3D Geometry through Linear Algebra

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Contents

***Abstract*—This manual introduces linear algebra by exploring 3D geometry through a problem solving approach.**

1 Lines and Planes

1.1 L_1 is the intersection of planes

$$\begin{pmatrix}2 & -2 & 3\end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = -1 \tag{1.1}$$

Find its equation.
Solution: (1.1) can be written in matrix form as

$$\begin{pmatrix}2 & -2 & 3\\1 & -1 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\-1\end{pmatrix}, \tag{1.2}$$

and solved using the augmented matrix as follows

$$\begin{pmatrix}2 & -2 & 3 & 2\\1 & -1 & 1 & -1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 1 & -1\\2 & -2 & 3 & 2\end{pmatrix} \tag{1.3}$$

$$\leftrightarrow \begin{pmatrix}1 & -1 & 1 & -1\\0 & 0 & 1 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 0 & -5\\0 & 0 & 1 & 4\end{pmatrix} \tag{1.4}$$

$$\implies \mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_2-5\\x_2\\4\end{pmatrix} = \begin{pmatrix}-5\\0\\4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{1.5}$$

which is the desired equation.

1.2 L_2 is the intersection of the planes

$$\begin{pmatrix}1 & 2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{1.6}$$
$$\begin{pmatrix}3 & -1 & 2\end{pmatrix}\mathbf{x} = 1 \tag{1.7}$$

Show that its equation is

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix}5\\8\\0\end{pmatrix} + \lambda_2\begin{pmatrix}-3\\5\\7\end{pmatrix} \tag{1.8}$$

1.3 Do L_1 and L_2 intersect? If so, find their point of intersection.
Solution: From (1.5),(1.8), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix}5\\8\\0\end{pmatrix} + \lambda_2\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}-5\\0\\4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{1.9}$$

$$\implies \begin{pmatrix}1 & 3\\1 & -5\\0 & -7\end{pmatrix}\mathbf{\Lambda} = \frac{1}{7}\begin{pmatrix}40\\8\\-28\end{pmatrix} \tag{1.10}$$

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This matrix equation can be solved as

$$\begin{pmatrix}1 & 3 & \frac{40}{7}\\ 1 & -5 & \frac{8}{7}\\ 0 & -7 & -4\end{pmatrix} \leftrightarrow \begin{pmatrix}8 & 0 & \frac{224}{7}\\ 0 & 1 & \frac{4}{7}\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \tag{1.11}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \implies \mathbf{\Lambda} = \begin{pmatrix}4\\ \frac{4}{7}\end{pmatrix} \tag{1.12}$$

Substituting $\lambda_1 = 4$ in (1.9)

$$\mathbf{x} = \begin{pmatrix}4\\4\\0\end{pmatrix} + \begin{pmatrix}-5\\0\\4\end{pmatrix} = \begin{pmatrix}-1\\4\\4\end{pmatrix} \tag{1.13}$$

2 Normal to a Plane

2.1 The cross product of $\mathbf{a}, \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{2.1}$$

From (1.5), (1.8), the direction vectors of L_1 and L_2 are

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}-3\\5\\7\end{pmatrix} \tag{2.2}$$

respectively. Find the direction vector of the normal to the plane spanned by L_1 and L_2.
Solution: The desired vector is obtained as

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}7\\-7\\8\end{pmatrix} = \mathbf{n} \tag{2.3}$$

2.2 Find the equation of the plane spanned by L_1 and L_2.
Solution: Let $\mathbf{x}_0$ be the intersection of L_1 and L_2. Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \tag{2.4}$$

$$\implies \mathbf{x}^T\mathbf{n} = \mathbf{x}_0^T\mathbf{n} \tag{2.5}$$

$$\implies \mathbf{x}^T\begin{pmatrix}7\\-7\\8\end{pmatrix} = \begin{pmatrix}-1 & 4 & 4\end{pmatrix}\begin{pmatrix}7\\-7\\8\end{pmatrix} = -3 \tag{2.6}$$

