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**Abstract**—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/ncert/codes
```

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### 1.1 Examples

1. Do the points  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$  form a triangle? If so, name the type of triangle formed.
2. Show that the points  $\begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \end{pmatrix}$  are the vertices of a square.
3. Verify if  $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$  are points on a line.
4. Find the condition for  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  to be equidistant from the points  $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ .
5. Find a point on the y-axis which is equidistant from the points  $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ .
6. Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 8.

**Solution:** Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \quad (1.1.6.1)$$

Then the point  $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k + 1} \quad (1.1.6.2)$$

divides  $AB$  in the ratio  $k : 1$ . For the given problem,  $k = \frac{5}{8}$ . The following code plots Fig. 1.1.6

```
codes/line/draw_section.py
```

7. Find a unit vector in the direction of  $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ .
8. Find the direction vector of  $PQ$ , where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \quad (1.1.8.1)$$

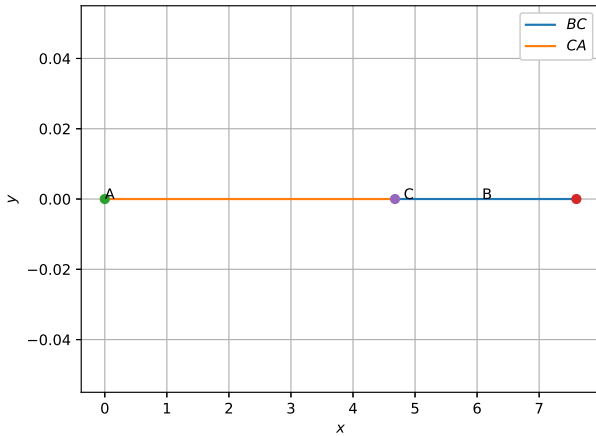


Fig. 1.1.6

9. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (1.1.9.1)$$

10. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \quad (1.1.10.1)$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (1.1.10.2)$$

11. Find a unit vector perpendicular to each of the vectors  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$ , where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (1.1.11.1)$$

12. Write down a unit vector in the  $xy$ -plane, making an angle of  $30^\circ$  with the positive direction of the  $x$ -axis.

13. Find the value of  $x$  for which  $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  is a unit vector.

14. Find the direction vectors and slopes of the lines passing through the points

a)  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ .

b)  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$ .

c)  $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .

- d) Making an inclination of  $60^\circ$  with the positive direction of the  $x$ -axis.

15. If the angle between two lines is  $\frac{\pi}{4}$  and the slope of one of the lines is  $\frac{1}{4}$  find the slope of the other line.

16. The line through the points  $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$  is perpendicular to the line through the points  $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$  and  $\begin{pmatrix} x \\ 24 \end{pmatrix}$ . Find the value of  $x$ .

17. Find the equations of the lines parallel to axes and passing through  $(-2, 3)$ .

18. Find the equation of the line through  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$  with slope  $-4$ .

19. Write the equation of the line through the points  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ .

20. Write the equation of the lines for which  $\tan \theta = \frac{1}{2}$ , where  $\theta$  is the inclination of the line and

a)  $y$ -intercept is  $-\frac{3}{2}$

b)  $x$ -intercept is 4.

21. Find the equation of the line, which makes intercepts  $-3$  and  $2$  on the  $x$  and  $y$  axes respectively.

22. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of  $x$ -axis is  $15^\circ$ .

23. Two positions of time and distance are recorded as, when  $T = 0, D = 2$  and when  $T = 3, D = 8$ . Using the concept of slope, find law of motion, i.e., how distance depends upon time.

24. The Fahrenheit temperature  $F$  and absolute temperature  $K$  satisfy a linear equation. Given  $K = 273$  when  $F = 32$  and that  $K = 373$  when  $F = 212$ , express  $K$  in terms of  $F$  and find the value of  $F$ , when  $K = 0$ .

25. Equation of a line is

$$(3 \quad -4) + 10 = 0. \quad (1.1.25.1)$$

Find its

a) slope,

b)  $x$  - and  $y$ -intercepts.

26. Find the angle between the lines

$$(1 \quad -\sqrt{3})\mathbf{x} = 5 \quad (1.1.26.1)$$

$$(\sqrt{3} \quad -1)\mathbf{x} = -6. \quad (1.1.26.2)$$

27. Find the equation of a line perpendicular to the line

$$(1 \quad -2)\mathbf{x} = 3 \quad (1.1.27.1)$$

and passes through the point  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ .

28. Find the distance of the point  $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$  from the line

$$(3 \quad -4)\mathbf{x} = 26 \quad (1.1.28.1)$$

29. If the lines

$$(2 \quad 1)\mathbf{x} = 3 \quad (1.1.29.1)$$

$$(5 \quad k)\mathbf{x} = 3 \quad (1.1.29.2)$$

$$(3 \quad 1)\mathbf{x} = 2 \quad (1.1.29.3)$$

are concurrent, find the value of  $k$ .

30. Find the distance of the line

$$(4 \quad 1)\mathbf{x} = 0 \quad (1.1.30.1)$$

from the point  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  measured along the line making an angle of  $135^\circ$  with the positive  $x$ -axis.

31. Assuming that straight lines work as a plane mirror for a point, find the image of the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  in the line

$$(1 \quad -3)\mathbf{x} = -4. \quad (1.1.31.1)$$

32. A line is such that its segment between the lines

$$(5 \quad -1)\mathbf{x} = -4 \quad (1.1.32.1)$$

$$(3 \quad 4)\mathbf{x} = 4 \quad (1.1.32.2)$$

is bisected at the point  $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ . Obtain its equation.

33. Show that the path of a moving point such that

its distances from two lines

$$(3 \quad -2)\mathbf{x} = 5 \quad (1.1.33.1)$$

$$(3 \quad 2)\mathbf{x} = 5 \quad (1.1.33.2)$$

are equal is a straight line.

## 1.2 Points and Vectors

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.2.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (1.2.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (1.2.1.3)$$

2. Find the distance between the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (1.2.2.1)$$

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

4. Without using the Pythagoras theorem, show that the points  $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$  are the vertices of a right angled triangle.

5. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.2.5.1)$$

are the vertices of an isosceles triangle.

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (1.2.6.1)$$

b)

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (1.2.6.2)$$

c)

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \quad (1.2.6.3)$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.2.6.4)$$

7. Find the angle between the x-axis and the line joining the points  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ .

8. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \quad (1.2.8.1)$$

9. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (1.2.9.1)$$

is 10 units.

10. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \quad (1.2.10.1)$$

11. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (1.2.11.1)$$

verify if

a)  $\|\mathbf{a}\| = \|\mathbf{b}\|$

b)  $\mathbf{a} = \mathbf{b}$

12. Find a vector  $\mathbf{x}$  in the direction of  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  such that  $\|\mathbf{x}\| = 7$ .

13. Find a unit vector in the direction of  $\mathbf{a} + \mathbf{b}$ , where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \quad (1.2.13.1)$$

14. Show that each of the given three vectors is a unit vector

$$\frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \quad (1.2.14.1)$$

Also, show that they are mutually perpendicular to each other.

15. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (1.2.15.1)$$

$(\mathbf{a} + \lambda \mathbf{b}) \perp \mathbf{c}$ . Find  $\lambda$ .

16. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (1.2.16.1)$$

find  $\|\mathbf{a} \times \mathbf{b}\|$ .

17. Find  $\mathbf{a} \times \mathbf{b}$  if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \quad (1.2.17.1)$$

18. Find a unit vector perpendicular to each of the vectors  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$ , where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad (1.2.18.1)$$

19. If  $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ , find a unit vector parallel to the vector  $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$ .

20. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors  $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} =$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

21. Show that the unit direction vector inclined equally to the coordinate axes is  $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ .

22. Let  $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$  and  $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ . Find a vector  $\mathbf{d}$  such that  $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$  and  $\mathbf{d}^T \mathbf{c} = 15$ .

