

# Linear Algebra through Coordinate Geometry

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**Abstract**—This manual introduces linear algebra through coordinate geometry using a problem solving approach.

## 1 THE STRAIGHT LINE

1.1 The equation of the line between two points **A** and **B** is given by

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{A} - \mathbf{B}) \quad (1.1)$$

Alternatively, it can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2)$$

where **n** is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{n} = 0 \quad (1.3)$$

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1.2 In  $\triangle ABC$ ,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.4)$$

and the equations of the medians through **B** and **C** are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \quad (1.5)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \quad (1.6)$$

Find the area of  $\triangle ABC$ .

**Solution:** The centroid **O** is the solution of (1.5), (1.6) and is obtained as the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad (1.7)$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.8)$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.9)$$

Let **AD** be the median through **A**. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \quad (1.10)$$

$$\Rightarrow \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.11)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.12)$$

From (1.6) and (1.12),

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} = 5 \quad (1.13)$$

$$\Rightarrow 5 + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 12 \quad (1.14)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 7 \quad (1.15)$$

From (1.15) and (1.6),  $\mathbf{C}$  can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad (1.16)$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.17)$$

$$\Rightarrow \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (1.18)$$

From (1.11),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.19)$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \quad (1.20)$$

1.3 Summarize all the above computations through a Python script and plot  $\triangle ABC$ .

**Solution:**

<https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/1.2.py>

## 2 ORTHOGONALITY

2.1  $\mathbf{u}^T \mathbf{x} = 0 \Rightarrow \mathbf{u} \perp \mathbf{x}$ . Show that

$$\mathbf{u}^T \mathbf{x} = \mathbf{P}^T \mathbf{x} = 0 \Rightarrow \mathbf{P} = \alpha \mathbf{u} \quad (2.1)$$

2.2 The foot of the perpendicular drawn from the origin on the line

$$AB : \mathbf{u}^T \mathbf{x} = \lambda \quad (2.2)$$

where

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.3)$$

is  $\mathbf{P}$ . The line meets the  $x$ -axis at  $\mathbf{A}$  and  $y$ -axis at  $\mathbf{B}$ . Show that  $\mathbf{P} = \alpha \mathbf{u}$  and find  $\alpha$ .

**Solution:** From (2.2),

$$\mathbf{u}^T \mathbf{A} = \mathbf{u}^T \mathbf{B} = \lambda \quad (2.4)$$

$$\Rightarrow \mathbf{u}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.5)$$

Since  $OP \perp AB$ ,

$$\mathbf{P}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.6)$$

Thus, from (2.1),

$$\mathbf{P} = \alpha \mathbf{u} \quad (2.7)$$

Since  $\mathbf{P}$  lies on (2.2),

$$\mathbf{u}^T \mathbf{P} = \alpha \mathbf{u}^T \mathbf{u} = \lambda \quad (2.8)$$

$$\Rightarrow \alpha = \frac{\lambda}{\mathbf{u}^T \mathbf{u}} = \frac{\lambda}{10}. \quad (2.9)$$

2.3 Find  $\mathbf{A}$ .

**Solution:** Let

$$\mathbf{A} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.10)$$

From (2.2),

$$\mathbf{u}^T \mathbf{A} = a \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \quad (2.11)$$

$$\Rightarrow a = \frac{\lambda}{3} \quad (2.12)$$

$$\text{and } \mathbf{A} = \frac{\lambda}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.13)$$

2.4 Find the ratio  $BP : PA$ .

**Solution:** Let

$$\frac{BP}{PA} = k \quad (2.14)$$

Then,

$$k\mathbf{A} + \mathbf{B} = (k+1)\mathbf{P} \quad (2.15)$$

$$\Rightarrow k\mathbf{A}^T \mathbf{A} + \mathbf{A}^T \mathbf{B} = (k+1)\mathbf{P}^T \mathbf{A} \quad (2.16)$$

$$\Rightarrow k\lambda^2 = \alpha(k+1)\lambda \quad (2.17)$$

using (2.7), (2.10), (2.2) and  $\mathbf{A} \perp \mathbf{B}$ . Substituting from (2.9) and (2.12),

$$\Rightarrow k \frac{\lambda^2}{9} = (k+1) \frac{\lambda^2}{10} \quad (2.18)$$

$$\Rightarrow k = 9 \quad (2.19)$$

## 3 MATRIX TRANSFORMATIONS

3.1 Find  $\mathbf{R}$ , the reflection of  $\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$  about the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (3.1)$$

