Geometry through Linear Algebra

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Contents

***Abstract*—This textbook introduces linear algebra by exploring Euclidean geometry.**

1 The Straight Line

1.1 The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.1}$$

$$\implies x_2 = mx_1 \tag{1.2}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.3}$$

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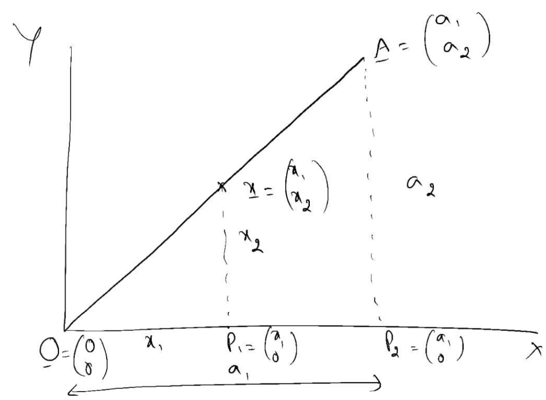

Fig. 1.1

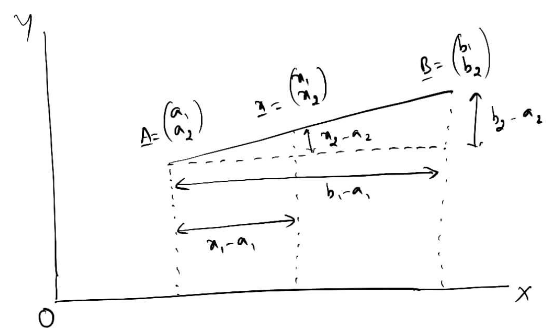

Fig. 1.2

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.4}$$

1.2 Find the equation of AB in Fig. 1.2

Solution: From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.5}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.6}$$

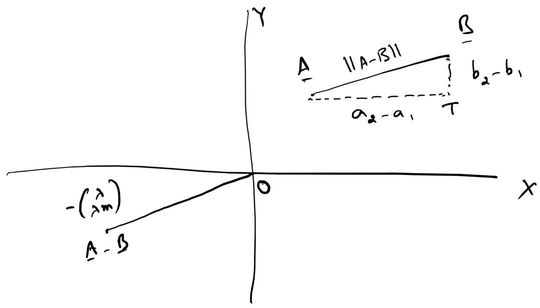

Fig. 1.4

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.7}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8}$$

$$= \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.9}$$

1.3 Find the length of $\mathbf{A}$ in Fig. 1.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.10}$$

Also, from (1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.11}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.

1.4 Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.4. From (1.9), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.12}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.13}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.4.

1.5 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

1.6 Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.14}$$

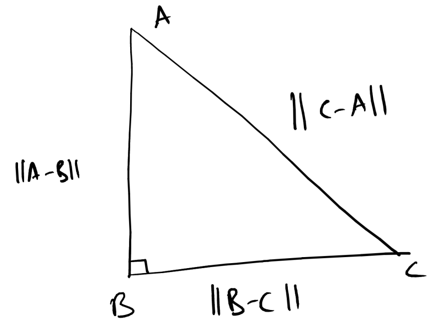

Fig. 2.1

2 Orthogonality

2.1 See Fig. 2.1. In $\triangle ABC, AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.1}$$

Solution: Using Baudhayana's theorem,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{2.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})$$

$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{2.3}$$

which can be simplified to obtain (2.1).

2.2 Let $\mathbf{x}$ be any point on AB in Fi.g 2.1. Show that

$$(\mathbf{x} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.4}$$

2.3 If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x} - \mathbf{y})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.5}$$

2.4 In Fig. 2.4, $BE \perp AC, CF \perp AB$. Show that $AD \perp BC$.

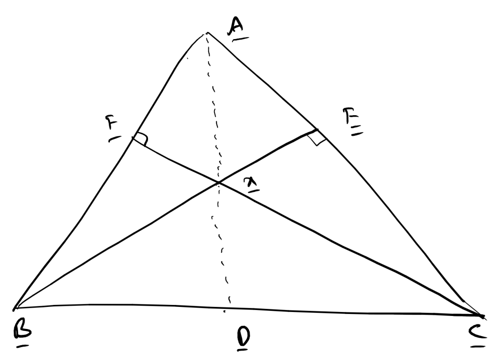

Fig. 2.4

Solution: Let $\mathbf{x}$ be the intersection of BE and

CF. Then, using (2.5),

$$\begin{aligned} (\mathbf{x}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) &= 0 \\ (\mathbf{x}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) &= 0 \end{aligned} \tag{2.6}$$

