# Linear Algebra through Coordinate Geometry

G V V Sharma*

Contents



*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. 

***Abstract*—This book provides a computational approach to linear algebra and matrices by solving problems in 2D and 3D coordinate geometry from IIT-JEE. An introduction to convex optimization is also provided in the process. Links to sample Python codes are available in the text. The book provides sufficient math basics for Machine Learning and is also recommended for high school students who wish to explore topics in Artificial Intelligence.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    linalg/book/codes
```

## 1 The Straight Line

### 1.1 Point

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as

$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$

2. The *norm* of a vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$

is defined as

$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$

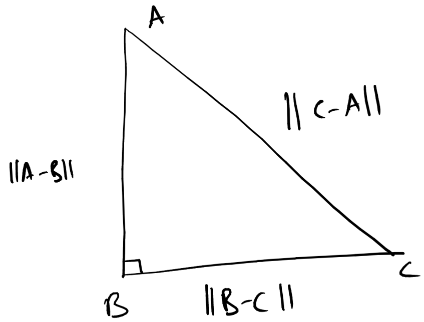


Fig. 1.1.7

3. The *length* of $PQ$ is defined as
$$\|\mathbf{P}-\mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line $PQ$ is defined as
$$\mathbf{P}-\mathbf{Q} = \begin{pmatrix} p_1-q_1 \\ p_2-q_2 \end{pmatrix} \tag{1.1.4}$$
5. The point dividing $PQ$ in the ratio $k:1$ is
$$\mathbf{R} = \frac{k\mathbf{P}+\mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.7}$$
**Solution:** Using Baudhayana's theorem,
$$\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{C}-\mathbf{A}\|^2 \tag{1.1.7}$$
$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) + (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A})$$
$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on $AB$ in Fi.g 1.1.7. Show that
$$(\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.8}$$
9. If $\mathbf{x}, \mathbf{y}$ are any two points on $AB$, show that
$$(\mathbf{x}-\mathbf{y})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1.9}$$

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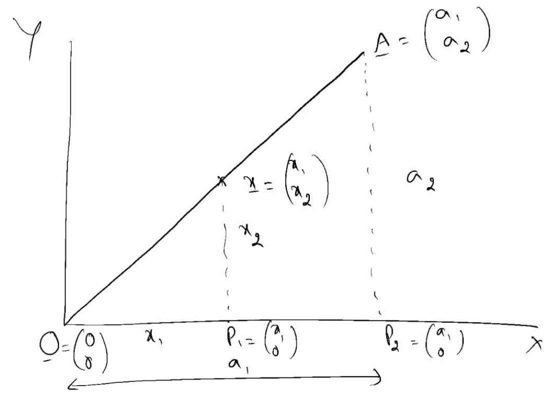


Fig. 1.2.1

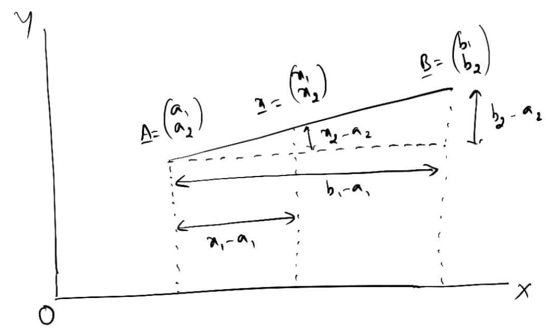


Fig. 1.2.2

### *1.2 Line*

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of $OA$.
**Solution:** Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on $OA$. Then, using similar triangles,
$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.2.1.1}$$
$$\implies x_2 = mx_1 \tag{1.2.1.2}$$
where $m$ is known as the slope of the line. Thus, the equation of the line is
$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ m \end{pmatrix} = x_1\mathbf{m} \tag{1.2.1.3}$$
In general, the above equation is written as
$$\mathbf{x} = \lambda\mathbf{m}, \tag{1.2.1.4}$$
where $\mathbf{m}$ is the direction vector of the line.
2. Find the equation of $AB$ in Fig. 1.2.2

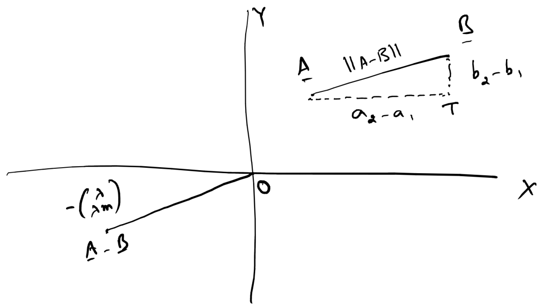


Fig. 1.2.5

**Solution:** From Fig. 1.2.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.2.2.1}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.2.2.2}$$

From (1.2.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.2.2.3}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2.4}$$

$$= \mathbf{A} + \lambda\mathbf{m} \tag{1.2.2.5}$$

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (1.2.2.5) is obtained from (1.2.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 1.2.1
**Solution:** Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.2.4.1}$$

Also, from (1.2.1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.2.4.2}$$

Note that $\lambda$ is the variable that determines the length of $\mathbf{A}$, since $m$ is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
**Solution:** See Fig. 1.2.5. From (1.2.2.5), for some $\lambda$,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.5.1}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.2.5.2}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.
6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of $AB$ is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.2.7.1}$$

8. The *normal* to the vector $\mathbf{m}$ is defined as

$$\mathbf{n}^T\mathbf{m} = 0 \tag{1.2.8.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{1.2.8.2}$$

9. From (1.2.7.1), the equation of a line can also be expressed as

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) \tag{1.2.9.1}$$

$$\implies \mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} = c \tag{1.2.9.2}$$

10. The unit vectors on the $x$ and $y$ axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.2.10.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.10.2}$$

11. If $a$ be the *intercept* of the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.11.1}$$

on the $x-$axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line. Thus,

$$\mathbf{n}^T\begin{pmatrix} a \\ 0 \end{pmatrix} = c \tag{1.2.11.2}$$

$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \tag{1.2.11.3}$$

12. The *rotation matrix* is defined as

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.2.12}$$

where $\theta$ is anti-clockwise.
13. 

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.13}$$

where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix.*
14. Find the equation of line $L$ in Fig. 1.2.14.
**Solution:** The equation of the $x-$axis is

$$\mathbf{x} = \lambda\mathbf{e}_1 \tag{1.2.14.1}$$

Translation by $p$ units along the $y-$axis results

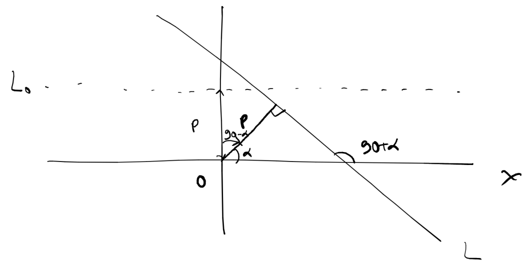


Fig. 1.2.14

in

$$L_0: \quad \mathbf{x} = \lambda \mathbf{e}_1 + p\mathbf{e}_2 \tag{1.2.14.2}$$

Rotation by $90° - \alpha$ in the anti-clockwise direction yields

$$L: \quad \mathbf{x} = \mathbf{Q}\left\{\lambda \mathbf{e}_1 + p\mathbf{e}_2\right\} \tag{1.2.14.3}$$

$$= \lambda \mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \tag{1.2.14.4}$$

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \tag{1.2.14.5}$$

$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \tag{1.2.14.6}$$

From (1.2.14.4),

$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda \mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$

$$= \lambda \mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \tag{1.2.14.7}$$

resulting in

$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \tag{1.2.14.8}$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.15.1}$$

is

$$p = \frac{c}{\|n\|} \tag{1.2.15.2}$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.16.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.16.2}$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \tag{1.2.16.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{1.2.16.4}$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{1.2.17.1}$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.18.1}$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \tag{1.2.18.2}$$

**Solution:** Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{1.2.18.3}$$

where

$$\mathbf{n}^T\mathbf{A} = c, \mathbf{n}^T\mathbf{m} = 0 \tag{1.2.18.4}$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line $L$ becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \tag{1.2.18.5}$$

$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \tag{1.2.18.6}$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.2.19.1}$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.2.19.2}$$

where

$$\mathbf{V} = \mathbf{V}^T \tag{1.2.19.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{1.2.19.4}$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.2.20.1}$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \tag{1.2.20.2}$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{1.2.21.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{1.2.21.2}$$

**Solution:** From (1.2.2.5),

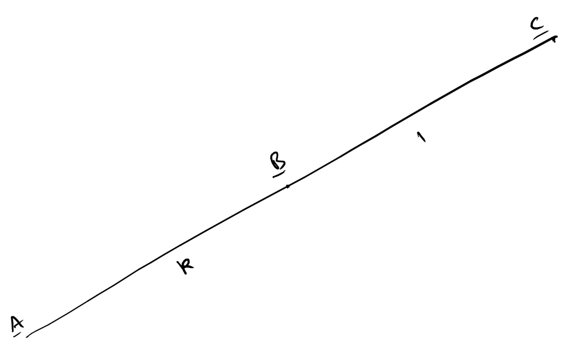


Fig. 1.2.21

$$\begin{aligned}\mathbf{B} &= \mathbf{A} + \lambda_1\mathbf{m}\\ \mathbf{B} &= \mathbf{C} - \lambda_2\mathbf{m}\end{aligned} \tag{1.2.21.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{1.2.21.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{1.2.21.5}$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A}-\mathbf{B} = k\left(\mathbf{B}-\mathbf{C}\right) \tag{1.2.21.6}$$

resulting in (1.2.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{1.2.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{1.2.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{1.2.23.2}$$
$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{1.2.23.3}$$

24. In $\Delta ABC$, Let $\mathbf{P}$ be a point on $BC$ such that $AP \perp BC$. Then $AP$ is defined to be an *altitude* of $\Delta ABC$.
25. Find the intersection of $AP$ and $BQ$.
**Solution:** The normal vector of $AP$ is $\mathbf{B}-\mathbf{C}$. From (1.2.8.1) and (1.2.9.2), the equation of $AP$ and $BQ$ are

$$\left(\mathbf{B}-\mathbf{C}\right)^T\left(\mathbf{x}-\mathbf{A}\right) = 0 \tag{1.2.25.1}$$
$$\left(\mathbf{C}-\mathbf{A}\right)^T\left(\mathbf{x}-\mathbf{B}\right) = 0 \tag{1.2.25.2}$$

which can be solved to obtain the intersection point using (1.2.16.3).

26. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.26.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.26.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{1.2.26.3}$$

27. Find the equation of a line passing through the intersection of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.27.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.27.2}$$

and passing through the point $\mathbf{p}$.
**Solution:** The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T}\mathbf{c} \tag{1.2.27.3}$$

where

$$\mathbf{N} = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix} \tag{1.2.27.4}$$
$$\mathbf{c} = \begin{pmatrix}c_1\\ c_2\end{pmatrix} \tag{1.2.27.5}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda\left(\mathbf{N}^{-T}\mathbf{c} - \mathbf{p}\right) \tag{1.2.27.6}$$
$$\implies \mathbf{N}^T\mathbf{x} = \mathbf{N}^T\mathbf{p} + \lambda\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right) \tag{1.2.27.7}$$

resulting in

$$\begin{aligned}&\left(\mathbf{c}-\mathbf{N}^T\mathbf{p}\right)^T\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{N}^T\mathbf{x}\\ &= \left(\mathbf{c}-\mathbf{N}^T\mathbf{p}\right)^T\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{N}^T\mathbf{p}\end{aligned} \tag{1.2.27.8}$$

28. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.28.1}$$

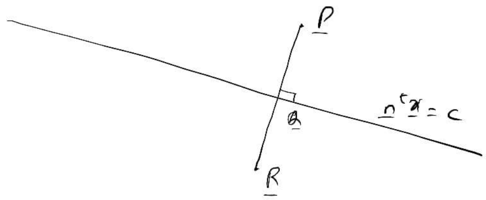


Fig. 1.2.28

**Solution:** Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on $L$, $\mathbf{Q}$ bisects $PR$. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.2.28.2}$$

$$\mathbf{n}^T\mathbf{Q} = c \tag{1.2.28.3}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{1.2.28.4}$$

where $\mathbf{m}$ is the direction vector of $L$. From (1.2.28.2) and (1.2.28.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{1.2.28.5}$$

From (1.2.28.5) and (1.2.28.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{1.2.28.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{1.2.28.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{1.2.28.9}$$

(1.2.28.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{1.2.28.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{1.2.28.12}$$

29. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.2.29.1}$$

## *1.3 Example*

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.3.1.1}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{1.3.1.2}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4 \tag{1.3.1.3}$$

Find the area of $\triangle ABC$.
**Solution:** The centroid $\mathbf{O}$ is the solution of (1.3.1.2),(1.3.1.3) and is obtained as the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} \tag{1.3.1.4}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix}1 & 1 & 5\\1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 5\\0 & 1 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & 1\end{pmatrix} \tag{1.3.1.5}$$

Thus,

$$\mathbf{O} = \begin{pmatrix}4\\1\end{pmatrix} \tag{1.3.1.6}$$

Let $AD$ be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.3.1.7}$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix}11\\1\end{pmatrix} \tag{1.3.1.8}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}11\\1\end{pmatrix} \tag{1.3.1.9}$$

From (1.3.1.3) and (1.3.1.9),

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} = 5 \tag{1.3.1.10}$$

$$\implies 5 + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 12 \tag{1.3.1.11}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 7 \tag{1.3.1.12}$$

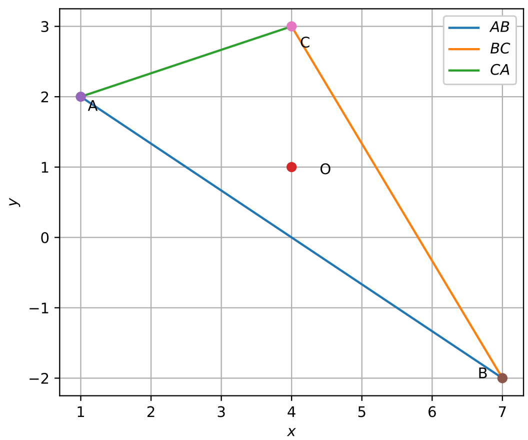


Fig. 1.3.2

From (1.3.1.12) and (1.3.1.3), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.3.1.13}$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.3.1.14}$$

$$\implies \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.3.1.15}$$

From (1.3.1.8),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.3.1.16}$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \tag{1.3.1.17}$$

2. Summarize all the above computations through a Python script and plot $\triangle ABC$.
**Solution:**

```
codes/2d/triang.py
```

### 1.4 Programming

1. Find the *orthocentre* of $\triangle ABC$.
**Solution:** The following code finds the required point using (1.2.25.1) and (1.2.25.2) .

