

# Linear Algebra through Coordinate Geometry

G V V Sharma\*

## CONTENTS

1	The Straight Line	1
2	Orthogonality	2
3	Matrix Transformations	3
4	Locus	3
5	Conics	4
6	Circle	5
7	Parabola	6
8	Affine Transformation	6
9	Ellipse	7
10	Hyperbola	8

**Abstract**—This manual introduces linear algebra through coordinate geometry using a problem solving approach.

## 1 THE STRAIGHT LINE

1.1 The equation of the line between two points **A** and **B** is given by

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{A} - \mathbf{B}) \quad (1.1)$$

Alternatively, it can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2)$$

where **n** is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{n} = 0 \quad (1.3)$$

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

1.2 In  $\triangle ABC$ ,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.4)$$

and the equations of the medians through **B** and **C** are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \quad (1.5)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \quad (1.6)$$

Find the area of  $\triangle ABC$ .

**Solution:** The centroid **O** is the solution of (1.5), (1.6) and is obtained as the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad (1.7)$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.8)$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.9)$$

Let **AD** be the median through **A**. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \quad (1.10)$$

$$\Rightarrow \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.11)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.12)$$

From (1.6) and (1.12),

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} = 5 \quad (1.13)$$

$$\Rightarrow 5 + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 12 \quad (1.14)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 7 \quad (1.15)$$

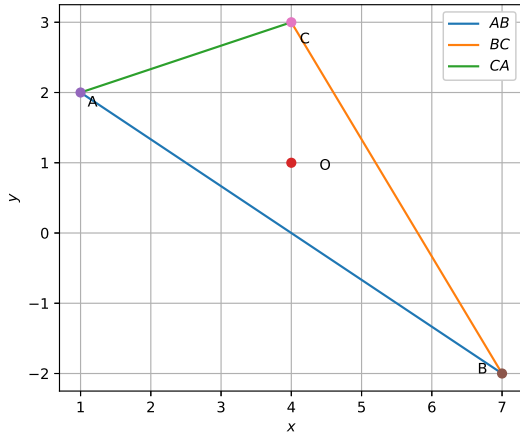


Fig. 1.3

From (1.15) and (1.6),  $\mathbf{C}$  can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad (1.16)$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.17)$$

$$\Rightarrow \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (1.18)$$

From (1.11),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.19)$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \quad (1.20)$$

1.3 Summarize all the above computations through a Python script and plot  $\triangle ABC$ .

**Solution:**

<https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/triang.py>

## 2 ORTHOGONALITY

2.1  $\mathbf{u}^T \mathbf{x} = 0 \Rightarrow \mathbf{u} \perp \mathbf{x}$ . Show that

$$\mathbf{u}^T \mathbf{x} = \mathbf{P}^T \mathbf{x} = 0 \Rightarrow \mathbf{P} = \alpha \mathbf{u} \quad (2.1)$$

2.2 The foot of the perpendicular drawn from the origin on the line

$$AB : \mathbf{u}^T \mathbf{x} = \lambda \quad (2.2)$$

where

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.3)$$

is  $\mathbf{P}$ . The line meets the  $x$ -axis at  $\mathbf{A}$  and  $y$ -axis at  $\mathbf{B}$ . Show that  $\mathbf{P} = \alpha \mathbf{u}$  and find  $\alpha$ .

**Solution:** From (2.2),

$$\mathbf{u}^T \mathbf{A} = \mathbf{u}^T \mathbf{B} = \lambda \quad (2.4)$$

$$\Rightarrow \mathbf{u}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.5)$$

Since  $OP \perp AB$ ,

$$\mathbf{P}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.6)$$

Thus, from (2.1),

$$\mathbf{P} = \alpha \mathbf{u} \quad (2.7)$$

Since  $\mathbf{P}$  lies on (2.2),

$$\mathbf{u}^T \mathbf{P} = \alpha \mathbf{u}^T \mathbf{u} = \lambda \quad (2.8)$$

$$\Rightarrow \alpha = \frac{\lambda}{\mathbf{u}^T \mathbf{u}} = \frac{\lambda}{10}. \quad (2.9)$$

