The Straight Line

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***Abstract*—Solved problems from JEE mains papers related to 2D lines in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.1}$$
$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.2}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{1.3}$$

Then (1.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.4}$$
$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.6}$$
$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.7}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \tag{1.8}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.9}$$
$$\mathbf{A} = -k\mathbf{B} \tag{1.10}$$

Substituting in (1.6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{1.11}$$
$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{1.12}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.13}$$

2 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{2.2}$$

by $2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (2.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \tag{2.4}$$

However,

$$PQ = 2\sqrt{3} \tag{2.5}$$
$$\implies \left\| \mathbf{P} - \mathbf{Q} \right\| = |\lambda| \left\| \mathbf{m} \right\| = 2\sqrt{3} \tag{2.6}$$
$$\implies \lambda = \pm \frac{2\sqrt{3}}{\left\| \mathbf{m} \right\|} = \pm\sqrt{6} \tag{2.7}$$

$$\because \left\| \mathbf{m} \right\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{2.8}$$

from (2.3). Since $\mathbf{Q}$ lies in the third quadrant,

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from (2.4) and (2.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{2.9}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \tag{2.10}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - \sqrt{6} \tag{2.11}$$

3 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{3.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{3.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (3.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{3.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{3.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{3.6}$$

$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{3.7}$$

Let the equation of AB be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \tag{3.8}$$

Since **R** lies on AB,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \tag{3.9}$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.10}$$

Substituting from (3.6) in (3.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \tag{3.11}$$

From (3.9) and (3.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \tag{3.12}$$

for some constant k. Multiplying both sides of (3.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.13}$$

$$\begin{aligned} \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{3.14}$$

$$\because \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{3.15}$$

which can be easily verified for any **R**. from (3.14),

$$\begin{aligned} &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0 \\ \implies &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0 \\ \implies &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{3.16}$$

which is the desired locus.

4 Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{4.1}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{4.2}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{4.3}$$

find its vertices.

Solution: From (4.1) and (4.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{4.4}$$

By row reducing the augmented matrix

$$\begin{pmatrix}1 & -1 & -1\\ 7 & -1 & 5\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & -1\\ 0 & 6 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & -1\\ 0 & 1 & 2\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{pmatrix} \implies \mathbf{A} = \begin{pmatrix}1\\ 2\end{pmatrix}, \tag{4.5}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A}+\mathbf{C}}{2}$$

$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix}-3\\ -6\end{pmatrix} \tag{4.6}$$

$$\because AD \parallel BC,$$

$$BC : \begin{pmatrix}7 & -1\end{pmatrix}(\mathbf{x} - \mathbf{C}) = 0$$

$$\implies \begin{pmatrix}7 & -1\end{pmatrix}\mathbf{x} = -15 \tag{4.7}$$

From (4.1) and (4.7),

$$\begin{pmatrix}7 & -1\\ 1 & -1\end{pmatrix}\mathbf{B} = \begin{pmatrix}-15\\ -1\end{pmatrix} \tag{4.8}$$

resulting in the augmented matrix

$$\begin{pmatrix}7 & -1 & -15\\ 1 & -1 & -1\end{pmatrix} \leftrightarrow \begin{pmatrix}7 & -1 & -15\\ 0 & 3 & -4\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}21 & 0 & -17\\ 0 & 3 & -4\end{pmatrix} \implies \mathbf{B} = -\frac{1}{21}\begin{pmatrix}17\\ 28\end{pmatrix} \tag{4.9}$$

$$\because AB \parallel CD,$$

$$CD : \begin{pmatrix}1 & -1\end{pmatrix}(\mathbf{x} - \mathbf{C}) = 0$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3 \tag{4.10}$$

From (4.2) and (4.10),

$$\begin{pmatrix}7 & -1\\ 1 & -1\end{pmatrix}\mathbf{D} = \begin{pmatrix}5\\ 3\end{pmatrix} \tag{4.11}$$

resulting in the augmented matrix

$$\begin{pmatrix}7 & -1 & 5\\ 1 & -1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}7 & -1 & 5\\ 0 & 3 & -8\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}3 & 0 & 1\\ 0 & 3 & -8\end{pmatrix} \implies \mathbf{D} = \frac{1}{3}\begin{pmatrix}1\\ -8\end{pmatrix} \tag{4.12}$$

5 Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix}k\\ -3k\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\ k\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-k\\ 2\end{pmatrix} \tag{5.1}$$

has area 28. Find the orthocentre of this triangle.

Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix}5+k\\ k-2\end{pmatrix}, \tag{5.2}$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T(\mathbf{x} - \mathbf{A}) = 0 \tag{5.3}$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T(\mathbf{x} - \mathbf{B}) = 0 \tag{5.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix}2k\\ -2-3k\end{pmatrix}, \tag{5.5}$$

The orthocentre is obtained by solving (5.3) and (5.4) using the matrix equation

$$\begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\end{pmatrix}^T \mathbf{x} = \begin{pmatrix}\mathbf{m}_1^T\mathbf{A}\\ \mathbf{m}_2^T\mathbf{B}\end{pmatrix} \tag{5.6}$$

which can be expressed using (5.2), (5.5), (5.3) and (5.4) as

$$\begin{pmatrix}5+k & k-2\\ 2k & -2-3k\end{pmatrix}\mathbf{x} = \begin{pmatrix}k^2+5k+6k-3k^2\\ 10k-2k-3k^2\end{pmatrix}$$

$$= k\begin{pmatrix}11-4k\\ 8-3k\end{pmatrix} \tag{5.7}$$

From (5.1), using the expression for the area of triangle,

$$\begin{vmatrix}k & 5 & -k\\ -3k & k & 2\\ 1 & 1 & 1\end{vmatrix} = 56$$

$$\implies \begin{vmatrix}k & 5-k & -2k\\ -3k & 4k & 2+3k\\ 1 & 0 & 0\end{vmatrix} = 56 \tag{5.8}$$

resulting in

$$(5-k)(2+3k) + 8k^2 = 56 \tag{5.9}$$

$$\implies 5k^2 + 13k - 46 = 0 \tag{5.10}$$

$$\text{or, } k = 2, -\frac{23}{5} \tag{5.11}$$

Substituting the above in (5.7) and solving yields the orthocentre.

6 If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2, \tag{6.1}$$

then find the area of this triangle.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0} \tag{6.2}$$

If AB be the line in (6.1), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{6.3}$$

is

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{6.4}$$

since CF passes through the origin and $CF \perp AB$. From (6.1) and (6.4),

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathbf{F} = \begin{pmatrix}2\\0\end{pmatrix} \tag{6.5}$$

Forming the augmented matrix,

$$\begin{pmatrix}1 & 1 & 2\\ 1 & -1 & 0\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} \implies \mathbf{F} = \begin{pmatrix}1\\1\end{pmatrix} \tag{6.6}$$

From (6.2),

$$\mathbf{C} = -(\mathbf{A}+\mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix}1\\1\end{pmatrix} \tag{6.7}$$

after substituting from (6.6). Thus,

$$CF = \|\mathbf{C}-\mathbf{F}\| = 3\sqrt{2} \tag{6.8}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = \sqrt{6} \tag{6.9}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 3\sqrt{3} \tag{6.10}$$

7 A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.

Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and y-axis respectively. Then

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D} = 4\begin{pmatrix}1\\1\end{pmatrix} \tag{7.1}$$

Multiplying (7.1) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{7.2}$$

$$\mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}) = 4\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$
$$= 4\begin{pmatrix}\cos\theta - \sin\theta\\ \cos\theta + \sin\theta\end{pmatrix} \tag{7.3}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{T}(\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D})$$
$$= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3}-1\right) \tag{7.4}$$

for $\theta = 30^\circ$.