

Problems in Linear Algebra

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1 THE STRAIGHT LINE

1.1 Point

1. The *inner product* of **P** and **Q** is defined as

$$\mathbf{P}^T \mathbf{Q} = p_1 q_1 + p_2 q_2 \quad (1.1.1)$$

2. The *norm* of a vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (1.1.2)$$

is defined as

$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \quad (1.1.2)$$

3. The *length* of PQ is defined as

$$\|\mathbf{P} - \mathbf{Q}\| \quad (1.1.3)$$

4. The *direction vector* of the line PQ is defined as

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \quad (1.1.4)$$

5. The point dividing PQ in the ratio $k : 1$ is

$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k + 1} \quad (1.1.5)$$

6. The *area* of $\triangle PQR$ is the *determinant*

$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \quad (1.1.6)$$

7. *Orthogonality*: See Fig. 1.1.7. In $\triangle ABC$, $AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.1.7)$$

Solution: Using Baudhayana's theorem,

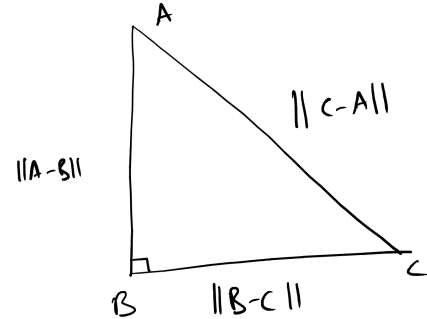


Fig. 1.1.7

$$\begin{aligned} \|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 &= \|\mathbf{C} - \mathbf{A}\|^2 \quad (1.1.7) \\ \Rightarrow (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) \\ &= (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) \\ \Rightarrow 2\mathbf{A}^T \mathbf{B} - 2\mathbf{B}^T \mathbf{B} + 2\mathbf{B}^T \mathbf{C} - 2\mathbf{A}^T \mathbf{C} &= 0 \quad (1.1.7) \end{aligned}$$

which can be simplified to obtain (1.1.7).

8. Let \mathbf{x} be any point on AB in Fig. 1.1.7. Show that

$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.1.8)$$

9. If \mathbf{x}, \mathbf{y} are any two points on AB , show that

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.1.9)$$

1.2 Line

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of OA .

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA . Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \quad (1.2.1.1)$$

$$\Rightarrow x_2 = m x_1 \quad (1.2.1.2)$$

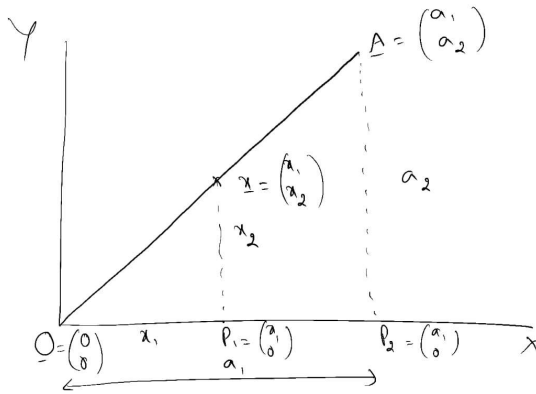


Fig. 1.2.1

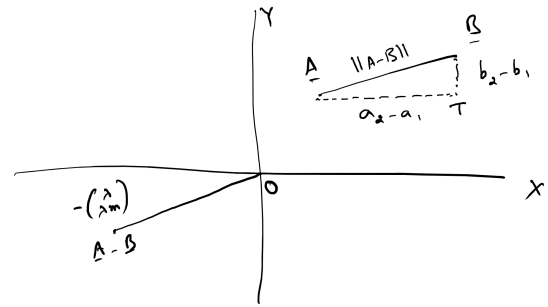


Fig. 1.2.5

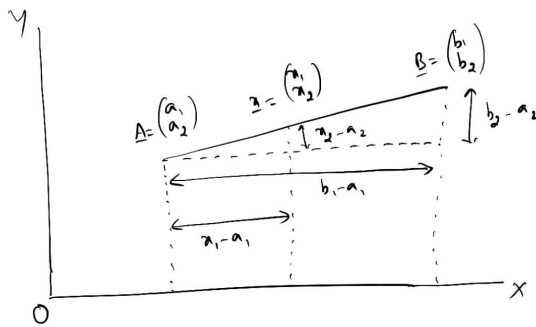


Fig. 1.2.2

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = x_1 \mathbf{m} \quad (1.2.1.3)$$

In general, the above equation is written as

$$\mathbf{x} = \lambda \mathbf{m}, \quad (1.2.1.4)$$

where \mathbf{m} is the direction vector of the line.

