Linear Algebra through Coordinate Geometry

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Contents

***Abstract*—This manual introduces linear algebra through coordinate geometry using a problem solving approach.**

1 The Straight Line

1.1 The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{A} - \mathbf{B}\right) \tag{1.1}$$

Alternatively, it can be expressed as

$$\mathbf{n}^T\left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2}$$

where $\mathbf{n}$ is the solution of

$$\left(\mathbf{A} - \mathbf{B}\right)^T\mathbf{n} = 0 \tag{1.3}$$

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1.2 In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.4}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{1.5}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \tag{1.6}$$

Find the area of $\triangle ABC$.

Solution: The centroid $\mathbf{O}$ is the solution of (1.5),(1.6) and is obtained as the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} \tag{1.7}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix}1 & 1 & 5\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 5\\0 & 1 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 1\end{pmatrix} \tag{1.8}$$

Thus,

$$\mathbf{O} = \begin{pmatrix}4\\1\end{pmatrix} \tag{1.9}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.10}$$
$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix}11\\1\end{pmatrix} \tag{1.11}$$
$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}11\\1\end{pmatrix} \tag{1.12}$$

From (1.6) and (1.12),

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} = 5 \tag{1.13}$$
$$\implies 5 + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 12 \tag{1.14}$$
$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 7 \tag{1.15}$$

From (1.15) and (1.6), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}7\\4\end{pmatrix} \tag{1.16}$$

using the augmented matrix as

$$\begin{pmatrix}1 & 1 & 7\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 7\\0 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 3\end{pmatrix} \tag{1.17}$$

$$\implies \mathbf{C} = \begin{pmatrix}4\\3\end{pmatrix} \tag{1.18}$$

From (1.11),

$$\mathbf{B} = \begin{pmatrix}11\\1\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}7\\-2\end{pmatrix} \tag{1.19}$$

Thus,

$$\frac{1}{2}\begin{vmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\\1 & 1 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}1 & 7 & 4\\2 & -2 & 3\\1 & 1 & 1\end{vmatrix} = 9 \tag{1.20}$$

2 Orthogonality

2.1 $\mathbf{u}^T\mathbf{x} = 0 \implies \mathbf{u} \perp \mathbf{x}$. Show that

$$\mathbf{u}^T\mathbf{x} = \mathbf{P}^T\mathbf{x} = 0 \implies \mathbf{P} = \alpha\mathbf{u} \tag{2.1}$$

2.2 The foot of the perpendicular drawn from the origin on the line

$$AB : \mathbf{u}^T\mathbf{x} = \lambda \tag{2.2}$$

where

$$\mathbf{u} = \begin{pmatrix}3\\1\end{pmatrix} \tag{2.3}$$

is $\mathbf{P}$. The line meets the x-axis at $\mathbf{A}$ and y-axis at $\mathbf{B}$. Show that $\mathbf{P} = \alpha\mathbf{u}$ and find α.
Solution: From (2.2),

$$\mathbf{u}^T\mathbf{A} = \mathbf{u}^T\mathbf{B} = \lambda \tag{2.4}$$

$$\implies \mathbf{u}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.5}$$

Since $OP \perp AB$,

$$\mathbf{P}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.6}$$

Thus, from (2.1),

$$\mathbf{P} = \alpha\mathbf{u} \tag{2.7}$$

Since $\mathbf{P}$ lies on (2.2),

$$\mathbf{u}^T\mathbf{P} = \alpha\mathbf{u}^T\mathbf{u} = \lambda \tag{2.8}$$

$$\implies \alpha = \frac{\lambda}{\mathbf{u}^T\mathbf{u}} = \frac{\lambda}{10}. \tag{2.9}$$

2.3 Find $\mathbf{A}$.
Solution: Let

$$\mathbf{A} = a\begin{pmatrix}1\\0\end{pmatrix} \tag{2.10}$$

From (2.2),

$$\mathbf{u}^T\mathbf{A} = a\begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \lambda \tag{2.11}$$

$$\implies a = \frac{\lambda}{3} \tag{2.12}$$

$$\text{and } \mathbf{A} = \frac{\lambda}{3}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.13}$$

