Algebraic Approach to School Geometry

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Contents

***Abstract*—This book provides an equation based approach to school geometry based on the NCERT textbooks from Class 6-12.**

1 The Right Angled Triangle

1. A right angled triangle looks like Fig. 1.0.1. with angles $\angle A, \angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle C$ which is defined to be 90°.

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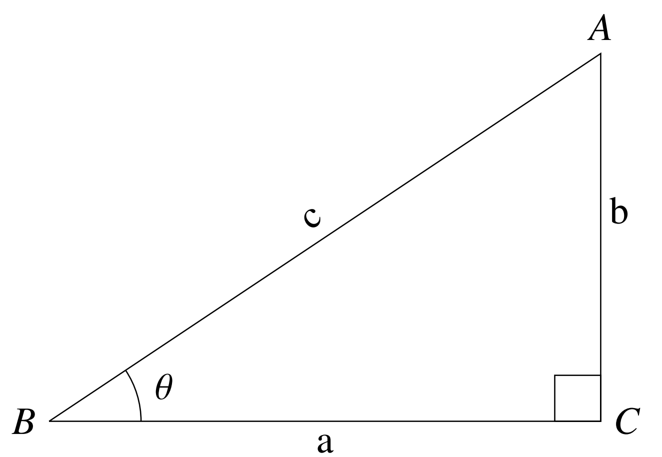

Fig. 1.0.1: Right Angled Triangle

2. For simplicity, let the greek letter $\theta = \angle B$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \frac{a}{c} & \cos\theta &= \frac{b}{c} \\ \tan\theta &= \frac{b}{a} & \cot\theta &= \frac{1}{\tan\theta} \\ \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} \end{aligned} \tag{1.0.2.1}$$

1.1 Sum of Angles

1. In Fig. 1.1.1, the sum of all the angles on the top or bottom side of the straight line XY is 180°.
2. In Fig. 1.1.1, the straight line making an angle of 90° to the side AC is said to be parallel to the side BC. Note there is an angle at A that is equal to θ. This is one property of parallel lines. Thus, $\angle YAZ = 90^\circ$.
3. Show that $\angle VAZ = 90^\circ - \theta$

   Proof. Considering the line XAZ,

$$\theta + 90^\circ + \angle VAZ = 180^\circ \tag{1.1.3.1}$$

$$\implies \angle VAZ = 90^\circ - \theta \tag{1.1.3.2}$$

   Show that $\angle BAC = 90^\circ - \theta$.

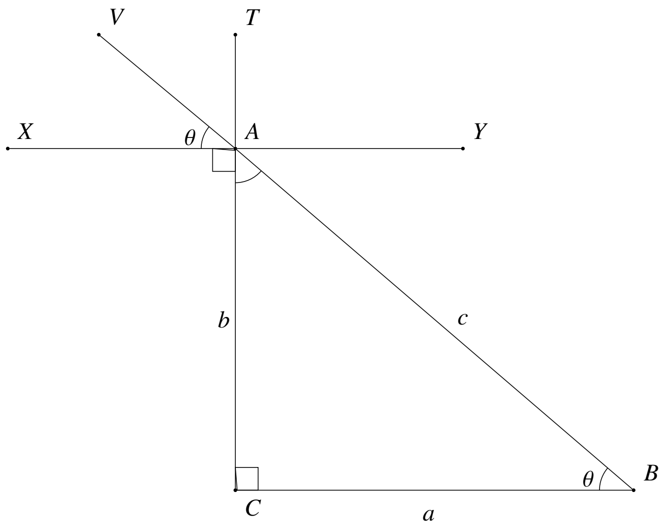

Fig. 1.1.1: Sum of angles of a triangle

Proof. Consider the line VAB and and use the approach in the previous problem. Note that this implies that $\angle VAZ = \angle BAC$. Such angles are known as vertically opposite angles.
Sum of the angles of a triangle is equal to 180°

1.2 Budhayana Theorem

1. Using Fig. 1.0.1, show that

$$\cos\theta = \sin\left(90^{\circ} - \theta\right) \tag{1.2.1.1}$$

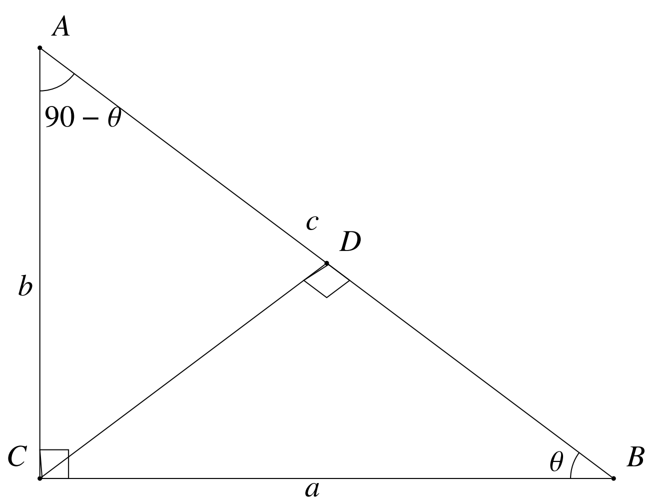

Fig. 1.2.1: Budhayana Theorem

Proof. From Problem 1.1.3 and (1.0.2.1)

$$\cos\left(90^{\circ} - \theta\right) = \frac{b}{c} = \sin\theta \tag{1.2.1.2}$$

Using Fig. 1.2.1, show that

$$c = a\cos\theta + b\sin\theta \tag{1.2.1.3}$$

Proof. We observe that

$$BD = a\cos\theta \tag{1.2.1.4}$$

$$AD = b\cos(90 - \theta) = b\sin\theta \quad \text{(From} \quad (1.1.3)) \tag{1.2.1.5}$$

Thus,

$$BD + AD = c = a\cos\theta + b\sin\theta \tag{1.2.1.6}$$

From (1.2.1.3), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.1.7}$$

Proof. Dividing both sides of (1.2.1.3) by c,

$$1 = \frac{a}{c}\cos\theta + \frac{b}{c}\sin\theta \tag{1.2.1.8}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.0.2.1)) \tag{1.2.1.9}$$

Using (1.2.1.3), show that

$$c^2 = a^2 + b^2 \tag{1.2.1.10}$$

(1.2.1.10) is known as the Budhayana theorem. It is also known as the Pythagoras theorem.
Proof. From (1.2.1.3),

$$c = a\frac{a}{c} + b\frac{b}{c} \quad \text{(from} \quad (1.0.2.1)) \tag{1.2.1.11}$$

$$\Rightarrow c^2 = a^2 + b^2 \tag{1.2.1.12}$$

2 Medians of a Triangle

2.1 Area of a Triangle

Definition 1. *The area of the rectangle ACBD shown in Fig. 1 is defined as ab. Note that all the angles in the rectangles are* 90°

