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CONTENTS

Abstract—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

svn co <https://github.com/gadepall/school/trunk/ncert/codes>

1 TRIANGLE

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90^\circ$, $a = 4$ and $b = 3$.

Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.1.1)$$

The following code plots Fig. 1.1.1

codes/triangle/rt_triangle.py

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.

Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.1.2.1)$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \quad (1.1.2.2)$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T \mathbf{A} = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad (1.1.2.3)$$

$$= p \times p + q \times q = p^2 + q^2 \quad (1.1.2.4)$$

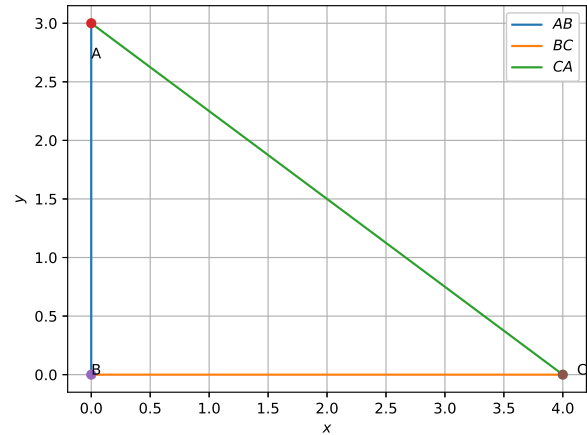


Fig. 1.1.1

Then

$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (1.1.2.5)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (1.1.2.6)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (1.1.2.7)$$

From (1.1.2.7),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (1.1.2.8)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{C}^T \mathbf{C} - \mathbf{A}^T \mathbf{C} - \mathbf{C}^T \mathbf{A} \quad (1.1.2.9)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} \quad (\because \mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}) \quad (1.1.2.10)$$

$$= a^2 + c^2 - 2ap \quad (1.1.2.11)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (1.1.2.12)$$

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From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \quad (1.1.2.13)$$

$$\implies q = \pm \sqrt{c^2 - p^2} \quad (1.1.2.14)$$

The following code plots Fig. 1.1.2

```
codes/triangle/draw_triangle.py
```

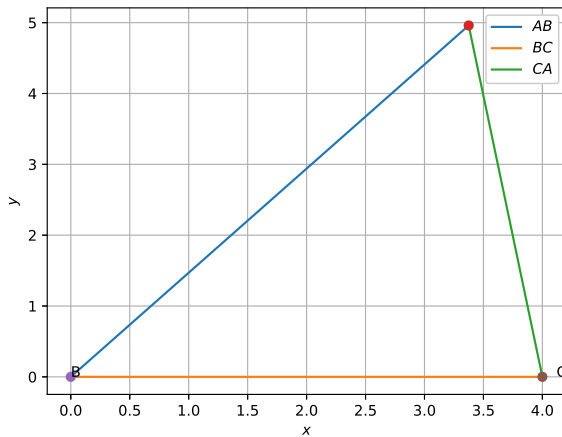


Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.

Solution: The sides of the similar triangle are $\frac{7}{5}a$, $\frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8\text{cm}$ and altitude $AD = h = 4\text{cm}$

Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \quad (1.1.4.1)$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.

Solution: From the given information,

$$a + b + c = 11 \quad (1.1.5.1)$$

$$b = c \quad (\because B = C = 45^\circ) \quad (1.1.5.2)$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \quad (1.1.5.3)$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \quad (1.1.5.4)$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \quad (1.1.5.5)$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \quad (1.1.5.6)$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times (-\sqrt{2}) - 2 \times 0}{1 \times (-\sqrt{2}) - 2 \times 1} \quad (1.1.5.7)$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \quad (1.1.5.8)$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \quad (1.1.5.9)$$

by expanding the determinants. The following code may be used to compute a, b and c .

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 1.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \quad (1.1.6.1)$$

$$a - b\sqrt{2} = 0 \quad (1.1.6.2)$$

$$b - c = 0 \quad (1.1.6.3)$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.6.4)$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.5)$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.6)$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.7)$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \quad (1.1.6.8)$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \quad (1.1.6.9)$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6$, $c = 5$ and $\angle B = 60^\circ$.

Solution: In Fig. 1.1.7, $AD \perp BC$.

$$\cos C = \frac{y}{b}, \quad (1.1.7.1)$$

$$\cos B = \frac{x}{c}, \quad (1.1.7.2)$$

Thus,

$$a = x + y = b \cos C + c \cos B, \quad (1.1.7.3)$$

$$b = c \cos A + a \cos C \quad (1.1.7.4)$$

$$c = b \cos A + a \cos B \quad (1.1.7.5)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.7.6)$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \quad (1.1.7.7)$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \quad (1.1.7.8)$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca \cos B \quad (1.1.7.9)$$

which is computed by the following code

codes/triangle/cos_form.py

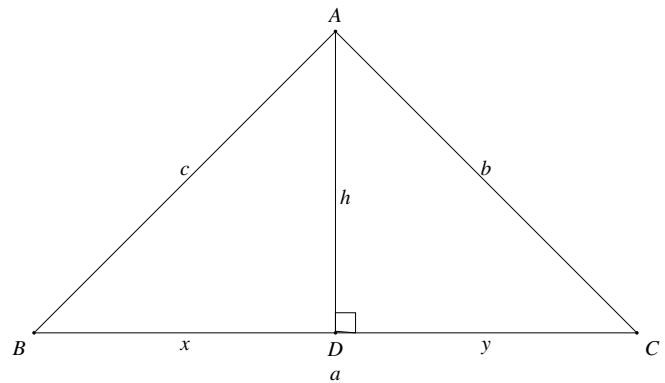


Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7$, $\angle B = 45^\circ$ and $\angle A = 105^\circ$.

Solution: In Fig. (1.1.7),

$$\sin B = \frac{h}{c} \quad (1.1.8.1)$$

$$\sin C = \frac{h}{b} \quad (1.1.8.2)$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.1.8.3)$$

Thus,

$$c = \frac{a \sin C}{\sin A} \quad (1.1.8.4)$$

where

$$C = 180 - A - B \quad (1.1.8.5)$$

9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30^\circ$.

Solution: From (1.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (1.1.9.1)$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \quad (1.1.9.2)$$

$$\therefore (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \quad (1.1.9.3)$$

(1.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \quad (1.1.9.4)$$

$$\Rightarrow (a - b \cos C)^2 = b^2 (1 - \cos^2 C) - c^2 \quad (1.1.9.5)$$

$$\text{or, } a = b \cos C \pm \sqrt{b^2 (1 - \cos^2 C) - c^2} \quad (1.1.9.6)$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \quad (1.1.10.1)$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (1.1.10.2)$$

Verify (1.1.9.6) using (1.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.

Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \quad (1.1.11.1)$$

$$\Rightarrow (18 - c)^2 = 12^2 + c^2 \quad (1.1.11.2)$$

which can be simplified to obtain

$$36c - 180 = 0 \quad (1.1.11.3)$$

$$\Rightarrow c = 5 \quad (1.1.11.4)$$

and $b = 13$

12. Find a simpler solution for Problem 1.1.5

Solution: Use cosine formula.

13. In $\triangle ABC$, $a = 7, \angle B = 75^\circ$ and $b + c = 13$. Alternatively,

$$a = b \cos C + c \cos B \quad (1.1.13.1)$$

$$b \sin C = c \sin B \quad (1.1.13.2)$$

$$a + b + c = 11 \quad (1.1.13.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (1.1.13.4)$$

Solving the equivalent matrix equation gives the desired answer.

1.2 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

Solution: The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \quad (1.2.1.1)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.2.1.2)$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \quad (1.2.1.3)$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T \quad (1.2.1.4)$$

If $\text{rank}(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow -5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \quad (1.2.1.5)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 2 \quad (1.2.1.6)$$

as the number of non zero rows is 2. The following code plots Fig. 1.2.1

codes/triangle/check_tri.py

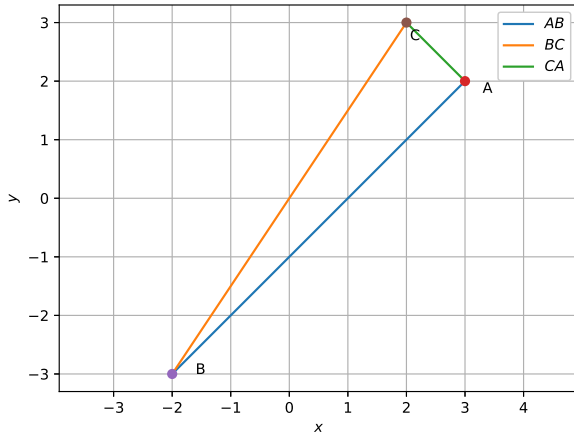


Fig. 1.2.1

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (1.2.1.7)$$

which, from (1.1.2.10) can be expressed as

$$\begin{aligned} \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} \\ = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \end{aligned} \quad (1.2.1.8)$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = 0 \quad (1.2.1.9)$$

after simplification. From (1.2.1.1) and (1.2.1.2), it is easy to verify that

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \quad (1.2.1.10)$$

satisfying (1.2.1.9). Thus, $\triangle ABC$ is right angled at \mathbf{A} .

2. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

codes/triangle/area_tri.py

3. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **Solution:** The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix \mathbf{M} in (1.2.1.4) as

$$\frac{1}{2} |\mathbf{M}| \quad (1.2.3.1)$$

The computation is done in **area_tri.py**

4. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \quad (1.2.4.1)$$

5. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.2.5.1)$$

as its vertices.

Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.2.5.2)$$

For any two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.2.5.3)$$

The following code computes the area using the vector product.

codes/triangle/area_tri_vec.py

6. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of \mathbf{A} and \mathbf{B} are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point \mathbf{C} .

Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.2.6.1)$$

Thus,

$$\mathbf{C} = 3\mathbf{O} - \mathbf{A} - \mathbf{B} \quad (1.2.6.2)$$

7. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.2.7.1)$$

are the vertices of a right angled triangle.

Solution: The following code plots Fig. 1.2.7

```
codes/triangle/triangle_3d.py
```

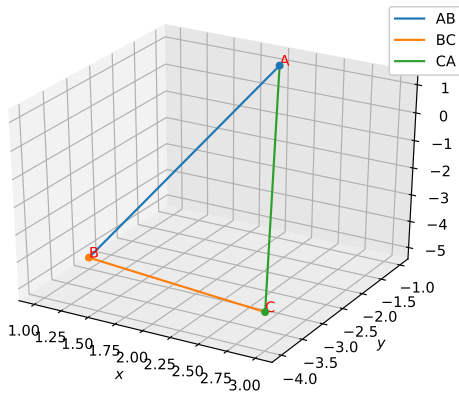


Fig. 1.2.7

From the figure, it appears that $\triangle ABC$ is right angled at \mathbf{C} . Since

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.2.7.2)$$

it is proved that the triangle is indeed right angled.

8. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \quad (1.2.8.1)$$

the vertices of a right angled triangle?

1.3 Triangle Exercises

1. Draw the graphs of the equations

$$(1 \ -1)\mathbf{x} + 1 = 0 \quad (1.3.1.1)$$

$$(3 \ 2)\mathbf{x} - 12 = 0 \quad (1.3.1.2)$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

2. In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

3. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.

4. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find the equation of the median through the vertex \mathbf{R} .

5. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the equation and length of the altitude from the vertex \mathbf{A} .

6. Find the area of the triangle whose vertices are

a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

7. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

8. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.

9. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (1.3.9.1)$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (1.3.9.2)$$

10. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the

vertices of $\triangle ABC$.

- The median from **A** meets BC at **D**. Find the coordinates of the point **D**.
- Find the coordinates of the point **P** on AD such that $AP : PD = 2 : 1$.
- Find the coordinates of the points **Q** and **R** on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

11. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.3.11.1)$$

12. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.3.12.1)$$

are the vertices of a right angled triangle.

13. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.

14. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.

15. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.

16. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

17. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$, $\mathbf{B} =$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$$

18. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.

19. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.3.19.1)$$

are the vertices of an isosceles triangle.

2 QUADRILATERAL

2.1 Construction Examples

- Draw $ABCD$ with $AB = a = 4.5$, $BC = b = 5.5$, $CD = c = 4$, $AD = d = 6$ and $AC = e = 7$.

Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting

$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.1.1.1)$$

it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ can be obtained by rotating an equivalent triangle with AC on the x -axis by an angle θ with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (2.1.1.2)$$

and

$$\cos \theta = \frac{a^2 + e^2 - b^2}{2ae} \quad (2.1.1.3)$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} \quad (2.1.1.4)$$

The coordinates of the rotated triangle ACD are

$$\mathbf{D} = \mathbf{P} \begin{pmatrix} h \\ k \end{pmatrix} \quad (2.1.1.5)$$

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.1.6)$$

$$\mathbf{C} = \mathbf{P} \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (2.1.1.7)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (2.1.1.8)$$

The following code plots quadrilateral $ABCD$

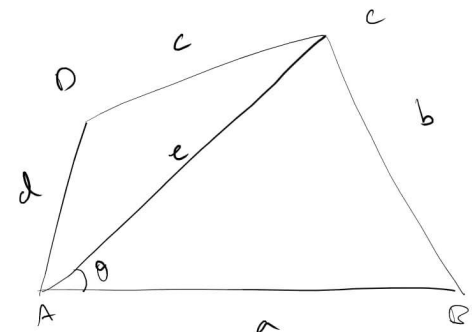


Fig. 2.1.1

in Fig. 2.1.1

codes/quad/draw_quad.py

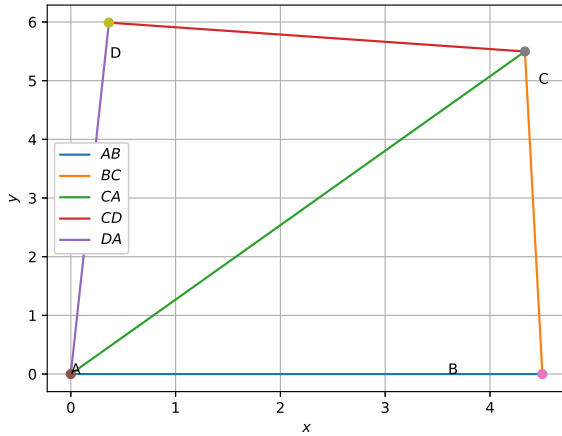


Fig. 2.1.1

2. Construct a kite $EASY$ if $AY = 8$, $EY = 4$ and $SY = 6$.

Solution: The diagonals of a kite are perpendicular to each other.

2.2 Quadrilateral Examples

1. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.

Solution: By inspection,

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (2.2.1.1)$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{D}) = 0 \quad (2.2.1.2)$$

$\Rightarrow AC \perp BD$. Hence $ABCD$ is a square.

2. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p .

Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p .

3. If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral $ABCD$.

Solution: The area of $ABCD$ is the sum of the areas of triangles ABD and CBD and is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| + \frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| \quad (2.2.3.1)$$

4. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$ are the vertices of a parallelogram $ABCD$ but it is not a rectangle.

Solution: Since the direction vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \quad (2.2.4.1)$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \quad (2.2.4.2)$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{D}) \neq 0 \quad (2.2.4.3)$$

Hence, it is not a rectangle. The following code plots Fig. 2.2.4

codes/triangle/quad_3d.py

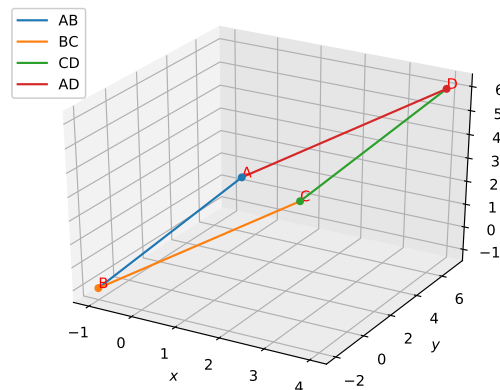


Fig. 2.2.4

5. Find the area of a parallelogram whose adja-

cent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The area is given by

$$\frac{1}{2} \left\| \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\| \quad (2.2.5.1)$$

2.3 Quadrilateral Geometry

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \quad (2.3.2.1)$$

$$\angle B = 3y - 5 \quad (2.3.2.2)$$

$$\angle C = -4x \quad (2.3.2.3)$$

$$\angle D = -7x + 5 \quad (2.3.2.4)$$

Find its angles.

3. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

Also, find its area.

4. Find the area of a rhombus if its vertices are $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ taken in order.
5. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.
6. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
7. The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.
8. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
 - a) square?
 - b) rectangle?
 - c) rhombus?

9. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

10. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.

11. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{D} =$

$$\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}.$$

12. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

3 LINE

3.1 Examples: Geometry

1. Check whether -2 and 2 are zeroes of the polynomial $x + 2$.
2. Find a zero of the polynomial $p(x) = 2x + 1$.
3. Verify whether the following are zeroes of the polynomial, indicated against them.
 - a) $p(x) = 3x + 1, x = \frac{1}{3}$
 - b) $p(x) = 5x - \pi, x = \frac{5}{\pi}$
 - c) $p(x) = 5lx + m, x = -\frac{m}{l}$
 - d) $p(x) = 2x + 1, x = \frac{1}{2}$
4. Find the zero of the polynomial in each of the following cases:
 - a) $p(x) = x + 5$
 - b) $p(x) = x - 5$
 - c) $p(x) = 2x + 5$
 - d) $p(x) = 3x - 2$
 - e) $p(x) = 3x$
 - f) $p(x) = ax, a \neq 0$
 - g) $p(x) = cx + d, c \neq 0, c, d$ are real numbers.
5. Find four different solutions of the equation

$$(1 \ 2)\mathbf{x} = 6 \quad (3.1.5.1)$$

6. Find two solutions for each of the following equations:
- $\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12$
 - $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$
 - $\begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 4$
7. Draw the graph of
- $$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \quad (3.1.7.1)$$
8. Draw the graphs of the following equations
- $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
 - $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$
 - $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$
 - $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1$
 - $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 4$
 - $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$
9. Two rails are represented by the equations
- $$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 4 = 0 \text{ and} \quad (3.1.9.1)$$
- $$\begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} - 12 = 0. \quad (3.1.9.2)$$
- Will the rails cross each other?
10. Check graphically whether the pair of equations
- $$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \text{ and} \quad (3.1.10.1)$$
- $$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \quad (3.1.10.2)$$
- is consistent. If so, solve them graphically.
11. Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:
- $$\begin{pmatrix} 5 & -8 \end{pmatrix} \mathbf{x} = -1 \text{ and} \quad (3.1.11.1)$$
- $$\begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\frac{3}{5} \quad (3.1.11.2)$$
12. Solve the following pair of equations
- $$\begin{pmatrix} 7 & -15 \end{pmatrix} \mathbf{x} = 2 \quad (3.1.12.1)$$
- $$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 3 \quad (3.1.12.2)$$
13. Find all possible solutions of
- $$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 8 \quad (3.1.13.1)$$
- $$\begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} = 7$$
14. For which values of p does the pair of equations given below has unique solution?
- $$\begin{pmatrix} 4 & p \end{pmatrix} \mathbf{x} = -8 \quad (3.1.14.1)$$
- $$\begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} = -2$$
15. For what values of k will the following pair of linear equations have infinitely many solutions?
- $$\begin{pmatrix} k & 3 \end{pmatrix} \mathbf{x} = k - 3$$
- $$\begin{pmatrix} 12 & k \end{pmatrix} \mathbf{x} = k \quad (3.1.15.1)$$
16. Find the values of x, y, z such that
- $$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \quad (3.1.16.1)$$
- Solution:** $x = 2, y = 2, z = 1$.
17. If
- $$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (3.1.17.1)$$
- verify if
- $\|\mathbf{a}\| = \|\mathbf{b}\|$
 - $\mathbf{a} = \mathbf{b}$
- Solution:**
- $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.
18. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.
- Solution:** The unit vector is given by
- $$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (3.1.18.1)$$
19. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.
20. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
21. Find a vector \mathbf{x} in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such

that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k| \left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = 7 \quad (3.1.21.1)$$

$$\Rightarrow |k| = \frac{7}{\sqrt{5}} \quad (3.1.21.2)$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (3.1.21.3)$$

22. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \quad (3.1.22.1)$$

23. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \quad (3.1.23.1)$$

24. Find the direction vector of PQ , where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \quad (3.1.24.1)$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \quad (3.1.24.2)$$

25. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.

Solution: Refer to Problem 1.2.1.

26. Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: From the given information,

$$\left\| \mathbf{x} - \begin{pmatrix} 7 \\ 1 \end{pmatrix} \right\|^2 = \left\| \mathbf{x} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\|^2 \quad (3.1.26.1)$$

$$\begin{aligned} \Rightarrow \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 7 \\ 1 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 7 & 1 \end{pmatrix} \mathbf{x} \\ = \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} \end{aligned} \quad (3.1.26.2)$$

which can be simplified to obtain

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (3.1.26.3)$$

which is the desired condition. The following code plots Fig. 3.1.26 clearly showing that the above equation is the perpendicular bisector of AB .

codes/line/line_perp_bisect.py

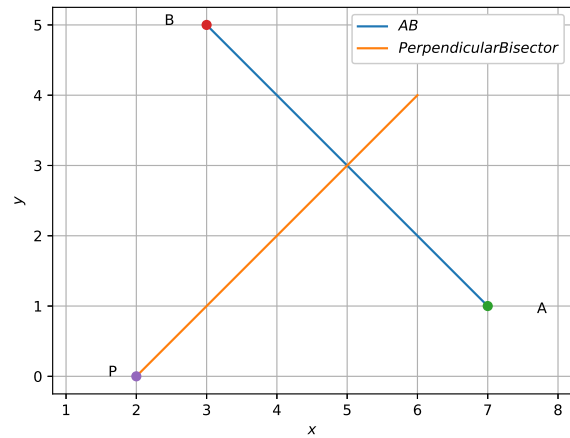


Fig. 3.1.26

27. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Solution: Choose $\mathbf{x} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and follow the approach in Problem (3.1.26). Solve for y .

28. Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 8.

Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \quad (3.1.28.1)$$

Then the point \mathbf{C}

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k + 1} \quad (3.1.28.2)$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 3.1.28

codes/line/draw_section.py

29. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and

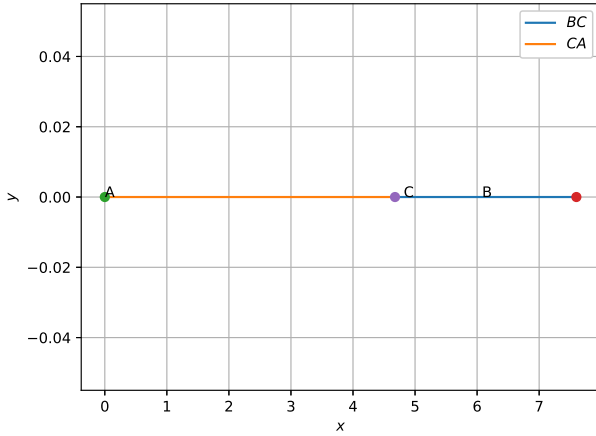


Fig. 3.1.28

$\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.

Solution: Using (3.1.28.2), the desired point is

$$\mathbf{P} = \frac{3 \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \quad (3.1.29.1)$$

30. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \quad (3.1.30.1)$$

Solution: Use (3.1.28.2).

31. Find the coordinates of the points of trisection of the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \quad (3.1.31.1)$$

Solution: Using (3.1.28.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \quad (3.1.31.2)$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \quad (3.1.31.3)$$

32. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.28.2),

the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k \begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (3.1.32.1)$$

$$\Rightarrow 0 = 5k - 1 \Rightarrow k = \frac{1}{5} \quad (3.1.32.2)$$

33. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.2.1.4),

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T = \begin{pmatrix} 2 & k - 3 \\ 4 & -6 \end{pmatrix} \quad (3.1.33.1)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k - 3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k - 3 \\ 0 & -k \end{pmatrix} \quad (3.1.33.2)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{ or } k = 0 \quad (3.1.33.3)$$

34. Find the direction vectors and slopes of the lines passing through the points

a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

d) Making an inclination of 60° with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (3.1.34.1)$$

the slope is m . Thus, the direction vector is

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \quad (3.1.34.2)$$

$$= \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \Rightarrow m = -\frac{3}{2} \quad (3.1.34.3)$$

b) The direction vector is

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (3.1.34.4)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow m = 0 \quad (3.1.34.5)$$

c) The direction vector is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (3.1.34.6)$$

$$= \begin{pmatrix} 1 \\ \infty \end{pmatrix} \Rightarrow m = \infty \quad (3.1.34.7)$$

d) The slope is $m = \tan 60^\circ = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (3.1.34.8)$$

35. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.

Solution: The angle θ between two lines is given by

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} \quad (3.1.35.1)$$

$$\Rightarrow 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \quad (3.1.35.2)$$

$$\text{or } m_1 = \frac{5}{3} \quad (3.1.35.3)$$

36. The line through the points $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} x \\ 24 \end{pmatrix}$. Find the value of x .

