

# Geometry through Trigonometry

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**Abstract**—The focus of this text is on investigating the properties of triangles and circles through trigonometry. Several proofs in geometry require a lot of constructions, which may not be easy for all the students in a class. This book provides an alternative approach to geometry which may be useful to those having difficulties with traditional methods. In the process, almost all the basic concepts of trigonometry are covered. Instead of teaching geometry and trigonometry as separate subjects in high school, this textbook shows how to develop a holistic approach for teaching math.

## 1 THE RIGHT ANGLED TRIANGLE

**Definition 1.1.** A right angled triangle looks like Fig. 1.0. with angles  $\angle A, \angle B$  and  $\angle C$  and sides  $a, b$

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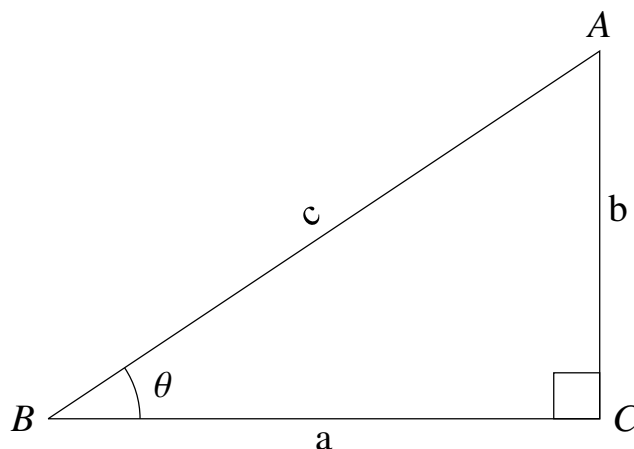


Fig. 1.0: Right Angled Triangle

and  $c$ . The unique feature of this triangle is  $\angle C$  which is defined to be  $90^\circ$ .

**Definition 1.2.** For simplicity, let the greek letter  $\theta = \angle B$ . We have the following definitions.

$$\begin{aligned} \sin \theta &= \frac{a}{c} & \cos \theta &= \frac{b}{c} \\ \tan \theta &= \frac{b}{a} & \cot \theta &= \frac{a}{b} \\ \csc \theta &= \frac{c}{a} & \sec \theta &= \frac{c}{b} \end{aligned} \quad (1.1)$$

### 1.1 Sum of Angles

**Definition 1.3.** In Fig. 1.0, the sum of all the angles on the top or bottom side of the straight line  $XY$  is  $180^\circ$ .

**Definition 1.4.** In Fig. 1.0, the straight line making an angle of  $90^\circ$  to the side  $AC$  is said to be parallel to the side  $BC$ . Note there is an angle at  $A$  that is equal to  $\theta$ . This is one property of parallel lines. Thus,  $\angle YAZ = 90^\circ$ .

**Problem 1.1.** Show that  $\angle VAZ = 90^\circ - \theta$

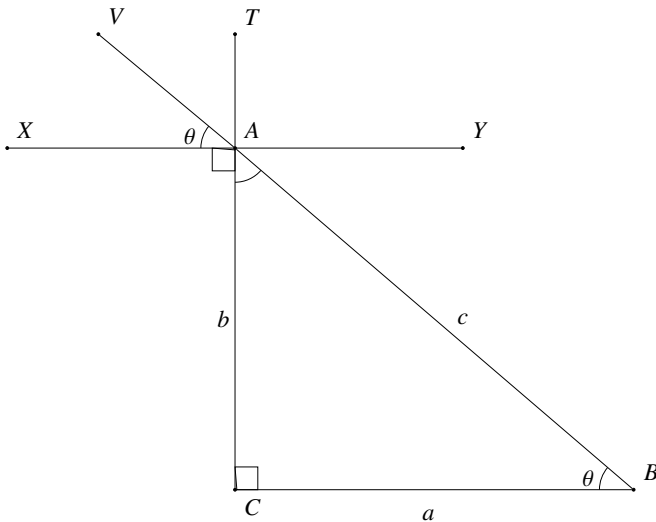


Fig. 1.0: Sum of angles of a triangle

*Proof.* Considering the line XAZ,

$$\theta + 90^\circ + \angle VAZ = 180^\circ \quad (1.2)$$

$$\Rightarrow \angle VAZ = 90^\circ - \theta \quad (1.3)$$

**Problem 1.2.** Show that  $\angle BAC = 90^\circ - \theta$ .

*Proof.* Consider the line VAB and use the approach in the previous problem. Note that this implies that  $\angle VAZ = \angle BAC$ . Such angles are known as vertically opposite angles.

**Problem 1.3.** Sum of the angles of a triangle is equal to  $180^\circ$

## 1.2 Budhayana Theorem

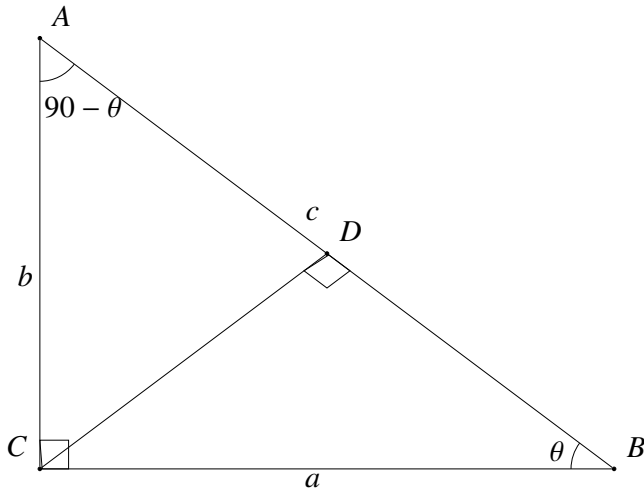


Fig. 1.3: Budhayana Theorem

**Problem 1.4.** Using Fig. 1.0, show that

$$\cos \theta = \sin(90^\circ - \theta) \quad (1.4)$$

*Proof.* From Problem 1.2 and (1.1)

$$\cos(90^\circ - \theta) = \frac{b}{c} = \sin \theta \quad (1.5)$$

**Problem 1.5.** Using Fig. 1.3, show that

$$c = a \cos \theta + b \sin \theta \quad (1.6)$$

*Proof.* We observe that

$$BD = a \cos \theta \quad (1.7)$$

$$AD = b \cos(90 - \theta) = b \sin \theta \quad (\text{From } (1.2)) \quad (1.8)$$

Thus,

$$BD + AD = c = a \cos \theta + b \sin \theta \quad (1.9)$$

**Problem 1.6.** From (1.6), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1.10)$$

*Proof.* Dividing both sides of (1.6) by  $c$ ,

$$1 = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta \quad (1.11)$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from } (1.1)) \quad (1.12)$$

**Problem 1.7.** Using (1.6), show that

$$c^2 = a^2 + b^2 \quad (1.13)$$

(1.13) is known as the Budhayana theorem. It is also known as the Pythagoras theorem.