**Solution:** The reflection of  $\mathbf{P}$  about  $L$  is given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (3.2)$$

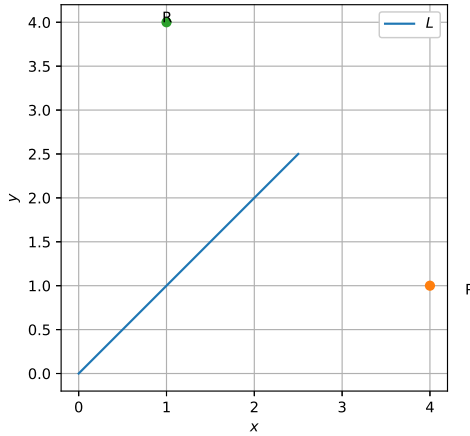


Fig. 3.2

where

$$L : \mathbf{n}^T \mathbf{x} = c \quad (3.3)$$

$$\mathbf{m}^T \mathbf{n} = 0 \quad (3.4)$$

$$\|\mathbf{m}\| = \|\mathbf{n}\| = 1 \quad (3.5)$$

Substituting

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c = 0 \quad (3.6)$$

in (3.2),

$$\frac{\mathbf{R}}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow \mathbf{R} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (3.7)$$

3.2 Summarize all the above computations through a Python script and plot  $L, P, R$ .

**Solution:** The following code generates Fig. 3.2.

```
wget https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/3.1.py
```

3.3  $\mathbf{R}$  is translated through a distance 2 units along the positive direction of x-axis to obtain  $\mathbf{S}$ . Find  $\mathbf{S}$ .

**Solution:**

$$\mathbf{S} = \mathbf{R} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.8)$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.9)$$

3.4 Rotate  $\mathbf{S}$  through an angle of  $\frac{\pi}{4}$  about the origin

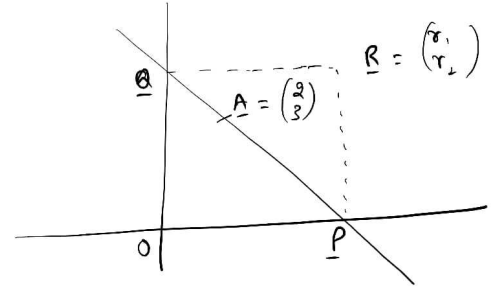


Fig. 4.1

in the counter clockwise direction to obtain  $\mathbf{T}$ .

**Solution:**

$$\mathbf{T} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{S} \quad (3.10)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.11)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 7 \end{pmatrix} \quad (3.12)$$

#### 4 Locus

4.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1)$$

intersects the coordinate axes at  $\mathbf{P}$  and  $\mathbf{Q}$ .  $\mathbf{O}$  is the origin and rectangle  $OPRQ$  is completed as shown in Fig. (4.1),

4.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (4.2)$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (4.3)$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \quad (4.4)$$

4.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{n} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{n} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{n} &= 0 \end{aligned} \quad (4.5)$$

**Solution:** Trivial using (1.2) and (1.3).

4.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{n} = 0 \quad (4.6)$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{n} = 0 \quad (4.7)$$

**Solution:** From (4.5) and (4.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{n} = 0 \quad (4.8)$$

resulting in (4.6). From (4.5) and (4.2),(4.3), (4.7) is obtained.

