Geometry through Linear Algebra

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Contents

***Abstract*—This textbook introduces linear algebra by exploring Euclidean geometry.**

1 The Straight Line

1.1 The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.1}$$

$$\implies x_2 = mx_1 \tag{1.2}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.3}$$

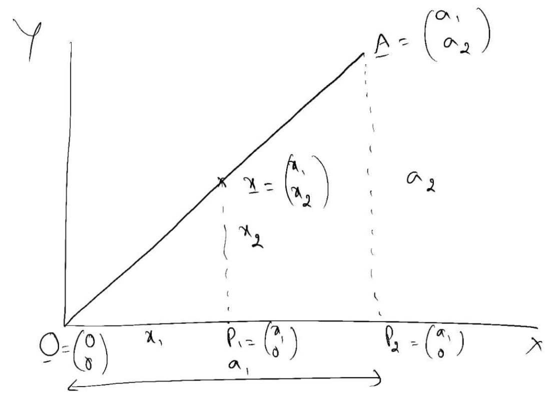

Fig. 1.1

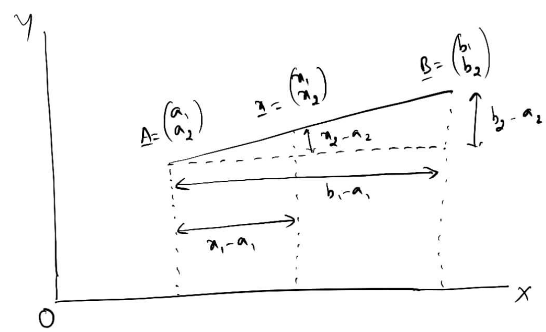

Fig. 1.2

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.4}$$

1.2 Find the equation of AB in Fig. 1.2

Solution: From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.5}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.6}$$

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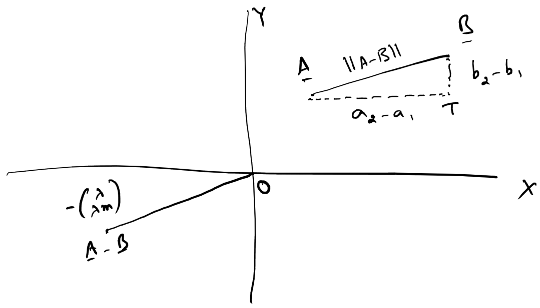

Fig. 1.4

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.7}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8}$$

$$= \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.9}$$

1.3 Find the length of **A** in Fig. 1.1
Solution: Using Baudhayana's theorem, the length of the vector **A** is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.10}$$

Also, from (1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.11}$$

Note that λ is the variable that determines the length of **A**, since m is constant for all points on the line.

1.4 Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.4. From (1.9), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.12}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.13}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.4.

1.5 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

2 Orthogonality

2.1 See Fig. 2.1. In $\triangle ABC, AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.1}$$

Solution: Using Baudhayana's theorem,

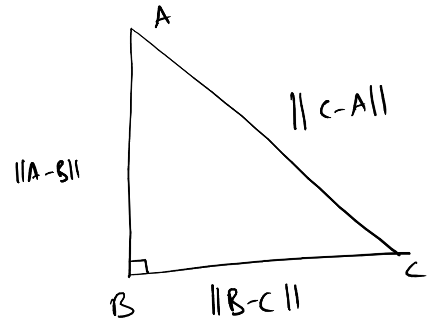

Fig. 2.1

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{2.2}$$

$$\begin{aligned} \implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) \\ = (\mathbf{C} - \mathbf{A})^T(\mathbf{C} - \mathbf{A}) \\ \implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \end{aligned} \tag{2.3}$$

which can be simplified to obtain (2.1).

2.2 Let **x** be any point on AB in Fi.g 2.1. Show that

$$(\mathbf{x} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.4}$$

2.3 If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x} - \mathbf{y})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.5}$$