Definition 2. *The area of the two triangles constituting the rectangle is the same.*

Definition 3. *The area of the rectangle is the sum of the areas of the two triangles inside.*

Problem 4. *Show that the area of* ΔABC *is* $\frac{ab}{2}$

Proof. From(3),

$$ar(ABCD) = ar(ACB) + ar(ADB) \tag{1.1}$$

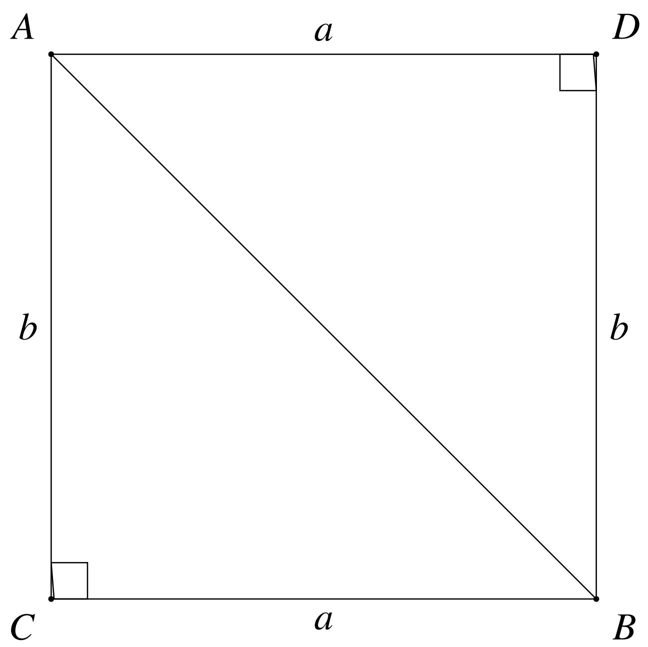

Fig. 1: Area of a Right Triangle

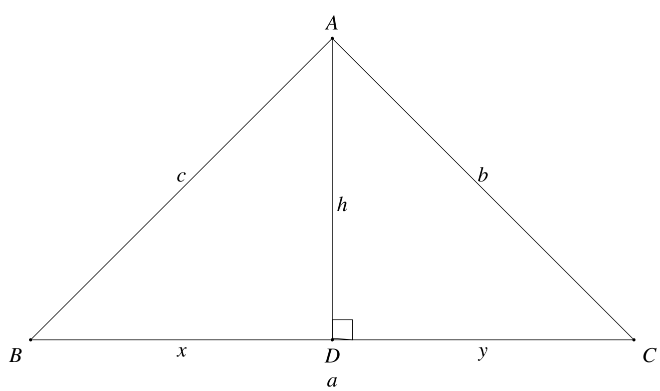

Fig. 1: Area of a Triangle

Also from (2),

$$ar(ACB) = ar(ADB) \tag{1.2}$$

From (1.1) and (1.2),

$$2ar(ACB) = ar(ABCD) = ab \,(\text{from} \quad (1)) \tag{1.3}$$

$$\Rightarrow ar(ACB) = \frac{ab}{2} \tag{1.4}$$

Problem 5. *Show that the area of ΔABC in Fig. 1 is $\frac{1}{2}ah$.*

Proof. In Fig. 1,

$$ar(\Delta ADC) = \frac{1}{2}hy \tag{1.5}$$

$$ar(\Delta ADB) = \frac{1}{2}hx \tag{1.6}$$

Thus,

$$ar(\Delta ABC) = ar(\Delta ADC) + ar(\Delta ADB) \tag{1.7}$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h(x+y) \tag{1.8}$$

$$= \frac{1}{2}ah \tag{1.9}$$

Problem 6. *Show that the area of ΔABC in Fig. 1 is $\frac{1}{2}ab\sin C$.*

Proof. We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{1.10}$$

Problem 7. *Show that*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.11}$$

Proof. Fig. 1 can be suitably modified to obtain

$$ar(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{s}bc\sin A = \frac{1}{2}ca\sin B \tag{1.12}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.13}$$

This is known as the sine formula.

2.2 Median

Definition 8. *The line AD in Fig. 1 that divides the side a in two equal halfs is known as the median.*

Problem 9. *Show that the median AD in Fig. 1 divides ΔABC into triangles ADB and ADC that have equal area.*

Proof. We have

$$ar(\Delta ADB) = \frac{1}{2}\frac{a}{2}c\sin B = \frac{1}{4}ac\sin B \tag{1.1}$$

$$ar(\Delta ADC) = \frac{1}{2}\frac{a}{2}b\sin C = \frac{1}{4}ab\sin C \tag{1.2}$$

Using the sine formula, $b\sin C = c\sin B$,

$$ar(\Delta ADB) = ar(\Delta ADC) \tag{1.3}$$

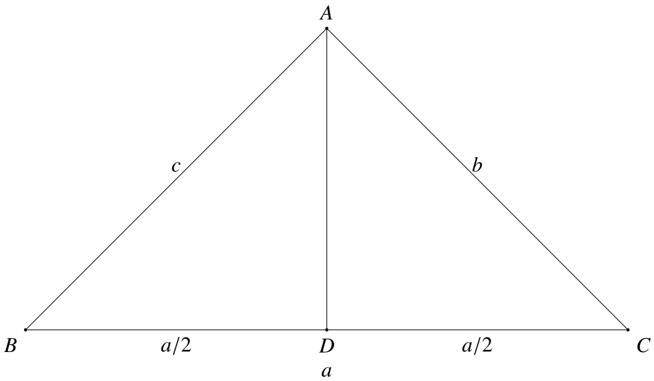

Fig. 1: Median of a Triangle

Problem 10. *BE and CF are the medians in Fig. 1. Show that*

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{1.4}$$

Proof. Since BE and CF are the medians,

$$ar(\Delta BFC) = \frac{1}{2} ar(\Delta ABC) \tag{1.5}$$

$$ar(\Delta BEC) = \frac{1}{2} ar(\Delta ABC) \tag{1.6}$$

From the above, we infer that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{1.7}$$

