

Abstract—A collection of problems from JEE papers related to matrices are available in this document. Verify your solutions using Python.

1 QUADRATIC FORM

1.1 Show that

$$\min_{a,b,c} |a + b\omega + c\omega^2|^2 \quad (1)$$

where $\omega^3 = 1, \omega \neq 1$ and a, b, c are distinct nonzero integers can be expressed as

$$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} \quad (2)$$

where

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \mathbf{A} = 2\mathbf{P}^T \mathbf{P}, \quad (3)$$

$$\mathbf{P} = \begin{pmatrix} 1 & \cos \theta & -\cos \theta \\ 0 & \sin \theta & \sin \theta \end{pmatrix}, \theta = \frac{\pi}{3} \quad (4)$$

1.2 Show that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \quad (5)$$

Solution:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \\ -\cos \theta & \sin \theta \end{pmatrix} \begin{pmatrix} 1 & \cos \theta & -\cos \theta \\ 0 & \sin \theta & \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cos \theta & -\cos \theta \\ \cos \theta & 1 & -\cos 2\theta \\ -\cos \theta & -\cos 2\theta & 1 \end{pmatrix}, \quad (6) \end{aligned}$$

resulting in (5).

$$\therefore \cos 2\theta = -\cos \theta = -\frac{1}{2} \quad (7)$$

1.3 Show that the characteristic equation of \mathbf{A} is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda \quad (8)$$

1.4 Show that the eigenvalues of \mathbf{A} are 0 and 3.

1.5 Verify that $\text{tr}(\mathbf{A})$ is the sum of its eigenvalues.

1.6 Verify that $\det(\mathbf{A})$ is the product of its eigenvalues.

1.7 Show that \mathbf{A} is positive definite.

1.8 Show that $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is convex.

1.9 Show that the unconstrained \mathbf{x} that minimizes $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is given by the line

$$\mathbf{x} = k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (9)$$

1.10 Find \mathbf{y} such that

$$\mathbf{A} \mathbf{y} = \lambda \mathbf{y} \quad (10)$$

where λ is an eigenvalue of \mathbf{A} .

1.11 Show that

$$\mathbf{A} = \mathbf{P}^{-1} \mathbf{D} \mathbf{P} \quad (11)$$

where \mathbf{D} is a diagonal matrix comprising of the eigenvalues of \mathbf{A} and the columns of \mathbf{P} are the corresponding eigenvectors.

1.12 Find \mathbf{U} such that

$$\mathbf{A} = \mathbf{U}^T \mathbf{D} \mathbf{U}, \mathbf{U}^T \mathbf{U} = \mathbf{I} \quad (12)$$

1.13 Show that

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = 3\mathbf{v}^T \mathbf{v}, \quad (13)$$

where

$$\mathbf{v} = \mathbf{U} \mathbf{x} \quad (14)$$

1.14 Show that when the entries of \mathbf{x} are unequal and integers, the solution of (2) can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (15)$$

2 MATRICES: CAYLEY-HAMILTON THEOREM

Thus,

$$\beta^* = -\frac{37}{16} \quad (28)$$

$$\Rightarrow \alpha^* + \beta^* = -\frac{29}{16} \quad (29)$$

2.1 Let

$$\mathbf{M} = \begin{pmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{pmatrix} = \alpha \mathbf{I} + \beta \mathbf{M}^{-1} \quad (16)$$

where α, β are real functions of θ and \mathbf{I} is the identity matrix. Find the characteristic equation of \mathbf{M} .

Solution: (16) can be expressed as

$$\mathbf{M}^2 - \alpha \mathbf{M} - \beta \mathbf{I} = 0 \quad (17)$$

which yields the characteristic equation of \mathbf{M} as

$$\lambda^2 - \alpha \lambda - \beta = 0 \quad (18)$$

2.2 Find α and β .

Solution: Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4 \theta + \cos^4 \theta \quad (19)$$

$$\beta = -\sin^4 \theta \cos^4 \theta + (1 + \sin^2 \theta)(1 + \cos^2 \theta) \quad (20)$$

2.3 If

$$\alpha^* = \min_{\theta} \alpha(\theta) \quad (21)$$

$$\beta^* = \min_{\theta} \beta(\theta), \quad (22)$$

find $\alpha^* + \beta^*$.

