Python with Linear Algebra: 2D

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***Abstract*—This manual introduces matrix computations using python and the properties of a triangle.**

1 Line

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \tag{1}$$

Find the equation of AB.
Solution: The desired equation is obtained as

$$AB: \quad \mathbf{x} = \mathbf{A} + \lambda_1 (\mathbf{B} - \mathbf{A}) \tag{2}$$

$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{3}$$

1.2 Find the direction vector and the normal vector for AB
Solution: The direction vector of AB is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{4}$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{5}$$

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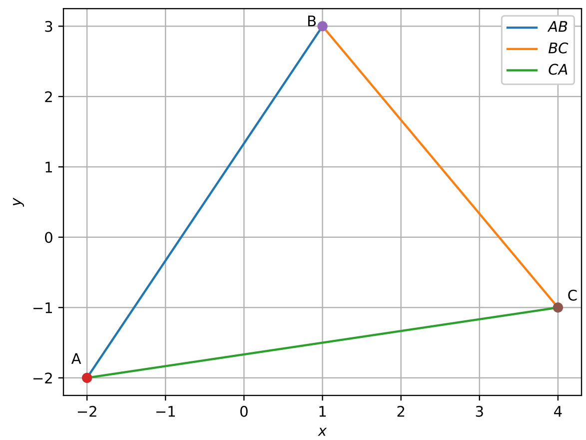

Fig. 1.2

The normal vector $\mathbf{n}$ is defined as

$$\mathbf{n}^T\mathbf{m} = 0 \tag{6}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{7}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{9}$$

Draw ΔABC.
Solution: The following codes yields the desired plot in Fig. 1.2

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/coeffs.py
```

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/draw_triangle.py
```

1.3 Find the equation of the line in terms of the normal vector.

Solution: The desired equation is

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = \mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \tag{10}$$

$$\implies \begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = -\begin{pmatrix}5 & -3\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = -4 \tag{11}$$

1.4 Find the equations of BC and CA.

2 Altitudes of a Triangle

2.1 In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of ΔABC.

2.2 Find the equation of AP.
Solution: The normal vector of AP is $\mathbf{B}-\mathbf{C}$. From (10), the equation of AP is

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{x}-\mathbf{A}) = 0 \tag{12}$$

$$\implies \begin{pmatrix}-3 & 4\end{pmatrix}\mathbf{x} = -\begin{pmatrix}-3 & 4\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = -2 \tag{13}$$

2.3 Find the equation of the altitude BQ.
Solution: The desired equation is

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{x}-\mathbf{B}) = 0 \tag{14}$$

$$\implies \begin{pmatrix}6 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}6 & 1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = 9 \tag{15}$$

2.4 Find the equation of the altitude CR.

2.5 Find the point of intersection of AP and BQ.
Solution: (12) and (14) can be stacked together into the matrix equation

$$\begin{pmatrix}-3 & 4\\6 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-2\\9\end{pmatrix} \tag{16}$$

The following code computes the point of intersection.

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    orthocentre.py
```

2.6 Find the point of intersection of and BQ and CR. Comment.

2.7 Find $\mathbf{P}$
Solution: The following code finds the required points.

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_foot.py
```