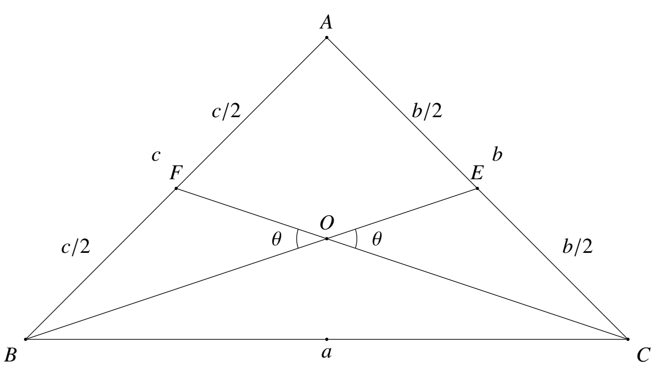

Fig. 1: O is the Intersection of Two Medians

Problem 11. *The medians BE and CF in Fig. 1 meet at point O. Show that*

$$\frac{OB}{OE} = \frac{OC}{OF} \tag{1.8}$$

Proof. From Problem 10,

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{1.9}$$

$$\Rightarrow ar(\Delta BOF) + ar(\Delta BOC) = ar(\Delta BOC) + ar(\Delta COE) \tag{1.10}$$

resulting in

$$ar(\Delta BOF) = ar(\Delta COE) \tag{1.11}$$

Using the sine formula for area of a triangle, the above equation can be expressed as

$$\frac{1}{2} OB\, OF \sin\theta = \frac{1}{2} OC\, OE \sin\theta \tag{1.12}$$

$$\Rightarrow \frac{OB}{OE} = \frac{OC}{OF} \tag{1.13}$$

Definition 12. *We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 1,*

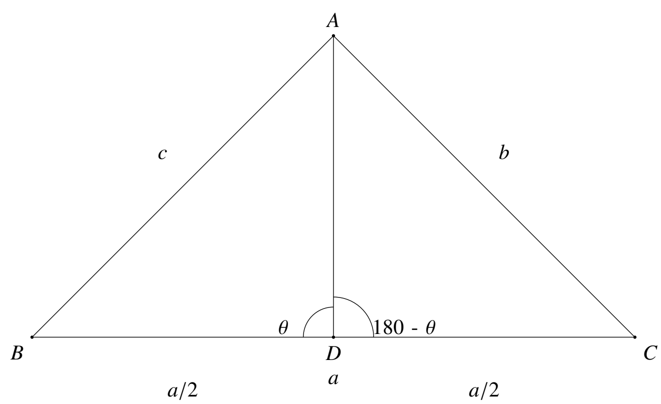

Fig. 1: $\sin\theta = \sin\left(180^{\circ} - \theta\right)$

$$\frac{1}{2}\frac{a}{2} AD \sin\theta = \frac{1}{2}\frac{a}{2} AD \sin\left(180^{\circ} - \theta\right) \tag{1.14}$$

$$\Rightarrow \sin\theta = \sin\left(180^{\circ} - \theta\right). \tag{1.15}$$

Note that our geometric definition of $\sin\theta$ *holds only for* $\theta < 90^{\circ}$. (1.15) *allows us to extend this definition for* $\angle ADC > 90^{\circ}$.

2.3 Similar Triangles

Problem 13. *In Fig. 1, show that*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1}$$

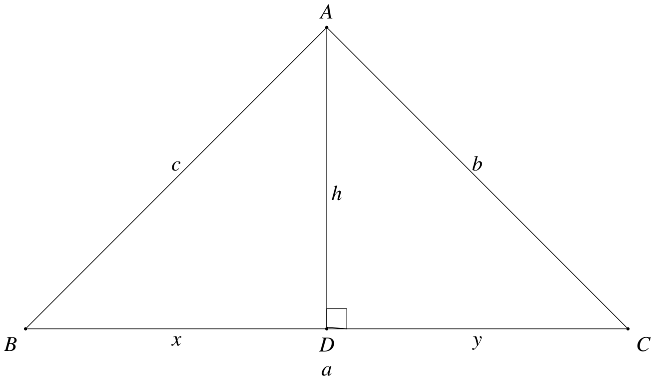

Fig. 1: The cosine formula

Proof. From the figure, the first of the following equations

$$a = b\cos C + c\cos B \tag{1.2}$$
$$b = c\cos A + a\cos C \tag{1.3}$$
$$c = b\cos A + a\cos B \tag{1.4}$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5}$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{1.6}$$

Problem 14. *In Fig. 1, show that $EF = \frac{a}{2}$.*

Proof. Using the cosine formula for ΔAEF,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \tag{1.7}$$
$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \tag{1.8}$$
$$= \frac{a^2}{4} \tag{1.9}$$
$$\Rightarrow EF = \frac{a}{2} \tag{1.10}$$

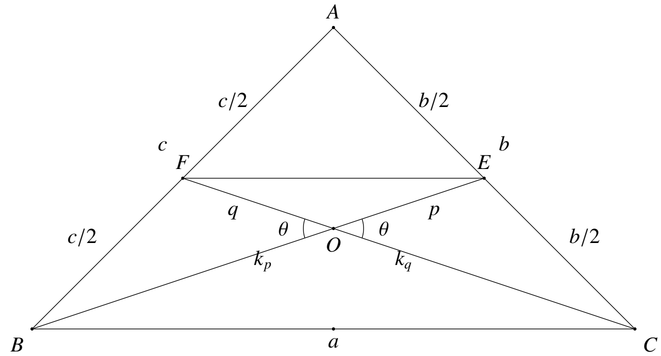

Fig. 1: Similar Triangles

Definition 15. *The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.*

Problem 16. *Show that similar triangles have the same angles.*

Proof. Use cosine formula and the proof is trivial.

Problem 17. *Show that in Fig. 1, $EF \| BC$.*

Proof. Since $\Delta AEF \sim \Delta ABC$, $\angle AEF = \angle ACB$. Hence the line $EF \| BC$

Problem 18. *Show that $\triangle OEF \sim \triangle OEC$.*

Problem 19. *Show that*

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \tag{1.11}$$

Problem 20. *In Fig. 1,BE is a median of ΔABC and $\frac{OB}{OE} = 2$. Show that AD is also a median.*