2.4 In Fig. 2.4, $BE \perp AC, CF \perp AB$. Show that $AD \perp BC$.

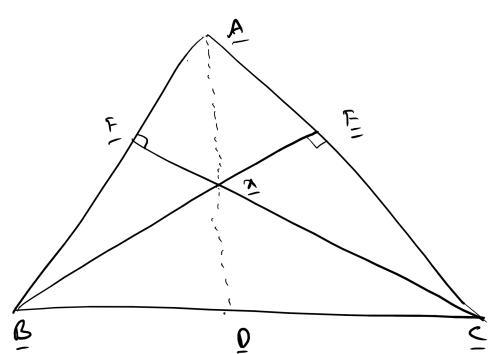

Fig. 2.4

Solution: Let **x** be the intersection of BE and

CF. Then, using (2.5),

$$\begin{aligned}(\mathbf{x}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) &= 0 \\ (\mathbf{x}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) &= 0\end{aligned} \tag{2.6}$$

$$\implies \mathbf{x}^T(\mathbf{A}-\mathbf{C}) - \mathbf{B}^T(\mathbf{A}-\mathbf{C}) = 0 \tag{2.7}$$

$$\text{and } \mathbf{x}^T(\mathbf{A}-\mathbf{B}) - \mathbf{C}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{2.8}$$

Subtracting (2.8) from ,

$$\mathbf{x}^T(\mathbf{B}-\mathbf{C}) + \mathbf{A}^T(\mathbf{C}-\mathbf{B}) = 0 \tag{2.9}$$

$$\implies \left(\mathbf{x}^T - \mathbf{A}^T\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{2.10}$$

$$\implies (\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.11}$$

which completes the proof.

$$\mathbf{x} = \mathbf{b} + \lambda\begin{pmatrix}1\\m\end{pmatrix}, \tag{2.12}$$

where

$$m = \frac{c_2 - c_1}{b_2 - b_1} \tag{2.13}$$

2.5

$$\frac{AB}{BC} = k \tag{2.14}$$

Show that

$$\mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \tag{2.15}$$

3 Medians of a triangle

Consider $\triangle ABC$ with vertices represented by the vectors $\mathbf{x}_1$

3.1 Get the **audio_source**

```
svn checkout https://github.com/gadepall/
    EE5347/trunk/audio_source
cd audio_source
```

3.2 Play the **signal_noise.wav** and **noise.wav** file. Comment.
Solution: signal_noise.wav contains a human voice along with an instrument sound in the background. This instrument sound is captured in **noise.wav**.

4 Problem Formulation

4.1 See Table 4.1. The goal is to extract the human voice $e(n)$ from $d(n)$ by suppressing the component of $\mathbf{X}(n)$. Formulate an equation for this.
Solution: The maximum component of $\mathbf{X}(n)$ in $d(n)$ can be estimated as

$$\mathbf{W}^T(n)\mathbf{X}(n) \tag{4.1}$$

Signal	Label	Type	Filename
Known	d(n)	Human+Instrument	signal noise.wav
	X(n)	Instrument	noise.wav
Unknown	e(n)	Human estimate	
	W(n)	Weight Vector	

TABLE 4.1

where

$$\mathbf{W}(n) = \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ .. \\ .. \\ w_{n-M+1}(n)\end{bmatrix}_{MX1} \tag{4.2}$$

Intuitively, the human voice $e(n)$ is obtained after removing as much of $\mathbf{X}(n)$ from $d(n)$ as possible. The first step in this direction is to estimate $\mathbf{W}$ in (4.1) using the metric

$$\min_{\mathbf{W}(n)} \|d(n) - \mathbf{W}^T(n)\mathbf{X}(n)\|^2 \tag{4.3}$$

The human voice can be then obtained as

$$e(n) = d(n) - \mathbf{W}^T(n)\mathbf{X}(n) \tag{4.4}$$

5 LMS Algorithm

5.1 Show using (4.4) that

$$\nabla_{\mathbf{W}(n)} e^2(n) = \frac{\partial e^2(n)}{\partial \mathbf{W}(n)} \tag{5.1}$$

$$= -2\mathbf{X}(n)d(n) + 2\mathbf{X}(n)X^T(n)\mathbf{W}(n) \tag{5.2}$$

5.2 Use the gradient descent method to obtain an algorithm for solving (4.3)
Solution: The desired algorithm can be expressed as

$$\mathbf{W}(n+1) = \mathbf{W}(n) - \bar{\mu}[\nabla_{\mathbf{W}(n)} e^2(n)] \tag{5.3}$$

$$\mathbf{W}(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \tag{5.4}$$

where $\mu = \bar{\mu}$.

