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Abstract—Solved problems from JEE mains papers related to conic sections in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.

- 1 Two parabolas with a common vertex and with axes along x -axis and y -axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.

Solution: The equation of a conic is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \quad (1.1)$$

For the standard parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2)$$

$$\mathbf{u} = -2a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.3)$$

$$F = 0 \quad (1.4)$$

The focus

$$\mathbf{F} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.5)$$

The Latus rectum is the line passing through \mathbf{F} with direction vector

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.6)$$

Thus, the equation of the Latus rectum is

$$\mathbf{x} = \mathbf{F} + \lambda \mathbf{m} \quad (1.7)$$

The intersection of the latus rectum and the parabola is obtained from (1.4), (1.7) and (1.1)

as

$$(\mathbf{F} + \lambda \mathbf{m})^T \mathbf{V} (\mathbf{F} + \lambda \mathbf{m}) + 2\mathbf{u}^T (\mathbf{F} + \lambda \mathbf{m}) = 0 \quad (1.8)$$

$$\Rightarrow (\mathbf{m}^T \mathbf{V} \mathbf{m}) \lambda^2 + 2(\mathbf{V} \mathbf{F} + \mathbf{u})^T \mathbf{m} \lambda + (\mathbf{V} \mathbf{F} + 2\mathbf{u})^T \mathbf{F} = 0 \quad (1.9)$$

From (1.2), (1.3), (1.5) and (1.6),

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 1 \quad (1.10)$$

$$(\mathbf{V} \mathbf{F} + \mathbf{u})^T \mathbf{m} = 0 \quad (1.11)$$

$$(\mathbf{V} \mathbf{F} + 2\mathbf{u})^T \mathbf{F} = -4a^2 \quad (1.12)$$

Substituting from (1.10), (1.11) and (1.12) in (1.9),

$$\lambda^2 - 4a^2 = 0 \quad (1.13)$$

$$\Rightarrow \lambda_1 = 2a, \lambda_2 = -2a \quad (1.14)$$

Thus, from (1.6), (1.7) and (1.14), the length of the latus rectum is

$$(\lambda_1 - \lambda_2) \|\mathbf{m}\| = 4a \quad (1.15)$$

From the given information, the two parabolas P_1, P_2 have parameters

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a \begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \quad (1.16)$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \quad (1.17)$$

$$4a = 3 \quad (1.18)$$

Let L be the common tangent for P_1, P_2 with \mathbf{c}, \mathbf{d} being the respective points of contact. The

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respective normal vectors are

$$\mathbf{n}_1 = \mathbf{V}_1 \mathbf{c} + \mathbf{u}_1 = -2a \begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} \quad (1.19)$$

$$\mathbf{n}_2 = \mathbf{V}_2 \mathbf{d} + \mathbf{u}_2 = d_1 \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \quad (1.20)$$

From the above equations, since both normals have the same direction vector,

$$\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \implies c_2 d_1 = 4a^2 \quad (1.21)$$

- 2 Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 225 \quad (2.1)$$

upon the tangent to it at the point

$$\frac{1}{2} \begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \quad (2.2)$$

Solution: For the ellipse in (2.1),

$$V = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}, \mathbf{u} = 0, F = -1 \quad (2.3)$$

The equation of the desired tangent is

$$(\mathbf{VP})^T \mathbf{x} = 1 \quad (2.4)$$

$$\implies \left(\frac{1}{3} \quad \frac{\sqrt{3}}{5} \right) \mathbf{x} = 2 \quad (2.5)$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (2.6)$$

The product of the perpendiculars is

$$\frac{\left| \left(\frac{1}{3} \quad \frac{\sqrt{3}}{5} \right) \begin{pmatrix} 0 \\ 4 \end{pmatrix} - 2 \right| \left| \left(\frac{1}{3} \quad \frac{\sqrt{3}}{5} \right) \begin{pmatrix} 0 \\ -4 \end{pmatrix} - 2 \right|}{\left\| \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix} \right\|^2} = 9 \quad (2.7)$$

- 3 Consider an ellipse, whose centre is at the origin and its major axis is along the x -axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.

