Computational Approach to School Mathematics

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Contents

***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/codes
```

1 line

1.1 Examples

1. Do the points $\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}-2\\-3\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

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2. Show that the points $\begin{pmatrix}1\\7\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
3. Verify if $\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\6\end{pmatrix}$ are points on a line.
4. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
5. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
6. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
   Solution: Let the end points of the line be
$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \tag{1.1.6.1}$$
   Then the point $\mathbf{C}$
$$\mathbf{C} = \frac{k\mathbf{A}+\mathbf{B}}{k+1} \tag{1.1.6.2}$$
   divides AB in the ration $k:1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 1.1.6

```
codes/line/draw_section.py
```

7. Find a unit vector in the direction of $\begin{pmatrix}2\\3\\1\end{pmatrix}$.
8. Find the direction vector of PQ, where
$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{1.1.8.1}$$
9. Find the angle between the vectors
$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{1.1.9.1}$$

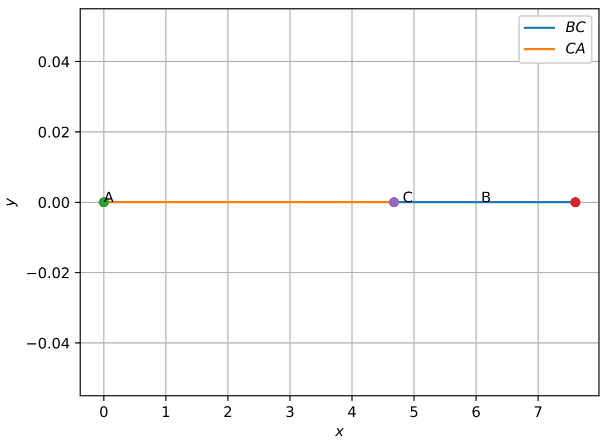

Fig. 1.1.6

10. Find the projection of the vector
$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{1.1.10.1}$$
on the vector
$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.1.10.2}$$
11. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where
$$\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}. \tag{1.1.11.1}$$
12. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.
13. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

1.2 Elementary Exercises

1. Find the distance between the following pairs of points
   a)
$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{1.2.1.1}$$
   b)
$$\begin{pmatrix}-5\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \tag{1.2.1.2}$$
   c)
$$\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}-1\\b\end{pmatrix} \tag{1.2.1.3}$$
2. Find the distance between the points
$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix} \tag{1.2.2.1}$$
3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
4. Determine if the points
$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{1.2.4.1}$$
are collinear.
5. Check whether
$$\begin{pmatrix}5\\-2\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \begin{pmatrix}7\\-2\end{pmatrix} \tag{1.2.5.1}$$
are the vertices of an isosceles triangle.
6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.
   a)
$$\begin{pmatrix}-1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-3\\0\end{pmatrix} \tag{1.2.6.1}$$
   b)
$$\begin{pmatrix}-3\\5\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}-1\\-4\end{pmatrix} \tag{1.2.6.2}$$
   c)
$$\begin{pmatrix}4\\5\end{pmatrix}, \begin{pmatrix}7\\6\end{pmatrix}, \tag{1.2.6.3}$$
$$\begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \tag{1.2.6.4}$$
7. Find the point on the x-axis which is equidistant from
$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{1.2.7.1}$$
8. Find the values of y for which the distance

between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \tag{1.2.8.1}$$

is 10 units.

9. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{1.2.9.1}$$

10. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{1.2.10.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

11. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$.

12. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \tag{1.2.12.1}$$

13. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \tag{1.2.13.1}$$

Also, show that they are mutually perpendicular to each other.

14. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \tag{1.2.14.1}$$

$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| \tag{1.2.14.2}$$

15. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \tag{1.2.15.1}$$

16. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{1.2.16.1}$$

$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{1.2.16.2}$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.

17. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \tag{1.2.17.1}$$

18. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \tag{1.2.18.1}$$

$(\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c}$. Find λ.

19. Show that

$$(\|\mathbf{a}\| \, \mathbf{b} + \|\mathbf{b}\| \, \mathbf{a}) \perp (\|\mathbf{a}\| \, \mathbf{b} - \|\mathbf{b}\| \, \mathbf{a}) \tag{1.2.19.1}$$

20. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

21. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \tag{1.2.21.1}$$

find the value of

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{1.2.21.2}$$

22. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if

a) $\lambda = 1$

b) $\lambda = -1$

c) $\|\mathbf{a}\| = |\lambda|$

d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

23. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{1.2.23.1}$$

find $\|\mathbf{a} \times \mathbf{b}\|$.

24. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \tag{1.2.24.1}$$

25. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \tag{1.2.25.1}$$

26. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

27. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2 (\mathbf{a} \times \mathbf{b}) \tag{1.2.27.1}$$

28. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

29. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

30. If $\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\-1\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.
31. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$,
32. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.
33. Let $\mathbf{a} = \begin{pmatrix}1\\4\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.
34. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.
35. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \qquad (1.2.35.1)$$
$$\iff \mathbf{a} \perp \mathbf{b}. \qquad (1.2.35.2)$$

36. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

a) $0 < \theta < \frac{\pi}{2}$
b) $0 \leq \theta \leq \frac{\pi}{2}$
c) $0 < \theta < \pi$
d) $0 \leq \theta \leq \pi$

37. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

38. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \left(\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\1\\0\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\0\\1\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}\right) \qquad (1.2.38.1)$$

is

a) 0
b) -1
c) 1
d) 3

39. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π.

40. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.
41. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \qquad (1.2.41.1)$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \qquad (1.2.41.2)$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

42. Let $\boldsymbol{\alpha} = \begin{pmatrix}3\\-1\\0\end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
43. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
44. Find a unit vector in the direction of $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.
45. Find a unit vector in the direction of the line passing through $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and 1
2
3.
46. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively.

47. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.
48. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\\6\end{pmatrix}$.
49. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.

1.3 Section Formula

1. Find the coordinates of the point which divides the join of
$$\begin{pmatrix}-1\\7\end{pmatrix}, = \begin{pmatrix}4\\-3\end{pmatrix} \tag{1.3.1.1}$$
in the ratio $2:3$.
2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}4\\y\end{pmatrix}$, $\begin{pmatrix}x\\6\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.
8. Find the value of k if the points $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\k\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}6\\-3\end{pmatrix}$ are collinear.
9. In each of the following, find the value of k for which the points are collinear
   a) $\begin{pmatrix}7\\-2\end{pmatrix}$, $\begin{pmatrix}5\\1\end{pmatrix}$, $\begin{pmatrix}3\\k\end{pmatrix}$
   b) $\begin{pmatrix}8\\1\end{pmatrix}$, $\begin{pmatrix}k\\-4\end{pmatrix}$, $\begin{pmatrix}2\\-5\end{pmatrix}$
10. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}7\\0\end{pmatrix}$ are collinear.
11. If
$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{1.3.11.1}$$
$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{1.3.11.2}$$
find $\mathbf{R}$, which divides PQ
   a) internally,
   b) externally.
12. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
14. Find $\mathbf{R}$ which divides the line joining the points
$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{1.3.14.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{1.3.14.2}$$
externally in the ratio $1:2$.
15. Show that $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\5\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}1\\-6\\-1\end{pmatrix}$, are collinear.

16. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.
17. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

1.4 Line Equation

1. Determine the ratio in which the line
$$\begin{pmatrix} 2 & 1 \end{pmatrix} - 4 = 0 \tag{1.4.1.1}$$
divides the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$
2. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.
3. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.
4. If
$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}, \tag{1.4.4.1}$$
find the equation of the line.
5. Find the angle between the pair of lines given by
$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.4.5.1}$$
$$\boldsymbol{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.4.5.2}$$
6. Find the angle between the pair of lines
$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{1.4.6.1}$$
7. Find the shortest distance between the lines
$$L_1 : \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.4.7.1}$$
$$L_2 : \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.4.7.2}$$
8. Find the distance between the lines
$$L_1 : \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.4.8.1}$$
$$L_2 : \quad \boldsymbol{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.4.8.2}$$
9. Find the equation of a line which passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.
10. Find the equaion off the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.
11. Find the equation of the line which passes through the point $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and parallel to the line given by
$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{1.4.11.1}$$
12. Find the equation of the line given by
$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \tag{1.4.12.1}$$
13. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.
14. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.
15. Find the angle between the following pair of lines:

a)

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.4.15.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.4.15.2}$$

b)

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{1.4.15.3}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{1.4.15.4}$$

16. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{1.4.16.1}$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{1.4.16.2}$$

17. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{1.4.17.1}$$

are at right angles.

18. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{1.4.18.1}$$

are perpendicular to each other.

19. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{1.4.19.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{1.4.19.2}$$

20. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{1.4.20.1}$$

21. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1.4.21.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.4.21.2}$$

22. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \tag{1.4.22.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \tag{1.4.22.2}$$

2 Triangle

2.1 Construction

1. Draw $\triangle ABC$ where $\angle B = 90°, a = 4$ and $b = 3$.
Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2.1.1.1}$$

The following code plots Fig. 2.1.1

```
codes/triangle/rt_triangle.py
```

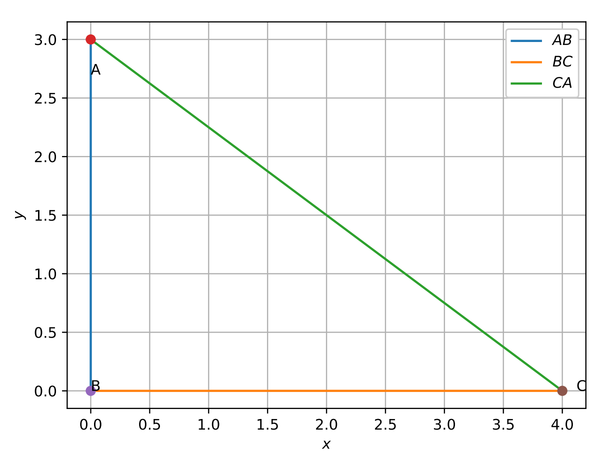

Fig. 2.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.2.1}$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \tag{2.1.2.2}$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{2.1.2.3}$$

$$= p \times p + q \times q = p^2 + q^2 \tag{2.1.2.4}$$

Then

$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{2.1.2.5}$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{2.1.2.6}$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \tag{2.1.2.7}$$

From (2.1.2.7),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \tag{2.1.2.8}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{2.1.2.9}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{2.1.2.10}$$

$$= a^2 + c^2 - 2ap \tag{2.1.2.11}$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{2.1.2.12}$$

From (2.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{2.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{2.1.2.14}$$

The following code plots Fig. 2.1.2

```
codes/triangle/draw_triangle.py
```

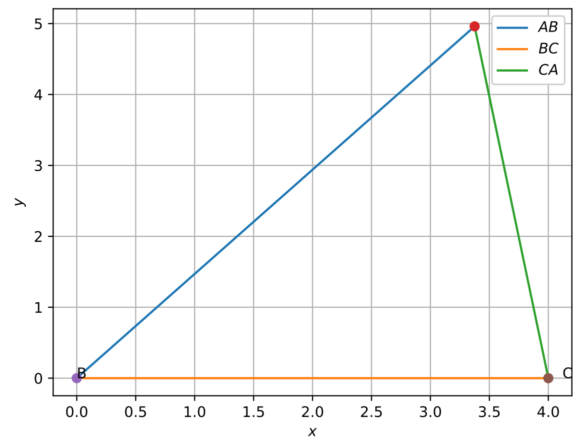

Fig. 2.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.
4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{2.1.4.1}$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.
Solution: From the given information,

$$a + b + c = 11 \tag{2.1.5.1}$$

$$b = c \quad (\because B = C = 45^\circ) \tag{2.1.5.2}$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{2.1.5.3}$$

From (2.1.5.1) and (2.1.5.2),

$$a + 2b = 11 \tag{2.1.5.4}$$

From (2.1.5.2) and (2.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{2.1.5.5}$$

(2.1.5.4) and (2.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{2.1.5.6}$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{2.1.5.7}$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{2.1.5.8}$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{2.1.5.9}$$

by expanding the determinants. The following code may be used to compute a, b and c.

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 2.1.5 using a single matrix equation.