23. The scalar product of  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  with a unit vector

along the sum of the vectors  $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$  is unity. Find the value of  $\lambda$ .

24. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (1.2.24.1)$$

is

- a) 0                                      c) 1  
b) -1                                      d) 3

25. Let  $\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ . Find  $\beta_1, \beta_2$  such that  $\beta = \beta_1 + \beta_2$ ,  $\beta_1 \parallel \alpha$  and  $\beta_2 \perp \alpha$ .

26. Find a unit vector that makes an angle of  $90^\circ, 60^\circ$  and  $30^\circ$  with the positive x, y and z axis respectively.

27. Find a unit vector in the direction of  $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ .

28. Find a unit vector in the direction of the line passing through  $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .

29. Find a unit vector that makes an angle of  $90^\circ, 135^\circ$  and  $45^\circ$  with the positive x, y and z axis respectively.

30. Show that the lines with direction vectors  $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$ ,

$\begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$  are mutually perpendicular.

31. Show that the line through the points  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ ,

$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$  is parallel to the line through the points  $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$ .

32. Show that the line through the points  $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

is parallel to the line through the points  $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ ,

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

33. Find a point on the x-axis, which is equidistant from the points  $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .

### 1.3 Points on a Line

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.3.1.1)$$

in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining  $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ .

3. Find the ratio in which the line segment joining the points  $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$  and  $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$  is divided by  $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$ .

4. Find the ratio in which the line segment joining  $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$  is divided by the x-axis. Also find the coordinates of the point of division.

5. If  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ y \end{pmatrix}$ ,  $\begin{pmatrix} x \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$  are the vertices of a parallelogram taken in order, find x and y.

6. If  $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$  respectively, find the coordinates of  $\mathbf{P}$  such that  $AP = \frac{3}{7}AB$  and  $\mathbf{P}$  lies on the line segment AB.

7. Find the coordinates of the points which divide the line segment joining  $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$  into four equal parts.

8. Find the value of k if the points  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{B} =$

$\begin{pmatrix} 4 \\ k \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$  are collinear.

9. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \quad (1.3.9.1)$$

are collinear.

10. By using the concept of equation of a line, prove that the three points  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$  are collinear.

11. Find the value of  $x$  for which the points  $\begin{pmatrix} x \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$  are collinear.
12. In each of the following, find the value of  $k$  for which the points are collinear
- $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ k \end{pmatrix}$
  - $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} k \\ -4 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$
13. Find a condition on  $\mathbf{x}$  such that the points  $\mathbf{x}$ ,  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$  are collinear.
14. Show that the points  $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$  are collinear.
15. Show that the points  $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$  are collinear, and find the ratio in which  $\mathbf{B}$  divides  $AC$ .
16. Show that  $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$  and  $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$  are collinear.
17. Show that  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$  are collinear.
18. Show that  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$  are collinear.
- 1.4 Lines and Planes
- Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points  $\mathbf{P} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ .
  - The slope of a line is double of the slope of another line. If the tangent of the angle between them is  $\frac{1}{3}$ , find the slopes of the lines.
  - Find the slope of the line, which makes an angle of  $30^\circ$  of y-axis measured anticlockwise.
  - Write the equations for the x and y axes.
  - Find the equation of the line satisfying the following conditions
    - passing through the point  $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$  with slope  $\frac{1}{2}$ .
    - passing through the point  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  with slope  $m$ .
    - passing through the point  $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$  and inclined with the x-axis at an angle of  $75^\circ$ .
    - Intersecting the x-axis at a distance of 3 units to the left of the origin with slope -2.
    - intersecting the y-axis at a distance of 2 units above the origin and making an angle of  $30^\circ$  with the positive direction of the x-axis.
    - passing through the points  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ .
    - perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is  $30^\circ$ .
  - Find the equation of the line passing through  $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$  and perpendicular to the line through the points  $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ .
  - Find the direction vectors and y-intercepts of the following lines
    - $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0$ .
    - $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5$ .
    - $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$ .
  - Find the intercepts of the following lines on the axes.
    - $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12$ .
    - $\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 6$ .
    - $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 0$ .
  - Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
    - $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8$ .
    - $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$ .
    - $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$ .
  - Find the distance of the point  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  from the line  $\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = -82$ .

11. Find the points on the x-axis, whose distances from the line

$$(4 \ 3)\mathbf{x} = 12 \quad (1.4.11.1)$$

are 4 units.

12. Find the distance between the parallel lines

$$(15 \ 8)\mathbf{x} = 34 \quad (1.4.12.1)$$

$$(15 \ 8)\mathbf{x} = -31 \quad (1.4.12.2)$$

13. Find the equation of the line parallel to the line

$$(3 \ -4)\mathbf{x} = -2 \quad (1.4.13.1)$$

and passing through the point  $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ .

14. Find the equation of a line perpendicular to the line

$$(1 \ -7)\mathbf{x} = -5 \quad (1.4.14.1)$$

and having x intercept 3.

15. Find angles between the lines

$$(\sqrt{3} \ 1)\mathbf{x} = 1 \quad (1.4.15.1)$$

$$(1 \ \sqrt{3})\mathbf{x} = 1 \quad (1.4.15.2)$$

16. The line through the points  $\begin{pmatrix} h \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  intersects the line

$$(7 \ -9)\mathbf{x} = 19 \quad (1.4.16.1)$$

at right angle. Find the value of  $h$ .

17. Two lines passing through the point  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  intersect each other at angle of  $60^\circ$ . If the slope of one line is 2, find the equation of the other line.

18. Find the equation of the right bisector of the line segment joining the points  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ .

19. Find the coordinates of the foot of the perpendicular from the point  $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$  to the line

$$(3 \ -4)\mathbf{x} = 16. \quad (1.4.19.1)$$

20. The perpendicular from the origin to the line

$$(-m \ 1)\mathbf{x} = c \quad (1.4.20.1)$$

meets it at the point  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ . Find the values of  $m$  and  $c$ .

21. Find  $\theta$  and  $p$  if

$$(\sqrt{3} \ 1)\mathbf{x} = -2 \quad (1.4.21.1)$$

is equivalent to

$$(\cos \theta \ \sin \theta)\mathbf{x} = p \quad (1.4.21.2)$$

22. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

23. Find the equation of the line parallel to the y-axis whose distance from the line

$$(4 \ 3)\mathbf{x} = 12 \quad (1.4.23.1)$$

4 units.

24. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$(1 \ -7)\mathbf{x} = -5 \quad (1.4.24.1)$$

$$(3 \ 1)\mathbf{x} = 0 \quad (1.4.24.2)$$

25. Find the value of  $p$  so that the three lines

$$(3 \ 1)\mathbf{x} = 2 \quad (1.4.25.1)$$

$$(p \ 2)\mathbf{x} = 3 \quad (1.4.25.2)$$

$$(2 \ -1)\mathbf{x} = 3 \quad (1.4.25.3)$$

may intersect at one point.

26. Find the equation of the lines through the point  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$  which make an angle of  $45^\circ$  with the line

$$(1 \ -2)\mathbf{x} = 3. \quad (1.4.26.1)$$

27. Find the equation of the line passing through the point of intersection of the lines

$$(4 \ 7)\mathbf{x} = 3 \quad (1.4.27.1)$$

$$(2 \ -3)\mathbf{x} = -1 \quad (1.4.27.2)$$

that has equal intercepts on the axes.

28. In what ratio is the line joining  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$  divided by the line

$$(1 \ 1)\mathbf{x} = 4 \quad (1.4.28.1)$$

29. Find the distance of the line

$$(4 \ 7)\mathbf{x} = -5 \quad (1.4.29.1)$$

from the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  along the line

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0. \quad (1.4.29.2)$$

30. Find the direction in which a straight line must be drawn through the point  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$  so that its point of intersection with the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4 \quad (1.4.30.1)$$

may be at a distance of 3 units from this point.

