JEE Advanced 2019: Paper 2

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CONTENTS

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***Abstract*—This manual has exercises based on problems in JEE advanced 2019.**

1 Matrices: Trace

1.1 Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^{6}$ from permutations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{1.1}$$

1.2 Let

$$\mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T. \tag{1.2}$$

Given

$$\mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{1.3}$$

is $\alpha = 30$?

Solution:

$$\because \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{aligned}
\mathbf{X}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \\
&= 30 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (1.4)
\end{aligned}$$

Thus, $\alpha = 30$.

1.3 Is $\mathbf{X}$ symmetric?
Solution: Yes. Trivial.

1.4 Show that

$$\mathbf{P}_k \mathbf{P}_k^T = \mathbf{I} \qquad (1.5)$$

Solution:

$$\mathbf{P}_k = \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} \qquad (1.6)$$

where $\mathbf{v}_{ki}, i = 1, 2, 3$ are from the standard basis. Then,

$$\mathbf{P}_k \mathbf{P}_k^T = \begin{pmatrix} \mathbf{v}_{k1}^T \\ \mathbf{v}_{k2}^T \\ \mathbf{v}_{k3}^T \end{pmatrix} \begin{pmatrix} \mathbf{v}_{k1} & \mathbf{v}_{k2} & \mathbf{v}_{k3} \end{pmatrix} \mathbf{I}$$

$$\because \mathbf{v}_{ji}^T \mathbf{v}_{kj} = \delta_{jk} \qquad (1.7)$$

1.5 For 2×2 matrices $\mathbf{A}, \mathbf{B}$, verify that

$$tr(AB) = tr(BA) \qquad (1.8)$$

Show that this is true for any square matrix.

1.6 Verify if the sum of the diagonal entries of X is 18.

Solution:

$$\begin{aligned}
\text{tr}(\mathbf{X}) &= \sum_{k=1}^{6} \text{tr} \left\{ \mathbf{P}_k \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k^T \right\} \\
&= \sum_{k=1}^{6} \text{tr} \left\{ \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P}_k \mathbf{P}_k^T \right\} \\
&= \sum_{k=1}^{6} \text{tr} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} = 6 \times 3 = 18 \qquad (1.9)
\end{aligned}$$

after substituting from (1.5).

1.7 Is $\mathbf{X} - 30\mathbf{I}$ invertible?
Solution: From (1.3),

$$\mathbf{X}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 30\mathbf{I}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\implies (\mathbf{X} - 30\mathbf{I})\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad (1.10)$$

If $(\mathbf{X} - 30\mathbf{I})^{-1}$ exists,

$$(\mathbf{X} - 30\mathbf{I})^{-1} (\mathbf{X} - 30\mathbf{I}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \qquad (1.11)$$

which is a contradiction. Hence, $\mathbf{X} - 30\mathbf{I}$ is not invertible.

2 Linear Algebra: Eigenvector and Null Space

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{pmatrix} \qquad (2.1)$$

2.1 Find x such that $PQ = QP$.
Solution:

$$\because \mathbf{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad (2.2)$$

$$\mathbf{PQ} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{pmatrix} \qquad (2.3)$$

and

$$\mathbf{QP} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 8 & 8 \\ 0 & 0 & 18 \end{pmatrix} + x \begin{pmatrix} 0 & 2 & 5 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \end{pmatrix} \tag{2.4}$$

Thus,

$$\mathbf{PQ} = \mathbf{QP} \implies \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = x \begin{pmatrix} -1 & 0 & 4 \\ -2 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \tag{2.5}$$

which has no solution.

2.2 If

$$\mathbf{R} = \mathbf{PQP}^{-1}, \tag{2.6}$$

verify whether

$$\det \mathbf{R} = \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8 \tag{2.7}$$

for all x.

