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Linear Algebra through Coordinate Geometry

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Contents

***Abstract*—This manual introduces linear algebra through coordinate geometry using a problem solving approach.**

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1 The Straight Line

1.1 The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{A} - \mathbf{B}) \tag{1.1}$$

Alternatively, it can be expressed as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{1.2}$$

where $\mathbf{n}$ is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{n} = 0 \tag{1.3}$$

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1.2 In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.4}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{1.5}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \tag{1.6}$$

Find the area of $\triangle ABC$.

Solution: The centroid $\mathbf{O}$ is the solution of (1.5),(1.6) and is obtained as the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{1.7}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.8}$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.9}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.10}$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.11}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.12}$$

From (1.6) and (1.12),

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{B} = 5 \tag{1.13}$$

$$\implies 5 + \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 12 \tag{1.14}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{C} = 7 \tag{1.15}$$

From (1.15) and (1.6), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.16}$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.17}$$

$$\implies \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.18}$$

From (1.11),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.19}$$

Thus,

$$\frac{1}{2} \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \tag{1.20}$$

2 Orthogonality

2.1 $\mathbf{u}^T\mathbf{x} = 0 \implies \mathbf{u} \perp \mathbf{x}$. Show that

$$\mathbf{u}^T\mathbf{x} = \mathbf{P}^T\mathbf{x} = 0 \implies \mathbf{P} = \alpha\mathbf{u} \tag{2.1}$$

2.2 The foot of the perpendicular drawn from the origin on the line

$$AB : \mathbf{u}^T\mathbf{x} = \lambda \tag{2.2}$$

where

$$\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.3}$$

is $\mathbf{P}$. The line meets the x-axis at $\mathbf{A}$ and y-axis at $\mathbf{B}$. Show that $\mathbf{P} = \alpha\mathbf{u}$ and find α.
Solution: From (2.2),

$$\mathbf{u}^T\mathbf{A} = \mathbf{u}^T\mathbf{B} = \lambda \tag{2.4}$$

$$\implies \mathbf{u}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.5}$$

Since $OP \perp AB$,

$$\mathbf{P}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.6}$$

Thus, from (2.1),

$$\mathbf{P} = \alpha\mathbf{u} \tag{2.7}$$

Since $\mathbf{P}$ lies on (2.2),

$$\mathbf{u}^T\mathbf{P} = \alpha\mathbf{u}^T\mathbf{u} = \lambda \tag{2.8}$$

$$\implies \alpha = \frac{\lambda}{\mathbf{u}^T\mathbf{u}} = \frac{\lambda}{10}. \tag{2.9}$$

2.3 Find $\mathbf{A}$.
Solution: Let

$$\mathbf{A} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.10}$$

From (2.2),

$$\mathbf{u}^T\mathbf{A} = a\begin{pmatrix} 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \tag{2.11}$$

$$\implies a = \frac{\lambda}{3} \tag{2.12}$$

$$\text{and } \mathbf{A} = \frac{\lambda}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.13}$$

2.4 Find the ratio $BP : PA$.
Solution: Let

$$\frac{BP}{PA} = k \tag{2.14}$$

Then,

$$k\mathbf{A} + \mathbf{B} = (k+1)\mathbf{P} \tag{2.15}$$

$$\implies k\mathbf{A}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} = (k+1)\mathbf{P}^T\mathbf{A} \tag{2.16}$$

$$\implies ka^2 = \alpha(k+1)\lambda \tag{2.17}$$

using (2.7), (2.10), (2.2) and $\mathbf{A} \perp \mathbf{B}$. Substituting from (2.9) and (2.12),

$$\implies k\frac{\lambda^2}{9} = (k+1)\frac{\lambda^2}{10} \tag{2.18}$$

$$\implies k = 9 \tag{2.19}$$

3 Locus

3.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{3.1}$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed as shown in Fig. (3.1),

