Linear Algebra through Coordinate Geometry

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Contents

***Abstract*—This manual introduces linear algebra through coordinate geometry using a problem solving approach.**

1 The Straight Line

1.1 The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{A} - \mathbf{B}\right) \tag{1.1}$$

Alternatively, it can be expressed as

$$\mathbf{m}^T\left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2}$$

where $\mathbf{m}$ is the solution of

$$\left(\mathbf{A} - \mathbf{B}\right)^T \mathbf{m} = 0 \tag{1.3}$$

1.2 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.4}$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. If $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed, find the locus of $\mathbf{R}$.

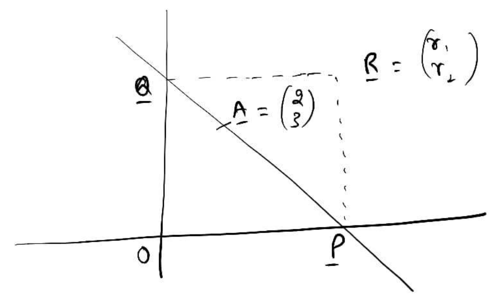

Fig. 1.2

Solution: From Fig. (1.2),

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} \tag{1.5}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R} \tag{1.6}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{1.7}$$

From (1.2) and (1.3),

$$\begin{aligned} \left(\mathbf{A} - \mathbf{P}\right)^T \mathbf{m} &= 0 \\ \left(\mathbf{A} - \mathbf{Q}\right)^T \mathbf{m} &= 0 \\ \left(\mathbf{P} - \mathbf{Q}\right)^T \mathbf{m} &= 0 \end{aligned} \tag{1.8}$$

From (1.8) and (**??**)

$$\left[2\mathbf{A} - \left(\mathbf{P} + \mathbf{Q}\right)\right]^T \mathbf{m} = 0 \tag{1.9}$$

$$\implies \left(2\mathbf{A} - \mathbf{R}\right)^T \mathbf{m} = 0 \tag{1.10}$$

From (1.8) and (1.5),(1.6)
Also,

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{1.11}$$

$$\implies \begin{pmatrix} \mathbf{P} & \mathbf{Q} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{R} \tag{1.12}$$

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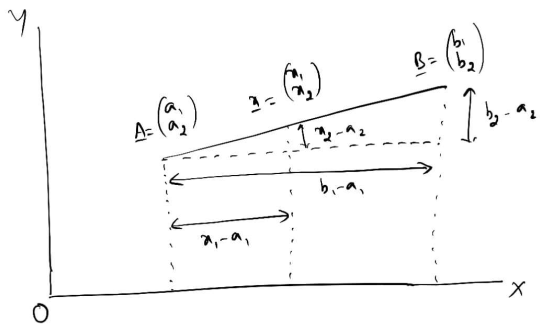

Fig. 1.3

From (**??**) and (1.13)

$$\begin{pmatrix}\mathbf{P} & \mathbf{Q}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -k\end{pmatrix} = \begin{pmatrix}\mathbf{R} & (1-k)\begin{pmatrix}2\\3\end{pmatrix}\end{pmatrix} \tag{1.13}$$

$$\implies \frac{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{R} - \begin{pmatrix}2\\3\end{pmatrix}}{\lambda_1} = \frac{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{R} - \begin{pmatrix}2\\3\end{pmatrix}}{\lambda_2} \tag{1.14}$$

$$\implies \begin{pmatrix}1 & 0\\ 0 & -k\end{pmatrix}\mathbf{R} = (1-k)\begin{pmatrix}2\\3\end{pmatrix} \tag{1.15}$$

Let $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.16}$$

$$\implies x_2 = mx_1 \tag{1.17}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix}x_1\\mx_1\end{pmatrix} = x_1\begin{pmatrix}1\\m\end{pmatrix} \tag{1.18}$$

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix}x_1\\mx_1\end{pmatrix} = \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{1.19}$$

1.3 Find the equation of AB in Fig. 1.3
Solution: From Fig. 1.3,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.20}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.21}$$

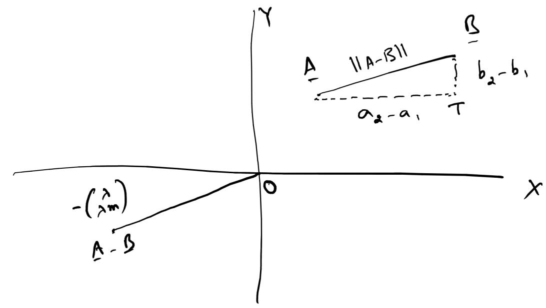

Fig. 1.5

From (1.21),

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\ mx_1 + a_2 - ma_1\end{pmatrix} \tag{1.22}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix}1\\m\end{pmatrix} \tag{1.23}$$

$$= \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{1.24}$$

1.4 Find the length of $\mathbf{A}$ in Fig. **??**
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.25}$$

Also, from (1.19),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.26}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.

1.5 Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.5. From (1.24), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{1.27}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix}1\\m\end{pmatrix}, \tag{1.28}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.5.

1.6 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

2 Orthogonality

2.1 See Fig. 2.1. In $\triangle ABC, AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.1}$$

Solution: Using Baudhayana's theorem,

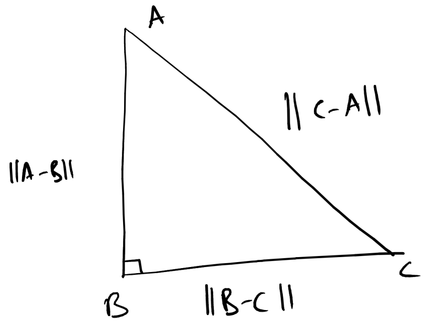

Fig. 2.1

$$\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{C}-\mathbf{A}\|^2 \tag{2.2}$$

$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) + (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A})$$

$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{2.3}$$

which can be simplified to obtain (2.1).

