Discrete Mathematics

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***Abstract*—This book provides a computational approach to discrete mathematics by solving problems in related areas from IIT-JEE. Links to sample C/Python codes are available in the text. The book provides sufficient math basics for Machine Learning and is also recommended for high school students who wish to explore topics in Artificial Intelligence.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    linalg/book/codes
```

1 Signal Processing: Z Transform

1. Let

$$a(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n) \tag{1.0.1.1}$$

$$b(n) = a(n-1) + a(n+1) - \delta(n) \tag{1.0.1.2}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{1.0.1.3}$$

and

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \ge 0 \end{cases} \tag{1.0.1.4}$$

$$\delta(n) = \begin{cases} 0, & n \ne 0 \\ 1, & n = 0 \end{cases} \tag{1.0.1.5}$$

2. Verify your results through a C program.
3. Show that the Z transform of $u(n)$

$$U(z) \triangleq \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{1.0.3.1}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{1.0.3.2}$$

4. Show that

$$A(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \tag{1.0.4.1}$$

5. Let

$$y(n) = a(n) * u(n) \triangleq \sum_{k=-\infty}^{\infty} a(k) u(n-k) \tag{1.0.5.1}$$

Show that

$$y(n) = \sum_{k=0}^{n} a(k) \tag{1.0.5.2}$$

6. Show that

$$Y(z) = A(z) U(z) \tag{1.0.6.1}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{1.0.6.2}$$

7. Show that

$$w(n) = \left[a(n+2) - 1\right] u(n-1) \tag{1.0.7.1}$$

$$= a(n+2) - u(n+1) + 2\delta(n) \tag{1.0.7.2}$$

8. Is $W(z) = Y(z)$?

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9. Verify if

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} = \frac{10}{89} \tag{1.0.9.1}$$

10. Verify if

$$\sum_{n=1}^{\infty} \frac{b(n)}{10^n} = \frac{8}{89} \tag{1.0.10.1}$$

2 Algebra: Modular Arithmetic

Let $AP(a; d)$ denote an A.P. with $d > 0$

1. Express $AP(a; d)$ in modulo arithmetic.
   Solution:

$$A \equiv a \pmod d \tag{2.0.10.1}$$

2. Express the intersection of $AP(1; 3), AP(2; 5)$ and $AP(3; 7)$ using modulo arithmetic.
   Solution: The desired AP can be expressed as

$$A \equiv 1 \pmod 3 \tag{2.0.10.2}$$
$$\equiv 2 \pmod 5 \tag{2.0.10.3}$$
$$\equiv 3 \pmod 7 \tag{2.0.10.4}$$

3. Two numbers are said to be coprime if their greatest common divisor (gcd) is 1. Verify if (3,5), (5,7) and (3,7) are pairwise coprime.
4. Does a solution for (2.0.10.2) exist?
   Solution: The Chinese remainder theorem guarantees that the system in (2.0.10.2) has a solution since 3,5,7 are pairwise coprime.
5. Simplify

$$(7 \times 5) \pmod 3 \tag{2.0.10.5}$$

   Solution: (2.0.10.5) can be expressed as

$$\begin{aligned}(7 \times 5) \pmod 3 &= 35 \pmod 3 \\ &= 2 \pmod 3\end{aligned} \tag{2.0.10.6}$$

6. Find x in

$$2x = 1 \pmod 3 \tag{2.0.10.7}$$

   Solution: By inspection, for $x = 2$,

$$2x = 2 \times 2 = 4 = 3 + 1 = 1 \pmod 3 \tag{2.0.10.8}$$

   Thus $x = 2$ is a solution of (2.0.10.7).
7. In general, x in

$$ax = 1 \pmod d \tag{2.0.10.9}$$

   is defined to be the modular multiplicative inverse of (2.0.10.1).
8. Show that the multiplicative inverse of

$$(3 \times 5) \pmod 7 = y = 1 \tag{2.0.10.10}$$

9. Show that the multiplicative inverse of

$$(3 \times 7) \pmod 5 = z = 1 \tag{2.0.10.11}$$

10. Find $a + d$.
   Solution:

$$\begin{aligned}&(5 \times 7 \times 1 \times x) + (3 \times 5 \times 3 \times y) \\ &\quad + (3 \times 7 \times 2 \times z) = 157\end{aligned} \tag{2.0.10.12}$$

