

Optimization through School

Geometry

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Abstract—This book provides an introduction to optimization based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

svn co <https://github.com/gadepall/school/trunk/ncert/optimization/codes>

1 CONSTRAINED OPTIMIZATION

- Express the problem of finding the distance of the point $\mathbf{P} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$L : (3 \ -4)\mathbf{x} = 26 \quad (1.1.1)$$

as an optimization problem.

Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \quad (1.1.2)$$

$$\text{s.t. } \mathbf{n}^T \mathbf{x} = c \quad (1.1.3)$$

where

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.1.4)$$

$$c = 26 \quad (1.1.5)$$

- Explain Problem 1.1 through a plot and find a graphical solution.

- Solve (1.1.2) using cvxpy.

Solution: The following code yields

codes/line_dist_cvx.py

$$\mathbf{x}_{\min} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix}, \quad (1.3.1)$$

$$g(\mathbf{x}_{\min}) = 0.6 \quad (1.3.2)$$

- Convert (1.1.2) to an *unconstrained* optimization problem.

Solution: L in (1.1.1) can be expressed in terms of the direction vector \mathbf{m} as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m}, \quad (1.4.1)$$

where \mathbf{A} is any point on the line and

$$\mathbf{m}^T \mathbf{n} = 0 \quad (1.4.2)$$

Substituting (1.4.1) in (1.1.2), an unconstrained optimization problem

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\|^2 \quad (1.4.3)$$

is obtained.

- Solve (1.4.3).

Solution:

$$f(\lambda) = (\lambda \mathbf{m} + \mathbf{A} - \mathbf{P})^T (\lambda \mathbf{m} + \mathbf{A} - \mathbf{P}) \quad (1.5.1)$$

$$= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^T (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \quad (1.5.2)$$

$$\therefore f^{(2)}(\lambda) = 2\|\mathbf{m}\|^2 > 0 \quad (1.5.3)$$

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the minimum value of $f(\lambda)$ is obtained when

$$f^{(1)}(\lambda) = 2\lambda \|\mathbf{m}\|^2 + 2\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0 \quad (1.5.4)$$

$$\Rightarrow \lambda_{\min} = -\frac{\mathbf{m}^T (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \quad (1.5.5)$$

Choosing \mathbf{A} such that

$$\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0, \quad (1.5.6)$$

substituting in (1.5.5),

$$\lambda_{\min} = 0 \quad \text{and} \quad (1.5.7)$$

$$\mathbf{A} - \mathbf{P} = \mu \mathbf{n} \quad (1.5.8)$$

for some constant μ . (1.5.8) is a consequence of (1.4.2) and (1.5.6). Also, from (1.5.8),

$$\mathbf{n}^T (\mathbf{A} - \mathbf{P}) = \mu \|\mathbf{n}\|^2 \quad (1.5.9)$$

$$\Rightarrow \mu = \frac{\mathbf{n}^T \mathbf{A} - \mathbf{n}^T \mathbf{P}}{\|\mathbf{n}\|^2} = \frac{c - \mathbf{n}^T \mathbf{P}}{\|\mathbf{n}\|^2} \quad (1.5.10)$$

from (1.1.3). Substituting $\lambda_{\min} = 0$ in (1.4.3),

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} - \mathbf{P}\|^2 = \mu^2 \|\mathbf{n}\|^2 \quad (1.5.11)$$

upon substituting from (1.5.8). The distance between \mathbf{P} and L is then obtained from (1.5.11) as

$$\|\mathbf{A} - \mathbf{P}\| = |\mu| \|\mathbf{n}\| \quad (1.5.12)$$

$$= \frac{|\mathbf{n}^T \mathbf{P} - c|}{\|\mathbf{n}\|} \quad (1.5.13)$$

after substituting for μ from (1.5.10). Using the corresponding values from Problem (1.1) in (1.5.13),

$$\min_{\lambda} f(\lambda) = 0.6 \quad (1.5.14)$$

where \mathbf{A} is the intercept of the line L in (1.1.1) on the x-axis and the points

$$\mathbf{U} = \begin{pmatrix} \lambda_1 \\ f(\lambda_1) \end{pmatrix}, \mathbf{V} = \begin{pmatrix} \lambda_2 \\ f(\lambda_2) \end{pmatrix} \quad (2.1.5)$$

$$\mathbf{X} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ f[t\lambda_1 + (1-t)\lambda_2] \end{pmatrix}, \quad (2.1.6)$$

$$\mathbf{Y} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ tf(\lambda_1) + (1-t)f(\lambda_2) \end{pmatrix} \quad (2.1.7)$$

