Geometry: Maths Olympiad

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Abstract—**This book provides a collection of the international maths olympiad problems in geometry.**

1. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.
2. An arbitrary point M is selected in the interior of the segment AB. The squares AMCD and MBEF are constructed on the same side of AB, with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N0 denote the point of intersection of the straight lines AF and BC.
   a) Prove that the points N and N0 coincide.
   b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M.
   c) Find the locus of the midpoints of the segments P Q as M varies between A and B.
3. Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid ABCD (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.
4. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

   $$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

   at least one is ≤ 2 and at least one is ≥ 2.

5. Construct triangle ABC if AC = b, AB = c and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^o$. Prove that a solution exists if and only if

   $$b \tan\frac{\omega}{2} \leq c < b.$$

   In what case does the equality hold?

6. Consider a plane ε and three non-collinear points A, B, C on the same side of ε; suppose the plane determined by these three points is not parallel to ε. In plane a take three arbitrary points A_0, B_0, C_0. Let L, M, N be the midpoints of segments AA', BB', CC'; let G be the centroid of triangle LMN. (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ε?
7. Consider the cube ABCD$A'B'C'D'$ (ABCD and $A'B'C'D'$ are the upper and lower bases, respectively, and edges AA', BB', CC', DD' are parallel). The point X moves at constant speed along the perimeter of the square ABCD in the direction ABCDA, and the point Y moves at the same rate along the perimeter of the square $B'C'$CB in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpoints of the segments XY.
8. On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the

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quadrilateral thus obtained.

9. Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r - 2\rho)}$$

10. The tetrahedron SABC has the following property: there exist five spheres, each tangent to the edges SA, SB, SC, BCCA, AB, or to their extensions.
    a) Prove that the tetrahedron SABC is regular.
    b) Prove conversely that for every regular tetrahedron five such spheres exist.
11. Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A, and the other side intersecting the segment BC.
12. Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$
13. A circle is inscribed in $\triangle ABC$ with sides a, b, c. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from $\triangle ABC$. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b,c).
14. Seventeen people correspond by mail with one another each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.
15. Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.
16. In tetrahedron ABCD, vertex D is connected with D_0 the centroid of $\triangle ABC$. Lines parallel to DD_0 are drawn through A, B and C. These lines intersect the planes BCD, CAD and ABD in points A_1, B_1 and C_1, respectively. Prove that the volume of ABCD is one third the volume of $A_1B_1C_1D_0$. Is the result true if point D_0 is selected anywhere within $\triangle ABC$?
17. Given the tetrahedron ABCD whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d, and the angle between them is ω. Tetrahedron ABCD is divided into two solids by plane ε, parallel to lines AB and CD. The ratio of the distances of ε from AB and CD is equal to k. Compute the ratio of the volumes of the two solids obtained.
18. Consider a triangle OAB with acute angle AOB. Through a point $M \neq O$ perpendiculars are drawn to OA and OB, the feet of which are P and Q respectively. The point of intersection of the altitudes of $\triangle OPQ$ is H. What is the locus of H if M is permitted to range over
    a) the side AB,
    b) the interior of $\triangle OAB$?
19. In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.
20. In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?
21. Let a, b, c be the lengths of the sides of a triangle, and α, β, γ, respectively, the angles opposite these sides. Prove that if

$$a + b = \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta)$$

the triangle is isosceles.

22. Prove the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.
23. In the interior of sides BC, CA, AB of triangle

ABC, any points K, L, M, respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC.

