Discrete Maths: Maths Olympiad

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Abstract—**This book provides a collection of the international maths olympiad problems in discrete mathematics.**

1. Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.
2. Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}T$.In what case does equality hold?
3. Find the smallest natural number n which has the following properties:
   a) Its decimal representation has 6 as the last digit.
   b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.
4. In an n-gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation
$$a_1 \geq a_2 \geq ...... \geq a_n. \tag{4.1}$$
Prove that
$$a_1 = a_2 = ... = a_n. \tag{4.2}$$
5. Five students, A, B, C, D, E, took part in a contest. One prediction was that the contestants would finish in the order ABCDE. This prediction was very poor. In fact no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order DAECB. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.
6. a) Find all positive integers n for which 2^n - 1 is divisible by 7.
   b) Prove that there is no positive integer n for which 2^n + 1 is divisible by 7.
7. Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.
8. Prove that for every natural number n, and for every real number x≠ $\frac{k\pi}{2^t}$(t = 0,1,..., n; k any integer)
$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + ...\frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$
9. In a sports contest, there were m medals awarded on n successive days $(n > 1)$. On the first day, one medal and $\frac{1}{7}$ of the remaining m-1 medals were awarded. On the second day, two medals and $\frac{1}{7}$ of the now remaining medals were awarded; and so on. On the n^{th} and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?
10. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation
$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2} \tag{10.1}$$
holds for all x.
    a) Prove that the function f is periodic (i.e., there exists a positive number b such that f(x + b) = f(x) for all x).
    b) For a = 1, give an example of a non-constant function with the required properties.
11. For every natural number n, evaluate the sum
$$\textstyle\sum_{k=1}^{\infty}[\frac{n+2^k}{2^{k+1}}] = [\frac{n+1}{2}] + [\frac{n+2}{4}]..........[\frac{n+2^k}{2^{k+1}}]+........$$

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(The symbol [x] denotes the greatest integer not exceeding x.)

12. For each value of k = 1, 2, 3, 4, 5, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a, and the remaining 6-k edges of length 1.

13. Let $a_1, a_2, ..., a_n$ be real constants, x a real variable, and

$f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + ... + \frac{1}{2^{n-1}}\cos(a_n + x)$.

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

14. In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

15. Find the set of all positive integers n with the property that the set [n, n + 1, n + 2, n + 3, n + 4, n + 5] can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

16. Prove that the set of integers of the form $2^k - 3$ (k = 2, 3,....) contains an infinite subset in which every two members are relatively prime.

17. Let A = (a_{ij})(i, j = 1, 2,....., n) be a square matrix whose elements are non negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the i^{th} row and the j^{th} column is ≥ n. Prove that the sum of all the elements of the matrix is $\geq \frac{n^2}{2}$

18. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

19. Let m and n be arbitrary non-negative integers. Prove that

$\frac{(2m!)(2n!)}{m!n!(m+n)!}$

is an integer(0! = 1)

20. Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x + y) + f(x - y) = 2f(x)g(y) \qquad (20.1)$$

for all x, y. Prove that if f(x) is not identically zero, and if $\left|f(x)\right| \leq 1$ for all x, then $\left|g(y)\right| \leq 1$ for all y.

21. A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path should he follow in order to travel the least possible distance and still accomplish his mission?

22. G is a set of non-constant functions of the real variable x of the form f(x) = ax + b, a and b are real numbers, and G has the following properties:
   a) If f and g are in G, then $g \circ f$ is in G; here $(g \circ f)$(x) = g[f(x)].
   b) If f is in G, then its inverse f^{-1} is in G; here the inverse of f(x) = ax +b is $f^{-1}(x) = \frac{(x-b)}{a}$.
   c) For every f in G, there exists a real number x_f such that $f(x_f) = x_f$. Prove that there exists a real number k such that f(k) = k for all f in G.

23. Three players A, B and C play the following game: On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players. This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

24. Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

25. Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:
26. Each rectangle has as many white squares as black squares.
27. If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < ..... < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p, determine all possible sequences $a_1, a_2, ..., a_p$.
28. Let x_i, y_i (i = 1, 2,....., n) be real numbers such that

$$x_1 \geq x_2 \geq ...... \geq x_n \text{ and } y_1 \geq y_2 \geq ...... \geq y_n.$$