11. Find a and d.
   Solution:

$$d = LCM(3, 5, 7) = 105 \tag{2.0.10.13}$$
$$A = 157 \pmod{105}$$
$$= 52 \pmod{105}$$
$$\implies a = 52 \tag{2.0.10.14}$$

12. Given the APs

$$a_1 \pmod{d_1} \tag{2.0.10.15}$$
$$a_2 \pmod{d_2} \tag{2.0.10.16}$$
$$a_3 \pmod{d_3}, \tag{2.0.10.17}$$

   such that

$$gcd(d_1, d_2) = gcd(d_2, d_3) = gcd(d_3, d_1) = 1, \tag{2.0.10.18}$$

   show that their intersection

$$a \pmod d \tag{2.0.10.19}$$

   is obtained through

$$\begin{aligned}a + d = &\\ &(d_1 \times d_2 \times a_3 \times x) + (d_2 \times d_3 \times a_1 \times y) \\ &\quad + (d_3 \times d_1 \times a_2 \times z)\end{aligned} \tag{2.0.10.20}$$

$$d = LCM(d_1, d_2, d_3), \tag{2.0.10.21}$$

   where x, y, z are the modular multiplicative inverses given by

$$x = [(d_1 \times d_2) \pmod{d_3}]^{-1} \quad (2.0.10.22)$$
$$y = [(d_2 \times d_3) \pmod{d_1}]^{-1} \quad (2.0.10.23)$$
$$z = [(d_3 \times d_1) \pmod{d_2}]^{-1} \quad (2.0.10.24)$$

respectively.

13. Write a C program to find x, y and z.

3 Discrete Fourier Transform

1. Show that

$$\sum_{k=0}^{n-1} e^{\jmath \frac{2\pi k}{n}} = \begin{cases} 1 & n = 1, \\ 0 & n > 1 \end{cases} \quad (3.0.1.1)$$

2. Show that

$$\sum_{k=0}^{n} \cos\left(\frac{2k+r}{n+2}\pi\right) = -\cos\left(\frac{r-2}{n+2}\pi\right) \quad (3.0.2.1)$$

Solution: From (3.0.1.1),

$$\sum_{k=0}^{n+1} e^{\jmath \frac{2k+r}{n+2}\pi} = 0$$
$$\implies \sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} + e^{\jmath \frac{2(n+1)+r}{n+2}\pi} = 0$$
$$\implies \sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} = -e^{\jmath \frac{2(n+2)+r-2}{n+2}\pi}$$
$$= -e^{\jmath \frac{r-2}{n+2}\pi} \quad (3.0.2.2)$$

Taking the real part on both sides yields (3.0.2.1).

3. Show that

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+2}{n+2}\pi\right)} \quad (3.0.3.1)$$
$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)}{n+\cos\left(\frac{2\pi}{n+2}\right)} \quad (3.0.3.2)$$

Solution: Let

$$\theta_n = \frac{\pi}{n+2} \quad (3.0.3.3)$$

$$\because \sin\{(k+1)\theta_n\}\sin\{(k+2)\theta_n\},$$
$$= \frac{1}{2}\left[\cos\theta_n - \cos\{(2k+3)\theta_n\}\right] \quad (3.0.3.4)$$

from (3.0.3.1) and (3.0.2.1),

$$f(n) = \frac{n\cos\theta_n - \sum_{k=0}^{n}\cos\{(2k+3)\theta_n\}}{n - \sum_{k=0}^{n}\cos\{(2k+4)\theta_n\}}$$
$$= \frac{n\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{n+\cos\left(\frac{2\pi}{n+2}\right)} \quad (3.0.3.5)$$

resulting in (3.0.3.2). Verify if

4.

$$f(4) = \frac{\sqrt{3}}{2} \quad (3.0.4.1)$$

5.

$$\lim_{n\to\infty} f(n) = \frac{1}{2} \quad (3.0.5.1)$$

6.