Solution:

$$\because \alpha = \sin^4 \theta + \cos^4 \theta = 1 - \frac{\sin^2 2\theta}{2}, \quad (23)$$

$$\alpha^* = \frac{1}{2}, \quad (24)$$

Similarly,

$$-\beta = \sin^4 \theta \cos^4 \theta + (1 + \sin^2 \theta)(1 + \cos^2 \theta) \quad (25)$$

$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \quad (26)$$

$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2} \right)^2 + \frac{7}{4} \quad (27)$$

3 MATRICES: ADJUGATE

Let

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}, \quad \text{adj}(\mathbf{M}) = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \quad (30)$$

3.1 Show that $a + b = 3$

Solution:

$$\because \mathbf{M} \text{adj}(\mathbf{M}) = \det(\mathbf{M}) \mathbf{I}, \quad (31)$$

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} = 0 \quad (32)$$

$$\begin{pmatrix} 3 & b & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = 0 \quad (33)$$

resulting in

$$a = 2, b = 1 \quad (34)$$

Hence, $a + b = 3$.

3.2 Verify if

$$(\text{adj}(\mathbf{M}))^{-1} + \text{adj}(\mathbf{M}^{-1}) = -\mathbf{M} \quad (35)$$

Solution: From (31)

$$(\text{adj}(\mathbf{M}))^{-1} = \frac{\mathbf{M}}{\det(\mathbf{M})} \quad (36)$$

and

$$(\text{adj}(\mathbf{M}^{-1})) = \frac{\mathbf{M}^{-1}}{\det(\mathbf{M}^{-1})} \quad (37)$$

$$= \mathbf{M}^{-1} \det(\mathbf{M}) \quad (38)$$

Thus,

$$\begin{aligned} & (\text{adj}(\mathbf{M}^{-1})) + \text{adj}(\mathbf{M}^{-1}) \\ &= \mathbf{M}^{-1} \det(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \\ &= \text{adj}(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \end{aligned} \quad (39)$$

From (31)

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = \det(\mathbf{M}) \quad (40)$$

$$\Rightarrow \det(\mathbf{M}) = 8 - 5a = -2 \quad (41)$$

If

$$\begin{aligned} (\text{adj}(\mathbf{M}^{-1})) + \text{adj}(\mathbf{M}^{-1}) &= -\mathbf{M}, \\ \text{adj}(\mathbf{M}) - \frac{\mathbf{M}}{2} &= -\mathbf{M} \\ \Rightarrow \mathbf{M} &= -\text{adj}(\mathbf{M}) \end{aligned}$$

which is incorrect.

3.3 Verify if

$$\det(\text{adj}(\mathbf{M}^2)) = 81 \quad (42)$$

Solution:

$$\text{adj}(\mathbf{M}^2) = \mathbf{M}^{-2} \det(\mathbf{M})^2 \quad (43)$$

$$= 4\mathbf{M}^{-2} \quad (44)$$

$$\Rightarrow \det(\text{adj}(\mathbf{M}^2)) = 4^3 \det(\mathbf{M})^{-2} \quad (45)$$

$$= 16 \neq 81 \quad (46)$$

3.4 If

$$\mathbf{M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (47)$$

show that

$$\alpha - \beta + \gamma = 3 \quad (48)$$

Solution:

$$\mathbf{M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (49)$$

$$\Rightarrow \text{adj}(\mathbf{M}) \mathbf{M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \text{adj}(\mathbf{M}) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (50)$$

which can be expressed as

$$\det(\mathbf{M}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \text{adj}(\mathbf{M}) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (51)$$

$$\text{or, } \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\frac{1}{2} \text{adj}(\mathbf{M}) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (52)$$

Thus,

$$\alpha - \beta + \gamma = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad (53)$$

$$= -\frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \text{adj}(\mathbf{M}) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (54)$$

$$= \begin{pmatrix} 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 3 \quad (55)$$

