Linear Algebra through Coordinate Geometry

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Contents

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***Abstract*—This book provides a computational approach to linear algebra and matrices by solving problems in 2D and 3D coordinate geometry from IIT-JEE. An introduction to convex optimization is also provided in the process. Links to sample Python codes are available in the text. The book provides sufficient math basics for Machine Learning and is also recommended for high school students who wish to explore topics in Artificial Intelligence.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    linalg/book/codes
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1 The Straight Line

1.1 Point

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$

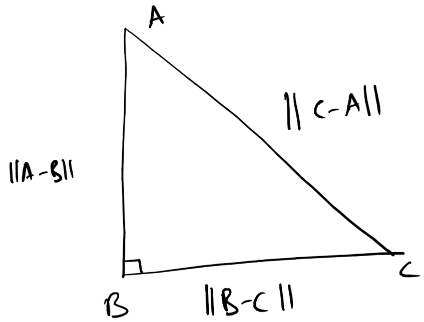

Fig. 1.1.7

3. The *length* of PQ is defined as
$$\|\mathbf{P}-\mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P}-\mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$
5. The point dividing PQ in the ratio $k:1$ is
$$\mathbf{R} = \frac{k\mathbf{P}+\mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.7}$$
Solution: Using Baudhayana's theorem,
$$\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{C}-\mathbf{A}\|^2 \tag{1.1.7}$$
$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) + (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A})$$
$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show that
$$(\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.8}$$
9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that
$$(\mathbf{x}-\mathbf{y})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.9}$$

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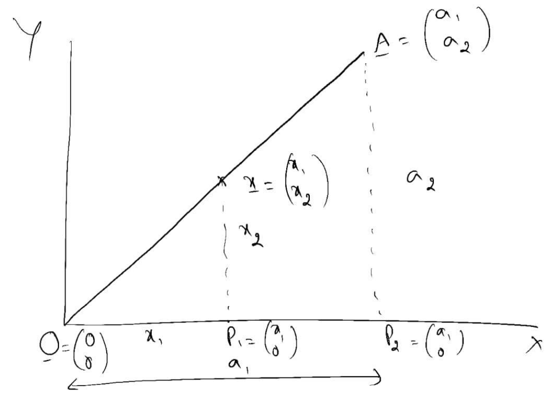

Fig. 1.2.1

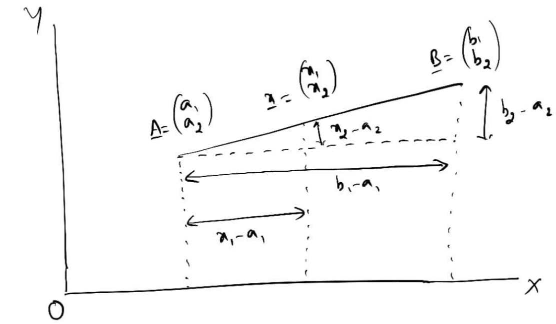

Fig. 1.2.2

1.2 Line

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of OA.
Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,
$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.2.1.1}$$
$$\implies x_2 = mx_1 \tag{1.2.1.2}$$
where m is known as the slope of the line. Thus, the equation of the line is
$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ m \end{pmatrix} = x_1\mathbf{m} \tag{1.2.1.3}$$
In general, the above equation is written as
$$\mathbf{x} = \lambda\mathbf{m}, \tag{1.2.1.4}$$
where $\mathbf{m}$ is the direction vector of the line.
2. Find the equation of AB in Fig. 1.2.2

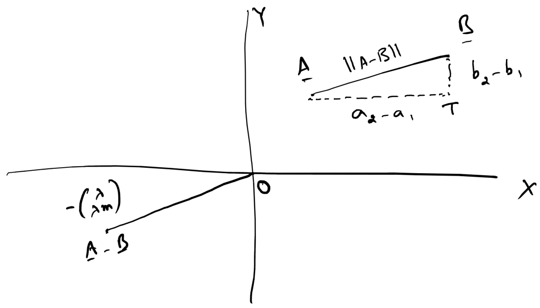

Fig. 1.2.5

Solution: From Fig. 1.2.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.2.2.1}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.2.2.2}$$

From (1.2.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.2.2.3}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2.4}$$

$$= \mathbf{A} + \lambda\mathbf{m} \tag{1.2.2.5}$$

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (1.2.2.5) is obtained from (1.2.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 1.2.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.2.4.1}$$

Also, from (1.2.1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.2.4.2}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.2.5. From (1.2.2.5), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.5.1}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.2.5.2}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.
6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.2.7.1}$$

8. The *normal* to the vector $\mathbf{m}$ is defined as

$$\mathbf{n}^T\mathbf{m} = 0 \tag{1.2.8.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{1.2.8.2}$$

9. From (1.2.7.1), the equation of a line can also be expressed as

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) \tag{1.2.9.1}$$

$$\implies \mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} = c \tag{1.2.9.2}$$

10. The unit vectors on the x and y axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.2.10.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.10.2}$$

11. If a be the *intercept* of the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.11.1}$$

on the $x-$axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line. Thus,

$$\mathbf{n}^T\begin{pmatrix} a \\ 0 \end{pmatrix} = c \tag{1.2.11.2}$$

$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \tag{1.2.11.3}$$

12. The *rotation matrix* is defined as

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.2.12}$$

where θ is anti-clockwise.
13.

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.13}$$

where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix.*
14. Find the equation of line L in Fig. 1.2.14.
Solution: The equation of the $x-$axis is

$$\mathbf{x} = \lambda\mathbf{e}_1 \tag{1.2.14.1}$$

Translation by p units along the $y-$axis results

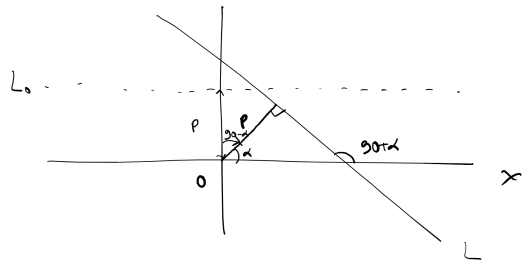

Fig. 1.2.14

in

$$L_0: \quad \mathbf{x} = \lambda \mathbf{e}_1 + p\mathbf{e}_2 \tag{1.2.14.2}$$

Rotation by $90° - \alpha$ in the anti-clockwise direction yields

$$L: \quad \mathbf{x} = \mathbf{Q}\{\lambda \mathbf{e}_1 + p\mathbf{e}_2\} \tag{1.2.14.3}$$
$$= \lambda \mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \tag{1.2.14.4}$$

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \tag{1.2.14.5}$$
$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \tag{1.2.14.6}$$

From (1.2.14.4),

$$L: \quad \mathbf{e}_2^T \mathbf{Q}^T \mathbf{x} = \lambda \mathbf{e}_2^T \mathbf{Q}^T \mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T \mathbf{Q}^T \mathbf{Q}\mathbf{e}_2$$
$$= \lambda \mathbf{e}_2^T \mathbf{e}_1 + p\mathbf{e}_2^T \mathbf{e}_2 \tag{1.2.14.7}$$

resulting in

$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \mathbf{x} = p \tag{1.2.14.8}$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T \mathbf{x} = c \tag{1.2.15.1}$$

is

$$p = \frac{c}{\|n\|} \tag{1.2.15.2}$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{1.2.16.1}$$
$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{1.2.16.2}$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1} \mathbf{c} \tag{1.2.16.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{1.2.16.4}$$

17. The *angle between two lines* is given by

$$\cos^{-1} \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} \tag{1.2.17.1}$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T \mathbf{x} = c \tag{1.2.18.1}$$

is

$$\frac{\left|\mathbf{n}^T \mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \tag{1.2.18.2}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.2.18.3}$$

where

$$\mathbf{n}^T \mathbf{A} = c, \mathbf{n}^T \mathbf{m} = 0 \tag{1.2.18.4}$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda \mathbf{m} \tag{1.2.18.5}$$
$$\implies \mathbf{n}^T \mathbf{x} = c - \mathbf{n}^T \mathbf{x}_0 \tag{1.2.18.6}$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.2.19.1}$$

can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{1.2.19.2}$$

where

$$\mathbf{V} = \mathbf{V}^T \tag{1.2.19.3}$$
$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{1.2.19.4}$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.2.20.1}$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \tag{1.2.20.2}$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \tag{1.2.21.1}$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k + 1} = \mathbf{B}. \tag{1.2.21.2}$$

Solution: From (1.2.2.5),

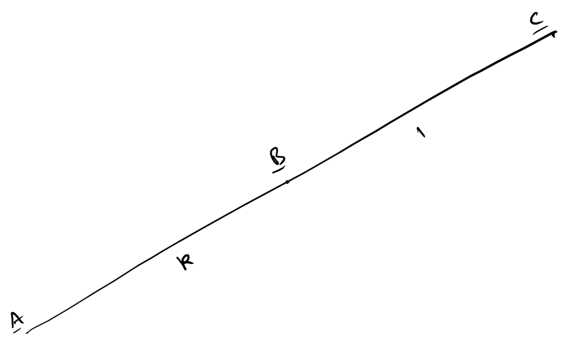

Fig. 1.2.21

$$\begin{aligned} \mathbf{B} &= \mathbf{A} + \lambda_1\mathbf{m} \\ \mathbf{B} &= \mathbf{C} - \lambda_2\mathbf{m} \end{aligned} \tag{1.2.21.3}$$

$$\implies \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{1.2.21.4}$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{1.2.21.5}$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{1.2.21.6}$$

resulting in (1.2.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{1.2.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{1.2.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{1.2.23.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{1.2.23.3}$$

24. In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of ΔABC.

25. Find the intersection of AP and BQ.
Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From (1.2.8.1) and (1.2.9.2), the equation of AP and BQ are

$$(\mathbf{B} - \mathbf{C})^T(\mathbf{x} - \mathbf{A}) = 0 \tag{1.2.25.1}$$

$$(\mathbf{C} - \mathbf{A})^T(\mathbf{x} - \mathbf{B}) = 0 \tag{1.2.25.2}$$

which can be solved to obtain the intersection point using (1.2.16.3).

26. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.26.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.26.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{1.2.26.3}$$

27. Find the equation of a line passing through the intersection of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.27.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.27.2}$$

and passing through the point $\mathbf{p}$.
Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T}\mathbf{c} \tag{1.2.27.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{1.2.27.4}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{1.2.27.5}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda\left(\mathbf{N}^{-T}\mathbf{c} - \mathbf{p}\right) \tag{1.2.27.6}$$

$$\implies \mathbf{N}^T\mathbf{x} = \mathbf{N}^T\mathbf{p} + \lambda\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right) \tag{1.2.27.7}$$

resulting in

$$\begin{aligned} &\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{x} \\ &\quad = \left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{p} \end{aligned} \tag{1.2.27.8}$$

28. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.28.1}$$

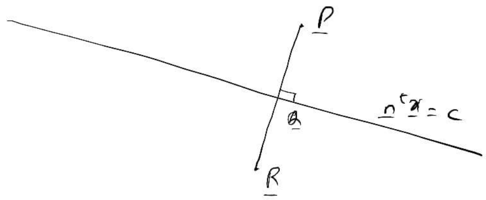

Fig. 1.2.28

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.2.28.2}$$

$$\mathbf{n}^T\mathbf{Q} = c \tag{1.2.28.3}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{1.2.28.4}$$

where $\mathbf{m}$ is the direction vector of L. From (1.2.28.2) and (1.2.28.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{1.2.28.5}$$

From (1.2.28.5) and (1.2.28.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{1.2.28.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{1.2.28.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{1.2.28.9}$$

(1.2.28.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{1.2.28.12}$$

29. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.2.29.1}$$

1.3 Example

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.3.1.1}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{1.3.1.2}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \tag{1.3.1.3}$$

Find the area of $\triangle ABC$.

Solution: The centroid $\mathbf{O}$ is the solution of (1.3.1.2),(1.3.1.3) and is obtained as the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} \tag{1.3.1.4}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix}1 & 1 & 5\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 5\\0 & 1 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 1\end{pmatrix} \tag{1.3.1.5}$$

Thus,

$$\mathbf{O} = \begin{pmatrix}4\\1\end{pmatrix} \tag{1.3.1.6}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.3.1.7}$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix}11\\1\end{pmatrix} \tag{1.3.1.8}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}11\\1\end{pmatrix} \tag{1.3.1.9}$$

From (1.3.1.3) and (1.3.1.9),

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} = 5 \tag{1.3.1.10}$$

$$\implies 5 + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 12 \tag{1.3.1.11}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 7 \tag{1.3.1.12}$$

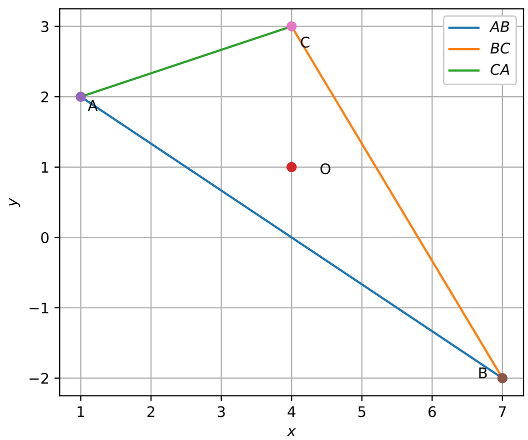

Fig. 1.3.2

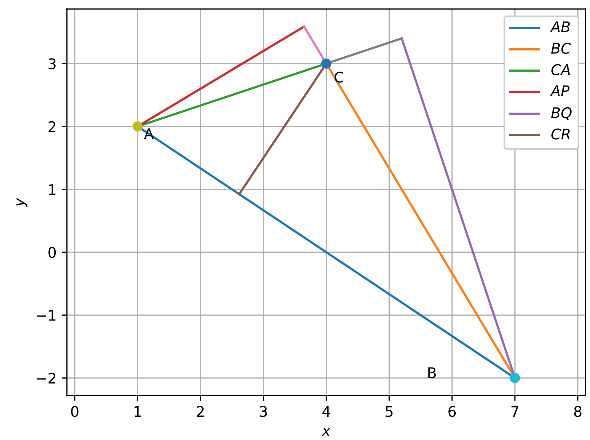

Fig. 1.4.4

From (1.3.1.12) and (1.3.1.3), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}7\\ 4\end{pmatrix} \tag{1.3.1.13}$$

using the augmented matrix as

$$\begin{pmatrix}1 & 1 & 7\\ 1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 7\\ 0 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & 3\end{pmatrix} \tag{1.3.1.14}$$

$$\implies \mathbf{C} = \begin{pmatrix}4\\ 3\end{pmatrix} \tag{1.3.1.15}$$

From (1.3.1.8),

$$\mathbf{B} = \begin{pmatrix}11\\ 1\end{pmatrix} - \begin{pmatrix}4\\ 3\end{pmatrix} = \begin{pmatrix}7\\ -2\end{pmatrix} \tag{1.3.1.16}$$

Thus,

$$\frac{1}{2}\begin{vmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\\ 1 & 1 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}1 & 7 & 4\\ 2 & -2 & 3\\ 1 & 1 & 1\end{vmatrix} = 9 \tag{1.3.1.17}$$

2. Summarize all the above computations through a Python script and plot $\triangle ABC$.
Solution:

```
codes/2d/triang.py
```

1.4 Programming

1. Find the *orthocentre* of $\triangle ABC$.
Solution: The following code finds the required point using (1.2.25.1) and (1.2.25.2) .

```
codes/2d/orthocentre.py
```

2. Find $\mathbf{P}$, the foot of the altitude from $\mathbf{A}$ upon BC.
Solution:

```
codes/2d/alt_foot.py
```

3. Find $\mathbf{Q}$ and $\mathbf{R}$.
4. Draw AP, BQ and CR and verify that they meet at a point $\mathbf{H}$.
Solution: The following code plots the altitudes in Fig. 1.4.4

```
codes/2d/alt_draw.py
```

5. Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.
6. Find the equations of AD, BE and CF.
7. Find the point of intersection of AD and CF.
8. Verify that $\mathbf{O}$ is the point of intersection of BE, CF as well.
9. Graphically show that the medians of ΔABC meet at the centroid.

1.5 Solved Problems

1. A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 10 \tag{1.5.1}$$

$$\begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 5 = 0 \tag{1.5.1}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.

Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.5.1}$$

Then (1.5.1) can be expressed as

$$\mathbf{n}^T\mathbf{x} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T\mathbf{x} = -5 \tag{1.5.1}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.5.1) respectively,

$$\mathbf{n}^T\mathbf{A} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T\mathbf{B} = -5 \tag{1.5.1}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \tag{1.5.1}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.5.1}$$
$$\mathbf{A} = -k\mathbf{B} \tag{1.5.1}$$

Substituting in (1.5.1), and simplifying,

$$\mathbf{n}^T\mathbf{B} = \frac{10}{-k} \tag{1.5.1}$$
$$\mathbf{n}^T\mathbf{B} = \frac{-5}{2} \tag{1.5.1}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.5.1}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.5.2}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 4 \tag{1.5.2}$$

by $d = 2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.

Solution: From (1.5.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.2}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \tag{1.5.2}$$

However,

$$PQ = d \tag{1.5.2}$$
$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda|\,\|\mathbf{m}\| = d \tag{1.5.2}$$
$$\implies \lambda = \pm\frac{d}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{1.5.2}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T\mathbf{m}} = \sqrt{2} \tag{1.5.2}$$

from (1.5.2). Since $\mathbf{Q}$ lies in the third quadrant, from (1.5.2) and (1.5.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{1.5.2}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{Q}) = 0 \tag{1.5.2}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 - 2\sqrt{6} \tag{1.5.2}$$

3. Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.5.3}$$
$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix}\mathbf{x} - 5 = 0. \tag{1.5.3}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{1.5.3}$$

find its vertices.

Solution: From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix}\mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{1.5.3}$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1.5.3}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2}$$
$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.5.3}$$

$$\because AD \parallel BC,$$
$$BC : \begin{pmatrix} 7 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} = -15 \quad (1.5.3)$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \quad (1.5.3)$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \implies \mathbf{B} = -\frac{1}{3} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad (1.5.3)$$

$$\because AB \parallel CD,$$
$$CD : \begin{pmatrix} 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \quad (1.5.3)$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.5.3)$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \implies \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \quad (1.5.3)$$

4. Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \quad (1.5.4)$$

has area 28. Find the orthocentre of this triangle.
Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \quad (1.5.4)$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.5.4)$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \quad (1.5.4)$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \quad (1.5.4)$$

The orthocentre is obtained by solving (1.5.4) and (1.5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \quad (1.5.4)$$

which can be expressed using (1.5.4), (1.5.4), (1.5.4) and (1.5.4) as

$$\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix}$$
$$= k \begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix} \quad (1.5.4)$$

From (1.5.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$
$$\implies \begin{vmatrix} k & 5 - k & -2k \\ -3k & 4k & 2 + 3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \quad (1.5.4)$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \quad (1.5.4)$$
$$\implies 5k^2 + 13k - 46 = 0 \quad (1.5.4)$$
$$\text{or, } k = 2, -\frac{23}{5} \quad (1.5.4)$$

Substituting the above in (1.5.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \quad (1.5.5)$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the

given information,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0} \tag{1.5.5}$$

If AB be the line in (1.5.5), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.5.5}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.5.5}$$

since CF passes through the origin and $CF \perp AB$. From (1.5.5) and (1.5.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.5.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

From (1.5.5),

$$\mathbf{C} = -(\mathbf{A}+\mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

after substituting from (1.5.5). Thus,

$$CF = \|\mathbf{C}-\mathbf{F}\| = 3\sqrt{2} \tag{1.5.5}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = 2\sqrt{6} \tag{1.5.5}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 6\sqrt{3} \tag{1.5.5}$$

6. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.
Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and y-axis respectively. Then

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.6}$$

Multiplying (1.5.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.5.6}$$

$$\mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}) = 4\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= 4\begin{pmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix} \tag{1.5.6}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D})$$
$$= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3}-1\right) \tag{1.5.6}$$

for $\theta = 30^\circ$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.

1.6 JEE Exercises

1. Find the area enclosed within the curve $|x| + |y| = 1$.
2. Find the equation of the line about which $y=10^x$ is the reflection of $y=\log_{10} x$.
3. If $3a+2b+4c = 0$, find the intersection of the set of lines

$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + c = 0. \tag{1.6.3.1}$$

4. Given the points $A = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$, find the equation of the locus of the point $P = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $|AP - BP|=6$.
5. If a,b and c are in A.P, show that the straight line

$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + c = 0 \tag{1.6.5.1}$$

will always pass through a fixed point and find its coordinates.
6. Find the quadrant in which the orthocentre of the triangle formed by the lines

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 1 \tag{1.6.6.1}$$
$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 6 \tag{1.6.6.2}$$
$$\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{x} + 4 = 0 \tag{1.6.6.3}$$

lies.

7. Let the algebraic sum of the perpendicular distances from the points $\begin{pmatrix}2\\0\end{pmatrix}$, $\begin{pmatrix}0\\2\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ to a variable straight line be zero. Show that the line passes through a fixed point and find its coordinates.
8. The vertices of a triangle are $\mathbf{A} = \begin{pmatrix}-1\\-7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}1\\4\end{pmatrix}$. Find the equation of the bisector of $\angle ABC$.
9. Verify if the straight line
$$\begin{pmatrix}5 & 4\end{pmatrix}\mathbf{x} = 0 \tag{1.6.9.1}$$
passes through the point of intersection of the straight lines
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} - 10 = 0 \tag{1.6.9.2}$$
and
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} + 5 = 0 \tag{1.6.9.3}$$
10. Do the lines
$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} + 19 = 0 \tag{1.6.10.1}$$
and
$$\begin{pmatrix}9 & 6\end{pmatrix}\mathbf{x} - 17 = 0 \tag{1.6.10.2}$$
cut the coordinate axes in concyclic points?
11. The points $\begin{pmatrix}-a\\b\end{pmatrix}$,$\begin{pmatrix}0\\0\end{pmatrix}$,$\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}a^2\\ab\end{pmatrix}$ are:
    a) Collinear
    b) Vertices of a parallelogram
    c) Vertices of a rectangle
    d) None of these
12. The point $\begin{pmatrix}4\\1\end{pmatrix}$ undergoes the following three transformations successively (i) Reflection about the line
$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.12.1}$$
(ii)Translation through a distance 2 units along the positive direction of x-axis (iii) Rotation through an angle $\frac{\pi}{4}$ about the origin in counter clockwise direction. Find the final position of the point.
13. The straight lines
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.13.1}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} - 4 = 0 \tag{1.6.13.2}$$
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} - 4 = 0 \tag{1.6.13.3}$$
form a triangle which is
    a) isosceles
    b) equilateral
    c) right angled
    d) none of these
14. If $\mathbf{P} = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix}-1\\0\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}2\\0\end{pmatrix}$ are three given points, then the locus of the point $\mathbf{S}$ satisfying the relation $SQ^2 + SR^2 = 2SP^2$, is
    a) a straight line parallel to X-axis
    b) a circle passing through the origin
    c) a circle with centre at the origin
    d) a straight line parallel to Y-axis.
15. Line L has intercepts a and b on the coordinate axes.When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q, then
    a) $a^2 + b^2 = p^2 + q^2$
    b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
    c) $a^2 + p^2 = b^2 + q^2$
    d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$
16. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
    a) Square
    b) Circle
    c) Straight line
    d) Two intersecting lines
17. The locus of a variable point whose distances from $\begin{pmatrix}-2\\0\end{pmatrix}$ is $\frac{2}{3}$ times its distance from the line $x = -\frac{9}{2}$ is
    a) Ellipse
    b) Parabola
    c) Hyperbola
    d) None of these
18. The equations of a pair of opposite sides of a parallelogram are
$$x^2 - 5x + 6 = 0 \tag{1.6.18.1}$$
$$y^2 - 6y + 5 = 0 \tag{1.6.18.2}$$

Find the equations of its diagonals.

19. Find the orthocentre of the triangle formed by the lines

$$\mathbf{x}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (1.6.19.1)$$

and

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \quad (1.6.19.2)$$

20. Let PQR be an isosceles triangle, right angled at $\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}$. If the equation of the line QR is

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3, \quad (1.6.20.1)$$

then find the equation representing the pair of lines PQ and PR is

21. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P with the same common ratio, then the points$\begin{pmatrix}x_1\\y_1\end{pmatrix}$,$\begin{pmatrix}x_2\\y_2\end{pmatrix}$ and $\begin{pmatrix}x_3\\y_3\end{pmatrix}$
   a) lie on a straight line
   b) lie on a ellipse
   c) lie on a circle
   d) are the vertices of a triangle

22. Let PS be the median of the triangle with vertices $\mathbf{P} = \begin{pmatrix}2\\2\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-1\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}7\\3\end{pmatrix}$. Find the equation of the line passing through $\begin{pmatrix}1\\-1\end{pmatrix}$ and parallel to PS.

23. Find the incentre of the triangle with vertices $\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.

24. The number of integer values of m, for which the x coordinate of the point of intersection of the lines $\begin{pmatrix}3\\4\end{pmatrix}\mathbf{x} = 9$ and $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x} - 1 = 0$ is also an integer, is
   a) 2
   b) 0
   c) 4
   d) 1

25. Find the area of the parallelogram formed by the lines $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x}$=0, $\begin{pmatrix}-m\\1\end{pmatrix}\mathbf{x}$+1=0 ,$\begin{pmatrix}-n\\1\end{pmatrix}\mathbf{x}$=0 and $\begin{pmatrix}-n\\1\end{pmatrix}\mathbf{x}$+ 1 =0.

26. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $\mathbf{P} = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}\cos(\alpha-\theta)\\\sin(\alpha-\theta)\end{pmatrix}$ then the $\mathbf{Q}$ is obtained from $\mathbf{P}$ by
   a) clockwise rotation around origin through an angleα
   b) anticlockwise rotation around origin through an angleα
   c) reflection in the line through origin with slope $\tan\alpha$
   d) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$

27. Let $\mathbf{P} = \begin{pmatrix}-1\\0\end{pmatrix}\mathbf{Q} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix}3\\3\sqrt{3}\end{pmatrix}$ be three points. Then find the equation of the bisector of the angle PQR.

28. A straight line through the origin $\mathbf{O}$ meets the parallel lines

$$\begin{pmatrix}4 & 2\end{pmatrix}\mathbf{x} = 9 \quad (1.6.28.1)$$

and

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} + 6 = 0 \quad (1.6.28.2)$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. Find the ratio in which $\mathbf{O}$ divides the segment PQ.