Solution:

$$\begin{aligned} \det(\mathbf{R}) &= \det(\mathbf{P}) \det(\mathbf{Q}) \det(\mathbf{P})^{-1} = \det(\mathbf{Q}) \\ &= 4\left(12 - x^2\right) \end{aligned} \tag{2.8}$$

Thus,

$$\begin{aligned} \det(\mathbf{R}) - \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} \\ = 4\left\{\left(12 - x^2\right) - \left(10 - x^2\right)\right\} \\ = 8 \end{aligned} \tag{2.9}$$

which is true.

2.3 For $x = 0$, if

$$\mathbf{R} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \tag{2.10}$$

then show that

$$a + b = 5. \tag{2.11}$$

Solution: For $x = 0$,

$$\mathbf{R} = \mathbf{PQP}^{-1}, \tag{2.12}$$

where $\mathbf{Q}$ is a diagonal matrix. This is the eigenvalue decomposition of $\mathbf{R}$. Thus,

$$\mathbf{R} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{2.13}$$

where

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.14}$$

is the eigenvector corresponding to the eigenvalue 6. Comparing with (2.13),

$$a = 2, b = 3 \implies a + b = 5. \tag{2.15}$$

2.4 For $x = 1$, verify if there exists a vector $\mathbf{y}$ for which $\mathbf{Ry} = \mathbf{0}$.

Solution:

$$\begin{aligned} \mathbf{Ry} = \mathbf{0} &\implies \mathbf{PQP}^{-1}\mathbf{y} = \mathbf{0} \\ &\implies \mathbf{Qz} = \mathbf{0}, \end{aligned} \tag{2.16}$$

where

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{y} \tag{2.17}$$

For $x = 1$, (2.1) and (2.16) yield

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} \mathbf{z} = \mathbf{0} \tag{2.18}$$

Using row reduction,

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 6 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 1 & 11 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 11 \end{pmatrix} \tag{2.19}$$

Thus, $\mathbf{Q}^{-1}$ exists and

$$\mathbf{z} = \mathbf{0} \implies \mathbf{y} = \mathbf{0} \tag{2.20}$$

upon substituting from (2.17). This implies that the null space of $\mathbf{R}$ is empty.

3 Discrete Fourier Transform

3.1 Show that

$$\sum_{k=0}^{n-1} e^{\jmath \frac{2\pi k}{n}} = \begin{cases} 1 & n = 1, \\ 0 & n > 1 \end{cases} \tag{3.1}$$

3.2 Show that

$$\sum_{k=0}^{n} \cos\left(\frac{2k+r}{n+2}\pi\right) = -\cos\left(\frac{r-2}{n+2}\pi\right) \tag{3.2}$$

Solution: From (3.1),

$$\sum_{k=0}^{n+1} e^{\jmath \frac{2k+r}{n+2}\pi} = 0$$

$$\implies \sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} + e^{\jmath \frac{2(n+1)+r}{n+2}\pi} = 0$$

$$\implies \sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} = -e^{\jmath \frac{2(n+2)+r-2}{n+2}\pi}$$

$$= -e^{\jmath \frac{r-2}{n+2}\pi} \tag{3.3}$$

Taking the real part on both sides yields (3.2).

3.3 Show that

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right) \sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+2}{n+2}\pi\right)} \tag{3.4}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)}{n + \cos\left(\frac{2\pi}{n+2}\right)} \tag{3.5}$$

Solution: Let

$$\theta_n = \frac{\pi}{n+2} \tag{3.6}$$

$$\because \sin\{(k+1)\theta_n\}\sin\{(k+2)\theta_n\},$$

$$= \frac{1}{2}\left[\cos\theta_n - \cos\{(2k+3)\theta_n\}\right] \tag{3.7}$$

from (3.4) and (3.2),

$$f(n) = \frac{n\cos\theta_n - \sum_{k=0}^{n} \cos\{(2k+3)\theta_n\}}{n - \sum_{k=0}^{n} \cos\{(2k+4)\theta_n\}}$$

$$= \frac{n\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{n + \cos\left(\frac{2\pi}{n+2}\right)} \tag{3.8}$$