Solution: The equations

$$a + 2b = 11 \tag{2.1.6.1}$$

$$a - b\sqrt{2} = 0 \tag{2.1.6.2}$$

$$b - c = 0 \tag{2.1.6.3}$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{2.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{2.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{2.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{2.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{2.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{2.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. (2.1.7), $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{2.1.7.1}$$

$$\cos B = \frac{x}{b}, \tag{2.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \qquad (2.1.7.3)$$
$$b = c\cos A + a\cos C \qquad (2.1.7.4)$$
$$c = b\cos A + a\cos B \qquad (2.1.7.5)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.1.7.6)$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \qquad (2.1.7.7)$$
$$= \frac{b^2 + c^2 - a^2}{2bc} \qquad (2.1.7.8)$$

From (2.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \qquad (2.1.7.9)$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

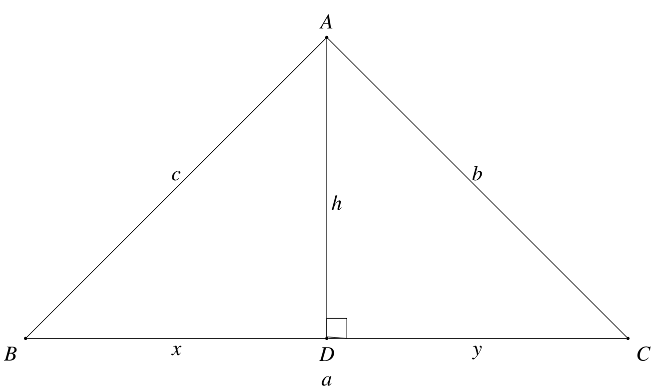

Fig. 2.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45^\circ$ and $\angle A = 105^\circ$.
Solution: In Fig. (2.1.7),

$$\sin B = \frac{h}{c} \qquad (2.1.8.1)$$
$$\sin C = \frac{h}{b} \qquad (2.1.8.2)$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad (2.1.8.3)$$

Thus,

$$c = \frac{a\sin C}{\sin A} \qquad (2.1.8.4)$$

where

$$C = 180 - A - B \qquad (2.1.8.5)$$

9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30^\circ$.
Solution: From (2.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad (2.1.9.1)$$

which can be expressed as

$$a^2 - 2ab\cos C + b^2 - c^2 = 0. \qquad (2.1.9.2)$$

$$\because (a - b\cos C)^2 = a^2 + b^2\cos^2 C - 2ab\cos C, \qquad (2.1.9.3)$$

(2.1.9.2) can be expressed as

$$(a - b\cos C)^2 - b^2\cos^2 C + b^2 - c^2 = 0 \qquad (2.1.9.4)$$
$$\implies (a - b\cos C)^2 = b^2\left(1 - \cos^2 C\right) - c^2 \qquad (2.1.9.5)$$
$$\text{or}, a = b\cos C \pm \sqrt{b^2\left(1 - \cos^2 C\right) - c^2} \qquad (2.1.9.6)$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \qquad (2.1.10.1)$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \qquad (2.1.10.2)$$

Verify (2.1.9.6) using (2.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b + c = 18$, find b, c and draw the triangle.
Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \qquad (2.1.11.1)$$
$$\implies (18 - c)^2 = 12^2 + c^2 \qquad (2.1.11.2)$$

which can be simplified to obtain

$$36c - 180 = 0 \quad (2.1.11.3)$$
$$\implies c = 5 \quad (2.1.11.4)$$

and $b = 13$

12. Find a simpler solution for Problem 2.1.5
Solution: Use cosine formula.
13. In $\triangle ABC$, $a = 7, \angle B = 75^{\circ}$ and $b + c = 13$. Alternatively,

$$a = b\cos C + c\cos B \quad (2.1.13.1)$$
$$b\sin C = c\sin B \quad (2.1.13.2)$$
$$a + b + c = 11 \quad (2.1.13.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (2.1.13.4)$$

Solving the equivalent matrix equation gives the desired answer.

2.2 *Construction Exercises*

1. In $\triangle ABC$, $a = 8, \angle B = 45^{\circ}$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6, \angle B = 60^{\circ}$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a + b + c = 11, \angle B = 30^{\circ}$ and $\angle C = 90^{\circ}$.
4. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60°$.
9. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90°$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
11. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60°$.
12. Construct $\triangle XYZ$ if $XY = 6, \angle X = 30°$ and $\angle Y = 100°$.
13. If $AC = 7, \angle A = 60°$ and $\angle B = 50°$, can you draw the triangle?
14. Construct $\triangle ABC$ given that $\angle A = 60°, \angle B = 30°$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5, \angle Q = 105°$ and $\angle R = 40°$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2, \angle E = 110°$ and $\angle F = 180°$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled $\triangle$ whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 2.2.21.

