Computational Approach to School Mathematics

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Contents

***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/codes
```

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1 Triangle

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90°, a = 4$ and $b = 3$.
Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1.1}$$

The following code plots Fig. 1.1.1

```
codes/triangle/rt_triangle.py
```

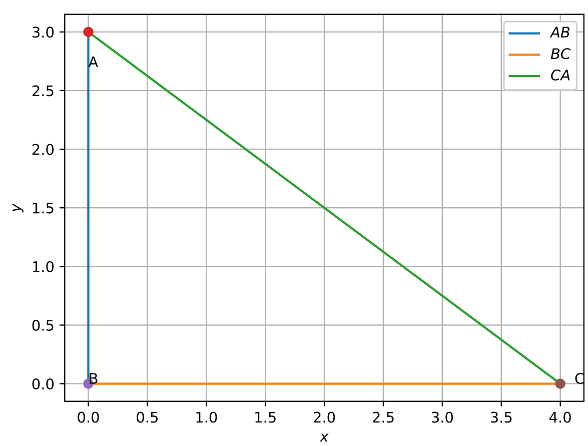

Fig. 1.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \tag{1.1.2.2}$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{1.1.2.3}$$

$$= p \times p + q \times q = p^2 + q^2 \tag{1.1.2.4}$$

Then

$$AB \triangleq \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{1.1.2.5}$$

$$BC = \|\mathbf{C}-\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{1.1.2.6}$$

$$AC = \|\mathbf{A}-\mathbf{C}\|^2 = b^2 \tag{1.1.2.7}$$

From (1.1.2.7),

$$b^2 = \|\mathbf{A}-\mathbf{C}\|^2 = \|\mathbf{A}-\mathbf{C}\|^T \|\mathbf{A}-\mathbf{C}\| \tag{1.1.2.8}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{1.1.2.9}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{1.1.2.10}$$

$$= a^2 + c^2 - 2ap \tag{1.1.2.11}$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{1.1.2.12}$$

From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.1.2.14}$$

The following code plots Fig. 1.1.2

codes/triangle/draw_triangle.py

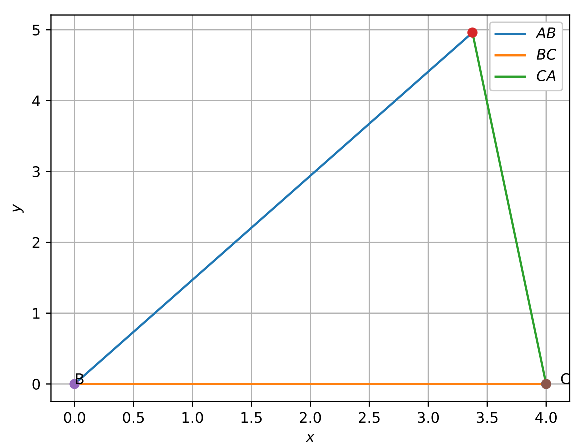

Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{1.1.4.1}$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.
Solution: From the given information,

$$a + b + c = 11 \tag{1.1.5.1}$$

$$b = c \quad (\because B = C = 45^\circ) \tag{1.1.5.2}$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{1.1.5.3}$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \tag{1.1.5.4}$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{1.1.5.5}$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{1.1.5.6}$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{1.1.5.7}$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{1.1.5.8}$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{1.1.5.9}$$

by expanding the determinants. The following code may be used to compute a, b and c.

codes/triangle/triangle_det.py

6. Repeat Problem 1.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \tag{1.1.6.1}$$
$$a - b\sqrt{2} = 0 \tag{1.1.6.2}$$
$$b - c = 0 \tag{1.1.6.3}$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{1.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{1.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. 1.1.7, $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{1.1.7.1}$$
$$\cos B = \frac{x}{b}, \tag{1.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \tag{1.1.7.3}$$
$$b = c\cos A + a\cos C \tag{1.1.7.4}$$
$$c = b\cos A + a\cos B \tag{1.1.7.5}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.7.6}$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.1.7.7}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1.7.8}$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \tag{1.1.7.9}$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

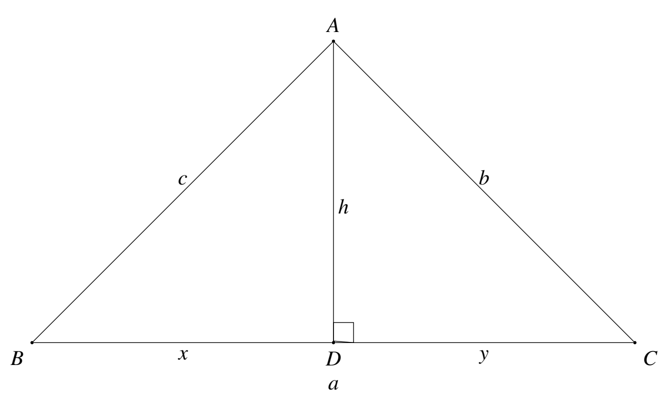

Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
Solution: In Fig. (1.1.7),
$$\sin B = \frac{h}{c} \quad (1.1.8.1)$$
$$\sin C = \frac{h}{b} \quad (1.1.8.2)$$
which can be used to show that
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.1.8.3)$$
Thus,
$$c = \frac{a \sin C}{\sin A} \quad (1.1.8.4)$$
where
$$C = 180 - A - B \quad (1.1.8.5)$$
9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30°$.
Solution: From (1.1.7.9),
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (1.1.9.1)$$
which can be expressed as
$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \quad (1.1.9.2)$$
$$\because (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \quad (1.1.9.3)$$
(1.1.9.2) can be expressed as
$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \quad (1.1.9.4)$$
$$\implies (a - b \cos C)^2 = b^2 \left(1 - \cos^2 C\right) - c^2 \quad (1.1.9.5)$$
$$\text{or}, a = b \cos C \pm \sqrt{b^2 \left(1 - \cos^2 C\right) - c^2} \quad (1.1.9.6)$$
Choose the value(s) for which $a > 0$.
10. The solution of a quadratic equation
$$\alpha x^2 + \beta x + \gamma = 0 \quad (1.1.10.1)$$
is given by
$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (1.1.10.2)$$
Verify (1.1.9.6) using (1.1.10.2).
11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.
Solution: From Baudhayana's theorem,
$$b^2 = a^2 + c^2 \quad (1.1.11.1)$$
$$\implies (18 - c)^2 = 12^2 + c^2 \quad (1.1.11.2)$$
which can be simplified to obtain
$$36c - 180 = 0 \quad (1.1.11.3)$$
$$\implies c = 5 \quad (1.1.11.4)$$
and $b = 13$
12. Find a simpler solution for Problem 1.1.5
Solution: Use cosine formula.
13. In $\triangle ABC$, $a = 7, \angle B = 75^{\circ}$ and $b + c = 13$. Alternatively,
$$a = b \cos C + c \cos B \quad (1.1.13.1)$$
$$b \sin C = c \sin B \quad (1.1.13.2)$$
$$a + b + c = 11 \quad (1.1.13.3)$$
resulting in the matrix equation
$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (1.1.13.4)$$
Solving the equivalent matrix equation gives the desired answer.

1.2 Construction Exercises

1. In $\triangle ABC$, $a = 8, \angle B = 45^{\circ}$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6, \angle B = 60^{\circ}$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a+b+c = 11, \angle B = 30^{\circ}$ and $\angle C = 90^{\circ}$.
4. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60°$.
9. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90°$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
11. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60°$.

12. Construct $\triangle XYZ$ if $XY = 6, \angle X = 30°$ and $\angle Y = 100°$.
13. If $AC = 7, \angle A = 60°$ and $\angle B = 50°$, can you draw the triangle?
14. Construct $\triangle ABC$ given that $\angle A = 60°, \angle B = 30°$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5, \angle Q = 105°$ and $\angle R = 40°$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2, \angle E = 110°$ and $\angle F = 180°$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled $\triangle$ whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 1.2.21.

S.No	**Triangle**	**Given Measurements**		
1	$\triangle$ABC	$\angle A = 85°$	$\angle B = 115°$	AB = 5
2	$\triangle$PQR	$\angle Q = 30°$	$\angle R = 60°$	QR = 4.7
3	$\triangle$ABC	$\angle A = 70°$	$\angle B = 50°$	AC = 3
4	$\triangle$LMN	$\angle L = 60°$	$\angle N = 120°$	LM = 5
5	$\triangle$ABC	BC = 2	AB = 4	AC = 2
6	$\triangle$PQR	PQ = 2.5	QR = 4	PR = 3.5
7	$\triangle$XYZ	XY = 3	YZ = 4	XZ = 5
8	$\triangle$DEF	DE = 4.5	EF = 5.5	DF = 4

TABLE 1.2.21

1.3 Triangle Examples

1. If three sides of one triangle are equal to three sides of the other triangle, then the two triangles are congruent (SSS Congruence Rule).
Solution: Using cosine formula in (1.1.7.9), it can be shown that the angles of the triangle are also equal, hnce they are congruent.
2. If two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle, then the two triangles are congruent (SAS Congruence Rule).
Solution: Use cosine formula in (1.1.7.9).
3. If two angles and the included side of one triangle are equal to two angles and the included side of the other triangle, then the two triangles are congruent (ASA Congruence Rule).
Solution: Use the sine formula in (1.1.8.3) to show that the corresponding sides opposite the angles are equal.
4. If two angles and one side of one triangle are equal to two angles and the corresponding side of the other triangle, then the two triangles are congruent (AAS Congruence Rule)
Solution: Use the sine formula in (1.1.8.3)
5. Angles opposite to equal sides of a triangle are equal.
Solution: Using the sine formula in (1.1.8.3),
$$\frac{\sin A}{a} = \frac{\sin B}{b} \tag{1.3.5.1}$$
Thus, if $A = B$, $\sin A = \sin B \implies a = b$.
6. Sides opposite to equal angles of a triangle are equal.
Solution: Use (1.1.8.3) and the argument in Problem 1.3.5
7. Each angle of an equilateral triangle is of 60°.
Solution: In an equilateral $\triangle$,
$$A = B = C. \tag{1.3.7.1}$$
$$\because A + B + C = 180°, 3A = 180° \tag{1.3.7.2}$$
$$\implies A = 60° \tag{1.3.7.3}$$
8. In a triangle, angle opposite to the longer side is larger (greater).
Solution: Consider a right triangle ABC right angled at B as in Fig. 1.3.8. From Baudhayana's theorem,
$$b^2 = a^2 + c^2 \tag{1.3.8.1}$$
$$p^2 = a^2 + x^2 \tag{1.3.8.2}$$
$\because c > x$, from (1.3.8.1) and (1.3.8.2), is obvious that
$$b > p \implies \frac{b}{p} > 1 \tag{1.3.8.3}$$
Also, from (1.1.8.1),
$$a = b \sin A = p \sin \theta \implies \frac{b}{p} = \frac{\sin \theta}{\sin A} > 1 \tag{1.3.8.4}$$
$$\text{or, } \sin \theta > \sin A \tag{1.3.8.5}$$
from (1.3.8.3). Note that this is always true. Thus,
$$\theta > A \iff \sin \theta > \sin A \tag{1.3.8.6}$$
In any $\triangle ABC$, using (1.1.8.3) and (1.3.8.6),

$$a > b \implies \frac{a}{b} = \frac{\sin A}{\sin B} > 1 \quad (1.3.8.7)$$

$$\text{or, } \sin A > \sin B \implies A > B \quad (1.3.8.8)$$

In any $\triangle ABC$, if $A > B$, using (1.1.8.3),

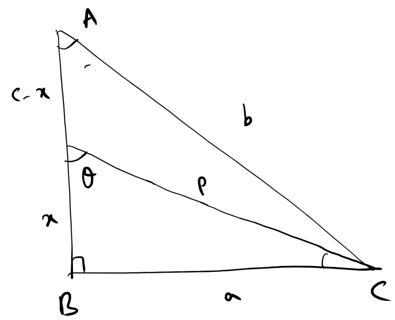

Fig. 1.3.8

9. In a triangle, side opposite to the larger (greater) angle is longer.
Solution: Use (1.1.8.3) and (1.3.8.6).
10. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.
Solution: The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \quad (1.3.10.1)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.3.10.2)$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \quad (1.3.10.3)$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \quad (1.3.10.4)$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \quad (1.3.10.5)$$

$$\implies rank(\mathbf{M}) = 2 \quad (1.3.10.6)$$

as the number of non zero rows is 2. The following code plots Fig. 1.3.10

```
codes/triangle/check_tri.py
```

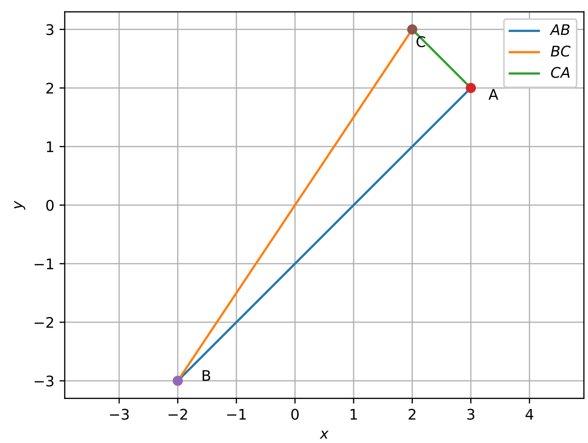

Fig. 1.3.10

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (1.3.10.7)$$

which, from (1.1.2.10) can be expressed as

$$\|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \quad (1.3.10.8)$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = 0 \quad (1.3.10.9)$$

after simplification. From (1.3.10.1) and (1.3.10.2), it is easy to verify that

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \quad (1.3.10.10)$$

satisfying (1.3.10.9). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.
