Convex Optimization in Python

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Contents

***Abstract*—This manual provides a simple introduction to various concepts in optimization.**

1 Convex Functions

A single variable function f is said to be convex if

$$f\left[\lambda x + (1-\lambda) y\right] \leq \lambda f(x) + (1-\lambda) f(y), \quad (1.1)$$

for $\quad 0 < \lambda < 1$.

1.1 Download and execute the following python script. Is $\ln x$ convex or concave?

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/1.1.py
```

1.2 Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/1.2.py
```

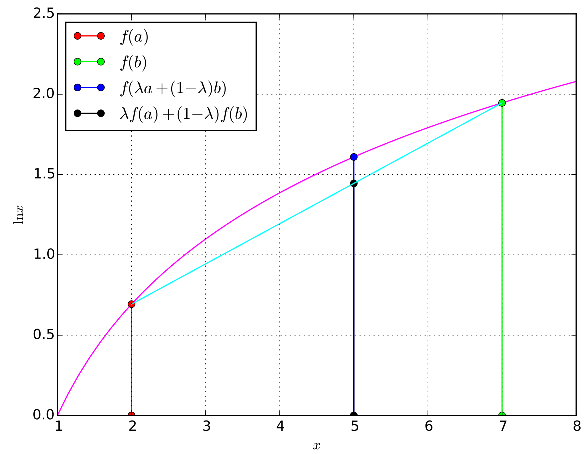

Fig. 1.1: $\ln x$ versus x

.

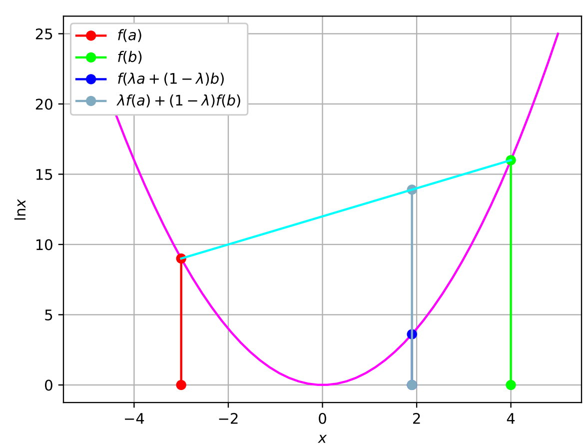

Fig. 1.2: x^2 versus x

.

1.3 Execute the following script to obtain Fig. 1.3. Comment.

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```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/1.3.py
```

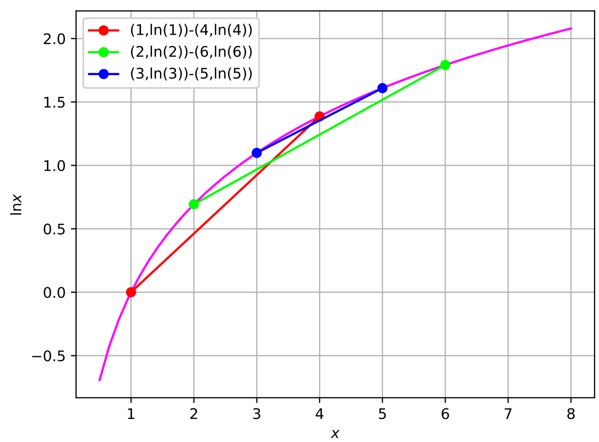

Fig. 1.3: Segments are below the curve

.

1.4 Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?

1.5 Let

$$f(\mathbf{x}) = x_1 x_2, \quad \mathbf{x} \in \mathbf{R}^2 \tag{1.2}$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex.

1.6 Show that

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} \tag{1.3}$$

and find $\mathbf{V}$.

1.7 Show that

$$\frac{1}{2}\nabla^2 f(\mathbf{x}) = \mathbf{V} \tag{1.4}$$

1.8 Use (1.1) to examine the convexity of $f(\mathbf{x})$.

1.9 How can you deduce the convexity of $f(\mathbf{x})$ using the eigenvalues of $\mathbf{V}$?

2 Gradient Descent Method

Consider the problem of finding the square root of a number c. This can be expressed as the equation

$$x^2 - c = 0 \tag{2.1}$$

2.1 Sketch the function for different values of c

$$f(x) = x^3 - 3xc \tag{2.2}$$

and comment upon its convexity.

2.2 Show that (2.1) results from

$$\min_x f(x) = x^3 - 3xc \tag{2.3}$$

2.3 Find a numerical solution for (2.1).

Solution: A numerical solution for (2.1) is obtained as

$$x_{n+1} = x_n - \mu f'(x) \tag{2.4}$$

$$= x_n - \mu\left(3x_n^2 - 3c\right) \tag{2.5}$$

where x_0 is an inital guess.

2.4 Write a program to implement (2.5).

Solution: Download and execute