$$\implies \mathbf{x}^T(\mathbf{A}-\mathbf{C}) - \mathbf{B}^T(\mathbf{A}-\mathbf{C}) = 0 \tag{2.7}$$

$$\text{and } \mathbf{x}^T(\mathbf{A}-\mathbf{B}) - \mathbf{C}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{2.8}$$

Subtracting (2.8) from (2.7),

$$\mathbf{x}^T(\mathbf{B}-\mathbf{C}) + \mathbf{A}^T(\mathbf{C}-\mathbf{B}) = 0 \tag{2.9}$$

$$\implies \left(\mathbf{x}^T - \mathbf{A}^T\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{2.10}$$

$$\implies (\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.11}$$

which completes the proof.

3 Medians of a triangle

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{3.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{3.2}$$

Solution: From (1.9),

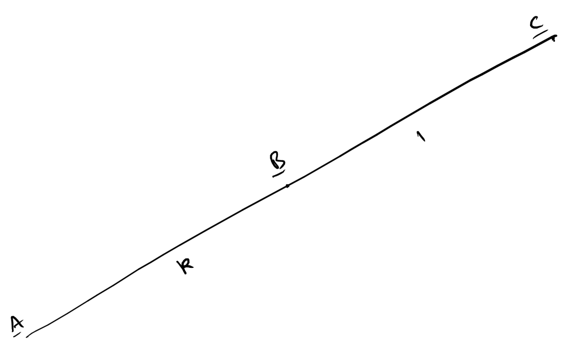

Fig. 3.1

$$\begin{aligned} \mathbf{B} &= \mathbf{A} + \lambda_1 \begin{pmatrix} 1 \\ m \end{pmatrix}, \\ \mathbf{B} &= \mathbf{C} - \lambda_2 \begin{pmatrix} 1 \\ m \end{pmatrix}. \end{aligned} \tag{3.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{3.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{3.5}$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A}-\mathbf{B} = k(\mathbf{B}-\mathbf{C}) \tag{3.6}$$

resulting in (3.2).

3.2 If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{3.7}$$

3.3 BE and CF are medians of $\triangle ABC$ intersecting at O as shown in Fig. 3.3. Show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \tag{3.8}$$

Solution: Let

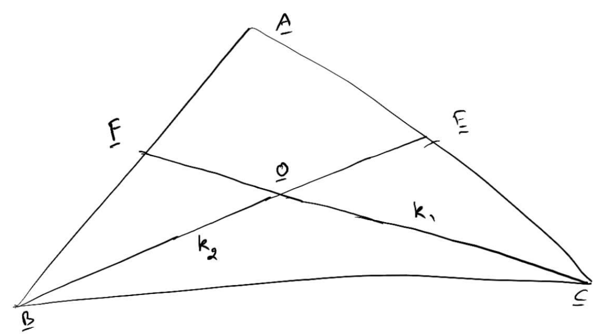

Fig. 3.3

$$\frac{CO}{OF} = k_1 \tag{3.9}$$

$$\frac{BO}{OE} = k_2 \tag{3.10}$$

Using (3.2),

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{3.11}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{3.12}$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F}+\mathbf{C}}{k_1+1} = \frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} \tag{3.13}$$

$$\mathbf{O} = \frac{k_2\mathbf{E}+\mathbf{B}}{k_2+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{3.14}$$

From (3.13) and (3.14),

$$\frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{3.15}$$

$$\implies \left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right]\mathbf{A} + \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} + \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \quad (3.16)$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = k_2 = 2, \quad (3.17)$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (3.14),

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \quad (3.18)$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \quad (3.19)$$

Note that $\mathbf{B}, \mathbf{C}$ are linearly independent. Substituting (3.19) in (3.16),

$$\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right][\alpha\mathbf{B} + \beta\mathbf{C}] + \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} + \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \quad (3.20)$$

$$\implies \begin{aligned}(k_1-k_2)\alpha + k_1k_2 - k_1 - 2 = 0\\ (k_1-k_2)\beta - k_1k_2 + k_2 + 2 = 0\end{aligned} \quad (3.21)$$

$$\implies (k_1-k_2)(\alpha+\beta-1) = 0 \quad (3.22)$$

If $\alpha + \beta = 1, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (3.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in (3.21), yields

