Discrete Maths: Maths Olympiad

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Abstract—**This book provides a collection of the international maths olympiad problems in discrete mathematics.**

1. Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.
2. Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$.In what case does equality hold?
3. In an n-gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation
$$a_1 \geq a_2 \geq ...... \geq a_n. \tag{3.1}$$
Prove that
$$a_1 = a_2 = ... = a_n. \tag{3.2}$$
4. Five students, A, B, C, D, E, took part in a contest. One prediction was that the contestants would finish in the order ABCDE. This prediction was very poor. In fact no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order DAECB. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.
5. a) Find all positive integers n for which 2^n - 1 is divisible by 7.
   b) Prove that there is no positive integer n for which 2^n + 1 is divisible by 7.
6. Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.
7. Prove that for every natural number n, and for every real number x≠ $\frac{k\pi}{2^t}$(t = 0,1,..., n; k any integer)
$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + ...\frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$
8. In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and $\frac{1}{7}$ of the remaining m-1 medals were awarded. On the second day, two medals and $\frac{1}{7}$ of the now remaining medals were awarded; and so on. On the n^{th} and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?
9. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation
$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2} \tag{9.1}$$
holds for all x.
   a) Prove that the function f is periodic (i.e., there exists a positive number b such that f(x + b) = f(x) for all x).
   b) For a = 1, give an example of a non-constant function with the required properties.
10. For every natural number n, evaluate the sum
$\sum_{k=1}^{\infty}[\frac{n+2^k}{2^{k+1}}] = [\frac{n+1}{2}] + [\frac{n+2}{4}]...........[\frac{n+2^k}{2^{k+1}}]+........$
(The symbol [x] denotes the greatest integer not exceeding x.)
11. For each value of k = 1, 2, 3, 4, 5, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a, and the remaining 6-k edges of length 1.

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12. Let $a_1, a_2, ..., a_n$ be real constants, x a real variable, and

$f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + ... + \frac{1}{2^n - 1}\cos(a_n + x)$.

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

13. In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

14. Find the set of all positive integers n with the property that the set [n, n + 1, n + 2, n + 3, n + 4, n + 5] can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.
15. Prove that the set of integers of the form $2^k - 3$ (k = 2, 3,....) contains an infinite subset in which every two members are relatively prime.

16. Let A = (a_{ij})(i, j = 1, 2,....., n) be a square matrix whose elements are non negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the i^{th} row and the j^{th} column is $\geq$ n. Prove that the sum of all the elements of the matrix is $\geq \frac{n^2}{2}$
17. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

18. Let m and n be arbitrary non-negative integers. Prove that

$\frac{(2m!)(2n!)}{m!n!(m+n)!}$

is an integer(0! = 1)

19. Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x + y) + f(x - y) = 2f(x)g(y) \qquad (19.1)$$

for all x, y. Prove that if f(x) is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.
20. A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path should he follow in order to travel the least possible distance and still accomplish his mission?

21. G is a set of non-constant functions of the real variable x of the form f(x) = ax + b, a and b are real numbers, and G has the following properties:
   a) If f and g are in G, then $g \circ f$ is in G; here $(g \circ f)$(x) = g[f(x)].
   b) If f is in G, then its inverse f^{-1} is in G; here the inverse of f(x) = ax +b is $f^{-1}(x) = \frac{(x-b)}{a}$.
   c) For every f in G, there exists a real number x_f such that $f(x_f) = x_f$. Prove that there exists a real number k such that f(k) = k for all f in G.
22. Three players A, B and C play the following game: On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players. This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

23. Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.
24. Consider decompositions of an 8 × 8 chessboard into p non-overlapping rectangles subject to the following conditions:
25. Each rectangle has as many white squares as black squares.
26. If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < ..... < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p,

determine all possible sequences $a_1, a_2, ..., a_p$.

