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Abstract—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/ncert/codes
```

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1.1 Examples

1. Do the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.
2. Show that the points $\begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.
3. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.
4. Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.
5. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.
6. Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 8.

Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \quad (1.1.6.1)$$

Then the point \mathbf{C}

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k + 1} \quad (1.1.6.2)$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 1.1.6

```
codes/line/draw_section.py
```

7. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.
8. Find the direction vector of PQ , where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \quad (1.1.8.1)$$

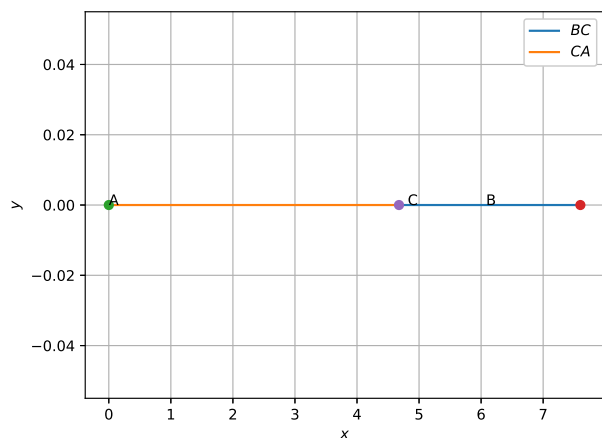


Fig. 1.1.6

9. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (1.1.9.1)$$

10. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \quad (1.1.10.1)$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (1.1.10.2)$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (1.1.11.1)$$

12. Write down a unit vector in the xy-plane, making an angle of 30° with the positive direction of the x-axis.

13. Find the value of x for which $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.

1.2 Elementary Exercises

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (1.2.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (1.2.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (1.2.1.3)$$

2. Find the distance between the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (1.2.2.1)$$

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

4. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \quad (1.2.4.1)$$

are collinear.

5. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.2.5.1)$$

are the vertices of an isosceles triangle.

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (1.2.6.1)$$

b)

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (1.2.6.2)$$

c)

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \quad (1.2.6.3)$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.2.6.4)$$

7. Find the point on the x -axis which is equidis-

tant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \quad (1.2.7.1)$$

8. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (1.2.8.1)$$

is 10 units.

9. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \quad (1.2.9.1)$$

10. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (1.2.10.1)$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

11. Find a vector \mathbf{x} in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$.

12. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \quad (1.2.12.1)$$

13. Show that each of the given three vectors is a unit vector

$$\frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \quad (1.2.13.1)$$

Also, show that they are mutually perpendicular to each other.

14. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (1.2.14.1)$$

$(\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c}$. Find λ .

15. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (1.2.15.1)$$

find $\|\mathbf{a} \times \mathbf{b}\|$.

16. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \quad (1.2.16.1)$$

17. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad (1.2.17.1)$$

18. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

19. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} =$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

20. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

21. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Find a vector \mathbf{d} such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T \mathbf{c} = 15$.

22. The scalar product of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with a unit vector

$$\text{along the sum of the vectors } \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \text{ and } \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix} \text{ is}$$

unity. Find the value of λ .

23. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (1.2.23.1)$$

is

a) 0

b) -1

c) 1

d) 3

24. Let $\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find β_1, β_2 such that $\beta = \beta_1 + \beta_2$, $\beta_1 \parallel \alpha$ and $\beta_2 \perp \alpha$.
25. Find a unit vector that makes an angle of $90^\circ, 60^\circ$ and 30° with the positive x, y and z axis respectively.
26. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.
27. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and 1
28. Find a unit vector that makes an angle of $90^\circ, 135^\circ$ and 45° with the positive x, y and z axis respectively.
29. Show that the lines with direction vectors $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$ are mutually perpendicular.
30. Show that the line through the points $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.
31. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.
- 1.3 Section Formula**
1. Find the coordinates of the point which divides the join of $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (1.3.1.1) in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.
4. Find the ratio in which the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of \mathbf{P} such that $AP = \frac{3}{7}AB$ and \mathbf{P} lies on the line segment AB .
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.
8. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.
9. In each of the following, find the value of k for which the points are collinear
- a) $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ k \end{pmatrix}$
- b) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$, $\begin{pmatrix} k \\ -4 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$
10. Find a condition on x such that the points \mathbf{x} , $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.
11. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$ are collinear.
12. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$ are collinear, and find the ratio in which \mathbf{B} divides AC .
13. Show that $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and

$\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, are collinear.

14. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.

15. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

1.4 Line Equation

1. Determine the ratio in which the line

$$(2 \ 1) - 4 = 0 \quad (1.4.1.1)$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

2. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.

3. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.

4. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}, \quad (1.4.4.1)$$

find the equation of the line.

5. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (1.4.5.1)$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (1.4.5.2)$$

6. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad (1.4.6.1)$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad (1.4.6.2)$$

7. Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (1.4.7.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (1.4.7.2)$$

8. Find the distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (1.4.8.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (1.4.8.2)$$

9. Find the equation of a line which passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and is parallel to the

vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

10. Find the equation of the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

11. Find the equation of the line which passes through the point $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \quad (1.4.11.1)$$

12. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \quad (1.4.12.1)$$

13. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

14. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

15. Find the angle between the following pair of lines:

a)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (1.4.15.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (1.4.15.2)$$

b)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (1.4.15.3)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \quad (1.4.15.4)$$

16. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \quad (1.4.16.1)$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \quad (1.4.16.2)$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \quad (1.4.16.3)$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \quad (1.4.16.4)$$

17. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \quad (1.4.17.1)$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \quad (1.4.17.2)$$

are at right angles.

18. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \quad (1.4.18.1)$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (1.4.18.2)$$

are perpendicular to each other.

19. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (1.4.19.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (1.4.19.2)$$

20. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad (1.4.20.1)$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad (1.4.20.2)$$

21. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (1.4.21.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.4.21.2)$$

22. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \quad (1.4.22.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \quad (1.4.22.2)$$

23. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{9}}$ from the origin and has normal

$$\text{vector} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}.$$

24. Find the unit normal vector of the plane

$$(6 \quad -3 \quad -2)\mathbf{x} = 1. \quad (1.4.24.1)$$

25. Find the distance of the plane

$$(2 \quad -3 \quad 4)\mathbf{x} - 6 = 0 \quad (1.4.25.1)$$

from the origin.

26. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$(2 \quad -3 \quad 4)\mathbf{x} - 6 = 0 \quad (1.4.26.1)$$

27. Find the equation of the plane which passes

$$\text{through the point} \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \text{ and perpendicular to}$$

the line with direction vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

28. Find the equation of the plane passing through

$$\mathbf{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} \text{ and } \mathbf{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}.$$

29. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
30. Find the equation of the plane passing through the intersection of the planes

$$(1 \ 1 \ 1)\mathbf{x} = 6 \quad (1.4.30.1)$$

$$(2 \ 3 \ 4)\mathbf{x} = -5 \quad (1.4.30.2)$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

31. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \quad (1.4.31.1)$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \quad (1.4.31.2)$$

are coplanar.

32. Find the angle between the two planes

$$(2 \ 1 \ -2)\mathbf{x} = 5 \quad (1.4.32.1)$$

$$(3 \ -6 \ -2)\mathbf{x} = 7. \quad (1.4.32.2)$$

33. Find the angle between the two planes

$$(2 \ 2 \ -2)\mathbf{x} = 5 \quad (1.4.33.1)$$

$$(3 \ -6 \ 2)\mathbf{x} = 7. \quad (1.4.33.2)$$

Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$(6 \ -3 \ 2)\mathbf{x} = 4 \quad (1.4.33.3)$$

Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \quad (1.4.33.4)$$

and the plane

$$(10 \ 2 \ -11)\mathbf{x} = 3 \quad (1.4.33.5)$$

34. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $(0 \ 0 \ 1)\mathbf{x} = 2$ c) $(0 \ 5 \ 0)\mathbf{x} = -8$

b) $(1 \ 1 \ 1)\mathbf{x} = 1$ d) $(2 \ 3 \ -1)\mathbf{x} = 5$

35. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

36. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $(2 \ 3 \ 4)\mathbf{x} = 12$ c) $(1 \ 1 \ 1)\mathbf{x} = 1$

b) $(3 \ 4 \ -6)\mathbf{x} = 0$ d) $(0 \ 5 \ 0)\mathbf{x} = -8$

37. Find the equation of the planes

- a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

- b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vector to the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

38. Find the equation of the planes that pass through three points

a) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

39. Find the intercepts cut off by the plane $(2 \ 1 \ 1)\mathbf{x} = 5$.

40. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOY plane.

