.8.120.2)$$

is semi-major axis. If the center of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is :

a) $\mathbf{x}^T\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 4$
b) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x} = 8$
c) $\mathbf{x}^T\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 8$
d) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x} = 16$

121. The equation of the circle passing through the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}\mathbf{x} = 1, \qquad (3.8.121.1)$$

and having centre at $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is

a) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -6 \end{pmatrix}\mathbf{x} - 7 = 0$
b) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -6 \end{pmatrix}\mathbf{x} + 7 = 0$
c) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -6 \end{pmatrix}\mathbf{x} - 5 = 0$
d) $\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -6 \end{pmatrix}\mathbf{x} + 5 = 0$

122. **Given:** A circle,

$$\mathbf{x}^T\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} = 5 \qquad (3.8.122.1)$$

and a parabola,

$$\mathbf{x}^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4\sqrt{5} & 0 \end{pmatrix}\mathbf{x} \qquad (3.8.122.2)$$

. **Statement-I:** An equation of a common tangent to these curve is

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + \sqrt{5} \tag{3.8.122.3}$$

. **Statement-2:** If the line,

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \frac{\sqrt{5}}{m}(m \neq 0) \tag{3.8.122.4}$$

is their common tangent,then m satisfies $m^4 - 3m^2 + 2 = 0$.

a) Statement-1 is true,Statement-2 is true;Statement-2 is correct explanation for Statement-1.
b) Statement-1 is true,Statement-2 is true;Statement-2 is not correct explanation for Statement-1.
c) Statement-1 is true,Statement-2 is false.
d) Statement-1 is false,Statement-2 is true.

123. The locus of the foot of perpendicular drawn from the centre of the ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}\mathbf{x} = 6 \tag{3.8.123.1}$$

on any tangent to it is

a) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
b) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
c) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
d) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

124. The slop of the line touching both the parabolas

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} \tag{3.8.124.1}$$

and

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & -32\end{pmatrix}\mathbf{x} \tag{3.8.124.2}$$

is

a) $\frac{1}{8}$
b) $\frac{2}{3}$
c) $\frac{1}{2}$
d) $\frac{3}{2}$

125. Let O be the vertex and Q be any point on the parabola,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 8\end{pmatrix}\mathbf{x}. \tag{3.8.125.1}$$

If the point **P** divides the lines segments OQ internally in the ratio 1:3, then locus of P is:

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x}$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x}$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}$
d) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}$

126. The normal to the curve,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\2 & -3\end{pmatrix}\mathbf{x} = 0, \tag{3.8.126.1}$$

at$\begin{pmatrix}1\\1\end{pmatrix}$

a) meets the curve again in the third quadrant.
b) meets the curve again in the fourth quadrant.
c) does not meet the curve again.
d) meets the curve again in the second quadrant.

127. The area(in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{5}\end{pmatrix}\mathbf{x} = 1 \tag{3.8.127.1}$$

is

a) $\frac{27}{2}$
b) 27
c) $\frac{27}{4}$
d) 18

128. Let **P** be the point on the parabola,

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} \tag{3.8.128.1}$$

which is at a minimum distance from the centre C of the circle

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 12\end{pmatrix}\mathbf{x} + 36 = 1, \tag{3.8.128.2}$$

Then the equation of the circle, passing through C and having its centre at **P** is:

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-\frac{1}{4} & 2\end{pmatrix}\mathbf{x} - 24 = 0$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & 9\end{pmatrix}\mathbf{x} + 18 = 0$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & 8\end{pmatrix}\mathbf{x} + 12 = 0$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} - 12 = 0$

129. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length

of its conjugate axis is equal to half of the distance between its foci, is :

a) $\frac{2}{\sqrt{3}}$
b) $\sqrt{3}$
c) $\frac{4}{3}$
d) $\frac{4}{\sqrt{3}}$

130. A hyperbola passes through the point $\mathbf{P} = \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix}$ and has foci at $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$. Then the tangent to this hyperbola at $\mathbf{P}$ also passes through the point:

a) $\begin{pmatrix} -\sqrt{2} \\ -\sqrt{3} \end{pmatrix}$
b) $\begin{pmatrix} 3\sqrt{2} \\ 2\sqrt{3} \end{pmatrix}$
c) $\begin{pmatrix} 2\sqrt{3} \\ 3\sqrt{3} \end{pmatrix}$
d) $\begin{pmatrix} \sqrt{3} \\ \sqrt{2} \end{pmatrix}$

131. The radius of a circle, having minimum area, which touches the curve

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 - \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.131.1}$$

and the lines,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} |1| & 0 \end{pmatrix} \mathbf{x} \tag{3.8.131.2}$$

is:

a) $4(\sqrt{2}+1)$
b) $2(\sqrt{2}+1)$
c) $2(\sqrt{2}-1)$
d) $4(\sqrt{2}-1)$

132. Tangents are drawn to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 36 \tag{3.8.132.1}$$

at the points $\mathbf{P}$ and $\mathbf{Q}$. if these tangents intersect at the point $\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ then the area(in sq.units) of the ΔPTQ is :

a) $54\sqrt{3}$
b) $60\sqrt{3}$
c) $36\sqrt{5}$
d) $45\sqrt{5}$

133. Tangents are normal are drawn at $\mathbf{P} = \begin{pmatrix} 16 \\ 16 \end{pmatrix}$ on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x}, \tag{3.8.133.1}$$

which intersect the axis of the parabola at **A** and **B**, respectively. If C is the centre of the circle through the points P,A and B and $\angle CPB = \theta$, then the value of $\tan\theta$ is :

a) 2
b) 3
c) $\frac{4}{3}$
d) $\frac{1}{2}$

134. Two sets A and B are as under :

$$A = \begin{pmatrix} a \\ b \end{pmatrix} \in RxR : |a-5| < 1 and |b-5| < 1 \tag{3.8.134.1}$$

$$B = \begin{pmatrix} a \\ b \end{pmatrix} \in RxR : 4(a-6)^2 + 9(b-5)^2 \leq 36. \tag{3.8.134.2}$$

Then:

a) $A \subset B$
b) $A \cap B = \phi$(an empty set)
c) neither $A \subset B$ nor $B \subset A$
d) $B \subset A$

135. If the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} - 6 \tag{3.8.135.1}$$

touches the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 16 & 12 \end{pmatrix} \mathbf{x} + c = 0 \tag{3.8.135.2}$$

then the value of c is :

a) 185
b) 85
c) 95
d) 195

136. Axis of a parabola lies along x-axis. If its vertex and focus are at distances 2 and 4 respectively from the origin, on the positive x-axis then which of the following points does not lie on it?

a) $\begin{pmatrix} 5 \\ 2\sqrt{6} \end{pmatrix}$
b) $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$

c) $\begin{pmatrix} 6 \\ 4\sqrt{2} \end{pmatrix}$
d) $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

137. Let $0 < \theta < \frac{\pi}{2}$. If The eccentricity of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\cos^2\theta} & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \mathbf{x} = 1 \qquad (3.8.137.1)$$

is greater than 2, then the length of its latus rectum lies in the interval:
a) $\begin{pmatrix} 3 \\ \infty \end{pmatrix}$
b) $\begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$
c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

138. Equation of a common tangent to the circle,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 6 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (3.8.138.1)$$

and the parabola,

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.138.2)$$

is :
a) $\begin{pmatrix} 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 & 0 \end{pmatrix} \mathbf{x} + 1$
b) $\begin{pmatrix} 0 & \sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 3$
c) $\begin{pmatrix} 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 12$
d) $\begin{pmatrix} 0 & \sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} + 1$

139. If the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix} \mathbf{x} + 7\sqrt{3} \qquad (3.8.139.1)$$

is normal to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{24} & 0 \\ 0 & -\frac{1}{18} \end{pmatrix} \mathbf{x} = 1, \qquad (3.8.139.2)$$

then a value of m is:
a) $\frac{\sqrt{5}}{2}$
b) $\frac{\sqrt{15}}{2}$
c) $\frac{2}{\sqrt{5}}$
d) $\frac{3}{\sqrt{5}}$

140. If one end of a focal chord of the parabola,

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.140.1)$$

is a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.Then the length of this focal chord is:
a) 25
b) 22
c) 24
d) 20

Match the Following DIRECTIONS(Q. 1-3) Each question contains statements given in two columns, which have to be matched. the statement in column-1 is labelled can A, B, C and D. while the three statements in column-2 are labelled p, q, r, s and t. any given statement in column-1 can have correct matching with ONE or MORE statements in column-2.

141. Match the following: $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is the pt, from which three normals are drawn to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.141.1}$$

which meet the parabola in the points **P**, **Q** and **R**. Then

Column-I	**Column-II**
(A) Area of ΔPQR	(p)2
(B) Radius of circum circle of ΔPQR	(q)$\frac{5}{2}$
(C) Centroid of ΔPQR	(r)$\begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}$
(D) circumcentre of ΔPQR	(s)$\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$

142. Match statements in the column I with the properties in Column II and indicate your answer by darkening the bubbles in 4 x 4 matrix given in the ORS.

Column-I	**Column-II**
(A) Two intersecting circles	(p) have a common tangents
(B) Two mutually external circles	(q) have a common normals
(C) Two circles,one strictly inside the other	(r) do not have a common tangents
(D) Two branches of a hyperbola	(s) do not have a common normals

143. Match the conics in Column I with the statement/expression in Column II

Column-I	**Column-II**
(A) Circle	(p) The focus of point $\begin{pmatrix} h \\ k \end{pmatrix}$ for which the line $\begin{pmatrix} h & k \end{pmatrix}\mathbf{x} = 1$ touches the circle $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 4$
(B) Parabola	(q) Point **z** in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(C) Ellipse	(r) Points of the conic have paramatric representation $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = \sqrt{3}(\frac{1-t^2}{1+t^2}), \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \leq x < \infty$

DIRECTIONS(Q.4) Following questions are matching lists.The codes for the list have choices(a),(b),(c)and (d)out of which ONLY ONE is correct.

144. A line L:$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix}+3$meets y-axis at $\mathbf{E} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and the are of the parabola $\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix}\mathbf{x}$, $0 \leq y \leq 6$at the point $\mathbf{F} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. The tangent to the parabola at $\mathbf{F} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$intersects the y-axis at $\mathbf{G} = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$.The slope m of the line L is chosen such that the area of the triangle EGF has a local maximum. Match the List I with List II and select the correct answer using the code given below the lists:

List-I	**List-II**
P. m=	1. $\frac{1}{2}$
Q. Maximum area of ΔEFG is	2. 4
R. y_0=	3. 2
S. y_1=	4. 1

codes:

	P	**Q**	**R**	**S**
(a)	4	1	2	3
(b)	3	4	1	2
(c)	1	3	2	4
(d)	1	3	4	2

Qs.5-7: By appropriately matching the information given in the three columns of the following table Column1,2 and 3 contains conics,equations of the tangents to the conics and points of contact,respectively.

Column-I	Column-II	Column-III
(I) $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = a^2$	(i) $\begin{pmatrix}0 & m\end{pmatrix}\mathbf{x} = \begin{pmatrix}m^2 & 0\end{pmatrix}\mathbf{x} + a$	(P) $\begin{pmatrix}\frac{a}{m^2}\\\frac{2a}{m}\end{pmatrix}$
(II) $\begin{pmatrix}1 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2$	(ii) $\begin{pmatrix}0 & m\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + a\sqrt{m^2+1}$	(Q) $\begin{pmatrix}-\frac{ma}{\sqrt{m^2+1}}\\\frac{a}{\sqrt{m^2+1}}\end{pmatrix}$
(III) $\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x}$	(iii) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \sqrt{a^2m^2-1}$	(R) $\begin{pmatrix}-\frac{a^2m}{\sqrt{a^2m^2+1}}\\\frac{1}{\sqrt{a^2m^2+1}}\end{pmatrix}$
(IV) $\begin{pmatrix}1 & 0\\0 & -a^2\end{pmatrix}\mathbf{x} = a^2$	(iv) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \sqrt{a^2m^2+1}$	(S) $\begin{pmatrix}-\frac{a^2m}{\sqrt{a^2m^2-1}}\\-\frac{1}{\sqrt{a^2m^2-1}}\end{pmatrix}$

145. For $\mathbf{a} = \sqrt{2}$,if a tangent is drawn to a suitable conic(Column 1)at the point of contact$\begin{pmatrix}-1\\1\end{pmatrix}$,then which of the following options is the only correct combination for obtaining its equation?
  a) (I) (i) (P)
  b) (I) (ii) (Q)
  c) (II) (ii) (Q)
  d) (III) (i) (P)
146. If a tangent to a suitable conic(Column 1)is founded to be $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 8$ and its point of contact is $\begin{pmatrix}8\\16\end{pmatrix}$,then which of the following options is the only correct combination?
  a) (I) (ii) (Q)
  b) (II) (iv) (R)
  c) (III) (i) (P)
  d) (III) (ii) (Q)
147. The tangent to a suitable conic(Column1)at $\begin{pmatrix}\sqrt{3}\\\frac{1}{2}\end{pmatrix}$ is found to be $\begin{pmatrix}\sqrt{3} & 2\end{pmatrix}\mathbf{x} = 4$,then which of the following options is the only correct combination?
  a) (IV) (iii) (S)
  b) (IV) (iv) (S)
  c) (II) (iii) (R)
  d) (II) (iii) (R)
148. Let **H**:

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & -\frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1, \qquad (3.8.148.1)$$

where $a > b > 0$,be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices

N.Let the area of the triangle LMN be $4\sqrt{3}$.