resulting in (3.5). Verify if

3.4

$$f(4) = \frac{\sqrt{3}}{2} \tag{3.9}$$

3.5

$$\lim_{n\to\infty} f(n) = \frac{1}{2} \tag{3.10}$$

3.6

$$\sin\left(7\cos^{-1} f(5)\right) = 0 \tag{3.11}$$

3.7 If

$$\alpha = \tan\left(\cos^{-1} f(6)\right) \tag{3.12}$$

verify if

$$\alpha^2 + 2\alpha - 1 = 0 \tag{3.13}$$

4 Calculus: Limits

Let

$$P_1 = \lim_{h\to 0} \frac{f(h) - f(0)}{\sqrt{|h|}} \tag{4.1}$$

$$P_2 = \lim_{h\to 0} \frac{f(h) - f(0)}{h^2} \tag{4.2}$$

4.1 Find P_1 for

$$f(x) = |x| \tag{4.3}$$

Solution: Substituting (4.3) in (4.1),

$$P_1 = \lim_{h\to 0} \frac{|h|}{\sqrt{h}} = 0 \tag{4.4}$$

4.2 Find P_1 for

$$f(x) = x^{\frac{2}{3}} \tag{4.5}$$

Solution: Substituting (4.5) in (4.1),

$$P_1 = \lim_{h\to 0} \frac{h^{\frac{2}{3}}}{\sqrt{h}} = h^{\frac{1}{3}} = 0 \tag{4.6}$$

4.3 Find P_2 for

$$f(x) = x|x| \tag{4.7}$$

Solution: Substituting (4.7) in (4.2),

$$\lim_{h\to 0+} \frac{h|h|}{h^2} = 1 \tag{4.8}$$

and

$$\lim_{h\to 0-} \frac{-h|h|}{h^2} = -1 \tag{4.9}$$

$$\because (4.8) \neq (4.9), \tag{4.10}$$

P_2 does not exist.

4.4 Find P_2 for

$$f(x) = \sin x \tag{4.11}$$

Solution: Substituting (4.12) in (4.2),

$$\lim_{h\to 0+} \frac{\sin h}{h^2} = \infty \tag{4.12}$$

Hence P_2 does not exist.

5 Calculus: Maxima and Minima

Let

$$f(x) = p(x)q(x), \quad x > 0, \text{ where} \tag{5.1}$$

$$p(x) = \sin \pi x \tag{5.2}$$

$$q(x) = \frac{1}{x^2} \tag{5.3}$$

5.1 Show that $q(x)$ is monotonically decreasing.

5.2 Show that $p(x)$ is oscillatory.

5.3 Find the regions where $f(x)$ is increasing and decreasing.

Solution:

$$\because f'(x) = p(x)q'(x) + p'(x)q(x), \tag{5.4}$$

$$q(x) > 0, \tag{5.5}$$

$$q'(x) < 0, \tag{5.6}$$

$$f'(x)\begin{cases} < 0 & p(x) > 0 \text{ and } p'(x) < 0 \\ > 0 & p(x) < 0 \text{ and } p'(x) > 0 \end{cases} \tag{5.7}$$

Table 5.3 computes the desired regions based on (5.7)

	> 0	< 0
$p(x)$	$x \in (2n, 2n+1)$	$x \in (2n+1, 2n+2)$
$q(x)$	$x > 0$	
$p'(x)$	$x \in \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right)$	$x \in \left(2n + \frac{1}{2}, 2n + \frac{3}{2}\right)$
$q'(x)$		$x > 0$
$f'(x)$	$x \in (2n+1, 2n+2) \cup x \in \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right) = x \in \left(2n - \frac{1}{2}, 2n\right)$	$x \in (2n, 2n+1) \cup x \in \left(2n + \frac{1}{2}, 2n + \frac{3}{2}\right) = x \in \left(2n + \frac{1}{2}, 2n + 1\right)$

