3D Geometry through Linear Algebra

G V V Sharma*

Contents

***Abstract*—This manual introduces linear algebra by exploring 3D geometry through a problem solving approach.**

1 Lines and Planes

1.1 L_1 is the intersection of planes

$$\begin{pmatrix} 2 & -2 & 3 \end{pmatrix} \mathbf{x} = 2$$
$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{1.1}$$

Find its equation.
Solution: (1.1) can be written in matrix form as

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \tag{1.2}$$

and solved using the augmented matrix as follows

$$\begin{pmatrix} 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & 2 \end{pmatrix} \tag{1.3}$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{1.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 5 \\ x_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.5}$$

which is the desired equation.

1.2 Summarize all the above computations through a Python script and plot L_1.
Solution: The following code generates Fig. 1.2.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/1.1.py
```

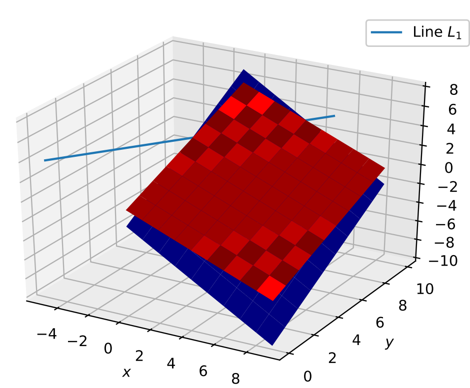

Fig. 1.2

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

1.3 L_2 is the intersection of the planes

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.6}$$

$$\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{1.7}$$

Show that its equation is

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \tag{1.8}$$

1.4 Plot L_2.
Solution: The following code generates Fig. 1.4.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/1.2.py
```

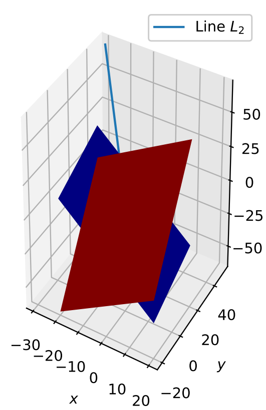

Fig. 1.4

1.5 Do L_1 and L_2 intersect? If so, find their point of intersection P.
Solution: From (1.5),(1.8), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.9}$$

$$\implies \begin{pmatrix} 1 & 3 \\ 1 & -5 \\ 0 & -7 \end{pmatrix} \mathbf{\Lambda} = \frac{1}{7}\begin{pmatrix} 40 \\ 8 \\ -28 \end{pmatrix} \tag{1.10}$$

This matrix equation can be solved as

$$\begin{pmatrix} 1 & 3 & \frac{40}{7} \\ 1 & -5 & \frac{8}{7} \\ 0 & -7 & -4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 8 & 0 & \frac{224}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 1 & \frac{4}{7} \end{pmatrix} \tag{1.11}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & \frac{4}{7} \end{pmatrix} \implies \mathbf{\Lambda} = \begin{pmatrix} 4 \\ \frac{4}{7} \end{pmatrix} \tag{1.12}$$

Substituting $\lambda_1 = 4$ in (1.9)

$$\mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \tag{1.13}$$

1.6 Plot P.
Solution: The following code generates Fig. 1.6.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/1.3.py
```

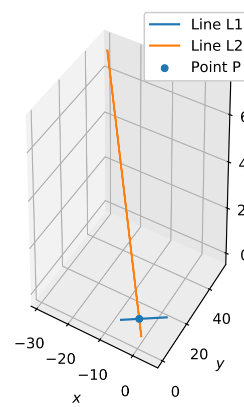

Fig. 1.6

2 Normal to a Plane

2.1 The cross product of $\mathbf{a}, \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2.1}$$

From (1.5), (1.8), the direction vectors of L_1 and L_2 are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \tag{2.2}$$

respectively. Find the direction vector of the normal to the plane spanned by L_1 and L_2.
Solution: The desired vector is obtained as

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\0 & 0 & -1\\-1 & 1 & 0\end{pmatrix}\begin{pmatrix}-3\\5\\7\end{pmatrix} = \begin{pmatrix}7\\-7\\8\end{pmatrix} = \mathbf{n} \tag{2.3}$$

2.2 Summarize all the above computations through a plot
Solution: The following code generates Fig. 2.2.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/2.1.py
```