```
wget
https://raw.githubusercontent.com/gadepall/
    optimization/master/manual/codes/
    square_root.py
```

3 Convex Optimization

3.1 *Lagrange Multipliers*

3.1 Find

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 = r^2 \tag{3.1}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{3.2}$$

by plotting the circles $f(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.

Solution: The following code plots Fig. 3.1

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.1.py
```

3.2 Show that

$$\min r = \frac{5}{\sqrt{2}} \tag{3.3}$$

3.3 Show that

$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.4}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{3.5}$$

3.4 Show that

$$\nabla f(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\} \tag{3.6}$$

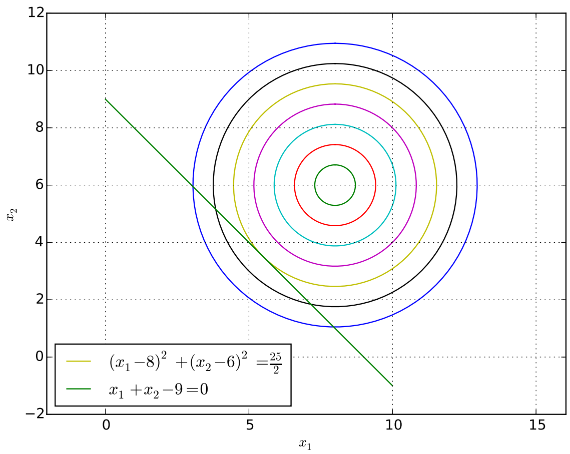

Fig. 3.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$

.

3.5 From Fig. 3.1, show that

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{3.7}$$

where $\mathbf{p}$ is the point of contact.

3.6 Use (3.7) and $\mathbf{g}(\mathbf{p}) = 0$ from (3.2) to obtain $\mathbf{p}$.

3.7 Define

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{3.8}$$

and show that $\mathbf{p}$ can also be obtained by solving the equations

$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{3.9}$$

What is the sign of λ? L is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

Solution:

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.3.py
```

3.2 Inequality Constraints

3.8 Modify the code in problem 3.1 to find a graphical solution for minimising

$$f(\mathbf{x}) \tag{3.10}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{3.11}$$

Solution: This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. 3.8 . It is clear that this radius is 0.

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.4.py
```

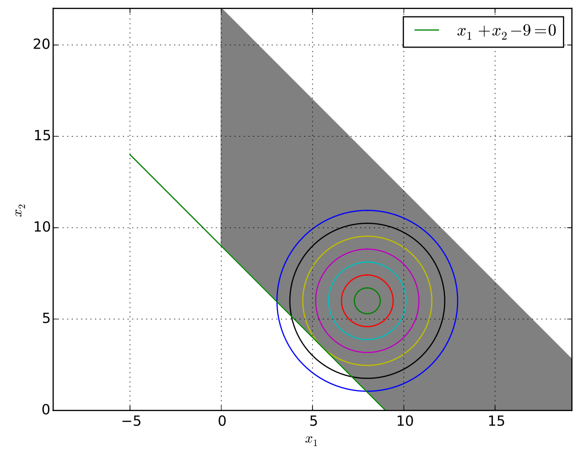

Fig. 3.8: Smallest circle in the shaded region is a point.

3.9 Now use the method of Lagrange multipliers to solve problem 3.8 and compare with the graphical solution. Comment.
Solution: Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem 3.8, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.

3.10 Repeat problem 3.9 by keeping $\lambda = 0$. Comment.
Solution: Keeping $\lambda = 0$ results in $\mathbf{x} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, which is the correct solution. The minimum value of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.