41. Find the equation of the plane through the intersection of the planes $(3 \ -1 \ 2)\mathbf{x} = 4$ and

$(1 \ 1 \ 1)\mathbf{x} = -2$ and the point $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

42. Find the equation of the plane passing through the intersection of the planes $(2 \ 2 \ -3)\mathbf{x} = 7$

and $(2 \ 5 \ 3)\mathbf{x} = 9$ and the point $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

43. Find the equation of the plane through the intersection of the planes $(1 \ 1 \ 1)\mathbf{x} = 1$ and $(2 \ 3 \ 4)\mathbf{x} = 5$ which is perpendicular to the plane $(1 \ -1 \ 1)\mathbf{x} = 0$.
44. Find the angle between the planes whose equations are $(2 \ 2 \ -3)\mathbf{x} = 5$ and $(3 \ -3 \ 5)\mathbf{x} = 3$.
45. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.
- $(7 \ 5 \ 6)\mathbf{x} = -30$ and $(3 \ -1 \ -10)\mathbf{x} = -4$
 - $(2 \ 1 \ 3)\mathbf{x} = 2$ and $(1 \ -2 \ 5)\mathbf{x} = 0$
 - $(2 \ -2 \ 4)\mathbf{x} = -5$ and $(3 \ -3 \ 6)\mathbf{x} = 1$
 - $(2 \ -1 \ 3)\mathbf{x} = 1$ and $(2 \ -1 \ 3)\mathbf{x} = -3$
 - $(4 \ 8 \ 1)\mathbf{x} = 8$ and $(0 \ 1 \ 1)\mathbf{x} = 4$
46. In the following cases, find the distance of each of the given points from the corresponding plane.
48. Find the distance between the point $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane determined by the points $\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.
49. Find the coordinates of the point where the lines through the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ crosses the XY plane.
50. Show that the line joining the origin to the point $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$.
51. If the coordinates of the points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, then find the angle between the lines AB and CD .

| Item | Point | Plane |
|------|--|-------------------------------|
| a) | $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ | $(3 \ -4 \ 12)\mathbf{x} = 3$ |
| b) | $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ | $(2 \ -1 \ 2)\mathbf{x} = -3$ |
| c) | $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ | $(1 \ 2 \ -2)\mathbf{x} = 9$ |
| d) | $\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$ | $(2 \ -3 \ 6)\mathbf{x} = 2$ |

TABLE 1.4.46

47. Find the equation of the plane that contains the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and is perpendicular to each of the planes

$$(2 \ 3 \ -2)\mathbf{x} = 5 \quad (1.4.47.1)$$

$$(1 \ 2 \ -3)\mathbf{x} = 8 \quad (1.4.47.2)$$

1.5 Miscellaneous

- If $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, then show that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1 n_2 - m_2 n_1 \\ n_1 l_2 - n_2 l_1 \\ l_1 m_2 - l_2 m_1 \end{pmatrix}$.
- A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}. \quad (1.5.2.1)$$

- Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \quad (1.5.3.1)$$

$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \quad (1.5.3.2)$$

are coplanar.

- If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (1.5.4.1)$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (1.5.4.2)$$

find \mathbf{R} , which divides PQ

- a) internally,
b) externally.

5. Find \mathbf{R} which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \quad (1.5.5.1)$$

$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \quad (1.5.5.2)$$

externally in the ratio 1 : 2.

6. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \quad (1.5.6.1)$$

$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| \quad (1.5.6.2)$$

7. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \quad (1.5.7.1)$$

8. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \quad (1.5.8.1)$$

$$\mathbf{a}^T \mathbf{b} = \frac{1}{2} \quad (1.5.8.2)$$

and the angle between \mathbf{a} and \mathbf{b} is 60° .

