Geometry through Linear Algebra

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Contents

***Abstract*—This textbook introduces linear algebra by exploring Euclidean geometry.**

1 The Straight Line

1.1 The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.1}$$

$$\implies x_2 = mx_1 \tag{1.2}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.3}$$

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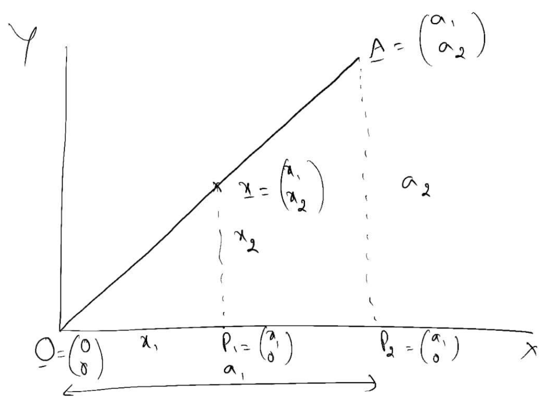

Fig. 1.1

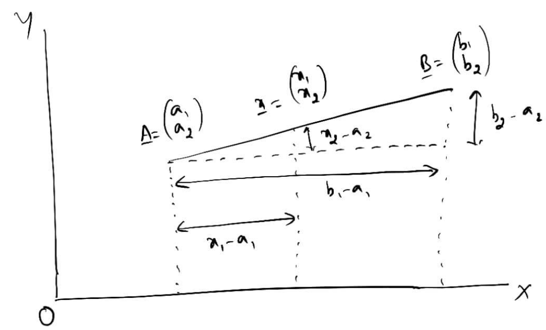

Fig. 1.3

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.4}$$

1.2 Find the length of $\mathbf{A}$.

Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.5}$$

1.3 Find the equation of AB.

Solution: From Fig. **??**, it is obvious that the desired equation is

$$\mathbf{x} = \mathbf{b} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.6}$$

where

$$m = \frac{c_2 - c_1}{b_2 - b_1} \tag{1.7}$$

1.4 In Fig. **??**,

$$\frac{AB}{BC} = k \tag{1.8}$$

Show that

$$\mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k + 1} \tag{1.9}$$

2 Medians of a triangle

Consider $\triangle ABC$ with vertices represented by the vectors $\mathbf{x}_1$

2.1 Get the **audio_source**

```
svn checkout https://github.com/gadepall/
    EE5347/trunk/audio_source
cd audio_source
```

2.2 Play the **signal_noise.wav** and **noise.wav** file. Comment.
Solution: signal_noise.wav contains a human voice along with an instrument sound in the background. This instrument sound is captured in **noise.wav**.

3 Problem Formulation

3.1 See Table 3.1. The goal is to extract the human voice $e(n)$ from $d(n)$ by suppressing the component of $\mathbf{X}(n)$. Formulate an equation for this.
Solution: The maximum component of $\mathbf{X}(n)$ in

Signal	Label	Type	Filename
Known	d(n)	Human+Instrument	signal noise.wav
	X(n)	Instrument	noise.wav
Unknown	e(n)	Human estimate	
	W(n)	Weight Vector	

TABLE 3.1

$d(n)$ can be estimated as

$$\mathbf{W}^T(n)\mathbf{X}(n) \tag{3.1}$$

where

$$\mathbf{W}(n) = \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ .. \\ .. \\ w_{n-M+1}(n) \end{bmatrix}_{MX1} \tag{3.2}$$

Intuitively, the human voice $e(n)$ is obtained after removing as much of $\mathbf{X}(n)$ from $d(n)$ as possible. The first step in this direction is to estimate $\mathbf{W}$ in (3.1) using the metric

$$\min_{\mathbf{W}(n)} \|d(n) - \mathbf{W}^T(n)\mathbf{X}(n)\|^2 \tag{3.3}$$

The human voice can be then obtained as

$$e(n) = d(n) - \mathbf{W}^T(n)\mathbf{X}(n) \tag{3.4}$$

4 LMS Algorithm

4.1 Show using (3.4) that

$$\nabla_{\mathbf{W}(n)} e^2(n) = \frac{\partial e^2(n)}{\partial \mathbf{W}(n)} \tag{4.1}$$

$$= -2\mathbf{X}(n)d(n) + 2\mathbf{X}(n)X^T(n)\mathbf{W}(n) \tag{4.2}$$

4.2 Use the gradient descent method to obtain an algorithm for solving (3.3)
Solution: The desired algorithm can be expressed as

$$\mathbf{W}(n+1) = \mathbf{W}(n) - \bar{\mu}[\nabla_{\mathbf{W}(n)} e^2(n)] \tag{4.3}$$

$$\mathbf{W}(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \tag{4.4}$$

where $\mu = \bar{\mu}$.