$$\sin\left(7\cos^{-1} f(5)\right) = 0 \quad (3.0.6.1)$$

7. If

$$\alpha = \tan\left(\cos^{-1} f(6)\right) \quad (3.0.7.1)$$

verify if

$$\alpha^2 + 2\alpha - 1 = 0 \quad (3.0.7.2)$$

4 Combinatorics

1. Find

$$\sum_{k=0}^{n} k \quad (4.0.1.1)$$

Solution: (4.0.1.1) can be expressed as

$$\frac{n(n+1)}{2} \quad (4.0.1.2)$$

2. Find

$$\sum_{k=0}^{n} {}^{n}C_k k^2 \quad (4.0.2.1)$$

Solution:

$$(1+x)^n = \sum_{k=0}^{n} {}^{n}C_k x^k \quad (4.0.2.2)$$
$$\implies n(1+x)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_k x^{k-1} \quad (4.0.2.3)$$

upon differentiation. Multiplying (4.0.2.3) by x

and differentiating,

$$\frac{d}{dx}\left[nx(1+x)^{n-1}\right] = \sum_{k=0}^{n} k^2\, {}^nC_k x^{k-1} \quad (4.0.2.4)$$

$$\implies n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1} = \sum_{k=0}^{n} k^2\, {}^nC_k x^{k-1} \quad (4.0.2.5)$$

Substituting $x = 1$ in (4.0.2.5),

$$\sum_{k=0}^{n} {}^nC_k k^2 = n(n-1)2^{n-2} + n2^{n-1} = n(n+1)2^{n-2} \quad (4.0.2.6)$$

3. Find

$$\sum_{k=0}^{n} {}^nC_k k \quad (4.0.3.1)$$

Solution: Substituting $x = 1$ in (4.0.2.3),

$$\sum_{k=0}^{n} {}^nC_k k = n2^{n-1} \quad (4.0.3.2)$$

4. Find

$$\sum_{k=0}^{n} {}^nC_k 3^k \quad (4.0.4.1)$$

Solution: Substituting $x = 2$ in (4.0.2.2),

$$\sum_{k=0}^{n} {}^nC_k 3^k = 4^n \quad (4.0.4.2)$$

5. If

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n+1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0 \quad (4.0.5.1)$$

for some n, find

$$\sum_{k=0}^{n} \frac{{}^nC_k}{k+1} \quad (4.0.5.2)$$

Solution: (4.0.5.1) can be expressed as

$$n(n+1)2^{2n-3}\begin{vmatrix} 1 & 1 \\ n & 4 \end{vmatrix} = 0 \quad (4.0.5.3)$$

$$\implies n = 4 \quad (4.0.5.4)$$

Integrating (4.0.2.2) from 0 to 1,

$$\frac{2^{n+1}}{n+1} = \sum_{k=0}^{n} \frac{{}^nC_k}{k+1} \quad (4.0.5.5)$$

Substituting $n = 4$ in the above,

$$\sum_{k=0}^{n} \frac{{}^nC_k}{k+1} = \frac{2^5-1}{5} = \frac{31}{5} \quad (4.0.5.6)$$

5 JEE Exercises

5.1 Sequences and Series

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is......................
2. The solution of the equation $log_7 log_5(\sqrt{x+5}+\sqrt{x}) = 0$ is ....
3. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 +$ ....is $\frac{n(n+1)^2}{2}$, when n is even.When n is odd,the sum is ....
4. Let the harmonic mean and geometric mean of two positive numbers be the ratio 4:5.Then the two numbers are in the ratio..................
5. For any odd integer n≥ 1,$n^3 - (n-1)^3 + ... + (-1)^{(n-1)}1^3$=............
6. Let p and q be roots of the equation

$$x^2 - 2x + A = 0 \quad (5.1.6.1)$$

and let r and s be the roots of the equation

$$x^2 - 18x + B = 0 \quad (5.1.6.2)$$

. If $p < q < r < s$ are in arithmetic progression,then A =.......... and B=........

MCQs with One Correct Answer

7. If x,y and z are p^th,q^th and r^th terms respectively of an A.P.and also of a G.P.,then $x^y - zy^z - xz^x - y$ is equal to:
   a) xyz
   b) 0
   c) 1
   d) None of these
8. The third term of a geometric progression is 4.The product of the first five terms is
   a) 4^3
   b) 4^5
   c) 4^4
   d) None of these
9. The rational number,which equals the number $2.\overline{357}$ with recurring decimal is

a) $\frac{2355}{1001}$
b) $\frac{2379}{997}$
c) $\frac{2355}{999}$
d) none of these

10. If a,b,c are in G.P.,then the equations

$$ax^2 + 2bx + c = 0 \quad (5.1.10.1)$$

and

$$dx^2 + 2ex + f = 0 \quad (5.1.10.2)$$

have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in......

a) A.P.
b) G.P.
c) H.P.
d) None of these

11. Sum of the first n terms of the series $\frac{1}{2}$+$\frac{3}{4}$+$\frac{7}{8}$+$\frac{15}{16}$+.....is equal to

a) $2^n - n - 1$
b) $1 - 2^- n$
c) $n + 2^- n - 1$
d) $2^n + 1$

12. The number $log_2 7$ is

a) an integer
b) a rational number
c) an irrational number
d) a prime number

13. If $ln(a+c)$, $ln(a-c)$, $ln(a-2b+c)$ are in A.P.,then

a) a,b,c are in A.P.
b) a^2, b^2, c^2 are in A.P.
c) a,b,c are in G.P.
d) a,b,c are in H.P.

