Python with Linear Algebra:
2D

G V V Sharma*

CONTENTS

Abstract—**This manual introduces matrix computations using python and the properties of a triangle.**

1 LINE

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad (1)$$

Find the direction vector and the normal vector for *AB*

Solution: The direction vector of *AB* is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \quad (2)$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 1 - (-2) \\ 3 - (-2) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (4)$$

1.2 Find the *normal* vector of *AB*.
Solution: The normal vector **n** is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \quad (5)$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.
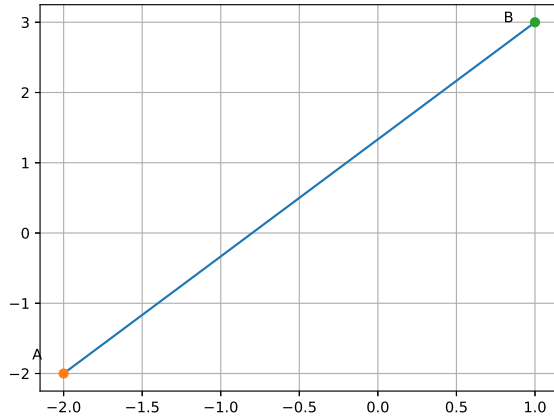
and can be obtained as

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (6)$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (7)$$

$$= \begin{pmatrix} 0 \times 3 + 1 \times 5 \\ -1 \times 3 + 0 \times 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \quad (8)$$

1.3 Verify (5).
Solution:

$$\mathbf{n}^T \mathbf{m} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (9)$$

$$= 5 \times 3 + (-3) \times 5 = 0 \quad (10)$$

1.4 Find the equation of *AB*.
Solution: The desired equation is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (11)$$

$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (12)$$

1.5 Draw the line *AB*
Solution: The following code plots *AB* in Fig. 1.5

```
#Plotting AB
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if


A = np.array([−2,−2])
B = np.array([1,3])
len =10

x_AB = np.zeros((2,len))
```
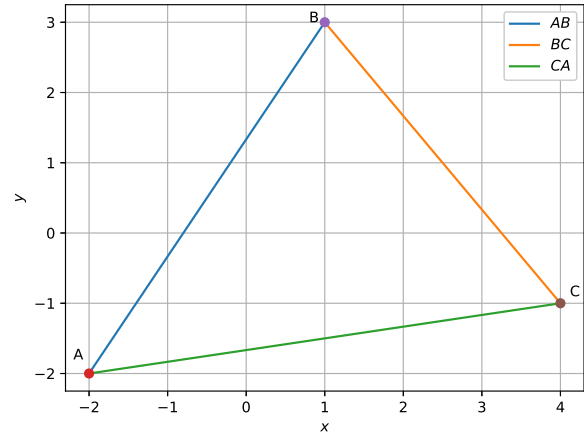
Fig. 1.5

Fig. 1.6

```
lam = np.linspace(0,1,len)
for i in range(len):
    temp1 = A + lam[i]*(B−A)
    x_AB[:,i]= temp1.T

plt.plot(x_AB[0,:],x_AB[1,:],label='$AB$')
plt.grid() # minor

plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 + 0.1), A[1] * (1 − 0.1) , '
    A')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 − 0.2), B[1] * (1) , 'B')

#if using termux
plt.savefig('../figs/draw_line.pdf')
plt.savefig('../figs/draw_line.eps')
subprocess.run(shlex.split("termux−open ../
    figs/draw_line.pdf"))
#else
#plt.show()
```

1.6 Draw $\triangle ABC$.
   Solution: The following codes yields the desired plot in Fig. 1.6

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    coeffs.py
```

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    draw_triangle.py
```

1.7 Find the equation of the line in terms of the normal vector.
   Solution: The desired equation is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \qquad (13)$$

$$\implies \begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = - \begin{pmatrix} 5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -4 \quad (14)$$

