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Abstract—This manual introduces linear algebra through coordinate geometry using a problem solving approach.

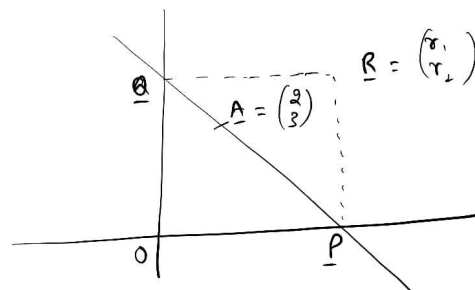


Fig. 2.1

1 THE STRAIGHT LINE

1.1 The equation of the line between two points **A** and **B** is given by

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{A} - \mathbf{B}) \quad (1.1)$$

Alternatively, it can be expressed as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2)$$

where **m** is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{m} = 0 \quad (1.3)$$

2 LOCUS

2.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.1)$$

intersects the coordinate axes at **P** and **Q**. **O** is the origin and rectangle **OPRQ** is completed as shown in Fig. (2.1),

2.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (2.2)$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (2.3)$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \quad (2.4)$$

2.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{m} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{m} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{m} &= 0 \end{aligned} \quad (2.5)$$

Solution: Trivial using (1.2) and (1.3).

2.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{m} = 0 \quad (2.6)$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{m} = 0 \quad (2.7)$$

Solution: From (2.5) and (2.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{m} = 0 \quad (2.8)$$

resulting in (2.6). From (2.5) and (2.2),(2.3), (2.7) is obtained.

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2.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \quad (2.9)$$

2.6 Find the locus of \mathbf{R} .

Solution: For \mathbf{m} to be unique in (2.6),(2.7),

$$\begin{aligned} (2\mathbf{A} - \mathbf{R}) &= k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (2\mathbf{A} - \mathbf{R}) \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (2.10) \end{aligned}$$

where k is some constant.

3 CONICS

3.1 The equation of quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \quad (3.1)$$

Show that (3.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \quad (3.2)$$

Find the matrix V and vector \mathbf{u} .

3.2 The tangent to (3.1) at a point \mathbf{p} on the curve is given by

$$(\mathbf{p}^T \quad 1) \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \quad (3.3)$$

Show that (3.3) can be expressed as

$$(\mathbf{p}^T V + \mathbf{u}^T) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \quad (3.4)$$

3.3 Find the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \quad (3.5)$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.6)$$

in (3.4), the desired equation is

$$\begin{aligned} &\left[\begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \end{pmatrix} \right] \mathbf{x} \\ &+ \frac{1}{2} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \quad (3.7) \end{aligned}$$

resulting in

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 5 \quad (3.8)$$

3.4 The line in (3.8) touches the circle

$$\mathbf{x}^T \mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \quad (3.9)$$

Find c .

Solution: Comparing (3.2) and (3.9),

$$\begin{aligned} V &= I, \\ \mathbf{u} &= 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned} \quad (3.10)$$

Comparing (3.4) and (3.8),

$$\mathbf{p} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.11)$$

$$\Rightarrow \mathbf{p} = - \begin{pmatrix} 6 \\ 7 \end{pmatrix} \quad (3.12)$$

and

$$c + \mathbf{p}^T \mathbf{u} = 5 \quad (3.13)$$

$$\Rightarrow c = 5 + 2 \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (3.14)$$

$$= 95 \quad (3.15)$$

3.5 Tangents are drawn to the hyperbola

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 36 \quad (3.16)$$

at points \mathbf{P} and \mathbf{Q} . If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad (3.17)$$

find the area of $\triangle PTQ$.

Solution:

4 JEE

4.1 Tangent and normal are drawn at

$$\mathbf{P} = \begin{pmatrix} 16 \\ 16 \end{pmatrix} \quad (4.1)$$

on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + (16 \ 0) \mathbf{x} = 0 \quad (4.2)$$

which intersect the axis of the parabola at **A** and **B** respectively. If **C** is the centre of the circle through the points **P**, **A** and **B**, find $\tan CPB$.