*Proof.* From (1.6),

$$c = a \frac{a}{c} + b \frac{b}{c} \quad (\text{from } (1.1)) \quad (1.14)$$

$$\Rightarrow c^2 = a^2 + b^2 \quad (1.15)$$

## 2 MEDIANS OF A TRIANGLE

### 2.1 Area of a Triangle

**Definition 2.1.** The area of the rectangle ACBD shown in Fig. 2.0 is defined as  $ab$ . Note that all the angles in the rectangles are  $90^\circ$

**Definition 2.2.** The area of the two triangles constituting the rectangle is the same.

**Definition 2.3.** The area of the rectangle is the sum of the areas of the two triangles inside.

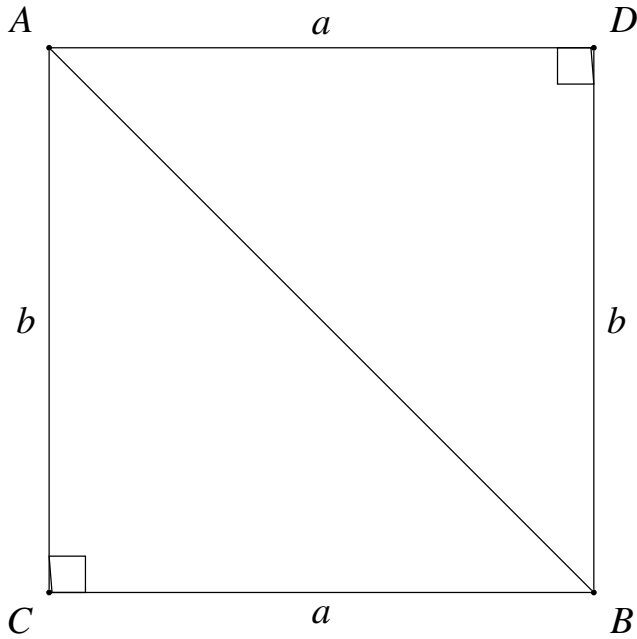


Fig. 2.0: Area of a Right Triangle

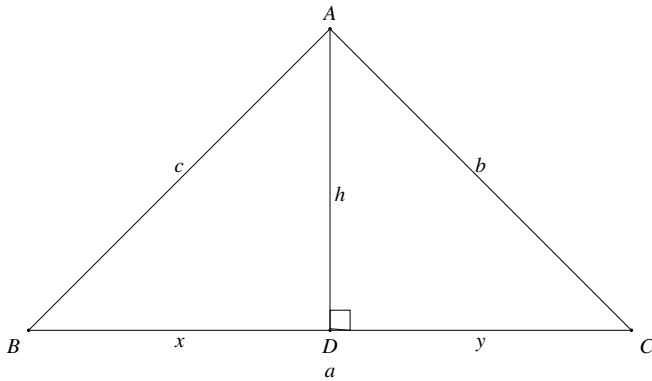


Fig. 2.1: Area of a Triangle

**Problem 2.1.** Show that the area of  $\triangle ABC$  is  $\frac{ab}{2}$

*Proof.* From (2.3),

$$ar(ABCD) = ar(ACB) + ar(ADB) \quad (2.1)$$

Also from (2.2),

$$ar(ACB) = ar(ADB) \quad (2.2)$$

From (2.1) and (2.2),

$$2ar(ACB) = ar(ABCD) = ab \text{ (from (2.0))} \quad (2.3)$$

$$\Rightarrow ar(ACB) = \frac{ab}{2} \quad (2.4)$$

**Problem 2.2.** Show that the area of  $\triangle ABC$  in Fig. 2.1 is  $\frac{1}{2}ah$ .

*Proof.* In Fig. 2.1,

$$ar(\triangle ADC) = \frac{1}{2}hy \quad (2.5)$$

$$ar(\triangle ADB) = \frac{1}{2}hx \quad (2.6)$$

Thus,

$$ar(\triangle ABC) = ar(\triangle ADC) + ar(\triangle ADB) \quad (2.7)$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h(x + y) \quad (2.8)$$

$$= \frac{1}{2}ah \quad (2.9)$$

**Problem 2.3.** Show that the area of  $\triangle ABC$  in Fig. 2.1 is  $\frac{1}{2}ab \sin C$ .

*Proof.* We have

$$ar(\triangle ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (2.10)$$

**Problem 2.4.** Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.11)$$

*Proof.* Fig. 2.1 can be suitably modified to obtain

$$ar(\triangle ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (2.12)$$

Dividing the above by  $abc$ , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.13)$$

This is known as the sine formula.

## 2.2 Median

**Definition 2.4.** The line AD in Fig. 2.4 that divides the side  $a$  in two equal halves is known as the median.

**Problem 2.5.** Show that the median AD in Fig. 2.4 divides  $\triangle ABC$  into triangles ADB and ADC that have equal area.

*Proof.* We have

$$ar(\triangle ADB) = \frac{1}{2} \frac{a}{2} c \sin B = \frac{1}{4} ac \sin B \quad (2.14)$$

$$ar(\triangle ADC) = \frac{1}{2} \frac{a}{2} b \sin C = \frac{1}{4} ab \sin C \quad (2.15)$$

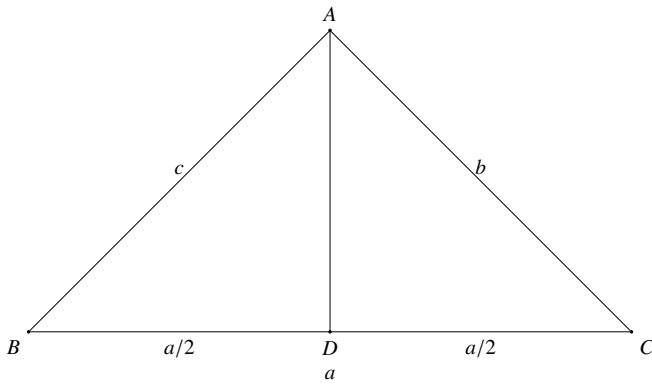


Fig. 2.4: Median of a Triangle

Using the sine formula,  $b \sin C = c \sin B$ ,

$$ar(\triangle ADB) = ar(\triangle ADC) \quad (2.16)$$

**Problem 2.6.**  $BE$  and  $CF$  are the medians in Fig. 2.6. Show that

$$ar(\triangle BFC) = ar(\triangle BEC) \quad (2.17)$$

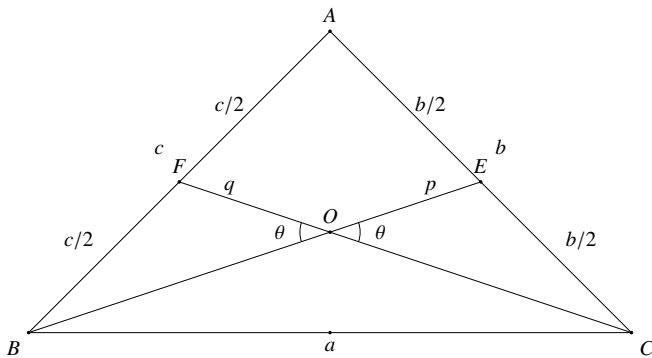
*Proof.* Since  $BE$  and  $CF$  are the medians,

$$ar(\triangle BFC) = \frac{1}{2} ar(\triangle ABC) \quad (2.18)$$

$$ar(\triangle BEC) = \frac{1}{2} ar(\triangle ABC) \quad (2.19)$$

From the above, we infer that

$$ar(\triangle BFC) = ar(\triangle BEC) \quad (2.20)$$

Fig. 2.6:  $O$  is the Intersection of Two Medians

**Problem 2.7.** The medians  $BE$  and  $CF$  in Fig. 2.6

meet at point  $O$ . Show that

$$\frac{OB}{OE} = \frac{OC}{OF} \quad (2.21)$$

*Proof.* From Problem 2.6,

$$ar(\triangle BFC) = ar(\triangle BEC) \quad (2.22)$$

$$\begin{aligned} &\Rightarrow ar(\triangle BOF) + ar(\triangle BOC) \\ &= ar(\triangle BOC) + ar(\triangle COE) \end{aligned} \quad (2.23)$$

resulting in

$$ar(\triangle BOF) = ar(\triangle COE) \quad (2.24)$$

Using the sine formula for area of a triangle, the above equation can be expressed as

$$\frac{1}{2} OB \cdot OF \sin \theta = \frac{1}{2} OC \cdot OE \sin \theta \quad (2.25)$$

