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Abstract—This textbook introduces linear algebra by exploring Euclidean geometry.

1 THE STRAIGHT LINE

1.1 The points $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA .

Solution: Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA . Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \quad (1.1)$$

$$\Rightarrow x_2 = mx_1 \quad (1.2)$$

where m is known as the slope of the line. Thus, the equation of the line is

$$x = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.3)$$

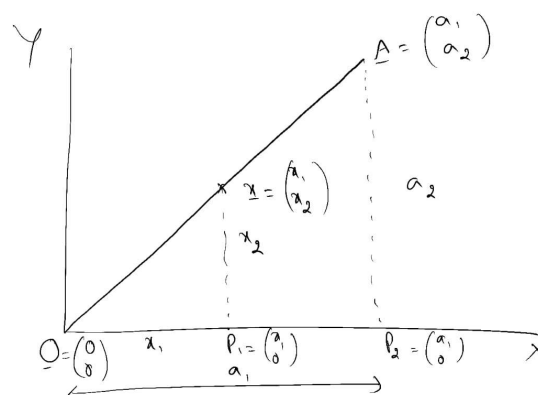


Fig. 1.1

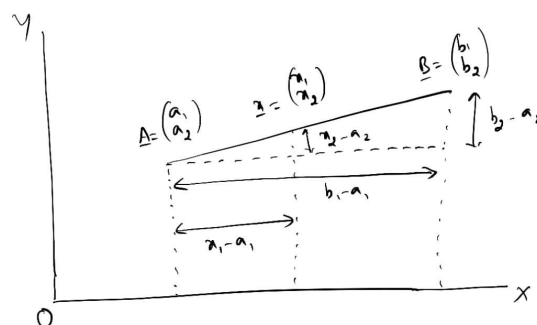


Fig. 1.2

In general, the above equation is written as

$$x = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.4)$$

1.2 Find the equation of AB in Fig. 1.2

Solution: From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \quad (1.5)$$

$$\Rightarrow x_2 = mx_1 + a_2 - ma_1 \quad (1.6)$$

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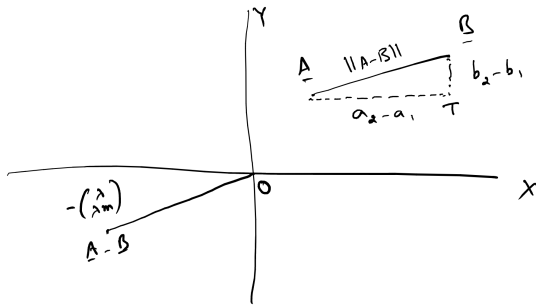


Fig. 1.4

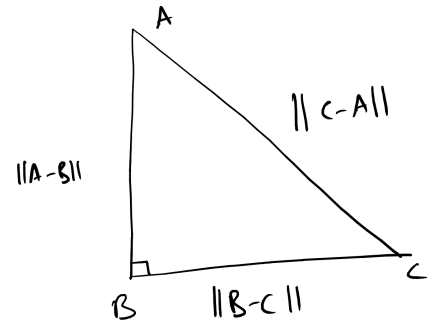


Fig. 2.1

2 ORTHOGONALITY

2.1 See Fig. 2.1. In $\triangle ABC$, $AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (2.1)$$

Solution: Using Baudhayana's theorem,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \quad (2.2)$$

$$\begin{aligned} \Rightarrow (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) &= (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) \\ \Rightarrow 2\mathbf{A}^T \mathbf{B} - 2\mathbf{B}^T \mathbf{B} + 2\mathbf{B}^T \mathbf{C} - 2\mathbf{A}^T \mathbf{C} &= 0 \end{aligned} \quad (2.3)$$

which can be simplified to obtain (2.1).

2.2 Let \mathbf{x} be any point on AB in Fig. 2.1. Show that

$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (2.4)$$

2.3 If \mathbf{x}, \mathbf{y} are any two points on AB , show that

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (2.5)$$

2.4 In Fig. 2.4, $BE \perp AC$, $CF \perp AB$. Show that $AD \perp BC$.

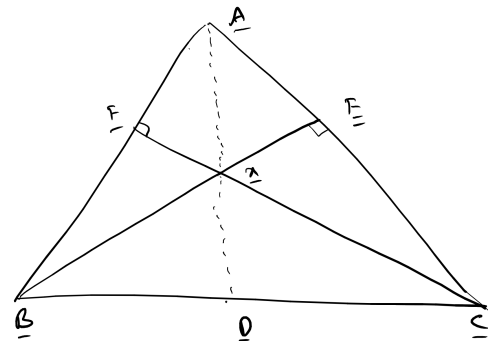


Fig. 2.4

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \quad (1.7)$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.8)$$

$$= \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.9)$$

1.3 Find the length of \mathbf{A} in Fig. 1.1

Solution: Using Baudhayana's theorem, the length of the vector \mathbf{A} is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \quad (1.10)$$