5.3 Write a program to suppress $\mathbf{X}(n)$ in $d(n)$.
Solution: Execute

```
wget https://raw.githubusercontent.com/
    gadepall/EE5347/master/lms/codes/
    LMS_NC_SPEECH.py
```

6 Wiener-Hopf Equation

6.1 Using (4.4), show that

$$E[e^2(n)] = r_{dd} - W^T(n)r_{xd} - r_{xd}^T\mathbf{W}(n) + W^T(n)R\mathbf{W}(n) \quad (6.1)$$

where

$$r_{dd} = E[d^2(n)] \quad (6.2)$$
$$r_{xd} = E[\mathbf{X}(n)d(n)] \quad (6.3)$$
$$R = E[\mathbf{X}(n)\mathbf{X}^T(n)] \quad (6.4)$$

6.2 By computing

$$\frac{\partial J(n)}{\partial \mathbf{W}(n)} = 0, \quad (6.5)$$

show that the optimal solution for

$$W^*(n) = \min_{\mathbf{W}(n)} E\left[e^2(n)\right] = R^{-1}r_{xd} \quad (6.6)$$

This is the Wiener optimal solution.

7 Convergence of the LMS Algorithm

7.1 Convergence in the Mean

7.1.1 Show that R in (6.4) is symmetric as well as positive definite.
Let

$$\tilde{W}(n) = \mathbf{W}(n) - W_* \quad (7.1)$$

where W_* is obtained in (6.6). Also, according to the LMS algorithm,

$$W(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \quad (7.2)$$
$$e(n) = d(n) - X^T(n)\mathbf{W}(n) \quad (7.3)$$

7.1.2 Show that

$$E\left[\tilde{W}(n+1)\right] = [I - \mu R]E\left[\tilde{W}(n)\right] \quad (7.4)$$

7.1.3 Show that

$$R = U\Lambda U^T \quad (7.5)$$

for some U, Λ, such that Λ is a diagonal matrix and $U^TU = I$.

7.1.4 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n+1)\right] = 0 \iff \lim_{n\to\infty}[I - \mu\Lambda]^n = 0 \quad (7.6)$$

7.1.5 Using (7.6), show that

$$0 < \mu < \frac{2}{\lambda_{\max}} \quad (7.7)$$

where $\lambda_{\max}$ is the largest entry of Λ.

7.2 Convergence in Mean-square sense

Let

$$\mathbf{X}(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \end{bmatrix} \tilde{W}(n) = \begin{bmatrix} \tilde{W}_1(n) \\ \tilde{W}_2(n) \end{bmatrix} \quad (7.8)$$

7.2.1 Show that

$$E[\tilde{W}^T(n)\mathbf{X}(n)X^T(n)\tilde{W}(n)] = E[\tilde{W}^T(n)R\tilde{W}(n)] \quad (7.9)$$

for R defined in (6.4).

7.2.2 Show that

$$J(n) = E[e^2(n)] = E[e_*^2(n)] + E[\tilde{W}(n)\mathbf{X}(n)\mathbf{X}(n)^T\tilde{W}(n)^T] - E[\tilde{W}(n)\mathbf{X}(n)e_*(n)] - E[e_*(n)X^T(n)\tilde{W}^T(n)] \quad (7.10)$$

where

$$\tilde{W}(n) = W(n) - W_* \quad (7.11)$$
$$e_*(n) = d(n) - W_*\mathbf{X}(n) \quad (7.12)$$

7.2.3 Show that

$$E\left[\tilde{W}(n)\mathbf{X}(n)e_*(n)\right] = E\left[e_*(n)X^T(n)\tilde{W}^T(n)\right] = 0 \quad (7.13)$$

7.2.4 Show that

$$E\left[\tilde{W}^T(n)R\tilde{W}(n)\right] = \text{trace}\left(E\left[\tilde{W}^T(n)R\tilde{W}(n)\right]\right) \quad (7.14)$$
$$= \text{trace}\left(E\left[\tilde{W}(n)\tilde{W}^T(n)\right]R\right) \quad (7.15)$$

7.2.5 Using (7.11), (7.2) and (7.12), show that

$$\tilde{W}(n+1) = \left[I - \mu\mathbf{X}(n)X^T(n)\right]\tilde{W}(n) + \mu\mathbf{X}(n)e_*(n) \quad (7.16)$$

7.2.6 Let $\mu^2 \to 0$. Using (7.5) and (6.6), show that

$$E\left[\tilde{W}(n+1)\tilde{W}^T(n+1)\right] = (I - 2\mu R)\,E\left[\tilde{W}(n)\tilde{W}^n(n)\right] \quad (7.17)$$

7.2.7 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n)\tilde{W}^T(n)\right] = 0 \iff 0 < \mu < \frac{1}{\lambda_{max}} \quad (7.18)$$

7.2.8 Find the value of the cost function at infinity i.e. $J(\infty)$

7.2.9 How can you choose the value of μ from the convergence of both in mean and mean-square sense?