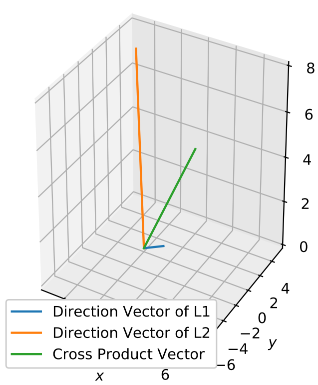

Fig. 2.2

2.3 Find the equation of the plane spanned by L_1 and L_2.
Solution: Let $\mathbf{x}_0$ be the intersection of L_1 and L_2. Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \tag{2.4}$$
$$\implies \mathbf{x}^T\mathbf{n} = \mathbf{x}_0^T\mathbf{n} \tag{2.5}$$
$$\implies \mathbf{x}^T \begin{pmatrix}7\\-7\\8\end{pmatrix} = \begin{pmatrix}-1 & 4 & 4\end{pmatrix}\begin{pmatrix}7\\-7\\8\end{pmatrix} = -3 \tag{2.6}$$

2.4 Summarize the above through a plot.
Solution: The following code generates Fig. 2.4.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/2.2.py
```

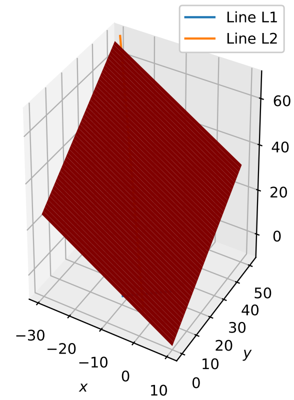

Fig. 2.4

2.5 Find the distance of the origin from the plane containing the lines L_1 and L_2.
Solution: The distance from the origin to the plane is given by

$$\frac{\left|\mathbf{x}_0^T\mathbf{n}\right|}{\|n\|} = \frac{1}{3\sqrt{2}} \tag{2.7}$$

3 Projection on a Plane

3.1 Find the equation of the line L joining the points

$$\mathbf{A} = \begin{pmatrix}5 & -1 & 4\end{pmatrix}^T \tag{3.1}$$
$$\mathbf{B} = \begin{pmatrix}4 & -1 & 3\end{pmatrix}^T \tag{3.2}$$

Solution: The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda(\mathbf{A} - \mathbf{B}) \tag{3.3}$$
$$= \begin{pmatrix}4\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{3.4}$$

3.2 Plot the above line.
Solution: The following code generates Fig. 3.2.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/3.1.py
```

3.3 Find the intersection of L and the plane P given by

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.5}$$

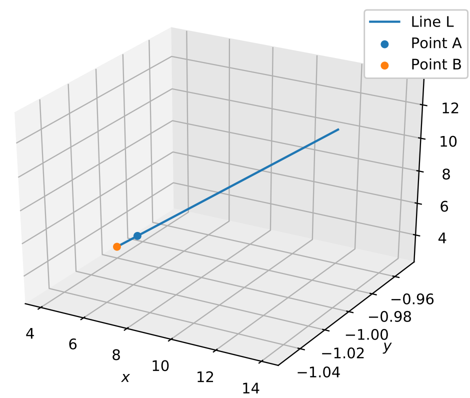

Fig. 3.2

Solution: From (3.4) and (3.5),

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}4\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=7 \tag{3.6}$$