S.No	**Triangle**	**Given Measurements**		
1	△ABC	$\angle A = 85°$	$\angle B = 115°$	AB = 5
2	△PQR	$\angle Q = 30°$	$\angle R = 60°$	QR = 4.7
3	△ABC	$\angle A = 70°$	$\angle B = 50°$	AC = 3
4	△LMN	$\angle L = 60°$	$\angle N = 120°$	LM = 5
5	△ABC	BC = 2	AB = 4	AC = 2
6	△PQR	PQ = 2.5	QR = 4	PR = 3.5
7	△XYZ	XY = 3	YZ = 4	XZ = 5
8	△DEF	DE = 4.5	EF = 5.5	DF = 4

TABLE 2.2.21

2.3 *Triangle Geometry*

1. Find the area of a triangle whose vertices are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
2. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.
3. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
4. Find the area of the triangle whose vertices are
   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
   b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
5. Find the area of the triangle formed by joining the mid points o the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

6. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
7. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{2.3.7.1}$$
Find
$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{2.3.7.2}$$
8. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the vertices of $\triangle ABC$.
   a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
   b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
   c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
9. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{2.3.9.1}$$
10. Show that the points
$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{2.3.10.1}$$
are the vertices of a right angled triangle.
11. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.
12. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.
13. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.
14. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$
15. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
16. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

3 Quadrilateral

3.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
Solution: Fig. 3.1.1 shows a rough sketch of $ABCD$. Letting
$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{3.1.1.1}$$
it is trivial to sketch $\triangle ABC$ from Problem 2.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x-axis by an angle θ with
$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{3.1.1.2}$$
and
$$\cos\theta = \frac{a^2 + e^2 - b^2}{2ae} \tag{3.1.1.3}$$
$$\sin\theta = \sqrt{1 - \cos^2\theta} \tag{3.1.1.4}$$
The coordinates of the rotated triangle ACD are
$$\mathbf{D} = \mathbf{P}\begin{pmatrix} h \\ k \end{pmatrix} \tag{3.1.1.5}$$
$$\mathbf{A} = \mathbf{P}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.1.1.6}$$
$$\mathbf{C} = \mathbf{P}\begin{pmatrix} e \\ 0 \end{pmatrix} \tag{3.1.1.7}$$
where
$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.1.1.8}$$
The following code plots quadrilateral $ABCD$ in Fig. 3.1.1

```
codes/quad/draw_quad.py
```

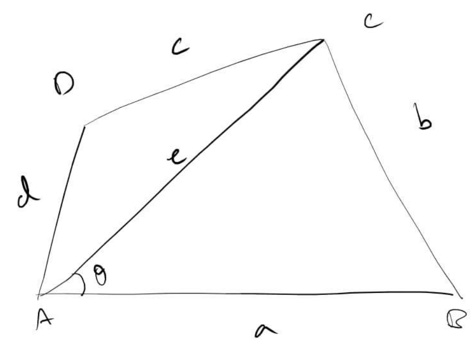

Fig. 3.1.1

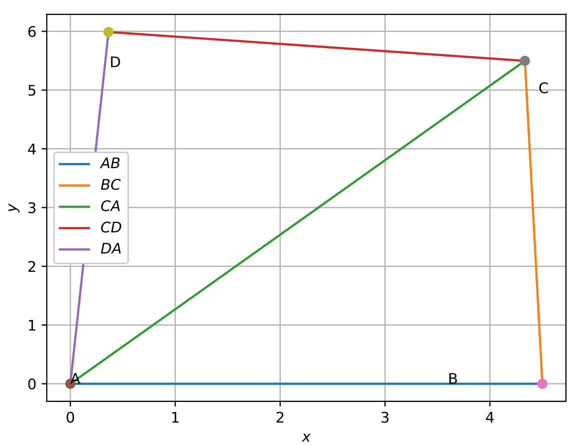

Fig. 3.1.1

2. Draw the parallelogram $MORE$ with $OR = 6, RE = 4.5$ and $EO = 7.5$.
   Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel .
3. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.
   Solution: The diagonals of a kite are perpendicular to each other.
4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
   Solution: Diagonals of a rhombus bisect each other at right angles.

3.2 Construction Exercises

1. Construct a quadrilateral $ABCD$ such that $AB = 5, \angle A = 50°, AC = 4, BD = 5$ and $AD = 6$.
2. Construct $PQRS$ where $PQ = 4, QR = 6, RS = 5, PS = 5.5$ and $PR = 7$.
3. Draw $JUMP$ with $JU = 3.5, UM = 4, MP = 5, PJ = 4.5$ and $PU = 6.5$
4. Construct a quadrilateral $ABCD$ such that $BC = 4.5, AC = 5.5, CD = 5, BD = 7$ and $AD = 5.5$.
5. Can you construct a quadrilateral $PQRS$ with $PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ = 4$?
6. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.
7. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.
8. DRAW rhombus $BEND$ such that $BN = 5.6, DE = 6.5$.
9. construct a quadrilateral MIST where $MI = 3.5, IS = 6.5, \angle M = 75°, \angle I = 105°$ and $\angle S = 120°$.
10. Can you construct the above quadrilateral MIST if $\angle M = 100°$ instead of $75°$.
11. Can you constrcut the quadrilateral PLAN if $PL = 6, LA = 9.5, \angle P = 75°, \angle L = 150°$ and $\angle A = 140°$?
12. Construct $MORE$ where $MO = 6, OR = 4.5, \angle M = 60°, \angle O = 105°, \angle R = 105°$.
13. Construct $PLAN$ where $PL = 4, LA = 6.5, \angle P = 90°, \angle A = 110°$ and $\angle N = 85°$.
14. Constrcut parallelogram $HEAR$ where $HE = 5, EA = 6, \angle R = 85°$.
15. Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.
16. Construct $ABCd$, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105°$ and $\angle C = 80°$.
17. Construct $DEAR$ with $DE = 4, EA = 5, AR = 4.5, \angle E = 60°$ and $\angle A = 90°$.
18. Construct $TRUE$ with $TR = 3.5, RU = 3, UE = 4 \angle R = 75°$ and $\angle U = 120°$.
19. Draw a square of side 4.5.
20. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
21. Draw a square $READ$ with $RE = 5.1$.
22. Draw a rhombus who diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

3.3 Quadrilateral Geometry

1. Find the area of a rhombus if its vertices are $\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}4\\5\end{pmatrix}, \begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\end{pmatrix}$ taken in order.

2. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
3. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}, \begin{pmatrix}-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$.
4. The two opposite vertices of a square are $\begin{pmatrix}--1\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
5. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?
6. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
7. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-7\\1\end{pmatrix}$.
8. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix}-1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\-\frac{1}{2}\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-\frac{1}{2}\\4\end{pmatrix}$.
9. The two adjacent sides of a parallelogram are $\begin{pmatrix}2\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

4 Circle

4.1 Construction Examples

1. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.
   Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC, AC \perp AB, a = 10$ and $c = 6$.
   $$b = \sqrt{a^2 - c^2} \tag{4.1.1.1}$$
   The following code plots Fig. 4.1.1

```
codes/circle/draw_circle_eg.py
```

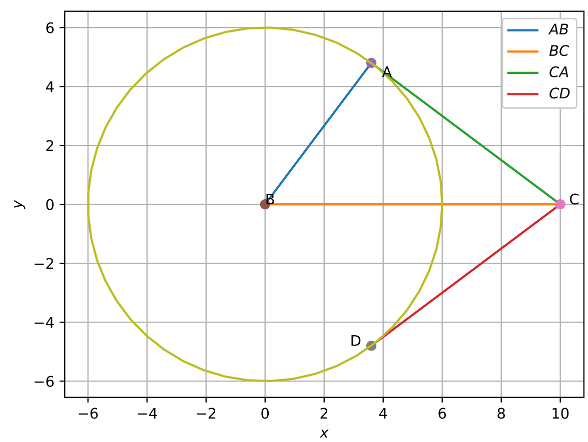

Fig. 4.1.1

2. Draw a circle of radius 3. Mark any point $\mathbf{A}$ on the circle, point $\mathbf{B}$ inside the circle and point $\mathbf{C}$ outside the circle.
   Solution: For any angle θ, a point on the circle with radius 3 has coordinates
   $$3\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix} \tag{4.1.2.1}$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre $\mathbf{O}$, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let $\mathbf{A}$ and $\mathbf{B}$ be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at $\mathbf{C}$ and $\mathbf{D}$. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
   Solution: Take the centre of both circles to be at the origin.

6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
   Solution: Take the diameter to be on the x-axis.
7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.
   Solution: The tangent is perpendicular to the radius.
8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.
   Solution: Let
   $$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \tag{4.2.2.1}$$
9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through $\mathbf{B}, \mathbf{C}, \mathbf{D}$ (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\left(2, -3\right)$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
2. Find the centre of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
3. Find the locus of all the unit vectors in the xy-plane.