3.11 Find a graphical solution for minimising

$$f(\mathbf{x}) \tag{3.12}$$

with constraint

$$g(\mathbf{x}) \leq 0 \tag{3.13}$$

Summarize your observations.
Solution: In Fig. 3.11, the shaded region represents the constraint. Thus, the solution is the same as the one in problem 3.8. This implies that the method of Lagrange multipliers can be

used to solve the optimization problem with this inequality constraint as well. Table 3.11 summarizes the conditions for this based on the observations so far.

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.7.py
```

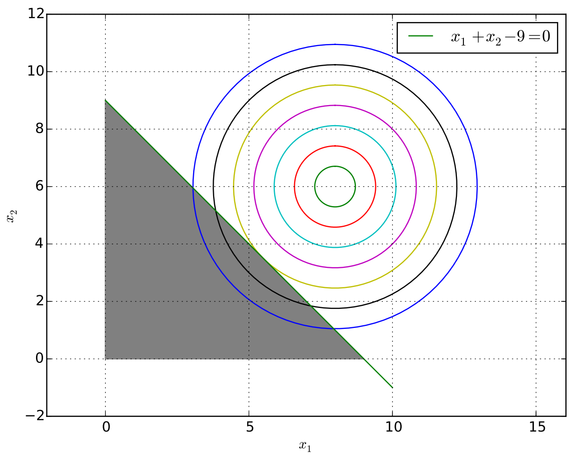

Fig. 3.11: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

TABLE 3.11: Summary of conditions.

Cost	Constraint	λ
$f(\mathbf{x})$	$g(\mathbf{x}) = 0$	< 0
	$g(\mathbf{x}) \geq 0$	0
	$g(\mathbf{x}) \leq 0$	< 0

3.12 Find a graphical solution for

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 \tag{3.14}$$

with constraint

$$g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 = 0 \tag{3.15}$$

Solution:

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.8.py
```

3.13 Repeat problem 3.12 using the method of Lagrange mutipliers. What is the sign of λ?
Solution: Using the following python script, λ is positive and the minimum value of f is 8.

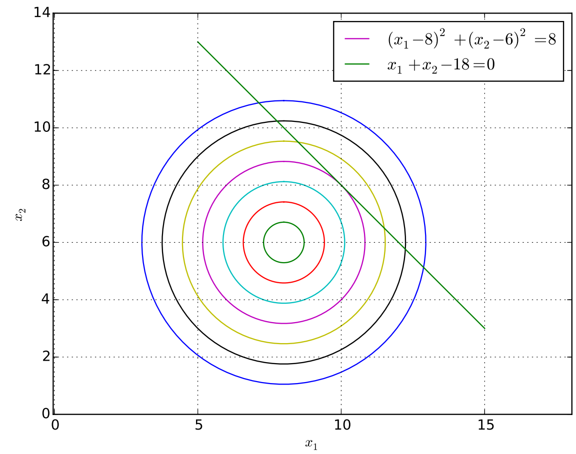

Fig. 3.12: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.9.py
```

3.14 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{3.16}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{3.17}$$

Solution: Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem 3.12.

3.15 Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \tag{3.18}$$

Solution: Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 \leq 0, \lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \tag{3.19}$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \tag{3.20}$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \tag{3.21}$$

else, $\lambda = 0$.

3.3 Karush Kuhn-Tucker Conditions

3.16 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \tag{3.22}$$

with constraints

$$g_1(\mathbf{x}) = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{3.23}$$

$$g_2(\mathbf{x}) = 15 - \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} \geq 0 \tag{3.24}$$

Solution: Considering the Lagrangian

$$\nabla L(\mathbf{x}, \lambda, \mu) = 0 \tag{3.25}$$

resulting in the matrix equation

$$\Rightarrow \begin{pmatrix} 8 & 0 & 3 & 2 \\ 0 & 4 & 1 & 4 \\ 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 15 \end{pmatrix} \tag{3.26}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \\ -3.12 \\ -2.12 \end{pmatrix} \tag{3.27}$$

using the following python script. The (incorrect) graphical solution is available in Fig. 3.16

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.12.py
```

Note that $\mu < 0$, contradicting the necessary condition in (3.21).