2.8 Find $\mathbf{Q}$ and $\mathbf{R}$.

2.9 Draw AP, BQ and CR and verify that they meet at a point $\mathbf{H}$.

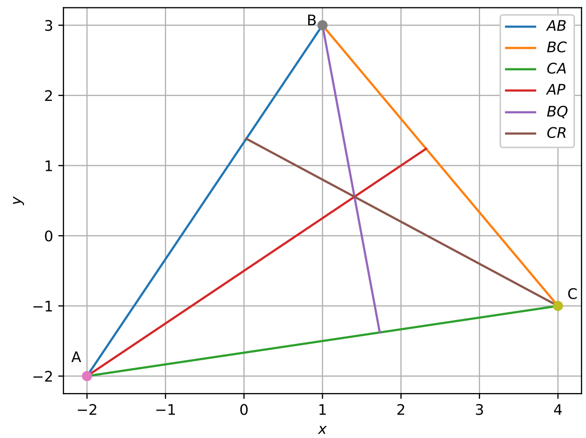

Fig. 2.9

Solution: The following code plots the altitudes in Fig. 2.9

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_draw.py
```

3 Circumcircle

3.1 Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be points on a circle with centre $\mathbf{O}$ and radius r.

3.2 Find $\mathbf{O}$.
Solution: The equation of the circle is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{17}$$

$$\implies \|\mathbf{x}-\mathbf{O}\|^2 = (\mathbf{x}-\mathbf{O})^T(\mathbf{x}-\mathbf{O}) = R^2 \tag{18}$$

From (17),

$$\|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \tag{19}$$

$$\implies (\mathbf{A}-\mathbf{O})^T(\mathbf{A}-\mathbf{O}) - (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = 0 \tag{20}$$

which can be simplified as

$$(\mathbf{A}-\mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{21}$$

Similarly,

$$(\mathbf{B}-\mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{22}$$

The following code computes $\mathbf{O}$ using the above two equations.

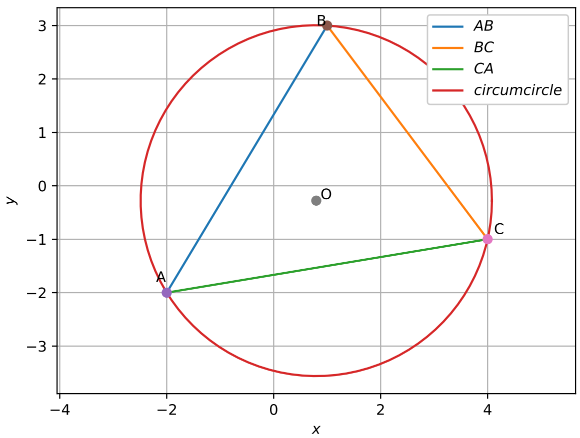

Fig. 3.4

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcentre.py

3.3 Find the radius R.

3.4 Plot the *circumcircle* of $\triangle ABC$.
Solution: The following code plots Fig. 3.4

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcircle.py

4 Medians of a Triangle

4.1 Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.

4.2 Find the equations of AD, BE and CF. These lines are the *medians* of ΔABC

4.3 Find the point of intersection of AD and CF.

4.4 Verify that $\mathbf{G}$ is the point of intersection of BE, CF as well as AD, BE. $\mathbf{G}$ is known as the *centroid* of ΔABC.

4.5 Graphically show that the medians of ΔABC meet at the centroid.

4.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{23}$$

5 Incircle

5.1 Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.

5.2 Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \tag{24}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{25}$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{26}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{U} - \mathbf{I}) = 0 \implies OD \perp BC \tag{27}$$

5.3 Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \tag{28}$$

$$\implies \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \tag{29}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{30}$$

$$\implies \mathbf{I} - \mathbf{U} = \lambda\mathbf{n} \tag{31}$$

$$\text{or} \quad r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \, \|\mathbf{n}\| \tag{32}$$

From (29) and (30)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{33}$$

$$\implies \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda\|\mathbf{n}\|^2 \tag{34}$$

$$\implies r = |\lambda| \, \|\mathbf{n}\| = \frac{\left|\mathbf{n}^T (\mathbf{I} - \mathbf{B})\right|}{\|\mathbf{n}\|} \tag{35}$$

from (32). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{36}$$

$$r = \left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| \tag{37}$$

5.4 Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (37),

$$\left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| = \left|\mathbf{n}_2^T (\mathbf{I} - \mathbf{C})\right| = \left|\mathbf{n}_3^T (\mathbf{I} - \mathbf{A})\right| \tag{38}$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (38) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \tag{39}$$

$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2\mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \tag{40}$$

where $k_1, k_2 = \pm 1$. The above equations can be

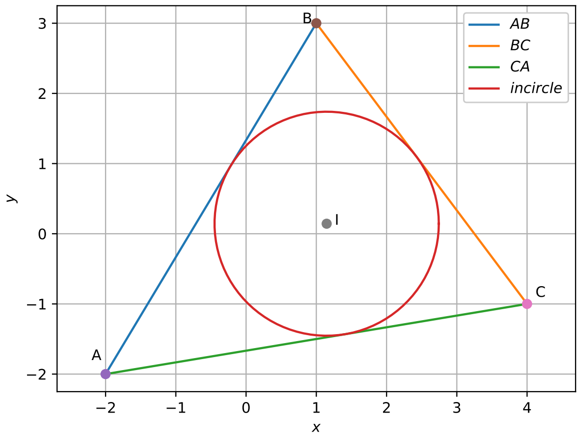

Fig. 5.7

expressed as the matrix equation

$$\begin{pmatrix}\mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3\end{pmatrix}^T \mathbf{I} = \begin{pmatrix}\mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A}\end{pmatrix} \tag{41}$$

5.5 Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$

5.6 Compute $\mathbf{I}$ and r.
Solution:

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/incentre.py

5.7 Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 5.7

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/incircle.py