Also, from (1.4),

$$\|\mathbf{A}\| = \lambda \sqrt{1 + m^2} \quad (1.11)$$

Note that λ is the variable that determines the length of \mathbf{A} , since m is constant for all points on the line.

1.4 Find $\mathbf{A} - \mathbf{B}$.

Solution: See Fig. 1.4. From (1.9), for some λ ,

$$\mathbf{B} = \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.12)$$

$$\Rightarrow \mathbf{A} - \mathbf{B} = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (1.13)$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.4.

1.5 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

1.6 Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (1.14)$$

Solution: Let \mathbf{x} be the intersection of BE and

CF. Then, using (2.5),

$$\begin{aligned} (\mathbf{x} - \mathbf{B})^T (\mathbf{A} - \mathbf{C}) &= 0 \\ (\mathbf{x} - \mathbf{C})^T (\mathbf{A} - \mathbf{B}) &= 0 \end{aligned} \quad (2.6)$$

$$\implies \mathbf{x}^T (\mathbf{A} - \mathbf{C}) - \mathbf{B}^T (\mathbf{A} - \mathbf{C}) = 0 \quad (2.7)$$

$$\text{and } \mathbf{x}^T (\mathbf{A} - \mathbf{B}) - \mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.8)$$

Subtracting (2.8) from (2.7),

$$\mathbf{x}^T (\mathbf{B} - \mathbf{C}) + \mathbf{A}^T (\mathbf{C} - \mathbf{B}) = 0 \quad (2.9)$$

$$\implies (\mathbf{x}^T - \mathbf{A}^T) (\mathbf{B} - \mathbf{C}) = 0 \quad (2.10)$$

$$\implies (\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (2.11)$$

which completes the proof.

3 MEDIANS OF A TRIANGLE

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \quad (3.1)$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k + 1} = \mathbf{B}. \quad (3.2)$$

Solution: From (1.9),

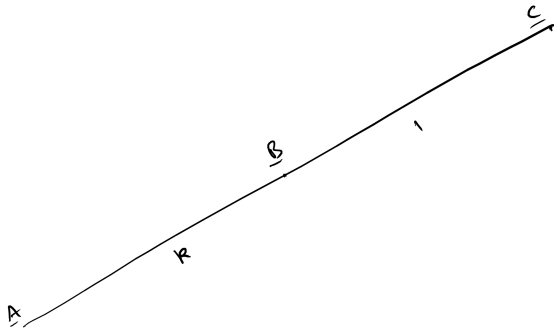


Fig. 3.1

$$\mathbf{B} = \mathbf{A} + \lambda_1 \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (3.3)$$

$$\mathbf{B} = \mathbf{C} - \lambda_2 \begin{pmatrix} 1 \\ m \end{pmatrix}.$$

$$\implies \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \quad (3.4)$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (3.5)$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \quad (3.6)$$

resulting in (3.2).

3.2 If \mathbf{A} and \mathbf{B} are linearly independent,

$$k_1 \mathbf{A} + k_2 \mathbf{B} = \mathbf{0} \implies k_1 = k_2 = 0 \quad (3.7)$$

3.3 BE and CF are medians of $\triangle ABC$ intersecting at O as shown in Fig. 3.3. Show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \quad (3.8)$$

