Algebra: Maths Olympiad

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1. For how many pairs of positive integers (x, y) is x + 3y = 100?
2. The letters R, M and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$ and $R + M \times O = 64$, What is the value of R + M + O?
3. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?
4. Suppose that $4^{x_1} = 5, 5^{x_2} = 6, 6^{x_3} = 7, ..., 126^{x_{123}} = 127, 127^{x_{124}} = 128$.
5. If
$$\frac{1}{\sqrt{2011 + \sqrt{2011^2 - 1}}} = \sqrt{m} - \sqrt{n}$$
where m and n are positive integers, What is the value of m + n ?
6. If $a = b - c$, $b = c - d$, $c = d - a$ and $abcd \neq 0$ then, What is the value of
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$
7. How many non-negative integral values of x satisfy the equation $[\frac{x}{5}] = [\frac{x}{7}]$? (Here [x] denotes the greatest integer less than or equal to x. For example [3, 4] = 3 and [-2, 3] = -3)
8. x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression
$$(x_1 + \frac{1}{x_1})(x_2 + \frac{1}{x_2})(x_3 + \frac{1}{x_3})$$
9. What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$? (Here [x] denotes the greatest integer less than or equal to x. For example [3, 4] = 3 and [-2, 3] = -3)
10. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$?
11. What is the smallest positive integer k such that
$$k(3^3 + 4^3 + 5^3) = a^n \tag{11.1}$$
for some positive integers a and n, with $n > 1$?
12. Let
$$S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}}.$$
What is the value of
$$\sum_{k=1}^{99} \frac{1}{S_n + S_{n-1}}$$
13. It is given that the equation
$$x^2 + ax + 20 = 0 \tag{13.1}$$
has integer roots. What is the sum of all possible values of a ?
14. Three real numbers x, y, z are such that
$$x^2 + 6y = -17, \tag{14.1}$$
$$y^2 + 4z = 1 \tag{14.2}$$
$$z^2 + 2x = 2 \tag{14.3}$$
What is the value of $x^2 + y^2 + z^2$?
15. Let
$$f(x) = x^3 - 3x + b$$
$$g(x) = x^2 + bx - 3$$
where b is a real number. What is the sum of all possible values of b for which the equations $f(x) = 0$ and $g(x) = 0$ have a common root ?
16. A natural k is such that $k^2 < 2014 < (k+1)^2$. What is the largest prime factor of k?
17. What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots?
18. If $x^{x^4} = 4$, What is the value of $x^{x^2} + x^{x^8}$?
19. Natural numbers k, l, p and q are such that if

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a and b are roots of $x^2 - kx + 1 = 0$ then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of $x^2 - px + q = 0$. What is the sum of all possible values of q?

20. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \leq y$ satisfy the equation xy = x + y + (x, y)?
21. If real numbers a, b, c, d, e satisfy
$$a + 1 = b + 2 = c + 3 = d + 4 = e + 5 = a + b + c + d + e + 3$$
What is the value of $a^2 + b^2 + c^2 + d^2 + e^2$?
22. Let $x_1, x_2, ..., x_{2014}$ be real numbers different from 1, such that
$$x_1 + x_2 + ... + x_{2014} = 1$$
and
$$\frac{x_1}{1 - x_1} + \frac{x_2}{1 - x_2} + ... + \frac{x_{2014}}{1 - x_{2014}} = 1$$
What is the value of
$$\frac{x_1^2}{1 - x_1} + \frac{x_2^2}{1 - x_2} + \frac{x_3^2}{1 - x_3} ..... + \frac{x_{2014}^2}{1 - x_{2014}}$$
23. Positive integers a and b are such that a + b = a/b + b/a. What is the value of $a^2 + b^2$?
24. The equations
$$x^2 - 4x + k = 0 \tag{24.1}$$
$$x^2 + kx - 4 = 0 \tag{24.2}$$
where k is a real number, have exactly one common root. What is the value of k?
25. Let P(x) be a non-zero polynomial with integer coefficients. If P(n) is divisible by n for each positive integer n, what is the value of P(0)?
26. Let a, b, and c be real numbers such that
$$a - 7b + 8c = 4$$
$$8a + 4b - c = 7$$
What is the value of $a^2 - b^2 + c^2$?
27. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y and 10x + y, where x and y are digits. What is the sum of the digits of n?
28. If
$$3^x + 2^y = 985 \tag{28.1}$$
$$3^x - 2^y = 473 \tag{28.2}$$
what is the value of xy?
29. Let a, b and c be such that a + b + c = 0 and
$$p = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$
is defined. What is the value of P?
30. Suppose a, b are positive real numbers such that
$$a\sqrt{a} + b\sqrt{b} = 183$$
$$a\sqrt{b} + b\sqrt{a} = 182$$
Find $\frac{9}{5}(a + b)$?
31. Let a, b be integers such that all the roots of the equation
$$(x^2 + ax + 10)(x^2 + 17x + b) = 0 \tag{31.1}$$
are negative integers. What is the smallest possible value of a + b?
32. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose
$$u^{40} = v^n = w^{60}$$
Find the value of n.
33. Let the sum
$$\sum_{n=1}^{9} \frac{1}{n(n + 1)(n + 2)}$$
written in its lowest terms be $\frac{p}{q}$. Find the value of q - p?
34. Find the number of positive integers n, such that $\sqrt{n} + \sqrt{n + 1} < 11$.
35. Suppose x is a positive real number such that {x}, [x] and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here [x] denotes the integer part of x and {x} = x - [x].)
36. Integers 1, 2, 3,.....,n, where $n > 2$, are written on a board. Two numbers m, k such that $1 < m < n$, $1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?
37. If the real numbers x, y, z are such that
$$x^2 + 4y^2 + 16z^2 = 48 \tag{37.1}$$

