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Abstract—This manual shows how to generate figures encountered in high school geometry using python. The process provides simple applications of coordinate geometry.

1 LINE

Problem 1. Let

$$A = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \quad (1)$$

Draw $\triangle ABC$.

Solution: The following code yields the desired plot in Fig. 1

```
#This program draws the triangle
ABC
import numpy as np
import matplotlib.pyplot as plt
```

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```
def slope_coeff(A,B):
    p = np.zeros((2,1))
    p[0] = (A[1]-B[1])/(A[0]-B
    [0])
    p[1] = (A[0]*B[1]-A[1]*B
    [0])/(A[0]-B[0])
    return p

A = np.matrix(' -2;-2 ')
B = np.matrix(' 1;3 ')
C = np.matrix(' 4;-1 ')

x = np.linspace(np.asscalar(A[0]),
    np.asscalar(B[0]),50)
p = slope_coeff(A,B)
y = p[0]*x + p[1]
plt.plot(x,y,label='$5x-3y+4=0$')

plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 + 0.1), A[1] *
    (1 - 0.1) , 'A')

x = np.linspace(np.asscalar(B[0]),
    np.asscalar(C[0]),50)
p = slope_coeff(B,C)
y = p[0]*x + p[1]
plt.plot(x,y,label='$4x+3y-13=0$')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 - 0.2), B[1] *
    (1) , 'B')

x = np.linspace(np.asscalar(C[0]),
    np.asscalar(A[0]),50)
p = slope_coeff(C,A)
y = p[0]*x + p[1]
plt.plot(x,y,label='$x-6y-10=0$')
plt.plot(C[0], C[1], 'o')
plt.text(C[0] * (1 + 0.03), C[1] *
```

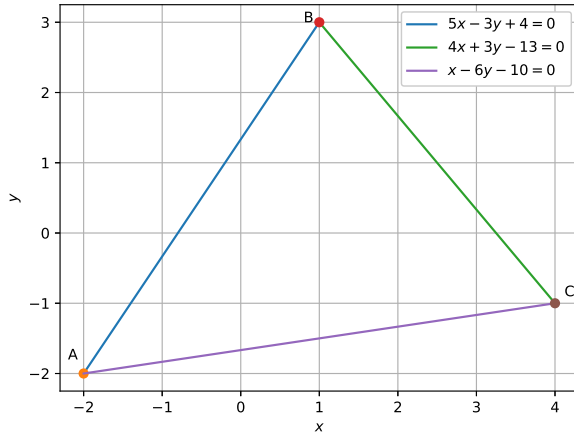


Fig. 1

(1 - 0.1) , 'C')

```
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend(loc='best')
#plt.savefig('../figs/triangle.eps')
plt.show()
```

Problem 2. Consider the line AB with

$$A = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad (2)$$

If AB is expressed by the equation

$$p_0x + p_1y + p_2 = 0, \quad (3)$$

find p_0, p_1 and p_2 .

Solution: Let

$$A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}, B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}, \quad (4)$$

The equation of AB is given by

$$\frac{y - A_1}{x - A_0} = \frac{A_1 - B_1}{A_0 - B_0} \quad (5)$$

$$\Rightarrow (A_1 - B_1)x + (B_0 - A_0)y + A_0B_1 - A_1B_0 = 0 \quad (6)$$

after some algebra. Thus,

$$p_0 = A_1 - B_1, p_1 = B_0 - A_0, p_2 = A_0B_1 - A_1B_0. \quad (7)$$

The following python code computes the numerical values and the equation for AB is

$$5x - 3y + 4 = 0 \quad (8)$$

```
import numpy as np
import matplotlib.pyplot as plt
```

```
A = np. matrix(' -2;-2 ')
B = np. matrix(' 1;3 ')
```

```
p = np.zeros((3,1))
p[0] = A[1]-B[1]
p[1] = B[0]-A[0]
p[2] = A[0]*B[1]- A[1]*B[0]
```

```
print (p)
```

Problem 3. Let

$$C = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \quad (9)$$

Find the equations of BC and CA

Solution: The following code yields the coefficients resulting in the respective equations

$$4x + 3y - 13 = 0 \quad (10)$$

$$x - 6y - 10 = 0 \quad (11)$$

```
#This program calculates the
#equations of
#all the sides of the triangle
import numpy as np
import matplotlib.pyplot as plt
```

```
def line_coeff(A,B):
    p = np.zeros((3,1))
    p[0] = A[1]-B[1]
    p[1] = B[0]-A[0]
    p[2] = A[0]*B[1]- A[1]*B
    [0]
    return p
```