29. The number of integral points (integral points means both the coordinate should be integer) exactly in the interior of the triangle with the vertices $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\21\end{pmatrix}$ and $\begin{pmatrix}21\\0\end{pmatrix}$ is
   a) 133
   b) 190
   c) 233
   d) 105

30. Find the orthocentre of a triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix}$,$\begin{pmatrix}3\\4\end{pmatrix}$,$\begin{pmatrix}4\\0\end{pmatrix}$

31. Find the area of the triangle formed by the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \quad (1.6.31.1)$$

and angle bisectors of the pair of straight lines

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} = 1. \quad (1.6.31.2)$$

32. Let $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{P} = \begin{pmatrix}3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\0\end{pmatrix}$ be the vertices of the triangle OPQ. The point $\mathbf{R}$ inside the triangle OPQ is such that the triangles OPR,PQR, OQR are of equal area. Find the coordinates of $\mathbf{R}$.

33. A straight line L through the point $\begin{pmatrix}3\\-2\end{pmatrix}$ is inclined at an angle of 60° to the line

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1. \tag{1.6.33.1}$$

If L also intersects the x-axis, then find the equation of L.

34. Three lines

$$\begin{pmatrix}p & q\end{pmatrix}\mathbf{x} + r = 0, \tag{1.6.34.1}$$

$$\begin{pmatrix}q & r\end{pmatrix}\mathbf{x} + p = 0 \tag{1.6.34.2}$$

and

$$\begin{pmatrix}r & p\end{pmatrix}\mathbf{x} + q = 0 \tag{1.6.34.3}$$

are concurrent if

a) $p + q + r = 0$
b) $p^2 + q^2 + r^2 = qr + rp + pq$
c) $p^3 + q^3 + r^3 = 3pqr$
d) none of these

35. The points $\begin{pmatrix}0\\\frac{8}{3}\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}82\\30\end{pmatrix}$ are vertices of

a) an obtuse angled triangle
b) an acute angled triangle
c) a right angled triangle
d) none of these

36. All points lying inside the triangle formed by the points $\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}5\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$ satisfy

a) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} \ge 0$
b) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 13 \ge 0$
c) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} - 12 \le 0$
d) $\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} \ge 0$
e) none of these

37. A vector $\mathbf{a} = \begin{pmatrix}2p\\1\end{pmatrix}$ with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counter clockwise sense. If, with respect to the new system, $\mathbf{a} = \begin{pmatrix}p+1\\1\end{pmatrix}$, then

a) p=0
b) p=1 or p=$-\frac{1}{3}$
c) p=-1 or p=$\frac{1}{3}$
d) p=1 or p=-1
e) none of these

38. If $\mathbf{P} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}4\\6\end{pmatrix}, \mathbf{R} = \begin{pmatrix}5\\7\end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix}a\\b\end{pmatrix}$ are the vertices of a parallelogram PQRS, find a and b.

39. The diagonals of a parallelogram PQRS are along the lines

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 4 \tag{1.6.39.1}$$

and

$$\begin{pmatrix}6 & -2\end{pmatrix}\mathbf{x} = 7 \tag{1.6.39.2}$$

. Then PQRS must be a

a) rectangle
b) square
c) cyclic quadrilateral
d) rhombus

40. If the vertices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ of a triangle PQR are rational points, which of the following points of the triangle PQR is (are) always rational point(s)?

a) centroid
b) incentre
c) circumcentre
d) orthocentre

41. Let L_1 be a straight line passing through the origin and L_2 be the straight line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1. \tag{1.6.41.1}$$

If the intercepts made by the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-1 & 3\end{pmatrix}\mathbf{x} = 0 \tag{1.6.41.2}$$

on L_1 and L_2 are equal, then which of the following equations can represent L_1?

a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 0$
d) $\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = 0$

42. For $a > b > c > 0$, the distance between $\begin{pmatrix}1\\1\end{pmatrix}$ and the point of intersection of the lines $\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} + c = 0$ and $\begin{pmatrix}b & a\end{pmatrix}\mathbf{x} + c = 0$ is less than $2\sqrt{2}$. Then

a) a+b-c>0
b) a-b+c<0
c) a-b+c>0
d) a+b-c<0

43. A straight line segment of length l, moves with its ends on two mutually perpendicular lines. Find the locus of the points which divides the

line segment in the ratio 1:2.

44. The area of triangle is 5. Two of its vertices are $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. The third vertex $\mathbf{C}$ lies on $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 3$. Find $\mathbf{C}$.
45. One side of a rectangle lies along the line
$$\begin{pmatrix} 4 & 7 \end{pmatrix} \mathbf{x} + 5 = 0. \tag{1.6.45.1}$$
Two of its vertices are $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the equations of the other three sides.
46. Two vertices of a triangle are $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$. If the orthocentre of the triangle is the origin, find the coordinates of the third point.
47. Find the equation of the line which bisects the obtuse angle between the lines
$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{1.6.47.1}$$
and
$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} - 2 = 0. \tag{1.6.47.2}$$
48. A straight line L is perpendicular to the line
$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = 1. \tag{1.6.48.1}$$
The area of the triangle formed by the line L and the coordinate axes is 5. Find the equation of the line L.
49. The end $\mathbf{A}, \mathbf{B}$ of a straight line segment of constant length c slide upon the fixed rectangular axis OX,OY respectively. If the rectangle OAPB be completed, then show that the locus of the foot of the perpendicular drawn from $\mathbf{P}$ to AB is $x^{2/3} + y^{2/3} = c^{2/3}$.
50. The vertices of a triangle are $\begin{pmatrix} at_1t_2 \\ a(t_1 + t_2) \end{pmatrix}, \begin{pmatrix} at_2t_3 \\ a(t_2 + t_3) \end{pmatrix}, \begin{pmatrix} at_3t_1 \\ a(t_3 + t_1) \end{pmatrix}$. Find the orthocentre of the triangle.
51. The coordinates of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $\begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ respectively, and $\mathbf{P}$ is any point $\mathbf{x}$. Show that the ratio of the area of the triangles $\triangle PBC$ and $\triangle ABC$ is $\frac{\left|\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 2\right|}{7}$
52. Two equal sides of an isosceles triangles are given by the equations
$$\begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{1.6.52.1}$$
and
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{1.6.52.2}$$
and its third side passes through the point $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$. Determine the equation of third side.
53. One of the diameters of the circle circumscribing the rectangle ABCD is
$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} = 7. \tag{1.6.53.1}$$
If A and B are the points $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ respectively, then find the area of the rectangle.
54. Two sides of a rhombus ABCD are parallel to the lines
$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{1.6.54.1}$$
and
$$\begin{pmatrix} -7 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{1.6.54.2}$$
. If the diagonals of the rhombus intersect at the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and vertex $\mathbf{A}$ is on the y axis. Find possible coordinates of $\mathbf{A}$.
55. Lines
$$L_1 \equiv \begin{pmatrix} a & b \end{pmatrix} \mathbf{x} + c = 0 \tag{1.6.55.1}$$
$$L_2 \equiv \begin{pmatrix} l & m \end{pmatrix} \mathbf{x} + n = 0 \tag{1.6.55.2}$$
intersect at the point $\mathbf{P}$ and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through $\mathbf{P}$ and makes the same angle θ with L_1.
56. Let ABC be a traingle with AB=AC. If $\mathbf{D}$ is the mid point of BC, $\mathbf{E}$ is the foot of the perpendicular drawn from $\mathbf{D}$ to AC and $\mathbf{F}$ the mid-point of DE, Prove that AF perpendicular to BE.
57. Straight lines
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5 \tag{1.6.57.1}$$
and
$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{1.6.57.2}$$
intersect at the point $\mathbf{A}$. Points $\mathbf{B}$ and $\mathbf{C}$ are chosen on these two lines such that AB=AC. Determine the possible equations of the lines

BC passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$

58. A line cuts the x-axis at $\mathbf{A} = \begin{pmatrix}7\\0\end{pmatrix}$ and the y-axis at $\mathbf{B} = \begin{pmatrix}0\\-5\end{pmatrix}$. A variable line PQ is draw perpendicular to AB cutting the x-axis in $\mathbf{P}$ and the y-axis in $\mathbf{Q}$. If AQ and BP intersect at $\mathbf{R}$, find the locus of $\mathbf{R}$.
59. Find the equation of the line passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ and making an intercept of length 2 units between the lines $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$=3 and $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$=5.
60. Show that all chords of the curve

$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-2 & 4\end{pmatrix}\mathbf{x} = 0 \quad (1.6.60.1)$$

which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point.

61. Determine all values of α for which the point $\begin{pmatrix}\alpha\\\alpha^2\end{pmatrix}$ lies inside the triangle formed by the lines

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 1 = 0 \quad (1.6.61.1)$$
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} - 3 = 0 \quad (1.6.61.2)$$
$$\begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} - 1 = 0 \quad (1.6.61.3)$$

62. The tangent at a point $\mathbf{P}_1$ (other than the origin) on the curve

$$y = x^3 \quad (1.6.62.1)$$

meets the curve again at $\mathbf{P}_2$. The tangent at $\mathbf{P}_2$ meets the curve at $\mathbf{P}_3$ and so on. Show that the abscissae of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \ldots \mathbf{P}_n$, form a G.P. Also find the ratio $\frac{\text{area}(\triangle P_1,P_2,P_3)}{\text{area}(\triangle P_2,P_3,P_4)}$.

63. A line through $\mathbf{A} = \begin{pmatrix}-5\\-4\end{pmatrix}$ meets the line $\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} + 2 = 0, \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}$ +4=0 and $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x}$ -5=0 at the points $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ respectively. If

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2, \quad (1.6.63.1)$$

find the equation of the line.

64. A rectangle PQRS has its side PQ parallel to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 0 \quad (1.6.64.1)$$

and vertices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{S}$ on the lines

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = a \quad (1.6.64.2)$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = b \quad (1.6.64.3)$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -b \quad (1.6.64.4)$$

respectively. Find the locus of vertex $\mathbf{R}$.

65. Using coordinate geometry, prove that the three altitudes of any triangle are concurrent.
66. For points $\mathbf{P} = \begin{pmatrix}x_1\\y_1\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$ of the coordinate plane, a new distance $d(\mathbf{P}, \mathbf{Q}) = |x_1 - x_2| + |y_1 - y_2|$. Let $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix}3\\2\end{pmatrix}$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from $\mathbf{O}$ and $\mathbf{A}$ consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.
67. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to the sides QR,RP,PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ to BC,CA,AB respectively are also concurrent.
68. Let a,b,c be real numbers with $a^2 + b^2 + c^2 = 1$. show that the equation

$$\begin{vmatrix}\begin{pmatrix}a & -b\end{pmatrix}\mathbf{x} - c & \begin{pmatrix}b & a\end{pmatrix}\mathbf{x} & \begin{pmatrix}c & 0\end{pmatrix}\mathbf{x} + a\\ \begin{pmatrix}b & a\end{pmatrix}\mathbf{x} & \begin{pmatrix}-a & b\end{pmatrix}\mathbf{x} - c & \begin{pmatrix}0 & c\end{pmatrix}\mathbf{x} + b\\ \begin{pmatrix}c & 0\end{pmatrix}\mathbf{x} + a & \begin{pmatrix}0 & c\end{pmatrix}\mathbf{x} + b & \begin{pmatrix}-a & -b\end{pmatrix}\mathbf{x} + c\end{vmatrix} = 0 \quad (1.6.68.1)$$

represents a straight line.

69. A straight line L through the origin meets the line and

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \quad (1.6.69.1)$$

at $\mathbf{P}$ and $\mathbf{Q}$ respectively. Through $\mathbf{P}$, straight lines L_1 and L_2 are drawn parallel to

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 5 \quad (1.6.69.2)$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 5 \quad (1.6.69.3)$$

respectively. Lines L_1 and L_2 intersect at that the locus of $\mathbf{R}$, as L varies, is a straight line.

70. A straight line L with negative slope passes through point $\begin{pmatrix}8\\2\end{pmatrix}$ and cuts the positive coordinates are **P** and **Q**. Find the absolute minimum value of OP varies, where **O** is origin.
71. The area of the triangle formed by the intersection of a line parallel to the x-axis and passing through $\mathbf{P} = \begin{pmatrix}h\\k\end{pmatrix}$ with the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.71.1}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.6.71.2}$$

is $4h^2$. Find the locus of the point.
72. Lines

$$L_1 : \begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.72.1}$$

$$L_2 : \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.72.2}$$

intersect the line

$$L_3 : \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + 2 = 0 \tag{1.6.72.3}$$

at **P** and **Q** respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at **R**.

Statement-1. The ratio PR:RQ equals $2\sqrt{2} : \sqrt{5}$

Statement-2. In any triangle, the bisector of an angle divides the triangle into two similar triangles.

a) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement-1
b) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement-1
c) Statement-1 is True,Statement False
d) Statement-1 is False,Statement True

73. For a point **P** in the plane, let $d_1(\mathbf{P})$ and $d_2(\mathbf{P})$ be the distance of the point **P** from the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.73.1}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.73.2}$$

respectively. Find the area of the region R consisting of all points **P** lying in the first quadrant of the plane and satisfying

$$2 \le d_1(\mathbf{P}) + d_2(\mathbf{P}) \le 4. \tag{1.6.73.3}$$

74. A triangle with vertices $\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ is

a) isosceles and right angled
b) isosceles but not right angled
c) right angled but not isosceles
d) neither right angled nor isosceles

75. Find the locus of mid points of the portion between the axis of

$$\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = 0 \tag{1.6.75.1}$$

where p is constant
76. If the pair of the lines

$$\mathbf{x}^T\begin{pmatrix}a & 2h\\0 & b\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x} + c = 0 \tag{1.6.76.1}$$

intersects the y-axis then

a) $2fgh = bg^2 + ch^2$
b) $bg^2 \neq ch^2$
c) $abc = 2fgh$
d) none of these

77. A pair of lines represented by

$$\mathbf{x}^T\begin{pmatrix}3a & 5\\0 & (a^2-2)\end{pmatrix}\mathbf{x} = 0 \tag{1.6.77.1}$$

are perpendicular to each other for

a) two values of a
b) $\forall$ a
c) for one value of a
d) for no values of a

78. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle $\alpha(0<\alpha<\pi/4)$ with the positive direction of the x-axis. Find the equation of its diagonal not passing through the origin.
79. If the pair of straight lines

$$\mathbf{x}^T\begin{pmatrix}1 & -2p\\0 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.79.1}$$

and

$$\mathbf{x}^T\begin{pmatrix}1 & -2q\\0 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.6.79.2}$$

be such that each pair bisects the angle between the other pair, then find the relation between p and q.
80. Find the locus of the centroid of the triangle whose vertices are $\begin{pmatrix}a\cos t\\a\sin t\end{pmatrix}, \begin{pmatrix}b\sin t\\-b\cos t\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$.
81. If x_1,x_2,x_3 and y_1,y_2,y_3 are both in G.P. with the same common ratio, then the common points

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

a) are verices of a triangle
b) lie on a straight line
c) lie on a ellipse
d) lie on a circle

82. If the equation of the locus of a point equidistant from the points $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is

$$\begin{pmatrix} a_1 - b_2 & a_1 - b_2 \end{pmatrix} \mathbf{x} + c = 0 \tag{1.6.82.1}$$

then find the value of c.

83. Let $\mathbf{A} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ be the vertices of a triangle ABC. If the centroid of the triangle moves on the line

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{1.6.83.1}$$

then find the locus of the vertex **C**.

84. Find the equation of the straight line passing through the point $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and making intercepts on the coordinate axis whose sum is -1.

85. If the sum of the slopes of the lines given by $\mathbf{x}^T \begin{pmatrix} 1 & -2c \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0$ is 4 times their product, then find c.

86. If one of the lines given by

$$\mathbf{x}^T \begin{pmatrix} 6 & -1 \\ 0 & 4c \end{pmatrix} \mathbf{x} = 0 \tag{1.6.86.1}$$

is

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.86.2}$$

then find c.

87. The line parallel to the x-axis and passing through the intersection of the lines

$$\begin{pmatrix} a & 2b \end{pmatrix} \mathbf{x} + 3b = 0 \tag{1.6.87.1}$$

$$\begin{pmatrix} b & -2a \end{pmatrix} \mathbf{x} - 3a = 0, \tag{1.6.87.2}$$

where $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

a) below the x-axis at a distance of 3/2 from it
b) below the x-axis at a distance of 2/3 from it
c) above the x-axis at a distance of 3/2 from it
d) above the x-axis at a distance of 2/3 from it

88. If a vertex of a triangle is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the mid points of two sides through this vertex are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, then find the centroid of the triangle.

89. A straight line through the point $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is such that its intercepts between the axes is bisected at **A**. Find its equation.

90. If $\begin{pmatrix} a \\ a^2 \end{pmatrix}$ falls inside the angle made by the lines

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = 0, \tag{1.6.90.1}$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.6.90.2}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} > 0, \tag{1.6.90.3}$$

then find the range of a.

91. Let $\mathbf{A} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be the vertices of a right angled triangle with AC has its hypotenuse. If the area of the triangle is 1 sq.unit, then find the range of k.

92. Let $\mathbf{P} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix}$ be three points. Find the equation of the bisector of the angle PQR.

93. If one of the lines of

$$\mathbf{x}^T \begin{pmatrix} -m & (1-m^2) \\ 0 & m \end{pmatrix} \mathbf{x} = 0 \tag{1.6.93.1}$$

is a bisector of the angle between the lines

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} = 0, \tag{1.6.93.2}$$

then find m.

94. The perpendicular bisector of the line segment joining $\mathbf{P} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} k \\ 3 \end{pmatrix}$ has y-intercept -4. Then a possible value of k is

a) 1
b) 2
c) -2
d) -4

95. Find the shortest distance between the line

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1.6.95.1}$$

and the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} = 0 \tag{1.6.95.2}$$

.

96. The lines
$$\begin{pmatrix}(p(p^2+1) & -1\end{pmatrix}\mathbf{x}+q=0 \quad (1.6.96.1)$$
$$\begin{pmatrix}(p^2+1)^2 & (p^2+1)\end{pmatrix}\mathbf{x}+2q=0 \quad (1.6.96.2)$$
are perpendicular to a common line for:
a) exactly one values of p
b) exactly two values of p
c) more than two values of p
d) no value of p

97. Three distinct points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are given in the two dimensional coordinates plane such that the ratio of the distance of any one of them from the point $\begin{pmatrix}1\\0\end{pmatrix}$ to the distance from the point $\begin{pmatrix}-1\\0\end{pmatrix}$ is equal to $\frac{1}{3}$. Find the circumcentre of the triangle ABC.

98. The line L given by
$$\begin{pmatrix}1/5 & 1/b\end{pmatrix}\mathbf{x}=1 \quad (1.6.98.1)$$
passes through the point $\begin{pmatrix}13\\32\end{pmatrix}$. The line K is parallel to L and has the equation
$$\begin{pmatrix}\frac{1}{c} & \frac{1}{3}\end{pmatrix}\mathbf{x}=1 \quad (1.6.98.2)$$
. Find the distance between L and K.

99. If the line
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}=k \quad (1.6.99.1)$$
passes through the point which divides the line segment joining the points$\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\end{pmatrix}$ in the ratio 3:2, then find k.

100. A ray of light along
$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x}=\sqrt{3} \quad (1.6.100.1)$$
gets reflected upon reaching the x-axis, then find the equation of the reflected ray.

101. Find the x coordinate of the incentre of the triangle that has the coordinates of the mid points of its sides as $\begin{pmatrix}0\\1\end{pmatrix}$,$\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$

102. Let PS be the median of the triangle with vertices $\mathbf{P}=\begin{pmatrix}2\\2\end{pmatrix}, \mathbf{Q}=\begin{pmatrix}6\\-1\end{pmatrix}$ and $\mathbf{R}=\begin{pmatrix}7\\3\end{pmatrix}$. Find the equation of the line passing through $\begin{pmatrix}1\\-1\end{pmatrix}$ and parallel to PS.

103. Let a,b,c and d be non zero numbers. If the point of intersection of the lines $\begin{pmatrix}4a & 2a\end{pmatrix}\mathbf{x}+c=0$ and $\begin{pmatrix}5b & 2b\end{pmatrix}\mathbf{x}+d=0$ lies in the fourth quadrant and is equidistant from the two axes, then
a) 3bc-2ad=0
b) 3bc+2ad=0
c) 2bc-3ad=0
d) 2bc+3ad=0

104. The number of points, having both coordinates as integers, that lie in the interior of the triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix}$,$\begin{pmatrix}0\\41\end{pmatrix}$ and$\begin{pmatrix}41\\0\end{pmatrix}$. is:
a) 820
b) 780
c) 901
d) 861

105. Two sides of a rhombus are along the lines, $\begin{pmatrix}1 & -1\end{pmatrix}$ $\mathbf{x}$+1=0 and $\begin{pmatrix}7 & -1\end{pmatrix}$$\mathbf{x}$-5=0. If its diagonals intersect at $\begin{pmatrix}-1\\-2\end{pmatrix}$ then which one of the following is a vertex of the rhombus?
a) $\begin{pmatrix}\frac{1}{3}\\\frac{8}{3}\end{pmatrix}$
b) $\begin{pmatrix}\frac{10}{3}\\\frac{7}{3}\end{pmatrix}$
c) $\begin{pmatrix}-3\\-9\end{pmatrix}$
d) $\begin{pmatrix}-3\\-8\end{pmatrix}$

106. A straight the through a fixed point $\begin{pmatrix}2\\3\end{pmatrix}$ intersects the coordinate axes at distinct points $\mathbf{P}$ and $\mathbf{Q}$. If $\mathbf{O}$ is the origin and the rectangle OPRQ is completed, then find the locus of $\mathbf{R}$.

107. Consider the set of all lines $\begin{pmatrix}p & q\end{pmatrix}$ $\mathbf{x}$+r=0 such that $3p+2q+4r=0$. Which one of the following statements is true?
a) The lines are concurrent at the point $\begin{pmatrix}\frac{3}{4}\\\frac{1}{2}\end{pmatrix}$
b) Each line passes through the origin
c) The lines are all parallel
d) The lines are not concurrent

108. Find the slope of a line passing through $\mathbf{P}=\begin{pmatrix}2\\3\end{pmatrix}$ and intersecting the line $\begin{pmatrix}1 & 1\end{pmatrix}$ $\mathbf{x}$=7 at a

distance of 4 units from $\mathbf{P}$.

2 The Circle

2.1 Definitions

1. The equation of a circle is

$$\|\mathbf{x} - \mathbf{c}\| = r \tag{2.1.1.1}$$

where $\mathbf{c}$ is the centre and r is the radius.

2. By expanding (2.1.1.1), the equation of a circle can also be expressed as

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.1.2.1}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} - r^2 = 0 \tag{2.1.2.2}$$

3. Find the equation of the *circumcircle* of $\triangle ABC$ in Fig. 2.1.4.
Solution: Let $\mathbf{O}$ be the centre and R the radius. From (2.1.2.2),

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = R^2 \tag{2.1.3.1}$$

$$\implies \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \tag{2.1.3.2}$$

which can be simplified to obtain

$$(\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad \text{and} \tag{2.1.3.3}$$

$$(\mathbf{A} - \mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{C}\|^2}{2} \tag{2.1.3.4}$$

Solving the two yields $\mathbf{O}$, which can then be used to obtain R.

4. Given $OD \perp BC$ as in Fig. 2.1.4. Show that

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{2.1.4.1}$$

Solution: From (2.1.1.1)

$$\|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = R^2 \tag{2.1.4.2}$$

$$\implies (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = (\mathbf{C} - \mathbf{O})^T (\mathbf{C} - \mathbf{O}) \tag{2.1.4.3}$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{O}\right) = 0 \tag{2.1.4.4}$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{D} - \mathbf{O}) = 0 \tag{2.1.4.5}$$

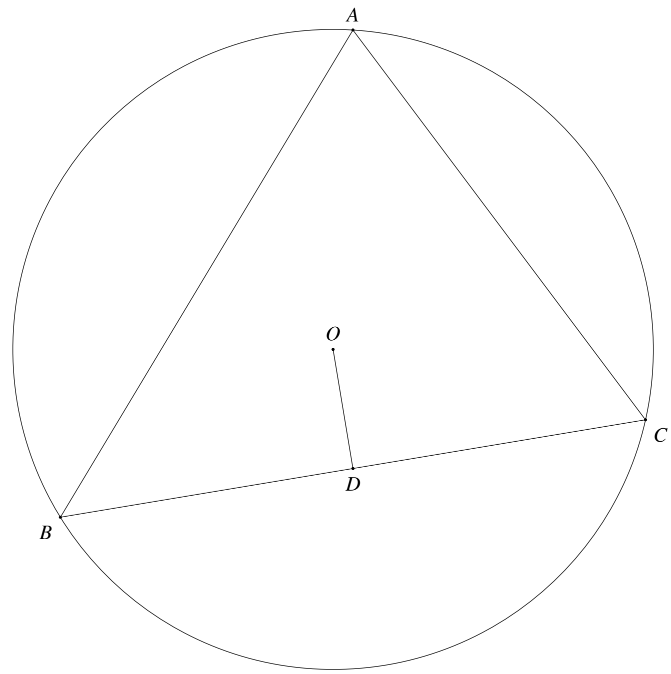

Fig. 2.1.4: Circumcircle.