2.4 Find the ratio $BP : PA$.
Solution: Let

$$\frac{BP}{PA} = k \tag{2.14}$$

Then,

$$k\mathbf{A} + \mathbf{B} = (k+1)\mathbf{P} \tag{2.15}$$

$$\implies k\mathbf{A}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} = (k+1)\mathbf{P}^T\mathbf{A} \tag{2.16}$$

$$\implies ka^2 = \alpha(k+1)\lambda \tag{2.17}$$

using (2.7), (2.10), (2.2) and $\mathbf{A} \perp \mathbf{B}$. Substituting from (2.9) and (2.12),

$$\implies k\frac{\lambda^2}{9} = (k+1)\frac{\lambda^2}{10} \tag{2.18}$$

$$\implies k = 9 \tag{2.19}$$

3 Matrix Transformations

3.1 Find $\mathbf{R}$, the reflection of $\mathbf{P} = \begin{pmatrix}4\\1\end{pmatrix}$ about the line

$$L : \quad \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.1}$$

Solution: The reflection of $\mathbf{P}$ about L is given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\mathbf{n} \tag{3.2}$$

where

$$L : \mathbf{n}^T\mathbf{x} = c \tag{3.3}$$

$$\mathbf{m}^T\mathbf{n} = 0 \tag{3.4}$$

Substituting

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c = 0 \tag{3.5}$$

in (3.6),

$$\frac{\mathbf{R}}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \implies \mathbf{R} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{3.6}$$

3.2 $\mathbf{R}$ is translated through a distance 2 units along the positive direction of x-axis to obtain $\mathbf{S}$. Find $\mathbf{S}$.
Solution:

$$\mathbf{S} = \mathbf{R} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.7}$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.8}$$

3.3 Rotate $\mathbf{S}$ through an angle of $\frac{\pi}{4}$ about the origin in the counter clockwise direction to obtain $\mathbf{T}$.
Solution:

$$\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{S} \tag{3.9}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.10}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 7 \end{pmatrix} \tag{3.11}$$

4 Locus

4.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.1}$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed as shown in Fig. (4.1),

4.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{4.2}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{4.3}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{4.4}$$

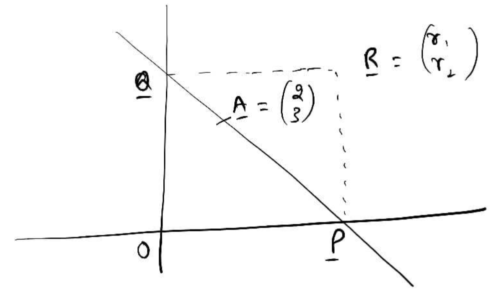

Fig. 4.1

4.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{n} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{n} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{n} &= 0 \end{aligned} \tag{4.5}$$

Solution: Trivial using (1.2) and (1.3).

4.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{n} = 0 \tag{4.6}$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{n} = 0 \tag{4.7}$$

Solution: From (4.5) and (4.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{n} = 0 \tag{4.8}$$

resulting in (4.6). From (4.5) and (4.2),(4.3), (4.7) is obtained.

4.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \tag{4.9}$$

4.6 Find the locus of $\mathbf{R}$.
Solution: For $\mathbf{n}$ to be unique in (4.6),(4.7),

$$\begin{aligned} (2\mathbf{A} - \mathbf{R}) &= k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &(2\mathbf{A} - \mathbf{R}) \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{4.10}$$

where k is some constant. Thus, the desired locus is

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) = 0 \tag{4.11}$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - 2\mathbf{A}^T\mathbf{R} = 0 \tag{4.12}$$

5 Conics

5.1 The equation of a quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{5.1}$$

Show that (5.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0 \tag{5.2}$$

Find the matrix V and vector $\mathbf{u}$.

