

Geometry: Maths Olympiad

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1. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?
2. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD ?
3. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect E . What is the area of $\triangle AEB$?
4. $ABCD$ is a square and $AB = 1$. Equilateral triangle AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY ?
5. O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Suppose O lies in the interior of $\triangle ABC$ and I lies on the circle passing through B , O , and C . What is the magnitude of $\angle BAC$ in degrees?
6. PS is a line segment of length 4 and O is the midpoint of PS . A semicircle arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS . What is the area of the region $QXROQ$ bounded by the two semicircular arcs?
7. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the midpoints of the diagonals AC and BD . If $AD = 16$ and $BC = 20$, What is the length of PQ ?
8. In a triangle ABC , let H , I and O be the orthocentre, incentre and circumcentre, respectively. If the points B , H , I , C lie on a circle, What is magnitude of $\angle BOC$ in degrees?
9. Let ABC be an equilateral triangle. Let P and S be points on AB and AC , respectively, and let Q and R be points on BC such that $PQRS$ is a triangle. If $PQ = \sqrt{3}PS$ and the area of $PQRS$ is $28\sqrt{3}$, What is the length of PC ?
10. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_1, B_2, C_2, D_2 be the midpoints of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$?
11. Let S be a circle with centre O . A chord AB , not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2 . Let S_1 be a circle with centre in R_1 , touching AB at X and S internally. Let S_2 be a circle with centre in R_2 , touching AB at Y , the circle S internally and passing through the centre of S . The point X lies on the diameter passing through the centre of S_2 and $\angle YXO = 30^\circ$. If the radius of S_2 is 100 then what is the radius of S_1 ?
12. In a triangle ABC with $\angle BCA = 90^\circ$, the perpendicular bisector of AB intersects segments AB and AC at X and Y , respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is $13 : 18$ and $BC = 12$ then what is the length of AC ?
13. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20$, $BC = 70$ and $CD = 90$, then what is the value of DA ?
14. In a triangle ABC , X and Y are points on the segments AB and AC , respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC ?
15. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BDC = 90^\circ$. Let the incircles of triangles ABD and BCD touch BD at P and Q , respectively, with P lying in between B and Q . If $AD = 999$ and $PQ = 200$ then what is the sum of the radii of the incircles of triangle ABD and BDC ?
16. Let XOY be a triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY , respectively. Suppose that $XN = 19$ and $YM =$

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22. What is XY ?
17. In a triangle ABC , let I denote the incenter. Let the lines AI , BI and CI intersect the incircle at P , Q and R , respectively. If $\angle BAC = 40^\circ$, what is the value of $\angle QPR$ in degrees?
18. In acute-angled triangle ABC , let D be the foot of the altitude from A , and E be the midpoint of BC . Let F be the midpoint of AC . Suppose $\angle BAE = 40^\circ$. If $\angle DAE = \angle DFE$, What is the magnitude of $\angle ADF$ in degrees?
19. The circle Ω touches the circle Ω internally at P . The centre O of Ω is outside Ω . Let XY be a diameter of Ω which is also tangent to Ω . Assume $PY > PX$. Let PY intersect Ω at Z . If $YZ = 2PZ$, What is the magnitude of $\angle PYX$ in degrees?
20. In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB , BPC and CPD , What is the value of $r_1 + r_2 + r_3$?
21. In a rectangle $ABCD$, E is the midpoint of AB ; F is a point on AC such that BF is perpendicular to AC ; and FE perpendicular to BD . Suppose $BC = 8\sqrt{3}$. Find AB ?
22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with nite areas) that can be formed?
23. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L . The line through P parallel to CA meets BC in M and BA in N . The line through P parallel to AB meets CA in S and CB in T . If KL , MN , ST are of equal lengths, find this common length.
24. Let $ABCD$ be a rectangle and let E and F be points on CD and BC respectively such that area $(ADE) = 16$, area $(CEF) = 9$ and area $(ABF) = 25$. What is the area of triangle AEF ?
25. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $\frac{m\pi+n}{k}$, where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n$?
26. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium $ABCD$. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.
27. In a quadrilateral $ABCD$, it is given that $AB = AD = 13$, $BC = CD = 20$, $BD = 24$. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r ?
28. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AD \perp AB$. Suppose $ABCD$ has an incircle which touches AB at Q and CD at P . Given that $PC = 36$ and $QB = 49$, find PQ ?
29. Let AB be a chord of a circle with centre O . Let C be a point on the circle such that $\angle ABC = 30^\circ$ and O lies inside the triangle ABC . Let D be a point on AB such that $\angle DCO = \angle OCB = 20^\circ$. Find the measure of $\angle CDO$ in degrees.
30. In a triangle ABC , right-angled at A , the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A .
31. Triangles ABC and DEF are such that $\angle A = \angle D$, $AB = DE = 17$, $BC = EF = 10$ and $AC - DF = 12$. What is $AC + DF$?
32. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC , HCA and HAB , respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC ?
33. Let D be an interior point of the side BC of a triangle ABC . Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60^\circ$, what is the measure of $\angle CI_2F$ in degrees?
34. In a triangle ABC , the median from B to CA is perpendicular to the median from C to AB . If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.
35. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?
36. Let ABC be a triangle and let Ω be its circum-

circle. The internal bisectors of angles A , B and C intersect Ω at A_1, B_1 , and C_1 , respectively, and the internal bisectors of angles A_1, B_1 , and C_1 , of the triangle $A_1B_1C_1$, intersect Ω at A_2, B_2 , and C_2 , respectively. If the smallest angle of triangle ABC is 40° , what is the magnitude of the smallest angle of triangle $A_2B_2C_2$, in degrees?

37. Let AB be a diameter of a circle and let C be a point on the segment AB such that $AC : CB = 6 : 7$. Let D be a point on the circle such that DC is perpendicular to AB . Let DE be the diameter through D . If $[XYZ]$ denotes the area of the triangle XYZ , find $[ABD]/[CDE]$ to the nearest integer.
38. Let $ABCD$ be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of $ABCD$ is the least. If $(PA, PB, PC, PD) = (3, 4, 6, 8)$, What is the maximum possible area of $ABCD$?
39. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of $\triangle ABC$. Draw tangents to Ω which are parallel to the sides of ABC . Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.
40. In a triangle ABC , the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If $AD = 7$ and $BE = 9$, find the integer nearest to the area of triangle ABC .