2.3 Find  $\mathbf{A}$ .

**Solution:** Let

$$\mathbf{A} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.10)$$

From (2.2),

$$\mathbf{u}^T \mathbf{A} = a \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \quad (2.11)$$

$$\Rightarrow a = \frac{\lambda}{3} \quad (2.12)$$

$$\text{and } \mathbf{A} = \frac{\lambda}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.13)$$

2.4 Find the ratio  $BP : PA$ .

**Solution:** Let

$$\frac{BP}{PA} = k \quad (2.14)$$

Then,

$$k\mathbf{A} + \mathbf{B} = (k+1)\mathbf{P} \quad (2.15)$$

$$\Rightarrow k\mathbf{A}^T \mathbf{A} + \mathbf{A}^T \mathbf{B} = (k+1)\mathbf{P}^T \mathbf{A} \quad (2.16)$$

$$\Rightarrow k\alpha^2 = \alpha(k+1)\lambda \quad (2.17)$$

using (2.7), (2.10), (2.2) and  $\mathbf{A} \perp \mathbf{B}$ . Substituting from (2.9) and (2.12),

$$\Rightarrow k \frac{\lambda^2}{9} = (k+1) \frac{\lambda^2}{10} \quad (2.18)$$

$$\Rightarrow k = 9 \quad (2.19)$$

### 3 MATRIX TRANSFORMATIONS

3.1 Find  $\mathbf{R}$ , the reflection of  $\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$  about the line

$$L: \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (3.1)$$

**Solution:** The reflection of  $\mathbf{P}$  about  $L$  is given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (3.2)$$

where

$$L: \mathbf{n}^T \mathbf{x} = c \quad (3.3)$$

$$\mathbf{m}^T \mathbf{n} = 0 \quad (3.4)$$

$$\|\mathbf{m}\| = \|\mathbf{n}\| = 1 \quad (3.5)$$

Substituting

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c = 0 \quad (3.6)$$

in (3.2),

$$\frac{\mathbf{R}}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow \mathbf{R} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (3.7)$$

3.2  $\mathbf{R}$  is translated through a distance 2 units along the positive direction of x-axis to obtain  $\mathbf{S}$ . Find  $\mathbf{S}$ .

**Solution:**

$$\mathbf{S} = \mathbf{R} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.8)$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.9)$$

3.3 Rotate  $\mathbf{S}$  through an angle of  $\frac{\pi}{4}$  about the origin in the counter clockwise direction to obtain  $\mathbf{T}$ .

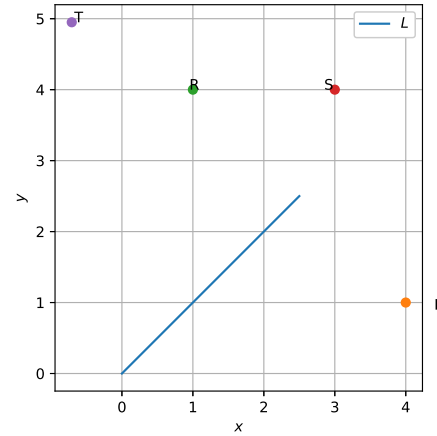


Fig. 3.4

**Solution:**

$$\mathbf{T} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{S} \quad (3.10)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.11)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 7 \end{pmatrix} \quad (3.12)$$

3.4 Summarize all the above computations through a Python script and plot  $L, P, R, S, T$ .

**Solution:** The following code generates Fig. 3.4.

```
wget https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/reflect.py
```

### 4 LOCUS

4.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1)$$

intersects the coordinate axes at  $\mathbf{P}$  and  $\mathbf{Q}$ .  $\mathbf{O}$  is the origin and rectangle  $OPRQ$  is completed as shown in Fig. (4.1),

4.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (4.2)$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (4.3)$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \quad (4.4)$$

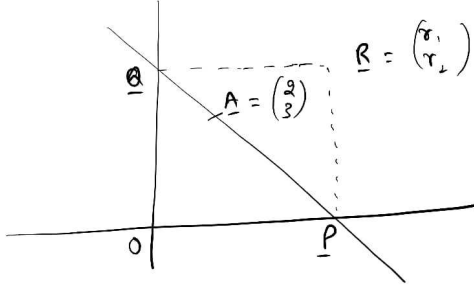


Fig. 4.1

4.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{n} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{n} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{n} &= 0 \end{aligned} \quad (4.5)$$

**Solution:** Trivial using (1.2) and (1.3).

4.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{n} = 0 \quad (4.6)$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{n} = 0 \quad (4.7)$$

**Solution:** From (4.5) and (4.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{n} = 0 \quad (4.8)$$

resulting in (4.6). From (4.5) and (4.2),(4.3), (4.7) is obtained.