5.2 The tangent to (5.1) at a point $\mathbf{p}$ on the curve is given by

$$\begin{pmatrix} \mathbf{p}^T & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{5.3}$$

Show that (5.3) can be expressed as

$$\left(\mathbf{p}^T V + \mathbf{u}^T\right)\mathbf{x} + \mathbf{p}^T\mathbf{u} + F = 0 \tag{5.4}$$

5.3 Classify the various conic sections based on (5.2).
Solution:

Curve	**Property**
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 5.3

6 Circle

6.1 Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{6.1}$$

Solution: let $\mathbf{c}$ be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{6.2}$$

$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \tag{6.3}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} = r^2 - \mathbf{c}^T\mathbf{c} \tag{6.4}$$

Comparing with (6.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6.5}$$

$$r^2 - \mathbf{c}^T\mathbf{c} = 1 \implies r = \sqrt{2} \tag{6.6}$$

6.2 Find the tangent to the circle C_1 at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Solution: From (5.3), the tangent T is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right]\mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{6.7}$$

$$\implies T : \mathbf{n}^T\mathbf{x} = 3 \tag{6.8}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.9}$$

6.3 The tangent T in (6.8) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{6.10}$$

Find $\mathbf{A} + \mathbf{B}$.
Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \tag{6.11}$$

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \tag{6.12}$$

Subtracting (6.12) from (6.11),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{6.13}$$

$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{6.14}$$

$\because$ $\mathbf{A}, \mathbf{B}$ lie on T, from (6.8),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{6.15}$$

$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \tag{6.16}$$

From (6.14) and (6.16)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{6.17}$$

$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{6.18}$$

$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{6.19}$$

$$\implies k = 2 \tag{6.20}$$

using (6.15). Substituting in (6.17)

$$\mathbf{A}+\mathbf{B} = 2\left(\mathbf{n}+\mathbf{C}\right) \tag{6.21}$$

6.4 If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.
Solution: From the given information,

$$\left\|\mathbf{A}-\mathbf{B}\right\|^2 = 4^2 \tag{6.22}$$

resulting in

$$\left\|\mathbf{A}+\mathbf{B}\right\|^2 - \left\|\mathbf{A}-\mathbf{B}\right\|^2 = 4\left\|\mathbf{n}+\mathbf{C}\right\|^2 - 4^2 \tag{6.23}$$

$$\implies \mathbf{A}^T\mathbf{B} = \left\|\mathbf{n}+\mathbf{C}\right\|^2 - 4 = 17 \tag{6.24}$$

using (6.21) and simplifying.

6.5 Show that

$$\left(\mathbf{A}-\mathbf{C}\right)^T\left(\mathbf{B}-\mathbf{C}\right) = 8 - r^2 \tag{6.25}$$

Solution:

$$\left\|\mathbf{A}-\mathbf{B}\right\|^2 = 4^2 \tag{6.26}$$

$$\implies \left(\mathbf{A}-\mathbf{B}\right)^T\left(\mathbf{A}-\mathbf{B}\right) = 4^2 \tag{6.27}$$

From (6.27),

$$\left[\left(\mathbf{A}-\mathbf{C}\right)-\left(\mathbf{B}-\mathbf{C}\right)\right]^T\left[\left(\mathbf{A}-\mathbf{C}\right)-\left(\mathbf{B}-\mathbf{C}\right)\right] = 4^2 \tag{6.28}$$

which can be expressed as

$$\left\|\mathbf{A}-\mathbf{C}\right\|^2 + \left\|\mathbf{B}-\mathbf{C}\right\|^2 + 2\left(\mathbf{A}-\mathbf{C}\right)^T\left(\mathbf{B}-\mathbf{C}\right) = 4^2 \tag{6.29}$$

Upon substituting from (6.12) and (6.11) and simplifying, (6.25) is obtained.