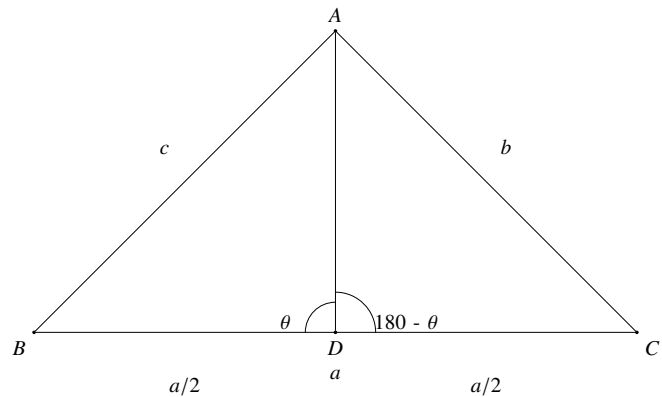
$$\Rightarrow \frac{OB}{OE} = \frac{OC}{OF} \quad (2.26)$$

**Definition 2.5.** We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 2.7,

$$\frac{1}{2} a AD \sin \theta = \frac{1}{2} a AD \sin(180^\circ - \theta) \quad (2.27)$$

$$\Rightarrow \sin \theta = \sin(180^\circ - \theta). \quad (2.28)$$

Note that our geometric definition of  $\sin \theta$  holds only for  $\theta < 90^\circ$ . (2.28) allows us to extend this definition for  $\angle ADC > 90^\circ$ .

Fig. 2.7:  $\sin \theta = \sin(180^\circ - \theta)$ 

*Proof.* Using the sine formula and (2.8), areas of some triangles in Fig. 2.9 are listed in the following

**Problem 2.8.** In Fig. 2.9, show that

$$ar(\Delta AFE) = \frac{1}{4} ar(\Delta ABC) \quad (2.29)$$

*Proof.* We have

$$ar(\Delta AFE) = \frac{1}{2} \frac{b}{2} \frac{c}{2} \sin A = \frac{1}{4} \cdot \frac{1}{2} bc \sin A \quad (2.30)$$

$$= \frac{1}{4} ar(\Delta ABC) \quad (2.31)$$

**Problem 2.9.** Using Fig. 2.9, show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \quad (2.32)$$

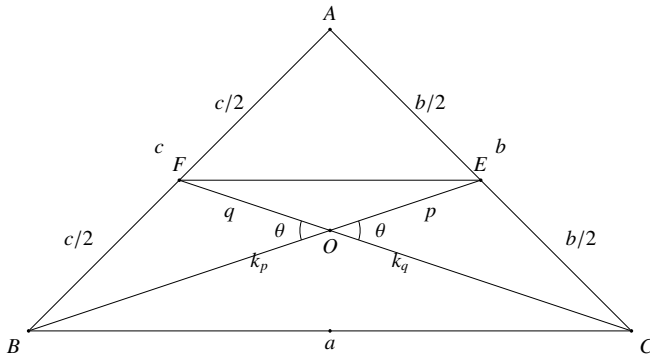


Fig. 2.9:  $O$  divides medians in the ratio 2 : 1

table

Triangle	Area
OFE	$\frac{1}{2} pq \sin \theta$
BOF	$\frac{k}{2} pq \sin \theta$
COE	$\frac{k}{2} pq \sin \theta$
BOC	$\frac{k^2}{2} pq \sin \theta$
BOC	$\frac{1}{2} ar(\Delta ABC)$

where we have used the fact that

$$\sin \angle BOC = \sin(180^\circ - \theta) = \sin \theta \quad (2.33)$$

Since  $BE$  is the median

$$ar(\Delta BEC) = \frac{1}{2} ar(\Delta ABC)$$

$$= ar(\Delta BOC) + ar(\Delta COE)$$

$$ar(\Delta BEA) = \frac{1}{2} ar(\Delta ABC)$$

$$= ar(\Delta AFE) + ar(\Delta BOF) + ar(\Delta FOE) \quad (2.34)$$

Substituting the respective values from the above table,

$$\frac{1}{2} ar(\Delta ABC) = \frac{k^2}{2} pq \sin \theta + \frac{k}{2} pq \sin \theta$$

$$\frac{1}{2} ar(\Delta ABC) = \frac{k}{2} pq \sin \theta + \frac{1}{2} pq \sin \theta + \frac{1}{4} ar(\Delta ABC) \quad (2.35)$$

Simplifying the above,

$$k(k+1) = 2(k+1) \quad (2.36)$$

Since  $k \neq -1$ ,  $k = 2$  and the proof is complete.

### 2.3 Similar Triangles

**Problem 2.10.** In Fig. 2.10, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (2.37)$$

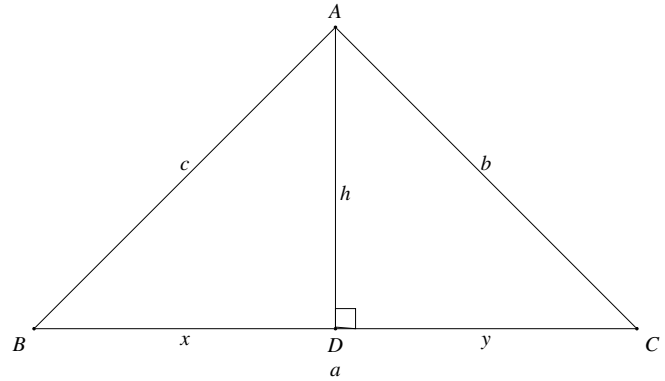


Fig. 2.10: The cosine formula

*Proof.* From the figure, the first of the following equations

$$a = b \cos C + c \cos B \quad (2.38)$$

$$b = c \cos A + a \cos C \quad (2.39)$$

$$c = b \cos A + a \cos B \quad (2.40)$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.41)$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \quad (2.42)$$

**Problem 2.11.** In Fig. 2.11, show that  $EF = \frac{a}{2}$ .

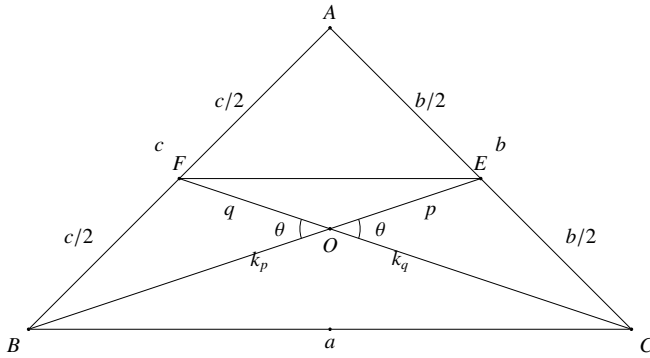


Fig. 2.11: Similar Triangles

*Proof.* Using the cosine formula for  $\triangle AEF$ ,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (2.43)$$

$$= \frac{b^2 + c^2 - 2bc \cos A}{4} \quad (2.44)$$

$$= \frac{a^2}{4} \quad (2.45)$$

$$\Rightarrow EF = \frac{a}{2} \quad (2.46)$$

**Definition 2.6.** The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.

**Problem 2.12.** Show that similar triangles have the same angles.

*Proof.* Use cosine formula and the proof is trivial.

**Problem 2.13.** Show that in Fig. 2.11,  $EF \parallel BC$ .

*Proof.* Since  $\triangle AEF \sim \triangle ABC$ ,  $\angle AEF = \angle ACB$ . Hence the line  $EF \parallel BC$ .