11. Area of a triangle is half the product of its base and the corresponding altitude.
Solution: First, we consider the right angled triangle in Fig1.3.11. By definition, the area of the rectangle $ABCD$ is ac. Also, The rectangle is a sum of two congruent triangles ABC and

ADC. Thus,

$$\text{ar}\triangle ABC = \text{ar}\triangle ADC = \frac{1}{2}ac \qquad (1.3.11.1)$$

For any $\triangle ABC$, as shown in Fig. 1.3.11, the

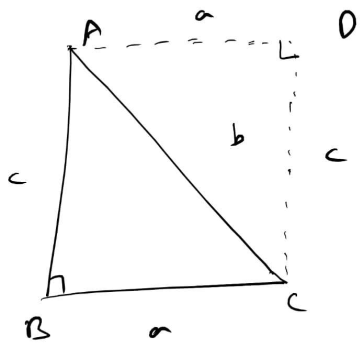

Fig. 1.3.11

area can be obtained as

$$\text{ar}\triangle ABC = \frac{1}{2}xh + \frac{1}{2}yh \qquad (1.3.11.2)$$

$$\frac{1}{2}(x+y)h = \frac{1}{2}ah \qquad (1.3.11.3)$$

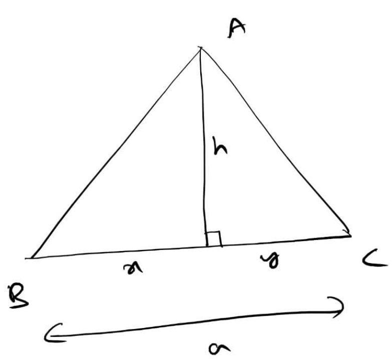

Fig. 1.3.11

12. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
13. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.
14. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
Solution: In Fig. 1.1.1, from Baudhayana's theorem,

$$b^2 = a^2 + c^2 \qquad (1.3.14.1)$$

$$= b^2\cos^2 C + b^2 \sin^2 C \qquad (1.3.14.2)$$

$$\implies \cos^2 C + \sin^2 C = 1 \qquad (1.3.14.3)$$

In Fig. 1.1.7, the area of $\triangle ABC$ is defined as

$$\frac{1}{2}ah = \frac{1}{2}ab\sin C \qquad (1.3.14.4)$$

$$= \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.3.15.1))} \qquad (1.3.14.5)$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.1.7.8))} \quad (1.3.14.6)$$

$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \qquad (1.3.14.7)$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \qquad (1.3.14.8)$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \qquad (1.3.14.9)$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \qquad (1.3.14.10)$$

Substituting

$$s = \frac{a+b+c}{2} \qquad (1.3.14.11)$$

in (1.3.15.10), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \qquad (1.3.14.12)$$

This is known as Hero's formula. The following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

15. A median of a triangle divides it into two triangles of equal areas.
Solution: In $\triangle ABC$, let AD
16. Show that the sum of any two sides of a triangle is greater than the third side.
Solution: In (1.3.15.12), all the factors inside the square root should be positive. Thus,

$$(s-a) > 0, (s-b) > 0\,(s-c) > 0 \quad (1.3.16.1)$$

$$(1.3.16.2)$$

$$(s-a) > 0 \implies \frac{a+b+c}{2} - a > 0 \quad (1.3.16.3)$$

$$\text{or, } b+c > a \qquad (1.3.16.4)$$

Similarly, it can be shown that $a+b > c, c+a > b$.
17. Draw a triangle whose sides are 8cm and 11cm

and the perimeter is 32 cm and find its area.
Solution: Use (1.3.15.12).

18. A triangular park ABC has sides 120m, 80m and 50m. A gardener Dhania has to put a fence all around it and also plant grass inside. Draw this park. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.
Solution: Use (1.3.15.12).
19. The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Draw the plot and find its area.
Solution: Use (1.3.15.12).
20. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix}5\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\-4\end{pmatrix}$.
Solution: The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.3.10.4) as
$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.3.20.1}$$
The computation is done in **area_tri.py**
21. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix}-1.5\\3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-2\end{pmatrix}, \mathbf{R} = \begin{pmatrix}-3\\4\end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is
$$\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{vmatrix} \tag{1.3.21.1}$$
22. Find the area of a triangle having the points
$$\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.3.22.1}$$
as its vertices.
Solution: The area of a triangle using the *vector product* is obtained as
$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.3.22.2}$$
For any two vectors $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,
$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.3.22.3}$$
The following code computes the area using the vector product.
```
codes/triangle/area_tri_vec.py
```
23. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.
24. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that
    a) $\triangle AOB \cong \triangle DOC$
    b) O is also the mid-point of BC.
25. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.
26. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.
27. In an isosceles $\triangle ABC$ with $AB =$ AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.
28. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.
29. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.
30. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$
31. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix}3\\-5\\7\end{pmatrix}$ and $\begin{pmatrix}-1\\7\\-6\end{pmatrix}$, respectively, find the coordinates of the point $\mathbf{C}$.
Solution: The centroid of $\triangle ABC$ is given by
$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.3.31.1}$$
Thus,
$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.3.31.2}$$
32. Show that the points
$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \tag{1.3.32.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.3.32

```
codes/triangle/triangle_3d.py
```

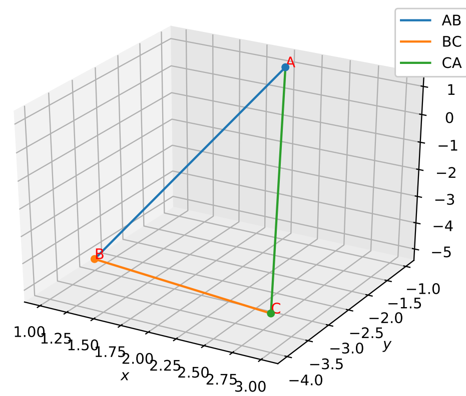

Fig. 1.3.32

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.3.32.2}$$

it is proved that the triangle is indeed right angled.

33. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{1.3.33.1}$$

the vertices of a right angled triangle?

34. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.
Solution: Fig. 1.3.34 summarizes the problem.

$$h = b\tan\theta = 15\tan 60^\circ = 15\sqrt{3} \tag{1.3.34.1}$$

35. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

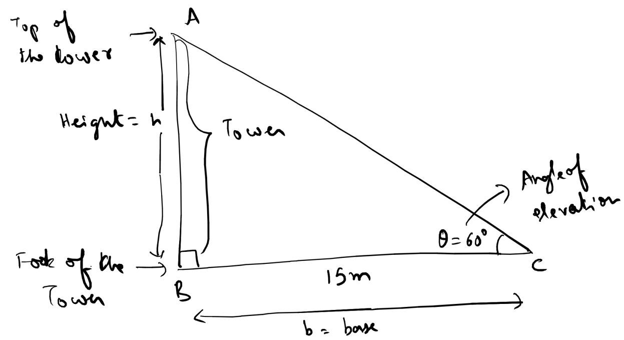

Fig. 1.3.34

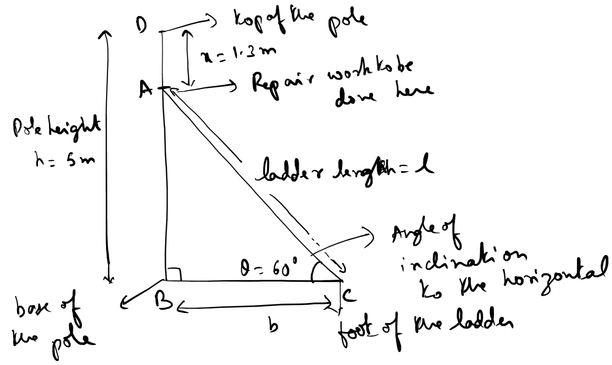

Fig. 1.3.35

Solution: Fig. 1.3.35 summarizes the problem. The objective is to find l and b. From the figure, if

$$\cot\theta = \frac{1}{\tan\theta}, \tag{1.3.35.1}$$

$$h - x = l\sin\theta = b\tan\theta \tag{1.3.35.2}$$

$$\implies l = (h-x)\csc\theta = 3.7\csc 60^\circ \tag{1.3.35.3}$$

$$\text{and } b = (h-x)\cot\theta = 3.7\cot^\circ \tag{1.3.35.4}$$

36. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?
Solution: Fig. 1.3.36 summarizes the problem. The objective is to find h. From the figure,

$$h - h_1 = b\tan\theta \tag{1.3.36.1}$$

$$\implies h = h_1 + b\tan\theta \tag{1.3.36.2}$$

$$= 1.5 + 28.5\tan 45^\circ \tag{1.3.36.3}$$

$$= 30m \tag{1.3.36.4}$$

37. From a point $\mathbf{P}$ on the ground the angle of

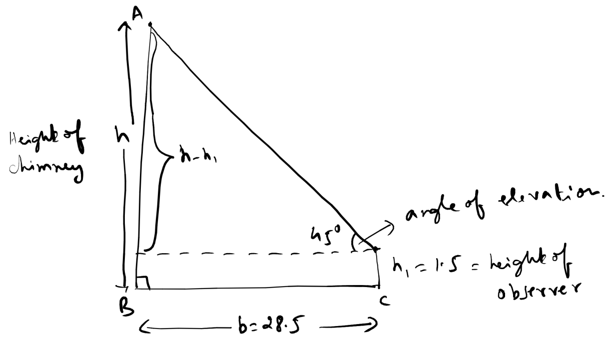

Fig. 1.3.36

elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

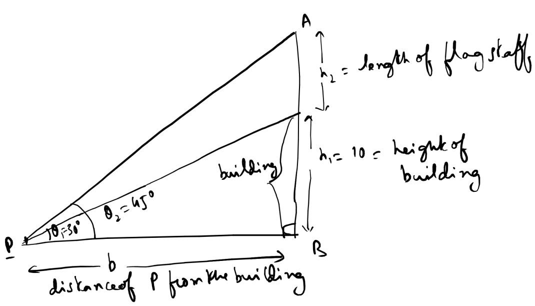

Fig. 1.3.37

Solution: Fig. 1.3.37 summarizes the problem. The objective is to find h_2 and b while h_1 is known. From the figure,

$$h_1 + h_2 = b\tan\theta_1 \tag{1.3.37.1}$$

$$h_1 = b\tan\theta_2 \tag{1.3.37.2}$$

This can be expressed as the matrix equation

$$\begin{pmatrix}\tan\theta_1 & -1\\ \tan\theta_2 & 0\end{pmatrix}\begin{pmatrix}b\\ h_2\end{pmatrix} = h_1\begin{pmatrix}1\\1\end{pmatrix} \tag{1.3.37.3}$$

and solved.

38. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.
Solution: Fig. 1.3.38 summarizes the problem.

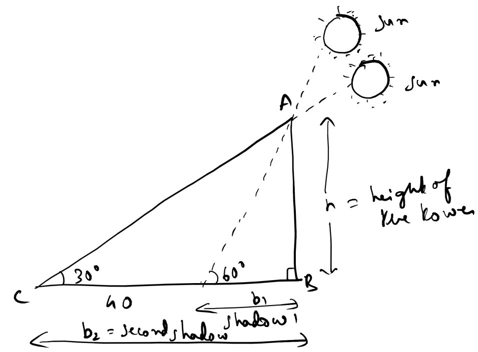

Fig. 1.3.38

The objective is to find h. from the figure,

$$b_1 = h\cot 60° \tag{1.3.38.1}$$

$$b_2 = h\cot 30° \tag{1.3.38.2}$$

$$b_2 - b_1 = 40 \tag{1.3.38.3}$$

$$\implies h\left(\cot 30° - \cot 60°\right) = 40 \tag{1.3.38.4}$$

$$\text{or } h = \frac{40}{\cot 30° - \cot 60°} \tag{1.3.38.5}$$

39. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

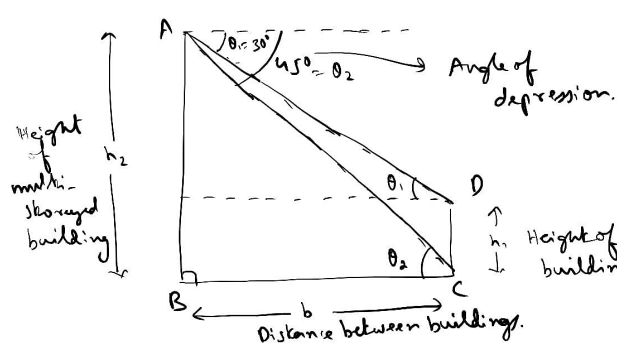

Fig. 1.3.39

Solution: Fig. 1.3.39 summarizes the problem. The objective is to find h_2 and b. From the figure,

$$h_2 = b \tan \theta_2 \tag{1.3.39.1}$$

$$h_2 - h_1 = b \tan \theta_1 \tag{1.3.39.2}$$

which can be expressed as

$$\begin{pmatrix} 1 & -\tan \theta_2 \\ 1 & -\tan \theta_1 \end{pmatrix} \begin{pmatrix} h_2 \\ b \end{pmatrix} = h_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.3.39.3}$$

and solved.

1.4 Triangle Exercises

1. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?
2. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m. The advertisements yield an earning of ₹5000 per m^2 per year. A company hired one o its walls for 3 months. How much rent did it pay?
3. There is a slide in a park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN". If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.
4. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.
5. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.
6. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.
7. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$ (see Fig. 7.16). Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?
8. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that
   a) $\triangle ABD \cong \triangle BAC$
   b) $BD = AC$
   c) $\angle ABD = \angle BAC$.
9. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.
10. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$ (see Fig. 7.20). Show that:
    a) $\triangle APB \cong \triangle AQB$
    b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.
11. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.
12. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:
    a) $\triangle AMC \cong \triangle BMD$
    b) $\angle DBC$ is a right angle.
    c) $\triangle DBC \cong \triangle ACB$
    d) $CM = \frac{1}{2}AB$
13. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
    a) $OB = OC$
    b) AO bisects $\angle A$
14. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.
15. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.
16. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that
    a) $\triangle ABE \cong \triangle ACF$
    b) $AB = AC$, i.e., ABC is an isosceles triangle.
17. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$.
18. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
    a) $\triangle ABD \cong \triangle ACD$
    b) $\triangle ABP \cong \triangle ACP$
    c) AP bisects $\angle A$ as well as $\angle D$.
    d) AP is the perpendicular bisector of BC.
19. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that
    a) AD bisects BC
    b) AD bisects $\angle A$.
20. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:
    a) $\triangle ABM \cong \triangle PQN$

b) $\triangle ABC \cong \triangle PQR$

21. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.
22. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.
23. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.
24. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.
25. Show that in a right angled triangle, the hypotenuse is the longest side.
26. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.
27. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.
28. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.
29. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.
30. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.
31. ABC is a triangle. Locate a point in the interior of $\triangle ABC$ which is equidistant from all the vertices of $\triangle ABC$.
32. In a triangle locate a point in its interior which is equidistant from all the sides of the triangle.
33. Draw the graphs of the equations

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.4.33.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix} - 12 = 0 \tag{1.4.33.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

34. In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
35. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
36. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-2\\3\end{pmatrix}, \mathbf{R} = \begin{pmatrix}4\\5\end{pmatrix}$. Find the equation of the median through the vertex $\mathbf{R}$.
37. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\2\end{pmatrix}$, find the equation and length of the altitude from the vertex $\mathbf{A}$.
38. Find the area of the triangle whose vertices are
    a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$
    b) $\begin{pmatrix}-5\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$
39. Find the area of the triangle formed by joining the mid points o the sides of a triangle whose vertices are $\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$.
40. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}4\\-6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\2\end{pmatrix}$ divides it into two triangles of equal areas.
41. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix}4\\6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7\\2\end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.4.41.1}$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{1.4.41.2}$$

42. Let $\mathbf{A} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}1\\4\end{pmatrix}$ be the vertices of $\triangle ABC$.
    a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
    b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
    c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
43. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.4.43.1}$$

44. Show that the points

$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \tag{1.4.44.1}$$

are the vertices of a right angled triangle.

45. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\1\\2\end{pmatrix}$. Find $\angle B$.
46. Show that the vectors $\begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-4\\-4\end{pmatrix}$ form the vertices of a right angled triangle.
47. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix}$ as its vertices.
48. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\3\\5\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}1\\5\\5\end{pmatrix}$
49. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
50. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix}3\\5\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}-5\\-5\\-2\end{pmatrix}$
51. Without using the Pythagoras theorem, show that the points $\begin{pmatrix}4\\4\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\end{pmatrix}$ are the vertices of a right angled triangle.
52. Check whether

$$\begin{pmatrix}5\\-2\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \begin{pmatrix}7\\-2\end{pmatrix} \tag{1.4.52.1}$$

are the vertices of an isosceles triangle.
53. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.
54. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.
55. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of 30° to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?
56. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is 30°. Find the height of the tower.
57. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.
58. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.
59. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.
60. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.
61. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.
62. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.
63. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of

elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.

64. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.
65. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.
66. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.
67. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.
68. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

2 Quadrilateral

2.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting

$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.1.1}$$

it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x-axis by an angle θ with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.2}$$

and

$$\cos\theta = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \tag{2.1.1.4}$$

The coordinates of the rotated triangle ACD are

$$\mathbf{D} = \mathbf{P}\begin{pmatrix} h \\ k \end{pmatrix} \tag{2.1.1.5}$$

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.6}$$

$$\mathbf{C} = \mathbf{P}\begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.7}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.1.8}$$

The following code plots quadrilateral $ABCD$

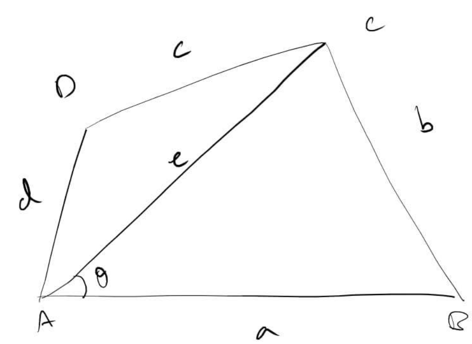

Fig. 2.1.1

in Fig. 2.1.1

```
codes/quad/draw_quad.py
```

2. Draw the parallelogram $MORE$ with $OR = 6, RE = 4.5$ and $EO = 7.5$.
Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel .
3. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.
Solution: The diagonals of a kite are perpendicular to each other.
4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
Solution: Diagonals of a rhombus bisect each other at right angles.

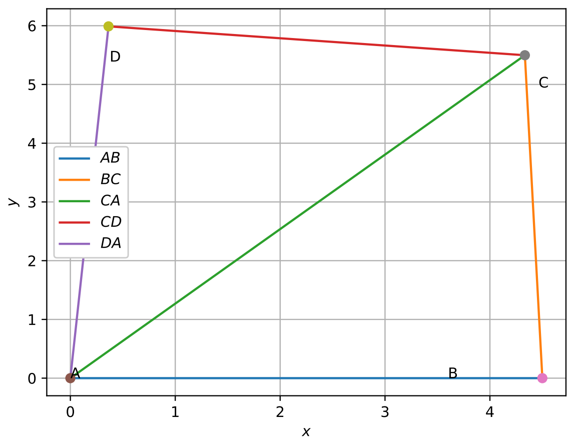

Fig. 2.1.1

2.2 Construction Exercises

1. Construct a quadrilateral $ABCD$ such that $AB = 5, \angle A = 50°, AC = 4, BD = 5$ and $AD = 6$.
2. Construct $PQRS$ where $PQ = 4, QR = 6, RS = 5, PS = 5.5$ and $PR = 7$.
3. Draw $JUMP$ with $JU = 3.5, UM = 4, MP = 5, PJ = 4.5$ and $PU = 6.5$
4. Construct a quadrilateral $ABCD$ such that $BC = 4.5, AC = 5.5, CD = 5, BD = 7$ and $AD = 5.5$.
5. Can you construct a quadrilateral $PQRS$ with $PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ = 4$?
6. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.
7. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.
8. DRAW rhombus $BEND$ such that $BN = 5.6$, $DE = 6.5$.
9. construct a quadrilateral MIST where $MI = 3.5, IS = 6.5, \angle M = 75°, \angle I = 105°$ and $\angle S = 120°$.
10. Can you construct the above quadrilateral MIST if $\angle M = 100°$ instead of 75°.
11. Can you constrcut the quadrilateral PLAN if $PL = 6, LA = 9.5, \angle P = 75°, \angle L = 150°$ and $\angle A = 140°$?
12. Construct $MORE$ where $MO = 6, OR = 4.5, \angle M = 60°, \angle O = 105°, \angle R = 105°$.
13. Construct $PLAN$ where $PL = 4, LA = 6.5, \angle P = 90°, \angle A = 110°$ and $\angle N = 85°$.
14. Constrcut parallelogram $HEAR$ where $HE = 5, EA = 6, \angle R = 85°$.
15. Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.
16. Construct $ABCd$, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105°$ and $\angle C = 80°$.
17. Construct $DEAR$ with $DE = 4, EA = 5, AR = 4.5, \angle E = 60°$ and $\angle A = 90°$.
18. Construct $TRUE$ with $TR = 3.5, RU = 3, UE = 4\angle R = 75°$ and $\angle U = 120°$.
19. Draw a square of side 4.5.
20. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
21. Draw a square $READ$ with $RE = 5.1$.
22. Draw a rhombus who diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

2.3 Quadrilateral Examples

1. Sum of the angles of a quadrilateral is 360°.
   Solution: Draw the diagonal and use the fact that sum of the angles of a triangle is 180°.
2. A diagonal of a parallelogram divides it into two congruent triangles.
   Solution: The alternate angles for the parallel sides are equal. The diagonal is common. Use ASA congruence.
3. In a parallelogram,
   a) opposite sides are equal
   b) opposite angles are equal
   c) diagonals bisect each other

   Solution: Since the diagonal divides the parallelogram into two congruent triangles, all the above results follow.
4. A quadrilateral is a parallelogram, if
   a) opposite sides are equal or
   b) opposite angles are equal or
   c) diagonals bisect each other or
   d) a pair of opposite sides is equal and parallel

   Solution: All the above lead to a quadrilateral that has two parallel sides, by showing that the alternate angles are equal.
5. A rectangle is a parallelogram with one angle that is 90°. Show that all angles of the rectangle are 90°.
   Solution: Draw a diagonal. Since the diagonal divides the rectangle into two congruent triangles, the angle opposite to the right angle is

also 90°. Using congruence, it can be shown that the other two angles are equal. Now use the fact that the sum of the angles of a quadrilateral is 360°.

6. Diagonals of a rectangle bisect each other and are equal and vice-versa.
   Solution: Use Baudhayana's theorem for equality of diagonals.
7. Diagonals of a rhombus bisect each other at right angles and vice-versa.
   Solution: The median of an isoceles triangle is also its perpendicular bisector.
8. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
   Solution: A square has the properties of a rectangle as well as a rhombus.
9. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
   Solution: If DE is the lie joining he mid points of $\triangle ABC$, use cosine formula to find the lengths of DE and BC. Then use cosine formula to show that all angles of $\triangle ADE$ are equal to the corresponding angles of $\triangle ABC$.
10. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.
    Solution: Use cosine formula.
11. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.
    Solution: Draw one diagonal and use Problem (2.3.9). Repeat for the other diagonal to show that the sides are parallel.
12. Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.
13. Show that the bisectors of angles of a parallelogram form a rectangle.
14. A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.
15. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:
    a) $APCQ$ is a parallelogram.
    b) $DPBQ$ is a parallelogram.
    c) $PSQR$ is a parallelogram.
16. In $\triangle ABC, D, E$ and F are respectively the mid-points of sides AB, BC and CA. Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F.
17. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.
18. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.
    Solution: By inspection,
    $$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.3.18.1}$$
    Hence, the diagonals AC and BD bisect each other. Also,
    $$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{D}) = 0 \tag{2.3.18.2}$$
    $\implies AC \perp BD$. Hence $ABCD$ is a square.
19. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
    Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.
20. If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral $ABCD$.
    Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by
    $$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D}) \right\| + \frac{1}{2} \left\| (\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D}) \right\| \tag{2.3.20.1}$$
21. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.

Solution: Since the direction vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{2.3.21.1}$$
$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{2.3.21.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{D}) \neq 0 \tag{2.3.21.3}$$

Hence, it is not a rectangle. The following code plots Fig. 2.3.21

```
codes/triangle/quad_3d.py
```

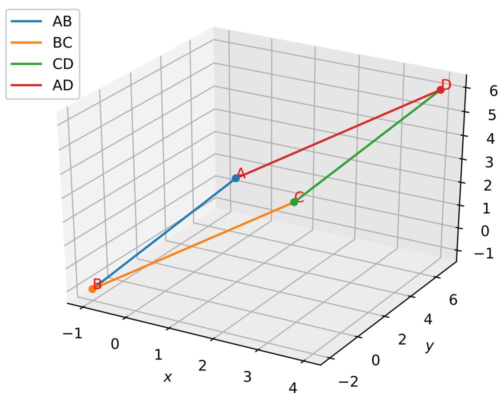

Fig. 2.3.21

22. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Solution: The area is given by
$$\frac{1}{2} \left\| \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\| \tag{2.3.22.1}$$
23. Kamla has a triangular field with sides 240 m, 200 m, 360 m, where she grew wheat. In another triangular field with sides 240 m, 320 m, 400 m adjacent to the previous field, she wanted to grow potatoes and onions. She divided the field in two parts by joining the mid-point of the longest side to the opposite vertex and grew patatoes in one part and onions in the other part. Draw the figure for this problem. How much area (in hectares) has been used for wheat, potatoes and onions? (1 hectare = 10000 m^2).
24. Students of a school staged a rally for cleanliness campaign. They walked through the lanes in two groups. One group walked through the lanes AB, BC and CA; while the other through AC, CD and DA. Then they cleaned the area enclosed within their lanes. If AB = 9 m, BC = 40 m, CD = 15 m, DA = 28 m and $\angle B = 90°$, which group cleaned more area and by how much? Draw the corresponding figure. Find the total area cleaned by the students (neglecting the width of the lanes).
25. Sanya has a piece of land which is in the shape of a rhombus. She wants her one daughter and one son to work on the land and produce different crops. She divided the land in two equal parts. If the perimeter of the land is 400 m and one of the diagonals is 160 m, how much area each of them will get for their crops? Draw the rhombus.
26. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
27. Area of a parallelogram is the product of its base and the corresponding altitude.
28. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.
29. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

2.4 *Quadrilateral Geometry*

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
4. Show that the diagonals of a square are equal and bisect each other at right angles.
5. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.

6. Diagonal AC of a parallelogram $ABCD$ bisects $\angle A$. Show that (i) it bisects $\angle C$ also, (ii) $ABCD$ is a rhombus.
7. $ABCD$ is a rhombus. Show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.
8. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that: (i) $ABCD$ is a square (ii) diagonal BD bisects $\angle B$ as well as $\angle D$.
9. In parallelogram $ABCD$, two points P and Q are taken on diagonal BD such that $DP = BQ$. Show that: (i) $\triangle APD \cong \triangle CQB$ (ii) $AP = CQ$ (iii) $\triangle AQB \cong \triangle CPD$ (iv) $AQ = CP$ (v) $APCQ$ is a parallelogram
10. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD . Show that (i) $\triangle APB \cong \triangle CQD$ (ii) $AP = CQ$
11. In $\triangle ABC$ and $\triangle DEF, AB = DE, ABparallelDE, BC = EF$ and $BCparallelEF$. Vertices A, B and C are joined to vertices D, E and F respectively. Show that (i) quadrilateral $ABED$ is a parallelogram (ii) quadrilateral $BEFC$ is a parallelogram (iii) $ADparallelCF$ and $AD = CF$ (iv) quadrilateral $ACFD$ is a parallelogram (v) $AC = DF$ (vi) $\triangle ABC \cong \triangle DEF$.
12. $ABCD$ is a trapezium in which AB *parallel* CD and $AD = BC$. Show that (i)$\angle A = \angle B$ (ii) $\angle C = \angle D$ (iii) $\triangle ABC \cong \triangle BAD$ (iv) diagonal $AC =$ diagonal BD
13. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA AC is a diagonal. Show that : (i) $SR \parallel AC$ and $SR = \frac{1}{2}AC$ (ii) $PQ = SR$ (iii) $PQRS$ is a parallelogram.
14. $ABCD$ is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rectangle.
15. $ABCD$ is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.
16. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and E is the mid-point of AD. A line is drawn through $E \parallel AB$ intersecting BC at F. Show that F is the mid-point of BC.
17. In a parallelogram $ABCD$, E and F are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.
18. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
19. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that (i) D is the mid-point of AC (ii) $MD \perp AC$ (iii) $CM = MA = \frac{1}{2}AB$
20. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \tag{2.4.20.1}$$
$$\angle B = 3y - 5 \tag{2.4.20.2}$$
$$\angle C = -4x \tag{2.4.20.3}$$
$$\angle D = -7x + 5 \tag{2.4.20.4}$$

Find its angles.
21. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}$, $\begin{pmatrix}0\\7\end{pmatrix}$, $\begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.
22. Find the area of a rhombus if its vertices are $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}4\\5\end{pmatrix}$, $\begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\end{pmatrix}$ taken in order.
23. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.
24. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-5\end{pmatrix}$, $\begin{pmatrix}3\\-2\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$.
25. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
26. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?
27. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and

$\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

28. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-7\\1\end{pmatrix}$.
29. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix}-1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\-\frac{1}{2}\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-\frac{1}{2}\\4\end{pmatrix}$.
30. The two adjacent sides of a parallelogram are $\begin{pmatrix}2\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.