14. Let $a_1, a_2, ...a_1 0$ be in A.P.and $h_1, h_2, ...h_{10}$ be in H.P.If $a_1 = h_1 = 2$ and $a_{10} = h_{10}$=3, then $a_4 h_7$ is

a) 2
b) 3
c) 5
d) 6

15. The harmonic mean of the roots of the equation

$$(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0 \quad (5.1.15.1)$$

is

a) 2
b) 4
c) 6
d) 8

16. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is $\frac{3}{4}$, then

a) a=$\frac{4}{7}$,r=$\frac{3}{7}$
b) a=2,r=$\frac{3}{8}$
c) a=$\frac{3}{2}$,r=$\frac{1}{2}$
d) a=3,r=$\frac{1}{4}$

17. Let α, β be the roots of

$$x^2 - x + p = 0 \quad (5.1.17.1)$$

and γ, δ be the roots of

$$x^2 - 4x + q = 0 \quad (5.1.17.2)$$

. If α, β, γ, δ are in G.P., then the integral values of p and q respectively, are

a) -2,-32
b) -2,3
c) -6,3
d) -6,-32

18. Let the positive numbers a,b,c,d be in A.P.Then abc,abd,acd,bcd are

a) NOT in A.P./G.P./H.P.
b) in A.P.
c) in G.P.
d) in H.P.

19. If the sum of the first 2n terms of the A.P.2,5,8,.....,is equal to the sum of the first n terms of the A.P.57,59,61,.....,then n equals

a) 10
b) 12
c) 11
d) 13

20. Suppose a,b,c are in A.P. and a^2, b^2, c^2 are in G.P.if $a < b < c$ and a+b+c=$\frac{3}{2}$, then the value of a is

a) $\frac{1}{2\sqrt{2}}$
b) $\frac{1}{2\sqrt{3}}$
c) $\frac{1}{2}$-$\frac{1}{\sqrt{3}}$
d) $\frac{1}{2}$-$\frac{1}{\sqrt{2}}$

21. An infinite G.P. has first term 'x' and sum '5', then x belongs to

a) $x < -10$
b) $-10 < x < 0$
c) $0 < x < 10$
d) $x > 10$

22. In the quadratic equation
$$ax^2 + bx + c = 0, \quad (5.1.22.1)$$
$\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$, are in G.P.where α, β are the root of $ax^2 + bx + c = 0$, then
   a) $\Delta \neq 0$
   b) $b\Delta = 0$
   c) $c\Delta 0$
   d) $\Delta = 0$
23. In the sum of first n terms of an A.P.is cn^2, then the sum of squares of these n terms is
   a) $\frac{n(4n^2-1)c^2}{6}$
   b) $\frac{n(4n^2+1)c^2}{3}$
   c) $\frac{n(4n^2-1)c^2}{3}$
   d) $\frac{n(4n^2+1)c^2}{6}$
24. Let a_1, a_2, a_3,........be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is
   a) 22
   b) 23
   c) 24
   d) 25
25. Let $b_1 > 1$ for $i = 1, 2, ..., 101$.Suppose $\log_e b_1, \log_e b_2, ... \log_e b_{101}$ are in Arithmetic progression(A.P) with the common difference $\log_e 2$. Suppose $a_1, ...a_{101}$ are in A.P such that $a_1 = b_1$ and $a_{51} = b_{51}$.If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ...a_{53}$, then
   a) $s > t$ and $a_{101} > b_{101}$
   b) $s > t$ and $a_{101} < b_{101}$
   c) $s < t$ and $a_{101} > b_{101}$
   d) $s < t$ and $a_{101} < b_{101}$

MCQs with One or More than One Correct

26. If the first and the $(2n-1)$st terms of an A.P., a G.P. and an H.P. are equal and their n-th terms are a,b and c respectively , then
   a) a=b=c
   b) $a \geq b \geq c$
   c) a+c=b
   d) $ac - b^2 = 0$.
27. For $0 < \phi < \frac{\Pi}{2}$, if $x = \sum_{n=0}^{\infty}(\cos^{2n})\phi$, $y = \sum_{n=0}^{\infty}(\sin^{2n})\phi$, $z = \sum_{n=0}^{\infty}(\cos^{2n})\phi(\sin^{2n})\phi$ then:
   a) xyz = xz+y
   b) xyz = xy+z
   c) xyz = x+y+z
   d) xyz = yz+x
28. Let n be an odd integer.If $\sin n\theta = \sum_{r=0}^{n}(b)_r \sin^r \theta$, for every value of θ,then
   a) $b_0 = 1, b_1 = 3$
   b) $b_0 = 0, b_1 = n$
   c) $b_0 = -1, b_1 = n$
   d) $b_0 = 0, b_1 = n^2 - 3n + 3$
29. Let T_r be the r^{th} term of an A.P.,for r=1,2,3,..... If for some positive integers m,n we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals
   a) $\frac{1}{mn}$
   b) $\frac{1}{m}+\frac{1}{n}$
   c) 1
   d) 0
30. If $x > 1, y > 1, z > 1$ are in G.P.,then $\frac{1}{1+lnx}, \frac{1}{1+lny}, \frac{1}{1+lnz}$ are in
   a) A.P.