Since D and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on BC, using (1.2.7.1),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \mathbf{B} + \lambda_1 (\mathbf{B} - \mathbf{C}) \tag{2.1.4.6}$$

$$\mathbf{D} = \mathbf{B} + \lambda_2 (\mathbf{B} - \mathbf{C}) \tag{2.1.4.7}$$

Multiplying (2.1.4.6) and (2.1.4.7) with $(\mathbf{B} - \mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{2.1.4.8}$$

5. Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.
6. The *incircle* with centre $\mathbf{I}$ and radius r in Fig.2.1.6 is inside $\triangle ABC$ and touches AB, BC and CA at $\mathbf{W}, \mathbf{U}$ and $\mathbf{V}$ respectively. AB, BC and CA are known as *tangents* to the circle.
7. Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \tag{2.1.7.1}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{2.1.7.2}$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{2.1.7.3}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the

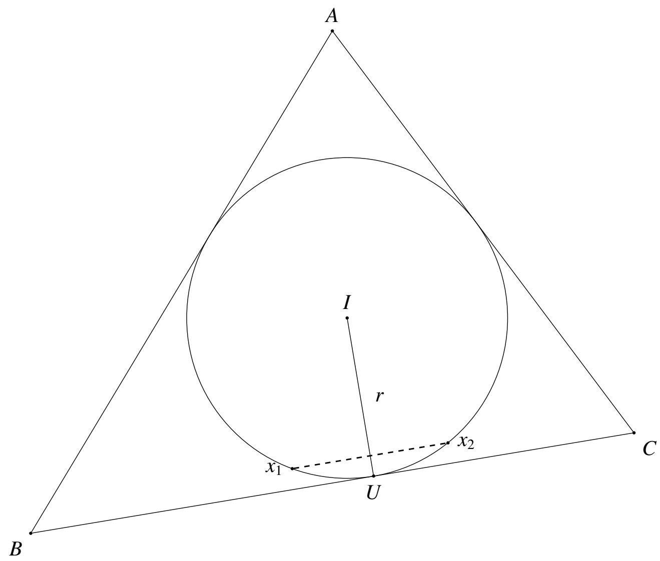

Fig. 2.1.6: Tangent and incircle.

above equation becomes

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{U}-\mathbf{I}) = 0 \implies OD \perp BC \tag{2.1.7.4}$$

8. Give an alternative proof for the above.
Solution: Let

$$\mathbf{B} = \mathbf{0} \tag{2.1.8.1}$$
$$\mathbf{U} = \lambda\mathbf{m} \tag{2.1.8.2}$$

Then

$$\|\mathbf{U}-\mathbf{I}\|^2 = r^2 \tag{2.1.8.3}$$
$$\implies \lambda^2\|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{I} + \|\mathbf{I}\|^2 = r^2 \tag{2.1.8.4}$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T\mathbf{I}}{\|\mathbf{m}\|^2} \tag{2.1.8.5}$$

Thus,

$$(\mathbf{U}-\mathbf{B})^T(\mathbf{U}-\mathbf{I}) = (\lambda\mathbf{m})^T(\lambda\mathbf{m}-\mathbf{I}) \tag{2.1.8.6}$$
$$= \lambda^2\|\mathbf{m}\|^2 - \lambda\mathbf{m}^T\mathbf{I} \tag{2.1.8.7}$$
$$= \mathbf{O} \text{ (from 2.1.8.5).} \tag{2.1.8.8}$$
$$\implies OD \perp BC \tag{2.1.8.9}$$

9. Find the equation of the tangent at $\mathbf{U}$.
Solution: The equation of the tangent is given by

$$(\mathbf{I}-\mathbf{U})^T(\mathbf{x}-\mathbf{U}) = 0 \tag{2.1.9.1}$$

10. The direction vector of *normal to the circle* in (2.1.2.2) at point $\mathbf{U}$ is

$$\mathbf{n} = \mathbf{U}-\mathbf{I} \tag{2.1.10.1}$$

11. Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \tag{2.1.11.1}$$
$$\implies \mathbf{n}^T(\mathbf{U}-\mathbf{B}) = 0 \tag{2.1.11.2}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{2.1.11.3}$$
$$\implies \mathbf{I}-\mathbf{U} = \lambda\mathbf{n} \tag{2.1.11.4}$$
$$\text{or} \quad r = \|\mathbf{I}-\mathbf{U}\| = |\lambda|\,\|\mathbf{n}\| \tag{2.1.11.5}$$

From (2.1.11.2) and (2.1.11.3)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{2.1.11.6}$$
$$\implies \mathbf{n}^T(\mathbf{I}-\mathbf{B}) = \lambda\|\mathbf{n}\|^2 \tag{2.1.11.7}$$
$$\implies r = |\lambda|\,\|\mathbf{n}\| = \frac{\left|\mathbf{n}^T(\mathbf{I}-\mathbf{B})\right|}{\|\mathbf{n}\|} \tag{2.1.11.8}$$

from (2.1.11.5). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{2.1.11.9}$$
$$r = \left|\mathbf{n}_1^T(\mathbf{I}-\mathbf{B})\right| \tag{2.1.11.10}$$

12. Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (2.1.11.10),

$$\left|\mathbf{n}_1^T(\mathbf{I}-\mathbf{B})\right| = \left|\mathbf{n}_2^T(\mathbf{I}-\mathbf{C})\right| = \left|\mathbf{n}_3^T(\mathbf{I}-\mathbf{A})\right| \tag{2.1.12.1}$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (2.1.12.1) can be expressed as

$$\mathbf{n}_1^T(\mathbf{I}-\mathbf{B}) = k_1\mathbf{n}_2^T(\mathbf{I}-\mathbf{C}) \tag{2.1.12.2}$$
$$\mathbf{n}_2^T(\mathbf{I}-\mathbf{C}) = k_2\mathbf{n}_3^T(\mathbf{I}-\mathbf{A}) \tag{2.1.12.3}$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3\end{pmatrix}^T\mathbf{I} = \begin{pmatrix}\mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A}\end{pmatrix} \tag{2.1.12.4}$$

13. Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$
14. Let $a = BC, b = CA$, $c = AB$, $x = BU = BW$, $y = CU = CV$, $z = AV = AW$. Find $\mathbf{U}$.

Solution: It is easy to verify that

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.1.14.1}$$

which can be used to obtain x and y. Using the section formula,

$$\mathbf{U} = \frac{x\mathbf{B} + \mathbf{A}}{x + y} \tag{2.1.14.2}$$

15. Show that the angle in a semi-circle is a right angle.

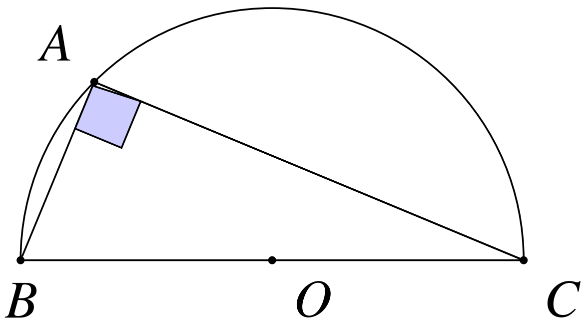

Fig. 2.1.15: Angle in a semi-circle.

Solution: Let

$$\mathbf{O} = 0 \tag{2.1.15.1}$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \tag{2.1.15.2}$$

$$\|\mathbf{B} - \mathbf{C}\|^2 = (2r)^2 \tag{2.1.15.3}$$

$$\mathbf{B} + \mathbf{C} = 0 \tag{2.1.15.4}$$

where r is the radius of the circle. Thus,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T(\mathbf{B} + \mathbf{C}) \tag{2.1.15.5}$$

From (2.1.15.4) and (2.1.15.2),

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 4r^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{2.1.15.6}$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

16. Show that $PA.PB = PC^2$, where PC is the tangent to the circle in Fig. 2.1.16.
Solution: Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda\|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda\mathbf{A}) \tag{2.1.16.1}$$

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \tag{2.1.16.2}$$

$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \tag{2.1.16.3}$$

$\because$

$$\|\mathbf{B} - \mathbf{O}\|^2 - \|\mathbf{A} - \mathbf{O}\|^2 = 0, \tag{2.1.16.4}$$

$$\left(\lambda^2 - 1\right)\|\mathbf{A}\|^2 - 2(\lambda - 1)\mathbf{A}^T\mathbf{O} = 0 \tag{2.1.16.5}$$

$$\implies PA.PB = \lambda\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{2.1.16.6}$$

after substituting from (2.1.16.1) and simplifying. From (2.1.16.3),

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \tag{2.1.16.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 = PC^2 \tag{2.1.16.8}$$

From (2.1.16.6) and (2.1.16.8),

$$PA.PB = PC^2 \tag{2.1.16.9}$$

17. In Fig. 2.1.17 show that $PA.PB = PC.PD$.
Solution: Let $\mathbf{P} = \mathbf{0}$. We then have the follow-

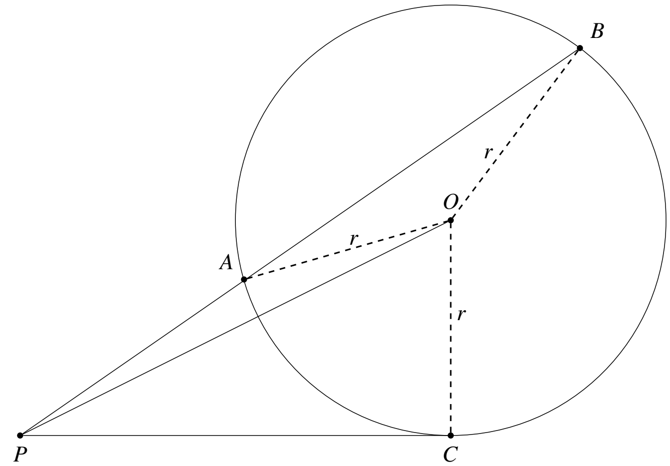

Fig. 2.1.16: $PA.PB = PC^2$.

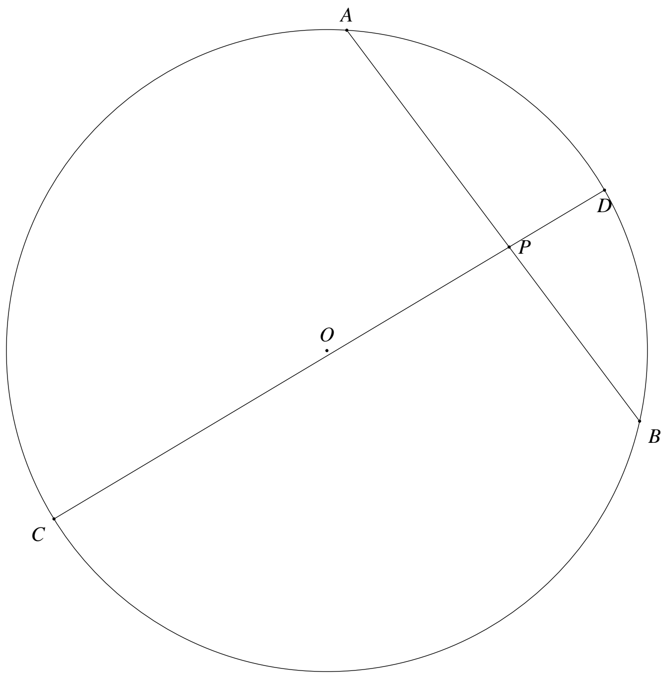

Fig. 2.1.17: Chords of a circle

ing equations

$$\mathbf{B} = k_1\mathbf{A}, k_1 = \frac{PB}{PA}$$
$$\mathbf{D} = k_2\mathbf{C}, k_2 = \frac{PD}{PC} \tag{2.1.17.1}$$

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2$$
$$= \|\mathbf{C}-\mathbf{O}\|^2 = \|\mathbf{D}-\mathbf{O}\|^2 = r^2 \tag{2.1.17.2}$$

where r is the radius of the circle and $\mathbf{O}$ is the centre. From (2.1.17.2),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 \tag{2.1.17.3}$$
$$\implies \|\mathbf{A}-\mathbf{O}\|^2 = \|k\mathbf{A}-\mathbf{O}\|^2 \quad \text{(from (2.1.17.1))} \tag{2.1.17.4}$$

which can be simplified to obtain

$$k_1 \|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{2.1.17.5}$$

Similarly,

$$k_2 \|\mathbf{C}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.6}$$

From (2.1.17.2), we also obtain

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 \tag{2.1.17.7}$$
$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.8}$$

after simplification. Using this result in (2.1.17.5) and (2.1.17.6),

$$k_1 \|\mathbf{A}\|^2 = k_2 \|\mathbf{C}\|^2 \tag{2.1.17.9}$$
$$\implies \|\mathbf{A}\| \, \|\mathbf{B}\| = \|\mathbf{C}\| \, \|\mathbf{D}\| \tag{2.1.17.10}$$

which completes the proof.

18. (Pole and Polar:) The polar of a point $\mathbf{x}$ with respect to the curve

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.18.1}$$

is the line

$$\mathbf{n}^T\mathbf{x} = c \tag{2.1.18.2}$$

where

$$\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ 1\end{pmatrix} \tag{2.1.18.3}$$

The pole of the line in (2.1.18.2) is obtained as $\frac{1}{x_3}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$, where

$$\begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} \tag{2.1.18.4}$$

19. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be conjugate points for (2.1.18.1) if $\mathbf{x}_2$ lies on the polar of $\mathbf{x}_1$ and vice-versa. A similar definition holds for conjugate lines as well.
20. Let $\mathbf{p}$ be a point of intersection of two circles with centres $\mathbf{c}_1$ and $\mathbf{c}_2$. The circles are said to be orthogonal if their tangents at $\mathbf{p}$ are perpendicular to each other. Show that if r_1 and r_2 are their respective radii,

$$\|\mathbf{c_1} - \mathbf{c}_2\|^2 = r_1^2 + r_2^2 \tag{2.1.20.1}$$

21. Show that the length of the tangent from a point $\mathbf{p}$ to the circle

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{2.1.21.1}$$

is

$$\mathbf{p}^T\mathbf{p} - 2\mathbf{c}^T\mathbf{p} + f \tag{2.1.21.2}$$

This length is also known as the *power* of the point $\mathbf{p}$ with respect to the circle.

22. The *radical axis* of the circles

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.22.1}$$
$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.2}$$

is the locus of the points from which lengths of the tangents to the circles are equal. From (2.1.25), this locus is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.3}$$

$$\implies 2\left(\mathbf{c}_1 - \mathbf{c}_2\right)^T \mathbf{x} + f_2 - f_1 = 0 \tag{2.1.22.4}$$

23. Show that the radical axis of the circles is perpendicular to the line joining their centres.
24. *Coaxal circles* have the same radical axis.
25. Obtain a family of coaxal circles from and find their *limit points*.
Solution: The family of circles is obtained as

$$\mathbf{x}^T\mathbf{x} - 2\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T \mathbf{x} + f_1 + \lambda f_2 = 0 \tag{2.1.25.1}$$

The limit points are the centres of those circles whose radii are 0. From (2.1.25.1) and (2.1.2.2), this results in

$$f_1 + \lambda f_2 = \left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T \left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right) \tag{2.1.25.2}$$

$$\implies \lambda^2 \left\|\mathbf{c}_2\right\|^2 + \lambda\left(2\mathbf{c}_1^T\mathbf{c}_2 - f_2\right) + \left\|\mathbf{c}_1\right\|^2 - f_1 = 0 \tag{2.1.25.3}$$

Solving for λ, the limit points are given by

$$\mathbf{c}_1 + \lambda\mathbf{c}_2 \tag{2.1.25.4}$$

2.2 *Programming*

1. Find the circumcentre $\mathbf{O}$ and radius R of $\triangle ABC$ in Fig. 1.3.2
Solution: $\mathbf{O}$ can be obtained from The following code computes $\mathbf{O}$ using (2.1.3.3) and (2.1.3.4)

```
codes/2d/circumcentre.py
```

2. Plot the circumcircle of $\triangle ABC$.
Solution: The following code plots Fig. 2.2.2

```
codes/2d/circumcircle.py
```

3. Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.
4. Compute $\mathbf{I}$ and r.
Solution: The following code uses (2.1.12.4) and (2.1.11.10) to compute $\mathbf{I}$ and r respectively.

```
codes/2d/incentre.py
```

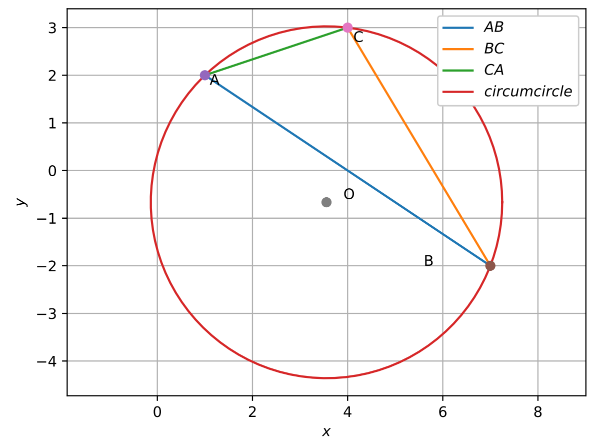

Fig. 2.2.2

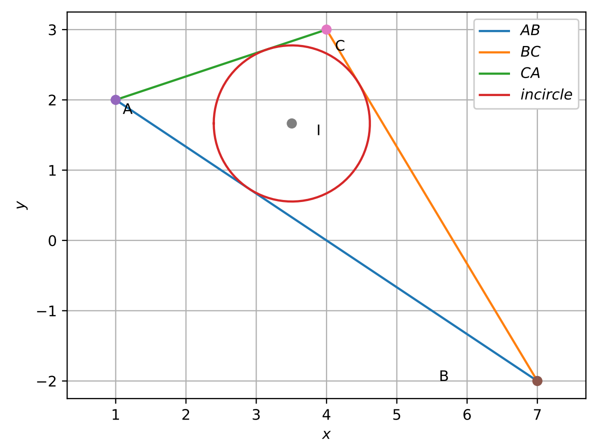

Fig. 2.2.5

5. Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 2.2.5

```
codes/2d/incircle.py
```

2.3 *Example*

1. Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{2.3.1.1}$$

Solution: let $\mathbf{c}$ be the centre of the circle. Then

$$\left\|\mathbf{x} - \mathbf{c}\right\|^2 = r^2 \tag{2.3.1.2}$$

$$\implies \left(\mathbf{x} - \mathbf{c}\right)^T \left(\mathbf{x} - \mathbf{c}\right) = r^2 \tag{2.3.1.3}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} = r^2 - \mathbf{c}^T\mathbf{c} \tag{2.3.1.4}$$

Comparing with (2.3.1.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.3.1.5}$$

$$r^2 - \mathbf{c}^T\mathbf{c} = 1 \implies r = \sqrt{2} \tag{2.3.1.6}$$

2. Find the tangent to the circle C_1 at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Solution: From (3.1.6), the tangent T is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right]\mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{2.3.2.1}$$

$$\implies T : \mathbf{n}^T\mathbf{x} = 3 \tag{2.3.2.2}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3.2.3}$$

3. The tangent T in (2.3.2.2) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{2.3.3.1}$$

Find $\mathbf{A} + \mathbf{B}$.
Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{C}) = r^2 \tag{2.3.3.2}$$

$$(\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = r^2 \tag{2.3.3.3}$$

Subtracting (2.3.3.3) from (2.3.3.2),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.4}$$

$$\implies (\mathbf{A} + \mathbf{B})^T(\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T(\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.5}$$

$\because \mathbf{A}, \mathbf{B}$ lie on T, from (2.3.2.2),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{2.3.3.6}$$

$$\implies \mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0, \tag{2.3.3.7}$$

From (2.3.3.5) and (2.3.3.7)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{2.3.3.8}$$

$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{2.3.3.9}$$

$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{2.3.3.10}$$

$$\implies k = 2 \tag{2.3.3.11}$$

using (2.3.3.6). Substituting in (2.3.3.8)

$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \tag{2.3.3.12}$$

4. If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.
Solution: From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.4.1}$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \tag{2.3.4.2}$$

$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \tag{2.3.4.3}$$

using (2.3.3.12) and simplifying.

5. Show that

$$(\mathbf{A} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = 8 - r^2 \tag{2.3.5.1}$$

Solution:

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.5.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) = 4^2 \tag{2.3.5.3}$$

From (2.3.5.3),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T\ [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \tag{2.3.5.4}$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2(\mathbf{A} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = 4^2 \tag{2.3.5.5}$$

Upon substituting from (2.3.3.3) and (2.3.3.2) and simplifying, (2.3.5.1) is obtained.

6. Find r.
Solution: (2.3.5.1) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T(\mathbf{A} + \mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{2.3.6.1}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T(\mathbf{A} + \mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \tag{2.3.6.2}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T(2\mathbf{n} + \mathbf{C}) = r^2 \tag{2.3.6.3}$$

$$\implies r = \sqrt{6}. \tag{2.3.6.4}$$

7. Summarize all the above computations through a Python script and plot the tangent and circle.
Solution: The following code generates Fig. 2.3.7.

```
wget
```

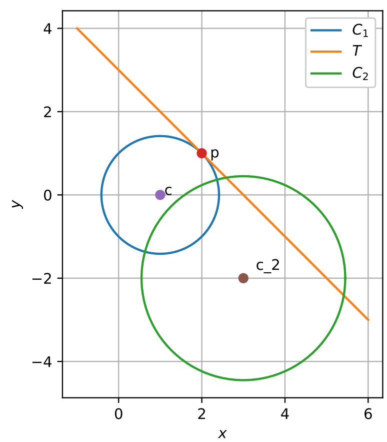

Fig. 2.3.7

codes/2d/circ.py

2.4 *Lagrange Multipliers*

1. Find
$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 = r^2 \tag{2.4.1.1}$$
$$\text{s.t.} \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{2.4.1.2}$$
by plotting the circles $f(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.
Solution: The following code plots Fig. 2.4.1

codes/optimization/2.1.py

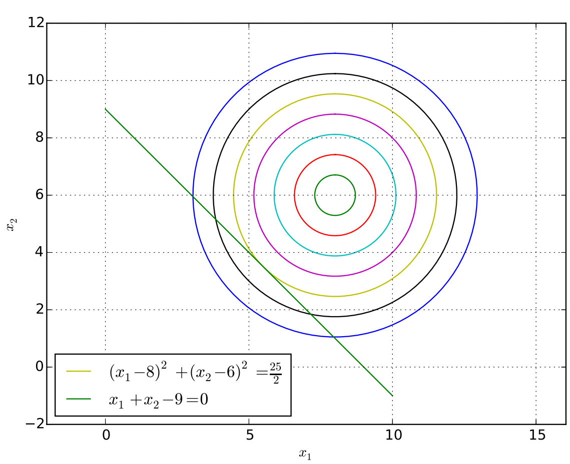

Fig. 2.4.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$
.

2. Show that
$$\min r = \frac{5}{\sqrt{2}} \tag{2.4.2.1}$$
3. Show that
$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.4.3.1}$$
where
$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{2.4.3.2}$$
4. Show that
$$\nabla f(\mathbf{x}) = 2\left\{ \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\} \tag{2.4.4.1}$$
is the direction vector of the normal at $\mathbf{x}$.
5. From Fig. 2.4.1, show that
$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{2.4.5.1}$$
where $\mathbf{p}$ is the point of contact.
6. Use (2.4.5.1) and $\mathbf{g(p)} = 0$ from (2.4.1.2) to obtain $\mathbf{p}$.
7. Define
$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{2.4.7.1}$$
and show that $\mathbf{p}$ can also be obtained by solving the equations
$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{2.4.7.2}$$
What is the sign of λ? L is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.
Solution:

codes/optimization/2.3.py

2.5 *Inequality Constraints*

1. Modify the code in problem 2.4.1 to find a graphical solution for minimising
$$f(\mathbf{x}) \tag{2.5.1.1}$$
with constraint
$$g(\mathbf{x}) \geq 0 \tag{2.5.1.2}$$
Solution: This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. 2.5.1 . It is clear that this radius is 0.

codes/optimization/2.4.py

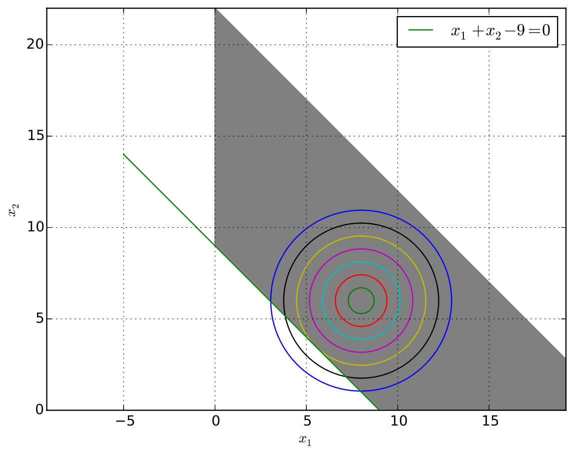

Fig. 2.5.1: Smallest circle in the shaded region is a point.

2. Now use the method of Lagrange multipliers to solve problem 2.5.1 and compare with the graphical solution. Comment.
Solution: Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem 2.5.1, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.
3. Repeat problem 2.5.2 by keeping $\lambda = 0$. Comment.
Solution: Keeping $\lambda = 0$ results in $\mathbf{x} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, which is the correct solution. The minimum value of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.
4. Find a graphical solution for minimising
$$f(\mathbf{x}) \tag{2.5.4.1}$$
with constraint
$$g(\mathbf{x}) \le 0 \tag{2.5.4.2}$$
Summarize your observations.
Solution: In Fig. 2.5.4, the shaded region represents the constraint. Thus, the solution is the same as the one in problem 2.5.1. This implies that the method of Lagrange multipliers can be used to solve the optimization problem with this inequality constraint as well. Table 2.5.4 summarizes the conditions for this based on the observations so far.

```
codes/optimization/2.7.py
```

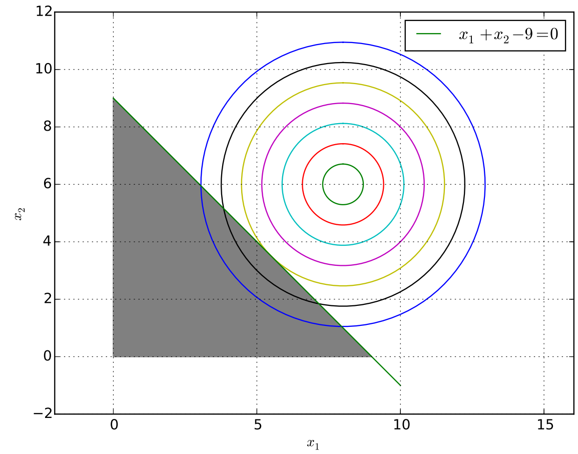

Fig. 2.5.4: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

TABLE 2.5.4: Summary of conditions.

Cost	**Constraint**	λ
$f(\mathbf{x})$	$g(\mathbf{x}) = 0$	< 0
	$g(\mathbf{x}) \ge 0$	0
	$g(\mathbf{x}) \le 0$	< 0

5. Find a graphical solution for
$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 \tag{2.5.5.1}$$
with constraint
$$g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 = 0 \tag{2.5.5.2}$$
Solution:

```
codes/optimization/2.8.py
```

6. Repeat problem 2.5.5 using the method of Lagrange mutipliers. What is the sign of λ?
Solution: Using the following python script, λ is positive and the minimum value of f is 8.

```
codes/optimization/2.9.py
```

7. Solve
$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{2.5.7.1}$$
with constraint
$$g(\mathbf{x}) \ge 0 \tag{2.5.7.2}$$

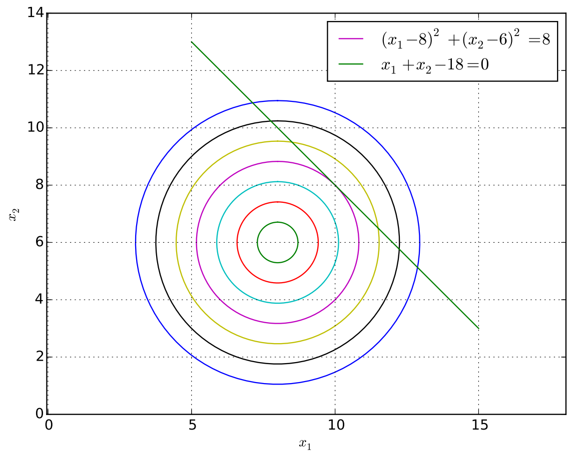

Fig. 2.5.5: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

Solution: Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem 2.5.5.

8. Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \tag{2.5.8.1}$$

Solution: Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 9 \leq 0, \lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \tag{2.5.8.2}$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \tag{2.5.8.3}$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \tag{2.5.8.4}$$

else, $\lambda = 0$.

9. Solve

$$\min_{\mathbf{x}} \quad x_1 + x_2 \tag{2.5.9.1}$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \tag{2.5.9.2}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solution:

Graphical solution:

```
codes/optimization/2.15.py
```

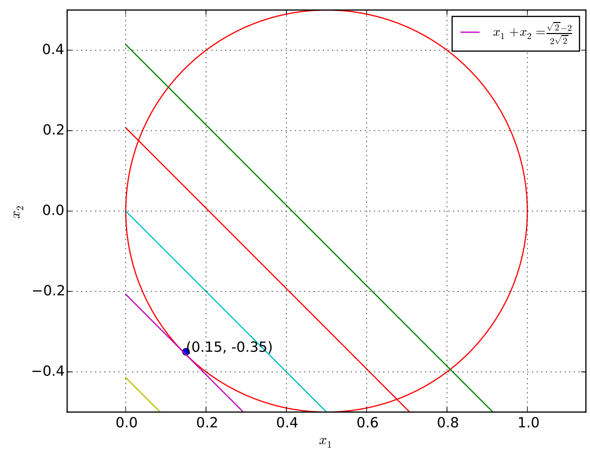

Fig. 2.5.9: Optimal solution is the lower tangent to the circle

2.6 *Solved Problems*

1. A circle passes through the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix} -1 & 4 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.6.1.1}$$

find its radius.
Solution: Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.6.1.2}$$

The direction vector of AB is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{2.6.1.3}$$

$$\because OC \perp AB,$$
$$OC : \mathbf{m}^T(\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 7 \tag{2.6.1.4}$$

Thus, $\mathbf{O}$ is the intersection of (2.6.1.1) and (2.6.1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{2.6.1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.6.1.6}$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \tag{2.6.1.7}$$

2. If a circle C_1, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix}\mathbf{x} = 4 \tag{2.6.2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.
Solution: From (2.6.2.1), the centre of C_2 is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.6.2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T\mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.6.2.3}$$

Since the radius of C_1 is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$
$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$
$$\implies \mathbf{O}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.6.2.4}$$

The intercepts of C_1 on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda\mathbf{m} \tag{2.6.2.5}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6.2.6}$$

Susbtituting in the equation for C_1,

$$\|\lambda\mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \tag{2.6.2.7}$$

which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$
$$\implies \lambda^2 - 10\lambda + 20 = 0 \tag{2.6.2.8}$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \tag{2.6.2.9}$$

after substituting from (2.6.2.6) and (2.6.2.4).

3. A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{2.6.3.1}$$

cuts the circle

$$C : \mathbf{x}^T\mathbf{x} = 9 \tag{2.6.3.2}$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$. Draw PAB for any two points $\mathbf{A}, \mathbf{B}$ on the circle.
Solution: Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line PAB can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{2.6.3.3}$$

for $\|\mathbf{m}\| = 1$. The intersection of L and C yields

$$(\mathbf{P} + \lambda\mathbf{m})^T (\mathbf{P} + \lambda\mathbf{m}) = 9$$
$$\implies \lambda^2 + 2\lambda\mathbf{m}^T\mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \tag{2.6.3.4}$$

The product of the roots in (2.6.3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \tag{2.6.3.5}$$

4. Find the equation of the circle C_2, which is the mirror image of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{2.6.4.1}$$

in the line

$$L : \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3. \tag{2.6.4.2}$$

Solution: From (2.6.4.1), circle C_1 has centre at

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6.4.3}$$

and radius

$$r_1 = \mathbf{O}_1^T\mathbf{O}_1 = 1 \tag{2.6.4.4}$$

The centre of C_2 is the reflection of $\mathbf{O}_1$ about L and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{O}_1 + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.6.4.5}$$

where the relevant parameters are obtained from (2.6.4.2) as

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 & -1 \end{pmatrix}, c = 3. \tag{2.6.4.6}$$

Substituting the above in (2.6.4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}}{4}\mathbf{O}_1 + c\frac{\mathbf{n}}{2}$$

$$\implies \mathbf{O}_2 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.6.4.7}$$

Thus

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix}3\\4\end{pmatrix}\right\| = 1 \tag{2.6.4.8}$$

5. One of the diameters of the circle, given by

$$C : \mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 12 \tag{2.6.5.1}$$

is a chord of a circle S, whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix}-3\\2\end{pmatrix}. \tag{2.6.5.2}$$

Find the radius of S.
Solution: From (2.6.5.1), the centre of C is

$$\mathbf{O}_1 = \begin{pmatrix}2\\-3\end{pmatrix} \tag{2.6.5.3}$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T\mathbf{O}_1 - 12} = 5 \tag{2.6.5.4}$$

From (2.6.5.3) and (2.6.5.2),

$$O_1O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$

$$\implies r_2 = \sqrt{O_1O_2^2 - r_1^2} = 5 \tag{2.6.5.5}$$

6. A circle C passes through

$$\mathbf{P} = \begin{pmatrix}-2\\4\end{pmatrix} \tag{2.6.6.1}$$

and touches the y-axis at

$$\mathbf{Q} = \begin{pmatrix}0\\2\end{pmatrix}. \tag{2.6.6.2}$$

Which one of the following equations can represent a diameter of this circle?

(i) $\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 6$ (iii) $\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 3$

(ii) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} + 10 = 0$(iv) $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} + 4 = 0$

Solution: Let **O** be the centre of C. Then the equation of the normal, OQ is

$$\begin{pmatrix}0 & 1\end{pmatrix}(\mathbf{O} - \mathbf{Q}) = 0$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{O} = 2 \tag{2.6.6.3}$$

Also,

$$\|\mathbf{O} - \mathbf{P}\|^2 = \|\mathbf{O} - \mathbf{Q}\|^2$$

$$\implies 2(\mathbf{P} - \mathbf{Q})^T\mathbf{O} = \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2$$

$$\text{or, } \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{O} = -4 \tag{2.6.6.4}$$

(2.6.6.3) and (2.6.6.4) result in the matrix equation

$$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\mathbf{O} = \begin{pmatrix}-4\\2\end{pmatrix} \tag{2.6.6.5}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & -4\\0 & 1 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -2\\0 & 1 & 2\end{pmatrix} \implies \mathbf{O} = \begin{pmatrix}-2\\2\end{pmatrix} \tag{2.6.6.6}$$

Hence, option ii) is correct.

7. Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix}1\\-1\end{pmatrix}, \tag{2.6.7.1}$$

whose centre **O** is the point of intersection of the straight lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.6.7.2}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{2.6.7.3}$$

Solution: From (2.6.7.2) and (2.6.7.3), we obtain the matrix equation

$$\begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}\mathbf{O} = \begin{pmatrix}3\\1\end{pmatrix} \tag{2.6.7.4}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & 1\\2 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 1\\0 & 3 & 1\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}3 & 0 & 4\\0 & 3 & 1\end{pmatrix} \implies \mathbf{O} = \frac{1}{3}\begin{pmatrix}4\\1\end{pmatrix} \tag{2.6.7.5}$$

Thus, the equation of the desired tangent is

$$(\mathbf{O} - \mathbf{P})^T(\mathbf{x} - \mathbf{P}) = 0$$

$$\implies \begin{pmatrix}1 & 4\end{pmatrix}\mathbf{x} = -3 \tag{2.6.7.6}$$

8. The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.1}$$

intersects the circle

$$\Omega : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{2.6.8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{R} = -\frac{3}{5} \tag{2.6.8.3}$$

Find m.

Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.4}$$

The intersection of (2.6.8.1) and (2.6.8.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{2.6.8.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T (\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \tag{2.6.8.6}$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1 \mathbf{m} \tag{2.6.8.7}$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2 \mathbf{m} \tag{2.6.8.8}$$

$$\implies \frac{\mathbf{P} + \mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2}\mathbf{m} \tag{2.6.8.9}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P} + \mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{m} \tag{2.6.8.10}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} - \frac{\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \tag{2.6.8.11}$$

using the sum of roots in (2.6.8.6). From (2.6.8.3) and (2.6.8.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5}\left(1 + m^2\right) \tag{2.6.8.12}$$

$$\implies m^2 - 5m + 6 = 0 \tag{2.6.8.13}$$

$$\implies m = 2 \text{ or } 3 \tag{2.6.8.14}$$

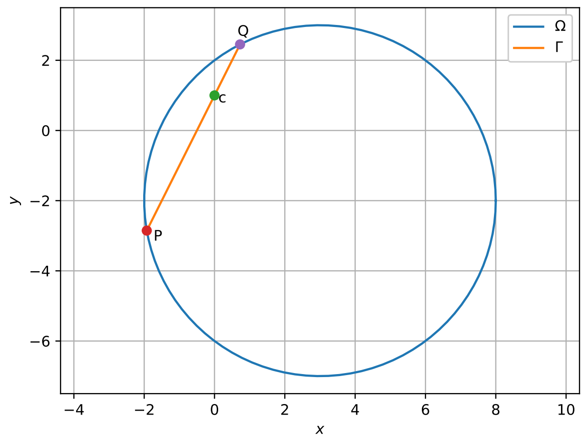

Fig. 2.6.8

From (2.6.8.6),

$$\lambda = \frac{-\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{(\mathbf{m}^T (\mathbf{c} - \mathbf{O}))^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{2.6.8.15}$$

Fig. 2.6.8 summarizes the solution for $m = 2$.

3 Conics

3.1 Definitions

1. From (1.2.19.2), the equation of a conic section is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3.1.1.1}$$

for

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} \neq 0 \tag{3.1.1.2}$$

2. Show that

$$\frac{d\left(\mathbf{u}^T \mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \tag{3.1.2.1}$$

3. Show that

$$\frac{d\left(\mathbf{x}^T \mathbf{V} \mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T \mathbf{x} \tag{3.1.3.1}$$

4. Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \tag{3.1.4.1}$$

5. Find the *normal* vector to the curve in (1.2.19.2) at point $\mathbf{p}$.
Solution: Differentiating (1.2.19.2) with respect to x_1,

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1}+\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1}=0 \quad (3.1.5.1)$$

$$\implies 2\mathbf{x}^T\mathbf{V}\mathbf{m}+2\mathbf{u}^T\mathbf{m}=0 \because \left(\frac{d\mathbf{x}}{dx_1}=\mathbf{m}\right) \quad (3.1.5.2)$$

Substituting $\mathbf{x}=\mathbf{p}$ and simplifying

$$(\mathbf{V}\mathbf{p}+\mathbf{u})^T\mathbf{m}=0 \quad (3.1.5.3)$$

$$\implies \mathbf{n}=\mathbf{V}\mathbf{p}+\mathbf{u} \quad (3.1.5.4)$$

6. The *tangent* to the curve at $\mathbf{p}$ is given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{p})=0 \quad (3.1.6)$$

This results in

$$\left(\mathbf{p}^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{x}+\mathbf{p}^T\mathbf{u}+f=0 \quad (3.1.6)$$

7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then

$$\mathbf{x}=\mathbf{P}\mathbf{y}+\mathbf{c}. \quad (3.1.7)$$

is known as an *affine* transformation.
8. Classify the various conic sections based on (1.2.19.2).
Solution:

Curve	Property
Circle	$V=kI$
Parabola	$\det(V)=0$
Ellipse	$\det(V)>0$
Hyperbola	$\det(V)<0$

TABLE 3.1.8

3.2 Parabola

1. Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola

$$\mathbf{x}^T\begin{pmatrix}1&0\\0&0\end{pmatrix}\mathbf{x}+\begin{pmatrix}0&-1\end{pmatrix}\mathbf{x}+6=0 \quad (3.2.1.1)$$

Solution: Substituting

$$\mathbf{p}=\begin{pmatrix}1\\7\end{pmatrix}, V=\begin{pmatrix}1&0\\0&0\end{pmatrix}, \mathbf{u}=\frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \quad (3.2.1.2)$$

in (3.1.6), the desired equation is

$$\left[\begin{pmatrix}1&7\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0&-1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1&7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}+6=0 \quad (3.2.1.3)$$

resulting in

$$\begin{pmatrix}2&-1\end{pmatrix}\mathbf{x}=-5 \quad (3.2.1.4)$$

2. The line in (3.2.1.4) touches the circle

$$\mathbf{x}^T\mathbf{x}+4\begin{pmatrix}4&3\end{pmatrix}\mathbf{x}+c=0 \quad (3.2.2.1)$$

Find c.
Solution: Comparing (1.2.19.2) and (3.2.2.1),

$$V=I,\quad \mathbf{u}=2\begin{pmatrix}4\\3\end{pmatrix} \quad (3.2.2.2)$$

Comparing (3.1.6) and (3.2.1.4),

$$\mathbf{p}+2\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix} \quad (3.2.2.3)$$

$$\implies \mathbf{p}=-\begin{pmatrix}6\\7\end{pmatrix} \quad (3.2.2.4)$$

and

$$c+\mathbf{p}^T\mathbf{u}=5 \quad (3.2.2.5)$$

$$\implies c=5+2\begin{pmatrix}6&7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \quad (3.2.2.6)$$

$$=95 \quad (3.2.2.7)$$

3. Summarize all the above computations through a Python script and plot the parabola, tangent and circle.
Solution: The following code generates Fig. 3.2.3.

```
wget
codes/2d/parab.py
```

3.3 Affine Transformation

1. In general, Fig. 3.2.3 was generated using an *affine transformation*.
2. Express

$$y_2=y_1^2 \quad (3.3.2.1)$$

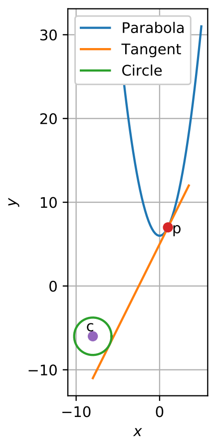

Fig. 3.2.3

as a matrix equation.
Solution: (3.3.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{g}^T\mathbf{y} = 0 \tag{3.3.2.2}$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.2.3}$$

3. Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0, \tag{3.3.3.1}$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \tag{3.3.3.2}$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.3.3.3}$$

(3.3.3.3) is known as an affine transformation. Show that

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P}$$
$$\mathbf{g} = \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) \tag{3.3.3.4}$$
$$F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0$$

Solution: Substituting (3.3.3.3) in (3.3.3.1),

$$\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right)^T\mathbf{V}\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + 2\mathbf{u}^T\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + F = 0, \tag{3.3.3.5}$$

which can be expressed as

$$\implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right)^T\mathbf{P}\mathbf{y} + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \tag{3.3.3.6}$$

Comparing (3.3.3.6) with (3.3.2.2) (3.3.3.4) is obtained.

4. Show that there exists a $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{3.3.4.1}$$

Find $\mathbf{P}$ using

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.3.4.2}$$

5. Find $\mathbf{c}$ from (3.3.3.4).
Solution:

$$\because \mathbf{g} = \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right), \tag{3.3.5.1}$$
$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{3.3.5.2}$$
$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\left(\mathbf{P}\mathbf{g} - \mathbf{u}\right) = -F - 2\mathbf{u}^T\mathbf{c} \tag{3.3.5.3}$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ \left(\mathbf{P}\mathbf{g} + \mathbf{u}\right)^T \end{pmatrix}\mathbf{c} = \begin{pmatrix} \mathbf{P}\mathbf{g} - \mathbf{u} \\ -F \end{pmatrix} \tag{3.3.5.4}$$

for computing $\mathbf{c}$.

3.4 Ellipse: Eigenvalues and Eigenvectors

1. Express the following equation in the form given in (1.2.19.2)

$$E : 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{3.4.1.1}$$

Solution: (3.4.1.1) can be expressed as

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{3.4.1.2}$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \tag{3.4.1.3}$$

2. Using the affine transformation in (3.3.3.3), show that (3.4.1.2) can be expressed as

$$\mathbf{y}^T D\mathbf{y} = 1 \tag{3.4.2.1}$$

where

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.2.2}$$
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.2.3}$$

for

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I} \tag{3.4.2.4}$$

3. Find $\mathbf{c}$

Solution:

$$\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.3.1}$$

4. If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.4.4.1}$$

$$P = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{3.4.4.2}$$

show that

$$V\mathbf{z} = \lambda\mathbf{z} \tag{3.4.4.3}$$

where $\lambda \in \{\lambda_1, \lambda_2\}, \mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$.

5. Find λ.
Solution: λ is obtained by solving the following equation.

$$|\lambda I - V| = 0 \tag{3.4.5.1}$$

$$\implies \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \tag{3.4.5.2}$$

$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{3.4.5.3}$$

$$\implies \lambda = 2, 8 \tag{3.4.5.4}$$

6. Sketch 3.4.2.1.
7. Find $\mathbf{P}_1$ and $\mathbf{P}_2$.
Solution: From (3.4.4.3)

$$V\mathbf{P}_1 = \lambda_1\mathbf{P}_1 \tag{3.4.7.1}$$

$$\implies (V - \lambda I)\,\mathbf{y} = 0 \tag{3.4.7.2}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.7.3}$$

$$\text{or, } \mathbf{P}_1 = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.4.7.4}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{P}_2 = 0 \tag{3.4.7.5}$$

$$\text{or, } \mathbf{P}_2 = k_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.4.7.6}$$

8. Find $\mathbf{P}$.
Solution: From (3.4.2.4) and (3.4.4.2),

$$k_1 = \frac{1}{\left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.1}$$

$$k_2 = \frac{1}{\left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.2}$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.4.8.3}$$

9. Find the equation of the major axis for E.
Solution: The major axis for (3.4.2.1) is the line

$$\mathbf{y} = \lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.4.9.1}$$

Using the affine transformation in (3.3.3.3)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.9.2}$$

$$\implies \mathbf{x} - \mathbf{c} = \lambda_1\mathbf{P}_1 \tag{3.4.9.3}$$

$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.9.4}$$

$$= -3 \tag{3.4.9.5}$$

since

$$P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.9.6}$$

which is the major axis of the ellipse E.

10. Find the minor axis of E.
11. Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{3.4.11.1}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

12. Summarize all the above computations through a Python script and plot the ellipses in (3.4.1.1) and (3.4.2.1).
Solution: The following script plots Fig. 3.4.12 using the principles of an affine transformation.

```
codes/2d/ellipse.py
```

3.5 Hyperbola

1. Tangents are drawn to the hyperbola

$$\mathbf{x}^T V\mathbf{x} = 36 \tag{3.5.1.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5.1.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{3.5.1.3}$$

find the equation of PQ.
Solution: The equations of the two tangents

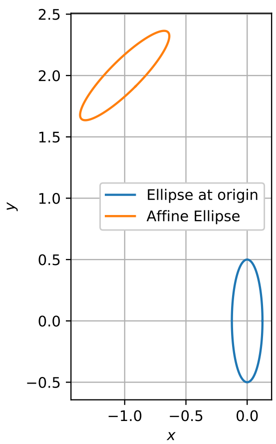

Fig. 3.4.12

are obtained using (3.1.6) as

$$\mathbf{P}^T V\mathbf{x} = 36 \tag{3.5.1.4}$$

$$\mathbf{Q}^T V\mathbf{x} = 36. \tag{3.5.1.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.6}$$

$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \tag{3.5.1.8}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{3.5.1.9}$$

which is the equation of PQ.

2. In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{3.5.2.1}$$

using (1.2.9.2) and (3.5.1.9), the equation of TD is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \tag{3.5.2.2}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.2.3}$$

3. Find D.
Solution: From (3.5.1.9) and (3.5.2.3),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.1}$$

$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.2}$$

4. Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \tag{3.5.4.1}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.4.2}$$

5. Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \tag{3.5.5.1}$$

can be expressed as

$$\lambda^2 \mathbf{m}^T V \mathbf{m} + 2\lambda \mathbf{m}^T V \mathbf{D} + \mathbf{D}^T V \mathbf{D} + F = 0 \tag{3.5.5.2}$$

6. Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (3.5.4.1) and (3.5.1.1) (3.5.5.2) is obtained. Substituting from (3.5.4.2), (3.5.1.2) and (3.5.3.2)

$$\mathbf{m}^T V \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \tag{3.5.6.1}$$

$$\mathbf{m}^T V \mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \tag{3.5.6.2}$$

$$\mathbf{D}^T V \mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \tag{3.5.6.3}$$

Substituting in (3.5.5.2)

$$4\lambda^2 - 144 = 36 \tag{3.5.6.4}$$

$$\implies \lambda = \pm 3\sqrt{5} \tag{3.5.6.5}$$

Substituting in (3.5.4.1),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3 \begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \tag{3.5.6.6}$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3 \begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \tag{3.5.6.7}$$

7. Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{3.5.7.1}$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{3.5.7.2}$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \tag{3.5.7.3}$$

8. Repeat the previous exercise using determinants.
9. Summarize all the above computations through a Python script and plot the hyperbola.

3.6 Karush-Kuhn-Tucker (KKT) Conditions

1. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \tag{3.6.1.1}$$

with constraints

$$g_1(\mathbf{x}) = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{3.6.1.2}$$

$$g_2(\mathbf{x}) = 15 - \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} \geq 0 \tag{3.6.1.3}$$

Solution: Considering the Lagrangian

$$\nabla L(\mathbf{x}, \lambda, \mu) = 0 \tag{3.6.1.4}$$

resulting in the matrix equation

$$\implies \begin{pmatrix} 8 & 0 & 3 & 2 \\ 0 & 4 & 1 & 4 \\ 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 15 \end{pmatrix} \tag{3.6.1.5}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \\ -3.12 \\ -2.12 \end{pmatrix} \tag{3.6.1.6}$$

using the following python script. The (incorrect) graphical solution is available in Fig. 3.6.1

```
codes/optimization/2.12.py
```

Note that $\mu < 0$, contradicting the necessary condition in (2.5.8.4).

2. Obtain the correct solution to the previous problem by considering $\mu = 0$.
3. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.3.1}$$

with constraints

$$g_1(\mathbf{x}) = 0 \tag{3.6.3.2}$$

$$g_2(\mathbf{x}) \leq 0 \tag{3.6.3.3}$$

4. Based on whatever you have done so far, list the steps that you would use in general for solving a convex optimization problem like (3.6.1.1) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.
Solution: For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.4.1}$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, .., m \tag{3.6.4.2}$$

$$\text{subject to } g_i(\mathbf{x}) \leq 0, \forall i = 1, .., n \tag{3.6.4.3}$$

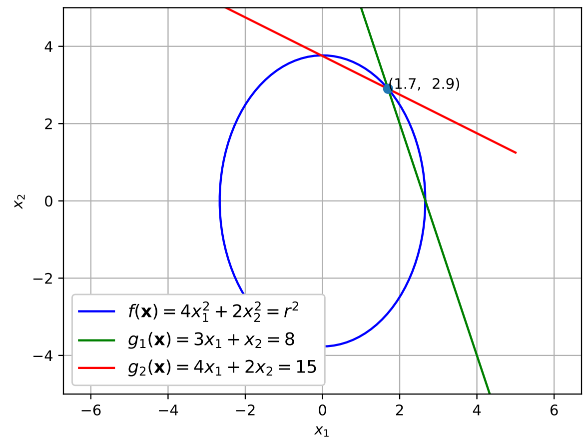

Fig. 3.6.1: Incorrect solution is at intersection of all curves $r = 5.33$

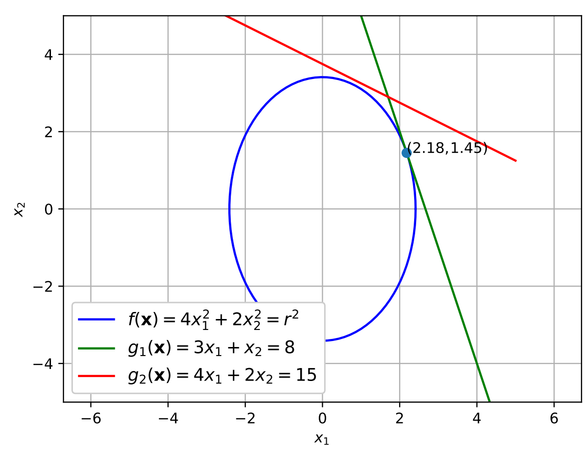

Fig. 3.6.2: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \tag{3.6.4.4}$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i g_i(\mathbf{x}), \tag{3.6.4.5}$$

using the KKT conditions

$$\Rightarrow \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \tag{3.6.4.6}$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, .., n \tag{3.6.4.7}$$

$$\text{and } \mu_i \geq 0, \forall i = 1, .., n \tag{3.6.4.8}$$

5. Maxmimize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \tag{3.6.5.1}$$

with the constraints

$$x_1^2 + x_2^2 \leq 5 \tag{3.6.5.2}$$

$$x_1 \geq 0, x_2 \geq 0 \tag{3.6.5.3}$$

3.7 Solved Problems

1. Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.
Solution: The equation of a conic is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{3.7.1.1}$$

For the standard parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.7.1.2}$$

$$\mathbf{u} = -2a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.1.3}$$

$$F = 0 \tag{3.7.1.4}$$

The focus

$$\mathbf{F} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.1.5}$$

The Latus rectum is the line passing through $\mathbf{F}$ with direction vector

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7.1.6}$$

Thus, the equation of the Latus rectum is

$$\mathbf{x} = \mathbf{F} + \lambda \mathbf{m} \tag{3.7.1.7}$$

The intersection of the latus rectum and the parabola is obtained from (3.7.1.4) , (3.7.1.7) and (3.7.1.1) as

$$(\mathbf{F} + \lambda \mathbf{m})^T \mathbf{V} (\mathbf{F} + \lambda \mathbf{m}) + 2\mathbf{u}^T (\mathbf{F} + \lambda \mathbf{m}) = 0 \tag{3.7.1.8}$$

$$\implies \left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right) \lambda^2 + 2 (\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m} \lambda + (\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = 0 \tag{3.7.1.9}$$

From (3.7.1.2), (3.7.1.3), (3.7.1.5) and (3.7.1.6),

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 1 \tag{3.7.1.10}$$

$$(\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m} = 0 \tag{3.7.1.11}$$

$$(\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = -4a^2 \tag{3.7.1.12}$$