4.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \quad (4.9)$$

4.6 Find the locus of  $\mathbf{R}$ .

**Solution:** For  $\mathbf{n}$  to be unique in (4.6),(4.7),

$$\begin{aligned} (2\mathbf{A} - \mathbf{R}) &= k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \quad (4.10)$$

where  $k$  is some constant. Thus, the desired

locus is

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) = 0 \quad (4.11)$$

$$\Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - 2\mathbf{A}^T \mathbf{R} = 0 \quad (4.12)$$

## 5 CONICS

5.1 The equation of a quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \quad (5.1)$$

Show that (5.1) can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \quad (5.2)$$

Find the matrix  $\mathbf{V}$  and vector  $\mathbf{u}$ .

5.2 Show that

$$\frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} = \mathbf{u} \quad (5.3)$$

5.3 Show that

$$\frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} = 2\mathbf{V}^T \mathbf{x} \quad (5.4)$$

5.4 Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \quad (5.5)$$

5.5 Find the normal vector to the curve in (5.2) at point  $\mathbf{p}$ .

**Solution:** Differentiating (5.2) with respect to  $x_1$ ,

$$\begin{aligned} \frac{d(\mathbf{x}^T \mathbf{V} \mathbf{x})}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} + \frac{d(\mathbf{u}^T \mathbf{x})}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} &= 0 \\ \Rightarrow 2\mathbf{x}^T \mathbf{V} \mathbf{m} + 2\mathbf{u}^T \mathbf{m} &= 0 \because \left( \frac{d\mathbf{x}}{dx_1} = \mathbf{m} \right) \end{aligned} \quad (5.6)$$

Substituting  $\mathbf{x} = \mathbf{p}$  and simplifying

$$(\mathbf{V} \mathbf{p} + \mathbf{u})^T \mathbf{m} = 0 \quad (5.8)$$

$$\Rightarrow \mathbf{n} = \mathbf{V} \mathbf{p} + \mathbf{u} \quad (5.9)$$

5.6 Show that the tangent to (5.1) at a point  $\mathbf{p}$  on the curve is given by

$$(\mathbf{p}^T \mathbf{V} + \mathbf{u}^T) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \quad (5.10)$$

5.7 Classify the various conic sections based on (5.2).

**Solution:**

Curve	Property
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 5.7

## 6 CIRCLE

6.1 Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - (2 \ 0) \mathbf{x} - 1 = 0 \quad (6.1)$$

**Solution:** let  $\mathbf{c}$  be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \quad (6.2)$$

$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \quad (6.3)$$

$$\implies \mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} = r^2 - \mathbf{c}^T \mathbf{c} \quad (6.4)$$

Comparing with (6.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.5)$$

$$r^2 - \mathbf{c}^T \mathbf{c} = 1 \implies r = \sqrt{2} \quad (6.6)$$

6.2 Find the tangent to the circle  $C_1$  at the point  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

**Solution:** From (5.10), the tangent  $T$  is given by

$$\left[ (2 \ 1) - (1 \ 0) \right] \mathbf{x} - (2 \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad (6.7)$$

$$\implies T : \mathbf{n}^T \mathbf{x} = 3 \quad (6.8)$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6.9)$$

6.3 The tangent  $T$  in (6.8) cuts off a chord  $AB$  from a circle  $C_2$  whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \quad (6.10)$$

Find  $\mathbf{A} + \mathbf{B}$ .