Prove that, if $z_1, z_2, ..., z_n$ is any permutation of $y_1, y_2, ..., y_n$, then
$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$
29. Let $a_1, a_2, a_3, ...$ be an infinite increasing sequence of positive integers. Prove that for every p $\geq$ 1 there are infinitely many a_m which can be written in the form $a_m = xa_p + ya_q$ with x, y positive integers and $q > p$.
30. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)
31. Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.
32. A sequence $[u_n]$ is defined by $u_0 = 2$, $u_1 = \frac{5}{2}$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$for n = 1,2,.....
Prove that for positive integers n, $[u_n] = 2^{\frac{[2^n-(-1)^n]}{3}}$ where [x] denotes the greatest integer $\leq$ x.
33. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.
34. Let n be a given integer > 2, and let V_n be the set of integers 1 + kn, where k = 1, 2,..... A number m $\in V_n$ is called indecomposable in V_n if there do not exist numbers p, q $\in V_n$ such that pq = m. Prove that there exists a number r $\in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)
35. Let f(n) be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then
f(n) = n for each n.
36. An international society has its members from six different countries. The list of members contains 1978 names, numbered 1, 2,....., 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.
37. The set of all positive integers is the union of two disjoint subsets

$$(f(1), f(2), .., f(n), ..), (g(1), g(2), .., g(n), ..),$$

where

$$f(1) < f(2) < ..... < f(n) < ....,$$
$$g(1) < g(2) < ...... < g(n) < ....,$$

and g(n) = f(f(n)) + 1 for all n $\geq$ 1. Determine f(240).
38. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$,

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), n = 1, 2, 3, ...,$$

where x = 2 + $\sqrt{2}$ and y = 2 - $\sqrt{2}$.
Note. A path of n jumps is a sequence of vertices $(P_0, ..., P_n)$ such that
   a) P_0 = A, P_n = E;
   b) for every i, $0 \leq i \leq n-1, P_i$ is distinct from E;
   c) for every i, $0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.
39. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set (1, 2,...., n). Each of these subsets has a smallest member. Let F (n, r)

denote the arithmetic mean of these smallest numbers; prove that

$$F(n, r) = \frac{n+1}{r+1}$$

40. The function f(x, y) satisfies
 a) f(0, y) = y + 1,
 b) f(x + 1, 0) = f(x, 1),
 c) f(x + 1, y + 1) = f(x, f(x + 1, y)), for all non-negative integers x, y. Determine f(4, 1981).
41. The function f(n) is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n
 f(m + n) - f(m) - f(n) = 0 or 1
 f(2) = 0, $f(3) > 0$, and f(9999) = 3333.
 Determine f(1982).
42. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, ..., A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.
43. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.
44. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
 a) f(xf(y)) = yf(x) for all positive x, y;
 b) f(x) $\to$ 0 as $x \to \infty$
45. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < [\frac{n}{2}][\frac{n+1}{2}] - 2$$

 where [x] denotes the greatest integer not exceeding x.
46. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.
47. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by x + y, -y, z + y respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.
48. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:
 a) f(xf(y))f(y) = f(x + y) for all x, y $\geq$ 0,
 b) f(2) = 0,
 c) f(x) $\neq$ 0 for $0 \leq x < 2$.
49. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?
50. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.
51. Prove that there is no function f from the set of non-negative integers into itself such that f(f(n)) = n + 1987 for every n.
52. Let $p_n(k)$ be the number of permutations of the set (1,........n), $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k.p_n(k) = n!.$$

53. Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

 is a union of disjoint intervals, the sum of whose lengths is 1988.

54. A function f is defined on the positive integers by

$$f(1) = 1, f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n + 1) = 2f(2n + 1) - f(n),$$
$$f(4n + 3) = 3f(2n + 1) - 2f(n)$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, for which f(n) = n.

55. A permutation $(x_1, x_2, ..., x_m)$ of the set (1, 2,........., 2n), where n is a positive integer, is said to have property P if $|x_i - x_{i+1}|$ = n for at least one i in (1,2,......, 2n - 1). Show that, for each n, there are more permutations with property P than without.

56. Prove that the set (1, 2,........, 1989) can be expressed as the disjoint union of subsets A_i (i = 1, 2,......., 117) such that:
   a) Each A_i contains 17 elements;
   b) The sum of all the elements in each A_i is the same.

57. Let $n \geq 3$ and consider a set E of 2n - 1 distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is ”good” if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

58. Let Q^+ be the set of positive rational numbers. Construct a function f : $Q^+ \to Q^+$ such that Q^+ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in Q^+.

59. Let S = 1, 2, 3,...., 280. Find the smallest integer n such that each nelement subset of S contains five numbers which are pairwise relatively prime.

60. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

61. Let R denote the set of all real numbers. Find all functions f : $R \to R$ such that

$f(x^2 + f(y)) = y + (f(x))^2$ for all x,y $\in$R.

62. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$\left|S^2\right| \leq \left|S_x\right| . \left|S_y\right| . \left|S_z\right|$$

where $|A|$ denotes the number of elements in the finite set $|A|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

63. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.