2. Find the equation of AB in Fig. 1.2.2

Solution: From Fig. 1.2.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \quad (1.2.2.1)$$

$$\Rightarrow x_2 = mx_1 + a_2 - ma_1 \quad (1.2.2.2)$$

From (1.2.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \quad (1.2.2.3)$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.2.2.4)$$

$$= \mathbf{A} + \lambda \mathbf{m} \quad (1.2.2.5)$$

3. *Translation:* If the line shifts from the origin by \mathbf{A} , (1.2.2.5) is obtained from (1.2.1.4) by adding \mathbf{A} .

4. Find the length of \mathbf{A} in Fig. 1.2.1

Solution: Using Baudhayana's theorem, the length of the vector \mathbf{A} is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \quad (1.2.4.1)$$

Also, from (1.2.1.4),

$$\|\mathbf{A}\| = \lambda \sqrt{1 + m^2} \quad (1.2.4.2)$$

Note that λ is the variable that determines the length of \mathbf{A} , since m is constant for all points on the line.

5. Find $\mathbf{A} - \mathbf{B}$.

Solution: See Fig. 1.2.5. From (1.2.2.5), for some λ ,

$$\mathbf{B} = \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.2.5.1)$$

$$\Rightarrow \mathbf{A} - \mathbf{B} = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (1.2.5.2)$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.

6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

7. Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (1.2.7.1)$$

8. The *normal* to the vector \mathbf{m} is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \quad (1.2.8.1)$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (1.2.8.2)$$

9. From (1.2.7.1), the equation of a line can also

be expressed as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A} + \lambda \mathbf{n}^T (\mathbf{B} - \mathbf{A}) \quad (1.2.9.1)$$

$$\Rightarrow \mathbf{n}^T \mathbf{x} = c \quad (1.2.9.2)$$

10. The unit vectors on the x and y axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1.2.10.1)$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.2.10.2)$$

11. If a be the *intercept* of the line

$$\mathbf{n}^T \mathbf{x} = c \quad (1.2.11.1)$$

on the x -axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line.

Thus,

$$\mathbf{n}^T \begin{pmatrix} a \\ 0 \end{pmatrix} = c \quad (1.2.11.2)$$

$$\Rightarrow a = \frac{c}{\mathbf{n}^T \mathbf{e}_1} \quad (1.2.11.3)$$

12. The *rotation matrix* is defined as

$$\mathbf{Q} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1.2.12)$$

where θ is anti-clockwise.

- 13.

$$\mathbf{Q}^T \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (1.2.13)$$

where \mathbf{I} is the *identity matrix*. The rotation matrix \mathbf{Q} is also an *orthogonal matrix*.

14. Find the equation of line L in Fig. 1.2.14.

Solution: The equation of the x -axis is

$$\mathbf{x} = \lambda \mathbf{e}_1 \quad (1.2.14.1)$$

Translation by p units along the y -axis results in

$$L_0 : \mathbf{x} = \lambda \mathbf{e}_1 + p \mathbf{e}_2 \quad (1.2.14.2)$$

Rotation by $90^\circ - \alpha$ in the anti-clockwise direction yields

$$L : \mathbf{x} = \mathbf{Q} \{ \lambda \mathbf{e}_1 + p \mathbf{e}_2 \} \quad (1.2.14.3)$$

$$= \lambda \mathbf{Q} \mathbf{e}_1 + p \mathbf{Q} \mathbf{e}_2 \quad (1.2.14.4)$$