TABLE 5.3

5.4 Find the points of local maxima x_i.

Solution: The maxima occur in the interval between $f'(x) > 0$ and $f'(x) < 0$. From Table 5.3,

$$x_i \in \left(2n, 2n + \frac{1}{2}\right), \quad n \geq 1 \tag{5.8}$$

5.5 Find the points of local minima y_i.

Solution: The minima occur in the interval between $f'(x) < 0$ and $f'(x) > 0$. From Table 5.3,

$$y_i \in \left(2n - 1, 2n - \frac{1}{2}\right), \quad n \geq 1 \tag{5.9}$$

5.6 Is

$$x_{n+1} - x_n > 2 \tag{5.10}$$

for every n?

Solution: From (5.8),

$$x_{n+1} - x_n > 2(n+1) - \left(2n + \frac{1}{2}\right) \tag{5.11}$$

$$= \frac{3}{2} < 2 \tag{5.12}$$

5.7 Show that

$$x_1 > y_1 \tag{5.13}$$

5.8 Verify if

$$|x_n - y_n| > 1 \tag{5.14}$$

for every n.

6 Definite Integral: Limit of a Sum

6.1 Show that

$$\lim_{n\to\infty} \frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{4}{3}}} = \int_0^1 x^{\frac{1}{3}}\, dx = \frac{3}{4} \tag{6.1}$$

6.2 Show that

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\left(a + \frac{1}{n}\right)^2} + \frac{1}{\left(a + \frac{1}{2}\right)^2} + \cdots + \frac{1}{(a+1)^2}\right]$$

$$= \int_0^1 \frac{1}{(a+x)^2}\, dx = \frac{1}{a(a+1)} \tag{6.2}$$

6.3 If

$$\lim_{n\to\infty} \left(\frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n^{\frac{7}{3}}\left\{\left[\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \cdots + \frac{1}{(an+n)^2}\right]\right\}}\right)$$

$$= 54, \quad |a| > 1, \tag{6.3}$$

find a.

Solution: Substituting from (6.1) and (6.2) in (6.3),

$$\frac{3}{4}a(a+1) = 54$$

$$a(a+1) = 72$$

$$\implies a = 8, -9. \tag{6.4}$$

7 Calculus: Integration

Let

$$F(x) = \int_0^x f(t)\,dt, \quad x > 0 \tag{7.1}$$

where

$$f(x) = (x-1)(x-2)(x-5) \tag{7.2}$$

7.1 Does $F(x)$ have a local minimum at $x = 1$?
Solution: The derivative of $F(x)$ is

$$F^{(1)}(x) = \lim_{\delta x \to 0} \frac{1}{\delta} \int_x^{x+\delta x} f(t)\,dt$$
$$= f(x) \tag{7.3}$$

Thus, from (7.3),

$$F^{(2)}(x) = f^{(1)}(x)$$
$$= (x-1)(x-2) + (x-2)(x-5) + (x-1)(x-5)$$
$$\implies F^{(2)}(1) = 4 > 0 \tag{7.4}$$

Since $f(1) = 0$, answer is yes.

7.2 Does $F(x)$ have a local maximum at $x = 2$?
Solution: From (7.4),

$$F^{(2)}(2) = -3 < 0 \tag{7.5}$$

Since $f(2) = 0$, answer is yes.

7.3 Does $F(x)$ have two local maxima and one local minimum in $(0, \infty)$?
Solution: From (7.4),

$$F^{(2)}(5) = 12 > 0 \tag{7.6}$$

Since $f(5) = 0$, $F(x)$ has two local minima and one minimum. So answer is false.

7.4 Verify if $F(x) \neq 0$ for all $x \in (0, 5)$.
Solution: From the previous solutions,

$$\min F(x) > 0, \quad x \in (0, 5) \tag{7.7}$$

Hence, the statement is correct.