9. Show that

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \perp (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a}) \quad (1.5.9.1)$$

10. If $\mathbf{a}^T \mathbf{a} = 0$ and $\mathbf{a}\mathbf{b} = 0$, what can be concluded about the vector \mathbf{b} ?

11. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (1.5.11.1)$$

find the value of

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \quad (1.5.11.2)$$

12. If $\mathbf{a} \neq \mathbf{0}$, $\lambda \neq 0$, then $\|\lambda \mathbf{a}\| = 1$ if

- a) $\lambda = 1$
b) $\lambda = -1$
c) $\|\mathbf{a}\| = |\lambda|$
d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

13. If a unit vector \mathbf{a} makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and \mathbf{a} .

14. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) \quad (1.5.14.1)$$

15. If $\mathbf{a}^T \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can you conclude about \mathbf{a} and \mathbf{b} ?

16. Find \mathbf{x} if \mathbf{a} is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \quad (1.5.16.1)$$

17. If $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between \mathbf{a} and \mathbf{b} is

- a) $\frac{\pi}{6}$ c) $\frac{\pi}{3}$
b) $\frac{\pi}{4}$ d) $\frac{\pi}{2}$

18. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (1.5.18.1)$$

$$\iff \mathbf{a} \perp \mathbf{b}. \quad (1.5.18.2)$$

19. If θ is the angle between two vectors \mathbf{a} and \mathbf{b} , then $\mathbf{a}^T \mathbf{b} \geq 0$ only when

- a) $0 < \theta < \frac{\pi}{2}$ c) $0 < \theta < \pi$
b) $0 \leq \theta \leq \frac{\pi}{2}$ d) $0 \leq \theta \leq \pi$

20. Let \mathbf{a} and \mathbf{b} be two unit vectors and θ be the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

- a) $\theta = \frac{\pi}{4}$ c) $\theta = \frac{\pi}{2}$
b) $\theta = \frac{\pi}{3}$ d) $\theta = \frac{2\pi}{3}$

21. If θ is the angle between any two vectors \mathbf{a} and \mathbf{b} , then $\|\mathbf{a}^T \mathbf{b}\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

- a) 0 c) $\frac{\pi}{2}$
b) $\frac{\pi}{4}$ d) π .

22. Let $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 4$, $\|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.

23. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (1.5.23.1)$$

evaluate

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}, \quad (1.5.23.2)$$

given that $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

24. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b - c \\ c - a \\ a - b \end{pmatrix}$.

25. Find the equation of a line parallel to the x-axis and passing through the origin.

26.

2 TRIANGLE

2.1 Construction

1. Draw $\triangle ABC$ where $\angle B = 90^\circ$, $a = 4$ and $b = 3$.

Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (2.1.1.1)$$

The following code plots Fig. 2.1.1

codes/triangle/rt_triangle.py

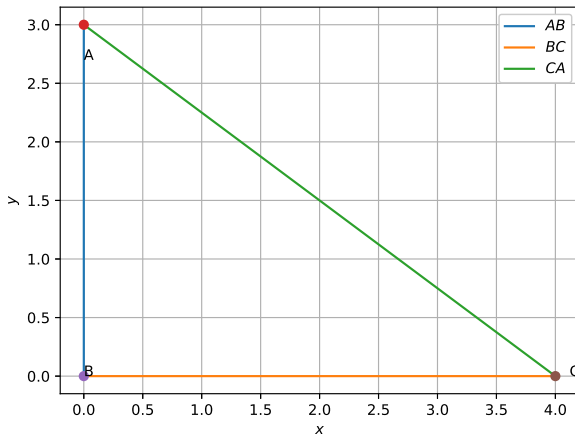


Fig. 2.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.

Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.1.2.1)$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \quad (2.1.2.2)$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T \mathbf{A} = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad (2.1.2.3)$$

$$= p \times p + q \times q = p^2 + q^2 \quad (2.1.2.4)$$

Then

$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (2.1.2.5)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (2.1.2.6)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (2.1.2.7)$$

From (2.1.2.7),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (2.1.2.8)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{C}^T \mathbf{C} - \mathbf{A}^T \mathbf{C} - \mathbf{C}^T \mathbf{A} \quad (2.1.2.9)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} \quad (\because \mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}) \quad (2.1.2.10)$$

$$= a^2 + c^2 - 2ap \quad (2.1.2.11)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (2.1.2.12)$$

From (2.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \quad (2.1.2.13)$$

$$\Rightarrow q = \pm \sqrt{c^2 - p^2} \quad (2.1.2.14)$$

The following code plots Fig. 2.1.2

codes/triangle/draw_triangle.py

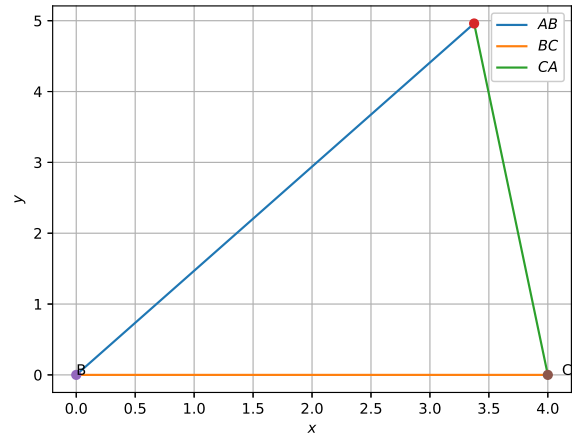


Fig. 2.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.