Substituting from (3.7.1.10), (3.7.1.11) and (3.7.1.12) in (3.7.1.9),

$$\lambda^2 - 4a^2 = 0 \tag{3.7.1.13}$$

$$\implies \lambda_1 = 2a, \lambda_2 = -2a \tag{3.7.1.14}$$

Thus, from (3.7.1.6), (3.7.1.7) and (3.7.1.14), the length of the latus rectum is

$$(\lambda_1 - \lambda_2) \|\mathbf{m}\| = 4a \tag{3.7.1.15}$$

From the given information, the two parabolas P_1, P_2 have parameters

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a \begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \tag{3.7.1.16}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \tag{3.7.1.17}$$

$$4a = 3 \tag{3.7.1.18}$$

Let L be the common tangent for P_1, P_2 with $\mathbf{c}, \mathbf{d}$ being the respective points of contact. The respectivel normal vectors are

$$\mathbf{n}_1 = \mathbf{V}_1\mathbf{c} + \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} \tag{3.7.1.19}$$

$$\mathbf{n}_2 = \mathbf{V}_2\mathbf{d} + \mathbf{u}_2 = d_1\begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \tag{3.7.1.20}$$

From the above equations, since both normals have the same direction vector,

$$\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \implies c_2 d_1 = 4a^2 \tag{3.7.1.21}$$

2. Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}\mathbf{x} = 225 \tag{3.7.2.1}$$

upon the tangent to it at the point

$$\frac{1}{2}\begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \tag{3.7.2.2}$$

Solution: For the ellipse in (3.7.2.1),

$$V = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}, \mathbf{u} = 0, F = -1 \tag{3.7.2.3}$$

The equation of the desired tangent is

$$(\mathbf{VP})^T\mathbf{x} = 1 \tag{3.7.2.4}$$

$$\implies \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\mathbf{x} = 2 \tag{3.7.2.5}$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{3.7.2.6}$$

The product of the perpendiculars is

$$\frac{\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} - 2\right|\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ -4 \end{pmatrix} - 2\right|}{\left\|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\right\|^2} = 9 \tag{3.7.2.7}$$

3. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.
Solution: If a and b be the semi-major and minor-axis respectively, the foci of the ellipse are

$$\mathbf{F}_1 = ae\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{F}_2 = -ae\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.3.1}$$

From the given information,

$$e = \frac{3}{5}, 2ae = 6 \tag{3.7.3.2}$$

$$\implies a = 5, b = a\sqrt{1-e^2} = 4 \tag{3.7.3.3}$$

Thus, the vertices of the ellipse are

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ -b \end{pmatrix} \tag{3.7.3.4}$$

and the area of the quadrilateral is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & b \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & -b \end{vmatrix} = 2ab = 40 \tag{3.7.3.5}$$

4. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{3.7.4.1}$$

If

$$\mathbf{S} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.7.4.2}$$

is a focus and

$$\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} = 9 \tag{3.7.4.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.
Solution: From (3.7.4.1),

$$(3e-1)(3e-5) = 0 \tag{3.7.4.4}$$

$$\implies e = \frac{5}{3}, \because e > 1 \tag{3.7.4.5}$$

for a hyperbola. Let $\mathbf{x}$ be a point on the hyperbola. From (3.7.4.3), its distance from the directrix is

$$\frac{\left|\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} - 9\right|}{5} \tag{3.7.4.6}$$

and from the focus is

$$\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right\| \tag{3.7.4.7}$$

From the definition of a hyperbola, the eccentricity is the ratio of these distances and

(3.7.4.5) , (3.7.4.6) and (3.7.4.7),

$$\frac{5\left\|\mathbf{x}-\begin{pmatrix}5\\0\end{pmatrix}\right\|}{\left|\begin{pmatrix}5 & 0\end{pmatrix}\mathbf{x}-9\right|}=\frac{5}{3} \tag{3.7.4.8}$$

$$\implies 9\left\{(x_1-5)^2+x_2^2\right\}=(5x_1-9)^2 \tag{3.7.4.9}$$

$$\text{or, } \mathbf{x}^T\begin{pmatrix}16 & 0\\0 & -9\end{pmatrix}\mathbf{x}=225 \tag{3.7.4.10}$$

which is the equation of the hyperbola. Thus,

$$a^2=\frac{225}{16}, b^2=\frac{225}{9} \tag{3.7.4.11}$$

$$\implies a^2-b^2=-\frac{175}{16} \tag{3.7.4.12}$$

5. A variable line drawn through the intersection of the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x}=12 \tag{3.7.5.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x}=12 \tag{3.7.5.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (3.7.5.1) is obtained through the matrix equation

$$\begin{pmatrix}4 & 3\\3 & 4\end{pmatrix}\mathbf{x}=\begin{pmatrix}12\\12\end{pmatrix} \tag{3.7.5.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix}4 & 3 & 12\\3 & 4 & 12\end{pmatrix}\leftrightarrow\begin{pmatrix}4 & 3 & 12\\0 & 7 & 12\end{pmatrix}\leftrightarrow\begin{pmatrix}28 & 0 & 48\\0 & 7 & 12\end{pmatrix}$$
$$\leftrightarrow\begin{pmatrix}7 & 0 & 12\\0 & 7 & 12\end{pmatrix} \tag{3.7.5.4}$$

resulting in

$$\mathbf{C}=\frac{1}{7}\begin{pmatrix}12\\12\end{pmatrix} \tag{3.7.5.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A}=2\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{R} \tag{3.7.5.6}$$

$$\mathbf{B}=2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{R} \tag{3.7.5.7}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x}-\mathbf{C})=0 \tag{3.7.5.8}$$

Since **R** lies on AB,

$$\mathbf{n}^T(\mathbf{R}-\mathbf{C})=0 \tag{3.7.5.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A}-\mathbf{B})=0 \tag{3.7.5.10}$$

Substituting from (3.7.5.6) in (3.7.5.10),

$$\mathbf{n}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.11}$$

From (3.7.5.9) and (3.7.5.11),

$$(\mathbf{R}-\mathbf{C})=k\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R} \tag{3.7.5.12}$$

for some constant k. Multiplying both sides of (3.7.5.12) by

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \tag{3.7.5.13}$$

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R}-\mathbf{C})=k\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R}$$
$$=k\mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.14}$$

$$\because \mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.15}$$

which can be easily verified for any **R**. from (3.7.5.14),

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R}-\mathbf{C})=0$$
$$\implies \mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R}-\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{C}=0$$
$$\implies \mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R}-\mathbf{C}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R}=0 \tag{3.7.5.16}$$

which is the desired locus.

3.8 JEE Exercises

1. Find the point of intersection of the tangents at the ends of the latusrectum of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.1.1}$$

2. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $\mathbf{P} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$. Its one directrix is the common tangent, nearer to the point $\mathbf{P}$, to the circle

$$\mathbf{x}^T \mathbf{x} = 1 \tag{3.8.2.1}$$

and the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 1. \tag{3.8.2.2}$$

Find the equation of the ellipse.

3. The equation

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{1-r} & 0 \\ 0 & -\frac{1}{1+r} \end{pmatrix} \mathbf{x} = 1, r > 1 \tag{3.8.3.1}$$

represents
   a) an ellipse
   b) a hyperbola
   c) a circle
   d) none of these

4. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ in the region, the point $\begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{y_1+y_2}{2} \end{pmatrix}$ is also in the region. Find the inequality defining this region.
   a) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \le 1$
   b) $\text{Max}\begin{pmatrix} |x| \\ |y| \end{pmatrix} \le 1$
   c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \le 1$
   d) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \le 0$

5. The equation

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 & -18 \end{pmatrix} \mathbf{x} + 35 = k \tag{3.8.5.1}$$

represents
   a) no locus if $k > 0$
   b) an ellipse if$k < 0$
   c) a point if k=0
   d) a hyperbola if $k > 0$

6. Let E be the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.6.1}$$

and C be the circle

$$\mathbf{x}^T \mathbf{x} = 9. \tag{3.8.6.2}$$

let $\mathbf{P}$ and $\mathbf{Q}$ be the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Then
   a) Q lies inside C but outside E.
   b) Q lies outside both C and E.
   c) P lies inside both C and E.
   d) P lies inside C but outside E.

7. Consider a circle with its center lying on the focus of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2p & 0 \end{pmatrix} \mathbf{x} \tag{3.8.7.1}$$

such that it touches the directrix of the parabola. Then find the point of intersection.

8. Find the radius of the circle passing through the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1, \tag{3.8.8.1}$$

and having its centre at $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

9. Let $\mathbf{P} = \begin{pmatrix} a \sec\theta \\ b \tan\theta \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} a \sec\phi \\ b \tan\phi \end{pmatrix}$ where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{-1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.9.1}$$

. If $\begin{pmatrix} h \\ k \end{pmatrix}$ is the point of intersection of the normals at $\mathbf{P}$ and $\mathbf{Q}$, then find k.

10. If

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.10.1}$$

is the chord of contact of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.10.2}$$

then find the equation of the corresponding pair of tangents.

11. The curve describes parametrically by

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = t^2 + t + 1 \quad (3.8.11.1)$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = t^2 - t + 1 \quad (3.8.11.2)$$

represents
   a) a pair of straight lines
   b) an ellipse
   c) a parabola
   d) a hyperbola

12. If

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = k \quad (3.8.12.1)$$

is normal to

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}12 & 0\end{pmatrix}\mathbf{x}, \quad (3.8.12.2)$$

then find k.

13. If the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 1 = 0 \quad (3.8.13.1)$$

is the directrix of the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}k & 0\end{pmatrix}\mathbf{x} + 8 = 0, \quad (3.8.13.2)$$

then find k.

14. Find the equation of the common tangent touching the circle

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}6 & 0\end{pmatrix}\mathbf{x} = 0 \quad (3.8.14.1)$$

and the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x}. \quad (3.8.14.2)$$

15. Find the equation of the directrix of the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}4 & 4\end{pmatrix}\mathbf{x} + 2 = 0. \quad (3.8.15.1)$$

16. If $a > 2b > 0$ then the positive value of m for which

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} - b\sqrt{1 + m^2} \quad (3.8.16.1)$$

is the common tangent to

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = b^2 \quad (3.8.16.2)$$

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}2a & 0\end{pmatrix}\mathbf{x} = a^2 - b^2 \quad (3.8.16.3)$$

is
   a) $\frac{2b}{\sqrt{a^2-4b^2}}$
   b) $\frac{\sqrt{a^2-4b^2}}{2b}$
   c) $\frac{2b}{a-2b}$
   d) $\frac{b}{a-2b}$

17. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x} \quad (3.8.17.1)$$

is another parabola with directrix
   a) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -a$
   b) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{-a}{2}$
   c) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$
   d) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{a}{2}$

18. Find the equation of the common tangent to the curves

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} \quad (3.8.18.1)$$

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\mathbf{x} = -1 \quad (3.8.18.2)$$

19. Find the area of the quadrilateral formed by the tangents at the end points of latusrectum to the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{5}\end{pmatrix}\mathbf{x} = 1. \quad (3.8.19.1)$$

20. The focal chord to

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}16 & 0\end{pmatrix}\mathbf{x} \quad (3.8.20.1)$$

is tangent to

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}12 & 0\end{pmatrix}\mathbf{x} + 36 = 0 \quad (3.8.20.2)$$

then the possible values of the slope of the chord,are
   a) $\begin{pmatrix}-1\\1\end{pmatrix}$

b) $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$
c) $\begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}$
d) $\begin{pmatrix} 2 \\ -\frac{1}{2} \end{pmatrix}$

21. For hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\cos^2\alpha} & 0 \\ 0 & -\frac{1}{\sin^2\alpha} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.21.1)$$

which of the following remains constant with change in $'\alpha'$

a) abscissae of vertices
b) abscissae of foci
c) eccentricity
d) directrix

22. If tangents are drawn to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 2 \quad (3.8.22.1)$$

then the locus of the mid point of the intercept made by the tangents between the coordinate axes is

a) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$
b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$
d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

23. Find the angle between the tangents drawn from the points$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.23.1)$$

24. If the line

$$\begin{pmatrix} 2 & \sqrt{6} \end{pmatrix} \mathbf{x} = 2 \quad (3.8.24.1)$$

touches the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} = 4 \quad (3.8.24.2)$$

then find the point of contact.

25. The minimum area of the triangle is formed by the tangent to the

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.25.1)$$

the coordinate axes is

a) ab sq.units
b) $\frac{a^2+b^2}{2}$sq.units
c) $\frac{(a+b)^2}{2}$sq.units
d) $\frac{a^2+ab+b^2}{3}$sq.units

26. Tangent to the curve

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 6 \quad (3.8.26.1)$$

at the points $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$touches the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 16 & 12 \end{pmatrix} \mathbf{x} + c = 0 \quad (3.8.26.2)$$

at a point $\mathbf{Q}$.Then the coordinates of $\mathbf{Q}$ are

a) $\begin{pmatrix} -6 \\ -11 \end{pmatrix}$
b) $\begin{pmatrix} -9 \\ -13 \end{pmatrix}$
c) $\begin{pmatrix} -10 \\ -15 \end{pmatrix}$
d) $\begin{pmatrix} -6 \\ -7 \end{pmatrix}$

27. The axis of a parabola is along the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.27.1)$$

and the distance of its vertex and focus from the origin are $\sqrt{2}$ and $2\sqrt{2}$ respectively.If vertex and focus both lies in the first quadrant,then find the equation of parabola.

28. A hyperbola, having the transverse axis of length $2\sin\theta$,is confocal with the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 12. \quad (3.8.28.1)$$

Then find its equation.

29. Let a and b be non zero real numbers,then the equation

$$(\mathbf{x}^T \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mathbf{x} + c)(\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ -5 & 6 \end{pmatrix} \mathbf{x}) = 0 \quad (3.8.29.1)$$

represents

a) four straight lines, when c=0 and a,b are of the same sign
b) two straight lines and a circle ,when a=b,and c is of sign opposite to that of a
c) two straight lines and a hyperbola ,when a and b are of the same sign and c is of sign opposite to that of a
d) a circle and an ellipse , when a and b are of the same sign and c is of sign opposite to

that of a

30. Consider a branch of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2\sqrt{2} & -4\sqrt{2} \end{pmatrix} \mathbf{x} - 6 = 0 \quad (3.8.30.1)$$

with the vertex at the point **A**.Let **B** be the one of the end points of its latusrectum.If **C** is the focus of the hyperbola nearer to the point **A**,find the area of the triangle ABC.

31. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 9 \quad (3.8.31.1)$$

meets its auxiliary circle at the point **M** then the area of the triangle with vertices at A, M and the origin O is

a) $\frac{31}{10}$
b) $\frac{29}{10}$
c) $\frac{21}{10}$
d) $\frac{27}{10}$

32. The normal at a point **P** on the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 16 \quad (3.8.32.1)$$

meets the x-axis at **Q**. If **M** is the mid point of the line segment PQ, then the locus of **M** intersects the latusrectums of the given ellipse at the points

a) $\begin{pmatrix} \pm\frac{3\sqrt{5}}{2} \\ \pm\frac{2}{7} \end{pmatrix}$
b) $\begin{pmatrix} \pm\frac{3\sqrt{5}}{2} \\ \pm\sqrt{\frac{19}{4}} \end{pmatrix}$
c) $\begin{pmatrix} \pm 2\sqrt{3} \\ \pm\frac{1}{7} \end{pmatrix}$
d) $\begin{pmatrix} \pm 2\sqrt{3} \\ \pm\frac{4\sqrt{3}}{7} \end{pmatrix}$

33. The locus of the orthocentre of the triangle formed by the lines

$$\begin{pmatrix} (1+p) & -p \end{pmatrix} \mathbf{x} + p(1+p) = 0 \quad (3.8.33.1)$$
$$\begin{pmatrix} (1+q) & -q \end{pmatrix} \mathbf{x} + q(1+q) = 0 \quad (3.8.33.2)$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.8.33.3)$$

, where $p \neq q$ is

a) a hyperbola
b) a parabola
c) an ellipse
d) a straight line

34. Let $\mathbf{P} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ be a points on the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.34.1)$$

If the normal at the points **P** intersects the x-axis at $\begin{pmatrix} 9 \\ 0 \end{pmatrix}$, then find the eccentricity of the hyperbola.

a) $\sqrt{\frac{5}{2}}$
b) $\sqrt{\frac{3}{2}}$
c) $\sqrt{2}$
d) $\sqrt{3}$

35. Let$\begin{pmatrix} x \\ y \end{pmatrix}$be any point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.35.1)$$

. Let **P** be the points that divides the lines segment from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$ in the ratio 1:3. Then the locus of P is

a) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}$
b) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x}$
c) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}$
d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x}$

36. The ellipse $\mathbf{E_1}$:

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.36.1)$$

is inscribed in a rectangle R whose sides are parallel to the coordinate axes.Another ellipse $\mathbf{E_2}$ passing through the points $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$circumscribes the rectangle R. Find the eccentricity of the ellipse $\mathbf{E_2}$.

37. The common tangents to the circle

$$\mathbf{x}^T \mathbf{x} = 2 \quad (3.8.37.1)$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.37.2)$$

touch the circle at the points $\mathbf{P}, \mathbf{Q}$ and the parabola at the points $\mathbf{R}, \mathbf{S}$. Then find the area of the quadrilateral PQRS.

38. The number of values of c such that the straight line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} + c \qquad (3.8.38.1)$$

touches the curve

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 1 \qquad (3.8.38.2)$$

is

a) 0
b) 1
c) 2
d) infinite.

39. If $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{F_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{F_2} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$and

$$\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & 25 \end{pmatrix} \mathbf{x} = 400, \qquad (3.8.39.1)$$

then $\mathbf{PF_1} + \mathbf{PF_2}$equals

a) 8
b) 6
c) 10
d) 12

40. On the ellipse

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 1, \qquad (3.8.40.1)$$

the points at which the tangents are parallel to the line

$$\begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 9 \end{pmatrix} \mathbf{x} \qquad (3.8.40.2)$$

are

a) $\begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$
b) $\begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$
c) $\begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$
d) $\begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5}) \end{pmatrix}$

41. The equation of the common tangents to the parabola

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.41.1)$$

and

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \mathbf{x} = 4 \qquad (3.8.41.2)$$

is/are

a) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 4{=}0$
b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$
c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} - 4 = 0$
d) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 30 & 0 \end{pmatrix} \mathbf{x} + 50 = 0$

42. Let the hyperbola passes through the focus of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1 \qquad (3.8.42.1)$$

The transverse and conjugate axes of this hyperbola coincides with the major and minor axis of the given ellipse also the product of eccentricities of given ellipse and hyperbola is 1, then

a) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
b) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{25} \end{pmatrix} \mathbf{x} = 1$
c) focus of hyperbola is $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
d) vertex of hyperbola is $\begin{pmatrix} 5\sqrt{3} \\ 0 \end{pmatrix}$

43. Let $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} and \mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, y_1 < 0, y_2 < 0$,be the end point of the latus rectum of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 4 \qquad (3.8.43.1)$$

.The equation of parabola with latus rectum PQ are

a) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = 3 + \sqrt{3}$
b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -2\sqrt{3} \end{pmatrix} \mathbf{x} = 3 + \sqrt{3}$
c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = 3 - \sqrt{3}$

d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -2\sqrt{3} \end{pmatrix} \mathbf{x} = 3 - \sqrt{3}$

44. In a triangle ABC with fixed base BC,the vertex A moves such that $\cos B + \cos C = 4\sin^2 \frac{A}{2}$. If a, b and c denote the lengths of the triangle A,B and C,respectively,then
   a) b+c=4a
   b) b+c=2a
   c) locus of point**A** is an ellipse
   d) locus of point **A** is a pair of straight lines

45. The tangent PT and the normal PN to the parabola

$$\mathbf{x}^T \begin{pmatrix} -4a & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.8.45.1}$$

   at a point **P** on it meet its axis at points **T** and **N**, respectively. The locus of the centroid of the triangle PTN is a parabola whose
   a) vertex is $\begin{pmatrix} \frac{2a}{3} \\ 0 \end{pmatrix}$
   b) directrix is $\begin{pmatrix} 1 & 0 \end{pmatrix}$=0
   c) latus rectum is $\frac{2a}{3}$
   d) focus is $\begin{pmatrix} a \\ 0 \end{pmatrix}$

46. An ellipse intersects the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.46.1}$$

   orthogonally.The eccentricity of the ellipse is reciprocal of that of the hyperbola.If the axes of the ellipse are along the coordinate axes,then
   a) equation of ellipse is$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 2$
   b) the foci of ellipse are $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$
   c) equation of ellipse is $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 4$
   d) the foci of ellipse are $\begin{pmatrix} \pm\sqrt{2} \\ 0 \end{pmatrix}$

47. Let**A** and **B** two distinct points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.47.1}$$

   If the axis of a parabola touches a circle of radius r, having AB as its diameter, then the slop of the line joining A and B can be
   a) $-\frac{1}{r}$
   b) $\frac{1}{r}$
   c) $\frac{2}{r}$
   d) $-\frac{2}{r}$

48. Let the eccentricity of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.48.1}$$

   . If the hyperbola passes to that of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{3.8.48.2}$$

   . If the hyperbola passing through a focus of the ellipse, then
   a) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{x} = 1$
   b) the focus of the hyperbola is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
   c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
   d) the equation of the hyperbola is $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} = 3$

49. Let L be a normal to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.49.1}$$

   . If **L** passes though the point$\begin{pmatrix} 9 \\ 6 \end{pmatrix}$, then **L** is given by
   a) $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} + 3 = 0$
   b) $\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 33 = 0$
   c) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 15 = 0$
   d) $\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + 12 = 0$

50. Tangents are drawn to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1, \tag{3.8.50.1}$$

   parallel to the straight line

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.50.2}$$

   . The point of contact of the tangents on the hyperbola are
   a) $\begin{pmatrix} \frac{9}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
   b) $\begin{pmatrix} \frac{9}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

c) $\begin{pmatrix} 3\sqrt{3} \\ -2\sqrt{2} \end{pmatrix}$
d) $\begin{pmatrix} -3\sqrt{3} \\ 2\sqrt{2} \end{pmatrix}$

51. Let **P** and **Q** be distinct points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x}. \quad (3.8.51.1)$$

such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are)the coordinates of **P**?

a) $\begin{pmatrix} 4 \\ 2\sqrt{2} \end{pmatrix}$
b) $\begin{pmatrix} 9 \\ 3\sqrt{2} \end{pmatrix}$
c) $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
d) $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

52. Let $\mathbf{E_1}$and $\mathbf{E_2}$be two ellipses whose centers are at the origin. The major axes of $\mathbf{E_1}$ and $\mathbf{E_2}$ lie along the x-axis and the y-axis, respectively. Let S be the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -2 \end{pmatrix} \mathbf{x} = 1 \quad (3.8.52.1)$$

. The straight line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (3.8.52.2)$$

touches the curves S. E_1 and E_2 at **P**,**Q** and **R** respectively. Suppose that PQ=PR=$\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 andE_2, respectively, Then the correct expression(s) is (are)

a) $e_1^2 + e_2^2 = \frac{43}{40}$
b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
c) $\mid e_1^2 - e_2^2 \mid = \frac{5}{8}$
d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

53. Consider a hyperbola H:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 1 \quad (3.8.53.1)$$

and a circle S with center $\mathbf{N} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$. Suppose that H and S touches each other at a point $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at **P** intersects the x-axis at point **P**. If$\begin{pmatrix} 1 \\ m \end{pmatrix}$ is the centroid of the triangle PMN, then the correct expression is(are)

a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2} for x_1 > 1$
b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{x_1^2 - 1})}$for $x_1 > 1$
c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}forx_1 > 1$
d) $\frac{dm}{dx_1} = \frac{1}{3}fory_1 > 0$

54. The circle C_1:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (3.8.54.1)$$

, with centre at O, intersects the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} \quad (3.8.54.2)$$

and centres $Q_2 Q_3$, respectively. If $Q_2 Q_3$ lie on the y-axis, then

a) $Q_2 Q_3 = 12$
b) $R_2\ R_3$=$4\sqrt{6}$
c) area of the triangle O$R_2 R_3$is $6\sqrt{2}$
d) area of the triangle P$Q_2 Q_3$is $4\sqrt{2}$

55. Let **P** be the point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.55.1)$$

which is at the shortest distance from the center S of the circle $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -16 \end{pmatrix} \mathbf{x} + 64 = 0$. Let **Q** be the point on the circle dividing the line segment SP internally. Then

a) SP=$2\sqrt{5}$
b) SQ:QP=$(\sqrt{5} + 1) : 2$
c) the x-intercept of the normal to the parabola at **P** is 6.
d) the slop of the tangent to the circle at **Q** is $\frac{1}{2}$.

56. If $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 1 = 0$ is a tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.56.1)$$

then which of the can not be sides of a right angled triangle ?

a) a,4,1
b) a,4,2
c) 2a,8,1
d) 2a,4,1

57. If a chord, which is not a tangent, of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.57.1}$$

has the equation $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = p$, and midpoint $\begin{pmatrix} h \\ k \end{pmatrix}$, then which of the following are possible values of p,h and k?

a) p=-2,h=2,k=-4
b) p=-1,h=1,k=-3
c) p=2,h=3,k=-4
d) p=5,h=4,k=-3

58. Consider two straight lines, each of which is tangents to both the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \tag{3.8.58.1}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.58.2}$$

. Let these lines intersect at a point **Q**. Consider the ellipse whose centre is at the origin $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and whose semi major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is(are) TRUE?

a) For the ellipse,the eccentricity is $\frac{1}{\sqrt{2}}$and the length of the latus rectum is 1
b) For the ellipse,the eccentricity is $\frac{1}{\sqrt{2}}$and the length of the latus rectum is $\frac{1}{2}$
c) the area of the region bounded by the ellipse between the lines $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{\sqrt{2}}$and $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1$ is $\frac{1}{4\sqrt{2}}(\Pi - 2)$
d) the area of the region bounded by the ellipse between the lines $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{1}{\sqrt{2}}$and $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1$ is $\frac{1}{16}(\Pi - 2)$

Subjective Problems

59. Suppose that the normals drawn at the different points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.59.1}$$

pass through the point $\begin{pmatrix} h \\ k \end{pmatrix}$. Show that $h > 2$.

60. **A** is a point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.60.1}$$

The normal A cuts the parabola again at the point B. if AB subtends a right angle at the vertex of the parabola. find the slop of AB.

61. Three normals are drawn from the point $\begin{pmatrix} c \\ 0 \end{pmatrix}$ to the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.61.1}$$

. Show that c must be greater than $\frac{1}{2}$. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other.

62. Through the vertex O of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x}, \tag{3.8.62.1}$$

chords OQ and OP are drawn at right angles to one other. Show that for all positions of **P**, PQ cuts the axis of the parabola at a fixed point. also find the locus of the middle point of PQ.

63. Show that the locus of point that divides a chord of slop 2 of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.63.1}$$

internally in the ratio 1:2 is a parabola. Find the vertex of this parabola.

64. Let $'d'$ be the perpendicular distance from the centre of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.64.1}$$

to the tangent drawn at a point **P** on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2(1 - \frac{b^2}{d^2})$.