for

$$\lambda_1 = -3, \lambda_2 = 4, t = 0.3 \quad (2.1.8)$$

in Fig. 2.1. Geometrically, this means that any point \mathbf{Y} between the points \mathbf{U}, \mathbf{V} on the line UV is always above the point \mathbf{X} on the curve $f(\lambda)$. Such a function f is defined to be *convex* function

codes/optimization/1.2.py

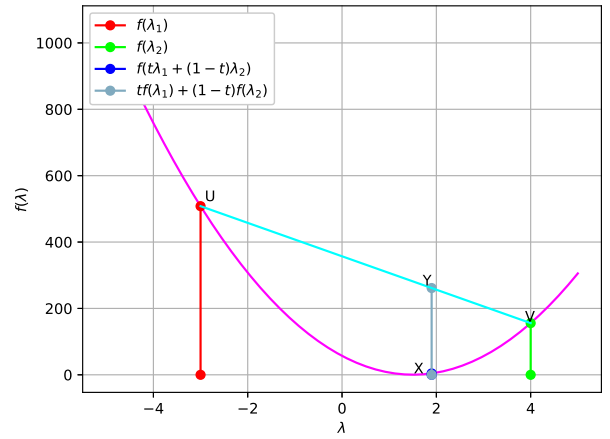


Fig. 2.1: $f(\lambda)$ versus λ

2 CONVEX FUNCTION

1. The following python script plots

$$f(\lambda) = a\lambda^2 + b\lambda + d \quad (2.1.1)$$

for

$$a = \|\mathbf{m}\|^2 > 0 \quad (2.1.2)$$

$$b = \mathbf{m}^T (\mathbf{A} - \mathbf{P}) \quad (2.1.3)$$

$$c = \|\mathbf{A} - \mathbf{P}\|^2 \quad (2.1.4)$$

2. Show that

$$f[t\lambda_1 + (1-t)\lambda_2] \leq tf(\lambda_1) + (1-t)f(\lambda_2) \quad (2.2.1)$$

for $0 < t < 1$. This is true for any convex function.

3. Show that

$$(2.2.1) \Rightarrow f^{(2)}(\lambda) > 0 \quad (2.3.1)$$

4. Show that a convex function has a unique minimum.

3 GRADIENT DESCENT

1. Find a numerical solution for (2.1.1)

Solution: A numerical solution for (2.1.1) is obtained as

$$\lambda_{n+1} = \lambda_n - \mu f'(\lambda_n) \quad (3.1.1)$$

$$= \lambda_n - \mu (2a\lambda_n + b) \quad (3.1.2)$$

where λ_0 is an initial guess and μ is a variable parameter. The choice of these parameters is very important since they decide how fast the algorithm converges.

2. Write a program to implement (3.1.2).

Solution: Download and execute

codes/optimization/gd.py

3. Find a closed form solution for λ_n in (3.1.2) using the one sided Z transform.
4. Find the condition for which (3.1.2) converges, i.e.

$$\lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0 \quad (3.4.1)$$

4 LAGRANGE MULTIPLIERS

1. Find

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 = r^2 \quad (4.1.1)$$

$$\text{s.t. } h(\mathbf{x}) = \mathbf{n}^T \mathbf{x} - c = 0 \quad (4.1.2)$$

by plotting the circles $g(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.

Solution: The following code plots Fig. 4.1

codes/concinc.py

2. By solving the quadratic equation obtained from (4.1.1), show that

$$\min_{\mathbf{x}} r = \frac{3}{5}, \mathbf{x}_{\min} = \mathbf{Q} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix} \quad (4.2.1)$$

In Fig. 4.1, it can be seen that \mathbf{Q} is the point of contact of the line L with the circle of minimum radius $r = \frac{3}{5}$.

3. Show that

$$\nabla h(\mathbf{x}) = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \mathbf{n} \quad (4.3.1)$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \quad (4.3.2)$$

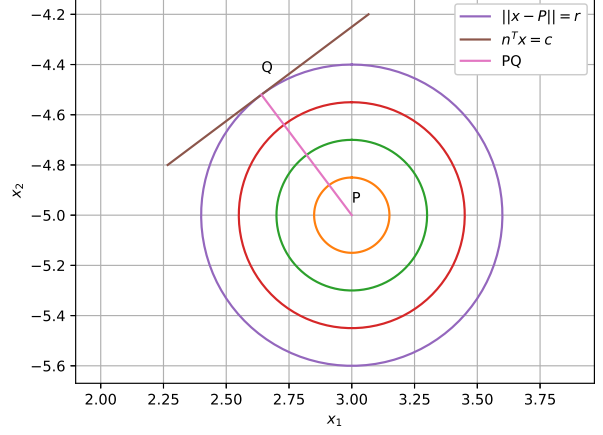


Fig. 4.1: Finding $\min_{\mathbf{x}} g(\mathbf{x})$

4. Show that

$$\nabla g(\mathbf{x}) = 2 \left\{ \mathbf{x} - \begin{pmatrix} 3 \\ -5 \end{pmatrix} \right\} = 2 \{\mathbf{x} - \mathbf{P}\} \quad (4.4.1)$$