64. Does there exist a function f : $N \to N$ such that f(1) = 2, f(f(n)) = f(n) + n for all $n \in N$, and $f(n) < f(n + 1)$ for all $n \in N$?

65. For any positive integer k, let f(k) be the number of elements in the set k+1, k+2, ... , 2k whose base 2 representation has precisely three 1s.
   a) Prove that, for each positive integer m, there exists at least one positive integer k such that f(k) = m.
   b) Determine all positive integers m for which there exists exactly one k with f(k) = m.

66. Let S be the set of real numbers strictly greater than -1. Find all functions f : $S \to S$ satisfying the two conditions:
   a) f(x+f(y)+xf(y)) = y+f(x)+yf(x) for all x and

y in S;

b) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$

67. Determine all integers $n > 3$ for which there exist n points $A_1, ..., A_n$ in the plane, no three collinear, and real numbers $r_1, ..., r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$.

68. Let p be an odd prime number. How many p-element subsets A of 1, 2, . . . 2p are there, the sum of whose elements is divisible by p?
69. Let S denote the set of non-negative integers. Find all functions f from S to itself such that

f(m + f(n)) = f(f(m)) + f(n) $\forall m, n \in S$.

70. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, ..., x_n)$ be an (n + 1) - tuple of integers satisfying the following conditions:

a) $x_0 = x_n = 0$.
b) For each i with $1 \leq i \leq n$, either $x_i - x_i - 1 = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

71. An n x n matrix whose entries come from the set S = 1, 2, ... , 2n - 1 is called a silver matrix if, for each i = 1, 2, ... , n, the ith row and the ith column together contain all elements of S. Show that

a) there is no silver matrix for n = 1997;
b) silver matrices exist for infinitely many values of n.

72. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that k/a ≥ (b - 1)/(2b).
73. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of f(1998).
74. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

75. Let n be a fixed integer, with $n \geq 2$.

a) Determine the least constant C such that the inequality

$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C (\sum_{1 \leq i \leq n} x_i)^4$

holds for all real numbers $x_1, ..., x_n \geq 0$

b) For this constant C, determine when equality holds.

76. Determine all functions f : $R \rightarrow R$ such that

f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 for all real numbers x, y.

77. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

78. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]
79. Twenty-one girls and twenty-one boys took part in a mathematical contest.

a) Each contestant solved at most six problems.
b) For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

80. Let n be an odd integer greater than 1, and let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations a = $(a_1, a_2, \ldots, a_n)$ of 1 , 2, . . . , n, let

$S(a) = \sum_{i=1}^{n} k_i a_i$.

Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of S(b)-S(c).

81. Find all real-valued functions on the reals such

that (f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu) for all x, y, u, v.

82. S is the set 1, 2, 3, . . . , 1000000. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $a + x_i/a \in A$ are all pairwise disjoint.

83. Given $n > 2$ and reals $x_1 \leq x_2 \leq .... \leq x_n$, show that $(\sum_{i,j} \left|x_i - x_j\right|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

84. We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.

85. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$of the contestants. Moreover, no contestant solved all the 6 problems.Show that there are at least 2 contestants who solved exactly 5 problems each.

86. Determine the least real number M such that the inequality

$$\left|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)\right| \leq M(a^2 + b^2 + c^2)$$

holds for all real numbers a, b and c.

87. Determine all pairs (x, y) of integers such that $1 + 2^x + 2^{2x+1} = y^2$.

88. Let P (x) be a polynomial of degree n > 1 with integer coefficients and let k be a positive integer. Consider the polynomial Q(x) = P (P (. . . P (P (x)) . . .)), where P occurs k times. Prove that there are at most n integers t such that Q(t) = t.

89. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

90. Find all functions f : (0, ∞) → (0,∞) (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying wx = yz.

91. Suppose that $s_1, s_2, s_3, ...$ is a strictly increasing sequence of positive integers such that the sub sequences $s_{s_1}, s_{s_2}, s_{s_3}, ...$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, ...$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, ...$is itself an arithmetic progression.

92. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths a, f(b) and f(b + f(a)-1).
(A triangle is non-degenerate if its vertices are not collinear.)

93. Determine all functions f : $R \to R$ such that the equality

$$f(\lfloor x \rfloor y = f(x) \lfloor f(y) \rfloor$$

holds for all x, y ∈R. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.

94. Let N be the set of positive integers. Determine all functions g : N → N such that (g(m) +n)(m+ g(n)) is a perfect square for all m, n ∈ N.

95. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:
Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.
Type 2: Choose a nonempty box B_k with 1 ≤ k ≤ 4. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

96. Given any set A = $\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i \leq j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

97. Let S be a finite set of at least two points in the plane. Assume that no three points of

S are collinear. A windmill is a process that starts with a line l going through a single point P $\in$ S. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

98. Let f : R $\to$ R be a real valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that f(x) = 0 for all x $\leq$ 0.

99. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference f(m)-f(n) is divisible by f(m-n). Prove that,for all integers m and n with $f(m) \leq f(n)$, the number f(n) is divisible by f(m).

100. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, ..., 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such away that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan,until all of the weights have been placed.Determine the number of ways in which this can be done.

101. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.At the start of the game A chooses integers x and N with 1 $\leq$x $\leq$ N. Player A keeps x secret,and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any k + 1 consecutive answers, at least one answer must be truthful.After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

a) If n $\leq 2^k$, then B can guarantee a win.
b) For all sufficiently large k, there exists an integer $n \leq 1.99^k$ such that B cannot guarantee a win.

102. Find all functions f : Z $\to$ Z such that, for all integers a, b, c that satisfy a + b + c = 0,the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a$$

(Here Z denotes the set of integers.)

103. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

a) no line passes through any point of the configuration;
b) no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

104. Let n $\geq$ 3 be an integer, and consider a circle with n + 1 equally spaced points marked on it. Consider all labellings of these points with the numbers 0, 1, . . . , n such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels a < b < c < d with a + d = b + c, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that x + y $\leq$ n and gcd(x, y) = 1. Prove that M = N + 1.

105. Let n $\geq$ 2 be an integer. Consider an n $\times$ n chessboard consisting of n^2 unit squares.A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks,there is a k $\times$ k square which does not contain a rook on any of its k^2 unit squares.
106. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$.Given a finite collection of such coins (of not necessarily different denominations) with total value at most 99 + $\frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.
107. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.
108. We say that a finite set S of the point in the plane is balanced if,for any two different points A and B in S,there is a point C in S such that AC=BC.We say that S is centre-free if for any three different point A,B,C in S,there is no point P in S such that PA=PB=PC.
    a) Show that for all integers n $\geq$ 3,there exist a balanced set consisting of n points.
    b) Determine all integers n $\geq$ 3 for which there exists a balanced centre-free set consisting of n points.
109. Let R be the set of Real numbers.Determine all functions f:R $\rightarrow$ R satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.
110. Find the positive integer n for which each cell od an n $\times$ n table can be filled with one of the letter I,M and O in such a way that:
    a) In each row and each column,one third of the entires are I,one third are M and one third are O;and
    b) In any diagonal,if the number of entries on the diagonal is a multiple of three,then one third of the entries are I,one third are M and one third are O.

NOTE: The row and column of an n $\times$ n table are each labelled 1 to n in a natural order.Thus each cell corresponding to a pair of positive integers(i, j) with 1 $\leq$ i,j $\leq$ n.For n > 1,the table has 4n-2 diagonlas of two types.A diagonal of first type consists of all cells(i,j) for which i+j is a constant,and a diagonal of the second type consists of all cells (i, j) for which i-j is a constant.
111. Let P=$A_1A_2...A_k$ be a convex polygon in the plane.The vertices $A_1, A_2...,A_k$ have integral coordinates and lie on a circle.Let S be the area of P.An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n.Prove that 2S is an integer divisible by n.
112. The equation

$$(x - 1)(x - 2)....(x - 2016) = (x - 1)(x - 2)....(x - 2016)$$

is written on the board,with 2016 linear factors on each side.What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solution?
113. There are n $\geq$ 2 line segments in the Plane such that every two segments,cross and no three segments meet at a point.Geoff has to choose an endpoint of each segment and place a frog on it,facing the other endpoint.Then he will clap his hand n-1 times.Every time he clap each frog will immediately jump forward to the next intersection point on its segment.Frogs never change the direction of their jumps.Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.
    a) Prove that Geoff can always fulfil his wish of n is odd.
    b) Prove that Geoff can never fulfil his wish of n is even.
114. Let R be the set of real numbers. Determine all functions f : R $\rightarrow$ R such that, for all real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

115. A hunter and an invisible rabbit play a game in

the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After n-1 rounds of the game,the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

a) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
b) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.
c) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 109 rounds she can ensure that the distance between her and the rabbit is at most 100?

116. An integer N $\geq$ 2 is given. A collection of N(N+1) soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove N(N-1) players from this row leaving a new row of 2N players in which the following N conditions hold:

a) no one stands between the two tallest players,
b) no one stands between the third and fourth tallest players,
c) (N) no one stands between the two shortest players.

Show that this is always possible.

117. An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to 1 + 2 +.......+ 2018?

118. A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone. Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.