JEE Problems in Discrete Mathematics

***Abstract*—A collection of problems from JEE papers related to matrices are available in this document. Verify your soluions using C.**

1 Signal Processing: Z Transform

1.1 Let

$$a(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n) \tag{1}$$
$$b(n) = a(n-1) + a(n+1) - \delta(n) \tag{2}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{3}$$

and

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{4}$$

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{5}$$

1.2 Verify your results through a C program.

1.3 Show that the Z transform of $u(n)$

$$U(z) \triangleq \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{6}$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{7}$$

1.4 Show that

$$A(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \tag{8}$$

1.5 Let

$$y(n) = a(n) * u(n) \triangleq \sum_{k=-\infty}^{\infty} a(k) u(n-k) \tag{9}$$

Show that

$$y(n) = \sum_{k=0}^{n} a(k) \tag{10}$$

1.6 Show that

$$Y(z) = A(z)U(z) \tag{11}$$
$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{12}$$

1.7 Show that

$$w(n) = \left[a(n+2) - 1\right] u(n-1) \tag{13}$$
$$= a(n+2) - u(n+1) + 2\delta(n) \tag{14}$$

1.8 Is $W(z) = Y(z)$?

1.9 Verify if

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} = \frac{10}{89} \tag{15}$$

1.10 Verify if

$$\sum_{n=1}^{\infty} \frac{b(n)}{10^n} = \frac{8}{89} \tag{16}$$

2 Algebra: Modular Arithmetic

Let $AP\,(a; d)$ denote an A.P. with $d > 0$

2.1 Express $AP\,(a; d)$ in modulo arithmetic.
Solution:

$$A \equiv a \pmod{d} \tag{17}$$

2.2 Express the intersection of $AP\,(1; 3)\,, AP\,(2; 5)$ and $AP\,(3; 7)$ using modulo arithmetic.
Solution: The desired AP can be expressed as

$$A \equiv 1 \pmod{3} \tag{18}$$
$$\equiv 2 \pmod{5} \tag{19}$$
$$\equiv 3 \pmod{7} \tag{20}$$

2.3 Two numbers are said to be coprime if their greatest common divisor (gcd) is 1. Verify if (3,5), (5,7) and (3,7) are pairwise coprime.

2.4 Does a solution for (18) exist?
Solution: The Chinese remainder theorem

guarantees that the system in (18) has a solution since 3,5,7 are pairwise coprime.

2.5 Simplify

$$(7 \times 5) \pmod 3 \tag{21}$$

Solution: (21) can be expressed as

$$\begin{aligned}(7 \times 5) \pmod 3 &= 35 \pmod 3 \\ &= 2 \pmod 3 \end{aligned}\tag{22}$$

2.6 Find x in

$$2x = 1 \pmod 3 \tag{23}$$

Solution: By inspection, for $x = 2$,

$$2x = 2 \times 2 = 4 = 3 + 1 = 1 \pmod 3 \tag{24}$$

Thus $x = 2$ is a solution of (23).

2.7 In general, x in

$$ax = 1 \pmod d \tag{25}$$

is defined to be the modular multiplicative inverse of (17).

2.8 Show that the multiplicative inverse of

$$(3 \times 5) \pmod 7 = y = 1 \tag{26}$$

2.9 Show that the multiplicative inverse of

$$(3 \times 7) \pmod 5 = z = 1 \tag{27}$$

2.10 Find $a + d$.
Solution:

$$\begin{aligned}&(5 \times 7 \times 1 \times x) + (3 \times 5 \times 3 \times y) \\ &\quad + (3 \times 7 \times 2 \times z) = 157\end{aligned}\tag{28}$$

2.11 Find a and d.
Solution:

$$d = LCM\,(3, 5, 7) = 105 \tag{29}$$

$$\begin{aligned}A &= 157 \pmod{105} \\ &= 52 \pmod{105} \\ \implies a &= 52\end{aligned}\tag{30}$$

