Algebraic Approach to School Geometry

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Contents

***Abstract*—This book provides an equation based approach to school geometry based on the NCERT textbooks from Class 6-12.**

1 Triangle

1.1 The Right Angled Triangle

1. A right angled triangle looks like Fig. 1.1.1. with angles $\angle A, \angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle C$ which is defined to be 90°.

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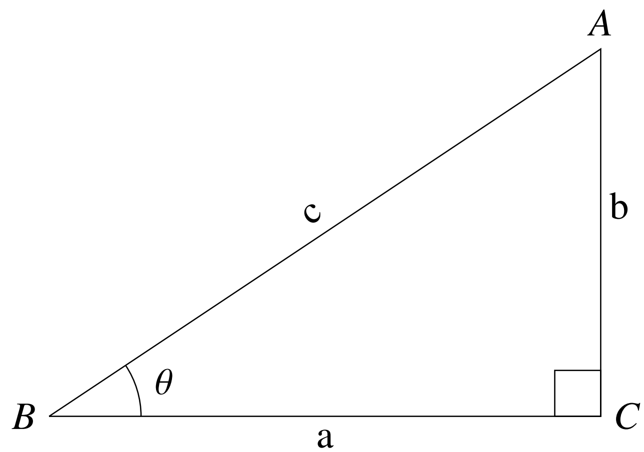

Fig. 1.1.1: Right Angled Triangle

2. For simplicity, let the greek letter $\theta = \angle B$. We have the following definitions.

$$\begin{matrix} \sin\theta = \frac{a}{c} & \cos\theta = \frac{b}{c} \\ \tan\theta = \frac{b}{a} & \cot\theta = \frac{1}{\tan\theta} \\ \csc\theta = \frac{1}{\sin\theta} & \sec\theta = \frac{1}{\cos\theta} \end{matrix} \tag{1.1.2.1}$$

1.2 Sum of Angles

1. In Fig. 1.2.1, the sum of all the angles on the top or bottom side of the straight line XY is 180°.
2. In Fig. 1.2.1, the straight line making an angle of 90° to the side AC is said to be parallel to the side BC. Note there is an angle at A that is equal to θ. This is one property of parallel lines. Thus, $\angle YAZ = 90^{\circ}$.
3. Show that $\angle VAZ = 90^{\circ} - \theta$
Solution: Considering the line XAZ,
$$\theta + 90^{\circ} + \angle VAZ = 180^{\circ} \tag{1.2.3.1}$$
$$\implies \angle VAZ = 90^{\circ} - \theta \tag{1.2.3.2}$$
4. Show that $\angle BAC = 90^{\circ} - \theta$.
Solution: Consider the line VAB and and use the approach in the previous problem. Note that

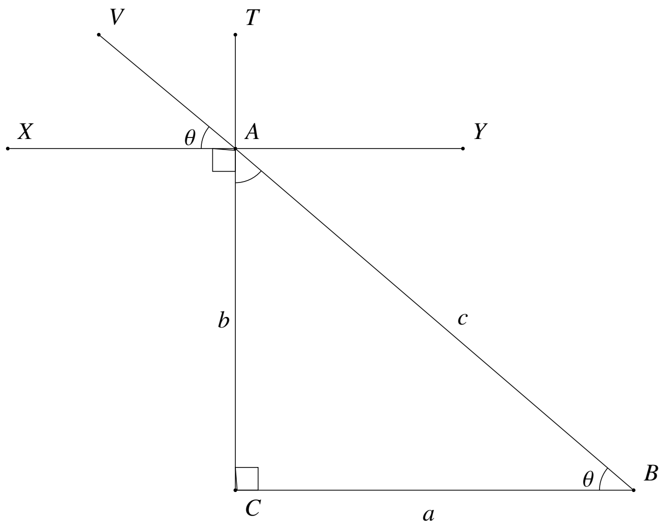

Fig. 1.2.1: Sum of angles of a triangle

this implies that $\angle VAZ = \angle BAC$. Such angles are known as vertically opposite angles.

5. Sum of the angles of a triangle is equal to 180°

1.3 Baudhayana Theorem

1. Using Fig. 1.1.1, show that

$$\cos\theta = \sin\left(90^\circ - \theta\right) \tag{1.3.1.1}$$

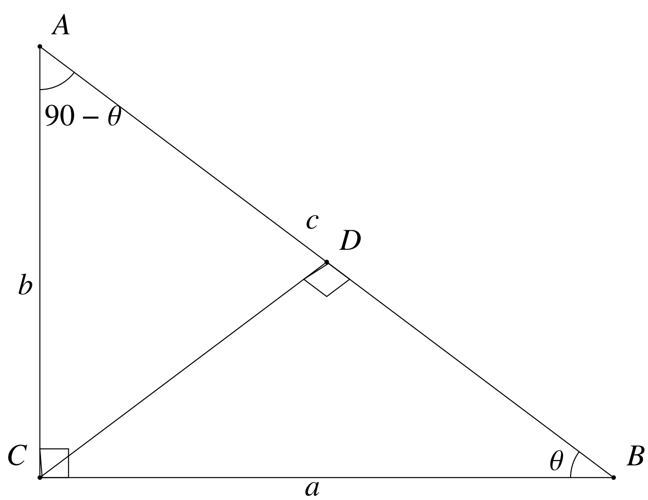

Fig. 1.3.1: Baudhayana Theorem

Solution: From Problem 1.2.4 and (1.1.2.1)

$$\cos\left(90^\circ - \theta\right) = \frac{b}{c} = \sin\theta \tag{1.3.1.2}$$

2. Using Fig. 1.3.1, show that

$$c = a\cos\theta + b\sin\theta \tag{1.3.2.1}$$

Solution: We observe that

$$BD = a\cos\theta \tag{1.3.2.2}$$

$$AD = b\cos(90-\theta) = b\sin\theta \quad \text{(From} \quad (1.2.4)) \tag{1.3.2.3}$$

Thus,

$$BD + AD = c = a\cos\theta + b\sin\theta \tag{1.3.2.4}$$

3. From (1.3.2.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.3.3.1}$$

Solution: Dividing both sides of (1.3.2.1) by c,

$$1 = \frac{a}{c}\cos\theta + \frac{b}{c}\sin\theta \tag{1.3.3.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.1.2.1)) \tag{1.3.3.3}$$

4. Using (1.3.2.1), show that

$$c^2 = a^2 + b^2 \tag{1.3.4.1}$$

(1.3.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.3.2.1),

$$c = a\frac{a}{c} + b\frac{b}{c} \quad \text{(from} \quad (1.1.2.1)) \tag{1.3.4.2}$$

$$\Rightarrow c^2 = a^2 + b^2 \tag{1.3.4.3}$$

1.4 Area of a Triangle

1. The area of the rectangle $ACBD$ shown in Fig. 1.4.1 is defined as ab. Note that all the angles in the rectangles are 90°
2. The area of the two triangles constituting the rectangle is the same.
3. The area of the rectangle is the sum of the areas of the two triangles inside.
4. Show that the area of ΔABC is $\frac{ab}{2}$
Solution: From(1.4.3),

$$ar(ABCD) = ar(ACB) + ar(ADB) \tag{1.4.4.1}$$

Also from (1.4.2),

$$ar(ACB) = ar(ADB) \tag{1.4.4.2}$$

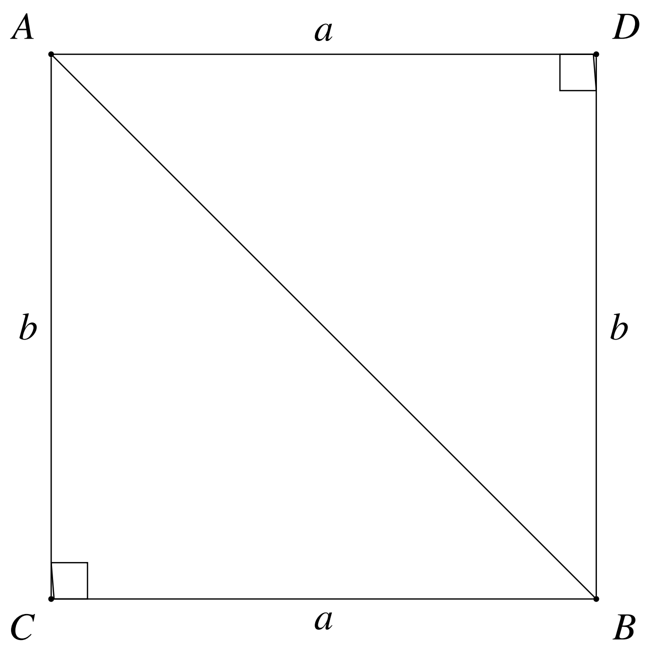

Fig. 1.4.1: Area of a Right Triangle

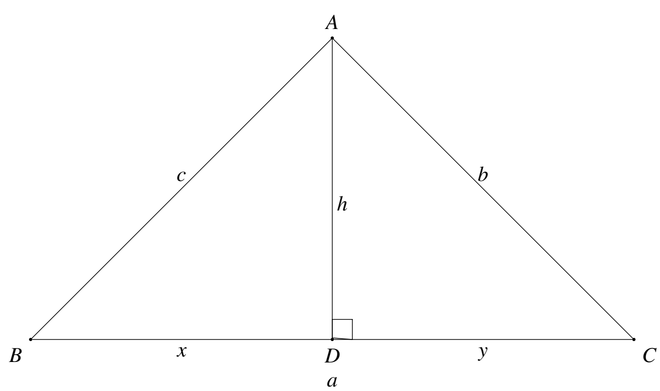

Fig. 1.4.4: Area of a Triangle

From (1.4.4.1) and (1.4.4.2),

$$2ar\left(ACB\right) = ar\left(ABCD\right) = ab \left(\text{from} \quad (1.4.1)\right) \tag{1.4.4.3}$$