**Problem 2.14.** In Fig. 2.14, the line AO cuts EF at G and is extended to meet the side BC at D. Show that

$$\frac{OA}{OD} = 2. \quad (2.47)$$

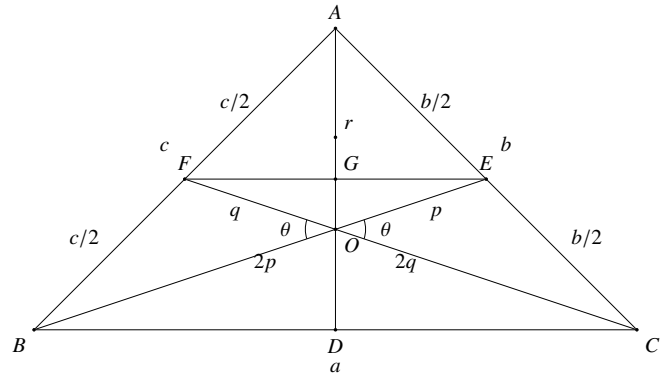


Fig. 2.14: Similar Triangles and Median

*Proof.* Since  $EF \parallel BC$ ,

$$\triangle AGE \sim \triangle ADC \Rightarrow AG = GD = \frac{AD}{2} \quad (2.48)$$

Also,

$$\triangle OGE \sim \triangle ODB \Rightarrow OD = 2r \quad (2.49)$$

Thus, we have the following relations

$$GD = OG + OD = r + 2r = 3r \quad (2.50)$$

$$AG = GD = 3r \quad (2.51)$$

$$OA = OG + AG = r + 3r = 4r \quad (2.52)$$

Hence

$$\frac{OA}{OD} = \frac{4r}{2r} = 2 \quad (2.53)$$

**Problem 2.15.** In Fig. 2.15, BE is a median of  $\triangle ABC$  and  $\frac{OA}{OD} = 2$ . Show that AD is also a median.

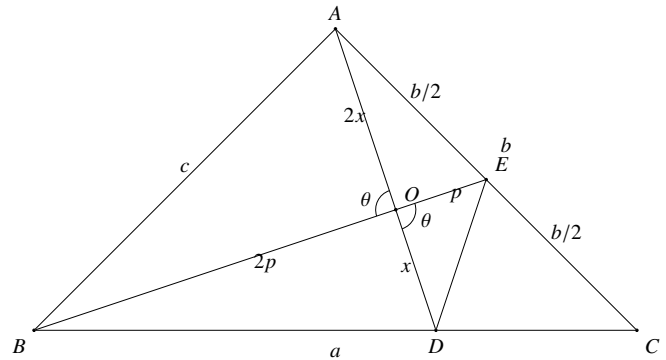


Fig. 2.15: Medians meet at a point

*Proof.* Using the cosine formula for  $\triangle ODE$  and

$\triangle OAB$ ,

$$DE^2 = p^2 + t^2 - 2pq \cos \theta \quad (2.54)$$

$$c^2 = (2p)^2 + (2t)^2 - 2(2p)(2q) \cos \theta \quad (2.55)$$

$$\Rightarrow DE = \frac{c}{2}. \quad (2.56)$$

Now using the cosine formula in  $\triangle CDE$  and  $\triangle ABC$ ,

$$\left(\frac{c}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + CD^2 - 2CD\left(\frac{b}{2}\right) \cos C \quad (2.57)$$

$$c^2 = b^2 + a^2 - 2ab \cos C, \quad (2.58)$$

which can be simplified to obtain

$$a^2 - 2ab \cos C = (2CD)^2 - 2(2CD)b \cos C \quad (2.59)$$

$$\begin{aligned} \Rightarrow (2CD)^2 - 2(2CD)b \cos C + (b \cos C)^2 \\ = a^2 - 2ab \cos C + (b \cos C)^2 \end{aligned} \quad (2.60)$$

$$\Rightarrow (2CD - b \cos C)^2 = (a - b \cos C)^2 \quad (2.61)$$

$$\Rightarrow 2CD - b \cos C = a - 2 \cos C \quad (2.62)$$

$$\Rightarrow CD = \frac{a}{2} \quad (2.63)$$

Thus,  $AD$  is a median of  $\triangle ABC$ .

**Conclusion:** The medians of a triangle meet at a point.

### 3 ANGLE AND PERPENDICULAR BISECTORS

#### 3.1 Angle Bisectors

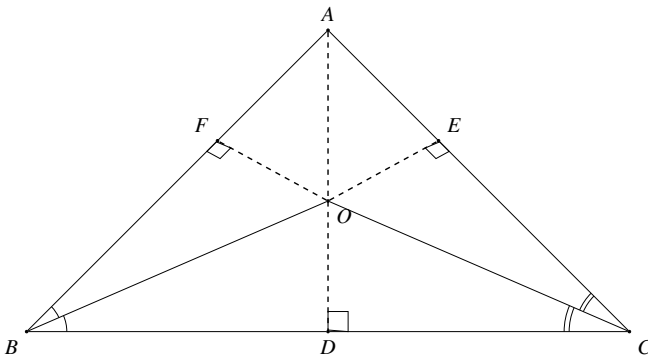


Fig. 3.0: Angle bisectors meet at a point

**Definition 3.1.** In Fig. 3.0,  $OB$  divides the  $\angle B$  into half, i.e.

$$\angle OBC = \angle OBA \quad (3.1)$$

$OB$  is known as an angle bisector.

$OB$  and  $OC$  are angle bisectors of angles  $B$  and  $C$ .  $OA$  is joined and  $OD, OF$  and  $OE$  are perpendiculars to sides  $a, b$  and  $c$ .

**Problem 3.1.** Show that  $OD = OE = OF$ .

*Proof.* In  $\triangle ODC$  and  $OEC$ ,

$$OD = OC \sin \frac{C}{2} \quad (3.2)$$

$$OE = OC \sin \frac{C}{2} \quad (3.3)$$

$$\Rightarrow OD = OE. \quad (3.4)$$

Similarly,

$$OD = OF. \quad (3.5)$$

**Problem 3.2.** Show that  $OA$  is the angle bisector of  $\angle A$

*Proof.* In  $\triangle OFA$  and  $OEA$ ,

$$OF = OE \quad (3.6)$$

$$\Rightarrow OA \sin OAF = OA \sin OAE \quad (3.7)$$

$$\Rightarrow \sin OAF = \sin OAE \quad (3.8)$$

$$\Rightarrow \angle OAF = \angle OAE \quad (3.9)$$

which proves that  $OA$  bisects  $\angle A$ . **Conclusion:** The angle bisectors of a triangle meet at a point.

#### 3.2 Congruent Triangles

**Problem 3.3.** Show that in  $\triangle ODC$  and  $OEC$ , corresponding sides and angles are equal.

**Definition 3.2.** Note that  $\triangle ODC$  and  $OEC$  are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.

**Problem 3.4.** SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.

**Problem 3.5.** ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.

**Problem 3.6.** SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.

**Problem 3.7.** RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

### 3.3 Perpendicular Bisectors

**Definition 3.3.** In Fig. 3.8,  $OD \perp BC$  and  $BD = DC$ .  $OD$  is defined as the perpendicular bisector of  $BC$ .

**Problem 3.8.** In Fig. 3.8, show that  $OA = OB = OC$ .

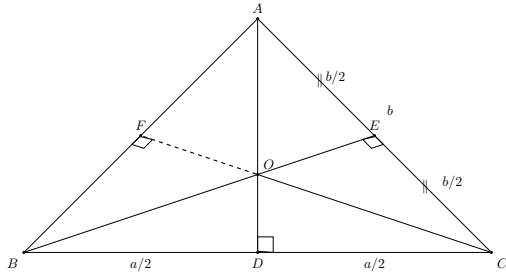


Fig. 3.8: Perpendicular bisectors meet at a point

*Proof.* In  $\Delta$ s  $ODB$  and  $ODC$ , using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \quad (3.10)$$

Since  $BD = DC = \frac{a}{2}$ ,  $OB = OC$ . Similarly, it can be shown that  $OA = OC$ . Thus,  $OA = OB = OC$ .