Solution: Using (1.2.1.9)

$$\left\{ \begin{pmatrix} -2 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \end{pmatrix} \right\}^T \left\{ \begin{pmatrix} 8 \\ 12 \end{pmatrix} - \begin{pmatrix} x \\ 24 \end{pmatrix} \right\} = 0 \quad (3.1.36.1)$$

which can be used to obtain x .

37. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \quad (3.1.37.1)$$

$$\Rightarrow \begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -6 \end{pmatrix} \quad (3.1.37.2)$$

which can be expressed as

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (3.1.37.3)$$

$$\Rightarrow \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} = -2 \quad (3.1.37.4)$$

$$\Rightarrow D = 2 + 2T \quad (3.1.37.5)$$

38. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Solution: The line parallel to the x -axis has direction vector $\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence, its equation is obtained as

$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.38.1)$$

Similarly, the equation of the line parallel to the y -axis can be obtained as

$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.1.38.2)$$

The following code plots Fig. 3.1.38

```
codes/line/line_parallel_axes.py
```

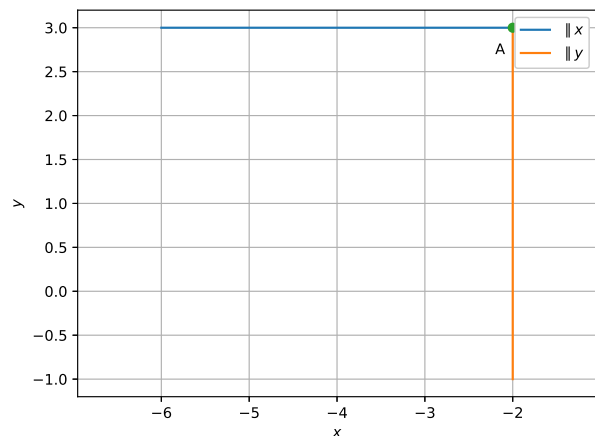


Fig. 3.1.38

39. Find the equation of the line through $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope -4 .

Solution: The direction vector is $\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \quad (3.1.39.1)$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.1.39.2)$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (3.1.39.3)$$

$$\Rightarrow (4 \ 1) \mathbf{x} = -5 \quad (3.1.39.4)$$

40. Write the equation of the line through the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: Use (3.1.38.1).

41. Write the equation of the lines for which $\tan \theta = \frac{1}{2}$, where θ is the inclination of the line and

- a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.

Solution: From the given information, $\tan \theta = \frac{1}{2} = m$.

- a) y-intercept is $-\frac{3}{2} \Rightarrow$ the line cuts through the y-axis at $\begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$.
b) x-intercept is 4 \Rightarrow the line cuts through the x-axis at $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

Use the above information get the equations for the lines.

42. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.

Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix} 5 & 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix} \quad (3.1.42.1)$$

43. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.

Solution: Using (3.1.37.1), the desired equa-

tion of the line is

$$\mathbf{x} = \begin{pmatrix} -1 & 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \quad (3.1.43.1)$$

$$= \begin{pmatrix} -1 & 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.43.2)$$

44. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \quad (3.1.44.1)$$

find the equation of the line.

Solution: The line can be expressed from (3.1.44.1) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 + 2\lambda \\ 5 + 4\lambda \\ -6 + 2\lambda \end{pmatrix} \quad (3.1.44.2)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \quad (3.1.44.3)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (3.1.44.4)$$

45. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.

Solution: See Problem 3.1.41. The line passes through the points $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

46. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15° .

Solution: In Fig. 3.1.46, the foot of the perpendicular P is the intersection of the lines L and M . Thus,

$$\mathbf{n}^T \mathbf{P} = c \quad (3.1.46.1)$$

$$\mathbf{P} = \mathbf{A} + \lambda \mathbf{n} \quad (3.1.46.2)$$

$$\text{or, } \mathbf{n}^T \mathbf{P} = \mathbf{n}^T \mathbf{A} + \lambda \|\mathbf{n}\|^2 = c \quad (3.1.46.3)$$

$$\Rightarrow -\lambda = \frac{\mathbf{n}^T \mathbf{A} - c}{\|\mathbf{n}\|^2} \quad (3.1.46.4)$$

Also, the distance between \mathbf{A} and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda \mathbf{n} \quad (3.1.46.5)$$

$$\Rightarrow \|\mathbf{P} - \mathbf{A}\| = |\lambda| \|\mathbf{n}\| \quad (3.1.46.6)$$

From (3.1.46.4) and (3.1.46.6)

$$\|\mathbf{P} - \mathbf{A}\| = \frac{|\mathbf{n}^T \mathbf{A} - c|}{\|\mathbf{n}\|} \quad (3.1.46.7)$$

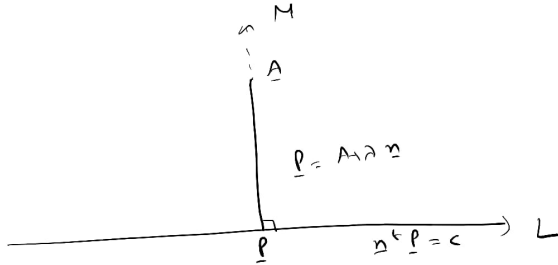


Fig. 3.1.46

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \quad (3.1.46.8)$$

$\therefore \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\|\mathbf{n}\|} \Rightarrow c = \pm 4 \sqrt{1 + \tan^2 15^\circ} \quad (3.1.46.9)$$

$$= \pm 4 \sec 15^\circ \quad (3.1.46.10)$$

where

$$\sec \theta = \frac{1}{\cos \theta} \quad (3.1.46.11)$$

This follows from (??), where

$$\cos^2 \theta + \sin^2 \theta = 1 \quad (3.1.46.12)$$

$$\Rightarrow 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \quad (3.1.46.13)$$

It is easy to verify that

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \quad (3.1.46.14)$$

$$\Rightarrow 1 + \tan^2 \theta = \sec^2 \theta \quad (3.1.46.15)$$

Thus, the equation of the line is

$$(1 \quad \tan 15^\circ) \mathbf{c} = \pm 4 \sec 15^\circ \quad (3.1.46.16)$$

47. The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F , when $K = 0$.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \quad (3.1.47.1)$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 212 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.39.3) and (3.1.39.1) as

$$(11 \quad -100) \mathbf{x} = (11 \quad -100) \begin{pmatrix} 273 \\ 32 \end{pmatrix} \quad (3.1.47.2)$$

$$\Rightarrow (11 \quad -100) \mathbf{x} = -197 \quad (3.1.47.3)$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$(11 \quad -100) \begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \Rightarrow F = -\frac{197}{11} \quad (3.1.47.4)$$

48. Equation of a line is

$$(3 \quad -4) \mathbf{x} + 10 = 0. \quad (3.1.48.1)$$

Find its

- slope,
- x - and y -intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad (3.1.48.2)$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad (3.1.48.3)$$

$$\text{a) } m = \frac{3}{4}$$

$$\text{b) } x\text{-intercept is } -\frac{10}{3} \text{ and } y\text{-intercept is } \frac{10}{4} = \frac{5}{2}.$$

49. Find the angle between two vectors \mathbf{a} and \mathbf{b} where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T \mathbf{b} = 1. \quad (3.1.49.1)$$

Solution: In Fig. 3.1.49, from the cosine formula in (1.1.7.9)

$$\cos \theta = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 - \|\mathbf{A} - \mathbf{C}\|^2}{2 \|\mathbf{A} - \mathbf{B}\| \|\mathbf{B} - \mathbf{C}\|} \quad (3.1.49.2)$$

Letting $\mathbf{a} = \mathbf{A} - \mathbf{B}$, $\mathbf{b} = \mathbf{B} - \mathbf{C}$,

$$\cos \theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a} + \mathbf{b}\|^2}{2 \|\mathbf{a}\| \|\mathbf{b}\|} \quad (3.1.49.3)$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - [\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T \mathbf{b}]}{2 \|\mathbf{a}\| \|\mathbf{b}\|} \quad (3.1.49.4)$$

$$\Rightarrow \cos \theta = \frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (3.1.49.5)$$

Thus, the angle θ between two vectors is given by

$$\cos \theta = \frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \quad (3.1.49.6)$$

$$= \frac{1}{2} \quad (3.1.49.7)$$

$$\Rightarrow \theta = 60^\circ \quad (3.1.49.8)$$

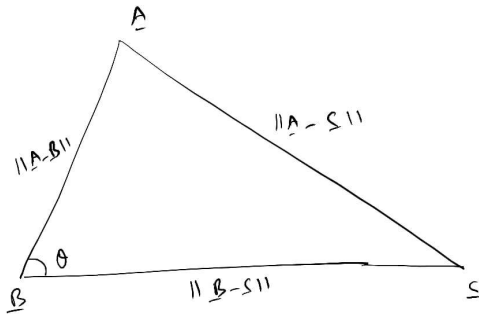


Fig. 3.1.49

50. Find the angle between the lines

$$(1 \quad -\sqrt{3})\mathbf{x} = 5 \quad (3.1.50.1)$$

$$(\sqrt{3} \quad -1)\mathbf{x} = -6. \quad (3.1.50.2)$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos \theta = \frac{\mathbf{n}_1 \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (3.1.50.3)$$

$$= \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ \quad (3.1.50.4)$$

51. Find the equation of a line perpendicular to the line

$$(1 \quad -2)\mathbf{x} = 3 \quad (3.1.51.1)$$

and passes through the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.1.51.2)$$

Thus, the desired equation of the line is

$$(2 \quad 1)\left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = 0 \quad (3.1.51.3)$$

$$\Rightarrow (2 \quad 1)\mathbf{x} = 0 \quad (3.1.51.4)$$

52. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$(3 \quad -4)\mathbf{x} = 26 \quad (3.1.52.1)$$

Solution: Use (3.1.46.7).

53. If the lines

$$(2 \quad 1)\mathbf{x} = 3 \quad (3.1.53.1)$$

$$(5 \quad k)\mathbf{x} = 3 \quad (3.1.53.2)$$

$$(3 \quad -1)\mathbf{x} = 2 \quad (3.1.53.3)$$

are concurrent, find the value of k .

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & k & 3 \\ 3 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 5 & k & 3 \end{pmatrix} \quad (3.1.53.4)$$

$$\xrightarrow{\begin{matrix} R_2 \leftrightarrow 2R_2 - 3R_1 \\ R_3 \leftrightarrow 2R_3 - 5R_1 \end{matrix}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & -5 \\ 0 & 2k - 5 & -9 \end{pmatrix} \quad (3.1.53.5)$$

$$\xrightarrow{R_2 \leftarrow -\frac{R_2}{5}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2k - 5 & -9 \end{pmatrix} \quad (3.1.53.6)$$

$$\xrightarrow{R_3 \leftarrow R_3 - (2k-5)R_2} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2k - 4 \end{pmatrix} \quad (3.1.53.7)$$

$$\Rightarrow k = -2 \quad (3.1.53.8)$$

54. Find the distance of the line

$$L_1 : (4 \quad 1)\mathbf{x} = 0 \quad (3.1.54.1)$$

from the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x -

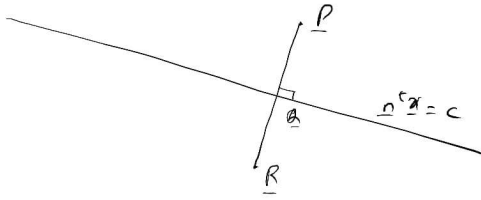


Fig. 3.1.55

axis.

Solution: Let P be the point of intersection of L_1 and L_2 . The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan 135^\circ \end{pmatrix} \quad (3.1.54.2)$$

Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on L_2 , the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \quad (3.1.54.3)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \quad (3.1.54.4)$$

$$\text{or, } \left\| \mathbf{P} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = d = |\lambda| \|\mathbf{m}\| \quad (3.1.54.5)$$

Since \mathbf{P} lies on L_1 , from (3.1.54.1),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{P} = 0 \quad (3.1.54.6)$$

Substituting from the above in (3.1.54.3),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m} = 0 \quad (3.1.54.7)$$

$$\Rightarrow \lambda = \frac{\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m}}{17} \quad (3.1.54.8)$$

substituting $|\lambda|$ in (3.1.54.5) gives the desired answer.

55. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$(1 \ -3)\mathbf{x} = -4. \quad (3.1.55.1)$$

Solution: Since \mathbf{R} is the reflection of \mathbf{P} and \mathbf{Q} lies on L , \mathbf{Q} bisects PR . This leads to the

following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (3.1.55.2)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad (3.1.55.3)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad (3.1.55.4)$$

where \mathbf{m} is the direction vector of L . From (3.1.55.2) and (3.1.55.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (3.1.55.5)$$

From (3.1.55.5) and (3.1.55.4),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (3.1.55.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \quad (3.1.55.7)$$

with the condition that \mathbf{m}, \mathbf{n} are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (3.1.55.8)$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.1.55.9)$$

(3.1.55.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (3.1.55.10)$$

$$\Rightarrow \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (3.1.55.11)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c \mathbf{n} \quad (3.1.55.12)$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (3.1.55.13)$$

The following code plots Fig. 3.1.55 while computing the reflection

codes/line/line_reflect.py

56. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Obtain

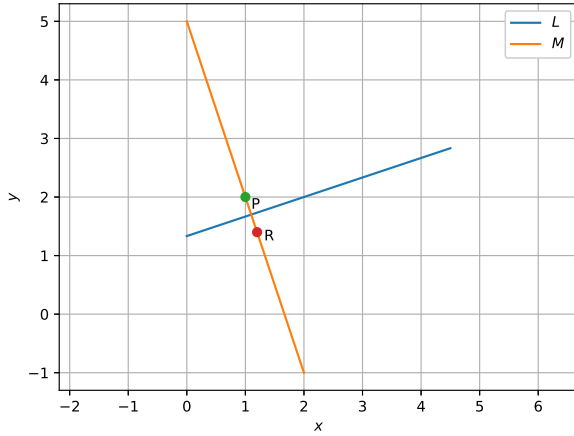


Fig. 3.1.55

its equation.

$$L_1 : (5 \ -1)\mathbf{x} = -4 \quad (3.1.56.1)$$

$$L_2 : (3 \ 4)\mathbf{x} = 4 \quad (3.1.56.2)$$

Solution: Let

$$L : \mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \quad (3.1.56.3)$$

If L intersects L_1 and L_2 at \mathbf{A} and \mathbf{B} respectively,

$$\mathbf{A} = \mathbf{P} + \lambda \mathbf{m} \quad (3.1.56.4)$$

$$\mathbf{B} = \mathbf{P} - \lambda \mathbf{m} \quad (3.1.56.5)$$

since \mathbf{P} bisects AB . Note that λ is a measure of the distance from P along the line L . From (3.1.56.1), (3.1.56.4) and (3.1.56.5),

$$(5 \ -1)\mathbf{A} = (5 \ -1)\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = -4 \quad (3.1.56.6)$$

$$(3 \ 4)\mathbf{B} = (3 \ 4)\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 4 \quad (3.1.56.7)$$

yielding

$$19(5 \ -1)\mathbf{m} = -4(3 \ 4)\mathbf{m} \quad (3.1.56.8)$$

$$\Rightarrow (107 \ -3)\mathbf{m} = 0 \quad (3.1.56.9)$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 107 \\ -3 \end{pmatrix} \quad (3.1.56.10)$$

after simplification. Thus, the equation of the

line is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \quad (3.1.56.11)$$

57. Show that the path of a moving point such that its distances from two lines

$$(3 \ -2)\mathbf{x} = 5 \quad (3.1.57.1)$$

$$(3 \ 2)\mathbf{x} = 5 \quad (3.1.57.2)$$

are equal is a straight line.

Solution: Using (3.1.46.7) the point \mathbf{x} satisfies

$$\frac{|(3 \ -2)\mathbf{x} - 5|}{\left\| \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\|} = \frac{|(3 \ 2)\mathbf{x} - 5|}{\left\| \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\|} \quad (3.1.57.3)$$

$$\Rightarrow |(3 \ -2)\mathbf{x} - 5| = |(3 \ 2)\mathbf{x} - 5| \quad (3.1.57.4)$$

resulting in

$$(3 \ -2)\mathbf{x} - 5 = \pm ((3 \ 2)\mathbf{x} - 5) \quad (3.1.57.5)$$

leading to the possible lines

$$L_1 : (0 \ 1)\mathbf{x} = 0 \quad (3.1.57.6)$$

$$L_2 : (1 \ 0)\mathbf{x} = \frac{5}{3} \quad (3.1.57.7)$$

58. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \quad (3.1.58.1)$$

Solution: The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

59. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.2.1.4)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.1.59.1)$$

$\Rightarrow \text{rank}(\mathbf{M}) = 1$. The following code plots Fig. 3.1.59 showing that the points are collinear.

codes/line/collinear_3d.py

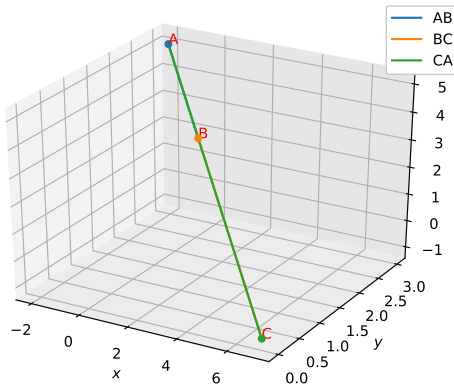


Fig. 3.1.59

60. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} =$

$\begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.

Solution: Use the approach in Problem (3.1.59).

61. Find the equation of set of points \mathbf{P} such that

$$PA^2 + PB^2 = 2k^2, \quad (3.1.61.1)$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \quad (3.1.61.2)$$

respectively.

62. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and

$\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3

- a) internally, and
b) externally.

Solution: Use (3.1.28.2).

63. Prove that the three points $\begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix}$ are collinear.

Solution: Use the approach in Problem 3.1.59.

64. Find the ratio in which the line segment joining

the points $\begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$ is divided by the YZ-plane.

Solution: Use (3.1.28.2). The YZ-plane has points $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$.

65. Find the equation of the set of points \mathbf{P} such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution: Use the approach in Problem 3.1.26.

66. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (3.1.66.1)$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (3.1.66.2)$$

find \mathbf{R} , which divides PQ in the ratio 2 : 1

- a) internally,
b) externally.

Solution: Use (3.1.28.2).

67. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: Use (3.1.49.6)

68. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (3.1.68.1)$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (3.1.68.2)$$

Solution: The direction vectors of the lines are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Using (3.1.49.6), the angle between the lines can be obtained.

69. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad (3.1.69.1)$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad (3.1.69.2)$$

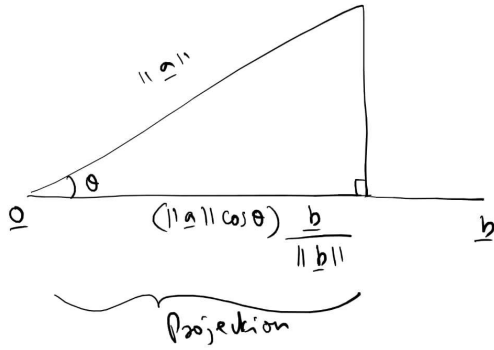


Fig. 3.1.71

Solution: From Problem 3.1.44, the direction vectors of the lines can be expressed as $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$

and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The angle between them can then be obtained from (3.1.49.6).

70. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Solution: Use (1.2.1.9).

71. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad (3.1.71.1)$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \quad (3.1.71.2)$$

Solution: The projection of \mathbf{a} on \mathbf{b} is shown in Fig. 3.1.71. It has magnitude $\|\mathbf{a}\| \cos \theta$ and is in the direction of \mathbf{b} . Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos \theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{(\mathbf{a}^T \mathbf{b}) \|\mathbf{a}\|}{\|\mathbf{b}\|} \mathbf{b} \quad (3.1.71.3)$$

72. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T \mathbf{b} = 4. \quad (3.1.72.1)$$

Solution:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T \mathbf{b} \quad (3.1.72.2)$$

73. If \mathbf{a} is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \quad (3.1.73.1)$$

then find \mathbf{x} .

Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \quad (3.1.73.2)$$

$$\Rightarrow \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \quad (3.1.73.3)$$

74. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (3.1.74.1)$$

find $\|\mathbf{a} \times \mathbf{b}\|$.

Solution: Use (1.2.5.3).

75. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (3.1.75.1)$$

Solution: If \mathbf{x} is the desired vector,

$$(\mathbf{a} + \mathbf{b})^T \mathbf{x} = 0 \quad (3.1.75.2)$$

$$(\mathbf{a} - \mathbf{b})^T \mathbf{x} = 0 \quad (3.1.75.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (3.1.75.4)$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \quad (3.1.75.5)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.75.6)$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.75.7)$$

76. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are collinear.

Solution: See Problem 3.1.59.

77. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.

Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \quad (3.1.77.1)$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \quad (3.1.77.2)$$

$$\therefore -2(\mathbf{A} - \mathbf{B}) = \mathbf{C} - \mathbf{D}, \quad (3.1.77.3)$$

$\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.

78. Let $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 4$, $\|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.

Solution: Given that

$$\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{c} = \mathbf{c}^T \mathbf{a} = 0. \quad (3.1.78.1)$$

Then,

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \\ &\quad + \mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \end{aligned} \quad (3.1.78.2)$$

which reduces to

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \quad (3.1.78.3)$$

using (3.1.78.1)

79. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (3.1.79.1)$$

evaluate

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}, \quad (3.1.79.2)$$

given that $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

Solution: Multiplying (3.1.79.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T \mathbf{b} + \mathbf{a}^T \mathbf{c} = 0 \quad (3.1.79.3)$$

$$\mathbf{a}^T \mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T \mathbf{c} = 0 \quad (3.1.79.4)$$

$$+\mathbf{c}^T \mathbf{a} + \mathbf{b}^T \mathbf{c} + \|\mathbf{c}\|^2 = 0 \quad (3.1.79.5)$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \quad (3.1.79.6)$$

80. Let $\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find β_1, β_2 such that $\beta = \beta_1 + \beta_2$, $\beta_1 \parallel \alpha$ and $\beta_2 \perp \alpha$.

Solution: Let $\beta_1 = k\alpha$. Then,

$$\beta = k\alpha + \beta_2 \quad (3.1.80.1)$$

$$\Rightarrow k = \frac{\alpha^T \beta}{\|\alpha\|^2} \quad (3.1.80.2)$$

and

$$\beta_2 = \beta - k\alpha \quad (3.1.80.3)$$

This process is known as *Gram-Schmidt orthogonalization*.

81. Find a unit vector that makes an angle of $90^\circ, 60^\circ$ and 30° with the positive x, y and z axis respectively.

Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90^\circ \\ \cos 60^\circ \\ \cos 30^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (3.1.81.1)$$

$\therefore \|\mathbf{x}\| = 1$, it is the desired unit vector.

82. Find the distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.82.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.82.2)$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

codes/line/line_dist_parallel.py

Fig. 3.1.82 From Fig. 3.1.82, the distance is

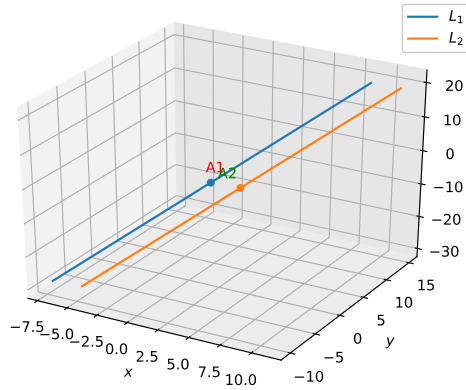


Fig. 3.1.82

$$\|\mathbf{A}_2 - \mathbf{A}_1\| \sin \theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \quad (3.1.82.3)$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.82.4)$$

83. Find the shortest distance between the lines

$$L_1: \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.83.1)$$

$$L_2: \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.83.2)$$

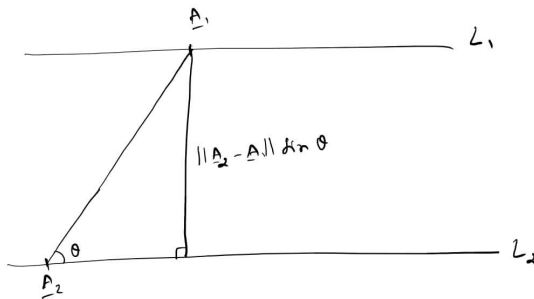


Fig. 3.1.82

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \quad (3.1.83.3)$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.83.4)$$

$$\Rightarrow \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (3.1.83.5)$$

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.1.83.6)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \quad (3.1.83.7)$$

$$\xrightarrow{R_2=R_1+R_2, R_3=2R_1-R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \quad (3.1.83.8)$$

$$\xrightarrow{R_3=3R_2+R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \quad (3.1.83.9)$$

The above matrix has $rank = 3$. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew lines*. The following code plots Fig. 3.1.83

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \quad (3.1.83.10)$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \quad (3.1.83.11)$$

The distance from \mathbf{A}_1 to the above line is then

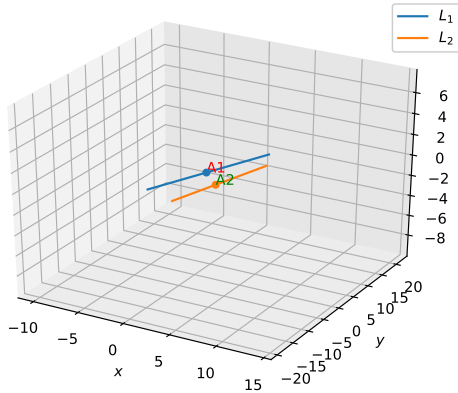


Fig. 3.1.83

obtained using (3.1.46.7) as

$$\frac{|\mathbf{n}^T (\mathbf{A}_2 - \mathbf{A}_1)|}{\|\mathbf{n}\|} = \frac{|(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2)|}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \quad (3.1.83.12)$$

84. Find the distance of the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (3.1.84.1)$$

from the origin.