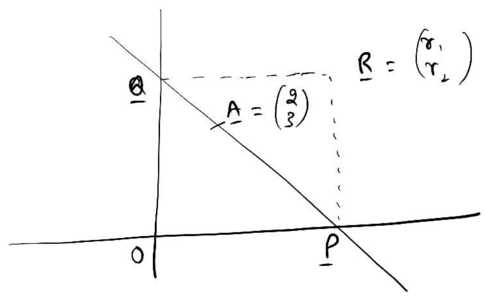

Fig. 3.1

3.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{3.2}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{3.3}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{3.4}$$

3.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{n} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{n} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{n} &= 0 \end{aligned} \tag{3.5}$$

Solution: Trivial using (1.2) and (1.3).

3.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{n} = 0 \tag{3.6}$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{n} = 0 \tag{3.7}$$

Solution: From (3.5) and (3.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{n} = 0 \tag{3.8}$$

resulting in (3.6). From (3.5) and (3.2),(3.3), (3.7) is obtained.

3.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \tag{3.9}$$

3.6 Find the locus of $\mathbf{R}$.

Solution: For $\mathbf{n}$ to be unique in (3.6),(3.7),

$$\begin{aligned} (2\mathbf{A} - \mathbf{R}) &= k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{3.10}$$

where k is some constant. Thus, the desired locus is

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) = 0 \tag{3.11}$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - 2\mathbf{A}^T \mathbf{R} = 0 \tag{3.12}$$

4 Conics

4.1 The equation of a quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{4.1}$$

Show that (4.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{4.2}$$

Find the matrix V and vector $\mathbf{u}$.

4.2 The tangent to (4.1) at a point $\mathbf{p}$ on the curve is given by

$$\begin{pmatrix} \mathbf{p}^T & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{4.3}$$

Show that (4.3) can be expressed as

$$\left(\mathbf{p}^T V + \mathbf{u}^T\right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \tag{4.4}$$

4.3 Classify the various conic sections based on (4.2).

Solution:

Curve	**Property**
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 4.3

5 Circle

5.1 Find the tangent to the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{5.1}$$

at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution: From (4.3), the tangent T is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right]\mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{5.2}$$

$$\implies T : \mathbf{n}^T\mathbf{x} = 3 \tag{5.3}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.4}$$

5.2 The tangent T in (5.3) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{5.5}$$

Find $\mathbf{A} + \mathbf{B}$.

Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \tag{5.6}$$

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \tag{5.7}$$

Subtracting (5.7) from (5.6),

$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{5.8}$$

$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{5.9}$$

$\because \mathbf{A}, \mathbf{B}$ lie on T, from (5.3),

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{5.10}$$

$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \tag{5.11}$$

From (5.9) and (5.11)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{5.12}$$

$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{5.13}$$

$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{5.14}$$

$$\implies k = 2 \tag{5.15}$$

using (5.10). Substituting in (5.12)

$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \tag{5.16}$$

5.3 If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.

Solution: From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{5.17}$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4\|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \tag{5.18}$$

$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \tag{5.19}$$

using (5.16) and simplifying.

5.4 Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \tag{5.20}$$

Solution:

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{5.21}$$

$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \tag{5.22}$$

From (5.22),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T \; [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \tag{5.23}$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \tag{5.24}$$

Upon substituting from (5.7) and (5.6) and simplifying, (5.20) is obtained.

5.5 Find r.

Solution: (5.20) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{5.25}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \tag{5.26}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \tag{5.27}$$

$$\implies r = \sqrt{6}. \tag{5.28}$$

6 Parabola

6.1 Find the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \tag{6.1}$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{6.2}$$

in (4.4), the desired equation is

$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} + 6 = 0 \tag{6.3}$$

resulting in

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 5 \tag{6.4}$$

6.2 The line in (6.4) touches the circle

$$\mathbf{x}^T\mathbf{x} + 4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + c = 0 \tag{6.5}$$