List-I	**List-II**
P. The length of the conjugate axis of H is	1. 8
Q. The eccentricity of H is	2. $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	3. $\frac{2}{\sqrt{3}}$
P. The length of the latus rectum of H is	4. 4

a) $P \rightarrow 4Q \rightarrow 2R \rightarrow 1S \rightarrow 3$
b) $P \rightarrow 4Q \rightarrow 3R \rightarrow 1S \rightarrow 2$
c) $P \rightarrow 4Q \rightarrow 1R \rightarrow 3S \rightarrow 2$
d) $P \rightarrow 3Q \rightarrow 4R \rightarrow 2S \rightarrow 1$

4 Solid Geometry

4.1 Lines and Planes

1. L_1 is the intersection of planes

$$\begin{pmatrix} 2 & -2 & 3 \end{pmatrix} \mathbf{x} = 2$$
$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{4.1.1.1}$$

   Find its equation.
   Solution: (4.1.1.1) can be written in matrix form as

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \tag{4.1.1.2}$$

   and solved using the augmented matrix as follows

$$\begin{pmatrix} 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & 2 \end{pmatrix} \tag{4.1.1.3}$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{4.1.1.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 5 \\ x_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.1.1.5}$$

   which is the desired equation.

2. Summarize all the above computations through a Python script and plot L_1.
   Solution: The following code generates Fig. 4.1.2.

```
codes/3d/1.1.py
```

3. L_2 is the intersection of the planes

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{4.1.3.1}$$
$$\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{4.1.3.2}$$

   Show that its equation is

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \tag{4.1.3.3}$$

4. Plot L_2.
   Solution: The following code generates Fig. 4.1.4.

```
codes/3d/1.2.py
```

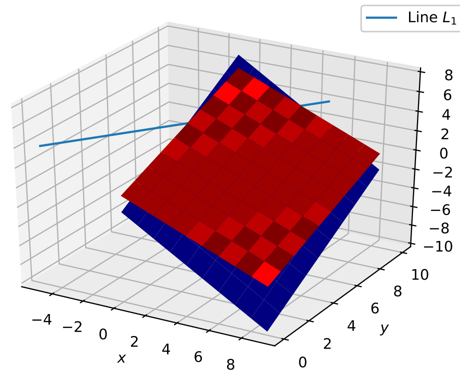

Fig. 4.1.2

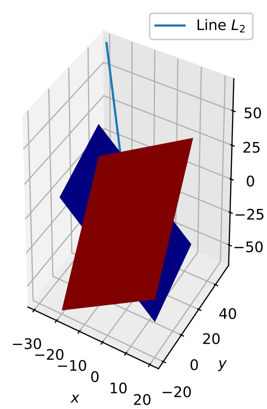

Fig. 4.1.4

5. Do L_1 and L_2 intersect? If so, find their point of intersection P.
   Solution: From (4.1.1.5),(4.1.3.3), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.1.5.1}$$

$$\implies \begin{pmatrix} 1 & 3 \\ 1 & -5 \\ 0 & -7 \end{pmatrix} \mathbf{\Lambda} = \frac{1}{7} \begin{pmatrix} 40 \\ 8 \\ -28 \end{pmatrix} \tag{4.1.5.2}$$

This matrix equation can be solved as

$$\begin{pmatrix}1 & 3 & \frac{40}{7}\\ 1 & -5 & \frac{8}{7}\\ 0 & -7 & -4\end{pmatrix} \leftrightarrow \begin{pmatrix}8 & 0 & \frac{224}{7}\\ 0 & 1 & \frac{4}{7}\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \tag{4.1.5.3}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \implies \boldsymbol{\Lambda} = \begin{pmatrix}4\\ \frac{4}{7}\end{pmatrix} \tag{4.1.5.4}$$

Substituting $\lambda_1 = 4$ in (4.1.5.1)

$$\mathbf{x} = \begin{pmatrix}4\\4\\0\end{pmatrix} + \begin{pmatrix}-5\\0\\4\end{pmatrix} = \begin{pmatrix}-1\\4\\4\end{pmatrix} \tag{4.1.5.5}$$

6. Plot P.
Solution: The following code generates Fig. 4.1.6.

```
codes/3d/1.3.py
```

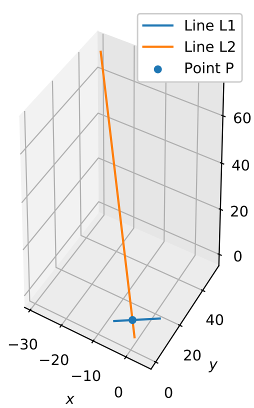

Fig. 4.1.6

4.2 Normal to a Plane

1. The cross product of $\mathbf{a}, \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{4.2.1.1}$$

From (4.1.1.5), (4.1.3.3), the direction vectors of L_1 and L_2 are

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}-3\\5\\7\end{pmatrix} \tag{4.2.1.2}$$

respectively. Find the direction vector of the normal to the plane spanned by L_1 and L_2.
Solution: The desired vector is obtained as

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}7\\-7\\8\end{pmatrix} = \mathbf{n} \tag{4.2.1.3}$$

2. Summarize all the above computations through a plot
Solution: The following code generates Fig. 4.2.2.

```
codes/3d/2.1.py
```

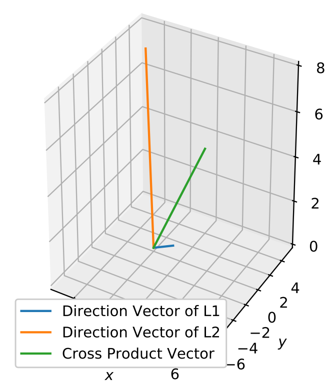

Fig. 4.2.2

3. Find the equation of the plane spanned by L_1 and L_2.
Solution: Let $\mathbf{x}_0$ be the intersection of L_1 and L_2. Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \tag{4.2.3.1}$$

$$\implies \mathbf{x}^T\mathbf{n} = \mathbf{x}_0^T\mathbf{n} \tag{4.2.3.2}$$

$$\implies \mathbf{x}^T\begin{pmatrix}7\\-7\\8\end{pmatrix} = \begin{pmatrix}-1 & 4 & 4\end{pmatrix}\begin{pmatrix}7\\-7\\8\end{pmatrix} = -3 \tag{4.2.3.3}$$

4. Summarize the above through a plot.
Solution: The following code generates Fig. 4.2.4.

```
codes/3d/2.2.py
```

5. Find the distance of the origin from the plane containing the lines L_1 and L_2.

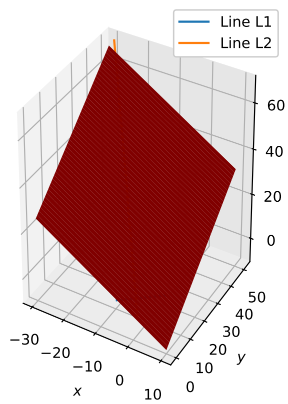

Fig. 4.2.4

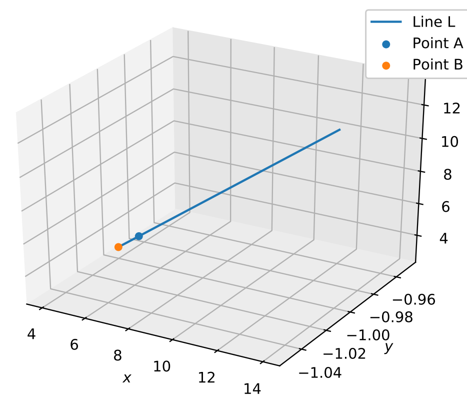

Fig. 4.3.2

Solution: The distance from the origin to the plane is given by

$$\frac{\left|\mathbf{x}_0^T \mathbf{n}\right|}{\|n\|} = \frac{1}{3\sqrt{2}} \tag{4.2.5.1}$$

4.3 Projection on a Plane

1. Find the equation of the line L joining the points

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \end{pmatrix}^T \tag{4.3.1.1}$$

$$\mathbf{B} = \begin{pmatrix} 4 & -1 & 3 \end{pmatrix}^T \tag{4.3.1.2}$$

Solution: The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda(\mathbf{A} - \mathbf{B}) \tag{4.3.1.3}$$

$$= \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.3.1.4}$$

2. Plot the above line.
Solution: The following code generates Fig. 4.3.2.

```
codes/3d/3.1.py
```

3. Find the intersection of L and the plane P given by

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{4.3.3.1}$$

Solution: From (4.3.1.4) and (4.3.3.1),

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 7 \tag{4.3.3.2}$$

$$\implies 6 + 2\lambda = 7 \tag{4.3.3.3}$$

$$\implies \lambda = \frac{1}{2} \tag{4.3.3.4}$$

Substituting in (4.3.1.4),

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} 9 & -1 & 7 \end{pmatrix} \tag{4.3.3.5}$$

4. Sketch the line, plane and the point of intersection.
Solution: The following code generates Fig. 4.3.4.

```
codes/3d/3.2.py
```

5. Find $\mathbf{C} \in P$ such that $AC \perp P$.
Solution: From (4.3.3.1), the direction vector of AC is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Hence, the equation of AC is

$$\mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.3.5.1}$$

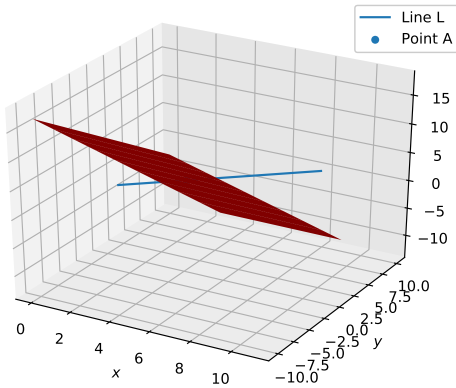

Fig. 4.3.4

Substituting in (4.3.3.1)

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 7 \tag{4.3.5.2}$$

$$\implies 8 + 3\lambda_1 = 7 \tag{4.3.5.3}$$

$$\implies \lambda_1 = -\frac{1}{3} \tag{4.3.5.4}$$

Thus,

$$\mathbf{C} = \frac{1}{3}\begin{pmatrix}14\\-4\\11\end{pmatrix} \tag{4.3.5.5}$$

6. Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D} = \frac{1}{3}\begin{pmatrix}13\\-2\\10\end{pmatrix} \tag{4.3.6.1}$$

7. Find the projection of AB on the plane P.
Solution: The projection is given by

$$CD = \|\mathbf{C} - \mathbf{D}\| = \sqrt{\frac{2}{3}} \tag{4.3.7.1}$$

8. Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \tag{4.3.8.1}$$

4.4 *Coplanar vectors*

1. If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that

$$\mathbf{u}^T(\mathbf{A} \times \mathbf{B}) = 0 \tag{4.4.1.1}$$

2. Find $\mathbf{A} \times \mathbf{B}$ given

$$\mathbf{A} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}^T \tag{4.4.2.1}$$

$$\mathbf{B} = \begin{pmatrix}0 & 1 & 1\end{pmatrix}^T \tag{4.4.2.2}$$

Solution: From (4.2.1.1),

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix}0 & 1 & 3\\-1 & 0 & -2\\-3 & 2 & 0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{4.4.2.3}$$

$$= \begin{pmatrix}4\\-2\\2\end{pmatrix} \tag{4.4.2.4}$$

3. Let $\mathbf{u}$ be coplanar with $\mathbf{A}$ such that $\mathbf{u} \perp \mathbf{A}$ and

$$\mathbf{u}^T\mathbf{B} = 24. \tag{4.4.3.1}$$

Find $\|\mathbf{u}\|^2$.
Solution: From (4.4.2.4) and the given information,

$$\mathbf{u}^T\begin{pmatrix}4 & -2 & 2\end{pmatrix} = 0 \tag{4.4.3.2}$$

$$\mathbf{u}^T\begin{pmatrix}2 & 3 & -1\end{pmatrix} = 0 \tag{4.4.3.3}$$

$$\mathbf{u}^T\begin{pmatrix}0 & 1 & 1\end{pmatrix} = 24 \tag{4.4.3.4}$$

$$\implies \begin{pmatrix}4 & -2 & 2\\2 & 3 & -1\\0 & 1 & 1\end{pmatrix}\mathbf{u} = \begin{pmatrix}0\\0\\24\end{pmatrix} \tag{4.4.3.5}$$

$$\implies \mathbf{u} = 4\begin{pmatrix}-1\\2\\4\end{pmatrix} \tag{4.4.3.6}$$

$$\implies \|\mathbf{u}\|^2 = 336 \tag{4.4.3.7}$$

4.5 *Orthogonality*

1. Let

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_1\begin{pmatrix}-1\\2\\2\end{pmatrix} \tag{4.5.1.1}$$

$$L_2 : \quad \mathbf{x} = \lambda_1\begin{pmatrix}2\\-1\\2\end{pmatrix} \tag{4.5.1.2}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.
Solution: Let

$$L_3: \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{4.5.1.3}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{4.5.1.4}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{4.5.1.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.6}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1: \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{4.5.1.7}$$

$$L_2: \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{4.5.1.8}$$

$$L_3: \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{4.5.1.9}$$

where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{4.5.1.10}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (4.5.1.7)-(4.5.1.9),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{4.5.1.11}$$

$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{4.5.1.12}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (4.5.1.11)-(4.5.1.12) can be expressed as

$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.13}$$

$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.14}$$

$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{4.5.1.15}$$

$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.16}$$

$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.17}$$

$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{4.5.1.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{4.5.1.19}$$