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = 2. \quad (3.23)$$

4 Matrix Transformations

4.1 Find $\mathbf{R}$, the reflection of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \quad (4.1)$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (4.2)$$

$$\mathbf{n}^T\mathbf{Q} = c \quad (4.3)$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \quad (4.4)$$

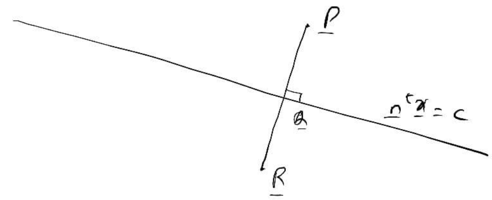

Fig. 4.1

where $\mathbf{m}$ is the direction vector of L. From (4.2) and (4.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \quad (4.5)$$

From (4.5) and (4.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \quad (4.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \quad (4.7)$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \quad (4.8)$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \quad (4.9)$$

(4.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \quad (4.10)$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \quad (4.11)$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \quad (4.12)$$

4.2 Rotate $\mathbf{P}$ through an angle of θ about the origin in the counter clockwise direction to obtain $\mathbf{S}$.
Solution:

$$\mathbf{S} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\mathbf{P} \quad (4.13)$$

5 Circumcircle

5.1 A circle with centre at $\mathbf{O}$ and radius r has the equation

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{5.1}$$
$$\implies (\mathbf{x} - \mathbf{O})^T (\mathbf{x} - \mathbf{O}) = r^2 \tag{5.2}$$

5.2 Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be three points on the circle and D be a point on BC such that $OD \perp BC$ as in Fig. 5.2. Show that

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{5.3}$$

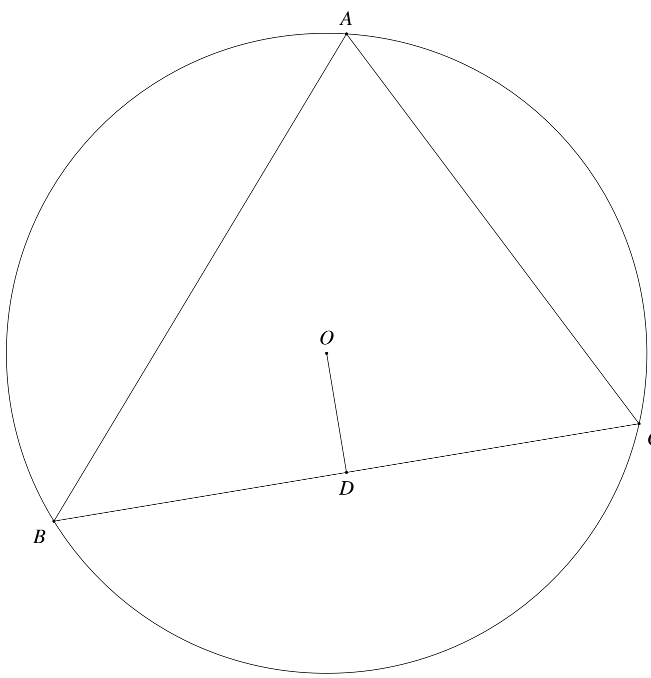

Fig. 5.2: Circumcircle.

Solution: From (5.2),

$$\|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \tag{5.4}$$
$$\implies (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = (\mathbf{C} - \mathbf{O})^T (\mathbf{C} - \mathbf{O}) \tag{5.5}$$
$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{O}\right) = 0 \tag{5.6}$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{D} - \mathbf{O}) = 0 \tag{5.7}$$

Since D and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on BC, using (1.14),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \mathbf{B} + \lambda_1 (\mathbf{B} - \mathbf{C}) \tag{5.8}$$
$$\mathbf{D} = \mathbf{B} + \lambda_2 (\mathbf{B} - \mathbf{C}) \tag{5.9}$$

Multiplying (5.8) and (5.9) with $(\mathbf{B} - \mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{5.10}$$

5.3 Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.

