The Circle

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***Abstract*—Solved problems from JEE mains papers related to 2D circles in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 A circle passes through the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{1.1}$$

find its radius.
Solution: Let

$$\mathbf{C} = \frac{\mathbf{A}+\mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.2}$$

The direction vector of AB is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{1.3}$$

$$\because OC \perp AB,$$
$$OC : \mathbf{m}^T \left(\mathbf{x} - \mathbf{C}\right) = 0$$
$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{1.4}$$

Thus, $\mathbf{O}$ is the intersection of (1.1) and (1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{1.6}$$

Thus the radius of the circle

$$OA = \left\|\mathbf{O} - \mathbf{A}\right\| = \sqrt{10} \tag{1.7}$$

2 If a circle C_1, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} = 4 \tag{2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.
Solution: From (2.1), the centre of C_2 is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T\mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.3}$$

Since the radius of C_1 is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$
$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$
$$\implies \mathbf{O}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.4}$$

The intercepts of C_1 on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda\mathbf{m} \tag{2.5}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6}$$

Susbtituting in the equation for C_1,

$$\left\|\lambda\mathbf{m} - \mathbf{O}_1\right\|^2 = r_1^2 \tag{2.7}$$

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which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda \mathbf{m}^T \mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$
$$\implies \lambda^2 - 10\lambda + 20 = 0 \quad (2.8)$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \quad (2.9)$$

after substituting from (2.6) and (2.4).

3 A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad (3.1)$$

cuts the circle

$$C : \mathbf{x}^T \mathbf{x} = 9 \quad (3.2)$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$. Draw PAB for any two points $\mathbf{A}, \mathbf{B}$ on the circle.
Solution: Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line PAB can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \quad (3.3)$$

for $\|\mathbf{m}\| = 1$. The intersection of L and C yields

$$(\mathbf{P} + \lambda \mathbf{m})^T (\mathbf{P} + \lambda \mathbf{m}) = 9$$
$$\implies \lambda^2 + 2\lambda \mathbf{m}^T \mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \quad (3.4)$$

The product of the roots in (3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \quad (3.5)$$

4 Find the equation of the circle C_2, which is the mirror image of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (4.1)$$

in the line

$$L : \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3. \quad (4.2)$$

Solution: From (4.1), circle C_1 has centre at

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.3)$$

and radius

$$r_1 = \mathbf{O}_1^T \mathbf{O}_1 = 1 \quad (4.4)$$

The centre of C_2 is the reflection of $\mathbf{O}_1$ about L and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}} \mathbf{O}_1 + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (4.5)$$

where the relevant parameters are obtained from (4.2) as

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 & -1 \end{pmatrix}, c = 3. \quad (4.6)$$

Substituting the above in (4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{4} \mathbf{O}_1 + c\frac{\mathbf{n}}{2}$$
$$\implies \mathbf{O}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (4.7)$$

Thus

$$C_2 : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| = 1 \quad (4.8)$$

5 One of the diameters of the circle, given by

$$C : \mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (5.1)$$

is a chord of a circle S, whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \quad (5.2)$$

Find the radius of S.
Solution: From (5.1), the centre of C is

$$\mathbf{O}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad (5.3)$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T \mathbf{O}_1 - 12} = 5 \quad (5.4)$$

From (5.3) and (5.2),

$$O_1 O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$
$$\implies r_2 = \sqrt{O_1 O_2^2 - r_1^2} = 5 \quad (5.5)$$

6 A circle C passes through

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \quad (6.1)$$

and touches the y-axis at

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (6.2)$$

Which one of the following equations can represent a diameter of this circle?

(i) $\begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} = 6$ (iii) $\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 3$

(ii) $\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} + 10 = 0$ (iv) $\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} + 4 = 0$

Solution: Let $\mathbf{O}$ be the centre of C. Then the equation of the normal, OQ is

$$\begin{pmatrix}0 & 1\end{pmatrix}(\mathbf{O}-\mathbf{Q}) = 0$$
$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{O} = 2 \tag{6.3}$$

Also,

$$\|\mathbf{O}-\mathbf{P}\|^2 = \|\mathbf{O}-\mathbf{Q}\|^2$$
$$\implies 2\left(\mathbf{P}-\mathbf{Q}\right)^T\mathbf{O} = \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2$$
$$\text{or, } \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{O} = -4 \tag{6.4}$$

(6.3) and (6.4) result in the matrix equation

$$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\mathbf{O} = \begin{pmatrix}-4\\2\end{pmatrix} \tag{6.5}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & -4\\0 & 1 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -2\\0 & 1 & 2\end{pmatrix} \implies \mathbf{O} = \begin{pmatrix}-2\\2\end{pmatrix} \tag{6.6}$$

Hence, option ii) is correct.

7 Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix}1\\-1\end{pmatrix}, \tag{7.1}$$

whose centre $\mathbf{O}$ is the point of intersection of the straight lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{7.2}$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{7.3}$$

Solution: From (7.2) and (7.3), we obtain the matrix equation

$$\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}\mathbf{O} = \begin{pmatrix}3\\1\end{pmatrix} \tag{7.4}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & 1\\2 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 1\\0 & 3 & 1\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}3 & 0 & 4\\0 & 3 & 1\end{pmatrix} \implies \mathbf{O} = \frac{1}{3}\begin{pmatrix}4\\1\end{pmatrix} \tag{7.5}$$

Thus, the equation of the desired tangent is

$$\left(\mathbf{O}-\mathbf{P}\right)^T\left(\mathbf{x}-\mathbf{P}\right) = 0$$
$$\implies \begin{pmatrix}1 & 4\end{pmatrix}\mathbf{x} = -3 \tag{7.6}$$

8 The line

$$\Gamma : \mathbf{x} = \begin{pmatrix}0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{8.1}$$

intersects the circle

$$\Omega : \left\|\mathbf{x} - \begin{pmatrix}3\\-2\end{pmatrix}\right\| = 5 \tag{8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{R} = -\frac{3}{5} \tag{8.3}$$

Find m.