```
A = np. matrix(' -2;-2 ')
B = np. matrix(' 1;3 ')
C = np. matrix(' 4;-1 ')
```

```
print (line_coeff(A,B))
print (line_coeff(B,C))
print (line_coeff(C,A))
```

Problem 4. An alternative equation of the line AB is

$$y = p_0x + p_1 \quad (12)$$

Find the equations of AB, BC and CA.

Solution: From (5),

$$p_0 = \frac{A_1 - B_1}{A_0 - B_0} \quad (13)$$

$$\begin{aligned} p_1 &= A_1 - A_0 \frac{A_1 - B_1}{A_0 - B_0} \\ &= \frac{A_0 B_1 - A_1 B_0}{A_0 - B_0} \end{aligned} \quad (14)$$

The following python code calculates the above coefficients resulting in the equations for AB, BC, CA

$$y = 1.67x + 1.33 \quad (15)$$

$$y = -1.33x + 4.33 \quad (16)$$

$$y = 0.16x - 1.67 \quad (17)$$

Problem 5. Draw the lines AB, BC and CA.

2 MEDIANS OF A TRIANGLE

Problem 6. Find the coordinates of D, E and F of the mid points of AB, BC and CA respectively for $\triangle ABC$.

Solution: The coordinates of the mid points are given by

$$D = \frac{B+C}{2}, E = \frac{C+A}{2}, F = \frac{A+B}{2} \quad (18)$$

The following code computes the values resulting in

$$D = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}, E = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}, F = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \quad (19)$$

```
#This program calculates the mid
point between
#any two coordinates
import numpy as np
import matplotlib.pyplot as plt

def mid_pt(B,C):
    D = (B+C)/2
```

```
return D
```

```
A = np. matrix(' -2; -2 ')
B = np. matrix(' 1; 3 ')
C = np. matrix(' 4; -1 ')

print (mid_pt(B,C))
print (mid_pt(C,A))
print (mid_pt(A,B))
```

Problem 7. Find the equations of AD, BE and CF. These lines are the medians of $\triangle ABC$

Solution: Using the code in Problem 2 and simplifying, the respective equations are

$$2x - 3y - 2 = 0 \quad (20)$$

$$x - 1 = 0 \quad (21)$$

$$x + 3y - 1 = 0 \quad (22)$$

Problem 8. Find the point of intersection of AD and CF.

Solution: Let the respective equations be

$$p_0x + p_1 + p_2 = 0 \quad (23)$$

$$q_0x + q_1 + q_2 = 0 \quad (24)$$

This can be written as the matrix equation

$$\begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \quad (25)$$

The following code yields the point of intersection

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (26)$$

```
#This program calculates the
#intersection of two lines
#given their equations
import numpy as np
import matplotlib.pyplot as plt

def line_intersect(p,q):
    P = np.column_stack((p
        [0:2],q[0:2])).transpose
    ()
    c = -np.row_stack((p[2],q
        [2]))
    return np.matmul(np.linalg
        .inv(P),c)
```

```
p = np.matrix('2;-3;-2')
q = np.matrix('1;3;-1')
r = np.matrix('1;0;-1')

print(line_intersect(p,r))
```

Problem 9. Using the code in Problem 8, verify that G is the point of intersection of BE, CF as well as AD, BE . G is known as the centroid of $\triangle ABC$.

Problem 10. Graphically show that the medians of $\triangle ABC$ meet at the centroid.

Problem 11. Verify that

$$G = \frac{A + B + C}{3} \quad (27)$$

3 ALTITUDES OF A TRIANGLE

Definition 12. In $\triangle ABC$, Let P be a point on BC such that $AP \perp BC$. Then AP is defined to be an altitude of $\triangle ABC$.

Problem 13. Find the equation of AP .

Solution: Let the equation for BC and AP be

$$y = p_0x + p_1 \quad (28)$$

$$y = q_0x + q_1 \quad (29)$$

respectively. Since, $AP \perp BC$,

$$p_0q_0 = -1 \quad (30)$$

The equation for AP is then obtained as

$$y - A_1 = q_0(x - A_0) \quad (31)$$

$$\implies y = q_0x + A_1 - q_0A_0 \quad (32)$$

From the following python code, AP can be expressed as

$$y = 0.75x - 0.5 \quad (33)$$