31. A park, in the shape of a quadrilateral ABCD, has $\angle C = 90°, AB = 9m, BC = 12m, CD = 5m$ and$AD = 8m$. How much area does it occupy? 2. Find the area of a quadrilateral ABCD in which $AB = 3cm, BC = 4cm, CD = 4cm, DA = 5cm$ and $AC = 5cm$.
32. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 26 cm, 28 cm and 30 cm, and the parallelogram stands on the base 28 cm, find the height of the parallelogram.
33. A rhombus shaped field has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48 m, how much area of grass field will each cow be getting?
34. A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The non-parallel sides are 14 m and 13 m. Find the area of the field.

3 LINE

3.1 Examples: Geometry

1. Find four different solutions of the equation
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6 \tag{3.1.1.1}$$
2. Find two solutions for each of the following equations:
a) $\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12$
b) $\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix}0 & 3\end{pmatrix}\mathbf{x} = 4$
3. Draw the graph of
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.1.3.1}$$
4. Draw the graphs of the following equations

a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$	d) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$
b) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$	e) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 4$
c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$	f) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4$

5. Two rails are represented by the equations
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x}–4 = 0 \text{ and} \tag{3.1.5.1}$$
$$\begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x}–12 = 0. \tag{3.1.5.2}$$
Will the rails cross each other?
6. Check graphically whether the pair of equations
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 6 \text{ and} \tag{3.1.6.1}$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 12 \tag{3.1.6.2}$$
is consistent. If so, solve them graphically.
7. Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:
$$\begin{pmatrix}5 & -8\end{pmatrix}\mathbf{x} = -1 \text{ and} \tag{3.1.7.1}$$
$$\begin{pmatrix}3 & -\frac{24}{5}\end{pmatrix}\mathbf{x} = -\frac{3}{5} \tag{3.1.7.2}$$
8. Solve the following pair of equations
$$\begin{pmatrix}7 & -15\end{pmatrix}\mathbf{x} = 2 \tag{3.1.8.1}$$
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.1.8.2}$$
9. Find all possibe solutions of
$$\begin{aligned}\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} &= 7\end{aligned} \tag{3.1.9.1}$$
10. For which values of p does the pair of equations given below has unique solution?
$$\begin{aligned}\begin{pmatrix}4 & p\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} &= -2\end{aligned} \tag{3.1.10.1}$$
11. For what values of k will the following pair of

linear equations have infinitely many solutions?

$$\begin{aligned} \begin{pmatrix} k & 3 \end{pmatrix}\mathbf{x} &= k-3 \\ \begin{pmatrix} 12 & k \end{pmatrix}\mathbf{x} &= k \end{aligned} \tag{3.1.11.1}$$

12. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{3.1.12.1}$$

Solution: $x = 2, y = 2, z = 1$.

13. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.1.13.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

14. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.1.14.1}$$

15. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

16. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

17. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\| = 7 \tag{3.1.17.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{3.1.17.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.1.17.3}$$

18. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \tag{3.1.18.1}$$

19. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{3.1.19.1}$$

20. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \tag{3.1.20.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \tag{3.1.20.2}$$

21. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.

Solution: Refer to Problem 1.3.10.

22. Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix} 7 \\ 1 \end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}\right\|^2 \tag{3.1.22.1}$$

$$\begin{aligned} \implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix} 7 \\ 1 \end{pmatrix}\right\|^2 - 2\begin{pmatrix} 7 & 1 \end{pmatrix}\mathbf{x} \\ = \|\mathbf{x}\|^2 + \left\|\begin{pmatrix} 3 \\ 5 \end{pmatrix}\right\|^2 - 2\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} \end{aligned} \tag{3.1.22.2}$$

which can be simplified to obtain

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 2 \tag{3.1.22.3}$$

which is the desired condition. The following code plots Fig. 3.1.22clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

23. Find a point on the y-axis which is equidistant

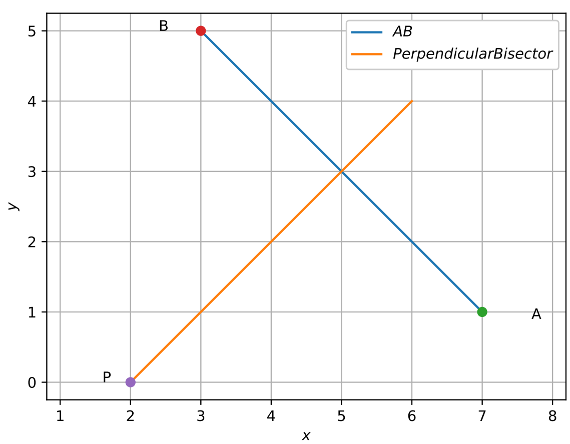

Fig. 3.1.22

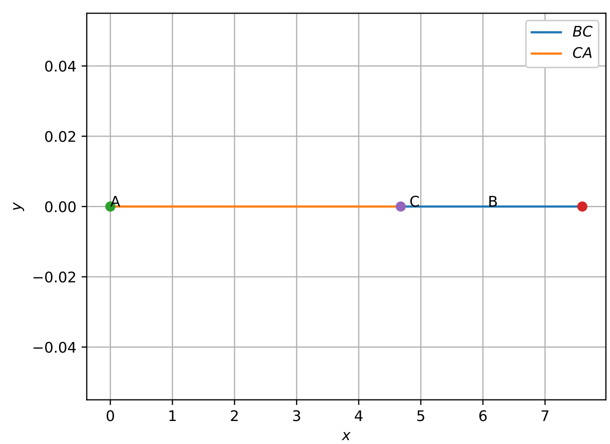

Fig. 3.1.24

from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Solution: Choose $\mathbf{x} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and follow the approach in Problem (3.1.22). Solve for y.

24. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \tag{3.1.24.1}$$

Then the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{3.1.24.2}$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 3.1.24

```
codes/line/draw_section.py
```

25. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.
Solution: Using (3.1.24.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \tag{3.1.25.1}$$

26. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \tag{3.1.26.1}$$

Solution: Use (3.1.24.2).

27. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \tag{3.1.27.1}$$

Solution: Using (3.1.24.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.1.27.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.1.27.3}$$

28. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.24.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.1.28.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.1.28.2}$$

29. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.3.10.4),

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{B}-\mathbf{A}\end{pmatrix}^T = \begin{pmatrix}2 & k-3\\4 & -6\end{pmatrix} \tag{3.1.29.1}$$

$$\xleftrightarrow{R_2\leftarrow\frac{R_2}{2}} \begin{pmatrix}2 & k-3\\2 & -3\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}2 & k-3\\0 & -k\end{pmatrix} \tag{3.1.29.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{3.1.29.3}$$

30. Find the direction vectors and slopes of the lines passing through the points
    a) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.
    b) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}7\\-2\end{pmatrix}$.
    c) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
    d) Making an inclination of 60° with the positive direction of the x-axis.

    Solution:
    a) If the direction vector is
    $$\begin{pmatrix}1\\m\end{pmatrix}, \tag{3.1.30.1}$$
    the slope is m. Thus, the direction vector is
    $$\begin{pmatrix}-1\\4\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix} = -\frac{1}{4}\begin{pmatrix}-4\\6\end{pmatrix} \tag{3.1.30.2}$$
    $$= \begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix} \implies m = -\frac{3}{2} \tag{3.1.30.3}$$
    b) The direction vector is
    $$\begin{pmatrix}7\\-2\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} \tag{3.1.30.4}$$
    $$= \begin{pmatrix}1\\0\end{pmatrix} \implies m = 0 \tag{3.1.30.5}$$
    c) The direction vector is
    $$\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix} \tag{3.1.30.6}$$
    $$= \begin{pmatrix}1\\\infty\end{pmatrix} \implies m = \infty \tag{3.1.30.7}$$
    d) The slope is $m = \tan 60° = \sqrt{3}$ and the direction vector is
    $$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{3.1.30.8}$$

31. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
    Solution: The angle θ between two lines is given by
    $$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{3.1.31.1}$$
    $$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{3.1.31.2}$$
    $$\text{or } m_1 = \frac{5}{3} \tag{3.1.31.3}$$

32. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
    Solution: Using (1.3.10.9)
    $$\left\{\begin{pmatrix}-2\\6\end{pmatrix} - \begin{pmatrix}4\\8\end{pmatrix}\right\}^T \left\{\begin{pmatrix}8\\12\end{pmatrix} - \begin{pmatrix}x\\24\end{pmatrix}\right\} = 0 \tag{3.1.32.1}$$
    which can be used to obtain x.

33. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
    Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3\\8\end{pmatrix}$ is obtained as
    $$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{3.1.33.1}$$
    $$\implies \begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} - \lambda\begin{pmatrix}-3\\-6\end{pmatrix} \tag{3.1.33.2}$$
    which can be expressed as
    $$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{3.1.33.3}$$
    $$\implies \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = -2 \tag{3.1.33.4}$$
    $$\implies D = 2 + 2T \tag{3.1.33.5}$$

34. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$.

Solution: The line parallel to the x-axis has direction vector $\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.34.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}0\\1\end{pmatrix} \tag{3.1.34.2}$$

The following code plots Fig. 3.1.34

```
codes/line/line_parallel_axes.py
```

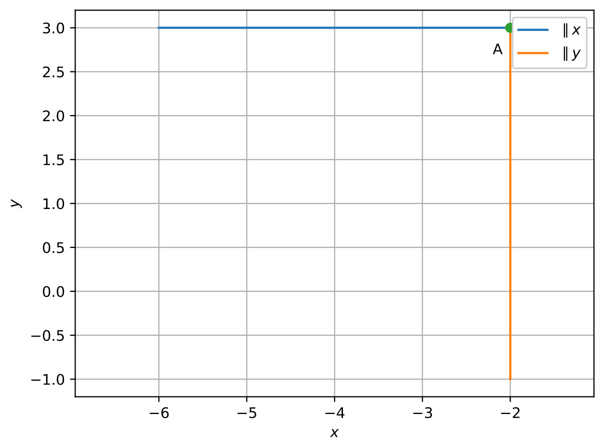

Fig. 3.1.34

35. Find the equation of the line through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$ with slope –4.

    Solution: The direction vector is $\mathbf{m} = \begin{pmatrix}1\\-4\end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{3.1.35.1}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} \tag{3.1.35.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.35.3}$$

$$\implies \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = -5 \tag{3.1.35.4}$$

36. Write the equation of the line through the points $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.

    Solution: Use (3.1.34.1).

37. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and

    a) y-intercept is $-\frac{3}{2}$
    b) x-intercept is 4.

    Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.

    a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$.
    b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix}4\\0\end{pmatrix}$.

    Use the above information get the equations for the lines.

38. Find the equation of a line through the point $\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}3\\2\\-8\end{pmatrix}$.

    Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix}5 & 2\\-4 & \end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-8\end{pmatrix} \tag{3.1.38.1}$$

39. Find the equation of a line passing through the points $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\\6\end{pmatrix}$.

    Solution: Using (3.1.33.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda\begin{pmatrix}4\\4\\4\end{pmatrix} \tag{3.1.39.1}$$

$$= \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.1.39.2}$$

40. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{3.1.40.1}$$

find the equation of the line.

Solution: The line can be expressed from

(3.1.40.1) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+2\lambda \\ 5+4\lambda \\ -6+2\lambda \end{pmatrix} \tag{3.1.40.2}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \tag{3.1.40.3}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{3.1.40.4}$$

41. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution: See Problem 3.1.37. The line passes through the points $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
42. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 3.1.42, the foot of the perpendicular P is the intersection of the lines L and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \tag{3.1.42.1}$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.42.2}$$

$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda\left\|\mathbf{n}\right\|^2 = c \tag{3.1.42.3}$$

$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \tag{3.1.42.4}$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.42.5}$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda|\left\|\mathbf{n}\right\| \tag{3.1.42.6}$$

From (3.1.42.4) and (3.1.42.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \tag{3.1.42.7}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15° \end{pmatrix} \tag{3.1.42.8}$$

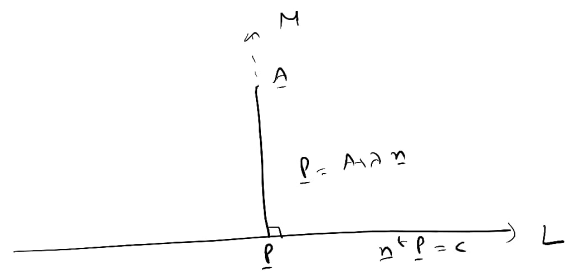

Fig. 3.1.42

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15°} \tag{3.1.42.9}$$

$$= \pm 4 \sec 15° \tag{3.1.42.10}$$

where

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.1.42.11}$$

This follows from (1.3.15.1), where

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.1.42.12}$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{3.1.42.13}$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{3.1.42.14}$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \tag{3.1.42.15}$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15° \end{pmatrix}\mathbf{c} = \pm 4 \sec 15° \tag{3.1.42.16}$$

43. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \tag{3.1.43.1}$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.35.3) and (3.1.35.1) as

$$\begin{pmatrix}11 & -100\end{pmatrix}\mathbf{x} = \begin{pmatrix}11 & -100\end{pmatrix}\begin{pmatrix}273\\32\end{pmatrix} \tag{3.1.43.2}$$

$$\implies \begin{pmatrix}11 & -100\end{pmatrix}\mathbf{x} = -197 \tag{3.1.43.3}$$

If $\begin{pmatrix}F\\0\end{pmatrix}$ is a point on the line,

$$\begin{pmatrix}11 & -100\end{pmatrix}\begin{pmatrix}F\\0\end{pmatrix} = -197 \implies F = -\frac{197}{11} \tag{3.1.43.4}$$

44. Equation of a line is

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 10 = 0. \tag{3.1.44.1}$$

Find its
a) slope,
b) x - and y-intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix}3\\-4\end{pmatrix}, \tag{3.1.44.2}$$

$$\mathbf{m} = \begin{pmatrix}4\\3\end{pmatrix}, \tag{3.1.44.3}$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

45. Find the angle between two vectors **a** and **b** where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T\mathbf{b} = 1. \tag{3.1.45.1}$$

Solution: In Fig. 3.1.45, from the cosine formula in (1.1.7.9)

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{3.1.45.2}$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.45.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.45.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.45.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.45.6}$$

$$= \frac{1}{2} \tag{3.1.45.7}$$

$$\implies \theta = 60° \tag{3.1.45.8}$$

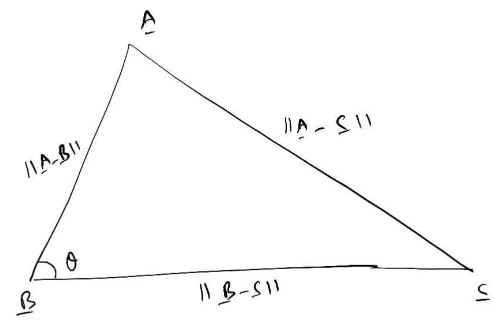

Fig. 3.1.45

46. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \tag{3.1.46.1}$$

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \tag{3.1.46.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{3.1.46.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30° \tag{3.1.46.4}$$

47. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3 \tag{3.1.47.1}$$

and passes through the point $\begin{pmatrix}1\\-2\end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix}2\\1\end{pmatrix} \tag{3.1.47.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix}2 & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\-2\end{pmatrix}\right) = 0 \tag{3.1.47.3}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.47.4}$$

48. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{3.1.48.1}$$

Solution: Use (3.1.42.7).