4.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \quad (4.9)$$

4.6 Find the locus of  $\mathbf{R}$ .

**Solution:** For  $\mathbf{n}$  to be unique in (4.6),(4.7),

$$\begin{aligned} (2\mathbf{A} - \mathbf{R}) &= k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \quad (4.10)$$

where  $k$  is some constant. Thus, the desired locus is

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) = 0 \quad (4.11)$$

$$\Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - 2\mathbf{A}^T \mathbf{R} = 0 \quad (4.12)$$

## 5 CONICS

5.1 The equation of a quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \quad (5.1)$$

Show that (5.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \quad (5.2)$$

Find the matrix  $V$  and vector  $\mathbf{u}$ .

5.2 The tangent to (5.1) at a point  $\mathbf{p}$  on the curve is given by

$$(\mathbf{p}^T \ 1) \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \quad (5.3)$$

Show that (5.3) can be expressed as

$$(\mathbf{p}^T V + \mathbf{u}^T) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \quad (5.4)$$

5.3 Classify the various conic sections based on (5.2).

**Solution:**

Curve	Property
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 5.3

## 6 CIRCLE

6.1 Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (6.1)$$

**Solution:** let  $\mathbf{c}$  be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \quad (6.2)$$

$$\Rightarrow (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \quad (6.3)$$

$$\Rightarrow \mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} = r^2 - \mathbf{c}^T \mathbf{c} \quad (6.4)$$

Comparing with (6.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.5)$$

$$r^2 - \mathbf{c}^T \mathbf{c} = 1 \Rightarrow r = \sqrt{2} \quad (6.6)$$

6.2 Find the tangent to the circle  $C_1$  at the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

**Solution:** From (5.3), the tangent  $T$  is given by

$$\left[ \begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix} \right] \mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad (6.7)$$

$$\Rightarrow T : \mathbf{n}^T \mathbf{x} = 3 \quad (6.8)$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6.9)$$

6.3 The tangent  $T$  in (6.8) cuts off a chord  $AB$  from a circle  $C_2$  whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \quad (6.10)$$

Find  $\mathbf{A} + \mathbf{B}$ .

**Solution:** Let the radius of  $C_2$  be  $r$ . From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \quad (6.11)$$

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \quad (6.12)$$

Subtracting (6.12) from (6.11),

$$\mathbf{A}^T \mathbf{A} - \mathbf{B}^T \mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (6.13)$$

$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (6.14)$$

$\therefore \mathbf{A}, \mathbf{B}$  lie on  $T$ , from (6.8),

$$\mathbf{n}^T \mathbf{A} = \mathbf{n}^T \mathbf{B} = 3 \quad (6.15)$$

$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \quad (6.16)$$

From (6.14) and (6.16)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \quad (6.17)$$

$$\implies \mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C} = k\mathbf{n}^T \mathbf{n} \quad (6.18)$$

$$\implies \frac{\mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C}}{\mathbf{n}^T \mathbf{n}} = k \quad (6.19)$$

$$\implies k = 2 \quad (6.20)$$

using (6.15). Substituting in (6.17)

$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \quad (6.21)$$

6.4 If  $AB = 4$ , find  $\mathbf{A}^T \mathbf{B}$ .

**Solution:** From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \quad (6.22)$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \quad (6.23)$$

$$\implies \mathbf{A}^T \mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \quad (6.24)$$

using (6.21) and simplifying.

6.5 Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \quad (6.25)$$

**Solution:**

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \quad (6.26)$$

$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \quad (6.27)$$

From (6.27),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \quad (6.28)$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \quad (6.29)$$

Upon substituting from (6.12) and (6.11) and simplifying, (6.25) is obtained.

6.6 Find  $r$ .

**Solution:** (6.25) can be expressed as

$$\mathbf{A}^T \mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T \mathbf{C} = 8 - r^2 \quad (6.30)$$

$$\implies 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T \mathbf{C} = r^2 \quad (6.31)$$

$$\implies 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \quad (6.32)$$

$$\implies r = \sqrt{6}. \quad (6.33)$$

## 7 PARABOLA

7.1 Find the tangent at  $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$  to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \quad (7.1)$$

**Solution:** Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (7.2)$$

in (5.4), the desired equation is

$$\left[ \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \end{pmatrix} \right] \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \quad (7.3)$$

resulting in

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -5 \quad (7.4)$$

7.2 The line in (7.4) touches the circle

$$\mathbf{x}^T \mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \quad (7.5)$$

Find  $c$ .