```
#This program calculates the
#equation of the altitude
import numpy as np
import matplotlib.pyplot as plt

def slope_coeff(A,B):
    p = np.zeros((2,1))
    p[0] = (A[1]-B[1])/(A[0]-B[0])
    p[1] = (A[0]*B[1]-A[1]*B[0])/(A[0]-B[0])
```

```
    return p

def alt_coeff(p,A):
    q = np.zeros((2,1))
    q[0] = -1/p[0]
    q[1] = A[1] - q[0]*A[0]
    return q
```

```
A = np.matrix('2;-2')
B = np.matrix('1;3')
C = np.matrix('4;-1')
```

```
p = slope_coeff(B,C)
q = alt_coeff(p,A)
print (q)
```

Problem 14. Find the equations of the altitudes BQ and CR .

Solution: Using the code in Problem 13, the respective equations are

$$y = -6x + 9 \quad (34)$$

$$y = -0.6x + 1.4 \quad (35)$$

Problem 15. Find the point of intersection of AP and BQ .

Solution: Using the code in Problem 8, the desired point of intersection is

$$H = \begin{pmatrix} 1.407 \\ 0.56 \end{pmatrix} \quad (36)$$

Interestingly, BQ and CR also intersect at the same point. Thus, the altitudes of a triangle meet at a single point known as the *orthocentre*

Problem 16. Find P, Q, R .

Solution: P is the intersection of AP and BC . Thus, the code in Problem 8 can be used to find P . The desired coordinates are

$$P = \begin{pmatrix} 2.32 \\ 1.24 \end{pmatrix}, Q = \begin{pmatrix} 1.73 \\ -1.38 \end{pmatrix}, R = \begin{pmatrix} 0.03 \\ 1.38 \end{pmatrix} \quad (37)$$

Problem 17. Draw AP, BQ and CR and verify that they meet at H .

4 ANGLE BISECTORS OF A TRIANGLE

Definition 18. In $\triangle ABC$, let U be a point on BC such that $\angle BAU = \angle CAU$. Then AU is known as the angle bisector.

Problem 19. Find the length of AB , BC and CA

Solution: The length of CA is given by

$$CA = \sqrt{(C_0 - A_0)^2 + (C_1 - A_1)^2} \quad (38)$$

The following code calculates the respective values as

$$AB = 5.83, BC = 5, CA = 6.08 \quad (39)$$

```
#This program calculates the
    distance between
#two points
import numpy as np
import matplotlib.pyplot as plt

def side_length(A,B):
    return np.sqrt((A[0]-B[0])
        **2 + (A[1]-B[1])**2)

A = np. matrix(' -2;-2 ')
B = np. matrix(' 1;3 ')
C = np. matrix(' 4;-1 ')

print (side_length(A,B))
print (side_length(B,C))
print (side_length(C,A))
```

Problem 20. If AU , BV and CW are the angle bisectors, find the coordinates of U , V and W .

Solution: Using the section formula,

$$W = \frac{AW.B + WB.A}{AW + WB} = \frac{\frac{AW}{WB}.B + A}{\frac{AW}{WB} + 1} \quad (40)$$

$$= \frac{\frac{CA}{BC}.B + A}{\frac{CA}{BC} + 1} \quad (41)$$

$$= \frac{CA \times B + BC \times A}{BC + CA} \quad (42)$$

since the angle bisector has the property that

$$\frac{AW}{WB} = \frac{CA}{AB} \quad (43)$$

The following code computes the coordinates as

$$U = \begin{pmatrix} 2.47 \\ 1.04 \end{pmatrix}, V = \begin{pmatrix} 1.23 \\ -1.46 \end{pmatrix} \approx \begin{pmatrix} -0.35 \\ 0.75 \end{pmatrix} \quad (44)$$

```
#This program calculates point
#where the angle bisector meets
the
```

#opposite side

```
import numpy as np
import matplotlib.pyplot as plt
```

```
def side_length(A,B):
    return np.sqrt((A[0]-B[0])
        **2 + (A[1]-B[1])**2)
```

```
def angle_bisect_coord(b,c,B,C):
    return np.multiply((np.
        multiply(b,B)+np.
        multiply(c,C)),1/(b+c))
```

```
A = np. matrix(' -2;-2 ')
B = np. matrix(' 1;3 ')
C = np. matrix(' 4;-1 ')

a = side_length(B,C)
b = side_length(C,A)
c = side_length(A,B)
```

```
U = angle_bisect_coord(b,c,B,C)
V = angle_bisect_coord(c,a,C,A)
W = angle_bisect_coord(a,b,A,B)
```