49. If the lines

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.49.1}$$
$$\begin{pmatrix} 5 & k \end{pmatrix} \mathbf{x} = 3 \tag{3.1.49.2}$$
$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{3.1.49.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & k & 3 \\ 3 & -1 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 5 & k & 3 \end{pmatrix} \tag{3.1.49.4}$$

$$\xleftrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftrightarrow 2R_2 - 3R_1} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & -5 \\ 0 & 2k-5 & -9 \end{pmatrix} \tag{3.1.49.5}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{5}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2k-5 & -9 \end{pmatrix} \tag{3.1.49.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (2k-5)R_2} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2k-4 \end{pmatrix} \tag{3.1.49.7}$$

$$\implies k = -2 \tag{3.1.49.8}$$

50. Find the distance of the line

$$L_1 : \quad \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.50.1}$$

from the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan 135° \end{pmatrix} \tag{3.1.50.2}$$

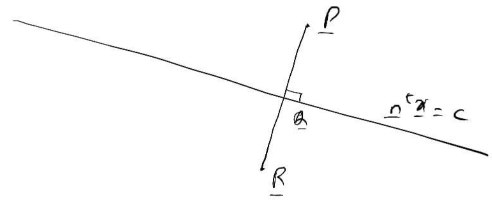

Fig. 3.1.51

Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.50.3}$$

$$\implies \mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.50.4}$$

$$\text{or,} \quad \left\| \mathbf{P} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = d = |\lambda| \, \|\mathbf{m}\| \tag{3.1.50.5}$$

Since $\mathbf{P}$ lies on L_1, from (3.1.50.1),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{P} = 0 \tag{3.1.50.6}$$

Substituting from the above in (3.1.50.3),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{3.1.50.7}$$

$$\implies \lambda = \frac{\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m}}{17} \tag{3.1.50.8}$$

substituting $|\lambda|$ in (3.1.50.5) gives the desired answer.

51. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = -4. \tag{3.1.51.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{3.1.51.2}$$
$$\mathbf{n}^T \mathbf{Q} = c \tag{3.1.51.3}$$
$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \tag{3.1.51.4}$$

where $\mathbf{m}$ is the direction vector of L. From

(3.1.51.2) and (3.1.51.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{3.1.51.5}$$

From (3.1.51.5) and (3.1.51.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.51.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{3.1.51.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{3.1.51.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{3.1.51.9}$$

(3.1.51.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.51.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.51.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{3.1.51.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{3.1.51.13}$$

The following code plots Fig. 3.1.51 while computing the reflection

codes/line/line_reflect.py

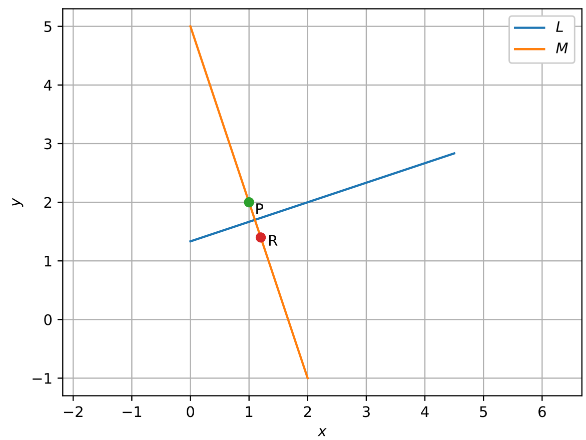

Fig. 3.1.51

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.52.4}$$

$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \tag{3.1.52.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (3.1.52.1), (3.1.52.4) and (3.1.52.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \tag{3.1.52.6}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \tag{3.1.52.7}$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \tag{3.1.52.8}$$

$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \tag{3.1.52.9}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \tag{3.1.52.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{3.1.52.11}$$

52. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix}1\\5\end{pmatrix}$. Obtain its equation.

$$L_1: \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.1.52.1}$$

$$L_2: \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.1.52.2}$$

Solution: Let

$$L: \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.52.3}$$

53. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.53.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.53.2}$$

are equal is a straight line.
Solution: Using (3.1.42.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{3.1.53.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right| \tag{3.1.53.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right) \tag{3.1.53.5}$$

leading to the possible lines

$$L_1: \quad \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.53.6}$$

$$L_2: \quad \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{3} \tag{3.1.53.7}$$

54. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix}1\\-3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{3.1.54.1}$$

Solution: The distance is given by $\|\mathbf{P}-\mathbf{Q}\|$

55. Show that the points $\mathbf{A} = \begin{pmatrix}-2\\3\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7\\0\\-1\end{pmatrix}$ are collinear.
Solution: Forming the matrix in (1.3.10.4)

$$\mathbf{M} = \begin{pmatrix}3 & -1 & -2\\9 & -3 & -6\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-3R_1} \begin{pmatrix}3 & -1 & -2\\0 & 0 & 0\end{pmatrix} \tag{3.1.55.1}$$

$\implies$ $rank(\mathbf{M}) = 1$. The following code plots Fig. 3.1.55 showing that the points are collinear.

```
codes/line/collinear_3d.py
```

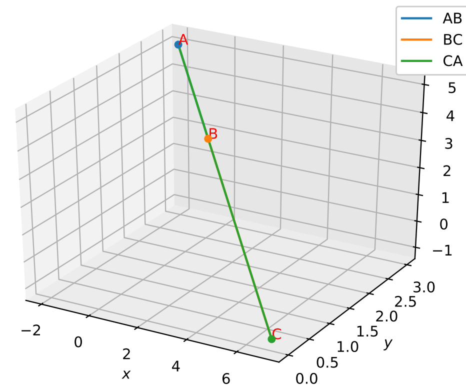

Fig. 3.1.55

56. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\8\\-11\end{pmatrix}$ are collinear.
Solution: Use the approach in Problem (3.1.55).

57. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{3.1.57.1}$$

$$\mathbf{A} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\3\\-7\end{pmatrix}, \tag{3.1.57.2}$$

respectively.

58. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\4\\-5\end{pmatrix}$ in the ratio 2 : 3
   a) internally, and
   b) externally.

   Solution: Use (3.1.24.2).

59. Prove that the three points $\begin{pmatrix}-4\\6\\10\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}$ and $\begin{pmatrix}14\\0\\-2\end{pmatrix}$ are collinear.
Solution: Use the approach in Problem 3.1.55.

60. Find the ratio in which the line segment joining the points $\begin{pmatrix}4\\8\\10\end{pmatrix}$ and $\begin{pmatrix}6\\10\\-8\end{pmatrix}$ is divided by the YZ-plane.
Solution: Use (3.1.24.2). The YZ-plane has points $\begin{pmatrix}0\\y\\z\end{pmatrix}$.

61. Find the equation of the set of points $\mathbf{P}$

such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution: Use the approach in Problem 3.1.22.

62. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{3.1.62.1}$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{3.1.62.2}$$

find $\mathbf{R}$, which divides PQ in the ratio $2 : 1$

a) internally,
b) externally.

Solution: Use (3.1.24.2).

63. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: Use (3.1.45.6)

64. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.1.64.1}$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.1.64.2}$$

Solution: The direction vectors of the lines are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Using(3.1.45.6), the angle between the lines can be obtained.

65. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{3.1.65.1}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{3.1.65.2}$$

Solution: From Problem 3.1.40, the direction vectors of the lines can be expressed as $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The angle between them can then be obtained from (3.1.45.6).

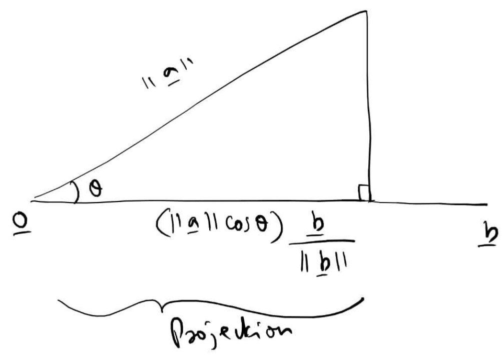

Fig. 3.1.67

66. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Solution: Use (1.3.10.9).

67. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{3.1.67.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{3.1.67.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. 3.1.67. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)\|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{3.1.67.3}$$

68. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{3.1.68.1}$$

Solution:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \tag{3.1.68.2}$$

69. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{3.1.69.1}$$

then find $\mathbf{x}$.

Solution:

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \quad (3.1.69.2)$$
$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \quad (3.1.69.3)$$

70. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (3.1.70.1)$$

find $\|\mathbf{a} \times \mathbf{b}\|$.
Solution: Use (1.3.22.3).

71. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (3.1.71.1)$$

Solution: If $\mathbf{x}$ is the desired vector,

$$(\mathbf{a}+\mathbf{b})^T \mathbf{x} = 0 \quad (3.1.71.2)$$
$$(\mathbf{a}-\mathbf{b})^T \mathbf{x} = 0 \quad (3.1.71.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (3.1.71.4)$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \quad (3.1.71.5)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.71.6)$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (3.1.71.7)$$

72. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are collinear.
Solution: See Problem 3.1.55.

73. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \quad (3.1.73.1)$$
$$\mathbf{C}-\mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \quad (3.1.73.2)$$

$$\because -2(\mathbf{A}-\mathbf{B}) = \mathbf{C}-\mathbf{D}, \quad (3.1.73.3)$$

$\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.

74. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \quad (3.1.74.1)$$

Then,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \quad (3.1.74.2)$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \quad (3.1.74.3)$$

using (3.1.74.1)

75. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \quad (3.1.75.1)$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \quad (3.1.75.2)$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (3.1.75.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \quad (3.1.75.3)$$
$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \quad (3.1.75.4)$$
$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \quad (3.1.75.5)$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \quad (3.1.75.6)$$

76. Let $\boldsymbol{\alpha} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.76.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{3.1.76.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.76.3}$$

This process is known as *Gram-Schmidth orthogonalization*.

77. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90° \\ \cos 60° \\ \cos 30° \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.1.77.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

78. Find the distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.78.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.78.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. 3.1.78 From Fig. 3.1.78, the distance is

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{3.1.78.3}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.78.4}$$

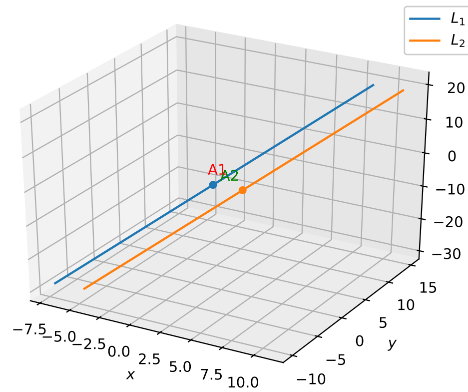

Fig. 3.1.78

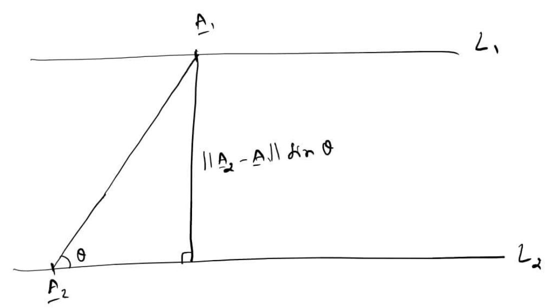

Fig. 3.1.78

79. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.79.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.79.2}$$

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{3.1.79.3}$$

The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}+\lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{3.1.79.4}$$

$$\implies \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix}-\lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{3.1.79.5}$$

$$\implies \begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{3.1.79.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}2 & 3 & 1\\-1 & -5 & 0\\1 & 2 & -1\end{pmatrix}\xleftrightarrow{R_3\leftrightarrow R_1}\begin{pmatrix}1 & 2 & -1\\-1 & -5 & 0\\2 & 3 & 1\end{pmatrix} \tag{3.1.79.7}$$

$$\xleftrightarrow[R_3=2R_1-R_3]{R_2=R_1+R_2}\begin{pmatrix}1 & 2 & -1\\0 & -3 & -1\\0 & 1 & -3\end{pmatrix}\xleftrightarrow{R_2\leftrightarrow R_3}\begin{pmatrix}1 & 2 & -1\\0 & 1 & -3\\0 & -3 & -1\end{pmatrix} \tag{3.1.79.8}$$

$$\xleftrightarrow{R_3=3R_2+R_3}\begin{pmatrix}1 & 2 & -1\\0 & 1 & -3\\0 & 0 & -10\end{pmatrix} \tag{3.1.79.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. 3.1.79

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n}=\mathbf{m}_1\times\mathbf{m}_2 \tag{3.1.79.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T\mathbf{x}=\mathbf{n}^T\mathbf{A}_2 \tag{3.1.79.11}$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (3.1.42.7) as

$$\frac{\left|\mathbf{n}^T\left(\mathbf{A}_2-\mathbf{A}_1\right)\right|}{\|\mathbf{n}\|}=\frac{\left|\left(\mathbf{A}_2-\mathbf{A}_1\right)^T\left(\mathbf{m}_1\times\mathbf{m}_2\right)\right|}{\|\mathbf{m}_1\times\mathbf{m}_2\|} \tag{3.1.79.12}$$

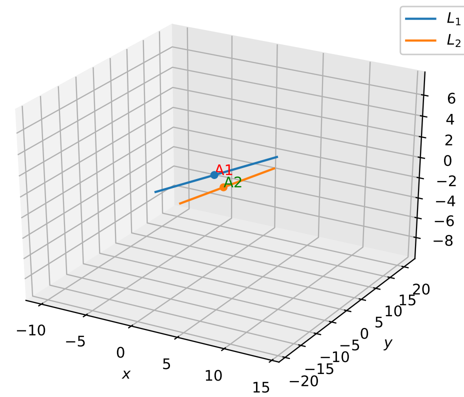

Fig. 3.1.79

80. Find the distance of the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x}-6=0 \tag{3.1.80.1}$$

from the origin.