   b) H.P.
   c) G.P.
   d) None of these
31. For a positive integer n, let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\frac{1}{(2^n)-1}$. Then
   a) $a(100) \leq 100$
   b) $a(100) > 100$
   c) $a(200) \leq 100$
   d) $a(200) > 100$
32. A straight line through the vertex P of a triangle PQR intersects the side QR at the points **S** and the circumcircle of the triangle PQR at the point **T**.If S is not the centre of the circumcircle, then
   a) $\frac{1}{(PS} + \frac{1}{(ST} < \frac{2}{\sqrt{QSXSR}}$
   b) $\frac{1}{(PS} + \frac{1}{(ST} > \frac{2}{\sqrt{QSXSR}}$
   c) $\frac{1}{(PS} + \frac{1}{(ST} < \frac{4}{QR}$
   d) $\frac{1}{(PS} + \frac{1}{(ST} > \frac{2}{QR}$
33. Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ for n=1,2,3......Then
34. Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(S)
   a) 1056
   b) 1088
   c) 1120
   d) 1332
35. Let α and β be the roots of $x^2 - x - 1 = 0$ with $\alpha > \beta$.For all positive integers n,define $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1 b_1$ and $b_n = a_{n-1} + a_{n+1}, n \geq 2$ Then which of the following options is /are correct?

a) $\sum_{n=1}^{\infty} \frac{\alpha_n}{10^n} = \frac{10}{89}$
b) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$
c) $a_1 + a_2 + a_3 + ...a_n = a_{n+2} - 1$ for all $n \geq 1$
d) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Subjective Problems

36. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation, $2A+G^2 = 27$. Find the two numbers.
37. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. Find the number of sides of the polygon.
38. Does there exists a geometric progression containing 27, 8 and 12 as three of its terms ? If it exits, how many such progressions are possible ?
39. Find three numbers a,b,c between 2 and 18 such that
    a) their sum is 25
    b) the numbers 2,a,b are consecutive terms of an A.P. and
    c) the numbers b,c,18 are consecutive terms of a G.P.
40. If $a > 0, b > 0$ and $c > 0$, prove that $(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq 9$
41. If n is a natural number such that $n = p_1^{\alpha}1.p_2^{\alpha}2.p_3^{\alpha}3....p_k^{\alpha}k$ and $p_1, p_2, ...p_k$ are distinct primes, then show that $lnn \geq kln2$
42. Find the sum of the series: $\sum_{r=0}^{n}(-1)^r nC_r[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}}$......upto m terms]
43. Solve for x the following equation: $log_{2x+3}(6x^2+23x+21) = 4-log_{3x+7}(4x^2+12x+9)$
44. If $log_3 2$, $log_3(2^x - 5)$, $log_3(2^x - \frac{7}{2})$ are in arithmetic progression, determine the value of x.
45. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $[\frac{n+1}{n-1}]^2$p.
46. If S_1, S_2, S_3,......., S_n are the sums of infinite geometric series whose first terms are 1, 2, 3,......, n and whose common ratios are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,..........$\frac{1}{n+1}$ respectively, then find the values of S_1^2+ S_2^2+ S_3^2,......., S_{2n-1}^2
47. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.
48. Let a,b,c,d are the real numbers in G.P. If u,v,w, satisfy the system of equations $u+2v+3w = 6$ $4u + 5v + 6w = 12$ $6u + 9v = 4$ then show that the root of the equation $(\frac{1}{u} + \frac{1}{v} + \frac{1}{w})x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]x + u + v + w = 0$ and $20x^2 + 10(a - d)^2 x - 9 = 0$ are reciprocals of the each other.
49. The fouth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.
50. Let a_1, a_2,.......,a_n be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of a_1, a_2,.......,a_n. Find an expression for the geometric mean of G_1, G_2,......, G_n, in terms of A_1, A_2,.......,A_n, H_1,H_2,......,H_n,
51. Let a, b be positive real numbers.If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)(a+2b)}{9ab}$.
52. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either a = b = c or a, b, $-\frac{c}{2}$ form a G.P.
53. If $a_n = \frac{3}{4} - [\frac{3}{4}]^2 + [\frac{3}{4}]^3 + ...(-1)^{n-1}[\frac{3}{4}]^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \forall n \geq n_0$.

