

# The Circle

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**Abstract**—Solved problems from JEE mains papers related to 2D circles in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.

- 1 A circle passes through the points  $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ . If its centre  $\mathbf{O}$  lies on the line

$$(-1 \ 4)\mathbf{x} - 3 = 0 \quad (1.1)$$

find its radius.

**Solution:** Let

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \Rightarrow \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (1.2)$$

The direction vector of  $AB$  is

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad (1.3)$$

$$\therefore OC \perp AB,$$

$$\begin{aligned} OC : \mathbf{m}^T (\mathbf{x} - \mathbf{C}) &= 0 \\ \Rightarrow (1 \ 1)\mathbf{x} &= 7 \end{aligned} \quad (1.4)$$

Thus,  $\mathbf{O}$  is the intersection of (1.1) and (1.4) and is the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} \quad (1.5)$$

From the augmented matrix,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & -3 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix} \\ \Rightarrow \mathbf{O} &= \begin{pmatrix} 5 \\ 2 \end{pmatrix} \end{aligned} \quad (1.6)$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \quad (1.7)$$

- 2 If a circle  $C_1$ , whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T \mathbf{x} + (2 \ -4)\mathbf{x} = 4 \quad (2.1)$$

at the point  $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ , then find the length of the intercept cut by this circle  $C$  on the  $x$ -axis.

**Solution:** From (2.1), the centre of  $C_2$  is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.2)$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T \mathbf{O}_2 = 4 \Rightarrow r_2 = 3 \quad (2.3)$$

Since the radius of  $C_1$  is  $r_1 = r_2 = 3$  and  $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$  are collinear,

$$\begin{aligned} \frac{\mathbf{O}_1 + \mathbf{O}_2}{2} &= \mathbf{P} \\ \Rightarrow \mathbf{O}_1 &= 2\mathbf{P} - \mathbf{O}_2 \\ \Rightarrow \mathbf{O}_1 &= \begin{pmatrix} 5 \\ 2 \end{pmatrix} \end{aligned} \quad (2.4)$$

The intercepts of  $C_1$  on the  $x$ -axis can be expressed as

$$\mathbf{x} = \lambda \mathbf{m} \quad (2.5)$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.6)$$

Substituting in the equation for  $C_1$ ,

$$\|\lambda \mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \quad (2.7)$$

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which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda \mathbf{m}^T \mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$

$$\implies \lambda^2 - 10\lambda + 20 = 0 \quad (2.8)$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \quad (2.9)$$

after substituting from (2.6) and (2.4).

3 A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad (3.1)$$

cuts the circle

$$\mathbf{x}^T \mathbf{x} = 9 \quad (3.2)$$

at the points **A** and **B**. Find  $PA.PB$ .

4 Find the equation of the circle, which is the mirror image of the circle

$$\mathbf{x}^T \mathbf{x} - (2 \ 0) \mathbf{x} = 0 \quad (4.1)$$

in the line

$$(1 \ 1) \mathbf{x} = 3. \quad (4.2)$$

5 Find the locus of the centres of those circles which touch the circle

$$\mathbf{x}^T \mathbf{x} - 8(1 \ 1) \mathbf{x} = 4 \quad (5.1)$$

and also touch the  $x$ -axis.

6 One of the diameters of the circle, given by

$$\mathbf{x}^T \mathbf{x} + 2(-2 \ 3) \mathbf{x} = 12 = 0 \quad (6.1)$$

is a chord of a circle  $S$ , whose centre is at

$$\begin{pmatrix} -3 \\ 2 \end{pmatrix}. \quad (6.2)$$

Find the radius of  $S$ .

7 A circle passes through

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix} \quad (7.1)$$

and touches the  $y$ -axis at

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (7.2)$$

Which one of the following equations can represent a diameter of this circle?

a)  $(4 \ 5) \mathbf{x} = 6$

b)  $(2 \ -3) \mathbf{x} + 10 = 0$

c)  $\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 3$

d)  $\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} + 4 = 0$

8 Find the equation of the tangent to the circle, at the point

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (8.1)$$

whose centre is the point of intersection of the straight lines

$$(2 \ 1) \mathbf{x} = 3 \quad (8.2)$$

$$(1 \ -1) \mathbf{x} = 1 \quad (8.3)$$

9 A straight line through the origin **O** meets the lines

$$(4 \ 3) \mathbf{x} = 10 \quad (9.1)$$

$$(8 \ 6) \mathbf{x} + 5 = 0 \quad (9.2)$$

at **A** and **B** respectively. Find the ratio in which **O** divides  $AB$ .