```
print (U)
print (V)
print (W)
```

Problem 21. Find the intersection of AU and BV .

Solution: Using the code in Problem 8, the desired point of intersection is

$$I = \begin{pmatrix} 1.15 \\ 0.14 \end{pmatrix} \quad (45)$$

It is easy to verify that even BV and CW meet at the same point. I is known as the *incentre* of $\triangle ABC$.

Problem 22. Draw AU , BV and CW and verify that they meet at a point I .

Problem 23. Verify that

$$I = \frac{BC.A + CA.B + AB.C}{AB + BC + CA} \quad (46)$$

Problem 24. Let the perpendiculars from I to AB , BC and CA be IX , IY , IZ . Verify that

$$IX = IY = IZ = r \quad (47)$$

r is known as the inradius of $\triangle ABC$.

Solution: The distance of a point (a, b) from the line $p_0x + p_1y + p_2 = 0$ is given by

$$\frac{|ap_0 + bp_1 + p_2|}{\sqrt{p_0^2 + p_1^2}} \quad (48)$$

The following code computes IX.

```
#This program calculates the
inradius

import numpy as np
import matplotlib.pyplot as plt

def line_dist(I,p):
    return np.abs((I[0]*p[0]+I
        [1]*p[1]+p[2])/(np.sqrt(
        p[0]**2+p[1]**2)))

I = np. matrix('1.15;0.14')
A = np. matrix('-2;-2')
B = np. matrix('1;3')
C = np. matrix('4;-1')
AB = line_coeff(A,B)
BC = line_coeff(B,C)
CA = line_coeff(C,A)

print(line_dist(I,AB))
```

```
r=1.6
a=1.15
b=0.14
x=np.linspace(a-r,a+r,1000)
y1=b+np.sqrt((r)**2-((x-a)**2))
y2=b-np.sqrt((r)**2-((x-a)**2))
plt.plot(x,y1)
plt.plot(x,y2)
plt.grid()
plt.axis("equal")
#plt.savefig(' ../figs/incircle.eps
')
plt.show()
```

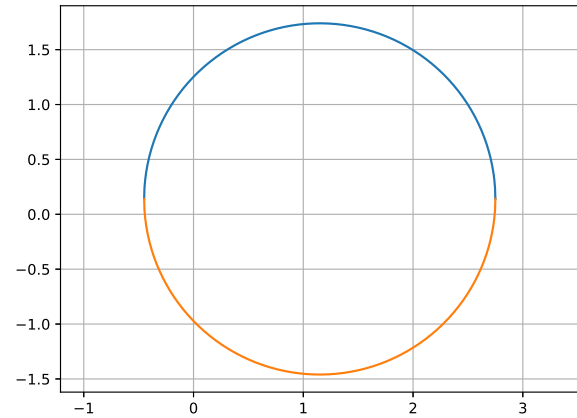


Fig. 26

5 CIRCLE

Definition 25. From Problem 24, it is obvious that a circle with centre at I and radius r passing through X, Y, Z can be drawn. The incircle is defined as the circle with centre at the incentre I and radius equal to the inradius.

Problem 26. Obtain the equation of the incircle of $\triangle ABC$ and draw it.

Solution: Letting $I = (a, b)$, the equation of the incircle is given by

$$(x - a)^2 + (y - b)^2 = r^2, \quad (49)$$

where r is the inradius. The following code plots this circle in Fig. 26

```
#This program plots the incircle
import numpy as np
import matplotlib.pyplot as plt
```

6 TANGENT AND DERIVATIVE

Definition 27. A line that meets the circle at exactly one point is known as a tangent to the circle.

Problem 28. Draw $\triangle ABC$ and its incircle in the same graph and verify that the lines AB, BC, CA are tangents to the incircle

Solution: Fig. 28 can be drawn using the codes in Problems 1 and 26. It is obvious from the figure that AB, BC and CA are tangents to the incircle.

Problem 29. Let the equation of AB be

$$p_0x + p_1y + p_2 = 0 \quad (50)$$

and the incircle be

$$(x - a)^2 + (y - b)^2 = r^2, \quad (51)$$

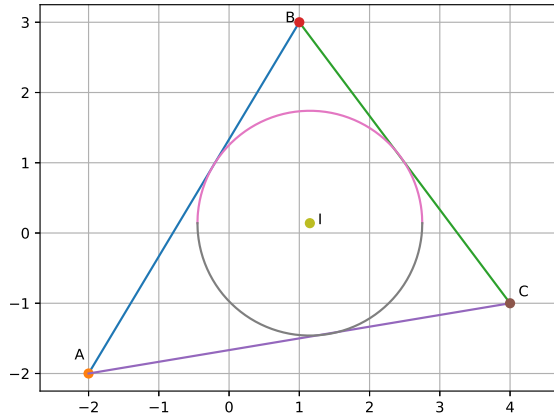


Fig. 28

Verify that

$$\left[p_0(p_2 + bp_1) - ap_1^2 \right]^2 = (p_0^2 + p_1^2) \left[p_1^2 a^2 + (p_2 + bp_1)^2 - p_1^2 r^2 \right] \quad (52)$$

and the point of contact

$$X = \left(\begin{array}{c} \frac{ap_1^2 - p_0(p_2 + bp_1)}{p_0^2 + p_1^2} \\ -\frac{p_2}{p_1} + \frac{p_0}{p_1} \frac{ap_1^2 - p_0(p_2 + bp_1)}{p_0^2 + p_1^2} \end{array} \right) \quad (53)$$

Solution: The following code computes the point of contact