31. The hypotenuse of a right angled triangle has its ends at the points  $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ . Find an equation of the legs of the triangle.

32. Find the image of the point  $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$  with respect to the line

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 7 \quad (1.4.32.1)$$

assuming the line to be a plane mirror.

33. If the lines

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 1 \quad (1.4.33.1)$$

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = 3 \quad (1.4.33.2)$$

are equally inclined to the line

$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 4, \quad (1.4.33.3)$$

find the value of  $m$ .

34. The sum of the perpendicular distances of a variable point  $\mathbf{P}$  from the lines

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (1.4.34.1)$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -7 \quad (1.4.34.2)$$

is always 10. Show that  $\mathbf{P}$  must move on a line.

35. Find the equation of the line which is equidistant from parallel lines

$$\begin{pmatrix} 9 & 7 \end{pmatrix} \mathbf{x} = 7 \quad (1.4.35.1)$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -6. \quad (1.4.35.2)$$

36. A ray of light passing through the point  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  reflects on the x-axis at point  $\mathbf{A}$  and the reflected ray passes through the point  $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ . Find the coordinates of  $\mathbf{A}$ .

37. A person standing at the junction of two

straight paths represented by the equations

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 4 \quad (1.4.37.1)$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5 \quad (1.4.37.2)$$

wants to reach the path whose equation is

$$\begin{pmatrix} 6 & -7 \end{pmatrix} \mathbf{x} = -8 \quad (1.4.37.3)$$

in the least time. Find the equation of the path that he should follow.

38. Find the equation of a line through the point  $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$  and parallel to the vector  $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$ .

39. Find the equation of a line passing through the points  $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$ .

40. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}, \quad (1.4.40.1)$$

find the equation of the line.

41. Determine the ratio in which the line

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (1.4.41.1)$$

divides the line segment joining the points  $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ .

42. A line perpendicular to the line segment joining the points  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  divides it in the ratio  $1 : n$ . Find the equation of the line.

43. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

44. Find the equation of the line passing through the point  $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$  and cutting off intercepts on the axes whose sum is 9.

45. Find the equation of the line through the point  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$  making an angle  $\frac{2\pi}{3}$  with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

46. The perpendicular from the origin to a line meets it at a point  $\begin{pmatrix} -2 \\ 9 \end{pmatrix}$ , find the equation of the line.



47. The length  $L$  (in cm) of a copper rod is a linear function of its Celsius temperature  $C$ . In an experiment, if  $L = 124.942$  when  $C = 20$  and  $L = 125.134$  when  $C = 110$ , express  $L$  in terms of  $C$ .

48. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?

49. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (1.4.49.1)$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (1.4.49.2)$$

50. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad (1.4.50.1)$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad (1.4.50.2)$$

51. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (1.4.51.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (1.4.51.2)$$

52. Find the distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (1.4.52.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (1.4.52.2)$$

53. Find the equation of a line which passes through the point  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and is parallel to the

vector  $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ .

54. Find the equation of the line that passes

through  $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$  and is in the direction  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ .

55. Find the equation of the line which passes through the point  $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$  and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \quad (1.4.55.1)$$

56. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \quad (1.4.56.1)$$

57. Find the equation of the line passing through the origin and the point  $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$ .

58. Find the equation of the line passing through the points  $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$ .

59. Find the angle between the following pair of lines:

a)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (1.4.59.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (1.4.59.2)$$

b)

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (1.4.59.3)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \quad (1.4.59.4)$$

60. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \quad (1.4.60.1)$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \quad (1.4.60.2)$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \quad (1.4.60.3)$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \quad (1.4.60.4)$$

61. Find the values of  $p$  so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \quad (1.4.61.1)$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \quad (1.4.61.2)$$

are at right angles.

62. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \quad (1.4.62.1)$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (1.4.62.2)$$

are perpendicular to each other.

63. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (1.4.63.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (1.4.63.2)$$

64. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad (1.4.64.1)$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad (1.4.64.2)$$

65. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (1.4.65.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.4.65.2)$$

66. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \quad (1.4.66.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \quad (1.4.66.2)$$

67. Find the equation of a plane which is at a

distance of  $\frac{6}{\sqrt{9}}$  from the origin and has normal vector  $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ .

68. Find the unit normal vector of the plane

$$(6 \ -3 \ -2)\mathbf{x} = 1. \quad (1.4.68.1)$$

69. Find the distance of the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (1.4.69.1)$$

from the origin.

70. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (1.4.70.1)$$

71. Find the equation of the plane which passes

through the point  $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$  and perpendicular tothe line with direction vector  $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ .

72. Find the equation of the plane passing through

 $\mathbf{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ ,  $\mathbf{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$  and  $\mathbf{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$ .

73. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.

74. Find the equation of the plane passing through the intersection of the planes

$$(1 \ 1 \ 1)\mathbf{x} = 6 \quad (1.4.74.1)$$

$$(2 \ 3 \ 4)\mathbf{x} = -5 \quad (1.4.74.2)$$

and the point  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

75. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \quad (1.4.75.1)$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \quad (1.4.75.2)$$

are coplanar.

76. Find the angle between the two planes

$$(2 \ 1 \ -2)\mathbf{x} = 5 \quad (1.4.76.1)$$

$$(3 \ -6 \ -2)\mathbf{x} = 7. \quad (1.4.76.2)$$

77. Find the angle between the two planes

$$(2 \ 2 \ -2)\mathbf{x} = 5 \quad (1.4.77.1)$$

$$(3 \ -6 \ 2)\mathbf{x} = 7. \quad (1.4.77.2)$$

Find the distance of a point  $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$  from the plane

$$(6 \ -3 \ 2)\mathbf{x} = 4 \quad (1.4.77.3)$$

Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \quad (1.4.77.4)$$

and the plane

$$(10 \ 2 \ -11)\mathbf{x} = 3 \quad (1.4.77.5)$$

78. In each of the following cases, determine the normal to the plane and the distance from the origin.

a)  $(0 \ 0 \ 1)\mathbf{x} = 2$       c)  $(0 \ 5 \ 0)\mathbf{x} = -8$

b)  $(1 \ 1 \ 1)\mathbf{x} = 1$       d)  $(2 \ 3 \ -1)\mathbf{x} = 5$

79. Find the equation of a plane which is at a distance of 7 units from the origin and normal to  $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$ .

80. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a)  $(2 \ 3 \ 4)\mathbf{x} = 12$       c)  $(1 \ 1 \ 1)\mathbf{x} = 1$

b)  $(3 \ 4 \ -6)\mathbf{x} = 0$       d)  $(0 \ 5 \ 0)\mathbf{x} = -8$

81. Find the equation of the planes

a) that passes through the point  $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$  and the

normal to the plane is  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .

b) that passes through the point  $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$  and the

normal vector the plane is  $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ .

82. Find the equation of the planes that passes through three points

a)  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b)  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ .

83. Find the intercepts cut off by the plane  $(2 \ 1 \ 1)\mathbf{x} = 5$ .

84. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOY plane.

85. Find the equation of the plane through the intersection of the planes  $(3 \ -1 \ 2)\mathbf{x} = 4$  and

$(1 \ 1 \ 1)\mathbf{x} = -2$  and the point  $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ .

86. Find the equation of the plane passing through the intersection of the planes  $(2 \ 2 \ -3)\mathbf{x} = 7$

and  $(2 \ 5 \ 3)\mathbf{x} = 9$  and the point  $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ .

87. Find the equation of the plane through the intersection of the planes  $(1 \ 1 \ 1)\mathbf{x} = 1$  and  $(2 \ 3 \ 4)\mathbf{x} = 5$  which is perpendicular to the plane  $(1 \ -1 \ 1)\mathbf{x} = 0$ .

