JEE Problems in Matrices

***Abstract*—A collection of problems from JEE papers related to matrices are available in this document. Verify your soluions using Python.**

1 Matrices: Cayley-Hamilton Theorem

1.1 Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{1}$$

where α, β are real functions of θ and $\mathbf{I}$ is the identity matrix. Find the characteristic equation of $\mathbf{M}$.

Solution: (1) can be expressed as

$$\mathbf{M}^2 - \alpha\mathbf{M} - \beta\mathbf{I} = 0 \tag{2}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{3}$$

1.2 Find α and β.

Solution: Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta + \cos^4\theta \tag{4}$$

$$\beta = -\sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{5}$$

1.3 If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{6}$$

$$\beta^* = \min_\theta \beta(\theta), \tag{7}$$

find $\alpha^* + \beta^*$.

Solution:

$$\because \alpha = \sin^4\theta + \cos^4\theta = 1 - \frac{\sin^2 2\theta}{2}, \tag{8}$$

$$\alpha^* = \frac{1}{2}, \tag{9}$$

Similarly,

$$-\beta = \sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{10}$$

$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \tag{11}$$

$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2}\right)^2 + \frac{7}{4} \tag{12}$$

Thus,

$$\beta^* = -\frac{37}{16} \tag{13}$$

$$\implies \alpha^* + \beta^* = -\frac{29}{16} \tag{14}$$

2 Matrices: Adjugate

Let

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}, \quad \operatorname{adj}(\mathbf{M}) = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \tag{15}$$

2.1 Show that $a + b = 3$

Solution:

$$\because \mathbf{M}\operatorname{adj}(\mathbf{M}) = \det(\mathbf{M})\,\mathbf{I}, \tag{16}$$

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix}\begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} = 0 \tag{17}$$

$$\begin{pmatrix} 3 & b & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = 0 \tag{18}$$

resulting in

$$a = 2, b = 1 \tag{19}$$

Hence, $a + b = 3$.

2.2 Verify if

$$\left(\operatorname{adj}(\mathbf{M})\right)^{-1} + \operatorname{adj}\left(\mathbf{M}^{-1}\right) = -\mathbf{M} \tag{20}$$

Solution: From (16)

$$\left(\text{adj}(\mathbf{M})\right)^{-1} = \frac{\mathbf{M}}{\det(\mathbf{M})} \tag{21}$$

and

$$\left(\text{adj}\left(\mathbf{M}^{-1}\right)\right) = \frac{\mathbf{M}^{-1}}{\det(\mathbf{M}^{-1})} \tag{22}$$

$$= \mathbf{M}^{-1}\det(\mathbf{M}) \tag{23}$$

Thus,

$$\begin{aligned} &\left(\text{adj}\left(\mathbf{M}^{-1}\right)\right) + \text{adj}\left(\mathbf{M}^{-1}\right) \\ &= \mathbf{M}^{-1}\det(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \\ &= \text{adj}(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \end{aligned} \tag{24}$$

From (16)

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = \det(\mathbf{M}) \tag{25}$$

$$\implies \det(\mathbf{M}) = 8 - 5a = -2 \tag{26}$$

If

$$\begin{aligned} \left(\text{adj}\left(\mathbf{M}^{-1}\right)\right) + \text{adj}\left(\mathbf{M}^{-1}\right) &= -\mathbf{M}, \\ \text{adj}(\mathbf{M}) - \frac{\mathbf{M}}{2} &= -\mathbf{M} \\ \implies \mathbf{M} &= -\text{adj}(\mathbf{M}) \end{aligned}$$

which is incorrect.