Solution: From (3.1.46.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{2^2 + 3^2 + 4^2}} \quad (3.1.84.2)$$

$$= \frac{6}{\sqrt{29}} \quad (3.1.84.3)$$

85. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Solution: From the previous problem, the desired equation is

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (3.1.85.1)$$

86. Find the unit normal vector of the plane

$$(6 \ -3 \ -2)\mathbf{x} = 1. \quad (3.1.86.1)$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \quad (3.1.86.2)$$

$$\therefore \|\mathbf{n}\| = 7, \quad (3.1.86.3)$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7} \begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \quad (3.1.86.4)$$

87. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (3.1.87.1)$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad (3.1.87.2)$$

Hence, the foot of the perpendicular from the origin is $\lambda \mathbf{n}$. Substituting in (3.1.87.1),

$$\lambda \|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \quad (3.1.87.3)$$

Thus, the foot of the perpendicular is

$$\frac{6}{29} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad (3.1.87.4)$$

88. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular

to the line with direction vector $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Solution: The normal vector to the plane is \mathbf{n} . Hence from (3.1.39.3), the equation of the plane is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (3.1.88.1)$$

$$\implies \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \quad (3.1.88.2)$$

$$= 20 \quad (3.1.88.3)$$

89. Find the equation of the plane passing through

$$\mathbf{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} \text{ and } \mathbf{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}.$$

Solution: If the equation of the plane be

$$\mathbf{n}^T \mathbf{x} = c, \quad (3.1.89.1)$$

$$\mathbf{n}^T \mathbf{R} = \mathbf{n}^T \mathbf{S} = \mathbf{n}^T \mathbf{T} = c, \quad (3.1.89.2)$$

$$\Rightarrow (\mathbf{R} - \mathbf{S} \quad \mathbf{S} - \mathbf{T})^T \mathbf{n} = 0 \quad (3.1.89.3)$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix} 4 & 8 & -8 \\ -7 & -6 & 8 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 2 & -2 \\ -7 & -6 & 8 \end{pmatrix} \quad (3.1.89.4)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 8 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{8}} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -\frac{3}{4} \end{pmatrix} \quad (3.1.89.5)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -\frac{3}{4} \end{pmatrix} \quad (3.1.89.6)$$

Thus,

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } \quad (3.1.89.7)$$

$$c = \mathbf{n}^T \mathbf{T} = 7 \quad (3.1.89.8)$$

Thus, the equation of the plane is

$$(2 \quad 3 \quad 4) \mathbf{n} = 7 \quad (3.1.89.9)$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \quad (3.1.89.10)$$

The equation of the plane is then obtained from (3.1.89.3) as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T (\mathbf{x} - \mathbf{T}) = 0 \quad (3.1.89.11)$$

90. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.

Solution: From the given information, the

plane passes through the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and

$\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ respectively. The equation can be obtained

using Problem 3.1.89.

91. Find the equation of the plane passing through

the intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (3.1.91.1)$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = -5 \quad (3.1.91.2)$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & -5 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -17 \end{pmatrix} \quad (3.1.91.3)$$

$$\xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 23 \\ 0 & 1 & 2 & -17 \end{pmatrix} \quad (3.1.91.4)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.91.5)$$

Thus, $\begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.91.6)$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ -22 & 18 & 1 \end{pmatrix} \xrightarrow{R_2 = R_2 + 22R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -26 & 23 \end{pmatrix} \quad (3.1.91.7)$$

$$\xleftrightarrow{R_1 = 13R_1 - R_2} \begin{pmatrix} 13 & 0 & -10 \\ 0 & -26 & 23 \end{pmatrix} \quad (3.1.91.8)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} \frac{10}{13} \\ \frac{23}{26} \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 23 \\ 26 \end{pmatrix} \quad (3.1.91.9)$$

Since the plane passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using

(3.1.39.3),

$$(20 \ 23 \ 26) \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \quad (3.1.91.10)$$

$$\Rightarrow (20 \ 23 \ 26) \mathbf{x} = 69 \quad (3.1.91.11)$$

Alternatively, the plane passing through the intersection of (3.1.91.1) and (3.1.91.2) has the form

$$(1 \ 1 \ 1) \mathbf{x} + \lambda (2 \ 3 \ 4) \mathbf{x} = 6 - 5\lambda \quad (3.1.91.12)$$

Substituting $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the above,

$$(1 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda (2 \ 3 \ 4) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6 - 5\lambda \quad (3.1.91.13)$$

$$\Rightarrow 3 + 9\lambda = 6 - 5\lambda \quad (3.1.91.14)$$

$$\Rightarrow \lambda = \frac{3}{14} \quad (3.1.91.15)$$

Substituting this value of λ in (3.1.91.12) yields the equation of the plane.

92. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \quad (3.1.92.1)$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \quad (3.1.92.2)$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.83), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \quad (3.1.92.3)$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \quad (3.1.92.4)$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix} -1 & 2 & 5 \\ -3 & 1 & 5 \end{pmatrix} \xrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix} \quad (3.1.92.5)$$

$$\xrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.92.6)$$

The LHS of (3.1.92.3) is

$$\begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \quad (3.1.92.7)$$

which completes the proof. Alternatively, the lines are coplanar if

$$|\mathbf{A}_1 - \mathbf{A}_2 \ \mathbf{m}_1 \ \mathbf{m}_2| = 0 \quad (3.1.92.8)$$

93. Find the angle between the two planes

$$(2 \ 1 \ -2) \mathbf{x} = 5 \quad (3.1.93.1)$$

$$(3 \ -6 \ -2) \mathbf{x} = 7. \quad (3.1.93.2)$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.49.6).

94. Find the angle between the two planes

$$(2 \ 2 \ -2) \mathbf{x} = 5 \quad (3.1.94.1)$$

$$(3 \ -6 \ 2) \mathbf{x} = 7. \quad (3.1.94.2)$$

Solution: See Problem (3.1.93).

95. Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$(6 \ -3 \ 2) \mathbf{x} = 4 \quad (3.1.95.1)$$

Solution: Use (3.1.46.7).

96. Find the angle between the line

$$L: \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \quad (3.1.96.1)$$

and the plane

$$P: (10 \ 2 \ -11) \mathbf{x} = 3 \quad (3.1.96.2)$$

Solution: The angle between the direction

vector of L and normal vector of P is

$$\cos \theta = \frac{\left| \begin{pmatrix} 10 & 2 & -11 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \right|}{\sqrt{225} \times \sqrt{49}} = \frac{8}{21} \quad (3.1.96.3)$$

Thus, the desired angle is $90^\circ - \theta$.

97. Find the equation of the plane that contains the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and is perpendicular to each of the planes

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \mathbf{x} = 5 \quad (3.1.97.1)$$

$$\begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \mathbf{x} = 8 \quad (3.1.97.2)$$

Solution: The normal vector to the desired plane is \perp the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \quad (3.1.97.3)$$

The equation of the plane is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (3.1.97.4)$$

98. Find the distance between the point $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$

and the plane determined by the points $\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.

Solution: Find the equation of the plane using Problem 3.1.89. Find the distance using (3.1.46.7).

99. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ crosses the XY plane.

Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (3.1.99.1)$$

$$= \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \quad (3.1.99.2)$$

The line crosses the XY plane for $x_3 = 0 \implies$

$\lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 13 \\ 23 \\ 0 \end{pmatrix} \quad (3.1.99.3)$$

100. Show that the function given by $f(x) = 7x - 3$ is increasing on \mathbf{R} .
101. Find the maximum and minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1). \quad (3.1.101.1)$$

102. Find all points of local maxima and local minima of the function f given by

$$f(x) = 3 + |x|, \quad x \in \mathbf{R} \quad (3.1.102.1)$$

103. Sketch the graph of $y = |x + 3|$ and evaluate its area for $-6 \leq x \leq 0$.

104. Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

105. Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.

106. Check the points where the constant function $f(x) = k$ is continuous.

107. Prove that the function $f(x) = x$ is continuous at every real number.

108. Is $f(x) = |x|$ a continuous function?

109. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x + 2 & x \leq 1 \\ x - 2 & x > 1 \end{cases} \quad (3.1.109.1)$$

110. Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x + 2 & x < 1 \\ 0 & x = 1 \\ x - 2 & x > 1 \end{cases} \quad (3.1.110.1)$$

111. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x + 2 & x < 0 \\ -x + 2 & x > 0 \end{cases} \quad (3.1.111.1)$$

112. Show that the function f defined by

$$f(x) = |1 - x + |x||, \quad (3.1.112.1)$$

where x is any real number, is a continuous function.

113. Find $\lim_{x \rightarrow 5} x + 10$

114. Find $\lim_{x \rightarrow 2} 3x$

115. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} 1 & x \leq 0 \\ 2 & x > 0 \end{cases} \quad (3.1.115.1)$$

116. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} x-2 & x < 0 \\ 0 & x = 0 \\ x+2 & x > 0 \end{cases} \quad (3.1.116.1)$$

117. Find $\frac{dy}{dx}$ if $x - y = \pi$.

118. Find the derivative at $x = 2$ of the function $f(x) = 3$.

119. Find the derivative of $f(x) = 3$ at $x = 0$ and $x = 3$.

120. Find the derivative of $f(x) = 10x$.

121. Find the derivative of $f(x) = a$ for a fixed real number a .

122. Form the differential equation representing the family of curves $y = mx$, where, m is an arbitrary constant.

8. $3(2-x) \leq 2(1-x)$.

9. $x + \frac{x}{2} + \frac{x}{3} < 11$.

10. $\frac{x}{3} \cdot \frac{x}{2} + 1$.

11. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$.

12. $\frac{1}{2}(\frac{3x}{5} + 4) \geq \frac{1}{3}(x - 6)$.

13. $2(2x+3)-10 < 6(x-2)$.

14. $37-(3x+5) \geq 9x-8(x-3)$.

15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$.

16. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$.

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x-2 < 2x+1$.

18. $5x-3 \geq 3x-5$.

19. $3(1-x) < 2(x+4)$.

20. $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$.

21. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

22. To receive Grade A in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.

23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

3.2 Linear Inequalities: Examples

1. Solve $24x < 100$, when
(i) x is a natural number.
(ii) x is an integer.

2. Solve $-12x > 30$, when
(i) x is a natural number.
(ii) x is an integer.

3. Solve $5x-3 < 7$, when
(i) x is an integer.
(ii) x is a real number.

4. Solve $3x+8 > 2$, when
(i) x is an integer.
(ii) x is a real number

Solve the inequalities in Exercises 5 to 16 for real x .

5. $4x+3 < 5x+7$.

6. $3x-7 > 5x-1$.

7. $3(x-1) \geq 2(x-3)$.

25. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.

26. A man wants to cut three lengths from a single piece of board of length 91cm. The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second? [Hint: If x is the length of the shortest board, then x , $(x + 3)$ and $2x$ are the lengths of the second and third piece, respectively. Thus, $x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$].

Solve the following inequalities graphically in two-dimensional plane:

27. $x + y < 5$.

28. $2x + y \geq 6$.

29. $3x + 4y \leq 12$.

30. $y + 8 \geq 2x$.

31. $x - y \leq 2$.

32. $2x - 3y > 6$.

33. $-3x + 2y \geq -6$.

34. $3y - 5x < 30$.

35. $y < -2$.

36. $x > -3$.

Solve the following system of inequalities graphically:

37. $x \geq 3$, $y \geq 2$.

38. $3x + 2y \leq 12$, $x \geq 1$, $y \geq 2$.

39. $2x + y \geq 6$, $3x + 4y \leq 12$.

40. $x + y \geq 4$, $2x - y < 0$.

41. $2x - y > 1$, $x - 2y < -1$.

42. $x + y \leq 6$, $x + y \geq 4$.

43. $2x + y \geq 8$, $x + 2y \geq 10$.

44. $x + y \leq 9$, $y > x$, $x \geq 0$.

45. $5x + 4y \leq 20$, $x \geq 1$, $y \geq 2$.

46. $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$.

47. $2x + y \geq 4$, $x + y \leq 3$, $2x - 3y \leq 6$.

48. $x - 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 1$.

49. $4x + 3y \leq 60$, $y \geq 2x$, $x \geq 3$, $x, y \geq 0$.

50. $3x + 2y \leq 150$, $x + 4y \leq 80$, $x \leq 15$, $y \geq 0$, $x \geq 0$.

51. $x + 2y \leq 10$, $x + y \geq 1$, $x - y \leq 0$, $x \geq 0$, $y \geq 0$.

Solve the inequalities in Exercises 52 to 57.

52. $2 \leq 3x - 4 \leq 5$.

53. $6 \leq -3(2x - 40) < 12$.

54. $-3 \leq 4 - \frac{7x}{2} \leq 18$.

55. $-15 < \frac{3(x-2)}{5} \leq 0$.

56. $-12 < 4 - \frac{3x}{-5} \leq 2$.

57. $7 \leq \frac{(3x+11)}{2} \leq 11$.

Solve the inequalities in Exercises 58 to 61 and represent the solution graphically on number line.

58. $5x + 1 > -24$, $5x - 1 < 24$.

59. $2(x - 1) < x + 5$, $3(x + 2) > 2 - x$.

60. $3x - 7 > 2(x - 6)$, $6 - x > 11 - 2x$.

61. $5(2x - 7) - 3(2x + 3) \leq 0$, $2x + 19 \leq 6x + 47$.

62. A solution is to be kept between 68°F and 77°F. What is the range in temperature in degree Celsius(C) if the Celsius / Fahrenheit(F) conversion formula is given by $F = \frac{9}{5}C + 32$?
63. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?
64. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
65. IQ of a person is given by the formula $IQ = \frac{MA}{CA} \times 100$, where MA is mental age and CA is chronological age. If $80 \leq IQ \leq 140$ for a group of 12 years old children, find the range of their mental age.
4. Determine graphically the minimum value of the objective function
 $Z = -50x + 20y$
 subject to the constraints:
 (i) $2x - y \geq -5$
 (ii) $3x + y \geq 3$
 (iii) $2x - 3y \leq 12$
 (iv) $x \geq 0, y \geq 0$
5. Minimise $Z = 3x + 2y$
 subject to the constraints:
 (i) $x + y \geq 8$
 (ii) $3x + 5y \leq 15$
 (iii) $x \geq 0, y \geq 0$
6. **(Diet problem):** A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.

3.3 Linear Programing: Examples

- Solve the linear programing problem graphically: Maximize $Z = 4x + y$
 subject to the constraints:
 (i) $x + y \leq 50$
 (ii) $3x + y \leq 90$
 (iii) $x \geq 0, y \geq 0$
- Solve the following linear programming problem graphically:
 Minimise $Z = 200x + 500y$
 subject to the constraints:
 (i) $x + 2y \geq 10$
 (ii) $3x + 4y \leq 24$
 (iii) $x \geq 0, y \geq 0$
- Solve the following problem graphically:
 Minimise and Maximise $Z = 3x + 9y$
 subject to the constraints:
 (i) $x + 3y \leq 60$
 (ii) $x + y \geq 10$
 (iii) $x \leq y$
 (iv) $x \geq 0, y \geq 0$
- (Allocation problem)** A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?
- (Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available

are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

9. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

10. **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Number of hours required on machines			
Items	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

11. **(Transportation problem)** There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C.

The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below where A,B,C are cost in ruppees:

From/To	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

3.4 Matrix Examples

1. Consider the following information regarding the number of men and women workers in the three factories I,II and III

	Men Workers	Women Workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

2. If a matrix has 8 elements, what are the possible orders it can have?
3. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2} |i - 3j|$

$$\begin{pmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{pmatrix}$$

Find the values of a,b,c,x,y and z.

5. Find the values of a,b,c and d from the following equation:

$$\begin{pmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 11 & 24 \end{pmatrix}$$

6. Given $A = \begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$, find $A+B$.

7. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$, then find $2A-B$.

8. If $A = \begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$, then find the matrix X , such that $2A + 3X = 5B$.
9. Find X and Y , if $X + Y = \begin{pmatrix} 5 & 2 \\ 0 & 9 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 6 \\ 0 & -1 \end{pmatrix}$.
10. Find the values of x and y from the following equation:

$$2 \begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$$
11. Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B .
 September Sales(in Rupees)
 Basmati Permal Naura

$$A = \begin{pmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{pmatrix} \begin{pmatrix} \text{Ramakishan} \\ \text{GurucharanSingh} \end{pmatrix}$$

 October sales (in Rupees)
 Basmati Permal Naura

$$B = \begin{pmatrix} 5,000 & 10,000 & 6,000 \\ 20,000 & 10,000 & 10,000 \end{pmatrix} \begin{pmatrix} \text{Ramkishan} \\ \text{GurucharanSingh} \end{pmatrix}$$

 (i) Find the combined sales in September and October for each farmer in each variety.
 (ii) Find the decrease in sales from September to October.
 (iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.
12. Find AB , if $A = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{pmatrix}$.
13. If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$, then find AB, BA . Show that $AB \neq BA$.
14. If $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then find AB, BA . Show that $AB \neq BA$.
15. Find AB , if $A = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$.
16. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{pmatrix}$, find $A(BC), (AB)C$ and show that $(AB)C = A(BC)$.
17. If $A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ Calculate AC, BC and $(A+B)C = AC + BC$.
18. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, then show that $A^3 - 23A - 40I = 0$.
19. In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as
 Cost per contact

$$A = \begin{pmatrix} 40 \\ 100 \\ 50 \end{pmatrix} \begin{pmatrix} \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{pmatrix}$$

 The number of contacts of each type made in two cities X and Y is given by
 Telephone Housecall Letter

$$B = \begin{pmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$
. Find the total amount spent by the group in the two cities X and Y .
20. If $A = \begin{pmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$, verify that
 (i) $(A')' = A$
 (ii) $(A + B)' = A' + B'$,
 (iii) $(kB)' = kB'$, where k is any constant.
21. If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, verify that $(AB)' = B'A'$.
22. Express the matrix $B = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.
23. By using elementary operations, find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

24. Obtain the inverse of the following matrix using elementary operations

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}.$$

25. Find P^{-1} , if it exists, given

$$P = \begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}.$$

26. If $A = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$,

then prove that $A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$, $n \in \mathbb{N}$.

27. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that $AB = BA$.

28. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. Find a matrix D such that $CD - AB = 0$.

a) $\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$
 b) $\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$

8. Find θ such that

$$\frac{\begin{pmatrix} 3 \\ 2 \sin \theta \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \sin \theta \end{pmatrix}} \quad (3.5.8.1)$$

is purely real.

9. Convert the complex number

$$z = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} \cos \frac{\pi}{3} \\ \sin \frac{\pi}{3} \end{pmatrix}} \quad (3.5.9.1)$$

in the polar form.

10. Simplify

$$z = \left[\frac{1}{\begin{pmatrix} 1 \\ -4 \end{pmatrix}} - \frac{2}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \right] \frac{\begin{pmatrix} 3 \\ -4 \end{pmatrix}}{\begin{pmatrix} 5 \\ 1 \end{pmatrix}} \quad (3.5.10.1)$$

3.5 Complex Numbers

1. Find $\begin{pmatrix} 5 \\ -3 \end{pmatrix}^3$

2. Find $\begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}.$

3. Find the multiplicative inverse of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}.$

4. Find

a) $\begin{pmatrix} 5 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ -2\sqrt{3} \end{pmatrix}.$

b) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{-35}.$

c) Show that the polar representation of $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ is $2\angle 60^\circ$.

5. Convert the complex number $-\frac{16}{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}}$

6. Find the conjugate of $\frac{\begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}.$

7. Find the modulus and argument of the complex numbers

11. Convert the following in the polar form:

a) $\frac{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}^2}.$

b) $\frac{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}.$

12. If $z_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $z_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\left\| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right\|$

13. Let $z_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $z_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Find

a) $\operatorname{Re} \left(\frac{z_1 z_2}{z_1^*} \right).$

b) $\operatorname{Im} \left(\frac{1}{z_1 z_1^*} \right).$

14. Find the modulus and argument of the complex

number $\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}.$

15. Find the real numbers x, y such that $\begin{pmatrix} x \\ -y \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is the conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$.

16. Find the modulus of $\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

3.6 Points and Vectors

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.6.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (3.6.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (3.6.1.3)$$

2. Find the distance between the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (3.6.2.1)$$

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (3.6.4.1)$$

b)

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (3.6.4.2)$$

c)

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \quad (3.6.4.3)$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.6.4.4)$$

5. Find the angle between the x -axis and the line joining the points $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

6. Find the point on the x -axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \quad (3.6.6.1)$$

7. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (3.6.7.1)$$

is 10 units.

8. Show that each of the given three vectors is a unit vector

$$\frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \quad (3.6.8.1)$$

Also, show that they are mutually perpendicular to each other.

9. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (3.6.9.1)$$

$(\mathbf{a} + \lambda \mathbf{b}) \perp \mathbf{c}$. Find λ .

10. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \quad (3.6.10.1)$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad (3.6.11.1)$$

12. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} =$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

14. Show that the unit direction vector inclined

equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

15. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Find a vector \mathbf{d} such that $\mathbf{d} \perp \mathbf{a}$, $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T \mathbf{c} = 15$.

16. The scalar product of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with a unit vector

along the sum of the vectors $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$ is

unity. Find the value of λ .

17. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (3.6.17.1)$$

is

- a) 0 c) 1
b) -1 d) 3

18. Find a unit vector that makes an angle of 90° , 135° and 45° with the positive x, y and z axis respectively.

19. Show that the lines with direction vectors $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$,

$\begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$ are mutually perpendicular.

20. Show that the line through the points $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$,

$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.

21. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$,

$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

23. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (3.6.23.1)$$

24. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \quad (3.6.24.1)$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (3.6.24.2)$$

25. Write down a unit vector in the xy-plane, making an angle of 30° with the positive direction of the x-axis.

26. Find the value of x for which $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.

3.7 Points on a Line

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (3.7.1.1)$$

in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

3. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

4. Find the ratio in which the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.

5. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y .
6. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of \mathbf{P} such that $AP = \frac{3}{7}AB$ and \mathbf{P} lies on the line segment AB .
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.
8. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \quad (3.7.8.1)$$

are collinear.

9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix} x \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
- a) $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ k \end{pmatrix}$
- b) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$, $\begin{pmatrix} k \\ -4 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$
12. Find a condition on \mathbf{x} such that the points \mathbf{x} , $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$ are collinear, and find the ratio in which \mathbf{B} divides AC .
15. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

3.8 Lines and Planes

1. Sketch the following lines

- a) $\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 9.35$ e) $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$
 b) $\begin{pmatrix} 1 & -\frac{1}{5} \end{pmatrix} \mathbf{x} = 10$ f) $\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = -2$
 c) $\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 6$ g) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$
 d) $\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 0$ h) $\begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 5$

2. Write four solutions for each of the following equations

- a) $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 7$
 b) $\begin{pmatrix} \pi & 1 \end{pmatrix} \mathbf{x} = 9$
 c) $\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = 0$

3. Check which of the following are solutions of the equation

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 4 \quad (3.8.3.1)$$

- a) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ d) $\begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$
 b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ e) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
 c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

4. Find the value of k , if $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a solution of the equation

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = k \quad (3.8.4.1)$$

5. Draw the graphs of the following equations

- a) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4$
 b) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 2$
 c) $\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 0$
 d) $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3$
 e) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$
 f) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
 g) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$
 h) $\begin{pmatrix} 7 & -3 \end{pmatrix} \mathbf{x} = 2$
 i) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
 j) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$
 k) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$
 l) $\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6$

6. Give the equations of two lines passing through $\begin{pmatrix} 2 \\ 14 \end{pmatrix}$. How many more such lines are there, and why?