Solution: From (3.1.42.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|}=\frac{6}{\sqrt{2^2+3^2+4^2}} \tag{3.1.80.2}$$

$$=\frac{6}{\sqrt{29}} \tag{3.1.80.3}$$

81. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n}=\begin{pmatrix}2\\-3\\4\end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x}-6=0 \tag{3.1.81.1}$$

82. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x}=1. \tag{3.1.82.1}$$

Solution: The normal vector is

$$\mathbf{n}=\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.82.2}$$

$$\because \|\mathbf{n}\|=7, \tag{3.1.82.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|}=\frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.82.4}$$

83. Find the coordinates of the foot of the perpen-

dicular drawn from the origin to the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.1.83.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{3.1.83.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda\mathbf{n}$. Substituting in (3.1.83.1),

$$\lambda \|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \tag{3.1.83.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{3.1.83.4}$$

84. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (3.1.35.3), the equation of the plane is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.84.1}$$

$$\implies \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \tag{3.1.84.2}$$

$$= 20 \tag{3.1.84.3}$$

85. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \boldsymbol{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$.
Solution: If the equation of the plane be

$$\mathbf{n}^T \mathbf{x} = c, \tag{3.1.85.1}$$

$$\mathbf{n}^T \mathbf{R} = \mathbf{n}^T \mathbf{S} = \mathbf{n}^T \mathbf{T} = c, \tag{3.1.85.2}$$

$$\implies \begin{pmatrix} \mathbf{R} - \mathbf{S} & \mathbf{S} - \mathbf{T} \end{pmatrix}^T \mathbf{n} = 0 \tag{3.1.85.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix} 4 & 8 & -8 \\ -7 & -6 & 8 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 2 & -2 \\ -7 & -6 & 8 \end{pmatrix} \tag{3.1.85.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 8 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.85.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.85.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and} \tag{3.1.85.7}$$

$$c = \mathbf{n}^T \mathbf{T} = 7 \tag{3.1.85.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{n} = 7 \tag{3.1.85.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{3.1.85.10}$$

The equation of the plane is then obtained from (3.1.35.3) as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T (\mathbf{x} - \mathbf{T}) = 0 \tag{3.1.85.11}$$

86. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
Solution: From the given information, the plane passes through the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ respectively. The equation can be obtained using Problem 3.1.85.
87. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{3.1.87.1}$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = -5 \tag{3.1.87.2}$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & -5 \end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.87.3}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & -1 & 23 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.87.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.87.5}$$

Thus, $\begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}\right) \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.87.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ -22 & 18 & 1 \end{pmatrix} \xleftrightarrow{R_2=R_2+22R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.87.7}$$

$$\xleftrightarrow{R_1=13R_1-R_2} \begin{pmatrix} 13 & 0 & -10 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.87.8}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{10}{13} \\ \frac{23}{26} \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 23 \\ 26 \end{pmatrix} \tag{3.1.87.9}$$

Since the plane passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using (3.1.35.3),

$$\begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = 0 \tag{3.1.87.10}$$

$$\implies \begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \mathbf{x} = 69 \tag{3.1.87.11}$$

Alternatively, the plane passing through the intersection of (3.1.87.1) and (3.1.87.2) has the form

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} + \lambda \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 6 - 5\lambda \tag{3.1.87.12}$$

Substituting $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the above,

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6 - 5\lambda \tag{3.1.87.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{3.1.87.14}$$

$$\implies \lambda = \frac{3}{14} \tag{3.1.87.15}$$

Substituting this value of λ in (3.1.87.12) yields the equation of the plane.

88. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{3.1.88.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{3.1.88.2}$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.79), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{3.1.88.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \tag{3.1.88.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix} -1 & 2 & 5 \\ -3 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2-3R_1}{5}} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix} \tag{3.1.88.5}$$

$$\xleftrightarrow{R_1=-R_1+2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.88.6}$$

The LHS of (3.1.88.3) is

$$\begin{pmatrix}-2 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0 \tag{3.1.88.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left|\mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2\right| = 0 \tag{3.1.88.8}$$

89. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.89.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.89.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.45.6).

90. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.90.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.90.2}$$

Solution: See Problem (3.1.89).

91. Find the distance of a point $\begin{pmatrix}2\\5\\-3\end{pmatrix}$ from the plane

$$\begin{pmatrix}6 & -3 & 2\end{pmatrix}\mathbf{x} = 4 \tag{3.1.91.1}$$

Solution: Use (3.1.42.7).

92. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{3.1.92.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{3.1.92.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225}\times\sqrt{49}} = \frac{8}{21} \tag{3.1.92.3}$$

Thus, the desired angle is $90° - \theta$.

93. Find the equation of the plane that contains the point $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix}2 & 3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.93.1}$$

$$\begin{pmatrix}1 & 2 & -3\end{pmatrix}\mathbf{x} = 8 \tag{3.1.93.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix}2\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\2\\-3\end{pmatrix} \tag{3.1.93.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{3.1.93.4}$$

94. Find the distance between the point $\mathbf{P} = \begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$.
Solution: Find the equation of the plane using Problem 3.1.85. Find the distance using (3.1.42.7).

95. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B}-\mathbf{A}) \tag{3.1.95.1}$$

$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{3.1.95.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \tag{3.1.95.3}$$

3.2 Points and Vectors

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{3.2.1.1}$$

b)

$$\begin{pmatrix}-5\\7\end{pmatrix},\begin{pmatrix}-1\\3\end{pmatrix} \tag{3.2.1.2}$$

c)

$$\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}-1\\b\end{pmatrix} \tag{3.2.1.3}$$

2. Find the distance between the points

$$\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}36\\15\end{pmatrix} \tag{3.2.2.1}$$

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\begin{pmatrix}-1\\-2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}-3\\0\end{pmatrix} \tag{3.2.4.1}$$

b)

$$\begin{pmatrix}-3\\5\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.2.4.2}$$

c)

$$\begin{pmatrix}4\\5\end{pmatrix},\begin{pmatrix}7\\6\end{pmatrix}, \tag{3.2.4.3}$$

$$\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix} \tag{3.2.4.4}$$

5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$.
6. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix},\begin{pmatrix}-2\\9\end{pmatrix}, \tag{3.2.6.1}$$

7. Find the values of y for which the distance between the points

$$\mathbf{P}=\begin{pmatrix}2\\-3\end{pmatrix},\mathbf{Q}=\begin{pmatrix}10\\y\end{pmatrix} \tag{3.2.7.1}$$

is 10 units.

8. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix},\frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix},\frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}. \tag{3.2.8.1}$$

Also, show that they are mutually perpendicular to each other.

9. For

$$\mathbf{a}=\begin{pmatrix}2\\2\\3\end{pmatrix},\mathbf{b}=\begin{pmatrix}-1\\2\\1\end{pmatrix},\mathbf{c}=\begin{pmatrix}3\\1\\0\end{pmatrix}, \tag{3.2.9.1}$$

$(\mathbf{a}+\lambda\mathbf{b})\perp\mathbf{c}$. Find λ.

10. Find $\mathbf{a}\times\mathbf{b}$ if

$$\mathbf{a}=\begin{pmatrix}1\\-7\\7\end{pmatrix},\mathbf{b}=\begin{pmatrix}3\\-2\\2\end{pmatrix}. \tag{3.2.10.1}$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a}=\begin{pmatrix}3\\2\\2\end{pmatrix},\mathbf{b}=\begin{pmatrix}1\\2\\-2\end{pmatrix}. \tag{3.2.11.1}$$

12. If $\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix},\mathbf{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix},\mathbf{c}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$.
13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix},\mathbf{b}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$,
14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.
15. Let $\mathbf{a}=\begin{pmatrix}1\\4\\2\end{pmatrix},\mathbf{b}=\begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d}\perp\mathbf{a},\mathbf{d}\perp\mathbf{b}$ and $\mathbf{d}^T\mathbf{c}=15$.
16. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.

17. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+\begin{pmatrix}0\\1\\0\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$
$$+\begin{pmatrix}0\\0\\1\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \tag{3.2.17.1}$$

is

a) 0
b) -1
c) 1
d) 3

18. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively.
19. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.
20. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}3\\5\\6\end{pmatrix}$.
21. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.
22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
23. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{3.2.23.1}$$

24. Find the projection of the vector

$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{3.2.24.1}$$

on the vector

$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.2.24.2}$$

25. Write down a unit vector in the xy-plane, makeing an angle of $30°$ with the positive direction of the x-axis.
26. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

3.3 Points on a Line

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix}-1\\7\end{pmatrix}, = \begin{pmatrix}4\\-3\end{pmatrix} \tag{3.3.1.1}$$

in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4\\y\end{pmatrix}, \begin{pmatrix}x\\6\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.
8. Determine if the points

$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{3.3.8.1}$$

are collinear.

9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix}x\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
    a) $\begin{pmatrix}7\\-2\end{pmatrix}$, $\begin{pmatrix}5\\1\end{pmatrix}$, $\begin{pmatrix}3\\k\end{pmatrix}$
    b) $\begin{pmatrix}8\\1\end{pmatrix}$, $\begin{pmatrix}k\\-4\end{pmatrix}$, $\begin{pmatrix}2\\-5\end{pmatrix}$
12. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}$, $\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}7\\0\end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
15. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\8\\7\end{pmatrix}$ are collinear.

3.4 Lines and Planes

1. Sketch the following lines

    a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 9.35$
    b) $\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\mathbf{x} = 10$
    c) $\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 6$
    d) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 0$
    e) $\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0$
    f) $\begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} = -2$
    g) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$
    h) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 5$

2. Write four solutions for each of the following equations
    a) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$
    b) $\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$
    c) $\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$
3. Check which of the following are solutions of the equation
$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 4 \tag{3.4.3.1}$$
    a) $\begin{pmatrix}0\\2\end{pmatrix}$
    b) $\begin{pmatrix}2\\0\end{pmatrix}$
    c) $\begin{pmatrix}4\\0\end{pmatrix}$
    d) $\begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix}$
    e) $\begin{pmatrix}1\\1\end{pmatrix}$
4. Find the value of k, if $\begin{pmatrix}2\\1\end{pmatrix}$ is a solution of the equation
$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = k \tag{3.4.4.1}$$
5. Draw the graphs of the following equations
    a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$
    b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2$
    c) $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 0$
    d) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3$
    e) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
    f) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
    g) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$
    h) $\begin{pmatrix}7 & -3\end{pmatrix}\mathbf{x} = 2$
    i) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
    j) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$
    k) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$
    l) $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6$
6. Give the equations of two lines passing through $\begin{pmatrix}2\\14\end{pmatrix}$. How many more such lines are there, and why?
7. If the point $\begin{pmatrix}3\\4\end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a
8. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident
    a)
$$\begin{aligned}\begin{pmatrix}5 & -4\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}7 & 6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.4.8.1}$$

b)

$$\begin{pmatrix}9 & 3\end{pmatrix}\mathbf{x} = -12$$
$$\begin{pmatrix}18 & 6\end{pmatrix}\mathbf{x} = -24 \tag{3.4.8.2}$$

c)

$$\begin{pmatrix}6 & -3\end{pmatrix}\mathbf{x} = -10$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -9 \tag{3.4.8.3}$$

9. Find out whether the following pair of linear equations are consistent, or inconsistent.

a)

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 7 \tag{3.4.9.1}$$

b)

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} = 9 \tag{3.4.9.2}$$

c)

$$\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\end{pmatrix}\mathbf{x} = 7$$
$$\begin{pmatrix}9 & -10\end{pmatrix}\mathbf{x} = 14 \tag{3.4.9.3}$$

d)

$$\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = 11$$
$$\begin{pmatrix}-10 & 6\end{pmatrix}\mathbf{x} = -22 \tag{3.4.9.4}$$

e)

$$\begin{pmatrix}\frac{4}{3} & 2\end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.4.9.5}$$

10. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} = 10 \tag{3.4.10.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} = 16 \tag{3.4.10.2}$$

c)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 6$$
$$\begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} = 4 \tag{3.4.10.3}$$

d)

$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix}4 & -4\end{pmatrix}\mathbf{x} = 5 \tag{3.4.10.4}$$

11. Given the linear equation $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x}-8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

a) intersecting lines c) coincident lines
b) parallel lines

12. Find the intersection of the following lines

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 14$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4 \tag{3.4.12.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = 6 \tag{3.4.12.2}$$

c)

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}9 & -3\end{pmatrix}\mathbf{x} = 9 \tag{3.4.12.3}$$

d)

$$\begin{pmatrix}0.2 & 0.3\end{pmatrix}\mathbf{x} = 1.3$$
$$\begin{pmatrix}0.4 & 0.5\end{pmatrix}\mathbf{x} = 2.3 \tag{3.4.12.4}$$

e)

$$\begin{pmatrix}\sqrt{2} & \sqrt{3}\end{pmatrix}\mathbf{x} = 0$$
$$\begin{pmatrix}\sqrt{3} & \sqrt{8}\end{pmatrix}\mathbf{x} = 0 \tag{3.4.12.5}$$

f)

$$\begin{pmatrix}\frac{3}{2} & -\frac{5}{3}\end{pmatrix}\mathbf{x} = -2$$
$$\begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \frac{13}{6} \tag{3.4.12.6}$$

13. Find m if

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 11$$
$$\begin{pmatrix}2 & -4\end{pmatrix}\mathbf{x} = -24 \qquad (3.4.13.1)$$
$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = -3$$

14. Solve the following

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \qquad (3.4.14.1)$$

b)

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 10$$
$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \qquad (3.4.14.2)$$

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix}9 & -2\end{pmatrix}\mathbf{x} = 7 \qquad (3.4.14.3)$$

$$\begin{pmatrix}\frac{1}{2} & \frac{2}{3}\end{pmatrix}\mathbf{x} = -1$$
$$\begin{pmatrix}1 & -\frac{1}{3}\end{pmatrix}\mathbf{x} = 3 \qquad (3.4.14.4)$$

15. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}3 & -9\end{pmatrix}\mathbf{x} = 2 \qquad (3.4.15.1)$$

b)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 8 \qquad (3.4.15.2)$$

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 20$$
$$\begin{pmatrix}6 & -10\end{pmatrix}\mathbf{x} = 40 \qquad (3.4.15.3)$$

d)

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 7$$
$$\begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} = 15 \qquad (3.4.15.4)$$

16. For which alues of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 7$$
$$\begin{pmatrix}a-b & a+b\end{pmatrix}\mathbf{x} = 3a+b-2 \qquad (3.4.16.1)$$

17. For which value of k will the following pair of linear equations have no solution?

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 1$$
$$\begin{pmatrix}2k-1 & k-1\end{pmatrix}\mathbf{x} = 2k+1 \qquad (3.4.17.1)$$

18. Solve the following pair of linear equations

$$\begin{pmatrix}8 & 5\end{pmatrix}\mathbf{x} = 9$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 4 \qquad (3.4.18.1)$$

19. Solve the following pair of linear equations

$$\begin{pmatrix}158 & -378\end{pmatrix}\mathbf{x} = -74$$
$$\begin{pmatrix}-378 & 152\end{pmatrix}\mathbf{x} = -604 \qquad (3.4.19.1)$$

20. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$.
21. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
22. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.
23. Write the equations for the x and y axes.
24. Find the equation of the line satisfying the following conditions
   a) passing through the point $\begin{pmatrix}-4\\3\end{pmatrix}$ with slope $\frac{1}{2}$.
   b) passing through the point $\begin{pmatrix}0\\0\end{pmatrix}$ with slope m.
   c) passing through the point $\begin{pmatrix}2\\2\sqrt{3}\end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
   d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.
   e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.
   f) passing through the points $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-4\end{pmatrix}$.
   g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.
25. Find the equation of the line passing through $\begin{pmatrix}-3\\5\end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\6\end{pmatrix}$.
26. Find the direction vectors and and y-intercepts of the following lines
   a) $\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 0$.
   b) $\begin{pmatrix}6 & 3\end{pmatrix}\mathbf{x} = 5$.
   c) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$.

27. Find the intercepts of the following lines on the axes.
    a) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 12.$
    b) $\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 6.$
    c) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 0.$
28. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
    a) $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = -8.$
    b) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2.$
    c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4.$
29. Find the distance of the point $\begin{pmatrix}1\\-1\end{pmatrix}$ from the line $\begin{pmatrix}12 & -5\end{pmatrix}\mathbf{x} = -82.$
30. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.4.30.1}$$

are 4 units.

31. Find the distance between the parallel lines

$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = 34 \tag{3.4.31.1}$$
$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = -31 \tag{3.4.31.2}$$

32. Find the equation of the line parallel to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = -2 \tag{3.4.32.1}$$

and passing through the point $\begin{pmatrix}-2\\3\end{pmatrix}$.

33. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.33.1}$$

and having x intercept 3.

34. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.4.34.1}$$
$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = 1 \tag{3.4.34.2}$$

35. The line through the points $\begin{pmatrix}h\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19 \tag{3.4.35.1}$$

at right angle. Find the value of h.

36. Two lines passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ intersect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.
37. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$.
38. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.4.38.1}$$

39. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.4.39.1}$$

meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.

40. Find θ and p if

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{3.4.40.1}$$

is equivalent to

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{3.4.40.2}$$

41. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.
42. Find the equation of the line parallel to the y-axis whose distance from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.4.42.1}$$

4 units.

43. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.43.1}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.4.43.2}$$

44. Find the alue of p so that the three lines

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{3.4.44.1}$$
$$\begin{pmatrix}p & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.4.44.2}$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{3.4.44.3}$$

may intersect at one point.

45. Find the equation of the lines through the point $\begin{pmatrix}3\\2\end{pmatrix}$ which make an angle of 45° with the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3. \tag{3.4.45.1}$$

46. Find the equation of the line passing through the point of intersection of the lines

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = 3 \tag{3.4.46.1}$$

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -1 \tag{3.4.46.2}$$

that has equal intercepts on the axes.

47. In what ratio is the line joining $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}5\\7\end{pmatrix}$ divided by the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.4.47.1}$$

48. Find the distance of the line

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5 \tag{3.4.48.1}$$

from the point $\begin{pmatrix}1\\2\end{pmatrix}$ along the line

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0. \tag{3.4.48.2}$$

49. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.4.49.1}$$

may be at a distance of 3 units from this point.

50. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Find an equation of the legs of the triangle.

51. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7 \tag{3.4.51.1}$$

assuming the line to be a plane mirror.

52. If the lines

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.4.52.1}$$

$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.4.52.2}$$

are equally inclined to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 4, \tag{3.4.52.3}$$

find the value of m.

53. The sum of the perpendicular distances of a variable point $\mathbf{P}$ from the lines

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.4.53.1}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.4.53.2}$$

is always 10. Show that $\mathbf{P}$ must move on a line.

54. Find the equation of the line which is equidistant from parallel lines

$$\begin{pmatrix}9 & 7\end{pmatrix}\mathbf{x} = 7 \tag{3.4.54.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = -6. \tag{3.4.54.2}$$

55. A ray of light passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\begin{pmatrix}5\\3\end{pmatrix}$. Find the coordinates of $\mathbf{A}$.

56. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.4.56.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.4.56.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{3.4.56.3}$$

in the least time. Find the equation of the path that he should follow.

57. Determine the ratio in which the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 4 = 0 \tag{3.4.57.1}$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\7\end{pmatrix}$.

58. A line perpendicular to the line segment joining the points $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.

59. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

60. Find the equation of the line passing through the point $\begin{pmatrix}2\\2\end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.

61. Find the equation of the line through the point $\begin{pmatrix}0\\2\end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it

and crossing the y-axis at a distance of 2 units below the origin.

62. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix}-2\\9\end{pmatrix}$, find the equation of the line.
63. The length L (in cm) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L in terms of C.
64. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
65. Find the equation of a line which passes through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}3\\2\\-2\end{pmatrix}$.
66. Find the equaion off the line that passes through $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and is in the direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.
67. Find the equation of the line which passes through the point $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and parallel to the line given by
$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{3.4.67.1}$$
68. Find the equation of the line given by
$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \tag{3.4.68.1}$$
69. Find the equation of the line passing through the origin and the point $\begin{pmatrix}5\\-2\\3\end{pmatrix}$.
70. Find the equation of the line passing through the points $\begin{pmatrix}3\\-2\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\\6\end{pmatrix}$.
71. Find the angle between the following pair of lines:

a)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix}2\\-5\\1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{3.4.71.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix}7\\-6\\0\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.4.71.2}$$

b)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\-2\end{pmatrix} \tag{3.4.71.3}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-56\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\-4\end{pmatrix} \tag{3.4.71.4}$$

72. Find the angle between the following pair of lines

a)
$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{3.4.72.1}$$
$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{3.4.72.2}$$

b)
$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{3.4.72.3}$$
$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{3.4.72.4}$$

73. Find the values of p so that the lines
$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{3.4.73.1}$$
$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{3.4.73.2}$$
are at right angles.
74. Show that the lines
$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{3.4.74.1}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{3.4.74.2}$$
are perpendicular to each other.

75. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} \qquad (3.4.75.1)$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix} \qquad (3.4.75.2)$$

76. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \qquad (3.4.76.1)$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \qquad (3.4.76.2)$$

77. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} \qquad (3.4.77.1)$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \qquad (3.4.77.2)$$

78. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix}1-t\\t-2\\3-2t\end{pmatrix} \qquad (3.4.78.1)$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}s+1\\2s-1\\-2s-1\end{pmatrix} \qquad (3.4.78.2)$$

79. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathbf{x} = 2$ c) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

b) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ d) $\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = 5$

80. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix}3\\5\\-6\end{pmatrix}$.

81. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 12$ c) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$

b) $\begin{pmatrix}3 & 4 & -6\end{pmatrix}\mathbf{x} = 0$ d) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

82. Find the equation of the planes

a) that passes through the point $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ and the normal to the plane is $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

b) that passes through the point $\begin{pmatrix}1\\4\\6\end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

83. Find the equation of the planes that passes through three points

a) $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}6\\4\\-5\end{pmatrix}, \begin{pmatrix}-4\\-2\\3\end{pmatrix}$

b) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\-1\end{pmatrix}$.

84. Find the intercepts cut off by the plane $\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 5$.

85. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

86. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}3 & -1 & 2\end{pmatrix}\mathbf{x} = 4$ and $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = -2$ and the pont $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

87. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix}2 & 5 & 3\end{pmatrix}\mathbf{x} = 9$ and the pont $\begin{pmatrix}2\\1\\3\end{pmatrix}$.

88. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 0$.

89. Find the angle between the planes whose equations are $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix}3 & -3 & 5\end{pmatrix}\mathbf{x} = 3$

90. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix}7 & 5 & 6\end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix}3 & -1 & -10\end{pmatrix}\mathbf{x} = -4$

b) $\begin{pmatrix}2 & 1 & 3\end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix}1 & -2 & 5\end{pmatrix}\mathbf{x} = 0$

c) $\begin{pmatrix}2 & -2 & 4\end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix}3 & -3 & 6\end{pmatrix}\mathbf{x} = 1$

d) $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = -3$

e) $\begin{pmatrix}4 & 8 & 1\end{pmatrix}\mathbf{x} = 8$ and $\begin{pmatrix}0 & 1 & 1\end{pmatrix}\mathbf{x} = 4$

91. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}3 & -4 & 12\end{pmatrix}\boldsymbol{x} = 3$
b)	$\begin{pmatrix}3\\-2\\1\end{pmatrix}$	$\begin{pmatrix}2 & -1 & 2\end{pmatrix}\boldsymbol{x} = -3$
c)	$\begin{pmatrix}2\\3\\-5\end{pmatrix}$	$\begin{pmatrix}1 & 2 & -2\end{pmatrix}\boldsymbol{x} = 9$
d)	$\begin{pmatrix}-6\\0\\0\end{pmatrix}$	$\begin{pmatrix}2 & -3 & 6\end{pmatrix}\boldsymbol{x} = 2$

TABLE 3.4.91

92. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.

93. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}, \begin{pmatrix}-4\\3\\-6\end{pmatrix}, \begin{pmatrix}2\\9\\2\end{pmatrix}$, then find the angle between the lines AB and CD.

94. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \qquad (3.4.94.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \qquad (3.4.94.2)$$

find the value of k.

95. Find the equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\mathbf{x} = -9 \qquad (3.4.95.1)$$

96. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}6\\2\\2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-2\\2\end{pmatrix} \text{ and} \qquad (3.4.96.1)$$

$$\mathbf{x} = \begin{pmatrix}-4\\0\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-2\\-2\end{pmatrix} \qquad (3.4.96.2)$$

97. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the YZ-plane.

98. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the ZX-plane.

99. Find the coordinates of the point where the line through $\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}-3\\1\end{pmatrix}$ crosses the plane

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \qquad (3.4.99.1)$$

100. Find the equation of the plane passing through the point $\begin{pmatrix}-1\\3\\2\end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 5 \qquad (3.4.100.1)$$

$$\begin{pmatrix}3 & 3 & 1\end{pmatrix}\mathbf{x} = 0 \qquad (3.4.100.2)$$

101. If the points $\begin{pmatrix}1\\1\\p\end{pmatrix}$ and $\begin{pmatrix}-3\\0\\1\end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix}3 & 4 & -12\end{pmatrix}\mathbf{x} = -13, \qquad (3.4.101.1)$$

then find the value of p.

102. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1 \text{ and} \qquad (3.4.102.1)$$

$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = -4 \qquad (3.4.102.2)$$

and parallel to the x-axis.

103. If **O** be the origin and the coordinates of **P** be $\begin{pmatrix}1\\2\\3\end{pmatrix}$, then find the equation of the plane passing through **P** and perpendicular to OP.

104. Find the equation of the plane which contains

the line of intersection of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 4 \quad (3.4.104.1)$$

$$\begin{pmatrix}2 & 1 & -1\end{pmatrix}\mathbf{x} = -5 \quad (3.4.104.2)$$

and which is perpendicular to the plane

$$\begin{pmatrix}5 & 3 & -6\end{pmatrix}\mathbf{x} = -8 \quad (3.4.104.3)$$

105. Find the distance of the point $\begin{pmatrix}-1\\-5\\-10\end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\2\end{pmatrix} \quad (3.4.105.1)$$

and the plane

$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 5 \quad (3.4.105.2)$$

106. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x} = 5 \quad (3.4.106.1)$$

$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \quad (3.4.106.2)$$

107. Find the vector equation of the line passing through the point $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \quad (3.4.107.1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad (3.4.107.2)$$

108. Distance between the two planes

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 4 \quad (3.4.108.1)$$

$$\begin{pmatrix}4 & 6 & 8\end{pmatrix}\mathbf{x} = 12 \quad (3.4.108.2)$$

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

109. The planes

$$\begin{pmatrix}2 & -1 & 4\end{pmatrix}\mathbf{x} = 5 \quad (3.4.109.1)$$

$$\begin{pmatrix}5 & -\frac{5}{2} & 10\end{pmatrix}\mathbf{x} = 6 \quad (3.4.109.2)$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix}0\\0\\\frac{5}{4}\end{pmatrix}$

3.5 Examples: Applications

1. The cost of a notebook is twice the cost of a pen. Write a linear equation in two variables to represent this statement.
2. The taxi fare in a city is as follows: For the first kilometre, the fare is ₹8 and for the subsequent distance it is ₹5 per km. Taking the distance covered as x km and total fare as ₹y, write a linear equation for this information, and draw its graph.
3. Yamini and Fatima, two students of Class IX of a school, together contributed ₹100 towards the Prime Minister's Relief Fund to help the earthquake victims. Write a linear equation which satisfies this data. (You may take their contributions as ₹x and ₹y.) Draw the graph of the same.
