Computational Approach to School Mathematics

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Contents

***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/codes
```

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1 Triangle

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90°, a = 4$ and $b = 3$.
   Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1.1}$$

   The following code plots Fig. 1.1.1

```
codes/triangle/rt_triangle.py
```

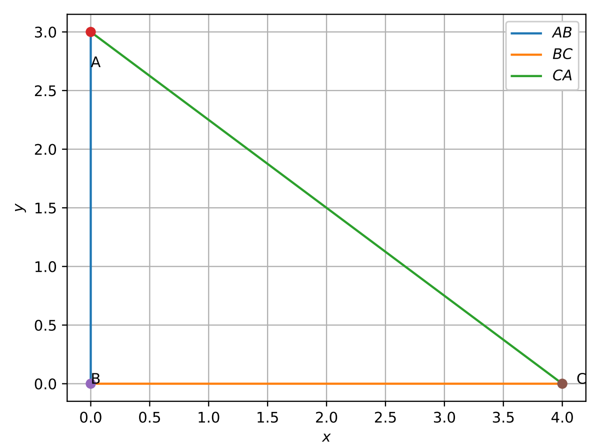

Fig. 1.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
   Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \tag{1.1.2.2}$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{1.1.2.3}$$

$$= p \times p + q \times q = p^2 + q^2 \tag{1.1.2.4}$$

Then

$$AB \triangleq \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{1.1.2.5}$$

$$BC = \|\mathbf{C}-\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{1.1.2.6}$$

$$AC = \|\mathbf{A}-\mathbf{C}\|^2 = b^2 \tag{1.1.2.7}$$

From (1.1.2.7),

$$b^2 = \|\mathbf{A}-\mathbf{C}\|^2 = \|\mathbf{A}-\mathbf{C}\|^T \|\mathbf{A}-\mathbf{C}\| \tag{1.1.2.8}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{1.1.2.9}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{1.1.2.10}$$

$$= a^2 + c^2 - 2ap \tag{1.1.2.11}$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{1.1.2.12}$$

From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.1.2.14}$$

The following code plots Fig. 1.1.2

```
codes/triangle/draw_triangle.py
```

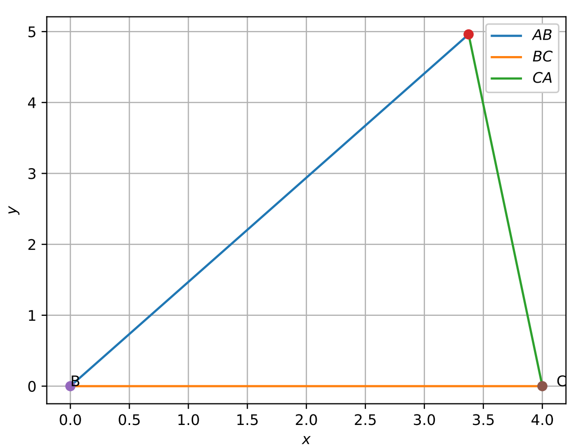

Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
   Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
   Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{1.1.4.1}$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.
   Solution: From the given information,

$$a + b + c = 11 \tag{1.1.5.1}$$

$$b = c \quad (\because B = C = 45^\circ) \tag{1.1.5.2}$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{1.1.5.3}$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \tag{1.1.5.4}$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{1.1.5.5}$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{1.1.5.6}$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{1.1.5.7}$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{1.1.5.8}$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{1.1.5.9}$$

by expanding the determinants. The following code may be used to compute a, b and c.

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 1.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \tag{1.1.6.1}$$

$$a - b\sqrt{2} = 0 \tag{1.1.6.2}$$

$$b - c = 0 \tag{1.1.6.3}$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{1.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{1.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. 1.1.7, $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{1.1.7.1}$$

$$\cos B = \frac{x}{b}, \tag{1.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \tag{1.1.7.3}$$

$$b = c\cos A + a\cos C \tag{1.1.7.4}$$

$$c = b\cos A + a\cos B \tag{1.1.7.5}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.7.6}$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.1.7.7}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1.7.8}$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \tag{1.1.7.9}$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

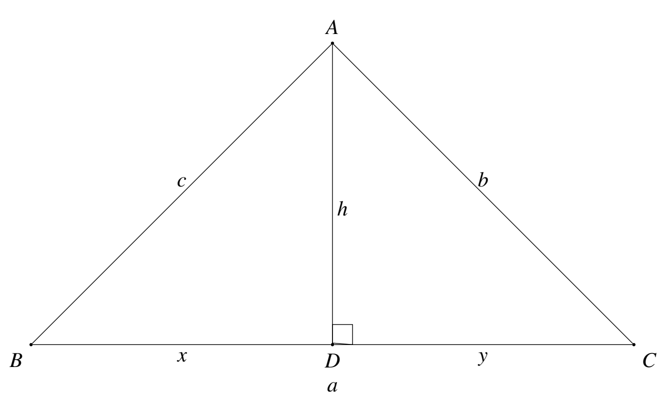

Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
Solution: In Fig. (1.1.7),

$$\sin B = \frac{h}{c} \tag{1.1.8.1}$$
$$\sin C = \frac{h}{b} \tag{1.1.8.2}$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.1.8.3}$$

Thus,

$$c = \frac{a \sin C}{\sin A} \tag{1.1.8.4}$$

where

$$C = 180 - A - B \tag{1.1.8.5}$$

9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30°$.
Solution: From (1.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.1.9.1}$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \tag{1.1.9.2}$$

$$\because (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \tag{1.1.9.3}$$

(1.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \tag{1.1.9.4}$$
$$\implies (a - b \cos C)^2 = b^2 \left(1 - \cos^2 C\right) - c^2 \tag{1.1.9.5}$$
$$\text{or}, a = b \cos C \pm \sqrt{b^2 \left(1 - \cos^2 C\right) - c^2} \tag{1.1.9.6}$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \tag{1.1.10.1}$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{1.1.10.2}$$

Verify (1.1.9.6) using (1.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b + c = 18$, find b, c and draw the triangle.
Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \tag{1.1.11.1}$$
$$\implies (18 - c)^2 = 12^2 + c^2 \tag{1.1.11.2}$$

which can be simplified to obtain

$$36c - 180 = 0 \tag{1.1.11.3}$$
$$\implies c = 5 \tag{1.1.11.4}$$

and $b = 13$

12. Find a simpler solution for Problem 1.1.5
Solution: Use cosine formula.

13. In $\triangle ABC$, $a = 7, \angle B = 75^{\circ}$ and $b + c = 13$. Alternatively,

$$a = b \cos C + c \cos B \tag{1.1.13.1}$$
$$b \sin C = c \sin B \tag{1.1.13.2}$$
$$a + b + c = 11 \tag{1.1.13.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \tag{1.1.13.4}$$

Solving the equivalent matrix equation gives the desired answer.

1.2 *Construction Exercises*

1. In $\triangle ABC$, $a = 8, \angle B = 45^{\circ}$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6, \angle B = 60^{\circ}$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a + b + c = 11, \angle B = 30^{\circ}$ and $\angle C = 90^{\circ}$.
4. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60°$.
9. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90°$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
11. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60°$.

12. Construct $\triangle XYZ$ if $XY = 6, \angle X = 30°$ and $\angle Y = 100°$.
13. If $AC = 7, \angle A = 60°$ and $\angle B = 50°$, can you draw the triangle?
14. Construct $\triangle ABC$ given that $\angle A = 60°, \angle B = 30°$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5, \angle Q = 105°$ and $\angle R = 40°$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2, \angle E = 110°$ and $\angle F = 180°$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled $\triangle$ whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 1.2.21.

S.No	**Triangle**	**Given Measurements**		
1	$\triangle$ABC	$\angle A = 85°$	$\angle B = 115°$	AB = 5
2	$\triangle$PQR	$\angle Q = 30°$	$\angle R = 60°$	QR = 4.7
3	$\triangle$ABC	$\angle A = 70°$	$\angle B = 50°$	AC = 3
4	$\triangle$LMN	$\angle L = 60°$	$\angle N = 120°$	LM = 5
5	$\triangle$ABC	BC = 2	AB = 4	AC = 2
6	$\triangle$PQR	PQ = 2.5	QR = 4	PR = 3.5
7	$\triangle$XYZ	XY = 3	YZ = 4	XZ = 5
8	$\triangle$DEF	DE = 4.5	EF = 5.5	DF = 4

TABLE 1.2.21

1.3 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

   Solution: The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{1.3.1.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.3.1.2}$$

Since

$$\mathbf{B} - \mathbf{A} \neq k (\mathbf{C} - \mathbf{A}), \tag{1.3.1.3}$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \tag{1.3.1.4}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \tag{1.3.1.5}$$

$$\implies rank(\mathbf{M}) = 2 \tag{1.3.1.6}$$

as the number of non zero rows is 2. The following code plots Fig. 1.3.1

```
codes/triangle/check_tri.py
```

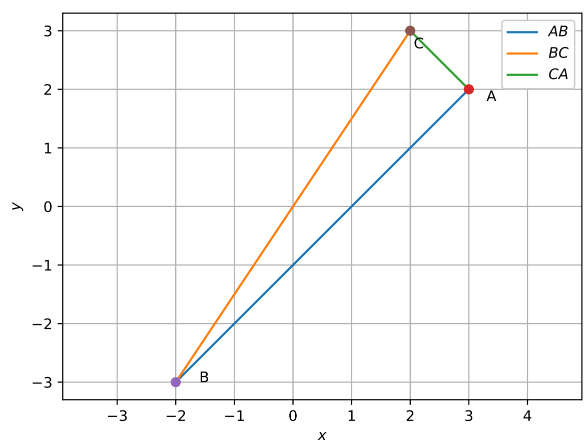

Fig. 1.3.1

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.3.1.7}$$

which, from (1.1.2.10) can be expressed as

$$\|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} \\ = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \tag{1.3.1.8}$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = 0 \tag{1.3.1.9}$$

after simplification. From (1.3.1.1) and (1.3.1.2), it is easy to verify that

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}-5 & -5\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 0 \tag{1.3.1.10}$$

satisfying (1.3.1.9). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.

2. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\6\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}-3\\-5\end{pmatrix}$.
Solution: In Fig. 1.1.1, from Baudhayana's theorem,

$$b^2 = a^2 + c^2 \tag{1.3.2.1}$$
$$= b^2\cos^2 C + b^2\sin^2 C \tag{1.3.2.2}$$
$$\implies \cos^2 C + \sin^2 C = 1 \tag{1.3.2.3}$$

In Fig. 1.1.7, the area of $\triangle ABC$ is defined as

$$\frac{1}{2}ah = \frac{1}{2}ab\sin C \tag{1.3.2.4}$$
$$= \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.3.2.1))} \tag{1.3.2.5}$$
$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.1.7.8))} \tag{1.3.2.6}$$
$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \tag{1.3.2.7}$$
$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.3.2.8}$$
$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.3.2.9}$$
$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.3.2.10}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.3.2.11}$$

in (1.3.2.10), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.3.2.12}$$

This is known as Hero's formula. The following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

3. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix}5\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\-4\end{pmatrix}$.
Solution: The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.3.1.4) as

$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.3.3.1}$$

The computation is done in **area_tri.py**

4. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix}-1.5\\3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-2\end{pmatrix}, \mathbf{R} = \begin{pmatrix}-3\\4\end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{vmatrix} \tag{1.3.4.1}$$

5. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.3.5.1}$$

as its vertices.
Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.3.5.2}$$

For any two vectors $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.3.5.3}$$

The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
```

6. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix}3\\-5\\7\end{pmatrix}$ and $\begin{pmatrix}-1\\7\\-6\end{pmatrix}$, respectively, find the coordinates of the point $\mathbf{C}$.
Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.3.6.1}$$

Thus,

$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.3.6.2}$$

7. Show that the points
$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.3.7.1}$$
are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.3.7

```
codes/triangle/triangle_3d.py
```

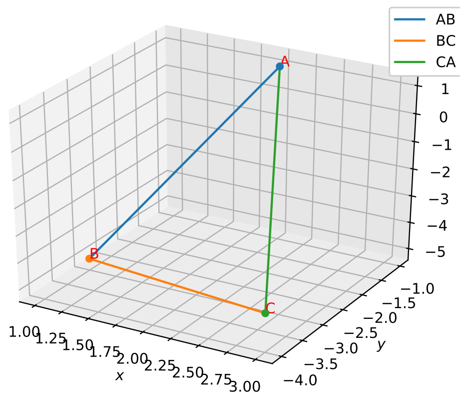

Fig. 1.3.7

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since
$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.3.7.2}$$
it is proved that the triangle is indeed right angled.

8. Are the points
$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{1.3.8.1}$$
the vertices of a right angled triangle?

9. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.
Solution: Fig. 1.3.9 summarizes the problem.

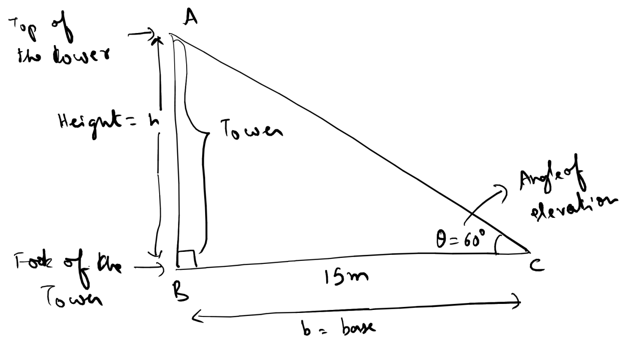

Fig. 1.3.9

$$h = b \tan\theta = 15 \tan 60^\circ = 15\sqrt{3} \tag{1.3.9.1}$$

10. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

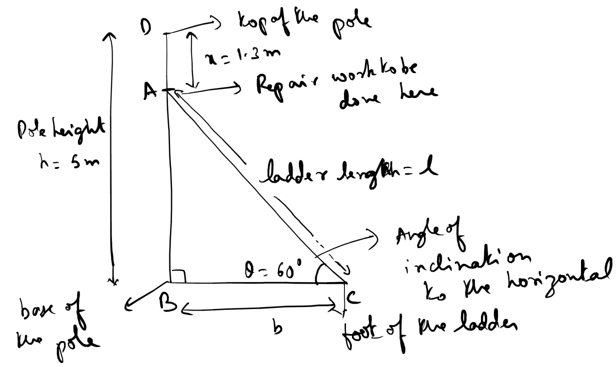

Fig. 1.3.10

Solution: Fig. 1.3.10 summarizes the problem. The objective is to find l and b. From the figure, if
$$\cot\theta = \frac{1}{\tan\theta}, \tag{1.3.10.1}$$
$$h - x = l\sin\theta = b\tan\theta \tag{1.3.10.2}$$
$$\implies l = (h - x)\csc\theta = 3.7\csc 60^\circ \tag{1.3.10.3}$$
$$\text{and } b = (h - x)\cot\theta = 3.7\cot^\circ \tag{1.3.10.4}$$

11. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?
Solution: Fig. 1.3.11 summarizes the problem.

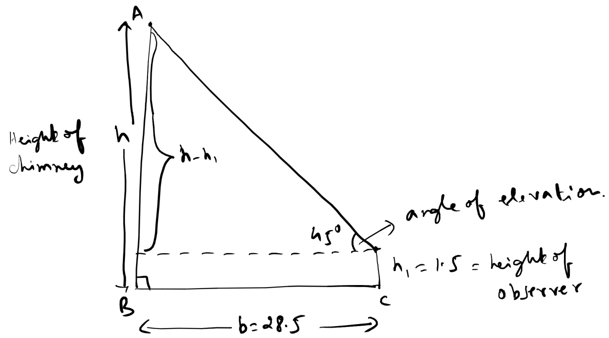

Fig. 1.3.11

The objective is to find h. From the figure,

$$h - h_1 = b \tan\theta \tag{1.3.11.1}$$
$$\implies h = h_1 + b \tan\theta \tag{1.3.11.2}$$
$$= 1.5 + 28.5 \tan 45^\circ \tag{1.3.11.3}$$
$$= 30m \tag{1.3.11.4}$$

12. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

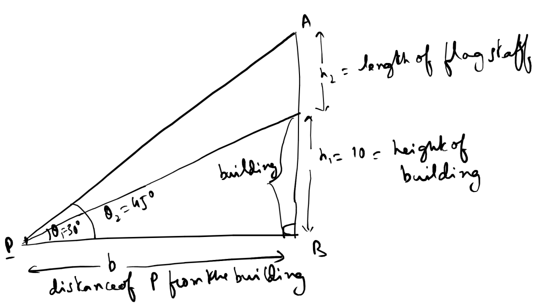

Fig. 1.3.12

Solution: Fig. 1.3.12 summarizes the problem. The objective is to find h_2 and b while h_1 is known. From the figure,

$$h_1 + h_2 = b \tan\theta_1 \tag{1.3.12.1}$$
$$h_1 = b \tan\theta_2 \tag{1.3.12.2}$$

This can be expressed as the matrix equation

$$\begin{pmatrix} \tan\theta_1 & -1 \\ \tan\theta_2 & 0 \end{pmatrix} \begin{pmatrix} b \\ h_2 \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.12.3}$$

and solved.

13. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

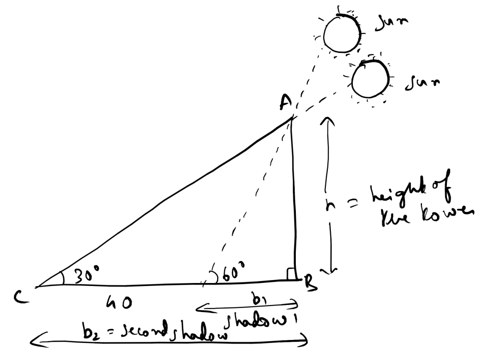

Fig. 1.3.13

Solution: Fig. 1.3.13 summarizes the problem. The objective is to find h. from the figure,

$$b_1 = h \cot 60^\circ \tag{1.3.13.1}$$
$$b_2 = h \cot 30^\circ \tag{1.3.13.2}$$
$$b_2 - b_1 = 40 \tag{1.3.13.3}$$
$$\implies h\left(\cot 30^\circ - \cot 60^\circ\right) = 40 \tag{1.3.13.4}$$
$$\text{or } h = \frac{40}{\cot 30^\circ - \cot 60^\circ} \tag{1.3.13.5}$$

14. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.
Solution: Fig. 1.3.14 summarizes the problem. The objective is to find h_2 and b. From the figure,

$$h_2 = b \tan\theta_2 \tag{1.3.14.1}$$
$$h_2 - h_1 = b \tan\theta_1 \tag{1.3.14.2}$$

which can be expressed as

$$\begin{pmatrix} 1 & -\tan\theta_2 \\ 1 & -\tan\theta_1 \end{pmatrix} \begin{pmatrix} h_2 \\ b \end{pmatrix} = h_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.3.14.3}$$

and solved.

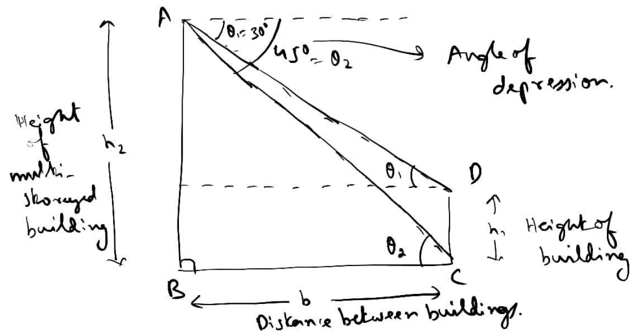

Fig. 1.3.14

1.4 Triangle Exercises

1. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find the equation of the median through the vertex $\mathbf{R}$.
2. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the equation and length of the altitude from the vertex $\mathbf{A}$.
3. Find the area of the triangle whose vertices are
   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
   b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
4. Find the area of the triangle formed by joining the mid points o the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
5. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
6. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.4.6.1}$$
   Find
$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{1.4.6.2}$$
7. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the vertices of $\triangle ABC$.
   a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
   b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
   c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
8. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.4.8.1}$$
9. Show that the points
$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.4.9.1}$$
   are the vertices of a right angled triangle.
10. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.
11. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.
12. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.
13. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$
14. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
15. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$
16. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.

17. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.4.17.1}$$

are the vertices of an isosceles triangle.
18. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.
19. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.
20. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of 30° to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?
21. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is 30°. Find the height of the tower.
22. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.
23. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.
24. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.
25. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.
26. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.
27. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.
28. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.
29. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.
30. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.
31. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.
32. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.
33. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9

m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

2 Quadrilateral

2.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
   Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting
$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.1.1}$$
   it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x-axis by an angle θ with
$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.2}$$
   and
$$\cos\theta = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$
$$\sin\theta = \sqrt{1 - \cos^2\theta} \tag{2.1.1.4}$$
   The coordinates of the rotated triangle ACD are
$$\mathbf{D} = \mathbf{P}\begin{pmatrix} h \\ k \end{pmatrix} \tag{2.1.1.5}$$
$$\mathbf{A} = \mathbf{P}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.6}$$
$$\mathbf{C} = \mathbf{P}\begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.7}$$
   where
$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.1.8}$$
   The following code plots quadrilateral $ABCD$ in Fig. 2.1.1

codes/quad/draw_quad.py

2. Draw the parallelogram $MORE$ with $OR = 6, RE = 4.5$ and $EO = 7.5$.
   Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel .
3. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.