3.17 Obtain the correct solution to the previous problem by considering $\mu = 0$.

3.18 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{3.28}$$

with constraints

$$g_1(\mathbf{x}) = 0 \tag{3.29}$$

$$g_2(\mathbf{x}) \leq 0 \tag{3.30}$$

3.19 Based on whatever you have done so far, list the steps that you would use in general for solving a convex optimization problem like (3.22) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.

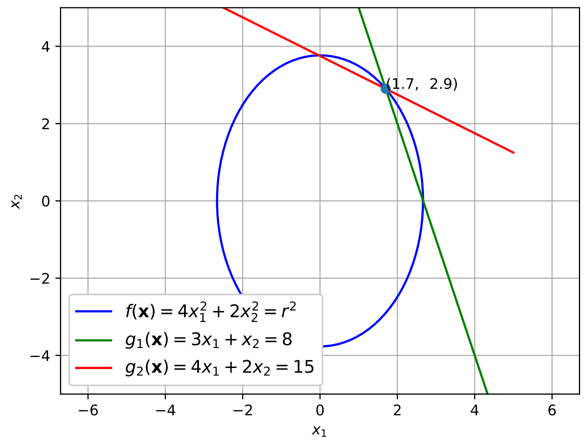

Fig. 3.16: Incorrect solution is at intersection of all curves $r = 5.33$

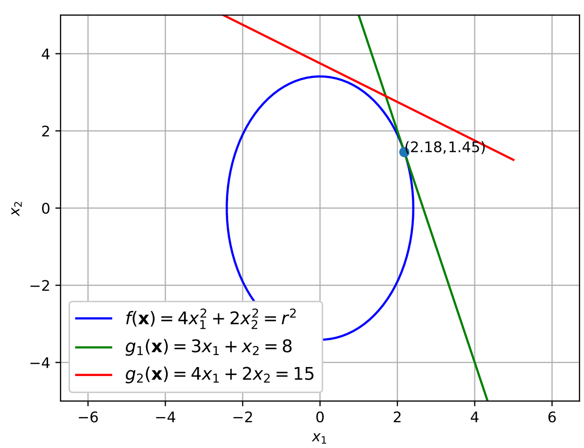

Fig. 3.17: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

Solution: For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \tag{3.31}$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, .., m \tag{3.32}$$

$$\text{subject to } g_i(\mathbf{x}) \leq 0, \forall i = 1, .., n \tag{3.33}$$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \tag{3.34}$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i g_i(\mathbf{x}), \tag{3.35}$$

using the KKT conditions

$$\Rightarrow \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \tag{3.36}$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, .., n \tag{3.37}$$

$$\text{and } \mu_i \geq 0, \forall i = 1, .., n \tag{3.38}$$

3.20 Maxmimize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \tag{3.39}$$

with the constraints

$$x_1^2 + x_2^2 \leq 5 \tag{3.40}$$

$$x_1 \geq 0, x_2 \geq 0 \tag{3.41}$$

3.21 Solve

$$\min_{\mathbf{x}} \quad x_1 + x_2 \tag{3.42}$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \tag{3.43}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solution:

Graphical solution:

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/2.15.py
```

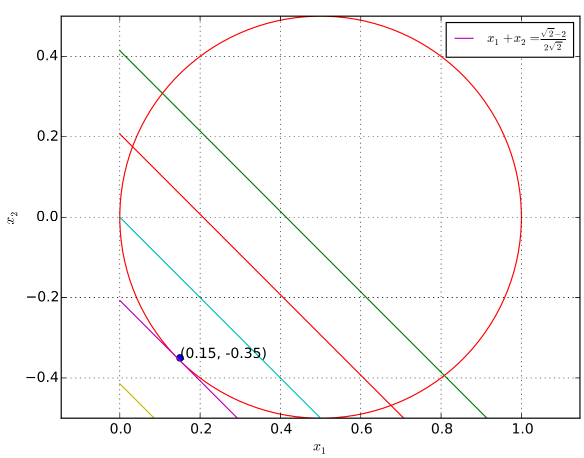

Fig. 3.21: Optimal solution is the lower tangent to the circle

4 Semi-definite Programming

4.1 The problem

$$\min_{\mathbf{X}} x_{11} + x_{12} \tag{4.1}$$

with constraints

$$x_{11} + x_{22} = 1 \tag{4.2}$$

$$\mathbf{X} \succeq 0 \quad (\succeq \text{ means positive definite}) \tag{4.3}$$

where

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \tag{4.4}$$

is known as a semi-definite program. Find a numerical solution to this problem. Compare with the solution in problem 3.21.

Solution: The *cvxopt* solver needs to be used in order to find a numerical solution. For this, the given problem has to be reformulated as

$$\min_{\mathbf{x}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \quad \text{s.t} \tag{4.5}$$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} = 1 \tag{4.6}$$

$$x_{11} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + x_{12} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_{22} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

The following script provides the solution to this problem.

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/3.1.py
```