2.3 Verify if

$$\det\left(\text{adj}\left(\mathbf{M}^2\right)\right) = 81 \tag{27}$$

Solution:

$$\text{adj}\left(\mathbf{M}^2\right) = \mathbf{M}^{-2}\det(\mathbf{M})^2 \tag{28}$$

$$= 4\mathbf{M}^{-2} \tag{29}$$

$$\implies \det\left(\text{adj}\left(\mathbf{M}^2\right)\right) = 4^3\det(\mathbf{M})^{-2} \tag{30}$$

$$= 16 \neq 81 \tag{31}$$

2.4 If

$$\mathbf{M}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{32}$$

show that

$$\alpha - \beta + \gamma = 3 \tag{33}$$

Solution:

$$\mathbf{M}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{34}$$

$$\implies \text{adj}(\mathbf{M})\,\mathbf{M}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \text{adj}(\mathbf{M})\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{35}$$

which can be expressed as

$$\det(\mathbf{M})\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \text{adj}(\mathbf{M})\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{36}$$

$$\text{or,} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\frac{1}{2}\,\text{adj}(\mathbf{M})\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{37}$$

Thus,

$$\alpha - \beta + \gamma = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{38}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\,\text{adj}(\mathbf{M})\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{39}$$

$$= \begin{pmatrix} 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 3 \tag{40}$$

3 Linear Algebra: Binary Matrices

Let S be the set of all 3×3 matrices whose entries are from $\{0, 1\}$ and

$$E_1 = \{\mathbf{A} \in S : \det(\mathbf{A}) = 0\} \tag{41}$$

and

$$E_2 = \{\mathbf{A} \in S : \text{ sum of entries of } A \text{ is } 7\} \tag{42}$$

3.1 Find $|E_2|$.
Solution:

$$|E_2| = \frac{9!}{7!2!} = 72 \tag{43}$$

3.2 Find $|(E_1|E_2)|$.
Solution: E_2 is the set of matrices with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the following combinations in Table 3.2. $\mathbf{e}_i, i = 1, 2, 3$ are the standard basis vectors. The equation

$$\mathbf{v}_1 = \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \tag{44}$$

has a solution only for the first combination in Table 3.2. Thus, $\det(A) = 0$ only for this combination. Thus

$$|(E_1|E_2)| = 3 \times 3 = 9 \tag{45}$$

$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\mathbf{e}_i$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$

TABLE 3.2

3.3 Find $\Pr(E_1|E_2)$.
Solution: From (43) and (45),

$$\Pr(E_1|E_2) = \frac{|(E_1|E_2)|}{|E_2|} = \frac{9}{72} = \frac{1}{8} \tag{46}$$

3.4 Verify using a python script.

4 Matrices: Trace

4.1 Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^{6}$ from permutations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{47}$$

4.2 Let

$$\mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix} \mathbf{P}_k^T. \tag{48}$$

Given

$$\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha \begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{49}$$

is $\alpha = 30$?
Solution:

$$\because \mathbf{P}_k^T \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\begin{aligned}
\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} &= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix} \mathbf{P}_k^T \begin{pmatrix}1\\1\\1\end{pmatrix} \\
&= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} \\
&= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}5\\3\\5\end{pmatrix} = 2 \begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix} \begin{pmatrix}6\\3\\6\end{pmatrix} \\
&= 30 \begin{pmatrix}1\\1\\1\end{pmatrix}
\end{aligned} \tag{50}$$

Thus, $\alpha = 30$.

4.3 Is $\mathbf{X}$ symmetric?
Solution: Yes. Trivial.

4.4 Show that

$$\mathbf{P}_k \mathbf{P}_k^T = \mathbf{I} \tag{51}$$

Solution:

$$\mathbf{P}_k = \begin{pmatrix}\mathbf{v}_{k1}^T\\ \mathbf{v}_{k2}^T\\ \mathbf{v}_{k3}^T\end{pmatrix} \tag{52}$$

where $\mathbf{v}_{ki}, i = 1, 2, 3$ are from the standard basis. Then,

$$\mathbf{P}_k \mathbf{P}_k^T = \begin{pmatrix}\mathbf{v}_{k1}^T\\ \mathbf{v}_{k2}^T\\ \mathbf{v}_{k3}^T\end{pmatrix} \begin{pmatrix}\mathbf{v}_{k1} & \mathbf{v}_{k2} & \mathbf{v}_{k3}\end{pmatrix} \mathbf{I}$$

$$\because \mathbf{v}_{ji}^T \mathbf{v}_{kj} = \delta_{jk} \tag{53}$$

4.5 For 2×2 matrices $\mathbf{A}, \mathbf{B}$, verify that

$$tr(AB) = tr(BA) \tag{54}$$

Show that this is true for any square matrix.