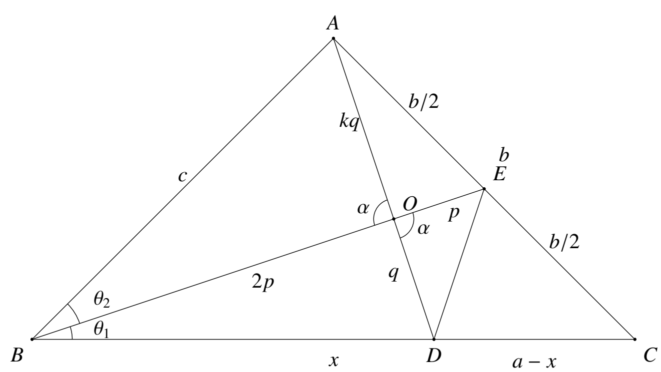

Fig. 1: Medians meet at a point

Proof. Since BE is a median, the areas of triangles $\triangle BEC$ and $\triangle BEA$ are equal. Hence,

$$\frac{1}{2}BE.BC\sin\theta_1 = \frac{1}{2}BE.AB\sin\theta_2 \tag{1.12}$$
$$\implies a\sin\theta_1 = c\sin\theta_2 \tag{1.13}$$

Using the sine formula in $\triangle OBD$ and $\triangle BOA$,

$$\frac{q}{\sin\theta_1} = \frac{x}{\sin\alpha} \tag{1.14}$$
$$\frac{c}{\sin\alpha} = \frac{kq}{\sin\theta_2} \tag{1.15}$$

Mutiplying (1.13),(1.14) and (1.15), we obtain

$$x = \frac{a}{k} \tag{1.16}$$

after simplification. Let the area of $\triangle ABC = \Delta$. Then, in terms of area,

$$\triangle ABD = \triangle OBD + \triangle OBA \tag{1.17}$$
$$\implies \frac{1}{2}(c)(x)\sin B = \frac{1}{2}(2p)(q)\sin\alpha \tag{1.18}$$
$$+ \frac{1}{2}(2p)(kq)\sin\alpha \tag{1.19}$$
$$\implies k(k+1)pq\sin\alpha = \frac{1}{2}ca\sin B = \Delta \tag{1.20}$$

Similarly,

$$\triangle ABE = \triangle AOB + \triangle AOE \tag{1.21}$$
$$\implies \frac{\Delta}{2} = \frac{1}{2}(2p)(kq)\sin\alpha \tag{1.22}$$
$$+ \frac{1}{2}(p)(kq)\sin\alpha \tag{1.23}$$
$$\implies 3kpq\sin\alpha = \Delta \tag{1.24}$$

after simplification. From (1.20) and (1.24),

$$k(k+1)pq\sin\alpha = 3kpq\sin\alpha \tag{1.25}$$
$$\implies k = 2. \tag{1.26}$$

Thus,

$$BD = DC \tag{1.27}$$
$$\frac{OA}{OD} = 2. \tag{1.28}$$

Thus, AD is a median.
Conculusion: The medians of a triangle meet at a point.

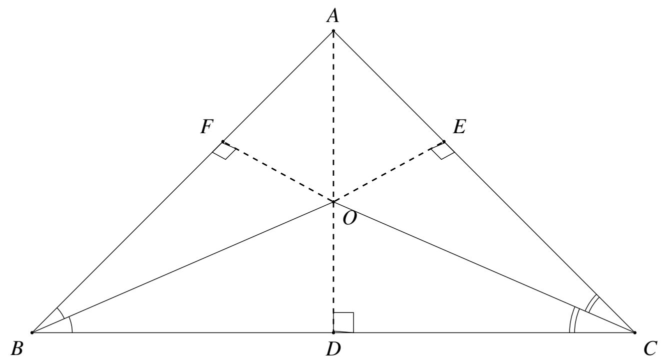

Fig. 1: Angle bisectors meet at a point

3 Angle and Perpendicular Bisectors

3.1 Angle Bisectors

Definition 21. *In Fig. 1, OB divides the $\angle B$ into half, i.e.*

$$\angle OBC = \angle OBA \tag{1.1}$$

OB is known as an angle bisector.

OB and OC are angle bisectors of angles B and C. OA is joined and OD, OF and OE are perpendiculars to sides a, b and c.

Problem 22. *Show that $OD = OE = OF$.*

Proof. In Δs ODC and OEC,

$$OD = OC\sin\frac{C}{2} \tag{1.2}$$
$$OE = OC\sin\frac{C}{2} \tag{1.3}$$
$$\Rightarrow OD = OE. \tag{1.4}$$

Similarly,

$$OD = OF. \tag{1.5}$$

Problem 23. *Show that OA is the angle bisector of $\angle A$*

Proof. In Δs OFA and OEA,

$$OF = OE \tag{1.6}$$
$$\Rightarrow OA\sin OAF = OA\sin OAE \tag{1.7}$$
$$\Rightarrow \sin OAF = \sin OAE \tag{1.8}$$
$$\Rightarrow \angle OAF = \angle OAE \tag{1.9}$$

which proves that OA bisects $\angle A$. *Conclusion:* The angle bisectors of a triangle meet at a point.

3.2 Congruent Triangles

Problem 24. *Show that in* Δ*s* ODC *and* OEC*, corresponding sides and angles are equal.*

Definition 25. *Note that* Δ*s* ODC *and* OEC *are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.*

Problem 26. *SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.*

Problem 27. *ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.*

Problem 28. *SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.*

Problem 29. *RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.*

3.3 Perpendicular Bisectors

Definition 30. *In Fig. 1,* $OD \perp BC$ *and* $BD = DC$*.* OD *is defined as the perpendicular bisector of* BC*.*

Problem 31. *In Fig. 1, show that* $OA = OB = OC$*.*

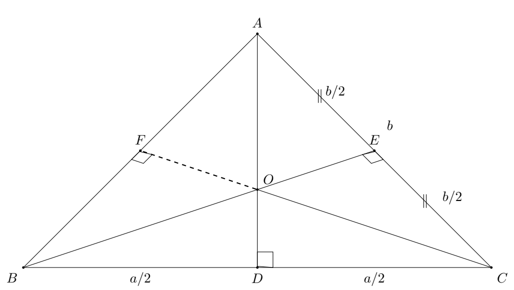

Fig. 1: Perpendicular bisectors meet at a point

Proof. In Δs ODB and ODC, using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \tag{1.1}$$

Since $BD = DC = \frac{a}{2}$, $OB = OC$. Similarly, it can be shown that $OA = OC$. Thus, $OA = OB = OC$.

Definition 32. *In* ΔAOB*,* $OA = OB$*. Such a triangle is known as an isoceles triangle.*

Problem 33. *Show that* $AF = BF$*.*

Proof. Trivial using Budhayana's theorem. This shows that OF is a perpendicular bisector of AB. *Conclusion:* The perpendicular bisectors of a triangle meet at a point.

3.4 Perpendiculars from Vertex to Opposite Side

In Fig. 1, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F.