```
#This program computes the point
#of contact between a circle
#and its tangent

import numpy as np
import matplotlib.pyplot as plt

def line_coeff(A,B):
    p = np.zeros((3,1))
    p[0] = A[1]-B[1]
    p[1] = B[0]-A[0]
    p[2] = A[0]*B[1]- A[1]*B
    [0]
    return p

A = np.matrix(' -2;-2')
B = np.matrix(' 1;3')
p = line_coeff(A,B)
r=1.6
a=1.15
```

b=0.14

```
print((p[0]*(p[2]+b*p[1])-a*p
[1]**2)**2)
print((p[0]**2+p[1]**2)*(p[1]**2*a
**2+(p[2]+b*p[1])**2-p[1]**2*r
**2))

X = np.zeros((2,1))
X[0] = -(p[0]*(p[2]+b*p[1])-a*p
[1]**2)/(p[0]**2+p[1]**2)
X[1] = -(p[2]+p[0]*X[0])/p[1]
print(X)
```

Problem 30. Verify that

$$AX = AZ \quad (54)$$

$$BX = BY \quad (55)$$

$$CY = CZ \quad (56)$$

Problem 31. Devise a method for calculating the slope of the tangent at X , given the equation of the circle and the point X .

Solution: In Fig. 31, it can be seen that the tangent at X has the same slope as the chord BC . From the equation of the circle,

$$(p_1 - a)^2 + (q_1 - b)^2 = r^2 \quad (57)$$

$$(p_2 - a)^2 + (q_2 - b)^2 = r^2 \quad (58)$$

which, after simplification, leads to the slope of the chord as

$$\frac{q_1 - q_2}{p_1 - p_2} = -\frac{p_1 + p_2 - 2a}{q_1 + q_2 - 2b} \quad (59)$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = -\frac{p_1 + p_2 - 2a}{q_1 + q_2 - 2b} \quad (60)$$

If we keep choosing smaller chords parallel to BC , p_1 and p_2 come closer while q_1 and q_2 come closer, without any change in the slope on the LHS. The limiting behaviour results in $p_1 = p_2 = p$ and $q_1 = q_2 = q$. This results in an expression for the slope of the tangent at X

$$\frac{dy}{dx} = -\frac{p - a}{q - b} \quad (61)$$

where

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}. \quad (62)$$

7 CONIC SECTIONS

Problem 33. Plot the circle

$$x^2 + y^2 = 1 \quad (63)$$

Solution:

```
#This program draws the unit
circle
import numpy as np
import matplotlib.pyplot as plt

r = 1
theta = np.linspace(-np.pi, np.pi, 50)
x = r*np.cos(theta)
y = r*np.sin(theta)

plt.plot(x,y)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
plt.show()
```

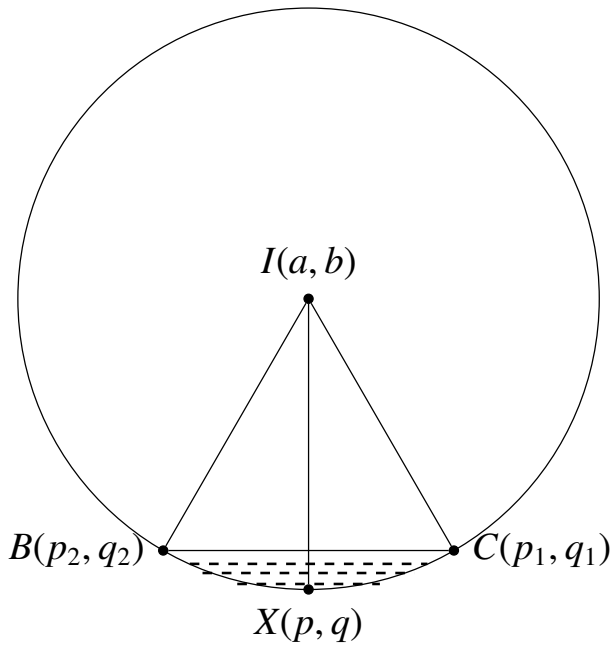


Fig. 31: Notion of the derivative.

Problem 32. Verify that the derivative of the circle at X is actually the slope of AB .**Solution:** The verification is done by the following program.

