Computational Approach to School Mathematics

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***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/codes
```

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1 Triangle

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90°, a = 4$ and $b = 3$.
Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1.1}$$

The following code plots Fig. 1.1.1

```
codes/triangle/rt_triangle.py
```

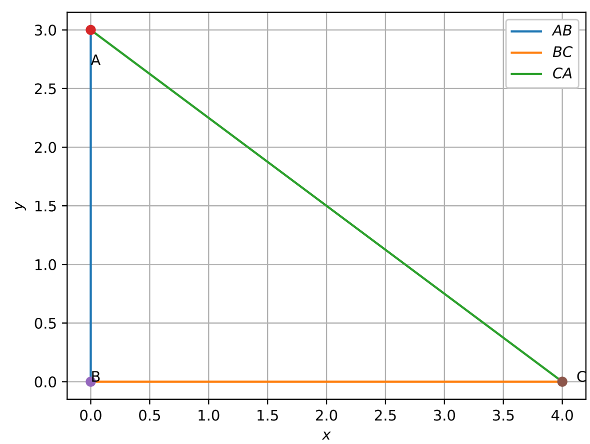

Fig. 1.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \tag{1.1.2.2}$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{1.1.2.3}$$

$$= p \times p + q \times q = p^2 + q^2 \tag{1.1.2.4}$$

Then

$$AB \triangleq \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{1.1.2.5}$$

$$BC = \|\mathbf{C}-\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{1.1.2.6}$$

$$AC = \|\mathbf{A}-\mathbf{C}\|^2 = b^2 \tag{1.1.2.7}$$

From (1.1.2.7),

$$b^2 = \|\mathbf{A}-\mathbf{C}\|^2 = \|\mathbf{A}-\mathbf{C}\|^T \|\mathbf{A}-\mathbf{C}\| \tag{1.1.2.8}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{1.1.2.9}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{1.1.2.10}$$

$$= a^2 + c^2 - 2ap \tag{1.1.2.11}$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{1.1.2.12}$$

From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.1.2.14}$$

The following code plots Fig. 1.1.2

```
codes/triangle/draw_triangle.py
```

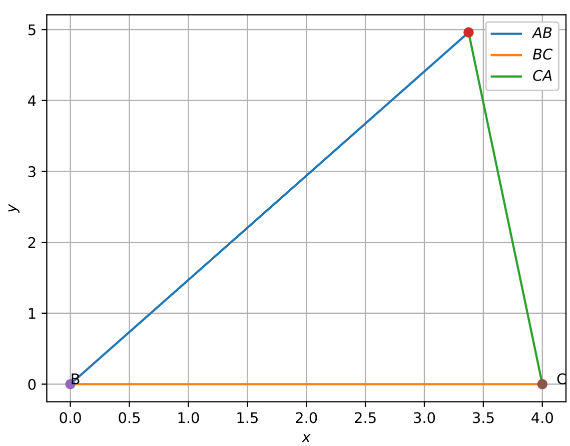

Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{1.1.4.1}$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.
Solution: From the given information,

$$a + b + c = 11 \tag{1.1.5.1}$$

$$b = c \quad (\because B = C = 45^\circ) \tag{1.1.5.2}$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{1.1.5.3}$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \tag{1.1.5.4}$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{1.1.5.5}$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{1.1.5.6}$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{1.1.5.7}$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{1.1.5.8}$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{1.1.5.9}$$

by expanding the determinants. The following code may be used to compute a, b and c.

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 1.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \tag{1.1.6.1}$$
$$a - b\sqrt{2} = 0 \tag{1.1.6.2}$$
$$b - c = 0 \tag{1.1.6.3}$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{1.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{1.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. (1.1.7), $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{1.1.7.1}$$
$$\cos B = \frac{x}{b}, \tag{1.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \tag{1.1.7.3}$$
$$b = c\cos A + a\cos C \tag{1.1.7.4}$$
$$c = b\cos A + a\cos B \tag{1.1.7.5}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.7.6}$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.1.7.7}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1.7.8}$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \tag{1.1.7.9}$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

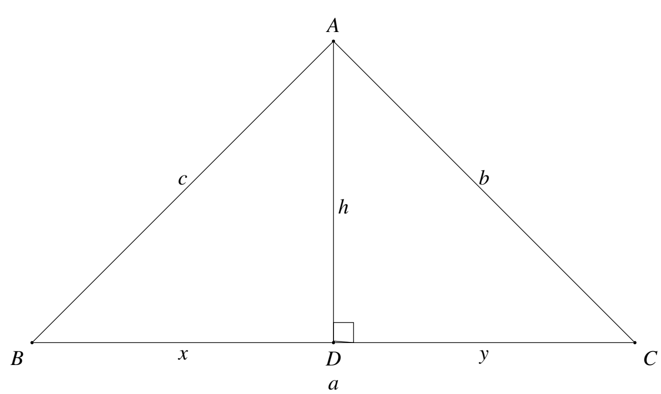

Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
   Solution: In Fig. (1.1.7),
$$\sin B = \frac{h}{c} \tag{1.1.8.1}$$
$$\sin C = \frac{h}{b} \tag{1.1.8.2}$$
   which can be used to show that
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.1.8.3}$$
   Thus,
$$c = \frac{a \sin C}{\sin A} \tag{1.1.8.4}$$
   where
$$C = 180 - A - B \tag{1.1.8.5}$$
9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30°$.
   Solution: From (1.1.7.9),
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.1.9.1}$$
   which can be expressed as
$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \tag{1.1.9.2}$$
$$\because (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \tag{1.1.9.3}$$
   (1.1.9.2) can be expressed as
$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \tag{1.1.9.4}$$
$$\implies (a - b \cos C)^2 = b^2 \left(1 - \cos^2 C\right) - c^2 \tag{1.1.9.5}$$
$$\text{or}, a = b \cos C \pm \sqrt{b^2 \left(1 - \cos^2 C\right) - c^2} \tag{1.1.9.6}$$
   Choose the value(s) for which $a > 0$.
10. The solution of a quadratic equation
$$\alpha x^2 + \beta x + \gamma = 0 \tag{1.1.10.1}$$
   is given by
$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{1.1.10.2}$$
   Verify (1.1.9.6) using (1.1.10.2).
11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.
   Solution: From Baudhayana's theorem,
$$b^2 = a^2 + c^2 \tag{1.1.11.1}$$
$$\implies (18 - c)^2 = 12^2 + c^2 \tag{1.1.11.2}$$
   which can be simplified to obtain
$$36c - 180 = 0 \tag{1.1.11.3}$$
$$\implies c = 5 \tag{1.1.11.4}$$
   and $b = 13$
12. Find a simpler solution for Problem 1.1.5
   Solution: Use cosine formula.
13. In $\triangle ABC$, $a = 7, \angle B = 75^\circ$ and $b + c = 13$. Alternatively,
$$a = b \cos C + c \cos B \tag{1.1.13.1}$$
$$b \sin C = c \sin B \tag{1.1.13.2}$$
$$a + b + c = 11 \tag{1.1.13.3}$$
   resulting in the matrix equation
$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \tag{1.1.13.4}$$
   Solving the equivalent matrix equation gives the desired answer.

1.2 Construction Exercises

1. In $\triangle ABC$, $a = 8, \angle B = 45^\circ$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6, \angle B = 60^\circ$ and $b-c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a+b+c = 11, \angle B = 30^\circ$ and $\angle C = 90^\circ$.
4. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60°$.
9. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90°$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
11. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60°$.