The Circle

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***Abstract*—Solved problems from JEE mains papers related to 2D circles in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 A circle passes through the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix} -1 & 4 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{1.1}$$

find its radius.
Solution: Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.2}$$

The direction vector of AB is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{1.3}$$

$$\because OC \perp AB,$$
$$OC : \mathbf{m}^T (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{1.4}$$

Thus, $\mathbf{O}$ is the intersection of (1.1) and (1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} \tag{1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{1.6}$$

Thus the radius of the circle

$$OA = \left\|\mathbf{O} - \mathbf{A}\right\| = \sqrt{10} \tag{1.7}$$

2 If a circle C_1, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} = 4 \tag{2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.
Solution: From (2.1), the centre of C_2 is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T \mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.3}$$

Since the radius of C_1 is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$
$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$
$$\implies \mathbf{O}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.4}$$

The intercepts of C_1 on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda \mathbf{m} \tag{2.5}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6}$$

Susbtituting in the equation for C_1,

$$\left\|\lambda \mathbf{m} - \mathbf{O}_1\right\|^2 = r_1^2 \tag{2.7}$$

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which can be expressed as

$$\lambda^2\|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$
$$\implies \lambda^2 - 10\lambda + 20 = 0 \quad (2.8)$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \quad (2.9)$$

after substituting from (2.6) and (2.4).

3 A line drawn through the point

$$\mathbf{P} = \begin{pmatrix}4\\7\end{pmatrix} \quad (3.1)$$

cuts the circle

$$C : \mathbf{x}^T\mathbf{x} = 9 \quad (3.2)$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$.

Solution: Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line PAB can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \quad (3.3)$$

for $\|\mathbf{m}\| = 1$. The intersection of L and C yields

$$(\mathbf{P} + \lambda\mathbf{m})(\mathbf{P} + \lambda\mathbf{m}) = 9$$
$$\implies \lambda^2 + 2\lambda\mathbf{m}^T\mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \quad (3.4)$$

The product of the roots in (3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \quad (3.5)$$

4 Find the equation of the circle, which is the mirror image of the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 0 \quad (4.1)$$

in the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3. \quad (4.2)$$

5 Find the locus of the centres of those circles which touch the circle

$$\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \quad (5.1)$$

and also touch the x-axis.

6 One of the diameters of the circle, given by

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 12 = 0 \quad (6.1)$$

is a chord of a circle S, whose centre is at

$$\begin{pmatrix}-3\\2\end{pmatrix}. \quad (6.2)$$

Find the radius of S.

7 A circle passes through

$$\begin{pmatrix}-2\\4\end{pmatrix} \quad (7.1)$$

and touches the y-axis at

$$\begin{pmatrix}0\\2\end{pmatrix}. \quad (7.2)$$

Which one of the following equations can represent a diameter of this circle?

a) $\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 6$
b) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} + 10 = 0$
c) $\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 3$
d) $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} + 4 = 0$

8 Find the equation of the tangent to the circle, at the point

$$\begin{pmatrix}1\\-1\end{pmatrix}, \quad (8.1)$$

whose centre is the point of intersection of the straight lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \quad (8.2)$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 1 \quad (8.3)$$

9 A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 10 \quad (9.1)$$
$$\begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 5 = 0 \quad (9.2)$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.

Solution: Let

$$\mathbf{n} = \begin{pmatrix}4 & 3\end{pmatrix} \quad (9.3)$$

Then (9.1) can be expressed as

$$\mathbf{n}^T\mathbf{x} = 10 \quad (9.4)$$
$$2\mathbf{n}^T\mathbf{x} = -5 \quad (9.5)$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (9.4) respectively,

$$\mathbf{n}^T\mathbf{A} = 10 \quad (9.6)$$
$$2\mathbf{n}^T\mathbf{B} = -5 \quad (9.7)$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (9.8)$$

$$\because \mathbf{O} = \mathbf{0}, \tag{9.9}$$
$$\mathbf{A} = -k\mathbf{B} \tag{9.10}$$

Substituting in (9.6), and simplifying,

$$\mathbf{n}^T\mathbf{B} = \frac{10}{-k} \tag{9.11}$$
$$\mathbf{n}^T\mathbf{B} = \frac{-5}{2} \tag{9.12}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{9.13}$$

10 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{10.1}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 4 \tag{10.2}$$

by $2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (10.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10.3}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \tag{10.4}$$

However,

$$PQ = 2\sqrt{3} \tag{10.5}$$
$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = 2\sqrt{3} \tag{10.6}$$
$$\implies \lambda = \pm\frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{10.7}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T\mathbf{m}} = \sqrt{2} \tag{10.8}$$

from (10.3). Since $\mathbf{Q}$ lies in the third quadrant, from (10.4) and (10.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{10.9}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{Q}) = 0 \tag{10.10}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 - \sqrt{6} \tag{10.11}$$

11 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{11.1}$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 12 \tag{11.2}$$

meets the coordinate axes at $\mathbf{A}$ and $\mathbf{B}$, then find the locus of the midpoint of AB.
Solution: The intersection of the lines in (11.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{11.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{11.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{11.5}$$

Let the $\mathbf{R}$ be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} \tag{11.6}$$
$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R} \tag{11.7}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x} - \mathbf{C}) = 0 \tag{11.8}$$

Since $\mathbf{R}$ lies on AB,

$$\mathbf{n}^T(\mathbf{R} - \mathbf{C}) = 0 \tag{11.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{11.10}$$

Substituting from (11.6) in (11.10),

$$\mathbf{n}^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} = 0 \tag{11.11}$$

From (11.9) and (11.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} \tag{11.12}$$

for some constant k. Multiplying both sides of

(11.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{11.13}$$

$$\begin{aligned}\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0\end{aligned} \tag{11.14}$$

$$\because \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{11.15}$$

which can be easily verified for any $\mathbf{R}$. from (11.14),

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{11.16}$$

which is the desired locus.