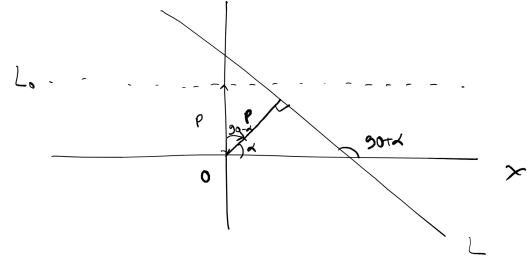


Fig. 1.2.14

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \quad (1.2.14.5)$$

$$= \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix} \quad (1.2.14.6)$$

From (1.2.14.4),

$$\begin{aligned} L : \mathbf{e}_2^T \mathbf{Q}^T \mathbf{x} &= \lambda \mathbf{e}_2^T \mathbf{Q}^T \mathbf{Q} \mathbf{e}_1 + p \mathbf{e}_2^T \mathbf{Q}^T \mathbf{Q} \mathbf{e}_2 \\ &= \lambda \mathbf{e}_2^T \mathbf{e}_1 + p \mathbf{e}_2^T \mathbf{e}_2 \end{aligned} \quad (1.2.14.7)$$

resulting in

$$L : (\cos \alpha \quad \sin \alpha) \mathbf{x} = p \quad (1.2.14.8)$$

15. Show that the distance from the origin to the line

$$\mathbf{n}^T \mathbf{x} = c \quad (1.2.15.1)$$

is

$$p = \frac{c}{\|\mathbf{n}\|} \quad (1.2.15.2)$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.2.16.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.2.16.2)$$

is given by

$$\mathbf{x} = (\mathbf{N}^T)^{-1} \mathbf{c} \quad (1.2.16.3)$$

where

$$\mathbf{N} = (\mathbf{n}_1 \quad \mathbf{n}_2) \quad (1.2.16.4)$$

17. The *angle between two lines* is given by

$$\cos^{-1} \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (1.2.17.1)$$

18. Show that the distance of a point \mathbf{x}_0 from the line

$$L: \mathbf{n}^T \mathbf{x} = c \quad (1.2.18.1)$$

is

$$\frac{|\mathbf{n}^T \mathbf{x}_0 - c|}{\|\mathbf{n}\|} \quad (1.2.18.2)$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (1.2.18.3)$$

where

$$\mathbf{n}^T \mathbf{A} = c, \mathbf{n}^T \mathbf{m} = 0 \quad (1.2.18.4)$$

If \mathbf{x}_0 is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda \mathbf{m} \quad (1.2.18.5)$$

$$\Rightarrow \mathbf{n}^T \mathbf{x} = c - \mathbf{n}^T \mathbf{x}_0 \quad (1.2.18.6)$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.2.19.1)$$

can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.2.19.2)$$

where

$$\mathbf{V} = \mathbf{V}^T \quad (1.2.19.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (1.2.19.4)$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (1.2.20.1)$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \quad (1.2.20.2)$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \quad (1.2.21.1)$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k+1} = \mathbf{B}. \quad (1.2.21.2)$$

Solution: From (1.2.2.5),

$$\mathbf{B} = \mathbf{A} + \lambda_1 \mathbf{m} \quad (1.2.21.3)$$

$$\mathbf{B} = \mathbf{C} - \lambda_2 \mathbf{m}$$

$$\Rightarrow \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \quad (1.2.21.4)$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \mathbf{m}, \quad (1.2.21.5)$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \quad (1.2.21.6)$$

resulting in (1.2.21.2)

22. If \mathbf{A} and \mathbf{B} are linearly independent,

$$k_1 \mathbf{A} + k_2 \mathbf{B} = 0 \Rightarrow k_1 = k_2 = 0 \quad (1.2.22.1)$$

23. Show that \mathbf{D} lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{C} \quad (1.2.23.1)$$

such that

$$0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1, \quad (1.2.23.2)$$

$$0 \leq \lambda_1 + \lambda_2 + \lambda_3 \leq 1, \quad (1.2.23.3)$$

24. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.2.24.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.2.24.2)$$

is

$$\frac{\mathbf{n}_1^T \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^T \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \quad (1.2.24.3)$$

25. Find the equation of a line passing through the

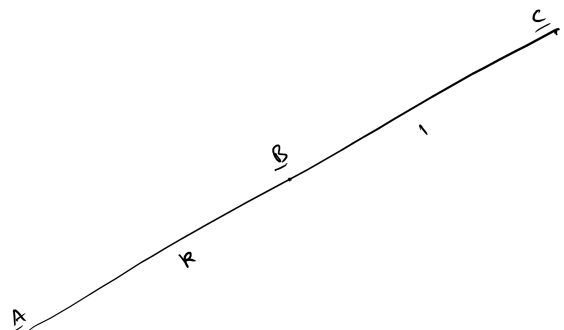


Fig. 1.2.21

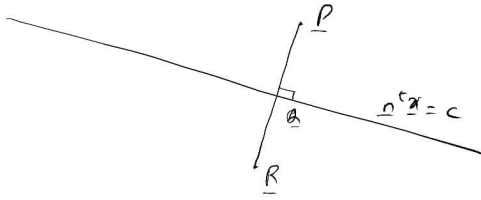


Fig. 1.2.26

intersection of the lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.2.25.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.2.25.2)$$

and passing through the point \mathbf{p} .

Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T} \mathbf{c} \quad (1.2.25.3)$$

where

$$\mathbf{N} = (\mathbf{n}_1 \quad \mathbf{n}_2) \quad (1.2.25.4)$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (1.2.25.5)$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda (\mathbf{N}^{-T} \mathbf{c} - \mathbf{p}) \quad (1.2.25.6)$$

$$\Rightarrow \mathbf{N}^T \mathbf{x} = \mathbf{N}^T \mathbf{p} + \lambda (\mathbf{c} - \mathbf{N}^T \mathbf{p}) \quad (1.2.25.7)$$

resulting in

$$\begin{aligned} & (\mathbf{c} - \mathbf{N}^T \mathbf{p})^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{x} \\ &= (\mathbf{c} - \mathbf{N}^T \mathbf{p})^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{p} \end{aligned} \quad (1.2.25.8)$$

26. Find \mathbf{R} , the reflection of \mathbf{P} about the line

$$L: \mathbf{n}^T \mathbf{x} = c \quad (1.2.26.1)$$

Solution: Since \mathbf{R} is the reflection of \mathbf{P} and \mathbf{Q} lies on L , \mathbf{Q} bisects PR . This leads to the following equations. Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (1.2.26.2)$$

$$\mathbf{n}^T \mathbf{Q} = c \quad (1.2.26.3)$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad (1.2.26.4)$$

where \mathbf{m} is the direction vector of L . From

(1.2.26.2) and (1.2.26.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (1.2.26.5)$$

From (1.2.26.5) and (1.2.26.4),

$$(\mathbf{m} \quad \mathbf{n})^T \mathbf{R} = (\mathbf{m} \quad -\mathbf{n})^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.26.6)$$

Letting

$$\mathbf{V} = (\mathbf{m} \quad \mathbf{n}) \quad (1.2.26.7)$$

with the condition that \mathbf{m}, \mathbf{n} are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (1.2.26.8)$$

Noting that

$$(\mathbf{m} \quad -\mathbf{n}) = (\mathbf{m} \quad \mathbf{n}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.2.26.9)$$

(1.2.26.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.26.10)$$

$$\Rightarrow \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.26.11)$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c \mathbf{n} \quad (1.2.26.12)$$

27. Show that, for any \mathbf{m}, \mathbf{n} , the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T \mathbf{m} + \mathbf{n}^T \mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \quad (1.2.27.1)$$

1.3 Example

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.3.1)$$

and the equations of the medians through \mathbf{B} and \mathbf{C} are respectively

$$(1 \quad 1)\mathbf{x} = 5 \quad (1.3.1)$$

$$(1 \quad 0)\mathbf{x} = 4 \quad (1.3.1)$$

Find the area of $\triangle ABC$.

Solution: The centroid \mathbf{O} is the solution of (1.3.1), (1.3.1) and is obtained as the solution

of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad (1.3.1)$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.3.1)$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.3.1)$$

Let AD be the median through A . Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \quad (1.3.1)$$

$$\Rightarrow \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.3.1)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \end{pmatrix} \quad (1.3.1)$$

From (1.3.1) and (1.3.1),

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} = 5 \quad (1.3.1)$$

$$\Rightarrow 5 + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 12 \quad (1.3.1)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 7 \quad (1.3.1)$$

From (1.3.1) and (1.3.1), \mathbf{C} can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad (1.3.1)$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.3.1)$$

$$\Rightarrow \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (1.3.1)$$

From (1.3.1),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.3.1)$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \quad (1.3.1)$$

2. Summarize all the above computations through

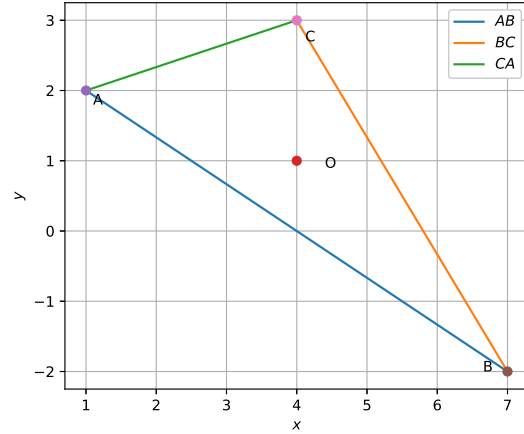


Fig. 1.3.2

a Python script and plot $\triangle ABC$.