Problem 34. *Show that*

$$OE = c \cos A \cot C \tag{1.1}$$

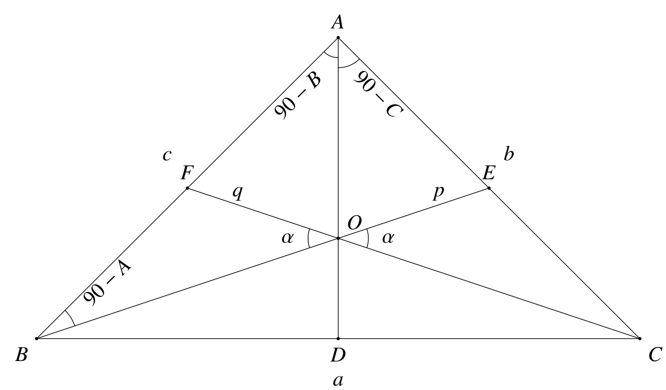

Fig. 1: Perpendiculars from vertex to opposite side meet at a point

Proof. In Δ s AEB and AEO,

$$AE = c \cos A \tag{1.2}$$

$$OE = AE \tan\left(90^{\circ} - C\right) (\because ADC \text{ is right angled}) \tag{1.3}$$

$$= AE \cot C \tag{1.4}$$

From both the above, we get the desired result.

Problem 35. *Show that* $\alpha = A$*.*

Proof. In ΔOEC,

$$CE = a \cos C \, (\because BEC \text{ is right angled}) \tag{1.5}$$

Hence,

$$
\begin{aligned}
\tan\alpha &= \frac{CE}{OE} \\
&= \frac{a\cos C}{c\cos A\cot C} \\
&= \frac{a\cos C\sin C}{c\cos A\cos C} \\
&= \frac{a\sin C}{c\cos A} \\
&= \frac{c\sin A}{c\cos A}\left(\because \frac{a}{\sin A} = \frac{c}{\sin C}\right) \\
&= \tan A
\end{aligned}
\tag{1.6}
$$

$$\Rightarrow \alpha = A$$

Problem 36. *Show that $CF \perp AB$*

Proof. Consider triangle OFB and the result of the previous problem. $\because$ the sum of the angles of a triangle is 180°, $\angle CFB = 90^\circ$. *Conclusion: The perperdiculars from the vertex of a triangle to the opposite side meet at a point.*

4 Circle

4.1 Chord of a Circle

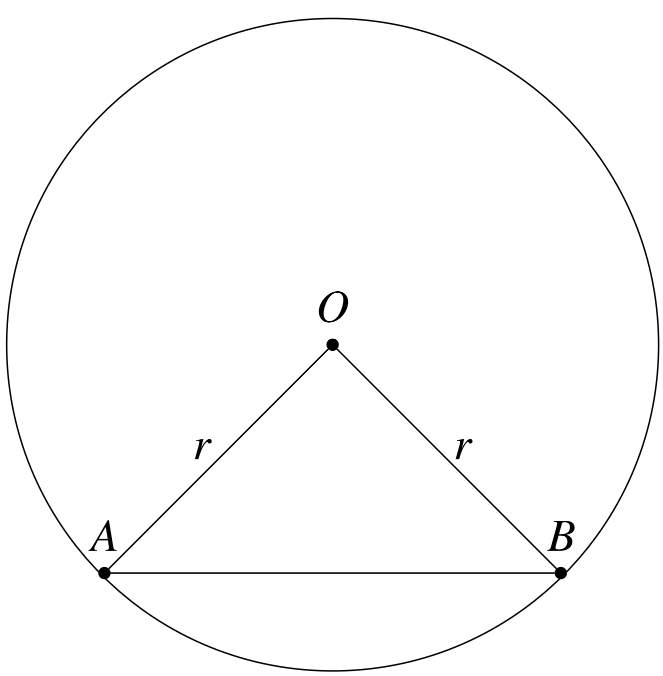

Fig. 1: Circle Definitions

Definition 37. *Fig. 1 represents a circle. The points in the circle are at a distance r from the centre O. r is known as the radius.*

4.2 Chords of a circle

Definition 38. *In Fig. 1, A and B are points on the circle. The line AB is known as a chord of the circle.*

Problem 39. *In Fig. 1 Show that $\angle AOB = 2\angle ACB$.*

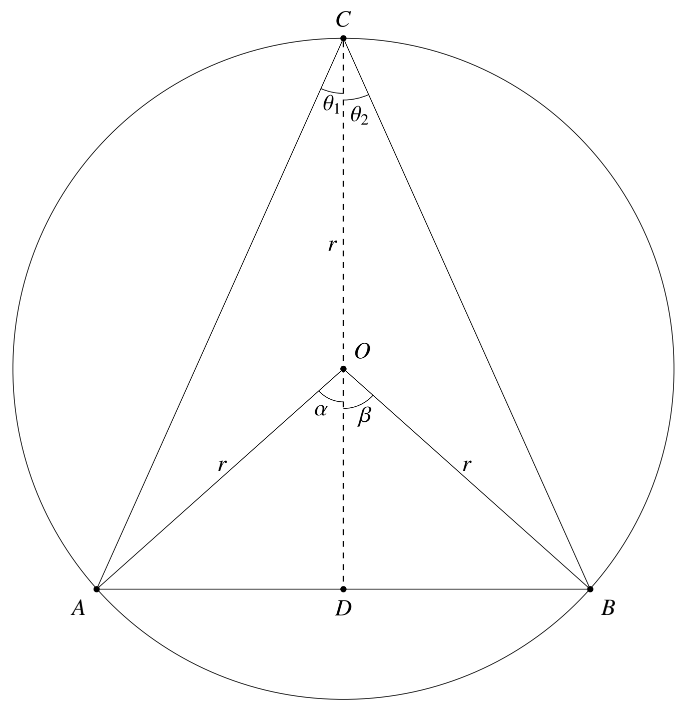

Fig. 1: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Proof. In Fig. 1, the triangeles OPA and OPB are isosceles. Hence,

$$\angle OCA = \angle OAC = \theta_1 \tag{1.1}$$
$$\angle OCB = \angle OBC = \theta_2 \tag{1.2}$$

Also, α and β are exterior angles corresponding to the triangle AOC and BOC respectively. Hence

$$\alpha = 2\theta_1 \tag{1.3}$$
$$\beta = 2\theta_2 \tag{1.4}$$

Thus,

$$
\begin{aligned}
\angle AOB &= \alpha + \beta &(1.5)\\
&= 2(\theta_1 + \theta_2) &(1.6)\\
&= 2\angle ACB &(1.7)
\end{aligned}
$$

Definition 40. *The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 1, AB is the diameter and passes through the centre O*

Problem 41. *In Fig. 1, show that* $\angle APB = 90^\circ$.

