

G V V Sharma*

CONTENTS

1	Line	1
2	Altitudes of a Triangle	2
3	Circumcircle	3
4	Medians of a Triangle	3
5	Incircle	4

Abstract—This manual introduces matrix computations using python and the properties of a triangle.

1 LINE

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad (1)$$

Find the direction vector and the normal vector for AB

Solution: The direction vector of AB is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \quad (2)$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} 1 - (-2) \\ 3 - (-2) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (4)$$

1.2 Find the normal vector of AB .

Solution: The normal vector \mathbf{n} is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \quad (5)$$

and can be obtained as

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (6)$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (7)$$

$$= \begin{pmatrix} 0 \times 3 + 1 \times 5 \\ -1 \times 3 + 0 \times 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \quad (8)$$

1.3 Verify (5).

Solution:

$$\mathbf{n}^T \mathbf{m} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (9)$$

$$= 5 \times 3 + (-3) \times 5 = 0 \quad (10)$$

1.4 Find the equation of AB .

Solution: The desired equation is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (11)$$

$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (12)$$

1.5 Draw the line AB

Solution: The following code plots AB in Fig. 1.5

```
#Plotting AB
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if

#Function for generating the line AB
def line_gen(A,B):
    len = 10
    x_AB = np.zeros((2,len))
    lam_1 = np.linspace(0,1,len)
    for i in range(len):
```

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

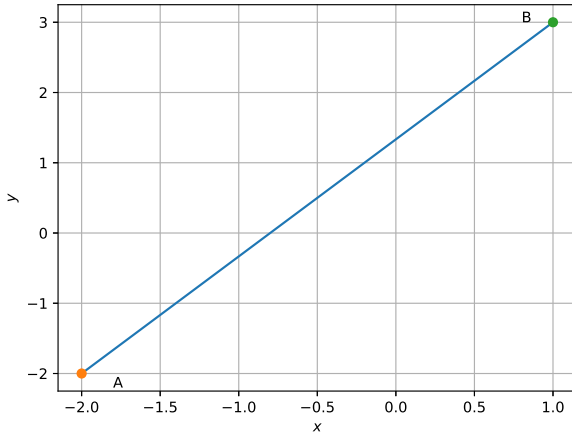


Fig. 1.5

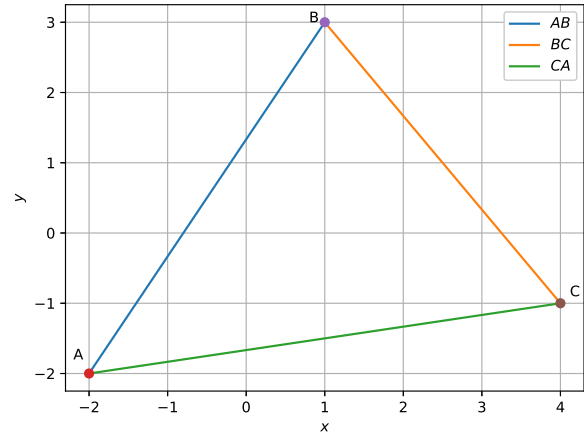


Fig. 1.6

```

temp1 = A + lam_1[i]*(B-A)
x_AB[:,i]= temp1.T
return x_AB

#Enter the points A and B
A = np.array([-2,-2])
B = np.array([1,3])
#Generate the line AB
x_AB=line_gen(A,B)
#Plot the line AB
plt.plot(x_AB[0,:],x_AB[1,:],label='$AB$')

#Figure details
plt.grid() # minor
plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 - 0.1), A[1] * (1 + 0.1) , '
    A')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 - 0.2), B[1] * (1) , 'B')
plt.xlabel('$x$')
plt.ylabel('$y$')

#if using termux
plt.savefig('../figs/draw_line.pdf')
plt.savefig('../figs/draw_line.eps')
subprocess.run(shlex.split("termux-open.../
    figs/draw_line.pdf"))
#else
#plt.show()

```

1.6 Draw $\triangle ABC$.

Solution: The following codes yields the desired plot in Fig. 1.6

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/coeffs.py

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/draw_triangle.py

1.7 Find the equation of the line in terms of the normal vector.

Solution: The desired equation is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (13)$$

$$\Rightarrow \begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = - \begin{pmatrix} 5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -4 \quad (14)$$