$$xy + 4yz + 2zx = 24 \qquad (37.2)$$

What is the value of $x^2 + y^2 + z^2$?

38. Suppose 1, 2, 3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.
39. What is the number of triples (a, b, c) of positive integers such that
    a) $a < b < c < 10$ and
    b) a, b, c, 10 form the sides of a quadrilateral?
40. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.
41. Suppose an integer x, a natural number n and a prime number p satisfy the equation

$$7x^2 + 44x + 12 = p^n$$

Find the largest value of p.

42. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of 3pq for all positive integers n. Find the least possible value of p + q.
43. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.
44. Integers a, b, c satisfy a + b - c = 1 and

$$a^2 + b^2 - c^2 = -1 \qquad (44.1)$$

What is the sum of all possible values of $a^2 + b^2 + c^2$?

45. Suppose a, b are integers and a + b is a root of

$$x^2 + ax + b = 0 \qquad (45.1)$$

What is the maximum possible value of b^2?

46. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, ..., \epsilon_8)$ such that $(\epsilon_1, \epsilon_2, ..., \epsilon_8) \in \{1, 1\}$ and $(\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ... + 8\epsilon_8)$ is a multiple of 3.
47. If

$$x = \cos 1^\circ \cos 2^\circ \cos 3^\circ .... \cos 89^\circ$$

$$y = \cos 2^\circ \cos 6^\circ \cos 10^\circ .... \cos 86^\circ$$

nearest to $\frac{2}{7} log_2(y/x)$?

48. Let a and b be natural numbers such that 2a - b, a - 2b and a + b are all distinct squares. hat is the smallest possible value of b?
49. If a, b, $c \geq 4$ are integers, not all equal, and 4abc = (a + 3)(b + 3)(c + 3), then what is the value of a + b + c?
50. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.
51. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29} + \frac{c}{31}} + \frac{b^2}{\frac{c}{29} + \frac{a}{31}} + \frac{c^2}{\frac{a}{29} + \frac{b}{31}} \geq n(a + b + c)$$

holds for all positive real numbers a, b, c.

52. Let

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

be a polynomial in which a_i is a non-negative integer for each i ∈ {0, 1, 2, 3,............ n}. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

53. Let $f(x) = x^2 + ax + b$. If for all nonzero real x

$$f(x + \frac{1}{x}) = f(x) + f(\frac{1}{x})$$

and the roots of f(x) = 0 are integers, what is the value of $a^2 + b^2$?

54. Let x_1 be a positive real number and for every integer $n \geq 1$ let

$$x_{n+1} 1 + x_1 x_2 .... x_{n-1} x_n$$

If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6?

55. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?
56. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with $b < c$ and $\frac{b}{c}$ in the reduced form. Whatis the value of a + b + c?
57. How many positive integers n are there such that $3 \leq n \leq 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?
58. Let the rational number $\frac{p}{q}$ be closest to but not equal to $\frac{22}{7}$ among all rational numbers with denominator < 100. What is the value of p - 3q?
59. A natural number $k > 1$ is called good if there exist natural numbers

$$a_1 < a_2 < ...... < a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + ... + \frac{1}{\sqrt{a_k}} = 1$$

Let f(n) be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which f(n + 5)/f(n) is an integer.

60. Each of the numbers $x_1, x_2, ..., x_{101}$ is $\pm$ 1. What is the smallest positive value of

$$\sum_{1 \leq i < j \leq 101} x_i x_j.$$

61. Find the smallest positive integer $n \geq 10$ such that n + 6 is a prime and 9n + 7 is a perfect square.
62. Find the number of ordered triples (a, b, c) of positive integers such that $30a+50b+70c \leq 343$
63. How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \leq a, b \leq 1000$ satisfy gcd(a, b) : lcm(a, b) = 1 : 495?
64. What is the greatest integer not exceeding the sum

$$\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}.$$

65. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set {1, 2, 3,......, n} can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.
66. Positive integers x, y, z satisfy xy + z = 160. Compute the smallest possible value of x + yz.