Algebraic Approach to School Geometry

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Contents

***Abstract*—This book provides an equation based approach to school geometry based on the NCERT textbooks from Class 6-12.**

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1 The Right Angled Triangle

1. A right angled triangle looks like Fig. 1.0.1. with angles $\angle A, \angle B$ and $\angle C$ and sides a, b and c.

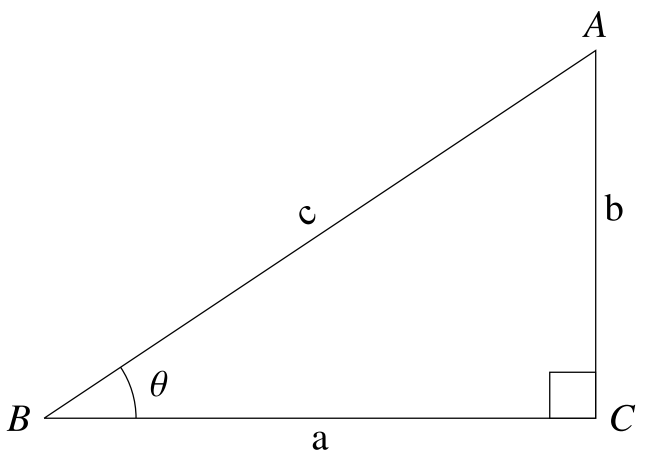

Fig. 1.0.1: Right Angled Triangle

The unique feature of this triangle is $\angle C$ which is defined to be 90°.

2. For simplicity, let the greek letter $\theta = \angle B$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \frac{a}{c} & \cos\theta &= \frac{b}{c} \\ \tan\theta &= \frac{b}{a} & \cot\theta &= \frac{1}{\tan\theta} \\ \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} \end{aligned} \tag{1.0.2.1}$$

1.1 Sum of Angles

1. In Fig. 1.1.1, the sum of all the angles on the top or bottom side of the straight line XY is 180°.
2. In Fig. 1.1.1, the straight line making an angle of 90° to the side AC is said to be parallel to the side BC. Note there is an angle at A that is equal to θ. This is one property of parallel lines. Thus, $\angle YAZ = 90^\circ$.
3. Show that $\angle VAZ = 90^\circ - \theta$

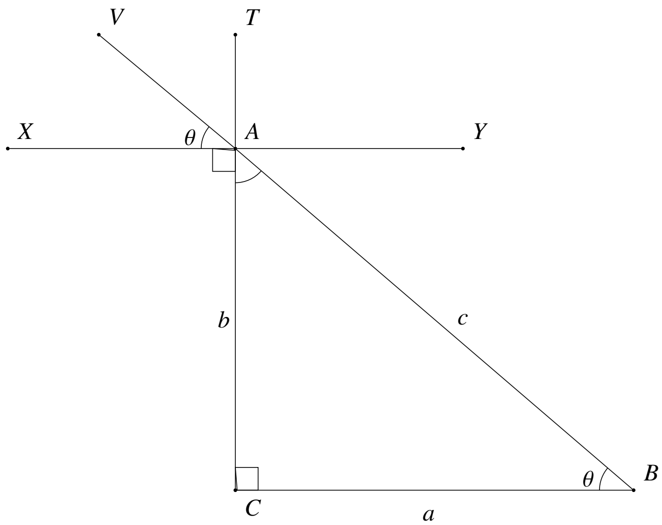

Fig. 1.1.1: Sum of angles of a triangle

Solution: Considering the line XAZ,

$$\theta + 90^{\circ} + \angle VAZ = 180^{\circ} \tag{1.1.3.1}$$
$$\implies \angle VAZ = 90^{\circ} - \theta \tag{1.1.3.2}$$

4. Show that $\angle BAC = 90^{\circ} - \theta$.
Solution: Consider the line VAB and and use the approach in the previous problem. Note that this implies that $\angle VAZ = \angle BAC$. Such angles are known as vertically opposite angles.
5. Sum of the angles of a triangle is equal to 180°