$$\Rightarrow ar\left(ACB\right) = \frac{ab}{2} \tag{1.4.4.4}$$

5. Show that the area of ΔABC in Fig. 1.4.4 is $\frac{1}{2}ah$.
Solution: In Fig. 1.4.4,

$$ar\left(\Delta ADC\right) = \frac{1}{2}hy \tag{1.4.5.1}$$

$$ar\left(\Delta ADB\right) = \frac{1}{2}hx \tag{1.4.5.2}$$

Thus,

$$ar\left(\Delta ABC\right) = ar\left(\Delta ADC\right) + ar\left(\Delta ADB\right) \tag{1.4.5.3}$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h\left(x+y\right) \tag{1.4.5.4}$$

$$= \frac{1}{2}ah \tag{1.4.5.5}$$

6. Show that the area of ΔABC in Fig. 1.4.4 is $\frac{1}{2}ab\sin C$.
Solution: We have

$$ar\left(\Delta ABC\right) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad \left(\because \quad h = b\sin C\right). \tag{1.4.6.1}$$

7. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.4.7.1}$$

Solution: Fig. 1.4.4 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{s}bc\sin A = \frac{1}{2}ca\sin B \tag{1.4.7.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.4.7.3}$$

This is known as the sine formula.

8. In Fig. 1.4.8, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.4.8.1}$$

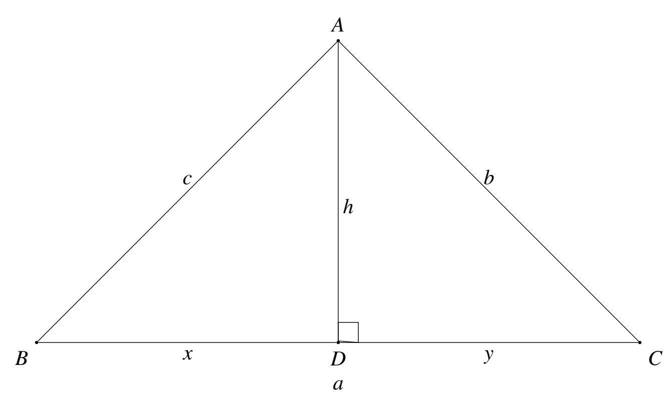

Fig. 1.4.8: The cosine formula

Solution: From the figure, the first of the following equations

$$a = b\cos C + c\cos B \tag{1.4.8.2}$$
$$b = c\cos A + a\cos C \tag{1.4.8.3}$$
$$c = b\cos A + a\cos B \tag{1.4.8.4}$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.4.8.5}$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.4.8.6}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} \tag{1.4.8.7}$$

9. Find Hero's formula for the area of a triangle. **Solution:** From (1.4.6), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.3.3.1))} \tag{1.4.9.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.4.8.1))} \tag{1.4.9.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2 - (a^2+b^2-c^2)} \tag{1.4.9.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.4.9.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.4.9.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.4.9.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.4.9.7}$$

in (1.4.9.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.4.9.8}$$

This is known as Hero's formula.

1.5 Median

1. The line AD in Fig. 1.5.1 that divides the side a in two equal halfs is known as the median.

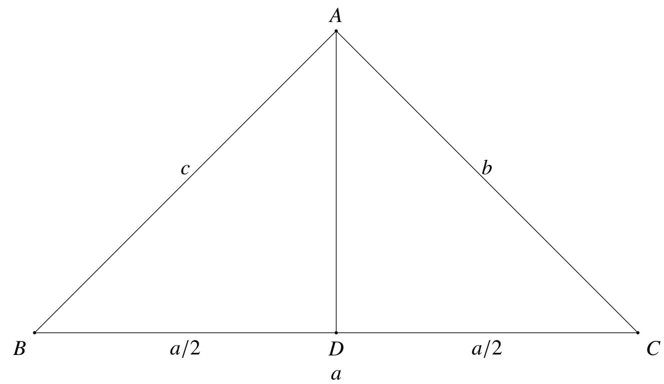

Fig. 1.5.1: Median of a Triangle

2. Show that the median AD in Fig. 1.5.1 divides ΔABC into triangles ADB and ADC that have equal area.
Solution: We have

$$ar(\Delta ADB) = \frac{1}{2}\frac{a}{2}c\sin B = \frac{1}{4}ac\sin B \tag{1.5.2.1}$$

$$ar(\Delta ADC) = \frac{1}{2}\frac{a}{2}b\sin C = \frac{1}{4}ab\sin C \tag{1.5.2.2}$$

Using the sine formula, $b\sin C = c\sin B$,

$$ar(\Delta ADB) = ar(\Delta ADC) \tag{1.5.2.3}$$

3. BE and CF are the medians in Fig. 1.5.3. Show that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{1.5.3.1}$$

Solution: Since BE and CF are the medians,

$$ar(\Delta BFC) = \frac{1}{2}ar(\Delta ABC) \tag{1.5.3.2}$$

$$ar(\Delta BEC) = \frac{1}{2}ar(\Delta ABC) \tag{1.5.3.3}$$

From the above, we infer that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{1.5.3.4}$$

4. We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 1.5.4,

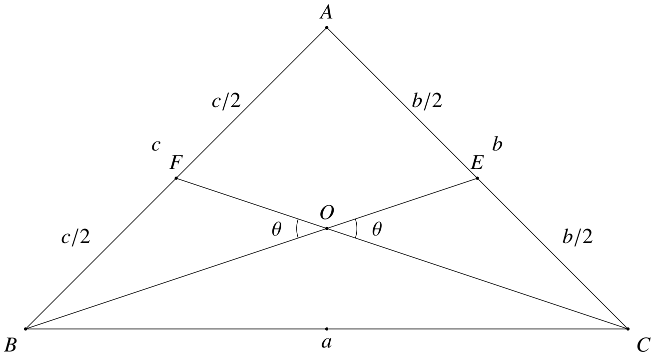

Fig. 1.5.3: O is the Intersection of Two Medians

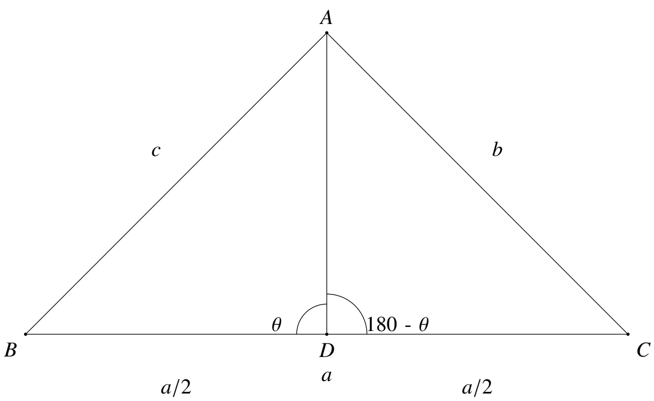

Fig. 1.5.4: $\sin\theta = \sin\left(180^{\circ} - \theta\right)$

$$\frac{1}{2}\frac{a}{2}AD\sin\theta = \frac{1}{2}\frac{a}{2}AD\sin\left(180^{\circ} - \theta\right) \quad (1.5.4.1)$$

$$\Rightarrow \sin\theta = \sin\left(180^{\circ} - \theta\right). \quad (1.5.4.2)$$

Note that our geometric definition of $\sin\theta$ holds only for $\theta < 90^{\circ}$. (1.5.4.2) allows us to extend this definition for $\angle ADC > 90^{\circ}$.

5. In Fig. 1.5.5, show that $EF = \frac{a}{2}$.

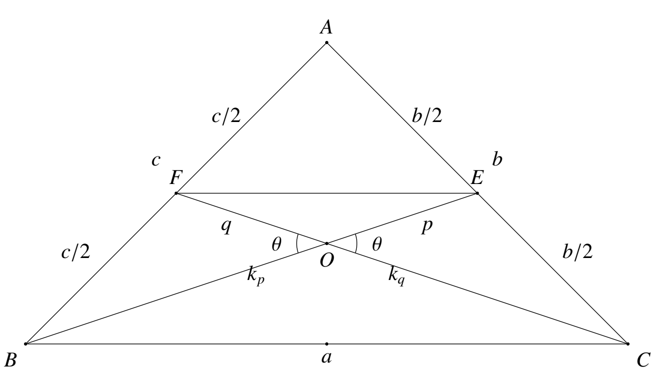

Fig. 1.5.5: Similar Triangles

Solution: Using the cosine formula for ΔAEF,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (1.5.5.1)$$

$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \quad (1.5.5.2)$$

$$= \frac{a^2}{4} \quad (1.5.5.3)$$

$$\Rightarrow EF = \frac{a}{2} \quad (1.5.5.4)$$

6. The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.
7. Show that similar triangles have the same angles.
   Solution: Use cosine formula and the proof is trivial.
8. Show that in Fig. 1.5.5, $EF \| BC$.
   Solution: Since $\Delta AEF \sim \Delta ABC$, $\angle AEF = \angle ACB$. Hence the line $EF \| BC$
9. Show that $\triangle OEF \sim \triangle OEC$.
10. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \quad (1.5.10.1)$$

11. Show that the medians of a triangle meet at a point.

1.6 Angle Bisectors

1. In Fig. 1.6.1, OB divides the $\angle B$ into half, i.e.

$$\angle OBC = \angle OBA \quad (1.6.1.1)$$

OB is known as an angle bisector.