Solution: If a and b be the semi-major and minor-axis respectively, the foci of the ellipse

are

$$\mathbf{F}_1 = ae \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{F}_2 = -ae \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1)$$

From the given information,

$$e = \frac{3}{5}, 2ae = 6 \quad (3.2)$$

$$\implies a = 5, b = a\sqrt{1 - e^2} = 4 \quad (3.3)$$

Thus, the vertices of the ellipse are

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ -b \end{pmatrix} \quad (3.4)$$

and the area of the quadrilateral is

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & b \end{vmatrix} + \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & -b \end{vmatrix} = 2ab = 40 \quad (3.5)$$

- 4 Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \quad (4.1)$$

If

$$\mathbf{S} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (4.2)$$

is a focus and

$$(5 \ 0) \mathbf{x} = 9 \quad (4.3)$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.

Solution: From (4.1),

$$(3e - 1)(3e - 5) = 0 \quad (4.4)$$

$$\implies e = \frac{5}{3}, \because e > 1 \quad (4.5)$$

for a hyperbola. Let \mathbf{x} be a point on the hyperbola. From (4.3), its distance from the directrix is

$$\frac{|(5 \ 0) \mathbf{x} - 9|}{5} \quad (4.6)$$

and from the focus is

$$\left\| \mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\| \quad (4.7)$$

From the definition of a hyperbola, the eccentricity is the ratio of these distances and (4.5)

, (4.6) and (4.7),

$$\frac{5 \left\| \mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\|}{\left| (5 \ 0) \mathbf{x} - 9 \right|} = \frac{5}{3} \quad (4.8)$$

$$\implies 9 \{(x_1 - 5)^2 + x_2^2\} = (5x_1 - 9)^2 \quad (4.9)$$

$$\text{or, } \mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 225 \quad (4.10)$$

which is the equation of the hyperbola. Thus,

$$a^2 = \frac{225}{16}, b^2 = \frac{225}{9} \quad (4.11)$$

$$\implies a^2 - b^2 = -\frac{175}{16} \quad (4.12)$$

5 A variable line drawn through the intersection of the lines

$$(4 \ 3) \mathbf{x} = 12 \quad (5.1)$$

$$(3 \ 4) \mathbf{x} = 12 \quad (5.2)$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of *AB*.

Solution: The intersection of the lines in (5.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (5.3)$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \quad (5.4)$$

resulting in

$$\mathbf{C} = \frac{1}{7} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (5.5)$$

Let the **R** be the mid point of *AB*. Then,

$$\mathbf{A} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (5.6)$$

$$\mathbf{B} = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (5.7)$$

Let the equation of *AB* be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \quad (5.8)$$

Since **R** lies on *AB*,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \quad (5.9)$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (5.10)$$

Substituting from (5.6) in (5.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \quad (5.11)$$

From (5.9) and (5.11),

$$(\mathbf{R} - \mathbf{C}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \quad (5.12)$$

for some constant *k*. Multiplying both sides of (5.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (5.13)$$

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ = k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (5.14)$$

$$\therefore \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (5.15)$$

which can be easily verified for any **R**. from (5.14),

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0 \\ \implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0 \\ \implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (5.16)$$

which is the desired locus.

1 ELLIPSE

Let the ellipse $E_1, n = 1, 2, \dots$ have the equation

$$\mathbf{x}^T \mathbf{D} \mathbf{x} = 1, \quad (5.17)$$

where

$$\mathbf{D} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \quad (5.18)$$

- 1.1 Let the largest rectangle inside E_1 with sides parallel to the axes be R_1 . Show that the coordinates of the R_1 have the form

$$\begin{pmatrix} \pm p_1 \\ \pm p_2 \end{pmatrix} \quad (1.1.1)$$

Solution: Let R_1 be the rectangle $PQRS$, where $PQ \parallel RS \parallel x$ -axis, $QR \parallel PS \parallel y$ -axis. Their corresponding equations are