Find c.
Solution: Comparing (4.2) and (6.5),

$$V = I, \quad \mathbf{u} = 2\begin{pmatrix}4\\3\end{pmatrix} \tag{6.6}$$

Comparing (4.4) and (6.4),

$$\mathbf{p} + 2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{6.7}$$

$$\implies \mathbf{p} = -\begin{pmatrix}6\\7\end{pmatrix} \tag{6.8}$$

and

$$c + \mathbf{p}^T\mathbf{u} = 5 \tag{6.9}$$

$$\implies c = 5 + 2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \tag{6.10}$$

$$= 95 \tag{6.11}$$

7 Ellipse

7.1 A tangent at a point on the ellipse

$$\mathbf{x}^T V\mathbf{x} = 51 \tag{7.1}$$

where

$$V = \begin{pmatrix}3 & 0\\0 & 27\end{pmatrix} \tag{7.2}$$

meets the coordinate axes at **A** and **B**. If **O** be the origin, find the minimum area of $\triangle OAB$.

8 Hyperbola

8.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V\mathbf{x} = 36 \tag{8.1}$$

where

$$V = \begin{pmatrix}4 & 0\\0 & -1\end{pmatrix} \tag{8.2}$$

at points **P** and **Q**. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix}0\\3\end{pmatrix}, \tag{8.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (4.4) as

$$\mathbf{P}^T V\mathbf{x} = 36 \tag{8.4}$$

$$\mathbf{Q}^T V\mathbf{x} = 36. \tag{8.5}$$

Since both pass through **T**

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T\begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{8.6}$$

$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T\begin{pmatrix}0\\-3\end{pmatrix} = 36 \tag{8.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix}0 & -3\end{pmatrix}\mathbf{x} = -36 \tag{8.8}$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = -12 \tag{8.9}$$

which is the equation of PQ.

8.2 In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0 \tag{8.10}$$

using (1.2) and (8.9), the equation of TD is

$$\begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{x} - \mathbf{T}) = 0 \tag{8.11}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{8.12}$$

8.3 Find D.
Solution: From (8.9) and (8.12),

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{8.13}$$

$$\implies \mathbf{D} = \begin{pmatrix}0\\-12\end{pmatrix} \tag{8.14}$$

8.4 Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda\mathbf{m} \tag{8.15}$$

where

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{8.16}$$

8.5 Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda\mathbf{m})^T V (\mathbf{D} + \lambda\mathbf{m}) + F = 0 \tag{8.17}$$

can be expressd as

$$\lambda^2\mathbf{m}^T V\mathbf{m} + 2\lambda\mathbf{m}^T V\mathbf{D} + \mathbf{D}^T V\mathbf{D} + F = 0 \tag{8.18}$$

8.6 Find **P** and **Q**.
Solution: From (8.15) and (8.1) (8.18) is obtained. Substituting from (8.16), (8.2) and (8.14)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 4 \tag{8.19}$$

$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = 0 \tag{8.20}$$

$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix}0 & -12\end{pmatrix}\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\-12\end{pmatrix} = -144 \tag{8.21}$$

Substituting in (8.18)

$$4\lambda^2 - 144 = 36 \tag{8.22}$$

$$\implies \lambda = \pm 3\sqrt{5} \tag{8.23}$$

Substituting in (8.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix}\sqrt{5}\\-4\end{pmatrix} \tag{8.24}$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix}\sqrt{5}\\4\end{pmatrix} \tag{8.25}$$

8.7 Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{8.26}$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{8.27}$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \tag{8.28}$$

8.8 Repeat the previous exercise using determinants.

9 JEE

9.1 Tangent and normal are drawn at

$$\mathbf{P} = \begin{pmatrix}16\\16\end{pmatrix} \tag{9.1}$$

on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}16 & 0\end{pmatrix}\mathbf{x} = 0 \tag{9.2}$$

which intersect the axis of the parabola at **A** and **B** respectively. If **C** is the centre of the circle through the ponts **P** **A** and **B**, find $\tan CPB$.

9.2 A circle passes through the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$. If its centre lies on the line

$$\begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} + 3 = 0 \tag{9.3}$$

find its radius.