4.2 Frame Problem 4.1 in terms of matrices.

Solution: It is easy to verify that

$$x_{11} + x_{12} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.8}$$

and

$$x_{11} + x_{22} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.9}$$

Thus, Problem 4.1 can be expressed as

$$\min_{\mathbf{X}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s.t$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1, \tag{4.10}$$
$$\mathbf{X} \geq 0$$

4.3 Solve (4.10) using *cvxpy*.
Solution:

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/3.1−cvx.py
```

4.4 Minimize

$$-x_{11} - 2x_{12} - 5x_{22} \tag{4.11}$$

subject to

$$2x_{11} + 3x_{12} + x_{22} = 7 \tag{4.12}$$
$$x_{11} + x_{12} \geq 1 \tag{4.13}$$
$$x_{11}, x_{12}, x_{22} \geq 0 \tag{4.14}$$
$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \geq 0 \tag{4.15}$$

using *cvxpy*.

4.5 Repeat the above exercise by converting the problem into a convex optimization problem in two variables and using graphical plots.

4.6 Solve the above problem using the KKT conditions. Comment.

5 Linear Programming

5.1 Graphically obtain a solution to the following

$$\max_{\mathbf{x}} 6x_1 + 5x_2 \tag{5.1}$$

with constraints

$$x_1 + x_2 \leq 5 \tag{5.2}$$
$$3x_1 + 2x_2 \leq 12 \tag{5.3}$$
$$\text{where } x_1, x_2 \geq 0 \tag{5.4}$$

Solution: The following program plots the solution in Fig. 5.1

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/4.1.py
```

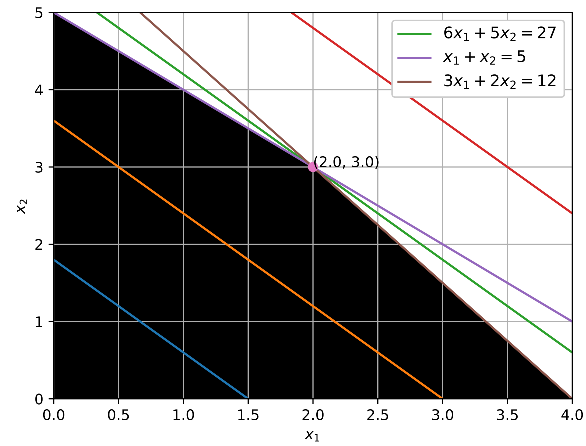

Fig. 5.1: The cost function intersects with the two constraints at $\mathbf{x} = (2, 3)$.

5.2 Now use *cvxpy* to obtain a solution to problem 5.1.
Solution: The given problem is expressed as follows

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad s.t. \tag{5.5}$$
$$\mathbf{A}\mathbf{x} \preceq \mathbf{b} \tag{5.6}$$

where

$$\mathbf{c} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{5.7}$$

The desired solution is then obtained using the following program.

```
wget https://raw.githubusercontent.com/
    gadepall/optimization/master/manual/
    codes/4.2−cvx.py
```

5.3 Verify your solution to the above problem using the method of Lagrange multipliers.

5.4 Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$\text{where } x_1, x_2 \geq 0$$

6 Convex Polygon

6.1 Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{C} \tag{6.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{6.2}$$
$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{6.3}$$

6.2 Prove that the point $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 24 \tag{6.4}$$
$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{6.5}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{6.6}$$

7 Complex Numbers: Optimization

7.1 Consider the optimization problem

$$\max_z \frac{1}{|z-1|} \tag{7.1}$$
$$s.t. \quad \left| z - 2 + \jmath \right| \ge \sqrt{5} \tag{7.2}$$

Show that it can be reframed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{c}_1\|^2 \tag{7.3}$$
$$s.t. \quad \|\mathbf{x} - \mathbf{c}_2\|^2 \ge 5 \tag{7.4}$$

where

$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{7.5}$$

7.2 Explain the optimization problem with a figure.
Solution: Fig. 6.2 explains (6.3) where z_0 is the set of points comprising of the intersection of the smallest circle Γ : with the largest circle Ω : $r_2 \ge \sqrt{5}$ with radii r_1 and $r_2 \ge \sqrt{5}$ respectively.