**Solution:** Let

$$\mathbf{n} = (4 \ 3) \quad (9.3)$$

Then (9.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \quad (9.4)$$

$$2\mathbf{n}^T \mathbf{x} = -5 \quad (9.5)$$

and since **A**, **B** satisfy (9.4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \quad (9.6)$$

$$2\mathbf{n}^T \mathbf{B} = -5 \quad (9.7)$$

Let **O** divide the segment  $AB$  in the ratio  $k : 1$ . Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (9.8)$$

$$\therefore \mathbf{O} = \mathbf{0}, \quad (9.9)$$

$$\mathbf{A} = -k\mathbf{B} \quad (9.10)$$

Substituting in (9.6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \quad (9.11)$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \quad (9.12)$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \quad (9.13)$$

10 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (10.1)$$

is translated parallel to the line

$$L : (1 \quad -1)\mathbf{x} = 4 \quad (10.2)$$

by  $2\sqrt{3}$  units. If the new point  $\mathbf{Q}$  lies in the third quadrant, then find the equation of the line passing through  $\mathbf{Q}$  and perpendicular to  $L$ .

**Solution:** From (10.2), the direction vector of  $L$  is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (10.3)$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \quad (10.4)$$

However,

$$PQ = 2\sqrt{3} \quad (10.5)$$

$$\Rightarrow \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = 2\sqrt{3} \quad (10.6)$$

$$\Rightarrow \lambda = \pm \frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm \sqrt{6} \quad (10.7)$$

$$\therefore \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \quad (10.8)$$

from (10.3). Since  $\mathbf{Q}$  lies in the third quadrant, from (10.4) and (10.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \quad (10.9)$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \quad (10.10)$$

$$(1 \quad 1)\mathbf{x} = 3 - \sqrt{6} \quad (10.11)$$

11 A variable line drawn through the intersection of the lines

$$(4 \quad 3)\mathbf{x} = 12 \quad (11.1)$$

$$(3 \quad 4)\mathbf{x} = 12 \quad (11.2)$$

meets the coordinate axes at  $\mathbf{A}$  and  $\mathbf{B}$ , then find the locus of the midpoint of  $AB$ .

**Solution:** The intersection of the lines in (11.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (11.3)$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \quad (11.4)$$

resulting in

$$\mathbf{C} = \frac{1}{7} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (11.5)$$

Let the  $\mathbf{R}$  be the mid point of  $AB$ . Then,

$$\mathbf{A} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (11.6)$$

$$\mathbf{B} = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (11.7)$$

Let the equation of  $AB$  be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \quad (11.8)$$

Since  $\mathbf{R}$  lies on  $AB$ ,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \quad (11.9)$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (11.10)$$

Substituting from (11.6) in (11.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \quad (11.11)$$

From (11.9) and (11.11),

$$(\mathbf{R} - \mathbf{C}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \quad (11.12)$$

for some constant  $k$ . Multiplying both sides of (11.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (11.13)$$

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ = k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (11.14)$$

$$\therefore \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (11.15)$$

which can be easily verified for any  $\mathbf{R}$ . from (11.14),

$$\begin{aligned} & \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0 \\ \Rightarrow & \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0 \\ \Rightarrow & \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (11.16) \end{aligned}$$

which is the desired locus.

12 Two sides of a rhombus are along the lines

$$AB : (1 \ -1)\mathbf{x} + 1 = 0 \quad (12.1)$$

$$AD : (7 \ -1)\mathbf{x} - 5 = 0. \quad (12.2)$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad (12.3)$$

find its vertices.

**Solution:** From (12.1) and (12.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad (12.4)$$

By row reducing the augmented matrix

$$\begin{aligned} \begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} & \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \\ & \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (12.5) \end{aligned}$$

From (12.3) and (12.5)

$$AP = \|\mathbf{A} - \mathbf{P}\| = 2\sqrt{5} = d(\text{say}) \quad (12.6)$$

The direction vector of  $AP$  is

$$\mathbf{m} = \mathbf{A} - \mathbf{P} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (12.7)$$

$$\Rightarrow \|\mathbf{m}\| = 2\sqrt{5} \quad (12.8)$$

Since the direction of  $AP$  is the same as  $AC$ ,

$$\begin{aligned} \mathbf{C} &= \mathbf{P} - d \frac{\mathbf{m}}{\|\mathbf{m}\|} \\ &= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \quad (12.9) \end{aligned}$$

Let  $\mathbf{n} \perp \mathbf{m}$ . Then

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (12.10)$$

and

$$\begin{aligned} \mathbf{B}, \mathbf{D} &= \mathbf{P} \pm d \frac{\mathbf{n}}{\|\mathbf{n}\|} \\ &= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \pm 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (12.11) \end{aligned}$$

13 Let  $k$  be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \quad (13.1)$$

has area 28. Find the orthocentre of this triangle.

