

# The Circle

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**Abstract**—Solved problems from JEE mains papers related to 2D circles in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.

- 1 A circle passes through the points  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ . If its centre  $\mathbf{O}$  lies on the line

$$(-1 \ 4)\mathbf{x} - 3 = 0 \quad (1.1)$$

find its radius.

**Solution:** Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \Rightarrow \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (1.2)$$

The direction vector of  $AB$  is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (1.3)$$

$$\therefore OC \perp AB,$$

$$\begin{aligned} OC : \mathbf{m}^T (\mathbf{x} - \mathbf{C}) &= 0 \\ \Rightarrow (1 \ 1)\mathbf{x} &= 7 \end{aligned} \quad (1.4)$$

Thus,  $\mathbf{O}$  is the intersection of (1.1) and (1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} \quad (1.5)$$

From the augmented matrix,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & -3 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix} \\ \Rightarrow \mathbf{O} &= \begin{pmatrix} 5 \\ 2 \end{pmatrix} \end{aligned} \quad (1.6)$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \quad (1.7)$$

- 2 If a circle  $C_1$ , whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T \mathbf{x} + (2 \ -4)\mathbf{x} = 4 \quad (2.1)$$

at the point  $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ , then find the length of the intercept cut by this circle  $C$  on the  $x$ -axis.

**Solution:** From (2.1), the centre of  $C_2$  is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.2)$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T \mathbf{O}_2 = 4 \Rightarrow r_2 = 3 \quad (2.3)$$

Since the radius of  $C_1$  is  $r_1 = r_2 = 3$  and  $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$  are collinear,

$$\begin{aligned} \frac{\mathbf{O}_1 + \mathbf{O}_2}{2} &= \mathbf{P} \\ \Rightarrow \mathbf{O}_1 &= 2\mathbf{P} - \mathbf{O}_2 \\ \Rightarrow \mathbf{O}_1 &= \begin{pmatrix} 5 \\ 2 \end{pmatrix} \end{aligned} \quad (2.4)$$

The intercepts of  $C_1$  on the  $x$ -axis can be expressed as

$$\mathbf{x} = \lambda \mathbf{m} \quad (2.5)$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.6)$$

Substituting in the equation for  $C_1$ ,

$$\|\lambda \mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \quad (2.7)$$

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which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda \mathbf{m}^T \mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$

$$\implies \lambda^2 - 10\lambda + 20 = 0 \quad (2.8)$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \quad (2.9)$$

after substituting from (2.6) and (2.4).

3 A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad (3.1)$$

cuts the circle

$$C : \mathbf{x}^T \mathbf{x} = 9 \quad (3.2)$$

at the points **A** and **B**. Find  $PA.PB$ .

**Solution:** Since the points **P**, **A**, **B** are collinear, the line  $PAB$  can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \quad (3.3)$$

for  $\|\mathbf{m}\| = 1$ . The intersection of  $L$  and  $C$  yields

$$(\mathbf{P} + \lambda \mathbf{m})^T (\mathbf{P} + \lambda \mathbf{m}) = 9$$

$$\implies \lambda^2 + 2\lambda \mathbf{m}^T \mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \quad (3.4)$$

The product of the roots in (3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \quad (3.5)$$

4 Find the equation of the circle  $C_2$ , which is the mirror image of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - (2 \ 0) \mathbf{x} = 0 \quad (4.1)$$

in the line

$$L : (1 \ 1) \mathbf{x} = 3. \quad (4.2)$$

**Solution:** From (4.1), circle  $C_1$  has centre at

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.3)$$

and radius

$$r_1 = \mathbf{O}_1^T \mathbf{O}_1 = 1 \quad (4.4)$$

The centre of  $C_2$  is the reflection of  $\mathbf{O}_1$  about  $L$  and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{O}_1 + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (4.5)$$

where the relevant parameters are obtained

from (4.2) as

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 & -1 \end{pmatrix}, c = 3. \quad (4.6)$$

Substituting the above in (4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4} \mathbf{O}_1 + c \frac{\mathbf{n}}{2}$$

$$\implies \mathbf{O}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.7)$$

Thus

$$C_2 : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| = 1 \quad (4.8)$$

5 One of the diameters of the circle, given by

$$C : \mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (5.1)$$

is a chord of a circle  $S$ , whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \quad (5.2)$$

Find the radius of  $S$ .

**Solution:** From (5.1), the centre of  $C$  is

$$\mathbf{O}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad (5.3)$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T \mathbf{O}_1 - 12} = 1 \quad (5.4)$$

From (5.3) and (5.2),

$$O_1 O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$

$$\implies r_2 = \sqrt{O_1 O_2^2 - r_1^2} = 7 \quad (5.5)$$

6 A circle  $C$  passes through

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \quad (6.1)$$

and touches the  $y$ -axis at

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (6.2)$$

Which one of the following equations can represent a diameter of this circle?

- a)  $(4 \ 5) \mathbf{x} = 6$
- b)  $(2 \ -3) \mathbf{x} + 10 = 0$
- c)  $(3 \ 4) \mathbf{x} = 3$
- d)  $(5 \ 2) \mathbf{x} + 4 = 0$

**Solution:** Let  $\mathbf{O}$  be the centre of  $C$ . Then the equation of the normal,  $OQ$  is

$$\begin{aligned} (0 \ 1)(\mathbf{O} - \mathbf{Q}) &= 0 \\ \Rightarrow (0 \ 1)\mathbf{O} &= 2 \end{aligned} \quad (6.3)$$

Also,

$$\begin{aligned} \|\mathbf{O} - \mathbf{P}\|^2 &= \|\mathbf{O} - \mathbf{Q}\|^2 \\ \Rightarrow 2(\mathbf{P} - \mathbf{Q})^T \mathbf{O} &= \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2 \\ \text{or, } (1 \ -1)\mathbf{O} &= -4 \end{aligned} \quad (6.4)$$

(6.3) and (6.4) result in the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{O} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (6.5)$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{O} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (6.6)$$

Hence, option b) is correct.

- 7 Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (7.1)$$

whose centre  $\mathbf{O}$  is the point of intersection of the straight lines

$$(2 \ 1)\mathbf{x} = 3 \quad (7.2)$$

$$(1 \ -1)\mathbf{x} = 1 \quad (7.3)$$

**Solution:** From (7.2) and (7.3), we obtain the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{O} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (7.4)$$

yielding the augmented matrix

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 1 \end{pmatrix} \Rightarrow \mathbf{O} = \frac{1}{3} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \end{aligned} \quad (7.5)$$

Thus, the equation of the desired tangent is

$$\begin{aligned} (\mathbf{O} - \mathbf{P})^T (\mathbf{x} - \mathbf{P}) &= 0 \\ \Rightarrow (1 \ 4)\mathbf{x} &= -3 \end{aligned} \quad (7.6)$$