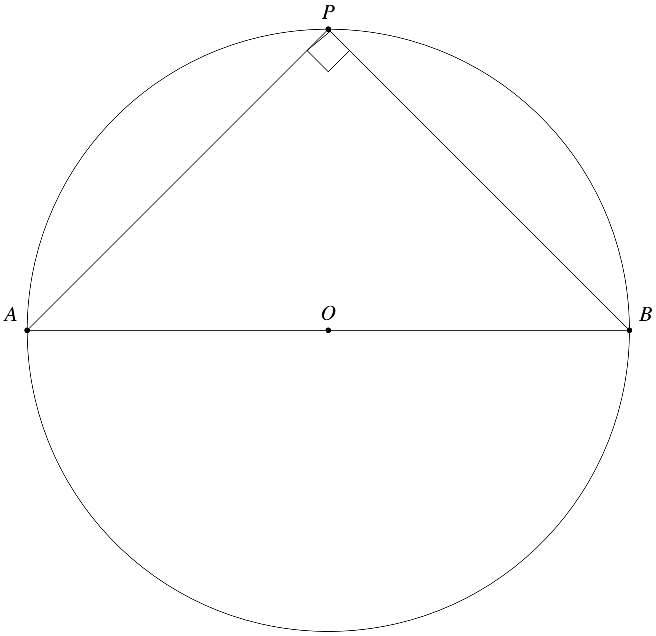

Fig. 1: Diameter of a circle.

Problem 42. *In Fig. 1, show that*

$$\begin{aligned}\angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB\end{aligned} \tag{1.8}$$

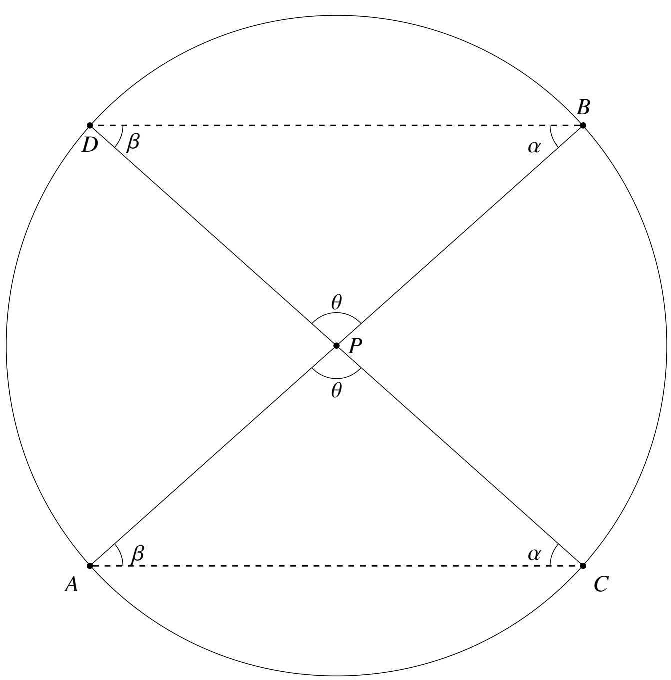

Fig. 1: $PA.PB = PC.PD$

Proof. Use Problem 39.

Problem 43. *In Fig. 1, show that the triangles PAB and PBD are similar*

Proof. Trivial using previous problem

Problem 44. *In Fig. 1, show that*

$$PA.PB = PC.PD \tag{1.9}$$

Proof. Since triangles PAC and PBD are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{1.10}$$

$$\Rightarrow PA.PB = PC.PD \tag{1.11}$$

Definition 45. *The line PX in Fig. 1 touches the circle at exactly one point P. It is known as the tangent to the circle.*

Problem 46. *Show that* $OP \perp PX$.

Proof. Without loss of generality, let $0 \le \theta \le 90^\circ$. Using the cosine formula in $\triangle OPP_n$,

$$(r + d_n)^2 > r^2, \tag{1.12}$$

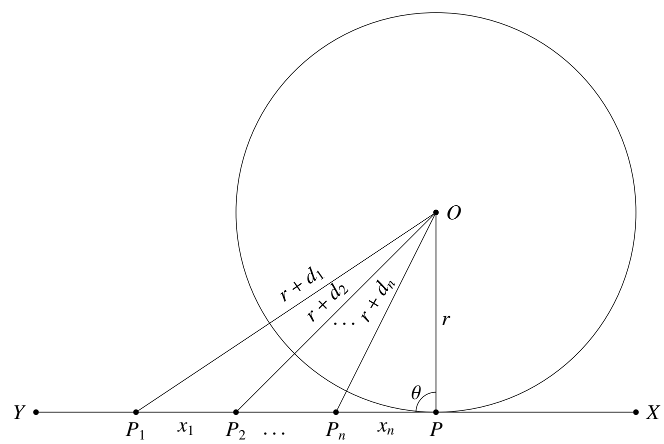

Fig. 1: Tangent to a Circle.

$$(r + d_n)^2 = r^2 + x_n^2 - 2rx_n \cos\theta > r^2 \tag{1.13}$$

$$\Longrightarrow 0 < \cos\theta < \frac{x_n}{2r}, \tag{1.14}$$

where x_n can be made as small as we choose. Thus,

$$\cos\theta = 0 \implies \theta = 90^\circ. \tag{1.15}$$

Problem 47. *In Fig. 1 show that*

$$\angle PCA = \angle PBC \tag{1.16}$$

O is the centre of the circle and PC is the tangent.

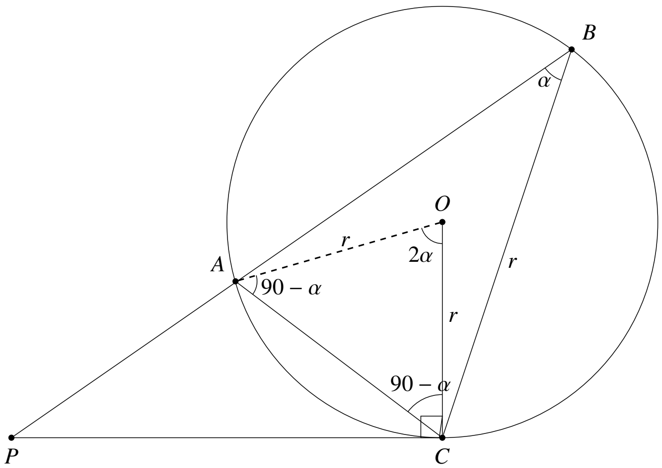

Fig. 1: $PA.PB = PC^2$.