6.6 Find r.
Solution: (6.25) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T\left(\mathbf{A}+\mathbf{B}\right) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{6.30}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T\left(\mathbf{A}+\mathbf{B}\right) - \mathbf{C}^T\mathbf{C} = r^2 \tag{6.31}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T\left(2\mathbf{n}+\mathbf{C}\right) = r^2 \tag{6.32}$$

$$\implies r = \sqrt{6}. \tag{6.33}$$

7 Parabola

7.1 Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} + 6 = 0 \tag{7.1}$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix}1\\7\end{pmatrix}, V = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \tag{7.2}$$

in (5.4), the desired equation is

$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} + 6 = 0 \tag{7.3}$$

resulting in

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 5 \tag{7.4}$$

7.2 The line in (7.4) touches the circle

$$\mathbf{x}^T\mathbf{x} + 4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + c = 0 \tag{7.5}$$

Find c.
Solution: Comparing (5.2) and (7.5),

$$V = I, \quad \mathbf{u} = 2\begin{pmatrix}4\\3\end{pmatrix} \tag{7.6}$$

Comparing (5.4) and (7.4),

$$\mathbf{p} + 2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{7.7}$$

$$\implies \mathbf{p} = -\begin{pmatrix}6\\7\end{pmatrix} \tag{7.8}$$

and

$$c + \mathbf{p}^T\mathbf{u} = 5 \tag{7.9}$$

$$\implies c = 5 + 2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \tag{7.10}$$

$$= 95 \tag{7.11}$$

8 Ellipse

8.1 Express the following equation in the form given in (5.1)

$$E : 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{8.1}$$

Solution: (8.1) can be expressed as

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{8.2}$$

where

$$V = \begin{pmatrix}5 & -3\\-3 & 5\end{pmatrix}, \mathbf{u} = \begin{pmatrix}11\\-13\end{pmatrix} \tag{8.3}$$

8.2 Find $\mathbf{c}$ and K such that

$$(\mathbf{x}-\mathbf{c})^T V (\mathbf{x}-\mathbf{c}) = K \tag{8.4}$$

Solution: (8.4) can be expressed as

$$\mathbf{x}^T V\mathbf{x} - 2\mathbf{c}^T V\mathbf{x} + \mathbf{c}^T V\mathbf{c} - K = 0. \tag{8.5}$$

Comparing with (8.2),

$$V\mathbf{c} = -\mathbf{u} \tag{8.6}$$
$$\mathbf{c}^T V\mathbf{c} - K = 29 \tag{8.7}$$
$$\implies \mathbf{c} = -V^{-1}\mathbf{u} = \begin{pmatrix}-1\\2\end{pmatrix}, \tag{8.8}$$
$$\text{and } K = 8 \tag{8.9}$$

8.3 Show that (8.4) can be expressed as

$$\mathbf{y}^T D\mathbf{y} = 1 \tag{8.10}$$

Solution: Let

$$PDP^T = V \tag{8.11}$$

For

$$\mathbf{y} = \frac{P^T(\mathbf{x}-\mathbf{c})}{\sqrt{K}}, \tag{8.12}$$

(8.4) transforms to (8.10).

8.4 If

$$D = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{8.13}$$
$$P = \begin{pmatrix}\mathbf{P}_1 & \mathbf{P}_2\end{pmatrix} \tag{8.14}$$

show that $(\lambda_1, \mathbf{P}_1)$ and $(\lambda_2, \mathbf{P}_2)$ satisfy

$$V\mathbf{y} = \lambda\mathbf{y} \tag{8.15}$$

if

$$P^T P = I \tag{8.16}$$

8.5 Find the length of the semi-major and semi-minor axes of E.
Solution: The values are given by

$$\sqrt{\frac{K}{\lambda}} \tag{8.17}$$

obtained by solving for λ in (8.15). Thus,

$$|\lambda I - V| = 0 \tag{8.18}$$
$$\implies \begin{vmatrix}\lambda-5 & 3\\3 & \lambda-5\end{vmatrix} = 0 \tag{8.19}$$
$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{8.20}$$
$$\implies \lambda = 2, 8 \tag{8.21}$$

Thus, the length of the semi-major axis is 2 and that of the semi-minor axis is 1.