**Solution:** Let the radius of  $C_2$  be  $r$ . From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \quad (6.11)$$

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \quad (6.12)$$

Subtracting (6.12) from (6.11),

$$\mathbf{A}^T \mathbf{A} - \mathbf{B}^T \mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (6.13)$$

$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (6.14)$$

$\therefore \mathbf{A}, \mathbf{B}$  lie on  $T$ , from (6.8),

$$\mathbf{n}^T \mathbf{A} = \mathbf{n}^T \mathbf{B} = 3 \quad (6.15)$$

$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \quad (6.16)$$

From (6.14) and (6.16)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \quad (6.17)$$

$$\implies \mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C} = k\mathbf{n}^T \mathbf{n} \quad (6.18)$$

$$\implies \frac{\mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C}}{\mathbf{n}^T \mathbf{n}} = k \quad (6.19)$$

$$\implies k = 2 \quad (6.20)$$

using (6.15). Substituting in (6.17)

$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \quad (6.21)$$

6.4 If  $AB = 4$ , find  $\mathbf{A}^T \mathbf{B}$ .

**Solution:** From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \quad (6.22)$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \quad (6.23)$$

$$\implies \mathbf{A}^T \mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \quad (6.24)$$

using (6.21) and simplifying.

6.5 Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \quad (6.25)$$

**Solution:**

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \quad (6.26)$$

$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \quad (6.27)$$

From (6.27),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \quad (6.28)$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \quad (6.29)$$

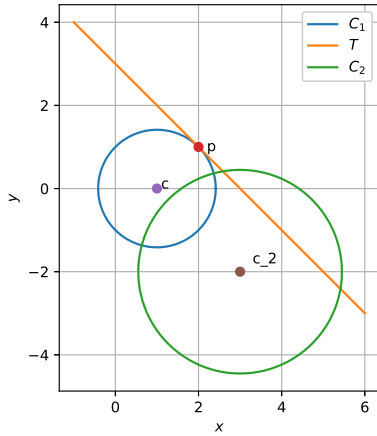


Fig. 6.7

Upon substituting from (6.12) and (6.11) and simplifying, (6.25) is obtained.

6.6 Find  $r$ .

**Solution:** (6.25) can be expressed as

$$\mathbf{A}^T \mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T \mathbf{C} = 8 - r^2 \quad (6.30)$$

$$\Rightarrow 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T \mathbf{C} = r^2 \quad (6.31)$$

$$\Rightarrow 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \quad (6.32)$$

$$\Rightarrow r = \sqrt{6}. \quad (6.33)$$

6.7 Summarize all the above computations through a Python script and plot the tangent and circle.

**Solution:** The following code generates Fig. 6.7.

```
wget
https://github.com/gadepall/school/raw/master/
linalg/2D/manual/codes/circ.py
```

## 7 PARABOLA

7.1 Find the tangent at  $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$  to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \quad (7.1)$$

**Solution:** Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (7.2)$$

in (5.10), the desired equation is

$$\left[ \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \end{pmatrix} \right] \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \quad (7.3)$$

resulting in

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -5 \quad (7.4)$$

7.2 The line in (7.4) touches the circle

$$\mathbf{x}^T \mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \quad (7.5)$$

Find  $c$ .

**Solution:** Comparing (5.2) and (7.5),

$$\mathbf{V} = \mathbf{I}, \quad \mathbf{u} = 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (7.6)$$

Comparing (5.10) and (7.4),

$$\mathbf{p} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (7.7)$$

$$\Rightarrow \mathbf{p} = - \begin{pmatrix} 6 \\ 7 \end{pmatrix} \quad (7.8)$$

and

$$c + \mathbf{p}^T \mathbf{u} = 5 \quad (7.9)$$

$$\Rightarrow c = 5 + 2 \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (7.10)$$

$$= 95 \quad (7.11)$$

7.3 Summarize all the above computations through a Python script and plot the parabola, tangent and circle.

**Solution:** The following code generates Fig. 7.3.

```
wget
https://github.com/gadepall/school/raw/master/
linalg/2D/manual/codes/parab.py
```

## 8 AFFINE TRANSFORMATION

8.1 In general, Fig. 7.3 was generated using an *affine transformation*.

8.2 Express

$$y_2 = y_1^2 \quad (8.1)$$

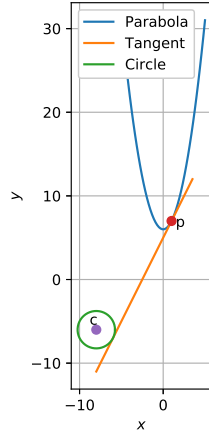


Fig. 7.3

as a matrix equation.