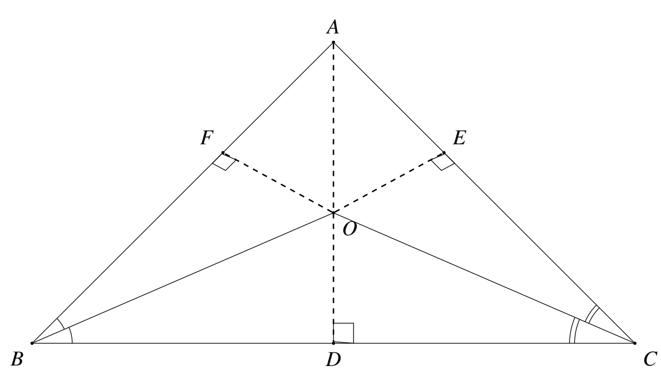

Fig. 1.6.1: Angle bisectors meet at a point

OB and OC are angle bisectors of angles B and C. OA is joined and OD, OF and OE are perpendiculars to sides a, b and c.

2. Show that $OD = OE = OF$. **Solution:** In Δs ODC and OEC,

$$OD = OC \sin \frac{C}{2} \tag{1.6.2.1}$$

$$OE = OC \sin \frac{C}{2} \tag{1.6.2.2}$$

$$\implies OD = OE. \tag{1.6.2.3}$$

Similarly,

$$OD = OF. \tag{1.6.2.4}$$

3. Show that OA is the angle bisector of $\angle A$ **Solution:** In Δs OFA and OEA,

$$OF = OE \tag{1.6.3.1}$$

$$\implies OA \sin OAF = OA \sin OAE \tag{1.6.3.2}$$

$$\implies \sin OAF = \sin OAE \tag{1.6.3.3}$$

$$\implies \angle OAF = \angle OAE \tag{1.6.3.4}$$

which proves that OA bisects $\angle A$. *Conclusion:* The angle bisectors of a triangle meet at a point.

1.7 Congruent Triangles

1. Show that in Δs ODC and OEC, corresponding sides and angles are equal.
2. Note that Δs ODC and OEC are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.
3. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.
4. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.
5. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.
6. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

1.8 Perpendicular Bisectors

1. In Fig. 1.8.2, OD $\perp BC$ and $BD = DC$. OD is defined as the perpendicular bisector of BC.

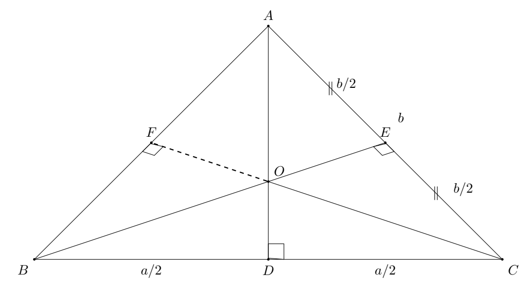

Fig. 1.8.2: Perpendicular bisectors meet at a point

2. In Fig. 1.8.2, show that $OA = OB = OC$. **Solution:** In Δs ODB and ODC, using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \tag{1.8.2.1}$$

Since $BD = DC = \frac{a}{2}$, $OB = OC$. Similarly, it can be shown that $OA = OC$. Thus, $OA = OB = OC$.

3. In ΔAOB, $OA = OB$. Such a triangle is known as an isoceles triangle.
4. Show that $AF = BF$. **Solution:** Trivial using Budhayana's theorem. This shows that OF is a perpendicular bisector of AB. *Conclusion:* The perpendicular bisectors of a triangle meet at a point.

1.9 Perpendiculars from Vertex to Opposite Side

1. In Fig. 1.9.1, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F. Show that

$$OE = c \cos A \cot C \tag{1.9.1.1}$$

Solution: In Δ s AEB and AEO,

$$AE = c \cos A \tag{1.9.1.2}$$

$$OE = AE \tan \left(90^{\circ} - C\right) (\because ADC \text{ is right angled}) \tag{1.9.1.3}$$

$$= AE \cot C \tag{1.9.1.4}$$

From both the above, we get the desired result.

2. Show that $\alpha = A$.

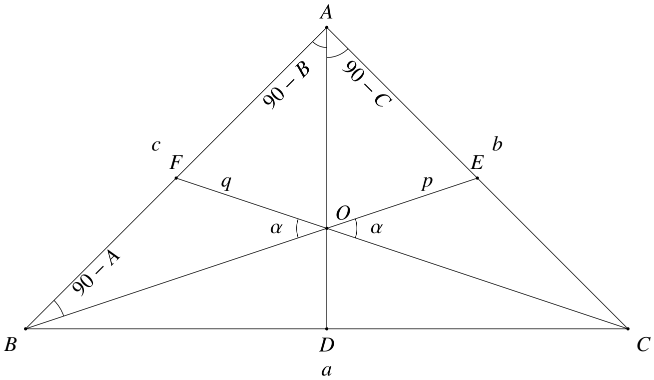

Fig. 1.9.1: Perpendiculars from vertex to opposite side meet at a point

Solution: In ΔOEC,

$$CE = a\cos C \,(\because BEC \text{ is right angled}) \tag{1.9.2.1}$$

Hence,

$$\begin{aligned}\tan\alpha &= \frac{CE}{OE}\\ &= \frac{a\cos C}{c\cos A\cot C}\\ &= \frac{a\cos C\sin C}{c\cos A\cos C}\\ &= \frac{a\sin C}{c\cos A}\\ &= \frac{c\sin A}{c\cos A}\left(\because \frac{a}{\sin A} = \frac{c}{\sin C}\right)\\ &= \tan A\end{aligned}$$

$$\Rightarrow \alpha = A \tag{1.9.2.2}$$

3. Show that $CF \perp AB$
Solution: Consider triangle OFB and the result of the previous problem. $\because$ the sum of the angles of a triangle is 180°, $\angle CFB = 90^{\circ}$.
Conclusion: The perperdiculars from the vertex of a triangle to the opposite side meet at a point.

1.10 Triangle Inequalities

1. Show that if

$$\theta_1 < \theta_2, \quad \sin\theta_1 < \sin\theta_2. \tag{1.10.1.1}$$

Solution: Using Baudhayana's theorem in

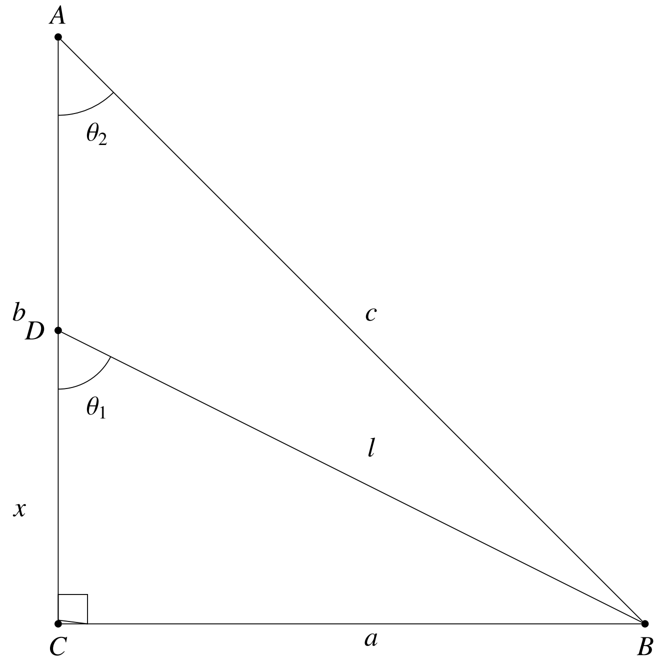

Fig. 1.10.1: $\theta_1 < \theta_2 \implies \sin\theta_1 < \sin\theta_2$.