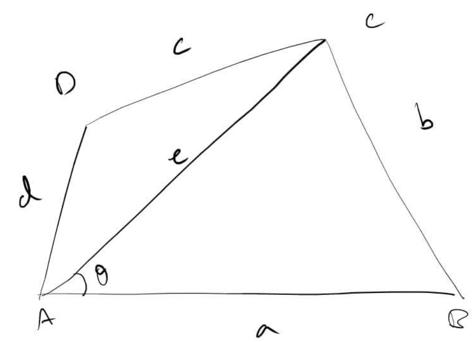

Fig. 2.1.1

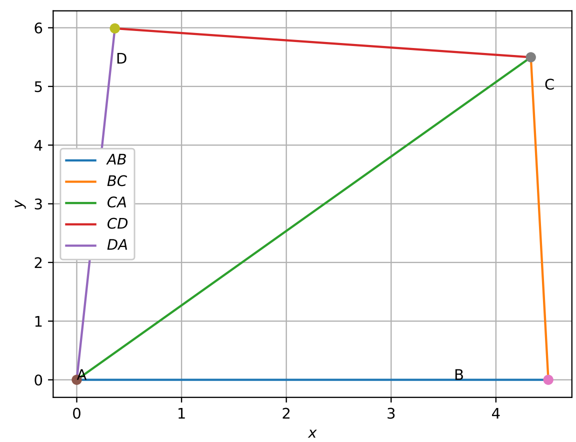

Fig. 2.1.1

   Solution: The diagonals of a kite are perpendicular to each other.
4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
   Solution: Diagonals of a rhombus bisect each other at right angles.

2.2 Construction Exercises

1. Construct a quadrilateral $ABCD$ such that $AB = 5, \angle A = 50°, AC = 4, BD = 5$ and $AD = 6$.
2. Construct $PQRS$ where $PQ = 4, QR = 6, RS = 5, PS = 5.5$ and $PR = 7$.
3. Draw $JUMP$ with $JU = 3.5, UM = 4, MP = 5, PJ = 4.5$ and $PU = 6.5$
4. Construct a quadrilateral $ABCD$ such that $BC = 4.5, AC = 5.5, CD = 5, BD = 7$ and $AD = 5.5$.
5. Can you construct a quadrilateral $PQRS$ with $PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ =$

4?

6. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.
7. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.
8. DRAW rhombus $BEND$ such that $BN = 5.6$, $DE = 6.5$.
9. construct a quadrilateral MIST where $MI = 3.5, IS = 6.5, \angle M = 75°, \angle I = 105°$ and $\angle S = 120°$.
10. Can you construct the above quadrilateral MIST if $\angle M = 100°$ instead of 75°.
11. Can you constrcut the quadrilateral PLAN if $PL = 6, LA = 9.5, \angle P = 75°, \angle L = 150°$ and $\angle A = 140°$?
12. Construct $MORE$ where $MO = 6, OR = 4.5, \angle M = 60°, \angle O = 105°, \angle R = 105°$.
13. Construct $PLAN$ where $PL = 4, LA = 6.5, \angle P = 90°, \angle A = 110°$ and $\angle N = 85°$.
14. Constrcut parallelogram $HEAR$ where $HE = 5, EA = 6, \angle R = 85°$.
15. Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.
16. Construct $ABCd$, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105°$ and $\angle C = 80°$.
17. Construct $DEAR$ with $DE = 4, EA = 5, AR = 4.5, \angle E = 60°$ and $\angle A = 90°$.
18. Construct $TRUE$ with $TR = 3.5, RU = 3, UE = 4\angle R = 75°$ and $\angle U = 120°$.
19. Draw a square of side 4.5.
20. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
21. Draw a square $READ$ with $RE = 5.1$.
22. Draw a rhombus who diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

2.3 Quadrilateral Examples

1. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,
$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.3.1.1}$$
Hence, the diagonals AC and BD bisect each other. Also,
$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.3.1.2}$$
$\implies AC \perp BD$. Hence $ABCD$ is a square.
2. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.
3. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by
$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\| + \frac{1}{2}\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\| \tag{2.3.3.1}$$
4. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors
$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{C} \tag{2.3.4.1}$$
$$\mathbf{A}-\mathbf{D} = \mathbf{B}-\mathbf{C} \tag{2.3.4.2}$$
$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \neq 0 \tag{2.3.4.3}$$
Hence, it is not a rectangle. The following code plots Fig. 2.3.4
```
codes/triangle/quad_3d.py
```
5. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

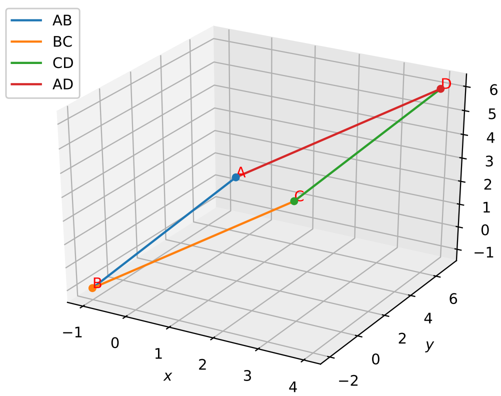

Fig. 2.3.4

Solution: The area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{2.3.5.1}$$

2.4 *Quadrilateral Geometry*

1. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}$, $\begin{pmatrix}0\\7\end{pmatrix}$, $\begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.
2. Find the area of a rhombus if its vertices are $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}4\\5\end{pmatrix}$, $\begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\end{pmatrix}$ taken in order.
3. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.
4. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-5\end{pmatrix}$, $\begin{pmatrix}3\\-2\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$.
5. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
6. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?
7. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
8. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-7\\1\end{pmatrix}$.
9. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix}-1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\-\frac{1}{2}\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-\frac{1}{2}\\4\end{pmatrix}$.
10. The two adjacent sides of a parallelogram are $\begin{pmatrix}2\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

3 Line

3.1 *Examples*

1. Find the values of x, y, z such that

$$\begin{pmatrix}x\\2\\z\end{pmatrix} = \begin{pmatrix}2\\y\\1\end{pmatrix} \tag{3.1.1.1}$$

   Solution: $x = 2, y = 2, z = 1$.
2. If

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}, \tag{3.1.2.1}$$

   verify if
   a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
   b) $\mathbf{a} = \mathbf{b}$

   Solution:
   a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.
3. Find a unit vector in the direction of $\begin{pmatrix}2\\3\\1\end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}}{\left\|\begin{pmatrix}2\\3\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{3.1.3.1}$$

4. Find a unit vector in the direction of $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.
5. Find a unit vector in the direction of the line passing through $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.
6. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix}1\\-2\end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \tag{3.1.6.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{3.1.6.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{3.1.6.3}$$

7. Find a unit vector in the direction of $\mathbf{a}+\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix}2\\2\\-5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\\3\end{pmatrix}. \tag{3.1.7.1}$$

8. Find a unit vector in the direction of

$$\begin{pmatrix}1\\1\\-2\end{pmatrix}. \tag{3.1.8.1}$$

9. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{3.1.9.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P}-\mathbf{Q} = \begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{3.1.9.2}$$

10. Verify if $\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\6\end{pmatrix}$ are points on a line.
Solution: Refer to Problem 1.3.1.
11. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
Solution: From the given information,

$$\left\|\mathbf{x}-\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x}-\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \tag{3.1.11.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$

$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \tag{3.1.11.2}$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.1.11.3}$$

which is the desired condition. The following code plots Fig. 3.1.11clearly showing that the above equation is the perpendicular bisector of AB.

codes/line/line_perp_bisect.py

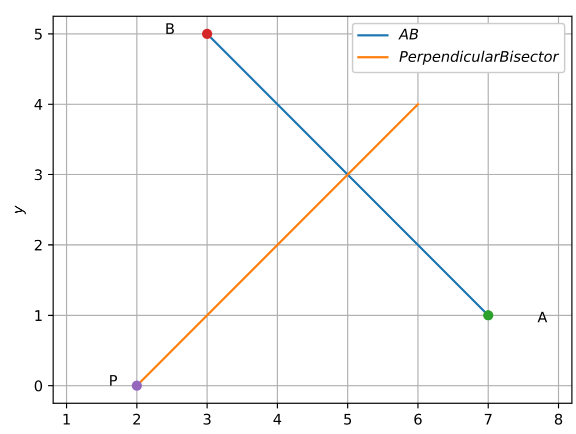

Fig. 3.1.11

12. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution: Choose $\mathbf{x} = \begin{pmatrix}0\\y\end{pmatrix}$ and follow the approach in Problem (3.1.11). Solve for y.

13. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \tag{3.1.13.1}$$

Then the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{3.1.13.2}$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 3.1.13

```
codes/line/draw_section.py
```

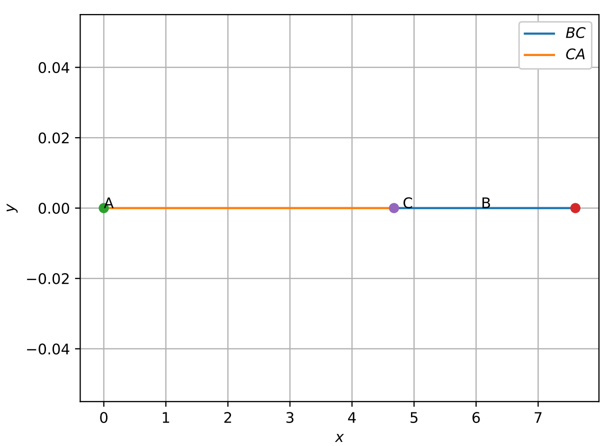

Fig. 3.1.13

14. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.
Solution: Using (3.1.13.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \tag{3.1.14.1}$$

15. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \tag{3.1.15.1}$$

Solution: Use (3.1.13.2).

16. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \tag{3.1.16.1}$$

Solution: Using (3.1.13.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.1.16.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.1.16.3}$$

17. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.13.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.1.17.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.1.17.2}$$

18. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.
Solution: Forming the matrix in (1.3.1.4),

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \tag{3.1.18.1}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \tag{3.1.18.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{3.1.18.3}$$

19. Find the direction vectors and slopes of the lines passing through the points
a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.
c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
d) Making an inclination of 60° with the positive direction of the x-axis.
Solution:

a) If the direction vector is

$$\begin{pmatrix}1\\m\end{pmatrix}, \tag{3.1.19.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}=-\frac{1}{4}\begin{pmatrix}-4\\6\end{pmatrix} \tag{3.1.19.2}$$

$$=\begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix} \implies m=-\frac{3}{2} \tag{3.1.19.3}$$

b) The direction vector is

$$\begin{pmatrix}7\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix} \tag{3.1.19.4}$$

$$=\begin{pmatrix}1\\0\end{pmatrix} \implies m=0 \tag{3.1.19.5}$$

c) The direction vector is

$$\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix} \tag{3.1.19.6}$$

$$=\begin{pmatrix}1\\\infty\end{pmatrix} \implies m=\infty \tag{3.1.19.7}$$

d) The slope is $m = \tan 60° = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{3.1.19.8}$$

20. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
Solution: The angle θ between two lines is given by

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \tag{3.1.20.1}$$

$$\implies 1=\frac{m_1-\frac{1}{4}}{1+\frac{m_1}{4}} \tag{3.1.20.2}$$

$$\text{or } m_1=\frac{5}{3} \tag{3.1.20.3}$$

21. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
Solution: Using (1.3.1.9)

$$\left\{\begin{pmatrix}-2\\6\end{pmatrix}-\begin{pmatrix}4\\8\end{pmatrix}\right\}^T\left\{\begin{pmatrix}8\\12\end{pmatrix}-\begin{pmatrix}x\\24\end{pmatrix}\right\}=0 \tag{3.1.21.1}$$

which can be used to obtain x.

22. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
Solution: The equation of the line joining the points $\mathbf{A}=\begin{pmatrix}0\\2\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3\\8\end{pmatrix}$ is obtained as

$$\mathbf{x}=\mathbf{A}+\lambda(\mathbf{B}-\mathbf{A}) \tag{3.1.22.1}$$

$$\implies \begin{pmatrix}T\\D\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}-\lambda\begin{pmatrix}-3\\-6\end{pmatrix} \tag{3.1.22.2}$$

which can be expressed as

$$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix}=\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{3.1.22.3}$$

$$\implies \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix}=-2 \tag{3.1.22.4}$$

$$\implies D=2+2T \tag{3.1.22.5}$$

23. Find the equations of the lines parallel to the axes and passing through $\mathbf{A}=\begin{pmatrix}-2\\3\end{pmatrix}$.
Solution: The line parallel to the x-axis has direction vector $\mathbf{m}=\begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x}=\begin{pmatrix}-2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.23.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x}=\begin{pmatrix}-2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}0\\1\end{pmatrix} \tag{3.1.23.2}$$

The following code plots Fig. 3.1.23

```
codes/line/line_parallel_axes.py
```

24. Find the equation of the line through $\mathbf{A}=\begin{pmatrix}-2\\3\end{pmatrix}$ with slope –4.
Solution: The direction vector is $\mathbf{m}=\begin{pmatrix}1\\-4\end{pmatrix}$.

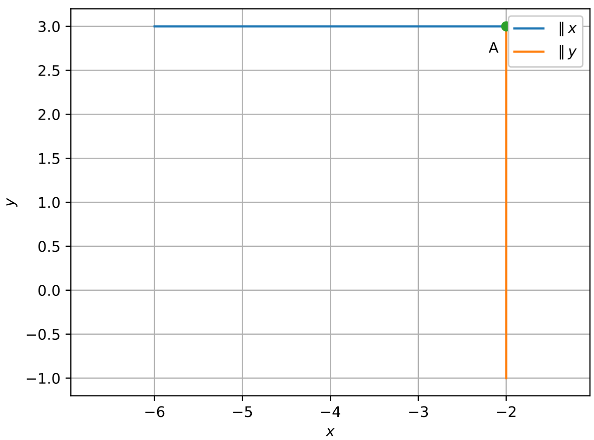

Fig. 3.1.23

Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{3.1.24.1}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} \tag{3.1.24.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.24.3}$$

$$\implies \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = -5 \tag{3.1.24.4}$$

25. Write the equation of the line through the points $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.
Solution: Use (3.1.23.1).
26. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and
    a) y-intercept is $-\frac{3}{2}$
    b) x-intercept is 4.

    Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.
    a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$.
    b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix}4\\0\end{pmatrix}$.

    Use the above information get the equations for the lines.
27. Find the equation of a line through the point $\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}3\\2\\-8\end{pmatrix}$.
Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix}5 & 2\\-4\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-8\end{pmatrix} \tag{3.1.27.1}$$

28. Find the equation of a line passing through the points $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\\6\end{pmatrix}$.
Solution: Using (3.1.22.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix}-1 & 0\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\4\\4\end{pmatrix} \tag{3.1.28.1}$$

$$= \begin{pmatrix}-1 & 0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.1.28.2}$$

29. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{3.1.29.1}$$

find the equation of the line.
Solution: The line can be expressed from (3.1.29.1) as

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3+2\lambda\\5+4\lambda\\-6+2\lambda\end{pmatrix} \tag{3.1.29.2}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\2\end{pmatrix} \tag{3.1.29.3}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\1\end{pmatrix} \tag{3.1.29.4}$$

30. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution: See Problem 3.1.26. The line passes through the points $\begin{pmatrix}-3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$.
31. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 3.1.31, the foot of the perpendicular P is the intersection of the lines L

and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \quad (3.1.31.1)$$
$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \quad (3.1.31.2)$$
$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda\left\|\mathbf{n}\right\|^2 = c \quad (3.1.31.3)$$
$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \quad (3.1.31.4)$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \quad (3.1.31.5)$$
$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda|\left\|\mathbf{n}\right\| \quad (3.1.31.6)$$

From (3.1.31.4) and (3.1.31.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \quad (3.1.31.7)$$

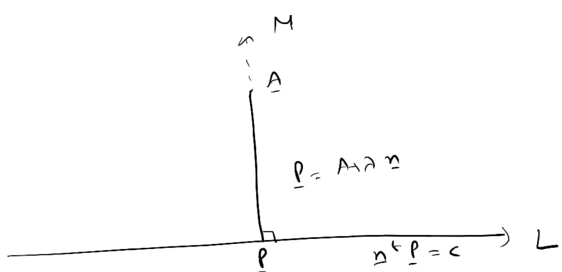

Fig. 3.1.31

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \quad (3.1.31.8)$$

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \quad (3.1.31.9)$$
$$= \pm 4 \sec 15^\circ \quad (3.1.31.10)$$

where

$$\sec\theta = \frac{1}{\cos\theta} \quad (3.1.31.11)$$

This follows from (1.3.2.1), where

$$\cos^2\theta + \sin^2\theta = 1 \quad (3.1.31.12)$$
$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad (3.1.31.13)$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \quad (3.1.31.14)$$
$$\implies 1 + \tan^2\theta = \sec^2\theta \quad (3.1.31.15)$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix}\mathbf{c} = \pm 4 \sec 15^\circ \quad (3.1.31.16)$$

32. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \quad (3.1.32.1)$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.24.3) and (3.1.24.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} 273 \\ 32 \end{pmatrix} \quad (3.1.32.2)$$
$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = -197 \quad (3.1.32.3)$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \quad (3.1.32.4)$$

33. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix}\mathbf{x} + 10 = 0. \quad (3.1.33.1)$$

Find its
a) slope,
b) x - and y-intercepts.
Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad (3.1.33.2)$$
$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad (3.1.33.3)$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

34. Find the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T\mathbf{b} = 1. \tag{3.1.34.1}$$

Solution: In Fig. 3.1.34, from the cosine formula in (1.1.7.9)

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\|\mathbf{A}-\mathbf{B}\|\|\mathbf{B}-\mathbf{C}\|} \tag{3.1.34.2}$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.34.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.34.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.34.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.34.6}$$

$$= \frac{1}{2} \tag{3.1.34.7}$$

$$\implies \theta = 60° \tag{3.1.34.8}$$

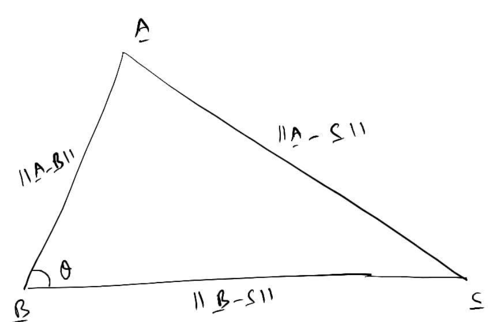

Fig. 3.1.34

35. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \tag{3.1.35.1}$$

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \tag{3.1.35.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{3.1.35.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30° \tag{3.1.35.4}$$

36. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3 \tag{3.1.36.1}$$

and passes through the point $\begin{pmatrix}1\\-2\end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix}2\\1\end{pmatrix} \tag{3.1.36.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix}2 & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\-2\end{pmatrix}\right) = 0 \tag{3.1.36.3}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.36.4}$$

37. Find the distance of the point $\begin{pmatrix}3\\-5\end{pmatrix}$ from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 26 \tag{3.1.37.1}$$

Solution: Use (3.1.31.7).

38. If the lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{3.1.38.1}$$

$$\begin{pmatrix}5 & k\end{pmatrix}\mathbf{x} = 3 \tag{3.1.38.2}$$

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.1.38.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix}2&1&3\\5&k&3\\3&-1&2\end{pmatrix}\xleftrightarrow{R_2\leftrightarrow R_3}\begin{pmatrix}2&1&3\\3&-1&2\\5&k&3\end{pmatrix} \tag{3.1.38.4}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftrightarrow 2R_2-3R_1}\begin{pmatrix}2&1&3\\0&-5&-5\\0&2k-5&-9\end{pmatrix} \tag{3.1.38.5}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{5}}\begin{pmatrix}2&1&3\\0&1&1\\0&2k-5&-9\end{pmatrix} \tag{3.1.38.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(2k-5)R_2}\begin{pmatrix}2&1&3\\0&1&1\\0&0&-2k-4\end{pmatrix} \tag{3.1.38.7}$$

$$\implies k=-2 \tag{3.1.38.8}$$

39. Find the distance of the line

$$L_1 : \quad \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.39.1}$$

from the point $\begin{pmatrix}4\\1\end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix}1\\ \tan 135^\circ\end{pmatrix} \tag{3.1.39.2}$$

Since $\begin{pmatrix}4\\1\end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix}4\\1\end{pmatrix} + \lambda\mathbf{m} \tag{3.1.39.3}$$

$$\implies \mathbf{P} = \begin{pmatrix}4\\1\end{pmatrix} + \lambda\mathbf{m} \tag{3.1.39.4}$$

$$\text{or, } \left\|\mathbf{P} - \begin{pmatrix}4\\1\end{pmatrix}\right\| = d = |\lambda|\,\|\mathbf{m}\| \tag{3.1.39.5}$$

Since $\mathbf{P}$ lies on L_1, from (3.1.39.1),

$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{P} = 0 \tag{3.1.39.6}$$

Substituting from the above in (3.1.39.3),

$$\begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} + \lambda\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{m} = 0 \tag{3.1.39.7}$$

$$\implies \lambda = \frac{\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{m}}{17} \tag{3.1.39.8}$$

substituting $|\lambda|$ in (3.1.39.5) gives the desired answer.