2.3 Find the distance of the origin from the plane containing the lines L_1 and L_2.
Solution: The distance from the origin to the plane is given by

$$\frac{\left|\mathbf{x}_0^T\mathbf{n}\right|}{\|n\|} = \frac{1}{3\sqrt{2}} \tag{2.7}$$

3 Projection on a Plane

3.1 Find the equation of the line L joining the points

$$\mathbf{A} = \begin{pmatrix}5 & -1 & 4\end{pmatrix}^T \tag{3.1}$$

$$\mathbf{B} = \begin{pmatrix}4 & -1 & 3\end{pmatrix}^T \tag{3.2}$$

Solution: The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda(\mathbf{A} - \mathbf{B}) \tag{3.3}$$

$$= \begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{3.4}$$

3.2 Find the intersection of L and the plane P given by

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.5}$$

Solution: From (3.4) and (3.5),

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 7 \tag{3.6}$$

$$\implies 6 + 2\lambda = 7 \tag{3.7}$$

$$\implies \lambda = \frac{1}{2} \tag{3.8}$$

Substituting in (3.4),

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix}9 & -1 & 7\end{pmatrix} \tag{3.9}$$

3.3 Find $\mathbf{C} \in P$ such that $AC \perp P$.
Solution: From (3.5), the direction vector of AC is $\begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$. Hence, the equation of AC is

$$\mathbf{x} = \begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.10}$$

Substituting in (3.5)

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 7 \tag{3.11}$$

$$\implies 8 + 3\lambda_1 = 7 \tag{3.12}$$

$$\implies \lambda_1 = -\frac{1}{3} \tag{3.13}$$

Thus,

$$\mathbf{C} = \frac{1}{3}\begin{pmatrix} 14 \\ -4 \\ 11 \end{pmatrix} \tag{3.14}$$

3.4 Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D} = \frac{1}{3}\begin{pmatrix} 13 \\ -2 \\ 10 \end{pmatrix} \tag{3.15}$$

3.5 Find the projection of AB on the plane P.
Solution: The projection is given by

$$CD = \|\mathbf{C} - \mathbf{D}\| = \sqrt{\frac{2}{3}} \tag{3.16}$$

4 Coplanar vectors

4.1 If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that

$$\mathbf{u}^T (\mathbf{A} \times \mathbf{B}) = 0 \tag{4.1}$$

4.2 Find $\mathbf{A} \times \mathbf{B}$ given

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}^T \tag{4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T \tag{4.3}$$

Solution: From (2.1),

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.4}$$

$$= \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \tag{4.5}$$

4.3 Let $\mathbf{u}$ be coplanar with such that $\mathbf{u} \perp \mathbf{A}$and

$$\mathbf{u}^T\mathbf{B} = 24. \tag{4.6}$$

Find $\|\mathbf{u}\|^2$.
Solution: From (4.5) and the given information,

$$\mathbf{u}^T \begin{pmatrix} 4 & -2 & 2 \end{pmatrix} = 0 \tag{4.7}$$

$$\mathbf{u}^T \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = 0 \tag{4.8}$$

$$\mathbf{u}^T \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 24 \tag{4.9}$$

$$\implies \begin{pmatrix} 4 & -2 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} \tag{4.10}$$

$$\implies \mathbf{u} = 4\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \tag{4.11}$$

$$\implies \|\mathbf{u}\|^2 = 336 \tag{4.12}$$

5 Least Squares

5.1 Find the equation of the plane P containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{5.1}$$

5.2 Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \tag{5.2}$$

lies outside P.

5.3 Find the point $\mathbf{w} \in P$ closest to $\mathbf{y}$.

5.4 Let

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}. \tag{5.3}$$

Show that

$$\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{w} = \mathbf{X}^T\mathbf{y} \tag{5.4}$$