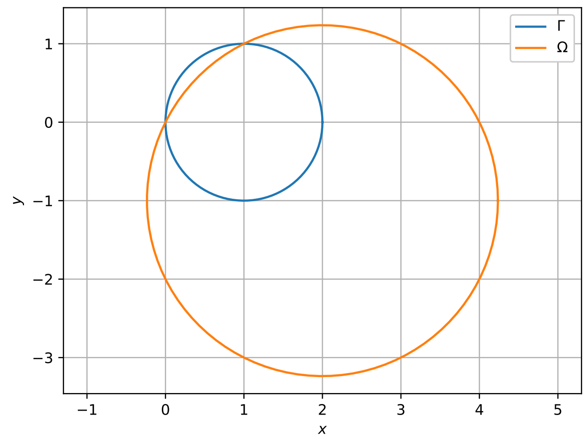

Fig. 7.2

7.3 Obtain the Lagrangian.
Solution: The Lagrangian is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x} - \mathbf{c}_1\|^2 - \lambda \left\{ \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 \right\} \tag{7.6}$$

7.4 Use the KKT conditions to obtain the minima.

Solution: From the KKT conditions,

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \tag{7.7}$$
$$\implies \mathbf{x} - \mathbf{c}_1 - \lambda(\mathbf{x} - \mathbf{c}_2) = 0 \tag{7.8}$$
$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} \tag{7.9}$$

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{7.10}$$
$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \tag{7.11}$$

Substituting from (6.9) in (6.11),

$$\left\| \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} - \mathbf{c}_2 \right\|^2 - r_2^2 = 0 \tag{7.12}$$
$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \tag{7.13}$$
$$= 1 \pm \sqrt{\frac{2}{5}} \tag{7.14}$$

Fig. 6.5 plots Γ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \tag{7.15}$$

7.5 If the maximum value is obtained at z_0, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{7.16}$$

Solution: From (6.9),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1-\lambda} \tag{7.17}$$

$$\implies z_0 = \frac{1}{1-\lambda}(1 - 2\lambda + \jmath\lambda) \tag{7.18}$$

$$\text{or, } \arg\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\{z_0\}}{\jmath(\Im\{z_0\}+1)} \tag{7.19}$$

$$= \frac{2(1-\lambda) - (1-2\lambda)}{\jmath} \tag{7.20}$$

$$= -\jmath \tag{7.21}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

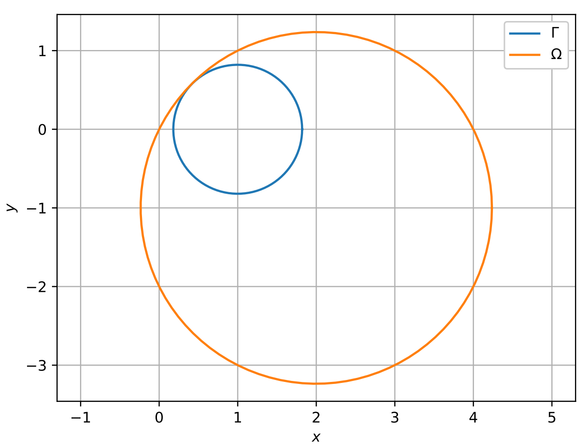

Fig. 7.5

7.6 Show that the set

$$D = \{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\}, r_2 > 0 \tag{7.22}$$

is nonconvex.
Solution: Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \tag{7.23}$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \tag{7.24}$$

$$\implies \mathbf{x}_2 \in D. \tag{7.25}$$

Suppose

$$\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \tag{7.26}$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \tag{7.27}$$

$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \tag{7.28}$$

$$\text{or, } \mathbf{x} \notin D \tag{7.29}$$

Thus, by definition, D is not a convex set.