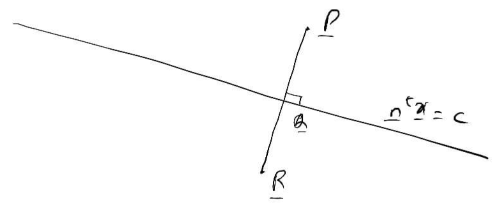

Fig. 3.1.40

40. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix}1\\2\end{pmatrix}$ in the line

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -4. \tag{3.1.40.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{3.1.40.2}$$

$$\mathbf{n}^T\mathbf{Q} = c \tag{3.1.40.3}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{3.1.40.4}$$

where $\mathbf{m}$ is the direction vector of L. From (3.1.40.2) and (3.1.40.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{3.1.40.5}$$

From (3.1.40.5) and (3.1.40.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.40.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{3.1.40.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{3.1.40.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{3.1.40.9}$$

(3.1.40.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \quad (3.1.40.10)$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \quad (3.1.40.11)$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \quad (3.1.40.12)$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (3.1.40.13)$$

The following code plots Fig. 3.1.40 while computing the reflection

```
codes/line/line_reflect.py
```

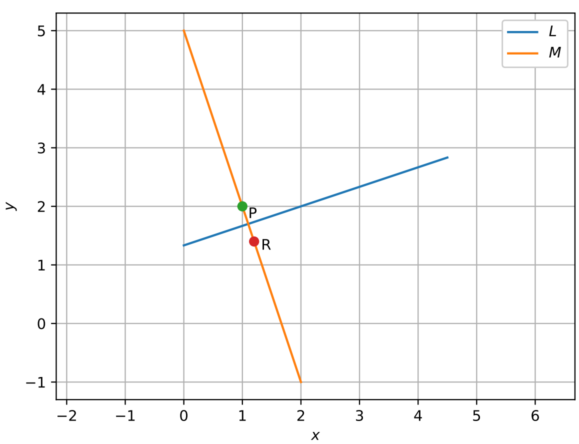

Fig. 3.1.40

41. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix}1\\5\end{pmatrix}$. Obtain its equation.

$$L_1 : \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \quad (3.1.41.1)$$

$$L_2 : \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \quad (3.1.41.2)$$

Solution: Let

$$L : \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \quad (3.1.41.3)$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \quad (3.1.41.4)$$

$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \quad (3.1.41.5)$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (3.1.41.1), (3.1.41.4) and (3.1.41.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \quad (3.1.41.6)$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \quad (3.1.41.7)$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \quad (3.1.41.8)$$

$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \quad (3.1.41.9)$$

$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \quad (3.1.41.10)$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \quad (3.1.41.11)$$

42. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \quad (3.1.42.1)$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \quad (3.1.42.2)$$

are equal is a straight line.
Solution: Using (3.1.31.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \quad (3.1.42.3)$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right| \quad (3.1.42.4)$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right) \quad (3.1.42.5)$$

leading to the possible lines

$$L_1: \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.42.6}$$

$$L_2: \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{5}{3} \tag{3.1.42.7}$$

43. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \tag{3.1.43.1}$$

Solution: The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

44. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.3.1.4)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1.44.1}$$

$\implies rank(\mathbf{M}) = 1$.

45. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.

Solution: Use the approach in Problem (3.1.44).

46. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{3.1.46.1}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \tag{3.1.46.2}$$

respectively.

47. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3

a) internally, and
b) externally.

Solution: Use (3.1.13.2).

48. Prove that the three points $\begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix}$ are collinear.

Solution: Use the approach in Problem 3.1.44.

49. Find the ratio in which the line segment joining the points $\begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$ is divided by the YZ-plane.

Solution: Use (3.1.13.2). The YZ-plane has points $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$.

50. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution: Use the approach in Problem 3.1.11.

51. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{3.1.51.1}$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{3.1.51.2}$$

find $\mathbf{R}$, which divides PQ in the ratio 2 : 1

a) internally,
b) externally.

Solution: Use (3.1.13.2).

52. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: Use (3.1.34.6)

53. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.1.53.1}$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.1.53.2}$$

Solution: The direction vectors of the lines are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Using(3.1.34.6), the angle between the lines can be obtained.

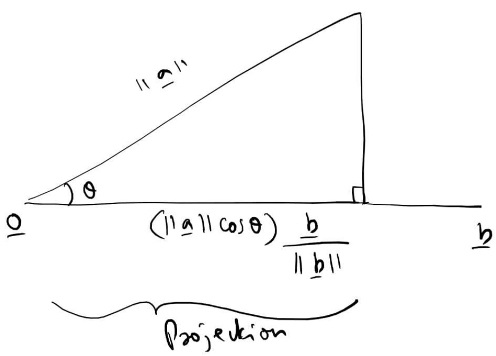

Fig. 3.1.56

54. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{3.1.54.1}$$
$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{3.1.54.2}$$

Solution: From Problem 3.1.29, the direction vectors of the lines can be expressed as $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The angle between them can then be obtained from (3.1.34.6).

55. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are perpendicular.
Solution: Use (1.3.1.9).

56. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{3.1.56.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{3.1.56.2}$$

Solution: The projection of **a** on **b** is shown in Fig. 3.1.56. It has magnitude $\|\mathbf{a}\|\cos\theta$ and is in the direction of **b**. Thus, the projection is defined as

$$(\|\mathbf{a}\|\cos\theta)\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)\|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{3.1.56.3}$$

57. Find $\|\mathbf{a}-\mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{3.1.57.1}$$

Solution:

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \tag{3.1.57.2}$$

58. If **a** is a unit vector and

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = 8, \tag{3.1.58.1}$$

then find **x**.
Solution:

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{3.1.58.2}$$
$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{3.1.58.3}$$

59. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{3.1.59.1}$$

find $\|\mathbf{a}\times\mathbf{b}\|$.
Solution: Use (1.3.5.3).

60. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{3.1.60.1}$$

Solution: If **x** is the desired vector,

$$(\mathbf{a}+\mathbf{b})^T\mathbf{x} = 0 \tag{3.1.60.2}$$
$$(\mathbf{a}-\mathbf{b})^T\mathbf{x} = 0 \tag{3.1.60.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix}\mathbf{x} = 0 \tag{3.1.60.4}$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_1\leftarrow R_1+3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \tag{3.1.60.5}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.60.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.60.7}$$

61. Show that $\mathbf{A} = \begin{pmatrix}-2\\3\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\0\\-1\end{pmatrix}$, are collinear.
Solution: See Problem 3.1.44.
62. If $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\5\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}1\\-6\\-1\end{pmatrix}$, show that $\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\-4\\1\end{pmatrix} \tag{3.1.62.1}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix}2\\8\\-2\end{pmatrix} \tag{3.1.62.2}$$

$$\because -2\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{C}-\mathbf{D}, \tag{3.1.62.3}$$

$\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
63. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{3.1.63.1}$$

Then,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.1.63.2}$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{3.1.63.3}$$

using (3.1.63.1)
64. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \tag{3.1.64.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{3.1.64.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (3.1.64.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{3.1.64.3}$$

$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{3.1.64.4}$$

$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{3.1.64.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{3.1.64.6}$$

65. Let $\boldsymbol{\alpha} = \begin{pmatrix}3\\-1\\0\end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.65.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{3.1.65.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.65.3}$$

This process is known as *Gram-Schmidth orthogonalization*.
66. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix}\cos 90°\\\cos 60°\\\cos 30°\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{3.1.66.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.
67. Find the distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix}1\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}2\\3\\6\end{pmatrix} \tag{3.1.67.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}3\\3\\-5\end{pmatrix} + \lambda_2\begin{pmatrix}2\\3\\6\end{pmatrix} \tag{3.1.67.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. From Fig. 3.1.67, the distance is

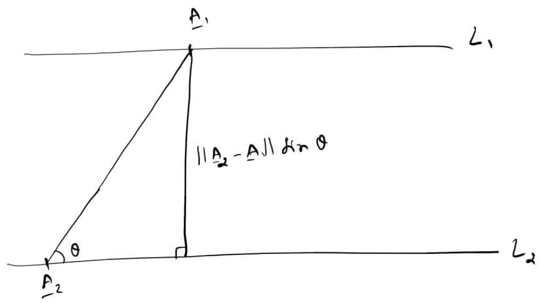

Fig. 3.1.67

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \quad (3.1.67.3)$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.67.4)$$

68. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.68.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.68.2)$$

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \quad (3.1.68.3)$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.68.4)$$

$$\implies \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (3.1.68.5)$$

$$\implies \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.1.68.6)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \quad (3.1.68.7)$$

$$\xleftrightarrow[R_3 = 2R_1 - R_3]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \quad (3.1.68.8)$$

$$\xleftrightarrow{R_3 = 3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \quad (3.1.68.9)$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \quad (3.1.68.10)$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \quad (3.1.68.11)$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (3.1.31.7) as

$$\frac{\left|\mathbf{n}^T (\mathbf{A}_2 - \mathbf{A}_1)\right|}{\|\mathbf{n}\|} = \frac{\left|(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2)\right|}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \quad (3.1.68.12)$$

69. Find the distance of the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \quad (3.1.69.1)$$

from the origin.

Solution: From (3.1.31.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{2^2 + 3^2 + 4^2}} \quad (3.1.69.2)$$

$$= \frac{6}{\sqrt{29}} \quad (3.1.69.3)$$

70. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Solution: From the previous problem, the desired equation is

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.1.70.1}$$

71. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x} = 1. \tag{3.1.71.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.71.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{3.1.71.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.71.4}$$

72. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.1.72.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.72.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda\mathbf{n}$. Substituting in (3.1.72.1),

$$\lambda\|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \tag{3.1.72.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29}\begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.72.4}$$

73. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix}5\\2\\-4\end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$.

Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (3.1.24.3), the equation of the plane is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.73.1}$$

$$\implies \begin{pmatrix}2\\3\\-1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}5\\2\\-4\end{pmatrix} \tag{3.1.73.2}$$

$$= 20 \tag{3.1.73.3}$$

74. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix}2\\5\\-3\end{pmatrix}, \boldsymbol{S} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix}5\\3\\-3\end{pmatrix}$.

Solution: If the equation of the plane be

$$\mathbf{n}^T\mathbf{x} = c, \tag{3.1.74.1}$$

$$\mathbf{n}^T\mathbf{R} = \mathbf{n}^T\mathbf{S} = \mathbf{n}^T\mathbf{T} = c, \tag{3.1.74.2}$$

$$\implies \begin{pmatrix}\mathbf{R} - \mathbf{S} & \mathbf{S} - \mathbf{T}\end{pmatrix}^T \mathbf{n} = 0 \tag{3.1.74.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix}4 & 8 & -8\\-7 & -6 & 8\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix}1 & 2 & -2\\-7 & -6 & 8\end{pmatrix} \tag{3.1.74.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix}1 & 2 & -2\\0 & 8 & -6\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}1 & 2 & -2\\0 & 4 & -3\end{pmatrix} \tag{3.1.74.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix}2 & 0 & -1\\0 & 4 & -3\end{pmatrix} \tag{3.1.74.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}\frac{1}{2}\\\frac{3}{4}\\1\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} \text{ and} \tag{3.1.74.7}$$

$$c = \mathbf{n}^T\mathbf{T} = 7 \tag{3.1.74.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{n} = 7 \tag{3.1.74.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{3.1.74.10}$$

The equation of the plane is then obtained from (3.1.24.3) as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T(\mathbf{x} - \mathbf{T}) = 0 \tag{3.1.74.11}$$

75. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.

Solution: From the given information, the plane passes through the points $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}$ and

$\begin{pmatrix}0\\0\\4\end{pmatrix}$ respectively. The equation can be obtained using Problem 3.1.74.

76. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{3.1.76.1}$$

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = -5 \tag{3.1.76.2}$$

and the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix}1 & 1 & 1 & 6\\2 & 3 & 4 & -5\end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & -17\end{pmatrix} \tag{3.1.76.3}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix}1 & 0 & -1 & 23\\0 & 1 & 2 & -17\end{pmatrix} \tag{3.1.76.4}$$

$$\implies \mathbf{x} = \begin{pmatrix}23\\-17\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.76.5}$$

Thus, $\begin{pmatrix}23\\-17\\0\end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}23\\-17\\0\end{pmatrix}\right) \times \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.76.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix}1 & -2 & 1\\-22 & 18 & 1\end{pmatrix} \xleftrightarrow{R_2=R_2+22R_1} \begin{pmatrix}1 & -2 & 1\\0 & -26 & 23\end{pmatrix} \tag{3.1.76.7}$$

$$\xleftrightarrow{R_1=13R_1-R_2} \begin{pmatrix}13 & 0 & -10\\0 & -26 & 23\end{pmatrix} \tag{3.1.76.8}$$

$$\implies \mathbf{n} = \begin{pmatrix}\frac{10}{13}\\\frac{23}{26}\\1\end{pmatrix} = \begin{pmatrix}20\\23\\26\end{pmatrix} \tag{3.1.76.9}$$

Since the plane passes through $\begin{pmatrix}1\\1\\1\end{pmatrix}$, using (3.1.24.3),

$$\begin{pmatrix}20 & 23 & 26\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 0 \tag{3.1.76.10}$$

$$\implies \begin{pmatrix}20 & 23 & 26\end{pmatrix}\mathbf{x} = 69 \tag{3.1.76.11}$$

Alternatively, the plane passing through the intersection of (3.1.76.1) and (3.1.76.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{3.1.76.12}$$

Substituting $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 6 - 5\lambda \tag{3.1.76.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{3.1.76.14}$$

$$\implies \lambda = \frac{3}{14} \tag{3.1.76.15}$$

Substituting this value of λ in (3.1.76.12) yields the equation of the plane.

77. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{3.1.77.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{3.1.77.2}$$

are coplanar.
Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.68), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{3.1.77.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \tag{3.1.77.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix} -1 & 2 & 5 \\ -3 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix} \tag{3.1.77.5}$$

$$\xleftrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.77.6}$$

The LHS of (3.1.77.3) is

$$\begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \tag{3.1.77.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left| \mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2 \right| = 0 \tag{3.1.77.8}$$

78. Find the angle between the two planes

$$\begin{pmatrix} 2 & 1 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.78.1}$$

$$\begin{pmatrix} 3 & -6 & -2 \end{pmatrix} \mathbf{x} = 7. \tag{3.1.78.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.34.6).

79. Find the angle between the two planes

$$\begin{pmatrix} 2 & 2 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.79.1}$$

$$\begin{pmatrix} 3 & -6 & 2 \end{pmatrix} \mathbf{x} = 7. \tag{3.1.79.2}$$

Solution: See Problem (3.1.78).

80. Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$\begin{pmatrix} 6 & -3 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{3.1.80.1}$$

Solution: Use (3.1.31.7).

81. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{3.1.81.1}$$

and the plane

$$P: \quad \begin{pmatrix} 10 & 2 & -11 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.81.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left| \begin{pmatrix} 10 & 2 & -11 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \right|}{\sqrt{225} \times \sqrt{49}} = \frac{8}{21} \tag{3.1.81.3}$$

Thus, the desired angle is $90° - \theta$.

82. Find the equation of the plane that contains the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.82.1}$$

$$\begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \mathbf{x} = 8 \tag{3.1.82.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tag{3.1.82.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.82.4}$$

83. Find the distance between the point $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.
Solution: Find the equation of the plane using Problem 3.1.74. Find the distance using (3.1.31.7).

84. Find the coordinates of the point where the line

through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{3.1.84.1}$$
$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{3.1.84.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \tag{3.1.84.3}$$

3.2 *Points and Vectors*

1. Find the distance between the following pairs of points
   a)
   $$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{3.2.1.1}$$
   b)
   $$\begin{pmatrix}-5\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \tag{3.2.1.2}$$
   c)
   $$\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}-1\\b\end{pmatrix} \tag{3.2.1.3}$$
2. Find the distance between the points
   $$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix} \tag{3.2.2.1}$$
3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.
   a)
   $$\begin{pmatrix}-1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-3\\0\end{pmatrix} \tag{3.2.4.1}$$
   b)
   $$\begin{pmatrix}-3\\5\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.2.4.2}$$
   c)
   $$\begin{pmatrix}4\\5\end{pmatrix}, \begin{pmatrix}7\\6\end{pmatrix}, \tag{3.2.4.3}$$
   $$\begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \tag{3.2.4.4}$$
5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$.
6. Find the point on the x-axis which is equidistant from
   $$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{3.2.6.1}$$
7. Find the values of y for which the distance between the points
   $$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{3.2.7.1}$$
   is 10 units.
8. Show that each of the given three vectors is a unit vector
   $$\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix}, \frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix}, \frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}. \tag{3.2.8.1}$$
   Also, show that they are mutually perpendicular to each other.
9. For
   $$\mathbf{a} = \begin{pmatrix}2\\2\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \mathbf{c} = \begin{pmatrix}3\\1\\0\end{pmatrix}, \tag{3.2.9.1}$$
   $(\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c}$. Find λ.
10. Find $\mathbf{a} \times \mathbf{b}$ if
    $$\mathbf{a} = \begin{pmatrix}1\\-7\\7\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\2\end{pmatrix}. \tag{3.2.10.1}$$
11. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where
    $$\mathbf{a} = \begin{pmatrix}3\\2\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\2\\-2\end{pmatrix}. \tag{3.2.11.1}$$

12. If $\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\-1\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.
13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$,
14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.
15. Let $\mathbf{a} = \begin{pmatrix}1\\4\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.
16. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.
17. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \left(\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\1\\0\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\0\\1\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}\right) \quad (3.2.17.1)$$

is

a) 0
b) -1
c) 1
d) 3

18. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively.
19. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.
20. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}3\\5\\6\end{pmatrix}$.
21. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.
22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
23. Find the angle between the vectors
$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \quad (3.2.23.1)$$
24. Find the projection of the vector
$$\begin{pmatrix}1\\3\\7\end{pmatrix} \quad (3.2.24.1)$$
on the vector
$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \quad (3.2.24.2)$$
25. Write down a unit vector in the xy-plane, makeing an angle of $30°$ with the positive direction of the x-axis.
26. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

3.3 Points on a Line

1. Find the coordinates of the point which divides the join of
$$\begin{pmatrix}-1\\7\end{pmatrix}, = \begin{pmatrix}4\\-3\end{pmatrix} \quad (3.3.1.1)$$
in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}4\\y\end{pmatrix}$, $\begin{pmatrix}x\\6\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.
8. Determine if the points
$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{3.3.8.1}$$
are collinear.
9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix}x\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
    a) $\begin{pmatrix}7\\-2\end{pmatrix}$, $\begin{pmatrix}5\\1\end{pmatrix}$, $\begin{pmatrix}3\\k\end{pmatrix}$
    b) $\begin{pmatrix}8\\1\end{pmatrix}$, $\begin{pmatrix}k\\-4\end{pmatrix}$, $\begin{pmatrix}2\\-5\end{pmatrix}$
12. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}7\\0\end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
15. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\8\\7\end{pmatrix}$ are collinear.

3.4 Lines and Planes

1. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$.
2. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
3. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.
4. Write the equations for the x and y axes.
5. Find the equation of the line satisfying the following conditions
    a) passing through the point $\begin{pmatrix}-4\\3\end{pmatrix}$ with slope $\frac{1}{2}$.
    b) passing through the point $\begin{pmatrix}0\\0\end{pmatrix}$ with slope m.
    c) passing through the point $\begin{pmatrix}2\\2\sqrt{3}\end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
    d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.
    e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.
    f) passing through the points $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-4\end{pmatrix}$.
    g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.
6. Find the equation of the line passing through $\begin{pmatrix}-3\\5\end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\6\end{pmatrix}$.
7. Find the direction vectors and and y-intercepts of the following lines
    a) $\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 0$.

b) $\begin{pmatrix}6 & 3\end{pmatrix}\mathbf{x} = 5$.
c) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$.