65. Points **A**,**B** and **C** lie on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.65.1}$$

The tangents to the parabola at A,B and C, taken in pairs, intersects at points **P**, **Q** and **R**. Determine the ratio of the areas of the triangles ABC and PQR.

66. From a point **A** common tangents are drawn to the circle
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \frac{a^2}{2} \tag{3.8.66.1}$$
and parabola
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x}. \tag{3.8.66.2}$$
Find the area of the quadrilateral formed by the common tangents, the chord of the contact of the circle and the chord of the contact of the parabola

67. A tangent to the ellipse
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.8.67.1}$$
meets the ellipse
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x} = 6 \tag{3.8.67.2}$$
at point **P** and **Q**. Prove that the tangents at point **P** and **Q** of the ellipse
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\mathbf{x} = 6 \tag{3.8.67.3}$$
are at right angles.

68. The angle between a pair of tangents drawn from a point **P** to the parabola
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x} \tag{3.8.68.1}$$
is 45°. Show that the locus of the point **P** is a hyperbola.

69. Consider the family of circles
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = r^2 \tag{3.8.69.1}$$
, $2 < r < 5$. If in the first quadrant, the common tangent to the circle of this family and the ellipse
$$\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 25\end{pmatrix}\mathbf{x} = 100 \tag{3.8.69.2}$$
meet the coordinate axes at A and B, then find the equation of the locus of the midpoint of AB.

70. Find co-ordinates of all the points **P** on the ellipse
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1, \tag{3.8.70.1}$$
for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P.

71. Let ABC be an equilateral triangle inscribed in the circle
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} = a^2. \tag{3.8.71.1}$$
Suppose perpendiculars from A,B,C to the major axis of the ellipse
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1, (a > b) \tag{3.8.71.2}$$
meets the ellipse respectively, at **P**, **Q**, **R**. So, that P,Q,R lie on the same side of the major axis as ABC respectively. Prove that the normals to the ellipse drawn at the points **P**,**Q** and **R** are concurrent.

72. Let C_1 and C_2 be respectively, the parabola
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} - 1 \tag{3.8.72.1}$$
and
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 1. \tag{3.8.72.2}$$
Let **P** be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflection of **P** and **Q**, respectively. with respect to the line
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}. \tag{3.8.72.3}$$
Prove that P_1 lies on C_2, Q_1 lies on C_1 and $PQ \geq \min\begin{pmatrix}PP_1\\QQ_1\end{pmatrix}$. Hence or otherwise determine points P_0 and Q_0 on parabolas C_1 and C_2 respectively such that $\left(P_0Q_0 \leq PQ\right)$ for all pairs points $\begin{pmatrix}P\\Q\end{pmatrix}$ with **P** on C_1 and **Q** on C_2.

73. Let**P** be a point on the ellipse
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1, \tag{3.8.73.1}$$

$0 < b < a$. Let the line parallel to y-axis passing through **P** meet the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2 \tag{3.8.73.2}$$

at the point **Q** such that **P** and **Q** are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point **R** on PQ such that $PR : RQ = r : s$ as **P** varies over ellipse.

74. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding dirctrix.
75. Normals are drawn from the point **P** with slopes m_1, m_2, m_3 to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.75.1}$$

. If locus of P with m_1,m_2=α. is a part of the parabola it self then find α.

76. Tangents is drawn to parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -2 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{3.8.76.1}$$

at a point **Q**. A point **R** is such that it divides QP externally in the ratio $\frac{1}{2} : 2$. Find the locus of point **R**.

77. Tangents are drawn from any point on the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.77.1}$$

to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 9. \tag{3.8.77.2}$$

Find the locus of mid-point of the chord of contact.

78. Find the equation of the common tangents in the 1st quadrant to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 16 \tag{3.8.78.1}$$

and the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1. \tag{3.8.78.2}$$

Also find the length of the intercept of the tangent between the coordinate axes.

Comprehension Based Questions

PASSAGE I

Consider the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 9 \tag{3.8.78.3}$$

and the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x}. \tag{3.8.78.4}$$

They intersects at **P** and **Q** in the first and fourth quadrants, respectively. Tangents to the circle at **P** and **Q** intersects the x-axis at **R** and tangents to the parabola at **P** and **Q** intersects the x-axis at **S**.

79. The ratio of the areas of the triangles PQS and PQR is
    a) $1 : \sqrt{2}$
    b) $1 : 2$
    c) $1 : 4$
    d) $1 : 8$
80. The radius of the circumcircle of the triangle PRS is
    a) 5
    b) $3\sqrt{3}$
    c) $3\sqrt{2}$
    d) $2\sqrt{3}$
81. The radius of the incircle of the triangle PQR is
    a) 4
    b) 3
    c) $\frac{8}{3}$
    d) 2

PASSAGE 2 The circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.8.81.1}$$

and hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.81.2}$$

intersect at the points **A** and **B**.

82. Equation of a common tangent with positive slop to the circle as well as to the hyperbola is
    a) $\begin{pmatrix} 2 & -\sqrt{5} \end{pmatrix} \mathbf{x} - 20 = 0$
    b) $\begin{pmatrix} 2 & -\sqrt{5} \end{pmatrix} \mathbf{x} + 4 = 0$

c) $\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 8 = 0$

d) $\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} + 4 = 0$

83. Equation of the circle with AB as its diameter is

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-12 & 0\end{pmatrix}\mathbf{x} + 24 = 0$

b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}12 & 0\end{pmatrix}\mathbf{x} + 24 = 0$

c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}24 & 0\end{pmatrix}\mathbf{x} - 12 = 0$

d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-24 & 0\end{pmatrix}\mathbf{x} - 12 = 0$

PASSAGE 3 Tangents are drawn from the point $\mathbf{P} = \begin{pmatrix}3\\4\end{pmatrix}$ to the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{4}\end{pmatrix}\mathbf{x} = 1 \tag{3.8.83.1}$$

touches the ellipse at points **A** and **B**.

84. The coordinates of A and B are

a) $\begin{pmatrix}3\\0\end{pmatrix} and \begin{pmatrix}0\\2\end{pmatrix}$

b) $\begin{pmatrix}-\frac{8}{5}\\\frac{2\sqrt{161}}{15}\end{pmatrix}$and $\begin{pmatrix}-\frac{9}{5}\\\frac{8}{5}\end{pmatrix}$

c) $\begin{pmatrix}-\frac{8}{5}\\\frac{2\sqrt{161}}{15}\end{pmatrix}and\begin{pmatrix}0\\2\end{pmatrix}$

d) $\begin{pmatrix}3\\0\end{pmatrix}and\begin{pmatrix}-\frac{9}{5}\\\frac{8}{5}\end{pmatrix}$

85. The orthocenter of the triangle PAB is

a) $\begin{pmatrix}5\\\frac{8}{7}\end{pmatrix}$

b) $\begin{pmatrix}\frac{7}{5}\\\frac{25}{8}\end{pmatrix}$

c) $\begin{pmatrix}\frac{11}{5}\\\frac{8}{5}\end{pmatrix}$

d) $\begin{pmatrix}\frac{8}{25}\\\frac{7}{5}\end{pmatrix}$

86. The equation of the locus of a point whose distances from the point **P** and the line AB are equal, is

a) $\mathbf{x}^T\begin{pmatrix}9 & 0\\-6 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-54 & -62\end{pmatrix}\mathbf{x} + 241 = 0$

b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\6 & 9\end{pmatrix}\mathbf{x} + \begin{pmatrix}-54 & 62\end{pmatrix}\mathbf{x} - 241 = 0$

c) $\mathbf{x}^T\begin{pmatrix}9 & 0\\-6 & 9\end{pmatrix}\mathbf{x} + \begin{pmatrix}-54 & -62\end{pmatrix}\mathbf{x} - 241 = 0$

d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}27 & 31\end{pmatrix}\mathbf{x} - 120 = 0$

PASSAGE 4

Let PQ be the focal chord of the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x}. \tag{3.8.86.1}$$

The tangents to the parabola at **P** and **Q** meet at a point lying on the line $\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} + a, a > 0$.

87. Length of chord PQ is

a) 7a

b) 5a

c) 2a

d) 3a

88. If the chord PQ subtends an angle θ at the vertex of

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x} \tag{3.8.88.1}$$

, then $\tan\theta$=

a) $\frac{2}{3}\sqrt{7}$

b) $-\frac{2}{3}\sqrt{7}$

c) $\frac{2}{3}\sqrt{5}$

d) $-\frac{2}{3}\sqrt{5}$

PASSAGE 5 Let a,r,s,t be the non zero real numbers. Let $\mathbf{P} = \begin{pmatrix}at^2\\2at\end{pmatrix}$,$\mathbf{Q}, \mathbf{R} = \begin{pmatrix}ar^2\\2ar\end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix}as^2\\2as\end{pmatrix}$ be distinct points on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x}. \tag{3.8.88.2}$$

Suppose that PQ is the focal chord and lines QR and PK are parallel, where $\mathbf{K} = \begin{pmatrix}2a\\0\end{pmatrix}$

89. The value of r is

a) $-\frac{1}{t}$

b) $\frac{t^2+1}{t}$

c) $\frac{1}{t}$

d) $\frac{t^2-1}{t}$

90. If st=1, then the tangent at **P** and the normal at **S** to the parabola meet at a point whose ordinate is

a) $\frac{(t^2+1)^2}{2t^3}$

b) $\frac{a(t^2+1)^2}{2t^3}$

c) $\frac{a(t^2+1)^2}{t^3}$

d) $\frac{a(t^2+2)^2}{t^3}$

PASSAGE 6

Let $F_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $F_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{8} \end{pmatrix} \mathbf{x} = 1. \quad (3.8.90.1)$$

Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point **M** in the first quadrant and at point **N** in the fourth quadrant.

91. The orthocentre of the triangle F_1MN is
    a) $\begin{pmatrix} -\frac{9}{10} \\ 0 \end{pmatrix}$
    b) $\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$
    c) $\begin{pmatrix} \frac{9}{10} \\ 0 \end{pmatrix}$
    d) $\begin{pmatrix} \frac{2}{3} \\ \sqrt{6} \end{pmatrix}$
92. If the tangents of the ellipse at **M** and **N** meet at **R** and the normals to the parabola at **M** meets the x-axis at **Q**, then the ratio of the triangle MQR to area of the quadrilateral MF_1NF_2 is
    a) 3:4
    b) 4:5
    c) 5:8
    d) 2:3

Assertion and Reason Type Questions

STATEMENT-I: The curve $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}=$

$$\mathbf{x}^T \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 1 \quad (3.8.92.1)$$

. because

STATEMENT-2:A parabola is symmetric about its axis.

a) Statement-1 is True,Statement-2 is True;Statement-2 is a correct explanation for Statement-1
b) Statement-1 is True,Statement-2 is True;Statement-2 is NOT correct explanation for Statement-1
c) Statement-1 is True,Statement-2 is False
d) Statement-1 is False,Statement-2 is True.

I Integer Value Correction Type

93. The line $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 1$ is tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.93.1)$$

If this line passes through the point of intersection of the nearest directrix and the x-axis,then the eccentricity of the hyperbola is

94. Consider the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.94.1)$$

. Let $\Delta_1$be the area of the triangle formed by the end points of its latus rectum and the point $\mathbf{P} = \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix}$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangents at **P** and at the end of the points of the latus rectum.Then $\frac{\Delta_1}{\Delta_2}$ is

95. Let S be the focus of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.95.1)$$

and let PQ be the common chord of the cicle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.8.95.2)$$

and the given parabola. The area of the triangle PQS is

96. A vertical line passing through the point $\begin{pmatrix} h \\ 0 \end{pmatrix}$ intersects the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.96.1)$$

. at the point **P** and **Q**. Let the tangents to the ellipse at **P** and **Q** meet at the point **R**. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max \frac{1}{2} < h < 1 \Delta(h)$ and $\Delta_2 = \min \frac{1}{2} < h < 1 \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$
    a) g(x) is continuous but not differentiable at a
    b) g(x) is differentiable on R
    c) g(x) is continuous but not differentiable at b
    d) g(x) is continuous but not differentiable at either (a)or (b) but not both.
97. If the normals of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \quad (3.8.97.1)$$

drawn at the end points of its latus rectum are tangents to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -6 & 4 \end{pmatrix} \mathbf{x} - 5 = r^2, \quad (3.8.97.2)$$

then the value of r^2 is

98. Let the curve C be the mirror image of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.98.1}$$

with respect to the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 4 = 0$. If **A** and **B** are the points of intersecting of **C** with the line $\begin{pmatrix} 0 & 1 \end{pmatrix} = -5$, then the distance between A and B is

99. Suppose that the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.99.1}$$

are $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} f_2 \\ 0 \end{pmatrix}$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and with foci at $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2f_2 \\ 0 \end{pmatrix}$ respectively. Let T_1 be a tangent to P_1 which passes through $\begin{pmatrix} 2f_2 \\ 0 \end{pmatrix}$ and T_2 be a tangent to P_2 which passes through $\begin{pmatrix} f_1 \\ 0 \end{pmatrix}$. If m_1 is the slope of the T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

Section-B
JEE Main/AIEEE

100. Two common tangents to the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 2a^2 \tag{3.8.100.1}$$

and parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.100.2}$$

are

a) $\begin{pmatrix} 1 & 0 \end{pmatrix} = \pm(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + 2a)$
b) $\begin{pmatrix} 0 & 1 \end{pmatrix} = \pm(\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2a)$
c) $\begin{pmatrix} 1 & 0 \end{pmatrix} = \pm(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + a)$
d) $\begin{pmatrix} 0 & 1 \end{pmatrix} = \pm(\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + a)$

101. The normals at the point $\begin{pmatrix} bt_1^2 \\ 2bt_1 \end{pmatrix}$ on a parabola meets the parabola again in the point $\begin{pmatrix} bt_2^2 \\ 2bt_2 \end{pmatrix}$, then

a) $t_2 = t_1 + \frac{2}{t_1}$
b) $t_2 = -t_1 - \frac{2}{t_1}$
c) $t_2 = -t_1 + \frac{2}{t_1}$
d) $t_2 = t_1 - \frac{2}{t_1}$

102. The foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.102.1}$$

and the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{144} & 0 \\ 0 & -\frac{1}{81} \end{pmatrix} \mathbf{x} = \frac{1}{25} \tag{3.8.102.2}$$

coincide.Then the value of b^2 is

a) 9
b) 1
c) 5
d) 7

103. If $a \neq 0$ and the line

$$\mathbf{x}^T \begin{pmatrix} 2b & 0 \\ 0 & 3c \end{pmatrix} \mathbf{x} + 4d = 0 \tag{3.8.103.1}$$

passes through the point of intersection of the parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4a & 0 \end{pmatrix} \mathbf{x} \tag{3.8.103.2}$$

and

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 4a \end{pmatrix} \mathbf{x} \tag{3.8.103.3}$$

, then

a) $d^2 + (3b - 2c)^2 = 0$
b) $d^2 + (3b + 2c)^2 = 0$
c) $d^2 + (2b - 3c)^2 = 0$
d) $d^2 + (2b + 3c)^2 = 0$

104. The eccentricity of an ellipse, with its centre at the origin, is $\frac{1}{2}$.If one of the directrices is $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4$, then the equation of the ellipse is:

a) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1$
b) $\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 12$
c) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 12$
d) $\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 1$

105. Let **P** be the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **Q** a point on the

locus

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.105.1}$$

the locus of mid point of PQ is

a) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$
b) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$
c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} + 2 = 0$
d) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 2 = 0$

106. The locus of a point $\mathbf{P} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ moving under the condition that the line $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \alpha & 0 \end{pmatrix} \mathbf{x} + \beta$ is a tangent to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.106.1}$$

is
a) an ellipse
b) a circle
c) a parabola
d) a hyperbola

107. An ellipse has OB as semi minor axis, F and F' it's foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is
a) $\frac{1}{\sqrt{2}}$
b) $\frac{1}{2}$
c) $\frac{1}{4}$
d) $\frac{1}{\sqrt{3}}$

108. The locus of the vertices of the family of parabolas

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x}^T \begin{pmatrix} \frac{a^3}{3} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{a^2}{2} & 0 \end{pmatrix} \mathbf{x} - 2a \tag{3.8.108.1}$$

is
a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{105}{64}$
b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{3}{4}$
c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{35}{16}$
d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \frac{64}{105}$

109. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then its eccentricity is
a) $\frac{3}{5}$
b) $\frac{1}{2}$
c) $\frac{4}{5}$
d) $\frac{1}{\sqrt{5}}$

110. Angle between the tangents to the curve $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -5 & 0 \end{pmatrix} \mathbf{x} + 6$ at the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is
a) π
b) $\frac{\pi}{2}$
c) $\frac{\pi}{6}$
d) $\frac{\pi}{4}$

111. For the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\cos^2 \alpha} & 0 \\ 0 & -\frac{1}{\sin^2 \alpha} \end{pmatrix} \mathbf{x} = 1 \tag{3.8.111.1}$$

, which of the following remains constant when α varies=?
a) abscissae of vertices
b) abscissae of foci
c) eccentricity
d) directrix.

112. The equation of a tangent to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.112.1}$$

is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2. \tag{3.8.112.2}$$

The point on this line from which the other tangents to the parabola is perpendicular to the given tangent is
a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
b) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
c) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$
d) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

113. The normal to a curve at $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$ meets the x-axis at G. If the distance G from the origin is twice the abscissa of $\mathbf{P}$, then the curve is a
a) circle
b) hyperbola

c) ellipse
d) parabola.

114. A focus of an ellipse is at the origin. The directrix is the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \tag{3.8.114.1}$$

and the eccentricity is $\frac{1}{2}$. Then the length of the semi major axis is

a) $\frac{8}{3}$
b) $\frac{2}{3}$
c) $\frac{4}{3}$
d) $\frac{5}{3}$

115. A parabola has the origin as its focus and the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 2 \tag{3.8.115.1}$$

as directrix. Then the vertex of the parabola is at

a) $\begin{pmatrix}0\\2\end{pmatrix}$
b) $\begin{pmatrix}1\\0\end{pmatrix}$
c) $\begin{pmatrix}0\\1\end{pmatrix}$
d) $\begin{pmatrix}2\\0\end{pmatrix}$

116. The ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.8.116.1}$$

is inscribed in a rectangular aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point $\begin{pmatrix}4\\0\end{pmatrix}$. Then the equation of the ellipse is :

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 12\end{pmatrix}\mathbf{x} = 16$
b) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 48\end{pmatrix}\mathbf{x} = 48$
c) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 64\end{pmatrix}\mathbf{x} = 48$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 16\end{pmatrix}\mathbf{x} = 16$

117. If two tangents drawn from a point **P** to the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} \tag{3.8.117.1}$$

are at right angles, then the locus of P is

a) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} + 1 = 0$
b) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -1$
c) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} - 1 = 0$
d) $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1$

118. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $\begin{pmatrix}-3\\1\end{pmatrix}$ and has eccentricity $\sqrt{\frac{2}{5}}$ is

a) $\mathbf{x}^T\begin{pmatrix}5 & 0\\0 & 3\end{pmatrix}\mathbf{x} - 48 = 0$
b) $\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 5\end{pmatrix}\mathbf{x} - 15 = 0$
c) $\mathbf{x}^T\begin{pmatrix}5 & 0\\0 & 3\end{pmatrix}\mathbf{x} - 32 = 0$
d) $\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 5\end{pmatrix}\mathbf{x} - 32 = 0$

119. **Statement-1:**An equation of a common tangent to the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}16\sqrt{3} & 0\end{pmatrix}\mathbf{x} \tag{3.8.119.1}$$

and the ellipse

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.8.119.2}$$

is

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} + 2\sqrt{3} \tag{3.8.119.3}$$

Statement-2: If the line

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \frac{4\sqrt{3}}{m}(m \neq 0) \tag{3.8.119.4}$$

, is a common tangent to the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}16\sqrt{3} & 0\end{pmatrix}\mathbf{x} \tag{3.8.119.5}$$

and the ellipse

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.8.119.6}$$

, then m satisfies $m^4 + 2m^2 = 24$

a) Statement-1 is false, Statement-2 is true.
b) Statement-1 is true, Statement-2 is true;Statement-2 is correct explanation for Statement-1.
c) Statement-1 is true, Statement-2 is

true;Statement-2 is NOT correct explanation for Statement-1.
d) Statement-1 is true, Statement-2 is false.

120. An ellipse is drawn by taking a diameter of the circle

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x}=0 \tag{3.8.120.1}$$

as its semi-minor axis and a diameter of the circle

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & 4\end{pmatrix}\mathbf{x}=0 \tag{3.8.120.2}$$

is semi-major axis. If the center of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is :

a) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}\mathbf{x}=4$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x}=8$
c) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}\mathbf{x}=8$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x}=16$

121. The equation of the circle passing through the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x}=1, \tag{3.8.121.1}$$

and having centre at $\begin{pmatrix}0\\3\end{pmatrix}$ is

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -6\end{pmatrix}\mathbf{x}-7=0$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -6\end{pmatrix}\mathbf{x}+7=0$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -6\end{pmatrix}\mathbf{x}-5=0$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -6\end{pmatrix}\mathbf{x}+5=0$

122. **Given:** A circle,

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\mathbf{x}=5 \tag{3.8.122.1}$$

and a parabola,

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}4\sqrt{5} & 0\end{pmatrix}\mathbf{x} \tag{3.8.122.2}$$

. **Statement-I:** An equation of a common tangent to these curve is

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}+\sqrt{5} \tag{3.8.122.3}$$

. **Statement-2:** If the line,

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}m & 0\end{pmatrix}\mathbf{x}+\frac{\sqrt{5}}{m}(m\neq 0) \tag{3.8.122.4}$$

is their common tangent,then m satisfies $m^4-3m^2+2=0$.

a) Statement-1 is true,Statement-2 is true;Statement-2 is correct explanation for Statement-1.
b) Statement-1 is true,Statement-2 is true;Statement-2 is not correct explanation for Statement-1.
c) Statement-1 is true,Statement-2 is false.
d) Statement-1 is false,Statement-2 is true.

123. The locus of the foot of perpendicular drawn from the centre of the ellipse

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}\mathbf{x}=6 \tag{3.8.123.1}$$

on any tangent to it is

a) $(x^2+y^2)^2=6x^2+2y^2$
b) $(x^2+y^2)^2=6x^2-2y^2$
c) $(x^2-y^2)^2=6x^2+2y^2$
d) $(x^2-y^2)^2=6x^2-2y^2$

124. The slop of the line touching both the parabolas

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} \tag{3.8.124.1}$$

and

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}=\begin{pmatrix}0 & -32\end{pmatrix}\mathbf{x} \tag{3.8.124.2}$$

is

a) $\frac{1}{8}$
b) $\frac{2}{3}$
c) $\frac{1}{2}$
d) $\frac{3}{2}$

125. Let O be the vertex and Q be any point on the parabola,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}=\begin{pmatrix}0 & 8\end{pmatrix}\mathbf{x}. \tag{3.8.125.1}$$

If the point **P** divides the lines segments OQ internally in the ratio 1:3, then locus of P is:

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x}$

b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x}$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}$
d) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}$

126. The normal to the curve,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\2 & -3\end{pmatrix}\mathbf{x} = 0, \tag{3.8.126.1}$$

at$\begin{pmatrix}1\\1\end{pmatrix}$

a) meets the curve again in the third quadrant.
b) meets the curve again in the fourth quadrant.
c) does not meet the curve again.
d) meets the curve again in the second quadrant.

127. The area(in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{5}\end{pmatrix}\mathbf{x} = 1 \tag{3.8.127.1}$$

is

a) $\frac{27}{2}$
b) 27
c) $\frac{27}{4}$
d) 18

128. Let $\mathbf{P}$ be the point on the parabola,

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} \tag{3.8.128.1}$$

which is at a minimum distance from the centre C of the circle

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 12\end{pmatrix}\mathbf{x} + 36 = 1, \tag{3.8.128.2}$$

Then the equation of the circle, passing through C and having its centre at $\mathbf{P}$ is:

a) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-\frac{1}{4} & 2\end{pmatrix}\mathbf{x} - 24 = 0$
b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & 9\end{pmatrix}\mathbf{x} + 18 = 0$
c) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-4 & 8\end{pmatrix}\mathbf{x} + 12 = 0$
d) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} - 12 = 0$

129. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is :

a) $\frac{2}{\sqrt{3}}$
b) $\sqrt{3}$
c) $\frac{4}{3}$
d) $\frac{4}{\sqrt{3}}$

130. A hyperbola passes through the point $\mathbf{P} = \begin{pmatrix}\sqrt{2}\\\sqrt{3}\end{pmatrix}$ and has foci at $\begin{pmatrix}\pm 2\\0\end{pmatrix}$. Then the tangent to this hyperbola at $\mathbf{P}$ also passes through the point:

a) $\begin{pmatrix}-\sqrt{2}\\-\sqrt{3}\end{pmatrix}$
b) $\begin{pmatrix}3\sqrt{2}\\2\sqrt{3}\end{pmatrix}$
c) $\begin{pmatrix}2\sqrt{3}\\3\sqrt{3}\end{pmatrix}$
d) $\begin{pmatrix}\sqrt{3}\\\sqrt{2}\end{pmatrix}$

131. The radius of a circle, having minimum area, which touches the curve

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 4 - \mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} \tag{3.8.131.1}$$

and the lines,

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}|1| & 0\end{pmatrix}\mathbf{x} \tag{3.8.131.2}$$

is:

a) $4(\sqrt{2}+1)$
b) $2(\sqrt{2}+1)$
c) $2(\sqrt{2}-1)$
d) $4(\sqrt{2}-1)$

132. Tangents are drawn to the hyperbola

$$\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 36 \tag{3.8.132.1}$$

at the points $\mathbf{P}$ and $\mathbf{Q}$. if these tangents intersect at the point $\mathbf{T} = \begin{pmatrix}0\\3\end{pmatrix}$ then the area(in sq.units) of the ΔPTQ is :

a) $54\sqrt{3}$
b) $60\sqrt{3}$
c) $36\sqrt{5}$
d) $45\sqrt{5}$

133. Tangents are normal are drawn at $\mathbf{P} = \begin{pmatrix}16\\16\end{pmatrix}$ on

the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x}, \qquad (3.8.133.1)$$

which intersect the axis of the parabola at **A** and **B**, respectively. If C is the centre of the circle through the points P,A and B and $\angle CPB = \theta$, then the value of $\tan\theta$ is :

a) 2
b) 3
c) $\frac{4}{3}$
d) $\frac{1}{2}$

134. Two sets A and B are as under :

$$A = \begin{pmatrix} a \\ b \end{pmatrix} \in RxR : |a-5| < 1 and |b-5| < 1 \qquad (3.8.134.1)$$

$$B = \begin{pmatrix} a \\ b \end{pmatrix} \in RxR : 4(a-6)^2 + 9(b-5)^2 \leq 36. \qquad (3.8.134.2)$$

Then:

a) $A \subset B$
b) $A \cap B = \phi$(an empty set)
c) neither $A \subset B$ nor $B \subset A$
d) $B \subset A$

135. If the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} - 6 \qquad (3.8.135.1)$$

touches the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 16 & 12 \end{pmatrix} \mathbf{x} + c = 0 \qquad (3.8.135.2)$$

then the value of c is :

a) 185
b) 85
c) 95
d) 195

136. Axis of a parabola lies along x-axis. If its vertex and focus are at distances 2 and 4 respectively from the origin, on the positive x-axis then which of the following points does not lie on it?

a) $\begin{pmatrix} 5 \\ 2\sqrt{6} \end{pmatrix}$
b) $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$
c) $\begin{pmatrix} 6 \\ 4\sqrt{2} \end{pmatrix}$
d) $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

137. Let $0 < \theta < \frac{\pi}{2}$. If The eccentricity of the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{\cos^2\theta} & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \mathbf{x} = 1 \qquad (3.8.137.1)$$

is greater than 2, then the length of its latus rectum lies in the interval:

a) $\begin{pmatrix} 3 \\ \infty \end{pmatrix}$
b) $\begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix}$
c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$

138. Equation of a common tangent to the circle,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 6 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (3.8.138.1)$$

and the parabola,

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.138.2)$$

is :

a) $\begin{pmatrix} 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 & 0 \end{pmatrix} \mathbf{x} + 1$
b) $\begin{pmatrix} 0 & \sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 3$
c) $\begin{pmatrix} 0 & 2\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 12$
d) $\begin{pmatrix} 0 & \sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} + 1$

139. If the line

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix} \mathbf{x} + 7\sqrt{3} \qquad (3.8.139.1)$$

is normal to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{24} & 0 \\ 0 & -\frac{1}{18} \end{pmatrix} \mathbf{x} = 1, \qquad (3.8.139.2)$$

then a value of m is:

a) $\frac{\sqrt{5}}{2}$
b) $\frac{\sqrt{15}}{2}$
c) $\frac{2}{\sqrt{5}}$
d) $\frac{3}{\sqrt{5}}$

140. If one end of a focal chord of the parabola,

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} \qquad (3.8.140.1)$$

is a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.Then the length of this focal chord is:

a) 25
b) 22
c) 24
d) 20

Match the Following DIRECTIONS(Q. 1-3) Each question contains statements given in two columns, which have to be matched. the statement in column-1 is labelled can A, B, C and D. while the three statements in column-2 are labelled p, q, r, s and t. any given statement in column-1 can have correct matching with ONE or MORE statements in column-2.