$\triangle ABC$ and $\triangle DBC$

$$l^2 = x^2 + a^2 \tag{1.10.1.2}$$

$$c^2 = b^2 + a^2 \tag{1.10.1.3}$$

$$\implies c > l \because b > x. \tag{1.10.1.4}$$

Also,

$$a = c\sin\theta_1 = l\sin\theta_2 \tag{1.10.1.5}$$

$$\implies \frac{\sin\theta_1}{\sin\theta_2} = \frac{l}{c} < 1 \quad \text{from (1.10.1.4)} \tag{1.10.1.6}$$

$$\text{or, } \sin\theta_1 < \sin\theta_2 \tag{1.10.1.7}$$

2. Show that if

$$\theta_1 < \theta_2, \cos\theta_1 > \cos\theta_2. \tag{1.10.2.1}$$

3. Show that in any $\triangle ABC$, $\angle A > \angle B \implies a > b$.
Solution: Use (1.4.7.3) and (1.10.1.7)
4. Show that the sum of any two sides of a triangle is greater than the third side.
Solution: In Hero's formula in (1.4.9.8), all the factors inside the square root should be positive. Thus,

$$(s-a) > 0, (s-b) > 0\,(s-c) > 0 \tag{1.10.4.1}$$

$$(1.10.4.2)$$

$$(s-a) > 0 \implies \frac{a+b+c}{2} - a > 0 \quad (1.10.4.3)$$

$$\text{or, } b+c > a \quad (1.10.4.4)$$

Similarly, it can be shown that $a+b > c, c+a > b$.

1.11 Triangle Exercises

1. Sides opposite to equal angles of a triangle are equal.
2. Each angle of an equilateral triangle is of 60°.
3. Using cosine formula in an equilateral $\triangle$, show that $\cos 60° = \frac{1}{2}$.
4. Using (1.3.3.1), show that $\sin 60° = \frac{\sqrt{3}}{2}$.
5. Find $\sin 30°$ and $\sin 30°$ using (1.3.1.2).
6. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
7. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.
8. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.
9. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.
10. In an isosceles $\triangle ABC$ with $AB =$ AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.
11. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.
12. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.
13. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$
14. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.
15. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that
    a) $\triangle AOB \cong \triangle DOC$
    b) O is also the mid-point of BC.
16. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?
17. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that
    a) $\triangle ABD \cong \triangle BAC$
    b) $BD = AC$
    c) $\angle ABD = \angle BAC$.
18. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.
19. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$ (see Fig. 7.20). Show that:
    a) $\triangle APB \cong \triangle AQB$
    b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.
20. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.
21. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:
    a) $\triangle AMC \cong \triangle BMD$
    b) $\angle DBC$ is a right angle.
    c) $\triangle DBC \cong \triangle ACB$
    d) $CM = \frac{1}{2}AB$
22. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
    a) $OB = OC$
    b) AO bisects $\angle A$
23. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.
24. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.
25. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that
    a) $\triangle ABE \cong \triangle ACF$
    b) $AB = AC$, i.e., ABC is an isosceles triangle.
26. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$.
27. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles

on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that

a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$.
d) AP is the perpendicular bisector of BC.

28. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that
    a) AD bisects BC
    b) AD bisects $\angle A$.
29. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:
    a) $\triangle ABM \cong \triangle PQN$
    b) $\triangle ABC \cong \triangle PQR$
30. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.
31. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.
32. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.
33. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.
34. Show that in a right angled triangle, the hypotenuse is the longest side.
35. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.
36. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.
37. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.
38. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.
39. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.
40. $ABCD$ is a trapezium with $AB \parallel DC$. E and F are points on non-parallel sides AD and BC respectively such that EF is parallel to AB . Show that $\frac{AE}{ED} = \frac{BF}{FC}$.
41. ST is a line joining two points on PQ and PR in $\triangle PQR$. If $\frac{PS}{SQ} = \frac{PT}{TR}$ and $\angle PST = \angle PRQ$, prove that PQR is an isosceles triangle.
42. If $LM \parallel CB$ and $LN \parallel CD$, prove that $\frac{AM}{AB} = \frac{AN}{AD}$.
43. D is a point on AB and E, F are points on BC such that $DE \parallel AC$ and $DF \parallel AE$. Prove that $\frac{BF}{FE} = \frac{BE}{EC}$.
44. O is a point in the interior of $\triangle PQR$. D is a point on OP. If $DE \parallel OQ$ and $DF \parallel OR$. Show that $EF \parallel QR$.
45. O is a point in the interior of $\triangle PQR$. $A, BandC$ are points on OP, OQ and OR respectively such that $AB \parallel PQ$ and $AC \parallel PR$. Show that $BC \parallel QR$.
46. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$
47. The diagonals of a quadrilateral $ABCD$ intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that $ABCD$ is a trapezium.
48. $PQ \parallel RS$ and PS intersects QR at O. Show that $\triangle OPQ \sim \triangle ORS$.
49. CM and RN are respectively the medians of $\triangle ABC$ and $\triangle PQR$. If $\triangle ABC \sim \triangle PQR$, prove that
    a) $\triangle AMC \sim \triangle PNR$
    b) $\frac{CM}{RN} = \frac{AB}{PQ}$
    c) $\triangle CMB \sim \triangle RNQ$
50. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. Using a similarity criterion for two triangles, show that $\frac{OA}{OC} = \frac{OB}{OD}$
51. In $\triangle PQR$, QP is extended to T and S is a point on QR such that $\frac{QR}{QS} = \frac{QT}{PR}$. If $\angle PRQ = \angle PQS$, show that that $\triangle PQS \sim \triangle TQR$.
52. S and T are points on sides PR and QR of $\triangle PQR$ such that $\angle P = \angle RTS$. Show that $\triangle RPQ \sim \triangle RTS$.
53. In $\triangle ABC$, D and E are points on the sides AB and AC respectively. If $\triangle ABE \cong \triangle ACD$, show that $\triangle ADE \sim \triangle ABC$.
54. Altitudes AD and CE of $\triangle ABC$ intersect each other at the point P. Show that:
    a) $\triangle AEP \sim \triangle CDP$
    b) $\triangle ABD \sim \triangle CBE$
    c) $\triangle AEP \sim \triangle ADB$
    d) $\triangle PDC \sim \triangle BEC$
55. E is a point on the side AD produced of a parallelogram $ABCD$ and BE intersects CD at F. Show that $\triangle ABE \sim \triangle CFB$.
56. ABC and AMP are two right triangles, right angled at B and M respectively. M lies on AC

and AB is extended to meet P. Prove that:

a) $\triangle ABC \sim \triangle AMP$
b) $\frac{CA}{PA} = \frac{BC}{MP}$

57. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of $\triangle ABC$ and $\triangle EFG$ respectively. If $\triangle ABC \sim \triangle FEG$, show that:
58. $\frac{CD}{GH} = \frac{AC}{FG}$
59. $\triangle DCB \sim \triangle HGE$
60. $\triangle DCA \sim \triangle HGF$
61. E is a point on side CB produced of an isosceles $\triangle ABC$ with $AB = AC$. If $AD \perp BC$ and $EF \perp AC$, prove that $\triangle ABD \sim \triangle ECF$.
62. Sides AB and BC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and QR and median PM of $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.
63. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that $CA^2 = CB.CD$.
64. Sides AB and AC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and PR and median PM of another $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.
65. If AD and PM are medians of $\triangle sABC$ and PQR, respectively where $\triangle ABC \sim \triangle PQR$, prove that $\frac{AB}{PQ} = \frac{AD}{PM}$
66. The line segment XY is parallel to side AC of $\triangle ABC$ and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$
67. Diagonals of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. If $AB = 2CD$, find the ratio of the areas of $\triangle sAOB$ and COD.
68. ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, show that $\frac{ar(ABC)}{ar(DBC)} = \frac{AO}{DO}$.
69. If the areas of two similar triangles are equal, prove that they are congruent.
70. D, E and F are respectively the mid-points of sides AB, BC and CA of $\triangle ABC$. Find the ratio of the areas of $\triangle DEF$ and $\triangle ABC$.
71. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.
72. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.
73. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. Find the ratio of the areas of triangles ABC and BDE.
74. The sides of two similar triangles are in the ratio 4 : 9. Find the ratio the area of these triangles are in the ratio
75. In $\triangle ABC, \angle ACB = 90^\circ$ and $CD \perp AB$. Prove that $\frac{BC^2}{AC^2} = \frac{BD}{AD}$.
76. In $\triangle ABC$, if $AD \perp BC$, prove that $AB^2 + CD^2 = BD^2 + AC^2$.
77. BL and CM are medians of a $\triangle ABC$ right angled at A. Prove that $4(BL^2 + CM^2) = 5BC2$.
78. O is any point inside a rectangle $ABCD$. Prove that $OB^2 + OD^2 = OA^2 + OC^2$.
79. PQR is a triangle right angled at P and M is a point on QR such that $PM \perp QR$. Show that $PM^2 = QM.MR$.
80. ABD is a triangle right angled at A and $AC \perp BD$. Show that

a) $AB^2 = BC.BD$
b) $AC^2 = BC.DC$
c) $AD^2 = BD.CD$

81. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC2$.
82. ABC is an isosceles triangle with $AC = BC$. If $AB^2 = 2AC2$, prove that ABC is a right triangle.
83. ABC is an equilateral triangle of side $2a$. Find each of its altitudes.
84. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.
85. O is a point in the interior of a $\triangle ABC, OD \perp BC, OE \perp ACandOF \perp AB$. Show that

a) $OA^2 + OB^2 + BD^2 – OD2 – OE2 – OF2 = AF^2 + BD^2 + CE^2$.
b) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