8. Find the intercepts of the following lines on the axes.
   a) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 12$.
   b) $\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 6$.
   c) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 0$.
9. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
   a) $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = -8$.
   b) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$.
   c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4$.
10. Find the distance of the point $\begin{pmatrix}1 \\ -1\end{pmatrix}$ from the line $\begin{pmatrix}12 & -5\end{pmatrix}\mathbf{x} = -82$.
11. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.4.11.1}$$

are 4 units.

12. Find the distance between the parallel lines

$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = 34 \tag{3.4.12.1}$$
$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = -31 \tag{3.4.12.2}$$

13. Find the equation of the line parallel to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = -2 \tag{3.4.13.1}$$

and passing through the point $\begin{pmatrix}-2 \\ 3\end{pmatrix}$.

14. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.14.1}$$

and having x intercept 3.

15. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.4.15.1}$$
$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = 1 \tag{3.4.15.2}$$

16. The line through the points $\begin{pmatrix}h \\ 3\end{pmatrix}$ and $\begin{pmatrix}4 \\ 1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19 \tag{3.4.16.1}$$

at right angle. Find the value of h.

17. Two lines passing through the point $\begin{pmatrix}2 \\ 3\end{pmatrix}$ intersect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.
18. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix}3 \\ 4\end{pmatrix}$ and $\begin{pmatrix}-1 \\ 2\end{pmatrix}$.
19. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1 \\ 3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.4.19.1}$$

20. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.4.20.1}$$

meets it at the point $\begin{pmatrix}-1 \\ 2\end{pmatrix}$. Find the values of m and c.

21. Find θ and p if

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{3.4.21.1}$$

is equivalent to

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{3.4.21.2}$$

22. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.
23. Find the equation of the line parallel to the y-axis whose distance from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.4.23.1}$$

4 units.

24. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.24.1}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.4.24.2}$$

25. Find the alue of p so that the three lines

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{3.4.25.1}$$
$$\begin{pmatrix}p & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.4.25.2}$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{3.4.25.3}$$

may intersect at one point.

26. Find the equation of the lines through the point

$\begin{pmatrix}3\\2\end{pmatrix}$ which make an angle of 45° with the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3. \tag{3.4.26.1}$$

27. Find the equation of the line passing through the point of intersection of the lines

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = 3 \tag{3.4.27.1}$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -1 \tag{3.4.27.2}$$

that has equal intercepts on the axes.

28. In what ratio is the line joining $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}5\\7\end{pmatrix}$ divided by the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.4.28.1}$$

29. Find the distance of the line

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.29.1}$$

from the point $\begin{pmatrix}1\\2\end{pmatrix}$ along the line

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0. \tag{3.4.29.2}$$

30. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.4.30.1}$$

may be at a distance of 3 units from this point.

31. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Find an equation of the legs of the triangle.

32. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7 \tag{3.4.32.1}$$

assuming the line to be a plane mirror.

33. If the lines

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.4.33.1}$$
$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.4.33.2}$$

are equally inclined to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 4, \tag{3.4.33.3}$$

find the value of m.

34. The sum of the perpendicular distances of a variable point **P** from the lines

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.4.34.1}$$
$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.4.34.2}$$

is always 10. Show that **P** must move on a line.

35. Find the equation of the line which is equidistant from parallel lines

$$\begin{pmatrix}9 & 7\end{pmatrix}\mathbf{x} = 7 \tag{3.4.35.1}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = -6. \tag{3.4.35.2}$$

36. A ray of light passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$ reflects on the x-axis at point **A** and the reflected ray passes through the point $\begin{pmatrix}5\\3\end{pmatrix}$. Find the coordinates of **A**.

37. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.4.37.1}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.4.37.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{3.4.37.3}$$

in the least time. Find the equation of the path that he should follow.

38. Determine the ratio in which the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 4 = 0 \tag{3.4.38.1}$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\7\end{pmatrix}$.

39. A line perpendicular to the line segment joining the points $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.

40. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

41. Find the equation of the line passing through the point $\begin{pmatrix}2\\2\end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.

42. Find the equation of the line through the point $\begin{pmatrix}0\\2\end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis.

Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

43. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix}-2\\9\end{pmatrix}$, find the equation of the line.
44. The length L (in cm) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L in terms of C.
45. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
46. Find the equation of a line which passes through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}3\\2\\-2\end{pmatrix}$.
47. Find the equaion off the line that passes through $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and is in the direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.
48. Find the equation of the line which passes through the point $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and parallel to the line given by
$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{3.4.48.1}$$
49. Find the equation of the line given by
$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \tag{3.4.49.1}$$
50. Find the equation of the line passing through the origin and the point $\begin{pmatrix}5\\-2\\3\end{pmatrix}$.
51. Find the equation of the line passing through the points $\begin{pmatrix}3\\-2\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\\6\end{pmatrix}$.
52. Find the angle between the following pair of lines:

a)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix}2\\-5\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}3\\2\\6\end{pmatrix} \tag{3.4.52.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix}7\\-6\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.4.52.2}$$

b)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\-2\end{pmatrix} \tag{3.4.52.3}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-56\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\-4\end{pmatrix} \tag{3.4.52.4}$$

53. Find the angle between the following pair of lines

a)
$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{3.4.53.1}$$
$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{3.4.53.2}$$

b)
$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{3.4.53.3}$$
$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{3.4.53.4}$$

54. Find the values of p so that the lines
$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{3.4.54.1}$$
$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{3.4.54.2}$$
are at right angles.
55. Show that the lines
$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{3.4.55.1}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{3.4.55.2}$$
are perpendicular to each other.

56. Find the shortest distance between the lines

$$L_1 : \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.4.56.1}$$

$$L_2 : \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{3.4.56.2}$$

57. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \tag{3.4.57.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{3.4.57.2}$$

58. Find the shortest distance between the lines

$$L_1 : \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{3.4.58.1}$$

$$L_2 : \quad \boldsymbol{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.4.58.2}$$

59. Find the shortest distance between the lines

$$L_1 : \quad \boldsymbol{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \tag{3.4.59.1}$$

$$L_2 : \quad \boldsymbol{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \tag{3.4.59.2}$$

60. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \boldsymbol{x} = 2$ c) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \boldsymbol{x} = -8$
b) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \boldsymbol{x} = 1$ d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \boldsymbol{x} = 5$

61. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

62. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \boldsymbol{x} = 12$ c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \boldsymbol{x} = 1$
b) $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix} \boldsymbol{x} = 0$ d) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \boldsymbol{x} = -8$

63. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

64. Find the equation of the planes that passes through three points

a) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

65. Find the intercepts cut off by the plane $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \boldsymbol{x} = 5$.

66. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

67. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \boldsymbol{x} = 4$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \boldsymbol{x} = -2$ and the pont $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

68. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix} \boldsymbol{x} = 7$ and $\begin{pmatrix} 2 & 5 & 3 \end{pmatrix} \boldsymbol{x} = 9$ and the pont $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

69. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \boldsymbol{x} = 1$ and $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \boldsymbol{x} = 5$ which is perpendicular to the plane $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \boldsymbol{x} = 0$.

70. Find the angle between the planes whose equations are $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix} \boldsymbol{x} = 5$ and $\begin{pmatrix} 3 & -3 & 5 \end{pmatrix} \boldsymbol{x} = 3$

71. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix} 7 & 5 & 6 \end{pmatrix} \boldsymbol{x} = -30$ and $\begin{pmatrix} 3 & -1 & -10 \end{pmatrix} \boldsymbol{x} = -4$
b) $\begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \boldsymbol{x} = 2$ and $\begin{pmatrix} 1 & -2 & 5 \end{pmatrix} \boldsymbol{x} = 0$
c) $\begin{pmatrix} 2 & -2 & 4 \end{pmatrix} \boldsymbol{x} = -5$ and $\begin{pmatrix} 3 & -3 & 6 \end{pmatrix} \boldsymbol{x} = 1$

d) $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\boldsymbol{x} = 1$ and $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\boldsymbol{x} = -3$

e) $\begin{pmatrix}4 & 8 & 1\end{pmatrix}\boldsymbol{x} = 8$ and $\begin{pmatrix}0 & 1 & 1\end{pmatrix}\boldsymbol{x} = 4$

72. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}3 & -4 & 12\end{pmatrix}\boldsymbol{x} = 3$
b)	$\begin{pmatrix}3\\-2\\1\end{pmatrix}$	$\begin{pmatrix}2 & -1 & 2\end{pmatrix}\boldsymbol{x} = -3$
c)	$\begin{pmatrix}2\\3\\-5\end{pmatrix}$	$\begin{pmatrix}1 & 2 & -2\end{pmatrix}\boldsymbol{x} = 9$
d)	$\begin{pmatrix}-6\\0\\0\end{pmatrix}$	$\begin{pmatrix}2 & -3 & 6\end{pmatrix}\boldsymbol{x} = 2$

TABLE 3.4.72

73. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.

74. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}, \begin{pmatrix}-4\\3\\-6\end{pmatrix}, \begin{pmatrix}2\\9\\2\end{pmatrix}$, then find the angle between the lines AB and CD.

75. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{3.4.75.1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{3.4.75.2}$$

find the value of k.

76. Find the equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\boldsymbol{x} = -9 \tag{3.4.76.1}$$

77. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}6\\2\\2\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-2\\2\end{pmatrix} \text{ and} \tag{3.4.77.1}$$

$$\mathbf{x} = \begin{pmatrix}-4\\0\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-2\\-2\end{pmatrix} \tag{3.4.77.2}$$

78. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the YZ-plane.

79. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the ZX-plane.

80. Find the coordinates of the point where the line through $\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}\\-3\\1\end{pmatrix}$ crosses the plane

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\boldsymbol{x} = 7 \tag{3.4.80.1}$$

81. Find the equation of the plane passing through the point $\begin{pmatrix}-1\\3\\2\end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\boldsymbol{x} = 5 \tag{3.4.81.1}$$

$$\begin{pmatrix}3 & 3 & 1\end{pmatrix}\boldsymbol{x} = 0 \tag{3.4.81.2}$$

82. If the points $\begin{pmatrix}1\\1\\p\end{pmatrix}$ and $\begin{pmatrix}-3\\0\\1\end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix}3 & 4 & -12\end{pmatrix}\boldsymbol{x} = -13, \tag{3.4.82.1}$$

then find the value of p.

83. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\boldsymbol{x} = 1 \text{ and} \tag{3.4.83.1}$$

$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\boldsymbol{x} = -4 \tag{3.4.83.2}$$

and parallel to the x-axis.