9.3 Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.

9.4 If the tangents drawn to the hyperbola

$$\mathbf{x}^T V\mathbf{x} + 1 = 0 \tag{9.4}$$

where

$$V = \begin{pmatrix}1 & 0\\0 & -4\end{pmatrix} \tag{9.5}$$

intersect the coordinate axes at the distinct points **A** and **B**, find the locus of the mid point of AB.

9.5 β is one of the angles between the normals to the ellipse

$$\mathbf{x}^T V\mathbf{x} = 9 \tag{9.6}$$

where

$$V = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} \tag{9.7}$$

at the points

$$\begin{pmatrix}3\cos\theta\\\sqrt{3}\sin\theta\end{pmatrix}, \begin{pmatrix}-3\sin\theta\\\sqrt{3}\cos\theta\end{pmatrix}, \quad \theta \in \left(0, \frac{\pi}{2}\right), \tag{9.8}$$

then find $\frac{2\cot\beta}{\sin 2\theta}$.

9.6 The sides of a rhombus ABC are parallel to the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 2 = 0 \tag{9.9}$$

$$\begin{pmatrix}7 & -1\end{pmatrix}\mathbf{x} + 3 = 0. \tag{9.10}$$

If the diagonals of the rhombus intersect at

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.11}$$

and the vertex **A** (different) from the origin is on the y-axis, then find the ordinate of A.

9.7 Tangents drawn from the point $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{9.12}$$

touch the parabola at **P** and **Q**. If **F** is the focus of the parabola, then find the area of $\triangle PFQ$.

9.8 A normal to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 36 \tag{9.13}$$

meets the coordinate axes x and y at **A** and **B** respectively. If the parallelogram $OABP$ is formed, find the locus of **P**.

9.9 Find the locus of the point of intersection of the lines

$$\begin{pmatrix} \sqrt{2} & -1 \end{pmatrix} \mathbf{x} + 4\sqrt{2}k = 0 \tag{9.14}$$

$$\begin{pmatrix} \sqrt{2}k & k \end{pmatrix} \mathbf{x} - 4\sqrt{2} = 0 \tag{9.15}$$

9.10 If a circle C,whose radius is 3, touches externally the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} = 4 \tag{9.16}$$

at the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.

9.11 Let **P** be the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{9.17}$$

Given that the distance of **P** from the centre of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{9.18}$$

is minimum. Find the equation of the tangent to the parabola at **P**.

9.12 The length of the latus rectum of an ellipse is 4 ad the distance between a focus and its nearest vertex on the major axis is $\frac{3}{2}$. Find its eccentricity.

9.13 A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.

9.14 A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{9.19}$$

cuts the circle

$$\mathbf{x}^T\mathbf{x} = 9 \tag{9.20}$$

at the points **P** and **P**. Find $PA.PB$.

9.15 Find the eccentricity of an ellipse having centre at the origin, axes along the coordinate axes and passing through the points

$$\mathbf{P} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}. \tag{9.21}$$

9.16 $\begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} + c = 0$ is the normal at a point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 = 0 \tag{9.22}$$

whose focal distance is 8. Find $|c|$.

9.17 Find the locus of the point of intersection of the straight lines

$$\begin{pmatrix} t & -2 \end{pmatrix} \mathbf{x} - 3t = 0 \tag{9.23}$$

$$\begin{pmatrix} 1 & -2t \end{pmatrix} \mathbf{x} + 3 = 0 \tag{9.24}$$

9.18 The common tangents to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \mathbf{x} = 0 = 0 \tag{9.25}$$

intersect at the point **P**. Find the distance of **P** from the origin.

9.19 Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.

9.20 Let k be an integer such that the triangle with vertices

$$\begin{pmatrix} k \\ -3k \end{pmatrix}, \begin{pmatrix} 5 \\ k \end{pmatrix}, \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{9.26}$$

has area 28. Find the orthocentre of this triangle.

