Algebra: International Maths Olympiad

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***Abstract*—This book provides a collection of the international maths olympiad problems in algebra.**

1. Solve for integers x, y, z: x + y = 1 - z, $x^3+y^3 = 1 - z^2$.
2. If a, b, c, x are real numbers such that abc ≠ 0 and
$$\frac{xb+(1-x)c}{a} = \frac{xc+(1-x)a}{b} = \frac{xa+(1-x)b}{c}$$
then prove that either a + b + c = 0 or a = b = c.
3. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex),then $\left|\lambda\right| \leq 1$.
4. Show that the equation
$$x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$$
has infinitely many solutions in integers x, y, z.
5. If a, b, c are positive real numbers such that abc = 1, prove that
$$a^{b+c}b^{c+a}c^{a+b} \leq 1.$$
6. Determine the least positive value taken by the expression
$$a^3 + b^3 + c^3 - 3abc$$
as a,b,c vary over all positive integers. Find also all triples(a,b,c) for which this least value is attained.
7. Let x,y be positive reals such that x + y= 2. Prove that
$$x^3y^3(x^3 + y^3) \leq 2.$$
8. Does there exist three distince positive real numbers a,b,c such that the numbers a, b, c, b + c - a, c + a - b, a + b - c and a + b + c from a 7-term arthemetic progression in some order?
9. Find all primes p and q and even numbers $n > 2$ satisfying the equation
$$p^n + p^{n-1} + ... + p + 1 = q^2 + q + 1.$$
10. Show that for every real number a the equation
$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$
has at least real non-real root and find the sum of all the non-real roots of the equation.
11. Suppose p is a prime greater than 3. Find all pairs of integers (a,b) satisfying the equation
$$a^2 + 3ab + 2p(a+b) + p^2 = 0$$
12. If α is real root of the equation $x^5-x^3+x-2 = 0$, prove that $[\alpha^6] = 3$.
13. Prove that the number of 5-tuples of positive integers (a,b,c,d,e) satisfying the equation
$$abcde = 5(bcde + acde + abde + abce + abcd)$$
14. Let α and β be positive integers such that
$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$$
Find the minimum possible value of β?
15. Let p,q,r be positive real numbers, not all equal, such that some two of the equations
$$px^2 + 2qx + r = 0 \tag{15.1}$$
$$qx^2 + 2rx + p = 0 \tag{15.2}$$
$$rx^2 + 2px + q = 0 \tag{15.3}$$
have a common root say α. Prove that

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a) α is a real and negative
b) the third equation has non-real roots.

16. Let x_1 be a given positive integer. A sequence

$$\sum_{n=1}^{\infty} = \{x_1, x_2, x_3, ...\}$$

of positive integers is such that x_n for $n \geq 2$ is obtained from x_{n-1} by adding some non-zero digit of x_{n-1}. Prove that

a) the sequence has an even number
b) the sequence has infinitely many even numbers.

17. Prove that for every positive integer n there exists a unique ordered pair (a,b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2) + a.$$

18. a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$

b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that

$$n < l^2 < (1 + \frac{1}{2005})n.$$

19. Let n be a natural number such that n = $a^2 + b^2 + c^2$, for some natural numbers a,b,c. Prove that

$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2$$

where $p_j's, q_j's, r_j's$ are all nonzero integers. Further, if 3 does not divide at least one of a,b,c, prove that 9n can be expresses $x^2+y^2+z^2$, where z,y,z are natural numbers none of which is divisible by 3.

20. Let m and n be a positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if [mα] + [mβ] is the square of an integer.

21. Let x,y,z are positive real numbers, Prove that

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

22. Define a sequence $\sum_{n=1}^{\infty}$ as follows:

$$a_n = \{0, ifthenumberofpositivedivisorsofnisodd, \\ 1, ifthenumberofpositivedivisorsofniseven\}$$

Let $x = 0.a_1a_2a_3....$ be the real number whose decimal expansion contains a_n in the n^{th} place $n \geq 1$. Determine with proof whether x is a rational or irrational.

23. Find all real numbers x such that

$$[x^2 + 2x] = [x^2] + 2[x]$$

24. Let a,b,c be positive real numbers such that $a^3 + b^3 + c^3$. Prove that

$$a^2 + b^2 + c^2 > 6(c-a)(c-b)$$

25. Call a natural number n faithful, if there exists natural numbers $a < b < c$ such that a divides b, b divides c and n = a + b + c.

a) Show that all but a finite number of natural numbers are faithful.
b) Find the sum of all natural numbers which are not faithful.

26. Consider two polynomials

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_1x + a_0 and$$
$$Q(x) = b_nx^n + b_{n-1}x^{n-1} + ...b_1x + b_0$$

with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_nb_0 - a_0b_n \neq 0$. Suppose there exists a rational number r such that P(r) = Q(r) = 0. Prove that r is an integer.

27. Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

28. Let a,b,c,x,y,z be positive real numbers such that a + b + c = x + y + z and abc = xyz. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that a=x, b=y and c=z.

29. Let a,b,c,d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation

$$x^4 - ax^3 - bx^2 - cx - d = 0$$

has no integer solution.

30. Find all positive integers m, n and primes $p \geq 5$ such that

$$m(4m^2 + m + 12) = 3(p^n - 1).$$

31. Let n be a natural number. Prove that
$$[\frac{1}{n}] + [\frac{n}{2}] + [\frac{n}{3}] + ...[\frac{n}{n}] + [\sqrt{n}]$$
is even.
32. Let a,b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by a+b. Prove that the quotient is at most (a+b)/4. When is this quotient exactly equal to (a+b)/4?
33. Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternatively in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.
34. From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that
$$a^2 + b^2 + c^2 = d^2 + e^2 + f^2.$$
35. For any natural number $n > 1$, write the infinite decimal expansion of 1/n (for example, we write 1/2 = 0.49 its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of $\frac{1}{n}$.
36. For positive real numbers a, b, c, which of the following statements necessarily implies a = b = c:
   a) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$,
   b) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$? Justify your answer.
37. Consider a nonconstant arithmetic progression $a_1, a_2, ..., a_n$, Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.
38. Suppose n ≥ 0 is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.
39. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set {1, 2, 3, 4, 5, 6, 7, 8, 9} such that $x^2 - a(x) + b = 0$, where [x] denotes the fractional part of the real number x. (For example (1.1) = 0.1 = (-0.9).)
40. Let n ≥ 1 be an integer and consider the sum
$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^{n-2}.3 + \binom{n}{4} 2^{n-4} 3^2 + ...$$
Show that 2x - 1, 2x, 2x + 1 form the sides of a triangle whose area and inradius are also integers.