**Solution:** (8.1) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2\mathbf{g}^T \mathbf{y} = 0 \quad (8.2)$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.3)$$

8.3 Given

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0, \quad (8.4)$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \quad (8.5)$$

and  $\mathbf{P}, \mathbf{c}$  such that

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c}. \quad (8.6)$$

(8.6) is known as an affine transformation. Show that

$$\begin{aligned} \mathbf{D} &= \mathbf{P}^T \mathbf{V} \mathbf{P} \\ \mathbf{g} &= \mathbf{P}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) \end{aligned} \quad (8.7)$$

$$F + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} = 0$$

**Solution:** Substituting (8.6) in (8.4),

$$(\mathbf{P} \mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P} \mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P} \mathbf{y} + \mathbf{c}) + F = 0, \quad (8.8)$$

which can be expressed as

$$\begin{aligned} \Rightarrow \mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} \\ + F + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} = 0 \end{aligned} \quad (8.9)$$

Comparing (8.9) with (8.2) (8.7) is obtained.

8.4 Show that there exists a  $\mathbf{P}$  such that

$$\mathbf{P}^T \mathbf{P} = \mathbf{I} \quad (8.10)$$

Find  $\mathbf{P}$  using

$$\mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (8.11)$$

8.5 Find  $\mathbf{c}$  from (8.7).

**Solution:**

$$\because \mathbf{g} = \mathbf{P}^T (\mathbf{V} \mathbf{c} + \mathbf{u}), \quad (8.12)$$

$$\mathbf{V} \mathbf{c} = \mathbf{P} \mathbf{g} - \mathbf{u} \quad (8.13)$$

$$\Rightarrow \mathbf{c}^T \mathbf{V} \mathbf{c} = \mathbf{c}^T (\mathbf{P} \mathbf{g} - \mathbf{u}) = -F - 2\mathbf{u}^T \mathbf{c} \quad (8.14)$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ (\mathbf{P} \mathbf{g} + \mathbf{u})^T \end{pmatrix} \mathbf{c} = \begin{pmatrix} \mathbf{P} \mathbf{g} - \mathbf{u} \\ -F \end{pmatrix} \quad (8.15)$$

for computing  $\mathbf{c}$ .

## 9 ELLIPSE

9.1 Express the following equation in the form given in (5.1)

$$E : 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \quad (9.1)$$

**Solution:** (9.1) can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + 29 = 0 \quad (9.2)$$

where

$$\mathbf{V} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \quad (9.3)$$

9.2 Using the affine transformation in (8.6), show that (9.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = 1 \quad (9.4)$$

where

$$\mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (9.5)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (9.6)$$

for

$$\mathbf{P}^T \mathbf{P} = \mathbf{I} \quad (9.7)$$

9.3 Find  $\mathbf{c}$

**Solution:**

$$\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (9.8)$$

9.4 If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (9.9)$$

$$P = (\mathbf{P}_1 \quad \mathbf{P}_2) \quad (9.10)$$

show that

$$V\mathbf{z} = \lambda\mathbf{z} \quad (9.11)$$

where  $\lambda \in \{\lambda_1, \lambda_2\}$ ,  $\mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$ .

9.5 Find  $\lambda$ .

**Solution:**  $\lambda$  is obtained by solving the following equation.

$$|\lambda I - V| = 0 \quad (9.12)$$

$$\Rightarrow \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \quad (9.13)$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0 \quad (9.14)$$

$$\Rightarrow \lambda = 2, 8 \quad (9.15)$$

9.6 Sketch 9.4.

9.7 Find  $\mathbf{P}_1$  and  $\mathbf{P}_2$ .

**Solution:** From (9.11)

$$V\mathbf{P}_1 = \lambda_1\mathbf{P}_1 \quad (9.16)$$

$$\Rightarrow (V - \lambda I)\mathbf{y} = 0 \quad (9.17)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \quad (9.18)$$

$$\text{or, } \mathbf{P}_1 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (9.19)$$

Similarly,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{P}_2 = 0 \quad (9.20)$$

$$\text{or, } \mathbf{P}_2 = k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (9.21)$$

9.8 Find  $\mathbf{P}$ .

**Solution:** From (9.7) and (9.10),

$$k_1 = \frac{1}{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \quad (9.22)$$

$$k_2 = \frac{1}{\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \quad (9.23)$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (9.24)$$

9.9 Find the equation of the major axis for  $E$ .

**Solution:** The major axis for (9.4) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (9.25)$$

Using the affine transformation in (8.6)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (9.26)$$

$$\Rightarrow \mathbf{x} - \mathbf{c} = \lambda_1 \mathbf{P}_1 \quad (9.27)$$

$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (9.28)$$

$$= -3 \quad (9.29)$$

since

$$P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \quad (9.30)$$

which is the major axis of the ellipse  $E$ .