```
import numpy as np

def slope_coeff(A,B):
    p = np.zeros((2,1))
    p[0] = (A[1]-B[1])/(A[0]-B[0])
    p[1] = (A[0]*B[1]-A[1]*B[0])/(A[0]-B[0])
    return p

A = np.matrix(' -2; -2')
B = np.matrix(' 1; 3')
p = slope_coeff(A,B)
print(p[0])
X=np.matrix(' -0.22; 0.96')
I=np.matrix(' 1.15; 0.14')
print(-(X[0]-I[0])/(X[1]-I[1]))

#C=np.matrix(' 2.43; 1.09')
```

Problem 34. Show that (63) can be expressed as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \quad (64)$$

Problem 35. Show that

$$\begin{pmatrix} x & y & 1 \end{pmatrix} M^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \quad (65)$$

for

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad (66)$$

can be expressed as

$$x^2 + 2xy + y^2 - 4x - 2y - 1 = 0 \quad (67)$$

Problem 36. Show that (67) results in the curve in Fig. 36. This is known as a parabola.**Problem 37.** Show that using

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (68)$$

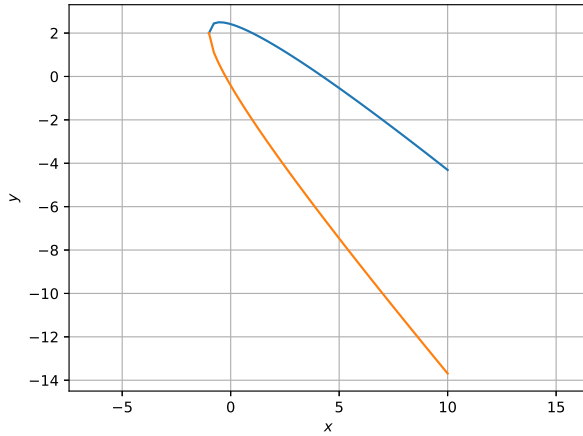


Fig. 36

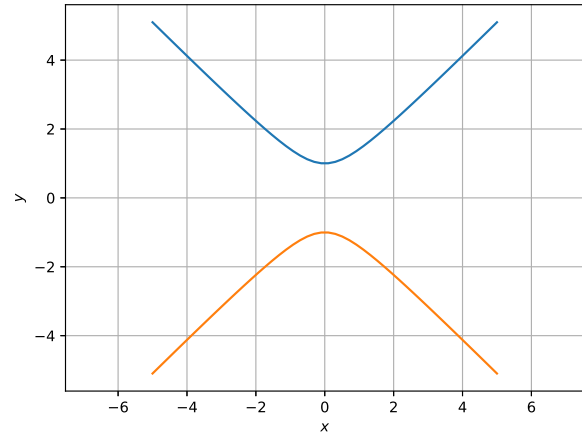


Fig. 38

in Problem 35 results in

$$y^2 - x^2 = 1 \quad (69)$$

Problem 38. Sketch (69) to obtain Fig. 38. This curve is known as a hyperbola

Solution:

```
#This program draws a hyperbola
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-5,5,50)
y1 = np.sqrt(1+x**2)
y2 = -np.sqrt(1+x**2)

plt.plot(x,y1,x,y2)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
plt.savefig(' ../figs/circle_
hyperbola.eps')
plt.show()
```

Problem 39. Generate the points

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \quad (70)$$

where x, y are the points generated in Problem 33. Plot y_1 with respect to x_1 . The figure that you obtain in Fig. 39 is known as an ellipse.

Solution:

#This program draws the triangle ABC

```
import numpy as np
import matplotlib.pyplot as plt
```

```
r = 1
theta = np.linspace(-np.pi,np.pi,50)
x = r*np.cos(theta)
y = r*np.sin(theta)
X = np.row_stack((x,y))
M = np.matrix('3,0;0,2')
Y = M*X
x1 = np.array(Y)[0]
y1 = np.array(Y)[1]
```