86. D and E are points on the sides CA and CB respectively of a $\triangle ABC$ right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.
87. The perpendicular from A on side BC of a $\triangle ABC$ intersects BC at D such that $DB = 3CD$. Prove that $2AB^2 = 2AC^2 + BC^2$.
88. In an equilateral $\triangle ABC$, D is a point on side BC such that $BD = \frac{1}{3}BC$. Prove that $9AD^2 = 7AB2$.
89. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.
90. PS is the bisector of $\angle QPR$ of $\triangle PQR$. Prove

that $\frac{QS}{SR} = \frac{PQ}{PR}$

91. D is a point on hypotenuse AC of $\triangle ABC$, such that $BD \perp AC, DM \perp BC$ and $DN \perp AB$. Prove that :
    a) DM2 = DN . MC
    b) DN2 = DM . AN
92. ABC is a triangle in which $\angle ABC > 90°$ and $AD \perp CB$ produced. Prove that $AC^2 = AB^2 + BC^2 + 2BC.BD$.
93. ABC is a triangle in which $\angle ABC < 90°$ and $AD \perp BC$. Prove that $AC^2 = AB2 + BC2–2BC.BD$.
94. AD is a median of a $\triangle ABC$ and $AM \perp BC$. Prove that :
    a) $AC2 = AD2 + BC.DM + \left(\frac{BC}{2}\right)^2$
    b) $AB2 = AD2–BC.DM + \left(\frac{BC}{2}\right)^2$
    c) $AC^2 + AB^2 = 2AD2 + \frac{1}{2}BC^2$
95. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.
96. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD}\frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

2 Quadrilaterals

2.1 Properties

1. Sum of the angles of a quadrilateral is 360°.
   Solution: Draw the diagonal and use the fact that sum of the angles of a triangle is 180°.
2. A diagonal of a parallelogram divides it into two congruent triangles.
   Solution: The alternate angles for the parallel sides are equal. The diagonal is common. Use ASA congruence.
3. In a parallelogram,
    a) opposite sides are equal
    b) opposite angles are equal
    c) diagonals bisect each other

   Solution: Since the diagonal divides the parallelogram into two congruent triangles, all the above results follow.
4. A quadrilateral is a parallelogram, if
    a) opposite sides are equal or
    b) opposite angles are equal or
    c) diagonals bisect each other or
    d) a pair of opposite sides is equal and parallel

   Solution: All the above lead to a quadrilateral that has two parallel sides, by showing that the alternate angles are equal.
5. A rectangle is a parallelogram with one angle that is 90°. Show that all angles of the rectangle are 90°.
   Solution: Draw a diagonal. Since the diagonal divides the rectangle into two congruent triangles, the angle opposite to the right angle is also 90°. Using congruence, it can be shown that the other two angles are equal. Now use the fact that the sum of the angles of a quadrilateral is 360°.
6. Diagonals of a rectangle bisect each other and are equal and vice-versa.
   Solution: Use Baudhayana's theorem for equality of diagonals.
7. Diagonals of a rhombus bisect each other at right angles and vice-versa.
   Solution: The median of an isoceles triangle is also its perpendicular bisector.
8. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
   Solution: A square has the properties of a rectangle as well as a rhombus.
9. The quadrilateral formed by joining the midpoints of the sides of a quadrilateral, in order, is a parallelogram.
10. Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.
11. Show that the bisectors of angles of a parallelogram form a rectangle.
12. A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.
13. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
14. Area of a parallelogram is the product of its base and the corresponding altitude.
15. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.
16. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
17. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
18. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

19. Show that the diagonals of a square are equal and bisect each other at right angles.
20. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.

2.2 *Quadrilateral Exercises*

1. In parallelogram $ABCD$, two points P and Q are taken on diagonal BD such that $DP = BQ$. show that
   a) $\triangle APD \cong \triangle CQB$
   b) $AP = CQ$
   c) $\triangle AQB \cong \triangle CPD$
   d) $AQ = CP$
   e) $APCQ$ is a parallelogram
2. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD . Show that
   a) $\triangle APB \cong \triangle CQD$
   b) $AP = CQ$
3. In $\triangle ABC$ and $\triangle DEF, AB = DE, AB \parallel DE, BC = EF$ and $BC \parallel EF$. Vertices A, B and C are joined to vertices D, E and F respectively. Show that
   a) quadrilateral $ABED$ is a parallelogram
   b) quadrilateral $BEFC$ is a parallelogram
   c) $AD \parallel CF$ and $AD = CF$
   d) quadrilateral $ACFD$ is a parallelogram
   e) $AC = DF$
   f) $\triangle ABC \cong \triangle DEF$.
4. $ABCD$ is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that
   a) $\angle A = \angle B$
   b) $\angle C = \angle D$
   c) $\triangle ABC \cong \triangle BAD$
   d) diagonal AC = diagonal BD
5. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA AC is a diagonal. Show that
   a) $SR \parallel AC$ and $SR = \frac{1}{2}AC$
   b) $PQ = SR$
   c) $PQRS$ is a parallelogram.
6. $ABCD$ is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rectangle.
7. $ABCD$ is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.
8. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and E is the mid-point of AD. A line is drawn through $E \parallel AB$ intersecting BC at F. Show that F is the mid-point of BC.
9. In a parallelogram $ABCD$, E and F are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.
10. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
11. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:
    a) $APCQ$ is a parallelogram.
    b) $DPBQ$ is a parallelogram.
    c) $PSQR$ is a parallelogram.
12. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.
13. Diagonal AC of a parallelogram $ABCD$ bisects $\angle A$. show that
    a) it bisects $\angle C$ also,
    b) $ABCD$ is a rhombus.
14. $ABCD$ is a rhombus. Show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.
15. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that
    a) $ABCD$ is a square
    b) diagonal BD bisects $\angle B$ as well as $\angle D$.
16. If E, F, G and H are respectively the mid-points of the sides of a parallelogram $ABCD$, show that
$$ar(EFGH) = \frac{1}{2} ar(ABCD). \qquad (2.2.16.1)$$
17. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$. Show that $ar(APB) = ar(BQC)$.
18. P is a point in the interior of a parallelogram $ABCD$. Show that
    a) $ar(APB) + ar(PCD) = \frac{1}{2}ar(ABCD)$
    b) $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

19. $PQRS$ and $ABRS$ are parallelograms and X is any point on side BR. show that
   a) $ar(PQRS) = ar(ABRS)$
   b) $ar(AXS) = \frac{1}{2}ar(PQRS)$
20. A farmer was having a field in the form of a parallelogram $PQRS$. She took any point A on RS and joined it to points P and Q. In how many parts the fields is divided? What are the shapes of these parts? The farmer wants to sow wheat and pulses in equal portions of the field separately. How should she do it?
21. $ABCD$ is a quadrilateral and $BE \parallel AC$ and also BE meets DC produced at E. Show that area of $\triangle ADE$ is equal to the area of the quadrilateral $ABCD$.
22. E is any point on median AD of a $\triangle ABC$. Show that $ar(ABE) = ar(ACE)$.
23. In a $\triangle ABC, E$ is the mid-point of median AD. Show that $ar(BED) = \frac{1}{4}ar(ABC)$.
24. Show that the diagonals of a parallelogram divide it into four triangles of equal area.
25. ABC and ABD are two triangles on the same base AB. If line- segment CD is bisected by AB at O, show that $ar(ABC) = ar(ABD)$.
26. D, E and F are respectively the mid-points of the sides BC, CA and AB of a $\triangle ABC$. show that
   a) $BDEF$ is a parallelogram.
   b) $ar(BDEF) = \frac{1}{2}ar(ABC)$
27. Diagonals AC and BD of quadrilateral $ABCD$ intersect at O such that $OB = OD$. If $AB = CD$, then show that
   a) $ar(DOC) = ar(AOB)$
   b) $ar(DCB) = ar(ACB)$
   c) $ar(DEF) = \frac{1}{4}ar(ABC)$
28. D and E are points on sides AB and AC respectively of $\triangle ABC$ such that $ar(DBC) = ar(EBC)$. Prove that $DE \parallel BC$.
29. XY is a line parallel to side BC of a $\triangle ABC$. If $BE \parallel AC$ and $CF \parallel AB$ meet XY at E and F respectively, show that $ar(ABE) = ar(ACF)$.
30. The side AB of a parallelogram $ABCD$ is produced to any point P. A line through A and parallel to CP meets CB produced at Q and then parallelogram $PBQR$ is completed. Show that $ar(\text{ABCD}) = ar(PBQR)$.
31. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at O. Prove that $ar(AOD) = ar(BOC)$.
32. $ABCDE$ is a pentagon. A line through B parallel to AC meets DC produced at F. Show that
   a) $ar(ACB) = ar(ACF)$
   b) $ar(AEDF) = ar(ABCDE)$.
33. A villager Itwaari has a plot of land of the shape of a quadrilateral. The Gram Panchayat of the village decided to take over some portion of his plot from one of the corners to construct a Health Centre. Itwaari agrees to the above proposal with the condition that he should be given equal amount of land in lieu of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented.
34. $ABCD$ is a trapezium with $AB \parallel DC$. A line parallel to AC intersects AB at X and BC at Y. Prove that $ar(ADX) = ar(ACY)$.
35. $AP \parallel BQ \parallel CR$. Prove that $ar(AQC) = ar(PBR)$.
36. Diagonals AC and BD of a quadrilateral $ABCD$ intersect at O in such a way that $ar(AOD) = ar(BOC)$. Prove that $ABCD$ is a trapezium.
37. $AB \parallel DC \parallel RP$. $ar(DRC) = ar(DPC)$ and $ar(BDP) = ar(ARC)$. Show that both the quadrilaterals $ABCD$ and $DCPR$ are trapeziums.
38. Parallelogram $ABCD$ and rectangle $ABEF$ are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.
39. In $\triangle ABC$, D and E are two points on BC such that $BD = DE = EC$. Show that $ar(ABD) = ar(ADE) = ar(AEC)$.
40. $ABCD, DCFE$ and $ABFE$ are parallelograms. Show that ar$(ADE) = ar(BCF)$.
41. $ABCD$ is a parallelogram and BC is produced to a point Q such that $AD = CQ$. If AQ intersect DC at P, show that $ar(BPC) = ar(DPQ)$. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. If AE intersects BC at$F$, show that
   a) $ar(BDE) = \frac{1}{4}ar(ABC)$
   b) $ar(BDE) = \frac{1}{2}ar(BAE)$
   c) $ar(ABC) = 2ar(BEC)$
   d) $ar(BFE) = ar(AFD)$
   e) $ar(BFE) = 2ar(FED)$
   f) $ar(FED) = \frac{1}{8}ar(AFC)$
42. Diagonals AC and BD of a quadrilateral $ABCD$ intersect each other at P. Show that $ar(APB) \times$