Comprehension Based Questions

PASSAGE - 1

Let V_r denote the sum of first r terms of an arithmetic progression(A.P.) whose first term is r and the common difference is (2r-1). Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for r=1,2,...

54. The sum $V_1 + V_2 + ... + V_n$ is
    a) $\frac{1}{12}n(n + 1)(3n^2 - n + 1)$
    b) $\frac{1}{12}n(n + 1)(3n^2 + n + 2)$
    c) $\frac{1}{2}n(2n^2 - n + 1)$
    d) $\frac{1}{3}(2n^3 - 2n + 3)$
55. T_r is always
    a) an odd number
    b) an even number
    c) a prime number
    d) a composite number
56. Which one of the following is a correct statement ?
    a) Q_1, Q_2, Q_3,.... are in A.P. with common difference 5

b) Q_1, Q_2, Q_3,.... are in A.P. with common difference 6

c) Q_1, Q_2, Q_3,.... are in A.P. with common difference 11

d) $Q_1 = Q_2 = Q_3$=........

PASSAGE - 2

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

57. Which one of the following statements is correct ?
   a) $G_1 > G_2 > G_3 >$ ....
   b) $G_1 < G_2 < G_3 <$ ....
   c) $G_1 = G_2 = G_3 =$ ....
   d) $G_1 < G_3 < G_5$ and $G_2 > G_4 > G_6 >$ ....
58. Which one of the following statements is correct ?
   a) $A_1 > A_2 > A_3 >$ ....
   b) $A_1 < A_2 < A_3 <$ ....
   c) $A_1 > A_3 > A_5 >$ .... and $A_2 < A_4 < A_6 <$ ....
   d) $A_1 < A_3 < A_5 <$ .... and $A_2 > A_4 > A_6 >$ ....
59. Which one of the following statements is correct ?
   a) $H_1 > H_2 > H_3 >$ ....
   b) $H_1 < H_2 < H_3 <$ ....
   c) $H_1 > H_3 > H_5 >$ .... and $H_2 < H_4 < H_6 <$ ....
   d) $H_1 < H_3 < H_6 <$ .... and $H_2 > H_4 > H_6 >$ ....

Assertion Reson type quations

60. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.
   STATEMENT - 1: The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. and
   STATEMENT - 2: The numbers b_1, b_2, b_3, b_4 are in H.P.
   a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1
   b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a NOT a correct explanation for STATEMENT - 1
   c) STATEMENT - 1 is True, STATEMENT - 2 is False
   d) STATEMENT - 1 is False, STATEMENT - 2 is True

Integer Value Correct Type

61. Let $S_k, k = 1, 2, ..., 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|$ is
62. $a_1, a_2, a_3, ..., a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for k= 3,4,.....,11, if $\frac{a_1^2+a_2^2+.....+a_{11}^2}{11} = 90$,then the value of $\frac{a_1+a_2+.....+a_{11}}{11}$ is equal to
63. Let a_1, a_2, a_3,.......,a_{100} be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \leq p \leq 100$.For any integer n with $1 \leq n \leq 20$, let m=5n .If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is......
64. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k-20=..........
65. Let a ,b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is b+2, then the value of $\frac{a^2+a-14}{a+1}$ is
66. Suppose that all the terms of an arithmetic progression(A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6:11 and the seventh term is lies in between 130 and 140, then the common difference of this A.P. is
67. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)......(1 + x^{100})$ is
68. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ?
69. Let X be the set consisting of the first 2018 terms of the arithmetic progression $1, 6, 11, ...$, and Y be the set consisting of the first 2018 terms of the arithmetic progression $9, 16, 23, ...$ Then, the number of elements in the set $X \cup Y$ is ....
70. Let AP(a;d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If AP(1;3) AP(2;5) AP(3;7)= AP(a;d) then a+d equals.....