9.21 A hyperbola passes through the point

$$\mathbf{P} = \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} \tag{9.27}$$

and has foci at $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$. Find the equation of the tangent to this hyperbola at $\mathbf{P}$.

9.22 If an equlateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{9.28}$$

then find the area of this triangle.

9.23 Find the equation of the circle, which is the mirror image of the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 0 = 0 \tag{9.29}$$

in the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3. \tag{9.30}$$

9.24 Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 225 \tag{9.31}$$

upon the tangent to it at the point

$$\frac{1}{2} \begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \tag{9.32}$$

9.25 Find the equation of the normal to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 144 \tag{9.33}$$

drawn at the point

$$\begin{pmatrix} 8 \\ 3\sqrt{3} \end{pmatrix} \tag{9.34}$$

9.26 Two sides of a rhombus are along the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{9.35}$$

$$\begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{9.36}$$

If its diagonals intersect at

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{9.37}$$

find its vertices.

9.27 Find the locus of the centres of those circles which touch the circle

$$\mathbf{x}^T\mathbf{x} - 8 \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{9.38}$$

and also touch the x-axis.

9.28 One of the diameters of the circle, given by

$$\mathbf{x}^T\mathbf{x} + 2 \begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 12 = 0 \tag{9.39}$$

is a chord of a circle S, whose centre is at

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix}. \tag{9.40}$$

Find the radius of S.

9.29 Let P be the point on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{9.41}$$

which is at a minimum distance from the centre C of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 0 & 12 \end{pmatrix} \mathbf{x} = 1 \tag{9.42}$$

Find the equation of the circle passing through C and having its centre at $\left(P\right)$.

9.30 Find the eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half the distance between its foci.

9.31 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{9.43}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{9.44}$$

meets the coordinate axes at $\mathbf{A}$ and $\mathbf{B}$, then find the locus of the midpoint of AB.

9.32 The point

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{9.45}$$

is translated parallel to the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{9.46}$$

by $2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.

9.33 A circle passes through

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \tag{9.47}$$

and touches the y-axis at

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{9.48}$$

Which one of the following equations can represent a diameter of this circle?

a) $\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 6$

b) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} + 10 = 0$

c) $\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 3$

d) $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} + 4 = 0$

9.34 Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{9.49}$$

If

$$\mathbf{S} = \begin{pmatrix}5\\0\end{pmatrix} \tag{9.50}$$

is a focus and

$$\begin{pmatrix}5 & 0\end{pmatrix}\mathbf{x} = 9 \tag{9.51}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.

9.35 A straight line through the origin **O** meets the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 10 \tag{9.52}$$

$$\begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 5 = 0 \tag{9.53}$$

at **A** and **B** respectively. Find the ratio in which **O** divides AB.

9.36 Find the equation of the tangent to the circle, at the point

$$\begin{pmatrix}1\\-1\end{pmatrix}, \tag{9.54}$$

whose centre is the point of intersection of the straight lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{9.55}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{9.56}$$

9.37 **P** and **Q** are two distinct points on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} = 0 \tag{9.57}$$

with parameters t and t_1 respectively. If the normal at **P** passes through **Q**, then find the minimum value of t_1^2.

9.38 A hyperbola whose transverse axis is along the major axis of the conic

$$\mathbf{x}^T V\mathbf{x} = 51 \tag{9.58}$$

where

$$V = \begin{pmatrix}3 & 0\\0 & 27\end{pmatrix} \tag{9.59}$$

and has vertices at the foci of this conic. If the eccentricity of the hyperbola is $\frac{3}{2}$, which of the following points doesnot lie on it?

a) $\begin{pmatrix}0\\2\end{pmatrix}$

b) $\begin{pmatrix}\sqrt{5}\\2\sqrt{2}\end{pmatrix}$

c) $\begin{pmatrix}\sqrt{10}\\2\sqrt{3}\end{pmatrix}$

d) $\begin{pmatrix}5\\2\sqrt{3}\end{pmatrix}$