7. If the point $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a

8. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{aligned} (5 \quad -4)\mathbf{x} &= -8 \\ (7 \quad 6)\mathbf{x} &= 9 \end{aligned} \quad (3.8.8.1)$$

b)

$$\begin{aligned} (9 \quad 3)\mathbf{x} &= -12 \\ (18 \quad 6)\mathbf{x} &= -24 \end{aligned} \quad (3.8.8.2)$$

c)

$$\begin{aligned} (6 \quad -3)\mathbf{x} &= -10 \\ (2 \quad -1)\mathbf{x} &= -9 \end{aligned} \quad (3.8.8.3)$$

9. Find out whether the following pair of linear equations are consistent, or inconsistent.

a)

$$\begin{aligned} (3 \quad 2)\mathbf{x} &= 5 \\ (2 \quad -3)\mathbf{x} &= 7 \end{aligned} \quad (3.8.9.1)$$

b)

$$\begin{aligned} (2 \quad -3)\mathbf{x} &= 8 \\ (4 \quad -6)\mathbf{x} &= 9 \end{aligned} \quad (3.8.9.2)$$

c)

$$\begin{aligned} \left(\frac{3}{2} \quad \frac{5}{3}\right)\mathbf{x} &= 7 \\ (9 \quad -10)\mathbf{x} &= 14 \end{aligned} \quad (3.8.9.3)$$

d)

$$\begin{aligned} (5 \quad -3)\mathbf{x} &= 11 \\ (-10 \quad 6)\mathbf{x} &= -22 \end{aligned} \quad (3.8.9.4)$$

e)

$$\begin{aligned} \left(\frac{4}{3} \quad 2\right)\mathbf{x} &= 8 \\ (2 \quad 3)\mathbf{x} &= 12 \end{aligned} \quad (3.8.9.5)$$

10. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{aligned} (1 \quad 1)\mathbf{x} &= 5 \\ (2 \quad 2)\mathbf{x} &= 10 \end{aligned} \quad (3.8.10.1)$$

b)

$$\begin{aligned} (1 \quad -1)\mathbf{x} &= 8 \\ (3 \quad -3)\mathbf{x} &= 16 \end{aligned} \quad (3.8.10.2)$$

c)

$$\begin{aligned} (2 \quad 1)\mathbf{x} &= 6 \\ (4 \quad -2)\mathbf{x} &= 4 \end{aligned} \quad (3.8.10.3)$$

d)

$$\begin{aligned} (2 \quad -2)\mathbf{x} &= 2 \\ (4 \quad -4)\mathbf{x} &= 5 \end{aligned} \quad (3.8.10.4)$$

11. Given the linear equation $(2 \quad 3)\mathbf{x} - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

a) intersecting lines c) coincident lines

b) parallel lines

12. Find the intersection of the following lines

a)

$$\begin{aligned} (1 \quad 1)\mathbf{x} &= 14 \\ (1 \quad -1)\mathbf{x} &= 4 \end{aligned} \quad (3.8.12.1)$$

b)

$$\begin{aligned} (1 \quad -1)\mathbf{x} &= 3 \\ \left(\frac{1}{3} \quad \frac{1}{2}\right)\mathbf{x} &= 6 \end{aligned} \quad (3.8.12.2)$$

c)

$$\begin{aligned} (3 \quad -1)\mathbf{x} &= 3 \\ (9 \quad -3)\mathbf{x} &= 9 \end{aligned} \quad (3.8.12.3)$$

d)

$$\begin{aligned} (0.2 \quad 0.3)\mathbf{x} &= 1.3 \\ (0.4 \quad 0.5)\mathbf{x} &= 2.3 \end{aligned} \quad (3.8.12.4)$$

e)

$$\begin{aligned} (\sqrt{2} \quad \sqrt{3})\mathbf{x} &= 0 \\ (\sqrt{3} \quad \sqrt{8})\mathbf{x} &= 0 \end{aligned} \quad (3.8.12.5)$$

f)

$$\begin{aligned} \left(\frac{3}{2} \quad -\frac{5}{3}\right)\mathbf{x} &= -2 \\ \left(\frac{1}{3} \quad \frac{1}{2}\right)\mathbf{x} &= \frac{13}{6} \end{aligned} \quad (3.8.12.6)$$

13. Find m if

$$\begin{aligned} (2 \quad 3)\mathbf{x} &= 11 \\ (2 \quad -4)\mathbf{x} &= -24 \\ (m \quad -1)\mathbf{x} &= -3 \end{aligned} \quad (3.8.13.1)$$

14. Solve the following

a)

$$\begin{aligned} (1 \quad 1)\mathbf{x} &= 5 \\ (2 \quad -3)\mathbf{x} &= 4 \end{aligned} \quad (3.8.14.1)$$

c)

$$\begin{aligned} (3 \quad -5)\mathbf{x} &= 4 \\ (9 \quad -2)\mathbf{x} &= 7 \end{aligned} \quad (3.8.14.3)$$

b)

$$\begin{aligned} (3 \quad 4)\mathbf{x} &= 10 \\ (2 \quad -2)\mathbf{x} &= 2 \end{aligned} \quad (3.8.14.2)$$

$$\begin{aligned} \left(\frac{1}{2} \quad \frac{2}{3}\right)\mathbf{x} &= -1 \\ \left(1 \quad -\frac{1}{3}\right)\mathbf{x} &= 3 \end{aligned} \quad (3.8.14.4)$$

15. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{aligned} (1 \quad -3)\mathbf{x} &= 3 \\ (3 \quad -9)\mathbf{x} &= 2 \end{aligned} \quad (3.8.15.1)$$

c)

$$\begin{aligned} (3 \quad -5)\mathbf{x} &= 20 \\ (6 \quad -10)\mathbf{x} &= 40 \end{aligned} \quad (3.8.15.3)$$

b)

$$\begin{aligned} (2 \quad 1)\mathbf{x} &= 5 \\ (3 \quad 2)\mathbf{x} &= 8 \end{aligned} \quad (3.8.15.2)$$

d)

$$\begin{aligned} (1 \quad -3)\mathbf{x} &= 7 \\ (3 \quad -3)\mathbf{x} &= 15 \end{aligned} \quad (3.8.15.4)$$

16. For which values of a and b does the following pair of linear equations have an infinite number

of solutions?

$$\begin{aligned} (2 \quad 3)\mathbf{x} &= 7 \\ (a-b \quad a+b)\mathbf{x} &= 3a+b-2 \end{aligned} \quad (3.8.16.1)$$

17. For which value of k will the following pair of linear equations have no solution?

$$\begin{aligned} (3 \quad 1)\mathbf{x} &= 1 \\ (2k-1 \quad k-1)\mathbf{x} &= 2k+1 \end{aligned} \quad (3.8.17.1)$$

18. Solve the following pair of linear equations

$$\begin{aligned} (8 \quad 5)\mathbf{x} &= 9 \\ (3 \quad 2)\mathbf{x} &= 4 \end{aligned} \quad (3.8.18.1)$$

19. Solve the following pair of linear equations

$$\begin{aligned} (158 \quad -378)\mathbf{x} &= -74 \\ (-378 \quad 152)\mathbf{x} &= -604 \end{aligned} \quad (3.8.19.1)$$

20. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$.21. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.22. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.

23. Write the equations for the x and y axes.

24. Find the equation of the line satisfying the following conditions

a) passing through the point $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ with slope $\frac{1}{2}$.b) passing through the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m .c) passing through the point $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75° .

d) Intersecting the x-axis at a distance of 3 units to the left of the origin with slope -2.

e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.f) passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30° .

25. Find the equation of the line passing through

$\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$.

26. Find the direction vectors and y-intercepts of the following lines

- a) $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0$.
 b) $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5$.
 c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$.

27. Find the intercepts of the following lines on the axes.

- a) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12$.
 b) $\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 6$.
 c) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 0$.

28. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.

- a) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8$.
 b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$.
 c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$.

29. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = -82$.

30. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (3.8.30.1)$$

are 4 units.

31. Find the distance between the parallel lines

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = 34 \quad (3.8.31.1)$$

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = -31 \quad (3.8.31.2)$$

32. Find the equation of the line parallel to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -2 \quad (3.8.32.1)$$

and passing through the point $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

33. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \quad (3.8.33.1)$$

and having x intercept 3.

34. Find angles between the lines

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (3.8.34.1)$$

$$\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 1 \quad (3.8.34.2)$$

35. The line through the points $\begin{pmatrix} h \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ intersects the line

$$\begin{pmatrix} 7 & -9 \end{pmatrix} \mathbf{x} = 19 \quad (3.8.35.1)$$

at right angle. Find the value of h .

36. Two lines passing through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ intersect each other at angle of 60° . If the slope of one line is 2, find the equation of the other line.

37. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

38. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 16. \quad (3.8.38.1)$$

39. The perpendicular from the origin to the line

$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = c \quad (3.8.39.1)$$

meets it at the point $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Find the values of m and c .

40. Find θ and p if

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \quad (3.8.40.1)$$

is equivalent to

$$\begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \mathbf{x} = p \quad (3.8.40.2)$$

41. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

42. Find the equation of the line parallel to the y-axis whose distance from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (3.8.42.1)$$

4 units.

43. Find the equation of the line parallel to the y-axis drawn through the point of intersection of

the lines

$$(1 \ -7)\mathbf{x} = -5 \quad (3.8.43.1)$$

$$(3 \ 1)\mathbf{x} = 0 \quad (3.8.43.2)$$

44. Find the value of p so that the three lines

$$(3 \ 1)\mathbf{x} = 2 \quad (3.8.44.1)$$

$$(p \ 2)\mathbf{x} = 3 \quad (3.8.44.2)$$

$$(2 \ -1)\mathbf{x} = 3 \quad (3.8.44.3)$$

may intersect at one point.

45. Find the equation of the lines through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which make an angle of 45° with the line

$$(1 \ -2)\mathbf{x} = 3. \quad (3.8.45.1)$$

46. Find the equation of the line passing through the point of intersection of the lines

$$(4 \ 7)\mathbf{x} = 3 \quad (3.8.46.1)$$

$$(2 \ -3)\mathbf{x} = -1 \quad (3.8.46.2)$$

that has equal intercepts on the axes.

47. In what ratio is the line joining $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ divided by the line

$$(1 \ 1)\mathbf{x} = 4 \quad (3.8.47.1)$$

48. Find the distance of the line

$$(4 \ 7)\mathbf{x} = -5 \quad (3.8.48.1)$$

from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ along the line

$$(2 \ -1)\mathbf{x} = 0. \quad (3.8.48.2)$$

49. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ so that its point of intersection with the line

$$(1 \ 1)\mathbf{x} = 4 \quad (3.8.49.1)$$

may be at a distance of 3 units from this point.

50. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Find an equation of the legs of the triangle.

51. Find the image of the point $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ with respect to

the line

$$(1 \ 3)\mathbf{x} = 7 \quad (3.8.51.1)$$

assuming the line to be a plane mirror.

52. If the lines

$$(-3 \ 1)\mathbf{x} = 1 \quad (3.8.52.1)$$

$$(-1 \ 2)\mathbf{x} = 3 \quad (3.8.52.2)$$

are equally inclined to the line

$$(-m \ 1)\mathbf{x} = 4, \quad (3.8.52.3)$$

find the value of m .

53. The sum of the perpendicular distances of a variable point \mathbf{P} from the lines

$$(1 \ 1)\mathbf{x} = 0 \quad (3.8.53.1)$$

$$(3 \ -2)\mathbf{x} = -7 \quad (3.8.53.2)$$

is always 10. Show that \mathbf{P} must move on a line.

54. Find the equation of the line which is equidistant from parallel lines

$$(9 \ 7)\mathbf{x} = 7 \quad (3.8.54.1)$$

$$(3 \ 2)\mathbf{x} = -6. \quad (3.8.54.2)$$

55. A ray of light passing through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ reflects on the x-axis at point \mathbf{A} and the reflected ray passes through the point $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Find the coordinates of \mathbf{A} .

56. A person standing at the junction of two straight paths represented by the equations

$$(2 \ -3)\mathbf{x} = 4 \quad (3.8.56.1)$$

$$(3 \ 4)\mathbf{x} = 5 \quad (3.8.56.2)$$

wants to reach the path whose equation is

$$(6 \ -7)\mathbf{x} = -8 \quad (3.8.56.3)$$

in the least time. Find the equation of the path that he should follow.

57. Determine the ratio in which the line

$$(2 \ 1)\mathbf{x} - 4 = 0 \quad (3.8.57.1)$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

58. A line perpendicular to the line segment joining

the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.

59. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

60. Find the equation of the line passing through the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.

61. Find the equation of the line through the point $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

62. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix} -2 \\ 9 \end{pmatrix}$, find the equation of the line.

63. Find the equation of a line which passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

64. Find the equation of the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

65. Find the equation of the line which passes through the point $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \quad (3.8.65.1)$$

66. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \quad (3.8.66.1)$$

67. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

68. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

69. Find the angle between the following pair of lines:

a)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (3.8.69.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (3.8.69.2)$$

b)

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (3.8.69.3)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \quad (3.8.69.4)$$

70. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \quad (3.8.70.1)$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \quad (3.8.70.2)$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \quad (3.8.70.3)$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \quad (3.8.70.4)$$

71. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \quad (3.8.71.1)$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \quad (3.8.71.2)$$

are at right angles.

72. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \quad (3.8.72.1)$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (3.8.72.2)$$

are perpendicular to each other.

73. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.8.73.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (3.8.73.2)$$

74. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad (3.8.74.1)$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad (3.8.74.2)$$

75. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (3.8.75.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (3.8.75.2)$$

76. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \quad (3.8.76.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \quad (3.8.76.2)$$

77. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $(0 \ 0 \ 1)\mathbf{x} = 2$ c) $(0 \ 5 \ 0)\mathbf{x} = -8$

b) $(1 \ 1 \ 1)\mathbf{x} = 1$ d) $(2 \ 3 \ -1)\mathbf{x} = 5$

78. Find the equation of a plane which is at a distance of 7 units from the origin and normal

to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

79. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $(2 \ 3 \ 4)\mathbf{x} = 12$ c) $(1 \ 1 \ 1)\mathbf{x} = 1$

b) $(3 \ 4 \ -6)\mathbf{x} = 0$ d) $(0 \ 5 \ 0)\mathbf{x} = -8$

80. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vector to the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

81. Find the equation of the planes that pass through three points

a) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

82. Find the intercepts cut off by the plane $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}\mathbf{x} = 5$.

83. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOY plane.

84. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 3 & -1 & 2 \end{pmatrix}\mathbf{x} = 4$ and

$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = -2$ and the point $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

85. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 7$

and $\begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\mathbf{x} = 9$ and the point $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

86. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\mathbf{x} = 0$.

87. Find the angle between the planes whose equations are $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix} 3 & -3 & 5 \end{pmatrix}\mathbf{x} = 3$

88. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix} 7 & 5 & 6 \end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix} 3 & -1 & -10 \end{pmatrix}\mathbf{x} = -4$

b) $\begin{pmatrix} 2 & 1 & 3 \end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix} 1 & -2 & 5 \end{pmatrix}\mathbf{x} = 0$

c) $\begin{pmatrix} 2 & -2 & 4 \end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix} 3 & -3 & 6 \end{pmatrix}\mathbf{x} = 1$

d) $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \mathbf{x} = 1$ and $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \mathbf{x} = -3$

e) $\begin{pmatrix} 4 & 8 & 1 \end{pmatrix} \mathbf{x} = 8$ and $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 4$

89. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -4 & 12 \end{pmatrix} \mathbf{x} = 3$
b)	$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} = -3$
c)	$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \mathbf{x} = 9$
d)	$\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 & 6 \end{pmatrix} \mathbf{x} = 2$

TABLE 3.8.89

90. Show that the line joining the origin to the point $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the line deter-

mined by the points $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$.

91. If the coordinates of the points A, B, C, D be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, then find the angle between the lines AB and CD .

92. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \quad (3.8.92.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \quad (3.8.92.2)$$

find the value of k .

93. Find the equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix} 1 & 2 & -5 \end{pmatrix} \mathbf{x} = -9 \quad (3.8.93.1)$$

94. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad (3.8.94.1)$$

$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad (3.8.94.2)$$

95. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the YZ -plane.

96. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the ZX -plane.

97. Find the coordinates of the point where the line through $\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ crosses the plane

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \mathbf{x} = 7 \quad (3.8.97.1)$$

98. Find the equation of the plane passing through the point $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \mathbf{x} = 5 \quad (3.8.98.1)$$

$$\begin{pmatrix} 3 & 3 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.8.98.2)$$

99. If the points $\begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix} 3 & 4 & -12 \end{pmatrix} \mathbf{x} = -13, \quad (3.8.99.1)$$

then find the value of p .

100. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1 \quad \text{and} \quad (3.8.100.1)$$

$$\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = -4 \quad (3.8.100.2)$$

and parallel to the x -axis.

101. If \mathbf{O} be the origin and the coordinates of \mathbf{P} be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then find the equation of the plane passing through \mathbf{P} and perpendicular to OP .

102. Find the equation of the plane which contains

the line of intersection of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \mathbf{x} = 4 \quad (3.8.102.1)$$

$$\begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \mathbf{x} = -5 \quad (3.8.102.2)$$

and which is perpendicular to the plane

$$\begin{pmatrix} 5 & 3 & -6 \end{pmatrix} \mathbf{x} = -8 \quad (3.8.102.3)$$

103. Find the distance of the point $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \quad (3.8.103.1)$$

and the plane

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5 \quad (3.8.103.2)$$

104. Find the vector equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \mathbf{x} = 5 \quad (3.8.104.1)$$

$$\begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (3.8.104.2)$$

105. Find the vector equation of the line passing through the point $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \quad (3.8.105.1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad (3.8.105.2)$$

106. Distance between the two planes

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 4 \quad (3.8.106.1)$$

$$\begin{pmatrix} 4 & 6 & 8 \end{pmatrix} \mathbf{x} = 12 \quad (3.8.106.2)$$

- a) 2 c) 8
b) 4 d) $\frac{2}{\sqrt{29}}$

107. The planes

$$\begin{pmatrix} 2 & -1 & 4 \end{pmatrix} \mathbf{x} = 5 \quad (3.8.107.1)$$

$$\begin{pmatrix} 5 & -\frac{5}{2} & 10 \end{pmatrix} \mathbf{x} = 6 \quad (3.8.107.2)$$

are

a) Perpendicular

b) Parallel

c) intersect y-axis

d) passes through $\begin{pmatrix} 0 \\ 0 \\ \frac{5}{4} \end{pmatrix}$

108. Find the maximum and minimum values, if any of the following functions given by

a) $f(x) = |x + 2| - 1$

b) $f(x) = -|x + 1| + 3$

c) $h(x) = x + 1, x \in (-1, 1)$.

109. Using integration find the area of region bounded by the triangle whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

110. Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.

111. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1$, $\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = -1$ and $x = 4$.

112. Find the area of the region bounded by the line $\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = -2$, the x-axis and the ordinates $x = -1, x = 1$.

113. Find the area bounded by the curve $|x| + |y| = 1$.

114. Using the method of integration find the area of $\triangle ABC$, whose vertices are $\mathbf{A} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix}$.

115. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 4$, $\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = 6$ and $\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = -5$.

116. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

117. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs ₹70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

118. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the

hypotenuse is

$$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad (3.8.118.1)$$

119. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

120. Examine the following functions for continuity.

a) $f(x) = x - 5$

b) $f(x) = |x - 1|$

121. Is the function defined by

$$f(x) = \begin{cases} x, & x \leq 1, \\ 5, & x > 1 \end{cases} \quad (3.8.121.1)$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

122. Find all points of discontinuity of f , where f is defined by

a) $f(x) = \begin{cases} 2x + 3, & x \leq 2, \\ 2x - 3, & x > 2 \end{cases}$

b) $f(x) = \begin{cases} |x| + 3, & x \leq -3, \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \geq 3 \end{cases}$

c) $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$

d) $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0, \\ -1, & x \geq 0, \end{cases}$

123. Is the function defined by

$$f(x) = \begin{cases} x + 5, & x \leq 1, \\ x - 5, & x > 1 \end{cases} \quad (3.8.123.1)$$

a continuous function?

124. Discuss the continuity of the function f , where f is defined by

a) $f(x) = \begin{cases} 3, & 0 \leq x \leq 1, \\ 4, & 0 < x \leq 3, \\ 5, & 3 \leq x \leq 10, \end{cases}$

b) $f(x) = \begin{cases} 2x, & x < 0, \\ 0, & 0 \leq x \leq 1 \\ 4x, & x > 1 \end{cases}$

c) $f(x) = \begin{cases} -2, & x < -1, \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$

125. Find the relationship between a and b so that

the function defined by

$$f(x) = \begin{cases} ax + 1, & x \leq 3, \\ bx + 3, & x > 3 \end{cases} \quad (3.8.125.1)$$

is continuous at $x = 3$

126. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x .

127. For what value of k is the following function continuous at the given point.

$$f(x) = \begin{cases} kx + 1, & x \leq 5, \\ 3x - 5, & x > 5, \end{cases} \quad x = 5 \quad (3.8.127.1)$$

128. Prove that the function f given by

$$f(x) = |x - 1|, x \in \mathbf{R} \quad (3.8.128.1)$$

is not differentiable at $x = 1$.

129. Prove that the greatest integer function defined by

$$f(x) = |x|, 0 < x < 3 \quad (3.8.129.1)$$

is not differentiable at $x = 1$ and $x = 2$.

130. Examine if Rolle's theorem is applicable to the following functions

a) $f(x) = [x], x \in [5, 9]$.

b) $f(x) = [x], x \in [-2, 2]$.

Can you say some thing about the converse of Rolle's theorem from this example?

131. Examine the applicability of the mean value theorem for all functions in Problem 3.8.130a.

132. Evaluate the following limits

a) $\lim_{x \rightarrow 3} x + 3$

b) $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7}\right)$

133. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (3.8.133.1)$$

134. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0, & x = 0 \end{cases} \quad (3.8.134.1)$$

135. Find $\lim_{x \rightarrow 5} |x| - 5$.

136. Suppose

$$f(x) = \begin{cases} a + bx & x \neq 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases} \quad (3.8.136.1)$$

and if $\lim_{x \rightarrow 1} f(x) = f(1)$, what are the possible values of a and b ?

137. If

$$f(x) = \begin{cases} |x| + 1 & x < 0 \\ 0, & x = 0 \\ |x| - 1 & x > 0 \end{cases} \quad (3.8.137.1)$$

for what value(s) of a does $\lim_{x \rightarrow a} f(x)$ exists?

138. Find the derivative of x at $x = 1$.

139. Find the derivative of $99x$ at $x = 100$.

140. Find the derivative of the following functions:

- $-x$
- $x + a$

141. Integrate the following as limit of sums:

- $\int_a^b x \, dx$
- $\int_0^5 (x + 1) \, dx$
- $\int_{-1}^1 (x + 1) \, dx$
- $\int_{-5}^5 |x + 2| \, dx$
- $\int_2^8 |x - 5| \, dx$
- $\int_0^4 |x - 1| \, dx$
- $\int_1^4 [|x - 1| + |x - 2| + |x - 3|] \, dx$

142. Form the differential equation representing the following family of curves

$$\left(\frac{1}{a} \quad \frac{1}{b}\right) \mathbf{x} = 1 \quad (3.8.142.1)$$

3.9 Matrix Exercises

1. In the matrix $A = \begin{pmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{pmatrix}$, write

- The order of the matrix
- The number of elements
- Write the elements $a_{31}, a_{21}, a_{33}, a_{24}, a_{23}$.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

$$(i) a_{ij} = \frac{(i+j)^2}{2} \quad (ii) a_{ij} = \frac{i}{j} \quad (iii) a_{ij} = \frac{(i+2j)^2}{2}$$

5. Construct a 3×4 matrix, whose elements are given by:

$$(i) a_{ij} = \frac{1}{2} |-3i + j| \quad (ii) a_{ij} = 2i - j$$

6. Find the values of x, y and z from the following equations:

$$(i) \begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix} \quad (ii) \begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$$

$$(iii) \begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

7. Find the values of a, b, c and d from the equations: $\begin{pmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & 13 \end{pmatrix}$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix}, \begin{pmatrix} 0 & y-2 \\ 8 & 4 \end{pmatrix}$$

$$(A) x = \frac{-1}{3}, y = 7$$

(B) Not possible to find

$$(C) y = 7, x = \frac{-2}{3}$$

$$(D) x = \frac{-1}{3}, y = \frac{-2}{3}$$

10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27 (B) 18 (C) 81 (D) 512

11. Let $A = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}, C = \begin{pmatrix} -2 & 5 \\ 3 & 4 \end{pmatrix}$ Find each of the following:

(i) $A+B$ (ii) $A-B$ (iii) $3A-C$ (iv) AB (v) BA

12. Compute the following:

$$(i) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$(ii) \begin{pmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{pmatrix} + \begin{pmatrix} 2ab & 2bc \\ -2ac & -2ab \end{pmatrix}$$

$$(iii) \begin{pmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{pmatrix} + \begin{pmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{pmatrix}$$

$$(iv) \begin{pmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{pmatrix} + \begin{pmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{pmatrix}$$

13. Compute the indicated products.

(i) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

(ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

14. If, $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$ and

$C = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$, then compute $(A+B)$ and $(B-C)$. Also, verify that $A+(B-C)=(A+B)-C$.

15. If $A = \begin{pmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{2}{3} & 2 & \frac{5}{3} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} & \frac{5}{5} \end{pmatrix}$, then compute $3A-5B$.

16. Simplify $\cos \theta \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} + \sin \theta \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix}$

17. Find X and Y, if

(i) $X+Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X-Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

(ii) $2X+3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $3X+2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$

18. Find X if $Y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $2X+Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$

19. Find x and y, if $2 \begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$

20. Solve the equation for x, y, z and t, if

$2 \begin{pmatrix} x & z \\ y & t \end{pmatrix} + 3 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 3 \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$

21. If $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$, find the values of x and y.

22. Given $3 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$, find

the values of x, y, z and w.

