

G V V Sharma\*

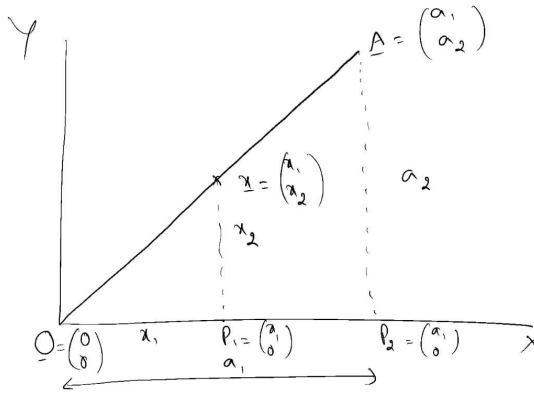


Fig. 1.1

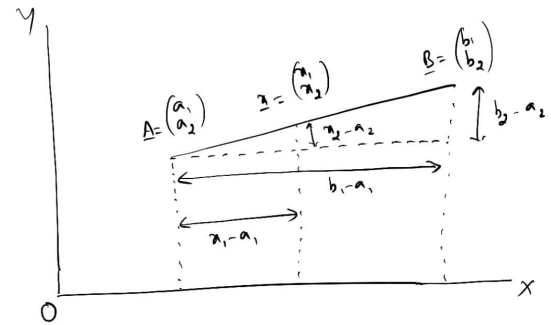


Fig. 1.2

## CONTENTS

**Abstract**—This textbook introduces linear algebra by exploring Euclidean geometry.

### 1 THE STRAIGHT LINE

1.1 The points  $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  are as shown in Fig. 1.1. Find the equation of  $OA$ .

**Solution:** Let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  be any point on  $OA$ . Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \quad (1.1)$$

$$\Rightarrow x_2 = mx_1 \quad (1.2)$$

where  $m$  is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.3)$$

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.4)$$

1.2 Find the equation of  $AB$  in Fig. 1.2

**Solution:** From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \quad (1.5)$$

$$\Rightarrow x_2 = mx_1 + a_2 - ma_1 \quad (1.6)$$

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \quad (1.7)$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.8)$$

$$= \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.9)$$

1.3 Find the length of  $\mathbf{A}$  in Fig. 1.1

**Solution:** Using Baudhayana's theorem, the length of the vector  $\mathbf{A}$  is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \quad (1.10)$$

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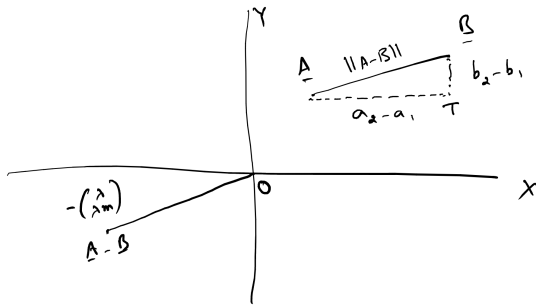


Fig. 1.4

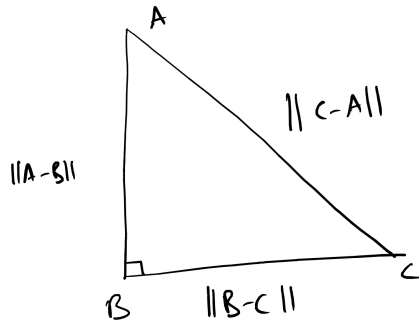


Fig. 2.1

Also, from (1.4),

$$\|A\| = \lambda \sqrt{1 + m^2} \quad (1.11)$$

Note that  $\lambda$  is the variable that determines the length of  $A$ , since  $m$  is constant for all points on the line.

1.4 Find  $A - B$ .

**Solution:** See Fig. 1.4. From (1.9), for some  $\lambda$ ,

$$B = A + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.12)$$

$$\Rightarrow A - B = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (1.13)$$

$A - B$  is marked in Fig. 1.4.

1.5 Show that  $AB = \|A - B\|$

## 2 ORTHOGONALITY

2.1 See Fig. 2.1. In  $\triangle ABC$ ,  $AB \perp BC$ . Show that

$$(A - B)^T (B - C) = 0 \quad (2.1)$$

**Solution:** Using Baudhayana's theorem,

$$\begin{aligned} \|A - B\|^2 + \|B - C\|^2 &= \|C - A\|^2 \quad (2.2) \\ \Rightarrow (A - B)^T (A - B) + (B - C)^T (B - C) \\ &= (C - A)^T (C - A) \\ \Rightarrow 2A^T B - 2B^T B + 2B^T C - 2A^T C &= 0 \quad (2.3) \end{aligned}$$

which can be simplified to obtain (2.1).