88. Find the angle between the planes whose equations are  $(2 \ 2 \ -3)\mathbf{x} = 5$  and  $(3 \ -3 \ 5)\mathbf{x} = 3$

89. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a)  $(7 \ 5 \ 6)\mathbf{x} = -30$  and  $(3 \ -1 \ -10)\mathbf{x} = -4$

b)  $(2 \ 1 \ 3)\mathbf{x} = 2$  and  $(1 \ -2 \ 5)\mathbf{x} = 0$

c)  $(2 \ -2 \ 4)\mathbf{x} = -5$  and  $(3 \ -3 \ 6)\mathbf{x} = 1$

d)  $(2 \ -1 \ 3)\mathbf{x} = 1$  and  $(2 \ -1 \ 3)\mathbf{x} = -3$

e)  $(4 \ 8 \ 1)\mathbf{x} = 8$  and  $(0 \ 1 \ 1)\mathbf{x} = 4$

90. In the following cases, find the distance of each of the given points from the corresponding plane.

91. Find the equation of the plane that contains the

point  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  and is perpendicular to each of the

Item	Point	Plane
a)	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$(3 \ -4 \ 12)\mathbf{x} = 3$
b)	$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$	$(2 \ -1 \ 2)\mathbf{x} = -3$
c)	$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$	$(1 \ 2 \ -2)\mathbf{x} = 9$
d)	$\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$	$(2 \ -3 \ 6)\mathbf{x} = 2$

TABLE 1.4.90

planes

$$(2 \ 3 \ -2)\mathbf{x} = 5 \quad (1.4.91.1)$$

$$(1 \ 2 \ -3)\mathbf{x} = 8 \quad (1.4.91.2)$$

92. Find the distance between the point  $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$  and the plane determined by the points  $\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$ .

93. Find the coordinates of the point where the lines through the points  $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$  crosses the XY plane.

94. Show that the line joining the origin to the point  $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  is perpendicular to the line determined by the points  $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ .

95. If the coordinates of the points  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  be  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$ ,  $\begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$ , then find the angle between the lines  $AB$  and  $CD$ .

96. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \quad (1.4.96.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \quad (1.4.96.2)$$

find the value of  $k$ .

97. Find the equation of the line passing through  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and perpendicular to the plane

$$(1 \ 2 \ -5)\mathbf{x} = -9 \quad (1.4.97.1)$$

98. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \text{ and } \quad (1.4.98.1)$$

$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad (1.4.98.2)$$

99. Find the coordinates of the point where the line through  $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$  crosses the YZ-plane.

100. Find the coordinates of the point where the line through  $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$  crosses the ZX-plane.

101. Find the coordinates of the point where the line through  $\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$  crosses the plane

$$(2 \ 1 \ 1)\mathbf{x} = 7 \quad (1.4.101.1)$$

102. Find the equation of the plane passing through the point  $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$  and perpendicular to each of the planes

$$(1 \ 2 \ 3)\mathbf{x} = 5 \quad (1.4.102.1)$$

$$(3 \ 3 \ 1)\mathbf{x} = 0 \quad (1.4.102.2)$$

103. If the points  $\begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$  be equidistant from the plane

$$(3 \ 4 \ -12)\mathbf{x} = -13, \quad (1.4.103.1)$$

then find the value of  $p$ .

104. Find the equation of the plane passing through the line of intersection of the planes

$$(1 \ 1 \ 1)\mathbf{x} = 1 \text{ and } (1.4.104.1)$$

$$(2 \ 3 \ -1)\mathbf{x} = -4 \quad (1.4.104.2)$$

and parallel to the  $x$ -axis.

105. If  $\mathbf{O}$  be the origin and the coordinates of  $\mathbf{P}$  be  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ , then find the equation of the plane passing through  $\mathbf{P}$  and perpendicular to  $OP$ .

106. Find the equation of the plane which contains the line of intersection of the planes

$$(1 \ 2 \ 3)\mathbf{x} = 4 \quad (1.4.106.1)$$

$$(2 \ 1 \ -1)\mathbf{x} = -5 \quad (1.4.106.2)$$

and which is perpendicular to the plane

$$(5 \ 3 \ -6)\mathbf{x} = -8 \quad (1.4.106.3)$$

107. Find the distance of the point  $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$  from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \quad (1.4.107.1)$$

and the plane

$$(1 \ -1 \ 1)\mathbf{x} = 5 \quad (1.4.107.2)$$

108. Find the vector equation of the line passing through  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and parallel to the planes

$$(1 \ -1 \ 2)\mathbf{x} = 5 \quad (1.4.108.1)$$

$$(3 \ 1 \ 1)\mathbf{x} = 6 \quad (1.4.108.2)$$

109. Find the vector equation of the line passing through the point  $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$  and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \quad (1.4.109.1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad (1.4.109.2)$$

110. Distance between the two planes

$$(2 \ 3 \ 4)\mathbf{x} = 4 \quad (1.4.110.1)$$

$$(4 \ 6 \ 8)\mathbf{x} = 12 \quad (1.4.110.2)$$

a) 2

b) 4

c) 8

d)  $\frac{2}{\sqrt{29}}$

111. The planes

$$(2 \ -1 \ 4)\mathbf{x} = 5 \quad (1.4.111.1)$$

$$(5 \ -\frac{5}{2} \ 10)\mathbf{x} = 6 \quad (1.4.111.2)$$

are

a) Perpendicular

b) Parallel

c) intersect  $y$ -axis

d) passes through  $\begin{pmatrix} 0 \\ 0 \\ \frac{5}{4} \end{pmatrix}$

### 1.5 Miscellaneous

1. The base of an equilateral triangle with side  $2a$  lies along the  $y$ -axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

2. Find the distance between  $\mathbf{P} = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}$  and  $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \\ k \end{pmatrix}$  when

a)  $PQ$  is parallel to the  $y$ -axis.

b)  $PQ$  is parallel to the  $x$ -axis.

3. If three points  $\begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} a \\ b \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ k \end{pmatrix}$  lie on a line, show that  $\frac{a}{h} + \frac{b}{k} = 1$ .

4.  $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$  is the mid-point of a line segment between axes. Show that equation of the line is

$$\left(\frac{1}{a} \ \frac{1}{b}\right)\mathbf{x} = 2 \quad (1.5.4.1)$$

5. Point  $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$  divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

6. Show that two lines

$$(a_1 \ b_1)\mathbf{x} + c_1 = 0 \quad (1.5.6.1)$$

$$(a_2 \ b_2)\mathbf{x} + c_2 = 0 \quad (1.5.6.2)$$

are

- a) parallel if  $\frac{a_1}{b_1} = \frac{a_2}{b_2}$  and  
 b) perpendicular if  $a_1a_2 - b_1b_2 = 0$ .

7. Find the distance between the parallel lines

$$l(1 \ 1)\mathbf{x} = -p \quad (1.5.7.1)$$

$$l(1 \ 1)\mathbf{x} = r \quad (1.5.7.2)$$

8. Find the equation of the line through the point  $\mathbf{x}_1$  and parallel to the line

$$(A \ B)\mathbf{x} = -C \quad (1.5.8.1)$$

9. If  $p$  and  $q$  are the lengths of perpendiculars from the origin to the lines

$$(\cos \theta \ \sin \theta)\mathbf{x} = k \cos 2\theta \quad (1.5.9.1)$$

$$(\sec \theta \ \operatorname{cosec} \theta)\mathbf{x} = k \quad (1.5.9.2)$$

respectively, prove that  $p^2 + 4q^2 = k^2$ .

10. If  $p$  is the length of the perpendicular from the origin to the line whose intercepts on the axes are  $a$  and  $b$ , then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \quad (1.5.10.1)$$

11. Show that the area of the triangle formed by the lines

$$(-m_1 \ 1)\mathbf{x} = c_1 \quad (1.5.11.1)$$

$$(-m_2 \ 1)\mathbf{x} = c_2 \quad (1.5.11.2)$$

$$(1 \ 0)\mathbf{x} = 0 \quad (1.5.11.3)$$

is  $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$ .