1.8 Verify that the above equation can be obtained by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \qquad (15)$$

1.9 Find the equations of BC and CA.

2 ALTITUDES OF A TRIANGLE

2.1 In $\triangle ABC$, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of $\triangle ABC$.

2.2 Find the equation of AP.
   Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From (12), the equation of AP is

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \qquad (16)$$

$$\implies \begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = - \begin{pmatrix} -3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -2 \quad (17)$$

2.3 Find the equation of the altitude BQ.
   Solution: The desired equation is

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{x} - \mathbf{B}) = 0 \qquad (18)$$

$$\implies \begin{pmatrix} 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9 \qquad (19)$$

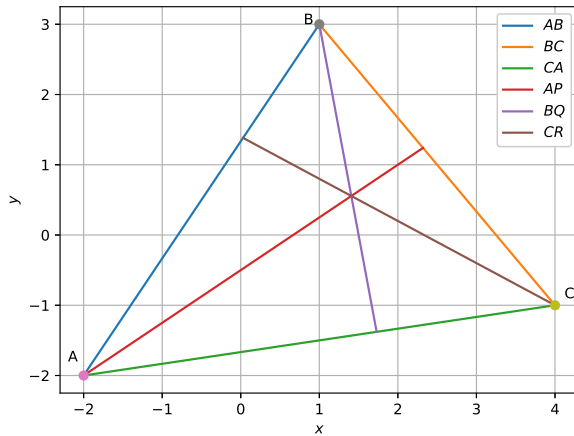2.4 Find the equation of the altitude CR.

Fig. 2.9

2.5 Find the point of intersection of *AP* and *BQ*.
Solution: (15) and (17) can be stacked together into the matrix equation

$$\begin{pmatrix} -3 & 4 \\ 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \quad (20)$$

The following code computes the point of intersection.

https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    orthocentre.py

2.6 Find the point of intersection of and *BQ* and *CR*. Comment.
2.7 Find **P**
Solution: The following code finds the required points.

https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_foot.py

2.8 Find **Q** and **R**.
2.9 Draw *AP*, *BQ* and *CR* and verify that they meet at a point **H**.
Solution: The following code plots the altitudes in Fig. 2.9

https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_draw.py

3 CIRCUMCIRCLE

3.1 Let **A**, **B** and **C** be points on a circle with centre **O** and radius *r*.
3.2 Find **O**.
Solution: The equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (21)$$
$$\implies \|\mathbf{x} - \mathbf{O}\|^2 = (\mathbf{x} - \mathbf{O})^T (\mathbf{x} - \mathbf{O}) = R^2 \quad (22)$$

From (20),

$$\|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \quad (23)$$

$$\implies (\mathbf{A} - \mathbf{O})^T (\mathbf{A} - \mathbf{O})$$
$$- (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \quad (24)$$

which can be simplified as

$$(\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (25)$$

Similarly,

$$(\mathbf{B} - \mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (26)$$

The following code computes **O** using the above two equations.

https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    circumcentre.py

3.3 Find the radius *R*.
3.4 Plot the *circumcircle* of △*ABC*.
Solution: The following code plots Fig. 3.4

https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    circumcircle.py

4 MEDIANS OF A TRIANGLE

4.1 Find the coordinates of **D**, **E** and **F** of the mid points of *AB*, *BC* and *CA* respectively for △*ABC*.
4.2 Find the equations of *AD*, *BE* and *CF*. These lines are the *medians* of △*ABC*
4.3 Find the point of intersection of *AD* and *CF*.
4.4 Verify that **G** is the point of intersection of *BE*, *CF* as well as *AD*, *BE*. **G** is known as the *centroid* of △*ABC*.
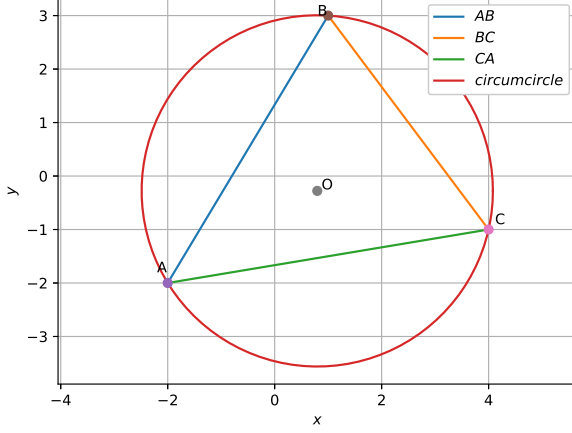4.5 Graphically show that the medians of △*ABC* meet at the centroid.