2.2 Let  $x$  be any point on  $AB$  in Fig. 2.1. Show that

$$(x - A)^T (B - C) = 0 \quad (2.4)$$

2.3 If  $x, y$  are any two points on  $AB$ , show that

$$(x - y)^T (B - C) = 0 \quad (2.5)$$

2.4 In Fig. 2.4,  $BE \perp AC$ ,  $CF \perp AB$ . Show that  $AD \perp BC$ .

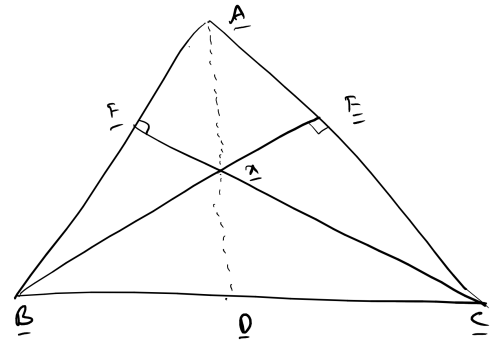


Fig. 2.4

**Solution:** Let  $x$  be the intersection of  $BE$  and  $CF$ . Then, using (2.5),

$$\begin{aligned} (x - B)^T (A - C) &= 0 \\ (x - C)^T (A - B) &= 0 \end{aligned} \quad (2.6)$$

$$\Rightarrow x^T (A - C) - B^T (A - C) = 0 \quad (2.7)$$

$$\text{and } x^T (A - B) - C^T (A - B) = 0 \quad (2.8)$$

Subtracting (2.8) from (2.7),

$$x^T (B - C) + A^T (C - B) = 0 \quad (2.9)$$

$$\Rightarrow (x^T - A^T) (B - C) = 0 \quad (2.10)$$

$$\Rightarrow (x - A)^T (B - C) = 0 \quad (2.11)$$

which completes the proof.

## 3 MEDIANS OF A TRIANGLE

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|A - B\|}{\|B - C\|} = k. \quad (3.1)$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k+1} = \mathbf{B}. \quad (3.2)$$

**Solution:** From (1.9),

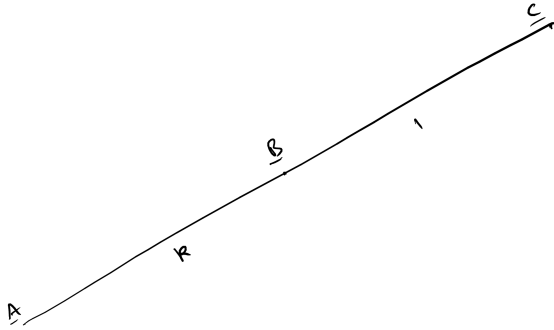


Fig. 3.1

$$\mathbf{B} = \mathbf{A} + \lambda_1 \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (3.3)$$

$$\mathbf{B} = \mathbf{C} - \lambda_2 \begin{pmatrix} 1 \\ m \end{pmatrix}.$$

$$\Rightarrow \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \quad (3.4)$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (3.5)$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \quad (3.6)$$

resulting in (3.2).

3.2 If  $\mathbf{A}$  and  $\mathbf{B}$  are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = \mathbf{0} \Rightarrow k_1 = k_2 = 0 \quad (3.7)$$

3.3  $BE$  and  $CF$  are medians of  $\triangle ABC$  intersecting at  $O$  as shown in Fig. 3.3. Show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \quad (3.8)$$

**Solution:** Let

$$\frac{CO}{OF} = k_1 \quad (3.9)$$

$$\frac{BO}{OE} = k_2 \quad (3.10)$$

Using (3.2),

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (3.11)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (3.12)$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F} + \mathbf{C}}{k_1 + 1} = \frac{k_1 \frac{\mathbf{A} + \mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} \quad (3.13)$$

$$\mathbf{O} = \frac{k_2\mathbf{E} + \mathbf{B}}{k_2 + 1} = \frac{k_2 \frac{\mathbf{A} + \mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \quad (3.14)$$

From (3.13) and (3.14),

$$\frac{k_1 \frac{\mathbf{A} + \mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} = \frac{k_2 \frac{\mathbf{A} + \mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \quad (3.15)$$

$$\begin{aligned} \Rightarrow & \left[ \frac{k_1(k_2 + 1)}{2} - \frac{k_2(k_1 + 1)}{2} \right] \mathbf{A} \\ & + \left[ \frac{k_1(k_2 + 1)}{2} - (k_1 + 1) \right] \mathbf{B} \\ & + \left[ (k_2 + 1) - \frac{k_2(k_1 + 1)}{2} \right] \mathbf{C} = \mathbf{0} \end{aligned} \quad (3.16)$$

resulting in  $k_1 = k_2$ ,

$$k_1^2 - k_1 - 2 = 0 \Rightarrow k_1 = k_2 = 2, \quad (3.17)$$

provided  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are linearly independent. Thus, substituting  $k_1 = 2$  in (3.14),