Proof. Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

Problem 48. *In Fig. 1, show that the triangles PAC and PBC are similar.*

Proof. From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

Problem 49. *Show that* $PA.PB = PC^2$

Proof. Since $\Delta PAC \sim \Delta PBC$, their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{1.17}$$

$$\Rightarrow PA.PB = PC^2 \tag{1.18}$$

Problem 50. *Given that* $PA.PB = PC^2$*, show that* PC *is a tangent to the circle.*

Problem 51. *In Fig. 1, show that*

$$PA.PB = PC.PD \tag{1.19}$$

Proof. Draw a tangent and use the previous problem.

5 Area of a Circle

5.1 The Regular Polygon

Definition 52. *In Fig. 1, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.*

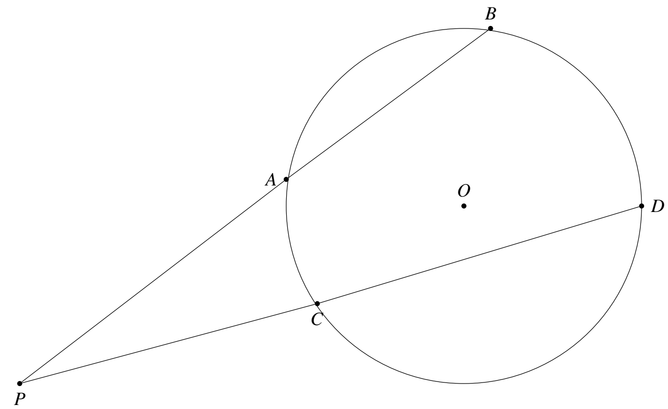

Fig. 1: $PA.PB = PC^2$.

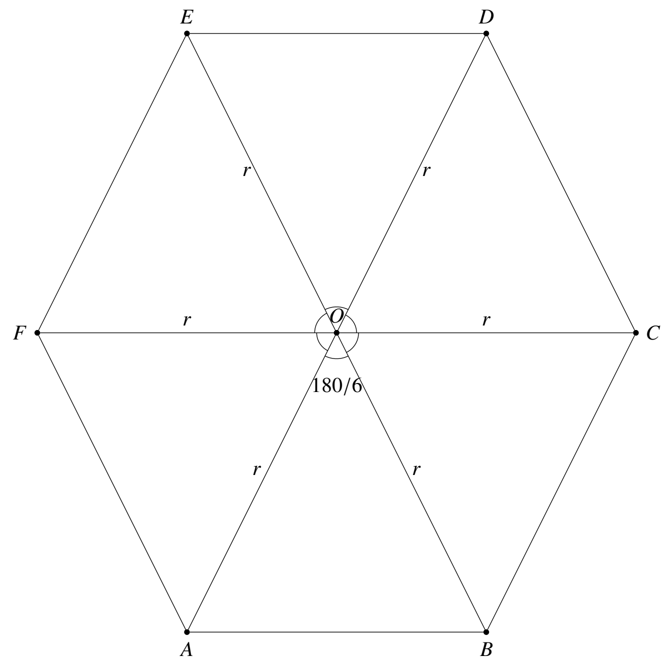

Fig. 1: Polygon Definition

Definition 53. *The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is* $\frac{360^\circ}{n}$.

Problem 54. *Show that the area of a regular polygon is given by*

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} \tag{1.1}$$

Proof. The triangle that forms the polygon of n sides is given in Fig. 1. Thus,

$$\begin{aligned} ar\left(polygon\right) &= nar\left(\Delta ABC\right) \\ &= \frac{n}{2}r^2 \sin\frac{360^\circ}{n} \end{aligned} \tag{1.2}$$

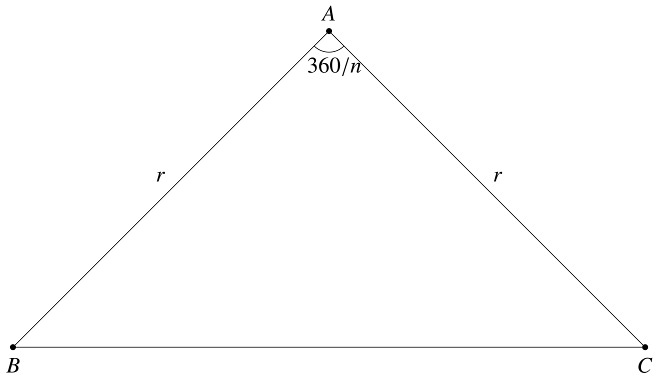

Fig. 1: Polygon Area

Problem 55. *Using Fig. 1, show that*

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} < \textit{ area of circle } < nr^2 \tan\frac{180^\circ}{n} \tag{1.3}$$

The portion of the circle visible in Fig. 1 is defined to be a sector of the circle.

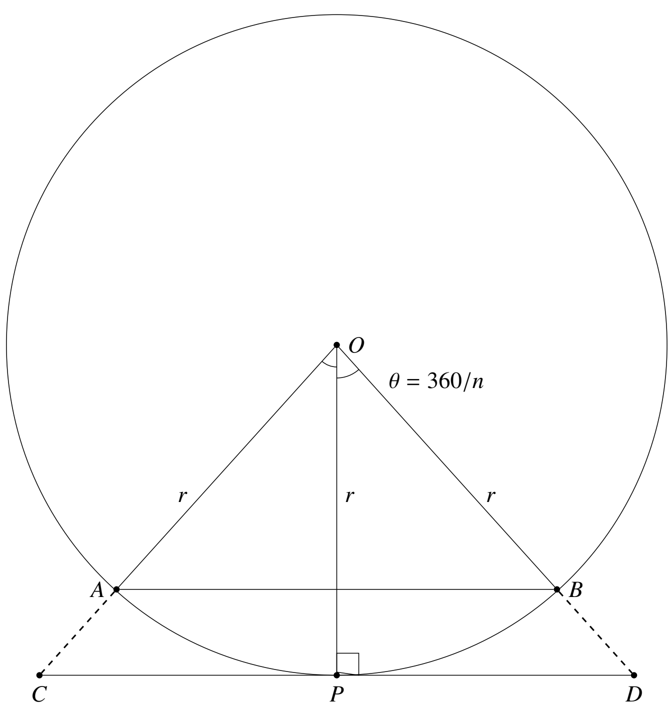

Fig. 1: Circle Area in between Area of Two Polygons

Proof. Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. 1, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$ar(\Delta OAB) = \frac{1}{2}r^2 \sin\frac{360^\circ}{n} \tag{1.4}$$

$$ar(\Delta OPQ) = 2 \times \frac{1}{2} \times r \tan\frac{360/n}{2} \times r \tag{1.5}$$

$$= r^2 \tan\frac{180^\circ}{n}, \tag{1.6}$$

we obtain (1.3).