Solution: Let

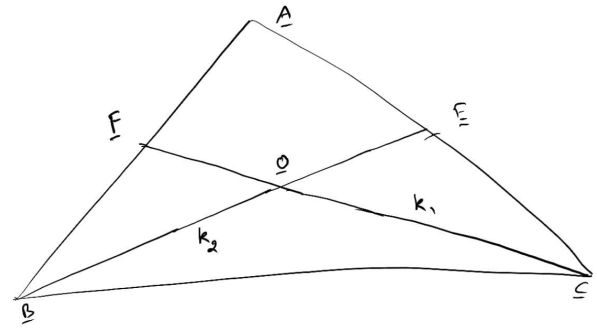


Fig. 3.3

$$\frac{CO}{OF} = k_1 \quad (3.9)$$

$$\frac{BO}{OE} = k_2 \quad (3.10)$$

Using (3.2),

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (3.11)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (3.12)$$

and

$$\mathbf{O} = \frac{k_1 \mathbf{F} + \mathbf{C}}{k_1 + 1} = \frac{k_1 \frac{\mathbf{A} + \mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} \quad (3.13)$$

$$\mathbf{O} = \frac{k_2 \mathbf{E} + \mathbf{B}}{k_2 + 1} = \frac{k_2 \frac{\mathbf{A} + \mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \quad (3.14)$$

From (3.13) and (3.14),

$$\frac{k_1 \frac{\mathbf{A} + \mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} = \frac{k_2 \frac{\mathbf{A} + \mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \quad (3.15)$$

$$\begin{aligned} \Rightarrow & \left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2} \right] \mathbf{A} \\ & + \left[\frac{k_1(k_2+1)}{2} - (k_1+1) \right] \mathbf{B} \\ & + \left[(k_2+1) - \frac{k_2(k_1+1)}{2} \right] \mathbf{C} = 0 \quad (3.16) \end{aligned}$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \Rightarrow k_1 = k_2 = 2, \quad (3.17)$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (3.14),

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.18)$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha \mathbf{B} + \beta \mathbf{C} \quad (3.19)$$

Note that \mathbf{B}, \mathbf{C} are linearly independent. Substituting (3.19) in (3.16),

$$\begin{aligned} & \left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2} \right] [\alpha \mathbf{B} + \beta \mathbf{C}] \\ & + \left[\frac{k_1(k_2+1)}{2} - (k_1+1) \right] \mathbf{B} \\ & + \left[(k_2+1) - \frac{k_2(k_1+1)}{2} \right] \mathbf{C} = 0 \quad (3.20) \end{aligned}$$

$$\begin{aligned} \Rightarrow & (k_1 - k_2)\alpha + k_1k_2 - k_1 - 2 = 0 \\ & (k_1 - k_2)\beta - k_1k_2 + k_2 + 2 = 0 \quad (3.21) \end{aligned}$$

$$\Rightarrow (k_1 - k_2)(\alpha + \beta - 1) = 0 \quad (3.22)$$

If $\alpha + \beta = 1$, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (3.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in (3.21), yields

$$k_1^2 - k_1 - 2 = 0 \Rightarrow k_1 = 2. \quad (3.23)$$

4 MATRIX TRANSFORMATIONS

4.1 Find \mathbf{R} , the reflection of \mathbf{P} about the line

$$L: \mathbf{n}^T \mathbf{x} = c \quad (4.1)$$

Solution: Since \mathbf{R} is the reflection of \mathbf{P} and \mathbf{Q} lies on L , \mathbf{Q} bisects PR . This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (4.2)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad (4.3)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad (4.4)$$

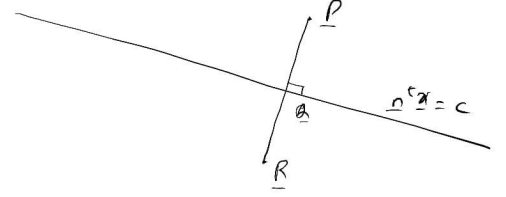


Fig. 4.1

where \mathbf{m} is the direction vector of L . From (4.2) and (4.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (4.5)$$

From (4.5) and (4.4),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \quad (4.7)$$

with the condition that \mathbf{m}, \mathbf{n} are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (4.8)$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.9)$$

(4.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.10)$$

$$\Rightarrow \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.11)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c \mathbf{n} \quad (4.12)$$

4.2 Rotate \mathbf{P} through an angle of θ about the origin in the counter clockwise direction to obtain \mathbf{S} .

Solution:

$$\mathbf{S} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{P} \quad (4.13)$$

5 CIRCUMCIRCLE

5.1 A circle with centre at \mathbf{O} and radius r has the equation

$$\|\mathbf{x} - \mathbf{O}\| = r \quad (5.1)$$

$$\implies (\mathbf{x} - \mathbf{O})^T (\mathbf{x} - \mathbf{O}) = r^2 \quad (5.2)$$

5.2 Let \mathbf{A}, \mathbf{B} and \mathbf{C} be three points on the circle and D be a point on BC such that $OD \perp BC$. Show that