Fig. 3.4

4.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \qquad (27)$$

5 Incircle

5.1 Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.

5.2 Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1 x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \qquad (28)$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I} \right) = 0 \qquad (29)$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I} \right) = 0 \qquad (30)$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, $x_1 x_2$ merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{U} - \mathbf{I}) = 0 \implies OD \perp BC \qquad (31)$$

5.3 Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \qquad (32)$$

$$\implies \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \qquad (33)$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda \mathbf{n} \qquad (34)$$

$$\implies \mathbf{I} - \mathbf{U} = \lambda \mathbf{n} \qquad (35)$$

$$\text{or} \quad r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \, \|\mathbf{n}\| \qquad (36)$$

From (32) and (33)

$$\mathbf{n}^T \mathbf{I} = \mathbf{n}^T \mathbf{B} + \lambda \mathbf{n}^T \mathbf{n} \qquad (37)$$

$$\implies \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda \|\mathbf{n}\|^2 \qquad (38)$$

$$\implies r = |\lambda| \, \|\mathbf{n}\| = \frac{\left| \mathbf{n}^T (\mathbf{I} - \mathbf{B}) \right|}{\|\mathbf{n}\|} \qquad (39)$$

from (35). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \qquad (40)$$

$$r = \left| \mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) \right| \qquad (41)$$

5.4 Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (40),

$$\left| \mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) \right| = \left| \mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \right| = \left| \mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \right| \quad (42)$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (41) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1 \mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \qquad (43)$$

$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2 \mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \qquad (44)$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\left(\mathbf{n}_1 - k_1 \mathbf{n}_2 \quad \mathbf{n}_2 - k_2 \mathbf{n}_3 \right)^T \mathbf{I} = \begin{pmatrix} \mathbf{n}_1^T \mathbf{B} - k_1 \mathbf{n}_2^T \mathbf{C} \\ \mathbf{n}_2^T \mathbf{C} - k_2 \mathbf{n}_3^T \mathbf{A} \end{pmatrix} \qquad (45)$$

5.5 Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$

5.6 Compute $\mathbf{I}$ and r.
Solution:

https://raw.githubusercontent.com/gadepall/
school/master/linalg/2D/python_2d/codes/
incentre.py

5.7 Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 5.7

https://raw.githubusercontent.com/gadepall/
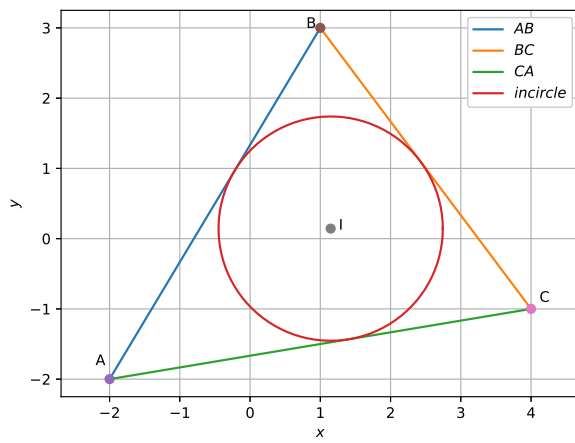school/master/linalg/2D/python_2d/codes/
incircle.py

Fig. 5.7