24. Let ABCD be a parallelogram with side lengths AB = a, AD = 1, and with $\angle BAD = \alpha$. If $\triangle ABD$ is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if

$$a \geq \cos\alpha + \sqrt{3}\sin\alpha.$$

25. Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq \frac{1}{8}$.
26. Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\triangle A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA, and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.
27. Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.
28. Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.
29. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.
30. A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B, and D is the foot of the perpendicular from C to AB. We consider three circles, γ_1, γ_2, γ_3, all tangent to the line AB. Of these, γ_1 is inscribed in $\triangle ABC$, while γ_2 and γ_3 are both tangent to CD and to γ, one on each side of CD. Prove that γ_1, γ_2 and γ_3 have a second tangent in common.
31. Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the inscribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1}.\frac{r_2}{q_2} = \frac{r}{q}$$

32. In the tetrahedron ABCD, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of $\triangle ABC$.Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

33. Consider a convex polyhedron P_1 with nine vertices $A_1A_2, ..., A_9$; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to A_i(i = 2, 3,......, 9). Prove that at least two of the polyhedra $P_1, P_2, ..., P_9$ have an interior point in common.
34. All the faces of tetrahedron ABCD are acute-angled triangles. We consider all closed polygonal paths of the form XY ZTX defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges BCCD, DA, respectively. Prove:
    a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.
    b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being 2AC sin $\frac{\alpha}{2}$ where $\alpha = \angle BAC + \angle CAD + \angle DAB$.
35. Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.
36. Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.
37. Point O lies on the line g:$\overrightarrow{OP_1}, \overrightarrow{OP_2}..........\overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, ..., P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd, $\left|\overrightarrow{OP_1} + \overrightarrow{OP_2}..... + \overrightarrow{OP_n}\right|$ Here $\left|\overrightarrow{OM}\right|$ denotes the length of vector $\overrightarrow{OM}$.
38. Determine whether or not there exists a finite set M of points in space not lying in the same plane such that, for any two points A and B of M, one can select two other points C and D of M so that lines AB and CD are parallel and not coincident.
39. In the triangle ABC, prove that there is a point

D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin \frac{C}{2}$$

40. Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.
41. On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45^o, \angle BCP = \angle ACQ = 30^o, \angle ABR = \angle BAR = 15^o$.Prove that $\angle QRP = 90^o$ and QR = RP.
42. In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.
43. A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.
44. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square ABCD. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AKBK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.
45. P is a given point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V, and W ; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P.
46. In triangle ABC, AB = AC. A circle is tangent internally to the circum circle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the in-circle of triangle ABC.
47. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.
48. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $\frac{(QP+PA)}{QR}$ is a maximum.
49. P is a point inside a given triangle ABC.D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.
50. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the in-center and the circum center of the triangle and the point O are collinear.
51. The diagonals AC and CE of the regular hexagon ABCDEF are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r$$

Determine r if B, M, and N are collinear.
52. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all i = 1, 2, 3, M_i is the midpoint of side a_i, and T_i. is the point where the in-circle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1, M_2S_2, and M_3S_3 are concurrent.
53. Let A be one of the two distinct points of intersection of two unequal co-planar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches O_1 atP_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.
54. Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.
55. Let ABCD be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent

to the circle on CD as diameter if and only if the lines BC and AD are parallel.

56. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by a(X) the measure of the angle between OA and OX in radians, counter clockwise from OA($0 \leq a(X) < 2\pi$). Let C(X) be the circle with center O and radius of length OX + a(X)/(OX). Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle C(Y).
57. A circle has center on the side AB of the cyclic quadrilateral ABCD.The other three sides are tangent to the circle. Prove that $AD+BC = AB$.
58. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.
59. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all s ≥ 4. We construct a set of points P_1, P_2, P_3,......, such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120^o clockwise (for k = 0,1,2,.....). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.
60. Let A, B be adjacent vertices of a regular n-gon (n ≥ 5) in the plane having center at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.
61. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circum circle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral AKNM and the triangle ABC have equal areas.
62. Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then A = P.)
    a) Find the set of values of $BC^2 + CA^2 + AB^2$.
    b) Find the locus of the midpoint of BC.
63. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that S≥ $2T$.
64. In an acute-angled triangle ABC the internal bisector of angle A meets the circum circle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:
    a) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.
    b) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.
65. Let ABCD be a convex quadrilateral such that the sides AB, AD, BC satisfy AB = AD + BC. There exists a point P inside the quadrilateral at a distance h from the line CD such that AP = h + AD and BP = h + BC. Show that:
$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$
66. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If $\frac{AM}{AB}$ = t, find $\frac{EG}{EF}$ interms of t.
67. Prove that there exists a convex 1990-gon with the following two properties:
    a) All angles are equal.
    b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, ..., 1990^2$ in some order.
68. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that $\frac{1}{4} < \frac{AI.BI.CI}{AA'.BB'.CC'} \leq \frac{8}{27}$.
69. Let ABC be a triangle and P an interior point