$$\implies 6+2\lambda=7 \tag{3.7}$$

$$\implies \lambda=\frac{1}{2} \tag{3.8}$$

Substituting in (3.4),

$$\mathbf{x}=\frac{1}{2}\begin{pmatrix}9 & -1 & 7\end{pmatrix} \tag{3.9}$$

3.4 Sketch the line, plane and the point of intersection.
Solution: The following code generates Fig. 3.4.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/3D/manual/codes/3.2.py
```

3.5 Find $\mathbf{C} \in P$ such that $AC \perp P$.
Solution: From (3.5), the direction vector of AC is $\begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$. Hence, the equation of AC is

$$\mathbf{x}=\begin{pmatrix}5\\-1\\4\end{pmatrix}+\lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.10}$$

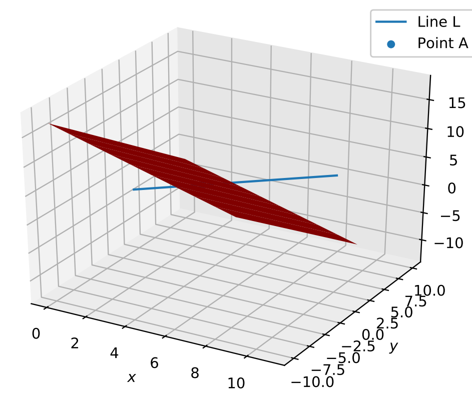

Fig. 3.4

Substituting in (3.5)

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}5\\-1\\4\end{pmatrix}+\lambda\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=7 \tag{3.11}$$

$$\implies 8+3\lambda_1=7 \tag{3.12}$$

$$\implies \lambda_1=-\frac{1}{3} \tag{3.13}$$

Thus,

$$\mathbf{C}=\frac{1}{3}\begin{pmatrix}14\\-4\\11\end{pmatrix} \tag{3.14}$$

3.6 Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D}=\frac{1}{3}\begin{pmatrix}13\\-2\\10\end{pmatrix} \tag{3.15}$$

3.7 Find the projection of AB on the plane P.
Solution: The projection is given by

$$CD=\|\mathbf{C}-\mathbf{D}\|=\sqrt{\frac{2}{3}} \tag{3.16}$$

4 Coplanar vectors

4.1 If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that

$$\mathbf{u}^T(\mathbf{A}\times\mathbf{B})=0 \tag{4.1}$$

4.2 Find $\mathbf{A} \times \mathbf{B}$ given

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}^T \tag{4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T \tag{4.3}$$

Solution: From (2.1),

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.4}$$

$$= \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \tag{4.5}$$

4.3 Let $\mathbf{u}$ be coplanar with such that $\mathbf{u} \perp \mathbf{A}$and

$$\mathbf{u}^T\mathbf{B} = 24. \tag{4.6}$$

Find $\|\mathbf{u}\|^2$.
Solution: From (4.5) and the given information,

$$\mathbf{u}^T \begin{pmatrix} 4 & -2 & 2 \end{pmatrix} = 0 \tag{4.7}$$

$$\mathbf{u}^T \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = 0 \tag{4.8}$$

$$\mathbf{u}^T \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 24 \tag{4.9}$$

$$\implies \begin{pmatrix} 4 & -2 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} \tag{4.10}$$

$$\implies \mathbf{u} = 4 \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \tag{4.11}$$

$$\implies \|\mathbf{u}\|^2 = 336 \tag{4.12}$$

5 Least Squares

5.1 Find the equation of the plane P containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \tag{5.1}$$

5.2 Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \tag{5.2}$$

lies outside P.

5.3 Find the point $\mathbf{w} \in P$ closest to $\mathbf{y}$.

5.4 Show that

$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 = \|\mathbf{y}\|^2 - \mathbf{w}^T\mathbf{X}^T\mathbf{y} \tag{5.3}$$

$$- \mathbf{y}^T A\mathbf{w} + \mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w} \tag{5.4}$$

5.5 Assuming 2×2 matrices and 2×1 vectors, show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T\mathbf{X}^T\mathbf{y} = \frac{\partial}{\partial \mathbf{w}} \mathbf{y}^T\mathbf{X}\mathbf{w} = \mathbf{y}^T\mathbf{X} \tag{5.5}$$

5.6 Show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w} = 2\mathbf{w}^T \left(\mathbf{X}^T\mathbf{X}\right) \tag{5.6}$$

5.7 Show that

$$\hat{\mathbf{w}} = \min_{\mathbf{w}} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 \tag{5.7}$$

$$= \left(\mathbf{X}^T\mathbf{X}\right)^{-1} \mathbf{X}^T\mathbf{y} \tag{5.8}$$

5.8 Let

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}. \tag{5.9}$$

from (5.1). Verify (5.8).

6 Linear Algebra: Orthogonality

6.1 Let

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \tag{6.1}$$

$$L_2: \quad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{6.2}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.
Solution: Let

$$L_3: \quad \mathbf{x} = \mathbf{c} + \lambda\mathbf{m}_3 \tag{6.3}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{6.4}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{6.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{6.6}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1: \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{6.7}$$
$$L_2: \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{6.8}$$
$$L_3: \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{6.9}$$

where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{6.10}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (6.7)-(6.9),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{6.11}$$
$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{6.12}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (6.11)-(6.12) can be expressed as

$$\mathbf{m}_1^T \mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T \mathbf{c}_3 \tag{6.13}$$
