Geometry through Linear Algebra

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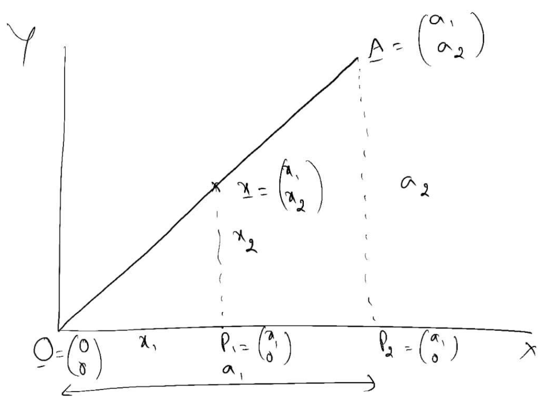

Fig. 1.1

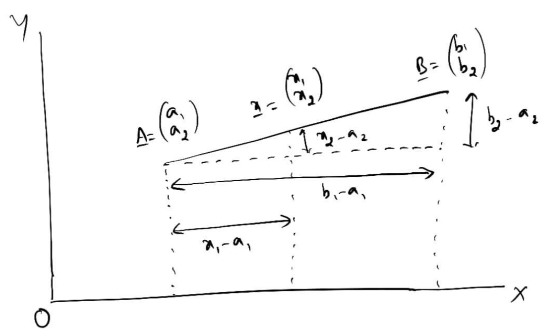

Fig. 1.2

Contents

***Abstract*—This textbook introduces linear algebra by exploring Euclidean geometry.**

1 The Straight Line

1.1 The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.1}$$

$$\implies x_2 = mx_1 \tag{1.2}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.3}$$

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.4}$$

1.2 Find the equation of AB in Fig. 1.2

Solution: From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.5}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.6}$$

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.7}$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8}$$

$$= \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.9}$$

1.3 Find the length of $\mathbf{A}$ in Fig. 1.1

Solution: Using Baudhayana's theorem, the

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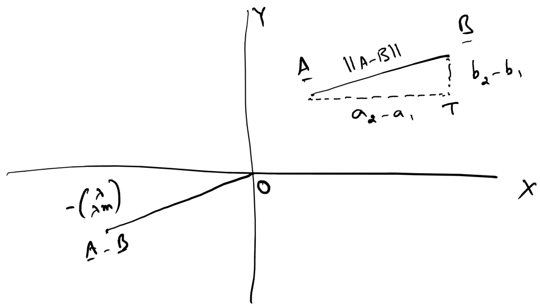

Fig. 1.4

length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.10}$$

Also, from (1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1+m^2} \tag{1.11}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.

1.4 Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.4. From (1.9), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{1.12}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix}1\\m\end{pmatrix}, \tag{1.13}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.4.

1.5 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

2 Orthogonality

2.1 See Fig. 2.1. In $\triangle ABC, AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.1}$$

Solution: Using Baudhayana's theorem,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{2.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})$$

$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{2.3}$$

which can be simplified to obtain (2.1).

2.2 Let $\mathbf{x}$ be any point on AB in Fi.g 2.1. Show that

$$(\mathbf{x} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.4}$$

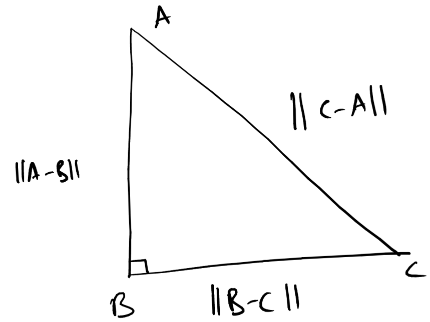

Fig. 2.1

2.3 If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x} - \mathbf{y})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.5}$$

2.4 In Fig. 2.4, $BE \perp AC, CF \perp AB$. Show that $AD \perp BC$.

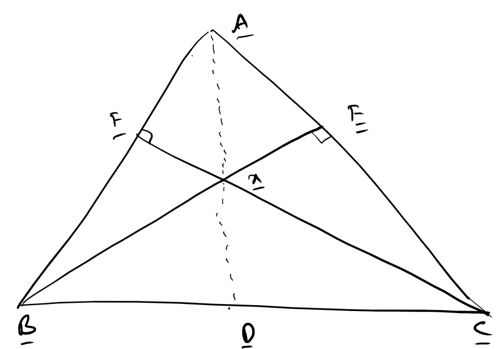

Fig. 2.4

Solution: Let $\mathbf{x}$ be the intersection of BE and CF. Then, using (2.5),

$$\begin{aligned}(\mathbf{x} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) &= 0\\ (\mathbf{x} - \mathbf{C})^T(\mathbf{A} - \mathbf{B}) &= 0\end{aligned} \tag{2.6}$$

$$\implies \mathbf{x}^T(\mathbf{A} - \mathbf{C}) - \mathbf{B}^T(\mathbf{A} - \mathbf{C}) = 0 \tag{2.7}$$

$$\text{and } \mathbf{x}^T(\mathbf{A} - \mathbf{B}) - \mathbf{C}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.8}$$

Subtracting (2.8) from ,

$$\mathbf{x}^T(\mathbf{B} - \mathbf{C}) + \mathbf{A}^T(\mathbf{C} - \mathbf{B}) = 0 \tag{2.9}$$

$$\implies \left(\mathbf{x}^T - \mathbf{A}^T\right)(\mathbf{B} - \mathbf{C}) = 0 \tag{2.10}$$

$$\implies (\mathbf{x} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.11}$$

which completes the proof.

3 Medians of a triangle

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \tag{3.1}$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k + 1} = \mathbf{B}. \tag{3.2}$$

Solution: From (1.9),

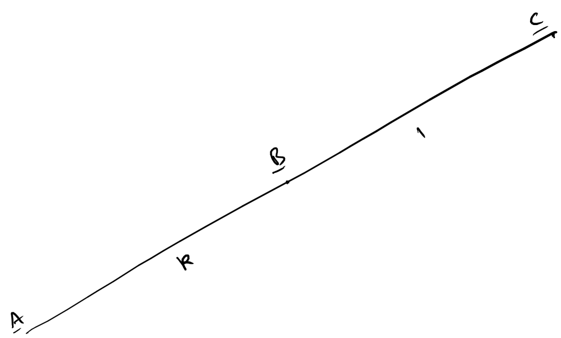

Fig. 3.1

$$\begin{aligned} \mathbf{B} &= \mathbf{A} + \lambda_1 \begin{pmatrix} 1 \\ m \end{pmatrix}, \\ \mathbf{B} &= \mathbf{C} - \lambda_2 \begin{pmatrix} 1 \\ m \end{pmatrix}. \end{aligned} \tag{3.3}$$

$$\implies \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{3.4}$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{3.5}$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{3.6}$$

resulting in (3.2).

3.2 If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{3.7}$$

Show that