23. If $F(x) = \begin{pmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix}$, show that $F(x)F(y) = F(x+y)$

24. Show that

(i) $\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \neq$

$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

25. Find $A^2 - 5A + 6I$, if $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

26. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

27. If $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find k so that $A^2 = kA - 2I$

28. If $A = \begin{pmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{pmatrix}$ and I is the identity matrix of order 2, show that

$I+A = (I-A) \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$

29. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) ₹1800 (b) ₹2000

30. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of orders $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively.

Choose the correct answer in Exercise 31 and 32.

31. The restriction on n, k and p so that $PY + WY$ will be defined are:

- (A) $k=3, p=n$
 (B) k is arbitrary, $p=2$
 (C) p is arbitrary, $k=3$
 (D) $k=2, p=3$

32. If $n=p$, then the order of the matrix $7X - 5Z$ is:

- (A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

33. Find the transpose of each of the following matrices:

(i) $\begin{pmatrix} 5 \\ \frac{1}{2} \\ -1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{pmatrix}$

34. If $A = \begin{pmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$, then

verify that

- (i) $(A + B)' = A' + B'$
 (ii) $(A - B)' = A' - B'$

35. If $A' = \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, then verify that

- (i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

36. If $A' = \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$, then find that $(A + 2B)'$

37. For the matrices A and B , verify that $(AB)' = B' A'$, where

(i) $A = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 5 & 7 \end{pmatrix}$

38. If (i) $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$, then verify that $A' A = I$

(ii) If $A = \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix}$, then verify that $A' A = I$

39. (i) Show that the matrix $A = \begin{pmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{pmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ is a skew symmetric matrix.

40. For the matrix $A = \begin{pmatrix} 1 & 5 \\ 6 & 7 \end{pmatrix}$, verify that

- (i) $(A + A')$ is a symmetric matrix
 (ii) $(A - A')$ is a skew symmetric matrix

41. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

42. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$

Choose the correct answer in question number 43 and 44

43. If A, B are symmetric matrices of same order, then $AB - BA$ is a

- (A) Skew symmetric matrix
 (B) Symmetric matrix
 (C) Zero matrix
 (D) Identity matrix

44. If $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$, and $A + A' = I$, then the value of α is

- (A) $\frac{\pi}{6}$
 (B) $\frac{\pi}{3}$
 (C) π

(D) $\frac{3\pi}{2}$

Using elementary transformations, find the inverse of each of the matrices, if it exists questions 45-61

45. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

46. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

47. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

48. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

49. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

50. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

51. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

52. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

53. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

54. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

55. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

56. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

57. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

58. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

59. $\begin{pmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}$

60. $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$

61. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

62. Matrices A and B will be inverse of each other only if
(A) $AB=BA$ (B) $AB=BA=0$

(C) $AB=0, BA=I$ (D) $AB=BA=I$

63. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,

prove that $A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}, n \in \mathbb{N}$

64. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show that

$(aI + bA)^n = a^n I + na^{n-1}bA$, where I is the identity matrix of order 2 and $n \in \mathbb{N}$

65. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$,

then prove that $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$, where n is any positive integer

66. If A and B are symmetric matrices, prove that $AB-BA$ is a skew symmetric matrix.

67. Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric

68. Find the values of x, y, z if the matrix

$A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ satisfy the equation $A'A = I$

69. For what values of x:

$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & x \end{pmatrix} = 0$?

70. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 5A + 7I = 0$

71. Find x, if $\begin{pmatrix} x & -5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix} = 0$

72. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	Products		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are ₹2.50, ₹1.50 and ₹1.00 respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit cost of the above three commodities are ₹2.00, ₹1.00 and 50 paise respectively. Find the gross profit.

73. Find the matrix X so that

$$X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$$

74. If A and B are square matrices of the same order such that $AB=BA$, then prove by induction that $AB^n = B^nA$. Further prove that $(AB)^n = A^nB^n$ for all $n \in \mathbb{N}$.

Choose the correct answer in the following questions:

75. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$, then
 (A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$
 (C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

76. If the matrix A is both symmetric and skew symmetric, then

- (A) A is a diagonal matrix
 (B) A is a zero matrix
 (C) A is a square matrix
 (D) None of these

77. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

- (A) A
 (B) I-A
 (C) I
 (D) 3A

3.10 Determinants

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$, then show that $|2A| = 4|A|$

4. If $A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$, then show that $|3A| = 27|A|$

5. Evaluate the determinants

a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$

7. Find the values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

- a) 6
 b) ± 6
 c) -6
 d) 0

9. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

10. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

11. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

12. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

13. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

14. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

15. $\begin{vmatrix} -a^2 & ab & ab \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

By Using properties of determinants, in Exercises 16 to 22, Show that;

16. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

17. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

$$18. (i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k)$$

$$19. (i) \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$(ii) \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

$$20. \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

$$21. \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

$$22. \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Choose the correct answer in Exercises 23 and 24.

23. Let A be a square matrix of order 3X3, then $|kA|$ is equal to

- a) $k|A|$
- b) $k^2|A|$
- c) $k^3|A|$
- d) $3k|A|$

24. Which of the following is correct

- a) Determinant is a square matrix.
- b) Determinant is a number associated to a matrix.
- c) Determinant is a number associated to a square matrix.
- d) None of these.

25. Find area of the triangle with vertices at the point given in each of the following :

- (i) $(1 \ 0), (6 \ 0), (4 \ 3)$
- (ii) $(2 \ 7), (1 \ 1), (10 \ 8)$
- (iii) $(-2 \ -3), (3 \ 2), (-1 \ -8)$

26. Show that points $A=(a \ b+c), B=(b \ c+a), C=(c \ a+b)$ are colinear.

27. Find values of k if area of triangle is 4sq.units and vertices are

- (i) $(k \ 0), (4 \ 0), (0 \ 2)$

- (ii) $(-2 \ 0), (0 \ 4), (0 \ k)$

28. (i) Find equation of line joining $(1 \ 2)$ and $(3 \ 6)$ using determinants.

(ii) Find equation of line joining $(3 \ 1)$ and $(9 \ 3)$ using determinants.

29. If the area of triangle is 35 sq.units with vertices $(2 \ -6), (5 \ 4)$ and $(k \ 4)$. then k is

- a) 12
- b) -2
- c) -12, -2
- d) 12, -2

Write Minors and Coactors of the elements of following determinants:

$$30. (i) \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$31. (i) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

32. Using Cofactors of elements of second

$$\text{row, evaluate } \Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

33. Using Cofactors of elements of third column

$$\text{, evaluate } \Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}.$$

34. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of

a_{ij} then value of Δ is given by

- a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
- b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
- c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
- d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Find adjoint of each of the matrices

$$35. \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$36. \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} \text{ Verify } A(\text{adj}A) = (\text{adj}A)A = |A| I$$

$$37. \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

$$38. \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$39. \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

$$40. \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

$$41. \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$42. \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

$$43. \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

$$44. \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$$

$$45. \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

$$46. \text{ Let } A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}. \text{ Verify that } (AB)^{-1} = B^{-1}A^{-1}$$

$$47. \text{ Let } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \text{ show that } A^2 - 5A + 7I = O. \text{ Hence find } A^{-1}$$

$$48. \text{ For the matrix } A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \text{ find the numbers } a \text{ and } b \text{ such that } A^2 + aA + bI = O.$$

$$49. \text{ For the matrix } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \text{ Show that } A^3 - 6A^2 + 9A - 4I = O \text{ and hence find } A^{-1}$$

$$50. \text{ Let } A \text{ be a nonsingular square matrix of order } 3 \times 3. \text{ Then } |adj A| \text{ is equal to}$$

- a) $|A|$
- b) $|A|^2$
- c) $|A|^3$
- d) $3|A|$

$$51. \text{ If } A \text{ is an invertible matrix of order } 2, \text{ then } \det(A^{-1}) \text{ is equal to}$$

- a) $\det(A)$
- b) $\frac{1}{\det(A)}$
- c) 1
- d) 0

Examine the consistency of the system of given Equations.

$$52. \quad x + 2y = 2$$

$$2x + 3y = 3$$

$$53. \quad 2x - y = 5$$

$$x + y = 4$$

$$54. \quad x + 3y = 5$$

$$2x + 6y = 8$$

$$55. \quad x + y + z = 1$$

$$2x + 3y + 2z = 2$$

$$ax + ay + 2az = 4$$

$$56. \quad 3x - y - 2z = 2$$

$$2y - z = -1$$

$$3x - 5y = 3$$

$$57. \quad 5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Solve the system linear equations, using matrix method.

$$58. \quad 5x + 2y = 4$$

$$7x + 3y = 5$$

$$59. \quad 2x - y = -2$$

$$3x + 4y = 3$$

$$60. \quad 4x - 3y = 3$$

$$3x - 5y = 7$$

$$61. \quad 5x + 2y = 3$$

$$3x + 2y = 5$$

$$62. \quad 2x + y + z = 1$$

$$x - 2y - z = \frac{3}{2}$$

$$3y - 5z = 9$$

$$63. \quad x - y + z = 4$$

$$2x + y - 3z = 0$$

$$x + y + z = 2$$

$$64. \quad 2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

$$65. \quad x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

$$66. \text{ If } A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, \text{ find } A^{-1}. \text{ Using } A^{-1} \text{ solve the system of equations } 2x - 3y + 5z = 11,$$

$$3x+2y-4z = -5,$$

$$x+y-2z = -3.$$

67. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find the cost of each item per kg by matrix method.

68. Prove that the determinant

$$\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$$
 is independent of θ

69. Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

70. Evaluate $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$.

71. If a, b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c=a & a=b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0, \text{ Show that}$$

either $a+b+c=0$ or $a=b=c$.

72. Solve the equation

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$$

73. Prove that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

74. If

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

find $(AB)^{-1}$

75. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adj A]^{-1} = adj(A)^{-1}$

(ii) $(A^{-1})^{-1} = A$

76. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

77. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ Using properties of

determinants, prove that:

$$78. \begin{vmatrix} \alpha & \alpha^2 & \beta + \gamma \\ \beta & \beta^2 & \gamma + \alpha \\ \gamma & \gamma^2 & \alpha + \beta \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$$

$$79. \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x),$$

where p is any scalar.

$$80. \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

$$81. \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

$$82. \begin{vmatrix} \sin \alpha & \cos \alpha & \cos(\alpha + \delta) \\ \sin \beta & \cos \beta & \cos(\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos(\gamma + \delta) \end{vmatrix} = 0$$

83. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

84. If a, b, c are in A.P, then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$
 is

a) 0

b) 1

c) x

d) 2x

85. If x, y, z are nonzero real numbers, then the

inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

b) $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

c) $\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

d) $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

86. Let $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq$

2 π . Then

- $\text{Det}(A)=0$
- $\text{Det}(A) \in (2, \infty)$
- $\text{Det}(A) \in (2, 4)$
- $\text{Det}(A) \in [2, 4]$

3.11 Linear Inequalities: Exercises

- Solve $30x < 200$ when
 - x is a natural number,
 - x is an integer.
- Solve $5x-3 < 3x+1$ when
 - x is an integer,
 - x is a real number.
- Solve $4x+3 < 6x+7$.
- Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.
- Solve $7x+3 < 5x+9$. Show the graph of the solutions on number line.
- Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.
- The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.
- Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.
- Solve $3x+2y > 6$ graphically.
- Solve $3x-6 \geq 0$ graphically in a two dimensional plane.
- Solve $y < 2$ graphically.
- Solve the following system of linear inequalities graphically.

$$\begin{aligned} x+y &\geq 5 \\ x-y &\leq 3 \end{aligned}$$
- Solve the following system of inequalities graphically.

$$\begin{aligned} 5x+4y &\leq 40 \\ x &\geq 2 \\ y &\geq 3 \end{aligned}$$
- Solve the following system of inequalities graphically.

$$\begin{aligned} 8x+3y &\leq 100 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$
- Solve the following system of inequalities graphically.

$$\begin{aligned} x+2y &\leq 8 \\ 2x+y &\leq 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$
- Solve $-8 \leq 5x-3 < 7$.
- Solve $-5 \leq \frac{5-3x}{2} \leq 8$.
- Solve the system inequalities:

$$\begin{aligned} 3x-7 &< 5+x \\ 11-5x &\leq 1 \end{aligned}$$
 and represent the solutions on the number line.
- In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by $C = \frac{5}{9}(F - 32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.
- A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

3.12 Linear Programing: Exercises

- Maximise $Z=3x+4y$
subject to the constraints : $x+y \leq 4$, $x \geq 0$, $y \geq 0$.

2. Minimise $Z = -3x + 4y$
subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.
3. Maximise $Z = 5x + 3y$ subject to $3x + 5y \leq 15$,
 $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.
4. Minimise $Z = 3x + 5y$ such that $x + 3y \geq 3$, $x + y \geq 2$,
 $x, y \geq 0$.
5. Maximise $Z = 3x + 2y$ subject to $x + 2y \leq 10$,
 $3x + y \leq 15$, $x, y \geq 0$.
6. Minimise $Z = x + 2y$ subject to $2x + y \geq 3$,
 $x + 2y \geq 6$, $x, y \geq 0$.
Show that the minimum of Z occurs at more than two points.
7. Minimise and Maximise $Z = 5x + 10y$ subject to
 $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$.
8. Minimise and Maximise $Z = x + 2y$ subject to
 $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.
9. Maximise $Z = -x + 2y$, subject to the constraints:
 $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.
10. Maximise $Z = x + y$, subject to $x - y \leq -1$, $-x + y \leq 0$,
 $x, y \geq 0$.
11. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
12. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
13. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
(i) What number of rackets and bats must be made if the factory is to work at full capacity?
(ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
14. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs 17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?
15. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.
16. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours

on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

17. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?
18. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.
19. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.
20. There are two types of fertilisers F_1 and F_2 . F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?
21. The corner points of the feasible region determined by the following system of linear inequalities: $2x+y \leq 10$, $x+3y \leq 15$, $x, y \geq 0$ are $(0,0)$, $(5,0)$, $(3,4)$ and $(0,5)$. Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both $(3,4)$ and $(0,5)$ is
 (A) $p = q$
 (B) $p = 2q$
 (C) $p = 3q$
 (D) $q = 3p$
22. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?
23. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?
24. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

25. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Machines			
Types of toys	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

26. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

27. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in Rs)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

28. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively.

The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

29. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	Brand Q
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

30. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

31. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the

company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

3.13 Miscellaneous

1. Solve the following pair of linear equations

a)

$$\begin{aligned}(p \quad q)\mathbf{x} &= p - q \\ (q \quad -p)\mathbf{x} &= p + q\end{aligned}\quad (3.13.1.1)$$

b)

$$\begin{aligned}(a \quad b)\mathbf{x} &= c \\ (b \quad a)\mathbf{x} &= 1 + c\end{aligned}\quad (3.13.1.2)$$

c)

$$\begin{aligned}\left(\frac{1}{a} \quad -\frac{1}{b}\right)\mathbf{x} &= 0 \\ (a \quad b)\mathbf{x} &= a^2 + b^2\end{aligned}\quad (3.13.1.3)$$

2. Solve the following pair of equations

$$\begin{aligned}(a - b \quad a + b)\mathbf{x} &= a^2 - 2ab - b^2 \\ (a + b \quad a + b)\mathbf{x} &= a^2 + b^2\end{aligned}\quad (3.13.2.1)$$

3. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.13.3.1)$$

4. (Cauchy-Schwarz Inequality:) Show that

$$|\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (3.13.4.1)$$

5. (Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (3.13.5.1)$$

6. The base of an equilateral triangle with side $2a$ lies along the y -axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

7. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ when

a) PQ is parallel to the y -axis.

b) PQ is parallel to the x -axis.

8. If three points $\begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

9. $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\left(\frac{1}{a} \quad \frac{1}{b}\right)\mathbf{x} = 2 \quad (3.13.9.1)$$

10. Point $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

11. Show that two lines

$$(a_1 \quad b_1)\mathbf{x} + c_1 = 0 \quad (3.13.11.1)$$

$$(a_2 \quad b_2)\mathbf{x} + c_2 = 0 \quad (3.13.11.2)$$

are

a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and

b) perpendicular if $a_1 a_2 - b_1 b_2 = 0$.

12. Find the distance between the parallel lines

$$l(1 \quad 1)\mathbf{x} = -p \quad (3.13.12.1)$$

$$l(1 \quad 1)\mathbf{x} = r \quad (3.13.12.2)$$

13. Find the equation of the line through the point \mathbf{x}_1 and parallel to the line

$$(A \quad B)\mathbf{x} = -C \quad (3.13.13.1)$$

14. If p and q are the lengths of perpendiculars from the origin to the lines

$$(\cos \theta \quad \sin \theta)\mathbf{x} = k \cos 2\theta \quad (3.13.14.1)$$

$$(\sec \theta \quad \csc \theta)\mathbf{x} = k \quad (3.13.14.2)$$

respectively, prove that $p^2 + 4q^2 = k^2$.

15. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b , then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \quad (3.13.15.1)$$

16. Show that the area of the triangle formed by the lines

$$(-m_1 \quad 1)\mathbf{x} = c_1 \quad (3.13.16.1)$$

$$(-m_2 \quad 1)\mathbf{x} = c_2 \quad (3.13.16.2)$$

$$(1 \quad 0)\mathbf{x} = 0 \quad (3.13.16.3)$$

is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

17. Find the values of k for which the line

$$(k-3 \quad -(4-k^2))\mathbf{x} + k^2 - 7k + 6 = 0 \quad (3.13.17.1)$$

is

- a) parallel to the x-axis
- b) parallel to the y-axis
- c) passing through the origin.

18. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and

$$\begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}.$$

19. Find the area of the triangle formed by the lines

$$(1 \quad -1)\mathbf{x} = 0 \quad (3.13.19.1)$$

$$(1 \quad 1)\mathbf{x} = 0 \quad (3.13.19.2)$$

$$(1 \quad 0)\mathbf{x} = k \quad (3.13.19.3)$$

20. If three lines whose equations are

$$(-m_1 \quad 1)\mathbf{x} = c_1 \quad (3.13.20.1)$$

$$(-m_2 \quad 1)\mathbf{x} = c_2 \quad (3.13.20.2)$$

$$(-m_3 \quad 1)\mathbf{x} = c_3 \quad (3.13.20.3)$$

are concurrent, show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \quad (3.13.20.4)$$

21. Find the equation of the line passing through the origin and making an angle θ with the line

$$(-m \quad 1)\mathbf{x} = c \quad (3.13.21.1)$$

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$

and $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$ to the line

$$\left(\frac{\cos \theta}{a} \quad \frac{\sin \theta}{b}\right)\mathbf{x} = 1 \quad (3.13.22.1)$$

is b^2 .

23. If $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1 n_2 - m_2 n_1 \\ n_1 l_2 - n_2 l_1 \\ l_1 m_2 - l_2 m_1 \end{pmatrix}$.

24. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}. \quad (3.13.24.1)$$

25. Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \quad (3.13.25.1)$$

$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \quad (3.13.25.2)$$

are coplanar.

26. Find \mathbf{R} which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \quad (3.13.26.1)$$

$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \quad (3.13.26.2)$$

externally in the ratio 1 : 2.

27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \quad (3.13.27.1)$$

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \quad (3.13.27.2)$$

28. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \quad (3.13.28.1)$$

29. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \quad (3.13.29.1)$$

$$\mathbf{a}^T \mathbf{b} = \frac{1}{2} \quad (3.13.29.2)$$

and the angle between \mathbf{a} and \mathbf{b} is 60° .

30. Show that

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \perp (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a}) \quad (3.13.30.1)$$

31. If $\mathbf{a}^T \mathbf{a} = 0$ and $\mathbf{a}\mathbf{b} = 0$, what can be concluded about the vector \mathbf{b} ?

32. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \quad (3.13.32.1)$$

find the value of

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \quad (3.13.32.2)$$

33. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda \mathbf{a}\| = 1$ if

- a) $\lambda = 1$
- b) $\lambda = -1$
- c) $\|\mathbf{a}\| = |\lambda|$
- d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

34. If a unit vector \mathbf{a} makes angles $\frac{\pi}{3}$ with the x-

axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and \mathbf{a} .

35. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) \quad (3.13.35.1)$$

36. If $\mathbf{a}^T \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = 0$, what can you conclude about \mathbf{a} and \mathbf{b} ?

37. Find \mathbf{x} if \mathbf{a} is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \quad (3.13.37.1)$$

38. If $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between \mathbf{a} and \mathbf{b} is

- | | |
|--------------------|--------------------|
| a) $\frac{\pi}{6}$ | c) $\frac{\pi}{3}$ |
| b) $\frac{\pi}{4}$ | d) $\frac{\pi}{2}$ |

39. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (3.13.39.1)$$

$$\iff \mathbf{a} \perp \mathbf{b}. \quad (3.13.39.2)$$

40. If θ is the angle between two vectors \mathbf{a} and \mathbf{b} , then $\mathbf{a}^T \mathbf{b} \geq 0$ only when

- | | |
|---------------------------------------|-----------------------------|
| a) $0 < \theta < \frac{\pi}{2}$ | c) $0 < \theta < \pi$ |
| b) $0 \leq \theta \leq \frac{\pi}{2}$ | d) $0 \leq \theta \leq \pi$ |

41. Let \mathbf{a} and \mathbf{b} be two unit vectors and θ be the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

- | | |
|-----------------------------|------------------------------|
| a) $\theta = \frac{\pi}{4}$ | c) $\theta = \frac{\pi}{2}$ |
| b) $\theta = \frac{\pi}{3}$ | d) $\theta = \frac{2\pi}{3}$ |

42. If θ is the angle between any two vectors \mathbf{a} and \mathbf{b} , then $\|\mathbf{a}^T \mathbf{b}\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

- | | |
|--------------------|--------------------|
| a) 0 | c) $\frac{\pi}{2}$ |
| b) $\frac{\pi}{4}$ | d) π . |

43. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.

44. Find the equation of a line parallel to the x-axis and passing through the origin.

45. Find the equation of a plane passing through

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (3.13.45.1)$$

46. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \quad (3.13.46.1)$$

4 CIRCLE

4.1 Construction Examples

1. ABC is a triangle. Locate a point in the interior of $\triangle ABC$ which is equidistant from all the vertices of $\triangle ABC$.

Solution: Let \mathbf{O} be the desired point. Then,

$$\|\mathbf{A} - \mathbf{O}\| = \|\mathbf{B} - \mathbf{O}\| = \|\mathbf{C} - \mathbf{O}\| = R \quad (4.1.1.1)$$

From (??),

$$\|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \quad (4.1.1.2)$$

$$\begin{aligned} \implies (\mathbf{A} - \mathbf{O})^T (\mathbf{A} - \mathbf{O}) \\ - (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \end{aligned} \quad (4.1.1.3)$$

which can be simplified as

$$(\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (4.1.1.4)$$

Similarly,

$$(\mathbf{B} - \mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (4.1.1.5)$$

From and , \mathbf{O} can be computed. A circle with centre \mathbf{O} can be drawn through $\mathbf{A}, \mathbf{B}, \mathbf{C}$. This circle is known as the *circumcircle*. The following code plots Fig. ??

codes/circle/circle_const_ccircle.py

2. In a triangle locate a point in its interior which is equidistant from all the sides of the triangle.

3. Draw a circle with centre \mathbf{B} and radius 6. If \mathbf{C} be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC$, $AC \perp AB$, $a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \quad (4.1.3.1)$$

The following code plots Fig. ??

codes/circle/draw_circle_eg.py

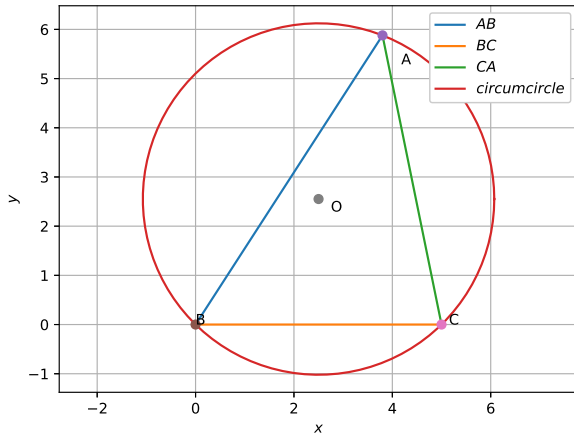


Fig. 4.1.1

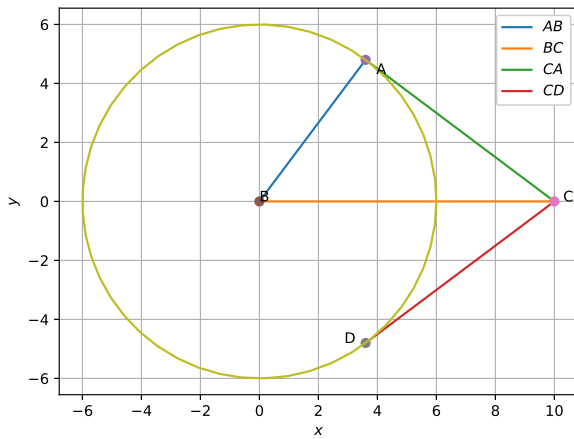


Fig. 4.1.3

4. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.