Solution:

<https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/triang.py>

3. Find the coordinates of \mathbf{D}, \mathbf{E} and \mathbf{F} of the mid points of AB, BC and CA respectively for $\triangle ABC$.
4. Find the equations of AD, BE and CF .
5. Find the point of intersection of AD and CF .
6. Verify that \mathbf{O} is the point of intersection of BE, CF as well as
7. Graphically show that the medians of $\triangle ABC$ meet at the centroid.
8. In $\triangle ABC$, Let \mathbf{P} be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of $\triangle ABC$.
9. Find the equation of AP .

Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From (??), the equation of AP is

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.3.9)$$

$$\Rightarrow \begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = -\begin{pmatrix} -3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = -2 \quad (1.3.9)$$

10. Find the equation of the altitude BQ .

Solution: The desired equation is

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{x} - \mathbf{B}) = 0 \quad (1.3.10)$$

$$\Rightarrow \begin{pmatrix} 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9 \quad (1.3.10)$$

11. Find the equation of the altitude CR .

12. Find the point of intersection of AP and BQ .
Solution: (1.3.9) and (1.3.10) can be stacked together into the matrix equation

$$\begin{pmatrix} -3 & 4 \\ 6 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \quad (1.3.12)$$

The following code computes the point of intersection.

```
https://raw.githubusercontent.com/gadepall/school/master/
linalg/2D/python_2d/codes/
orthocentre.py
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13. Find the point of intersection of AP and BQ and CR . Comment.
 14. Find \mathbf{P}

Solution: The following code finds the required points.

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https://raw.githubusercontent.com/gadepall/school/master/
linalg/2D/python_2d/codes/
alt_foot.py
```

15. Find \mathbf{Q} and \mathbf{R} .
 16. Draw AP , BQ and CR and verify that they meet at a point \mathbf{H} .

Solution: The following code plots the altitudes in Fig. ??