27. Let x_i, y_i (i = 1, 2,....., n) be real numbers such that

$$x_1 \geq x_2 \geq ...... \geq x_n \text{ and } y_1 \geq y_2 \geq ...... \geq y_n.$$

Prove that, if $z_1, z_2, ..., z_n$ is any permutation of $y_1, y_2, ..., y_n$, then $\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$

28. Let $a_1, a_2, a_3, ...$ be an infinite increasing sequence of positive integers. Prove that for every p ≥ 1 there are infinitely many a_m which can be written in the form $a_m = xa_p + ya_q$ with x, y positive integers and $q > p$.
29. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)
30. Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.
31. A sequence $[u_n]$ is defined by $u_0 = 2$, $u_1 = \frac{5}{2}$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$for n = 1,2,.....
Prove that for positive integers n, $[u_n] = 2^{\frac{[2^n-(-1)^n]}{3}}$ where [x] denotes the greatest integer ≤ x.
32. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.
33. Let n be a given integer > 2, and let V_n be the set of integers 1 + kn, where k = 1, 2,..... A number m ∈ V_n is called indecomposable in V_n if there do not exist numbers p, q ∈ V_n such that pq = m. Prove that there exists a number r ∈ V_n that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)

34. Let f(n) be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then
f(n) = n for each n.

35. Let S = 1, 2, 3,...., 280. Find the smallest integer n such that each nelement subset of S contains five numbers which are pairwise relatively prime.

36. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.
37. Let R denote the set of all real numbers. Find all functions f : $R \to R$ such that

$f(x^2 + f(y)) = y + (f(x))^2$ for all x,y ∈R.

38. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$\left|S^2\right| \leq \left|S_x\right| . \left|S_y\right| . \left|S_z\right|$$

where $\left|A\right|$ denotes the number of elements in the finite set $\left|A\right|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

39. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.
40. Does there exist a function f : $N \to N$ such that f(1) = 2, f(f(n)) = f(n) + n for all $n \in N$, and $f(n) < f(n+1)$ for all $n \in N$?
41. For any positive integer k, let f(k) be the number of elements in the set k+1, k+2, ... , 2k whose base 2 representation has precisely three 1s.

a) Prove that, for each positive integer m, there exists at least one positive integer k such that f(k) = m.

b) Determine all positive integers m for which there exists exactly one k with f(k) = m.

42. Let S be the set of real numbers strictly greater than -1. Find all functions f : $S \to S$ satisfying the two conditions:

a) f(x+f(y)+xf(y)) = y+f(x)+yf(x) for all x and y in S;
b) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$

43. Determine all integers $n > 3$ for which there exist n points $A_1, ..., A_n$ in the plane, no three collinear, and real numbers $r_1, ..., r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

44. Let p be an odd prime number. How many p-element subsets A of 1, 2, . . . 2p are there, the sum of whose elements is divisible by p?

45. Let S denote the set of non-negative integers. Find all functions f from S to itself such that

f(m + f(n)) = f(f(m)) + f(n) $\forall m, n \in S$.

46. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, ..., x_n)$ be an (n + 1) - tuple of integers satisfying the following conditions:

a) $x_0 = x_n = 0$.
b) For each i with $1 \leq i \leq n$, either $x_i - x_i - 1 = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

47. An n x n matrix whose entries come from the set S = 1, 2, ... , 2n - 1 is called a silver matrix if, for each i = 1, 2, ... , n, the ith row and the ith column together contain all elements of S. Show that

a) there is no silver matrix for n = 1997;
b) silver matrices exist for infinitely many values of n.

48. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that k/a ≥ (b - 1)/(2b).

49. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of f(1998).

50. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

51. Let n be a fixed integer, with $n \geq 2$.

a) Determine the least constant C such that the inequality

$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C(\sum_{1 \leq i \leq n} x_i)^4$

holds for all real numbers $x_1, ..., x_n \geq 0$

b) For this constant C, determine when equality holds.

52. Determine all functions f : $R \to R$ such that

f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 for all real numbers x, y.

53. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

54. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

55. Twenty-one girls and twenty-one boys took part in a mathematical contest.

a) Each contestant solved at most six problems.
b) For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

56. Let n be an odd integer greater than 1, and let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations a = $(a_1, a_2, \ldots, a_n)$ of 1 , 2, $\ldots$, n, let

$S(a) = \sum_{i=1}^{n} k_i a_i$.

Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of S(b)-S(c).

57. Find all real-valued functions on the reals such that (f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu) for all x, y, u, v.

58. S is the set 1, 2, 3, . . . , 1000000. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $a + x_i/a \in A$ are all pairwise disjoint.

59. Given $n > 2$ and reals $x_1 \leq x_2 \leq .... \leq x_n$, show that $(\sum_{i,j} \left|x_i - x_j\right|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

60. We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.