**Definition 3.4.** In  $\Delta AOB$ ,  $OA = OB$ . Such a triangle is known as an isocetes triangle.

**Problem 3.9.** Show that  $AF = BF$ .

*Proof.* Trivial using Budhayana's theorem. This shows that  $OF$  is a perpendicular bisector of  $AB$ .  
**Conclusion:** The perpendicular bisectors of a triangle meet at a point.

### 3.4 Perpendiculars from Vertex to Opposite Side

In Fig. 3.10,  $AD \perp BC$  and  $BE \perp AC$ .  $CF$  passes through  $O$  and meets  $AB$  at  $F$ .

**Problem 3.10.** Show that

$$OE = c \cos A \cot C \quad (3.11)$$

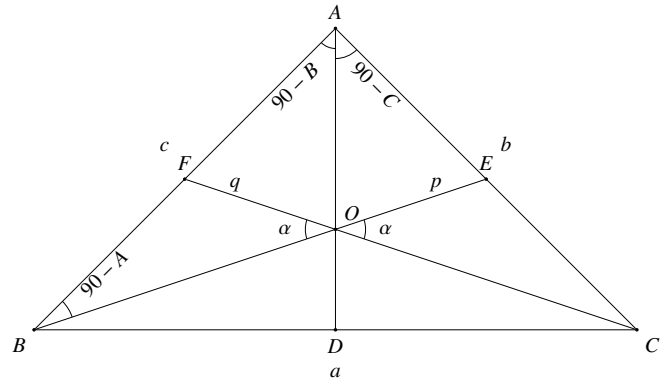


Fig. 3.10: Perpendiculars from vertex to opposite side meet at a point

*Proof.* In  $\Delta$ s  $AEB$  and  $AEO$ ,

$$AE = c \cos A \quad (3.12)$$

$$OE = AE \tan(90^\circ - C) (\because ADC \text{ is right angled}) \quad (3.13)$$

$$= AE \cot C \quad (3.14)$$

From both the above, we get the desired result.

**Problem 3.11.** Show that  $\alpha = A$ .

*Proof.* In  $\Delta OEC$ ,

$$CE = a \cos C (\because BEC \text{ is right angled}) \quad (3.15)$$

Hence,

$$\begin{aligned} \tan \alpha &= \frac{CE}{OE} \\ &= \frac{a \cos C}{c \cos A \cot C} \\ &= \frac{a \cos C \sin C}{c \cos A \cos C} \\ &= \frac{a \sin C}{c \cos A} \\ &= \frac{c \sin A}{c \cos A} \left( \because \frac{a}{\sin A} = \frac{c}{\sin C} \right) \\ &= \tan A \end{aligned} \quad (3.16)$$

$$\Rightarrow \alpha = A$$

**Problem 3.12.** Show that  $CF \perp AB$

*Proof.* Consider triangle OFB and the result of the previous problem.  $\therefore$  the sum of the angles of a triangle is  $180^\circ$ ,  $\angle CFB = 90^\circ$ . **Conclusion:** The perpendiculars from the vertex of a triangle to the opposite side meet at a point.



## 4 CIRCLE

## 4.1 Chord of a Circle

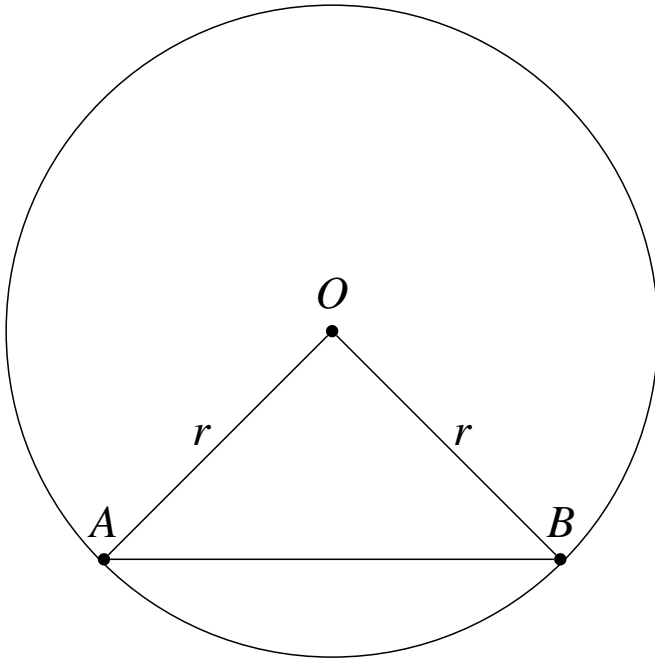


Fig. 4.0: Circle Definitions

**Definition 4.1.** Fig. 4.0 represents a circle. The points in the circle are at a distance  $r$  from the centre  $O$ .  $r$  is known as the radius.

## 4.2 Chords of a circle

**Definition 4.2.** In Fig. 4.0,  $A$  and  $B$  are points on the circle. The line  $AB$  is known as a chord of the circle.

**Problem 4.1.** In Fig. 4.1 Show that  $\angle OAB = 2\angle APB$ .

*Proof.* In Fig. 4.1, the triangles  $OPA$  and  $OPB$  are isosceles. Hence,

$$\angle OPB = \angle OBP = \theta_1 \quad (4.1)$$

$$\angle OPA = \angle OAP = \theta_2 \quad (4.2)$$

Also,  $\alpha$  and  $\beta$  are exterior angles corresponding to the triangle  $OPB$  and  $OPA$  respectively. Hence

$$\alpha = 2\theta_1 \quad (4.3)$$

$$\beta = 2\theta_2 \quad (4.4)$$

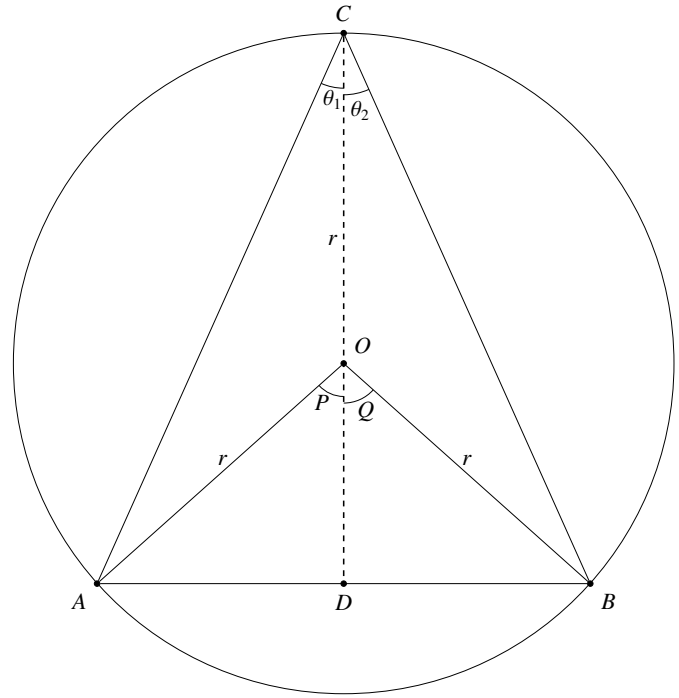


Fig. 4.1: Angle subtended by chord  $AB$  at the centre  $O$  is twice the angle subtended at  $P$ .