Solution: Let

$$\mathbf{c} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{O} = \begin{pmatrix}3\\-2\end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix} \tag{8.4}$$

The intersection of (8.1) and (8.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{8.5}$$

$$\implies \lambda^2\|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T\left(\mathbf{c}-\mathbf{O}\right) + \|\mathbf{c}-\mathbf{O}\|^2 - 25 = 0 \tag{8.6}$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1\mathbf{m} \tag{8.7}$$
$$\mathbf{Q} = \mathbf{c} + \lambda_2\mathbf{m} \tag{8.8}$$
$$\implies \frac{\mathbf{P}+\mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1+\lambda_2}{2}\mathbf{m} \tag{8.9}$$
$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{c} + \frac{\lambda_1+\lambda_2}{2}\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{m} \tag{8.10}$$
$$= \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{c} - \frac{\mathbf{m}^T\left(\mathbf{c}-\mathbf{O}\right)}{\|\mathbf{m}\|^2} \tag{8.11}$$

using the sum of roots in (8.6). From (8.3) and (8.4),

$$-\begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}-3\\3\end{pmatrix} = -\frac{3}{5}\left(1+m^2\right) \tag{8.12}$$
$$\implies m^2 - 5m + 6 = 0 \tag{8.13}$$
$$\implies m = 2 \text{ or } 3 \tag{8.14}$$

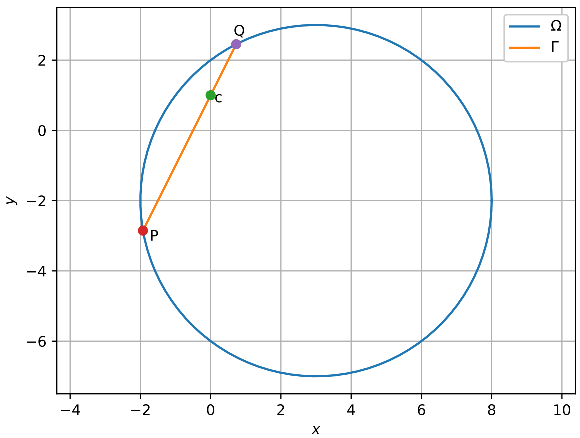

Fig. 8

From (8.6),

$$\lambda = \frac{-\mathbf{m}^T(\mathbf{c}-\mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{\left(\mathbf{m}^T(\mathbf{c}-\mathbf{O})\right)^2 - \|\mathbf{c}-\mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{8.15}$$

Fig. 8 summarizes the solution for $m = 2$.

1 Linear Algebra: Reflection

1.1 Let $\mathbf{B}$ be the reflection of $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}$ with respect to the line

$$L : \begin{pmatrix}8 & -6\end{pmatrix}\mathbf{x} = 23 \tag{1.1.1}$$

Show that $\mathbf{B} = \begin{pmatrix}6\\0\end{pmatrix}$.

1.2 Find the equation of AB.
Solution: The normal vector of L is

$$\mathbf{n} = \begin{pmatrix}6\\8\end{pmatrix} \tag{1.2.1}$$

Thus, the equation of AB is

$$AB : \mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.2.2}$$

$$\implies \begin{pmatrix}6 & 8\end{pmatrix}\mathbf{x} = 36 \tag{1.2.3}$$

1.3 Let Γ_A and Γ_B be circles of radii $r_1 = 2, r_2 = 1$ with centres at $\mathbf{A}$ and $\mathbf{B}$ respectively. Let T be the common tangent to both the circles such that they are on the same side of T.

1.4 Find the point $\mathbf{C}$ where AB meets T.
Solution: Let $\mathbf{D}, \mathbf{E}$ be the points of contact for T with Γ_A and $4\Gamma_B$ respectively. It is obvious that $\triangle ADC \sim \triangle BEC$. Hence,

$$AB = BC \tag{1.4.1}$$

$$\implies \mathbf{C} = 2\mathbf{B} - A = \begin{pmatrix}10\\-3\end{pmatrix} \tag{1.4.2}$$

1.5 Find AC.
Solution:

$$AC = \|\mathbf{A}-\mathbf{C}\| = 10 \tag{1.5.1}$$

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1.6 Find $\mathbf{D}$ and $\mathbf{E}$.

2 Linear Algebra: Coordinate Geometry

2.1 Find the points $\mathbf{X}, \mathbf{Y}$ where

$$C_1 : \|\mathbf{x}\| = 3 \tag{2.1.1}$$

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix}3\\4\end{pmatrix}\right\| = 4 \tag{2.1.2}$$

intersect.

2.2 Find the centre $\mathbf{O}_3$ and radius r of C_3 such that
  a) $\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3$ are collinear.
  b) C_1, C_2 lie inside C_3 and
  c) C_3 touches C_1 at $\mathbf{M}$ and C_2 at $\mathbf{N}$.

2.3 Find the equation of XY.

2.4 Find $\mathbf{Z}, \mathbf{W}$ the points of intersection of XY and C_3.

2.5 A common tangent of C_1 and C_3 is also a tangent to

$$P : \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} - 2\alpha\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.5.1}$$

Find α.

3 Linear Algebra: Coordinate Geometry

Find the following based on the previous problem

3.1 $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{O}_3$

3.2 $\frac{ZW}{XY}$

3.3 $\frac{\operatorname{ar}(\triangle MZN)}{\operatorname{ar}(\triangle ZMW)}$