Solution: For any angle θ , a point on the circle with radius 3 has coordinates

$$3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (4.1.4.1)$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?

4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?

5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

Solution: Take the centre of both circles to be at the origin.

6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.

Solution: Take the diameter to be on the x -axis.

7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60° .

Solution: The tangent is perpendicular to the radius.

8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (4.2.4.1)$$

9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through **B, C, D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.

10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

5. Form the differential equation representing the family of curves given by

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} - (2a \ 0) \mathbf{x} = 0, \quad (4.2.5.1)$$

where a is an arbitrary constant.

6. Form the differential equation of the family of circles in the first quadrant which touch the

coordinate axes.

4.3 Circle Geometry Examples

- Find the equation of a circle with centre $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and radius 4.
- Find the centre and radius of the circle

$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 8 \\ 10 \end{pmatrix} \mathbf{x} - 8 = 0 \quad (4.3.2.1)$$
- Find the equation of the circle which passes through the points $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.
- Find the area enclosed by the circle $\|\mathbf{x}\| = a$
- Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
- Find the area of the region enclosed between the two circles: $\mathbf{x}^T \mathbf{x} = 4$ and $\left\| \mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\| = 2$.
- Form the differential equation of the family of circles touching the x-axis at the origin.
- Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

4.4 Circle Geometry Exercises

- Find the coordinates of a point **A**, where **AB** is the diameter of a circle whose centre is $(2, -3)$ and **B** = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
- Find the centre *O* of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
- Sketch the circles with
 - centre $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2
 - centre $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4
 - centre $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$.
 - centre $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$.
 - centre $\begin{pmatrix} -a \\ -b \end{pmatrix}$ and radius $\sqrt{a^2 - b^2}$.
-
- Sketch the circles with equation

$$\text{a) } \left\| \mathbf{x} - \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right\|^2 = 36$$

$$\text{b) } \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 4 \\ 8 \end{pmatrix} \mathbf{x} - 45 = 0$$

$$\text{c) } \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix} \mathbf{x} - 12 = 0$$

$$\text{d) } 2\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{x} = 0$$

- Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$.
- Find the equation of the circle passing through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 11$.
- Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
- Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts *a* and *b* on the coordinate axes.
- Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.
- Does the point $\begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T \mathbf{x} = 25$?
- Find the locus of all the unit vectors in the xy-plane.
- Find the points on the curve $\mathbf{x}^T \mathbf{x} - 2\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
- Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.
- Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
- Find the area of the circle $4\mathbf{x}^T \mathbf{x} = 9$.
- Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$
- Find the smaller area enclosed by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.
- The sum of the perimeter of a circle and square

is k , where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

20. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.
21. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{(1 + y_2)^{\frac{3}{2}}}{y_2} \quad (4.4.21.1)$$

is a constant independent of a and b .

22. Form the differential equation of the family of circles touching the y -axis at origin.
23. Form the differential equation of the family of circles having centre on y -axis and radius 3 units.

5 CONICS

5.1 Examples

1. Find the value of the following polynomial at the indicated value of variables

$$p(x) = 5x^2 - 3x + 7 \text{ at } x = 1. \quad (5.1.1.1)$$

2. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.
3. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
- a) $p(y) = y^2$.
- b) $p(x) = (x-1)(x+1)$.
4. Find the roots of the equation $2x^2 - 5x + 3 = 0$.
5. Find the roots of the quadratic equation $6x^2 - x - 2 = 0$.
6. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
7. Factorise $6x^2 + 17x + 5$.
8. Factorise $y^2 - 5y + 6$.
9. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$ and verify the relationship between the zeroes and the coefficients.
10. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.
11. Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2 , respectively.
12. Find the roots of the equation $5x^2 - 6x - 2 = 0$.
13. Find the roots of $4x^2 + 3x + 5 = 0$.

14. Find the roots of the following quadratic equations, if they exist.

a) $3x^2 - 5x + 2 = 0$

b) $x^2 + 4x + 5 = 0$

c) $2x^2 - 2\sqrt{2}x + 1 = 0$

15. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.

16. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.

17. Solve $x^2 + 2 = 0$.

18. Solve $x^2 + x + 1 = 0$.

19. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.

20. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

21. Find the equation of the parabola with focus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and directrix $(1 \ 0) \mathbf{x} = -2$.

22. Find the equation of the parabola with vertex at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and focus at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

23. Find the equation of the parabola which is symmetric about the y -axis, and passes through the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

24. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1 \quad (5.1.24.1)$$

25. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36 \quad (5.1.25.1)$$

26. Find the equation of the ellipse whose vertices are $\begin{pmatrix} \pm 13 \\ 0 \end{pmatrix}$ and foci are $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.

27. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

28. Find the equation of the ellipse, with major axis along the x -axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

29. Find the coordinates of the foci and the ver-

tices, the eccentricity, the length of the latus rectum of the hyperbolas

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1$

b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 16$

30. Find the equation of the hyperbola with vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

31. Find the equation of the hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of latus rectum 36.

32. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \quad (5.1.32.1)$$

33. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

34. Find the roots of the following equations:

a) $x + \frac{1}{x} = 3, x \neq 0$

b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq 0, 2$

35. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$ at which the tangents are

a) parallel to x-axis

b) parallel to y-axis

36. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

37. Find the area enclosed by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$

38. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

39. Find the area bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$.

40. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.

41. Find the area of the region

$$\{(x, y) = 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\} \quad (5.1.41.1)$$

42. Find the intervals in which the function

$$f(x) = x^2 - 4x + 6 \quad (5.1.42.1)$$

is

a) increasing

b) decreasing.

43. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \leq c \leq 5$.

44. An apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$, wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

45. Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

46. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x & x \geq 0 \\ x^2 & x < 0 \end{cases} \quad (5.1.46.1)$$

47. Verify Rolle's theorem for the function $y = x^2 + 2, a = -2$ and $b = 2$.

48. Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, -4]$.

49. Find the derivative of $f(x) = x^2$.

50. Find the derivative of $x^2 - 2$ at $x = 10$.

51. Find the derivative of $(x-1)(x-2)$.

52. Find

$$\int_0^2 (x^2 + 1) dx \quad (5.1.52.1)$$

as a limit of a sum.

53. Evaluate the following integral:

$$\int_2^3 x^2 dx \quad (5.1.53.1)$$

54. Form the differential equation representing the family of ellipses having foci on x-axis and centre at the origin.

55. Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

56. Form a differential equation representing the following family of curves

$$y^2 = a(b^2 - x^2) \quad (5.1.56.1)$$

5.2 Exercises

1. Verify whether the following are zeroes of the polynomial, indicated against them.

a) $p(x) = x^2 - 1, x = 1, -1$

- b) $p(x) = (x + 1)(x - 2)$, $x = -1, 2$
 c) $p(x) = x^2$, $x = 0$.
 d) $p(x) = 3x^2 - 1$, $x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$.
2. Find the value of k , if $x - 1$ is a factor of $p(x)$ in each of the following cases:
 a) $p(x) = 2x^3 + x^2 - 2x - 1$, $g(x) = x + 1$
 b) $p(x) = x^3 + 3x^2 + 3x + 1$, $g(x) = x + 2$
 c) $x^4 - 4x^2 + x + 6$, $g(x) = x - 3$
3. Factorise :
 a) $12x^2 - 7x + 1$
 b) $6x^2 + 5x - 6$
 c) $2x^2 + 7x + 3$
 d) $3x^2 - x - 4$
4. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
 a) $x^2 - 2x - 8$
 b) $4u^2 + 8u$
 c) $4s^2 - 4s + 1$
 d) $t^2 - 15$
 e) $6x^2 - 3 - 7x$
 f) $3x^2 - x - 4$
5. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.
 a) $-1, \frac{1}{4}$
 b) $1, 1$
 c) $0, \sqrt{5}$
 d) $4, 1$
 e) $\frac{1}{4}, \frac{1}{4}$
 f) $\sqrt{2}, \frac{1}{3}$
6. Find the roots of the following quadratic equations:
 a) $x^2 - 3x - 10 = 0$
 b) $2x^2 + x - 6 = 0$
 c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
 d) $2x^2 - x + \frac{1}{8} = 0$
 e) $100x^2 - 20x + 1 = 0$
7. Find the roots of the following quadratic equations
 a) $2x^2 - 7x + 3 = 0$
 b) $2x^2 + x - 4 = 0$
 c) $4x^2 + 4\sqrt{3}x + 3 = 0$
 d) $2x^2 + x + 4 = 0$
8. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
 a) $2x^2 - 3x + 5 = 0$
 b) $2x^2 - 6x + 3 = 0$
 c) $3x^2 - 4\sqrt{3}x + 4 = 0$
9. Solve each of the following equations
 a) $x^2 + 3 = 0$
 b) $2x^2 + x + 1 = 0$
 c) $x^2 + 3x + 9 = 0$
 d) $-x^2 + x - 2 = 0$
 e) $x^2 + 3x + 5 = 0$
 f) $x^2 - 3x + 2 = 0$
 g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
 h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
 i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
 j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$
10. Solve each of the following equations
 a) $3x^2 - 4x + \frac{20}{3} = 0$
 b) $x^2 - 2x + \frac{3}{2} = 0$
 c) $27x^2 - 10x + 1 = 0$
 d) $21x^2 - 28x + 10 = 0$
11. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum
 a) $y^2 = 12x$
 b) $x^2 = 6y$
 c) $y^2 = -8x$
 d) $x^2 = -16y$
 e) $y^2 = 10x$
 f) $x^2 = -9y$
12. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:
 a) Focus $\left(\frac{6}{0}\right)$, directrix $(1 \ 0) = -6$.
 b) Focus $\left(\frac{0}{-3}\right)$, directrix $(0 \ 1) = 3$.
 c) Focus $\left(\frac{3}{0}\right)$, vertex $(0 \ 0)$.
 d) Focus $\left(\frac{-2}{0}\right)$, vertex $(0 \ 0)$.
 e) vertex $(0 \ 0)$ passing through $\left(\frac{2}{2}\right)$ and axis is along the x-axis
 f) vertex $(0 \ 0)$ passing through $\left(\frac{5}{2}\right)$ and symmetric with respect to the y-axis.
13. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis,

the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$

b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$

c) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

d) $\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{100} \end{pmatrix} \mathbf{x} = 1$

e) $\mathbf{x}^T \begin{pmatrix} \frac{1}{49} & 0 \\ 0 & \frac{1}{36} \end{pmatrix} \mathbf{x} = 1$

f) $\mathbf{x}^T \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$

g) $\mathbf{x}^T \begin{pmatrix} 36 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 144$

h) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 16$

i) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$

14. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$

b) Vertices $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

c) Vertices $\begin{pmatrix} \pm 6 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$

d) Ends of major axis $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} 0 \\ \pm 2 \end{pmatrix}$

e) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$

f) Length of major axis 26, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$

g) Length of minor axis 16, foci $\begin{pmatrix} 0 \\ \pm 6 \end{pmatrix}$

h) Foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, $a = 4$

i) $b = 3$, $c = 4$, centre at the origin; foci on the x axis.

j) Centre at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

k) Major axis on the x-axis and passes through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

15. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{27} \end{pmatrix} \mathbf{x} = 1$

c) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} = 36$

d) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 576$

e) $\mathbf{x}^T \begin{pmatrix} 5 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 36$

f) $\mathbf{x}^T \begin{pmatrix} 49 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 784$

16. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$

b) Vertices $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 8 \end{pmatrix}$

c) Vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

d) Transverse axis length 8, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.

e) Conjugate axis length 24, foci $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$.

f) Latus rectum length 8, foci $\begin{pmatrix} \pm 3\sqrt{5} \\ 0 \end{pmatrix}$.

g) Latus rectum length 12, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$.

h) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$

i) Vertices $\begin{pmatrix} \pm 7 \\ 0 \end{pmatrix}$, $e = \frac{4}{3}$

j) Foci $\begin{pmatrix} 0 \\ \pm \sqrt{10} \end{pmatrix}$, passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

17. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

18. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the

points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

19. Find the equation of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$
20. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
21. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are
 - a) parallel to x-axis
 - b) parallel to y-axis.
22. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
23. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
 - a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$
 - b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.
24. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.
25. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
26. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m .
27. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
28. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
29. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
30. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
31. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
32. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
33. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
34. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a .
35. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
36. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
37. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
38. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
39. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
40. Find the area lying above x-axis and included between the circle $\mathbf{x}^T \mathbf{x} - 8 \begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
41. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB .
42. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
43. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
44. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
45. Find the area between $y = x^2$ and $y = x$.
46. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
47. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
48. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
49. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
50. Find the area bounded by the curves

$$\{(x, y) : y > x^2, y = |x|\} \quad (5.2.50.1)$$
51. Find the area of the region

$$\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T \mathbf{x} = 9\} \quad (5.2.51.1)$$
52. Find the area of the circle $\mathbf{x}^T \mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.
53. Find the intervals in which the function given

by

$$f(x) = 2x^2 - 3x \quad (5.2.53.1)$$

is

- a) increasing
- b) decreasing.

54. Find the intervals in which the following functions are strictly increasing or decreasing

- a) $x^2 + 2x - 5$
- b) $10 - 6x - 2x^2$
- c) $6 - 9x - x^2$

55. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(1, -1)$.

56. Find the maximum and minimum values, if any, of the following functions given by

- a) $f(x) = (2x - 1)^2 + 3$
- b) $f(x) = 9x^2 + 12x + 2$
- c) $f(x) = -(x - 1)^2 + 10$
- d) $f(x) = x^2$.

57. Find the absolute maximum and absolute minimum value of the following functions in the given intervals

- a) $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$
- b) $f(x) = (x - 1)^2 + 3, x \in (-3, 1)$

58. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2 \quad (5.2.58.1)$$

59. Find the point on the curve $x^2 = 2y$ which is nearest to the point $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

60. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \quad (5.2.60.1)$$

with its vertex at one end of the major axis.

61. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

62. Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} x + 1, & x \geq 1, \\ x^2 + 1, & x < 1, \end{cases} \quad (5.2.62.1)$$

63. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & x \leq 0, \\ 4x + 1, & x > 0 \end{cases} \quad (5.2.63.1)$$

continuous at $x = 0$? What about continuity at $x = 1$?

64. For what value of k is the following function continuous at the given point.

$$f(x) = \begin{cases} kx^2, & x \leq 2, \\ 3, & x > 2, \end{cases} \quad x = 2 \quad (5.2.64.1)$$

65. Find $\frac{dy}{dx}$ in the following

$$x^2 + xy + y^2 = 100 \quad (5.2.65.1)$$

66. Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

67. Examine if Rolle's theorem is applicable to the following function $f(x) = x^2 - 1, x \in [1, 2]$. Can you say something about the converse of Rolle's theorem from this example?

68. Examine the applicability of the mean value theorem for the function in Problem ??.

69. Find $\lim_{x \rightarrow 1} \pi r^2$.

70. Find $\lim_{x \rightarrow 0} f(x)$ where

$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases} \quad (5.2.70.1)$$

71. For some constants a and b , find the derivative of

$$(x - a)(x - b) \quad (5.2.71.1)$$

72. Integrate the following as limit of sums:

$$(i) \int_2^3 x^2 dx$$

$$(ii) \int_1^4 (x^2 - x) dx$$

73. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

74. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

75. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

6 CURVES

6.1 Examples

- Find the value of each of the following polynomials at the indicated value of variables:
 - $q(y) = 3y^3 - 4y + 11$ at $y = 2$.
 - $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$.
- Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
 - $p(t) = 2 + t + 2t^2 - t^3$
 - $p(x) = x^3$
- Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.
- Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.
- Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.
- Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.
- Factorize $x^3 - 23x^2 + 142x - 120$.
- Verify that $3, -1, \frac{1}{3}$, are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.
- Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3 & x \neq 0 \\ 1, & x = 0 \end{cases} \quad (6.1.9.1)$$
 is not continuous at $x = 0$.
- Discuss the continuity of the function f defined by $f(x) = x^2 + x + 1$.
- Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}, x \neq 0$.
- Show that every polynomial function is continuous.
- Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x .
- Discuss the continuity of sine function.
- Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.
- Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$
- Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$
- Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ at $\left(\frac{1}{8}, \frac{1}{8}\right)$.

- Find the equation of the tangent to the curve $\begin{pmatrix} a \sin^3 t \\ b \cos^3 t \end{pmatrix}$ at $t = \frac{\pi}{2}$.
- Find the equation of tangents to the curve $y = \cos(x+y)$, $-2\pi \leq x \leq 2\pi$ that are parallel to the line $\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 0$.
- Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.
- Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.
- Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R} \quad (6.1.23.1)$$

is increasing on \mathbf{R} .

- Prove that the function given by $f(x) = \cos x$ is
 - decreasing in $(0, \pi)$.
 - increasing in $(\pi, 2\pi)$ and
- Find the intervals in which the function

$$f(x) = 4x^3 - 6x^2 - 72x + 30 \quad (6.1.25.1)$$

is

- increasing
 - decreasing.
- Find the intervals in which the function given by

$$f(x) = \sin x, x \in \left[0, \frac{\pi}{2}\right] \quad (6.1.26.1)$$

is

- increasing
 - decreasing.
- Find the intervals in which the function given by

$$f(x) = \sin x + \cos x, x \in [0, 2\pi] \quad (6.1.27.1)$$

is increasing or decreasing.

- Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3 \quad (6.1.28.1)$$

- Find all points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5 \quad (6.1.29.1)$$

- Find the local maxima and minima of the

function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12 \quad (6.1.30.1)$$

31. Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1, \quad x \in [1, 5]. \quad (6.1.31.1)$$

32. Find the absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, \quad x \in [1, 1]. \quad (6.1.32.1)$$

33. A car starts from a point P at time $t = 0$ seconds and stops at point Q . The distance x , in metres, covered by it, in t seconds is given by

$$x = t^2 \left(2 - \frac{t}{3} \right). \quad (6.1.33.1)$$

Find the time taken by it to reach Q and also find the distance between P and Q .

34. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.
35. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.
36. Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5} + 11 \quad (6.1.36.1)$$

is

- a) decreasing
b) increasing

37. Show that the function f given by

$$f(x) = \tan^{-1}(\sin x + \cos x), \quad x > 0 \quad (6.1.37.1)$$

is always an increasing function in $(0, \frac{\pi}{4})$.

38. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

39. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

40. A manufacturer can sell x items at a price of $\left(5 - \frac{x}{500}\right)$ each. The cost price of x items is $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

41. Find the limits

- a) $\lim_{x \rightarrow 1} x^3 - x^2 + 1$
b) $\lim_{x \rightarrow 1} x(x+1)$
c) $\lim_{x \rightarrow 1} 1 + x + x^2 + \dots + x^{10}$

42. Find the limits

- a) $\lim_{x \rightarrow 1} \frac{x^2+1}{x^3+100}$
b) $\lim_{x \rightarrow 2} \frac{x^3-4x^2+4x}{x^2-4}$
c) $\lim_{x \rightarrow 1} \frac{x^3-4x^2+4x}{x^3-2x^2}$
d) $\lim_{x \rightarrow 1} \frac{x^2-5x+6}{x^2-x}$
e) $\lim_{x \rightarrow 1} \left[\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} \right]$

43. Evaluate

- a) $\lim_{x \rightarrow 1} \frac{x^5-1}{x^3-1}$
b) $\lim_{x \rightarrow 2} \frac{\sqrt{1+x}}{x}$

44. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 2x} \quad (6.1.44.1)$$

45. Evaluate

$$\int_{-1}^{\frac{3}{2}} |x \sin(\pi x)| dx \quad (6.1.45.1)$$

46. Evaluate

$$\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad (6.1.46.1)$$

47. Evaluate

$$\int_0^2 e^x dx \quad (6.1.47.1)$$

as a limit of a sum.

48. Evaluate the following integrals:

- a) $\int_2^3 x^2 dx$
b) $\int_4^9 \frac{\sqrt{x}}{(30-x^{\frac{3}{2}})^2} dx$
c) $\int_1^2 \frac{x}{(x+1)(x+2)} dx$
d) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t dt$

49. Evaluate

$$\int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx \quad (6.1.49.1)$$

50. Evaluate

$$\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx \quad (6.1.50.1)$$

51. Evaluate

$$\int_{-1}^2 |x^3 - x| dx \quad (6.1.51.1)$$

52. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx \quad (6.1.52.1)$$

53. Evaluate

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad (6.1.53.1)$$

54. Evaluate

$$\int_{-1}^1 \sin^5 x \cos^4 x dx \quad (6.1.54.1)$$

55. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 + \cos^4 x} dx \quad (6.1.55.1)$$

56. Evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} \quad (6.1.56.1)$$

57. Evaluate

$$\int_0^{\frac{\pi}{2}} \log \sin x dx \quad (6.1.57.1)$$

58. Solve the differential equation

$$y_1 = -4xy^2, \quad y(0) = 1 \quad (6.1.58.1)$$

59. Find the equation of the curve passing through the point $\left(\frac{1}{1}\right)$, whose differential equation is $xdy = (2x^2 + 1)dx$ ($x \neq 0$).

60. Find the equation of a curve passing through the point $\left(\frac{-2}{3}\right)$ given that the slope of the tangent to the curve at any point $\left(\frac{x}{y}\right)$ is $\frac{2x}{y^2}$.

61. In a bank, principal increases continuously at the rate of 5% per year. In how many years

will 100 double itself?

62. Solve

$$2ye^{\frac{x}{y}} dx + \left(y - 2xe^{\frac{x}{y}}\right) dy = 0, \quad y(0) = 1 \quad (6.1.62.1)$$

63. Solve

$$y_1 + y \cot x = 2x + x^2 \cot x \quad (x \neq 0), \quad y\left(\frac{\pi}{2}\right) = 0 \quad (6.1.63.1)$$

64. Find the equation of a curve passing through the point $\left(\frac{0}{1}\right)$. If the slope of the tangent to the curve at any point $\left(\frac{x}{y}\right)$ is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

65. Solve

$$\log y_1 = 3x + 4y, \quad y(0) = 0 \quad (6.1.65.1)$$

6.2 Exercises

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

- $x + 1$
- $x - \frac{1}{2}$
- x
- $x + \pi$
- $5 + 2x$

2. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

3. Determine which of the following polynomials has $(x + 1)$ as a factor:

- $x^3 + x^2 + x + 1$
- $x^4 + x^3 + x^2 + x + 1$
- $x^4 + 3x^3 + 3x^2 + x + 1$
- $x^3 - x^2 - (2 + \sqrt{2}) + \sqrt{2}$.

4. Determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

- $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
- $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
- $p(x) = x^4 - 4x^2 + x + 6, g(x) = x - 3$

5. Factorise :

- $x^3 - 2x^2 - x + 2$
- $x^3 - 3x^2 - 9x - 5$
- $x^3 + 13x^2 + 32x + 20$
- $2y^3 + y^2 - 2y - 1$

6. Find the roots of the following equations:

- $x - \frac{1}{x} = 3, x \neq 0$

- b) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}$, $x \neq -4, 7$
7. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
8. Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 2.
9. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
10. Find the slope of the normal to the curve $\mathbf{x} = a \begin{pmatrix} \cos^3 \theta \\ \sin^3 \theta \end{pmatrix}$ at $\theta = \frac{\pi}{4}$.
11. Find the slope of the normal to the curve $\mathbf{x} = \begin{pmatrix} 1 - a \sin \theta \\ b \cos^2 \theta \end{pmatrix}$ at $\theta = \frac{\pi}{2}$.
12. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.
13. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $(1 \ -1)\mathbf{x} = 11$.
14. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.
15. Find the equations of the tangent and normal to the given curves at the indicated points:
- a) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.
- b) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
- c) $y = x^3$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3 - 2x^5$ find all the points at which the tangent passes through the origin.
19. Find the equation of the normal at the point $\begin{pmatrix} am^2 \\ am^3 \end{pmatrix}$ for the curve $ay^2 = x^3$
20. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $(1 \ 14)\mathbf{x} + 4 = 0$.
21. Find the slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$. Show that the normal at any point θ to the curve $\mathbf{x} = \begin{pmatrix} a \cos \theta + a\theta \sin \theta \\ a \sin \theta - a\theta \cos \theta \end{pmatrix}$ is at a constant distance from the origin.
22. Find the slope of the tangent to the curve $\mathbf{x} = \begin{pmatrix} t^2 + 3t - 8 \\ 2t^2 - 2t - 5 \end{pmatrix}$ at the point $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
23. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes.
24. Find the area under $y = x^4$, $x = 1$, $x = 5$ and x-axis.
25. Find the area bounded by the curve $y = x^3$, $x = -2$, $x = 1$ and the x-axis.
26. Find the area bounded by the curve $y = x|x|$, $x = -1$, $x = 1$ and the x-axis.
27. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \leq x \leq \frac{\pi}{2}$.
28. Show that the function given by $f(x) = 3x + 17$ is increasing on \mathbf{R} .
29. Show that the function given by $f(x) = e^{2x}$ is increasing on \mathbf{R} .
30. Show that the function given by
- $$f(x) = \sin x \quad (6.2.30.1)$$
- is
- a) increasing in $(0, \frac{\pi}{2})$
- b) decreasing in $(\frac{\pi}{2}, \pi)$
31. Find the intervals in which the function given by
- $$f(x) = 2x^3 - 3x^2 - 36x + 7 \quad (6.2.31.1)$$
- is
- a) increasing
- b) decreasing.
32. Find the intervals in which the following functions are strictly increasing or decreasing
- a) $(x + 1)^3 (x - 3)^3$
- b) $-2x^3 - 9x^2 - 12x + 1$
33. Show that
- $$y = \log(1 + x) - \frac{2x}{2 + x}, x > -1, \quad (6.2.33.1)$$
- is an increasing function of x throughout its domain.
34. Find the values of x for which $y = x(x - 2)^2$ is an increasing function.
35. Prove that
- $$y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta \quad (6.2.35.1)$$
- is an increasing function of θ in $[0, \frac{\pi}{2}]$.
36. Prove that the logarithmic function is increasing on $(0, \infty)$.
37. Which of the following functions are decreasing on $[0, \frac{\pi}{2}]$?