Problem 56. *Using Fig. 1, show that*

$$\sin\theta_1 = \sin(\theta_1 + \theta_2)\cos\theta_2 - \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{1.7}$$

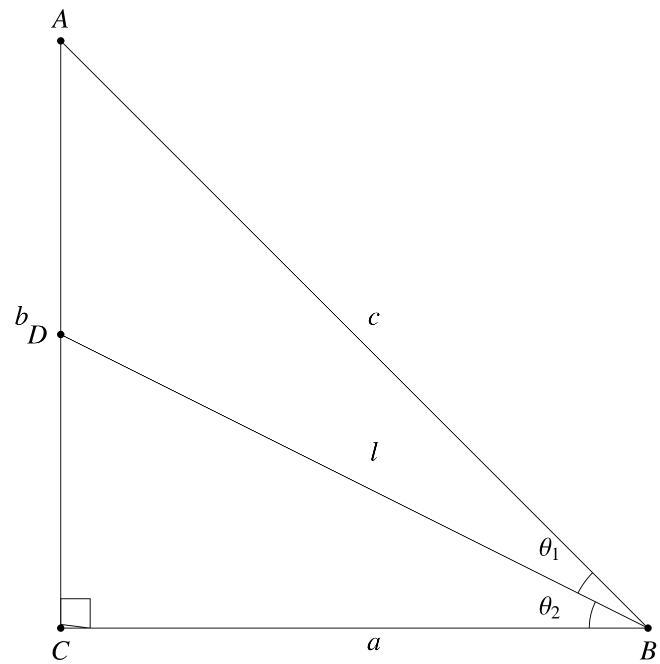

Fig. 1: $\sin 2\theta = 2\sin\theta\cos\theta$

Proof. The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{1.8}$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{1.9}$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{1.10}$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{1.11}$$

($\because l = a\sec\theta_2$). From the above,

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \quad (1.12)$$

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \quad (1.13)$$

Multiplying both sides by $\cos\theta_2$,

$$\Rightarrow \sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \quad (1.14)$$

resulting in

$$\Rightarrow \sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \quad (1.15)$$

Problem 57. *Prove the following identities*

1)

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \quad (1.1)$$

2)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \quad (2.1)$$

Proof. In (1.7), let

$$\begin{aligned}\theta_1+\theta_2 &= \alpha\\ \theta_2 &= \beta\end{aligned} \quad (2.2)$$

This gives (1.1). In (1.1), replace α by $90^\circ - \alpha$. This results in

$$\begin{aligned}&\sin\left(90^\circ - \alpha - \beta\right)\\ &\quad = \sin\left(90^\circ-\alpha\right)\cos\beta - \cos\left(90^\circ-\alpha\right)\sin\beta\\ &\quad \Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad (2.3)\end{aligned}$$

Problem 58. *Using* (1.7) *and* (2.1)*, show that*

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \quad (2.4)$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \quad (2.5)$$

Proof. From (1.7),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \quad (2.6)$$

Using (2.1) in the above,

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 &= \sin\theta_1 + (\cos\theta_1\cos\theta_2\\ &\quad - \sin\theta_1\sin\theta_2)\sin\theta_2 \quad (2.7)\end{aligned}$$

which can be expressed as

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 &= \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2\\ &\quad - \sin\theta_1\sin^2\theta_2 \quad (2.8)\end{aligned}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \quad (2.9)$$

we obtain

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 &= \cos\theta_1\cos\theta_2\sin\theta_2\\ &\quad + \sin\theta_1\cos^2\theta_2 \quad (2.10)\end{aligned}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \quad (2.11)$$

after factoring out $\cos\theta_2$. Using a similar approach, (2.5) can also be proved.

Problem 59. *Show that*

$$\sin 2\theta = 2\sin\theta\cos\theta \quad (2.12)$$

Problem 60. *Show that*

$$\cos^2\frac{180^\circ}{n} < \frac{\textit{area of circle}}{nr^2\tan\frac{180^\circ}{n}} < 1 \quad (2.13)$$

Proof. From (1.3) and (2.12),

$$\begin{aligned}&\frac{n}{2}r^2\sin\frac{360^\circ}{n} < \text{ area of circle}\\ &\qquad < nr^2\tan\frac{180^\circ}{n}\\ &\Rightarrow nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} < \text{ area of circle}\\ &\qquad < nr^2\tan\frac{180^\circ}{n} \quad (2.14)\end{aligned}$$

Problem 61. *Show that if*

$$\theta_1 < \theta_2, \sin\theta_1 < \sin\theta_2. \quad (2.15)$$

Proof. Trivial using the definition and choosing angles θ_1 and θ_2 appropriately in a right angled triangle.

Problem 62. *Show that if*

$$\theta_1 < \theta_2, \cos\theta_1 > \cos\theta_2. \quad (2.16)$$

Problem 63. *Show that*

$$\sin 0^\circ = 0 \quad (2.17)$$

Proof. Follows from (2.16).

Problem 64. *Show that*

$$\cos 0^\circ = 1 \tag{2.18}$$

Problem 65. *Show that for large values of n*

$$\cos^2 \frac{180^\circ}{n} = 1 \tag{2.19}$$

Proof. Follows from previous problem.

Definition 66. *The previous result can be expressed as*

$$\lim_{n\to\infty} \cos^2 \frac{180^\circ}{n} = 1 \tag{2.20}$$

Problem 67. *Show that*

$$\textit{area of circle } = r^2 \lim_{n\to\infty} n \tan \frac{180^\circ}{n} \tag{2.21}$$

Definition 68.

$$\pi = \lim_{n\to\infty} n \tan \frac{180^\circ}{n} \tag{2.22}$$

Thus, the area of a circle is πr^2.

Definition 69. *The radian is a unit of angle defined by*

$$1 \textit{ radian} = \frac{360^\circ}{2\pi} \tag{2.23}$$

Problem 70. *Show that the circumference of a circle is $2\pi r$.*

Problem 71. *Show that the area of a sector with angle θ in radians is $\frac{1}{2}r^2\theta$.*

Problem 72. *Show that*

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \tag{2.24}$$