$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{4.5.1.20}$$

Substituting in (4.5.1.11) and (4.5.1.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ or} \tag{4.5.1.21}$$

$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.22}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

4.6 Least Squares

1. Find the equation of the plane P containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{4.6.1.1}$$

2. Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \tag{4.6.2.1}$$

lies outside P.
3. Find the point $\mathbf{w} \in P$ closest to $\mathbf{y}$.
4. Show that

$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 = \|\mathbf{y}\|^2 - \mathbf{w}^T \mathbf{X}^T \mathbf{y} \tag{4.6.4.1}$$

$$- \mathbf{y}^T A \mathbf{w} + \mathbf{w}^T \mathbf{X}^T \mathbf{X} \mathbf{w} \tag{4.6.4.2}$$

5. Assuming 2×2 matrices and 2×1 vectors, show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{y} = \frac{\partial}{\partial \mathbf{w}} \mathbf{y}^T \mathbf{X} \mathbf{w} = \mathbf{y}^T \mathbf{X} \tag{4.6.5.1}$$

6. Show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{X} \mathbf{w} = 2 \mathbf{w}^T \left(\mathbf{X}^T \mathbf{X} \right) \tag{4.6.6.1}$$

7. Show that

$$\hat{\mathbf{w}} = \min_{\mathbf{w}} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 \tag{4.6.7.1}$$

$$= \left(\mathbf{X}^T \mathbf{X} \right)^{-1} \mathbf{X}^T \mathbf{y} \tag{4.6.7.2}$$

8. Let
$$\mathbf{X} = \begin{pmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{pmatrix}. \tag{4.6.8.1}$$
from (4.6.1.1). Verify (4.6.7.2).
9. Run the following Python code and comment on the output for different values of $\mathbf{x}$

```
codes/matrix/Prob1_4.py
```

10. Compare the results obtained by typing the following code with the results in the previous problem.

```
codes/matrix/Prob1_6.py
```

11. Type the following code in Python and run. Comment.

```
codes/matrix/Prob1_7.py
```

4.7 Singular Value Decomposition

1. Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C} = \mathbf{X}^T\mathbf{X}$.
2. Obtain $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{v}_1, \mathbf{v}_2$ through the following equations.
$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \tag{4.7.2.1}$$
$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{4.7.2.2}$$
$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \tag{4.7.2.3}$$
This procedure is known as Gram-Schmidt orthogonalization.
3. Stack the vectors $\mathbf{u}_1, \mathbf{u}_2$ in columns to obtain the matrix $\mathbf{Q}$. Show that $\mathbf{Q}$ is orthogonal.
4. From the Gram-Schmidt process, show that $\mathbf{C} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix. This is known as the $\mathbf{Q}-\mathbf{R}$ decomposition.
5. Find an orthonormal basis for $\mathbf{X}^T\mathbf{X}$ comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix $\mathbf{V}$. This is known as *Orthogonal Diagonalization*.
6. Find the singular values of $\mathbf{X}^T\mathbf{X}$. The singular values are obtained by taking the square roots of its eigenvalues.
7. Stack the singular values of $\mathbf{X}^T\mathbf{X}$ diagonally to obtain a matrix $\mathbf{\Sigma}$.
8. Obtain the matrix $\mathbf{XV}$. Verify if the columns of this matrix are orthogonal.
9. Extend the columns of $\mathbf{XV}$ if necessary, to obtain an orthogonal matrix $\mathbf{U}$.
10. Find $\mathbf{U\Sigma V}^T$. Comment.

4.8 JEE Exercises

1. Let $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ be vectors of length 3, 4, 5 respectively. Let $\overrightarrow{A}$ be perpendicular to $\overrightarrow{B} + \overrightarrow{C}$, $\overrightarrow{B}$ to $\overrightarrow{C} + \overrightarrow{A}$ and $\overrightarrow{C}$ to $\overrightarrow{A} + \overrightarrow{B}$. Then the length of vector $\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ is.............
2. The unit vector perpendicular to the plane determined by P(1, -1, 2), Q(2, 0, -1) and R(0, 2, 1) is.........
3. The area of the triangle whose vertices are A(1, -1, 2), B(2, 1, -1) and C(3, -1, 2) is........
4. A, B, C, and D are four points in a plane with position vectors a, b, c, and d respectively such that
$$(\overrightarrow{a} - \overrightarrow{d})(\overrightarrow{b} - \overrightarrow{c}) = (\overrightarrow{b} - \overrightarrow{d})(\overrightarrow{c} - \overrightarrow{a}) = 0$$
The Point D, then is the.................of the triangle ABC.
5. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\overrightarrow{A} = (1, a, a^2)$, $\overrightarrow{B} = (1, b, b^2)$, $\overrightarrow{C} = (1, c, c^2)$ are non-olar, then the product abc = ...
6. If $\overrightarrow{A}\overrightarrow{B}\overrightarrow{C}$ are three non-polar vectors, then $\frac{\overrightarrow{A}.\overrightarrow{B}\times\overrightarrow{C}}{\overrightarrow{C}\times\overrightarrow{A}.\overrightarrow{B}} + \frac{\overrightarrow{B}.\overrightarrow{A}\times\overrightarrow{C}}{\overrightarrow{C}.\overrightarrow{A}\times\overrightarrow{B}}$ = ...
7. If $\overrightarrow{A}$ = (1, 1, 1), $\overrightarrow{C}$ = (0, 1, -1) are given vectors, then a vector B satisfying the equations $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{C}$ and $\overrightarrow{A}.\overrightarrow{B} = 3$.............
8. If the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{j}$, $\hat{i} + \hat{j} + c\hat{k}$ $(a \neq b \neq c \neq 1)$ are co-planar, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ = ...
9. Let b = $4\vec{i} + 3\vec{j}$ and $\overrightarrow{c}$ be two vectors perpendicular to each other in the xy - plane. All vectors in the same plane having projections 1 and 2 along $\overrightarrow{b}$ and $\overrightarrow{c}$, respectively are given by..........
10. The components of a vector $\overrightarrow{a}$ along and perpendicular to a non-zero vector $\overrightarrow{b}$ are............ and ....respectively.
11. Given that $\overrightarrow{a}$ = (1, 1, 1), $\overrightarrow{c}$ = (0, 1, -1), $\overrightarrow{a}\overrightarrow{b}$ = 3 and $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{c}$, then $\overrightarrow{b}$ = ...
12. A unit vector co-planr with $\overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$ and $\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$ and perpendicular to $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$ is...........

13. A unit vector perpendicular to the plane determined by the points P(1, -1, 2), Q(2, 0, -1) and R(0, 2, 1) is..........
14. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}$, $\hat{i} + \hat{j}$ and plane determined by the vectors $\hat{i} - \hat{j}$, $\hat{i} + \hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is...........
15. If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector $(\vec{a}.\vec{b})\vec{b} + (\vec{a}.\vec{c})\vec{c} + \frac{\vec{a}.(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}(\vec{b}\times\vec{c})$ = ...
16. Let OA = a, OB = 10a + 2b and OC = b where O, A, and C are non-collinear points. Let p denote thea area of the quadrilateral OABC, and let q denote the area of the parallelgram with OA and OC as adjacent sides. If p = kq, then k = ...

(B). True/False

17. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be unit vectors suppose that $\vec{A}.\vec{B} = \vec{A}.\vec{C} = 0$, and the angle between $\vec{B}$ and $\vec{C}$ is $\frac{\pi}{6}$. Then $\vec{A} = \pm 2(\vec{B}\times\vec{C})$.
18. If X.A = 0, X.B = 0, X.C = 0 for some non-zero vector X, then [A B C] = 0.
19. The points with position vectors a + b, a - b, and a + kb are collinear for all real values of k.
20. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $(\vec{a}\,\vec{b}).(\vec{b}-\vec{c})\times(\vec{c}-\vec{a}) = 2\vec{a}.(\vec{b}\times\vec{c})$.

(C). MCQs with One Correct Answer

21. The scalar $\vec{A}.(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals:
    a) 0
    b) $[\vec{A}\,\vec{B}\,\vec{C}] + [\vec{B}\,\vec{C}\,\vec{A}]$
    c) $[\vec{A}\,\vec{B}\,\vec{C}]$
    d) None of these
22. For non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $|(\vec{a}\times\vec{b}).\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds if and only if
    a) $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$
    b) $\vec{b}.\vec{c} = 0$, $\vec{c}.\vec{a} = 0$
    c) $\vec{c}.\vec{a} = 0$, $\vec{a}.\vec{b} = 0$
    d) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$
23. The volume of the parallelopiped whose sides are given by $\overrightarrow{OA}$ = 2i - 2j, $\overrightarrow{OB}$ = i + j - k, $\overrightarrow{OC}$ = 3i - k, is
    a) $\frac{4}{13}$
    b) 4
    c) $\frac{2}{7}$
    d) None of these
24. The points with position vectors 60i + 3j, 40i - 8j, ai - 52j are collinear if
    a) a = -40
    b) a = 40
    c) a = 20
    d) None of these
25. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-coplanar vectors and $\vec{p}$, $\vec{q}$, $\vec{r}$, are vectors defined by the relations

$$\vec{p} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{q} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{r} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

    then the value of the expression $(\vec{a}+\vec{b}).\vec{p} + (\vec{b}+\vec{c}).\vec{q} + (\vec{c}+\vec{a}).\vec{r}$ is eqaul to
    a) 0
    b) 1
    c) 2
    d) 3
26. Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is
    a) the arithmetic mean of a and b
    b) the geomemetic mean of a and b
    c) the harmonic mean of a and b
    d) equal to zero
27. Let $\vec{p}$ and $\vec{q}$ be the position vectors of P and Q respectively, with respect to O $|\vec{p}|$ = p, $|\vec{q}|$ = q. The points R and S divide PQ internally and externally in the ratio 2:3 respectively. If OR and OS are perpendicular then
    a) $9q^2 = 4q^2$
    b) $4p^2 = 9q^2$
    c) 9p = 4q
    d) 4p = 9q
28. Let α, β, γ be distinct real numbers. The points with postion vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$
    a) are collinear
    b) form an equilateral triangle
    c) form an scalene triangle
    d) form a right angled triangle
29. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$, $\vec{c} = \hat{k} - \hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a}\,.\,\vec{d} = 0 = [\vec{b}\,\vec{c}\,\vec{d}]$, then $\vec{d}$ equals
    a) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$

b) $\pm \frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$
c) $\pm \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{3}}$
d) $\pm\hat{k}$

30. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non coplanar vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{(\vec{b}+\vec{c})}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is
a) $\frac{3\pi}{4}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

31. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$, $|\vec{w}| = 5$, then $\vec{u}.\vec{v} + \vec{v}.\vec{w} + \vec{w}.\vec{u}$ is
a) 47
b) -25
c) 0
d) 25

32. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non polar vectors, then $\vec{a} + \vec{b} + \vec{c}$. $[(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})]$ equals
a) 0
b) $1[\vec{a}\ \vec{b}\ \vec{c}]$
c) $2[\vec{a}\ \vec{b}\ \vec{c}]$
d) $-[\vec{a}\ \vec{b}\ \vec{c}]$

33. Let a = 2i + j - 2k and b = i + j. If c is a vector such that a.c = $|c|$, $|c-a| = 2\sqrt{2}$ and the angle between $(a \times b)$ and c is 30°, then $|(a \times b) \times c|$ =
a) $\frac{2}{3}$
b) $\frac{3}{2}$
c) 2
d) 3

34. a = 2i + j + k, b = i + 2j - k and a unit vector c be coplanar. If c is perpendicular to a, then c =
a) $\frac{1}{\sqrt{2}}$(-j + k)
b) $\frac{1}{\sqrt{3}}$(-i - j - k)
c) $\frac{1}{\sqrt{5}}$(i - 2j)
d) $\frac{1}{\sqrt{3}}$(i - j - k)

35. If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the sides BC, CA, AB respectively of a triangle ABC, then
a) $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0$
b) $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$
c) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$
d) $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

36. Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be such that

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0.$$

Let P_1 and P_2 be planes determined by the pairs of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{d}$ respectively. Then the angle between P_1 and P_2 is
a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

37. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit co-planar vectors, then the scalar triple product $[2\vec{a} - \vec{b}, 2\vec{b} - \vec{c}, 2\vec{c} - \vec{a}]$ =
a) 0
b) 1
c) $-\sqrt{3}$
d) $\sqrt{3}$

38. Let

$$\vec{a} = \vec{i} - \vec{k}$$

$$\vec{b} = x\vec{i} + \vec{j} + (1-x)\vec{k}$$

$$\vec{c} = y\vec{i} + x\vec{j} + (1+x-y)\vec{k}$$

Then $[\vec{a}\vec{b}\vec{c}]$ depends on
a) only x
b) only y
c) Neither x nor y
d) both x and y

39. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors, then

$$|\vec{a}-\vec{b}|^2 + |\vec{b}-\vec{c}|^2 + |\vec{c}-\vec{a}|^2$$

does not exceed.
a) 4
b) 9
c) 8
d) 6

40. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + 2\vec{b}$ and $5\vec{a} - 4\vec{b}$ are perpendicular to each other then the angle between $\vec{a}$ and $\vec{b}$ is
a) 45°
b) 60°
c) $\cos^{-1}(\frac{1}{3})$
d) $\cos^{-1}(\frac{2}{7})$