Solution: The sides of the similar triangle are $\frac{7}{5}a$, $\frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8\text{cm}$ and altitude $AD = h = 4\text{cm}$

Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \quad (2.1.4.1)$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.

Solution: From the given information,

$$a + b + c = 11 \quad (2.1.5.1)$$

$$b = c \quad (\because B = C = 45^\circ) \quad (2.1.5.2)$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \quad (2.1.5.3)$$

From (2.1.5.1) and (2.1.5.2),

$$a + 2b = 11 \quad (2.1.5.4)$$

From (2.1.5.2) and (2.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \quad (2.1.5.5)$$

(2.1.5.4) and (2.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \quad (2.1.5.6)$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times (-\sqrt{2}) - 2 \times 0}{1 \times (-\sqrt{2}) - 2 \times 1} \quad (2.1.5.7)$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \quad (2.1.5.8)$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \quad (2.1.5.9)$$

by expanding the determinants. The following code may be used to compute a, b and c .

codes/triangle/triangle_det.py

6. Repeat Problem 2.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \quad (2.1.6.1)$$

$$a - b\sqrt{2} = 0 \quad (2.1.6.2)$$

$$b - c = 0 \quad (2.1.6.3)$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \quad (2.1.6.4)$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.5)$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.6)$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (2.1.6.7)$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \quad (2.1.6.8)$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \quad (2.1.6.9)$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60^\circ$.

Solution: In Fig. (2.1.7), $AD \perp BC$.

$$\cos C = \frac{y}{b}, \quad (2.1.7.1)$$

$$\cos B = \frac{x}{b}, \quad (2.1.7.2)$$

Thus,

$$a = x + y = b \cos C + c \cos B, \quad (2.1.7.3)$$

$$b = c \cos A + a \cos C \quad (2.1.7.4)$$

$$c = b \cos A + a \cos B \quad (2.1.7.5)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.1.7.6)$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \quad (2.1.7.7)$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \quad (2.1.7.8)$$

From (2.1.7.8)

$$b^2 = c^2 + a^2 - 2ca \cos B \quad (2.1.7.9)$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

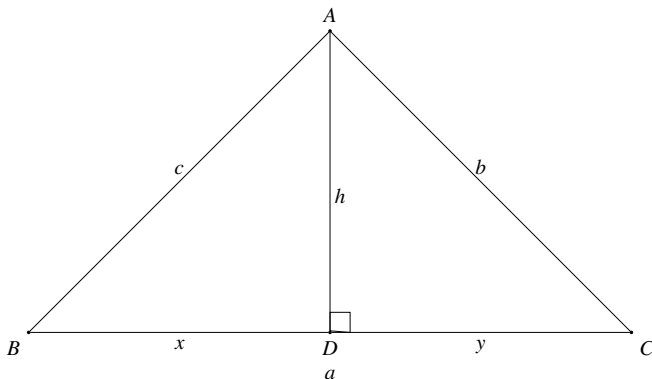


Fig. 2.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7$, $\angle B = 45^\circ$ and $\angle A = 105^\circ$.