```
plt.plot(x1,y1)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
#plt.savefig(' ../figs/ellipse_
transform.eps')
plt.show()
```

Problem 40. Draw the curve

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad (71)$$

Comment.

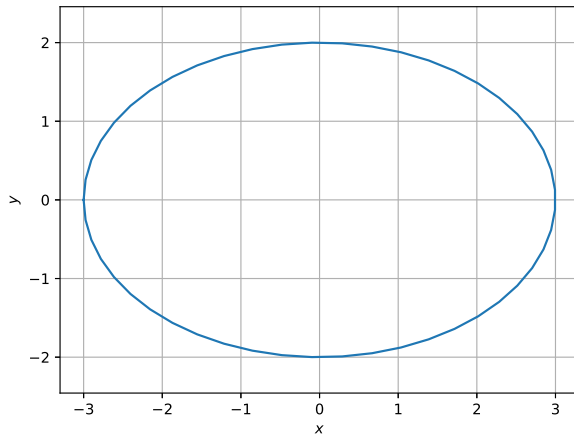


Fig. 39

8 AREA WITHIN A PARABOLA

Problem 41. Sketch the parabola

$$y^2 = x \quad (72)$$

Problem 42. Using n rectangles of equal width as shown in Fig. 42, find the limiting area of the parabola in $x \in (0, 1)$.

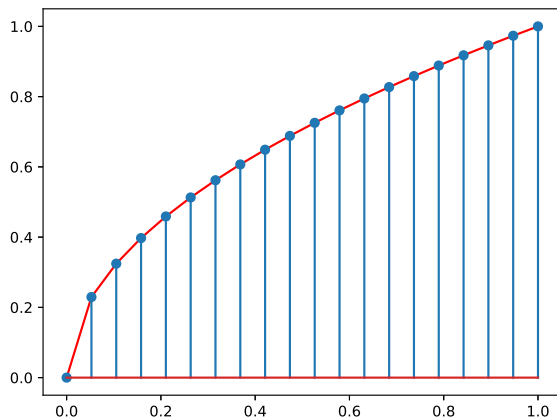


Fig. 42

Solution: Considering the width of the rectangle as $h = \frac{1}{n}$, $n = 100$, the approximate area of the parabola can be computed as

$$A = h * (\sqrt{h} + \sqrt{2h} + \dots + \sqrt{100h}) \quad (73)$$

$$\approx 0.67 \quad (74)$$

using the following program

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt

n = 100
h = 1/n
x = np.linspace(1,n,n)
y = np.sqrt(h*x)
A = h*np.sum(y)
print(A)
```

8.1 Arithmetic Progression

Problem 43. Plot

$$y = x^2 \quad (75)$$

and verify that the area under this parabola for $x \in (0, 1)$ is $A_0 = 1 - A$.