4 LINEAR ALGEBRA: BINARY MATRICES

Let S be the set of all 3×3 matrices whose entries are from $\{0, 1\}$ and

$$E_1 = \{\mathbf{A} \in S : \det(\mathbf{A}) = 0\} \quad (56)$$

and

$$E_2 = \{\mathbf{A} \in S : \text{sum of entries of } \mathbf{A} \text{ is } 7\} \quad (57)$$

4.1 Find $|E_2|$.

Solution:

$$|E_2| = \frac{9!}{7!2!} = 72 \quad (58)$$

4.2 Find $|(E_1|E_2)|$.

Solution: E_2 is the set of matrices with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the following combinations in Table 4.2. $\mathbf{e}_i, i = 1, 2, 3$ are the standard basis vectors. The equation

$$\mathbf{v}_1 = \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 \quad (59)$$

has a solution only for the first combination in Table 4.2. Thus, $\det(\mathbf{A}) = 0$ only for this combination. Thus

$$|(E_1|E_2)| = 3 \times 3 = 9 \quad (60)$$

4.3 Find $\Pr(E_1|E_2)$.

Solution: From (58) and (60),

$$\Pr(E_1|E_2) = \frac{|(E_1|E_2)|}{|E_2|} = \frac{9}{72} = \frac{1}{8} \quad (61)$$

4.4 Verify using a python script.

\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	\mathbf{e}_i
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

TABLE 4.2

5 MATRICES: TRACE

5.1 Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^6$ from permutations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (62)$$

5.2 Let

$$\mathbf{X} = \sum_{k=1}^6 \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T. \quad (63)$$

Given

$$\mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (64)$$

is $\alpha = 30$?

Solution:

$$\begin{aligned} \because \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\ \mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \sum_{k=1}^6 \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \sum_{k=1}^6 \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \sum_{k=1}^6 \mathbf{P}_k \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \\ &= 30 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \quad (65)$$

Thus, $\alpha = 30$.

5.3 Is \mathbf{X} symmetric?

Solution: Yes. Trivial.

5.4 Show that

$$\mathbf{P}_k \mathbf{P}_k^T = \mathbf{I} \quad (66)$$

Solution:

$$\mathbf{P}_k = \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} \quad (67)$$

where $\mathbf{v}_{ki}, i = 1, 2, 3$ are from the standard basis. Then,

$$\begin{aligned} \mathbf{P}_k \mathbf{P}_k^T &= \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} (\mathbf{v}_{k1} \quad \mathbf{v}_{k2} \quad \mathbf{v}_{k3}) \mathbf{I} \\ \because \mathbf{v}_{ji}^T \mathbf{v}_{kj} &= \delta_{jk} \end{aligned} \quad (68)$$

5.5 For 2×2 matrices \mathbf{A}, \mathbf{B} , verify that

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad (69)$$

Show that this is true for any square matrix.

5.6 Verify if the sum of the diagonal entries of \mathbf{X} is 18.

Solution:

$$\begin{aligned}
 \text{tr}(\mathbf{X}) &= \sum_{k=1}^6 \text{tr} \left\{ \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \right\} \\
 &= \sum_{k=1}^6 \text{tr} \left\{ \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k \mathbf{P}_k^T \right\} \\
 &= \sum_{k=1}^6 \text{tr} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} = 6 \times 3 = 18 \quad (70)
 \end{aligned}$$

after substituting from (66).

5.7 Is $\mathbf{X} - 30\mathbf{I}$ invertible?

Solution: From (64),

$$\begin{aligned}
 \mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= 30\mathbf{I} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
 \Rightarrow (\mathbf{X} - 30\mathbf{I}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= 0 \quad (71)
 \end{aligned}$$

If $(\mathbf{X} - 30\mathbf{I})^{-1}$ exists,

$$\begin{aligned}
 (\mathbf{X} - 30\mathbf{I})^{-1} (\mathbf{X} - 30\mathbf{I}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= 0 \\
 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \mathbf{0} \quad (72)
 \end{aligned}$$

which is a contradiction. Hence, $\mathbf{X} - 30\mathbf{I}$ is not invertible.