12. Find the values of  $k$  for which the line

$$(k - 3 \ -(4 - k^2))\mathbf{x} + k^2 - 7k + 6 = 0 \quad (1.5.12.1)$$

is

- a) parallel to the x-axis  
 b) parallel to the y-axis  
 c) passing through the origin.

13. Find the perpendicular distance from the origin to the line joining the points  $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$  and

$$\begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}.$$

14. Find the area of the triangle formed by the lines

$$(1 \ -1)\mathbf{x} = 0 \quad (1.5.14.1)$$

$$(1 \ 1)\mathbf{x} = 0 \quad (1.5.14.2)$$

$$(1 \ 0)\mathbf{x} = k \quad (1.5.14.3)$$

15. If three lines whose equations are

$$(-m_1 \ 1)\mathbf{x} = c_1 \quad (1.5.15.1)$$

$$(-m_2 \ 1)\mathbf{x} = c_2 \quad (1.5.15.2)$$

$$(-m_3 \ 1)\mathbf{x} = c_3 \quad (1.5.15.3)$$

are concurrent, show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \quad (1.5.15.4)$$

16. Find the equation of the line passing through the origin and making an angle  $\theta$  with the line

$$(-m \ 1)\mathbf{x} = c \quad (1.5.16.1)$$

17. Prove that the product of the lengths of the perpendiculars drawn from the points  $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$

and  $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$  to the line

$$\left(\frac{\cos \theta}{a} \ \frac{\sin \theta}{b}\right)\mathbf{x} = 1 \quad (1.5.17.1)$$

is  $b^2$ .

18. If  $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$  and  $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$  are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is

$$\begin{pmatrix} m_1n_2 - m_2n_1 \\ n_1l_2 - n_2l_1 \\ l_1m_2 - l_2m_1 \end{pmatrix}.$$

19. A line makes angles  $\alpha, \beta, \gamma, \delta$  with the diagonals of a cube, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}. \quad (1.5.19.1)$$

20. Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}, \quad (1.5.20.1)$$

$$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma} \quad (1.5.20.2)$$

are coplanar.

21. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (1.5.21.1)$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (1.5.21.2)$$

find  $\mathbf{R}$ , which divides  $PQ$

- a) internally,
- b) externally.

22. Find  $\mathbf{R}$  which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \quad (1.5.22.1)$$

$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \quad (1.5.22.2)$$

externally in the ratio 1 : 2.

23. Find  $\|\mathbf{a}\|$  and  $\|\mathbf{b}\|$  if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \quad (1.5.23.1)$$

$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| \quad (1.5.23.2)$$

24. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \quad (1.5.24.1)$$

25. Find  $\|\mathbf{a}\|$  and  $\|\mathbf{b}\|$ , if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \quad (1.5.25.1)$$

$$\mathbf{a}^T \mathbf{b} = \frac{1}{2} \quad (1.5.25.2)$$

and the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $60^\circ$ .

26. Show that

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \perp (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a}) \quad (1.5.26.1)$$

27. If  $\mathbf{a}^T \mathbf{a} = 0$  and  $\mathbf{a}\mathbf{b} = 0$ , what can be concluded about the vector  $\mathbf{b}$ ?

28. If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \quad (1.5.28.1)$$

find the value of

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \quad (1.5.28.2)$$

29. If  $\mathbf{a} \neq \mathbf{0}$ ,  $\lambda \neq 0$ , then  $\|\lambda \mathbf{a}\| = 1$  if

- a)  $\lambda = 1$
- b)  $\lambda = -1$
- c)  $\|\mathbf{a}\| = |\lambda|$
- d)  $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

30. If a unit vector  $\mathbf{a}$  makes angles  $\frac{\pi}{3}$  with the x-axis and  $\frac{\pi}{4}$  with the y-axis and an acute angle  $\theta$  with the z-axis, find  $\theta$  and  $\mathbf{a}$ .

31. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) \quad (1.5.31.1)$$

32. If  $\mathbf{a}^T \mathbf{b} = 0$  and  $\mathbf{a} \times \mathbf{b} = 0$ , what can you conclude about  $\mathbf{a}$  and  $\mathbf{b}$ ?

33. Find  $\mathbf{x}$  if  $\mathbf{a}$  is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \quad (1.5.33.1)$$

34. If  $\|\mathbf{a}\| = 3$ ,  $\|\mathbf{b}\| = \frac{\sqrt{2}}{3}$ , then  $\mathbf{a} \times \mathbf{b}$  is a unit vector if the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is

- a)  $\frac{\pi}{6}$
- b)  $\frac{\pi}{4}$
- c)  $\frac{\pi}{3}$
- d)  $\frac{\pi}{2}$

35. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (1.5.35.1)$$

$$\iff \mathbf{a} \perp \mathbf{b}. \quad (1.5.35.2)$$

36. If  $\theta$  is the angle between two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then  $\mathbf{a}^T \mathbf{b} \geq 0$  only when

- a)  $0 < \theta < \frac{\pi}{2}$
- b)  $0 \leq \theta \leq \frac{\pi}{2}$
- c)  $0 < \theta < \pi$
- d)  $0 \leq \theta \leq \pi$

37. Let  $\mathbf{a}$  and  $\mathbf{b}$  be two unit vectors and  $\theta$  be the angle between them. Then  $\mathbf{a} + \mathbf{b}$  is a unit vector if

- a)  $\theta = \frac{\pi}{4}$
- b)  $\theta = \frac{\pi}{3}$
- c)  $\theta = \frac{\pi}{2}$
- d)  $\theta = \frac{2\pi}{3}$

38. If  $\theta$  is the angle between any two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then  $\|\mathbf{a}^T \mathbf{b}\| = \|\mathbf{a} \times \mathbf{b}\|$  when  $\theta$  is equal to

- a) 0
- b)  $\frac{\pi}{4}$
- c)  $\frac{\pi}{2}$
- d)  $\pi$ .

39. Let  $\|\mathbf{a}\| = 3$ ,  $\|\mathbf{b}\| = 4$ ,  $\|\mathbf{c}\| = 5$  such that each vector is perpendicular to the other two. Find  $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$ .

40. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (1.5.40.1)$$

evaluate

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}, \quad (1.5.40.2)$$

given that  $\|\mathbf{a}\| = 3$ ,  $\|\mathbf{b}\| = 4$  and  $\|\mathbf{c}\| = 2$ .

41. Find the angle between the lines whose direc-

tion vectors are  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$  and  $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$ .

42. Find the equation of a line parallel to the x-axis and passing through the origin.

43. Find the equation of a plane passing through  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$  and parallel to the plane

$$(1 \ 1 \ 1)\mathbf{x} = 2 \quad (1.5.43.1)$$

44. Prove that if a plane has the intercepts  $a, b, c$  and is at a distance of  $p$  units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \quad (1.5.44.1)$$

## 2 TRIANGLE

### 2.1 Construction

1. Draw  $\triangle ABC$  where  $\angle B = 90^\circ$ ,  $a = 4$  and  $b = 3$ .

**Solution:** The vertices of  $\triangle ABC$  are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (2.1.1.1)$$

The following code plots Fig. 2.1.1

```
codes/triangle/rt_triangle.py
```

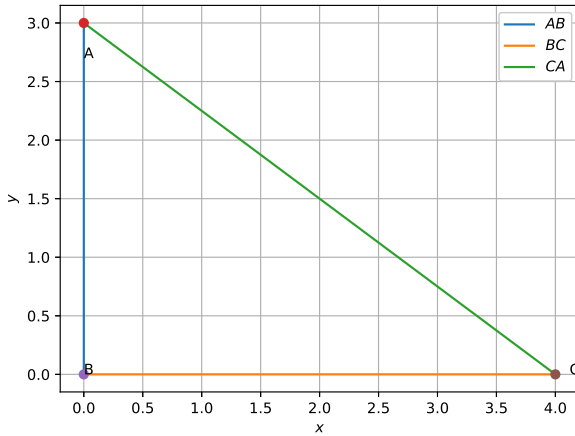


Fig. 2.1.1