$ar(CPD) = ar(APD) \times ar(BPC)$.

43. P and Q are respectively the mid-points of sides AB and BC of a $\triangle ABC$ and R is the mid-point of AP, show that
    a) $ar(PRQ) = \frac{1}{2}ar(ARC)$
    b) $ar(PBQ) = ar(ARC)$
    c) $ar(RQC) = \frac{3}{8}ar(ABC)$
44. ABC is a right triangle right angled at A. $BCED$, $ACFG$ and $ABMN$ are squares on the sides BC, CA and AB respectively. Line segment $AX \perp DE$ meets BC at Y. Show that
    a) $\triangle MBC \cong \triangle ABD$
    b) $ar(BYXD) = ar(ABMN)$
    c) $ar(CYXE) = 2ar(FCB)$
    d) $ar(BYXD) = 2ar(MBC)$
    e) $\triangle FCB \cong \triangle ACE$
    f) $ar(CYXE) = ar(ACFG)$
    g) $ar(BCED) = ar(ABMN) + ar(ACFG)$
45. L is a point on the diagonal AC of quadrilateral $ABCD$. If LM —— CB and LN —— CD, prove that $\frac{AM}{AB} = \frac{AN}{AD}$

3 Circle

3.1 Properties

1. Fig. 3.1.1 represents a circle. The points in the circle are at a distance r from the centre O. r is known as the radius.

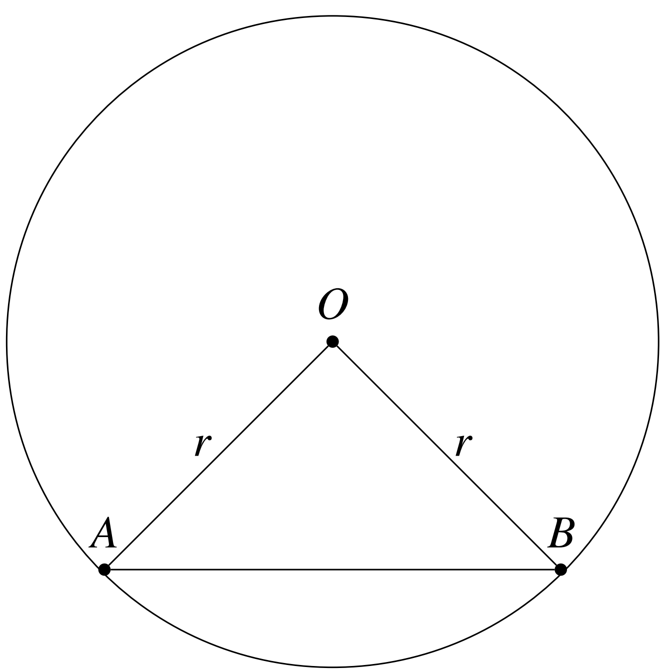

Fig. 3.1.1: Circle Definitions

2. In Fig. 3.1.1, A and B are points on the circle. The line AB is known as a chord of the circle.
3. In Fig. 3.1.3 Show that $\angle AOB = 2\angle ACB$.

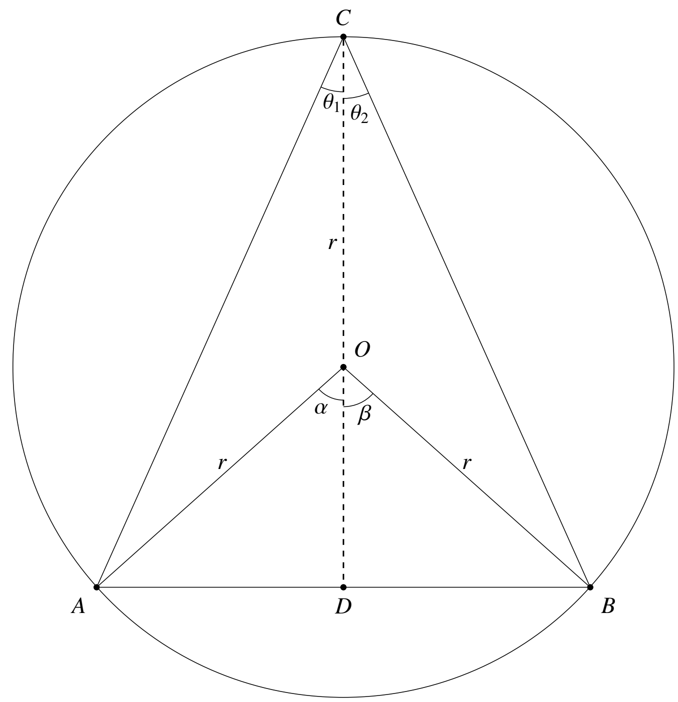

Fig. 3.1.3: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Solution: In Fig. 3.1.3, the triangeles OPA and OPB are isosceles. Hence,

$$\angle OCA = \angle OAC = \theta_1 \tag{3.1.3.1}$$
$$\angle OCB = \angle OBC = \theta_2 \tag{3.1.3.2}$$

Also, α and β are exterior angles corresponding to the triangle AOC and BOC respectively. Hence

$$\alpha = 2\theta_1 \tag{3.1.3.3}$$
$$\beta = 2\theta_2 \tag{3.1.3.4}$$

Thus,

$$\angle AOB = \alpha + \beta \tag{3.1.3.5}$$
$$= 2\left(\theta_1 + \theta_2\right) \tag{3.1.3.6}$$
$$= 2\angle ACB \tag{3.1.3.7}$$

4. The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 3.1.5, AB is the diameter and passes through the centre O
5. In Fig. 3.1.5, show that $\angle APB = 90^\circ$.

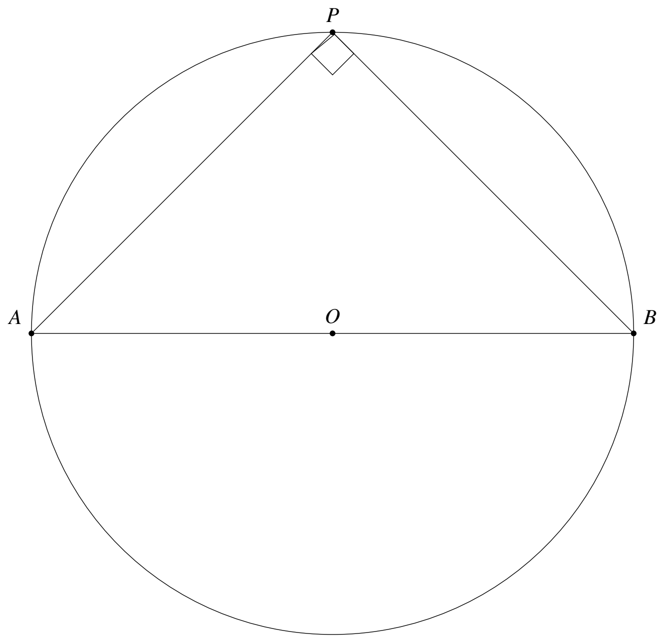

Fig. 3.1.5: Diameter of a circle.