5. From Fig. 4.1, show that

$$\nabla g(\mathbf{Q}) = \lambda \nabla h(\mathbf{Q}), \quad (4.5.1)$$

Solution: In Fig. 4.1, PQ is the normal to the line L , represented by $h(\mathbf{x})$. \therefore the normal vector of L is in the same direction as PQ , for some constant k ,

$$(\mathbf{Q} - \mathbf{P}) = k\mathbf{n} \quad (4.5.2)$$

which is the same as (4.5.1) after substituting from (4.3.1). and (4.4.1).

6. Use (4.5.1) and $\mathbf{h}(\mathbf{Q}) = 0$ from (4.1.2) to obtain \mathbf{Q} .

Solution: From the given equations, we obtain

$$(\mathbf{Q} - \mathbf{P}) - \lambda \mathbf{n} = 0 \quad (4.6.1)$$

$$\mathbf{n}^T \mathbf{Q} - c = 0 \quad (4.6.2)$$

which can be simplified to obtain

$$\begin{pmatrix} \mathbf{I} & -\mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ c \end{pmatrix} \quad (4.6.3)$$

The following code computes the solution to (4.6.3)

codes/lagmul.py

7. Define

$$C(\mathbf{x}, \lambda) = g(\mathbf{x}) - \lambda h(\mathbf{x}) \quad (4.7.1)$$

and show that \mathbf{Q} can also be obtained by solving the equations

$$\nabla C(\mathbf{x}, \lambda) = 0. \quad (4.7.2)$$

What is the sign of λ ? C is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

8. Obtain \mathbf{Q} using gradient descent.

Solution:

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codes/gd_lagrange.py
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imum distance as 2.236 and the nearest point on the curve as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad (5.4.1)$$

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codes/qp_cvx.py
```

5. Solve (5.3.1) using the method of Lagrange multipliers.

6. Graphically verify the solution to Problem 5.1.

Solution: The following code plots Fig.

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codes/qp_parab.py
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5 QUADRATIC PROGRAMMING

1. An apache helicopter of the enemy is flying along the curve given by

$$y = x^2 + 7 \quad (5.1.1)$$

A soldier, placed at

$$\mathbf{P} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}. \quad (5.1.2)$$

wants to shoot the helicopter when it is nearest to him. Express this as an optimization problem.

Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{P}\|^2 \quad (5.1.3)$$

$$\text{s.t. } \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{u}^T \mathbf{x} + d = 0 \quad (5.1.4)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.1.5)$$

$$\mathbf{u} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.1.6)$$

$$d = 7 \quad (5.1.7)$$

2. Show that the constraint in 5.1.3 is nonconvex.
3. Show that the following *relaxation* makes (5.1.3) a convex optimization problem.

$$\min_{\mathbf{x}} (\mathbf{x} - \mathbf{P})^T (\mathbf{x} - \mathbf{P}) \quad (5.3.1)$$

$$\text{s.t. } \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{u}^T \mathbf{x} \leq 0 \quad (5.3.2)$$

4. Solve (5.3.1) using cvxpy.

Solution: The following code yields the min-

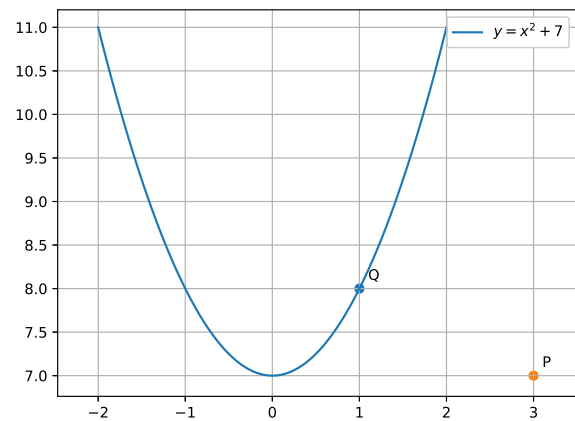


Fig. 5.6: \mathbf{Q} is closest to \mathbf{P}

7. Solve (5.3.1) using gradient descent.