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (5.3)$$

Solution: From (5.2),

$$\|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \quad (5.4)$$

$$\implies (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = (\mathbf{C} - \mathbf{O})^T (\mathbf{C} - \mathbf{O}) \quad (5.5)$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{O} \right) = 0 \quad (5.6)$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{D} - \mathbf{O}) = 0 \quad (5.7)$$

Since D and $\frac{\mathbf{B} + \mathbf{C}}{2}$ lie on BC , using (1.14),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \mathbf{B} + \lambda_1 (\mathbf{B} - \mathbf{C}) \quad (5.8)$$

$$\mathbf{D} = \mathbf{B} + \lambda_2 (\mathbf{B} - \mathbf{C}) \quad (5.9)$$

Multiplying (5.8) and (5.9) with $(\mathbf{B} - \mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (5.10)$$

5.3 Let \mathbf{D} be the mid point of BC . Show that $OD \perp BC$.

6 INCIRCLE

6.1 The circle with centre \mathbf{O} and radius r in Fig. is inside $\triangle ABC$ and touches AB, BC and CA at \mathbf{F}, \mathbf{D} and \mathbf{E} respectively. AB, BC and CA are known as *tangents* to the circle.

6.2 Show that $OD \perp BC$.

Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle

such that $x_1 x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{O}\|^2 - \|\mathbf{x}_2 - \mathbf{O}\|^2 = 0 \quad (6.1)$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{O} \right) = 0 \quad (6.2)$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{O} \right) = 0 \quad (6.3)$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{D}$, $x_1 x_2$ merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{D} - \mathbf{O}) = 0 \implies OD \perp BC \quad (6.4)$$

6.3 Find the equation of the tangent at \mathbf{D} .

Solution: The equation of the tangent is given by

$$(\mathbf{O} - \mathbf{D})^T (\mathbf{x} - \mathbf{D}) = 0 \quad (6.5)$$

7 MISCELLANEOUS

7.1 Show that the angle in a semi-circle is a right angle.

8 PARABOLA

8.1 Express

$$y_2 = y_1^2 \quad (8.1)$$

as a matrix equation.

Solution: (8.1) can be expressed as

$$\mathbf{y}^T \mathbf{W} \mathbf{y} + 2\mathbf{g}^T \mathbf{y} = 0 \quad (8.2)$$

where

$$\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.3)$$

8.2 Given

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0, \quad (8.4)$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \quad (8.5)$$

and \mathbf{P}, \mathbf{c} such that

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c}. \quad (8.6)$$

Show that

$$\mathbf{W} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (8.7)$$

$$\mathbf{g} = \mathbf{P}^T (\mathbf{V} \mathbf{c} + \mathbf{u})$$

$$F + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} = 0$$

Solution: Substituting (8.6) in (8.4),

$$(\mathbf{Py} + \mathbf{c})^T \mathbf{V} (\mathbf{Py} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{Py} + \mathbf{c}) + F = 0, \quad (8.8)$$

which can be expressed as

$$\begin{aligned} \Rightarrow \mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{Vc} + \mathbf{u})^T \mathbf{P} \mathbf{y} \\ + F + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} = 0 \end{aligned} \quad (8.9)$$

Comparing (8.9) with (8.2) (8.7) is obtained.

8.3 Show that there exists a \mathbf{P} such that

$$\mathbf{P}^T \mathbf{P} = \mathbf{I} \quad (8.10)$$

Find \mathbf{P} using

$$\mathbf{W} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (8.11)$$

8.4 Find \mathbf{c} from (8.7).

Solution:

$$\because \mathbf{g} = \mathbf{P}^T (\mathbf{Vc} + \mathbf{u}), \quad (8.12)$$

$$\mathbf{Vc} = \mathbf{Pg} - \mathbf{u} \quad (8.13)$$

$$\Rightarrow \mathbf{c}^T \mathbf{V} \mathbf{c} = \mathbf{c}^T (\mathbf{Pg} - \mathbf{u}) = -F - 2\mathbf{u}^T \mathbf{c} \quad (8.14)$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ (\mathbf{Pg} + \mathbf{u})^T \end{pmatrix} \mathbf{c} = \begin{pmatrix} \mathbf{Pg} - \mathbf{u} \\ -F \end{pmatrix} \quad (8.15)$$

for computing \mathbf{c} .