2. Construct a triangle of sides  $a = 4$ ,  $b = 5$  and  $c = 6$ .

**Solution:** Let the vertices of  $\triangle ABC$  be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.1.2.1)$$

$$\mathbf{A}^T \triangleq (p \ q) \quad (2.1.2.2)$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T \mathbf{A} = (p \ q) \begin{pmatrix} p \\ q \end{pmatrix} \quad (2.1.2.3)$$

$$= p \times p + q \times q = p^2 + q^2 \quad (2.1.2.4)$$

Then

$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (2.1.2.5)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (2.1.2.6)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (2.1.2.7)$$

From (2.1.2.7),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (2.1.2.8)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{C}^T \mathbf{C} - \mathbf{A}^T \mathbf{C} - \mathbf{C}^T \mathbf{A} \quad (2.1.2.9)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} \quad (\because \mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}) \quad (2.1.2.10)$$

$$= a^2 + c^2 - 2ap \quad (2.1.2.11)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (2.1.2.12)$$

From (2.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \quad (2.1.2.13)$$

$$\Rightarrow q = \pm \sqrt{c^2 - p^2} \quad (2.1.2.14)$$

The following code plots Fig. 2.1.2

```
codes/triangle/draw_triangle.py
```

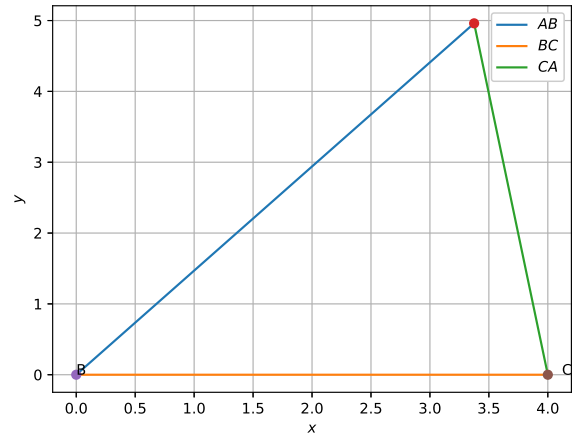


Fig. 2.1.2



3. Construct a triangle of sides  $a = 5$ ,  $b = 6$  and  $c = 7$ . Construct a similar triangle whose sides are  $\frac{7}{5}$  times the corresponding sides of the first triangle.

**Solution:** The sides of the similar triangle are  $\frac{7}{5}a$ ,  $\frac{7}{5}b$  and  $\frac{7}{5}c$ .

4. Construct an isosceles triangle whose base is  $a = 8\text{cm}$  and altitude  $AD = h = 4\text{cm}$

**Solution:** Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \quad (2.1.4.1)$$

5. In  $\triangle ABC$ , given that  $a+b+c = 11$ ,  $\angle B = 45^\circ$  and  $\angle C = 45^\circ$ , find  $a, b, c$  and sketch the triangle.

**Solution:** From the given information,

$$a + b + c = 11 \quad (2.1.5.1)$$

$$b = c \quad (\because B = C = 45^\circ) \quad (2.1.5.2)$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \quad (2.1.5.3)$$

From (2.1.5.1) and (2.1.5.2),

$$a + 2b = 11 \quad (2.1.5.4)$$

From (2.1.5.2) and (2.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \quad (2.1.5.5)$$

(2.1.5.4) and (2.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \quad (2.1.5.6)$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times (-\sqrt{2}) - 2 \times 0}{1 \times (-\sqrt{2}) - 2 \times 1} \quad (2.1.5.7)$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \quad (2.1.5.8)$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \quad (2.1.5.9)$$

by expanding the determinants. The following code may be used to compute  $a, b$  and  $c$ .

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 2.1.5 using a single matrix equation.

**Solution:** The equations

$$a + 2b = 11 \quad (2.1.6.1)$$

$$a - b\sqrt{2} = 0 \quad (2.1.6.2)$$

$$b - c = 0 \quad (2.1.6.3)$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \quad (2.1.6.4)$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.5)$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.6)$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.7)$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \quad (2.1.6.8)$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \quad (2.1.6.9)$$

The determinants of larger matrices can be expressed similarly.

7. Draw  $\triangle ABC$  with  $a = 6, c = 5$  and  $\angle B = 60^\circ$ .

**Solution:** In Fig. (2.1.7),  $AD \perp BC$ .

$$\cos C = \frac{y}{b}, \quad (2.1.7.1)$$

$$\cos B = \frac{x}{c}, \quad (2.1.7.2)$$

Thus,

$$a = x + y = b \cos C + c \cos B, \quad (2.1.7.3)$$

$$b = c \cos A + a \cos C \quad (2.1.7.4)$$

$$c = b \cos A + a \cos B \quad (2.1.7.5)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.1.7.6)$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \quad (2.1.7.7)$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \quad (2.1.7.8)$$

From (2.1.7.8)

$$b^2 = c^2 + a^2 - 2ca \cos B \quad (2.1.7.9)$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

8. Draw  $\triangle ABC$  with  $a = 7, \angle B = 45^\circ$  and  $\angle A = 105^\circ$ .

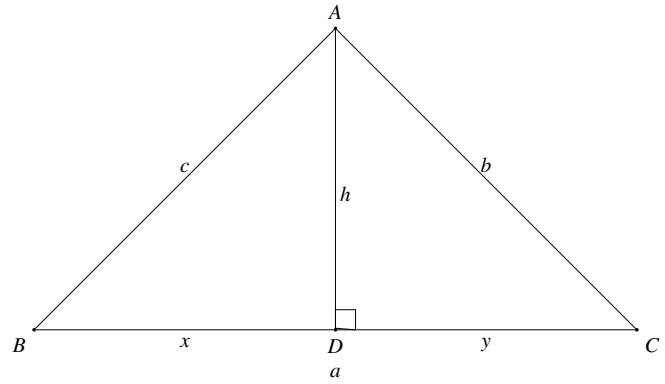


Fig. 2.1.7: The cosine formula

**Solution:** In Fig. (2.1.7),

$$\sin B = \frac{h}{c} \quad (2.1.8.1)$$

$$\sin C = \frac{h}{b} \quad (2.1.8.2)$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.1.8.3)$$

Thus,

$$c = \frac{a \sin C}{\sin A} \quad (2.1.8.4)$$

where

$$C = 180 - A - B \quad (2.1.8.5)$$

9. Draw  $\triangle ABC$  if  $AB = 3, AC = 5$  and  $\angle C = 30^\circ$ .

**Solution:** From (2.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (2.1.9.1)$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \quad (2.1.9.2)$$

$$\therefore (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \quad (2.1.9.3)$$

(2.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \quad (2.1.9.4)$$

$$\Rightarrow (a - b \cos C)^2 = b^2 (1 - \cos^2 C) - c^2 \quad (2.1.9.5)$$

$$\text{or, } a = b \cos C \pm \sqrt{b^2 (1 - \cos^2 C) - c^2} \quad (2.1.9.6)$$

Choose the value(s) for which  $a > 0$ .

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \quad (2.1.10.1)$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (2.1.10.2)$$

Verify (2.1.9.6) using (2.1.10.2).

11.  $\triangle ABC$  is right angled at **B**. If  $a = 12$  and  $b+c = 18$ , find  $b, c$  and draw the triangle.

**Solution:** From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \quad (2.1.11.1)$$

$$\Rightarrow (18 - c)^2 = 12^2 + c^2 \quad (2.1.11.2)$$

which can be simplified to obtain

$$36c - 180 = 0 \quad (2.1.11.3)$$

$$\Rightarrow c = 5 \quad (2.1.11.4)$$

and  $b = 13$

12. Find a simpler solution for Problem 2.1.5

**Solution:** Use cosine formula.

13. In  $\triangle ABC$ ,  $a = 7, \angle B = 75^\circ$  and  $b + c = 13$ . Alternatively,

$$a = b \cos C + c \cos B \quad (2.1.13.1)$$

$$b \sin C = c \sin B \quad (2.1.13.2)$$

$$a + b + c = 11 \quad (2.1.13.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (2.1.13.4)$$

Solving the equivalent matrix equation gives the desired answer.