Thus,

$$\angle AOB = \alpha + \beta \quad (4.5)$$

$$= \theta_1 + \theta_2 \quad (4.6)$$

$$= \angle APB \quad (4.7)$$

**Definition 4.3.** The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 4.2,  $AB$  is the diameter and passes through the centre  $O$

**Problem 4.2.** In Fig. 4.2, show that  $\angle APB = 90^\circ$ .

**Problem 4.3.** In Fig. 4.3, show that

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \quad (4.8)$$

*Proof.* Use Problem 4.1.

**Problem 4.4.** In Fig. 4.3, show that the triangles  $PAB$  and  $PBD$  are similar

*Proof.* Trivial using previous problem

**Problem 4.5.** In Fig. 4.3, show that

$$PA.PB = PC.PD \quad (4.9)$$

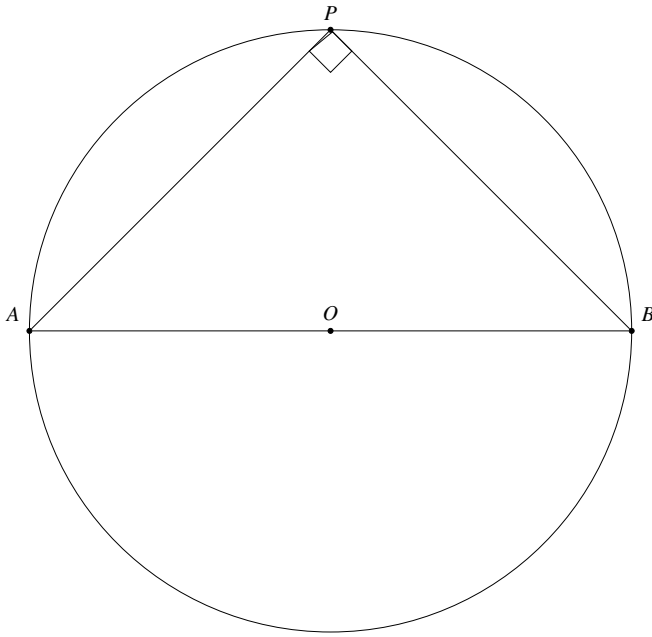
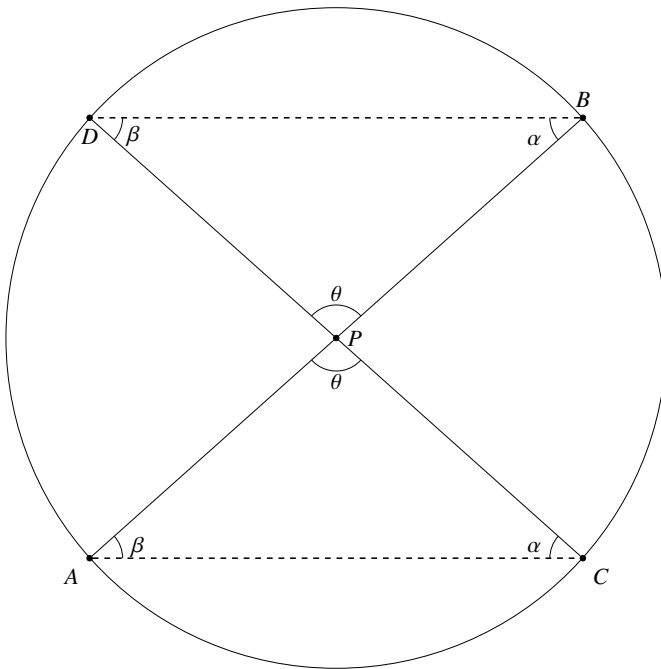


Fig. 4.2: Diameter of a circle.

Fig. 4.3:  $PA.PB = PC.PD$ 

*Proof.* Since triangles  $PAC$  and  $PBD$  are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \quad (4.10)$$

$$\Rightarrow PA.PB = PC.PD \quad (4.11)$$

**Definition 4.4.** The line  $PX$  in Fig. 4.6 touches the circle at exactly one point  $P$ . It is known as the tangent to the circle.

**Problem 4.6.**  $OP$  is the perpendicular to the line  $PX$  as shown in the Fig. 4.6. Show that  $OP$  is the shortest distance between the point  $O$  and the line  $PX$ .

*Proof.* Let  $P_1$  be a point on the line  $PX$ . Then  $OPP_1$  is a right angled triangle. Using Budhayana's theorem,

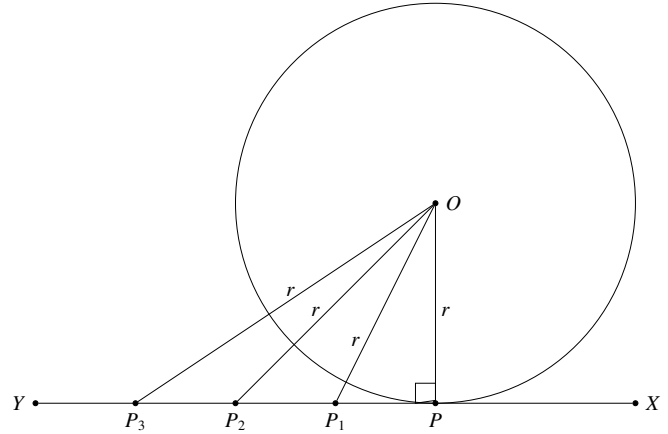
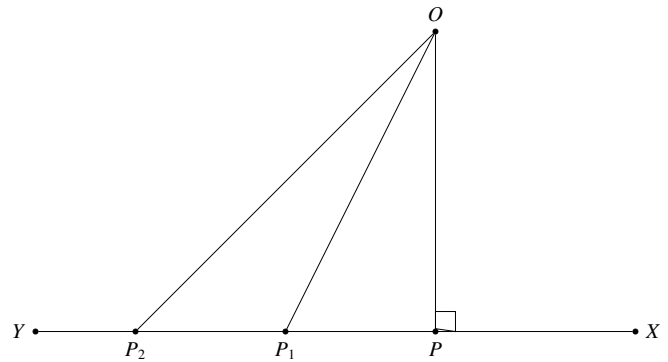


Fig. 4.6: Tangent to a Circle.

Fig. 4.6: Shortest distance from  $O$  to line  $PX$ 

$$\begin{aligned} OP_1^2 &= OP^2 + PP_1^2 \\ \Rightarrow OP_1 &> OP \end{aligned} \quad (4.12)$$

Thus,  $OP$  is the shortest distance between  $O$  and line  $PX$ .

**Problem 4.7.** Show that  $\angle OPX = 90^\circ$

*Proof.* In Fig. 4.6, we can see that  $OP$  is the radius of the circle and the length of all line segments from  $O$  to the line  $PX > r$ . Using the result of the previous problem, it is obvious that  $OP \perp PX$ .

**Problem 4.8.** In Fig. 4.8 show that

$$\angle PCA = \angle PBC \quad (4.13)$$

$O$  is the centre of the circle and  $PC$  is the tangent.

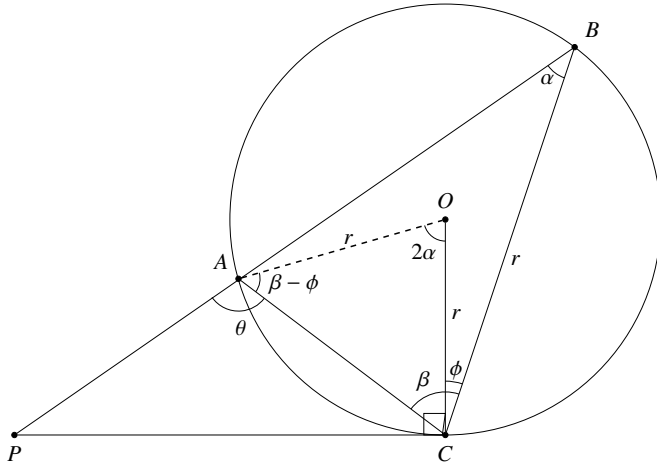


Fig. 4.8:  $PA.PB = PC^2$ .

