भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

JEE Problems in Calculus

Teaching Learning Center
IIT Hyderabad

***Abstract*—A collection of problems from JEE papers related to calculus are available in this document. Verify your soluions using numerical techniques for integration and differentiation.**

1 Calculus: Integration

1.1 Sketch the region

$$\begin{pmatrix} x \\ y \end{pmatrix} : xy \le 8, 1 \le y \le x^2 \tag{1}$$

1.2 Find the area of the region.
Solution: The intersection of $y = 1, y = x^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2}$$

The intersection of $y = 1, xy = 8$ is

$$\mathbf{B} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{3}$$

The intersection of $y = x^2, xy = 8$ is

$$\mathbf{C} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{4}$$

The desired region is enclosed by the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ Thus, the area is obtained as

$$\int_1^2 x^2 \, dx + \int_2^8 \frac{8}{x} \, dx = \left[\frac{x^3}{3}\right]_1^2 + 8 \left[\ln x\right]_2^8 - 7 \tag{5}$$

$$= 16 \ln 2 - \frac{14}{3} \tag{6}$$

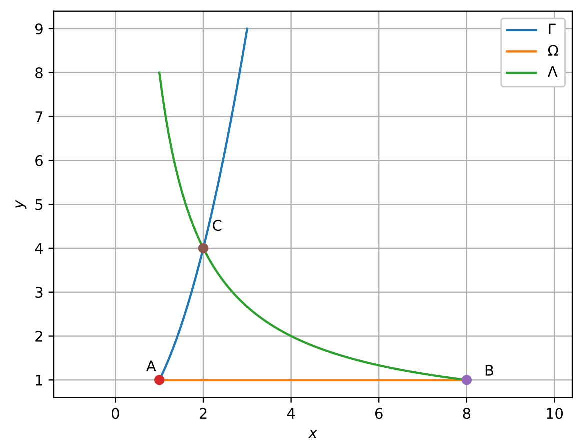

Fig. 1.2

2 Calculus: Differentiation

Let

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x-2)\ln(x-2) - x + \frac{10}{3} & x \ge 3 \end{cases} \tag{7}$$

2.1 Is f increasing in $(-\infty, 0)$?
Solution:

$$\begin{aligned} f'(x) &= 5x^4 + 20x^3 + 30x^2 + 20x + 3 \quad x < 0 \\ \implies f'(-1) &= 5 - 20 + 30 - 20 + 3 = -2 < 0 \end{aligned} \tag{8}$$

Hence $f'(x)$ is non-increasing.

2.2 Does f' have a local maximum at $x = 1$?
Solution:

$$f'(x) = \begin{cases} 2x - 1 > 0, & \frac{1}{2} < x < 1, \\ 2(x-2)^2 - 1 < 0 & 1 \le x < 3 \end{cases} \tag{9}$$

Hence, f is increasing in $\left(\frac{1}{2}, 1\right)$ and decreasing between $(1, 3) \implies f$ has a local maximum at $x = 1$

2.3 Show that f' is differentiable at $x = 1$.
Solution: Since

$$f'(1-) = f'(1) = 1, \tag{10}$$

f is differentiable at $x = 1$.

2.4 Is f onto?

2.5 Sketch $f(x)$ in Python to verify your answeres.

3 Calculus: Differential Equations

Γ is a curve in the first qudrant and

$$\mathbf{R} = \begin{pmatrix}1\\0\end{pmatrix} \tag{11}$$

lies on it. The tangent to Γ at $\mathbf{P}$ intersects the $y-$axis at $\mathbf{Y}_P$. The line segment $PY_P = 1$.

3.1 Find the differential equation of Γ.
Solution: Let

$$\mathbf{P} = \begin{pmatrix}x\\y\end{pmatrix}, \mathbf{Y}_P = \begin{pmatrix}0\\c\end{pmatrix}. \tag{12}$$