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.18)$$

If  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \quad (3.19)$$

Note that  $\mathbf{B}, \mathbf{C}$  are linearly independent. Sub-

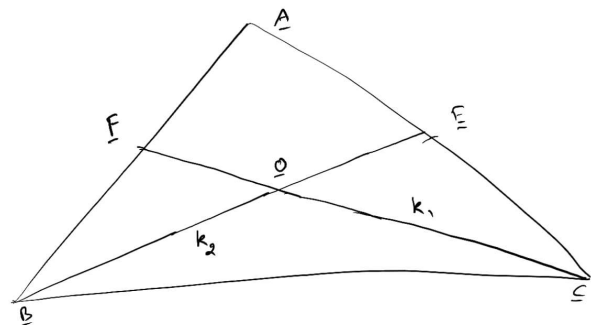


Fig. 3.3

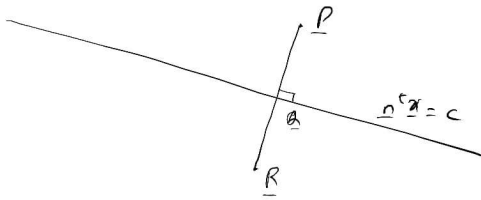


Fig. 4.1

stituting (3.19) in (3.16),

$$\left[ \frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2} \right] [\alpha \mathbf{B} + \beta \mathbf{C}] + \left[ \frac{k_1(k_2+1)}{2} - (k_1+1) \right] \mathbf{B} + \left[ (k_2+1) - \frac{k_2(k_1+1)}{2} \right] \mathbf{C} = 0 \quad (3.20)$$

$$\begin{aligned} \Rightarrow (k_1 - k_2)\alpha + k_1k_2 - k_1 - 2 &= 0 \\ (k_1 - k_2)\beta - k_1k_2 + k_2 + 2 &= 0 \end{aligned} \quad (3.21)$$

$$\Rightarrow (k_1 - k_2)(\alpha + \beta - 1) = 0 \quad (3.22)$$

If  $\alpha + \beta = 1$ ,  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are collinear according to (3.2) resulting in a contradiction. Hence,  $k_1 = k_2$ , which, upon substitution in (3.21), yields

$$k_1^2 - k_1 - 2 = 0 \Rightarrow k_1 = 2. \quad (3.23)$$

#### 4 MATRIX TRANSFORMATIONS

4.1 Find  $\mathbf{R}$ , the reflection of  $\mathbf{P}$  about the line

$$L: \mathbf{n}^T \mathbf{x} = c \quad (4.1)$$

**Solution:** Since  $\mathbf{R}$  is the reflection of  $\mathbf{P}$  and  $\mathbf{Q}$  lies on  $L$ ,  $\mathbf{Q}$  bisects  $PR$ . This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (4.2)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad (4.3)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad (4.4)$$

where  $\mathbf{m}$  is the direction vector of  $L$ . From (4.2) and (4.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (4.5)$$

From (4.5) and (4.4),

$$(\mathbf{m} \ \mathbf{n})^T \mathbf{R} = (\mathbf{m} \ -\mathbf{n})^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.6)$$

Letting

$$\mathbf{V} = (\mathbf{m} \ \mathbf{n}) \quad (4.7)$$

with the condition that  $\mathbf{m}, \mathbf{n}$  are orthonormal,

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (4.8)$$

Noting that

$$(\mathbf{m} \ -\mathbf{n}) = (\mathbf{m} \ \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.9)$$

(4.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[ \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.10)$$

$$\Rightarrow \mathbf{R} = \left[ \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (4.11)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \quad (4.12)$$

4.2 Rotate  $\mathbf{P}$  through an angle of  $\theta$  about the origin in the counter clockwise direction to obtain  $\mathbf{S}$ .  
**Solution:**

$$\mathbf{S} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{P} \quad (4.13)$$

#### 5 CIRCUMCIRCLE

5.1 A circle with centre at  $\mathbf{c}$  and radius  $r$  has the equation

$$\|\mathbf{x} - \mathbf{O}\| = r \quad (5.1)$$

$$\Rightarrow (\mathbf{x} - \mathbf{O})^T (\mathbf{x} - \mathbf{O}) = r^2 \quad (5.2)$$

5.2 Let  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$  be three points on the circle and  $OD \perp BC$ . Show that

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (5.3)$$

5.3 Let  $\mathbf{D}$  be the mid point of  $AB$ . Show that  $OD \perp AB$ .

5.4 Show that the direction vector of the normal to the tangent at  $\mathbf{P}$  is  $\mathbf{P}$ .

5.5 Let  $\mathbf{A}$  be any point on the tangent. Show that  $OP \perp AP$ .

5.6 Show that the angle in a semi-circle is a right angle.