2.2 Let $\mathbf{x}$ be any point on AB in Fi.g 2.1. Show that

$$(\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.4}$$

2.3 If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x}-\mathbf{y})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.5}$$

2.4 In Fig. 2.4, $BE \perp AC, CF \perp AB$. Show that $AD \perp BC$.

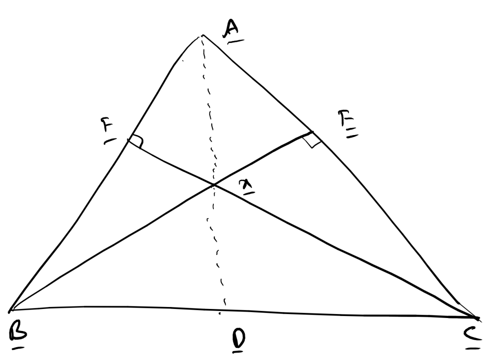

Fig. 2.4

Solution: Let $\mathbf{x}$ be the intersection of BE and CF. Then, using (2.5),

$$\begin{aligned}(\mathbf{x}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) &= 0 \\ (\mathbf{x}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) &= 0\end{aligned} \tag{2.6}$$

$$\implies \mathbf{x}^T(\mathbf{A}-\mathbf{C}) - \mathbf{B}^T(\mathbf{A}-\mathbf{C}) = 0 \tag{2.7}$$

$$\text{and } \mathbf{x}^T(\mathbf{A}-\mathbf{B}) - \mathbf{C}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{2.8}$$

Subtracting (2.8) from (2.7),

$$\mathbf{x}^T(\mathbf{B}-\mathbf{C}) + \mathbf{A}^T(\mathbf{C}-\mathbf{B}) = 0 \tag{2.9}$$

$$\implies \left(\mathbf{x}^T - \mathbf{A}^T\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{2.10}$$

$$\implies (\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.11}$$

which completes the proof.

3 Medians of a triangle

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{3.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{3.2}$$

Solution: From (1.24),

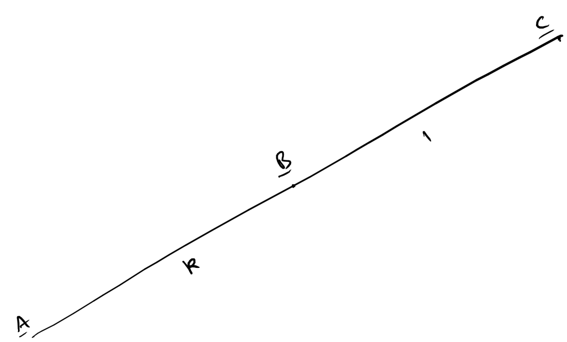

Fig. 3.1

$$\begin{aligned}\mathbf{B} &= \mathbf{A} + \lambda_1\begin{pmatrix}1\\m\end{pmatrix}, \\ \mathbf{B} &= \mathbf{C} - \lambda_2\begin{pmatrix}1\\m\end{pmatrix}.\end{aligned} \tag{3.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{3.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \begin{pmatrix}1\\m\end{pmatrix}, \tag{3.5}$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A}-\mathbf{B} = k(\mathbf{B}-\mathbf{C}) \tag{3.6}$$

resulting in (3.2).

3.2 If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{3.7}$$

3.3 BE and CF are medians of $\triangle ABC$ intersecting at O as shown in Fig. 3.3. Show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \tag{3.8}$$

Solution: Let

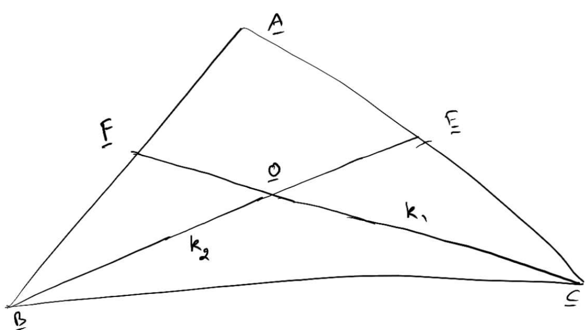

Fig. 3.3

$$\frac{CO}{OF} = k_1 \tag{3.9}$$

$$\frac{BO}{OE} = k_2 \tag{3.10}$$

Using (3.2),

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{3.11}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{3.12}$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F} + \mathbf{C}}{k_1 + 1} = \frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} \tag{3.13}$$

$$\mathbf{O} = \frac{k_2\mathbf{E} + \mathbf{B}}{k_2 + 1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \tag{3.14}$$

From (3.13) and (3.14),

$$\frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \tag{3.15}$$

$$\begin{aligned}\implies &\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right]\mathbf{A} \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0\end{aligned} \tag{3.16}$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = k_2 = 2, \tag{3.17}$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (3.14),

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.18}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \tag{3.19}$$

Note that $\mathbf{B}, \mathbf{C}$ are linearly independent. Substituting (3.19) in (3.16),

$$\begin{aligned}&\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right][\alpha\mathbf{B} + \beta\mathbf{C}] \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0\end{aligned} \tag{3.20}$$

$$\implies \begin{matrix}(k_1 - k_2)\alpha + k_1k_2 - k_1 - 2 = 0 \\ (k_1 - k_2)\beta - k_1k_2 + k_2 + 2 = 0\end{matrix} \tag{3.21}$$

$$\implies (k_1 - k_2)(\alpha + \beta - 1) = 0 \tag{3.22}$$

If $\alpha + \beta = 1, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (3.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in (3.21), yields

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = 2. \tag{3.23}$$