1.8 Verify that the above equation can be obtained by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \quad (15)$$

1.9 Find the equations of BC and CA .

2 ALTITUDES OF A TRIANGLE

2.1 In $\triangle ABC$, Let \mathbf{P} be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of $\triangle ABC$.

2.2 Find the equation of AP .

Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From (13), the equation of AP is

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (16)$$

$$\Rightarrow \begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = - \begin{pmatrix} -3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -2 \quad (17)$$

2.3 Find the equation of the altitude BQ .

Solution: The desired equation is

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{x} - \mathbf{B}) = 0 \quad (18)$$

$$\Rightarrow \begin{pmatrix} 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9 \quad (19)$$

2.4 Find the equation of the altitude CR .

2.5 Find the point of intersection of AP and BQ .

Solution: (16) and (18) can be stacked together into the matrix equation

$$\begin{pmatrix} -3 & 4 \\ 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \quad (20)$$

The following code computes the point of intersection.

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/orthocentre.py
```

2.6 Find the point of intersection of BQ and CR . Comment.

2.7 Find \mathbf{P}

Solution: The following code finds the required points.

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/alt_foot.py
```

2.8 Find \mathbf{Q} and \mathbf{R} .

2.9 Draw AP , BQ and CR and verify that they meet at a point \mathbf{H} .

Solution: The following code plots the altitudes in Fig. 2.9

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/alt_draw.py
```

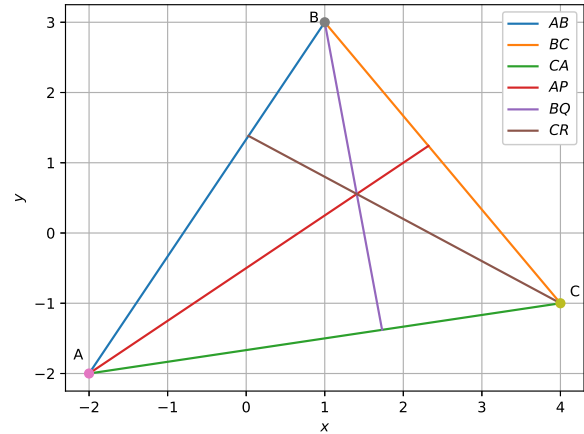


Fig. 2.9

$$\Rightarrow (\mathbf{A} - \mathbf{O})^T (\mathbf{A} - \mathbf{O}) - (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \quad (24)$$

which can be simplified as

$$(\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (25)$$

Similarly,

$$(\mathbf{B} - \mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (26)$$

The following code computes \mathbf{O} using the above two equations.

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcentre.py
```

