Algebra: Maths Olympiad

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***Abstract*—This book provides a collection of the international maths olympiad problems in algebra.**

1. For what real values of x is
$$\sqrt{(x+\sqrt{2x-1}}+\sqrt{(x-\sqrt{2x-1}}=A \quad (1.1)$$
given
   a) $A=\sqrt{2}$,
   b) $A=1$,
   c) $A=2$

   where only non-negative real numbers are admitted for square roots?
2. Let a,b,c be real numbers. Consider the quadratic equation in $\cos x$:
$$a\cos^2 x+b\cos x+c=0. \quad (2.1)$$
Using the numbers a,b,c, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4,b=2,c=-1$.
3. Solve the system of equations:

   x + y + z = a

   $x^2+y^2+z^2=b^2$

   $xy=z^2$

   where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

4. Solve the equation $cos^n x - sin^n x = 1$, where n is a natural number.

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5. Find all real roots of the equation
$$\sqrt{x^2-p}+2\sqrt{x^2-1}=x \quad (5.1)$$
where p is a real parameter.
6. Find all solutions x_1,x_2,x_3,x_4,x_5 of the system
$$x_5+x_2=yx_1 \quad (6.1)$$
$$x_1+x_3=yx_2 \quad (6.2)$$
$$x_2+x_4=yx_3 \quad (6.3)$$
$$x_3+x_5=yx_4 \quad (6.4)$$
$$x_4+x_1=yx_5 \quad (6.5)$$
where y is a parameter.
7. Suppose a, b, c are the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\le 3abc$$
8. Determine all values x in the interval $0\le x\le 2\pi$ which satisfy the inequality
$$2\cos x\left|\sqrt{(1+\sin 2x)}-\sqrt{(1-\sin 2x)}\right|$$
9. Consider the system of equations
$$a_{11}x_1+a_{12}x_2+a_{13}x_3=0 \quad (9.1)$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=0 \quad (9.2)$$
$$a_{31}x_1+a_{32}x_2+a_{33}x_3=0 \quad (9.3)$$
with unknowns x_1,x_2,x_3. The coefficients satisfy the conditions:

   a) a_{11},a_{22},a_{33} are positive numbers;
   b) the remaining coefficients are negative numbers;
   c) in each equation, the sum of the coefficients is positive.

   Prove that the given system has only the solution

$$x_1 = x_2 = x_3 = 0$$

10. Solve the system of equations

$$\begin{aligned} \left|a_1 - a_2\right| x_2 + \left|a_1 - a_3\right| x_3 + \left|a_1 - a_4\right| x_4 &= 1 \\ \left|a_2 - a_1\right| x_1 + \left|a_2 - a_3\right| x_3 + \left|a_2 - a_3\right| x_4 &= 1 \\ \left|a_3 - a_1\right| x_1 + \left|a_3 - a_2\right| x_2 &= 1 \\ \left|a_4 - a_1\right| x_1 + \left|a_4 - a_2\right| x_2 + \left|a_4 - a_3\right| x_3 &= 1 \end{aligned}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

11. Consider the sequence $[c_n]$, where

$$\begin{aligned} c_1 &= a_1 + a_2 + \ldots.. + a_8 \\ c_2 &= a_1^2 + a_2^2 + \ldots.. + a_8^2 \\ &\ldots.. \\ c_2 &= a_1^n + a_2^n + \ldots.. + a_8^n \end{aligned}$$

in which $a_1, a_2, \ldots\ldots., a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $[c_n]$ are equal to zero. Find all natural numbers n for which $c_n = 0$.

12. Let k, m, n be natural numbers such that m + k + 1 is a prime greater than n + 1. Let c_s = s(s + 1). Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k)\ldots..(c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \ldots.. c_n$.

13. Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

14. Consider the system of equations

$$ax_1^2 + bx_1 + c = x_2$$

$$ax_2^2 + bx_2 + c = x_3$$

....

$$ax_{n-1}^2 + bx_{n-1} + c = x_n$$

$$ax_n^2 + bx_n + c = x_1$$

with unknowns $x_1, x_2, \ldots\ldots, x_n$, where a, b, c are real and a≠0. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system

a) if $\Delta < 0$, there is no solution,
b) if $\Delta = 0$, there is exactly one solution,
c) if $\Delta > 0$, there is more than one solution.

15. Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0, x_2 > 0, x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1-z_1^2} + \frac{1}{x_2y_2-z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

16. Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n.

17. Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and B_n are numbers in the system with base b; these are related as follows:

$$A_n = x_n x_{n-1} \ldots\ldots. x_0, A_{n-1} = x_{n-1} x_{n-2} \ldots\ldots.. x_0,$$

$$B_n = x_n x_{n-1} \ldots\ldots. x_0, B_{n-1} = x_{n-1} x_{n-2} \ldots\ldots.. x_0,$$

$$x_n \neq 0, x_{n-1} \neq 0$$

Prove:
$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ if and only if $a > b$

18. The real numbers $a_0, a_1, \ldots\ldots, a_n, \ldots\ldots$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \ldots.. \leq a_n \leq \ldots\ldots..$$

The numbers $b_1, b_2, \ldots\ldots, b_n, \ldots\ldots.$ are defined by

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right)\left(\frac{1}{\sqrt{a_k}}\right)$$

a) Prove that $0 \leq bn < 2$ for all n.
b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, \ldots..$ with the above properties such that $b_n > c$ for large enough n.

19. Prove that for every natural number m, there

exists a finite set S of points in a plane with the following property: For every point A in S, there are exactly m points in S which are at unit distance from A.

20. Prove that the following assertion is true for n = 3 and n = 5, and that it is false for every other natural number $n > 2$: If $a_1, a_2, ..., a_n$ are arbitrary real numbers, then $(a_1 - a_2)(a_1 - a_3).....(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)......(a_2 - a_n) + ... + (a_n - a_1)(a_n - a_2).......(a_n - a_{n-1} \geq 0$
21. Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities
$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0$

$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq 0$

$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \leq 0$

$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq 0$

$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \leq 0$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.
22. Let a and b be real numbers for which the equation

$x^4 + ax^3 + bx^2 + ax + 1 = 0$

has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

23. Let $a_1, a_2, ..., a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, ..., b_n$ for which

a) $a_k < b_k$ for k = 1, 2,......, n,
b) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for k = 1, 2, ..., n - 1,
c) $b_1 + b_2 + ... + b_n < \frac{1+q}{1-q}(a_1 + a_2 + ... + a_n)$.

24. Determine all possible values of

$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$

where a, b, c, d are arbitrary positive numbers.

25. Let P be a non-constant polynomial with integer coefficients. If n(P) is the number of distinct integers k such that $(P(k))^2 = 1$, prove that n(P) - deg(P) $\leq$ 2, where deg(P) denotes the degree of the polynomial P.
26. Find all polynomials P, in two variables, with the following properties:
a) for a positive integer n and all real t, x, y
P(tx, ty) = t^nP(x, y)
(that is, P is homogeneous of degree n),
b) for all real a, b, c,
P(b + c, a) + P(c + a, b) + P(a + b, c) = 0,
c) P(1,0) = 1.
27. Consider the system of p equations in q = 2p unknowns $x_1, x_2, ..., x_q$:

$a_{11}x_1 + a_{12}x_2 + ... + a_{1q}x_q = 0$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2q}x_q = 0$

....

$a_{p1}x_1 + a_{p2}x_2 + ... + a_{pq}x_q = 0$

with every coefficient a_{ij} member of the set [-1,0,1]. Prove that the system has a solution $(x_1, x_2, ..., x_q)$ such that
a) all x_j (j = 1,2, ..., q) are integers,
b) there is at least one value of j for which $x_j \neq 0$,
c) $|x_j| \leq$q(j = 1,2, ..., q).

28. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j = 2,3, $\cdots$. Show that, for any positive integer n, the roots of the equation $P_n(x)$ = x are real and distinct.
29. Let a and b be positive integers. When $a^2 + b^2$ is divided by a+b, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

30. Four real constants a, b, A, B are given, and

$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$

Prove that if $f(\theta) \geq 0$ for all real θ, then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.
31. Let $n > 6$ be an integer and $a_1, a_2, ...a_k$ be all the natural numbers less than n and relatively prime to n. If

$a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0$
prove that n must be either a prime number or a power of 2.

32. An infinite sequence x_0, x_1, x_2,....... of real numbers is said to be bounded if there is a constant C such that $\left|x_i\right| \leq C$ for every i ≥ 0. Given any real number $a > 1$, construct a bounded infinite sequence x_0, x_1, x_2,.......such that

$$\left|x_i - x_j\right| \left|i - j\right|^a \geq 1$$

for every pair of distinct non negative integers i, j.