4. In countries like USA and Canada, temperature is measured in Fahrenheit, whereas in countries like India, it is measured in Celsius. Here is a linear equation that converts Fahrenheit to Celsius:

$$F = \frac{9}{5}C + 32 \quad (3.5.4.1)$$

   a) Draw the graph of the linear equation above using Celsius for x-axis and Fahrenheit for y-axis.
   b) If the temperature is 30°C, what is the temperature in Fahrenheit?
   c) If the temperature is 95°F, what is the temperature in Celsius?
   d) If the temperature is 0°C, what is the temperature in Fahrenheit and if the temperature is 0°F, what is the temperature in Celsius?
   e) Is there a temperature which is numerically the same in both Fahrenheit and Celsius? If yes, find it.
5. Romila went to a stationery shop and purchased 2 pencils and 3 erasers for ₹9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹18. Represent this situation algebraically and graphically. Find the cost of each pencil and eraser.

6. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically. Find their respective ages.
7. The coach of a cricket team buys 3 bats and 6 balls for ₹3900. Later, she buys another bat and 3 more balls of the same kind for ₹1300. Represent this situation algebraically and geometrically.Find the cost of each bat and ball.
8. The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹300. Represent the situation algebraically and geometrically. Find the cost of apples and grape.
9. Form the pair of linear equations in the following problems, and find their solutions.
10. 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
11. 5 pencils and 7 pens together cost ₹50, whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.
12. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.
13. Form the pair of linear equations for the following problems and find their solution
14. The difference between two numbers is 26 and one number is three times the other. Find them.
15. The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.
16. The coach of a cricket team buys 7 bats and 6 balls for ₹3800. Later, she buys 3 bats and 5 balls for ₹1750. Find the cost of each bat and each ball.
17. The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹105 and for a journey of 15 km, the charge paid is ₹155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?
18. A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. . Find the fraction.
19. Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages
20. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly incomes.
21. The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?
22. If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$. if we only add 1 to the denominator. What is the fraction?
23. Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?
24. The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.
25. Meena went to a bank to withdraw ₹2000. She asked the cashier to give her ₹50 and ₹100 notes only. Meena got 25 notes in all. Find how many notes of ₹50 and ₹100 she received.
26. A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹27 for a book kept for seven days, while Susy paid ₹21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.
27. The cost of 5 oranges and 3 apples is ₹35 and the cost of 2 oranges and 4 apples is ₹28. Let us find the cost of an orange and an apple.
28. From a bus stand in Bangalore , if we buy 2 tickets to Malleswaram and 3 tickets to Yeshwanthpur, the total cost is ₹46; but if we buy 3 tickets to Malleswaram and 5 tickets to Yeshwanthpur the total cost is ₹74. Find the fares from the bus stand to Malleswaram, and to Yeshwanthpur.
29. A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹1000 as hostel charges whereas a student B, who

takes food for 26 days, pays ₹1180 as hostel charges. Find the fixed charges and the cost of food per day.

30. A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes when 8 is added to its denominator. Find the fraction.
31. Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?
32. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?
33. The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.
34. Solve the pair of equations:

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = 13 \quad \begin{pmatrix} 5 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = -2 \tag{3.5.34.1}$$

35. Solve the pair of equations by reducing them to a pair of linear equations

$$\begin{pmatrix} 5 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{x-1} \\ \frac{1}{y-2} \end{pmatrix} = 2 \quad \begin{pmatrix} 6 & -3 \end{pmatrix}\begin{pmatrix} \frac{1}{x-1} \\ \frac{1}{y-2} \end{pmatrix} = 1 \tag{3.5.35.1}$$

36. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.
37. Solve the following pairs of equations

a)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = 2 \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \frac{13}{6} \tag{3.5.37.1}$$

b)

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{x}} \\ \frac{1}{\sqrt{y}} \end{pmatrix} = 2 \quad \begin{pmatrix} 4 & -9 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{x}} \\ \frac{1}{\sqrt{y}} \end{pmatrix} = -1 \tag{3.5.37.2}$$

c)

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ y \end{pmatrix} = 14 \quad \begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ y \end{pmatrix} = 23 \tag{3.5.37.3}$$

d)

$$\begin{pmatrix} 10 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{x+y} \\ \frac{1}{x-y} \end{pmatrix} = 4 \quad \begin{pmatrix} 15 & -5 \end{pmatrix}\begin{pmatrix} \frac{1}{x+y} \\ \frac{1}{x-y} \end{pmatrix} = -2 \tag{3.5.37.4}$$

e)

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3x+y} \\ \frac{1}{3x-y} \end{pmatrix} = \frac{3}{4} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3x+y} \\ \frac{1}{3x-y} \end{pmatrix} = -\frac{1}{8} \tag{3.5.37.5}$$

38. Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.
39. 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.
40. Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

41. The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju is twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju.
42. One says, "Give me a hundred, friend! I shall then become twice as rich as you". The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their (respective) capital? [From the Bijaganita of Bhaskara II].
43. A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.
44. The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.

3.6 Miscellaneous

1. Solve the following pair of linear equations

a)

$$\begin{pmatrix} p & q \end{pmatrix} \mathbf{x} = p - q$$
$$\begin{pmatrix} q & -p \end{pmatrix} \mathbf{x} = p + q \tag{3.6.1.1}$$

b)

$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} = c$$
$$\begin{pmatrix} b & a \end{pmatrix} \mathbf{x} = 1 + c \tag{3.6.1.2}$$

c)

$$\begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \end{pmatrix} \mathbf{x} = 0$$
$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} = a^2 + b^2 \tag{3.6.1.3}$$

2. Solve the following pair of equations

$$\begin{aligned} \begin{pmatrix} a-b & a+b \end{pmatrix} \mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix} a+b & a+b \end{pmatrix} \mathbf{x} &= a^2 + b^2 \end{aligned} \tag{3.6.2.1}$$

3. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.6.3.1}$$

4. (Cauchy-Schwarz Inequality:) Show that

$$\left|\mathbf{a}^T \mathbf{b}\right| \le \|\mathbf{a}\| \, \|\mathbf{b}\| \tag{3.6.4.1}$$

5. (Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{3.6.5.1}$$

6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
7. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.
8. If three points $\begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
9. $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 2 \tag{3.6.9.1}$$

10. Point $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
11. Show that two lines

$$\begin{pmatrix} a_1 & b_1 \end{pmatrix} \mathbf{x} + c_1 = 0 \tag{3.6.11.1}$$
$$\begin{pmatrix} a_2 & b_2 \end{pmatrix} \mathbf{x} + c_2 = 0 \tag{3.6.11.2}$$

are
   a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
   b) perpendicular if $a_1 a_2 - b_1 b_2 = 0$.
12. Find the distance between the parallel lines

$$l\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -p \tag{3.6.12.1}$$
$$l\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = r \tag{3.6.12.2}$$

13. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line

$$\begin{pmatrix} A & B \end{pmatrix} \mathbf{x} = -C \tag{3.6.13.1}$$

14. If p and q are the lengths of perpendiculars from the origin to the lines

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = k \cos 2\theta \tag{3.6.14.1}$$
$$\begin{pmatrix} \sec\theta & \operatorname{cosec}\theta \end{pmatrix} \mathbf{x} = k \tag{3.6.14.2}$$

respectively, prove that $p^2 + 4q^2 = k^2$.

15. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{3.6.15.1}$$

16. Show that the area of the triangle formed by the lines
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix} \mathbf{x} = c_1 \tag{3.6.16.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{3.6.16.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.6.16.3}$$
is $\frac{(c_1-c_2)^2}{2|m_1-m_2|}$.

17. Find the values of k for which the line
$$\begin{pmatrix} k-3 & -\left(4-k^2\right) \end{pmatrix} \mathbf{x} + k^2 - 7k + 6 = 0 \tag{3.6.17.1}$$
is
   a) parallel to the x-axis
   b) parallel to the y-axis
   c) passing through the origin.

18. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$.

19. Find the area of the triangle formed by the lines
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.6.19.1}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.6.19.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{3.6.19.3}$$

20. If three lines whose equations are
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix} \mathbf{x} = c_1 \tag{3.6.20.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{3.6.20.2}$$
$$\begin{pmatrix} -m_3 & 1 \end{pmatrix} \mathbf{x} = c_3 \tag{3.6.20.3}$$
are concurrent, show that
$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \tag{3.6.20.4}$$

21. Find the equation of the line passing through the origin and making an angle θ with the line
$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = c \tag{3.6.21.1}$$

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ to the line
$$\begin{pmatrix} \frac{\cos\theta}{a} & \frac{\sin\theta}{b} \end{pmatrix} \mathbf{x} = 1 \tag{3.6.22.1}$$
is b^2.

23. If $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1n_2 - m_2n_1 \\ n_1l_2 - n_2l_1 \\ l_1m_2 - l_2m_1 \end{pmatrix}$.

24. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{3.6.24.1}$$

25. Show that the lines
$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \tag{3.6.25.1}$$
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{3.6.25.2}$$
are coplanar.

26. Find $\mathbf{R}$ which divides the line joining the points
$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{3.6.26.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{3.6.26.2}$$
externally in the ratio 1 : 2.

27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if
$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \tag{3.6.27.1}$$
$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \tag{3.6.27.2}$$

28. Evaluate the product
$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \tag{3.6.28.1}$$

29. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if
$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{3.6.29.1}$$
$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{3.6.29.2}$$
and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

30. Show that
$$(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a}) \perp (\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a}) \quad (3.6.30.1)$$
31. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?
32. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that
$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \quad (3.6.32.1)$$
find the value of
$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \quad (3.6.32.2)$$
33. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if
    a) $\lambda = 1$
    b) $\lambda = -1$
    c) $\|\mathbf{a}\| = |\lambda|$
    d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$
34. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.
35. Show that
$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\,(\mathbf{a} \times \mathbf{b}) \quad (3.6.35.1)$$
36. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?
37. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that
$$(\mathbf{x} - \mathbf{a})^T(\mathbf{x} + \mathbf{a}) = 12. \quad (3.6.37.1)$$
38. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a}\times\mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is
    a) $\frac{\pi}{6}$ c) $\frac{\pi}{3}$
    b) $\frac{\pi}{4}$ d) $\frac{\pi}{2}$
39. Prove that
$$(\mathbf{a} + \mathbf{b})^T(\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (3.6.39.1)$$
$$\iff \mathbf{a} \perp \mathbf{b}. \quad (3.6.39.2)$$
40. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when
    a) $0 < \theta < \frac{\pi}{2}$ c) $0 < \theta < \pi$
    b) $0 \leq \theta \leq \frac{\pi}{2}$ d) $0 \leq \theta \leq \pi$
41. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if
    a) $\theta = \frac{\pi}{4}$ c) $\theta = \frac{\pi}{2}$
    b) $\theta = \frac{\pi}{3}$ d) $\theta = \frac{2\pi}{3}$
42. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to
    a) 0 c) $\frac{\pi}{2}$
    b) $\frac{\pi}{4}$ d) π.
43. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.
44. Find the equation of a line parallel to the x-axis and passing through the origin.
45. Find the equation of a plane passing through $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane
$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x}x = 2 \quad (3.6.45.1)$$
46. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \quad (3.6.46.1)$$

4 Circle

4.1 Construction Examples

1. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.
   Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC, AC \perp AB, a = 10$ and $c = 6$.
$$b = \sqrt{a^2 - c^2} \quad (4.1.1.1)$$
   The following code plots Fig. 4.1.1
```
codes/circle/draw_circle_eg.py
```
2. Draw a circle of radius 3. Mark any point $\mathbf{A}$ on the circle, point $\mathbf{B}$ inside the circle and point $\mathbf{C}$ outside the circle.
   Solution: For any angle θ, a point on the circle with radius 3 has coordinates
$$3\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad (4.1.2.1)$$

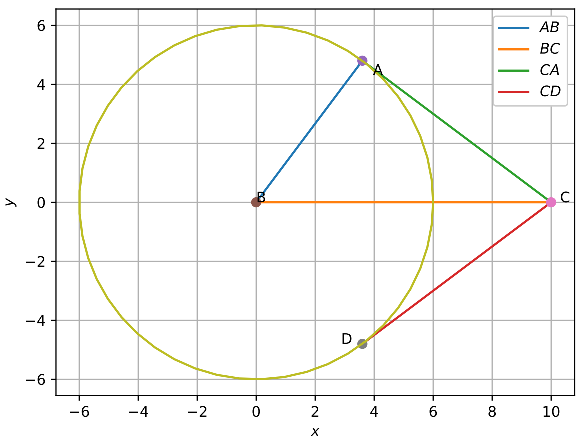

Fig. 4.1.1

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
   Solution: Take the centre of both circles to be at the origin.
6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
   Solution: Take the diameter to be on the x-axis.
7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.
   Solution: The tangent is perpendicular to the radius.
8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.
   Solution: Let
   $$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \tag{4.2.2.1}$$
9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^{\circ}$. BD is the perpendicular from **B** on AC (altitude). The circle through **B**, **C**, **D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry Examples

1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3. The perpendicular from the centre of a circle to a chord bisects the chord.
4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
5. There is one and only one circle passing through three non-collinear points.
6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
9. Congruent arcs of a circle subtend equal angles at the centre.
10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
11. Angles in the same segment of a circle are equal.
12. Angle in a semicircle is a right angle.

13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
15. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.
16. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.
17. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$
18. Two circles intersect at two points $AandB$. AD and AC are diameters to the two circles (see Fig.10.34). Prove that B lies on the line segment DC.
19. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

4.4 Circle Geometry Exercises

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\left(2, -3\right)$ and $\mathbf{B} = \begin{pmatrix}1\\4\end{pmatrix}$.
2. Find the centre of a circle passing through the points $\begin{pmatrix}6\\-6\end{pmatrix}$, $\begin{pmatrix}3\\-7\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$.
3. Find the locus of all the unit vectors in the xy-plane.
4. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
5. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
6. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
7. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.
8. A,B and C are three points on a circle with centre O such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.
9. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
10. $\angle PQR = 100°$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$. Fig. 10.37
11. A, B, C, D are points on a circle such that $\angle ABC = 69°$, $\angle ACB = 31°$, find $\angle BDC$.
12. A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130°$ and $\angle ECD = 20°$. Find $\angle BAC$.
13. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70°, \angle BACis30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.
14. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
15. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
16. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.
17. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
18. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.
19. Prove that a cyclic parallelogram is a rectangle.
20. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
21. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.
22. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?
23. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and

DE at the centre.

24. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.
25. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.
26. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.
27. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90°-\frac{A}{2}, 90°-\frac{B}{2}$ and $90°-\frac{C}{2}$.
28. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.
29. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.

4.5 Circle Applications

1. Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?
2. A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.