1.2 Baudhayana Theorem

1. Using Fig. 1.0.1, show that
$$\cos\theta = \sin\left(90^{\circ} - \theta\right) \tag{1.2.1.1}$$
Solution: From Problem 1.1.4 and (1.0.2.1)
$$\cos\left(90^{\circ} - \theta\right) = \frac{b}{c} = \sin\theta \tag{1.2.1.2}$$
2. Using Fig. 1.2.1, show that
$$c = a\cos\theta + b\sin\theta \tag{1.2.2.1}$$
Solution: We observe that
$$BD = a\cos\theta \tag{1.2.2.2}$$
$$AD = b\cos\left(90 - \theta\right) = b\sin\theta \quad \text{(From} \quad \text{(1.1.4))} \tag{1.2.2.3}$$
Thus,
$$BD + AD = c = a\cos\theta + b\sin\theta \tag{1.2.2.4}$$

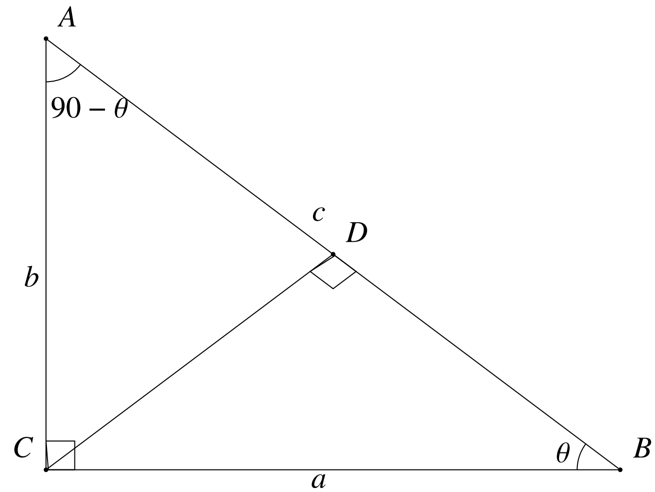

Fig. 1.2.1: Baudhayana Theorem

3. From (1.2.2.1), show that
$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.3.1}$$
Solution: Dividing both sides of (1.2.2.1) by c,
$$1 = \frac{a}{c}\cos\theta + \frac{b}{c}\sin\theta \tag{1.2.3.2}$$
$$\implies \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad \text{(1.0.2.1))} \tag{1.2.3.3}$$
4. Using (1.2.2.1), show that
$$c^2 = a^2 + b^2 \tag{1.2.4.1}$$
(1.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.2.2.1),
$$c = a\frac{a}{c} + b\frac{b}{c} \quad \text{(from} \quad \text{(1.0.2.1))} \tag{1.2.4.2}$$
$$\implies c^2 = a^2 + b^2 \tag{1.2.4.3}$$

1.3 Area of a Triangle

1. The area of the rectangle $ACBD$ shown in Fig. 1.3.1 is defined as ab. Note that all the angles in the rectangles are 90°
2. The area of the two triangles constituting the rectangle is the same.
3. The area of the rectangle is the sum of the areas of the two triangles inside.
4. Show that the area of ΔABC is $\frac{ab}{2}$