**Solution:** Comparing (5.2) and (7.5),

$$\begin{aligned} V &= I, \\ \mathbf{u} &= 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned} \quad (7.6)$$

Comparing (5.4) and (7.4),

$$\mathbf{p} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (7.7)$$

$$\Rightarrow \mathbf{p} = - \begin{pmatrix} 6 \\ 7 \end{pmatrix} \quad (7.8)$$

and

$$c + \mathbf{p}^T \mathbf{u} = 5 \quad (7.9)$$

$$\Rightarrow c = 5 + 2 \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (7.10)$$

$$= 95 \quad (7.11)$$

## 8 ELLIPSE

8.1 Express the following equation in the form given in (5.1)

$$E : 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \quad (8.1)$$

**Solution:** (8.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + 29 = 0 \quad (8.2)$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \quad (8.3)$$

8.2 Find  $\mathbf{c}$  and  $K$  such that

$$(\mathbf{x} - \mathbf{c})^T V (\mathbf{x} - \mathbf{c}) = K \quad (8.4)$$

**Solution:** (8.4) can be expressed as

$$\mathbf{x}^T V \mathbf{x} - 2\mathbf{c}^T V \mathbf{x} + \mathbf{c}^T V \mathbf{c} - K = 0. \quad (8.5)$$

Comparing with (8.2),

$$V\mathbf{c} = -\mathbf{u} \quad (8.6)$$

$$\mathbf{c}^T V \mathbf{c} - K = 29 \quad (8.7)$$

$$\Rightarrow \mathbf{c} = -V^{-1}\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad (8.8)$$

$$\text{and } K = 8 \quad (8.9)$$

8.3 Show that (8.4) can be expressed as

$$\mathbf{y}^T D \mathbf{y} = 1 \quad (8.10)$$

**Solution:** Let

$$PDP^T = V \quad (8.11)$$

For

$$\mathbf{y} = \frac{P^T (\mathbf{x} - \mathbf{c})}{\sqrt{K}}, \quad (8.12)$$

(8.4) transforms to (8.10).

8.4 If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (8.13)$$

$$P = (\mathbf{P}_1 \quad \mathbf{P}_2) \quad (8.14)$$

show that  $(\lambda_1, \mathbf{P}_1)$  and  $(\lambda_2, \mathbf{P}_2)$  satisfy

$$V\mathbf{y} = \lambda\mathbf{y} \quad (8.15)$$

if

$$P^T P = I \quad (8.16)$$

8.5 Find the length of the semi-major and semi-minor axes of  $E$ .

**Solution:** The values are given by

$$\sqrt{\frac{K}{\lambda}} \quad (8.17)$$

obtained by solving for  $\lambda$  in (8.15). Thus,

$$|\lambda I - V| = 0 \quad (8.18)$$

$$\Rightarrow \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \quad (8.19)$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0 \quad (8.20)$$

$$\Rightarrow \lambda = 2, 8 \quad (8.21)$$

Thus, the length of the semi-major axis is 2 and that of the semi-minor axis is 1.