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https://raw.githubusercontent.com/gadepall/school/master/
linalg/2D/python_2d/codes/
alt_draw.py
```

1.4 Exercises

1. A straight line through the origin \mathbf{O} meets the lines

$$(4 \ 3)\mathbf{x} = 10 \quad (1.4.1)$$

$$(8 \ 6)\mathbf{x} + 5 = 0 \quad (1.4.1)$$

at \mathbf{A} and \mathbf{B} respectively. Find the ratio in which \mathbf{O} divides AB .

Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (1.4.1)$$

Then (1.4.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \quad (1.4.1)$$

$$2\mathbf{n}^T \mathbf{x} = -5 \quad (1.4.1)$$

and since \mathbf{A}, \mathbf{B} satisfy (1.4.1) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \quad (1.4.1)$$

$$2\mathbf{n}^T \mathbf{B} = -5 \quad (1.4.1)$$

Let \mathbf{O} divide the segment AB in the ratio $k : 1$.
 Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (1.4.1)$$

$$\therefore \mathbf{O} = \mathbf{0}, \quad (1.4.1)$$

$$\mathbf{A} = -k\mathbf{B} \quad (1.4.1)$$

Substituting in (1.4.1), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \quad (1.4.1)$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \quad (1.4.1)$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \quad (1.4.1)$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.4.2)$$

is translated parallel to the line

$$L : (1 \ -1)\mathbf{x} = 4 \quad (1.4.2)$$

by $d = 2\sqrt{3}$ units. If the new point \mathbf{Q} lies in the third quadrant, then find the equation of the line passing through \mathbf{Q} and perpendicular to L .

Solution: From (1.4.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.4.2)$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \quad (1.4.2)$$

However,

$$PQ = d \quad (1.4.2)$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = d \quad (1.4.2)$$

$$\implies \lambda = \pm \frac{d}{\|\mathbf{m}\|} = \pm \sqrt{6} \quad (1.4.2)$$

$$\therefore \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \quad (1.4.2)$$

from (1.4.2). Since \mathbf{Q} lies in the third quadrant, from (1.4.2) and (1.4.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \quad (1.4.2)$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \quad (1.4.2)$$

$$(1 \ 1)\mathbf{x} = 3 - 2\sqrt{6} \quad (1.4.2)$$

3. Two sides of a rhombus are along the lines

$$AB : (1 \ -1)\mathbf{x} + 1 = 0 \quad (1.4.3)$$

$$AD : (7 \ -1)\mathbf{x} - 5 = 0. \quad (1.4.3)$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad (1.4.3)$$

find its vertices.

Solution: From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad (1.4.3)$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (1.4.3)$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2} \\ \mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \quad (1.4.3)$$

$$\therefore AD \parallel BC,$$

$$BC : (7 \ -1)(\mathbf{x} - \mathbf{C}) = 0 \\ \Rightarrow (7 \ -1)\mathbf{x} = -15 \quad (1.4.3)$$

From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \quad (1.4.3)$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \Rightarrow \mathbf{B} = -\frac{1}{3} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad (1.4.3)$$

$$\therefore AB \parallel CD,$$

$$CD : (1 \ -1)(\mathbf{x} - \mathbf{C}) = 0 \\ \Rightarrow (1 \ -1)\mathbf{x} = 3 \quad (1.4.3)$$

From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.4.3)$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \Rightarrow \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \quad (1.4.3)$$

4. Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \quad (1.4.4)$$

has area 28. Find the orthocentre of this triangle.

Solution: Let \mathbf{m}_1 be the direction vector of BC . Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5+k \\ k-2 \end{pmatrix}, \quad (1.4.4)$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.4.4)$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \quad (1.4.4)$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2-3k \end{pmatrix}, \quad (1.4.4)$$

The orthocentre is obtained by solving (1.4.4)

and (1.4.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \quad (1.4.4)$$

which can be expressed using (1.4.4), (1.4.4), (1.4.4) and (1.4.4) as

$$\begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2+5k+6k-3k^2 \\ 10k-2k-3k^2 \end{pmatrix} \\ = k \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \quad (1.4.4)$$

From (1.4.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56 \\ \Rightarrow \begin{vmatrix} k & 5-k & -2k \\ -3k & 4k & 2+3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \quad (1.4.4)$$

resulting in

$$(5-k)(2+3k)+8k^2=56 \quad (1.4.4)$$

$$\Rightarrow 5k^2+13k-46=0 \quad (1.4.4)$$

$$\text{or, } k=2, -\frac{23}{5} \quad (1.4.4)$$

Substituting the above in (1.4.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$(1 \ 1) \mathbf{x} = 2, \quad (1.4.5)$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.

Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{0} \\ \Rightarrow \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0} \quad (1.4.5)$$

If AB be the line in (1.4.5), the equation of CF , where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (1.4.5)$$

is

$$(1 \ -1) \mathbf{x} = 0 \quad (1.4.5)$$

since CF passes through the origin and $CF \perp AB$. From (1.4.5) and (1.4.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.4.5)$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.4.5)$$

From (1.4.5),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.4.5)$$

after substituting from (1.4.5). Thus,

$$CF = \|\mathbf{C} - \mathbf{F}\| = 3\sqrt{2} \quad (1.4.5)$$

$$\Rightarrow AB = CF \frac{2}{\sqrt{3}} = 2\sqrt{6} \quad (1.4.5)$$

and the area of the triangle is

$$\frac{1}{2} AB \times CF = 6\sqrt{3} \quad (1.4.5)$$

6. A square, of each side 2, lies above the x -axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x -axis, then find the sum of the x -coordinates of the vertices of the square.

Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that \mathbf{B} and \mathbf{D} lie on the x and y -axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.4.6)$$

Multiplying (1.4.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (1.4.6)$$

$$\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 4 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ = 4 \begin{pmatrix} \cos \theta - \sin \theta \\ \cos \theta + \sin \theta \end{pmatrix} \quad (1.4.6)$$

$$\begin{aligned} &\Rightarrow (1 \ 0) \mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) \\ &= 4(\cos \theta - \sin \theta) = 2(\sqrt{3} - 1) \quad (1.4.6) \end{aligned}$$

for $\theta = 30^\circ$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.