Then using the equation of a line,

$$\mathbf{Y}_P = \mathbf{P} + \lambda\mathbf{m}, \tag{13}$$

where

$$\mathbf{m} = \begin{pmatrix}1\\y'\end{pmatrix}. \tag{14}$$

Thus,

$$\begin{pmatrix}0\\c\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \lambda\begin{pmatrix}1\\y'\end{pmatrix} \tag{15}$$

$$\implies \lambda = -x. \tag{16}$$

$$\because PY_P = \|\mathbf{P} - \mathbf{Y}_P\| = |\lambda|\,\|\mathbf{m}\| = 1, \tag{17}$$

$$x^2\left(1 + (y')^2\right) = 1 \tag{18}$$

$$\implies xy' \pm \sqrt{1 - x^2} = 0 \tag{19}$$

3.2 Find the equation of Γ.
Solution: From (19),

$$dy = \pm\frac{\sqrt{1-x^2}}{x}\,dx \tag{20}$$

$$\implies \int dy = \pm\int \frac{\sqrt{1-x^2}}{x}\,dx \tag{21}$$

Letting

$$z = \sqrt{1-x^2}, dz = -\frac{x}{\sqrt{1-x^2}}\,dx$$

$$\begin{aligned}\implies \int \frac{\sqrt{1-x^2}}{x}\,dx &= -\int \frac{z^2}{1-z^2}\,dz\\ &= \int dz - \int \frac{1}{1-z^2}\,dz\\ &= z + \frac{1}{2}\ln\frac{1-z}{1+z} + C\end{aligned} \tag{22}$$

Thus,

$$y = \pm\left(\sqrt{1-x^2} + \frac{1}{2}\ln\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right) \tag{23}$$

since $C = 0$ after substituting $x = 0, y = 1$.

3.3 Verify your result through a python sketch.

4 Calculus: Definite Integral

4.1 If

$$I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}, \tag{24}$$

find $27I^2$.
Solution: Substituting $-x$ for x,

$$I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{-\sin x})(2-\cos 2x)}, \tag{25}$$

Adding (24) and (25),

$$2I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2-\cos 2x)}\left[\frac{1}{(1+e^{\sin x})} + \frac{1}{(1+e^{-\sin x})}\right], \tag{26}$$

which can be simplified to obtain

$$\begin{aligned}I &= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2-\cos 2x)}\frac{\left(1+e^{\sin x}+1+e^{-\sin x}\right)}{(1+e^{\sin x}+e^{-\sin x}+1)}\\ &= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(2-\cos 2x)}\end{aligned} \tag{27}$$

Substituting

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2}, \tag{28}$$

in (27) and simplifying,

$$\begin{aligned}I &= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1+3\tan^2 x)}\,dx\\ &= \frac{1}{\pi\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\\ &= \frac{2}{3\sqrt{3}}\end{aligned} \tag{29}$$

resulting in

$$27I^2 = 4 \tag{30}$$

5 Calculus: Limits

Let

$$P_1 = \lim_{h \to 0} \frac{f(h) - f(0)}{\sqrt{|h|}} \tag{31}$$

$$P_2 = \lim_{h \to 0} \frac{f(h) - f(0)}{h^2} \tag{32}$$

5.1 Find P_1 for

$$f(x) = |x| \tag{33}$$

Solution: Substituting (33) in (31),

$$P_1 = \lim_{h \to 0} \frac{|h|}{\sqrt{h}} = 0 \tag{34}$$

5.2 Find P_1 for

$$f(x) = x^{\frac{2}{3}} \tag{35}$$

Solution: Substituting (35) in (31),

$$P_1 = \lim_{h \to 0} \frac{h^{\frac{2}{3}}}{\sqrt{h}} = h^{\frac{1}{3}} = 0 \tag{36}$$

5.3 Find P_2 for

$$f(x) = x\,|x| \tag{37}$$

Solution: Substituting (37) in (32),

$$\lim_{h \to 0+} \frac{h\,|h|}{h^2} = 1 \tag{38}$$

and

$$\lim_{h \to 0-} \frac{-h\,|h|}{h^2} = -1 \tag{39}$$

$$\because (38) \neq (39), \tag{40}$$

P_2 does not exist.

5.4 Find P_2 for

$$f(x) = \sin x \tag{41}$$

Solution: Substituting (42) in (32),

$$\lim_{h \to 0+} \frac{\sin h}{h^2} = \infty \tag{42}$$

Hence P_2 does not exist.