```
codes/2d/orthocentre.py
```

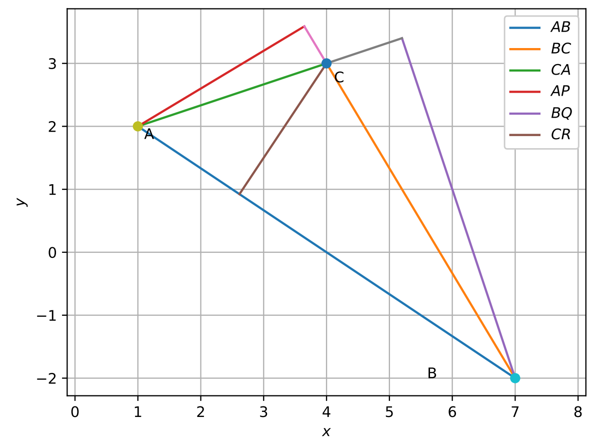


Fig. 1.4.4

2. Find $\mathbf{P}$, the foot of the altitude from $\mathbf{A}$ upon BC.
**Solution:**

```
codes/2d/alt_foot.py
```

3. Find $\mathbf{Q}$ and $\mathbf{R}$.
4. Draw $AP, BQ$ and $CR$ and verify that they meet at a point $\mathbf{H}$.
**Solution:** The following code plots the altitudes in Fig. 1.4.4

```
codes/2d/alt_draw.py
```

5. Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of $AB, BC$ and $CA$ respectively for $\Delta ABC$.
6. Find the equations of $AD, BE$ and $CF$.
7. Find the point of intersection of $AD$ and $CF$.
8. Verify that $\mathbf{O}$ is the point of intersection of $BE, CF$ as well.
9. Graphically show that the medians of $\Delta ABC$ meet at the centroid.

### 1.5 Solved Problems

1. A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.5.1}$$

$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.5.1}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides $AB$.

**Solution:** Let

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.5.1}$$

Then (1.5.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.5.1}$$

$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5.1}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.5.1) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.5.1}$$

$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.5.1}$$

Let $\mathbf{O}$ divide the segment $AB$ in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.5.1}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.5.1}$$

$$\mathbf{A} = -k\mathbf{B} \tag{1.5.1}$$

Substituting in (1.5.1), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{1.5.1}$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{1.5.1}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.5.1}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.5.2}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{1.5.2}$$

by $d = 2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to $L$.
**Solution:** From (1.5.2), the direction vector of $L$ is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.2}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \tag{1.5.2}$$

However,

$$PQ = d \tag{1.5.2}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \, \|\mathbf{m}\| = d \tag{1.5.2}$$

$$\implies \lambda = \pm \frac{d}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{1.5.2}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{1.5.2}$$

from (1.5.2). Since $\mathbf{Q}$ lies in the third quadrant, from (1.5.2) and (1.5.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{1.5.2}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \tag{1.5.2}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - 2\sqrt{6} \tag{1.5.2}$$

3. Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.5.3}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{1.5.3}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{1.5.3}$$

find its vertices.
**Solution:** From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{1.5.3}$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1.5.3}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2}$$

$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.5.3}$$

$$\because AD \parallel BC,$$
$$BC : \begin{pmatrix} 7 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} = -15 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \implies \mathbf{B} = -\frac{1}{3} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.5.3}$$

$$\because AB \parallel CD,$$
$$CD : \begin{pmatrix} 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \implies \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \tag{1.5.3}$$

4. Let $k$ be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{1.5.4}$$

has area 28. Find the orthocentre of this triangle.
**Solution:** Let $\mathbf{m}_1$ be the direction vector of $BC$. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \tag{1.5.4}$$

If $AD$ be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.5.4}$$

Similarly, considering the side $AC$ the equation of the altitude $BE$ is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \tag{1.5.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \tag{1.5.4}$$

The orthocentre is obtained by solving (1.5.4) and (1.5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \tag{1.5.4}$$

which can be expressed using (1.5.4), (1.5.4), (1.5.4) and (1.5.4) as

$$\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix}$$
$$= k \begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix} \tag{1.5.4}$$

From (1.5.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$
$$\implies \begin{vmatrix} k & 5 - k & -2k \\ -3k & 4k & 2 + 3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \tag{1.5.4}$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \tag{1.5.4}$$
$$\implies 5k^2 + 13k - 46 = 0 \tag{1.5.4}$$
$$\text{or, } k = 2, -\frac{23}{5} \tag{1.5.4}$$

Substituting the above in (1.5.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.5.5}$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.
**Solution:** Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the

given information,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0} \tag{1.5.5}$$

If $AB$ be the line in (1.5.5), the equation of $CF$, where

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.5.5}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.5.5}$$

since $CF$ passes through the origin and $CF \perp AB$. From (1.5.5) and (1.5.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.5.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

From (1.5.5),

$$\mathbf{C} = -(\mathbf{A}+\mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

after substituting from (1.5.5). Thus,

$$CF = \|\mathbf{C}-\mathbf{F}\| = 3\sqrt{2} \tag{1.5.5}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = 2\sqrt{6} \tag{1.5.5}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 6\sqrt{3} \tag{1.5.5}$$

6. A square, of each side 2, lies above the $x$-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle $30^\circ$ with the positive direction of the $x$-axis, then find the sum of the $x$-coordinates of the vertices of the square.
**Solution:** Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the $x$ and $y$-axis respectively. Then

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.6}$$

Multiplying (1.5.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.5.6}$$

$$\mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}) = 4\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= 4\begin{pmatrix} \cos\theta - \sin\theta \\ \cos\theta + \sin\theta \end{pmatrix} \tag{1.5.6}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D})$$
$$= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3}-1\right) \tag{1.5.6}$$

for $\theta = 30^\circ$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.

## 2 The Circle

### 2.1 Definitions

1. The equation of a circle is

$$\|\mathbf{x}-\mathbf{c}\| = r \tag{2.1.1.1}$$

where $\mathbf{c}$ is the centre and $r$ is the radius.

2. By expanding (2.1.1.1), the equation of a circle can also be expressed as

$$\|\mathbf{x}-\mathbf{c}\|^2 = r^2 \tag{2.1.2.1}$$
$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} - r^2 = 0 \tag{2.1.2.2}$$

3. Find the equation of the *circumcircle* of $\triangle ABC$ in Fig. 2.1.4.
**Solution:** Let $\mathbf{O}$ be the centre and $R$ the radius. From (2.1.2.2),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = R^2 \tag{2.1.3.1}$$
$$\implies \|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \tag{2.1.3.2}$$

which can be simplified to obtain

$$(\mathbf{A}-\mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad \text{and} \tag{2.1.3.3}$$
$$(\mathbf{A}-\mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{C}\|^2}{2} \tag{2.1.3.4}$$

Solving the two yields $\mathbf{O}$, which can then be used to obtain $R$.

4. Given $OD \perp BC$ as in Fig. 2.1.4. Show that

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{2.1.4.1}$$

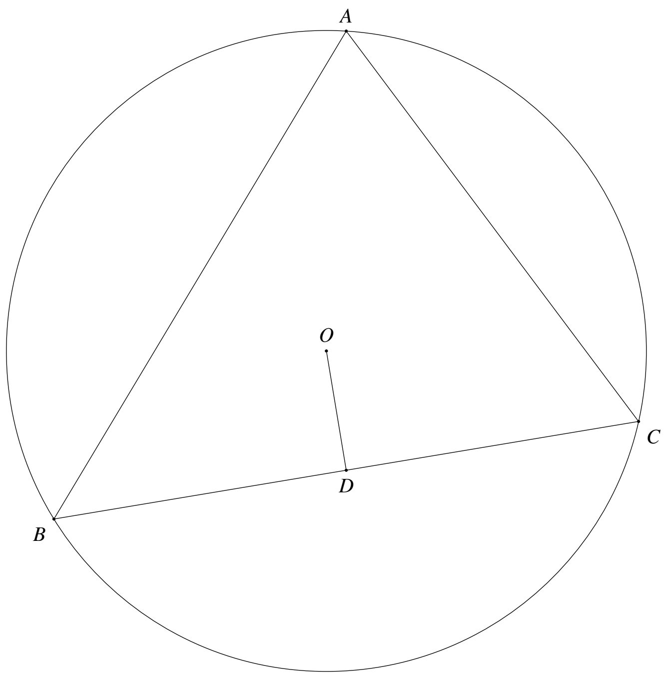


Fig. 2.1.4: Circumcircle.

**Solution:** From (2.1.1.1)

$$\|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = R^2 \tag{2.1.4.2}$$

$$\implies (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = (\mathbf{C}-\mathbf{O})^T(\mathbf{C}-\mathbf{O}) \tag{2.1.4.3}$$

$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{B}+\mathbf{C}}{2}-\mathbf{O}\right) = 0 \tag{2.1.4.4}$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{O}) = 0 \tag{2.1.4.5}$$

Since $D$ and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on $BC$, using (1.2.7.1),

$$\frac{\mathbf{B}+\mathbf{C}}{2} = \mathbf{B} + \lambda_1(\mathbf{B}-\mathbf{C}) \tag{2.1.4.6}$$

$$\mathbf{D} = \mathbf{B} + \lambda_2(\mathbf{B}-\mathbf{C}) \tag{2.1.4.7}$$

Multiplying (2.1.4.6) and (2.1.4.7) with $(\mathbf{B}-\mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{2.1.4.8}$$

5. Let $\mathbf{D}$ be the mid point of $BC$. Show that $OD \perp BC$.
6. The *incircle* with centre $\mathbf{I}$ and radius $r$ in Fig.2.1.6 is inside $\triangle ABC$ and touches $AB, BC$ and $CA$ at $\mathbf{W}, \mathbf{U}$ and $\mathbf{V}$ respectively. $AB, BC$ and $CA$ are known as *tangents* to the circle.

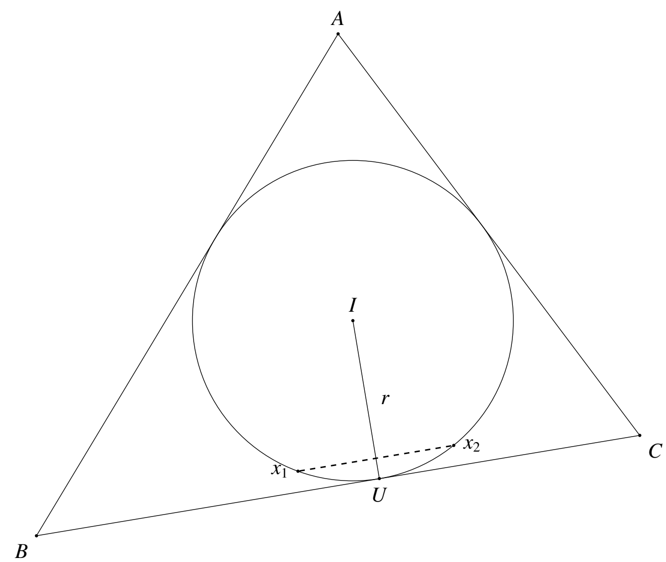


Fig. 2.1.6: Tangent and incircle.

7. Show that $IU \perp BC$.
**Solution:** Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1-\mathbf{I}\|^2 - \|\mathbf{x}_2-\mathbf{I}\|^2 = 0 \tag{2.1.7.1}$$

$$\implies (\mathbf{x}_1-\mathbf{x}_2)^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{I}\right) = 0 \tag{2.1.7.2}$$

$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{I}\right) = 0 \tag{2.1.7.3}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, $x_1x_2$ merges into $BC$ and the above equation becomes

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{U}-\mathbf{I}) = 0 \implies OD \perp BC \tag{2.1.7.4}$$

8. Give an alternative proof for the above.
**Solution:** Let

$$\mathbf{B} = \mathbf{0} \tag{2.1.8.1}$$

$$\mathbf{U} = \lambda\mathbf{m} \tag{2.1.8.2}$$

Then

$$\|\mathbf{U}-\mathbf{I}\|^2 = r^2 \quad (2.1.8.3)$$
$$\implies \lambda^2\|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{I} + \|\mathbf{I}\|^2 = r^2 \quad (2.1.8.4)$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T\mathbf{I}}{\|\mathbf{m}\|^2} \quad (2.1.8.5)$$

Thus,

$$(\mathbf{U}-\mathbf{B})^T(\mathbf{U}-\mathbf{I}) = (\lambda\mathbf{m})^T(\lambda\mathbf{m}-\mathbf{I}) \quad (2.1.8.6)$$
$$= \lambda^2\|\mathbf{m}\|^2 - \lambda\mathbf{m}^T\mathbf{I} \quad (2.1.8.7)$$
$$= \mathbf{O} \text{ (from 2.1.8.5).} \quad (2.1.8.8)$$
$$\implies OD \perp BC \quad (2.1.8.9)$$

9. Find the equation of the tangent at $\mathbf{U}$.
**Solution:** The equation of the tangent is given by

$$(\mathbf{I}-\mathbf{U})^T(\mathbf{x}-\mathbf{U}) = 0 \quad (2.1.9.1)$$

10. The direction vector of *normal to the circle* in (2.1.2.2) at point $\mathbf{U}$ is

$$\mathbf{n} = \mathbf{U}-\mathbf{I} \quad (2.1.10.1)$$

11. Find an expression for $r$ if $\mathbf{I}$ is known.
**Solution:** Let $\mathbf{n}$ be the normal vector of $BC$. The equation for $BC$ is then given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \quad (2.1.11.1)$$
$$\implies \mathbf{n}^T(\mathbf{U}-\mathbf{B}) = 0 \quad (2.1.11.2)$$

since $\mathbf{U}$ lies on $BC$. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \quad (2.1.11.3)$$
$$\implies \mathbf{I}-\mathbf{U} = \lambda\mathbf{n} \quad (2.1.11.4)$$
$$\text{or} \quad r = \|\mathbf{I}-\mathbf{U}\| = |\lambda|\,\|\mathbf{n}\| \quad (2.1.11.5)$$

From (2.1.11.2) and (2.1.11.3)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \quad (2.1.11.6)$$
$$\implies \mathbf{n}^T(\mathbf{I}-\mathbf{B}) = \lambda\|\mathbf{n}\|^2 \quad (2.1.11.7)$$
$$\implies r = |\lambda|\,\|\mathbf{n}\| = \frac{\left|\mathbf{n}^T(\mathbf{I}-\mathbf{B})\right|}{\|\mathbf{n}\|} \quad (2.1.11.8)$$

from (2.1.11.5). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \quad (2.1.11.9)$$
$$r = \left|\mathbf{n}_1^T(\mathbf{I}-\mathbf{B})\right| \quad (2.1.11.10)$$

12. Find $\mathbf{I}$.
**Solution:** Since $r = IU = IV = IW$, from (2.1.11.10),

$$\left|\mathbf{n}_1^T(\mathbf{I}-\mathbf{B})\right| = \left|\mathbf{n}_2^T(\mathbf{I}-\mathbf{C})\right| = \left|\mathbf{n}_3^T(\mathbf{I}-\mathbf{A})\right| \quad (2.1.12.1)$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of $CA, AB$ respectively. (2.1.12.1) can be expressed as

$$\mathbf{n}_1^T(\mathbf{I}-\mathbf{B}) = k_1\mathbf{n}_2^T(\mathbf{I}-\mathbf{C}) \quad (2.1.12.2)$$
$$\mathbf{n}_2^T(\mathbf{I}-\mathbf{C}) = k_2\mathbf{n}_3^T(\mathbf{I}-\mathbf{A}) \quad (2.1.12.3)$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3\end{pmatrix}^T \mathbf{I} = \begin{pmatrix}\mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A}\end{pmatrix} \quad (2.1.12.4)$$

13. Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$
14. Let $a = BC, b = CA$, $c = AB$, $x = BU = BW$, $y = CU = CV$, $z = AV = AW$. Find $\mathbf{U}$.
**Solution:** It is easy to verify that

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \quad (2.1.14.1)$$

which can be used to obtain $x$ and $y$. Using the section formula,

$$\mathbf{U} = \frac{x\mathbf{B}+\mathbf{A}}{x+y} \quad (2.1.14.2)$$

15. Show that the angle in a semi-circle is a right angle.

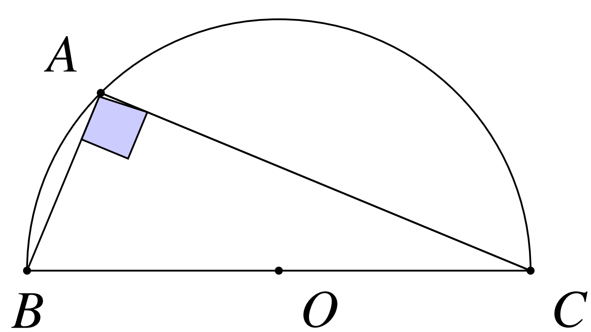


Fig. 2.1.15: Angle in a semi-circle.

**Solution:** Let

$$\mathbf{O} = 0 \quad (2.1.15.1)$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \tag{2.1.15.2}$$
$$\|\mathbf{B}-\mathbf{C}\|^2 = (2r)^2 \tag{2.1.15.3}$$
$$\mathbf{B}+\mathbf{C} = 0 \tag{2.1.15.4}$$

where $r$ is the radius of the circle. Thus,

$$\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T(\mathbf{B}+\mathbf{C}) \tag{2.1.15.5}$$

From (2.1.15.4) and (2.1.15.2),

$$\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2 = 4r^2 = \|\mathbf{B}-\mathbf{C}\|^2 \tag{2.1.15.6}$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

16. Show that $PA.PB = PC^2$, where $PC$ is the tangent to the circle in Fig. 2.1.16.

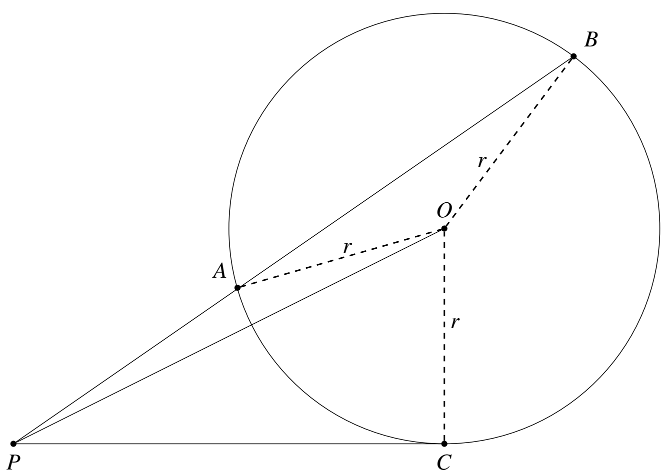


Fig. 2.1.16: $PA.PB = PC^2$.

**Solution:** Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda\|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda\mathbf{A}) \tag{2.1.16.1}$$
$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = r^2 \tag{2.1.16.2}$$
$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \tag{2.1.16.3}$$

$\because$

$$\|\mathbf{B}-\mathbf{O}\|^2 - \|\mathbf{A}-\mathbf{O}\|^2 = 0, \tag{2.1.16.4}$$
$$\left(\lambda^2 - 1\right)\|\mathbf{A}\|^2 - 2(\lambda-1)\mathbf{A}^T\mathbf{O} = 0 \tag{2.1.16.5}$$
$$\implies PA.PB = \lambda\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{2.1.16.6}$$

after substituting from (2.1.16.1) and simplifying. From (2.1.16.3),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \tag{2.1.16.7}$$
$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 = PC^2 \tag{2.1.16.8}$$

From (2.1.16.6) and (2.1.16.8),

$$PA.PB = PC^2 \tag{2.1.16.9}$$

17. In Fig. 2.1.17 show that $PA.PB = PC.PD$.

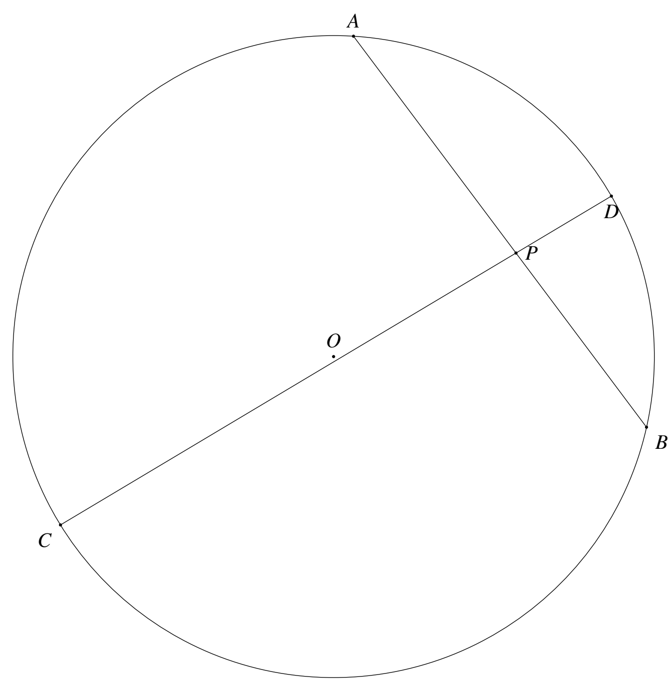


Fig. 2.1.17: Chords of a circle

**Solution:** Let $\mathbf{P} = \mathbf{0}$. We then have the following equations

$$\mathbf{B} = k_1\mathbf{A}, k_1 = \frac{PB}{PA} \quad \mathbf{D} = k_2\mathbf{C}, k_2 = \frac{PD}{PC} \tag{2.1.17.1}$$

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = \|\mathbf{D}-\mathbf{O}\|^2 = r^2 \tag{2.1.17.2}$$

where $r$ is the radius of the circle and $\mathbf{O}$ is the centre. From (2.1.17.2),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 \tag{2.1.17.3}$$

$$\implies \|\mathbf{A}-\mathbf{O}\|^2 = \|k\mathbf{A}-\mathbf{O}\|^2 \quad \text{(from (2.1.17.1))} \tag{2.1.17.4}$$

which can be simplified to obtain

$$k_1 \|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{2.1.17.5}$$

Similarly,

$$k_2 \|\mathbf{C}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.6}$$

From (2.1.17.2), we also obtain

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 \tag{2.1.17.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{2.1.17.8}$$

after simplification. Using this result in (2.1.17.5) and (2.1.17.6),

$$k_1 \|\mathbf{A}\|^2 = k_2 \|\mathbf{C}\|^2 \tag{2.1.17.9}$$

$$\implies \|\mathbf{A}\| \|\mathbf{B}\| = \|\mathbf{C}\| \|\mathbf{D}\| \tag{2.1.17.10}$$

which completes the proof.

18. (Pole and Polar:) The polar of a point $\mathbf{x}$ with respect to the curve

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.18.1}$$

is the line

$$\mathbf{n}^T\mathbf{x} = c \tag{2.1.18.2}$$

where

$$\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ 1\end{pmatrix} \tag{2.1.18.3}$$

The pole of the line in (2.1.18.2) is obtained as $\frac{1}{x_3}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$, where

$$\begin{pmatrix}x_1 \\ x_2 \\ x_3\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{n}^T \\ -c\end{pmatrix} \tag{2.1.18.4}$$

19. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be conjugate points for (2.1.18.1) if $\mathbf{x}_2$ lies on the polar of $\mathbf{x}_1$ and vice-versa. A similar definition holds for conjugate lines as well.
20. Let $\mathbf{p}$ be a point of intersection of two circles with centres $\mathbf{c}_1$ and $\mathbf{c}_2$. The circles are said to be orthogonal if their tangents at $\mathbf{p}$ are perpendicular to each other. Show that if $r_1$ and $r_2$ are their respective radii,

$$\|\mathbf{c_1} - \mathbf{c}_2\|^2 = r_1^2 + r_2^2 \tag{2.1.20.1}$$

21. Show that the length of the tangent from a point $\mathbf{p}$ to the circle

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{2.1.21.1}$$

is

$$\mathbf{p}^T\mathbf{p} - 2\mathbf{c}^T\mathbf{p} + f \tag{2.1.21.2}$$

This length is also known as the *power* of the point $\mathbf{p}$ with respect to the circle.

22. The *radical axis* of the circles

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.22.1}$$

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.2}$$

is the locus of the points from which lengths of the tangents to the circles are equal. From (2.1.25), this locus is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.3}$$

$$\implies 2\left(\mathbf{c}_1 - \mathbf{c}_2\right)^T\mathbf{x} + f_2 - f_1 = 0 \tag{2.1.22.4}$$

23. Show that the radical axis of the circles is perpendicular to the line joining their centres.
24. *Coaxal circles* have the same radical axis.
25. Obtain a family of coaxal circles from and find their *limit points*.

**Solution:** The family of circles is obtained as

$$\mathbf{x}^T\mathbf{x} - 2\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\mathbf{x} + f_1 + \lambda f_2 = 0 \tag{2.1.25.1}$$

The limit points are the centres of those cir-

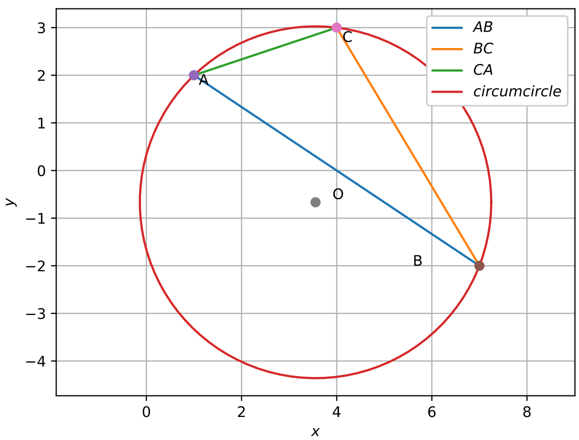


Fig. 2.2.2

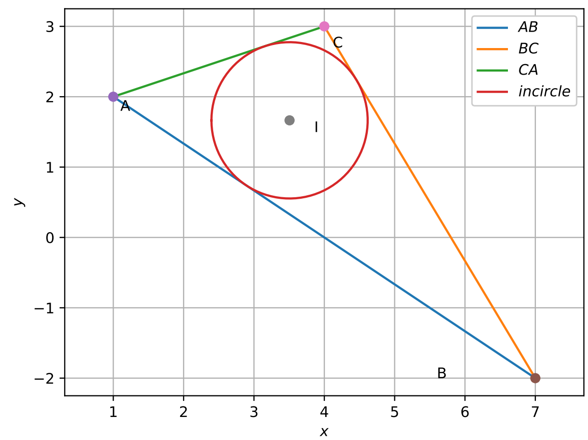


Fig. 2.2.5

cles whose radii are 0. From (2.1.25.1) and (2.1.2.2), this results in

$$f_1 + \lambda f_2 = (\mathbf{c}_1 + \lambda \mathbf{c}_2)^T (\mathbf{c}_1 + \lambda \mathbf{c}_2) \tag{2.1.25.2}$$

$$\implies \lambda^2 \|\mathbf{c}_2\|^2 + \lambda \left(2\mathbf{c}_1^T \mathbf{c}_2 - f_2\right) + \|\mathbf{c}_1\|^2 - f_1 = 0 \tag{2.1.25.3}$$

Solving for $\lambda$, the limit points are given by

$$\mathbf{c}_1 + \lambda \mathbf{c}_2 \tag{2.1.25.4}$$

## 2.2 Programming

1. Find the circumcentre $\mathbf{O}$ and radius $R$ of $\triangle ABC$ in Fig. 1.3.2
   **Solution:** $\mathbf{O}$ can be obtained from The following code computes $\mathbf{O}$ using (2.1.3.3) and (2.1.3.4)