12 Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{12.1}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{12.2}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{12.3}$$

find its vertices.
Solution: From (12.1) and (12.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{12.4}$$

By row reducing the augmented matrix

$$\begin{aligned}\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\end{aligned} \tag{12.5}$$

From (12.3) and (12.5)

$$AP = \|\mathbf{A} - \mathbf{P}\| = 2\sqrt{5} = d(say) \tag{12.6}$$

The direction vector of AP is

$$\mathbf{m} = \mathbf{A} - \mathbf{P} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{12.7}$$

$$\implies \|\mathbf{m}\| = 2\sqrt{5} \tag{12.8}$$

Since the direction of AP is the same as AC,

$$\begin{aligned}\mathbf{C} &= \mathbf{P} - d\frac{\mathbf{m}}{\|\mathbf{m}\|} \\ &= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}\end{aligned} \tag{12.9}$$

Let $\mathbf{n} \perp \mathbf{m}$. Then

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{12.10}$$

and

$$\begin{aligned}\mathbf{B}, \mathbf{D} &= \mathbf{P} \pm d\frac{\mathbf{n}}{\|\mathbf{n}\|} \\ &= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \pm 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}\end{aligned} \tag{12.11}$$

13 Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{13.1}$$

has area 28. Find the orthocentre of this triangle.
Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \tag{13.2}$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \tag{13.3}$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \tag{13.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \tag{13.5}$$

The orthocentre is obtained by solving (13.3) and (13.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \tag{13.6}$$

which can be expressed using (13.2), (13.5), (13.3) and (13.4) as

$$\begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2+5k+6k-3k^2 \\ 10k-2k-3k^2 \end{pmatrix}$$

$$= k \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \tag{13.7}$$

The solution to the above is

$$\begin{aligned}
\mathbf{x} &= k \begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix}^{-1} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
&= \frac{k}{-3k^2-17k-10-2k^2+4k} \\
&\quad \times \begin{pmatrix} -2-3k & -k+2 \\ -2k & 5+k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
&= \frac{k}{5k^2+13k+10} \\
&\quad \times \begin{pmatrix} 2+3k & k-2 \\ 2k & -5-k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
&= \frac{k}{5k^2+13k+10} \\
&\quad \times \begin{pmatrix} 22-12k^2-8k+33k-3k^2+6k+8k-16 \\ 22k-8k^2-40+15k-8k+3k^2 \end{pmatrix} \\
&= \frac{k}{5k^2+13k+10} \begin{pmatrix} 6+39k-15k^2 \\ -40+29k-5k^2 \end{pmatrix} \quad (13.8)
\end{aligned}$$

From the given information,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$

$$\Longrightarrow \begin{vmatrix} k & 5-k & -2k \\ -3k & 4k & 2+3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \tag{13.9}$$

resulting in

$$(5-k)(2+3k)+8k^2 = 56 \tag{13.10}$$

$$\Longrightarrow 5k^2+13k-46 = 0 \tag{13.11}$$

$$\text{or, } k = 2, -\frac{23}{5} \tag{13.12}$$

14 If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{14.1}$$

then find the area of this triangle.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{0}$$

$$\Longrightarrow \mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0} \tag{14.2}$$

If AB be the line in (14.1), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{14.3}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{14.4}$$

since CF passes through the origin and $CF \perp AB$. From (14.1) and (14.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{14.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Longrightarrow \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{14.6}$$

From (14.2),

$$\mathbf{C} = -(\mathbf{A}+\mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{14.7}$$

after substituting from (14.6). Thus,

$$CF = \|\mathbf{C}-\mathbf{F}\| = 3\sqrt{2} \tag{14.8}$$

$$\Longrightarrow AB = CF\frac{2}{\sqrt{3}} = \sqrt{6} \tag{14.9}$$

and the area of the triangle is

$$\frac{1}{2} AB \times CF = 3\sqrt{3} \tag{14.10}$$

15 A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.
Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and

y-axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4\begin{pmatrix}1\\1\end{pmatrix} \tag{15.1}$$

Multiplying (15.1) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{15.2}$$

$$\begin{aligned}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) &= 4\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &= 4\begin{pmatrix}\cos\theta - \sin\theta\\ \cos\theta + \sin\theta\end{pmatrix} \quad (15.3)\end{aligned}$$

$$\begin{aligned}&\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})\\ &\quad = 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3} - 1\right) \quad (15.4)\end{aligned}$$

for $\theta = 30^{\circ}$.