6 Calculus: Maxima and Minima

Let

$$f(x) = p(x)q(x), \quad x > 0, \text{ where} \tag{43}$$

$$p(x) = \sin \pi x \tag{44}$$

$$q(x) = \frac{1}{x^2} \tag{45}$$

6.1 Show that $q(x)$ is monotonically decreasing.

6.2 Show that $p(x)$ is oscillatory.

6.3 Find the regions where $f(x)$ is increasing and decreasing.

Solution:

$$\because f'(x) = p(x)q'(x) + p'(x)q(x), \tag{46}$$

$$q(x) > 0, \tag{47}$$

$$q'(x) < 0, \tag{48}$$

$$f'(x) \begin{cases} < 0 & p(x) > 0 \text{ and } p'(x) < 0 \\ > 0 & p(x) < 0 \text{ and } p'(x) > 0 \end{cases} \tag{49}$$

Table 6.3 computes the desired regions based on (49)

	> 0	< 0
$p(x)$	$x \in (2n, 2n+1)$	$x \in (2n+1, 2n+2)$
$q(x)$	$x > 0$	
$p'(x)$	$x \in \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right)$	$x \in \left(2n + \frac{1}{2}, 2n + \frac{3}{2}\right)$
$q'(x)$		$x > 0$
$f'(x)$	$x \in (2n+1, 2n+2) \cup x \in \left(2n - \frac{1}{2}, 2n + \frac{1}{2}\right) = x \in \left(2n - \frac{1}{2}, 2n\right)$	$x \in (2n, 2n+1) \cup x \in \left(2n + \frac{1}{2}, 2n + \frac{3}{2}\right) = x \in \left(2n + \frac{1}{2}, 2n + 1\right)$

TABLE 6.3

6.4 Find the points of local maxima x_i.

Solution: The maxima occur in the interval between $f'(x) > 0$ and $f'(x) < 0$. From Table 6.3,

$$x_i \in \left(2n, 2n + \frac{1}{2}\right), \quad n \geq 1 \tag{50}$$

6.5 Find the points of local minima y_i.

Solution: The minima occur in the interval between $f'(x) < 0$ and $f'(x) > 0$. From Table 6.3,

$$y_i \in \left(2n - 1, 2n - \frac{1}{2}\right), \quad n \geq 1 \tag{51}$$

6.6 Is

$$x_{n+1} - x_n > 2 \tag{52}$$

for every n?
Solution: From (50),

$$x_{n+1} - x_n > 2(n+1) - \left(2n + \frac{1}{2}\right) \tag{53}$$

$$= \frac{3}{2} < 2 \tag{54}$$

6.7 Show that

$$x_1 > y_1 \tag{55}$$

6.8 Verify if

$$|x_n - y_n| > 1 \tag{56}$$

for every n.

7 Calculus: Integration

Let

$$F(x) = \int_0^x f(t)\,dt, \quad x > 0 \tag{57}$$

where

$$f(x) = (x-1)(x-2)(x-5) \tag{58}$$

7.1 Does $F(x)$ have a local minimum at $x = 1$?
Solution: The derivative of $F(x)$ is

$$F^{(1)}(x) = \lim_{\delta x \to 0} \frac{1}{\delta} \int_x^{x+\delta x} f(t)\,dt$$
$$= f(x) \tag{59}$$

Thus, from (59),

$$F^{(2)}(x) = f^{(1)}(x)$$
$$= (x-1)(x-2) + (x-2)(x-5)$$
$$+ (x-1)(x-5)$$
$$\implies F^{(2)}(1) = 4 > 0 \tag{60}$$

Since $f(1) = 0$, answer is yes.
7.2 Does $F(x)$ have a local maximum at $x = 2$?
Solution: From (60),

$$F^{(2)}(2) = -3 < 0 \tag{61}$$

Since $f(2) = 0$, answer is yes.
7.3 Does $F(x)$ have two local maxima and one local minimum in $(0, \infty)$?
Solution: From (60),

$$F^{(2)}(5) = 12 > 0 \tag{62}$$

Since $f(5) = 0$, $F(x)$ has two local minima and one minimum. So answer is false.
7.4 Verify if $F(x) \neq 0$ for all $x \in (0, 5)$.
Solution: From the previous solutions,

$$\min F(x) > 0, \quad x \in (0, 5) \tag{63}$$

Hence, the statement is correct.

8 Trigonometry

8.1 Find

$$\sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right) \tag{64}$$

Solution: (64) can be expressed as

$$\frac{2}{\cos\left(\frac{7\pi}{6} + \frac{(2k+1)\pi}{2}\right) + \cos\frac{\pi}{2}}$$
$$= \frac{2}{\cos\left((k+1)\pi + \frac{\pi}{6} + \frac{\pi}{2}\right)} = 4(-1)^{k+1} \tag{65}$$

after simplification.
8.2 Find the value of

$$\theta = \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\right.$$
$$\left.\times \sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right] \tag{66}$$

in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
Solution: Substituting from (65) in (66) results in

$$\theta = \sec^{-1}(1) \implies \theta = 0 \tag{67}$$

in the given interval.