$$PQ : \mathbf{x} = \mathbf{P} + \lambda_1 \mathbf{m}_1 \quad (1.1.2)$$

$$PS : \mathbf{x} = \mathbf{P} + \lambda_2 \mathbf{m}_2 \quad (1.1.3)$$

$$QR : \mathbf{x} = \mathbf{Q} + \lambda_3 \mathbf{m}_2 \quad (1.1.4)$$

where

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.5)$$

The intersection of PQ with E_1 is

$$\begin{aligned} & [\mathbf{P} + \lambda_1 \mathbf{m}_1]^T \mathbf{D} [\mathbf{P} + \lambda_1 \mathbf{m}_1] = 1 \\ \Rightarrow & \lambda_1^2 \|\mathbf{m}_1\|^2 + 2\lambda_1 \mathbf{m}_1^T \mathbf{P} + \mathbf{P}^T \mathbf{D} \mathbf{P} - 1 = 0 \end{aligned}$$

$$\begin{aligned} \because & \mathbf{P} \in E_1, \|\mathbf{m}_1\|^2 = 1, \mathbf{P}^T \mathbf{D} \mathbf{P} - 1 = 0 \\ \Rightarrow & \lambda_1 = 0, -2\mathbf{m}_1^T \mathbf{P} \quad (1.1.6) \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{Q} &= \mathbf{P} - 2\mathbf{m}_1^T \mathbf{P} \mathbf{m}_1 \\ &= \begin{pmatrix} -p_1 \\ p_2 \end{pmatrix} \quad (1.1.7) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{S} &= \mathbf{P} - 2\mathbf{m}_2^T \mathbf{P} \mathbf{m}_2 \\ &= \begin{pmatrix} p_1 \\ -p_2 \end{pmatrix} \quad (1.1.8) \end{aligned}$$

and

$$\begin{aligned} \mathbf{R} &= \mathbf{Q} - 2\mathbf{m}_2^T \mathbf{Q} \mathbf{m}_2 \\ &= \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix} \quad (1.1.9) \end{aligned}$$

- 1.2 Find an expression for the square of the area of R_1 .

Solution:

$$\begin{aligned} \because & \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1, \\ p_2 &= b \sqrt{1 - \frac{p_1^2}{a^2}}. \quad (1.2.1) \end{aligned}$$

Hence the desired expression is

$$F = (PQ \times QR)^2 = 16p_1^2 p_2^2 = 16p_1^2 b^2 \left(1 - \frac{p_1^2}{a^2}\right). \quad (1.2.2)$$

- 1.3 Find p_1 that maximises F .

Solution: (1.2.2) can be expressed as

$$\begin{aligned} F &= a^2 b^2 (16a^2 p_1^2 - 16p_1^4) \\ &= a^2 b^2 \{a^4 - (a^2 - 4p_1^2)^2\} \quad (1.3.1) \end{aligned}$$

Thus, F is maximum when

$$\begin{aligned} (a^2 - 4p_1^2)^2 &= 0 \\ \Rightarrow p_1 &= \pm \frac{a}{2} \quad (1.3.2) \end{aligned}$$

- 1.4 Verify the above result graphically.

- 1.5 Find p_2 .

Solution: From (1.2.1)

$$p_2 = \pm \frac{\sqrt{3}}{2} b \quad (1.5.1)$$

- 1.6 Find E_2 , the largest ellipse within R_1 .

Solution: From (1.3.2) and (1.5.1), the semi-major/minor axes of E_2 are

$$E_2 : \left(\frac{a}{2}, \frac{\sqrt{3}}{2} b \right) \quad (1.6.1)$$

- 1.7 find E_n and R_n ,

Solution: From (1.3.2) and (1.5.1), the vertices of R_n and semi-major/minor axes of E_n are

$$\begin{aligned} R_n &: \left\{ \pm \frac{a}{2^n}, \pm \left(\frac{\sqrt{3}}{2} \right)^n b \right\} \\ E_n &: \left\{ \frac{a}{2^{n-1}}, \left(\frac{\sqrt{3}}{2} \right)^{n-1} b \right\} \quad (1.7.1) \end{aligned}$$

In the following questions, $a = 3, b = 2$. Use a computer program.

- 1.8 Is the eccentricity $e_1 8 = e_1 9$?

1.9 Verify if

$$\sum_{n=1}^N (\text{Area of } R_n) < 24, \quad (1.9.1)$$

for each positive integer N .

1.10 Is the length of the latus rectum of $E_9 = \frac{1}{6}$?

1.11 Is the distance of a focus from the centre in $E_9 = \frac{\sqrt{5}}{32}$?