of ABC . Show that at least one of the angles ∠PAB, ∠PBC, ∠PCA is less than or equal to 30^o.

70. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

71. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.
72. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \frac{\pi}{2}$ and AC · BD = AD · BC.
    a) Calculate the ratio (AB · CD)/(AC · BD).
    b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.
73. ABC is an isosceles triangle with AB = AC. Suppose that
    a) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
    b) Q is an arbitrary point on the segment BC different from B and C;
    c) E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.
74. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

75. Let ABCDEF be a convex hexagon with AB = BC = CD and DE = EF = F A, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that AG + GB + GH + DH + $HE \geq CF$.
76. We are given a positive integer r and a rectangular board ABCD with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20 × 12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.
    a) Show that the task cannot be done if r is divisible by 2 or 3.
    b) Prove that the task is possible when r = 73.
    c) Can the task be done when r = 97?
77. Let P be a point inside triangle ABC such that

    $\angle APB - \angle ACB = \angle APC - \angle ABC.$

    Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

78. Let ABCDEF be a convex hexagon such that AB is parallel to DE, BC is parallel to EF , and CD is parallel to F A. Let RA, RC, RE denote the circumradii of triangles F AB, BCD, DEF ,respectively, and let P denote the perimeter of the hexagon. Prove that

    $R_A + R_C + R_E \geq \frac{P}{2}$.
79. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.
    Let S1 be the total area of the black part of the triangle and S2 be the total area of the white part. Let

    f(m, n) = $|S_1 - S_2|$.
    a) Calculate f(m, n) for all positive integers m and n which are either both even or both odd.
    b) Prove that $f(m, n) \leq \frac{1}{2} \max m, n$ for all m and n.

c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

80. The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W , respectively. The lines BV and CW meet at T . Show that

    AU = TB + TC.

81. In the convex quadrilateral ABCD, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P , where the perpendicular bisectors of AB and DC meet, is inside ABCD. Prove that ABCD is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

82. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

83. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.

84. AB is tangent to the circles CAMN and NMBD. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that PE = QE.

85. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

86. Let ABC be an acute-angled triangle with circumcentre O.Let P on BC be the foot of the altitude from A.
    Suppose that $\angle BCA \geq \angle ABC + 30^o$.
    Prove that $\angle CAB + \angle COP < 90^o$.

87. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^o$ and that AB + BP = AQ + QB. What are the possible angles of triangle ABC?

88. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF .

89. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, ..., O_n$. Show that $\sum_{i<j} \frac{1}{O_iO_j} \leq (n-1)\frac{\pi}{4}$.

90. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

91. ABCD is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC if RP=RQ.

92. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

93. In a convex quadrilateral ABCD the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside ABCD andsatisfies

    $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

    Prove that ABCD is a cyclic quadrilateral if and only if AP=CP .

94. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.
95. Let ABCD be a fixed convex quadrilateral with BC = DA and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy BE = DF. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.
96. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$
Show that AP ≥ AI, and that equality holds if and only if P = I.
97. Let P be a regular 2006-gon. A diagonal of P is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P .The sides of P are also called good. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P . Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.
98. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P . Show that the sum of the areas assigned to the sides of P is at least twice the area of P.
99. Consider five points A, B, C, D and E such that ABCD is a parallelogram and BCED is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that EF = EG = EC. Prove that l is the bisector of angle DAB.
100. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.
101. An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at $A_1 and A_2$. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at $B_1 and B_2$, and the circle passing through H with centre the midpoint of AB intersects the line AB at $C_1 and C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.
102. Let ABCD be a convex quadrilateral with $|BA| \neq |BC|$. Denote the in circles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.Prove that the common external tangents of ω_1 and ω_2 intersect on ω.
103. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that OP = OQ.
104. Let ABC be a triangle with AB = AC. The angle bisectors of ∠ CAB and ∠ ABC meet the sides BC and CA at D and E, respectively. Let K be the in centre of triangle ADC.Suppose that ∠BEK = 45^o. Find all possible values of ∠ CAB.
105. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that
$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$
Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.
106. Let P be a point inside the triangle ABC. The lines AP , BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that SC = SP . Prove that MK = ML.
107. Let ABC be an acute triangle with circumcircle Γ. Let l be a tangent line to Γ, and let l_a, l_b and

l_c be the lines obtained by reflecting l in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines l_a, l_b and l_c is tangent to the circle Γ.

108. Given triangle ABC the point J is the centre of the excircle opposite the vertex A.This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively.The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC.Prove that M is the midpoint of ST. (The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC,to the ray AB beyond B, and to the ray AC beyond C.)
109. Let ABC be a triangle with ∠BCA = 90^o, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that BK = BC. Similarly, let L be the point on the segment BX such that AL = AC. Let M be the point of intersection of AL and BK.Show that MK = ML.
110. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.
111. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.
112. Convex quadri lateral ABCD has ∠ABC = ∠CDA = 90^o. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^o, \angle THC - \angle DTC = 90^o$$

Prove that line BD is tangent to the circumcircle of triangle TSH.
113. Points P and Q lie on side BC of acute-angled triangle ABC so that ∠PAB = ∠BCA and ∠CAQ = ∠ ABC. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.
114. Let ABC be an acute triangle with AB > AC.Let Γ be its circumcircle,H its orthocentre,and F the foot of the altitude from A.Let M be the midpoint of BC.Let Q be the point on Γ suchn that ∠HQA=90,and let K be the point on Γ such that ∠HKQ=90.Assume that the point A,B,C,K and Q are all different,and lie on Γ in this order. Prove that the circumcircle of triangle KQH and FKM are tangent to each other.
115. Triangle ABC has cicumcircle Ω and circumcenter O.A circle Γ with centre A intersects the segment BC at point D and E,such that B,D,E and C are all different and lie on the line BC in this order .Let F and G be the points of intersection of Γ and Ω,such that A,F,B,C and G lie on Ω in this order.Let K be the second point of intersection of the circumcircle of the triangle BDF and segment AB.Let L be the second point of intersection of circumcircle of triangle CGE and the segment CA. Suppose that lines FK and Gl are different and intersects at the point X.Prove that X lies on the line AO.
116. Trianlge BCF has a right angle at B.Let A be the point on line CF such that FA=FB and F lies between A and C.Point D is chosen such that DA=DC and AC is the bisector of ∠DAB.Point E is chosen such that EA=ED and AD is the bisector of ∠EAC.Let M be the midpoint of CF.Let X be the point such that AMXE is a parallelogram (where AM ||EM and AE || MX).Prove that lines BD,FX,and ME are concurrent.
117. Let R and S be different points on a circle Ω such that RS is not a diameter. Let l be the tangent line to Ω at R. Point T is such that S

is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects l at two distinct points. Let A be the common point of Γ and l that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

118. Let Γbe the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that AD = AE. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

119. A convex quadrilateral ABCD satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside ABCD so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.
Prove that $\angle BXA + \angle DXC = 180^o$.