Geometry: Maths Olympiad

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Abstract—**This book provides a collection of the international maths olympiad problems in geometry.**

1. Let BE and CF be the altitude if an acute angle triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parllel to EN.
2. In a triangle ABC, D is point on BC such that AD is internal bisector of $\angle$A. Suppose $\angle$B = 2$\angle$C and CD = AB. Prove that $\angle$A = 72^o.
3. In an acute angle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that
$$\frac{CD}{CE} = \frac{CA}{CB}, \frac{AE}{AF} = \frac{AB}{AC}, \frac{BF}{BD} = \frac{BC}{BA}$$
Prove that AD, BE, CF are the altitude of ABC.
4. The circumference of a circle is divided into eight arcs by a convex quadrilateral ABCD, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose p + r = q + s. Prove that ABCD is a cyclic quadrilateral.
5. Consider in plane circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a unique point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.
6. Let ABCD be a quadrilateral X and Y be the mid points of AC and BD respectively and the lines through X and Y respectively parllel to BD, AC meet in O. Let P, Q, R, S be the mid points of AB, BC, CD, DA respectively. Prove that
   a) quadrilaterals APOS and APXS have the same area.
   b) the areeas is the quadrilateral APOS, BQOP, CROQ, DSOR are all equal.
7. Let ABCD be a convex quadrilateral; P, Q, R, S be the mid point of AB, BC, CD, DA respectively such that triangle AQR and CSP are equilateral. Prove that ABCD in a rhombus. Determine the angle.
8. In triangle ABC, let D be the midpoint of BC. If $\angle$ADB = 45^o and $\angle$ACD = 30^o, Determine $\angle$BAD.
9. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendicular from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are collinear.
10. Let ABCD be a quadrilateral in which AB is parallel to CD and perpendicular to AD : AB = 3CD; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.
11. Let ABC be an acute-angled triangle;AD be the bisector of $\angle$ BAC with D on BC;and BE be the altitude from B on AC. Show that $\angle$CED > 45^o.
12. A trapezium ABCD in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M,and OM = 2.
   a) If $\angle$AMB is 60^o, determine with proof,the difference between the length of the parallel

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sides.

b) If $\angle$AMD is 60^o, find all the difference between the length of the parallel sides.

13. Let ABC be an acute angled triangle; let D, F be the mid-point of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N. Prove that ND is equal to the circum radius of ABC.
14. Let ABC be a triangle in which AB = AC and let I be its incentre. Suppose BC = AB + AI. Find $\angle$BAC.
15. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.
    a) Prove that the distance between any two points on the boundary of Γ does not exceed 1.
    b) If X and Y are two distinct points inside Γ, Prove that there exists a point Z on the boundary of Γ such that XZ + YZ $\leq$ 1.
16. Let ABCDEF be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of triangle OAF is the geometric mean of those of OAB and OEF;and the area of triangle OBC is geometric mean those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.
17. Let ABC be a triangle in which $\angle$A = 60^o. Let BE and CF be the bisectors of the angles $\angle$B and $\angle$C with E on the AC and F on AB. Let M be the reflection of A on the line EF. Prove that M lies on BC.
18. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose BD/DC = BF/FA and $\angle$ADB = $\angle$AFC. Prove that $\angle$ABE = $\angle$CAD.
19. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisector of $\angle$B, $\angle$C with B_1 on AC and C_1 on AB. Let E, F be the feet of perpendiculars drawn from A onto BB_1,CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that AD = EF.
20. Let ABC be a triangle and D be a point on the segment BC such that DC = 2BD. Let E be the mid-point of AC. Let AD and BE intersect in P. Determine the ratios BP/PE and AP/PD.
21. Let ABC be a triangle. Let BE and CF be internal angle bisector of $\angle$B and $\angle$C respectively with E on AC and F on AB. Suppose X is a point on the segment CF such that AX $\perp$ CF;and Y is a point on the segment BE such that AY $\perp$ BE. Prove that XY = (b + c - a)/2 where BC = a, CA = b and AB = c.
22. Let ABC be an acute-angle triangle. The circle Γ with BC as diameter intersects AB and AC again at P and Q,respectively. Determine $\angle$BAC given that the orthocentre of triangle APQ lies on Γ.
23. Let ABC be a triangle with $\angle$A = 90^o and AB = AC. Let D and E be points on the segement BC such that BD:DE:EC=3:5:4 Prove that $\angle$DAE = 45^o.
24. In a cyclic quadrilateral ABCD, let the diagonals AC and BD intersect at X. Let the circumcircles of triangles AXD and BXD intersect again at Y. If X is the incentre of triangle ABY, Show that $\angle$CAD = 90^o.
25. Let ABC be a right triangle with $\angle$ B = 90^o. Let E and F be respectively the mid-point of AB and AC. Suppose the incentre I of triangle ABC lies on the circumcircle of triangle AEF. Find the ratio of BC/AB.
26. Let ABC be a right-angled triangle with $\angle$b = 90^o. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD prove that ID = $\sqrt{b(b-a)}$ where BC=a and CA=b.
27. Let ABC be a triangle with centroid G. Let the circumcircle of triangle AGB intersect the line BC in X different from B;and the ciecumcircle of triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of triangle AXY.
28. Let ABC be a right-angled triangle with $\angle$B = 90^o. Let I be the incentre of ABC. Let AI extended intersect BC at F. Draw a perpendicular to AI at I. Let it intersect AC at E. Prove that IE = IF.
29. Let ABC be right-angle triangle with $\angle$B = 90^o. Let AD be the bisector of $\angle$a with D on BC. Let the circumcircle of triangle ACD intersect AB again in E; and let the circumcircle of triangle ABD intersect AC again in F. Let K be the reflection of E in the line BC. Prove that FK = BC.

30. Let ABC be a triangle and D be the mid-point of BC. Suppose the angle bisector of $\angle ADC$ is tangent to the circumcircle of triangle ABD at D. Prove that $\angle A = 90^o$.
31. Let ABC be a right-angle triangle with $\angle B = 90^o$. Let I be the incentre of ABC. Extend AI and CI;let them intersect BC in D and AB in E respectively. Draw a line perpendicular to AI at I to meet AC in J; draw a line perpendicular to CI at I to meet AC in K. Suppose DJ = EK. Prove that BA = BC.
32. Let ABC be the Isosceles triangle with AB = AC. Let Γ be its circumcircle and let O be the center of Γ. Let CO meet Γ in D. Draw line parllel to AC through D. Let it intresect AB at E. Suppose AE : EB = 2 : 1. Prove that ABC is an equilateral triangle.
33. Let a, b, c be positive real number such that

$$\frac{ac}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1$$

Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$$

34. Let ABC be a triangle, AD an altitude and AE a median. Assume B, D, E, C lie in that order on the line BC. Suppose the incentre of triangle ABE lies on AD and the incentre ADC lies on AE. Find, with proof, the angles of triangle ABC.
35. Let AOB be a given angle less than 180^o and let P be an interior point of the angular region determined by $\angle AOB$. Show with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray QA and D lies on the ray OB, and CP:PD=1:2.
36. Let Ω be the circle with chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise ,let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersect Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is diameter of Ω.