```
codes/2d/circumcentre.py
```

2. Plot the circumcircle of $\triangle ABC$.
   **Solution:** The following code plots Fig. 2.2.2

```
codes/2d/circumcircle.py
```

3. Consider a circle with centre $\mathbf{I}$ and radius $r$ that lies within $\triangle ABC$ and touches $BC, CA$ and $AB$ at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.
4. Compute $\mathbf{I}$ and $r$.
   **Solution:** The following code uses (2.1.12.4) and (2.1.11.10) to compute $\mathbf{I}$ and $r$ respectively.

```
codes/2d/incentre.py
```

5. Plot the incircle of $\triangle ABC$
   **Solution:** The following code plots the incircle in Fig. 2.2.5

```
codes/2d/incircle.py
```

## 2.3 Example

1. Find the centre and radius of the circle

$$C_1 : \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{2.3.1.1}$$

**Solution:** let $\mathbf{c}$ be the centre of the circle. Then

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.3.1.2}$$

$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \tag{2.3.1.3}$$

$$\implies \mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} = r^2 - \mathbf{c}^T \mathbf{c} \tag{2.3.1.4}$$

Comparing with (2.3.1.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.3.1.5}$$

$$r^2 - \mathbf{c}^T \mathbf{c} = 1 \implies r = \sqrt{2} \tag{2.3.1.6}$$

2. Find the tangent to the circle $C_1$ at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
   **Solution:** From (3.1.6), the tangent $T$ is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right] \mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{2.3.2.1}$$

$$\implies T : \mathbf{n}^T \mathbf{x} = 3 \tag{2.3.2.2}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3.2.3}$$

3. The tangent $T$ in (2.3.2.2) cuts off a chord $AB$ from a circle $C_2$ whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{2.3.3.1}$$

Find $\mathbf{A} + \mathbf{B}$.
**Solution:** Let the radius of $C_2$ be $r$. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \tag{2.3.3.2}$$
$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \tag{2.3.3.3}$$

Subtracting (2.3.3.3) from (2.3.3.2),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.4}$$
$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$
$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.5}$$

$\because \mathbf{A}, \mathbf{B}$ lie on $T$, from (2.3.2.2),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{2.3.3.6}$$
$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \tag{2.3.3.7}$$

From (2.3.3.5) and (2.3.3.7)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{2.3.3.8}$$
$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{2.3.3.9}$$
$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{2.3.3.10}$$
$$\implies k = 2 \tag{2.3.3.11}$$

using (2.3.3.6). Substituting in (2.3.3.8)

$$\mathbf{A} + \mathbf{B} = 2 (\mathbf{n} + \mathbf{C}) \tag{2.3.3.12}$$

4. If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.
**Solution:** From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.4.1}$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4 \|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \tag{2.3.4.2}$$
$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \tag{2.3.4.3}$$

using (2.3.3.12) and simplifying.

5. Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \tag{2.3.5.1}$$

**Solution:**

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.5.2}$$
$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \tag{2.3.5.3}$$

From (2.3.5.3),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T \; [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \tag{2.3.5.4}$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2 (\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \tag{2.3.5.5}$$

Upon substituting from (2.3.3.3) and (2.3.3.2) and simplifying, (2.3.5.1) is obtained.

6. Find $r$.
**Solution:** (2.3.5.1) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{2.3.6.1}$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \tag{2.3.6.2}$$
$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \tag{2.3.6.3}$$
$$\implies r = \sqrt{6}. \tag{2.3.6.4}$$

7. Summarize all the above computations through a Python script and plot the tangent and circle.
**Solution:** The following code generates Fig. 2.3.7.

```
wget
codes/2d/circ.py
```

### 2.4 Lagrange Multipliers

1. Find

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 = r^2 \tag{2.4.1.1}$$
$$\text{s.t.} \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{2.4.1.2}$$

by plotting the circles $f(\mathbf{x})$ for different values of $r$ along with the line $g(\mathbf{x})$.
**Solution:** The following code plots Fig. 2.4.1

```
codes/optimization/2.1.py
```

2. Show that

$$\min r = \frac{5}{\sqrt{2}} \tag{2.4.2.1}$$

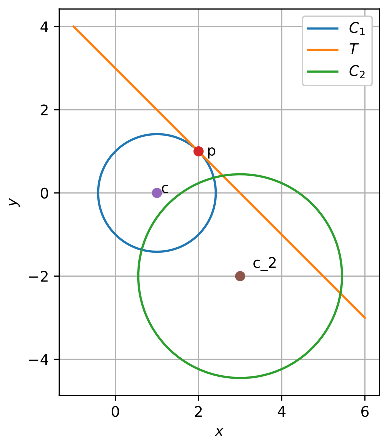


Fig. 2.3.7

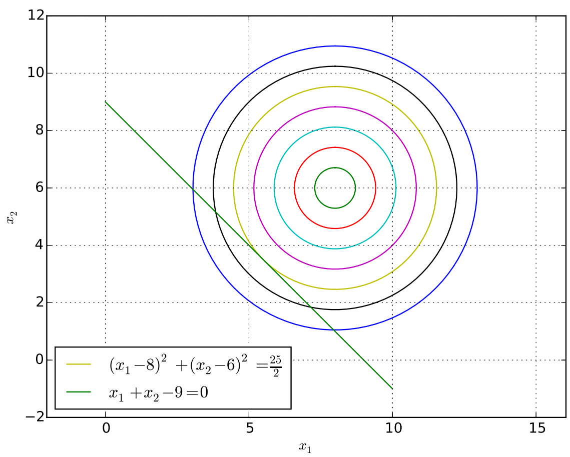


Fig. 2.4.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$

.

3. Show that

$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.4.3.1}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{2.4.3.2}$$

4. Show that

$$\nabla f(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\} \tag{2.4.4.1}$$

is the direction vector of the normal at $\mathbf{x}$.

5. From Fig. 2.4.1, show that

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{2.4.5.1}$$

where $\mathbf{p}$ is the point of contact.

6. Use (2.4.5.1) and $\mathbf{g}(\mathbf{p}) = 0$ from (2.4.1.2) to obtain $\mathbf{p}$.
7. Define

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{2.4.7.1}$$

and show that $\mathbf{p}$ can also be obtained by solving the equations

$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{2.4.7.2}$$

What is the sign of $\lambda$? $L$ is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

**Solution:**

```
codes/optimization/2.3.py
```

### *2.5 Inequality Constraints*

1. Modify the code in problem 2.4.1 to find a graphical solution for minimising

$$f(\mathbf{x}) \tag{2.5.1.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{2.5.1.2}$$

**Solution:** This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. 2.5.1 . It is clear that this radius is 0.

```
codes/optimization/2.4.py
```

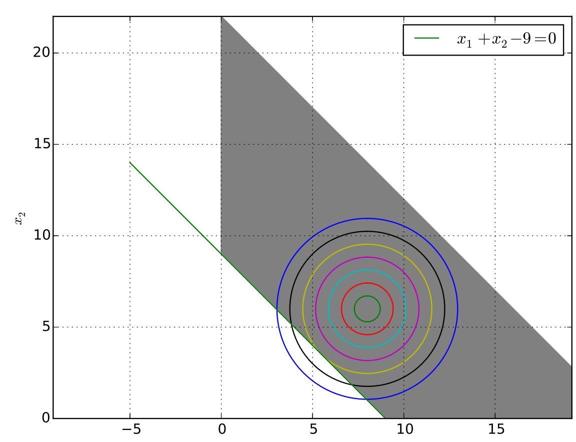


Fig. 2.5.1: Smallest circle in the shaded region is a point.

2. Now use the method of Lagrange multipliers to solve problem 2.5.1 and compare with the graphical solution. Comment.
**Solution:** Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem 2.5.1, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.
3. Repeat problem 2.5.2 by keeping $\lambda = 0$. Comment.
**Solution:** Keeping $\lambda = 0$ results in $\mathbf{x} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, which is the correct solution. The minimum value of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.
4. Find a graphical solution for minimising

$$f(\mathbf{x}) \tag{2.5.4.1}$$

with constraint

$$g(\mathbf{x}) \leq 0 \tag{2.5.4.2}$$

Summarize your observations.
**Solution:** In Fig. 2.5.4, the shaded region represents the constraint. Thus, the solution is the same as the one in problem 2.5.1. This implies that the method of Lagrange multipliers can be used to solve the optimization problem with this inequality constraint as well. Table 2.5.4 summarizes the conditions for this based on the observations so far.

