Discrete Maths: Maths Olympiad

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Abstract—**This book provides a collection of the international maths olympiad problems in discrete mathematics.**

1. Find the number of positive integers x which satisfy the condition
$$[\frac{x}{99}] = [\frac{x}{101}]$$
{Here [z] denotes,for any real z, the largest integer not exceeding z; e.g.[7/4]=1.}
2. Consider an n × n array of numbers:
$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$
Suppose each row consists of the n numbers 1, 2, 3,...,n in some order and $a_{ij} = a_{ji}$ for i = 1, 2, 3,...,n and j = 1, 2, 3,..,n If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}...a_{nn}$ are 1, 2, 3,...n in some order.
3. If x, y, z are the sides of a triangle,then prove that
$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz$$
4. Prove that the product of the first 1000 positive even integers differ from the product of the first 1000 positive odd integers by a multiple of 2001.
5. The circumference of a circle is divided into eight arcs by a convex quadrilateral ABCD, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose p + r = q + s. Prove that ABCD is a cyclic quadrilateral.
6. For any natural number n > 1, Prove the inequality:
$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + ... + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}$$
7. Find all the integers a, b, c, d satisfying following relations:
   a) 1 ≤ a ≤ b ≤ c ≤ d;
   b) ab + cd = a + b + c + d + 3
8. Positive integers are written on all the faces of a cube, one on each. At each corner of the cube. The product of the number on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the number on all the faces, find all the possible values of T.
9. Prove that the number of triples (A, B, C) when A, B, C are subsets of $\{1, 2, 3, ...n\}$ such that A ∩ B ∩ C = ϕ,A ∩ B = ϕ, B ∩ C ≠ 0 is $7^n - 2.6^n + 5^n$.
10. Let $\langle\ p_1, p_2, p_3, ..., p_n, ..\ \rangle$ be a squence of primes definite by $p_1 = 2$ and for n ≥ 1,p_{n+1} is largest prime factor of $p_1, p_2, p_3, ..., p_n$+1,(Thus $p_2 = 3$, $p_3 = 7$).Prove that $p_n \neq 5$ for any n.
11. Find the number of all 5-digit numbers each of which contains the block 15 and is divisible by 15.
12. Find the least possible value of a + b, where a, b are positive integers such that 11 divides a + 13b and 13 divides a+11b.
13. A 6 × 6 square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides, Prove that there are always two congruent rectangles.
14. Let ABCD be a quadrilateral in which AB is parallel to CD and perpendicular to AD ; AB = 3CD;and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of

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the quadrilateral, find its radius.

15. How many 6 digit numbers are there such that:
    a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$
    b) any digit that appears in the number appears at least twice?
16. Prove that:
    a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$
    b) $8 < \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$
    c) $n < \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.
17. Let ABC be an acute angled triangle; let D, F be the mid-point of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N. Prove that ND is equal to the circum radius of ABC.
18. Find number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0, 1, 2, 3 occurs at least once in them.
19. There nonzero real numbers a, b and c are said to be harmonic progression if

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

find all three-term harmonic progression a, b, c strictly increasing positive integers in which a = 20 and b divides c.

20. Find the number of all integer-sides isosceles obtuse-angled triangles with perimeter 2008.
21. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.
22. In a book with page number from 1 to 100, some pages are not torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?
23. Find the number of 4-digit numbers having non-zero digits and which are divisible by 4 but not 8.
24. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.
25. For each integer $n \geq 1$, define $a_n=[\frac{n}{\sqrt{n}}]$, where [x] denotes the largest integer not exceeding x,for any real number x. Find the number of all n in the set $\{1, 2, 3, ...2010\}$ for which $a_n > a_{n+1}$
26. Let $(a_1, a_2, a_3, ...a_{2011})$ be a permutation of the numbers 1, 2, 3,....2011. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.
27. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n. Prove that n-kl is a perfect square.
28. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, A_3...., A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them.
29. Let a, b, c be positive integers such that a divides b^3, b divides c^3 and c divides a^3. Prove that abc divides $(a + b + c)^{13}$
30. Let $x = \{1, 2, 3, ...10\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$.
31. Find all the primes p and q such that p divides q^2 - 4 and q divides $p^2 - 1$.
32. Find all fractions which can be written simultaneously in the form of forms $\frac{7k-5}{5k-3}$ and $\frac{6l-1}{4l-3}$ for some integers k, l.
33. Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?
34. Find all natural numbers n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n = 25(n)^2$.
35. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.
36. Let $\{a_1, a_2, a_3...\}$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.
37. For any natural number n, express in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.
38. How many 6-digit natural numbers containing only the digits 1, 2, 3 are there in which 3 occur exactly twice and the number is divisible by 9?
39. Show that the infinite arithmetic progression $\{1, 4, 7, 10, ..\}$ has infinitely many 3-term sub sequence in harmonic progression such that for

any two such triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ in harmonic progression,one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$$

40. Let a, b, c, d, e, f be positive integers such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}.$$

Suppose af - be = -1. Show that d ≥ b + f.

41. There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language,and no language is common to all 100 countries, what is the minimum possible value of n?

42. a) Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.
    b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

43. The present ages in years of two brothers A and B and their father C are three distinct positive integers a, b,and c respectively. Suppose $\frac{b-1}{a-1}$ and $\frac{b+1}{a+1}$ are two consecutive integers, and $\frac{c-1}{b-1}$ and $\frac{c+1}{b+1}$ are two consecutive integers. If a + b + c ≤ 150 determine a, b and c.

44. A box contain 4032 answer scripts out of which exactly half have odd numbers of marks. We chose 2 scripts randomly and,if the score on both of them are odd number, we add one mark to one of them, put the scrip back in the box and keep the other scrip outside. If both scrips have even score, we put back the one of the scripts and keep other outside. If there is one script with even and other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find,with proof,the number of scripts with odd score among the three left.

45. a) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.
    b) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

46. Consider n^2 units squares in the xy-plane centred at point (i, j) with integer co-ordinates, 1 ≤ i ≤ n, 1 ≤ j ≤ n. It is required to colour each unit square in such a way that when ever 1 ≤ i < j ≤ n and 1 ≤ k < l ≤ n,the three squares with centres at (i, k),(j, k),(j, l)have distinct colours. What is the least possible number of colours needed?

47. Let x, y, z be real numbers each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{y-1}{z-1} + \frac{z-1}{x-1}.$$