Geometry: Maths Olympiad

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***Abstract*—This book provides a collection of the international maths olympiad problems in geometry.**

1. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.
2. An arbitrary point M is selected in the interior of the segment AB. The squares AMCD and MBEF are constructed on the same side of AB, with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N0 denote the point of intersection of the straight lines AF and BC.
   a) Prove that the points N and N0 coincide.
   b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M.
   c) Find the locus of the midpoints of the segments P Q as M varies between A and B.
3. Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid ABCD (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.
4. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

   $\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$

   at least one is ≤ 2 and at least one is ≥ 2.

5. Construct triangle ABC if AC = b, AB = c and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^o$. Prove that a solution exists if and only if

   $btan\frac{\omega}{2} \leq c < b.$

   In what case does the equality hold?

6. Consider a plane ε and three non-collinear points A, B, C on the same side of ε; suppose the plane determined by these three points is not parallel to ε. In plane a take three arbitrary points A_0, B_0, C_0. Let L, M, N be the midpoints of segments $AA^{'}$, $BB^{'}$, $CC^{'}$; let G be the centroid of triangle LMN. (We will not consider positions of the points $A^{'}$, $B^{'}$, $C^{'}$ such that the points L, M, N do not form a triangle.) What is the locus of point G as $A^{'}$, $B^{'}$, $C^{'}$ range independently over the plane ε?
7. Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A, and the other side intersecting the segment BC.
8. Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$
9. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in $A^{'}, B^{'}, C^{'}$ respectively. Prove that
   $\frac{1}{4} < \frac{AI.BI.CI}{AA^{'}.BB^{'}.CC^{'}} \leq \frac{8}{27}$.
10. Let ABC be a triangle and P an interior point of ABC . Show that at least one of the angles ∠PAB, ∠PBC, ∠PCA is less than or equal to 30^o.
11. Consider nine points in space, no four of

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which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

12. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.
13. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \frac{\pi}{2}$ and AC · BD = AD · BC.
    a) Calculate the ratio (AB · CD)/(AC · BD).
    b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.
14. ABC is an isosceles triangle with AB = AC. Suppose that
    a) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
    b) Q is an arbitrary point on the segment BC different from B and C;
    c) E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.
15. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

16. Let ABCDEF be a convex hexagon with AB = BC = CD and DE = EF = F A, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that AG + GB + GH + DH + $HE \geq CF$.
17. We are given a positive integer r and a rectangular board ABCD with dimensions $\left|AB\right| = 20$, $\left|BC\right| = 12$. The rectangle is divided into a grid of 20 × 12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.
    a) Show that the task cannot be done if r is divisible by 2 or 3.
    b) Prove that the task is possible when r = 73.
    c) Can the task be done when r = 97?
18. Let P be a point inside triangle ABC such that

    $$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

    Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

19. Let ABCDEF be a convex hexagon such that AB is parallel to DE, BC is parallel to EF , and CD is parallel to F A. Let RA, RC, RE denote the circumradii of triangles F AB, BCD, DEF ,respectively, and let P denote the perimeter of the hexagon. Prove that

    $$R_A + R_C + R_E \geq \frac{P}{2}.$$
20. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.
    Let S1 be the total area of the black part of the triangle and S2 be the total area of the white part. Let

    f(m, n) = $\left|S_1 - S_2\right|$.
    a) Calculate f(m, n) for all positive integers m and n which are either both even or both odd.
    b) Prove that $f(m,n) \leq \frac{1}{2}\max m, n$ for all m and n.
    c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.
21. The angle at A is the smallest angle of triangle ABC. The points B and C divide the

circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W , respectively. The lines BV and CW meet at T . Show that

AU = TB + TC.

22. In the convex quadrilateral ABCD, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P , where the perpendicular bisectors of AB and DC meet, is inside ABCD. Prove that ABCD is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

23. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.
24. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.
25. AB is tangent to the circles CAMN and NMBD. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that PE = QE.

26. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.
27. Let ABC be an acute-angled triangle with circumcentre O.Let P on BC be the foot of the altitude from A.
Suppose that $\angle BCA \geq \angle ABC + 30^o$.
Prove that $\angle CAB + \angle COP < 90^o$.
28. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^o$ and that AB + BP = AQ + QB. What are the possible angles of triangle ABC?
29. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF .

30. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, ..., O_n$. Show that $\sum_{i<j} \frac{1}{O_iO_j} \leq (n-1)\frac{\pi}{4}$.
31. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

32. ABCD is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC if RP=RQ.
33. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

34. In a convex quadrilateral ABCD the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside ABCD andsatisfies

$\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

Prove that ABCD is a cyclic quadrilateral if and only if AP=CP .