6. In Fig. 3.1.6, show that

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \tag{3.1.6.1}$$

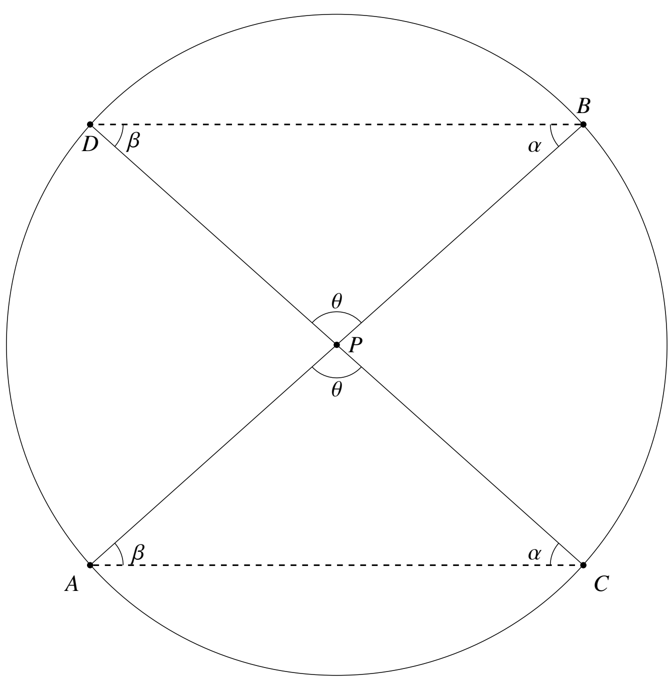

Fig. 3.1.6: *PA.PB* = *PC.PD*

Solution: Use Problem 3.1.3.

7. In Fig. 3.1.6, show that the triangles *PAB* and *PBD* are similar
Solution: Trivial using previous problem

8. In Fig. 3.1.6, show that

$$PA.PB = PC.PD \tag{3.1.8.1}$$

Solution: Since triangles *PAC* and *PBD* are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{3.1.8.2}$$

$$\Rightarrow PA.PB = PC.PD \tag{3.1.8.3}$$

9. Show that

$$\sin 0^{\circ} = 0 \tag{3.1.9.1}$$

Solution: From (1.1.2.1), $\theta \to 0° \implies a \to 0 \implies \sin\theta$ and

10. Show that

$$\cos 0^{\circ} = 1 \tag{3.1.10.1}$$

Solution: Follows from the fact that $\sin 0 = 0$ and (1.3.3.1).

11. Show that

$$\cos 90^{\circ} = 0 \tag{3.1.11.1}$$

Solution: Follows from the fact that $\cos 90° = \sin(90° - 90°) = 0$ using (1.3.1.2).

12. The line *PX* in Fig. 3.1.13 touches the circle at exactly one point *P*. It is known as the tangent to the circle.
13. Show that $OP \perp PX$.
Solution: Without loss of generality, let $0 \leq \theta \leq 90^{\circ}$. Using the cosine formula in $\triangle OPP_n$,

$$(r + d_n)^2 > r^2, \tag{3.1.13.1}$$

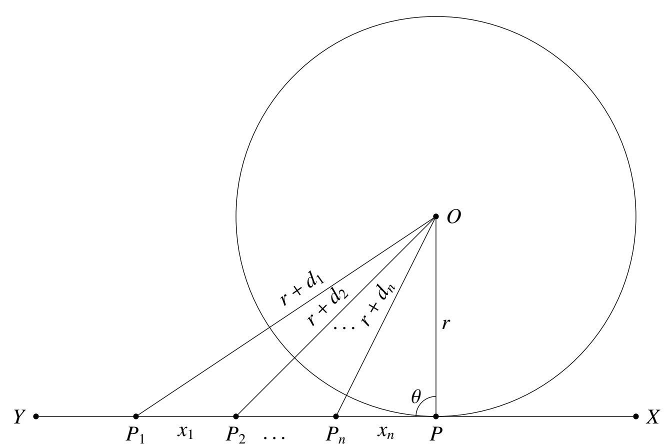

Fig. 3.1.13: Tangent to a Circle.

$$(r + d_n)^2 = r^2 + x_n^2 - 2rx_n \cos\theta > r^2 \tag{3.1.13.2}$$

$$\implies 0 < \cos\theta < \frac{x_n}{2r}, \tag{3.1.13.3}$$

where x_n can be made as small as we choose. Thus,

$$\cos\theta = 0 \implies \theta = 90^\circ. \tag{3.1.13.4}$$

14. In Fig. 3.1.14 show that

$$\angle PCA = \angle PBC \tag{3.1.14.1}$$

O is the centre of the circle and PC is the tangent.

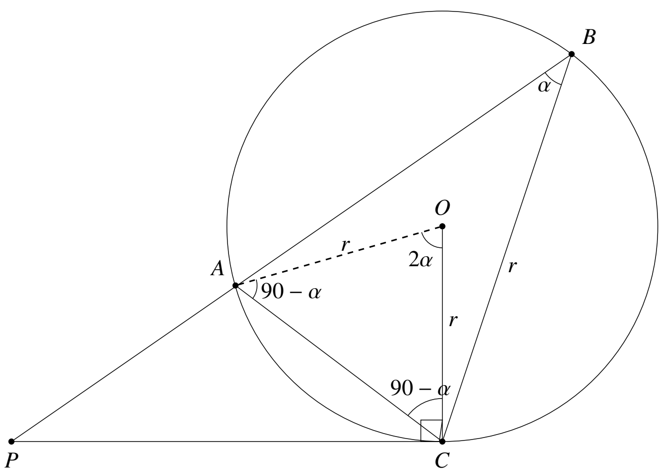

Fig. 3.1.14: $PA.PB = PC^2$.

Solution: Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

15. In Fig. 3.1.14, show that the triangles PAC and PBC are similar.
Solution: From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.
16. Show that $PA.PB = PC^2$
Solution: Since $\Delta PAC \sim \Delta PBC$, their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{3.1.16.1}$$

$$\Rightarrow PA.PB = PC^2 \tag{3.1.16.2}$$

17. Given that $PA.PB = PC^2$, show that PC is a tangent to the circle.
18. In Fig. 3.1.18, show that

$$PA.PB = PC.PD \tag{3.1.18.1}$$

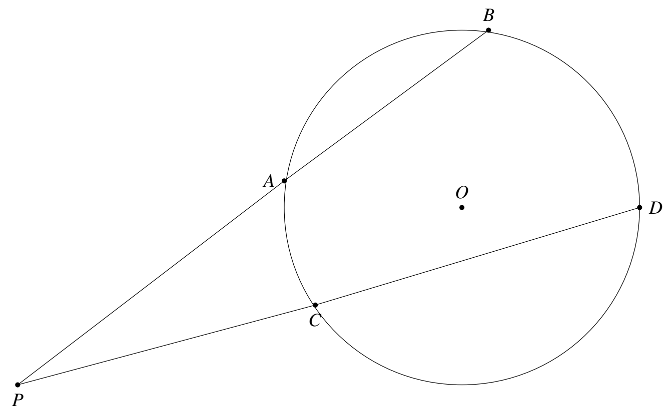

Fig. 3.1.18: $PA.PB = PC^2$.

Solution: Draw a tangent and use the previous problem.

3.2 Area of a Circle

1. In Fig. 3.2.1, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.

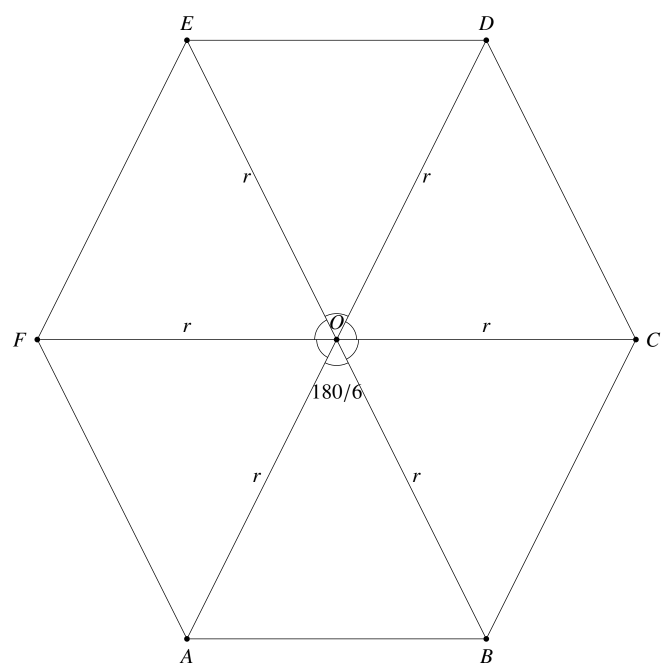

Fig. 3.2.1: Polygon Definition

2. The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{360^\circ}{n}$.
3. Show that the perimeter of a regular polygon is given by
$$2rn\sin\frac{180^\circ}{n} \tag{3.2.3.1}$$
4. Show that the area of a regular polygon is given by
$$\frac{n}{2}r^2\sin\frac{360^\circ}{n} \tag{3.2.4.1}$$

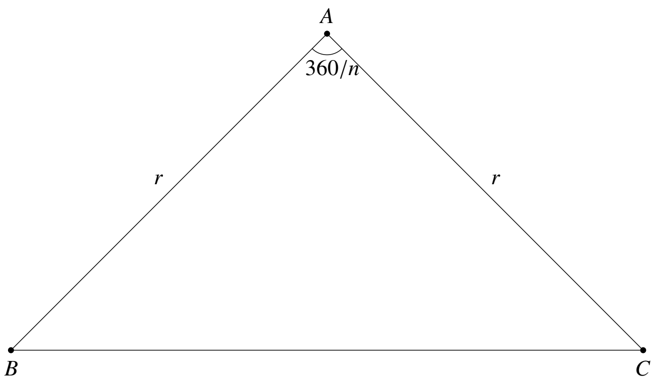

Fig. 3.2.4: Polygon Area

Solution: The triangle that forms the polygon of n sides is given in Fig. 3.2.3. Thus,
$$ar\,(polygon) = n \times ar\,(\Delta ABC)$$
$$= \frac{n}{2}r^2\sin\frac{360^\circ}{n} \tag{3.2.4.2}$$