2.12 Given the APs

$$a_1 \pmod{d_1} \tag{31}$$

$$a_2 \pmod{d_2} \tag{32}$$

$$a_3 \pmod{d_3}, \tag{33}$$

such that

$$gcd(d_1, d_2) = gcd(d_2, d_3) = gcd(d_3, d_1) = 1, \tag{34}$$

show that their intersection

$$a \pmod d \tag{35}$$

is obtained through

$$\begin{aligned}&a + d = \\ &\quad (d_1 \times d_2 \times a_3 \times x) + (d_2 \times d_3 \times a_1 \times y) \\ &\quad\quad + (d_3 \times d_1 \times a_2 \times z)\end{aligned}\tag{36}$$

$$d = LCM\,(d_1, d_2, d_3)\,, \tag{37}$$

where x, y, z are the modular multiplicative inverses given by

$$x = [(d_1 \times d_2) \pmod{d_3}]^{-1} \tag{38}$$

$$y = [(d_2 \times d_3) \pmod{d_1}]^{-1} \tag{39}$$

$$z = [(d_3 \times d_1) \pmod{d_2}]^{-1} \tag{40}$$

respectively.

2.13 Write a C program to find x, y and z.

3 Discrete Fourier Transform

3.1 Show that

$$\sum_{k=0}^{n-1} e^{\jmath \frac{2\pi k}{n}} = \begin{cases} 1 & n = 1, \\ 0 & n > 1 \end{cases} \tag{41}$$

3.2 Show that

$$\sum_{k=0}^{n} \cos\left(\frac{2k + r}{n + 2}\pi\right) = -\cos\left(\frac{r - 2}{n + 2}\pi\right) \tag{42}$$

Solution: From (41),

$$\begin{aligned}&\sum_{k=0}^{n+1} e^{\jmath \frac{2k+r}{n+2}\pi} = 0 \\ \implies &\sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} + e^{\jmath \frac{2(n+1)+r}{n+2}\pi} = 0 \\ \implies &\sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} = -e^{\jmath \frac{2(n+2)+r-2}{n+2}\pi} \\ &= -e^{\jmath \frac{r-2}{n+2}\pi}\end{aligned}\tag{43}$$

Taking the real part on both sides yields (42).

3.3 Show that

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+2}{n+2}\pi\right)} \tag{44}$$

$$=\frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)}{n+\cos\left(\frac{2\pi}{n+2}\right)} \tag{45}$$

Solution: Let

$$\theta_n=\frac{\pi}{n+2} \tag{46}$$

$$\because \sin\left\{(k+1)\theta_n\right\}\sin\left\{(k+2)\theta_n\right\},$$

$$=\frac{1}{2}\left[\cos\theta_n-\cos\left\{(2k+3)\theta_n\right\}\right] \tag{47}$$

from (44) and (42),

$$f(n)=\frac{n\cos\theta_n-\sum_{k=0}^{n}\cos\left\{(2k+3)\theta_n\right\}}{n-\sum_{k=0}^{n}\cos\left\{(2k+4)\theta_n\right\}}$$

$$=\frac{n\cos\left(\frac{\pi}{n+2}\right)+\cos\left(\frac{\pi}{n+2}\right)}{n+\cos\left(\frac{2\pi}{n+2}\right)} \tag{48}$$

resulting in (45). Verify if

3.4

$$f(4)=\frac{\sqrt{3}}{2} \tag{49}$$

3.5

$$\lim_{n\to\infty}f(n)=\frac{1}{2} \tag{50}$$

3.6

$$\sin\left(7\cos^{-1}f(5)\right)=0 \tag{51}$$

3.7 If

$$\alpha=\tan\left(\cos^{-1}f(6)\right) \tag{52}$$

verify if

$$\alpha^2+2\alpha-1=0 \tag{53}$$

4 Combinatorics

4.1 Find

$$\sum_{k=0}^{n}k \tag{54}$$

Solution: (54) can be expressed as

$$\frac{n(n+1)}{2} \tag{55}$$

4.2 Find

$$\sum_{k=0}^{n}{}^{n}C_k k^2 \tag{56}$$

Solution:

$$(1+x)^n=\sum_{k=0}^{n}{}^{n}C_k x^k \tag{57}$$

$$\implies n(1+x)^{n-1}=\sum_{k=0}^{n}k\,{}^{n}C_k x^{k-1} \tag{58}$$

upon differentiation. Multiplying (58) by x and differentiating,

$$\frac{d}{dx}\left[nx(1+x)^{n-1}\right]=\sum_{k=0}^{n}k^2\,{}^{n}C_k x^{k-1} \tag{59}$$

$$\implies n(n-1)x(1+x)^{n-2}+n(1+x)^{n-1}$$
$$=\sum_{k=0}^{n}k^2\,{}^{n}C_k x^{k-1} \tag{60}$$