8 Linear Algebra: Linear Dependence

8.1 Let

$$L_1 : \quad \mathbf{r} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{8.1}$$

$$L_2 : \quad \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{8.2}$$

$$L_3 : \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.3}$$

Let $\mathbf{P} \in L_1, \mathbf{Q} \in L_2, \mathbf{R} \in L_3$. Given that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear, If $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear, find $\mathbf{Q}$.
Solution:

$$\frac{PQ}{QR} = k, \tag{8.4}$$
$$(k+1)\,\mathbf{Q} = k\mathbf{P} + \mathbf{R}, \tag{8.5}$$

From (8.1), (8.2) and (8.3),

$$k\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (k+1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (k+1)\,\lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{8.6}$$

which can be expressed as

$$\begin{pmatrix} k & 0 & 0 \\ 0 & -(k+1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ k+1 \end{pmatrix} \tag{8.7}$$

Thus,

$$\mathbf{Q} = \begin{pmatrix} 0 \\ \frac{1}{k+1} \\ 1 \end{pmatrix} \tag{8.8}$$

8.2 Verify if $\mathbf{Q}$ can be

a) $\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

9 Combinatorics

9.1 Find

$$\sum_{k=0}^{n} k \tag{9.1}$$

Solution: (9.1) can be expressed as

$$\frac{n(n+1)}{2} \tag{9.2}$$

9.2 Find

$$\sum_{k=0}^{n} {}^{n}C_k k^2 \tag{9.3}$$

Solution:

$$(1+x)^n = \sum_{k=0}^{n} {}^{n}C_k x^k \tag{9.4}$$

$$\implies n(1+x)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_k x^{k-1} \tag{9.5}$$

upon differentiation. Multiplying (9.5) by x and differentiating,

$$\frac{d}{dx}\left[nx(1+x)^{n-1}\right] = \sum_{k=0}^{n} k^2\,{}^{n}C_k x^{k-1} \tag{9.6}$$

$$\begin{aligned}\implies n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1} \\ = \sum_{k=0}^{n} k^2\,{}^{n}C_k x^{k-1}\end{aligned} \tag{9.7}$$

Substituting $x = 1$ in (9.7),

$$\begin{aligned}\sum_{k=0}^{n} {}^{n}C_k k^2 &= n(n-1)2^{n-2} + n2^{n-1} \\ &= n(n+1)2^{n-2}\end{aligned} \tag{9.8}$$

9.3 Find

$$\sum_{k=0}^{n} {}^{n}C_k k \tag{9.9}$$

Solution: Substituting $x = 1$ in (9.5),

$$\sum_{k=0}^{n} {}^{n}C_k k = n2^{n-1} \tag{9.10}$$

9.4 Find

$$\sum_{k=0}^{n} {}^{n}C_k 3^k \tag{9.11}$$

Solution: Substituting $x = 2$ in (9.4),

$$\sum_{k=0}^{n} {}^{n}C_k 3^k = 4^n \tag{9.12}$$

9.5 If

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n+1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0 \tag{9.13}$$

for some n, find

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{9.14}$$

Solution: (9.13) can be expressed as

$$n(n+1)2^{2n-3}\begin{vmatrix} 1 & 1 \\ n & 4 \end{vmatrix} = 0 \tag{9.15}$$

$$\implies n = 4 \tag{9.16}$$

Integrating (9.4) from 0 to 1,

$$\frac{2^{n+1}}{n+1} = \sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{9.17}$$

Substituting $n = 4$ in the above,

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} = \frac{2^5-1}{5} = \frac{31}{5} \tag{9.18}$$