84. If $\mathbf{O}$ be the origin and the coordinates of $\mathbf{P}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}$, then find the equation of the plane passing through $\mathbf{P}$ and perpendicular to OP.

85. Find the equation of the plane which contains

the line of intersection of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\boldsymbol{x} = 4 \tag{3.4.85.1}$$

$$\begin{pmatrix}2 & 1 & -1\end{pmatrix}\boldsymbol{x} = -5 \tag{3.4.85.2}$$

and which is perpendicular to the plane

$$\begin{pmatrix}5 & 3 & -6\end{pmatrix}\boldsymbol{x} = -8 \tag{3.4.85.3}$$

86. Find the distance of the point $\begin{pmatrix}-1\\-5\\-10\end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\2\end{pmatrix} \tag{3.4.86.1}$$

and the plane

$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\boldsymbol{x} = 5 \tag{3.4.86.2}$$

87. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\boldsymbol{x} = 5 \tag{3.4.87.1}$$

$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\boldsymbol{x} = 6 \tag{3.4.87.2}$$

88. Find the vector equation of the line passing through the point $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \tag{3.4.88.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{3.4.88.2}$$

89. Distance between the two planes

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\boldsymbol{x} = 4 \tag{3.4.89.1}$$

$$\begin{pmatrix}4 & 6 & 8\end{pmatrix}\boldsymbol{x} = 12 \tag{3.4.89.2}$$

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

90. The planes

$$\begin{pmatrix}2 & -1 & 4\end{pmatrix}\boldsymbol{x} = 5 \tag{3.4.90.1}$$

$$\begin{pmatrix}5 & -\frac{5}{2} & 10\end{pmatrix}\boldsymbol{x} = 6 \tag{3.4.90.2}$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix}0\\0\\\frac{5}{4}\end{pmatrix}$

3.5 Miscellaneous

1. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{3.5.1.1}$$

2. (Cauchy-Schwarz Inequality:) Show that

$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{3.5.2.1}$$

3. (Triangle Inequality:) Show that

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{3.5.3.1}$$

4. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
5. Find the distance between $\mathbf{P} = \begin{pmatrix}x_1\\y_1\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.
6. If three points $\begin{pmatrix}h\\0\end{pmatrix}$, $\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}0\\k\end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
7. $\mathbf{P} = \begin{pmatrix}a\\b\end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\begin{pmatrix}\frac{1}{a} & \frac{1}{b}\end{pmatrix}\mathbf{x} = 2 \tag{3.5.7.1}$$

8. Point $\mathbf{R} = \begin{pmatrix}h\\k\end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
9. Show that two lines

$$\begin{pmatrix}a_1 & b_1\end{pmatrix}\mathbf{x} + c_1 = 0 \tag{3.5.9.1}$$

$$\begin{pmatrix}a_2 & b_2\end{pmatrix}\mathbf{x} + c_2 = 0 \tag{3.5.9.2}$$

are
   a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
   b) perpendicular if $a_1a_2 - b_1b_2 = 0$.

10. Find the distance between the parallel lines

$$l\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -p \tag{3.5.10.1}$$
$$l\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = r \tag{3.5.10.2}$$

11. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line

$$\begin{pmatrix}A & B\end{pmatrix}\mathbf{x} = -C \tag{3.5.11.1}$$

12. If p and q are the lengths of perpendiculars from the origin to the lines

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = k\cos 2\theta \tag{3.5.12.1}$$
$$\begin{pmatrix}\sec\theta & \operatorname{cosec}\theta\end{pmatrix}\mathbf{x} = k \tag{3.5.12.2}$$

respectively, prove that $p^2 + 4q^2 = k^2$.

13. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{3.5.13.1}$$

14. Show that the area of the triangle formed by the lines

$$\begin{pmatrix}-m_1 & 1\end{pmatrix}\mathbf{x} = c_1 \tag{3.5.14.1}$$
$$\begin{pmatrix}-m_2 & 1\end{pmatrix}\mathbf{x} = c_2 \tag{3.5.14.2}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.5.14.3}$$

is $\frac{(c_1-c_2)^2}{2|m_1-m_2|}$.

15. Find the values of k for which the line

$$\begin{pmatrix}k-3 & -\left(4-k^2\right)\end{pmatrix}\mathbf{x} + k^2 - 7k + 6 = 0 \tag{3.5.15.1}$$

is

a) parallel to the x-axis
b) parallel to the y-axis
c) passing through the origin.

16. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$.

17. Find the area of the triangle formed by the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.17.1}$$
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.17.2}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = k \tag{3.5.17.3}$$

18. If three lines whose equations are

$$\begin{pmatrix}-m_1 & 1\end{pmatrix}\mathbf{x} = c_1 \tag{3.5.18.1}$$
$$\begin{pmatrix}-m_2 & 1\end{pmatrix}\mathbf{x} = c_2 \tag{3.5.18.2}$$
$$\begin{pmatrix}-m_3 & 1\end{pmatrix}\mathbf{x} = c_3 \tag{3.5.18.3}$$

are concurrent, show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \tag{3.5.18.4}$$

19. Find the equation of the line passing through the origin and making an angle θ with the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.5.19.1}$$

20. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix}\sqrt{a^2-b^2} \\ 0\end{pmatrix}$ and $\begin{pmatrix}\sqrt{a^2-b^2} \\ 0\end{pmatrix}$ to the line

$$\begin{pmatrix}\frac{\cos\theta}{a} & \frac{\sin\theta}{b}\end{pmatrix}\mathbf{x} = 1 \tag{3.5.20.1}$$

is b^2.

21. If $\begin{pmatrix}l_1 \\ m_1 \\ n_1\end{pmatrix}$ and $\begin{pmatrix}l_2 \\ m_2 \\ n_2\end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix}m_1n_2 - m_2n_1 \\ n_1l_2 - n_2l_1 \\ l_1m_2 - l_2m_1\end{pmatrix}$.

22. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{3.5.22.1}$$

23. Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \tag{3.5.23.1}$$
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{3.5.23.2}$$

are coplanar.

24. Find $\mathbf{R}$ which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{3.5.24.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{3.5.24.2}$$

externally in the ratio $1:2$.

25. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}-\mathbf{b}) = 8 \tag{3.5.25.1}$$
$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \tag{3.5.25.2}$$

26. Evaluate the product

$$(3\mathbf{a}-5\mathbf{b})^T(2\mathbf{a}+7\mathbf{b}) \tag{3.5.26.1}$$

27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{3.5.27.1}$$
$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{3.5.27.2}$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

28. Show that

$$(\|\mathbf{a}\|\mathbf{b}+\|\mathbf{b}\|\mathbf{a}) \perp (\|\mathbf{a}\|\mathbf{b}-\|\mathbf{b}\|\mathbf{a}) \tag{3.5.28.1}$$

29. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

30. If $\mathbf{a},\mathbf{b},\mathbf{c}$ are unit vectors such that

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0, \tag{3.5.30.1}$$

find the value of

$$\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a}. \tag{3.5.30.2}$$

31. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if
    a) $\lambda = 1$
    b) $\lambda = -1$
    c) $\|\mathbf{a}\| = |\lambda|$
    d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

32. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

33. Show that

$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b}) \tag{3.5.33.1}$$

34. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

35. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x}-\mathbf{a})^T(\mathbf{x}+\mathbf{a}) = 12. \tag{3.5.35.1}$$

36. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a}\times\mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

    a) $\frac{\pi}{6}$　　c) $\frac{\pi}{3}$
    b) $\frac{\pi}{4}$　　d) $\frac{\pi}{2}$

37. Prove that

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \tag{3.5.37.1}$$
$$\iff \mathbf{a}\perp\mathbf{b}. \tag{3.5.37.2}$$

38. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

    a) $0 < \theta < \frac{\pi}{2}$　　c) $0 < \theta < \pi$
    b) $0 \leq \theta \leq \frac{\pi}{2}$　　d) $0 \leq \theta \leq \pi$

39. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

    a) $\theta = \frac{\pi}{4}$　　c) $\theta = \frac{\pi}{2}$
    b) $\theta = \frac{\pi}{3}$　　d) $\theta = \frac{2\pi}{3}$

40. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a}\times\mathbf{b}\|$ when θ is equal to

    a) 0　　c) $\frac{\pi}{2}$
    b) $\frac{\pi}{4}$　　d) π.

41. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.

42. Find the equation of a line parallel to the x-axis and passing through the origin.

43. Find the equation of a plane passing through $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\boldsymbol{x} = 2 \tag{3.5.43.1}$$

44. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{p^2} \tag{3.5.44.1}$$

4 Circle

4.1 Construction Examples

1. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.
   Solution: The tangent is perpendicular to

the radius. From the given information, in $\triangle ABC, AC \perp AB, a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \tag{4.1.1.1}$$

The following code plots Fig. 4.1.1

```
codes/circle/draw_circle_eg.py
```

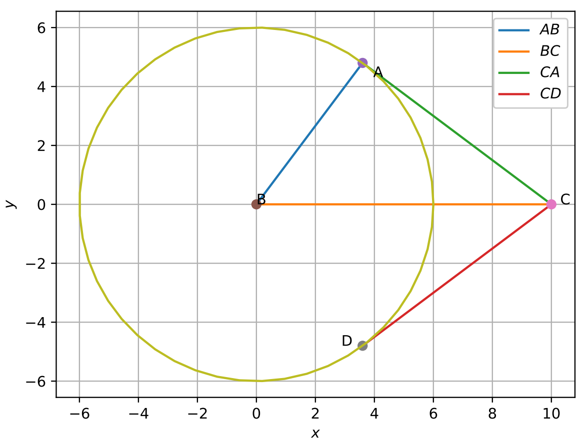

Fig. 4.1.1

2. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.
Solution: For any angle θ, a point on the circle with radius 3 has coordinates
$$3\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \tag{4.1.2.1}$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
Solution: Take the centre of both circles to be at the origin.
6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
Solution: Take the diameter to be on the x-axis.
7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.
Solution: The tangent is perpendicular to the radius.
8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.
Solution: Let
$$\mathbf{A} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8 \\ 0\end{pmatrix}. \tag{4.2.2.1}$$
9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through **B**, **C**, **D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\begin{pmatrix}2, -3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}1 \\ 4\end{pmatrix}$.
2. Find the centre of a circle passing through the points $\begin{pmatrix}6 \\ -6\end{pmatrix}, \begin{pmatrix}3 \\ -7\end{pmatrix}$ and $\begin{pmatrix}3 \\ 3\end{pmatrix}$.
3. Find the locus of all the unit vectors in the xy-plane.