41. Let $\vec{V} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{W} = \vec{i} + 3\vec{k}$. If

$\overrightarrow{U}$ is a unit vector, then the maximum value of the scalar triple product $|\overrightarrow{U}\overrightarrow{V}\overrightarrow{W}|$ is

a) -1
b) $\sqrt{10} + \sqrt{6}$
c) $\sqrt{59}$
d) $\sqrt{60}$

42. The value of k such that $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane 2x - 4y + z = 7, is

a) 7
b) -7
c) no real value
d) 4

43. The value of $'a'$ so that the volume of parallelopiped formed by $\overrightarrow{i}$ + a$\overrightarrow{j}$ + $\overrightarrow{k}$, $\overrightarrow{j}$ + a$\overrightarrow{k}$ and a$\overrightarrow{i}$ + $\overrightarrow{k}$ becomes minimum is

a) -3
b) 3
c) $\frac{1}{\sqrt{3}}$
d) $\sqrt{3}$

44. If $\overrightarrow{a} = (\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k})$, $\overrightarrow{a} \cdot \overrightarrow{b} = 1$ and $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{j} - \overrightarrow{k}$, then $\overrightarrow{b}$ is

a) $\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$
b) $2\overrightarrow{j} - \overrightarrow{k}$
c) $\overrightarrow{i}$
d) $2\overrightarrow{i}$

45. If the lines

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} and \frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$$

intersect, then the value of k is

a) 3/2
b) 9/2
c) -2/9
d) -3/2

46. The unit vector which is orthogonal to the vector $3\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$ and is co-planar with the vectors $2\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$ and $\overrightarrow{i} - \overrightarrow{j} + \overrightarrow{k}$ is

a) $\frac{2\overrightarrow{i}-6\overrightarrow{j}+\overrightarrow{k}}{\sqrt{41}}$
b) $\frac{2\overrightarrow{i}-3\overrightarrow{j}}{\sqrt{13}}$
c) $\frac{3\overrightarrow{i}-\overrightarrow{k}}{10}$
d) $\frac{4\overrightarrow{i}+3\overrightarrow{j}-3\overrightarrow{k}}{34}$

47. A variable plane at a distance of the one unit from the origin cuts the coordinates axes at A, B, and C. If the centroid D(x, y, z) of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$$

then the value of k is

a) 3
b) 1
c) $\frac{1}{3}$
d) 9

48. If $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ are three non-zero, non-polar vectors and

$$\overrightarrow{b_1} = \overrightarrow{b} - \frac{\overrightarrow{b}.\overrightarrow{a}}{|\overrightarrow{a}|^2}\overrightarrow{a},$$

$$\overrightarrow{b_2} = \overrightarrow{b} + \frac{\overrightarrow{b}.\overrightarrow{a}}{|\overrightarrow{a}|^2}\overrightarrow{a}$$

$$\overrightarrow{c_1} = \overrightarrow{c} - \frac{\overrightarrow{c}.\overrightarrow{a}}{|\overrightarrow{a}|^2}\overrightarrow{a} + \frac{\overrightarrow{b}.\overrightarrow{c}}{|\overrightarrow{c}|^2}\overrightarrow{b_1}$$

$$\overrightarrow{c_2} = \overrightarrow{c} - \frac{\overrightarrow{c}.\overrightarrow{a}}{|\overrightarrow{a}|^2}\overrightarrow{a} - \frac{\overrightarrow{b_1}.\overrightarrow{c}}{|\overrightarrow{b_1}|^2}\overrightarrow{b_1}$$

$$\overrightarrow{c_3} = \overrightarrow{c} - \frac{\overrightarrow{c}.\overrightarrow{a}}{|\overrightarrow{c}|^2}\overrightarrow{a} + \frac{\overrightarrow{b}.\overrightarrow{c}}{|\overrightarrow{c}|^2}\overrightarrow{b_1}$$

$$\overrightarrow{c_4} = \overrightarrow{c} - \frac{\overrightarrow{c}.\overrightarrow{a}}{|\overrightarrow{c}|^2}\overrightarrow{a} = \frac{\overrightarrow{b}.\overrightarrow{c}}{|\overrightarrow{b}|^2}\overrightarrow{b_1}$$

then the set of orthogonal vectors ts

a) $(\overrightarrow{a}, \overrightarrow{b_1}, \overrightarrow{c_3})$
b) $(\overrightarrow{a}, \overrightarrow{b_1}, \overrightarrow{c_2})$
c) $(\overrightarrow{a}, \overrightarrow{b_1}, \overrightarrow{c_1})$
d) $(\overrightarrow{a}, \overrightarrow{b_2}, \overrightarrow{c_2})$

49. A plane which is perpendicular to two planes

$$2x - 2y + z = 0 \tag{4.8.49.1}$$

$$x - y + 2z = 4 \tag{4.8.49.2}$$

passes through (1, -2 ,1). The distance of the plane from the point (1, 2, 2) is

a) 0
b) 1
c) $\sqrt{2}$

d) $2\sqrt{2}$

50. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$ is
   a) $4\hat{i} - \hat{j} + 4\hat{k}$
   b) $3\hat{i} + \hat{j} - 3\hat{k}$
   c) $2\hat{i} + \hat{j} - 2\hat{k}$
   d) $4\hat{i} + \hat{j} - 4\hat{k}$

51. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \lambda^2\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - \lambda^2\hat{k}$ are coplanar is
   a) 0
   b) 1
   c) 2
   d) 3

52. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Which one of the following is correct?
   a) $(\vec{a} \times \vec{b}) = (b \times \vec{c}) = (\vec{c} \times \vec{a}) = \vec{0}$
   b) $(\vec{a} \times \vec{b}) = (b \times \vec{c}) = (\vec{c} \times \vec{a}) \neq \vec{0}$
   c) $(\vec{a} \times \vec{b}) = (b \times \vec{c}) = (\vec{a} \times \vec{c}) \neq \vec{0}$
   d) $(\vec{a} \times \vec{b})$, $(b \times \vec{c})$, $(\vec{c} \times \vec{a})$ are mutually perpendicular.

53. The edges of a parallelopiped are of unit length and are to parallel to non-coplanar unit vectors $\hat{a}$, $\hat{b}$, $\hat{c}$ such that $\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = \frac{1}{2}$. Then, the volume of the parallelopiped is
   a) $\frac{1}{\sqrt{2}}$
   b) $\frac{1}{2\sqrt{2}}$
   c) $\frac{\sqrt{3}}{2}$
   d) $\frac{1}{\sqrt{3}}$

54. Let two non-collinear vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overrightarrow{OP}$(where O is the origin) is given by $\hat{a}\cos t + \hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Then,
   a) $\hat{u} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + \hat{a}.\hat{b})^{1/2}$
   b) $\hat{u} = \frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = (1 + \hat{a}.\hat{b})^{1/2}$
   c) $\hat{u} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + 2\hat{a}.\hat{b})^{1/2}$
   d) $\hat{u} = \frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = (1 + 2\hat{a}.\hat{b})^{1/2}$

55. Let P(3, 2, 6) be a point in a space and Q be a point on the line

$$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$$

Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane x - 4y + 3z = 1 is
   a) $\frac{1}{4}$
   b) $\frac{-1}{4}$
   c) $\frac{1}{8}$
   d) $\frac{-1}{8}$

56. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a} \times \vec{b}) . (\vec{c} \times \vec{d}) = 1$ and $\vec{a}.\vec{c} = \frac{1}{2}$, then
   a) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-polar
   b) $\vec{b}$, $\vec{c}$, $\vec{d}$ are non-polar
   c) $\vec{b}$, $\vec{d}$ are non-parallel
   d) $\vec{a}$, $\vec{d}$ are parallel and $\vec{b}$, $\vec{c}$ are parallel

57. A line with positive direction cosines passes through the point P(2, -1, 2) and make equal angles with the coordinate axes. The line meets the plane 2x + y + z = 9 at a point Q. The length of the line segment PQ equals
   a) 1
   b) $\sqrt{2}$
   c) $\sqrt{3}$
   d) 2

58. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The qudrilateral PQRS must be a
   a) parallelgram, which is neither a rhombus nor a rectangle
   b) square
   c) rectangle, but not a square
   d) rhombus, but a square

59. Equation of the plane containing the straight line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$$

and perpendicular to the plane containing the straight lines

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2} and \frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

is
   a) x + 2y - 2z = 0
   b) 3x + 2y - 2z = 0
   c) x - 2y + z = 0
   d) 5x + 2y - 4z = 0

60. If the distance of the point P(1, -2, 1) from the plane x + 2y - 2z = α, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to

the plane is

a) $(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3})$
b) $(\frac{4}{3}, \frac{-4}{3}, \frac{1}{3})$
c) $(\frac{1}{3}, \frac{2}{3}, \frac{10}{3})$
d) $(\frac{2}{3}, \frac{-1}{3}, \frac{5}{2})$

61. Two adjacent sides of a parallelgram ABCD are given by $\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = \hat{i} + 2\hat{j} + 2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelgram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α os given by

a) $\frac{8}{9}$
b) $\frac{\sqrt{17}}{9}$
c) $\frac{1}{9}$
d) $\frac{4\sqrt{5}}{9}$

62. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

a) $\hat{i} - 3\hat{j} + 3\hat{k}$
b) $-3\hat{i} - 3\hat{j} - \hat{k}$
c) $3\hat{i} - \hat{j} + 3\hat{k}$
d) $\hat{i} + 3\hat{j} - 3\hat{k}$

63. The point P is the intersection of the straight line joining the points Q(2, 3, 5) and R(1, -1, 4) with the plane 5x - 4y - z = 1. If S is the foot of the perpendicular drawn from the point T(2, 1, 4) to QR, then the length of the line segment PS is

a) $\frac{1}{\sqrt{2}}$
b) $\sqrt{2}$
c) 2
d) $2\sqrt{2}$

64. The equation of the plane passing through the line of intersection of the planes

$$x + 2y + 3z = 2$$

$$x - y + z = 3$$

and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, -1) is

a) 5x - 11y + z = 17
b) $\sqrt{3}x + y = 3\sqrt{2} - 1$
c) $x + y + z = \sqrt{3}$
d) $x - \sqrt{2}y = 1 - \sqrt{2}$

65. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a} + \vec{b}| = \sqrt{29}$and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a} + \vec{b}).(-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

a) 0
b) 3
c) 4
d) 8

66. Let P be the image of the point(3, 1, 7) with respect to the plane x - y + z = 3. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{z} = \frac{z}{1}$ is

a) x + y - 3z = 0
b) 3x + z = 0
c) x - 4y + 7z = 0
d) 2x - y = 0

67. The equation of the plane passing through the point(1, 1, 1) and perpendicular to the planes 2x + y - 2z = 0 and 3x - 6y - 2z = 7, is

a) 14x + 2y - 15z = 1
b) 14x - 2y + 15z = 27
c) 14x + 2y + 15z = 31
d) -14x + 2y + 15z = 3

68. Let O be the origin and let PQR be an arbitrary triangle. The points S is such that

$$\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS}$$
$$= \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$$

Then the triangle PQR has S as its

a) Centroid
b) Circumcentre
c) Incentre
d) Orthocentre

(D). MCQs with One or More than One Correct

69. Let $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$, $\vec{c} = c_1 i + c_2 j + c_3 k$, be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

is equal to

a) 0
b) 1

c) $\frac{1}{4}(a_1^2 + a_2^2 + a_2^3)(b_1^2 + b_2^2 + b_3^2)$
d) $\frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^3)$

70. The number of vectors of unit length perpendicular to vectors $\overrightarrow{a}$ = (1, 1, 0) and $\overrightarrow{b}$ = (0, 1, 1) is
 a) 0
 b) 1
 c) 2
 d) 3
 e) ∞
 f) None of these

71. Let $\overrightarrow{a}$ = $2\hat{i}$ - $\hat{j}$ + $\hat{k}$, $\overrightarrow{b}$ = $\hat{i}$ + $2\hat{j}$ - $\hat{k}$ and $\overrightarrow{c}$ = $\hat{i}$ + $\hat{j}$ - $2\hat{k}$ be three vectors. A vector in the plane of $\overrightarrow{b}$ and $\overrightarrow{c}$, whose projection on $\overrightarrow{a}$ is magnitudu of $\sqrt{2/3}$, is
 a) $2\hat{i}$ + $3\hat{j}$ - $3\hat{k}$
 b) $2\hat{i}$ + $3\hat{j}$ + $3\hat{k}$
 c) -$2\hat{i}$ - $\hat{j}$ + $5\hat{k}$
 d) $2\hat{i}$ + $\hat{j}$ + $5\hat{k}$

72. The vector $\frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$ is
 a) a unit vector
 b) makes an angle $\frac{\pi}{3}$ with the vector ($2\hat{i}$ - $4\hat{j}$ + $3\hat{k}$)
 c) parallel to the vector (-$\hat{i}$ + $\hat{j}$ - $\frac{1}{2}\hat{k}$)
 d) perpendicular to the vector ($3\hat{i}$ + $2\hat{j}$ - $2\hat{k}$)

73. If a = i + j + k, $\overrightarrow{b}$ = 4i + 3j + 4k and c = i + αj + βk are linearly dependent vectors and $|c| = \sqrt{3}$, then
 a) $\alpha = 1, \beta = -1$
 b) $\alpha = 1, \beta = \pm 1$
 c) $\alpha = -1, \beta = \pm 1$
 d) $\alpha = \pm 1, \beta = 1$

74. For three vectors u, v, w which of the following expression in not equal to any one of the remaining three?
 a) u.(v × w)
 b) (v × w).u
 c) v.(u × w)
 d) (u × v).w