9.10 Find the minor axis of  $E$ .

9.11 Let  $\mathbf{F}_1, \mathbf{F}_2$  be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \quad (9.31)$$

Find  $\mathbf{F}_1, \mathbf{F}_2$  and  $k$ .

9.12 Summarize all the above computations through a Python script and plot the ellipses in (9.1) and (9.4).

**Solution:** The following script plots Fig. 9.12 using the principles of an affine transformation.

<https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/ellipse.py>

## 10 HYPERBOLA

10.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \quad (10.1)$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \quad (10.2)$$

at points  $\mathbf{P}$  and  $\mathbf{Q}$ . If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad (10.3)$$



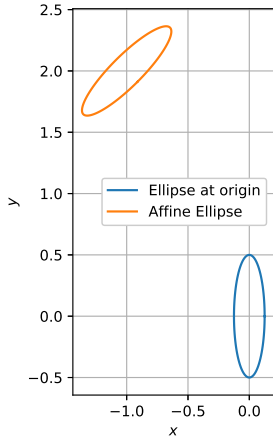


Fig. 9.12

find the equation of  $PQ$ .

**Solution:** The equations of the two tangents are obtained using (5.10) as

$$\mathbf{P}^T V \mathbf{x} = 36 \quad (10.4)$$

$$\mathbf{Q}^T V \mathbf{x} = 36. \quad (10.5)$$

Since both pass through  $\mathbf{T}$

$$\mathbf{P}^T V \mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (10.6)$$

$$\mathbf{Q}^T V \mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (10.7)$$

Thus,  $\mathbf{P}, \mathbf{Q}$  satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \quad (10.8)$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \quad (10.9)$$

which is the equation of  $PQ$ .

10.2 In  $\triangle PTQ$ , find the equation of the altitude  $TD \perp PQ$ .

**Solution:** Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad (10.10)$$

using (1.2) and (10.9), the equation of  $TD$  is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \quad (10.11)$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (10.12)$$

10.3 Find  $D$ .

**Solution:** From (10.9) and (10.12),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (10.13)$$

$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (10.14)$$

10.4 Show that the equation of  $PQ$  can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \quad (10.15)$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (10.16)$$

10.5 Show that for  $\mathbf{V}^T = \mathbf{V}$ ,

$$(\mathbf{D} + \lambda \mathbf{m})^T \mathbf{V} (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \quad (10.17)$$

can be expressed as

$$\lambda^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\lambda \mathbf{m}^T \mathbf{V} \mathbf{D} + \mathbf{D}^T \mathbf{V} \mathbf{D} + F = 0 \quad (10.18)$$

10.6 Find  $\mathbf{P}$  and  $\mathbf{Q}$ .

**Solution:** From (10.15) and (10.1) (10.18) is obtained. Substituting from (10.16), (10.2) and (10.14)

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \quad (10.19)$$

$$\mathbf{m}^T \mathbf{V} \mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \quad (10.20)$$

$$\mathbf{D}^T \mathbf{V} \mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \quad (10.21)$$

Substituting in (10.18)

$$4\lambda^2 - 144 = 36 \quad (10.22)$$

$$\implies \lambda = \pm 3\sqrt{5} \quad (10.23)$$

Substituting in (10.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3 \begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \quad (10.24)$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3 \begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \quad (10.25)$$

10.7 Find the area of  $\triangle PTQ$ .

**Solution:** Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \quad (10.26)$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \quad (10.27)$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \quad (10.28)$$

- 10.8 Repeat the previous exercise using determinants.
- 10.9 Summarize all the above computations through a Python script and plot the hyperbola.