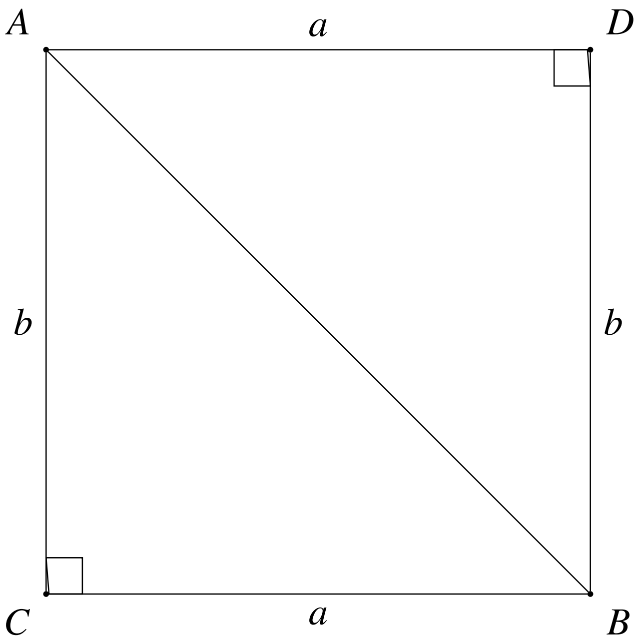

Fig. 1.3.1: Area of a Right Triangle

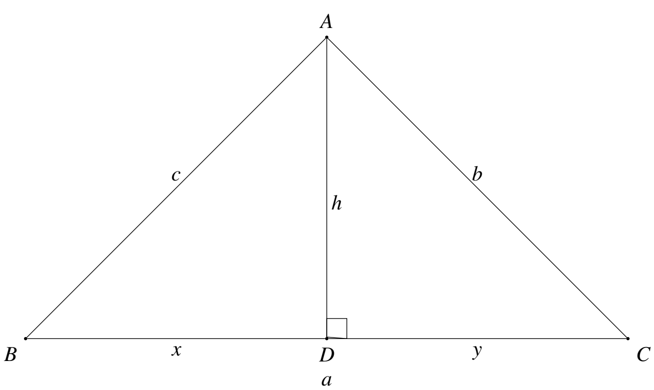

Fig. 1.3.4: Area of a Triangle

Solution: From(1.3.3),

$$ar\left(ABCD\right) = ar\left(ACB\right) + ar\left(ADB\right) \quad (1.3.4.1)$$

Also from (1.3.2),

$$ar\left(ACB\right) = ar\left(ADB\right) \quad (1.3.4.2)$$

From (1.3.4.1) and (1.3.4.2),

$$2ar\left(ACB\right) = ar\left(ABCD\right) = ab\,(\text{from} \quad (1.3.1)) \quad (1.3.4.3)$$

$$\Rightarrow ar\left(ACB\right) = \frac{ab}{2} \quad (1.3.4.4)$$

5. Show that the area of ΔABC in Fig. 1.3.4 is $\frac{1}{2}ah$.

Solution: In Fig. 1.3.4,

$$ar\left(\Delta ADC\right) = \frac{1}{2}hy \quad (1.3.5.1)$$

$$ar\left(\Delta ADB\right) = \frac{1}{2}hx \quad (1.3.5.2)$$

Thus,

$$ar\left(\Delta ABC\right) = ar\left(\Delta ADC\right) + ar\left(\Delta ADB\right) \quad (1.3.5.3)$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h\left(x+y\right) \quad (1.3.5.4)$$

$$= \frac{1}{2}ah \quad (1.3.5.5)$$

6. Show that the area of ΔABC in Fig. 1.3.4 is $\frac{1}{2}ab\sin C$.

Solution: We have

$$ar\left(\Delta ABC\right) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad \left(\because \quad h = b\sin C\right). \quad (1.3.6.1)$$

7. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.3.7.1)$$

Solution: Fig. 1.3.4 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{s}bc\sin A = \frac{1}{2}ca\sin B \quad (1.3.7.2)$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.3.7.3)$$

This is known as the sine formula.

8. In Fig. 1.3.8, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (1.3.8.1)$$

Solution: From the figure, the first of the following equations

$$a = b\cos C + c\cos B \quad (1.3.8.2)$$

$$b = c\cos A + a\cos C \quad (1.3.8.3)$$

$$c = b\cos A + a\cos B \quad (1.3.8.4)$$

is obvious and the other two can be similarly obtained. The above equations can be

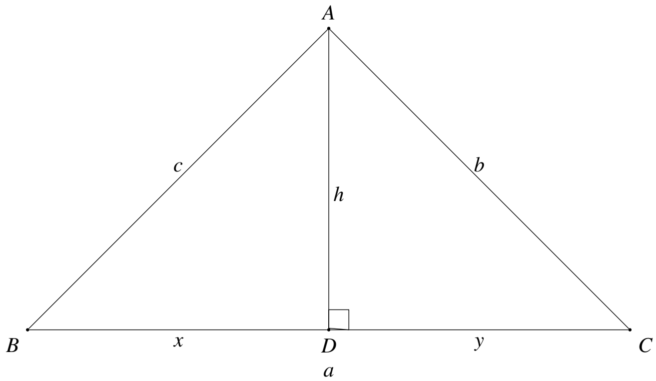

Fig. 1.3.8: The cosine formula

expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.3.8.5}$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.3.8.6}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} \tag{1.3.8.7}$$

9. Find Hero's formula for the area of a triangle.
Solution: From (1.3.6), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.2.3.1))} \tag{1.3.9.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.3.8.1))} \tag{1.3.9.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2 - (a^2+b^2-c^2)} \tag{1.3.9.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.3.9.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.3.9.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.3.9.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.3.9.7}$$

in (1.3.9.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.3.9.8}$$

This is known as Hero's formula.