10 Combinatorics

10.1 Five persons A, B, C, D and E are seated in a circuar arrangment. Let A, C, E be given the colour green. Find the number of ways to distribute blue and red to B and D.
Solution: Both B and D can be given either blue or red. The number of possible ways is

$$2 \times 2 = 4 \tag{10.1}$$

10.2 Repeat the above exercise with B, D, A having the colour green.

10.3 Find the total number of ways in which the colour green can be distributed to alternate persons so that persons seated in adjacent seats get different coloured hats.
Solution: The number of such ways is

$$2 \times 2 \times 2 = 8 \tag{10.2}$$

10.4 If each person is given a hat of one of 3 colours red, blue and green, then find the number of ways of distributing the hats such that the

persons seated in adjacent seats get different coloured hats.
Solution: The number of ways is

$$2 \times 2 \times 2 \times 3 = 24 \tag{10.3}$$

11 Combinatorics

11.1

12 Trigonometry

12.1 Find

$$\sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right) \tag{12.1}$$

Solution: (12.1) can be expressed as

$$\frac{2}{\cos\left(\frac{7\pi}{6} + \frac{(2k+1)\pi}{2}\right) + \cos\frac{\pi}{2}} = \frac{2}{\cos\left((k+1)\pi + \frac{\pi}{6} + \frac{\pi}{2}\right)} = 4(-1)^{k+1} \tag{12.2}$$

after simplification.

12.2 Find the value of

$$\theta = \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right) \times \sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right] \tag{12.3}$$

in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
Solution: Substituting from (12.2) in (12.3) results in

$$\theta = \sec^{-1}(1) \implies \theta = 0 \tag{12.4}$$

in the given interval.

13 Definite Integral

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5}\, d\theta \tag{13.1}$$

13.1 Show that

$$I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5}\, d\theta \tag{13.2}$$

13.2 Show that

$$I = \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4}\, d\theta \tag{13.3}$$

13.3 Show that

$$I = 3\int_0^{\frac{\pi}{4}} \frac{1}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4}\, d\theta \tag{13.4}$$

13.4 Find I.
Solution: From (13.4)

$$I = 3\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(1 + \sqrt{\tan\theta}\right)^4}\, d\theta \tag{13.5}$$

which, after substituting

$$t = 1 + \sqrt{\tan\theta} \tag{13.6}$$

results in

$$\begin{aligned} I &= 3\int_1^2 \frac{2(t-1)}{t^4}\, dt \\ &= 6\left[\frac{1}{3t^3} - \frac{1}{2t^2}\right]_1^2 \\ &= \left[2\left(\frac{1}{8} - 1\right) - 3\left(\frac{1}{4} - 1\right)\right] = \frac{1}{2} \end{aligned} \tag{13.7}$$

14 Linear Algebra: Projection

14.1 Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \tag{14.1}$$

14.2 Given

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}, \tag{14.2}$$

show that

$$(\mathbf{c} - (\mathbf{a} \times \mathbf{b}))^T\,\mathbf{c} = \|\mathbf{c}\|^2 \tag{14.3}$$

14.3 Find

$$\min_{\mathbf{u}} \|\mathbf{c}\|^2 \tag{14.4}$$

$$s.t \quad \left\|\text{proj}_{\mathbf{a}+\mathbf{b}}\mathbf{c}\right\| = 3\sqrt{2} \tag{14.5}$$

by using the fact that

$$\mathbf{c} = \mathbf{P}\mathbf{u} \tag{14.6}$$

$$\mathbf{a} + \mathbf{b} = \mathbf{P}\mathbf{1} \tag{14.7}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \tag{14.8}$$

$$\mathbf{u} = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \tag{14.9}$$

$$\mathbf{1} = \begin{pmatrix}1 \\ 1\end{pmatrix} \tag{14.10}$$

15 Calculus: Differentiation

For $x > 0$,

$$f(x) = \sin(\pi \cos x) \tag{15.1}$$

$$g(x) = \cos(2\pi \sin x) \tag{15.2}$$

15.1 Find

$$X = \{x : f(x) = 0\} \tag{15.3}$$

Solution:

$$\sin(\pi \cos x) = 0$$

$$\implies \pi \cos x = k\pi \text{ or, } \cos x = k \tag{15.4}$$

where k is an integer.