6 Incircle

6.1 The circle with centre $\mathbf{O}$ and radius r in Fig.6.1 is inside $\triangle ABC$ and touches AB, BC and CA at $\mathbf{F}, \mathbf{D}$ and $\mathbf{E}$ respectively. AB, BC and CA are known as *tangents* to the circle.

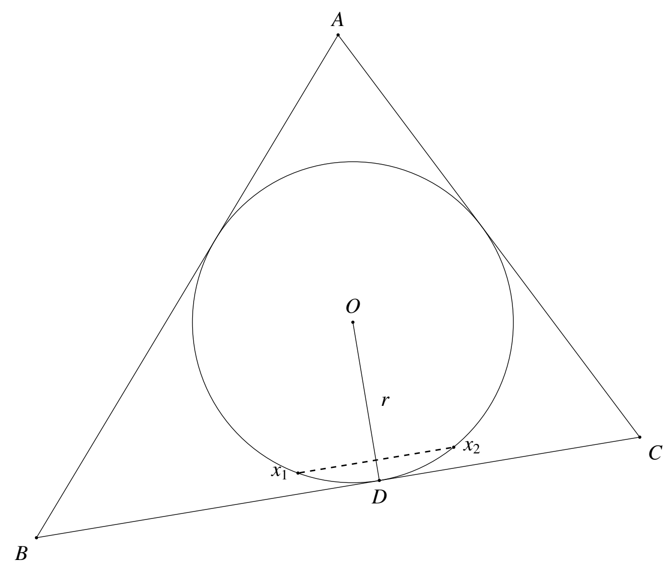

Fig. 6.1: Tangent and incircle.

6.2 Show that $OD \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1 x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{O}\|^2 - \|\mathbf{x}_2 - \mathbf{O}\|^2 = 0 \tag{6.1}$$
$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{O}\right) = 0 \tag{6.2}$$
$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{O}\right) = 0 \tag{6.3}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{D}$, $x_1 x_2$ merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{D} - \mathbf{O}) = 0 \implies OD \perp BC \tag{6.4}$$

6.3 Give an alternative proof for the above.
Solution: Let

$$\mathbf{B} = \mathbf{0} \tag{6.5}$$
$$\mathbf{D} = \lambda\mathbf{m} \tag{6.6}$$

Then

$$\|\mathbf{D} - \mathbf{O}\|^2 = r^2 \tag{6.7}$$
$$\implies \lambda^2\|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O} + \|\mathbf{O}\|^2 = r^2 \tag{6.8}$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T\mathbf{O}}{\|\mathbf{m}\|^2} \tag{6.9}$$

Thus,

$$(\mathbf{D} - \mathbf{B})^T(\mathbf{D} - \mathbf{O}) = (\lambda\mathbf{m})^T(\lambda\mathbf{m} - \mathbf{O}) \tag{6.10}$$
$$= \lambda^2\|\mathbf{m}\|^2 - \lambda\mathbf{m}^T\mathbf{O} \tag{6.11}$$
$$= \mathbf{O} \text{ (from 6.9).} \tag{6.12}$$
$$\implies OD \perp BC \tag{6.13}$$

6.4 Find the equation of the tangent at $\mathbf{D}$.
Solution: The equation of the tangent is given by

$$(\mathbf{O} - \mathbf{D})^T(\mathbf{x} - \mathbf{D}) = 0 \tag{6.14}$$

7 Miscellaneous

7.1 Show that the angle in a semi-circle is a right angle. **Solution:** Let

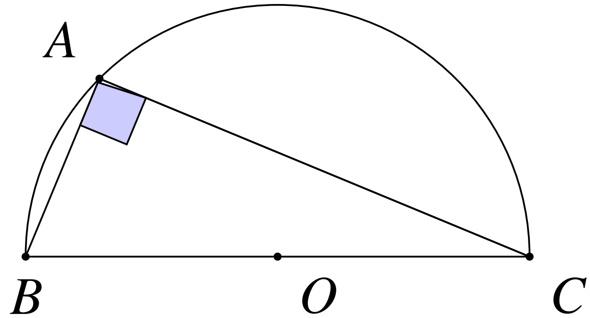

Fig. 7.1: The straight line.