```
codes/optimization/2.7.py
```

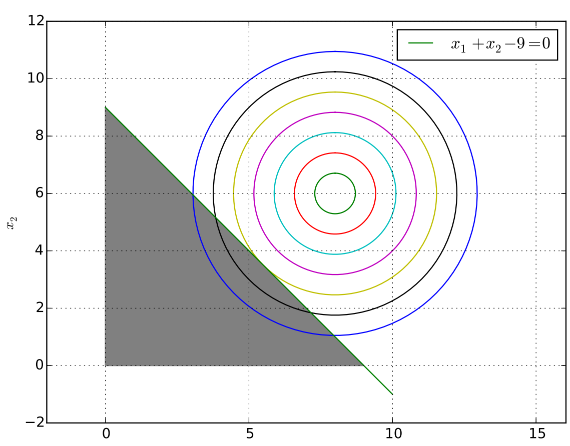


Fig. 2.5.4: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

TABLE 2.5.4: Summary of conditions.

| Cost | Constraint | $\lambda$ |
|---|---|---|
| $f(\mathbf{x})$ | $g(\mathbf{x}) = 0$ | $< 0$ |
| | $g(\mathbf{x}) \geq 0$ | 0 |
| | $g(\mathbf{x}) \leq 0$ | $< 0$ |

5. Find a graphical solution for

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 \tag{2.5.5.1}$$

with constraint

$$g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 = 0 \tag{2.5.5.2}$$

**Solution:**

```
codes/optimization/2.8.py
```

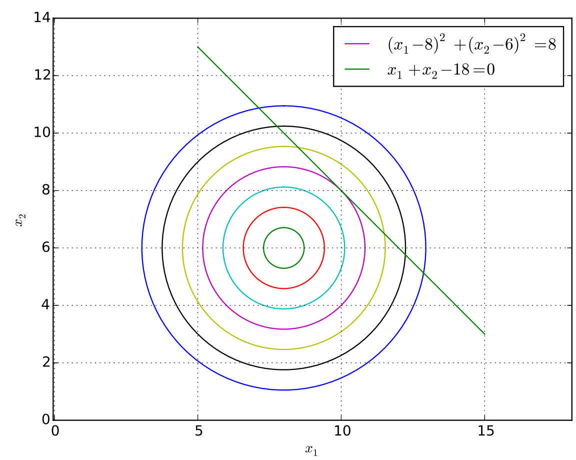


Fig. 2.5.5: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

6. Repeat problem 2.5.5 using the method of Lagrange mutipliers. What is the sign of $\lambda$?
**Solution:** Using the following python script, $\lambda$ is positive and the minimum value of $f$ is 8.

```
codes/optimization/2.9.py
```

7. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{2.5.7.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{2.5.7.2}$$

**Solution:** Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem 2.5.5.

8. Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \tag{2.5.8.1}$$

**Solution:** Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 \leq 0, \lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \tag{2.5.8.2}$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \tag{2.5.8.3}$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \tag{2.5.8.4}$$

else, $\lambda = 0$.

9. Solve

$$\min_{\mathbf{x}} \quad x_1 + x_2 \tag{2.5.9.1}$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \tag{2.5.9.2}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

**Solution:**

*Graphical solution:*

```
codes/optimization/2.15.py
```

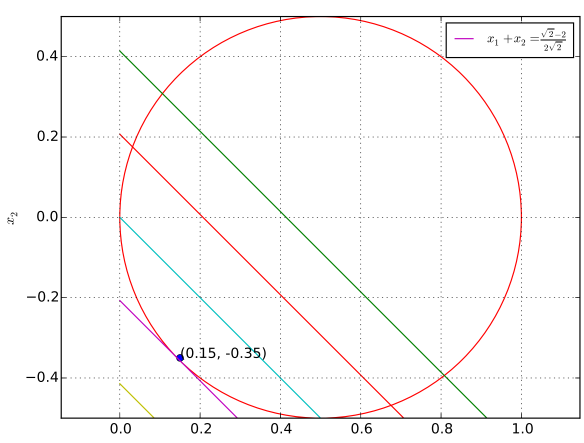


Fig. 2.5.9: Optimal solution is the lower tangent to the circle

*2.6 Solved Problems*

1. A circle passes through the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{2.6.1.1}$$

find its radius.

**Solution:** Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.6.1.2}$$

The direction vector of $AB$ is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{2.6.1.3}$$

$$\because OC \perp AB,$$

$$OC : \mathbf{m}^T (\mathbf{x} - \mathbf{C}) = 0$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{2.6.1.4}$$

Thus, $\mathbf{O}$ is the intersection of (2.6.1.1) and (2.6.1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{2.6.1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.6.1.6}$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \tag{2.6.1.7}$$

2. If a circle $C_1$, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} = 4 \tag{2.6.2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle $C$ on the $x$-axis.

**Solution:** From (2.6.2.1), the centre of $C_2$ is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.6.2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T\mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.6.2.3}$$

Since the radius of $C_1$ is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$
$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$
$$\implies \mathbf{O}_1 = \begin{pmatrix}5\\2\end{pmatrix} \tag{2.6.2.4}$$

The intercepts of $C_1$ on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda\mathbf{m} \tag{2.6.2.5}$$

where

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.6.2.6}$$

Susbtituting in the equation for $C_1$,

$$\|\lambda\mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \tag{2.6.2.7}$$

which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$
$$\implies \lambda^2 - 10\lambda + 20 = 0 \tag{2.6.2.8}$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \tag{2.6.2.9}$$

after substituting from (2.6.2.6) and (2.6.2.4).

3. A line drawn through the point

$$\mathbf{P} = \begin{pmatrix}4\\7\end{pmatrix} \tag{2.6.3.1}$$

cuts the circle

$$C : \mathbf{x}^T\mathbf{x} = 9 \tag{2.6.3.2}$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$. Draw $PAB$ for any two points $\mathbf{A}, \mathbf{B}$ on the circle.

**Solution:** Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line $PAB$ can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{2.6.3.3}$$

for $\|\mathbf{m}\| = 1$. The intersection of $L$ and $C$ yields

$$(\mathbf{P} + \lambda\mathbf{m})^T (\mathbf{P} + \lambda\mathbf{m}) = 9$$
$$\implies \lambda^2 + 2\lambda\mathbf{m}^T\mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \tag{2.6.3.4}$$

The product of the roots in (2.6.3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \tag{2.6.3.5}$$

4. Find the equation of the circle $C_2$, which is the mirror image of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.6.4.1}$$

in the line

$$L : \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3. \tag{2.6.4.2}$$

**Solution:** From (2.6.4.1), circle $C_1$ has centre at

$$\mathbf{O}_1 = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.6.4.3}$$

and radius

$$r_1 = \mathbf{O}_1^T\mathbf{O}_1 = 1 \tag{2.6.4.4}$$

The centre of $C_2$ is the reflection of $\mathbf{O}_1$ about $L$ and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{O}_1 + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.6.4.5}$$

where the relevant parameters are obtained from (2.6.4.2) as

$$\mathbf{n} = \begin{pmatrix}1 & 1\end{pmatrix}, \mathbf{m} = \begin{pmatrix}1 & -1\end{pmatrix}, c = 3. \tag{2.6.4.6}$$

Substituting the above in (2.6.4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}}{4}\mathbf{O}_1 + c\frac{\mathbf{n}}{2}$$
$$\implies \mathbf{O}_2 = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.6.4.7}$$

Thus

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix}3\\4\end{pmatrix}\right\| = 1 \tag{2.6.4.8}$$

5. One of the diameters of the circle, given by

$$C : \mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 12 \tag{2.6.5.1}$$

is a chord of a circle $S$, whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix}-3\\2\end{pmatrix}. \tag{2.6.5.2}$$

Find the radius of $S$.

**Solution:** From (2.6.5.1), the centre of $C$ is

$$\mathbf{O}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{2.6.5.3}$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T\mathbf{O}_1 - 12} = 5 \tag{2.6.5.4}$$

From (2.6.5.3) and (2.6.5.2),

$$O_1O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$
$$\implies r_2 = \sqrt{O_1O_2^2 - r_1^2} = 5 \tag{2.6.5.5}$$

6. A circle $C$ passes through

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \tag{2.6.6.1}$$

and touches the $y$-axis at

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{2.6.6.2}$$

Which one of the following equations can represent a diameter of this circle?

(i) $\begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} = 6$ (iii) $\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 3$

(ii) $\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} + 10 = 0$(iv) $\begin{pmatrix} 5 & 2 \end{pmatrix}\mathbf{x} + 4 = 0$

**Solution:** Let $\mathbf{O}$ be the centre of $C$. Then the equation of the normal, OQ is

$$\begin{pmatrix} 0 & 1 \end{pmatrix}(\mathbf{O} - \mathbf{Q}) = 0$$
$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{O} = 2 \tag{2.6.6.3}$$

Also,

$$\|\mathbf{O} - \mathbf{P}\|^2 = \|\mathbf{O} - \mathbf{Q}\|^2$$
$$\implies 2(\mathbf{P} - \mathbf{Q})^T\mathbf{O} = \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2$$
$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{O} = -4 \tag{2.6.6.4}$$

(2.6.6.3) and (2.6.6.4) result in the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\mathbf{O} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{2.6.6.5}$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & -4 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{O} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{2.6.6.6}$$

Hence, option ii) is correct.

7. Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{2.6.7.1}$$

whose centre $\mathbf{O}$ is the point of intersection of the straight lines

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{2.6.7.2}$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 1 \tag{2.6.7.3}$$

**Solution:** From (2.6.7.2) and (2.6.7.3), we obtain the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{O} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.6.7.4}$$

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 1 \end{pmatrix} \implies \mathbf{O} = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{2.6.7.5}$$

Thus, the equation of the desired tangent is

$$(\mathbf{O} - \mathbf{P})^T(\mathbf{x} - \mathbf{P}) = 0$$
$$\implies \begin{pmatrix} 1 & 4 \end{pmatrix}\mathbf{x} = -3 \tag{2.6.7.6}$$

8. The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.1}$$

intersects the circle

$$\Omega : \left\|\mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right\| = 5 \tag{2.6.8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of $PQ$ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{R} = -\frac{3}{5} \tag{2.6.8.3}$$

Find $m$.
**Solution:** Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.6.8.4}$$

The intersection of (2.6.8.1) and (2.6.8.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{2.6.8.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^T (\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \quad (2.6.8.6)$$

Since $\mathbf{P}, \mathbf{Q}$ lie on $\Gamma$,

$$\mathbf{P} = \mathbf{c} + \lambda_1 \mathbf{m} \quad (2.6.8.7)$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2 \mathbf{m} \quad (2.6.8.8)$$

$$\implies \frac{\mathbf{P} + \mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \mathbf{m} \quad (2.6.8.9)$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P} + \mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{m} \quad (2.6.8.10)$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} - \frac{\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \quad (2.6.8.11)$$

using the sum of roots in (2.6.8.6). From (2.6.8.3) and (2.6.8.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5} \left(1 + m^2\right) \quad (2.6.8.12)$$

$$\implies m^2 - 5m + 6 = 0 \quad (2.6.8.13)$$

$$\implies m = 2 \text{ or } 3 \quad (2.6.8.14)$$

From (2.6.8.6),

$$\lambda = \frac{-\mathbf{m}^T (\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{(\mathbf{m}^T (\mathbf{c} - \mathbf{O}))^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \quad (2.6.8.15)$$

Fig. 2.6.8 summarizes the solution for $m = 2$.

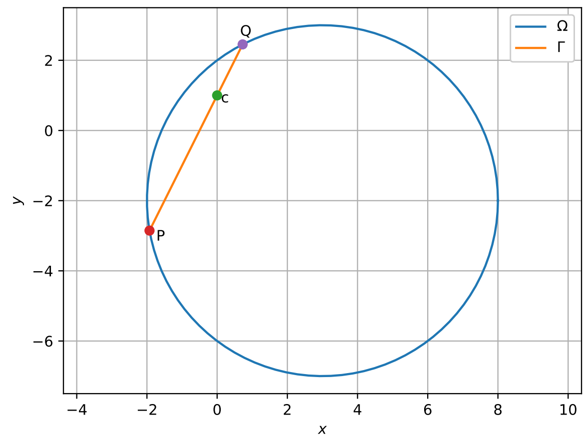


Fig. 2.6.8

## 3 Conics

### 3.1 Definitions

1. From (1.2.19.2), the equation of a conic section is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.1.1.1)$$

for

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} \neq 0 \quad (3.1.1.2)$$

2. Show that

$$\frac{d\left(\mathbf{u}^T \mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \quad (3.1.2.1)$$

3. Show that

$$\frac{d\left(\mathbf{x}^T \mathbf{V} \mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T \mathbf{x} \quad (3.1.3.1)$$

4. Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \quad (3.1.4.1)$$

5. Find the *normal* vector to the curve in (1.2.19.2) at point $\mathbf{p}$.
**Solution:** Differentiating (1.2.19.2) with respect to $x_1$,

$$\frac{d\left(\mathbf{x}^T \mathbf{V} \mathbf{x}\right)}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} + \frac{d\left(\mathbf{u}^T \mathbf{x}\right)}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} = 0 \quad (3.1.5.1)$$

$$\implies 2\mathbf{x}^T \mathbf{V} \mathbf{m} + 2\mathbf{u}^T \mathbf{m} = 0 \because \left(\frac{d\mathbf{x}}{dx_1} = \mathbf{m}\right) \quad (3.1.5.2)$$

Substituting $\mathbf{x} = \mathbf{p}$ and simplifying

$$(\mathbf{V}\mathbf{p} + \mathbf{u})^T \mathbf{m} = 0 \quad (3.1.5.3)$$

$$\implies \mathbf{n} = \mathbf{V}\mathbf{p} + \mathbf{u} \quad (3.1.5.4)$$

6. The *tangent* to the curve at $\mathbf{p}$ is given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{p}) = 0 \quad (3.1.6)$$

This results in

$$\left(\mathbf{p}^T \mathbf{V} + \mathbf{u}^T\right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + f = 0 \quad (3.1.6)$$

7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \quad (3.1.7)$$

is known as an *affine* transformation.

8. Classify the various conic sections based on (1.2.19.2).
**Solution:**

| **Curve** | **Property** |
|---|---|
| Circle | $V = kI$ |
| Parabola | $\det(V) = 0$ |
| Ellipse | $\det(V) > 0$ |
| Hyperbola | $\det(V) < 0$ |

TABLE 3.1.8

### 3.2 *Parabola*

1. Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x}+6=0 \tag{3.2.1.1}$$

**Solution:** Substituting

$$\mathbf{p}=\begin{pmatrix}1\\7\end{pmatrix}, V=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u}=\frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \tag{3.2.1.2}$$

in (3.1.6), the desired equation is

$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}+6=0 \tag{3.2.1.3}$$

resulting in

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x}=-5 \tag{3.2.1.4}$$

2. The line in (3.2.1.4) touches the circle

$$\mathbf{x}^T\mathbf{x}+4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x}+c=0 \tag{3.2.2.1}$$

Find $c$.
**Solution:** Comparing (1.2.19.2) and (3.2.2.1),

$$V=I, \quad \mathbf{u}=2\begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.2.2}$$

Comparing (3.1.6) and (3.2.1.4),

$$\mathbf{p}+2\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix} \tag{3.2.2.3}$$

$$\implies \mathbf{p}=-\begin{pmatrix}6\\7\end{pmatrix} \tag{3.2.2.4}$$

and

$$c+\mathbf{p}^T\mathbf{u}=5 \tag{3.2.2.5}$$

$$\implies c=5+2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.2.6}$$

$$=95 \tag{3.2.2.7}$$

3. Summarize all the above computations through a Python script and plot the parabola, tangent and circle.
**Solution:** The following code generates Fig. 3.2.3.