4.6 Verify if the sum of the diagonal entries of X is 18.

Solution:

$$\begin{aligned}\mathrm{tr}\,(\mathbf{X}) &= \sum_{k=1}^{6} \mathrm{tr}\left\{\mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix}\mathbf{P}_k^T\right\}\\ &= \sum_{k=1}^{6} \mathrm{tr}\left\{\begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix}\mathbf{P}_k\mathbf{P}_k^T\right\}\\ &= \sum_{k=1}^{6} \mathrm{tr}\begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix} = 6 \times 3 = 18 \end{aligned} \quad (55)$$

after substituting from (51).

4.7 Is $\mathbf{X} - 30\mathbf{I}$ invertible?
Solution: From (49),

$$\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = 30\mathbf{I}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\implies (\mathbf{X} - 30\mathbf{I})\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \quad (56)$$

If $(\mathbf{X} - 30\mathbf{I})^{-1}$ exists,

$$(\mathbf{X} - 30\mathbf{I})^{-1}(\mathbf{X} - 30\mathbf{I})\begin{pmatrix}1\\1\\1\end{pmatrix} = 0$$

$$\implies \begin{pmatrix}1\\1\\1\end{pmatrix} = \mathbf{0} \quad (57)$$

which is a contradiction. Hence, $\mathbf{X} - 30\mathbf{I}$ is not invertible.

5 Linear Algebra: Eigenvector and Null Space

Let

$$\mathbf{P} = \begin{pmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}2 & x & x\\0 & 4 & 0\\x & x & 6\end{pmatrix} \quad (58)$$

5.1 Find x such that $PQ = QP$.
Solution:

$$\because \mathbf{Q} = \begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 6\end{pmatrix} + x\begin{pmatrix}0 & 1 & 1\\0 & 0 & 0\\1 & 1 & 0\end{pmatrix}, \quad (59)$$

$$\mathbf{PQ} = \begin{pmatrix}2 & 4 & 6\\0 & 8 & 12\\0 & 0 & 18\end{pmatrix} + x\begin{pmatrix}1 & 2 & 1\\2 & 2 & 0\\3 & 3 & 0\end{pmatrix} \quad (60)$$

and

$$\mathbf{QP} = \begin{pmatrix}2 & 2 & 2\\0 & 8 & 8\\0 & 0 & 18\end{pmatrix} + x\begin{pmatrix}0 & 2 & 5\\0 & 0 & 0\\1 & 3 & 3\end{pmatrix} \quad (61)$$

Thus,

$$\mathbf{PQ} = \mathbf{QP} \implies \begin{pmatrix}0 & 2 & 4\\0 & 0 & 4\\0 & 0 & 0\end{pmatrix} = x\begin{pmatrix}-1 & 0 & 4\\-2 & -2 & 0\\-2 & 0 & 3\end{pmatrix} \quad (62)$$

which has no solution.

5.2 If

$$\mathbf{R} = \mathbf{PQP}^{-1}, \quad (63)$$

verify whether

$$\det\mathbf{R} = \det\begin{pmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{pmatrix} + 8 \quad (64)$$

for all x.
Solution:

$$\begin{aligned}\det(\mathbf{R}) &= \det(\mathbf{P})\det(\mathbf{Q})\det(\mathbf{P})^{-1} = \det(\mathbf{Q})\\ &= 4\left(12 - x^2\right)\end{aligned} \quad (65)$$

Thus,

$$\begin{aligned}\det(\mathbf{R}) - \det\begin{pmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{pmatrix} &= 4\left\{\left(12 - x^2\right) - \left(10 - x^2\right)\right\}\\ &= 8\end{aligned} \quad (66)$$

which is true.