33. Find all integers a, b, c with $1 < a < b < c$ such that

(a - 1)(b - 1)(c - 1) is a divisor of abc - 1.

34. For each positive integer n, S(n) is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.
   a) Prove that $(n) \leq n^2 - 14$ for each $n \geq 4$.
   b) Find an integer n such that $(n) = n^2 - 14$.
   c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.
35. Let f(x) = $x^n + 5x^{n-1}$+3, where $n > 1$ is an integer. Prove that f(x) cannot be expressed as the product of two nonconstant polynomials with integer coefficients.
36. There are n lamps L_0, ... , L_{n-1} in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps s_0, s_1, ... as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:
   a) There is a positive integer M(n) such that after M(n) steps all the lamps are on again;
   b) If n = 2^k, we can take M(n) = n^2 - 1;
   c) If n = 2^k + 1, we can take M(n) = n^2-n+1.
37. Let m and n be positive integers. Let a_1, a_2, ... ,a_m be distinct elements of 1, 2, ... , n such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i \leq j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1+a_2+...+a_m}{m} \geq \frac{n+1}{2}.$$

38. . Determine all ordered pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.
39. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin$ A each of which is a product of k distinct elements of S for some $k \geq 2$.
40. Let a, b, c be positive real numbers such that abc = 1. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(b+a)} \geq \frac{3}{2}$$

41. Find the maximum value of x_0 for which there exists a sequence x_0, x_1 ... , x_{1995} of positive reals with $x_0 = x_{1995}$, such that for i = 1, ... , 1995,

$$x_{i-1}+\frac{2}{x_{i-1}}=2x_i+\frac{1}{x_i}$$

42. The positive integers a and b are such that the numbers 15a + 16b and 16a - 15b are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?
43. Let $x_1, x_2, ..., x_n$ be real numbers satisfying the conditions

$$\left|x_1 + x_2 + ... + x_n\right| = 1$$

and

$$\left|x_i\right| \leq \frac{n+1}{2} \quad i = 1, 2, . . . , n$$

Show that there exists a permutation $y_1, y_2, ..., y_n$ of $x_1, x_2, ..., x_n$ such that

$$\left|y_1 + 2y_2 + ... + ny_n\right| \leq \frac{n+1}{2}$$

44. Find all pairs (a, b) of integers a, $b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

45. For each positive integer n , let f(n) denote the number of ways of representing n as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, f(4) = 4, because the number 4 can be represented in the following four ways:

    4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.

    Prove that, for any integer $n \geq 3$,

    $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$
46. For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n)$ = k for some n.

47. Determine all pairs (a, b) of positive integers such that ab^2 + b + 7 divides $a^2b + a + b$.
48. Consider an n x n square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

    N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

    Determine the smallest possible value of N.

49. Determine all pairs (n, p) of positive integers such that

    p is a prime,

    n not exceeded 2p, and

    $(p-1)^n + 1$ is divisible by n^{p-1}.
50. A, B, C are positive reals with product 1. Prove that $(A-1+\frac{1}{B})(B-1+\frac{1}{C})(C-1+\frac{1}{A}) \leq 1$.

51. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A move is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B0 to the right of A such that $AB' = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves?
52. Prove that

    $$\frac{a}{\sqrt{(a^2+8bc)}}+\frac{b}{\sqrt{(b^2+8ac)}}+\frac{c}{\sqrt{(c^2+8ba)}} \geq 1$$

    for all positive real numbers a,b and c.

53. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

    ac + bd = (b + d a - c) (b + d - a + c).

    Prove that ab + cd is not prime.
54. S is the set of all (h, k) with h, k non-negative integers such that h + $k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \leq h, k' \leq k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

55. Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which k^n+k^2-1 divides k^m+k+1.

56. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < ... < d_k$, so that $d_1 = 1$, $d_k = n$. Let $d = d_1d_2 + d_2d_3 + ... + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.
57. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2-n^3+1}$ is a positive integer.

58. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.
59. Find all polynomials f with real coefficients such that for all reals a,b,c such that ab + bc + ca = 0 we have the following relations

    f(a - b)+f(b - c)+f(c - a) = 2f(a+b+c).

60. Let $n \geq 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + ... + t_n)(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n}).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.