Solution: In Fig. (2.1.7),

$$\sin B = \frac{h}{c} \quad (2.1.8.1)$$

$$\sin C = \frac{h}{b} \quad (2.1.8.2)$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.1.8.3)$$

Thus,

$$c = \frac{a \sin C}{\sin A} \quad (2.1.8.4)$$

where

$$C = 180 - A - B \quad (2.1.8.5)$$

9. Draw $\triangle ABC$ if $AB = 3$, $AC = 5$ and $\angle C = 30^\circ$.

Solution: From (2.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (2.1.9.1)$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \quad (2.1.9.2)$$

$$\therefore (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \quad (2.1.9.3)$$

(2.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \quad (2.1.9.4)$$

$$\Rightarrow (a - b \cos C)^2 = b^2 (1 - \cos^2 C) - c^2 \quad (2.1.9.5)$$

$$\text{or, } a = b \cos C \pm \sqrt{b^2 (1 - \cos^2 C) - c^2} \quad (2.1.9.6)$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \quad (2.1.10.1)$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (2.1.10.2)$$

Verify (2.1.9.6) using (2.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.

Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \quad (2.1.11.1)$$

$$\Rightarrow (18 - c)^2 = 12^2 + c^2 \quad (2.1.11.2)$$

which can be simplified to obtain

$$36c - 180 = 0 \quad (2.1.11.3)$$

$$\Rightarrow c = 5 \quad (2.1.11.4)$$

and $b = 13$

12. Find a simpler solution for Problem 2.1.5

Solution: Use cosine formula.

13. In $\triangle ABC$, $a = 7$, $\angle B = 75^\circ$ and $b + c = 13$.
Alternatively,

$$a = b \cos C + c \cos B \quad (2.1.13.1)$$

$$b \sin C = c \sin B \quad (2.1.13.2)$$

$$a + b + c = 11 \quad (2.1.13.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (2.1.13.4)$$

Solving the equivalent matrix equation gives the desired answer.

2.2 Construction Exercises

1. In $\triangle ABC$, $a = 8$, $\angle B = 45^\circ$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6$, $\angle B = 60^\circ$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a + b + c = 11$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$.
4. Construct $\triangle xyz$ where $xy = 4.5$, $yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4$, $QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5$, $BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3$, $QR = 5.5$ and $\angle PQR = 60^\circ$.
9. Construct $\triangle DEF$ such that $DE = 5$, $DF = 3$ and $\angle D = 90^\circ$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110° .
11. Construct $\triangle ABC$ with $BC = 7.5$, $AC = 5$ and $\angle C = 60^\circ$.

12. Construct $\triangle XYZ$ if $XY = 6$, $\angle X = 30^\circ$ and $\angle Y = 100^\circ$.
13. If $AC = 7$, $\angle A = 60^\circ$ and $\angle B = 50^\circ$, can you draw the triangle?
14. Construct $\triangle ABC$ given that $\angle A = 60^\circ$, $\angle B = 30^\circ$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5$, $\angle Q = 105^\circ$ and $\angle R = 40^\circ$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2$, $\angle E = 110^\circ$ and $\angle F = 180^\circ$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled \triangle whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 2.2.21.

| S.No | Triangle | Given Measurements | | |
|------|-----------------|-----------------------|------------------------|------------|
| 1 | $\triangle ABC$ | $\angle A = 85^\circ$ | $\angle B = 115^\circ$ | $AB = 5$ |
| 2 | $\triangle PQR$ | $\angle Q = 30^\circ$ | $\angle R = 60^\circ$ | $QR = 4.7$ |
| 3 | $\triangle ABC$ | $\angle A = 70^\circ$ | $\angle B = 50^\circ$ | $AC = 3$ |
| 4 | $\triangle LMN$ | $\angle L = 60^\circ$ | $\angle N = 120^\circ$ | $LM = 5$ |
| 5 | $\triangle ABC$ | $BC = 2$ | $AB = 4$ | $AC = 2$ |
| 6 | $\triangle PQR$ | $PQ = 2.5$ | $QR = 4$ | $PR = 3.5$ |
| 7 | $\triangle XYZ$ | $XY = 3$ | $YZ = 4$ | $XZ = 5$ |
| 8 | $\triangle DEF$ | $DE = 4.5$ | $EF = 5.5$ | $DF = 4$ |

TABLE 2.2.21

2.3 Triangle Geometry

1. Find the area of a triangle whose vertices are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
2. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.
3. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
4. Find the area of the triangle whose vertices are
 - a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$
 - b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

5. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
6. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
7. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that
- $$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (2.3.7.1)$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (2.3.7.2)$$

8. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the vertices of $\triangle ABC$.