75. Which of the following expressions are meaningful?
 a) u(v × w)
 b) (u . v).w
 c) (u . v)w
 d) u ×(v . w)

76. Let a and b two non-collinear unit vectors. If u = a - (a.b)b and v = a × b, then $|v|$ is
 a) $|u|$
 b) $|u| + |u.a|$
 c) $|u| + |u.b|$
 d) $|u| + u.(a + b)$

77. Let $\overrightarrow{A}$ be a parallel to line of intersection of planes P_1 and P_2. Plane P_1 is parallel to the vectors $2\hat{j}$ + $3\hat{k}$ and $4\hat{j}$ - $3\hat{k}$ and that P_2 is parallel to $\hat{j}$ - $\hat{k}$ and $3\hat{i}$ + $3\hat{j}$, then the angle between vector $\overrightarrow{A}$ and a given vector $2\hat{i}$ + $\hat{j}$ - $2\hat{k}$ is
 a) $\frac{\pi}{2}$
 b) $\frac{\pi}{4}$
 c) $\frac{\pi}{6}$
 d) $\frac{3\pi}{4}$

78. The vectors which are coplanr with vectors $\hat{i}$ + $\hat{j}$ + $2\hat{k}$ and $\hat{i}$ + $2\hat{j}$ + $\hat{k}$, and perpendicular to the vector $\hat{i}$ + $\hat{j}$ + $\hat{k}$ are
 a) $\hat{j}$ - $\hat{k}$
 b) -$\hat{i}$ - $\hat{j}$
 c) $\hat{i}$ - $\hat{j}$
 d) -$\hat{j}$ + $\hat{k}$

79. If the straight lines
$$\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2} and \frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$$
are co-planar, then the plane(s) containing these two lines is(are)
 a) y + 2z = -1
 b) y + z = -1
 c) y - z = -1
 d) y - 2z = -1

80. A line l passing through the origin is perpendicular to the lines
$$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, -\infty < t < \infty$$
$$l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, -\infty < s < \infty$$
Then, the coordinates of the points on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)
 a) $(\frac{7}{5}, \frac{7}{3}, \frac{5}{3})$
 b) (-1, -1, 0)
 c) (1, 1, 1)
 d) $(\frac{7}{9}, \frac{7}{9}, \frac{8}{9})$

81. Two lines
$$L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$$

$$L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$$

are coplanar. Then α can take value(s)

a) 1
b) 2
c) 3
d) 4

82. Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then

a) $\vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$
b) $\vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$
c) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$
d) $\vec{a} = (\vec{a}.\vec{y})(\vec{z}-\vec{y})$

83. From a point P($\lambda, \lambda, \lambda$) perpendicular to PQ and PR are drawn respectively on the lines y = x, z = 1. If P is such that $\angle$ QPR is a right angle, then the possible value(s) of λ is(are)

a) $\sqrt{2}$
b) 1
c) -1
d) - $\sqrt{2}$

84. In R^3, consider the planes P_1: y = 0 and P_2: X + Z = 1. Let P_3 be the plane, different from P_1 and P_2 , which passes through the intersection of P_1 and P_2. If the distance the point (0, 1, 0) from P_3 is 1 and the distance of a point (α,β,γ) from P_3 is 2, then which of the following relation is(are) true?

a) $2\alpha + \beta + 2\gamma + 2 = 0$
b) $2\alpha - \beta + 2\gamma + 4 = 0$
c) $2\alpha + \beta - 2\gamma - 10 = 0$
d) $2\alpha - \beta + 2\gamma - 8 = 0$

85. In R^3,let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes

$$P_1 : x + 2y - z + 1 = 0$$

$$P_2 : 2x - y + z - 1 = 0$$

Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

a) $(0, \frac{-5}{6}, \frac{-2}{3})$
b) $(\frac{-1}{6}, \frac{-1}{3}, \frac{1}{6})$
c) $(\frac{-5}{6}, 0, \frac{1}{3})$
d) $\frac{-1}{3}, 0, \frac{2}{3})$

86. Let Δ PQR be a triangle. Let $\vec{a} = \vec{QR}$, $\vec{b} = \vec{RP}$ and $\vec{c} = \vec{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$, $\vec{b}.\vec{c} = 24$, then the which of the following is(are) true?

a) $\frac{|c|^2}{2} - |\vec{a}| = 12$
b) $\frac{|c|^2}{2} + |\vec{a}| = 30$
c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$
d) $\vec{a}.\vec{b} = -72$

87. Consider a pyramid OPQRS located in the first octant(x $\geq$ 0, y $\geq$ 0, z $\geq$ 0) with O as origin, and OP and OR along x-axis and along y-axis respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point, T of diagonal OQ such that TS = 3. Then

a) the acute angle between OQ and OS is $\frac{\pi}{3}$
b) the equation of the plane containing the triangle OQS is x-y = 0
c) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$
d) The perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$.

88. Let $\hat{u} = u_1 i + u_2 j + u_3 k$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector $\vec{v}$ in following R^3 such that $|\vec{u} \times \vec{v}| = 1$ and $\vec{w}(\vec{u} \times \vec{v}) = 1$. Which of the following statement(s) is(are) correct?

a) There is exactly one choice for such $\vec{v}$
b) There are infinitely many choices for such $\vec{v}$
c) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$
d) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

89. Let

$$P_1 : 2x + y - z = 3$$

$$P_2 : x + 2y + z = 2$$

be two planes. Then, which of the following statement(s) is(are) correct?

a) The line of intersection of P_1 and P_2 has direction ratios 1, 2, -1
b) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2.
c) The acute angle between P_1 and P_2 is 60°

d) If P_3 is the plane passing through the point(4, 2, -2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$.

90. Let L_1 and L_2 denote the lines

$$\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in R$$

$$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in R$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

a) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R
b) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R
c) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R
d) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})$, t ∈ R

91. Three lines

$$L_1 : \vec{r} = \lambda\hat{i}, \lambda \in R$$

$$L_2 : \vec{r} = \hat{k} + \mu\hat{j}, \mu \in R$$

$$L_3 : \vec{r} = \hat{i} + \hat{j} + v\hat{k}, v \in R$$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P,Q and R are collinear?

a) $\hat{k}$ - $\frac{1}{2}\hat{j}$
b) $\hat{k}$
c) $\hat{k}$ + $\hat{j}$
d) $\hat{k}$ + $\frac{1}{2}\hat{j}$

(E). Subjective Problems

92. From a point O inside a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB respectively. Prove that the perpendiculars from A, B, C to the sides EF, FD, DE are concurrent.

93. A_1, A_2,............A_n are the vertices of a regular plane polygon with n sides and O is its centre. Show that

$$\sum_{i=1}^{n-1}(\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = (1 - n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$$

94. Find all values of λ such that x, y, z ≠ (0, 0, 0) and

$$(\vec{i} + \vec{j} + 3\vec{k})x + (3\vec{i} - 3\vec{j} + \vec{k})y + (-4\vec{i} + 5\vec{j})z = \lambda(x\vec{i} \times \vec{j}y + \vec{k}z)$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along the coordinate axes.

95. A vector $\vec{A}$ has components $\vec{A_1}$, $\vec{A_2}$, $\vec{A_3}$ ina right-handed rectangular Cartesian coordinate system oxyz. The coordinate system is rotated about the x-axis through an angle of $\frac{\pi}{2}$. Find the components of A in the new coordinate system, interms of $\vec{A_1}$, $\vec{A_2}$, $\vec{A_3}$.

96. The position vectors of the points A, B, C and D are ($3\hat{i}$ - $2\hat{j}$ - $\hat{k}$), ($2\hat{i}$ + $3\hat{j}$ - $4\hat{k}$), (-$\hat{i}$ + $\hat{j}$ + $2\hat{k}$) and ($4\hat{i}$ + $5\hat{j}$ + $\lambda\hat{k}$) respectively. If the points A, B, C and D lie on a plane, find the value of λ?

97. If A, B, C, D are any four points in space, Prove that

$$|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}| = 4$$

(area of triangle ABC)

98. Let OACB be a parallelgram with O at the origin and OC a diagonal. Let D be the mid-point of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio.

99. If vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, show that

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = \vec{0}$$

100. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, Determine the ratio OP : PD using vectors methods?

101. Let $\vec{A}$ = $2\hat{i}$ + $\hat{k}$, $\vec{B}$ = $\hat{i}$ + $\hat{j}$ + $\hat{k}$, and $\vec{C}$ = $4\hat{i}$ - $3\hat{j}$ + $7\hat{k}$. Determine a vector $\vec{R}$. Satisfying $\vec{R} \times \vec{B}$ = $\vec{C} \times \vec{B}$ and $\vec{R}.\vec{A}$ = 0

102. Determine the value of 'c' so that for all real x, the vector cx$\hat{i}$ - 6$\hat{j}$ - 3$\hat{k}$ and x$\hat{i}$ + 2$\hat{j}$ + 2cx$\hat{k}$ make an obtuse angle with each other.

103. In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods.

104. If the vectors $\vec{b}$, $\vec{c}$, $\vec{d}$ are not coplanar, then

prove that the vector

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$$

is parallel to $\vec{a}$.

105. The position vectors of the vertices A, B, C of a tetrahedron ABCD are $\hat{i} + \hat{j} + \hat{k}$, $\hat{i}$ and $3\hat{i}$ respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of tetrahedron is $\frac{2\sqrt{2}}{3}$. find the position vector of the point E for all its possible positions.

106. If A, B and C vectors such that $|B| = |C|$, Prove that

$$[(A + B) \times (A + C)] \times (B \times C)(B + C) = 0.$$

107. Prove, by vector methods or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides. (You may assume that the trapezium is not a parallelgram.)

108. For any two vectors u and v, prove that
   a) $(u.v)^2 + |u \times v|^2 = |u|^2|v|^2$
   b) $(1+|u|^2)(1+|v|^2) = (1-u.v)^2+|u+v+(u\times v)|^2$

109. Let u and v be unit vectors. If w is a vector such that w + (w × u) = v, then prove that $|u \times v|.w \leq 1/2$ and that the equality holds if and onlt if u is perpendicular to v.

110. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point concurrency in terms of the position vectors of the vertices.

111. Find 3-dimensional vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ satisfying

$$\vec{v_1}.\vec{v_1} = 4, \vec{v_1}.\vec{v_2} = -2, \vec{v_1}.\vec{v_3} = 6,$$

$$\vec{v_2}.\vec{v_2} = 2, \vec{v_2}.\vec{v_3} = -5, \vec{v_3}.\vec{v_3} = 29$$

112. Let

$$\overrightarrow{A(t)} = f_1(t)\hat{i} + f_2(t)\hat{j}$$

$$\overrightarrow{B(t)} = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0, 1]$$

where f_1, f_2, g_1, g_2 are continuous functions. If $\overrightarrow{A(t)}$ and $\overrightarrow{B(t)}$ are non-zero vectors for all t and $\overrightarrow{A(0)} = 2\hat{i} + 3\hat{j}$, $\overrightarrow{A(1)} = 6\hat{i} + 2\hat{j}$, $\overrightarrow{B(0)} = 3\hat{i} + 2\hat{j}$, $\overrightarrow{B(1)} = 2\hat{i} + 6\hat{j}$. Then show that $\overrightarrow{A(t)}$ and $\overrightarrow{B(t)}$ are parallel for some t.

113. Let V be the volume of the parallelopiped formed by the vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

If a_r, b_r, c_r where r = 1, 2, 3, are non-negative real numbers and

$$\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$$

Show that $V \leq L^3$.

114. a) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).
   b) If P is the point (2, 1, 6) then find the point Q such that PQ is perpendicular to the plane in (i) and the midpoint of PQ lies on it.

115. If $\vec{u}, \vec{v}, \vec{w}$ are three non-coplanar unit vectors and α, β, γ are the angles between $\vec{u}$ and $\vec{v}$ and $\vec{w}$, $\vec{w}$ and $\vec{u}$ respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisection of the angles α, β, γ respectively. Prove that

$$[(\vec{x} \times \vec{y})(\vec{y} \times \vec{z})(\vec{z} \times \vec{x})] = \frac{1}{16}[\vec{u}\,\vec{v}\,\vec{w}]^2 \sec^2\frac{\alpha}{2} \sec^2\frac{\beta}{2} \sec^2\frac{\gamma}{2}.$$

116. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$. Prove that

$$(\vec{a} - \vec{d}).(\vec{b} - \vec{c}) \neq 0$$

$$(i.e, \vec{a}.\vec{b} + \vec{d}.\vec{c} \neq \vec{d}.\vec{b} + \vec{a}.\vec{c})$$

117. Find the equation of the plane passing through (1, 1, 1) and parallel to the lines L_1, L_2 having direction ratios (1, 0, -1), (1, -1, 0). Find the volume of tetrahedron formed by origin and the points where these planes intersect the coordinate axes.

118. A parallelopiped $'S'$ has base points A, B, C and D and upper face points A', B', C', D'. This parallelopiped is compressed by upper face $A'B'C'D'$ to form a new parallelopiped

$'T'$ having upper face points A'', B'', C'', D''. Volume of parallelopiped T is 90 perecnt of the volume of the parallelopiped S. Prove that the locus of a A'', is a plane.

119. P_1 and P_2 are planes passing through origin. L_1 and L_2 are two lines on P_1 and P_2 respectively such that their intersection is origin. Show that there exists points A, B, C whose permutation A', B', C' can be chosen such that
   a) A is on L_1, B on P_1 but not on L_1 and C not on P_1
   b) A' is on L_2, B' on P_2 but not on L_2 and C' not on P_2

120. Find the equation of the plane containing the line

$$2x - y + z - 3 = 0$$

$$3x + y + z = 5$$

and at a distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, -1).