- a) $\cos x$
- b) $\cos 2x$
- c) $\cos 3x$
- d) $\tan x$

38. Find the intervals on which

$$f(x) = x^{100} + \sin x - 1 \quad (6.2.38.1)$$

is decreasing.

39. Let I be any interval disjoint from $[1, -1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I .
40. Prove that the function f given by $f(x) = \log \sin x$ is increasing on $(0, \frac{\pi}{2})$ and decreasing on $(\frac{\pi}{2}, \pi)$.
41. Prove that the function f given by $f(x) = \log |\cos x|$ is decreasing on $(0, \frac{\pi}{2})$ and increasing on $(\frac{3\pi}{2}, 2\pi)$.
42. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in \mathbf{R} .
43. Find the interval(s) in which $f(x) = x^2 e^{-x}$ is increasing.
44. Find the maximum and minimum values, if any, of $g(x) = x^3 + 1$.
45. Find the maximum and minimum values, if any of the following functions given by
- a) $h(x) = \sin(2x) + 5$
 - b) $f(x) = |\sin(4x) + 3|$
46. Find the local maximum and minima, if any, of the following functions. Find also the local maximum and local minimum values, as the case may be
- a) $g(x) = x^3 - 3x$
 - b) $h(x) = \sin x + \cos x, x \in (0, \frac{\pi}{2})$
 - c) $f(x) = \sin x - \cos x, x \in (0, 2\pi)$
 - d) $f(x) = x^3 - 6x^2 + 9x + 15$
 - e) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
 - f) $g(x) = \frac{1}{x^2+2}$
 - g) $f(x) = x\sqrt{1-x}, 0 < x < 1$.
47. Prove that the following functions do not have maxima or minima:
- a) $f(x) = e^x$
 - b) $g(x) = \log x$
 - c) $h(x) = x^3 + x^2 + x + 1$
48. Find the absolute maximum and absolute minimum value of the following functions in the given intervals
- a) $f(x) = x^3, x \in (-2, 2)$

b) $f(x) = \sin x + \cos x, x \in (0, \pi)$.

49. Find both the maximum value and the minimum value of

$$3x^4 - 8x^3 + 12x^2 - 48x + 25, x \in [0, 3]. \quad (6.2.49.1)$$

50. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?
51. What is the maximum value of the function $\sin x + \cos x$?
52. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, 1]$.
53. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Find the value of a .
54. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.
55. For all real values of x , the minimum value of

$$\frac{1 - x + x^2}{1 + x + x^2}. \quad (6.2.55.1)$$

56. Find the maximum value of

$$[x(x-1)]^{\frac{1}{3}}. \quad (6.2.56.1)$$

57. Using differentials, find the approximate value of each of the following

- a) $(\frac{17}{81})^{\frac{1}{4}}$
- b) $(33)^{-\frac{1}{5}}$

58. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = 3$.

59. Find the intervals in which the function f given by

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x} \quad (6.2.59.1)$$

is

- a) increasing
- b) decreasing

60. Find the intervals in which the function f given by

$$f(x) = x^3 + \frac{1}{x^3}, \quad x \neq 0 \quad (6.2.60.1)$$

is

- a) increasing
- b) decreasing

61. Find the absolute maximum and minimum val-

ues of the function f given by

$$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi] \quad (6.2.61.1)$$

62. Find the points at which the function f given by

$$f(x) = (x-2)^4(x+1)^3 \quad (6.2.62.1)$$

has

- a) local maxima
- b) local minima
- c) point of inflexion

63. Examine the following functions for continuity.

- a) $f(x) = \frac{1}{x-5}$
- b) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

64. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

- a) $f(x) = \begin{cases} x^3 - 3, & x \leq 2, \\ x^2 + 1, & x > 2 \end{cases}$
- b) $f(x) = \begin{cases} x^4 - 1, & x \leq 1, \\ x^2, & x > 1 \end{cases}$

65. Discuss the continuity of the following functions:

- a) $f(x) = \sin x + \cos x$
- b) $f(x) = \sin x - \cos x$
- c) $f(x) = \sin x \cos x$

66. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

67. Find all points of discontinuity of f , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0, \\ x + 1, & x \geq 0 \end{cases} \quad (6.2.67.1)$$

68. Determine if

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases} \quad (6.2.68.1)$$

is a continuous function.

69. Examine the continuity of

$$f(x) = \begin{cases} \sin x - \cos x, & x \neq 0, \\ -1, & x = 0 \end{cases} \quad (6.2.69.1)$$

70. Find values of k so that the following functions are continuous at the points indicated

- a) $\begin{cases} \frac{k \cos x}{\pi - 2x} & x \neq \frac{\pi}{2}, \\ 3, & x = \frac{\pi}{2} \end{cases}, \quad x = \frac{\pi}{2}$

$$b) \begin{cases} kx + 1 & x \leq \pi, \\ \cos x, & x > \pi, \end{cases} \quad x = \pi$$

71. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

72. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

73. Examine that $\sin |x|$ is a continuous function.

74. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

75. Evaluate the following limits

$$a) \lim_{x \rightarrow 4} \frac{4x+3}{x-2}$$

$$b) \lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x-1}$$

$$c) \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$$

$$d) \lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$$

$$e) \lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

$$f) \lim_{x \rightarrow 0} \frac{ax+b}{cx+1}$$

$$g) \lim_{z \rightarrow 1} \frac{z^3 - 1}{z^6 - 1}$$

$$h) \lim_{x \rightarrow 1} \frac{ax^2 + bx + 3}{cx^2 + bx + a}, \quad a + b + c \neq 0$$

$$i) \lim_{x \rightarrow 2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2}$$

$$j) \lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

$$k) \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, \quad a, b \neq 0$$

$$l) \lim_{x \rightarrow \pi} \frac{\sin(\pi-x)}{\pi(\pi-x)}, \quad a, b \neq 0$$

$$m) \lim_{x \rightarrow 0} \frac{\cos x}{\pi-x}$$

$$n) \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

$$o) \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

$$p) \lim_{x \rightarrow 0} x \sec x$$

$$q) \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}, \quad a, b, a + b \neq 0$$

$$r) \lim_{x \rightarrow 0} \csc - \cot x$$

$$s) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

76. Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ where

$$f(x) = \begin{cases} 2x + 3 & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases} \quad (6.2.76.1)$$

77. Let a_1, a_2, \dots, a_n be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n) \quad (6.2.77.1)$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$, compute $\lim_{x \rightarrow a} f(x)$.

78. If

$$\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi, \quad (6.2.78.1)$$

evaluate $\lim_{x \rightarrow 1} f(x)$.

79. If

$$f(x) = \begin{cases} mx^2 + n & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1, \end{cases} \quad (6.2.79.1)$$

for what integers m and n does both $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ exist?

80. Integrate the following as limit of sums:

- (i) $\int_{-1}^1 e^x dx$
 (ii) $\int_{-1}^1 (x - e^{2x}) dx$

81. Evaluate the following definite integrals

- (i) $\int_2^3 \frac{1}{x} dx$
 (ii) $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$
 (iii) $\int_0^{\frac{\pi}{4}} \sin 2x dx$
 (iv) $\int_0^{\frac{\pi}{2}} \cos 2x dx$
 (v) $\int_4^5 e^x dx$
 (vi) $\int_0^{\frac{\pi}{4}} \tan 2x dx$
 (vii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc 2x dx$
 (viii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} dx$
 (ix) $\int_0^1 \frac{dx}{1+x^2} dx$
 (x) $\int_2^3 \frac{dx}{x^2-1} dx$
 (xi) $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
 (xii) $\int_2^3 \frac{x}{1+x^2} dx$
 (xiii) $\int_0^1 \frac{2x+3}{5x^2+1} dx$
 (xiv) $\int_0^1 xe^{x^2} dx$
 (xv) $\int_1^2 \frac{5x^2}{x^2+4x+3} dx$
 (xvi) $\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$
 (xvii) $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$
 (xviii) $\int_0^2 \frac{6x+3}{x^2+4} dx$
 (xix) $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$

82. Find $\int_{\frac{1}{2}}^{\sqrt{3}} \frac{dx}{1+x^2}$

83. Find $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$

84. Evaluate the following definite integrals

- (i) $\int_0^{\frac{\pi}{2}} \cos^2 x dx$
 (ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$
 (iii) $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$
 (iv) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$
 (v) $\int_0^1 x(1-x)^n dx$
 (vi) $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

(vii) $\int_0^1 x \sqrt{2-x} dx$

(viii) $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx$

(ix) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$

(x) $\int_0^{\pi} \frac{x}{1+\sin x} dx$

(xi) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$

(xii) $\int_0^{2\pi} \cos^5 x dx$

(xiii) $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

(xiv) $\int_0^{\pi} \log(1 + \cos x) dx$

(xv) $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$

85. Find the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx \quad (6.2.85.1)$$

86. Find the value of

$$\int_0^{\frac{\pi}{2}} \log \left(\frac{4 + 3 \sin x}{4 + 3 \cos x} \right) dx \quad (6.2.86.1)$$

87. Evaluate the following definite integrals

- (i) $\int_{\frac{\pi}{4}}^{\pi} \left(\frac{1-\sin x}{1-\cos x} \right) dx$
 (ii) $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$
 (iii) $\int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$
 (iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$
 (v) $\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$
 (vi) $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$
 (vii) $\int_{\frac{\pi}{2}}^{\pi} \sin 2x \tan^{-1}(\sin x) dx$
 (viii) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

88. Prove that

- (i) $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$
 (ii) $\int_0^1 e^x dx = 1$
 (iii) $\int_{-1}^1 x^{17} \cos^4 x dx = 0$
 (iv) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$
 (v) $\int_{\frac{\pi}{4}}^{\pi} 2 \tan^3 x dx = 1 - \log 2$
 (vi) $\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$

89. Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum.

90. Find the value of $\int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx = \frac{\pi}{2} - 1$

91. Solve

- (i) $(x^3 + x^2 + x + 1)y_1 = 2x^2 + x \quad y(0) = 1$
 (ii) $(x(x^2 - 1))y_1 = 2x^2 + x \quad y(2) = 0$
 (iii) $\cos(y_1) = y \tan x; y(0) = 1$

92. Find the equation of a curve passing through the origin and whose differential equation is

$$y_1 = e^x \sin x$$

93. For the differential equation $xyy_1 = (x+2)(y+2)$, find the solution curve passing through the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
94. Find the equation of a curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.
95. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$.
96. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.
97. In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if ₹100 double itself in 10 years.
98. In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹1000 is deposited with this bank, how much will it worth after 10 years.
99. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?
100. Solve
- $(x+y) dy + (x-y) dx = 0, y(1) = 1$
 - $x^2 dy + (xy + y^2) dx = 0, y(1) = 1$
 - $\left[x \sin^2 \left(\frac{y}{x} - y \right) \right] dx + x dy = 0, y(1) = \frac{\pi}{4}$
 - $y_1 - \frac{y}{x} + \csc \left(\frac{y}{x} \right) = 0, y(1) = 0$
 - $2xy + y^2 - 2x^2 y_1 = 0, y(1) = 2$
101. Solve
- $y_1 + 2y \tan x = \sin x, y \left(\frac{\pi}{3} \right) = 0$
 - $(1+x^2)y_1 + 2xy = \frac{1}{1+x^2}, y(0) = 1$
 - $y_1 - 3y \cot x = \sin 2x, y \left(\frac{\pi}{2} \right) = 2$
102. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point $\begin{pmatrix} x \\ y \end{pmatrix}$ is equal to the sum of the coordinates of the point.
103. Find the equation of a curve passing through the point $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.
104. Find the equation of the curve passing through the point $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$ whose differential equation is $\sin x \cos y dx + \cos x \sin y dy = 0$
105. Solve
- $$(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0, \quad y(0) = 1 \quad (6.2.105.1)$$
106. Solve
- $$(x - y)(dx - dy) = dx - dy, \quad y(0) = -1 \quad (6.2.106.1)$$
107. Solve
- $$y_1 + y \cot x = 4x \csc x \quad y \left(\frac{\pi}{2} \right) = 0 \quad (6.2.107.1)$$
108. Solve
- $$(x + 1)y_1 = 2e^{-y} - 1 \quad y(0) = 0 \quad (6.2.108.1)$$
109. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

7 MISCELLANEOUS EXERCISES

- If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
- An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?
- The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
- An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.
7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.
9. Prove that the curves $x = y^2$ and $kx = y$ cut at right angles if $8k^2 = 1$
10. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $\left(\frac{at^2}{2}, 2at\right)$.
11. Find the equations of the tangent and normal to the hyperbola $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ at the point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.
12. Find the area of the smaller part of the circle $\mathbf{x}^x = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
13. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.
14. The focus of a parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB.
15. A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?
16. A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P is taken on the rod in such a way that $AP = 6$ cm. Show that the locus of P is an ellipse
17. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.
18. Find the rate of change of the area of a circle per second with respect to its radius when $r = 5$ cm.
19. The volume of a cube is increasing at a rate of 9 cu cm per second. How fast is the surface area increasing when the length of an edge is 10 cm?
20. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
21. The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2 cm/minute. When $x = 10$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.
22. The total cost $C(x)$ in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
23. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.
24. Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm
25. The volume of a cube is increasing at the rate of $8 \text{ cm}^3/\text{s}$. How fast is the surface area increasing when the length of an edge is 12 cm?
26. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
27. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
28. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?
29. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

30. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
31. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
32. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
33. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
34. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y -coordinate is changing 8 times as fast as the x -coordinate.
35. The radius of an air bubble is increasing at the rate of 12 cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
36. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}2x + 1$. Find the rate of change of its volume with respect to x .
37. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
38. The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.
39. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.
40. Find the rate of change of the area of a circle with respect to its radius r at $r = 6$ cm.
41. The total revenue in | received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x =$
- 15.
42. For what vaues of a the function given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?
43. Let AP and BQ be two vertical poles at points A and B respectively. If $AP = 16$ m, $BQ = 22$ m, and $AB = 20$ m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.
44. If length of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum.
45. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.
46. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.
47. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.
48. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.
49. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?
50. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.
51. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.
52. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area.
53. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
54. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$

of the volume of the sphere.

55. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.
56. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1} \sqrt{2}$.
57. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1} \frac{1}{3}$.
58. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.
59. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.
60. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Find the rate at which the depth of the wheat is increasing.
61. Let f be a function defined on $[a, b]$ such that $f'(x) = 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b) .
62. Prove that every rational function is continuous.
63. Prove that the function defined by $f(x) = \tan x$ is a continuous function.

8 CALCULUS

8.1 Examples

1. Find the derivative of the function given by $f(x) = \sin(x^2)$.
2. Find the derivative of $\tan(2x + 3)$.
3. Find $\frac{dy}{dx}$ if $y + \sin y = \cos x$.
4. Find the derivative of $f(x) = \sin^{-1} x$ assuming it exists.
5. Find the derivative of $f(x) = \tan^{-1} x$ assuming it exists.
6. Differentiate the following with respect to x .
 - a) e^x
 - b) $\sin(\log x)$, $x > 0$
 - c) $\cos^{-1}(e^x)$
 - d) $e^{\cos x}$.
7. Differentiate

$$\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} \quad (8.1.7.1)$$

8. Differentiate a^x w.r.t. x , where a is a positive constant.
9. Differentiate $x^{\sin x}$, $x > 0$ w.r.t. x .
10. Find $\frac{dy}{dx}$, if $Y^x + x^y + x^x = a^b$.
11. Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.
12. Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.
13. Find $\frac{dy}{dx}$, if $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$.
14. Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
15. Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.
16. If $y = A \sin x + B \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.
17. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.
18. If $y = \sin^{-1} x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.
19. Differentiate the following with respect to x .
 - a) $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$
 - b) $e^{\sec^2 x} + 3 \cos^{-1} x$
 - c) $\log_7(\log x)$
 - d) $\cos^{-1}(\sin x)$
 - e) $\tan^{-1}\left(\frac{1}{1+\cos x}\right)$
 - f) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$
20. Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $x \in (0, \pi)$.
21. For a positive constant a , find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, x = \left(t + \frac{1}{t}\right)^a \quad (8.1.21.1)$$

22. Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.
23. Find the derivative of $\sin x$ at $x = 0$.
24. Find the derivative of $f(x) = \frac{1}{x}$.
25. Find the derivative of $f(x) = 1 + x + x^2 + x^3 + \dots + x^{50}$ at $x = 1$.
26. Find the derivative of $f(x) = \frac{x+1}{x}$.
27. Find the derivative of $\sin x$.
28. Find the derivative of $\tan x$.
29. Find the derivative of $f(x) = \sin^2 x$.
30. Find the derivative of f from the first principle, where f is given by
 - a) $f(x) = \frac{2x+3}{x-2}$
 - b) $f(x) = x + \frac{1}{x}$
31. Find the derivative of f from the first principle, where $f(x)$ is
 - a) $\sin x + \cos x$
 - b) $x \sin x$
32. Compute the derivative of
 - a) $f(x) = \sin 2x$
 - b) $g(x) = \cot x$
33. Find the derivative of

- a) $\frac{x^5 - \cos x}{\sin x}$
 b) $\frac{x + \cos x}{\sin x}$
34. Write an an anti-derivative for each of the following functions using the method of inspection:
- a) $\cos 2x$
 b) $3x^2 + 4x^3$
 c) $\frac{1}{x}, x \neq 0$
35. Find the following integrals:
- a) $\int \frac{x^3 - 1}{x^2} dx$
 b) $\int (x^{\frac{3}{2}} + 1) x^2 dx$
 c) $\int (x^{\frac{2}{3}} + 2e^x - \frac{1}{x}) x^2 dx$
36. Find the following integrals:
- a) $\int (\sin x + \cos x) dx$
 b) $\int \csc x (\csc x + \cot x) dx$
 c) $\int \frac{1 - \sin x}{\cos^2 x} dx$
37. Find an anti-derivative F of f defined by $f(x) = 4x^3 - 6$, where $F(0) = 3$.
38. Integrate the following functions w.r.t x :
- a) $\sin mx$
 b) $2x \sin(x^2 + 1)$
 c) $\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$
 d) $\frac{\sin(\tan^{-1} x)}{1 + x^2}$
39. Find the following integrals:
- a) $\int \sin^3 x \cos^2 x dx$
 b) $\int \frac{\sin x}{\sin(x+a)} dx$
 c) $\int \frac{1}{1 + \tan x} dx$
40. Find
- a) $\int \cos^2 x dx$
 b) $\int \sin 2x \cos 3x dx$
 c) $\int \sin^3 x dx$
41. Find the following integrals
- a) $\int \frac{dx}{x^2 - 16}$
 b) $\int \frac{dx}{\sqrt{2x - x^2}}$
42. Find the following integrals
- a) $\int \frac{dx}{x^2 - 6x + 13}$
 b) $\int \frac{dx}{3x^2 + 13x - 10}$
 c) $\int \frac{dx}{\sqrt{5x^2 - 2x}}$
43. Find the following integrals
- a) $\int \frac{x+2}{2x^2 + 6x + 5} dx$
 b) $\int \frac{x+3}{\sqrt{5-4x-x^2}} dx$
44. Find

$$\int \frac{dx}{(x+1)(x+2)} \quad (8.1.44.1)$$

45. Find

$$\int \frac{x^2 + 1}{x^2 - 5x + 6} dx \quad (8.1.45.1)$$

46. Find

$$\int \frac{3x - 2}{(x+1)^2 (x+3)} dx \quad (8.1.46.1)$$

47. Find

$$\int \frac{x^2}{(x^2 + 1)^2 (x^2 + 4)} dx \quad (8.1.47.1)$$

48. Find

$$\int \frac{(3 \sin \phi - 2) \cos \phi}{5 - \cos^2 \phi - 4 \sin \phi} d\phi \quad (8.1.48.1)$$

49. Find

$$\int \frac{x^2 + x + 1}{(x+2)(x^2 + 1)} dx \quad (8.1.49.1)$$

50. Find

$$\int x \cos x dx \quad (8.1.50.1)$$

51. Find

$$\int \log x dx \quad (8.1.51.1)$$

52. Find

$$\int x e^x dx \quad (8.1.52.1)$$

53. Find

$$\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx \quad (8.1.53.1)$$

54. Find

$$\int e^x \sin x dx \quad (8.1.54.1)$$

55. Find

a) $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$
 b) $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

56. Find

$$\int \sqrt{x^2 + 2x + 5} dx \quad (8.1.56.1)$$

57. Find

$$\int \sqrt{3 - 2x - x^2} dx \quad (8.1.57.1)$$

58. Find

$$\int \cos 6x \sqrt{1 + \sin 6x} dx \quad (8.1.58.1)$$

59. Find

$$\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx \quad (8.1.59.1)$$

60. Find

$$\int \frac{x^4}{(x-1)(x^2+1)} dx \quad (8.1.60.1)$$

61. Find

$$\int [\log(\log x)] + \frac{1}{(\log x)^2} dx \quad (8.1.61.1)$$

62. Find

$$\int [\sqrt{\cot x} + \sqrt{\tan x}] dx \quad (8.1.62.1)$$

63. Find

$$\int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4(2x)}} dx \quad (8.1.63.1)$$

64. Verify that $y = e^{-3x}$ is a solution of the differential equation

$$y_2 + y_1 - 6y = 0 \quad (8.1.64.1)$$

65. Verify that $y = a \cos x + b \sin x$ is a solution of the differential equation

$$y_2 + y = 0 \quad (8.1.65.1)$$

66. Form the differential equation representing the family of curves $y = a \sin(x + b)$, where a, b are arbitrary constants.

67. Find the general solution of the differential equation

$$y_1 = \frac{x+1}{2-y} \quad (8.1.67.1)$$

68. Find the general solution of the differential equation

$$y_1 = \frac{1+y^2}{1+x^2} \quad (8.1.68.1)$$

69. Show that the differential equation $(x-y)y_1 = x+2y$ is homogeneous and solve it.70. Solve $x \cos\left(\frac{x}{y}\right) y_1 = y \cos\left(\frac{y}{x}\right) + x$.

71. Show that the family of curves for which the

slope of the tangent at any point $\left(\frac{x}{y}\right)$ on it is $\frac{x^2+y^2}{xy}$, is given by $x^2 - y^2 = c$.

72. Solve

$$y_1 - y = \cos x \quad (8.1.72.1)$$

73. Solve

$$xy_1 + 2y = x^2 \quad (8.1.73.1)$$

74. Solve

$$y dx - (x + 2y^2) dy = 0 \quad (8.1.74.1)$$

75. Solve

$$y dx - (x + 2y^2) dy = 0 \quad (8.1.75.1)$$

76. Verify that $y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$y_2 - 2ay_1 + (a^2 + b^2)y = 0 \quad (8.1.76.1)$$

77. Solve

$$(x dy - y dx) y \sin\left(\frac{y}{x}\right) = (y dx + x dy) x \cos\left(\frac{y}{x}\right) \quad (8.1.77.1)$$

78. Solve the differential equation

$$(\tan^{-1} x - x) dy = (1 + y^2) dx \quad (8.1.78.1)$$

8.2 Exercises

1. Differentiate the following functions with respect to x

a) $\sin(x^2 + 5)$

b) $\cos(\sin x)$

c) $\sin(ax + b)$

d) $\sec(\tan \sqrt{x})$

e) $\frac{\sin(ax+b)}{\cos(cx+d)}$

f) $\cos x^3 \sin^2(x^5)$

g) $2\sqrt{\cot(x^2)}$

h) $\cos(\sqrt{x})$

2. Find $\frac{dy}{dx}$ in the following:

a) $2x + 3y = \sin x$

b) $2x + 3y = \sin y$

c) $ax + by^2 = \cos y$

d) $xy + y^2 = \tan x + y$

e) $x^3 + x^2y + xy^2 + y^3 = 81$

f) $\sin^2 y + \cos xy = \kappa$

g) $\sin^2 x + \cos^2 y = 1$

- h) $y = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$
 i) $y = \tan^{-1} \left(\frac{3x-x^2}{1-3x^2} \right), x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$
 j) $y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), 0 < x < 1$
 k) $y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right) < x < 1$
 l) $y = \cos^{-1} \left(\frac{2x}{1+x^2} \right), -1 < x < 1$
 m) $y = \sin^{-1} \left(2x \sqrt{1-x^2} \right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
 n) $y = \sec^{-1} \left(\frac{1}{2x^2-1} \right), 0 < x < \frac{1}{\sqrt{2}}$

3. Differentiate the following w.r.t. x :

- a) $\frac{e^x}{\sin x}$
 b) $e^{\sin^{-1} x}$
 c) e^{x^3}
 d) $\sin(\tan^{-1} e^{-x})$
 e) $\log(\cos e^x)$
 f) $e^x + e^{x^2} + \dots + e^{x^5}$
 g) $\sqrt{e^{\sqrt{x}}}, x > 0$
 h) $\log(\log x), x > 1$
 i) $\frac{\cos x}{\log x}, x > 0$
 j) $\cos(\log x + e^x), x > 0$

4. Differentiate the following w.r.t. x

- a) $\cos x \cos 2x \cos 3x$
 b) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
 c) $(\log x)^{\cos x}$
 d) $x^x - 2^{\sin x}$
 e) $(x+3)^2 (x+4)^3 (x+5)^4$
 f) $\left(x + \frac{1}{x}\right)^x + x^{1+\frac{1}{x}}$
 g) $(\log x)^x + (\sin x)^{\cos x}$
 h) $(\sin x)^x + \sin^{-1} \sqrt{x}$
 i) $x^{\sin x} + (\sin x)^{\cos x}$
 j) $x^{\cos x} + \frac{x^2+1}{x^2-1}$
 k) $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$
 l) $x^y + y^x = 1$
 m) $y^x = x^y$
 n) $(\cos x)^y = (\cos y)^x$
 o) $xy = e^{x-y}$

5. Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

6. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:

- a) by using product rule
 b) by expanding the product to obtain a single polynomial
 c) by logarithmic differentiation.