8.6 Find $\mathbf{P}_1$ and $\mathbf{P}_2$.
Solution: From (8.15)

$$V\mathbf{P}_1 = \lambda_1\mathbf{y} \tag{8.22}$$
$$\implies (V-\lambda I)\,\mathbf{y} = 0 \tag{8.23}$$
$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{y} = 0 \tag{8.24}$$
$$\implies \mathbf{n}_1^T\mathbf{y} = 0 \tag{8.25}$$

where

$$\mathbf{n}_1 = \begin{pmatrix}1\\-1\end{pmatrix} \tag{8.26}$$

Similarly, $\mathbf{P}_2$ is a point on

$$\mathbf{n}_2^T\mathbf{y} = 0 \tag{8.27}$$

$$\mathbf{n}_2 = \begin{pmatrix}1\\1\end{pmatrix} \tag{8.28}$$

8.7 Find the equation of the major axis for E.
Solution: The major axis for (8.10) is the line

$$\mathbf{y} = \lambda_1\begin{pmatrix}1\\0\end{pmatrix}. \tag{8.29}$$

From (8.12), letting

$$P\mathbf{y} = \frac{(\mathbf{x}-\mathbf{c})}{\sqrt{K}}, \tag{8.30}$$
$$\implies \lambda_1\mathbf{P}_1 = \frac{\mathbf{x}-\mathbf{c}}{\sqrt{K}} \tag{8.31}$$

since

$$P\begin{pmatrix}1\\0\end{pmatrix} = \mathbf{P}_1 \text{ and } P^T P = I \tag{8.32}$$

Also, from (8.25),

$$\lambda_1\mathbf{n}_1^T\mathbf{P}_1 = \frac{\mathbf{n}_1(\mathbf{x}-\mathbf{c})}{\sqrt{K}} = 0 \tag{8.33}$$

resulting in

$$\mathbf{n}_1^T\mathbf{x} = \mathbf{n}_1^T\mathbf{c} = -3 \tag{8.34}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 3 = 0 \tag{8.35}$$

which is the major axis of E.

8.8 Find the minor axis of E.

8.9 Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{8.36}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

9 Hyperbola

9.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V\mathbf{x} = 36 \tag{9.1}$$

where

$$V = \begin{pmatrix}4 & 0\\0 & -1\end{pmatrix} \tag{9.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix}0\\3\end{pmatrix}, \tag{9.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (5.4) as

$$\mathbf{P}^T V\mathbf{x} = 36 \tag{9.4}$$

$$\mathbf{Q}^T V\mathbf{x} = 36. \tag{9.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T\begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{9.6}$$

$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T\begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{9.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix}0 & -3\end{pmatrix}\mathbf{x} = -36 \tag{9.8}$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = -12 \tag{9.9}$$

which is the equation of PQ.

9.2 In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0 \tag{9.10}$$

using (1.2) and (9.9), the equation of TD is

$$\begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{x} - \mathbf{T}) = 0 \tag{9.11}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{9.12}$$

9.3 Find D.
Solution: From (9.9) and (9.12),

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{9.13}$$

$$\implies \mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{9.14}$$

9.4 Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda\mathbf{m} \tag{9.15}$$

where

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{9.16}$$

9.5 Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda\mathbf{m})^T V(\mathbf{D} + \lambda\mathbf{m}) + F = 0 \tag{9.17}$$

can be expressed as

$$\lambda^2\mathbf{m}^T V\mathbf{m} + 2\lambda\mathbf{m}^T V\mathbf{D} + \mathbf{D}^T V\mathbf{D} + F = 0 \tag{9.18}$$

9.6 Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (9.15) and (9.1) (9.18) is obtained. Substituting from (9.16), (9.2) and (9.14)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 4 \tag{9.19}$$

$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = 0 \tag{9.20}$$

$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix}0 & -12\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = -144 \tag{9.21}$$

Substituting in (9.18)

$$4\lambda^2 - 144 = 36 \tag{9.22}$$

$$\implies \lambda = \pm 3\sqrt{5} \tag{9.23}$$

Substituting in (9.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix}\sqrt{5}\\-4\end{pmatrix} \tag{9.24}$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix}\sqrt{5}\\4\end{pmatrix} \tag{9.25}$$

9.7 Find the area of $\triangle PTQ$.

Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \qquad (9.26)$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \qquad (9.27)$$

the desired area is

$$\frac{1}{2} PQ \times TD = 45\sqrt{5} \qquad (9.28)$$

9.8 Repeat the previous exercise using determinants.