121. If the incident ray on a surface is along the unit vector $\hat{v}$ the reflected ray is along the unit vector $\hat{w}$ and the normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.

(F). Match the following

122. Match the following:

Column I	Column II
(A) Two rays x+y=$\lvert a\rvert$ and ax-y=1 intersects each other in the first quadrant in the interval a $\in (a_0, \infty)$, the value of a_0 is	(p) 2
(B) Point (α, β, γ) lies on the plane x+y+z=2. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\hat{k} \times (\hat{k} \times \hat{a})$=0, then γ=	(q) $\frac{4}{3}$
(C) $\lvert \int_0^1 (1 - y^2) dy \rvert + \lvert \int_1^0 (y^2 - 1) dy \rvert$	(r) $\lvert \int_0^1 \sqrt{1 - x} dx \rvert + \lvert \int_{-1}^0 \sqrt{1 + x} dx \rvert$
(D) If $\sin A \sin B \sin C + \cos A \cos B = 1$, then value of $\sin C$=	(s) 1

123. Match the statements/expressions in **Column-I** with the values given in **Column-II**.

Column I

(A) Roots of the equation $2\sin^2\theta + \sin^2 2\theta$

(B) Points of discontinuity of the unction f(x)=$[\frac{6x}{\pi}]\cos[\frac{3x}{\pi}]$

(C) Volume of the parallelopiped with its edges represented by the vectors $\hat{i} + \hat{j}, \hat{i} + 2\hat{j}$ and $\hat{i} + \hat{j} + \pi\hat{k}$

(D) Angle between vector $\vec{a}$ and $\vec{b}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a} + \vec{b} + \sqrt{3}\vec{c} = 0$

Column II

124. Consider the following linear equations

$$ax + by + cz = 0$$

$$bx + cy + az = 0$$

$$cx + ay + bz = 0$$

Match the conditions/expressions in **Column-I** with statements in **Column-II** and indicate your answer by darkening the bubbles in the 4 × 4 matrix given in the *ORS*.

Column I	Column II
(A) a+b+c $\neq$ 0 and $a^2 + b^2 + c^2$=ab+bc+ca	(p) the equation represents planes meeting only at single point
(B) a+b+c=0 and $a^2 + b^2 + c^2 \neq$ ab+bc+ca	(q) the equation represents the line x=y=z
(C) a+b+c $\neq$ 0 and $a^2 + b^2 + c^2 \neq$ ab+bc+ca	(r) the equation represents identical planes
(D) a+b+c=0 and $a^2 + b^2 + c^2$=ab+bc+ca	(s) the equation represents the whole of the 3 dimensional space

125. Mtach the statements/expressions given in **Column-I** with the values in given in **Column-II**.

Column I	Column II
(A) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $(0, \frac{\pi}{2})$	(p) 1
(B) Values of k for which the planes kx+4y+z=0, 4x+ky+2z=0 and 2x+2y+z=0 intersect in a straight line	(q) 2
(C) Values of k for which $\|x-1\| + \|x-2\| + \|x+1\|\|x+2\| = 4k$ has integer solutions	(r) 3
(D) If $y' = y + 1$ and y(0)=1, then values of y(ln2)	(s) 4

126. Mtach the statement in **Column-I** with the values in given in **Column-II**.

Column I	Colu
(A) A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ and $\frac{x-8/3}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ=d, then d^2	(p)
(B) The values of x for $\tan^{-1}(x+3) - \tan^{-1}(x-3)$ =$\sin^{-1}(\frac{3}{5})$ are	(q)
(C) Non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy $\vec{a}.\vec{b} = 0$ $(\vec{b} - \vec{a}).(\vec{b} + \vec{c})$ and $2\|\vec{b} + \vec{c}\| = \|\vec{b} - \vec{a}\|$. If $\vec{a} = \mu\vec{b} + 4\vec{c}$, then the possible values of μ are	(r)
(D) Let f be the function in $[-\pi, \pi]$ given by f(0)=9 and $f(x) = \sin(\frac{9x}{2})/\sin(\frac{x}{2})$ for $x \neq 0$ The value of $\frac{2}{\pi}\int_{\pi}^{\pi} f(x)dx$ is	(s)

127. Mtach the statement in **Column-I** with the values in given in **Column-II**.

Column I	**Column II**
(A) If $\overrightarrow{a} = \hat{j} + \sqrt{3}\hat{k}, \overrightarrow{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\overrightarrow{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f(\frac{\pi}{6})$	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{ln3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\lvert Arg(\frac{1}{1-z})\rvert$ for $\lvert z\rvert=1$, $z \neq 1$ is given by	(s) π

128. Mtach the List-I with List-II and select the correct answer using the code given the below lists..

Column I	**Column II**
(A) Volume of parallelopiped determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and $\overrightarrow{c}$ is 2. Then the volume the parallelopiped determined by the vectors $2(\overrightarrow{a}+\overrightarrow{b}), 3(\overrightarrow{b}+\overrightarrow{c})$ and $2(\overrightarrow{c}+\overrightarrow{a})$ is	(p) 100
(B) Volume of parallelopiped determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and $\overrightarrow{c}$ is 5. Then the volume the parallelopiped determined by the vectors $3(\overrightarrow{a}+\overrightarrow{b}), 3(\overrightarrow{b}+\overrightarrow{c})$ and $2(\overrightarrow{c}+\overrightarrow{a})$ is	(p) 30
(C) Area of the triangle with adjacent sides determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\overrightarrow{a}+3\overrightarrow{b})$ and $\overrightarrow{a}-\overrightarrow{b}$ is	(r) 24
(D) Area of the parallelgram with adjacent sides determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is 30. Then the area of the parallelgram with adjacent sides determined by vectors $(\overrightarrow{a}+\overrightarrow{b})$ and $\overrightarrow{a}$ is	(s) 60

codes:

	P	Q	R	S
(a)	4	2	3	1
(b)	2	3	1	4
(c)	3	4	1	2
(d)	1	4	3	2

129. Mtach the statements/expressions given in **Column-I** with the values in given in **Column-II**.

Column I

(A) In R^2, if the magnitude of the projection vector o $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, the value of $\lvert\alpha\rvert$ is

(B) Let a and b real numbers such that the fu $f(x) = -3ax^2 - 2, x < 1$ and $f(x) = bx + a^2, x \geq 1$ if differen Then possible value of a is(are)

(C) Let $\omega \neq 1$ be a complex cube root of uni $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega +$ then possible value(s) of n is(are)

(D) Let the harmonic mean of two positive real number If q is a positive real number such that a,5,q,b is ar progression, then the value(s) of $\lvert q - a\rvert$ is(a

130. Consider the lines

$$L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$$

$$L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$$

and the planes $P_1 : 7x+y+2z = 3$, $P_2 : 3x+5y-6z = 4$. Let ax+by+cz=d be the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to the planes P_1 and P_2.

Column I	**Column II**
(P) a=	(1) 13
(Q) b=	(2) -3
(R) c=	(3) 1
(S) d=	(4) -2

codes:

	P	Q	R	S
(a)	3	2	4	1
(b)	1	3	4	2
(c)	3	2	1	4
(d)	2	4	1	3

131. Mtach the statements/expressions given in **Column-I** with the values in given in **Column-II**.

Column I	**Column II**
(A) In a triangle ΔXYZ, let a,b,c be the lengths of the sides opposite to the angles X,Y,Z respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is(are)	(p) 1
(B) In a triangle ΔXYZ, let a,b,c be the lengths of the sides opposite to the angles X,Y,Z respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X$ then possible values of $\frac{a}{b}$ is(are)	(q) 2
(C) In a R^2 let $\sqrt{3}\hat{i} + \hat{j}, \hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X,Y and Z w.r.t. to the origin O respectively. If the distance Z from the bisector of the acute angle of $\overrightarrow{OX}$ with $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible values of $\lvert\beta\rvert$ is(are)	(r) 3
(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by x=0, x=2, $y^2 = 4x$ and $y = \lvert\alpha x - 1\rvert + \lvert\alpha x - 2\rvert + \alpha x$, where $\alpha \in (0, 1)$ Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0, 1$ is(are)	(s) 5

(G). Comprehension Based Questions:
Consider the lines

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

132. The unit vector perpendicular to both L_1 and L_2 is
   a) $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$
   b) $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$
   c) $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$
   d) $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$
133. The shortest distance between L_1 and L_2 is
   a) 0
   b) $\frac{17}{\sqrt{3}}$
   c) $\frac{41}{5\sqrt{3}}$
   d) $\frac{17}{5\sqrt{3}}$
134. The distance of the point (1, 1, 1) from the plane passing through the point (-1, -2, -1) and whose normal is perpendicular to both the lines L_1 and L_2 is
   a) $\frac{2}{\sqrt{75}}$
   b) $\frac{7}{\sqrt{75}}$
   c) $\frac{13}{\sqrt{75}}$
   d) $\frac{23}{\sqrt{75}}$

(H). Assertion and Reason Type Questions

135. Consider the planes 3x - 6y - 2z = 15 and 2x + y - 2z = 5.
   STATEMENT-1: The parametric equations of the line of intersection of the given planes are x = 3 + 14t, y = 1 + 2t, z = 15t.
   STATEMENT-2: The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of given planes.
   a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
   b) Statement-1 is true, Statement-2 is false
   c) Statement-1 is false, Statement-2 is true
   d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
136. Let the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$,$\overrightarrow{ST}$, $\overrightarrow{TU}$ and $\overrightarrow{UP}$ represent the sides of a regular hexagon.
   STATEMENT-1: $\overrightarrow{PQ} \times (\overrightarrow{RS} + \overrightarrow{ST} \neq \vec{0}$
   STATEMENT-2: $\overrightarrow{PQ}\times\overrightarrow{RS} = \vec{0}$ and $\overrightarrow{PQ}\times\overrightarrow{ST} \neq \vec{0}$.
   a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
   b) Statement-1 is true, Statement-2 is false
   c) Statement-1 is false, Statement-2 is true
   d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
137. Consider three planes

$$P_1 : x - y + z = 1$$

$$P_2 : x + y - z = 1$$

$$P_3 : x - 3y + 3z = 2$$

   Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, P_1 and P_2 respectively.
   STATEMENT-1: At least two of the lines L_1, L_2, L_3 are non-parallel
   STATEMENT-2: The three planes does not have a common point.
   a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
   b) Statement-1 is true, Statement-2 is false
   c) Statement-1 is false, Statement-2 is true
   d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(I). Integer Value Correct Type:

138. If $\vec{a}$ and $\vec{b}$ are vectors in space given by

$$\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$$

$$\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$$

   then find the value of $(2\vec{a} + \vec{b}).[(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})]$.
139. If the distance between the plane Ax - 2y + z

= d and the plane containing the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

is $\sqrt{6}$, then find $|d|$.

140. Let $\vec{a} = -\vec{i} - \vec{k}$, $\vec{b} = -\vec{i} + \vec{j}$ and $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$, then the value of $\vec{r}.\vec{b}$ is

141. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying

$$|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9,$$

then $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ is

142. Consider the set of eight vectors

$$V = \{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\}$$

Three non-copolanar vectors can be chosen from V in 2^p ways. Then p is

143. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k - 20 =

144. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-copolar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If

$$(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) = p\vec{a} + q\vec{b} + r\vec{c},$$

where p, q, r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is

145. Suppose that $\vec{p}$, $\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}$, $\vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r})$, $(\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively. then the value of 2x+y+z is

146. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b}=0$. For some x, y $\in$ R, let

$$\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$$

If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is

147. Let P be a point in the first octant, whose image Q in the plane x + y = 3(that is the line segment PQ is perpendicular to the plane x + y = 3 and mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is

148. Consider the cube in the first octant with sides OP, OQ, OR of length 1, along the x-axis, y-axis, z-axis respectively, where O(0, 0, 0) is the origin. Let S($\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$) be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is

149. Three lines are given by $\vec{r} = \lambda\hat{i}$, $\lambda \in$ R; $\vec{r} = \mu(\hat{i} + \hat{j})$, $\mu \in R$ and $\vec{r} = v(\hat{i} + \hat{j} + \hat{k})$, v $\in$ R. Let the lines cut the plane x+y+z=1 at the points A, B, C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals

150. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\hat{a} + \beta\hat{b}$, $\alpha, \beta \in R$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ equals

Section-B

151. A plane which passes through the point (3, 2, 0) and the line

$$\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$$

is

a) x - y + z = 1
b) x + y + z = 5
c) x + 2y -z = 1
d) 2x - y + z = 5

152. If $|\vec{a}| = 4$, $|\vec{b}| = 2$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $(\vec{a} \times \vec{b})^2$ is equal to

a) 48
b) 16
c) $\vec{a}$
d) none of these

153. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors show that $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 7$, $|\vec{b}| = 5$, $|\vec{c}| = 3$ then angle between vector $\vec{b}$ and $\vec{c}$ is

a) 60^o

b) 30^o
c) 45^o
d) 90^o

154. If $|\vec{a}| = 5$, $|\vec{b}| = 4$, $|\vec{c}| = 3$ thus what will be the value of $|\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}|$, given that $\vec{a} + \vec{b} + \vec{c} = 0$
a) 25
b) 50
c) -25
d) -50

155. If the vectors $\vec{c}$, $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{b} = \hat{j}$ are such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right handed system then $\vec{c}$ is
a) $z\hat{i} - x\hat{k}$
b) 0
c) $y\hat{j}$
d) $-z\hat{i} + x\hat{k}$