2 Medians of a Triangle

2.1 Median

1. The line AD in Fig. 2.1.1 that divides the side a in two equal halfs is known as the median.

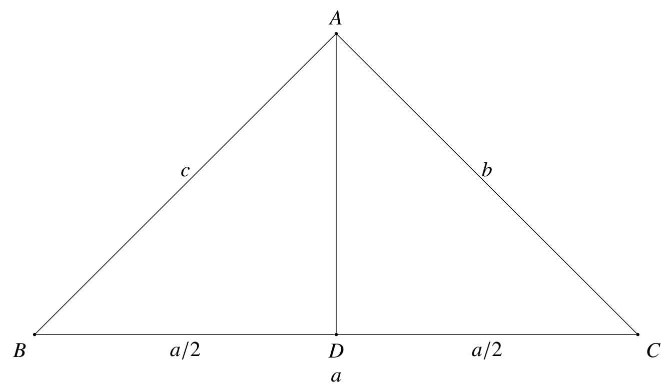

Fig. 2.1.1: Median of a Triangle

2. Show that the median AD in Fig. 2.1.1 divides ΔABC into triangles ADB and ADC that have equal area.
Solution: We have

$$ar(\Delta ADB) = \frac{1}{2}\frac{a}{2}c\sin B = \frac{1}{4}ac\sin B \tag{2.1.2.1}$$

$$ar(\Delta ADC) = \frac{1}{2}\frac{a}{2}b\sin C = \frac{1}{4}ab\sin C \tag{2.1.2.2}$$

Using the sine formula, $b\sin C = c\sin B$,

$$ar(\Delta ADB) = ar(\Delta ADC) \tag{2.1.2.3}$$

3. BE and CF are the medians in Fig. 2.1.3. Show that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.1.3.1}$$

Solution: Since BE and CF are the medians,

$$ar(\Delta BFC) = \frac{1}{2}ar(\Delta ABC) \tag{2.1.3.2}$$

$$ar(\Delta BEC) = \frac{1}{2}ar(\Delta ABC) \tag{2.1.3.3}$$

From the above, we infer that

$$ar(\Delta BFC) = ar(\Delta BEC) \quad (2.1.3.4)$$

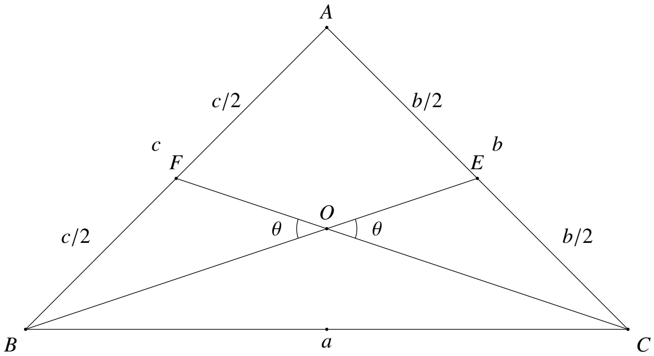

Fig. 2.1.3: O is the Intersection of Two Medians

4. We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 2.1.4,

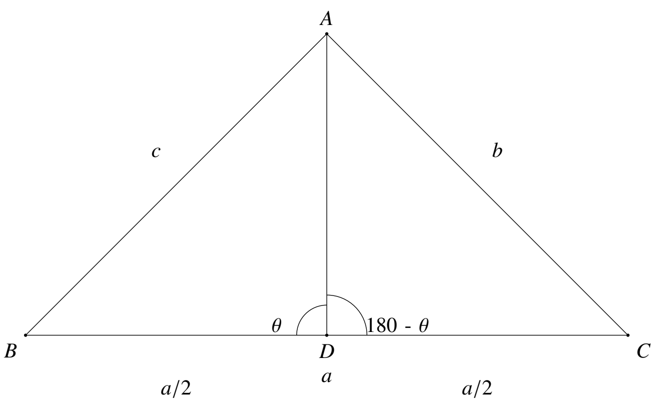

Fig. 2.1.4: $\sin\theta = \sin\left(180^\circ - \theta\right)$

$$\frac{1}{2}\frac{a}{2}AD\sin\theta = \frac{1}{2}\frac{a}{2}AD\sin\left(180^\circ - \theta\right) \quad (2.1.4.1)$$

$$\Rightarrow \sin\theta = \sin\left(180^\circ - \theta\right). \quad (2.1.4.2)$$

Note that our geometric definition of $\sin\theta$ holds only for $\theta < 90^\circ$. (2.1.4.2) allows us to extend this definition for $\angle ADC > 90^\circ$.