```
wget
codes/2d/parab.py
```

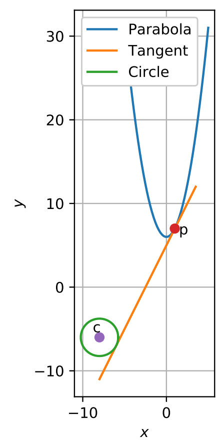


Fig. 3.2.3

### 3.3 *Affine Transformation*

1. In general, Fig. 3.2.3 was generated using an *affine transformation*.
2. Express

$$y_2=y_1^2 \tag{3.3.2.1}$$

as a matrix equation.
**Solution:** (3.3.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y}+2\mathbf{g}^T\mathbf{y}=0 \tag{3.3.2.2}$$

where

$$\mathbf{D}=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{g}=-\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.3.2.3}$$

3. Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+F=0, \tag{3.3.3.1}$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \tag{3.3.3.2}$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.3.3.3}$$

(3.3.3.3) is known as an affine transformation. Show that

$$\begin{aligned} \mathbf{D} &= \mathbf{P}^T\mathbf{V}\mathbf{P} \\ \mathbf{g} &= \mathbf{P}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) \qquad (3.3.3.4) \\ F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} &= 0 \end{aligned}$$

**Solution:** Substituting (3.3.3.3) in (3.3.3.1),

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + F = 0, \tag{3.3.3.5}$$

which can be expressed as

$$\begin{aligned} \implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} \\ + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \end{aligned} \tag{3.3.3.6}$$

Comparing (3.3.3.6) with (3.3.2.2) (3.3.3.4) is obtained.

4. Show that there exists a $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{3.3.4.1}$$

Find $\mathbf{P}$ using

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.3.4.2}$$

5. Find $\mathbf{c}$ from (3.3.3.4).
**Solution:**

$$\because \mathbf{g} = \mathbf{P}^T(\mathbf{V}\mathbf{c} + \mathbf{u}), \tag{3.3.5.1}$$

$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{3.3.5.2}$$

$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T(\mathbf{P}\mathbf{g} - \mathbf{u}) = -F - 2\mathbf{u}^T\mathbf{c} \tag{3.3.5.3}$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ (\mathbf{P}\mathbf{g} + \mathbf{u})^T \end{pmatrix} \mathbf{c} = \begin{pmatrix} \mathbf{P}\mathbf{g} - \mathbf{u} \\ -F \end{pmatrix} \tag{3.3.5.4}$$

for computing $\mathbf{c}$.

### 3.4 Ellipse: Eigenvalues and Eigenvectors

1. Express the following equation in the form given in (1.2.19.2)

$$E: 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{3.4.1.1}$$

**Solution:** (3.4.1.1) can be expressed as

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{3.4.1.2}$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \tag{3.4.1.3}$$

2. Using the affine transformation in (3.3.3.3), show that (3.4.1.2) can be expressed as

$$\mathbf{y}^T D\mathbf{y} = 1 \tag{3.4.2.1}$$

where

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.2.2}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.2.3}$$

for

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I} \tag{3.4.2.4}$$

3. Find $\mathbf{c}$
**Solution:**

$$\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.3.1}$$

4. If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.4.4.1}$$

$$P = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{3.4.4.2}$$

show that

$$V\mathbf{z} = \lambda\mathbf{z} \tag{3.4.4.3}$$

where $\lambda \in \{\lambda_1, \lambda_2\}, \mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$.

5. Find $\lambda$.
**Solution:** $\lambda$ is obtained by solving the following equation.

$$|\lambda I - V| = 0 \tag{3.4.5.1}$$

$$\implies \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \tag{3.4.5.2}$$

$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{3.4.5.3}$$

$$\implies \lambda = 2, 8 \tag{3.4.5.4}$$

6. Sketch 3.4.2.1.
7. Find $\mathbf{P}_1$ and $\mathbf{P}_2$.

**Solution:** From (3.4.4.3)

$$V\mathbf{P}_1 = \lambda_1 \mathbf{P}_1 \tag{3.4.7.1}$$
$$\implies (V - \lambda I)\,\mathbf{y} = 0 \tag{3.4.7.2}$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \tag{3.4.7.3}$$
$$\text{or, } \mathbf{P}_1 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.4.7.4}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{P}_2 = 0 \tag{3.4.7.5}$$
$$\text{or, } \mathbf{P}_2 = k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.4.7.6}$$

8. Find $\mathbf{P}$.
**Solution:** From (3.4.2.4) and (3.4.4.2),

$$k_1 = \frac{1}{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.1}$$
$$k_2 = \frac{1}{\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.2}$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.4.8.3}$$

9. Find the equation of the major axis for $E$.
**Solution:** The major axis for (3.4.2.1) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.4.9.1}$$

Using the affine transformation in (3.3.3.3)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.9.2}$$
$$\implies \mathbf{x} - \mathbf{c} = \lambda_1 \mathbf{P}_1 \tag{3.4.9.3}$$
$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.9.4}$$
$$= -3 \tag{3.4.9.5}$$

since

$$P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \tag{3.4.9.6}$$

which is the major axis of the ellipse $E$.

10. Find the minor axis of $E$.
11. Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{3.4.11.1}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and $k$.

12. Summarize all the above computations through a Python script and plot the ellipses in (3.4.1.1) and (3.4.2.1).
**Solution:** The following script plots Fig. 3.4.12 using the principles of an affine transformation.

```
codes/2d/ellipse.py
```

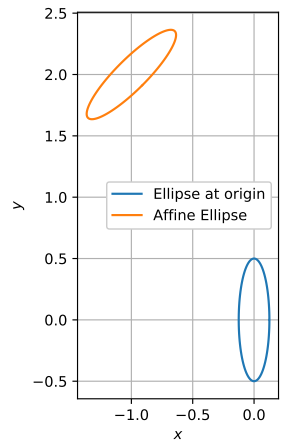


Fig. 3.4.12

### 3.5 Hyperbola

1. Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \tag{3.5.1.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5.1.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{3.5.1.3}$$

find the equation of $PQ$.
**Solution:** The equations of the two tangents are obtained using (3.1.6) as

$$\mathbf{P}^T V \mathbf{x} = 36 \tag{3.5.1.4}$$
$$\mathbf{Q}^T V \mathbf{x} = 36. \tag{3.5.1.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V \mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.6}$$
$$\mathbf{Q}^T V \mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \tag{3.5.1.8}$$
$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{3.5.1.9}$$

which is the equation of $PQ$.

2. In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
**Solution:** Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{3.5.2.1}$$

using (1.2.9.2) and (3.5.1.9), the equation of $TD$ is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \tag{3.5.2.2}$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.2.3}$$

3. Find $D$.
**Solution:** From (3.5.1.9) and (3.5.2.3),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.1}$$
$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.2}$$

4. Show that the equation of $PQ$ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \tag{3.5.4.1}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.4.2}$$

5. Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \tag{3.5.5.1}$$

can be expressed as

$$\lambda^2 \mathbf{m}^T V \mathbf{m} + 2\lambda \mathbf{m}^T V \mathbf{D} + \mathbf{D}^T V \mathbf{D} + F = 0 \tag{3.5.5.2}$$

6. Find $\mathbf{P}$ and $\mathbf{Q}$.
**Solution:** From (3.5.4.1) and (3.5.1.1) (3.5.5.2) is obtained. Substituting from (3.5.4.2), (3.5.1.2) and (3.5.3.2)

$$\mathbf{m}^T V \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \tag{3.5.6.1}$$
$$\mathbf{m}^T V \mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \tag{3.5.6.2}$$
$$\mathbf{D}^T V \mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \tag{3.5.6.3}$$

Substituting in (3.5.5.2)

$$4\lambda^2 - 144 = 36 \tag{3.5.6.4}$$
$$\implies \lambda = \pm 3\sqrt{5} \tag{3.5.6.5}$$

Substituting in (3.5.4.1),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \tag{3.5.6.6}$$
$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \tag{3.5.6.7}$$

7. Find the area of $\triangle PTQ$.
**Solution:** Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{3.5.7.1}$$
$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{3.5.7.2}$$

the desired area is

$$\frac{1}{2} PQ \times TD = 45\sqrt{5} \tag{3.5.7.3}$$

8. Repeat the previous exercise using determinants.
9. Summarize all the above computations through a Python script and plot the hyperbola.

### 3.6 Karush-Kuhn-Tucker (KKT) Conditions

1. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \tag{3.6.1.1}$$

with constraints

$$g_1(\mathbf{x}) = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{3.6.1.2}$$
$$g_2(\mathbf{x}) = 15 - \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} \ge 0 \tag{3.6.1.3}$$

**Solution:** Considering the Lagrangian

$$\nabla L(\mathbf{x}, \lambda, \mu) = 0 \tag{3.6.1.4}$$

resulting in the matrix equation

$$\Rightarrow \begin{pmatrix} 8 & 0 & 3 & 2 \\ 0 & 4 & 1 & 4 \\ 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 15 \end{pmatrix} \tag{3.6.1.5}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \\ -3.12 \\ -2.12 \end{pmatrix} \tag{3.6.1.6}$$

using the following python script. The (incorrect) graphical solution is available in Fig. 3.6.1

```
codes/optimization/2.12.py
```

Note that $\mu < 0$, contradicting the necessary condition in (2.5.8.4).

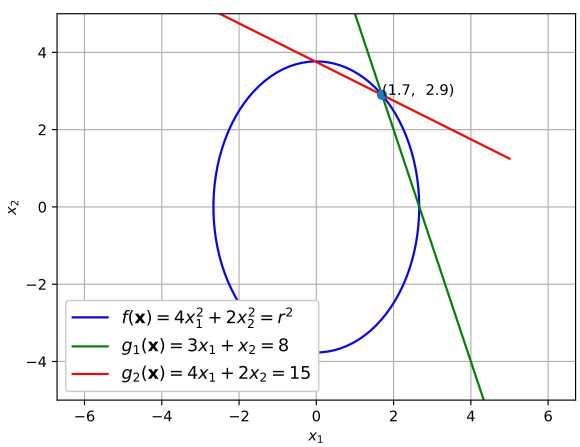


Fig. 3.6.1: Incorrect solution is at intersection of all curves $r = 5.33$

2. Obtain the correct solution to the previous problem by considering $\mu = 0$.
3. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.3.1}$$

with constraints

$$g_1(\mathbf{x}) = 0 \tag{3.6.3.2}$$

$$g_2(\mathbf{x}) \le 0 \tag{3.6.3.3}$$

4. Based on whatever you have done so far, list the steps that you would use in general for solving a convex optimization problem like (3.6.1.1) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.

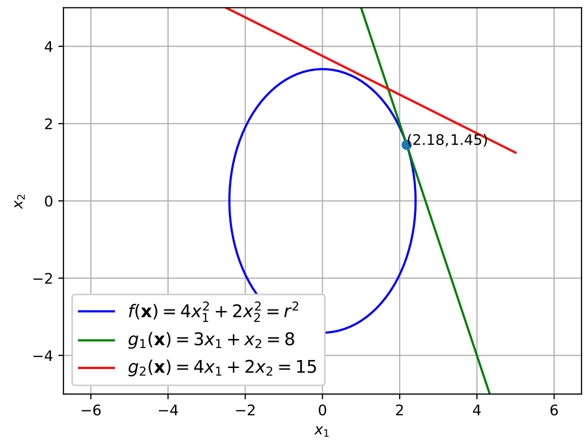


Fig. 3.6.2: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

**Solution:** For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \tag{3.6.4.1}$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, .., m \tag{3.6.4.2}$$

$$\text{subject to } g_i(\mathbf{x}) \le 0, \forall i = 1, .., n \tag{3.6.4.3}$$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \tag{3.6.4.4}$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i g_i(\mathbf{x}), \tag{3.6.4.5}$$

using the KKT conditions

$$\Rightarrow \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \tag{3.6.4.6}$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, .., n \tag{3.6.4.7}$$

$$\text{and } \mu_i \ge 0, \forall i = 1, .., n \tag{3.6.4.8}$$

5. Maxmimize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \tag{3.6.5.1}$$

with the constraints

$$x_1^2 + x_2^2 \le 5 \tag{3.6.5.2}$$

$$x_1 \ge 0, x_2 \ge 0 \tag{3.6.5.3}$$

### 3.7 Solved Problems

1. Two parabolas with a common vertex and with axes along $x$-axis and $y$-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.
   **Solution:** The equation of a conic is given by
   $$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0 \quad (3.7.1.1)$$
   For the standard parabola,
   $$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.7.1.2)$$
   $$\mathbf{u} = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.7.1.3)$$
   $$F = 0 \quad (3.7.1.4)$$
   The focus
   $$\mathbf{F} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.7.1.5)$$
   The Latus rectum is the line passing through $\mathbf{F}$ with direction vector
   $$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.7.1.6)$$
   Thus, the equation of the Latus rectum is
   $$\mathbf{x} = \mathbf{F} + \lambda\mathbf{m} \quad (3.7.1.7)$$
   The intersection of the latus rectum and the parabola is obtained from (3.7.1.4) , (3.7.1.7) and (3.7.1.1) as
   $$(\mathbf{F} + \lambda\mathbf{m})^T \mathbf{V} (\mathbf{F} + \lambda\mathbf{m}) + 2\mathbf{u}^T (\mathbf{F} + \lambda\mathbf{m}) = 0 \quad (3.7.1.8)$$
   $$\implies \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\lambda^2 + 2(\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m}\lambda + (\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = 0 \quad (3.7.1.9)$$
   From (3.7.1.2), (3.7.1.3), (3.7.1.5) and (3.7.1.6),
   $$\mathbf{m}^T\mathbf{V}\mathbf{m} = 1 \quad (3.7.1.10)$$
   $$(\mathbf{V}\mathbf{F} + \mathbf{u})^T \mathbf{m} = 0 \quad (3.7.1.11)$$
   $$(\mathbf{V}\mathbf{F} + 2\mathbf{u})^T \mathbf{F} = -4a^2 \quad (3.7.1.12)$$
   Substituting from (3.7.1.10), (3.7.1.11) and (3.7.1.12) in (3.7.1.9),
   $$\lambda^2 - 4a^2 = 0 \quad (3.7.1.13)$$
   $$\implies \lambda_1 = 2a, \lambda_2 = -2a \quad (3.7.1.14)$$
   Thus, from (3.7.1.6), (3.7.1.7) and (3.7.1.14), the length of the latus rectum is
   $$(\lambda_1 - \lambda_2)\left\|\mathbf{m}\right\| = 4a \quad (3.7.1.15)$$
   From the given information, the two parabolas $P_1, P_2$ have parameters
   $$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \quad (3.7.1.16)$$
   $$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a\begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \quad (3.7.1.17)$$
   $$4a = 3 \quad (3.7.1.18)$$
   Let $L$ be the common tangent for $P_1, P_2$ with $\mathbf{c}, \mathbf{d}$ being the respective points of contact. The respectivel normal vectors are
   $$\mathbf{n}_1 = \mathbf{V}_1\mathbf{c} + \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} \quad (3.7.1.19)$$
   $$\mathbf{n}_2 = \mathbf{V}_2\mathbf{d} + \mathbf{u}_2 = d_1\begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \quad (3.7.1.20)$$
   From the above equations, since both normals have the same direction vector,
   $$\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \implies c_2d_1 = 4a^2 \quad (3.7.1.21)$$
2. Find the product of the perpendiculars drawn from the foci of the ellipse
   $$\mathbf{x}^T\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}\mathbf{x} = 225 \quad (3.7.2.1)$$
   upon the tangent to it at the point
   $$\frac{1}{2}\begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \quad (3.7.2.2)$$
   **Solution:** For the ellipse in (3.7.2.1),
   $$V = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}, \mathbf{u} = 0, F = -1 \quad (3.7.2.3)$$
   The equation of the desired tangent is
   $$(\mathbf{V}\mathbf{P})^T \mathbf{x} = 1 \quad (3.7.2.4)$$
   $$\implies \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\mathbf{x} = 2 \quad (3.7.2.5)$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{3.7.2.6}$$

The product of the perpendiculars is

$$\frac{\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} - 2\right|\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ -4 \end{pmatrix} - 2\right|}{\left\|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\right\|^2} = 9 \tag{3.7.2.7}$$

3. Consider an ellipse, whose centre is at the origin and its major axis is along the $x$-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.
**Solution:** If $a$ and $b$ be the semi-major and minor-axis respectively, the foci of the ellipse are

$$\mathbf{F}_1 = ae\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{F}_2 = -ae\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.3.1}$$

From the given information,

$$e = \frac{3}{5}, 2ae = 6 \tag{3.7.3.2}$$

$$\implies a = 5, b = a\sqrt{1 - e^2} = 4 \tag{3.7.3.3}$$

Thus, the vertices of the ellipse are

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ -b \end{pmatrix} \tag{3.7.3.4}$$

and the area of the quadrilateral is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & b \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & -b \end{vmatrix} = 2ab = 40 \tag{3.7.3.5}$$

4. Let $a$ and $b$ respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{3.7.4.1}$$

If

$$\mathbf{S} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.7.4.2}$$

is a focus and

$$\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} = 9 \tag{3.7.4.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.
**Solution:** From (3.7.4.1),

$$(3e - 1)(3e - 5) = 0 \tag{3.7.4.4}$$

$$\implies e = \frac{5}{3}, \because e > 1 \tag{3.7.4.5}$$

for a hyperbola. Let $\mathbf{x}$ be a point on the hyperbola. From (3.7.4.3), its distance from the directrix is

$$\frac{\left|\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} - 9\right|}{5} \tag{3.7.4.6}$$

and from the focus is

$$\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right\| \tag{3.7.4.7}$$

From the definition of a hyperbola, the eccentricity is the ratio of these distances and (3.7.4.5) , (3.7.4.6) and (3.7.4.7),

$$\frac{5\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right\|}{\left|\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} - 9\right|} = \frac{5}{3} \tag{3.7.4.8}$$

$$\implies 9\left\{(x_1 - 5)^2 + x_2^2\right\} = (5x_1 - 9)^2 \tag{3.7.4.9}$$

$$\text{or, } \mathbf{x}^T\begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix}\mathbf{x} = 225 \tag{3.7.4.10}$$

which is the equation of the hyperbola. Thus,

$$a^2 = \frac{225}{16}, b^2 = \frac{225}{9} \tag{3.7.4.11}$$

$$\implies a^2 - b^2 = -\frac{175}{16} \tag{3.7.4.12}$$

5. A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{3.7.5.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 12 \tag{3.7.5.2}$$

meets the coordinate axes at $\mathbf{A}$ and $\mathbf{B}$, then find the locus of the midpoint of $AB$.
**Solution:** The intersection of the lines in (3.7.5.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.7.5.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{3.7.5.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.7.5.5}$$

Let the $\mathbf{R}$ be the mid point of $AB$. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} \tag{3.7.5.6}$$

$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R} \tag{3.7.5.7}$$

Let the equation of $AB$ be

$$\mathbf{n}^T(\mathbf{x} - \mathbf{C}) = 0 \tag{3.7.5.8}$$

Since $\mathbf{R}$ lies on $AB$,

$$\mathbf{n}^T(\mathbf{R} - \mathbf{C}) = 0 \tag{3.7.5.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{3.7.5.10}$$

Substituting from (3.7.5.6) in (3.7.5.10),

$$\mathbf{n}^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} = 0 \tag{3.7.5.11}$$

From (3.7.5.9) and (3.7.5.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} \tag{3.7.5.12}$$

for some constant $k$. Multiplying both sides of (3.7.5.12) by

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.7.5.13}$$

$$\begin{aligned} \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} \\ &= k\mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \end{aligned} \tag{3.7.5.14}$$

$$\because \mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \tag{3.7.5.15}$$

which can be easily verified for any $\mathbf{R}$. from (3.7.5.14),

$$\begin{aligned} \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R} - \mathbf{C}) &= 0 \\ \implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{C} &= 0 \\ \implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{C}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} &= 0 \end{aligned} \tag{3.7.5.16}$$

which is the desired locus.

### *3.8 Miscellaneous*

1. What does the equation

$$\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 4 & 6 \end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.8.1.1}$$

   become when the origin is moved to the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$?

2. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T\begin{pmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 10 & -19 \end{pmatrix}\mathbf{x} + 23 = 0 \tag{3.8.2.1}$$

   may become

$$\mathbf{x}^T\begin{pmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{pmatrix}\mathbf{x} = 1 \tag{3.8.2.2}$$

3. Show that the equation

$$\mathbf{x}^T\mathbf{x} = a^2 \tag{3.8.3.1}$$

   remains unaltered by any rotation of the axes.

4. What does the equation

$$\mathbf{x}^T\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\mathbf{x} = 2a^2 \tag{3.8.4.1}$$

   become when the axes are turned through 30°?

5. What does the equation

$$\mathbf{x}^T\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{x} - 4\sqrt{2}a\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{3.8.5.1}$$

   become when the axes are turned through 45°?

6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix}\mathbf{x} = 0 \tag{3.8.6.1}$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.6.2}$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \tag{3.8.6.3}$$

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.7.1}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c \tag{3.8.7.2}$$

where $c$ is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 13 = 0 \tag{3.8.8.1}$$

can be transformed to

$$8\mathbf{x}^T\mathbf{x} = 89 \tag{3.8.8.2}$$

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.9.1}$$

can be reduced to

$$\sqrt{85}\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{3.8.9.2}$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \tag{3.8.10.1}$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.8.10.2}$$

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \tag{3.8.11.1}$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.11.2}$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{3.8.11.3}$$

## 4 Solid Geometry

### 4.1 Lines and Planes

1. $L_1$ is the intersection of planes

$$\begin{aligned} \begin{pmatrix} 2 & -2 & 3 \end{pmatrix} \mathbf{x} &= 2 \\ \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} &= -1 \end{aligned} \tag{4.1.1.1}$$

Find its equation.
**Solution:** (4.1.1.1) can be written in matrix form as

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \tag{4.1.1.2}$$

and solved using the augmented matrix as follows

$$\begin{pmatrix} 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & 2 \end{pmatrix} \tag{4.1.1.3}$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{4.1.1.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 5 \\ x_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.1.1.5}$$

which is the desired equation.

2. Summarize all the above computations through a Python script and plot $L_1$.
**Solution:** The following code generates Fig. 4.1.2.