Solution: The following code

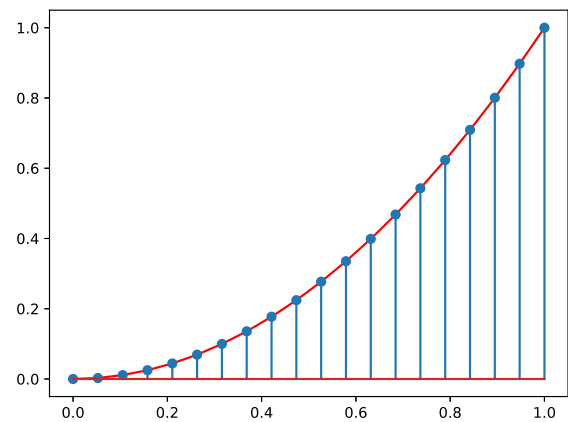


Fig. 43

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt

n = 100
h = 1/n
x = np.linspace(1,n,n)
y = (h*x)**2
A_1 = h*np.sum(y)
print(A_1)
```

yields the area in Fig.43 as $A_0 \approx 0.33 = 1 - A = 1 - 0.67$.

Problem 44. Show that the limiting area in Problem 43 is

$$A_0 = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^3 \sum_{k=1}^n k^2 = \frac{1}{3} \quad (76)$$

Solution: We have

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1 \quad (77)$$

$$\Rightarrow n^3 = 3 \sum_{k=1}^n k^2 - 3 \sum_{k=1}^n k + n \quad (78)$$

$$\Rightarrow \sum_{k=1}^n k^2 = \frac{1}{3} \left[n^3 + 3 \sum_{k=1}^n k - n \right] \quad (79)$$

Letting

$$S_n = 1 + 2 + \cdots + n \quad (80)$$

$$S_n = n + n - 1 + \cdots + 1 \quad (81)$$

$$\Rightarrow 2S_n = n(n+1) \quad (82)$$

$$\Rightarrow S_n = \frac{n(n+1)}{2} \quad (83)$$

Thus,

$$\sum_{k=1}^n k^2 = \frac{1}{3} \left[n^3 + 3 \frac{n(n+1)}{2} - n \right] \quad (84)$$

$$= \frac{n}{6} [2n^2 + 3n + 1] \quad (85)$$

$$= \frac{n(n+1)(2n+1)}{6} \quad (86)$$

and

$$A_0 = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^3 \sum_{k=1}^n k^2 \quad (87)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \quad (88)$$

$$= \frac{1}{3} \quad (89)$$

This process of finding the area under a curve is known as *integration*. Integration is the opposite of differentiation. The sequence that is summed in S_n is known as an *Arithmetic Progression*.

8.2 Geometric Progression

Problem 45. Plot the parabola

$$y = x^2 \quad (90)$$

for $x \in (0, 1)$ with points $(r^k, 0), k = 0, 1, \dots, n$ for $r = 0.8, n = 10$.

Solution: The following code

```
import numpy as np
import matplotlib.pyplot as plt

n = 10
r = 0.8
temp = [ r ** (k - 1) for k in
         range(1, n + 1)]
x = np.array(temp)
y = x**2
plt.plot(x, y, 'r')
plt.stem(x, y)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('../figs/parabola_area_
            _gp.eps')
plt.show()
```

plots Fig. 45

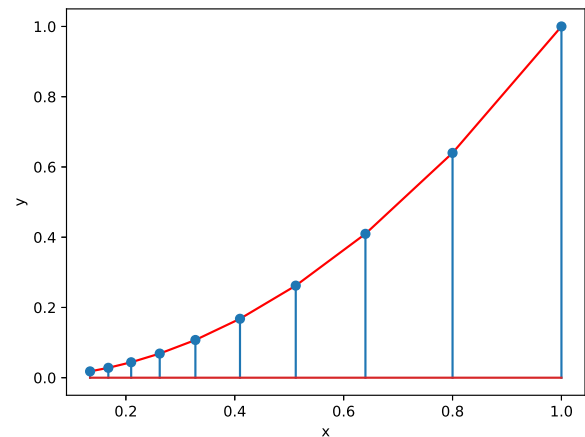


Fig. 45

Problem 46. Calculate the area of the parabola using Fig. 45 with $n = 100, r = 0.98$.

Solution: The intervals are of width $r^{k-1}(1-r), k = 1, \dots, n$. The corresponding heights are r^{2k-2} . Thus, the area is

$$A_0 = \sum_{k=1}^n r^{k-1} (1-r) r^{2k-2} \quad (91)$$

$$= (1-r) \sum_{k=1}^n r^{3k-3} \quad (92)$$

The following code calculates the desired area as 0.33

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt

n = 100
r = 0.98
k = np.linspace(1,n,n)
y = r** (3*k-3)
A = (1-r)*np.sum(y)
print(A)
```

Problem 47. Obtain the area in the previous problem as the limit of a sum.

Solution: Let

$$A_0 = \lim_{\substack{r \rightarrow 1 \\ n \rightarrow \infty}} (1-r) \sum_{k=1}^n r^{3k-3}, \quad r < 1 \quad (93)$$

If $p = r^3$,

$$S_n = \sum_{k=1}^n p^{k-1} \quad (94)$$

$$\Rightarrow pS_n = \sum_{k=1}^n p^k \quad (95)$$

$$\Rightarrow (1-p)S_n = 1 - p^n \quad (96)$$

$$\Rightarrow S_n = \frac{1 - p^n}{1 - p} \quad (97)$$

The sequence $p^{k-1}, k = 1, \dots, n$ is known as a *Geometric Progression*. Substituting in (93),

$$A_0 = \lim_{\substack{r \rightarrow 1 \\ n \rightarrow \infty}} (1-r) \frac{1 - r^{3n}}{1 - r^3} \quad (98)$$

$$= \lim_{\substack{r \rightarrow 1 \\ n \rightarrow \infty}} \frac{1 - r^{3n}}{1 + r + r^2} \quad (99)$$

$$= \frac{1}{3} \quad (\because r^{3n} \rightarrow 0) \quad (100)$$