$$\mathbf{m}_2^T \mathbf{c}_1 = \mathbf{m}_2^T \mathbf{c}_3 \tag{6.14}$$
$$\mathbf{m}_3^T \mathbf{c}_1 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{6.15}$$
$$0 = \mathbf{m}_1^T \mathbf{c}_3 \tag{6.16}$$
$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T \mathbf{c}_3 \tag{6.17}$$
$$0 = \mathbf{m}_3^T \mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{6.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T \mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{6.19}$$
$$\lambda_2 = \frac{\mathbf{m}_2^T \mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{6.20}$$

Substituting in (6.11) and (6.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ or} \tag{6.21}$$
$$L_3: \quad \mathbf{x} = \frac{2}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{6.22}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

7 Linear Algebra: Area of a Triangle

7.1 Let the lines

$$L_1: \mathbf{x} = \lambda_1 \mathbf{a} \tag{7.1}$$
$$L_2: \mathbf{x} = \lambda_2 \mathbf{b} \tag{7.2}$$
$$L_3: \mathbf{x} = \lambda_3 \mathbf{c} \tag{7.3}$$

intersect the plane

$$P: \mathbf{n}^T \mathbf{x} = c \tag{7.4}$$

at the points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ respectively. Find λ_1, λ_2 and λ_3.

Solution: From the given information, $\mathbf{A} \in P, L_1$. $\because \mathbf{A} = \lambda_1 \mathbf{a}$,

$$\lambda_1 \mathbf{n}^T \mathbf{a} = c \implies \lambda_1 = \frac{c}{\mathbf{n}^T \mathbf{a}} \tag{7.5}$$

Similarly, λ_2 and λ_3 are obtained.

7.2 Find the area Δ of $\triangle ABC$

Solution:

$$\begin{aligned} \Delta &= \|\frac{1}{2} (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \\ &= \frac{1}{2} \|(\lambda_1 \mathbf{a} - \lambda_2 \mathbf{b}) \times (\lambda_1 \mathbf{a} - \lambda_3 \mathbf{c})\| \\ &= \frac{1}{2} \|\lambda_1 \lambda_2 (\mathbf{a} \times \mathbf{b}) + \lambda_2 \lambda_3 (\mathbf{b} \times \mathbf{c}) \\ &\quad + \lambda_1 \lambda_3 (\mathbf{c} \times \mathbf{a}) \| \end{aligned} \tag{7.6}$$

7.3 Find $(6\Delta)^2$ given

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{7.7}$$
$$\mathbf{n} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, c = 1 \tag{7.8}$$

Solution:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \tag{7.9}$$

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{7.10}$$

$$\mathbf{c} \times \mathbf{a} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{7.11}$$

Using (7.5),

$$\lambda_1 = 1, \lambda_2 = \frac{1}{2}, \lambda_3 = \frac{1}{3} \tag{7.12}$$

Thus,

$$\Delta = \frac{1}{2}\|\frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix} + \frac{1}{6}\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\\-1\end{pmatrix}\| \tag{7.13}$$

$$= \frac{1}{12}\|\begin{pmatrix}1\\1\\1\end{pmatrix}\| = \frac{\sqrt{3}}{12} \tag{7.14}$$

Hence,

$$(6\Delta)^2 = \frac{3}{2} \tag{7.15}$$

8 Linear Algebra: Linear Dependence

8.1 Let

$$L_1 : \quad \mathbf{r} = \lambda_1 \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{8.1}$$

$$L_2 : \quad \mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{8.2}$$

$$L_3 : \quad \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{8.3}$$

Let $\mathbf{P} \in L_1, \mathbf{Q} \in L_2, \mathbf{R} \in L_3$. Given that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear, If $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear, find $\mathbf{Q}$.
Solution:

$$\frac{PQ}{QR} = k, \tag{8.4}$$

$$(k+1)\,\mathbf{Q} = k\mathbf{P} + \mathbf{R}, \tag{8.5}$$

From (8.1), (8.2) and (8.3),

$$k\lambda_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = (k+1)\begin{pmatrix}0\\0\\1\end{pmatrix} + (k+1)\,\lambda_2 \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{8.6}$$

which can be expressed as

$$\begin{pmatrix}k & 0 & 0\\0 & -(k+1) & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}-1\\-1\\k+1\end{pmatrix} \tag{8.7}$$

Thus,

$$\mathbf{Q} = \begin{pmatrix}0\\\frac{1}{k+1}\\1\end{pmatrix} \tag{8.8}$$

8.2 Verify if $\mathbf{Q}$ can be

a) $\begin{pmatrix}0\\-\frac{1}{2}\\1\end{pmatrix}$

c) $\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix}$

b) $\begin{pmatrix}0\\0\\1\end{pmatrix}$

d) $\begin{pmatrix}0\\1\\1\end{pmatrix}$

9 Linear Algebra: Projection

9.1 Show that the projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \tag{9.1}$$

9.2 Given

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}, \tag{9.2}$$

show that

$$(\mathbf{c} - (\mathbf{a} \times \mathbf{b}))^T\,\mathbf{c} = \|\mathbf{c}\|^2 \tag{9.3}$$

9.3 Find

$$\min_{\mathbf{u}} \|\mathbf{c}\|^2 \tag{9.4}$$

$$s.t \quad \|\text{proj}_{\mathbf{a+b}}\mathbf{c}\| = 3\sqrt{2} \tag{9.5}$$

by using the fact that

$$\mathbf{c} = \mathbf{Pu} \tag{9.6}$$

$$\mathbf{a} + \mathbf{b} = \mathbf{P1} \tag{9.7}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \tag{9.8}$$

$$\mathbf{u} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \tag{9.9}$$

$$\mathbf{1} = \begin{pmatrix}1\\1\end{pmatrix} \tag{9.10}$$