Section-B JEE Main/AIEEE

71. If 1, $log_9(3^{1-x}+2), log_3(4.3^x-1)$ are in A.P. then x equals
   a) $log_3 4$
   b) $1-log_3 4$
   c) $1-log_4 3$
   d) $log_4 3$
72. l, m, n are the p^{th}, q^{th} , and r^{th} term of a G.P. all positive, then $\begin{pmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log nn & r & 1 \end{pmatrix}$ equals
   a) -1
   b) 2
   c) 1
   d) 0
73. The value of $2^{\frac{1}{4}}.4^{\frac{1}{8}}.8^{\frac{1}{16}}$.........$\infty$ is
   a) 1
   b) 2
   c) $\frac{3}{2}$
   d) 4
74. Fifth term of a G.P. is 2, then the product of its 9 terms is
   a) 256
   b) 512
   c) 1024
   d) none of these
75. Sum of infinite number of terms of GP is 20 and sum of their square is 100. The common ratio of GP is
   a) 5
   b) $\frac{3}{5}$
   c) $\frac{8}{5}$
   d) $\frac{1}{5}$
76. $1^3-2^3+3^3-4^3+....+9^3$=
   a) 425
   b) -425
   c) 475
   d) -475
77. The sum of the series $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}$....... upto ∞ is equal to
   a) $log_e\frac{4}{e}$
   b) $2log_e 2$
   c) $log_e 2-1$
   d) $log_e 2$
78. If $S_n=\sum_{r=0}^{n}\frac{1}{nC_r}$ and $t_n=\sum_{r=0}^{n}\frac{r}{nC_r}$, then $\frac{t_n}{S_n}$ is equal to
   a) $\frac{2n-1}{2}$
   b) $\frac{1}{2}$n-1
   c) n-1
   d) $\frac{1}{2}$n
79. Let T_r be the r^{th} term of an A.P.whose first term is a and common difference is d. If for some positive integers m, n, $m\neq n$, $T_m=\frac{1}{n}$ and $T_n=\frac{1}{m}$, then a-d equals
   a) $\frac{1}{m}+\frac{1}{n}$
   b) 1
   c) $\frac{1}{mn}$
   d) 0
80. The sum of the first n terms of the series $1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+.....$ is $\frac{n(n+1)^2}{2}$ when n is even. When n is odd the sum is
   a) $[\frac{n(n+1)}{2}]^2$
   b) $\frac{n^2(n+1)}{2}$
   c) $\frac{n(n+1)^2}{4}$
   d) $\frac{3n(n+1)}{2}$
81. The sum of series $\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}$..... is
   a) $\frac{(e^2-2)}{e}$
   b) $\frac{(e-1)^2}{2e}$
   c) $\frac{(e^2-1)}{2e}$
   d) $\frac{(e^2-1)}{2}$
82. If the coefficient of r^{th},$(r+1)^th$ and $(r+2)^{th}$ terms in the binomial expansion of $(1+y)^m$ are in A.P., then m and r satisfy the equation
   a) $m^2-m(4r-1)+4r^2-2=0$
   b) $m^2-m(4r+1)+4r^2+2=0$
   c) $m^2-m(4r+1)+4r^2-2=0$
   d) $m^2-m(4r-1)+4r^2+2=0$
83. If $x=\sum_{n=0}^{\infty}a^n$, $y=\sum_{n=0}^{\infty}b^n$, $z=\sum_{n=0}^{\infty}c^n$ where a, b, c are in A.P. and $|a|<1, |b|<1, |c|<1$ then x, y, z are in
   a) G.P.
   b) A.P.
   c) Arithmetic - Geometric Progression
   d) H.P.
84. The sum of series $1+\frac{1}{4.2!}+\frac{1}{16.4!}+\frac{1}{64.6!}$.....$\infty$ is
   a) $\frac{(e-1)}{\sqrt{e}}$
   b) $\frac{(e+1)}{\sqrt{e}}$
   c) $\frac{(e-1)}{2\sqrt{e}}$
   d) $\frac{(e+1)}{2\sqrt{e}}$
85. Let a_1, a_2, a_3.....be terms in A.P. If $\frac{a_1+a_2+......+a_p}{a_1+a_2+......+a_q}=\frac{p^2}{q^2}$, $p\neq q$, then $\frac{a_6}{a_{21}}$ equals
   a) $\frac{41}{11}$
   b) $\frac{7}{2}$.

c) $\frac{2}{7}$
d) $\frac{11}{41}$

86. If a_1, a_2, a_3.....a_n are in H.P., then the expression $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n$ is equal to
a) $n(a_1 - a_n)$
b) $(n-1)(a_1 - a_n)$
c) $n(a_1a_n)$
d) $(n-1)(a_1a_n)$

87. The sum of series $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$..... upto ∞ is
a) $e^{-\frac{1}{2}}$
b) $e^{+\frac{1}{2}}$
c) e^{-2}
d) e^{-1}

88. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of its progression is equals
a) $\sqrt{5}$
b) $\frac{1}{2}(\sqrt{5} - 1)$
c) $\frac{1}{2}(1 - \sqrt{5})$
d) $\frac{1}{2}\sqrt{5}$

89. The first two terms of a geometric progression add up to 12. the sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is
a) -4
b) -12
c) 12
d) 4

90. The sum to infinite term of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...$ is
a) 3
b) 4
c) 6
d) 2

91. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = ... = a_{10} = 150$ and $a_{10}, a_{11}, ...$ are in A.P. with common difference -2, then the time taken by him to count all notes is
a) 34 minutes
b) 125 minutes
c) 135 minutes
d) 24 minutes

92. A man saves 200 in each of the first three months of his service. In each of the subsequent months his saving increases by 40 more than the saving of immediately previous month. His total savings from the start of service will be 11040 after
a) 19 months
b) 20 months
c) 21 months
d) 18 months

93. **Statement - 1:** The sum of the series 1+(1+2+4)+(4+6+9)+(9+12+16)+.....+(361+380+400) is 8000.