## 2.2 Construction Exercises

1. In  $\triangle ABC$ ,  $a = 8, \angle B = 45^\circ$  and  $c - b = 3.5$ . Sketch  $\triangle ABC$ .
2. In  $\triangle ABC$ ,  $a = 6, \angle B = 60^\circ$  and  $b - c = 2$ . Sketch  $\triangle ABC$ .
3. Draw  $\triangle ABC$ , given that  $a + b + c = 11, \angle B = 30^\circ$  and  $\angle C = 90^\circ$ .
4. Construct  $\triangle xyz$  where  $xy = 4.5, yz = 5$  and  $zx = 6$ .
5. Draw an equilateral triangle of side 5.5.
6. Draw  $\triangle PQR$  with  $PQ = 4, QR = 3.5$  and  $PR = 4$ . What type of triangle is this?
7. Construct  $\triangle ABC$  such that  $AB = 2.5, BC = 6$  and  $AC = 6.5$ . Find  $\angle B$ .
8. Construct  $\triangle PQR$ , given that  $PQ = 3, QR = 5.5$  and  $\angle PQR = 60^\circ$ .
9. Construct  $\triangle DEF$  such that  $DE = 5, DF = 3$  and  $\angle D = 90^\circ$ .
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is  $110^\circ$ .
11. Construct  $\triangle ABC$  with  $BC = 7.5, AC = 5$  and  $\angle C = 60^\circ$ .
12. Construct  $\triangle XYZ$  if  $XY = 6, \angle X = 30^\circ$  and  $\angle Y = 100^\circ$ .
13. If  $AC = 7, \angle A = 60^\circ$  and  $\angle B = 50^\circ$ , can you draw the triangle?
14. Construct  $\triangle ABC$  given that  $\angle A = 60^\circ, \angle B = 30^\circ$  and  $AB = 5.8$ .
15. Construct  $\triangle PQR$  if  $PQ = 5, \angle Q = 105^\circ$  and  $\angle R = 40^\circ$ .
16. Can you construct  $\triangle DEF$  such that  $EF = 7.2, \angle E = 110^\circ$  and  $\angle F = 180^\circ$ ?
17. Construct  $\triangle LMN$  right angled at  $M$  such that  $LN = 5$  and  $MN = 3$ .
18. Construct  $\triangle PQR$  right angled at  $Q$  such that  $QR = 8$  and  $PR = 10$ .
19. Construct right angled  $\triangle$  whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled  $\triangle ABC$  right angled at  $C$  such  $AC = 6$ .
21. Construct the triangles in Table 2.2.21.

## 2.3 Triangle Geometry

1. The vertices of  $\triangle PQR$  are

$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ . Find the equation of the median through the vertex **R**.

S.No	Triangle	Given Measurements		
1	$\triangle ABC$	$\angle A = 85^\circ$	$\angle B = 115^\circ$	$AB = 5$
2	$\triangle PQR$	$\angle Q = 30^\circ$	$\angle R = 60^\circ$	$QR = 4.7$
3	$\triangle ABC$	$\angle A = 70^\circ$	$\angle B = 50^\circ$	$AC = 3$
4	$\triangle LMN$	$\angle L = 60^\circ$	$\angle N = 120^\circ$	$LM = 5$
5	$\triangle ABC$	$BC = 2$	$AB = 4$	$AC = 2$
6	$\triangle PQR$	$PQ = 2.5$	$QR = 4$	$PR = 3.5$
7	$\triangle XYZ$	$XY = 3$	$YZ = 4$	$XZ = 5$
8	$\triangle DEF$	$DE = 4.5$	$EF = 5.5$	$DF = 4$

TABLE 2.2.21

- In the  $\triangle ABC$  with vertices  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ , find the equation and length of the altitude from the vertex  $\mathbf{A}$ .
- Find the area of a triangle whose vertices are  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$ .
- Find the area of a triangle formed by the vertices  $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$ .
- Find the area of a triangle formed by the points  $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$ ,  $\mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ ,  $\mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ .
- Find the area of the triangle whose vertices are
  - $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$
  - $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$
- Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are  $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ .
- Verify that the median of  $\triangle ABC$  with vertices  $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$  divides it into two triangles of equal areas.
- The vertices of  $\triangle ABC$  are  $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ . A line is drawn to intersect sides  $AB$  and  $AC$  at  $D$  and  $E$  respectively, such that
 
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (2.3.9.1)$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (2.3.9.2)$$

- Let  $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$  and  $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$  be the vertices of  $\triangle ABC$ .
  - The median from  $\mathbf{A}$  meets  $BC$  at  $\mathbf{D}$ . Find the coordinates of the point  $\mathbf{D}$ .
  - Find the coordinates of the point  $\mathbf{P}$  on  $AD$  such that  $AP : PD = 2 : 1$ .
  - Find the coordinates of the points  $\mathbf{Q}$  and  $\mathbf{R}$  on medians  $BE$  and  $CF$  respectively such that  $BQ : QE = 2 : 1$  and  $CR : RF = 2 : 1$ .
- In  $\triangle ABC$ , Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (2.3.11.1)$$

- Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (2.3.12.1)$$

are the vertices of a right angled triangle.

- In  $\triangle ABC$ ,  $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ . Find  $\angle B$ .
- Show that the vectors  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$  form the vertices of a right angled triangle.
- Find the area of a triangle having the points  $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ , and  $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$  as its vertices.
- Find the area of a triangle with vertices  $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ , and  $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$ .
- A girl walks 4km west, then she walks 3km in a direction  $30^\circ$  east of north and stops. Determine the girl's displacement from her initial point of departure.
- Find the direction vectors of the sides of a triangle with vertices  $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$ ,  $\mathbf{B} =$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$$

### 3 QUADRILATERAL

#### 3.1 Construction Examples

1. Draw  $ABCD$  with  $AB = a = 4.5$ ,  $BC = b = 5.5$ ,  $CD = c = 4$ ,  $AD = d = 6$  and  $AC = e = 7$ .

**Solution:** Fig. 3.1.1 shows a rough sketch of  $ABCD$ . Letting

$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (3.1.1.1)$$

it is trivial to sketch  $\triangle ABC$  from Problem 2.1.2.  $\triangle ACD$  is can be obtained by rotating an equivalent triangle with  $AC$  on the  $x$ -axis by an angle  $\theta$  with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (3.1.1.2)$$

and

$$\cos \theta = \frac{a^2 + e^2 - b^2}{2ae} \quad (3.1.1.3)$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} \quad (3.1.1.4)$$

The coordinates of the rotated triangle  $ACD$  are

$$\mathbf{D} = \mathbf{P} \begin{pmatrix} h \\ k \end{pmatrix} \quad (3.1.1.5)$$

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.1.1.6)$$

$$\mathbf{C} = \mathbf{P} \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (3.1.1.7)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (3.1.1.8)$$

The following code plots quadrilateral  $ABCD$

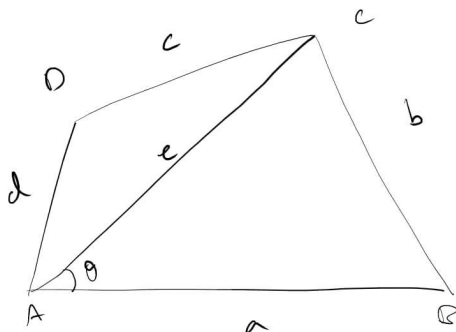


Fig. 3.1.1

in Fig. 3.1.1

codes/quad/draw\_quad.py

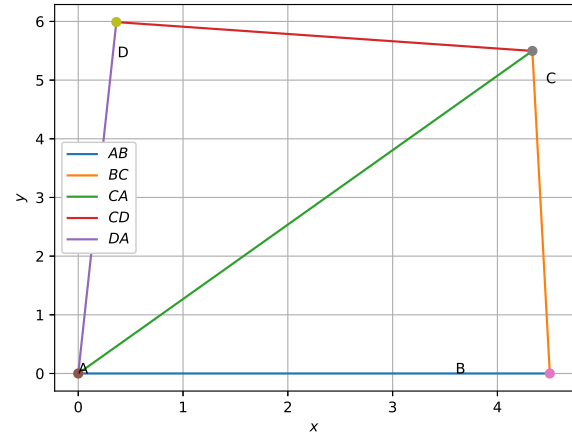


Fig. 3.1.1