9 Definite Integral

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5}\,d\theta \tag{68}$$

9.1 Show that

$$I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5}\,d\theta \tag{69}$$

9.2 Show that

$$I = \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4}\, d\theta \tag{70}$$

9.3 Show that

$$I = 3\int_0^{\frac{\pi}{4}} \frac{1}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4}\, d\theta \tag{71}$$

9.4 Find I.
Solution: From (71)

$$I = 3\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(1 + \sqrt{\tan\theta}\right)^4}\, d\theta \tag{72}$$

which, after substituting

$$t = 1 + \sqrt{\tan\theta} \tag{73}$$

results in

$$\begin{aligned} I &= 3\int_1^2 \frac{2(t-1)}{t^4}\, dt \\ &= 6\left[\frac{1}{3t^3} - \frac{1}{2t^2}\right]_1^2 \\ &= \left[2\left(\frac{1}{8} - 1\right) - 3\left(\frac{1}{4} - 1\right)\right] = \frac{1}{2} \end{aligned} \tag{74}$$

10 Calculus: Differentiation

For $x > 0$,

$$f(x) = \sin(\pi\cos x) \tag{75}$$
$$g(x) = \cos(2\pi\sin x) \tag{76}$$

10.1 Find

$$X = \{x : f(x) = 0\} \tag{77}$$

Solution:

$$\begin{aligned} &\sin(\pi\cos x) = 0 \\ &\implies \pi\cos x = k\pi \text{ or, } \cos x = k \end{aligned} \tag{78}$$

where k is an integer.

$$\begin{aligned} \because |\cos x| \le 1,& \\ X &= \{x : \cos x = -1, 0, 1\} \\ \implies X &= k\pi \end{aligned} \tag{79}$$

10.2 Find

$$Y = \{x : f'(x) = 0\}, \tag{80}$$

Solution:

$$\begin{aligned} Y &= \{x : \sin x \cos(\pi\cos x) = 0\}, \\ &= \{x : \sin x = 0\} \cup \{x : \cos(\pi\cos x) = 0\}, \\ &= \{k\pi\} \cup \left\{x : \cos x = \left(2m \pm \frac{1}{2}\right)\right\}, \\ &= \{k\pi\} \cup \left\{x : \cos x = \pm\frac{1}{2}\right\}, \end{aligned} \tag{81}$$

which can be expressed as

$$\begin{aligned} Y &= \{k\pi\} \cup \left\{2m\pi \pm \frac{\pi}{3}\right\} \cup \left\{2m\pi \pm \frac{2\pi}{3}\right\} \\ &= \frac{k\pi}{3} \end{aligned} \tag{82}$$

10.3 Find

$$Z = \{x : g(x) = 0\} \tag{83}$$

Solution:

$$\begin{aligned} Z &= \{x : \cos(2\pi\sin x) = 0\} \\ &= \left\{x : \sin x = k \pm \frac{1}{4}\right\} \\ &= \left\{x : \sin x = \pm\frac{1}{4}, \pm\frac{3}{4}\right\} \\ &= \left\{k\pi \pm \sin^{-1}\frac{1}{4}\right\} \cup \left\{k\pi \pm \sin^{-1}\frac{3}{4}\right\} \end{aligned} \tag{84}$$

10.4 Find

$$W = \{x : g'(x) = 0\}. \tag{85}$$

Solution:

$$\begin{aligned} W &= \{x : \cos x \sin(2\pi\sin x) = 0\} \\ &= \{x : \cos x \sin(2\pi\sin x) = 0\} \\ &= \{x : \cos x = 0\} \cup \{x : \sin(2\pi\sin x) = 0\} \\ &= \left\{\left(2k \pm \frac{1}{2}\right)\pi\right\} \cup \left\{x : \sin x = \frac{k}{2}\right\} \\ &= \left\{m\pi \pm \frac{\pi}{6}\right\} \cup \{r\pi\} \cup \left\{n\pi \pm \frac{\pi}{2}\right\} \end{aligned} \tag{86}$$

11 Numbers

11.1 Given (refer to previous problem)

List I		*List II*	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

which of the following is the only CORRECT combination?

(A) (I), (P), (R)
(B) (II), (Q), (T)
(C) (I), (Q), (U)
(D) (II), (R), (S)

11.2 Which of the following is the only CORRECT combination?

(A) (III), (R), (U)
(B) (IV), (P), (R), (S)
(C) (III), (P), (Q), (U)
(D) (IV), (Q), (T)