*Proof.* For convenience, greek letters are used for representing certain angles. Since  $\triangle OAC$  is isosceles,

$$2\alpha + 2(\beta - \phi) = 180^\circ \quad (4.14)$$

$$\Rightarrow \alpha + (\beta - \phi) = 90^\circ \quad (4.15)$$

$$\Rightarrow \alpha + \beta = 90^\circ + \phi \quad (4.16)$$

Since  $\theta$  is an exterior angle for the  $\triangle ABC$ ,

$$\theta = \alpha + \beta \quad (4.17)$$

From both the above equations

$$\theta = 90^\circ + \phi \quad (4.18)$$

Since  $PC$  is the tangent,

$$\angle PCB = 90^\circ + \phi = \theta \quad (4.19)$$

Considering the sum of angles in  $\triangle PAC$  and  $\triangle PBC$ ,

$$\angle P + \theta + \angle PCA = 180^\circ \quad (4.20)$$

$$\angle P + \theta + \alpha = 180^\circ \quad (4.21)$$

Hence,

$$\angle PCA = \alpha \quad (4.22)$$

**Problem 4.9.** In Fig. 4.8, show that the triangles  $PAC$  and  $PBC$  are similar.

*Proof.* From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

**Problem 4.10.** Show that  $PA.PB = PC^2$

*Proof.* Since  $\triangle PAC \sim \triangle PBC$ , their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \quad (4.23)$$

$$\Rightarrow PA.PB = PC^2 \quad (4.24)$$

**Problem 4.11.** In Fig. 4.11, show that

$$PA.PB = PC.PD \quad (4.25)$$

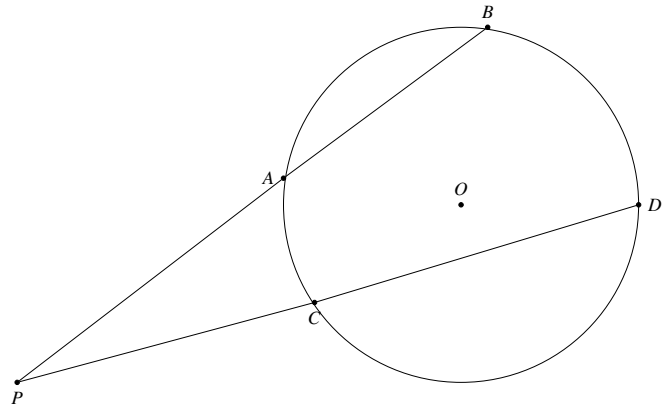


Fig. 4.11:  $PA.PB = PC^2$ .

*Proof.* Draw a tangent and use the previous problem.

## 5 AREA OF A CIRCLE

### 5.1 The Regular Polygon

**Definition 5.1.** In Fig. 5.0, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general,  $n$  number of triangles can be arranged to form a regular polygon.

**Definition 5.2.** The angle formed by each of the congruent triangles at the centre of a regular polygon of  $n$  sides is  $\frac{360^\circ}{n}$ .

**Problem 5.1.** Show that the area of a regular polygon is given by

$$\frac{n}{2} r^2 \sin \frac{360^\circ}{n} \quad (5.1)$$

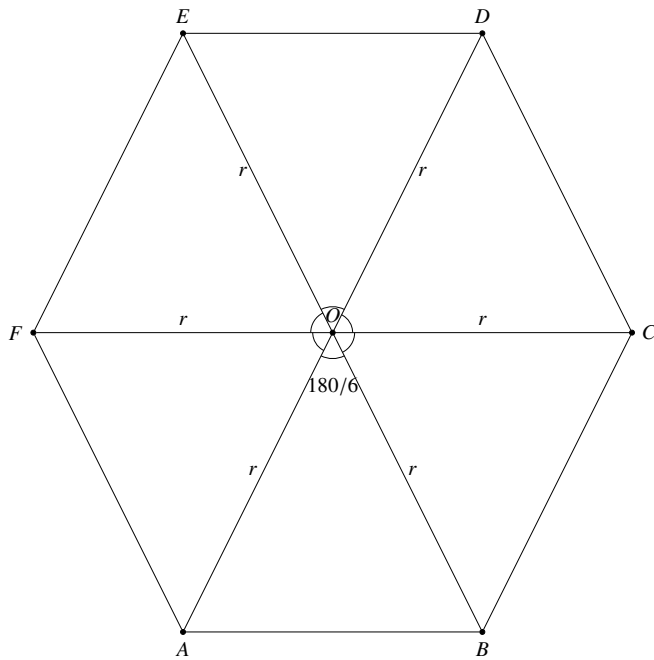


Fig. 5.0: Polygon Definition

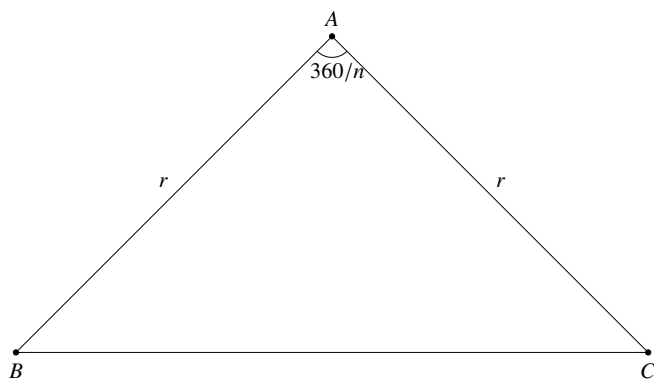


Fig. 5.1: Polygon Area

*Proof.* The triangle that forms the polygon of  $n$  sides is given in Fig. 5.1. Thus,

$$\begin{aligned} \text{ar}(\text{polygon}) &= n \text{ar}(\triangle ABC) \\ &= \frac{n}{2} r^2 \sin \frac{360^\circ}{n} \end{aligned} \quad (5.2)$$

**Problem 5.2.** Using Fig. 5.2, show that

$$\frac{n}{2} r^2 \sin \frac{360^\circ}{n} < \text{area of circle} < n r^2 \tan \frac{180^\circ}{n} \quad (5.3)$$

The portion of the circle visible in Fig. 5.2 is defined to be a sector of the circle.