4.3 Write a program to suppress $\mathbf{X}(n)$ in $d(n)$.
Solution: Execute

```
wget https://raw.githubusercontent.com/
    gadepall/EE5347/master/lms/codes/
    LMS_NC_SPEECH.py
```

5 Wiener-Hopf Equation

5.1 Using (3.4), show that

$$E[e^2(n)] = r_{dd} - W^T(n)r_{xd} - r_{xd}^T\mathbf{W}(n) + W^T(n)R\mathbf{W}(n) \tag{5.1}$$

where

$$r_{dd} = E[d^2(n)] \tag{5.2}$$
$$r_{xd} = E[\mathbf{X}(n)d(n)] \tag{5.3}$$
$$R = E[\mathbf{X}(n)\mathbf{X}^T(n)] \tag{5.4}$$

5.2 By computing

$$\frac{\partial J(n)}{\partial \mathbf{W}(n)} = 0, \tag{5.5}$$

show that the optimal solution for

$$W^*(n) = \min_{\mathbf{W}(n)} E\left[e^2(n)\right] = R^{-1}r_{xd} \tag{5.6}$$

This is the Wiener optimal solution.

6 Convergence of the LMS Algorithm

6.1 Convergence in the Mean

6.1.1 Show that R in (5.4) is symmetric as well as positive definite.
Let

$$\tilde{W}(n) = \mathbf{W}(n) - W_* \tag{6.1}$$

where W_* is obtained in (5.6). Also, according to the LMS algorithm,

$$W(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \tag{6.2}$$
$$e(n) = d(n) - X^T(n)\mathbf{W}(n) \tag{6.3}$$

6.1.2 Show that

$$E\left[\tilde{W}(n+1)\right] = [I - \mu R]E\left[\tilde{W}(n)\right] \tag{6.4}$$

6.1.3 Show that

$$R = U\Lambda U^T \tag{6.5}$$

for some U, Λ, such that Λ is a diagonal matrix and $U^T U = I$.

6.1.4 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n+1)\right] = 0 \iff \lim_{n\to\infty}[I - \mu\Lambda]^n = 0 \tag{6.6}$$

6.1.5 Using (6.6), show that

$$0 < \mu < \frac{2}{\lambda_{\max}} \tag{6.7}$$

where $\lambda_{\max}$ is the largest entry of Λ.

6.2 Convergence in Mean-square sense

Let

$$\mathbf{X}(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \end{bmatrix} \tilde{W}(n) = \begin{bmatrix} \tilde{W}_1(n) \\ \tilde{W}_2(n) \end{bmatrix} \tag{6.8}$$

6.2.1 Show that

$$E[\tilde{W}^T(n)\mathbf{X}(n)X^T(n)\tilde{W}(n)] = E[\tilde{W}^T(n)R\tilde{W}(n)] \tag{6.9}$$

for R defined in (5.4).

6.2.2 Show that

$$\begin{aligned} J(n) = E[e^2(n)] = E[e_*^2(n)] \\ +E[\tilde{W}(n)\mathbf{X}(n)\mathbf{X}(n)^T\tilde{W}(n)^T] - E[\tilde{W}(n)\mathbf{X}(n)e_*(n)] \\ - E[e_*(n)X^T(n)\tilde{W}^T(n)] \end{aligned} \tag{6.10}$$

where

$$\tilde{W}(n) = W(n) - W_* \tag{6.11}$$
$$e_*(n) = d(n) - W_*\mathbf{X}(n) \tag{6.12}$$

6.2.3 Show that

$$\begin{aligned} E\left[\tilde{W}(n)\mathbf{X}(n)e_*(n)\right] &= E\left[e_*(n)X^T(n)\tilde{W}^T(n)\right] \\ &= 0 \end{aligned} \tag{6.13}$$

6.2.4 Show that

$$E\left[\tilde{W}^T(n)R\tilde{W}(n)\right] = \text{trace}\left(E\left[\tilde{W}^T(n)R\tilde{W}(n)\right]\right) \tag{6.14}$$
$$= \text{trace}\left(E\left[\tilde{W}(n)\tilde{W}^T(n)\right]R\right) \tag{6.15}$$

6.2.5 Using (6.11), (6.2) and (6.12), show that

$$\tilde{W}(n+1) = \left[I - \mu\mathbf{X}(n)X^T(n)\right]\tilde{W}(n) + \mu\mathbf{X}(n)e_*(n) \tag{6.16}$$

6.2.6 Let $\mu^2 \to 0$. Using (6.5) and (5.6), show that

$$\begin{aligned} &E\left[\tilde{W}(n+1)\tilde{W}^T(n+1)\right] \\ &= (I - 2\mu R)\,E\left[\tilde{W}(n)\tilde{W}^n(n)\right] \end{aligned} \tag{6.17}$$

6.2.7 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n)\tilde{W}^T(n)\right] = 0 \iff 0 < \mu < \frac{1}{\lambda_{max}} \tag{6.18}$$

6.2.8 Find the value of the cost function at infinity i.e. $J(\infty)$

6.2.9 How can you choose the value of μ from the convergence of both in mean and mean-square sense?