141. Match the following: $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is the pt, from which three normals are drawn to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} \tag{3.8.141.1}$$

which meet the parabola in the points **P**, **Q** and **R**. Then

Column-I	**Column-II**
(A) Area of ΔPQR	(p)2
(B) Radius of circum circle of ΔPQR	(q)$\frac{5}{2}$
(C) Centroid of ΔPQR	(r)$\begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}$
(D) circumcentre of ΔPQR	(s)$\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$

142. Match statements in the column I with the properties in Column II and indicate your answer by darkening the bubbles in 4 x 4 matrix given in the ORS.

Column-I	**Column-II**
(A) Two intersecting circles	(p) have a common tangents
(B) Two mutually external circles	(q) have a common normals
(C) Two circles,one strictly inside the other	(r) do not have a common tangents
(D) Two branches of a hyperbola	(s) do not have a common normals

143. Match the conics in Column I with the statement/expression in Column II

Column-I	Column-II
(A) Circle	(p) The focus of point $\begin{pmatrix} h \\ k \end{pmatrix}$ for which the line $\begin{pmatrix} h & k \end{pmatrix}\mathbf{x} = 1$touches the circle $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 4$
(B) Parabola	(q) Point **z** in the complex plane satisfying $\|z+2\| - \|z-2\| = \pm 3$
(C) Ellipse	(r) Points of the conic have paramatric representation $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = \sqrt{3}(\frac{1-t^2}{1+t^2}), \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$

DIRECTIONS(Q.4) Following questions are matching lists.The codes for the list have choices(a),(b),(c)and (d)out of which ONLY ONE is correct.

144. A line L:$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} m & 0 \end{pmatrix}+3$meets y-axis at $\mathbf{E} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and the are of the parabola $\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 16 & 0 \end{pmatrix}\mathbf{x}$, $0 \le y \le 6$at the point $\mathbf{F} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. The tangent to the parabola at $\mathbf{F} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$intersects the y-axis at $\mathbf{G} = \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$.The slope m of the line L is chosen such that the area of the triangle EGF has a local maximum. Match the List I with List II and select the correct answer using the code given below the lists:

List-I	List-II
P. m=	1. $\frac{1}{2}$
Q. Maximum area of ΔEFG is	2. 4
R. y_0=	3. 2
S. y_1=	4. 1

codes:

	P	Q	R	S
(a)	4	1	2	3
(b)	3	4	1	2
(c)	1	3	2	4
(d)	1	3	4	2

Qs.5-7: By appropriately matching the information given in the three columns of the following table Column1,2 and 3 contains conics,equations of the tangents to the conics and points of contact,respectively.

Column-I	**Column-II**	**Column-III**
(I) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = a^2$	(i) $\begin{pmatrix}0 & m\end{pmatrix}\mathbf{x} = \begin{pmatrix}m^2 & 0\end{pmatrix}\mathbf{x} + a$	(P) $\begin{pmatrix}\frac{a}{m^2}\\ \frac{2a}{m}\end{pmatrix}$
(II) $\begin{pmatrix}1 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2$	(ii) $\begin{pmatrix}0 & m\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + a\sqrt{m^2+1}$	(Q) $\begin{pmatrix}-\frac{ma}{\sqrt{m^2+1}}\\ \frac{a}{\sqrt{m^2+1}}\end{pmatrix}$
(III) $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}4a & 0\end{pmatrix}\mathbf{x}$	(iii) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \sqrt{a^2m^2-1}$	(R) $\begin{pmatrix}-\frac{a^2m}{\sqrt{a^2m^2+1}}\\ \frac{1}{\sqrt{a^2m^2+1}}\end{pmatrix}$
(IV) $\begin{pmatrix}1 & 0\\ 0 & -a^2\end{pmatrix}\mathbf{x} = a^2$	(iv) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}m & 0\end{pmatrix}\mathbf{x} + \sqrt{a^2m^2+1}$	(S) $\begin{pmatrix}-\frac{a^2m}{\sqrt{a^2m^2-1}}\\ -\frac{1}{\sqrt{a^2m^2-1}}\end{pmatrix}$

145. For $\mathbf{a} = \sqrt{2}$,if a tangent is drawn to a suitable conic(Column 1)at the point of contact$\begin{pmatrix}-1\\ 1\end{pmatrix}$,then which of the following options is the only correct combination for obtaining its equation?
   a) (I) (i) (P)
   b) (I) (ii) (Q)
   c) (II) (ii) (Q)
   d) (III) (i) (P)
146. If a tangent to a suitable conic(Column 1)is founded to be $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 8$ and its point of contact is $\begin{pmatrix}8\\ 16\end{pmatrix}$,then which of the following options is the only correct combination?
   a) (I) (ii) (Q)
   b) (II) (iv) (R)
   c) (III) (i) (P)
   d) (III) (ii) (Q)
147. The tangent to a suitable conic(Column1)at $\begin{pmatrix}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ is found to be $\begin{pmatrix}\sqrt{3} & 2\end{pmatrix}\mathbf{x} = 4$,then which of the following options is the only correct combination?
   a) (IV) (iii) (S)
   b) (IV) (iv) (S)
   c) (II) (iii) (R)
   d) (II) (iii) (R)
148. Let $\mathbf{H}$:

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\ 0 & -\frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1, \qquad (3.8.148.1)$$

where $a > b > 0$,be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of $60°$ at one of its vertices

N.Let the area of the triangle LMN be $4\sqrt{3}$.

List-I	**List-II**
P. The length of the conjugate axis of H is	1. 8
Q. The eccentricity of H is	2. $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	3. $\frac{2}{\sqrt{3}}$
P. The length of the latus rectum of H is	4. 4

a) $P \to 4 Q \to 2 R \to 1 S \to 3$
b) $P \to 4 Q \to 3 R \to 1 S \to 2$
c) $P \to 4 Q \to 1 R \to 3 S \to 2$
d) $P \to 3 Q \to 4 R \to 2 S \to 1$

4 Solid Geometry

4.1 Lines and Planes

1. L_1 is the intersection of planes

$$\begin{aligned} \begin{pmatrix} 2 & -2 & 3 \end{pmatrix} \mathbf{x} &= 2 \\ \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} &= -1 \end{aligned} \tag{4.1.1.1}$$

   Find its equation.
   Solution: (4.1.1.1) can be written in matrix form as

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \tag{4.1.1.2}$$

   and solved using the augmented matrix as follows

$$\begin{pmatrix} 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & 2 \end{pmatrix} \tag{4.1.1.3}$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{4.1.1.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 5 \\ x_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.1.1.5}$$

   which is the desired equation.
2. Summarize all the above computations through a Python script and plot L_1.
   Solution: The following code generates Fig. 4.1.2.

```
codes/3d/1.1.py
```

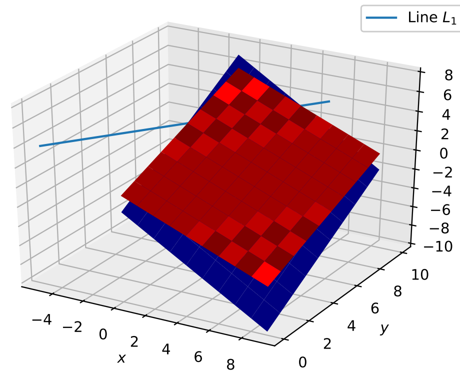

Fig. 4.1.2

3. L_2 is the intersection of the planes

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{4.1.3.1}$$

$$\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{4.1.3.2}$$

   Show that its equation is

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \tag{4.1.3.3}$$

4. Plot L_2.
   Solution: The following code generates Fig. 4.1.4.

```
codes/3d/1.2.py
```

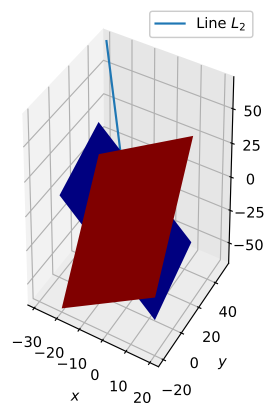

Fig. 4.1.4

5. Do L_1 and L_2 intersect? If so, find their point of intersection P.
   Solution: From (4.1.1.5),(4.1.3.3), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.1.5.1}$$

$$\implies \begin{pmatrix} 1 & 3 \\ 1 & -5 \\ 0 & -7 \end{pmatrix} \boldsymbol{\Lambda} = \frac{1}{7} \begin{pmatrix} 40 \\ 8 \\ -28 \end{pmatrix} \tag{4.1.5.2}$$

This matrix equation can be solved as

$$\begin{pmatrix}1 & 3 & \frac{40}{7}\\ 1 & -5 & \frac{8}{7}\\ 0 & -7 & -4\end{pmatrix} \leftrightarrow \begin{pmatrix}8 & 0 & \frac{224}{7}\\ 0 & 1 & \frac{4}{7}\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \tag{4.1.5.3}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & \frac{4}{7}\end{pmatrix} \implies \mathbf{\Lambda} = \begin{pmatrix}4\\ \frac{4}{7}\end{pmatrix} \tag{4.1.5.4}$$

Substituting $\lambda_1 = 4$ in (4.1.5.1)

$$\mathbf{x} = \begin{pmatrix}4\\4\\0\end{pmatrix} + \begin{pmatrix}-5\\0\\4\end{pmatrix} = \begin{pmatrix}-1\\4\\4\end{pmatrix} \tag{4.1.5.5}$$

6. Plot P.
Solution: The following code generates Fig. 4.1.6.

```
codes/3d/1.3.py
```

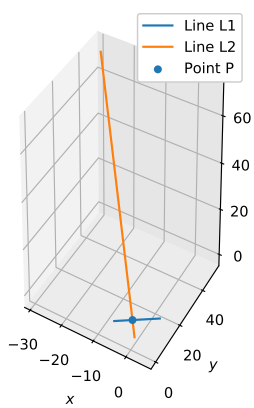

Fig. 4.1.6

4.2 Normal to a Plane

1. The cross product of $\mathbf{a}, \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{4.2.1.1}$$

From (4.1.1.5), (4.1.3.3), the direction vectors of L_1 and L_2 are

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}-3\\5\\7\end{pmatrix} \tag{4.2.1.2}$$

respectively. Find the direction vector of the normal to the plane spanned by L_1 and L_2.
Solution: The desired vector is obtained as

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}7\\-7\\8\end{pmatrix} = \mathbf{n} \tag{4.2.1.3}$$

2. Summarize all the above computations through a plot
Solution: The following code generates Fig. 4.2.2.

```
codes/3d/2.1.py
```

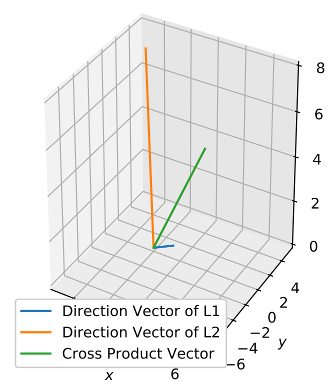

Fig. 4.2.2

3. Find the equation of the plane spanned by L_1 and L_2.
Solution: Let $\mathbf{x}_0$ be the intersection of L_1 and L_2. Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \tag{4.2.3.1}$$

$$\implies \mathbf{x}^T\mathbf{n} = \mathbf{x}_0^T\mathbf{n} \tag{4.2.3.2}$$

$$\implies \mathbf{x}^T\begin{pmatrix}7\\-7\\8\end{pmatrix} = \begin{pmatrix}-1 & 4 & 4\end{pmatrix}\begin{pmatrix}7\\-7\\8\end{pmatrix} = -3 \tag{4.2.3.3}$$

4. Summarize the above through a plot.
Solution: The following code generates Fig. 4.2.4.

```
codes/3d/2.2.py
```

5. Find the distance of the origin from the plane containing the lines L_1 and L_2.

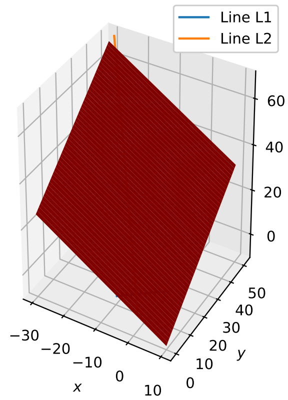

Fig. 4.2.4

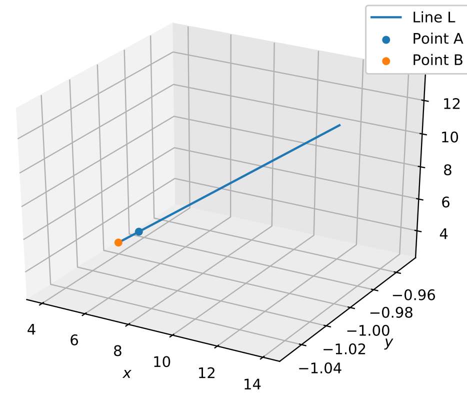

Fig. 4.3.2

Solution: The distance from the origin to the plane is given by

$$\frac{\left|\mathbf{x}_0^T\mathbf{n}\right|}{\|n\|} = \frac{1}{3\sqrt{2}} \tag{4.2.5.1}$$

4.3 Projection on a Plane

1. Find the equation of the line L joining the points

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \end{pmatrix}^T \tag{4.3.1.1}$$

$$\mathbf{B} = \begin{pmatrix} 4 & -1 & 3 \end{pmatrix}^T \tag{4.3.1.2}$$

Solution: The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda(\mathbf{A} - \mathbf{B}) \tag{4.3.1.3}$$

$$= \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.3.1.4}$$

2. Plot the above line.
Solution: The following code generates Fig. 4.3.2.

```
codes/3d/3.1.py
```

3. Find the intersection of L and the plane P given by

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 7 \tag{4.3.3.1}$$

Solution: From (4.3.1.4) and (4.3.3.1),

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 7 \tag{4.3.3.2}$$

$$\implies 6 + 2\lambda = 7 \tag{4.3.3.3}$$

$$\implies \lambda = \frac{1}{2} \tag{4.3.3.4}$$

Substituting in (4.3.1.4),

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} 9 & -1 & 7 \end{pmatrix} \tag{4.3.3.5}$$

4. Sketch the line, plane and the point of intersection.
Solution: The following code generates Fig. 4.3.4.

```
codes/3d/3.2.py
```

5. Find $\mathbf{C} \in P$ such that $AC \perp P$.
Solution: From (4.3.3.1), the direction vector of AC is $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Hence, the equation of AC is

$$\mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.3.5.1}$$

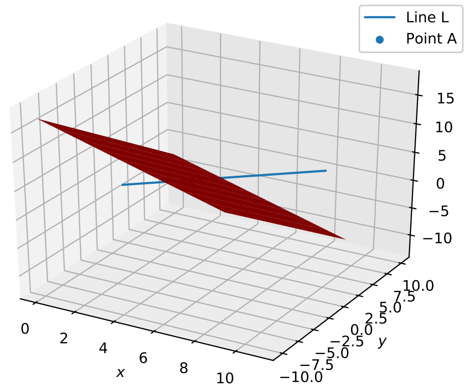

Fig. 4.3.4

Substituting in (4.3.3.1)

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 7 \tag{4.3.5.2}$$

$$\implies 8 + 3\lambda_1 = 7 \tag{4.3.5.3}$$

$$\implies \lambda_1 = -\frac{1}{3} \tag{4.3.5.4}$$

Thus,

$$\mathbf{C} = \frac{1}{3}\begin{pmatrix}14\\-4\\11\end{pmatrix} \tag{4.3.5.5}$$

6. Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D} = \frac{1}{3}\begin{pmatrix}13\\-2\\10\end{pmatrix} \tag{4.3.6.1}$$

7. Find the projection of AB on the plane P.
Solution: The projection is given by

$$CD = \|\mathbf{C} - \mathbf{D}\| = \sqrt{\frac{2}{3}} \tag{4.3.7.1}$$

8. Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \tag{4.3.8.1}$$

4.4 Coplanar vectors

1. If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that

$$\mathbf{u}^T(\mathbf{A} \times \mathbf{B}) = 0 \tag{4.4.1.1}$$

2. Find $\mathbf{A} \times \mathbf{B}$ given

$$\mathbf{A} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}^T \tag{4.4.2.1}$$

$$\mathbf{B} = \begin{pmatrix}0 & 1 & 1\end{pmatrix}^T \tag{4.4.2.2}$$

Solution: From (4.2.1.1),

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix}0 & 1 & 3\\-1 & 0 & -2\\-3 & 2 & 0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{4.4.2.3}$$

$$= \begin{pmatrix}4\\-2\\2\end{pmatrix} \tag{4.4.2.4}$$

3. Let $\mathbf{u}$ be coplanar with $\mathbf{A}$ such that $\mathbf{u} \perp \mathbf{A}$ and

$$\mathbf{u}^T\mathbf{B} = 24. \tag{4.4.3.1}$$

Find $\|\mathbf{u}\|^2$.
Solution: From (4.4.2.4) and the given information,

$$\mathbf{u}^T\begin{pmatrix}4 & -2 & 2\end{pmatrix} = 0 \tag{4.4.3.2}$$

$$\mathbf{u}^T\begin{pmatrix}2 & 3 & -1\end{pmatrix} = 0 \tag{4.4.3.3}$$

$$\mathbf{u}^T\begin{pmatrix}0 & 1 & 1\end{pmatrix} = 24 \tag{4.4.3.4}$$

$$\implies \begin{pmatrix}4 & -2 & 2\\2 & 3 & -1\\0 & 1 & 1\end{pmatrix}\mathbf{u} = \begin{pmatrix}0\\0\\24\end{pmatrix} \tag{4.4.3.5}$$

$$\implies \mathbf{u} = 4\begin{pmatrix}-1\\2\\4\end{pmatrix} \tag{4.4.3.6}$$

$$\implies \|\mathbf{u}\|^2 = 336 \tag{4.4.3.7}$$

4.5 Orthogonality

1. Let

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_1\begin{pmatrix}-1\\2\\2\end{pmatrix} \tag{4.5.1.1}$$

$$L_2 : \quad \mathbf{x} = \lambda_1\begin{pmatrix}2\\-1\\2\end{pmatrix} \tag{4.5.1.2}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.
Solution: Let

$$L_3: \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{4.5.1.3}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{4.5.1.4}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{4.5.1.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.6}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1: \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{4.5.1.7}$$
$$L_2: \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{4.5.1.8}$$
$$L_3: \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{4.5.1.9}$$

where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{4.5.1.10}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (4.5.1.7)-(4.5.1.9),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{4.5.1.11}$$
$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{4.5.1.12}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (4.5.1.11)-(4.5.1.12) can be expressed as

$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.13}$$
$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.14}$$
$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{4.5.1.15}$$
$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.16}$$
$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.17}$$
$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{4.5.1.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{4.5.1.19}$$
$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{4.5.1.20}$$

Substituting in (4.5.1.11) and (4.5.1.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \text{or} \tag{4.5.1.21}$$

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.22}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

4.6 Least Squares

1. Find the equation of the plane P containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{4.6.1.1}$$

2. Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \tag{4.6.2.1}$$

lies outside P.

3. Find the point $\mathbf{w} \in P$ closest to $\mathbf{y}$.
4. Show that

$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 = \|\mathbf{y}\|^2 - \mathbf{w}^T \mathbf{X}^T \mathbf{y} \tag{4.6.4.1}$$
$$- \mathbf{y}^T A \mathbf{w} + \mathbf{w}^T \mathbf{X}^T \mathbf{X} \mathbf{w} \tag{4.6.4.2}$$

5. Assuming 2×2 matrices and 2×1 vectors, show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{y} = \frac{\partial}{\partial \mathbf{w}} \mathbf{y}^T \mathbf{X} \mathbf{w} = \mathbf{y}^T \mathbf{X} \tag{4.6.5.1}$$

6. Show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{X} \mathbf{w} = 2\mathbf{w}^T \left(\mathbf{X}^T \mathbf{X}\right) \tag{4.6.6.1}$$

7. Show that

$$\hat{\mathbf{w}} = \min_{\mathbf{w}} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 \tag{4.6.7.1}$$
$$= \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{y} \tag{4.6.7.2}$$

8. Let

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}. \tag{4.6.8.1}$$

from (4.6.1.1). Verify (4.6.7.2).

9. Run the following Python code and comment on the output for different values of $\mathbf{x}$

```
codes/matrix/Prob1_4.py
```

10. Compare the results obtained by typing the following code with the results in the previous problem.

```
codes/matrix/Prob1_6.py
```

11. Type the following code in Python and run. Comment.

```
codes/matrix/Prob1_7.py
```

4.7 Singular Value Decomposition

1. Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C} = \mathbf{X}^T\mathbf{X}$.
2. Obtain $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{v}_1, \mathbf{v}_2$ through the following equations.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \tag{4.7.2.1}$$

$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{4.7.2.2}$$

$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \tag{4.7.2.3}$$

This procedure is known as Gram-Schmidt orthogonalization.

3. Stack the vectors $\mathbf{u}_1, \mathbf{u}_2$ in columns to obtain the matrix $\mathbf{Q}$. Show that $\mathbf{Q}$ is orthogonal.
4. From the Gram-Schmidt process, show that $\mathbf{C} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix. This is known as the $\mathbf{Q}-\mathbf{R}$ decomposition.
5. Find an orthonormal basis for $\mathbf{X}^T\mathbf{X}$ comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix $\mathbf{V}$. This is known as *Orthogonal Diagonalization*.
6. Find the singular values of $\mathbf{X}^T\mathbf{X}$. The singular values are obtained by taking the square roots of its eigenvalues.
7. Stack the singular values of $\mathbf{X}^T\mathbf{X}$ diagonally to obtain a matrix $\mathbf{\Sigma}$.
8. Obtain the matrix $\mathbf{XV}$. Verify if the columns of this matrix are orthogonal.
9. Extend the columns of $\mathbf{XV}$ if necessary, to obtain an orthogonal matrix $\mathbf{U}$.
10. Find $\mathbf{U\Sigma V}^T$. Comment.

5 Optimization

5.1 Convex Functions

1. A single variable function f is said to be convex if

$$f\left[\lambda x + (1-\lambda)\,y\right] \le \lambda f(x) + (1-\lambda)\,f(y)\,, \tag{5.1.1.1}$$

for $\quad 0 < \lambda < 1$.

2. Download and execute the following python script. Is $\ln x$ convex or concave?

```
codes/optimization/1.1.py
```

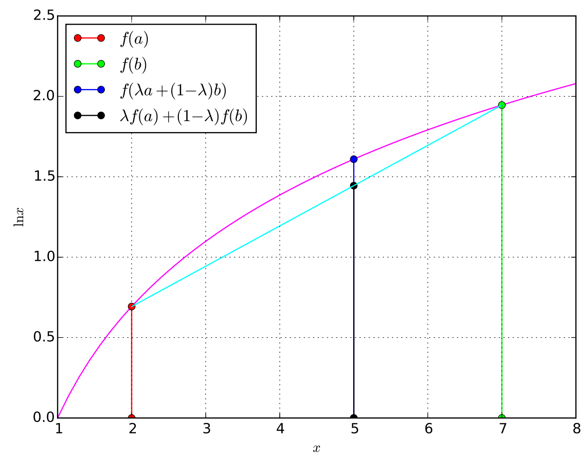

Fig. 5.1.2: $\ln x$ versus x

.

3. Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
codes/optimization/1.2.py
```

4. Execute the following script to obtain Fig. 5.1.4. Comment.