$$\mathbf{O} = 0 \tag{7.1}$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \tag{7.2}$$
$$\|\mathbf{B} - \mathbf{C}\|^2 = (2r)^2 \tag{7.3}$$
$$\mathbf{B} + \mathbf{C} = 0 \tag{7.4}$$

where r is the radius of the circle. Thus,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T(\mathbf{B} + \mathbf{C}) \tag{7.5}$$

From (7.4) and (7.2),

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 4r^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{7.6}$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

7.2 Show that $PA.PB = PC^2$, where PC is the tangent to the circle in Fig. 7.2.

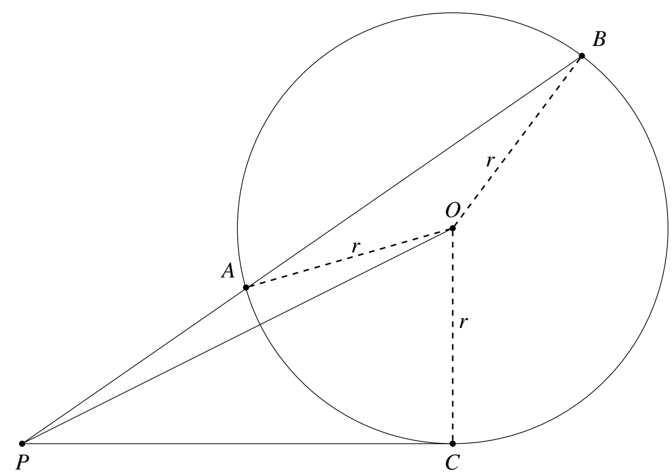

Fig. 7.2: $PA.PB = PC^2$.

Solution: Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda\|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda\mathbf{A}) \tag{7.7}$$
$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \tag{7.8}$$
$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \tag{7.9}$$

$\because$

$$\|\mathbf{B} - \mathbf{O}\|^2 - \|\mathbf{A} - \mathbf{O}\|^2 = 0, \tag{7.10}$$
$$\left(\lambda^2 - 1\right)\|\mathbf{A}\|^2 - 2(\lambda - 1)\mathbf{A}^T\mathbf{O} = 0 \tag{7.11}$$
$$\implies PA.PB = \lambda\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{7.12}$$

after substituting from (7.7) and simplifying.

From (7.9),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \tag{7.13}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 = PC^2 \tag{7.14}$$

From (7.12) and (7.14),

$$PA.PB = PC^2 \tag{7.15}$$

8 Parabola

8.1 Express

$$y_2 = y_1^2 \tag{8.1}$$

as a matrix equation.
Solution: (8.1) can be expressed as

$$\mathbf{y}^T\mathbf{W}\mathbf{y} + 2\mathbf{g}^T\mathbf{y} = 0 \tag{8.2}$$

where

$$\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.3}$$

8.2 Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0, \tag{8.4}$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \tag{8.5}$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{8.6}$$

Show that

$$\begin{aligned} \mathbf{W} &= \mathbf{P}^T\mathbf{V}\mathbf{P} \\ \mathbf{g} &= \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) \\ F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} &= 0 \end{aligned} \tag{8.7}$$

Solution: Substituting (8.6) in (8.4),

$$(\mathbf{P}\mathbf{y}+\mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y}+\mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y}+\mathbf{c}) + F = 0, \tag{8.8}$$

which can be expressed as

$$\begin{aligned} \implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)^T\mathbf{P}\mathbf{y} \\ + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \end{aligned} \tag{8.9}$$

Comparing (8.9) with (8.2) (8.7) is obtained.

8.3 Show that there exists a $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{8.10}$$

Find $\mathbf{P}$ using

$$\mathbf{W} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{8.11}$$

8.4 Find $\mathbf{c}$ from (8.7).
Solution:

$$\because \mathbf{g} = \mathbf{P}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right), \tag{8.12}$$

$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{8.13}$$

$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\left(\mathbf{P}\mathbf{g}-\mathbf{u}\right) = -F - 2\mathbf{u}^T\mathbf{c} \tag{8.14}$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ (\mathbf{P}\mathbf{g}+\mathbf{u})^T \end{pmatrix}\mathbf{c} = \begin{pmatrix} \mathbf{P}\mathbf{g}-\mathbf{u} \\ -F \end{pmatrix} \tag{8.15}$$

for computing $\mathbf{c}$.