**Solution:** Let  $\mathbf{m}_1$  be the direction vector of  $BC$ . Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5+k \\ k-2 \end{pmatrix}, \quad (13.2)$$

If  $AD$  be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \quad (13.3)$$

Similarly, considering the side  $AC$  the equation of the altitude  $BE$  is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \quad (13.4)$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2-3k \end{pmatrix}, \quad (13.5)$$

The orthocentre is obtained by solving (13.3) and (13.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \quad (13.6)$$

which can be expressed using (13.2), (13.5), (13.3) and (13.4) as

$$\begin{aligned} \begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix} \mathbf{x} &= \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix} \\ &= k \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \quad (13.7) \end{aligned}$$

The solution to the above is

$$\begin{aligned}
 \mathbf{x} &= k \begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix}^{-1} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
 &= \frac{k}{-3k^2-17k-10-2k^2+4k} \\
 &\times \begin{pmatrix} -2-3k & -k+2 \\ -2k & 5+k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
 &= \frac{k}{5k^2+13k+10} \\
 &\times \begin{pmatrix} 2+3k & k-2 \\ 2k & -5-k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\
 &= \frac{k}{5k^2+13k+10} \\
 &\times \begin{pmatrix} 22-12k^2-8k+33k-3k^2+6k+8k-16 \\ 22k-8k^2-40+15k-8k+3k^2 \end{pmatrix} \\
 &= \frac{k}{5k^2+13k+10} \begin{pmatrix} 6+39k-15k^2 \\ -40+29k-5k^2 \end{pmatrix} \quad (13.8)
 \end{aligned}$$

From the given information,

$$\begin{aligned}
 &\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56 \\
 \Rightarrow &\begin{vmatrix} k & 5-k & -2k \\ -3k & 4k & 2+3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \quad (13.9)
 \end{aligned}$$

resulting in

$$(5-k)(2+3k)+8k^2=56 \quad (13.10)$$

$$\Rightarrow 5k^2+13k-46=0 \quad (13.11)$$

$$\text{or, } k=2, -\frac{23}{5} \quad (13.12)$$

- 14 If an equilateral triangle, having centroid at the origin, has a side along the line

$$(1 \ 1)\mathbf{x} = 2, \quad (14.1)$$

then find the area of this triangle.

**Solution:** Let the vertices be  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ . From the given information,

$$\begin{aligned}
 &\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{0} \\
 \Rightarrow &\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0} \quad (14.2)
 \end{aligned}$$

If  $AB$  be the line in (14.1), the equation of  $CF$ , where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (14.3)$$

is

$$(1 \ -1)\mathbf{x} = 0 \quad (14.4)$$

since  $CF$  passes through the origin and  $CF \perp AB$ . From (14.1) and (14.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (14.5)$$

Forming the augmented matrix,

$$\begin{aligned}
 \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\
 &\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (14.6)
 \end{aligned}$$

From (14.2),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (14.7)$$

after substituting from (14.6). Thus,

$$CF = \|\mathbf{C} - \mathbf{F}\| = 3\sqrt{2} \quad (14.8)$$

$$\Rightarrow AB = CF \frac{2}{\sqrt{3}} = \sqrt{6} \quad (14.9)$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 3\sqrt{3} \quad (14.10)$$

- 15 A square, of each side 2, lies above the  $x$ -axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle  $30^\circ$  with the positive direction of the  $x$ -axis, then find the sum of the  $x$ -coordinates of the vertices of the square.

**Solution:** Consider the square  $ABCD$  with  $\mathbf{A} = \mathbf{0}, AB = 2$  such that  $\mathbf{B}$  and  $\mathbf{D}$  lie on the  $x$  and  $y$ -axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (15.1)$$

Multiplying (15.1) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (15.2)$$

$$\begin{aligned}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) &= 4 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 4 \begin{pmatrix} \cos \theta - \sin \theta \\ \cos \theta + \sin \theta \end{pmatrix} \quad (15.3)\end{aligned}$$

$$\begin{aligned}\Rightarrow (1 \quad 0) \mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) \\ = 4(\cos \theta - \sin \theta) = 2(\sqrt{3} - 1) \quad (15.4)\end{aligned}$$

for  $\theta = 30^\circ$ .