Do they all give the same answer?

7. Without eliminating the parameter, find $\frac{dy}{dx}$ in the following

- a) $x = 2at^2, y = at^4$
 b) $x = a \cos \theta, y = b \cos \theta$
 c) $x = \sin t, y = \cos t$
 d) $x = 4t, y = \frac{4}{t}$
 e) $x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$
 f) $x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$
 g) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
 h) $x = a \left(\cos t + \log \tan \frac{t}{2} \right), y = a \sin t$
 i) $x = a \sec \theta, y = b \tan \theta$
 j) $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$
 k) If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$

show that $\frac{dy}{dx} = -\frac{y}{x}$.

8. Find the second order derivatives of the following functions

- a) $x^2 + 3x + 2$
 b) x^{20}
 c) $x \cos x$
 d) $\log x$
 e) $x^3 \log x$
 f) $x^x \sin 5x$
 g) $e^{6x} \cos 3x$
 h) $\tan^{-1} x$
 i) $\log(\log x)$
 j) $\sin(\log x)$

9. If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2 y}{dx^2} + y = 0$

10. If $y = \cos^{-1} x$, find $\frac{d^2 y}{dx^2}$ in terms of y .

11. If $y = 3 \cos(\log x) + 4 \sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

12. If $y = Ae^{mx} + Be^{nx}$, show that $y_2 - (m+n)y_1 + mny = 0$.

13. If $y = 500e^{7x} + 600e^{-7x}$, show that $y_2 = 49y$

14. If $e^y (x+1) = 1$, show that $y_2 = y_1^2$

15. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)y_2 + 2x(x^2 + 1)y_1 = 2$

16. If $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

17. Verify mean value theorem, if $f(x) = x^3 - 5x^2 - 3x, x \in [a, b]$ where $a = 1, b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$

18. Differentiate the following functions w.r.t x

- a) $(3x^2 - 9x + 5)^9$
 b) $\sin^3 x + \cos^6 x$
 c) $(5x)^{3 \cos 2x}$
 d) $\sin^{-1}(x\sqrt{x}), 0 \leq x \leq 1$
 e) $\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$

f) $\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$

g) $(\log x)^{\log x}, x > 1$

h) $\cos(a \cos x + b \sin x)$, for some constant a and b .

i) $(\sin x - \cos x)^{\sin x - \cos x}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

j) $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

k) $x_+^{x^2-3} (x-3)^{x^2}$, for $x > 3$.

19. Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < x < \frac{\pi}{2}$.

20. Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}, 0 < x < 1$

21. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} \quad (8.2.21.1)$$

22. If $\cos y = x \cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

23. if $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find y_2

24. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

25. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n .

26. Using the fact that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad (8.2.26.1)$$

show that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad (8.2.26.2)$$

27. If

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}, \quad (8.2.27.1)$$

prove that

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} \quad (8.2.27.2)$$

28. If $y = e^{a \cos^{-1} x}, -1 \leq x \leq 1$, show that

$$(1-x^2)y_2 - xy_1 - ay^2 = 0. \quad (8.2.28.1)$$

29. Find the derivative of the following functions from the first principle:

a) $x^3 - 27$

b) $\frac{1}{x^2}$
c) $\frac{x+1}{x-1}$

30. For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1. \quad (8.2.30.1)$$

prove that $f'(1) = 100f'(0)$.

31. Find the derivative of

$$x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n \quad (8.2.31.1)$$

for some fixed real number a .

32. For some constants a and b , find the derivative of

a) $(ax^2 + b)^2$

b) $\frac{x-a}{x-b}$

33. Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a .

34. Find the derivative of

a) $2x - \frac{3}{4}$

b) $(5x^3 + 3x - 1)(x - 1)$

c) $x^{-3}(3 - 4x^{-5})$

d) $x^5(x - 6x^{-9})$

e) $x^{-4}(3 - 4x^{-5})$

f) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

35. Find the derivative of $\cos x$ from the first principle.

36. Find the derivative of the following functions:

a) $\sin x \cos x$

b) $\sec x$

c) $5 \sec x + 4 \cos x$

d) $\csc x$

e) $3 \cot x + 5 \csc x$

f) $5 \sin x - 6 \cos x + 7$

g) $2 \tan x - 7 \sec x$

37. Find the derivative of the following functions:

(i) $(-x)^{-1}$

(ii) $\sin(x+1)$

(iii) $\cos\left(x - \frac{\pi}{8}\right)$

(iv) $\frac{ax+b}{cx+d}$

(v) $(px+q)\left(\frac{r}{x} + s\right)$

(vi) $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

(vii) $\frac{1}{ax^2+bx+c}$

(viii) $\frac{ax+b}{px^2+qx+r}$

(ix) $\frac{px^2+qx+r}{ax+b}$

(x) $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

- (xi) $4\sqrt{x} - 2$
 (xii) $(ax + b)^n$
 (xiii) $(ax + b)^n (cx + d)^m$
 (xiv) $\sin(x + a)$
 (xv) $\csc x \cot x$
 (xvi) $\frac{\cos x}{1 + \sin x}$
 (xvii) $\frac{\sin x - \cos x}{\sec x - 1}$
 (xviii) $\frac{\sec x + 1}{\sin^n x}$
 (xix) $\sin^n x$
 (xx) $\frac{a+b \sin x}{c+d \cos x}$
 (xxi) $\frac{\cos x}{\sin(x+a)}$
 (xxii) $x^4 (5 \sin x - 3 \cos x)$
 (xxiii) $(x^2 + 1) \cos x$
 (xxiv) $(ax^2 + \sin x)(p + q \cos x)$
 (xxv) $(x - \tan x)(x + \cos x)$
 (xxvi) $\frac{4x+5 \sin x}{3x+7 \cos x}$
 (xxvii) $\frac{x^2 \cos(\frac{\pi}{4})}{\sin x}$
 (xxviii) $(x)(1 + \tan x)$
 (xxix) $(x + \sec x)(x - \tan x)$
 (xxx) $\frac{x}{\sin^n x}$

38. Find anti-derivative of each of the following functions

- a) $\sin 2x$
 b) $\cos 2x$
 c) e^{2x}
 d) $(ax + b)^2$
 e) $\sin 2x - 4e^{2x}$

39. Find the following integrals:

- a) $\int 4e^{3x} + 1, dx$
 b) $\int x^2 \left(1 - \frac{1}{x^2}\right), dx$
 c) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$
 d) $\int (ax^2 + bx + c), dx$
 e) $\int (2x^2 + e^x), dx$
 f) $\int \frac{x^3 + 5x^2 - 4}{x^2}, dx$
 g) $\int \frac{x^3 - x^2 + x - 1}{x - 1}, dx$
 h) $\int (1 - x) \sqrt{x}, dx$
 i) $\int \sqrt{x} (3x^2 + 2x + 3), dx$
 j) $\int (2x - 3 \cos x + e^x), dx$
 k) $\int (2x^2 - 3 \sin x + 5 \sqrt{x}), dx$
 l) $\int \sec x (\sec x + \tan x), dx$
 m) $\int \frac{\sec^2 x}{\csc^2 x}, dx$

40. Find anti-derivative of

$$\sqrt{x} + \frac{1}{\sqrt{x}}$$

41. If

$$\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}, \quad f(2) = 0 \quad (8.2.41.1)$$

Find $f(x)$.

42. Integrate the following functions:

- (i) $\frac{2x}{1+x^2}$
 (ii) $\frac{(\log x)^2}{x}$
 (iii) $\frac{1}{x+x \log x}$
 (iv) $\sin x \sin(\cos x)$
 (v) $\sin(ax + b) \cos ax + b$
 (vi) $\sqrt{ax + b}$
 (vii) $x \sqrt{x + 2}$
 (viii) $x \sqrt{1 + 2x^2}$
 (ix) $(4x + 2) \sqrt{x^2 + x + 1}$
 (x) $\frac{1}{x - \sqrt{x}}$
 (xi) $\frac{x}{\sqrt{x+4}}, \quad x > 0$
 (xii) $(x^3 - 1)^{\frac{1}{3}} x^5$
 (xiii) $\frac{x^2}{(2+3x^3)^3}$
 (xiv) $\frac{1}{x(\log x)^m} \quad x > 0, m \neq 1$
 (xv) $\frac{x}{9-4x^2}$
 (xvi) e^{2x+3}
 (xvii) $\frac{x}{e^{x^2}}$
 (xviii) $e^{\frac{\tan^{-1} x}{1+x^2}}$
 (xix) $\frac{e^{2x}-1}{e^{2x}x+1}$
 (xx) $\frac{e^{2x}-e^{-2x}}{e^2 x + e^{-2x}}$
 (xxi) $\tan^2(2x - 3)$
 (xxii) $\sec^2(7 - 4x)$
 (xxiii) $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$
 (xxiv) $\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$
 (xxv) $\frac{1}{\cos^2 x (1 - \tan x)^2}$
 (xxvi) $\frac{\cos \sqrt{x}}{\sqrt{x}}$
 (xxvii) $\sqrt{\sin 2x} \cos 2x$
 (xxviii) $\frac{\cos x}{\sqrt{1+\sin x}}$
 (xxix) $\cot x \log x \sin x$
 (xxx) $\frac{\sin x}{1+\cos x}$
 (xxxix) $\frac{\sin x}{(1+\cos x)^2}$
 (xxxii) $\frac{1}{1+\cot x}$
 (xxxiii) $\frac{1}{1-\tan x}$
 (xxxiv) $\frac{\sqrt{\tan x}}{\sin x \cos x}$
 (xxxv) $\frac{x}{(1+\log x)^2}$
 (xxxvi) $\frac{x}{(x+1)(x+\log x)^2}$
 (xxxvii) $\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$

43. Find $\int \frac{10x^9 + 10^x \ln 10}{x^{10} + 10^x}, dx$

44. Find $\int \frac{dx}{\sin^2 x \cos^2 x}, dx$

45. Find the integrals of the following functions:

- (i) $\sin^2 (2x + 5)$
- (ii) $\sin 3x \cos 4x$
- (iii) $\cos 2x \cos 4x \cos 6x$
- (iv) $\sin^3 (2x + 1)$
- (v) $\sin^3 x \cos^3 x$
- (vi) $\sin x \sin 2x \sin 3x$
- (vii) $\sin 4x \sin 8x$
- (viii) $\frac{1 - \cos x}{1 + \cos x}$
- (ix) $\frac{\cos x}{1 + \cos x}$
- (x) $\sin^4 x$
- (xi) $\cos^4 x$
- (xii) $\frac{\sin^2 x}{1 + \cos x}$
- (xiii) $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
- (xiv) $\frac{\cos x - \sin x}{1 + \sin 2x}$
- (xv) $\tan^3 2x \sec 2x$
- (xvi) $\tan^4 x$
- (xvii) $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
- (xviii) $\frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}$
- (xix) $\frac{\cos^2 x}{(\cos x + \sin x)^2}$
- (xx) $\sin^{-1} (\cos x)$
- (xxi) $\frac{1}{\cos(x-a) \cos(x-b)}$

46. Find $\frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}$

47. Find $\frac{e^x(1+x)}{\cos^2(e^x)}$

48. Integrate the following functions:

- (i) $\frac{3x^2}{x^6 + 1}$
- (ii) $\frac{1}{\sqrt{1+4x^2}}$
- (iii) $\frac{1}{\sqrt{1+(2-x)^2}}$
- (iv) $\frac{1}{\sqrt{9-25x^2}}$
- (v) $\frac{3x}{1+2x^4}$
- (vi) $\frac{x^2}{1-x^6}$
- (vii) $\frac{x-1}{\sqrt{x^2-1}}$
- (viii) $\frac{x^2}{\sqrt{x^6+6^6}}$
- (ix) $\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$
- (x) $\frac{1}{\sqrt{x^2+2x+2}}$
- (xi) $\frac{1}{9x^2+6x+5}$
- (xii) $\frac{1}{\sqrt{7-6x-x^2}}$
- (xiii) $\frac{1}{\sqrt{(x-1)(x-2)}}$
- (xiv) $\frac{1}{\sqrt{8+3x-x^2}}$
- (xv) $\frac{1}{\sqrt{(x-a)(x-b)}}$
- (xvi) $\frac{4x+1}{\sqrt{2x^2+x-3}}$
- (xvii) $\frac{x+2}{\sqrt{x^2-1}}$
- (xviii) $\frac{5x-2}{1+2x+3x^2}$
- (xix) $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

(xx) $\frac{x+2}{\sqrt{4x-x^2}}$

(xxi) $\frac{x+2}{\sqrt{x^2+2x+3}}$

(xxii) $\frac{x+3}{3x^2-2x-5}$

(xxiii) $\frac{5x+3}{\sqrt{x^2+4x+10}}$

49. Find $\int \frac{dx}{x^2+2x+2} dx$

50. Find $\int \frac{dx}{\sqrt{9x-4x^2}} dx$

51. Integrate the following:

- (i) $\frac{x}{(x+1)(x+2)}$
- (ii) $\frac{1}{x^2-9}$
- (iii) $\frac{3x-1}{(x-1)(x-2)(x-3)}$
- (iv) $\frac{x}{(x-1)(x-2)(x-3)}$
- (v) $\frac{1-x^2}{x^2+3x+2}$
- (vi) $\frac{1-x^2}{x(1-2x)}$
- (vii) $\frac{x}{(x^2+1)(x-1)}$
- (viii) $\frac{(x+2)(x-1)^2}{3x+5}$
- (ix) $\frac{x^3-x^2-x+1}{2x-3}$
- (x) $\frac{5x}{(x^2-1)(2x+3)}$
- (xi) $\frac{5x}{(x^2-4)(x+1)}$
- (xii) $\frac{x^3+x+1}{x^2-1}$
- (xiii) $\frac{1}{(1-x)(1+x^2)}$
- (xiv) $\frac{3x-1}{(x+2)^2}$
- (xv) $\frac{1}{x^4-1}$
- (xvi) $\frac{x}{x^n+1}$
- (xvii) $\frac{\cos x}{(1 \sin x)(2-\sin x)}$
- (xviii) $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$
- (xix) $\frac{1}{e^x-1}$

52. Find $\int \frac{x dx}{(x-1)(x-2)}$

53. Find $\int \frac{dx}{x(x^2+1)}$

54. Integrate the following functions:

- (i) $x \sin x$
- (ii) $x \sin 3x$
- (iii) $x^2 e^x$
- (iv) $x \log x$
- (v) $x \log 2x$
- (vi) $x^2 \log x$
- (vii) $x \sin^{-1} x$
- (viii) $x \tan^{-1} x$
- (ix) $x \cos^{-1} x$
- (x) $(\sin^{-1} x)^2$
- (xi) $\frac{\cos^{-1} x}{\sqrt{1-x^2}}$
- (xii) $x \sec^2 x$
- (xiii) $\tan^{-1} x$
- (xiv) $x (\log x)^2$
- (xv) $(x^2 + 1) \log x$

- (xvi) $e^x (\sin x + \cos x)$
 (xvii) $\frac{xe^x}{(1+x)^2}$
 (xviii) $e^x \left(\frac{1+\sin x}{1+\cos x} \right)$
 (xix) $e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$
 (xx) $\frac{(x-3)e^x}{(x-1)^3}$
 (xxi) $e^2 x \sin x$
 (xxii) $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$

55. Find $\int x^2 e^{x^3} dx$

56. Find $\int e^x \sec x (1 + \tan x) dx$

57. Integrate the following functions:

- (i) $\sqrt{4-x^2}$
 (ii) $\sqrt{1-4x^2}$
 (iii) $\sqrt{x^2+4x+6}$
 (iv) $\sqrt{x^2+4x+1}$
 (v) $\sqrt{1-4x-x^2}$
 (vi) $\sqrt{x^2+4x-5}$
 (vii) $\sqrt{1+3x-x^2}$
 (viii) $\sqrt{x^2+3x}$
 (ix) $\sqrt{1+\frac{x^2}{9}}$

58. Integrate $\int \sqrt{1+x^2} dx$

59. Integrate $\int \sqrt{x^2-8x+7} dx$

60. Show that

$$\int_0^a f(x)g(x) dx = 2 \int_0^a f(x) dx \quad (8.2.60.1)$$

if

$$f(x) = f(a-x)g(x) + g(a-x) = 4 \quad (8.2.60.2)$$

61. Integrate the following functions:

- (i) $\frac{1}{x-x^3}$
 (ii) $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$
 (iii) $\frac{1}{x\sqrt{ax-x^2}}$
 (iv) $\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$
 (v) $\frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{2}}}$
 (vi) $\frac{1}{(x+1)(x^2+9)}$
 (vii) $\frac{\sin x}{\sin(x-a)}$
 (viii) $\frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$
 (ix) $\frac{\cos x}{\sqrt{4-\sin^2 x}}$
 (x) $\frac{\sin^8 x - \cos^8 x}{1-2 \sin^2 x \cos^2 x}$
 (xi) $\frac{1}{\cos(x+a) \cos(x+b)}$
 (xii) $\frac{x^3}{\sqrt{1-x^8}}$
 (xiii) $\frac{e^x}{\sqrt{2+e^x}}$
 (xiv) $\frac{1}{(x^2+1)(x^2+4)}$

- (xv) $\cos^3 x e^{\log \sin x}$
 (xvi) $e^{3 \log x} (x^4 + 1)^{-1}$
 (xvii) $\frac{1}{\sqrt{\sin^3 x \sin(x+a)}}$
 (xviii) $\frac{\sin^{-1} \sqrt{x-\cos^{-1} \sqrt{x}}}{\sin^{-1} \sqrt{x+\cos^{-1} \sqrt{x}}}, x \in [0, 1]$
 (xix) $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$
 (xx) $\frac{2+\sin 2x}{1+\cos 2x} e^x$
 (xxi) $\frac{x^2+x+1}{(x+1)^2(x+2)}$
 (xxii) $\tan^{-1} \sqrt{\frac{1-x}{1+x}}$
 (xxiii) $\frac{\sqrt{x^2+1} \log(x^2+1) - 2 \log x}{x^4}$

62. Find $\int \frac{dx}{e^x + e^{-x}} dx$

63. Find $\int \frac{\cos 2x}{\sin x + \cos x} dx$

64. Verify that the given functions is a solution of the corresponding differential equation:

- (i) $y = e^x + 1; y_2 - y_1 = 0$
 (ii) $y = x^2 + 2x + C; y_1 - 2x - 2 = 0$
 (iii) $y = \cos x + C; y_1 + \sin x = 0$
 (iv) $y = \sqrt{1+x^2}; y_1 = \frac{xy}{1+x^2}$
 (v) $y = Ax; xy_1 = y, x \neq 0$
 (vi) $y = x \sin x;$

$$xy_1 = y + x \sqrt{x^2 - y^2}, (x \neq 0, x > y \text{ or } x < -y)$$

(vii) $xy = \log y + C; y_1 = \frac{y^2}{1-xy}, (xy \neq 1)$

(viii) $y - \cos y = x; y^2 y_1 + y^2 + 1 = 0$

(ix) $y = \sqrt{a^2 - x^2}, x \in (a, -a); x + yy_1 = 0, (y \neq 0)$

65. Form the differential equation representing the following family of curves where a, b are arbitrary constants.

- (i) $y = ae^{3x} + be^{-2x}$
 (ii) $y = e^{2x} (a + bx)$
 (iii) $y = e^x (a \cos x + b \sin x)$

66. Find the general solution for each of the following differential equations

- (i) $y_1 = \frac{1-\cos x}{1+\cos x}$
 (ii) $y_1 = \sqrt{4-y^2} \quad (|y| < 2)$
 (iii) $y_1 + y = 1 \quad (y \neq 1)$
 (iv) $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$
 (v) $(e^x + e^{-x}) dy - (e^y + e^{-y}) dx = 0$
 (vi) $y_1 = (1+x^2)(1+y^2)$
 (vii) $y \log y dx - x dy = 0$
 (viii) $x^5 y_1 = -y^5$
 (ix) $y_1 = \sin^{-1} x$
 (x) $e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$

67. Find the general solution of $y_1 = e^{x+y}$

68. Solve

$$(i) (x^2 + xy) dy = (x^2 + y^2)$$

- (ii) $y_1 = \frac{x+y}{x}$
 (iii) $(x-y) dy - (x+y) dx = 0$
 (iv) $(x^2 - y^2) dx + 2xy dy = 0$
 (v) $x^2 y_1 = x^2 - 2y^2 + xy$
 (vi) $x dy - y dx = \sqrt{x^2 + y^2} dx$
 (vii) $\left\{ x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right) \right\} y dx$
 $\left\{ y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right) \right\} x dx$
 (viii) $xy_1 - y + x \sin\left(\frac{y}{x}\right) = 0$
 (ix) $y dx + x \log\left(\frac{y}{x}\right) dy - 2x dy = 0$
 (x) $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

69. Solve

- (i) $y_1 + 2y = \sin x$
 (ii) $y_1 + 3y = e^{-2x}$
 (iii) $y_1 + \frac{y}{x} = x^2$
 (iv) $y_1 + y \sec x = \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$
 (v) $\cos^2 xy_1 + y = \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$
 (vi) $xy_1 + 2y = x^2 \log x$
 (vii) $x \log xy_1 + y = \frac{2}{x} \log x$
 (viii) $(1 + x^2) dy + 2xy dx = \cot x dx$
 (ix) $xy_1 + y - x + xy \cot x = 0$
 (x) $(x+y)y_1 = 1$
 (xi) $y dx + (x - y^2) dy = 0$
 (xii) $(x + 3y^2)y_1 = y, \quad y > 0$

70. Solve

$$xy_1 - y = 2x^2 \quad (8.2.70.1)$$

71. Solve

$$(1 - y^2)y_1 + xy = ay \quad (-1 < y < 1) \quad (8.2.71.1)$$

72. For each of the exercises below, verify that the given function is a solution of the corresponding differential equation:

73. Solve

- (i) $xy = ae^x + be^{-x} + x^2; \quad xy_2 + 2y_1 - xy + x^2 - 2 = 0$
 (ii) $y = e^x (a \cos x + b \sin x); \quad y_2 - 2y_1 + 2y = 0$
 (iii) $y = x \sin 3x; \quad y_2 + 9y_1 - 6 \cos 3x = 0$
 (iv) $x^2 = 2y^2 \log y; \quad (x^2 + y^2)y_1 - xy = 0$

74. Prove that $x^2 - y^2 = c(x^2 y^2)^2$ is the general solution of differential equation

$$(x^3 - 3xy^2) dx = (y^3 - 3x^2 y) dy, \quad (8.2.74.1)$$

where c is a parameter.

75. Find the general solution of the differential

equation

$$y_1 + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \quad (8.2.75.1)$$

76. Show that the general solution of the differential equation

$$y_1 + \frac{y^2 + y + 1}{x^2 + x + 1} = 0 \quad (8.2.76.1)$$

is

$$(x + y + 1) = A(1 - x - y - 2xy), \quad (8.2.76.2)$$

where A is a parameter.

77. Solve

$$ye^{\frac{x}{y}} dx = (xe^{\frac{x}{y}} + y^2) dy \quad (8.2.77.1)$$

78. Solve

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right] \quad (8.2.78.1)$$

79. Solve

$$\frac{y dx - x dy}{y} = 0 \quad (8.2.79.1)$$

80. Solve

$$e^x dy + (ye^x + 2x) dx = 0 \quad (8.2.80.1)$$