5. In Fig. 2.1.5, show that $EF = \frac{a}{2}$.

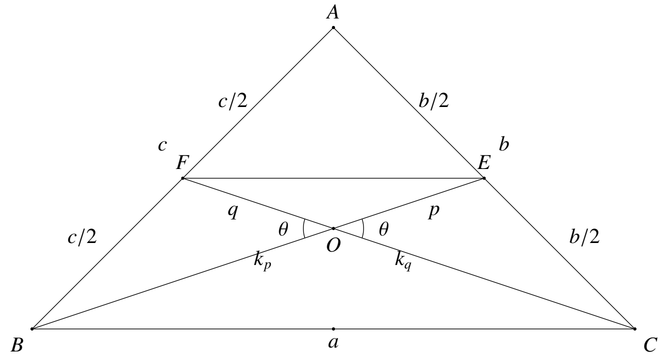

Fig. 2.1.5: Similar Triangles

Solution: Using the cosine formula for ΔAEF,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (2.1.5.1)$$

$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \quad (2.1.5.2)$$

$$= \frac{a^2}{4} \quad (2.1.5.3)$$

$$\Rightarrow EF = \frac{a}{2} \quad (2.1.5.4)$$

6. The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.
7. Show that similar triangles have the same angles.
   Solution: Use cosine formula and the proof is trivial.
8. Show that in Fig. 2.1.5, $EF \| BC$.
   Solution: Since $\Delta AEF \sim \Delta ABC$, $\angle AEF = \angle ACB$. Hence the line $EF \| BC$
9. Show that $\triangle OEF \sim \triangle OEC$.
10. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \quad (2.1.10.1)$$

11. Show that the medians of a triangle meet at a point.

3 Angle and Perpendicular Bisectors

3.1 Angle Bisectors

1. In Fig. 3.1.1, OB divides the $\angle B$ into half, i.e.

$$\angle OBC = \angle OBA \quad (3.1.1.1)$$

OB is known as an angle bisector.

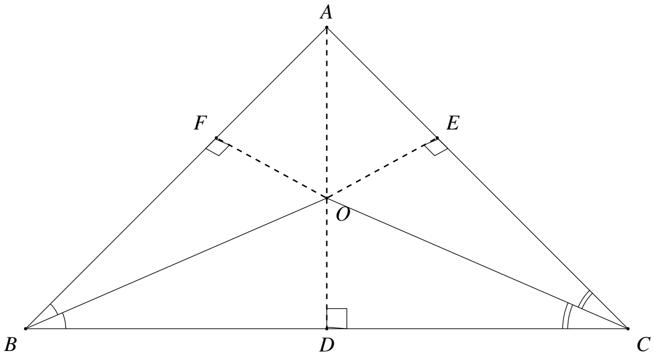

Fig. 3.1.1: Angle bisectors meet at a point

OB and OC are angle bisectors of angles B and C. OA is joined and OD, OF and OE are perpendiculars to sides a, b and c.

2. Show that $OD = OE = OF$. **Solution:** In Δs ODC and OEC,

$$OD = OC \sin\frac{C}{2} \tag{3.1.2.1}$$
$$OE = OC \sin\frac{C}{2} \tag{3.1.2.2}$$
$$\implies OD = OE. \tag{3.1.2.3}$$

Similarly,

$$OD = OF. \tag{3.1.2.4}$$

3. Show that OA is the angle bisector of $\angle A$
Solution: In Δs OFA and OEA,

$$OF = OE \tag{3.1.3.1}$$
$$\implies OA \sin OAF = OA \sin OAE \tag{3.1.3.2}$$
$$\implies \sin OAF = \sin OAE \tag{3.1.3.3}$$
$$\implies \angle OAF = \angle OAE \tag{3.1.3.4}$$

which proves that OA bisects $\angle A$. *Conclusion:* The angle bisectors of a triangle meet at a point.