5. Using Fig. 3.2.4, show that
$$\frac{n}{2}r^2\sin\frac{360^\circ}{n} < \text{ area of circle } < nr^2\tan\frac{180^\circ}{n} \tag{3.2.5.1}$$
The portion of the circle visible in Fig. 3.2.4 is defined to be a sector of the circle.
Solution: Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. 3.2.4, the area of the circle should be in between the areas of the inner and outer polygons. Since
$$ar\,(\Delta OAB) = \frac{1}{2}r^2\sin\frac{360^\circ}{n} \tag{3.2.5.2}$$
$$ar\,(\Delta OPQ) = 2 \times \frac{1}{2} \times r\tan\frac{360/n}{2} \times r \tag{3.2.5.3}$$
$$= r^2\tan\frac{180^\circ}{n}, \tag{3.2.5.4}$$

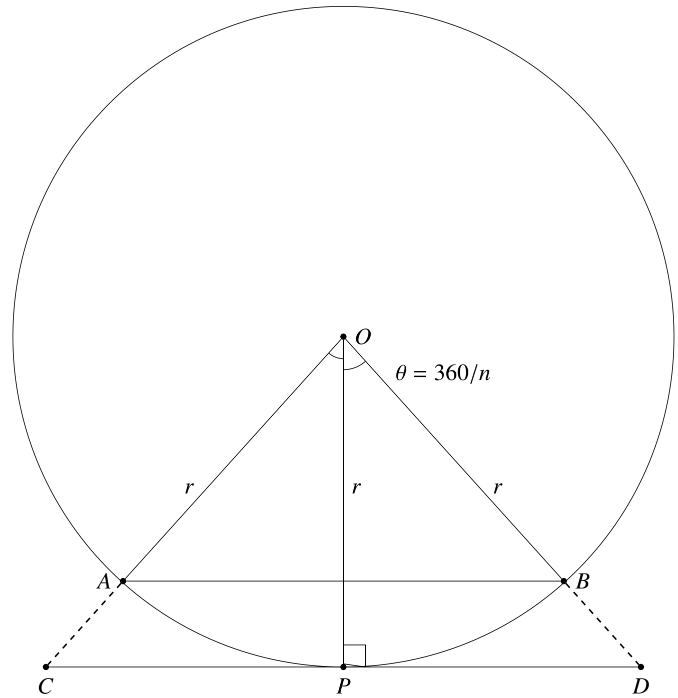

Fig. 3.2.5: Circle Area in between Area of Two Polygons

we obtain (3.2.4.1).

6. Using Fig. 3.2.5, show that
$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{3.2.6.1}$$

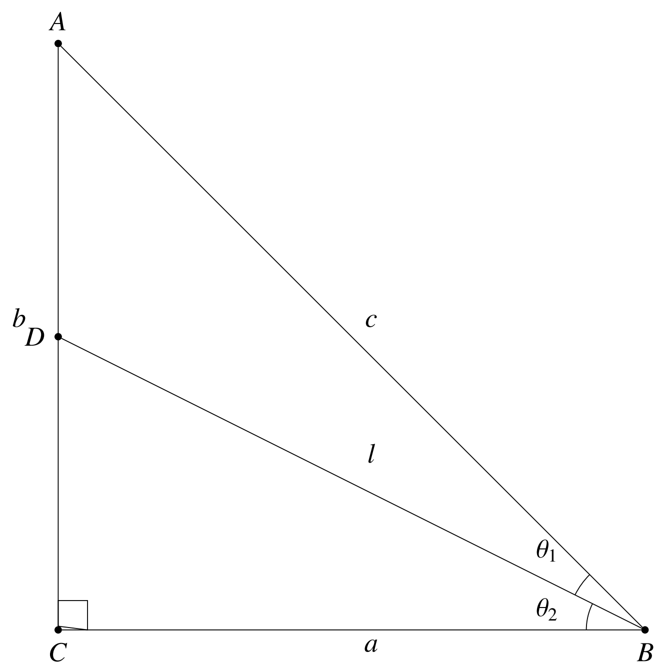

Fig. 3.2.6: $\sin 2\theta = 2\sin\theta\cos\theta$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1+\theta_2) \tag{3.2.6.2}$$
$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{3.2.6.3}$$
$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{3.2.6.4}$$
$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{3.2.6.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\implies \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{3.2.6.6}$$
$$\implies \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \tag{3.2.6.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\implies \sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{3.2.6.8}$$

resulting in

$$\implies \sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{3.2.6.9}$$

7. Prove the following identities

a)

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{3.2.7.1}$$

b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{3.2.7.2}$$

Solution: In (3.2.5.1), let

$$\begin{aligned}\theta_1+\theta_2 &= \alpha\\ \theta_2 &= \beta\end{aligned} \tag{3.2.7.3}$$

This gives (3.2.6.1). In (3.2.6.1), replace α by $90° - \alpha$. This results in

$$\sin(90° - \alpha - \beta)$$
$$= \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta$$
$$\implies \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3.2.7.4}$$

8. Using (3.2.5.1) and (3.2.6.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{3.2.8.1}$$
$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{3.2.8.2}$$

Solution: From (3.2.5.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{3.2.8.3}$$

Using (3.2.6.2) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{3.2.8.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{3.2.8.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{3.2.8.6}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{3.2.8.7}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{3.2.8.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (3.2.7.2) can also be proved.

9. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{3.2.9.1}$$

10. Show that

$$\cos^2\frac{180°}{n} < \frac{\text{area of circle}}{nr^2\tan\frac{180°}{n}} < 1 \tag{3.2.10.1}$$

Solution: From (3.2.4.1) and (3.2.8.1),

$$\frac{n}{2} r^2 \sin \frac{360^\circ}{n} < \text{ area of circle}$$
$$< nr^2 \tan \frac{180^\circ}{n}$$
$$\Rightarrow nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} < \text{ area of circle}$$
$$< nr^2 \tan \frac{180^\circ}{n} \quad (3.2.10.2)$$

11. Show that for large values of n

$$\cos^2 \frac{180^\circ}{n} = 1 \qquad (3.2.11.1)$$

Solution: Follows from previous problem.

12. The previous result can be expressed as

$$\lim_{n\to\infty} \cos^2 \frac{180^\circ}{n} = 1 \qquad (3.2.12.1)$$

13. Show that

$$\text{area of circle } = r^2 \lim_{n\to\infty} n \tan \frac{180^\circ}{n} \qquad (3.2.13.1)$$

14.

$$\pi = \lim_{n\to\infty} n \tan \frac{180^\circ}{n} \qquad (3.2.14.1)$$

Thus, the area of a circle is πr^2.

15. The radian is a unit of angle defined by

$$1 \text{ radian} = \frac{360^\circ}{2\pi} \qquad (3.2.15.1)$$

16. Show that the circumference of a circle is $2\pi r$.
17. Show that the area of a sector with angle θ in radians is $\frac{1}{2} r^2 \theta$.
18. Show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (3.2.18.1)$$

3.3 Circle Exercises

1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3. The perpendicular from the centre of a circle to a chord bisects the chord.
4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
5. There is one and only one circle passing through three non-collinear points.
6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
9. Congruent arcs of a circle subtend equal angles at the centre.
10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
11. Angles in the same segment of a circle are equal.
12. Angle in a semicircle is a right angle.
13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
15. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.
16. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60^\circ$.
17. Two circles intersect at two points A and B. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.
18. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
19. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
20. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
21. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.

22. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
23. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
24. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
25. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.
26. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
27. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.
28. Prove that a cyclic parallelogram is a rectangle.
29. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
30. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.
31. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.
32. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.
33. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.
34. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90°-\frac{A}{2}, 90°-\frac{B}{2}$ and $90°-\frac{C}{2}$.
35. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.
36. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.
37. The lengths of tangents drawn from an external point to a circle are equal.
38. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.
39. Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2\angle OPQ$.
40. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.
41. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.
42. A quadrilateral $ABCD$ is drawn to circumscribe a circle. Prove that $AB + CD = AD + BC$.
43. XY and $X'Y'$ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and $X'Y'$ at B. Prove that $\angle AOB = 90°$
44. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.
45. Prove that the parallelogram circumscribing a circle is a rhombus.
46. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
47. Find the area of a sector of angle p (in degrees) of a circle with radius R.
48. Two chords AB and CD intersect each other at the point P. Prove that :
    a) $\triangle APC \sim \triangle DPB$
    b) $AP.PB = CP.DP$
49. Two chords AB and CD of a circle intersect each other at the point P (when produced) outside the circle. Prove that
    a) $\triangle PAC \sim \triangle PDB$
    b) $PA.PB = PC.PD$