```
codes/3d/1.1.py
```

3. $L_2$ is the intersection of the planes

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{4.1.3.1}$$

$$\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{4.1.3.2}$$

Show that its equation is

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \tag{4.1.3.3}$$

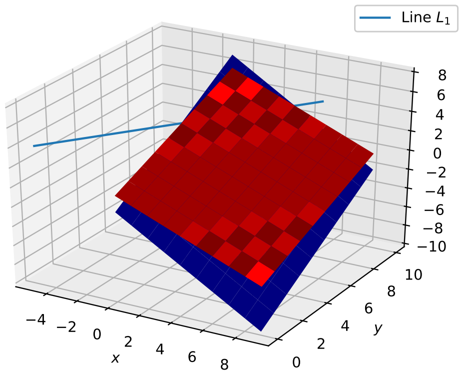


Fig. 4.1.2

4. Plot $L_2$.
**Solution:** The following code generates Fig. 4.1.4.

```
codes/3d/1.2.py
```

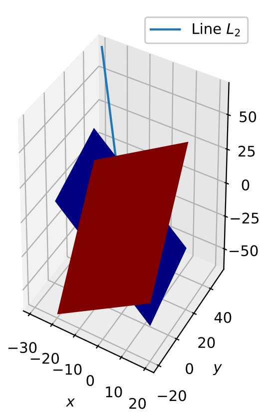


Fig. 4.1.4

5. Do $L_1$ and $L_2$ intersect? If so, find their point of intersection $P$.
**Solution:** From (4.1.1.5),(4.1.3.3), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix}5\\8\\0\end{pmatrix} + \lambda_2\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}-5\\0\\4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{4.1.5.1}$$

$$\implies \begin{pmatrix}1 & 3\\1 & -5\\0 & -7\end{pmatrix}\mathbf{\Lambda} = \frac{1}{7}\begin{pmatrix}40\\8\\-28\end{pmatrix} \tag{4.1.5.2}$$

This matrix equation can be solved as

$$\begin{pmatrix}1 & 3 & \frac{40}{7}\\1 & -5 & \frac{8}{7}\\0 & -7 & -4\end{pmatrix} \leftrightarrow \begin{pmatrix}8 & 0 & \frac{224}{7}\\0 & 1 & \frac{4}{7}\\0 & 1 & \frac{4}{7}\end{pmatrix} \tag{4.1.5.3}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 4\\0 & 1 & \frac{4}{7}\end{pmatrix} \implies \mathbf{\Lambda} = \begin{pmatrix}4\\\frac{4}{7}\end{pmatrix} \tag{4.1.5.4}$$

Substituting $\lambda_1 = 4$ in (4.1.5.1)

$$\mathbf{x} = \begin{pmatrix}4\\4\\0\end{pmatrix} + \begin{pmatrix}-5\\0\\4\end{pmatrix} = \begin{pmatrix}-1\\4\\4\end{pmatrix} \tag{4.1.5.5}$$

6. Plot $P$.
**Solution:** The following code generates Fig. 4.1.6.

```
codes/3d/1.3.py
```

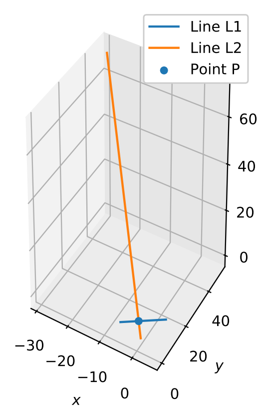


Fig. 4.1.6

### *4.2 Normal to a Plane*

1. The cross product of $\mathbf{a}, \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\a_3 & 0 & -a_1\\-a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{4.2.1.1}$$

From (4.1.1.5), (4.1.3.3), the direction vectors of $L_1$ and $L_2$ are

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}-3\\5\\7\end{pmatrix} \tag{4.2.1.2}$$

respectively. Find the direction vector of the normal to the plane spanned by $L_1$ and $L_2$.
**Solution:** The desired vector is obtained as

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\0 & 0 & -1\\-1 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}7\\-7\\8\end{pmatrix} = \mathbf{n} \tag{4.2.1.3}$$

2. Summarize all the above computations through a plot
**Solution:** The following code generates Fig. 4.2.2.

```
codes/3d/2.1.py
```

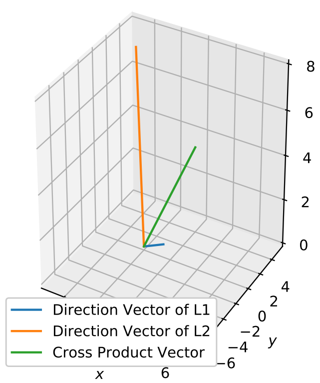


Fig. 4.2.2

3. Find the equation of the plane spanned by $L_1$ and $L_2$.
**Solution:** Let $\mathbf{x}_0$ be the intersection of $L_1$ and $L_2$. Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \tag{4.2.3.1}$$
$$\implies \mathbf{x}^T\mathbf{n} = \mathbf{x}_0^T\mathbf{n} \tag{4.2.3.2}$$
$$\implies \mathbf{x}^T \begin{pmatrix}7\\-7\\8\end{pmatrix} = \begin{pmatrix}-1 & 4 & 4\end{pmatrix}\begin{pmatrix}7\\-7\\8\end{pmatrix} = -3 \tag{4.2.3.3}$$

4. Summarize the above through a plot.
**Solution:** The following code generates Fig. 4.2.4.

```
codes/3d/2.2.py
```

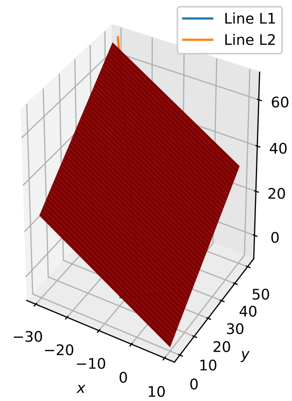


Fig. 4.2.4

5. Find the distance of the origin from the plane containing the lines $L_1$ and $L_2$.
**Solution:** The distance from the origin to the plane is given by

$$\frac{\left|\mathbf{x}_0^T\mathbf{n}\right|}{\|n\|} = \frac{1}{3\sqrt{2}} \tag{4.2.5.1}$$

### 4.3 *Projection on a Plane*

1. Find the equation of the line $L$ joining the points

$$\mathbf{A} = \begin{pmatrix}5 & -1 & 4\end{pmatrix}^T \tag{4.3.1.1}$$
$$\mathbf{B} = \begin{pmatrix}4 & -1 & 3\end{pmatrix}^T \tag{4.3.1.2}$$

**Solution:** The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda(\mathbf{A} - \mathbf{B}) \tag{4.3.1.3}$$
$$= \begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{4.3.1.4}$$

2. Plot the above line.
**Solution:** The following code generates Fig. 4.3.2.

```
codes/3d/3.1.py
```

3. Find the intersection of $L$ and the plane $P$ given by

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{4.3.3.1}$$

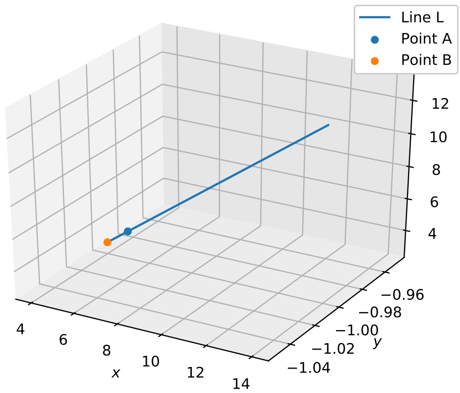


Fig. 4.3.2

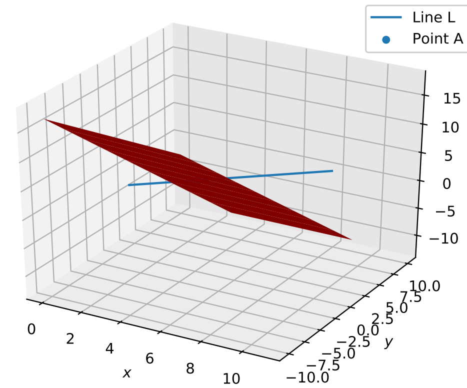


Fig. 4.3.4

**Solution:** From (4.3.1.4) and (4.3.3.1),

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 7 \tag{4.3.3.2}$$

$$\implies 6 + 2\lambda = 7 \tag{4.3.3.3}$$

$$\implies \lambda = \frac{1}{2} \tag{4.3.3.4}$$

Substituting in (4.3.1.4),

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix}9 & -1 & 7\end{pmatrix} \tag{4.3.3.5}$$

4. Sketch the line, plane and the point of intersection.
**Solution:** The following code generates Fig. 4.3.4.

```
codes/3d/3.2.py
```

5. Find $\mathbf{C} \in P$ such that $AC \perp P$.
**Solution:** From (4.3.3.1), the direction vector of $AC$ is $\begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$. Hence, the equation of $AC$ is

$$\mathbf{x} = \begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{4.3.5.1}$$

Substituting in (4.3.3.1)

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 7 \tag{4.3.5.2}$$

$$\implies 8 + 3\lambda_1 = 7 \tag{4.3.5.3}$$

$$\implies \lambda_1 = -\frac{1}{3} \tag{4.3.5.4}$$

Thus,

$$\mathbf{C} = \frac{1}{3}\begin{pmatrix}14\\-4\\11\end{pmatrix} \tag{4.3.5.5}$$

6. Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D} = \frac{1}{3}\begin{pmatrix}13\\-2\\10\end{pmatrix} \tag{4.3.6.1}$$

7. Find the projection of $AB$ on the plane $P$.
**Solution:** The projection is given by

$$CD = \|\mathbf{C} - \mathbf{D}\| = \sqrt{\frac{2}{3}} \tag{4.3.7.1}$$

8. Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \tag{4.3.8.1}$$

### 4.4 Coplanar vectors

1. If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that
$$\mathbf{u}^T (\mathbf{A} \times \mathbf{B}) = 0 \tag{4.4.1.1}$$

2. Find $\mathbf{A} \times \mathbf{B}$ given
$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}^T \tag{4.4.2.1}$$
$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T \tag{4.4.2.2}$$
**Solution:** From (4.2.1.1),
$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.4.2.3}$$
$$= \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \tag{4.4.2.4}$$

3. Let $\mathbf{u}$ be coplanar with $\mathbf{A}$ such that $\mathbf{u} \perp \mathbf{A}$ and
$$\mathbf{u}^T \mathbf{B} = 24. \tag{4.4.3.1}$$
Find $\|\mathbf{u}\|^2$.
**Solution:** From (4.4.2.4) and the given information,
$$\mathbf{u}^T \begin{pmatrix} 4 & -2 & 2 \end{pmatrix} = 0 \tag{4.4.3.2}$$
$$\mathbf{u}^T \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = 0 \tag{4.4.3.3}$$
$$\mathbf{u}^T \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 24 \tag{4.4.3.4}$$
$$\implies \begin{pmatrix} 4 & -2 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} \tag{4.4.3.5}$$
$$\implies \mathbf{u} = 4 \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \tag{4.4.3.6}$$
$$\implies \|\mathbf{u}\|^2 = 336 \tag{4.4.3.7}$$

### 4.5 Orthogonality

1. Let
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \tag{4.5.1.1}$$
$$L_2 : \quad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{4.5.1.2}$$
Given that $L_3 \perp L_1, L_3 \perp L_2$, find $L_3$.
**Solution:** Let
$$L_3 : \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{4.5.1.3}$$
Then
$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{4.5.1.4}$$
Row reducing the coefficient matrix,
$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{4.5.1.5}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.6}$$
Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as
$$L_1 : \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{4.5.1.7}$$
$$L_2 : \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{4.5.1.8}$$
$$L_3 : \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{4.5.1.9}$$
where
$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{4.5.1.10}$$
The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (4.5.1.7)-(4.5.1.9),
$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{4.5.1.11}$$
$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{4.5.1.12}$$
Using the fact that $L_1 \perp L_2 \perp L_3$, (4.5.1.11)-(4.5.1.12) can be expressed as
$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.13}$$
$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.14}$$
$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{4.5.1.15}$$
$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{4.5.1.16}$$
$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{4.5.1.17}$$
$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{4.5.1.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{4.5.1.19}$$

$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{4.5.1.20}$$

Substituting in (4.5.1.11) and (4.5.1.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ or} \tag{4.5.1.21}$$

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{4.5.1.22}$$

The key concept in this question is that orthogonality of $L_1$ and $L_2$ doesnot mean that they intersect. They are skew lines.

### *4.6 Least Squares*

1. Find the equation of the plane $P$ containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{4.6.1.1}$$

2. Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \tag{4.6.2.1}$$

lies outside $P$.

3. Find the point $\mathbf{w} \in P$ closest to $\mathbf{y}$.
4. Show that

$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 = \|\mathbf{y}\|^2 - \mathbf{w}^T\mathbf{X}^T\mathbf{y} \tag{4.6.4.1}$$

$$- \mathbf{y}^T A\mathbf{w} + \mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w} \tag{4.6.4.2}$$

5. Assuming 2×2 matrices and 2×1 vectors, show that

$$\frac{\partial}{\partial \mathbf{w}}\mathbf{w}^T\mathbf{X}^T\mathbf{y} = \frac{\partial}{\partial \mathbf{w}}\mathbf{y}^T\mathbf{X}\mathbf{w} = \mathbf{y}^T\mathbf{X} \tag{4.6.5.1}$$

6. Show that

$$\frac{\partial}{\partial \mathbf{w}}\mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w} = 2\mathbf{w}^T\left(\mathbf{X}^T\mathbf{X}\right) \tag{4.6.6.1}$$

7. Show that

$$\hat{\mathbf{w}} = \min_{\mathbf{w}} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 \tag{4.6.7.1}$$

$$= \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y} \tag{4.6.7.2}$$

8. Let

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}. \tag{4.6.8.1}$$

from (4.6.1.1). Verify (4.6.7.2).

9. Run the following Python code and comment on the output for different values of $\mathbf{x}$

```
codes/matrix/Prob1_4.py
```

10. Compare the results obtained by typing the following code with the results in the previous problem.

```
codes/matrix/Prob1_6.py
```

11. Type the following code in Python and run. Comment.