3.2 Congruent Triangles

1. Show that in Δs ODC and OEC, corresponding sides and angles are equal.
2. Note that Δs ODC and OEC are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.
3. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.
4. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.
5. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.
6. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

3.3 Perpendicular Bisectors

1. In Fig. 3.3.2, OD $\perp BC$ and $BD = DC$. OD is defined as the perpendicular bisector of BC.
2. In Fig. 3.3.2, show that $OA = OB = OC$.

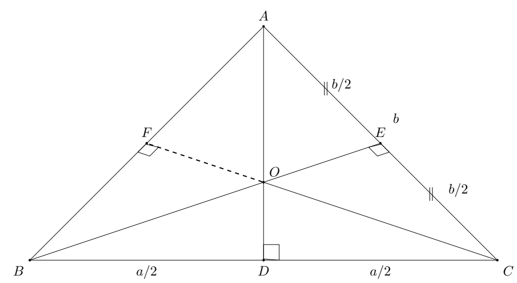

Fig. 3.3.2: Perpendicular bisectors meet at a point

Solution: In Δs ODB and ODC, using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \tag{3.3.2.1}$$

Since $BD = DC = \frac{a}{2}$, $OB = OC$. Similarly, it can be shown that $OA = OC$. Thus, $OA = OB = OC$.

3. In ΔAOB, $OA = OB$. Such a triangle is known as an isoceles triangle.
4. Show that $AF = BF$.
Solution: Trivial using Budhayana's theorem. This shows that OF is a perpendicular bisector of AB. *Conclusion:* The perpendicular bisectors of a triangle meet at a point.

3.4 Perpendiculars from Vertex to Opposite Side

1. In Fig. 3.4.1, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F. Show

that

$$OE = c\cos A\cot C \tag{3.4.1.1}$$

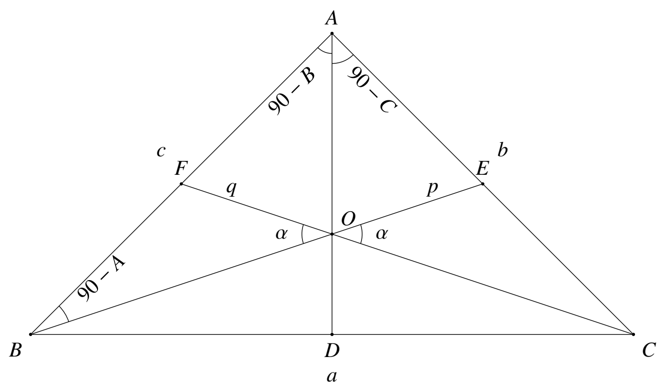

Fig. 3.4.1: Perpendiculars from vertex to opposite side meet at a point

Solution: In Δ s AEB and AEO,

$$AE = c\cos A \tag{3.4.1.2}$$

$$OE = AE\tan\left(90^\circ - C\right)(\because ADC \text{ is right angled}) \tag{3.4.1.3}$$

$$= AE\cot C \tag{3.4.1.4}$$

From both the above, we get the desired result.

2. Show that $\alpha = A$.
Solution: In ΔOEC,

$$CE = a\cos C\,(\because BEC \text{ is right angled}) \tag{3.4.2.1}$$

Hence,

$$\begin{aligned}\tan\alpha &= \frac{CE}{OE}\\ &= \frac{a\cos C}{c\cos A\cot C}\\ &= \frac{a\cos C\sin C}{c\cos A\cos C}\\ &= \frac{a\sin C}{c\cos A}\\ &= \frac{c\sin A}{c\cos A}\left(\because \frac{a}{\sin A} = \frac{c}{\sin C}\right)\\ &= \tan A\end{aligned}$$

$$\Rightarrow \alpha = A \tag{3.4.2.2}$$

3. Show that $CF \perp AB$
Solution: Consider triangle OFB and the result of the previous problem. $\because$ the sum of the angles of a triangle is 180°, $\angle CFB = 90^\circ$.

Conclusion: The perperdiculars from the vertex of a triangle to the opposite side meet at a point.