$$\because |\cos x| \le 1,$$

$$X = \{x : \cos x = -1, 0, 1\}$$

$$\implies X = k\pi \tag{15.5}$$

15.2 Find

$$Y = \{x : f'(x) = 0\}, \tag{15.6}$$

Solution:

$$\begin{aligned} Y &= \{x : \sin x \cos(\pi \cos x) = 0\}, \\ &= \{x : \sin x = 0\} \cup \{x : \cos(\pi \cos x) = 0\}, \\ &= \{k\pi\} \cup \left\{x : \cos x = \left(2m \pm \frac{1}{2}\right)\right\}, \\ &= \{k\pi\} \cup \left\{x : \cos x = \pm\frac{1}{2}\right\}, \end{aligned} \tag{15.7}$$

which can be expressed as

$$\begin{aligned} Y &= \{k\pi\} \cup \left\{2m\pi \pm \frac{\pi}{3}\right\} \cup \left\{2m\pi \pm \frac{2\pi}{3}\right\} \\ &= \frac{k\pi}{3} \end{aligned} \tag{15.8}$$

15.3 Find

$$Z = \{x : g(x) = 0\} \tag{15.9}$$

Solution:

$$\begin{aligned} Z &= \{x : \cos(2\pi \sin x) = 0\} \\ &= \left\{x : \sin x = k \pm \frac{1}{4}\right\} \\ &= \left\{x : \sin x = \pm\frac{1}{4}, \pm\frac{3}{4}\right\} \\ &= \left\{k\pi \pm \sin^{-1}\frac{1}{4}\right\} \cup \left\{k\pi \pm \sin^{-1}\frac{3}{4}\right\} \end{aligned} \tag{15.10}$$

15.4 Find

$$W = \{x : g'(x) = 0\}. \tag{15.11}$$

Solution:

$$\begin{aligned} W &= \{x : \cos x \sin(2\pi \sin x) = 0\} \\ &= \{x : \cos x \sin(2\pi \sin x) = 0\} \\ &= \{x : \cos x = 0\} \cup \{x : \sin(2\pi \sin x) = 0\} \\ &= \left\{\left(2k \pm \frac{1}{2}\right)\pi\right\} \cup \left\{x : \sin x = \frac{k}{2}\right\} \\ &= \left\{m\pi \pm \frac{\pi}{6}\right\} \cup \{r\pi\} \cup \left\{n\pi \pm \frac{\pi}{2}\right\} \end{aligned} \tag{15.12}$$

16 Numbers

16.1 Given (refer to previous problem)

List I		*List II*	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W		
		(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

which of the following is the only CORRECT combination?

(A) (I), (P), (R) (C) (I), (Q), (U)

(B) (II), (Q), (T) (D) (II), (R), (S)

16.2 Which of the following is the only CORRECT combination?

(A) (III), (R), (U) (C) (III), (P), (Q), (U)

(B) (IV), (P), (R), (S) (D) (IV), (Q), (T)

17 Linear Algebra: Coordinate Geometry

17.1 Find the points $\mathbf{X}, \mathbf{Y}$ where

$$C_1 : \|\mathbf{x}\| = 3 \tag{17.1}$$

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = 4 \tag{17.2}$$

intersect.

17.2 Find the centre $\mathbf{O}_3$ and radius r of C_3 such that

a) $\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3$ are collinear.

b) C_1, C_2 lie inside C_3 and

c) C_3 touches C_1 at $\mathbf{M}$ and C_2 at $\mathbf{N}$.

17.3 Find the equation of XY.

17.4 Find $\mathbf{Z}, \mathbf{W}$ the points of intersection of XY and C_3.

17.5 A common tangent of C_1 and C_3 is also a tangent to

$$P : \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - 2\alpha \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{17.3}$$

Find α.

18 Linear Algebra: Coordinate Geometry

Find the following based on the previous problem

18.1 $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{O}_3$

18.2 $\frac{ZW}{XY}$

18.3 $\frac{\text{ar}(\triangle MZN)}{\text{ar}(\triangle ZMW)}$