Substituting $x=1$ in (60),

$$\sum_{k=0}^{n}{}^{n}C_k k^2=n(n-1)2^{n-2}+n2^{n-1}$$
$$=n(n+1)2^{n-2} \tag{61}$$

4.3 Find

$$\sum_{k=0}^{n}{}^{n}C_k k \tag{62}$$

Solution: Substituting $x=1$ in (58),

$$\sum_{k=0}^{n}{}^{n}C_k k=n2^{n-1} \tag{63}$$

4.4 Find

$$\sum_{k=0}^{n}{}^{n}C_k 3^k \tag{64}$$

Solution: Substituting $x=2$ in (57),

$$\sum_{k=0}^{n}{}^{n}C_k 3^k=4^n \tag{65}$$

4.5 If

$$\begin{vmatrix}\frac{n(n+1)}{2} & n(n+1)2^{n-2}\\ n2^{n-1} & 4^n\end{vmatrix}=0 \tag{66}$$

for some n, find

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{67}$$

Solution: (66) can be expressed as

$$n(n+1)2^{2n-3} \begin{vmatrix} 1 & 1 \\ n & 4 \end{vmatrix} = 0 \tag{68}$$

$$\implies n = 4 \tag{69}$$

Integrating (57) from 0 to 1,

$$\frac{2^{n+1}}{n+1} = \sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{70}$$

Substituting $n = 4$ in the above,

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} = \frac{2^5 - 1}{5} = \frac{31}{5} \tag{71}$$

5 Probability

Table 5 lists the number of red (R) and green (G) balls in bags B_1, B_2 and B_3. Also listed are the probabilities of each bag.

Bag	**R**	**G**	**Probability**
B_1	5	5	$\Pr(B_1) = \frac{3}{10}$
B_2	3	5	$\Pr(B_2) = \frac{3}{10}$
B_3	5	3	$\Pr(B_3) = \frac{4}{10}$

TABLE 5

5.1 Show that

$$\Pr(G|B_3) = \frac{3}{8} \tag{72}$$

5.2 Show that

$$\Pr(G) = \frac{39}{80} \tag{73}$$

Solution:

$$\because \Pr(G|B_1) = \frac{1}{2}, \Pr(G|B_2) = \frac{5}{8}, \Pr(G|B_3) = \frac{3}{8},$$

$$\Pr(G) = \sum_{i=1}^{3} \Pr(G|B_i) \Pr(B_i) \tag{74}$$

$$= \frac{1}{2} \times \frac{3}{10} + \frac{5}{8} \times \frac{3}{10} + \frac{3}{8} \times \frac{4}{10} \tag{75}$$

$$= \frac{39}{80} \tag{76}$$

5.3 Is

$$\Pr(B_3|G) = \frac{5}{13}\,? \tag{77}$$

Solution:

$$\Pr(B_3|G) = \frac{\Pr(G|B_3)\Pr(B_3)}{\Pr(G)} \tag{78}$$

$$= \frac{\frac{3}{8} \times \frac{4}{10}}{\frac{39}{80}} = \frac{4}{13} \neq \frac{5}{13} \tag{79}$$

5.4 Is

$$\Pr(B_3 \cap G) = \frac{3}{10}\,? \tag{80}$$

Solution:

$$\Pr(B_3 \cap G) = \Pr(G|B_3)\Pr(B_3) \tag{81}$$

$$\tag{82}$$

$$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20} \neq \frac{3}{10} \tag{83}$$

6 Combinatorics

6.1 Five persons A, B, C, D and E are seated in a circuar arrangment. Let A, C, E be given the colour green. Find the number of ways to distribute blue and red to B and D.
Solution: Both B and D can be given either blue or red. The number of possible ways is

$$2 \times 2 = 4 \tag{84}$$

6.2 Repeat the above exercise with B, D, A having the colour green.

6.3 Find the total number of ways in which the colour green can be distributed to alternate persons so that persons seated in adjacent seats get different coloured hats.
Solution: The number of such ways is

$$2 \times 2 \times 2 = 8 \tag{85}$$

6.4 If each person is given a hat of one of 3 colours red, blue and green, then find the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats.
Solution: The number of ways is

$$2 \times 2 \times 2 \times 3 = 24 \tag{86}$$