*Proof.* Note that the circle is squeezed between the

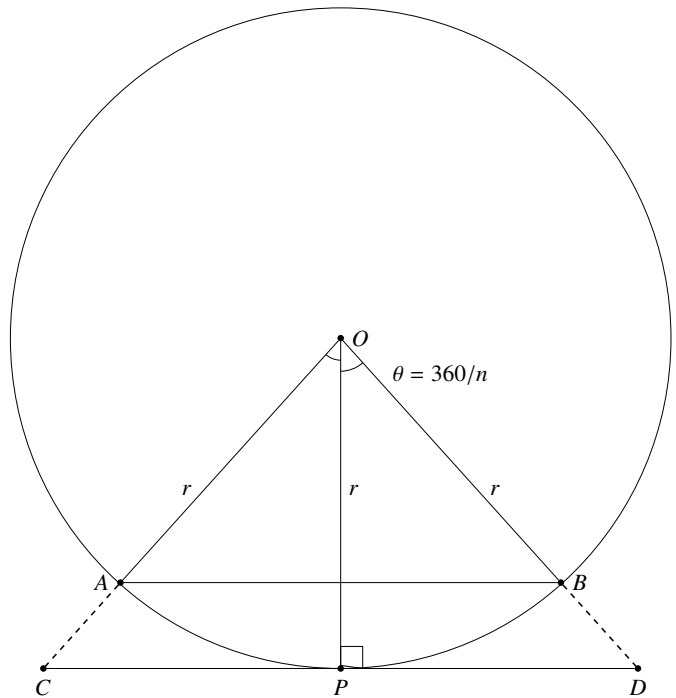


Fig. 5.2: Circle Area in between Area of Two Polygons

inner and outer regular polygons. As we can see from Fig. 5.2, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$\text{ar}(\triangle OAB) = \frac{1}{2} r^2 \sin \frac{360^\circ}{n} \quad (5.4)$$

$$\text{ar}(\triangle OPQ) = 2 \times \frac{1}{2} \times r \tan \frac{360/n}{2} \times r \quad (5.5)$$

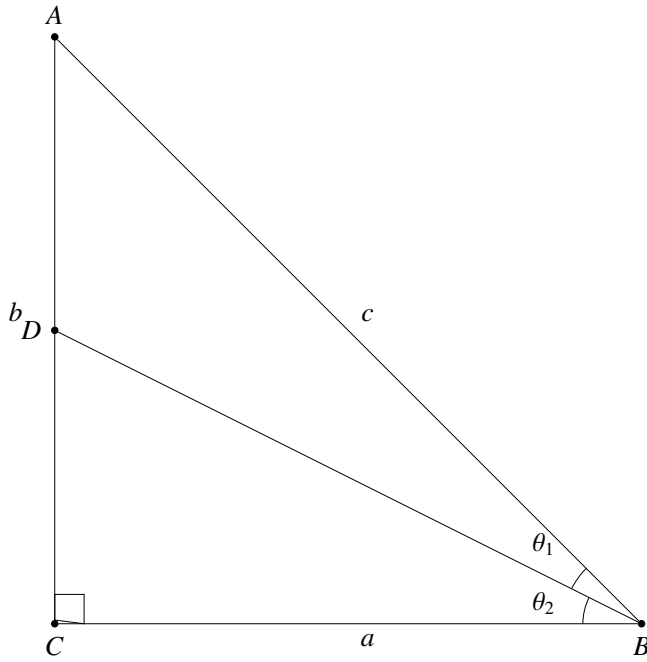
$$= r^2 \tan \frac{180^\circ}{n}, \quad (5.6)$$

we obtain (5.3).

**Problem 5.3.** Using Fig. 5.3, show that

$$\sin \theta_1 = \sin(\theta_1 + \theta_2) \cos \theta_2 - \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (5.7)$$

*Proof.* The following equations can be obtained from the figure using the formula for the area of a

Fig. 5.3:  $\sin 2\theta = 2 \sin \theta \cos \theta$ 

triangle

$$ar(\triangle ABC) = \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (5.8)$$

$$= ar(\triangle BDC) + ar(\triangle ADB) \quad (5.9)$$

$$= \frac{1}{2}cl \sin \theta_1 + \frac{1}{2}al \sin \theta_2 \quad (5.10)$$

$$= \frac{1}{2}ac \sin \theta_1 \sec \theta_2 + \frac{1}{2}a^2 \tan \theta_2 \quad (5.11)$$

( $\because l = a \sec \theta_2$ ). From the above,

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (5.12)$$

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \cos(\theta_1 + \theta_2) \tan \theta_2 \quad (5.13)$$

Multiplying both sides by  $\cos \theta_2$ ,

$$\Rightarrow \sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (5.14)$$

resulting in

$$\Rightarrow \sin \theta_1 = \sin(\theta_1 + \theta_2) \cos \theta_2 - \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (5.15)$$

**Problem 5.4.** Prove the following identities

1)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (5.16)$$

2)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (5.17)$$

*Proof.* In (5.7), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \quad (5.18)$$

This gives (5.16). In (5.16), replace  $\alpha$  by  $90^\circ - \alpha$ . This results in

$$\begin{aligned} \sin(90^\circ - \alpha - \beta) \\ &= \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \\ \Rightarrow \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned} \quad (5.19)$$

**Problem 5.5.** Using (5.7) and (5.17), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (5.20)$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \quad (5.21)$$

*Proof.* From (5.7),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (5.22)$$

Using (5.17) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + (\cos \theta_1 \cos \theta_2 \\ &\quad - \sin \theta_1 \sin \theta_2) \sin \theta_2 \end{aligned} \quad (5.23)$$

which can be expressed as

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + \cos \theta_1 \cos \theta_2 \sin \theta_2 \\ &\quad - \sin \theta_1 \sin^2 \theta_2 \end{aligned} \quad (5.24)$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (5.25)$$

we obtain

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \cos \theta_1 \cos \theta_2 \sin \theta_2 \\ &\quad + \sin \theta_1 \cos^2 \theta_2 \end{aligned} \quad (5.26)$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (5.27)$$

after factoring out  $\cos \theta_2$ . Using a similar approach, (5.21) can also be proved.

**Problem 5.6.** Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (5.28)$$

**Problem 5.7.** Show that

$$\cos^2 \frac{180^\circ}{n} < \frac{\text{area of circle}}{nr^2 \tan \frac{180^\circ}{n}} < 1 \quad (5.29)$$

*Proof.* From (5.3) and (5.28),

$$\begin{aligned} \frac{n}{2} r^2 \sin \frac{360^\circ}{n} &< \text{area of circle} \\ &< nr^2 \tan \frac{180^\circ}{n} \\ \Rightarrow nr^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n} &< \text{area of circle} \\ &< nr^2 \tan \frac{180^\circ}{n} \quad (5.30) \end{aligned}$$

**Problem 5.8.** Show that if

$$\theta_1 < \theta_2, \sin \theta_1 < \sin \theta_2. \quad (5.31)$$

*Proof.* Trivial using the definition and choosing angles  $\theta_1$  and  $\theta_2$  appropriately in a right angled triangle.

**Problem 5.9.** Show that if

$$\theta_1 < \theta_2, \cos \theta_1 > \cos \theta_2. \quad (5.32)$$

**Problem 5.10.** Show that

$$\sin 0^\circ = 0 \quad (5.33)$$

*Proof.* Follows from (5.32).

**Problem 5.11.** Show that

$$\cos 0^\circ = 1 \quad (5.34)$$

**Problem 5.12.** Show that for large values of  $n$

$$\cos^2 \frac{180^\circ}{n} = 1 \quad (5.35)$$

*Proof.* Follows from previous problem.

**Definition 5.3.** The previous result can be expressed as

$$\lim_{n \rightarrow \infty} \cos^2 \frac{180^\circ}{n} = 1 \quad (5.36)$$

**Problem 5.13.** Show that

$$\text{area of circle} = r^2 \lim_{n \rightarrow \infty} n \tan \frac{180^\circ}{n} \quad (5.37)$$

**Definition 5.4.**

$$\pi = \lim_{n \rightarrow \infty} n \tan \frac{180^\circ}{n} \quad (5.38)$$

Thus, the area of a circle is  $\pi r^2$ .

**Definition 5.5.** The radian is a unit of angle defined by

$$1 \text{ radian} = \frac{360^\circ}{2\pi} \quad (5.39)$$

**Problem 5.14.** Show that the circumference of a circle is  $2\pi r$ .

**Problem 5.15.** Show that the area of a sector with angle  $\theta$  in radians is  $\frac{1}{2}r^2\theta$ .

**Problem 5.16.** Show that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (5.40)$$