6 LINEAR ALGEBRA: EIGENVECTOR AND NULL SPACE

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{pmatrix} \quad (73)$$

6.1 Find x such that $PQ = QP$.

Solution:

$$\therefore \mathbf{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad (74)$$

$$\mathbf{PQ} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{pmatrix} \quad (75)$$

and

$$\mathbf{QP} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 8 & 8 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \end{pmatrix} \quad (76)$$

Thus,

$$\mathbf{PQ} = \mathbf{QP} \Rightarrow \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = x \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \quad (77)$$

which has no solution.

6.2 If

$$\mathbf{R} = \mathbf{PQP}^{-1}, \quad (78)$$

verify whether

$$\det \mathbf{R} = \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8 \quad (79)$$

for all x .

Solution:

$$\begin{aligned}
 \det(\mathbf{R}) &= \det(\mathbf{P}) \det(\mathbf{Q}) \det(\mathbf{P})^{-1} = \det(\mathbf{Q}) \\
 &= 4(12 - x^2) \quad (80)
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \det(\mathbf{R}) - \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} &= 4 \{ (12 - x^2) - (10 - x^2) \} \\
 &= 8 \quad (81)
 \end{aligned}$$

which is true.

6.3 For $x = 0$, if

$$\mathbf{R} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \quad (82)$$

then show that

$$a + b = 5. \quad (83)$$

Solution: For $x = 0$,

$$\mathbf{R} = \mathbf{PQP}^{-1}, \quad (84)$$

where \mathbf{Q} is a diagonal matrix. This is the

eigenvalue decomposition of \mathbf{R} . Thus,

$$\mathbf{R} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad (85)$$

where

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (86)$$

is the eigenvector corresponding to the eigenvalue 6. Comparing with (85),

$$a = 2, b = 3 \implies a + b = 5. \quad (87)$$

6.4 For $x = 1$, verify if there exists a vector \mathbf{y} for which $\mathbf{R}\mathbf{y} = \mathbf{0}$.

Solution:

$$\begin{aligned} \mathbf{R}\mathbf{y} = \mathbf{0} &\implies \mathbf{PQP}^{-1}\mathbf{y} = \mathbf{0} \\ \implies \mathbf{Qz} = \mathbf{0}, \end{aligned} \quad (88)$$

where

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{y} \quad (89)$$

For $x = 1$, (73) and (88) yield

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} \mathbf{z} = \mathbf{0} \quad (90)$$

Using row reduction,

$$\begin{aligned} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow \\ \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{pmatrix} \end{aligned} \quad (91)$$

Thus, \mathbf{Q}^{-1} exists and

$$\mathbf{z} = \mathbf{0} \implies \mathbf{y} = \mathbf{0} \quad (92)$$

upon substituting from (89). This implies that the null space of \mathbf{R} is empty.

7.2 Show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{\left(a + \frac{1}{n}\right)^2} + \frac{1}{\left(a + \frac{1}{2}\right)^2} + \cdots + \frac{1}{(a+1)^2} \right] \\ = \int_0^1 \frac{1}{(a+x)^2} dx = \frac{1}{a(a+1)} \end{aligned} \quad (94)$$

7.3 If

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{7}{3}} \left\{ \left[\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \cdots + \frac{1}{(an+n)^2} \right] \right\}} \right) \\ = 54, \quad |a| > 1, \end{aligned} \quad (95)$$

find a .

Solution: Substituting from (93) and (94) in (95),

$$\begin{aligned} \frac{3}{4}a(a+1) &= 54 \\ a(a+1) &= 72 \\ \implies a &= 8, -9. \end{aligned} \quad (96)$$

8 EXERCISES

- For any two 3×3 matrices A and B , let $A+B = 2B^T$ and $3A+2B = I_3$. Which of the following is true?
 - $5A + 10B = 2I_3$.
 - $10A + 5B = 3I_3$.
 - $2A + B = 3I_3$.
 - $3A + 6B = 2I_3$.

7 DEFINITE INTEGRAL: LIMIT OF A SUM

7.1 Show that

$$\lim_{n \rightarrow \infty} \frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{4}{3}}} = \int_0^1 x^{\frac{1}{3}} dx = \frac{3}{4} \quad (93)$$