```
codes/matrix/Prob1_7.py
```

### *4.7 Singular Value Decomposition*

1. Let $\mathbf{v}_1, \mathbf{v}_2$ be the columns of $\mathbf{C} = \mathbf{X}^T\mathbf{X}$.
2. Obtain $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{v}_1, \mathbf{v}_2$ through the following equations.

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \tag{4.7.2.1}$$

$$\hat{\mathbf{u}}_2 = \mathbf{v}_2 - (\mathbf{v}_2, \mathbf{u}_1)\,\mathbf{u}_1 \tag{4.7.2.2}$$

$$\mathbf{u}_2 = \frac{\hat{\mathbf{u}}_2}{\|\hat{\mathbf{u}}_2\|} \tag{4.7.2.3}$$

This procedure is known as Gram-Schmidt orthogonalization.

3. Stack the vectors $\mathbf{u}_1, \mathbf{u}_2$ in columns to obtain the matrix $\mathbf{Q}$. Show that $\mathbf{Q}$ is orthogonal.
4. From the Gram-Schmidt process, show that $\mathbf{C} = \mathbf{Q}\mathbf{R}$, where $\mathbf{R}$ is an upper triangular matrix. This is known as the $\mathbf{Q}-\mathbf{R}$ decomposition.
5. Find an orthonormal basis for $\mathbf{X}^T\mathbf{X}$ comprising of the eigenvectors. Stack these orthonormal eigenvectors in a matrix $\mathbf{V}$. This is known as *Orthogonal Diagonalization*.
6. Find the singular values of $\mathbf{X}^T\mathbf{X}$. The singular values are obtained by taking the square roots of its eigenvalues.
7. Stack the singular values of $\mathbf{X}^T\mathbf{X}$ diagonally to obtain a matrix $\mathbf{\Sigma}$.
8. Obtain the matrix $\mathbf{X}\mathbf{V}$. Verify if the columns of this matrix are orthogonal.

9. Extend the columns of $\mathbf{XV}$ if necessary, to obtain an orthogonal matrix $\mathbf{U}$.
10. Find $\mathbf{U\Sigma V}^T$. Comment.

## 5 Optimization

### 5.1 Convex Functions

1. A single variable function $f$ is said to be convex if

$$f\left[\lambda x + (1-\lambda) y\right] \le \lambda f(x) + (1-\lambda) f(y), \tag{5.1.1.1}$$

for $\quad 0 < \lambda < 1$.

2. Download and execute the following python script. Is $\ln x$ convex or concave?

```
codes/optimization/1.1.py
```

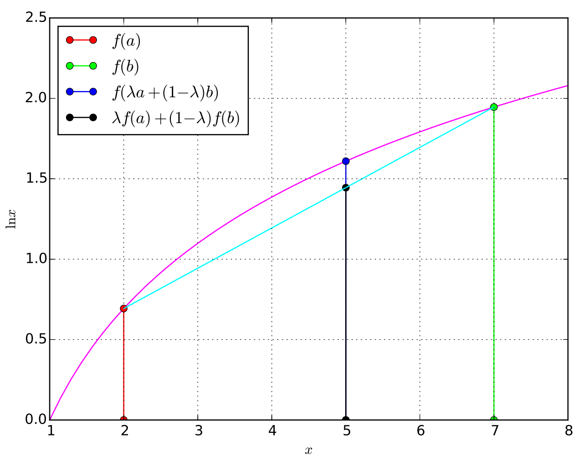


Fig. 5.1.2: $\ln x$ versus $x$

.

3. Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
codes/optimization/1.2.py
```

4. Execute the following script to obtain Fig. 5.1.4. Comment.

```
codes/optimization/1.3.py
```

5. Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?
6. Let

$$f(\mathbf{x}) = x_1 x_2, \quad \mathbf{x} \in \mathbf{R}^2 \tag{5.1.6.1}$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex.

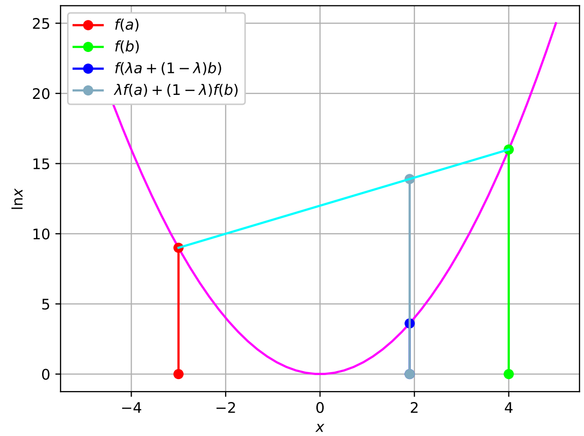


Fig. 5.1.3: $x^2$ versus $x$

.

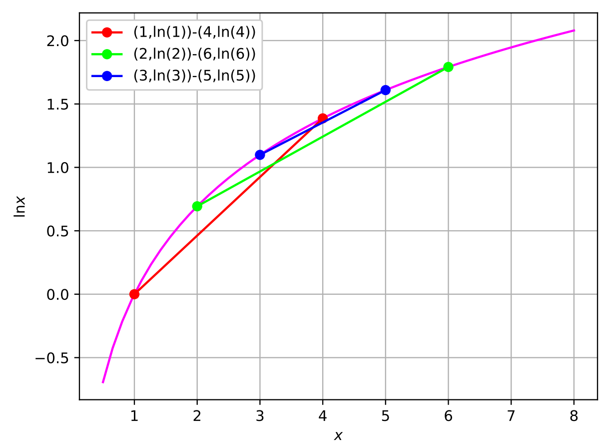


Fig. 5.1.4: Segments are below the curve

.

7. Show that

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} \tag{5.1.7.1}$$

and find $\mathbf{V}$.

8. Show that

$$\frac{1}{2} \nabla^2 f(\mathbf{x}) = \mathbf{V} \tag{5.1.8.1}$$

9. Use (5.1.1.1) to examine the convexity of $f(\mathbf{x})$.
10. How can you deduce the convexity of $f(\mathbf{x})$ using the eigenvalues of $\mathbf{V}$?
11. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{C} \tag{5.1.11.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{5.1.11.2}$$
$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{5.1.11.3}$$

12. Prove that the point $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 24 \tag{5.1.12.1}$$
$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{5.1.12.2}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.12.3}$$

## *5.2 More on Convexity*

1. Consider the optimization problem

$$\max_z \frac{1}{|z-1|} \tag{5.2.1.1}$$
$$s.t. \quad \left|z - 2 + \jmath\right| \ge \sqrt{5} \tag{5.2.1.2}$$

Show that it can be reframed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{c}_1\|^2 \tag{5.2.1.3}$$
$$s.t. \quad \|\mathbf{x} - \mathbf{c}_2\|^2 \ge 5 \tag{5.2.1.4}$$

where

$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{5.2.1.5}$$

2. Explain the optimization problem with a figure.
**Solution:** Fig. 5.2.2 explains (5.2.1.3) where $z_0$ is the set of points comprising of the intersection of the smallest circle $\Gamma$ : with the largest circle $\Omega : r_2 \ge \sqrt{5}$ with radii $r_1$ and $r_2 \ge \sqrt{5}$ respectively.

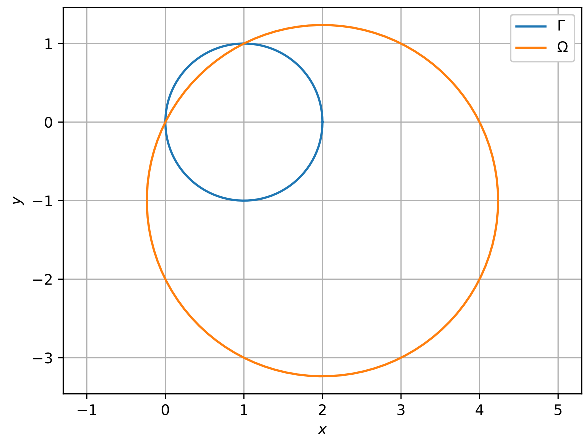


Fig. 5.2.2

3. Obtain the Lagrangian.
**Solution:** The Lagrangian is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x} - \mathbf{c}_1\|^2 - \lambda \left\{ \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 \right\} \tag{5.2.3.1}$$

4. Use the KKT conditions to obtain the minima.
**Solution:** From the KKT conditions,

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \tag{5.2.4.1}$$
$$\implies \mathbf{x} - \mathbf{c}_1 - \lambda(\mathbf{x} - \mathbf{c}_2) = 0 \tag{5.2.4.2}$$
$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} \tag{5.2.4.3}$$

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{5.2.4.4}$$
$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \tag{5.2.4.5}$$

Substituting from (5.2.4.3) in (5.2.4.5),

$$\left\| \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} - \mathbf{c}_2 \right\|^2 - r_2^2 = 0 \tag{5.2.4.6}$$
$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \tag{5.2.4.7}$$
$$= 1 \pm \sqrt{\frac{2}{5}} \tag{5.2.4.8}$$

Fig. 5.2.5 plots $\Gamma$ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \tag{5.2.4.9}$$

5. If the maximum value is obtained at $z_0$, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{5.2.5.1}$$

**Solution:** From (5.2.4.3),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1-\lambda} \tag{5.2.5.2}$$

$$\implies z_0 = \frac{1}{1-\lambda}\left(1 - 2\lambda + \jmath\lambda\right) \tag{5.2.5.3}$$

$$\text{or, } \arg \frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\{z_0\}}{\jmath\left(\Im\{z_0\} + 1\right)} \tag{5.2.5.4}$$

$$= \frac{2(1-\lambda) - (1-2\lambda)}{\jmath} \tag{5.2.5.5}$$

$$= -\jmath \tag{5.2.5.6}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

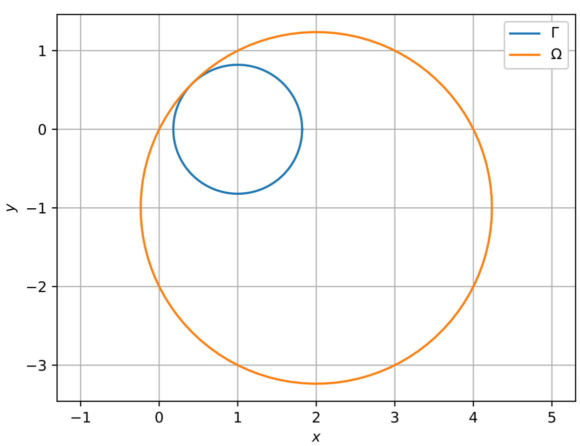


Fig. 5.2.5

6. Show that the set

$$D = \{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\}, r_2 > 0 \tag{5.2.6.1}$$

is nonconvex.
**Solution:** Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \tag{5.2.6.2}$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \tag{5.2.6.3}$$

$$\implies \mathbf{x}_2 \in D. \tag{5.2.6.4}$$

Suppose

$$\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \tag{5.2.6.5}$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \tag{5.2.6.6}$$

$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \tag{5.2.6.7}$$

$$\text{or, } \mathbf{x} \notin D \tag{5.2.6.8}$$

Thus, by definition, $D$ is not a convex set.

### 5.3 *Gradient Descent Method*

1. Consider the problem of finding the square root of a number $c$. This can be expressed as the equation
$$x^2 - c = 0 \tag{5.3.1.1}$$
2. Sketch the function for different values of $c$
$$f(x) = x^3 - 3xc \tag{5.3.2.1}$$
and comment upon its convexity.
3. Show that (5.3.1.1) results from
$$\min_x f(x) = x^3 - 3xc \tag{5.3.3.1}$$
4. Find a numerical solution for (5.3.1.1).
**Solution:** A numerical solution for (5.3.1.1) is obtained as
$$x_{n+1} = x_n - \mu f'(x) \tag{5.3.4.1}$$
$$= x_n - \mu\left(3x_n^2 - 3c\right) \tag{5.3.4.2}$$
where $x_0$ is an inital guess.
5. Write a program to implement (5.3.4.2).
**Solution:** Download and execute

```
codes/optimization/square_root.py
```

### 5.4 *Semi-definite Programming*

1. The problem
$$\min_{\mathbf{X}} x_{11} + x_{12} \tag{5.4.1.1}$$
with constraints
$$x_{11} + x_{22} = 1 \tag{5.4.1.2}$$
$$\mathbf{X} \succeq 0 \quad (\succeq \text{ means positive definite}) \tag{5.4.1.3}$$
where
$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \tag{5.4.1.4}$$
is known as a semi-definite program. Find a numerical solution to this problem. Compare with the solution in problem 2.5.9.

**Solution:** The *cvxopt* solver needs to be used in order to find a numerical solution. For this, the given problem has to be reformulated as

$$\min_{\mathbf{x}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \quad \text{s.t} \tag{5.4.1.5}$$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} = 1 \tag{5.4.1.6}$$

$$x_{11} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + x_{12} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_{22} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.4.1.7}$$

The following script provides the solution to this problem.

```
codes/optimization/3.1.py
```

2. Frame Problem 5.4.1 in terms of matrices.
**Solution:** It is easy to verify that

$$x_{11} + x_{12} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.4.2.1}$$

and

$$x_{11} + x_{22} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.2.2}$$

Thus, Problem 5.4.1 can be expressed as

$$\begin{gathered} \min_{\mathbf{X}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s.t \\ \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1, \\ \mathbf{X} \succeq 0 \end{gathered} \tag{5.4.2.3}$$

3. Solve (5.4.2.3) using *cvxpy*.
**Solution:**

```
wget https://raw.githubusercontent.com/
    gadepall/school/master/linalg/book/
    optimizationcodes/3.1−cvx.py
```

4. Minimize

$$-x_{11} - 2x_{12} - 5x_{22} \tag{5.4.4.1}$$

subject to

$$2x_{11} + 3x_{12} + x_{22} = 7 \tag{5.4.4.2}$$

$$x_{11} + x_{12} \geq 1 \tag{5.4.4.3}$$

$$x_{11}, x_{12}, x_{22} \geq 0 \tag{5.4.4.4}$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \succeq 0 \tag{5.4.4.5}$$

using *cvxpy*.

5. Repeat the above exercise by converting the problem into a convex optimization problem in two variables and using graphical plots.
6. Solve the above problem using the KKT conditions. Comment.

## 5.5 Linear Programming

1. Graphically obtain a solution to the following

$$\max_{\mathbf{x}} 6x_1 + 5x_2 \tag{5.5.1.1}$$

with constraints

$$x_1 + x_2 \leq 5 \tag{5.5.1.2}$$

$$3x_1 + 2x_2 \leq 12 \tag{5.5.1.3}$$

$$\text{where } x_1, x_2 \geq 0 \tag{5.5.1.4}$$

**Solution:** The following program plots the solution in Fig. 5.5.1

```
codes/optimization/4.1.py
```

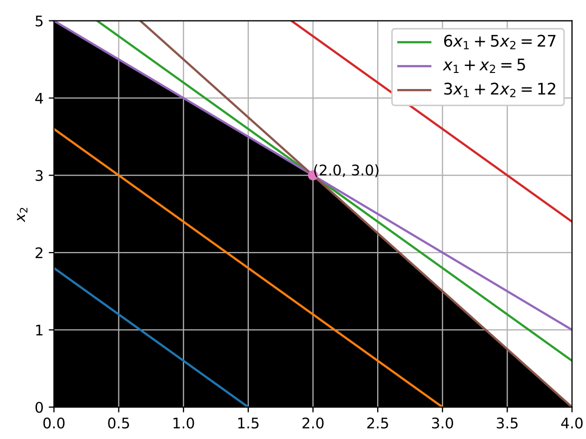


Fig. 5.5.1: The cost function intersects with the two constraints at $\mathbf{x} = (2, 3)$.

2. Now use *cvxpy* to obtain a solution to problem 5.5.1.

**Solution:** The given problem is expressed as follows

$$\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad s.t. \tag{5.5.2.1}$$
$$\mathbf{A}\mathbf{x} \le \mathbf{b} \tag{5.5.2.2}$$

where

$$\mathbf{c} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{5.5.2.3}$$

The desired solution is then obtained using the following program.

```
codes/optimization/4.2−cvx.py
```

3. Verify your solution to the above problem using the method of Lagrange multipliers.
4. Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$x_1 + x_2 \le 2$$
$$5x_1 + 2x_2 \le 10$$
$$3x_1 + 8x_2 \le 12$$
$$\text{where } x_1, x_2 \ge 0$$

### 5.6 *Exercises*

1. Let **P** be the point on the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.1.1}$$
which is at a minimum distance from the centre **C** of the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 0 & 12 \end{pmatrix} \mathbf{x} = 1 \tag{5.6.1.2}$$
Find the equation of the circle passing through **C** and having its centre at **P**.
2. Let **P** be a point on the parabola
$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.2.1}$$
Given that the distance of **P** from the centre of the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{5.6.2.2}$$
is minimum. Find the equation of the tangent to the parabola at **P**.
3. Find the eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half the distance between its foci.
4. **P** and **Q** are two distinct points on the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.4.1}$$
with parameters $t$ and $t_1$ respectively. If the normal at **P** passes through **Q**, then find the minimum value of $t_1^2$ using a descent algorithm.
5. A tangent at a point on the ellipse
$$\mathbf{x}^T V \mathbf{x} = 51 \tag{5.6.5.1}$$
where
$$V = \begin{pmatrix} 3 & 0 \\ 0 & 27 \end{pmatrix} \tag{5.6.5.2}$$
meets the coordinate axes at **A** and **B**. If **O** be the origin, find the minimum area of $\triangle OAB$.
6. Find the shortest distance between the line
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{5.6.6.1}$$
and the curve
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{5.6.6.2}$$
7. Let $S$ be the set of all complex numbers $z$ satisfying $\left|z - 2 + \jmath\right| \ge \sqrt{5}$. If the complex number $z_0$ is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, find the principal argument of
$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{5.6.7.1}$$
8. Let $\omega \ne 1$ be a cube root of unity. Find the minimum of the set
$$\left|a + b\omega + c\omega^2\right|, \tag{5.6.8.1}$$
where $a, b, c$ are distinct nonzero integers.
9. Let
$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{5.6.9.1}$$
where $\alpha, \beta$ are real functions of $\theta$ and **I** is the identity matrix. If
$$\alpha^* = \min_{\theta} \alpha(\theta) \tag{5.6.9.2}$$
$$\beta^* = \min_{\theta} \beta(\theta), \tag{5.6.9.3}$$
find $\alpha^* + \beta^*$.