8.6 Find  $\mathbf{P}_1$  and  $\mathbf{P}_2$ .

**Solution:** From (8.15)

$$V\mathbf{P}_1 = \lambda_1 \mathbf{y} \quad (8.22)$$

$$\Rightarrow (V - \lambda I) \mathbf{y} = 0 \quad (8.23)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{y} = 0 \quad (8.24)$$

$$\Rightarrow \mathbf{n}_1^T \mathbf{y} = 0 \quad (8.25)$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (8.26)$$

Similarly,  $\mathbf{P}_2$  is a point on

$$\mathbf{n}_2^T \mathbf{y} = 0 \quad (8.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (8.28)$$

8.7 Find the equation of the major axis for  $E$ .

**Solution:** The major axis for (8.10) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (8.29)$$

From (8.12), letting

$$P\mathbf{y} = \frac{(\mathbf{x} - \mathbf{c})}{\sqrt{K}}, \quad (8.30)$$

$$\Rightarrow \lambda_1 \mathbf{P}_1 = \frac{\mathbf{x} - \mathbf{c}}{\sqrt{K}} \quad (8.31)$$

since

$$P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } P^T P = I \quad (8.32)$$

Also, from (8.25),

$$\lambda_1 \mathbf{n}_1^T \mathbf{P}_1 = \frac{\mathbf{n}_1^T (\mathbf{x} - \mathbf{c})}{\sqrt{K}} = 0 \quad (8.33)$$

resulting in

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{n}_1^T \mathbf{c} = -3 \quad (8.34)$$

$$\Rightarrow (1 \ -1) \mathbf{x} + 3 = 0 \quad (8.35)$$

which is the major axis of  $E$ .

8.8 Find the minor axis of  $E$ .

8.9 Let  $\mathbf{F}_1, \mathbf{F}_2$  be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \quad (8.36)$$

Find  $\mathbf{F}_1, \mathbf{F}_2$  and  $k$ .

## 9 HYPERBOLA

9.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \quad (9.1)$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \quad (9.2)$$

at points  $\mathbf{P}$  and  $\mathbf{Q}$ . If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad (9.3)$$

find the equation of  $PQ$ .

**Solution:** The equations of the two tangents are obtained using (5.4) as

$$\mathbf{P}^T V \mathbf{x} = 36 \quad (9.4)$$

$$\mathbf{Q}^T V \mathbf{x} = 36. \quad (9.5)$$

Since both pass through  $\mathbf{T}$

$$\mathbf{P}^T V \mathbf{T} = 36 \Rightarrow \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (9.6)$$

$$\mathbf{Q}^T V \mathbf{T} = 36 \Rightarrow \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (9.7)$$

Thus,  $\mathbf{P}, \mathbf{Q}$  satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \quad (9.8)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \quad (9.9)$$

which is the equation of  $PQ$ .

9.2 In  $\triangle PTQ$ , find the equation of the altitude  $TD \perp PQ$ .

**Solution:** Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad (9.10)$$

using (1.2) and (9.9), the equation of  $TD$  is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \quad (9.11)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (9.12)$$

9.3 Find  $D$ .

**Solution:** From (9.9) and (9.12),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (9.13)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (9.14)$$

9.4 Show that the equation of  $PQ$  can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \quad (9.15)$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9.16)$$

9.5 Show that for  $\mathbf{V}^T = \mathbf{V}$ ,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \quad (9.17)$$

can be expressed as

$$\lambda^2 \mathbf{m}^T V \mathbf{m} + 2\lambda \mathbf{m}^T V \mathbf{D} + \mathbf{D}^T V \mathbf{D} + F = 0 \quad (9.18)$$

9.6 Find  $\mathbf{P}$  and  $\mathbf{Q}$ .

**Solution:** From (9.15) and (9.1) (9.18) is obtained. Substituting from (9.16), (9.2) and (9.14)

$$\mathbf{m}^T V \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \quad (9.19)$$

$$\mathbf{m}^T V \mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \quad (9.20)$$

$$\mathbf{D}^T V \mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \quad (9.21)$$

Substituting in (9.18)

$$4\lambda^2 - 144 = 36 \quad (9.22)$$

$$\implies \lambda = \pm 3\sqrt{5} \quad (9.23)$$

Substituting in (9.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3 \begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \quad (9.24)$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3 \begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \quad (9.25)$$

9.7 Find the area of  $\triangle PTQ$ .

**Solution:** Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \quad (9.26)$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \quad (9.27)$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \quad (9.28)$$

9.8 Repeat the previous exercise using determinants.