- The median from \mathbf{A} meets BC at \mathbf{D} . Find the coordinates of the point \mathbf{D} .
 - Find the coordinates of the point \mathbf{P} on AD such that $AP : PD = 2 : 1$.
 - Find the coordinates of the points \mathbf{Q} and \mathbf{R} on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
9. In $\triangle ABC$, Show that the centroid
- $$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (2.3.9.1)$$
10. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (2.3.10.1)$$

are the vertices of a right angled triangle.

11. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.
12. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.
13. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.

14. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$
15. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
16. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

3 QUADRILATERAL

3.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
Solution: Fig. 3.1.1 shows a rough sketch of $ABCD$. Letting

$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (3.1.1.1)$$

it is trivial to sketch $\triangle ABC$ from Problem 2.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x -axis by an angle θ with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (3.1.1.2)$$

and

$$\cos \theta = \frac{a^2 + e^2 - b^2}{2ae} \quad (3.1.1.3)$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} \quad (3.1.1.4)$$

The coordinates of the rotated triangle ACD are

$$\mathbf{D} = \mathbf{P} \begin{pmatrix} h \\ k \end{pmatrix} \quad (3.1.1.5)$$

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.1.1.6)$$

$$\mathbf{C} = \mathbf{P} \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (3.1.1.7)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (3.1.1.8)$$

The following code plots quadrilateral $ABCD$

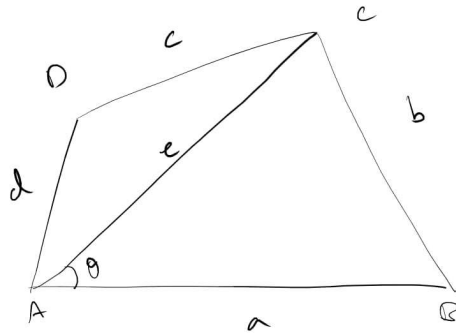


Fig. 3.1.1

in Fig. 3.1.1

```
codes/quad/draw_quad.py
```

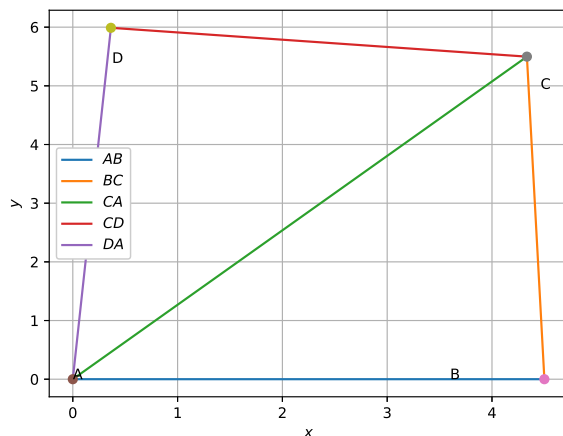


Fig. 3.1.1

3.2 Construction Exercises

- Construct a quadrilateral $ABCD$ such that $AB = 5$, $\angle A = 50^\circ$, $AC = 4$, $BD = 5$ and $AD = 6$.
- Construct $PQRS$ where $PQ = 4$, $QR = 6$, $RS = 5$, $PS = 5.5$ and $PR = 7$.
- Draw $JUMP$ with $JU = 3.5$, $UM = 4$, $MP = 5$, $PJ = 4.5$ and $PU = 6.5$.
- Construct a quadrilateral $ABCD$ such that $BC = 4.5$, $AC = 5.5$, $CD = 5$, $BD = 7$ and $AD = 5.5$.
- Can you construct a quadrilateral $PQRS$ with $PQ = 3$, $RS = 3$, $PS = 7.5$, $PR = 8$ and $SQ = 4$?
- Construct $LIFT$ such that $LI = 4$, $IF = 3$, $TL = 2.5$, $LF = 4.5$, $IT = 4$.
- Draw $GOLD$ such that $OL = 7.5$, $GL = 6$, $GD = 6$, $LD = 5$, $OD = 10$.
- DRAW rhombus $BEND$ such that $BN = 5.6$, $DE = 6.5$.
- construct a quadrilateral $MIST$ where $MI = 3.5$, $IS = 6.5$, $\angle M = 75^\circ$, $\angle I = 105^\circ$ and $\angle S = 120^\circ$.
- Can you construct the above quadrilateral $MIST$ if $\angle M = 100^\circ$ instead of 75° ?