10. Find the minimum value of
$$\cos(P+Q)\cos(Q+R)\cos(R+P) \tag{5.6.10.1}$$
in $\triangle PQR$.
11. Find the minimum value of $\alpha$ for which
$$4\alpha x^2 + \frac{1}{x} \ge 1, x > 0. \tag{5.6.11.1}$$
12. Let
$$S = S_1 \cap S_2 \cap S_3, \tag{5.6.12.1}$$
where
$$S_1 = \{z \in C : |z| < 4\} \tag{5.6.12.2}$$
$$S_2 = \left\{z \in C : \Im\left[\frac{z - 1 + \jmath\sqrt{3}}{1 - \jmath\sqrt{3}}\right]\right\}, \tag{5.6.12.3}$$
$$S_3 = \{z \in C : \Re(z) > 0\} \tag{5.6.12.4}$$
Find
$$\min_{z\in S}\left|1 - 3\jmath - z\right| \tag{5.6.12.5}$$
13. A line
$$L : \begin{pmatrix} m & -1 \end{pmatrix}\mathbf{x} = -3 \tag{5.6.13.1}$$
passes through
$$\mathbf{E} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{5.6.13.2}$$
and
$$\mathbf{x}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 16 & 0 \end{pmatrix}\mathbf{x} = 0, 0 \le \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} \le 6 \tag{5.6.13.3}$$
at the point $\mathbf{F}$. Find $m$ such that the area of $\triangle EFG$ is maximum.
14. If $|z - 3 - 2j| \le 2$, find
$$\min_z \left|2z - 6 + 5\jmath\right| \tag{5.6.14.1}$$
15. Find
$$\max_z \left|Arg\left(\frac{1}{1-z}\right)\right| \tag{5.6.15.1}$$
$$s.t \quad |z| = 1, z \ne 1. \tag{5.6.15.2}$$
16. Find the maximum value of the function
$$f(x) = 2x^3 - 15x^2 + 36x - 48 \tag{5.6.16.1}$$
on the set
$$A = \left\{x : x^2 + 20 \le 9x\right\} \tag{5.6.16.2}$$
17. Find the minimum distance of a point on the curve
$$\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix}\mathbf{x} = 4 \tag{5.6.17.1}$$
from the origin.
18. Find
$$\min_z \left|z + \frac{1}{2}\right| \tag{5.6.18.1}$$
$$s.t \quad |z| \ge 2 \tag{5.6.18.2}$$
19. Find the minimum value of
$$\tan A + \tan B \tag{5.6.19.1}$$
$$s.t \quad A + B = 6, \tag{5.6.19.2}$$
$$A, B \ge 0 \tag{5.6.19.3}$$
20. Show that
$$\sin A_1 + \sin A_2 + \cdots + \sin A_n \le n\sin\left(\frac{A_1 + A_2 + \cdots + A_n}{n}\right)$$
$$0 < A_i < \pi, i = 1, 2, \ldots, n, n \ge 1 \tag{5.6.20.1}$$
21. Let $\mathbf{F}_1, \mathbf{F}_2$ be the foci of the standard ellipse with parameters $a$ and $b$. If $\mathbf{P}$ be any point on the ellipse, find the maximum area of $\triangle PF_1F_2$.
22. A circle $\|\mathbf{x}\| = 1$ intersects the $X-$ axis at $\mathbf{P}$ and $\mathbf{Q}$. Another circle with centre $\mathbf{Q}$ intersects this circle above the $X-$ axis at $\mathbf{R}$ and the line segent $PQ$ at $\mathbf{S}$. Find the maximum area of $\triangle QSR$.
23. Let $\mathbf{M}$ be a fixed point in the first quadrant. A line through $\mathbf{M}$ intersects the positive axes at $\mathbf{P}, \mathbf{Q}$ respectively. If $\mathbf{O}$ be the origin, find the minimum area of $\triangle OPQ$.

# 6 Matrices

## 6.1 Properties

1. Show that
$$\min_{a,b,c}\left|a + b\omega + c\omega^2\right|^2 \tag{6.1.1.1}$$
where $\omega^3 = 1, \omega \ne 1$ and $a, b, c$ are distinct nonzero integers can be expressed as
$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} \tag{6.1.1.2}$$

where

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mathbf{A} = 2\mathbf{P}^T\mathbf{P}, \tag{6.1.1.3}$$

$$\mathbf{P} = \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}, \theta = \frac{\pi}{3} \tag{6.1.1.4}$$

2. Show that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \tag{6.1.2.1}$$

**Solution:**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ 0 & \sin\theta & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cos\theta & -\cos\theta \\ \cos\theta & 1 & -\cos 2\theta \\ -\cos\theta & -\cos 2\theta & 1 \end{pmatrix}, \tag{6.1.2.2}$$

resulting in (6.1.2.1).

$$\because \cos 2\theta = -\cos\theta = -\frac{1}{2} \tag{6.1.2.3}$$

3. Show that the characteristic equation of $\mathbf{A}$ is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda \tag{6.1.3.1}$$

4. Show that the eigenvalues of $\mathbf{A}$ are 0 and 3.
5. Verify that $tr(\mathbf{A})$ is the sum of its eigenvalues.
6. Verify that $\det(\mathbf{A})$ is the product of its eigenvalues.
7. Show that $\mathbf{A}$ is positive definite.
8. Show that $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is convex.
9. Show that the unconstrained $\mathbf{x}$ that minimizes $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is given by the line

$$\mathbf{x} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{6.1.9.1}$$

10. Find $\mathbf{y}$ such that

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y} \tag{6.1.10.1}$$

where $\lambda$ is an eigenvalue of $\mathbf{A}$.

11. Show that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P} \tag{6.1.11.1}$$

where $\mathbf{D}$ is a diagonal matrix comprising of the eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{P}$ are the corresponding eigenvectors.

12. Find $\mathbf{U}$ such that

$$\mathbf{A} = \mathbf{U}^T\mathbf{D}\mathbf{U}, \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{6.1.12.1}$$

13. Show that

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 3\mathbf{v}^T\mathbf{v}, \tag{6.1.13.1}$$

where

$$\mathbf{v} = \mathbf{U}\mathbf{x} \tag{6.1.13.2}$$

14. Show that when the entries of $\mathbf{x}$ are unequal and integers, the solution of (6.1.1.2) can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{6.1.14.1}$$

### *6.2 Cayley-Hamilton Theorem*

1. Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{6.2.1.1}$$

where $\alpha, \beta$ are real functions of $\theta$ and $\mathbf{I}$ is the identity matrix. Find the characteristic equation of $\mathbf{M}$.
**Solution:** (6.2.1.1) can be expressed as

$$\mathbf{M}^2 - \alpha\mathbf{M} - \beta\mathbf{I} = 0 \tag{6.2.1.2}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{6.2.1.3}$$

2. Find $\alpha$ and $\beta$.
**Solution:** Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta + \cos^4\theta \tag{6.2.2.1}$$

$$\beta = -\sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{6.2.2.2}$$

3. If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{6.2.3.1}$$

$$\beta^* = \min_\theta \beta(\theta), \tag{6.2.3.2}$$

find $\alpha^* + \beta^*$.

**Solution:**

$$\because \alpha = \sin^4\theta + \cos^4\theta = 1 - \frac{\sin^2 2\theta}{2}, \quad (6.2.3.3)$$

$$\alpha^* = \frac{1}{2}, \quad (6.2.3.4)$$

Similarly,

$$-\beta = \sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \quad (6.2.3.5)$$

$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \quad (6.2.3.6)$$

$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2}\right)^2 + \frac{7}{4} \quad (6.2.3.7)$$

Thus,

$$\beta^* = -\frac{37}{16} \quad (6.2.3.8)$$

$$\implies \alpha^* + \beta^* = -\frac{29}{16} \quad (6.2.3.9)$$

*6.3 Trace*

1. Obtain the $3\times 3$ matrices $\{\mathbf{P}_k\}_{k=1}^6$ from permutations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad (6.3.1.1)$$

2. Let

$$\mathbf{X} = \sum_{k=1}^{6}\mathbf{P}_k\begin{pmatrix}2&1&3\\1&0&2\\3&2&1\end{pmatrix}\mathbf{P}_k^T. \quad (6.3.2.1)$$

Given

$$\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad (6.3.2.2)$$

is $\alpha = 30$?

**Solution:**

$$\because \mathbf{P}_k^T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = \sum_{k=1}^{6}\mathbf{P}_k\begin{pmatrix}2&1&3\\1&0&2\\3&2&1\end{pmatrix}\mathbf{P}_k^T\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \sum_{k=1}^{6}\mathbf{P}_k\begin{pmatrix}2&1&3\\1&0&2\\3&2&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \sum_{k=1}^{6}\mathbf{P}_k\begin{pmatrix}5\\3\\5\end{pmatrix} = 2\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}6\\3\\6\end{pmatrix}$$

$$= 30\begin{pmatrix}1\\1\\1\end{pmatrix} \quad (6.3.2.3)$$

Thus, $\alpha = 30$.

3. Is $\mathbf{X}$ symmetric?
**Solution:** Yes. Trivial.
4. Show that

$$\mathbf{P}_k\mathbf{P}_k^T = \mathbf{I} \quad (6.3.4.1)$$

**Solution:**

$$\mathbf{P}_k = \begin{pmatrix}\mathbf{v}_{k1}^T\\\mathbf{v}_{k2}^T\\\mathbf{v}_{k3}^T\end{pmatrix} \quad (6.3.4.2)$$

where $\mathbf{v}_{ki}, i = 1,2,3$ are from the standard basis. Then,

$$\mathbf{P}_k\mathbf{P}_k^T = \begin{pmatrix}\mathbf{v}_{k1}^T\\\mathbf{v}_{k2}^T\\\mathbf{v}_{k3}^T\end{pmatrix}\begin{pmatrix}\mathbf{v}_{k1} & \mathbf{v}_{k2} & \mathbf{v}_{k3}\end{pmatrix}\mathbf{I}$$

$$\because \mathbf{v}_{ji}^T\mathbf{v}_{kj} = \delta_{jk} \quad (6.3.4.3)$$

5. For $2\times 2$ matrices $\mathbf{A}, \mathbf{B}$, verify that

$$tr(AB) = tr(BA) \quad (6.3.5.1)$$

Show that this is true for any square matrix.
6. Verify if the sum of the diagonal entries of $X$ is 18.

**Solution:**

$$\begin{aligned}\operatorname{tr}(\mathbf{X}) &= \sum_{k=1}^{6} \operatorname{tr}\left\{\mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 2\\ 3 & 2 & 1\end{pmatrix}\mathbf{P}_k^T\right\}\\ &= \sum_{k=1}^{6} \operatorname{tr}\left\{\begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 2\\ 3 & 2 & 1\end{pmatrix}\mathbf{P}_k\mathbf{P}_k^T\right\}\\ &= \sum_{k=1}^{6} \operatorname{tr}\begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 2\\ 3 & 2 & 1\end{pmatrix} = 6\times 3 = 18\end{aligned} \tag{6.3.6.1}$$

after substituting from (6.3.4.1).

7. Is $\mathbf{X} - 30\mathbf{I}$ invertible?
**Solution:** From (6.3.2.2),

$$\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = 30\mathbf{I}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\implies (\mathbf{X} - 30\mathbf{I})\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \tag{6.3.7.1}$$

If $(\mathbf{X} - 30\mathbf{I})^{-1}$ exists,

$$(\mathbf{X} - 30\mathbf{I})^{-1}(\mathbf{X} - 30\mathbf{I})\begin{pmatrix}1\\1\\1\end{pmatrix} = 0$$

$$\implies \begin{pmatrix}1\\1\\1\end{pmatrix} = \mathbf{0} \tag{6.3.7.2}$$

which is a contradiction. Hence, $\mathbf{X} - 30\mathbf{I}$ is not invertible.

8. For any two $3\times 3$ matrices $A$ and $B$, let $A + B = 2B^T$ and $3A + 2B = I_3$. Which of the following is true?
a) $5A + 10B = 2I_3$.
b) $10A + 5B = 3I_3$.
c) $2A + B = 3I_3$.
d) $3A + 6B = 2I_3$.

### 6.4 Eigenvector and Null Space

1. Let

$$\mathbf{P} = \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 6\end{pmatrix} \tag{6.4.1.1}$$

2. Find $x$ such that $PQ = QP$.
**Solution:**

$$\because \mathbf{Q} = \begin{pmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 6\end{pmatrix} + x\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 1 & 1 & 0\end{pmatrix}, \tag{6.4.2.1}$$

$$\mathbf{PQ} = \begin{pmatrix}2 & 4 & 6\\ 0 & 8 & 12\\ 0 & 0 & 18\end{pmatrix} + x\begin{pmatrix}1 & 2 & 1\\ 2 & 2 & 0\\ 3 & 3 & 0\end{pmatrix} \tag{6.4.2.2}$$

and

$$\mathbf{QP} = \begin{pmatrix}2 & 2 & 2\\ 0 & 8 & 8\\ 0 & 0 & 18\end{pmatrix} + x\begin{pmatrix}0 & 2 & 5\\ 0 & 0 & 0\\ 1 & 3 & 3\end{pmatrix} \tag{6.4.2.3}$$

Thus,

$$\mathbf{PQ} = \mathbf{QP} \implies \begin{pmatrix}0 & 2 & 4\\ 0 & 0 & 4\\ 0 & 0 & 0\end{pmatrix} = x\begin{pmatrix}-1 & 0 & 4\\ -2 & -2 & 0\\ -2 & 0 & 3\end{pmatrix} \tag{6.4.2.4}$$

which has no solution.

3. If

$$\mathbf{R} = \mathbf{PQP}^{-1}, \tag{6.4.3.1}$$

verify whether

$$\det\mathbf{R} = \det\begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 5\end{pmatrix} + 8 \tag{6.4.3.2}$$

for all $x$.
**Solution:**

$$\begin{aligned}\det(\mathbf{R}) &= \det(\mathbf{P})\det(\mathbf{Q})\det(\mathbf{P})^{-1} = \det(\mathbf{Q})\\ &= 4\left(12 - x^2\right)\end{aligned} \tag{6.4.3.3}$$

Thus,

$$\begin{aligned}\det(\mathbf{R}) - \det\begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 5\end{pmatrix} &= 4\left\{\left(12 - x^2\right) - \left(10 - x^2\right)\right\}\\ &= 8\end{aligned} \tag{6.4.3.4}$$

which is true.

4. For $x = 0$, if

$$\mathbf{R}\begin{pmatrix}1\\a\\b\end{pmatrix} = 6\begin{pmatrix}1\\a\\b\end{pmatrix}, \tag{6.4.4.1}$$

then show that

$$a + b = 5. \tag{6.4.4.2}$$

**Solution:** For $x = 0$,

$$\mathbf{R} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}, \tag{6.4.4.3}$$

where $\mathbf{Q}$ is a diagonal matrix. This is the eigenvalue decomposition of $\mathbf{R}$. Thus,

$$\mathbf{R}\begin{pmatrix}1\\2\\3\end{pmatrix} = 6\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.4.4.4}$$

where

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{6.4.4.5}$$

is the eigenvector corresponding to the eigenvalue 6. Comparing with (6.4.4.4),

$$a = 2, b = 3 \implies a + b = 5. \tag{6.4.4.6}$$

5. For $x = 1$, verify if there exists a vector $\mathbf{y}$ for which $\mathbf{R}\mathbf{y} = \mathbf{0}$.
**Solution:**

$$\begin{aligned} \mathbf{R}\mathbf{y} = \mathbf{0} &\implies \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}\mathbf{y} = \mathbf{0} \\ &\implies \mathbf{Q}\mathbf{z} = \mathbf{0}, \end{aligned} \tag{6.4.5.1}$$

where

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{y} \tag{6.4.5.2}$$

For $x = 1$, (6.4.1.1) and (6.4.5.1) yield

$$\begin{pmatrix}2 & 1 & 1\\0 & 4 & 0\\1 & 1 & 6\end{pmatrix}\mathbf{z} = \mathbf{0} \tag{6.4.5.3}$$

Using row reduction,

$$\begin{pmatrix}2 & 1 & 1\\0 & 4 & 0\\1 & 1 & 6\end{pmatrix} \leftrightarrow \begin{pmatrix}2 & 1 & 1\\0 & 1 & 0\\0 & 1 & 11\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 1 & 11\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -5\\0 & 1 & 0\\0 & 0 & 11\end{pmatrix} \tag{6.4.5.4}$$

Thus, $\mathbf{Q}^{-1}$ exists and

$$\mathbf{z} = \mathbf{0} \implies \mathbf{y} = \mathbf{0} \tag{6.4.5.5}$$

upon substituting from (6.4.5.2). This implies that the null space of $\mathbf{R}$ is empty.