Statement - 2: $\sum_{k=1}^{n}(k^3 - (k-1)^3) = n^3$, for any natural number n.
a) Statement - 1 is false, Statement - 2 is true.
b) Statement - 1 is true, Statement - 2 is true, Statement - 2 is a correct explanation for Statement - 1
c) Statement - 1 is true, Statement - 2 is true, Statement - 2 is a not a correct explanation for Statement - 1
d) Statement - 1 is true, Statement - 2 is false.

94. The sum of the first 20 terms of sequence 0.7,0.77,0.777,....,is
a) $\frac{7}{81}(179 - 10^{-20})$
b) $\frac{7}{9}(99 - 10^{-20})$
c) $\frac{7}{81}(179 + 10^{-20})$
d) $\frac{7}{9}(99 + 10^{-20})$

95. If $(10)^9 + 2(11)^1(10^8) + 3(11)^2(10)^7 + ... + 10(11)^9 = k(10)^9$, then k is equal to :
a) 100
b) 110
c) $\frac{122}{10}$
d) $\frac{441}{100}$

96. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. then the common ration of the G.P. is:
a) $2 - \sqrt{3}$
b) $2 + \sqrt{3}$
c) $\sqrt{2} + \sqrt{3}$
d) $3 + \sqrt{2}$

97. The sum of the first 9 terms of the series.$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$
a) 142
b) 192
c) 71
d) 96

98. If m is the A.M. of two distinct real numbers l and $n(l, n > 1)$ and G_1, G_2 and G_3 are the three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals:
   a) $4lmn^2$
   b) $4l^2m^2n^2$
   c) $4l^2mn$
   d) $4lm^2n$
99. If the $2^{nd}, 5^{th}$ and 9^{th} terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:
   a) 1
   b) $\frac{7}{4}$
   c) $\frac{8}{5}$
   d) $\frac{4}{3}$
100. If the sum of the first ten terms of the series $(1\frac{3}{5})^2 + (2\frac{2}{5})^2 + (3\frac{1}{5})^2 + 4^2 + (4\frac{4}{5})^2 + ...$, is $\frac{16}{5}$m then m is equal to :
   a) 100
   b) 99
   c) 102
   d) 101
101. If, for a positive integer n, the quadratic equation, $x(x+1) + (x+1)(x+2) + ... + (x+\overline{n-1})(x+n) = 10n$ has two consecutive integral solutions, then n is equal to :
   a) 11
   b) 12
   c) 9
   d) 10
102. For any three positive real numbers a, b and c,
$$9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c) \tag{5.1.102.1}$$
   Then:
   a) a, b and c are in G.P.
   b) b, c and a are in G.P.
   c) b, c and a are in A.P.
   d) a, b and c are in A.P.
103. Let a, b, c ∈ R. If $f(x) = ax^2+bx+c$ is such that $a+b+c = 3$ $f(x+y) = f(x)+f(y)+xy, \forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to :
   a) 255
   b) 330
   c) 165
   d) 190
104. Let $a_1, a_2, a_3, ..., a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$.If $a_1^2 + a_2^2 + ... + a_{17}^2 = 140m$, then m is equal to
   a) 68
   b) 34
   c) 33
   d) 66
105. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ If $B - 2A = 100\Lambda$, then Λ is equal to :
   a) 248
   b) 464
   c) 496
   d) 232
106. If a, b, and c be three distinct real numbers in G.P. and a+b+c=xb, then x cannot be :
   a) -2
   b) -3
   c) 4
   d) 2
107. Let $a_1, a_2, a_3, ..., a_{30}$ be in A.P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{2i-1}$. If $a_5 = 27$ and S-2T=75, Then a_{10} is equal to :
   a) 52
   b) 57
   c) 47
   d) 42
108. Three circles of radii a, b, c $(a < b < c)$ touch each other externally. If they have X-axis as a common tangent, then :
   a) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$
   b) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$
   c) a, b, c are in A.P.
   d) $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are in A.P.
109. Let the sum of the first n terms of a non-constant A.P., $a_1, a_2, a_3, ...$ be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this A.P.,then the ordered pair (d, a_{10}) is equal to:
   a) (50, 50+46A)
   b) (50, 50+45A)
   c) (A, 50+45A)
   d) (A, 50+46A)