156. $\vec{a} = 3\hat{i} - 5\hat{j}$ and $\vec{b} = 6\hat{i} + 3\hat{j}$ are two vectors and $\vec{c}$ is a vector such that $\vec{c} = \vec{a} \times \vec{b}$ then $|\vec{a}| : |\vec{b}| : |\vec{c}|$
a) $\sqrt{34} : \sqrt{45} : \sqrt{39}$
b) $\sqrt{34} : \sqrt{45} : 39$
c) 34 : 39 : 45
d) 39 : 35 : 34

157. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ then $\vec{a} + \vec{b} + \vec{c} =$
a) abc
b) -1
c) 0
d) 2

158. The d.r. of a normal to the plane through (1,0,0), (0,1,0) which makes an angle $\pi/4$ with plane x+y=3 are
a) $1, \sqrt{2}, 1$
b) $1, 1, \sqrt{2}$
c) 1, 1, 2
d) $\sqrt{2}, 1, 1$

159. Let $\vec{u} = \hat{i}+\hat{j}$, $\vec{v} = \hat{i}-\hat{j}$ and $\vec{w} = \hat{i}+2\hat{j}+3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}.\hat{n} = 0$ and $\vec{v}, \hat{n} = 0$, then $|\vec{w}.\hat{n}|$ is equal to
a) 3
b) 0
c) 1
d) 2

160. A particle acted on by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ is displaced from the point $\hat{i} + 2\hat{j} - 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. The total work done the by the forces is
a) 50 units
b) 20 units
c) 30 units
d) 40 units

161. The vectors $\vec{AB} = 3\hat{i}+4\hat{k}$ and $\vec{AC} = 5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a triangle ABC. The length of the median through A is
a) $\sqrt{288}$
b) $\sqrt{18}$
c) $\sqrt{72}$
d) $\sqrt{33}$

162. The shortest distance from the plane

$$12x + 4y + 3z = 327 \qquad (4.8.162.1)$$

to the sphere

$$x^2 + y^2 + z^2 + 4x - 2y - 6z = 155 \qquad (4.8.162.2)$$

is
a) 39
b) 26
c) $11\frac{4}{13}$
d) 13

163. The two lines

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$$

and

$$\frac{x-1}{k} = \frac{y-4}{1} = \frac{z-5}{1}$$

are coplanar if
a) k = 3 or -2
b) k = 0 or -1
c) k = 1 or -1
d) k = 0 or -3

164. $\vec{a}, \vec{b}, \vec{c}$ are 3 vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 1$, $|\vec{b}| = 2$, $\vec{c} = 3$, then $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0$ is equal to
a) 1
b) 0
c) -7
d) 7

165. The radius of the circle in which the sphere

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0 \qquad (4.8.165.1)$$

is cut by the plane

$$x + 2y + 2z + 7 = 0 \quad (4.8.165.2)$$

is

a) 4
b) 1
c) 2
d) 3

166. A tetrahedron has vertices O(0,0,0), A(1,2,1), B(2,1,3) and C(-1,1,2). Then the angle between the faces OAB and ABC will be

a) 90^o
b) $\cos^{-1}(\frac{19}{35})$
c) $\cos^{-1}(\frac{17}{31})$
d) 30^o

167. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$=0 and vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then the product abc equals

a) 0
b) 2
c) -1
d) 1

168. Consider a points A, B, C, D with position vectors $7\hat{i} - 4\hat{j} + 7\hat{k}$, $\hat{i} - 6\hat{j} + 10\hat{k}$, $-\hat{i} - 3\hat{j} + 4\hat{k}$ and $5\hat{i} - \hat{j} + 5\hat{k}$ respectively. Then ABCD is a

a) parallelogram but not a rhombus
b) square
c) rhombus
d) rectangle

169. If $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ are three non-coplanar vectors, then

$$(\overrightarrow{u} + \overrightarrow{v} - \overrightarrow{w}).(\overrightarrow{u} - \overrightarrow{v}) \times (\overrightarrow{v} - \overrightarrow{w})$$

equals

a) $3\overrightarrow{u}.\overrightarrow{v} \times \overrightarrow{w}$
b) 0
c) $\overrightarrow{u}.\overrightarrow{v} \times \overrightarrow{w}$
d) $\overrightarrow{u}.\overrightarrow{w} \times \overrightarrow{v}$

170. Two system of rectangle axes have the same origin. If a plane cuts them at distances a,b,c and a', b', c' from the origin then

a) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} = 0$
b) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 0$
c) $\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} = 0$
d) $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} = 0$

171. Distance between two parallel planes

$$2x + y + 2z = 8 \quad (4.8.171.1)$$

$$4x + 2y + 4z + 5 = 0 \quad (4.8.171.2)$$

is

a) $\frac{9}{2}$
b) $\frac{5}{2}$
c) $\frac{7}{2}$
d) $\frac{3}{2}$

172. A line with direction cosines proportional to 2, 1, 2 meets each of the lines

$$x = y + a = z \quad (4.8.172.1)$$

$$x + a = 2y = 2z \quad (4.8.172.2)$$

The coordinates os each of the points of intersection are given by

a) (2a,3a,3a), (2a,a,a)
b) (3a,2a,3a), (a,a,a)
c) (3a,2a,3a), (a,a,2a)
d) (3a,3a,3a), (a,a,a)

173. If the straight lines

$$x = 1 + s, y = -3 - \lambda s, z = 1 + \lambda s \quad (4.8.173.1)$$

$$x = \frac{t}{2}, y = 1 + t, z = 2 - t \quad (4.8.173.2)$$

a) 0
b) -1
c) $\frac{-1}{2}$
d) -2

174. The intersection of the spheres

$$x^2 + y^2 + z^2 + 7x - 2y - z = 13 \quad (4.8.174.1)$$

$$x^2 + y^2 + z^2 - 3x + 3y + 4z = 8 \quad (4.8.174.2)$$

is the same as the intersection of one of the sphere and the plane

a) 2x-y-z=1
b) x-2y-z=1
c) x-y-2z=1
d) x-y-z=1

175. Let $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ be three non zero vectors such that no two of these are collinear. If the vector $\overrightarrow{a} + 2\overrightarrow{b}$ is collinear with $\overrightarrow{c}$ and $\overrightarrow{b} + 3\overrightarrow{c}$ is collinear with $\overrightarrow{a}$(λ being some non-zero scalar)

then $\overrightarrow{a} + 2\overrightarrow{b} + 6\overrightarrow{c}$ equals

a) 0
b) $\lambda\overrightarrow{b}$
c) $\lambda\overrightarrow{c}$
d) $\lambda\overrightarrow{a}$

176. A particle is acted upon by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ which displace it from a point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. The work done in standard units by the forces is given by

a) 15
b) 30
c) 25
d) 40

177. If $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ are non-coplanar vectors and λ is a real number, then the vctors $\overrightarrow{a} + 2\overrightarrow{b} + 3\overrightarrow{c}$, $\lambda\overrightarrow{b} + 4\overrightarrow{c}$ and $(2\lambda - 1)\overrightarrow{c}$ are non coplanar for

a) no values of λ
b) all except one value of λ
c) all except two values of λ
d) all value of λ

178. Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be such that $\lceil\overrightarrow{u}|=1$, $\lceil\overrightarrow{v}|=2$, $\lceil\overrightarrow{w}|=3$. If the projection $\overrightarrow{v}$ along $\lceil\overrightarrow{u}|$ is equal to that of $\overrightarrow{w}$ along $\overrightarrow{u}$ and $\overrightarrow{v}, \overrightarrow{w}$ are perpendicular to each other then $\lceil\overrightarrow{u} - \overrightarrow{v} + \overrightarrow{w}|$ equals

a) 14
b) $\sqrt{7}$
c) $\sqrt{14}$
d) 2

179. Let $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ be non-zero vectors such that $(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c} = \frac{1}{3}|\overrightarrow{b}|\lceil\overrightarrow{c}\lceil\overrightarrow{a}$. If θ is the acute angle between the vectors $\overrightarrow{b}$ and $\overrightarrow{c}$, then $\sin\theta$ equals

a) $\frac{2\sqrt{2}}{3}$
b) $\frac{\sqrt{2}}{3}$
c) $\frac{2}{3}$
d) $\frac{1}{3}$

180. If C is the mid-point of AB and P is any point outside AB, then

a) $\overrightarrow{AB} + \overrightarrow{PB} = 2\overrightarrow{PC}$
b) $\overrightarrow{PA} + \overrightarrow{PB} = \overrightarrow{PC}$
c) $\overrightarrow{PA} + \overrightarrow{PB} + 2\overrightarrow{PC} = \overrightarrow{0}$
d) $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{0}$

181. If the angle θ between the line

$$\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$$

and the plane $2x - y + \sqrt{\lambda}z + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$ then the value of λ is

a) $\frac{5}{3}$
b) $\frac{-3}{5}$
c) $\frac{3}{4}$
d) $\frac{-4}{3}$

182. The angle between the lines

$$2x = 3y = -z$$

$$6x = -y = -4z$$

is

a) 0^o
b) 90^o
c) 45^o
d) 30^o

183. If the plane

$$2ax - 3ay + 4az + 6 = 0$$

passes through the midpoint of the line joining the centres of the spheres

$$x^2 + y^2 + z^2 + 6x - 8y - 2z = 13 \quad (4.8.183.1)$$

$$x^2 + y^2 + z^2 - 10x + 4y - 2z = 8 \quad (4.8.183.2)$$

the a equals

a) -1
b) 1
c) -2
d) 2

184. The distance between the lines

$$\overrightarrow{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(i - j + 4k)$$

and the plane

$$\overrightarrow{r}.(\hat{i} + 5\hat{j} + \hat{k}) = 5$$

is

a) $\frac{10}{9}$
b) $\frac{10}{3\sqrt{3}}$
c) $\frac{3}{10}$
d) $\frac{10}{3}$

185. For any vector $\overrightarrow{a}$, the value of

$$(\overrightarrow{a} \times \hat{i})^2 + (\overrightarrow{a} \times \hat{j})^2 + (\overrightarrow{a} \times \hat{k})^2$$

is equal to

a) $3\overrightarrow{a}^2$

b) $\overrightarrow{a}^2$
c) $2\overrightarrow{a}^2$
d) $4\overrightarrow{a}^2$

186. If non-zero numbers a,b,c are in H.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point. The point is
a) (-1, 2)
b) (-1 -2)
c) (1, -2)
d) $(1, \frac{-1}{2}$

187. Let a, b and c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is
a) the Geometric Mean of a and b
b) the Arithmetic Mean of a and b
c) equal to zero
d) the Harmonic Mean of a and b

188. If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are non-coplanar vectors and λ is a real number the

$$\lambda(\overrightarrow{a}+\overrightarrow{b})\lambda^2\overrightarrow{b}\lambda\overrightarrow{c}=[\overrightarrow{a}(\overrightarrow{b}+\overrightarrow{c})\overrightarrow{b}]$$

for
a) exactly one value of λ
b) no value of λ
c) exactly three values of λ
d) exactly two values of λ

189. Let $\overrightarrow{a}=\hat{i}-\hat{k}$, $\overrightarrow{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and $\overrightarrow{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$. Then $[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}]$ depends on
a) only y
b) only x
c) both x and y
d) neither x nor y

190. The plane

$$x+2y-z=4$$

cuts the sphere

$$x^2+y^2+z^2-x+z-2=0 \qquad (4.8.190.1)$$

in a circle of radius
a) 3
b) 1
c) 2
d) $\sqrt{2}$

191. If $(\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c}=\overrightarrow{a}\times(\overrightarrow{b}\times\overrightarrow{c})$ where $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are any three vectors such that $\overrightarrow{a}.\overrightarrow{b}\neq 0$, $\overrightarrow{b}.\overrightarrow{c}\neq 0$ then $\overrightarrow{a}$ and $\overrightarrow{c}$ are
a) inclined at an angle of $\frac{\pi}{3}$ between them
b) inclined at an angle of $\frac{\pi}{6}$ between them
c) perpendicular
d) parallel

192. The values of a , for which points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $a\hat{i}-3\hat{j}+\hat{k}$ respectively are the vertices of a right angled triangle with C=$\frac{\pi}{2}$ are
a) 2 and 1
b) -2 and -1
c) -2 and 1
d) 2 and -1

193. The two lines x=ay+b, z=cy+d; and x=$a'y+b'$, z=$c'y+d'$ are perpendicular to each other if
a) $aa'+cc'=-1$
b) $aa'+cc'=1$
c) $\frac{a}{a'}+\frac{c}{c'}$=-1
d) $\frac{a}{a'}+\frac{c}{c'}$=1

194. The image of the point (-1,3,4) in the plane x-2y=0 is
a) $(\frac{-17}{3}, \frac{-19}{3}), 4$
b) (15,11,4)
c) $(\frac{-17}{3}, \frac{-19}{3}), 1$
d) None of these

195. If a line makes an angle of $\pi/4$ with the positive directions of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is
a) $\frac{\pi}{4}$
b) $\frac{\pi}{2}$
c) $\frac{\pi}{6}$
d) $\frac{\pi}{3}$

196. If $\hat{u}$ and $\hat{v}$ are unit vectors and θ is the acute angle between them, then $2\hat{u}\times 3\hat{v}$ is a unit vector for
a) no value of θ
b) exactly one value of θ
c) exactly two values of θ
d) more than two values of θ