2. Draw the parallelogram  $MORE$  with  $OR = 6$ ,  $RE = 4.5$  and  $EO = 7.5$ .

**Solution:** Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel.

3. Construct a kite  $EASY$  if  $AY = 8$ ,  $EY = 4$  and  $SY = 6$ .

**Solution:** The diagonals of a kite are perpendicular to each other.

4. Draw the rhombus  $BEST$  with  $BE = 4.5$  and  $ET = 6$ .

**Solution:** Diagonals of a rhombus bisect each other at right angles.

#### 3.2 Construction Exercises

1. Construct a quadrilateral  $ABCD$  such that  $AB = 5$ ,  $\angle A = 50^\circ$ ,  $AC = 4$ ,  $BD = 5$  and  $AD = 6$ .
2. Construct  $PQRS$  where  $PQ = 4$ ,  $QR = 6$ ,  $RS = 5$ ,  $PS = 5.5$  and  $PR = 7$ .
3. Draw  $JUMP$  with  $JU = 3.5$ ,  $UM = 4$ ,  $MP = 5$ ,  $PJ = 4.5$  and  $PU = 6.5$ .
4. Construct a quadrilateral  $ABCD$  such that  $BC = 4.5$ ,  $AC = 5.5$ ,  $CD = 5$ ,  $BD = 7$  and  $AD = 5.5$ .
5. Can you construct a quadrilateral  $PQRS$  with  $PQ = 3$ ,  $RS = 3$ ,  $PS = 7.5$ ,  $PR = 8$  and  $SQ = 4$ ?
6. Construct  $LIFT$  such that  $LI = 4$ ,  $IF = 3$ ,  $TL = 2.5$ ,  $LF = 4.5$ ,  $IT = 4$ .

7. Draw *GOLD* such that  $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$ .
8. DRAW rhombus *BEND* such that  $BN = 5.6, DE = 6.5$ .
9. construct a quadrilateral *MIST* where  $MI = 3.5, IS = 6.5, \angle M = 75^\circ, \angle I = 105^\circ$  and  $\angle S = 120^\circ$ .
10. Can you construct the above quadrilateral *MIST* if  $\angle M = 100^\circ$  instead of  $75^\circ$ .
11. Can you construct the quadrilateral *PLAN* if  $PL = 6, LA = 9.5, \angle P = 75^\circ, \angle L = 150^\circ$  and  $\angle A = 140^\circ$ ?
12. Construct *MORE* where  $MO = 6, OR = 4.5, \angle M = 60^\circ, \angle O = 105^\circ, \angle R = 105^\circ$ .
13. Construct *PLAN* where  $PL = 4, LA = 6.5, \angle P = 90^\circ, \angle A = 110^\circ$  and  $\angle N = 85^\circ$ .
14. Construct parallelogram *HEAR* where  $HE = 5, EA = 6, \angle R = 85^\circ$ .
15. Draw rectangle *OKAY* with  $OK = 7$  and  $KA = 5$ .
16. Construct *ABCD*, where  $AB = 4, BC = 5, CD = 6.5, \angle B = 105^\circ$  and  $\angle C = 80^\circ$ .
17. Construct *DEAR* with  $DE = 4, EA = 5, AR = 4.5, \angle E = 60^\circ$  and  $\angle A = 90^\circ$ .
18. Construct *TRUE* with  $TR = 3.5, RU = 3, UE = 4, \angle R = 75^\circ$  and  $\angle U = 120^\circ$ .
19. Draw a square of side 4.5.
20. Can you construct a rhombus *ABCD* with  $AC = 6$  and  $BD = 7$ ?
21. Draw a square *READ* with  $RE = 5.1$ .
22. Draw a rhombus whose diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram *OKAY* with  $OK = 5.5$  and  $KA = 4.2$ .

### 3.3 Quadrilateral Geometry

1. Draw a quadrilateral in the Cartesian plane, whose vertices are  $\begin{pmatrix} -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ . Also, find its area.
2. Find the area of a rhombus if its vertices are  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$  taken in order.
3. If  $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ , find the area of the quadrilateral *ABCD*.
4. Without using distance formula, show that points  $\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$  are the vertices

of a parallelogram.

5. Find the area of the quadrilateral whose vertices, taken in order, are  $\begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .
6. The two opposite vertices of a square are  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ . Find the coordinates of the other two vertices.
7. *ABCD* is a rectangle formed by the points  $\mathbf{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ .  $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$  are the mid points of *AB, BC, CD, DA* respectively. Is the quadrilateral *PQRS* a
  - a) square?
  - b) rectangle?
  - c) rhombus?
8. Find the area of a parallelogram whose adjacent sides are given by the vectors  $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ .
9. Find the area of a parallelogram whose adjacent sides are determined by the vectors  $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$ .
10. Find the area of a rectangle *ABCD* with vertices  $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$ .
11. The two adjacent sides of a parallelogram are  $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ . Find the unit vector parallel to its diagonal. Also, find its area.

## 4 CIRCLE

### 4.1 Construction Examples

1. Draw a circle with centre **B** and radius 6. If **C** be a point 10 units away from its centre, construct the pair of tangents *AC* and *CD* to the circle.

**Solution:** The tangent is perpendicular to

the radius. From the given information, in  $\triangle ABC$ ,  $AC \perp AB$ ,  $a = 10$  and  $c = 6$ .

$$b = \sqrt{a^2 - c^2} \quad (4.1.1.1)$$

The following code plots Fig. 4.1.1

codes/circle/draw\_circle\_eg.py

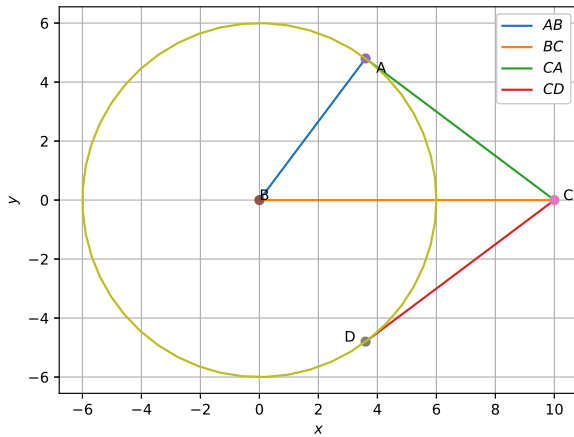


Fig. 4.1.1

2. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.

**Solution:** For any angle  $\theta$ , a point on the circle with radius 3 has coordinates

$$3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (4.1.2.1)$$

## 4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is  $AB \perp CD$ ?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

**Solution:** Take the centre of both circles to be at the origin.

6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.

**Solution:** Take the diameter to be on the  $x$ -axis.

7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of  $60^\circ$ .

**Solution:** The tangent is perpendicular to the radius.

8. Draw a line segment **AB** of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (4.2.2.1)$$

9. Let **ABC** be a right triangle in which  $a = 8$ ,  $c = 6$  and  $\angle B = 90^\circ$ . **BD** is the perpendicular from **B** on **AC** (altitude). The circle through **B**, **C**, **D** (circumcircle of  $\triangle BCD$ ) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

## 4.3 Circle Geometry

1. Find the coordinates of a point **A**, where **AB** is the diameter of a circle whose centre is  $(2, -3)$  and  $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ .
2. Find the centre of a circle passing through the points  $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ .
3. Find the locus of all the unit vectors in the  $xy$ -plane.