```
codes/optimization/1.3.py
```

5. Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?
6. Let

$$f(\mathbf{x}) = x_1x_2, \quad \mathbf{x} \in \mathbf{R}^2 \tag{5.1.6.1}$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex.

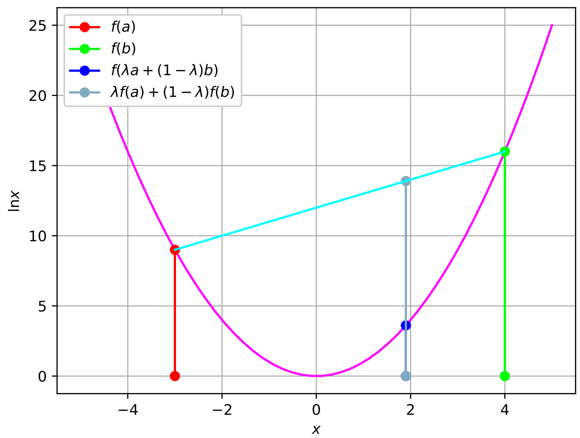

Fig. 5.1.3: x^2 versus x

.

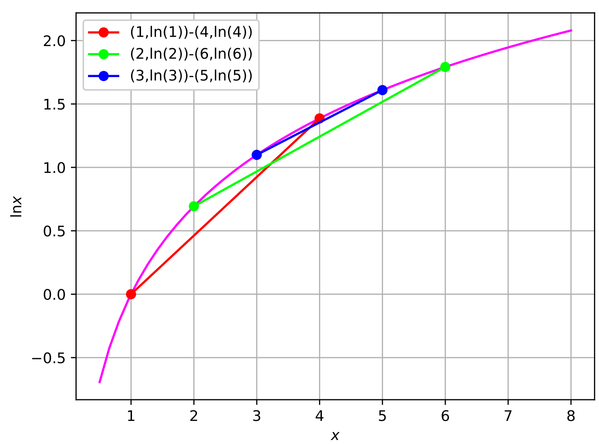

Fig. 5.1.4: Segments are below the curve

.

7. Show that
$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} \tag{5.1.7.1}$$
and find $\mathbf{V}$.
8. Show that
$$\frac{1}{2}\nabla^2 f(\mathbf{x}) = \mathbf{V} \tag{5.1.8.1}$$
9. Use (5.1.1.1) to examine the convexity of $f(\mathbf{x})$.
10. How can you deduce the convexity of $f(\mathbf{x})$ using the eigenvalues of $\mathbf{V}$?
11. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff
$$\mathbf{D} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{C} \tag{5.1.11.1}$$
such that
$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{5.1.11.2}$$
$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{5.1.11.3}$$
12. Prove that the point $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ lies outside the triangle whose sides are the lines
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 24 \tag{5.1.12.1}$$
$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{5.1.12.2}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.12.3}$$

5.2 More on Convexity

1. Consider the optimization problem
$$\max_z \frac{1}{|z-1|} \tag{5.2.1.1}$$
$$s.t. \quad \left|z - 2 + \jmath\right| \ge \sqrt{5} \tag{5.2.1.2}$$
Show that it can be reframed as
$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{c}_1\right\|^2 \tag{5.2.1.3}$$
$$s.t. \quad \left\|\mathbf{x} - \mathbf{c}_2\right\|^2 \ge 5 \tag{5.2.1.4}$$
where
$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{5.2.1.5}$$
2. Explain the optimization problem with a figure.
Solution: Fig. 5.2.2 explains (5.2.1.3) where z_0 is the set of points comprising of the intersection of the smallest circle Γ : with the largest circle Ω : $r_2 \ge \sqrt{5}$ with radii r_1 and $r_2 \ge \sqrt{5}$ respectively.
3. Obtain the Lagrangian.
Solution: The Lagrangian is
$$L(\mathbf{x}, \lambda) = \left\|\mathbf{x} - \mathbf{c}_1\right\|^2 - \lambda\left\{\left\|\mathbf{x} - \mathbf{c}_2\right\|^2 - r_2^2\right\} \tag{5.2.3.1}$$
4. Use the KKT conditions to obtain the minima.
Solution: From the KKT conditions,
$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \tag{5.2.4.1}$$
$$\implies \mathbf{x} - \mathbf{c}_1 - \lambda(\mathbf{x} - \mathbf{c}_2) = 0 \tag{5.2.4.2}$$
$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1 - \lambda} \tag{5.2.4.3}$$

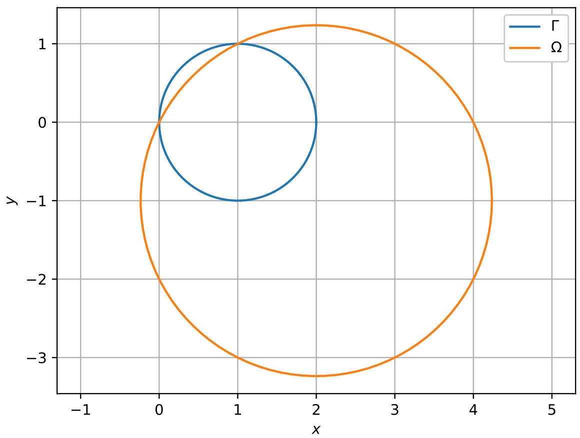

Fig. 5.2.2

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{5.2.4.4}$$

$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \tag{5.2.4.5}$$

Substituting from (5.2.4.3) in (5.2.4.5),

$$\left\|\frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1-\lambda} - \mathbf{c}_2\right\|^2 - r_2^2 = 0 \tag{5.2.4.6}$$

$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \tag{5.2.4.7}$$

$$= 1 \pm \sqrt{\frac{2}{5}} \tag{5.2.4.8}$$

Fig. 5.2.5 plots Γ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \tag{5.2.4.9}$$

5. If the maximum value is obtained at z_0, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{5.2.5.1}$$

Solution: From (5.2.4.3),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1-\lambda} \tag{5.2.5.2}$$

$$\implies z_0 = \frac{1}{1-\lambda}\left(1 - 2\lambda + \jmath\lambda\right) \tag{5.2.5.3}$$

$$\text{or,} \quad \arg \frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\left\{z_0\right\}}{\jmath\left(\Im\left\{z_0\right\} + 1\right)} \tag{5.2.5.4}$$

$$= \frac{2\left(1-\lambda\right) - \left(1 - 2\lambda\right)}{\jmath} \tag{5.2.5.5}$$

$$= -\jmath \tag{5.2.5.6}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

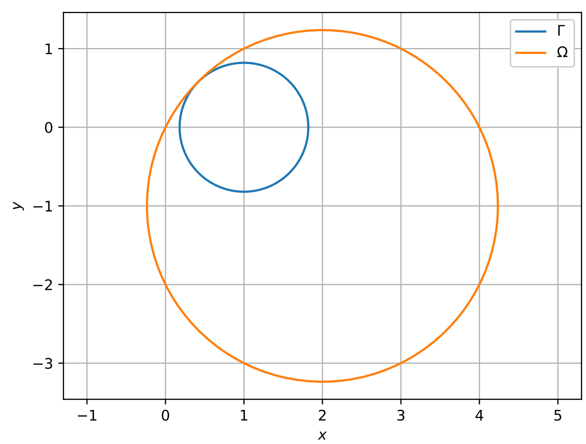

Fig. 5.2.5

6. Show that the set

$$D = \left\{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\right\}, r_2 > 0 \tag{5.2.6.1}$$

is nonconvex.
Solution: Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \tag{5.2.6.2}$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \tag{5.2.6.3}$$

$$\implies \mathbf{x}_2 \in D. \tag{5.2.6.4}$$

Suppose

$$\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\,\mathbf{x}_2 \tag{5.2.6.5}$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \tag{5.2.6.6}$$
$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \tag{5.2.6.7}$$
$$\text{or, } \mathbf{x} \notin D \tag{5.2.6.8}$$

Thus, by definition, D is not a convex set.

5.3 Gradient Descent Method

1. Consider the problem of finding the square root of a number c. This can be expressed as the equation
$$x^2 - c = 0 \tag{5.3.1.1}$$
2. Sketch the function for different values of c
$$f(x) = x^3 - 3xc \tag{5.3.2.1}$$
and comment upon its convexity.
3. Show that (5.3.1.1) results from
$$\min_x f(x) = x^3 - 3xc \tag{5.3.3.1}$$
4. Find a numerical solution for (5.3.1.1).
Solution: A numerical solution for (5.3.1.1) is obtained as
$$x_{n+1} = x_n - \mu f'(x) \tag{5.3.4.1}$$
$$= x_n - \mu\left(3x_n^2 - 3c\right) \tag{5.3.4.2}$$
where x_0 is an inital guess.
5. Write a program to implement (5.3.4.2).
Solution: Download and execute
```
codes/optimization/square_root.py
```

5.4 Semi-definite Programming

1. The problem
$$\min_{\mathbf{X}} x_{11} + x_{12} \tag{5.4.1.1}$$
with constraints
$$x_{11} + x_{22} = 1 \tag{5.4.1.2}$$
$$\mathbf{X} \geq 0 \quad (\geq \text{ means positive definite}) \tag{5.4.1.3}$$
where
$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \tag{5.4.1.4}$$
is known as a semi-definite program. Find a numerical solution to this problem. Compare with the solution in problem 2.5.9.
Solution: The *cvxopt* solver needs to be used in order to find a numerical solution. For this, the given problem has to be reformulated as
$$\min_{\mathbf{x}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \quad \text{s.t} \tag{5.4.1.5}$$
$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} = 1 \tag{5.4.1.6}$$
$$x_{11}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + x_{12}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_{22}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.4.1.7}$$
The following script provides the solution to this problem.
```
codes/optimization/3.1.py
```
2. Frame Problem 5.4.1 in terms of matrices.
Solution: It is easy to verify that
$$x_{11} + x_{12} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.4.2.1}$$
and
$$x_{11} + x_{22} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.2.2}$$
Thus, Problem 5.4.1 can be expressed as
$$\min_{\mathbf{X}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s.t$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1, \tag{5.4.2.3}$$
$$\mathbf{X} \geq 0$$
3. Solve (5.4.2.3) using *cvxpy*.
Solution:
```
wget https://raw.githubusercontent.com/
    gadepall/school/master/linalg/book/
    optimizationcodes/3.1−cvx.py
```
4. Minimize
$$-x_{11} - 2x_{12} - 5x_{22} \tag{5.4.4.1}$$

subject to

$$2x_{11} + 3x_{12} + x_{22} = 7 \tag{5.4.4.2}$$
$$x_{11} + x_{12} \geq 1 \tag{5.4.4.3}$$
$$x_{11}, x_{12}, x_{22} \geq 0 \tag{5.4.4.4}$$
$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \geq 0 \tag{5.4.4.5}$$

using *cvxpy*.

5. Repeat the above exercise by converting the problem into a convex optimization problem in two variables and using graphical plots.
6. Solve the above problem using the KKT conditions. Comment.

5.5 Linear Programming

1. Graphically obtain a solution to the following

$$\max_{\mathbf{x}} 6x_1 + 5x_2 \tag{5.5.1.1}$$

with constraints

$$x_1 + x_2 \leq 5 \tag{5.5.1.2}$$
$$3x_1 + 2x_2 \leq 12 \tag{5.5.1.3}$$
$$\text{where } x_1, x_2 \geq 0 \tag{5.5.1.4}$$

Solution: The following program plots the solution in Fig. 5.5.1

```
codes/optimization/4.1.py
```

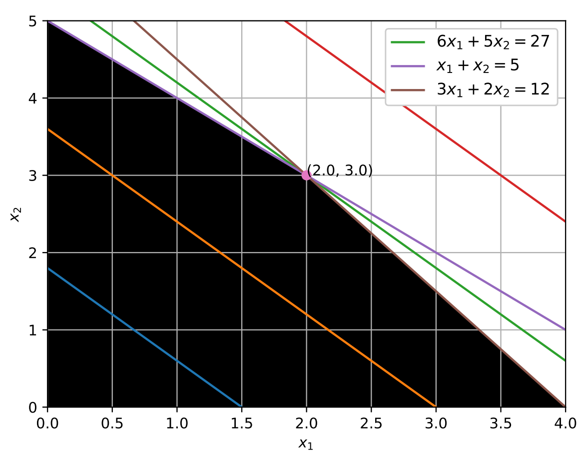

Fig. 5.5.1: The cost function intersects with the two constraints at $\mathbf{x} = (2, 3)$.

2. Now use *cvxpy* to obtain a solution to problem 5.5.1.

Solution: The given problem is expressed as follows

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad s.t. \tag{5.5.2.1}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{5.5.2.2}$$

where

$$\mathbf{c} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{5.5.2.3}$$

The desired solution is then obtained using the following program.

```
codes/optimization/4.2-cvx.py
```

3. Verify your solution to the above problem using the method of Lagrange multipliers.
4. Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$\text{where } x_1, x_2 \geq 0$$

5.6 Exercises

1. Let $\mathbf{P}$ be the point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.1.1}$$

which is at a minimum distance from the centre $\mathbf{C}$ of the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 0 & 12 \end{pmatrix} \mathbf{x} = 1 \tag{5.6.1.2}$$

Find the equation of the circle passing through $\mathbf{C}$ and having its centre at $\mathbf{P}$.

2. Let $\mathbf{P}$ be a point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.2.1}$$

Given that the distance of $\mathbf{P}$ from the centre of the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{5.6.2.2}$$

is minimum. Find the equation of the tangent to the parabola at $\mathbf{P}$.

3. Find the eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the

length of its conjugate axis is equal to half the distance between its foci.

4. **P** and **Q** are two distinct points on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.6.4.1)$$

with parameters t and t_1 respectively. If the normal at **P** passes through **Q**, then find the minimum value of t_1^2 using a descent algorithm.

5. A tangent at a point on the ellipse

$$\mathbf{x}^T V \mathbf{x} = 51 \quad (5.6.5.1)$$

where

$$V = \begin{pmatrix} 3 & 0 \\ 0 & 27 \end{pmatrix} \quad (5.6.5.2)$$

meets the coordinate axes at **A** and **B**. If **O** be the origin, find the minimum area of $\triangle OAB$.

6. Find the shortest distance between the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (5.6.6.1)$$

and the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \quad (5.6.6.2)$$

7. Let S be the set of all complex numbers z satisfying $\left|z - 2 + \jmath\right| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \quad (5.6.7.1)$$

8. Let $\omega \neq 1$ be a cube root of unity. Find the minimum of the set

$$\left|a + b\omega + c\omega^2\right|, \quad (5.6.8.1)$$

where a, b, c are distinct nonzero integers.

9. Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \quad (5.6.9.1)$$

where α, β are real functions of θ and **I** is the identity matrix. If

$$\alpha^* = \min_{\theta} \alpha(\theta) \quad (5.6.9.2)$$

$$\beta^* = \min_{\theta} \beta(\theta), \quad (5.6.9.3)$$

find $\alpha^* + \beta^*$.

10. Find the minimum value of

$$\cos(P + Q)\cos(Q + R)\cos(R + P) \quad (5.6.10.1)$$

in $\triangle PQR$.

11. Find the minimum value of α for which

$$4\alpha x^2 + \frac{1}{x} \geq 1, x > 0. \quad (5.6.11.1)$$

12. Let

$$S = S_1 \cap S_2 \cap S_3, \quad (5.6.12.1)$$

where

$$S_1 = \{z \in C : |z| < 4\} \quad (5.6.12.2)$$

$$S_2 = \left\{z \in C : \Im\left[\frac{z - 1 + \jmath\sqrt{3}}{1 - \jmath\sqrt{3}}\right]\right\}, \quad (5.6.12.3)$$

$$S_3 = \{z \in C : \Re(z) > 0\} \quad (5.6.12.4)$$

Find

$$\min_{z \in S} \left|1 - 3\jmath - z\right| \quad (5.6.12.5)$$

13. A line

$$L : \begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = -3 \quad (5.6.13.1)$$

passes through

$$\mathbf{E} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (5.6.13.2)$$

and

$$\mathbf{x} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} = 0, 0 \leq \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \leq 6 \quad (5.6.13.3)$$

at the point **F**. Find m such that the area of $\triangle EFG$ is maximum.

14. If $|z - 3 - 2j| \leq 2$, find

$$\min_{z} \left|2z - 6 + 5\jmath\right| \quad (5.6.14.1)$$

15. Find

$$\max_{z} \left|Arg\left(\frac{1}{1 - z}\right)\right| \quad (5.6.15.1)$$

$$s.t \quad |z| = 1, z \neq 1. \quad (5.6.15.2)$$

16. Find the maximum value of the function

$$f(x) = 2x^3 - 15x^2 + 36x - 48 \quad (5.6.16.1)$$

on the set

$$A = \left\{x : x^2 + 20 \le 9x\right\} \tag{5.6.16.2}$$

17. Find the minimum distance of a point on the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{5.6.17.1}$$

from the origin.

18. Find

$$\min_z \left| z + \frac{1}{2} \right| \tag{5.6.18.1}$$

$$s.t \quad |z| \ge 2 \tag{5.6.18.2}$$

19. Find the minimum value of

$$\tan A + \tan B \tag{5.6.19.1}$$

$$s.t \quad A + B = 6, \tag{5.6.19.2}$$

$$A, B \ge 0 \tag{5.6.19.3}$$

20. Show that

$$\sin A_1 + \sin A_2 + \cdots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \cdots + A_n}{n}\right)$$
$$0 < A_i < \pi, i = 1, 2, \ldots, n, n \ge 1 \tag{5.6.20.1}$$

21. Let $\mathbf{F}_1, \mathbf{F}_2$ be the foci of the standard ellipse with parameters a and b. If $\mathbf{P}$ be any point on the ellipse, find the maximum area of $\triangle PF_1F_2$.
22. A circle $\|\mathbf{x}\| = 1$ intersects the $X-$ axis at $\mathbf{P}$ and $\mathbf{Q}$. Another circle with centre $\mathbf{Q}$ intersects this circle above the $X-$ axis at $\mathbf{R}$ and the line segent PQ at $\mathbf{S}$. Find the maximum area of $\triangle QSR$.
23. Let $\mathbf{M}$ be a fixed point in the first quadrant. A line through $\mathbf{M}$ intersects the positive axes at $\mathbf{P}, \mathbf{Q}$ respectively. If $\mathbf{O}$ be the origin, find the minimum area of $\triangle OPQ$.

6 Matrices

6.1 Properties

1. Show that

$$\min_{a,b,c} \left| a + b\omega + c\omega^2 \right|^2 \tag{6.1.1.1}$$

where $\omega^3 = 1, \omega \ne 1$ and a, b, c are distinct nonzero integers can be expressed as

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{6.1.1.2}$$

where

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mathbf{A} = 2\mathbf{P}^T\mathbf{P}, \tag{6.1.1.3}$$

$$\mathbf{P} = \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}, \theta = \frac{\pi}{3} \tag{6.1.1.4}$$

2. Show that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \tag{6.1.2.1}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ \cos\theta & 1 & -\cos 2\theta \\ -\cos\theta & -\cos 2\theta & 1 \end{pmatrix}, \tag{6.1.2.2}$$

resulting in (6.1.2.1).

$$\because \cos 2\theta = -\cos\theta = -\frac{1}{2} \tag{6.1.2.3}$$

3. Show that the characteristic equation of $\mathbf{A}$ is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda \tag{6.1.3.1}$$

4. Show that the eigenvalues of $\mathbf{A}$ are 0 and 3.
5. Verify that $tr(\mathbf{A})$ is the sum of its eigenvalues.
6. Verify that $\det(\mathbf{A})$ is the product of its eigenvalues.
7. Show that $\mathbf{A}$ is positive definite.
8. Show that $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is convex.
9. Show that the unconstrained $\mathbf{x}$ that minimizes $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is given by the line

$$\mathbf{x} = k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{6.1.9.1}$$

10. Find $\mathbf{y}$ such that

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y} \tag{6.1.10.1}$$

where λ is an eigenvalue of $\mathbf{A}$.

11. Show that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P} \tag{6.1.11.1}$$

where $\mathbf{D}$ is a diagonal matrix comprising of the eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{P}$ are the corresponding eigenvectors.

12. Find $\mathbf{U}$ such that

$$\mathbf{A} = \mathbf{U}^T\mathbf{D}\mathbf{U}, \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{6.1.12.1}$$

13. Show that

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 3\mathbf{v}^T\mathbf{v}, \tag{6.1.13.1}$$

where

$$\mathbf{v} = \mathbf{U}\mathbf{x} \tag{6.1.13.2}$$

14. Show that when the entries of $\mathbf{x}$ are unequal and integers, the solution of (6.1.1.2) can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{6.1.14.1}$$

6.2 Cayley-Hamilton Theorem

1. Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{6.2.1.1}$$

where α, β are real functions of θ and $\mathbf{I}$ is the identity matrix. Find the characteristic equation of $\mathbf{M}$.
Solution: (6.2.1.1) can be expressed as

$$\mathbf{M}^2 - \alpha\mathbf{M} - \beta\mathbf{I} = 0 \tag{6.2.1.2}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{6.2.1.3}$$

2. Find α and β.
Solution: Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta + \cos^4\theta \tag{6.2.2.1}$$
$$\beta = -\sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{6.2.2.2}$$

3. If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{6.2.3.1}$$
$$\beta^* = \min_\theta \beta(\theta), \tag{6.2.3.2}$$

find $\alpha^* + \beta^*$.

Solution:

$$\because \alpha = \sin^4\theta + \cos^4\theta = 1 - \frac{\sin^2 2\theta}{2}, \tag{6.2.3.3}$$
$$\alpha^* = \frac{1}{2}, \tag{6.2.3.4}$$

Similarly,

$$-\beta = \sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{6.2.3.5}$$
$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \tag{6.2.3.6}$$
$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2}\right)^2 + \frac{7}{4} \tag{6.2.3.7}$$

Thus,

$$\beta^* = -\frac{37}{16} \tag{6.2.3.8}$$
$$\implies \alpha^* + \beta^* = -\frac{29}{16} \tag{6.2.3.9}$$

6.3 Trace

1. Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^6$ from permutations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.3.1.1}$$

2. Let

$$\mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T. \tag{6.3.2.1}$$

Given

$$\mathbf{X}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{6.3.2.2}$$

is $\alpha = 30$?

Solution:

$$\because \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{aligned} \mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \\ &= 30 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \tag{6.3.2.3}$$

Thus, $\alpha = 30$.

3. Is $\mathbf{X}$ symmetric?
   Solution: Yes. Trivial.
4. Show that

$$\mathbf{P}_k \mathbf{P}_k^T = \mathbf{I} \tag{6.3.4.1}$$

Solution:

$$\mathbf{P}_k = \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} \tag{6.3.4.2}$$

where $\mathbf{v}_{ki}, i = 1, 2, 3$ are from the standard basis. Then,

$$\mathbf{P}_k \mathbf{P}_k^T = \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} \begin{pmatrix} \mathbf{v}_{k1} & \mathbf{v}_{k2} & \mathbf{v}_{k3} \end{pmatrix} \mathbf{I}$$

$$\because \mathbf{v}_{ji}^T \mathbf{v}_{kj} = \delta_{jk} \tag{6.3.4.3}$$

5. For 2×2 matrices $\mathbf{A}, \mathbf{B}$, verify that

$$tr(AB) = tr(BA) \tag{6.3.5.1}$$

Show that this is true for any square matrix.

6. Verify if the sum of the diagonal entries of X is 18.

Solution:

$$\begin{aligned} \text{tr}(\mathbf{X}) &= \sum_{k=1}^{6} \text{tr} \left\{ \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \right\} \\ &= \sum_{k=1}^{6} \text{tr} \left\{ \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k \mathbf{P}_k^T \right\} \\ &= \sum_{k=1}^{6} \text{tr} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} = 6 \times 3 = 18 \end{aligned} \tag{6.3.6.1}$$

after substituting from (6.3.4.1).

7. Is $\mathbf{X} - 30\mathbf{I}$ invertible?
   Solution: From (6.3.2.2),

$$\mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 30\mathbf{I} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\implies (\mathbf{X} - 30\mathbf{I}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \tag{6.3.7.1}$$

If $(\mathbf{X} - 30\mathbf{I})^{-1}$ exists,

$$(\mathbf{X} - 30\mathbf{I})^{-1} (\mathbf{X} - 30\mathbf{I}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \tag{6.3.7.2}$$

which is a contradiction. Hence, $\mathbf{X} - 30\mathbf{I}$ is not invertible.

8. For any two 3×3 matrices A and B, let $A + B = 2B^T$ and $3A + 2B = I_3$. Which of the following is true?
   a) $5A + 10B = 2I_3$.
   b) $10A + 5B = 3I_3$.
   c) $2A + B = 3I_3$.
   d) $3A + 6B = 2I_3$.

6.4 Eigenvector and Null Space

1. Let

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{pmatrix} \tag{6.4.1.1}$$

2. Find x such that $PQ = QP$.

Solution:

$$\because \mathbf{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad (6.4.2.1)$$

$$\mathbf{PQ} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{pmatrix} \qquad (6.4.2.2)$$

and

$$\mathbf{QP} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 8 & 8 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \end{pmatrix} \qquad (6.4.2.3)$$

Thus,

$$\mathbf{PQ} = \mathbf{QP} \implies \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = x \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \qquad (6.4.2.4)$$

which has no solution.

3. If

$$\mathbf{R} = \mathbf{PQP}^{-1}, \qquad (6.4.3.1)$$

verify whether

$$\det \mathbf{R} = \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8 \qquad (6.4.3.2)$$

for all x.

Solution:

$$\det(\mathbf{R}) = \det(\mathbf{P}) \det(\mathbf{Q}) \det(\mathbf{P})^{-1} = \det(\mathbf{Q})$$
$$= 4\left(12 - x^2\right) \qquad (6.4.3.3)$$

Thus,

$$\det(\mathbf{R}) - \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix}$$
$$= 4\left\{\left(12 - x^2\right) - \left(10 - x^2\right)\right\}$$
$$= 8 \qquad (6.4.3.4)$$

which is true.

4. For $x = 0$, if

$$\mathbf{R} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \qquad (6.4.4.1)$$

then show that

$$a + b = 5. \qquad (6.4.4.2)$$

Solution: For $x = 0$,

$$\mathbf{R} = \mathbf{PQP}^{-1}, \qquad (6.4.4.3)$$

where $\mathbf{Q}$ is a diagonal matrix. This is the eigenvalue decomposition of $\mathbf{R}$. Thus,

$$\mathbf{R} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad (6.4.4.4)$$

where

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad (6.4.4.5)$$

is the eigenvector corresponding to the eigenvalue 6. Comparing with (6.4.4.4),

$$a = 2, b = 3 \implies a + b = 5. \qquad (6.4.4.6)$$

5. For $x = 1$, verify if there exists a vector $\mathbf{y}$ for which $\mathbf{Ry} = \mathbf{0}$.

Solution:

$$\mathbf{Ry} = \mathbf{0} \implies \mathbf{PQP}^{-1}\mathbf{y} = \mathbf{0}$$
$$\implies \mathbf{Qz} = \mathbf{0}, \qquad (6.4.5.1)$$

where

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{y} \qquad (6.4.5.2)$$

For $x = 1$, (6.4.1.1) and (6.4.5.1) yield

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} \mathbf{z} = \mathbf{0} \qquad (6.4.5.3)$$

Using row reduction,

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow$$
$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{pmatrix} \qquad (6.4.5.4)$$

Thus, $\mathbf{Q}^{-1}$ exists and

$$\mathbf{z} = \mathbf{0} \implies \mathbf{y} = \mathbf{0} \qquad (6.4.5.5)$$

upon substituting from (6.4.5.2). This implies that the null space of $\mathbf{R}$ is empty.