3.3 Find the radius R .

3.4 Plot the *circumcircle* of $\triangle ABC$.

Solution: The following code plots Fig. 3.4

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcircle.py
```

3 CIRCUMCIRCLE

3.1 Let \mathbf{A} , \mathbf{B} and \mathbf{C} be points on a circle with centre \mathbf{O} and radius r .

3.2 Find \mathbf{O} .

Solution: The equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (21)$$

$$\Rightarrow \|\mathbf{x} - \mathbf{O}\|^2 = (\mathbf{x} - \mathbf{O})^T (\mathbf{x} - \mathbf{O}) = R^2 \quad (22)$$

From (21),

$$\|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \quad (23)$$

4 MEDIANS OF A TRIANGLE

4.1 Find the coordinates of \mathbf{D} , \mathbf{E} and \mathbf{F} of the mid points of AB , BC and CA respectively for $\triangle ABC$.

4.2 Find the equations of AD , BE and CF . These lines are the *medians* of $\triangle ABC$

4.3 Find the point of intersection of AD and CF .

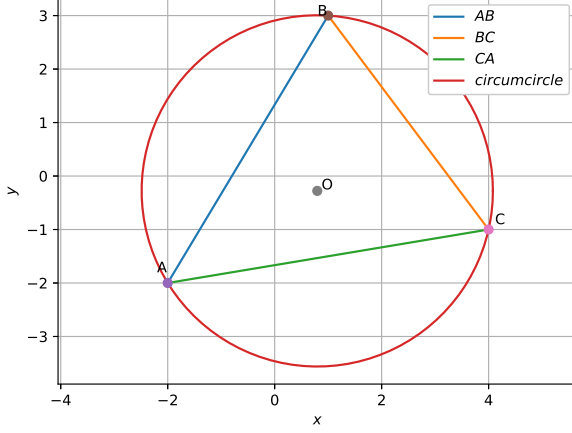


Fig. 3.4

- 4.4 Verify that \mathbf{G} is the point of intersection of BE, CF as well as AD, BE . \mathbf{G} is known as the *centroid* of $\triangle ABC$.
- 4.5 Graphically show that the medians of $\triangle ABC$ meet at the centroid.
- 4.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (27)$$

5 INCIRCLE

- 5.1 Consider a circle with centre \mathbf{I} and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at \mathbf{U}, \mathbf{V} and \mathbf{W} respectively.
- 5.2 Show that $IU \perp BC$.

Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $\mathbf{x}_1\mathbf{x}_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \quad (28)$$

$$\Rightarrow (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I} \right) = 0 \quad (29)$$

$$\Rightarrow (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I} \right) = 0 \quad (30)$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, $\mathbf{x}_1\mathbf{x}_2$ merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{U} - \mathbf{I}) = 0 \Rightarrow OD \perp BC \quad (31)$$

- 5.3 Find an expression for r if \mathbf{I} is known.

Solution: Let \mathbf{n} be the normal vector of BC . The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \quad (32)$$

$$\Rightarrow \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \quad (33)$$

since \mathbf{U} lies on BC . Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda \mathbf{n} \quad (34)$$

$$\Rightarrow \mathbf{I} - \mathbf{U} = \lambda \mathbf{n} \quad (35)$$

$$\text{or } r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \|\mathbf{n}\| \quad (36)$$

From (33) and (34)

$$\mathbf{n}^T \mathbf{I} = \mathbf{n}^T \mathbf{B} + \lambda \mathbf{n}^T \mathbf{n} \quad (37)$$

$$\Rightarrow \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda \|\mathbf{n}\|^2 \quad (38)$$

$$\Rightarrow r = |\lambda| \|\mathbf{n}\| = \frac{|\mathbf{n}^T (\mathbf{I} - \mathbf{B})|}{\|\mathbf{n}\|} \quad (39)$$

from (36). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \quad (40)$$

$$r = |\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})| \quad (41)$$

- 5.4 Find \mathbf{I} .

Solution: Since $r = IU = IV = IW$, from (41),

$$|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})| = |\mathbf{n}_2^T (\mathbf{I} - \mathbf{C})| = |\mathbf{n}_3^T (\mathbf{I} - \mathbf{A})| \quad (42)$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (42) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1 \mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \quad (43)$$

$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2 \mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \quad (44)$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{n}_1 - k_1 \mathbf{n}_2 & \mathbf{n}_2 - k_2 \mathbf{n}_3 \end{pmatrix}^T \mathbf{I} = \begin{pmatrix} \mathbf{n}_1^T \mathbf{B} - k_1 \mathbf{n}_2^T \mathbf{C} \\ \mathbf{n}_2^T \mathbf{C} - k_2 \mathbf{n}_3^T \mathbf{A} \end{pmatrix} \quad (45)$$

- 5.5 Show that \mathbf{I} lies inside $\triangle ABC$ for $k_1 = k_2 = 1$

- 5.6 Compute \mathbf{I} and r .

Solution:

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/incircle.py

- 5.7 Plot the incircle of $\triangle ABC$

Solution: The following code plots the incircle in Fig. 5.7

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/incircle.py

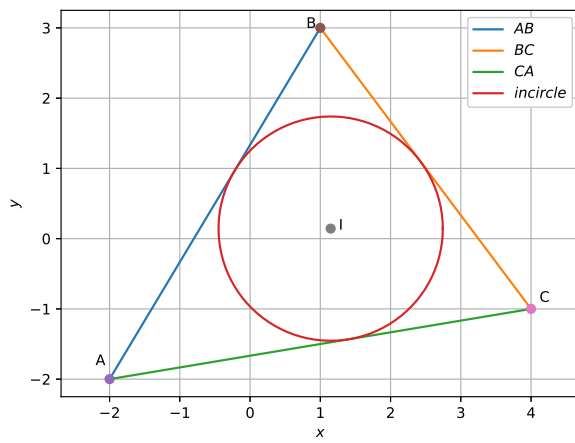


Fig. 5.7