The Straight Line

G V V Sharma*

***Abstract*—Solved problems from JEE mains papers related to 2D lines in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.1}$$

$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.2}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{1.3}$$

Then (1.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.4}$$

$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.6}$$

$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.7}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.8}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.9}$$

$$\mathbf{A} = -k\mathbf{B} \tag{1.10}$$

Substituting in (1.6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{1.11}$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{1.12}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.13}$$

2 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{2.2}$$

by $2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (2.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \tag{2.4}$$

However,

$$PQ = 2\sqrt{3} \tag{2.5}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = 2\sqrt{3} \tag{2.6}$$

$$\implies \lambda = \pm \frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{2.7}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{2.8}$$

from (2.3). Since $\mathbf{Q}$ lies in the third quadrant,

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

from (2.4) and (2.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{2.9}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \tag{2.10}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - \sqrt{6} \tag{2.11}$$

3 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{3.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{3.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (3.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{3.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{3.6}$$

$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{3.7}$$

Let the equation of AB be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \tag{3.8}$$

Since **R** lies on AB,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \tag{3.9}$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.10}$$

Substituting from (3.6) in (3.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \tag{3.11}$$

From (3.9) and (3.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \tag{3.12}$$

for some constant k. Multiplying both sides of (3.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.13}$$

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}$$
$$= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{3.14}$$

$$\because \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{3.15}$$

which can be easily verified for any **R**. from (3.14),

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0$$
$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0$$
$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{3.16}$$

which is the desired locus.

4 Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{4.1}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{4.2}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{4.3}$$

find its vertices.

Solution: From (4.1) and (4.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{4.4}$$

By row reducing the augmented matrix

$$\begin{pmatrix}1 & -1 & -1\\ 7 & -1 & 5\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & -1\\ 0 & 6 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & -1\\ 0 & 1 & 2\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{pmatrix} \implies \mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix}, \tag{4.5}$$

From (4.3) and (4.5)

$$AP = \|\mathbf{A} - \mathbf{P}\| = 2\sqrt{5} = d(say) \tag{4.6}$$

The direction vector of AP is

$$\mathbf{m} = \mathbf{A} - \mathbf{P} = 2\begin{pmatrix}1\\2\end{pmatrix} \tag{4.7}$$

$$\implies \|\mathbf{m}\| = 2\sqrt{5} \tag{4.8}$$

Since the direction of AP is the same as AC,

$$\mathbf{C} = \mathbf{P} - d\frac{\mathbf{m}}{\|\mathbf{m}\|}$$

$$= -\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-3\\-6\end{pmatrix} \tag{4.9}$$

Let $\mathbf{n} \perp \mathbf{m}$. Then

$$\mathbf{n} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{4.10}$$

and

$$\mathbf{B}, \mathbf{D} = \mathbf{P} \pm d\frac{\mathbf{n}}{\|\mathbf{n}\|}$$

$$= -\begin{pmatrix}1\\2\end{pmatrix} \pm 2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}, \begin{pmatrix}-3\\4\end{pmatrix} \tag{4.11}$$

5 Let k be an integer such that the triangle with vertices

$$\begin{pmatrix}k\\-3k\end{pmatrix}, \begin{pmatrix}5\\k\end{pmatrix}, \begin{pmatrix}-k\\2\end{pmatrix} \tag{5.1}$$

has area 28. Find the orthocentre of this triangle.

6 If an equlateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2, \tag{6.1}$$

then find the area of this triangle.

7 Find the locus of the point of intersection of the straight lines

$$\begin{pmatrix}t & -2\end{pmatrix}\mathbf{x} - 3t = 0 \tag{7.1}$$

$$\begin{pmatrix}1 & -2t\end{pmatrix}\mathbf{x} + 3 = 0 \tag{7.2}$$

8 A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.

9 Find the locus of the point of intersection of the lines

$$\begin{pmatrix}\sqrt{2} & -1\end{pmatrix}\mathbf{x} + 4\sqrt{2}k = 0 \tag{9.1}$$

$$\begin{pmatrix}\sqrt{2}k & k\end{pmatrix}\mathbf{x} - 4\sqrt{2} = 0 \tag{9.2}$$

10 The sides of a rhombus ABC are parallel to the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 2 = 0 \tag{10.1}$$

$$\begin{pmatrix}7 & -1\end{pmatrix}\mathbf{x} + 3 = 0. \tag{10.2}$$

If the diagonals of the rhombus intersect at

$$\mathbf{P} = \begin{pmatrix}1\\2\end{pmatrix} \tag{10.3}$$

and the vertex $\mathbf{A}$ (different) from the origin is on the y-axis, then find the ordinate of A.