197. If (2,3,5) is one end of a diameter of the sphere

$$x^2+y^2+z^2-6x-12y-2z+20=0 \qquad (4.8.197.1)$$

then the coordinates of the other end of the diameter are
a) (4,3,5)
b) (4,3,-3)

c) (4,9,-3)
d) (4,-3,3)

198. Let $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}$, $\overrightarrow{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\overrightarrow{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$. If the vectors $\overrightarrow{c}$ lies in the plane of $\overrightarrow{a}$ and $\overrightarrow{b}$, then x equals
a) -4
b) -2
c) 0
d) 1

199. If L be the line of intersection of the planes
$$2x + 3y + z = 1$$
$$x + 3y + 2z = 2$$
If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals
a) 1
b) $\frac{1}{\sqrt{2}}$
c) $\frac{1}{\sqrt{3}}$
d) $\frac{1}{2}$

200. The vector $\overrightarrow{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$ lies in the plane of the vectors $\overrightarrow{b} = \hat{i} + \hat{j}$ and $\overrightarrow{c} = \hat{j} + \hat{k}$ and bisects the angle between $\overrightarrow{b}$ and $\overrightarrow{c}$. Then whoch one of the following fives possible values of α and β?
a) $\alpha = 2, \beta = 2$
b) $\alpha = 1, \beta = 2$
c) $\alpha = 2, \beta = 1$
d) $\alpha = 1, \beta = 1$

201. The non-zero vectors are $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ are related by $\overrightarrow{a} = 8\overrightarrow{b}$ and $\overrightarrow{c} = 7\overrightarrow{b}$. Then the angle between $\overrightarrow{a}$ and $\overrightarrow{c}$ is
a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

202. The line passing through the points (5,1,a) and (3,b,1) crosses the yz-plane at the point $(0, \frac{17}{2}, \frac{-13}{2})$. Then
a) a=2, b=8
b) a=4, b=6
c) a=6, b=4
d) a=8, b=2

203. If the straight lines
$$\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$$
and
$$\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$$
intersect at a point, then the integer k is equal to
a) -5
b) 5
c) 2
d) -2

204. Let the line
$$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$$
lie in the plane x+3y-αz+β=0. Then (α, β) equals
a) (-6,7)
b) (5,-15)
c) (-5,5)
d) (6,-17)

205. The projections of a vector om the three co-ordinates axis are 6, -3, 2 respectively. The direction cosnies of the vector are:
a) $\frac{6}{5}, \frac{-3}{5}, \frac{2}{5}$
b) $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$
c) $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}$
d) 6, -3, 2

206. If $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ are non=coplanar vectors and p,q are real numbers then the equality
$$[3\overrightarrow{u}p\overrightarrow{v}p\overrightarrow{w}] - [p\overrightarrow{v}\overrightarrow{w}q\overrightarrow{u}] - [2\overrightarrow{w}q\overrightarrow{v}q\overrightarrow{u}] = 0$$
holds for:
a) exactly two values of (p,q)
b) more than two but not all values of (p,q)
c) all values of (p,q)
d) exactly one value of (p,q)

207. Let $\overrightarrow{a} = \hat{j} - \hat{k}$ and $\overrightarrow{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\overrightarrow{b}$ satisfying $\overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$ and $\overrightarrow{a}.\overrightarrow{b} = 3$
a) $2\hat{i} - \hat{j} + 2\hat{k}$
b) $\hat{i} - \hat{j} - 2\hat{k}$
c) $\hat{i} + \hat{j} - 2\hat{k}$
d) $-\hat{i} + \hat{j} - 2\hat{k}$

208. If the vectors $\overrightarrow{a} = \hat{i} - \hat{j} + 22\hat{k}$, $\overrightarrow{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\overrightarrow{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then (λ, μ)=
a) 2,-3
b) -2,3
c) 3,-2

d) -3,2

209. **Statement-1**: The point A(3,1,6) is the mirror image of the point B(1,3,4) in the plane x-y+z=5

Statement-2: The plane x-y+z=5 bisects the line segment joining A(3,1,6) and B(1,3,4).

a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
b) Statement-1 is true, Statement-2 is false
c) Statement-1 is false, Statement-2 is true
d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

210. A line AB in three-dimensional space makes angle 45^o and 120^o with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals

a) 45^o
b) 60^o
c) 75^o
d) 30^o

211. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane x+2y+3z=4 is $\cos^{-1}(\sqrt{\frac{5}{14}})$, then λ equals

a) $\frac{3}{2}$
b) $\frac{2}{5}$
c) $\frac{5}{3}$
d) $\frac{2}{3}$

212. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i}+\hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$, then the value of $(2\vec{a} - \vec{b})[(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})]$ is

a) -3
b) 5
c) -3
d) -5

213. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to

a) $\vec{c} + (\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}})\vec{b}$
b) $\vec{b} + (\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}})\vec{c}$
c) $\vec{c} - (\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}})\vec{b}$
d) $\vec{b} - (\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}})\vec{c}$

214. **Statement-1**: The point A(1,0,7) is the mirror image of the point B(1,6,3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Statement-2: The plane $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining A(1,0,7) and B(1,6,3).

a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
b) Statement-1 is true, Statement-2 is false
c) Statement-1 is false, Statement-2 is true
d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

215. Let $\vec{a}$ and $\vec{b}$ be two unit vectors. If the vectors $\vec{c} = \vec{a} + 2\vec{b}$ and $\vec{d} = 5\vec{a} - 4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is:

a) $\frac{\pi}{6}$
b) $\frac{\pi}{2}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{4}$

216. A equation of a plane parallel to the plane

$$x - 2y + 2z - 5 = 0 \qquad (4.8.216.1)$$

and at a unit distance from the origin is:

a) x-2y+2z-3=0
b) x-2y+2z+1=0
c) x-2y+2z-1=0
d) x-2y+2z+5=0

217. If the line

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$$

and

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$$

intersect, then k is equal to

a) -1
b) $\frac{2}{9}$
c) $\frac{9}{2}$
d) 0

218. Let ABCD be a parallelogram such that $\vec{AB} = \vec{q}$, $\vec{AD} = \vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coinside with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by:

a) $\vec{r} = 3\vec{q} - \frac{3(\vec{p}.\vec{q})}{(\vec{p}.\vec{p})}\vec{p}$

b) $\vec{r} = -\vec{q} + \frac{(\vec{p}.\vec{q})}{(\vec{p}.\vec{p})}\vec{p}$
c) $\vec{r} = \vec{q} - \frac{(\vec{p}.\vec{q})}{(\vec{p}.\vec{p})}\vec{p}$
d) $\vec{r} = -3\vec{q} - \frac{3(\vec{p}.\vec{q})}{(\vec{p}.\vec{p})}\vec{p}$

219. Distance between the two parallel planes

$$2x + y + 2z = 8$$

$$4x + 2y + 4z + 5 = 0$$

is
a) $\frac{3}{2}$
b) $\frac{5}{2}$
c) $\frac{7}{2}$
d) $\frac{9}{2}$

220. If the lines

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$$

and

$$\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$$

are coplanr, then k can have
a) any value
b) exactly one value
c) exactly two values
d) exactly three values

221. If the vectors $\overrightarrow{AB} = 3\hat{i}+4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is
a) $\sqrt{18}$
b) $\sqrt{72}$
c) $\sqrt{33}$
d) $\sqrt{45}$

222. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane 2x-y+z+3=0 is the line:
a) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$
b) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$
c) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$
d) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

223. The angle between the lines whose direction cosines satisfy the equations

$$l + m + n = 0$$

$$l^2 = m^2 + n^2$$

is
a) $\frac{\pi}{6}$
b) $\frac{\pi}{2}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{4}$

224. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and

$$(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is:
a) $\frac{2}{3}$
b) $\frac{-2\sqrt{3}}{3}$
c) $\frac{2\sqrt{2}}{3}$
d) $\frac{-\sqrt{2}}{3}$

225. The equation of the plane containing the line

$$2x - 5y + z = 3$$

$$x + y + 4z = 5$$

and parallel to the plane x+3y+6z=1 is:
a) x+3y+6z=7
b) 2x+6y+12z=-13
c) 2x+6y+12z=13
d) x+3y+6z=-7

226. The distance of the point (1,0,2) from the point of intersection of the line

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$$

and the plane x-y+z=16, is
a) $3\sqrt{21}$
b) 13
c) $2\sqrt{14}$
d) 8

227. If the line

$$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$$

lies in the plane, lx+my-z=9, then $l^2 + m^2$ is equal to:
a) 5
b) 2
c) 26
d) 18

228. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three unit vectors such that

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$$

If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is:

a) $\frac{2\pi}{3}$
b) $\frac{5\pi}{6}$
c) $\frac{3\pi}{4}$
d) $\frac{\pi}{2}$

229. The distance of the point (1,-5,9) from the plane x-y+z=5 measured along the line x=y=z is:

a) $\frac{10}{\sqrt{3}}$
b) $\frac{20}{3}$
c) $3\sqrt{10}$
d) $10\sqrt{3}$

230. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that

$$|\vec{c} - \vec{a}| = 3, |(\vec{a} \times \vec{b}) \times \vec{c}| = 3$$

and the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$ be 30^o. Then $\vec{a}.\vec{c}$ is equal to:

a) $\frac{1}{8}$
b) $\frac{25}{8}$
c) 2
d) 5

231. If the image of the point P(1,-2,3) in the plane 2x+3y-4z+22=0 measured parallel to line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to

a) $6\sqrt{5}$
b) $3\sqrt{5}$
c) $2\sqrt{42}$
d) $\sqrt{42}$

232. The distance of the point (1,3,-7) from the plane passing through the point (1,-1,-1) having normal perpendicular to both the lines

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$$

and

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$$

is

a) $\frac{10}{\sqrt{74}}$
b) $\frac{20}{\sqrt{74}}$
c) $\frac{10}{\sqrt{83}}$
d) $\frac{5}{\sqrt{83}}$

233. Let $\vec{u}$ be a vector coplanr with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b} - 24$, then $|\vec{u}|^2$ is equal to:

a) 315
b) 256
c) 84
d) 336

234. The length of the projection of the line segment joining the points (5,-1,4) and (4,-1,3) on the plane x+y+z=7 is:

a) $\frac{2}{3}$
b) $\frac{1}{3}$
c) $\sqrt{\frac{2}{3}}$
d) $\frac{2}{\sqrt{3}}$

235. If L_1 is the line of intersection of the planes

$$2x - 2y + 3z - 2 = 0$$

$$x - y + z + 1 = 0$$

and L_2 is the line of intersection of the planes

$$x + 2y - z - 3 = 0$$

$$3x - y + 2z - 1 = 0$$

then the distance of the origin from the plane, containing the lines L_1 and L_2 is

a) $\frac{1}{3\sqrt{2}}$
b) $\frac{1}{2\sqrt{2}}$
c) $\frac{1}{\sqrt{2}}$
d) $\frac{1}{4\sqrt{2}}$

236. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c}$ be a vector such that $\vec{a} \times \vec{c} + \vec{b} = \vec{0}$ and $\vec{a}.\vec{c} = 4$, then $|\vec{c}|^2$ is equal to

a) $\frac{19}{2}$
b) 9
c) 8
d) $\frac{17}{2}$

237. The equation of the line passing through the point (-4,3,1), parallel to the plane x+2y-z-5=0 and intersecting the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z-2}{-1}$ is

a) $\frac{x-4}{-3} = \frac{y+3}{1} = \frac{z+1}{4}$
b) $\frac{x+4}{1} = \frac{y-3}{1} = \frac{z-1}{3}$
c) $\frac{x+4}{3} = \frac{y-3}{-1} = \frac{z-1}{1}$
d) $\frac{x+4}{-1} = \frac{y-3}{1} = \frac{z-1}{1}$

238. The plane through the inersection of the planes x+y+z=1 and 2x+3y-z+4=0 and parallel to y-axis also passes through the point:

a) (-3,0,-1)
b) (-3,1,1)
c) (3,3,-1)
d) (3,2,1)

239. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ meets the plane, x+2y+3z=15 at a point P, then the distance of P from the origin is

a) $\frac{\sqrt{5}}{2}$
b) $2\sqrt{5}$
c) $\frac{9}{2}$
d) $\frac{7}{2}$

240. A plane passing through the points (0, -1, 0) and (0, 0, 1) and making an angle $\frac{\pi}{4}$ with the plane y - z + 5 = 0, also passes through the point:

a) $(-\sqrt{2}, 1, -4)$
b) $(\sqrt{2}, -1, 4)$
c) $(-\sqrt{2}, -1, -4)$
d) $(\sqrt{2}, 1, 4)$

241. Let $\overrightarrow{\alpha} = 3i + \hat{j}$ and $\overrightarrow{\beta} = 2\hat{i} - \hat{j} + 3\hat{k}$. If $\overrightarrow{\beta} = \overrightarrow{\beta_1} - \overrightarrow{\beta_2}$, where $\overrightarrow{\beta_1}$ is parallel to $\overrightarrow{\alpha}$ and $\overrightarrow{\beta_2}$ is perpendicular to $\overrightarrow{\alpha}$, then $\overrightarrow{\beta_1} \times \overrightarrow{\beta_2}$ is equal to:

a) $-3\hat{i} + 9\hat{j} + 5\hat{k}$
b) $3\hat{i} - 9\hat{j} - 5\hat{k}$
c) $\frac{1}{2}(-3\hat{i} + 9\hat{j} + 5\hat{k})$
d) $\frac{1}{2}(3\hat{i} - 9\hat{j} + 5\hat{k})$