- Can you construct the quadrilateral $PLAN$ if $PL = 6$, $LA = 9.5$, $\angle P = 75^\circ$, $\angle L = 150^\circ$ and $\angle A = 140^\circ$?
- Construct $MORE$ where $MO = 6$, $OR = 4.5$, $\angle M = 60^\circ$, $\angle O = 105^\circ$, $\angle R = 105^\circ$.
- Construct $PLAN$ where $PL = 4$, $LA = 6.5$, $\angle P = 90^\circ$, $\angle A = 110^\circ$ and $\angle N = 85^\circ$.
- Construct parallelogram $HEAR$ where $HE = 5$, $EA = 6$, $\angle R = 85^\circ$.
- Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.
- Construct $ABCD$, where $AB = 4$, $BC = 5$, $CD = 6.5$, $\angle B = 105^\circ$ and $\angle C = 80^\circ$.
- Construct $DEAR$ with $DE = 4$, $EA = 5$, $AR = 4.5$, $\angle E = 60^\circ$ and $\angle A = 90^\circ$.
- Construct $TRUE$ with $TR = 3.5$, $RU = 3$, $UE = 4$, $\angle R = 75^\circ$ and $\angle U = 120^\circ$.
- Draw a square of side 4.5.
- Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
- Draw a square $READ$ with $RE = 5.1$.
- Draw a rhombus whose diagonals are 5.2 and 6.4.
- Draw a rectangle with adjacent sides 5 and 4.
- Draw a parallelogram $OKAY$ with $OK = 5.5$.

- Draw the parallelogram $MORE$ with $OR = 6$, $RE = 4.5$ and $EO = 7.5$.

Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel.

- Construct a kite $EASY$ if $AY = 8$, $EY = 4$ and $SY = 6$.

Solution: The diagonals of a kite are perpendicular to each other.

- Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.

Solution: Diagonals of a rhombus bisect each other at right angles.

and $KA = 4.2$.

3.3 Quadrilateral Geometry

- Find the area of a rhombus if its vertices are $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ taken in order.
- If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral $ABCD$.
- Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
- The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.
- $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
 - square?
 - rectangle?
 - rhombus?
- Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
- Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.
- Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$.
- The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

4 CIRCLE

4.1 Construction Examples

- Draw a circle with centre \mathbf{B} and radius 6. If \mathbf{C} be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC$, $AC \perp AB$, $a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \quad (4.1.1.1)$$

The following code plots Fig. 4.1.1

codes/circle/draw_circle_eg.py

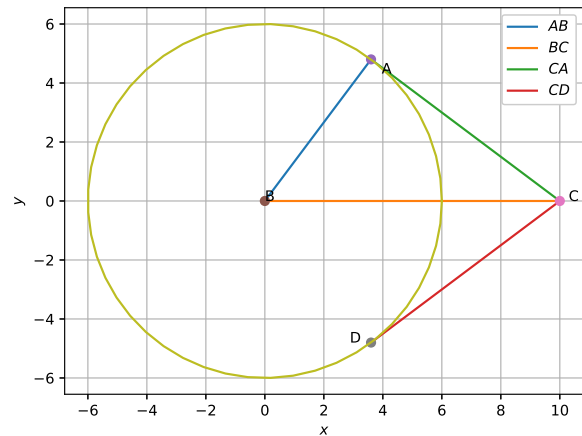


Fig. 4.1.1

- Draw a circle of radius 3. Mark any point \mathbf{A} on the circle, point \mathbf{B} inside the circle and point \mathbf{C} outside the circle.

Solution: For any angle θ , a point on the circle with radius 3 has coordinates

$$3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (4.1.2.1)$$

4.2 Construction Exercises

- Draw a circle of diameter 6.1
- With the same centre \mathbf{O} , draw two circles of radii 4 and 2.5
- Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
- Let \mathbf{A} and \mathbf{B} be two circles of equal radii 3 such that each one of them passes through

the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?

5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

Solution: Take the centre of both circles to be at the origin.

6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.

Solution: Take the diameter to be on the x -axis.

7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60° .

Solution: The tangent is perpendicular to the radius.

8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (4.2.2.1)$$

9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through **B, C, D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.

10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry

- Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $(2, -3)$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
- Find the centre of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
- Find the locus of all the unit vectors in the xy -plane.