5.3 For $x = 0$, if

$$\mathbf{R}\begin{pmatrix}1\\a\\b\end{pmatrix} = 6\begin{pmatrix}1\\a\\b\end{pmatrix}, \quad (67)$$

then show that

$$a + b = 5. \quad (68)$$

Solution: For $x = 0$,

$$\mathbf{R} = \mathbf{PQP}^{-1}, \quad (69)$$

where $\mathbf{Q}$ is a diagonal matrix. This is the eigenvalue decomposition of $\mathbf{R}$. Thus,

$$\mathbf{R}\begin{pmatrix}1\\2\\3\end{pmatrix} = 6\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{70}$$

where

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{71}$$

is the eigenvector corresponding to the eigenvalue 6. Comparing with (70),

$$a = 2, b = 3 \implies a + b = 5. \tag{72}$$

5.4 For $x = 1$, verify if there exists a vector $\mathbf{y}$ for which $\mathbf{Ry} = \mathbf{0}$.
Solution:

$$\begin{aligned} \mathbf{Ry} = \mathbf{0} &\implies \mathbf{PQP}^{-1}\mathbf{y} = \mathbf{0} \\ &\implies \mathbf{Qz} = \mathbf{0}, \end{aligned} \tag{73}$$

where

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{y} \tag{74}$$

For $x = 1$, (58) and (73) yield

$$\begin{pmatrix}2 & 1 & 1\\0 & 4 & 0\\1 & 1 & 6\end{pmatrix}\mathbf{z} = \mathbf{0} \tag{75}$$

Using row reduction,

$$\begin{pmatrix}2 & 1 & 1\\0 & 4 & 0\\1 & 1 & 6\end{pmatrix} \leftrightarrow \begin{pmatrix}2 & 1 & 1\\0 & 1 & 0\\0 & 1 & 11\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 1 & 11\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 0 & 11\end{pmatrix} \tag{76}$$

Thus, $\mathbf{Q}^{-1}$ exists and

$$\mathbf{z} = \mathbf{0} \implies \mathbf{y} = \mathbf{0} \tag{77}$$

upon substituting from (74). This implies that the null space of $\mathbf{R}$ is empty.

6 Definite Integral: Limit of a Sum

6.1 Show that

$$\lim_{n\to\infty} \frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{4}{3}}} = \int_0^1 x^{\frac{1}{3}}\, dx = \frac{3}{4} \tag{78}$$

6.2 Show that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\left(a + \frac{1}{n}\right)^2} + \frac{1}{\left(a + \frac{1}{2}\right)^2} + \cdots + \frac{1}{(a+1)^2}\right] \\ &= \int_0^1 \frac{1}{(a+x)^2}\, dx = \frac{1}{a(a+1)} \end{aligned} \tag{79}$$

6.3 If

$$\begin{aligned} &\lim_{n\to\infty}\left(\frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{7}{3}}\left\{\left[\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \cdots + \frac{1}{(an+n)^2}\right]\right\}}\right) \\ &= 54, \quad |a| > 1, \end{aligned} \tag{80}$$

find a.
Solution: Substituting from (78) and (79) in (80),

$$\begin{aligned} \frac{3}{4}a(a+1) &= 54 \\ a(a+1) &= 72 \\ \implies a &= 8, -9. \end{aligned} \tag{81}$$

7 Exercises

1. For any two 3×3 matrices A and B, let $A + B = 2B^T$ and $3A + 2B = I_3$. Which of the following is true?
   a) $5A + 10B = 2I_3$.
   b) $10A + 5B = 3I_3$.
   c) $2A + B = 3I_3$.
   d) $3A + 6B = 2I_3$.