4 Triangle Inequalities

1. Show that if

$$\theta_1 < \theta_2, \quad \sin\theta_1 < \sin\theta_2. \tag{4.0.1.1}$$

Solution: Using Baudhayana's theorem in

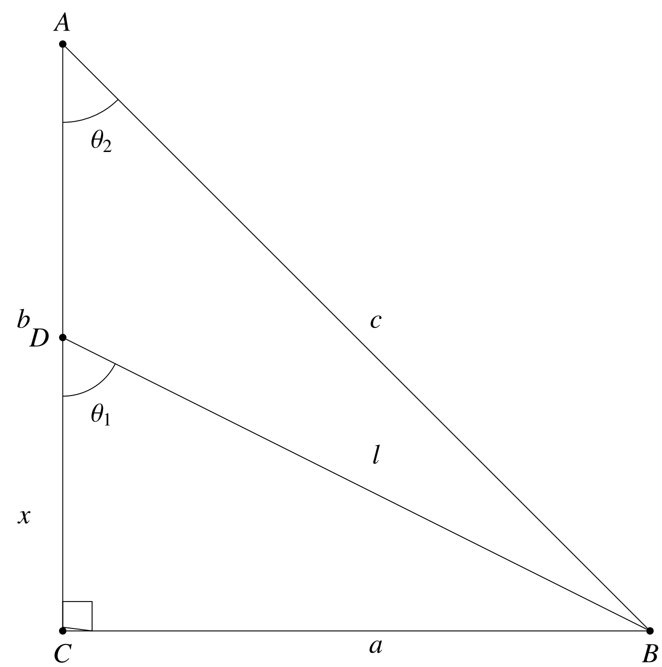

Fig. 4.0.1: $\theta_1 < \theta_2 \implies \sin\theta_1 < \sin\theta_2$.

$\triangle ABC$ and $\triangle DBC$

$$l^2 = x^2 + a^2 \tag{4.0.1.2}$$

$$c^2 = b^2 + a^2 \tag{4.0.1.3}$$

$$\implies c > l \because b > x. \tag{4.0.1.4}$$

Also,

$$a = c\sin\theta_1 = l\sin\theta_2 \tag{4.0.1.5}$$

$$\implies \frac{\sin\theta_1}{\sin\theta_2} = \frac{l}{c} < 1 \quad \text{from (4.0.1.4)} \tag{4.0.1.6}$$

$$\text{or, } \sin\theta_1 < \sin\theta_2 \tag{4.0.1.7}$$

2. Show that if

$$\theta_1 < \theta_2, \cos\theta_1 > \cos\theta_2. \tag{4.0.2.1}$$

3. Show that in any $\triangle ABC$, $\angle A > \angle B \implies a > b$.
Solution: Use (1.3.7.3) and (4.0.1.7)

4. Show that the sum of any two sides of a triangle is greater than the third side.
Solution: In Hero's formula in (1.3.9.8), all the factors inside the square root should be positive. Thus,

$$(s-a) > 0, (s-b) > 0 \, (s-c) > 0 \quad (4.0.4.1)$$

$$(4.0.4.2)$$

$$(s-a) > 0 \implies \frac{a+b+c}{2} - a > 0 \quad (4.0.4.3)$$

$$\text{or, } b+c > a \quad (4.0.4.4)$$

Similarly, it can be shown that $a+b > c, c+a > b$.

5 Triangle Exercises

1. Angles opposite to equal sides of a triangle are equal.
Solution: Using the sine formula in (1.3.7.3),
$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad (5.0.1.1)$$
Thus, if $A = B$, $\sin A = \sin B \implies a = b$.
2. Sides opposite to equal angles of a triangle are equal.
3. Each angle of an equilateral triangle is of 60°.
4. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
5. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.
6. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.
7. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.
8. In an isosceles $\triangle ABC$ with $AB =$ AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.
9. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.
10. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.
11. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$
12. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.
13. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that
    a) $\triangle AOB \cong \triangle DOC$
    b) O is also the mid-point of BC.
14. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?
15. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that
    a) $\triangle ABD \cong \triangle BAC$
    b) $BD = AC$
    c) $\angle ABD = \angle BAC$.
16. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.
17. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$ (see Fig. 7.20). Show that:
    a) $\triangle APB \cong \triangle AQB$
    b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.
18. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.
19. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:
    a) $\triangle AMC \cong \triangle BMD$
    b) $\angle DBC$ is a right angle.
    c) $\triangle DBC \cong \triangle ACB$
    d) $CM = \frac{1}{2}AB$
20. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
    a) $OB = OC$
    b) AO bisects $\angle A$
21. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.
22. ABC is an isosceles triangle in which altitudes

BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.

23. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that
    a) $\triangle ABE \cong \triangle ACF$
    b) $AB = AC$, i.e., ABC is an isosceles triangle.
24. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$.
25. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
    a) $\triangle ABD \cong \triangle ACD$
    b) $\triangle ABP \cong \triangle ACP$
    c) AP bisects $\angle A$ as well as $\angle D$.
    d) AP is the perpendicular bisector of BC.
26. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that
    a) AD bisects BC
    b) AD bisects $\angle A$.
27. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:
    a) $\triangle ABM \cong \triangle PQN$
    b) $\triangle ABC \cong \triangle PQR$
28. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.
29. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.
30. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.
31. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.
32. Show that in a right angled triangle, the hypotenuse is the longest side.
33. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.
34. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.
35. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.
36. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.
37. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.

6 Quadrilateral Exercises

1. Sum of the angles of a quadrilateral is 360°.
   Solution: Draw the diagonal and use the fact that sum of the angles of a triangle is 180°.
2. A diagonal of a parallelogram divides it into two congruent triangles.
   Solution: The alternate angles for the parallel sides are equal. The diagonal is common. Use ASA congruence.
3. In a parallelogram,
    a) opposite sides are equal
    b) opposite angles are equal
    c) diagonals bisect each other

   Solution: Since the diagonal divides the parallelogram into two congruent triangles, all the above results follow.
4. A quadrilateral is a parallelogram, if
    a) opposite sides are equal or
    b) opposite angles are equal or
    c) diagonals bisect each other or
    d) a pair of opposite sides is equal and parallel

   Solution: All the above lead to a quadrilateral that has two parallel sides, by showing that the alternate angles are equal.
5. A rectangle is a parallelogram with one angle that is 90°. Show that all angles of the rectangle are 90°.
   Solution: Draw a diagonal. Since the diagonal divides the rectangle into two congruent triangles, the angle opposite to the right angle is also 90°. Using congruence, it can be shown that the other two angles are equal. Now use the fact that the sum of the angles of a quadrilateral is 360°.
6. Diagonals of a rectangle bisect each other and are equal and vice-versa.
   Solution: Use Baudhayana's theorem for equality of diagonals.
7. Diagonals of a rhombus bisect each other at right angles and vice-versa.
   Solution: The median of an isoceles triangle is also its perpendicular bisector.
8. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.

Solution: A square has the properties of a rectangle as well as a rhombus.

9. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.
   Solution: Draw one diagonal and use Problem **??**. Repeat for the other diagonal to show that the sides are parallel.
10. Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.
11. Show that the bisectors of angles of a parallelogram form a rectangle.
12. A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.
13. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:
    a) $APCQ$ is a parallelogram.
    b) $DPBQ$ is a parallelogram.
    c) $PSQR$ is a parallelogram.
14. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.
15. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
16. Area of a parallelogram is the product of its base and the corresponding altitude.
17. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.
18. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
19. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
20. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
21. Show that the diagonals of a square are equal and bisect each other at right angles.
22. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.
23. Diagonal AC of a parallelogram $ABCD$ bisects $\angle A$. Show that (i) it bisects $\angle C$ also, (ii) $ABCD$ is a rhombus.
24. $ABCD$ is a rhombus. Show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.
25. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that: (i) $ABCD$ is a square (ii) diagonal BD bisects $\angle B$ as well as $\angle D$.
26. In parallelogram $ABCD$, two points P and Q are taken on diagonal BD such that $DP = BQ$. Show that: (i) $\triangle APD \cong \triangle CQB$ (ii) $AP = CQ$ (iii) $\triangle AQB \cong \triangle CPD$ (iv) $AQ = CP$ (v) $APCQ$ is a parallelogram
27. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD . Show that (i) $\triangle APB \cong \triangle CQD$ (ii) $AP = CQ$
28. In $\triangle ABC$ and $\triangle DEF, AB = DE, AB parallel DE, BC = EF$ and $BC parallel EF$. Vertices A, B and C are joined to vertices D, E and F respectively. Show that (i) quadrilateral $ABED$ is a parallelogram (ii) quadrilateral $BEFC$ is a parallelogram (iii) $AD parallel CF$ and $AD = CF$ (iv) quadrilateral $ACFD$ is a parallelogram (v) $AC = DF$ (vi) $\triangle ABC \cong \triangle DEF$.
29. $ABCD$ is a trapezium in which AB *parallel* CD and $AD = BC$. Show that (i)$\angle A = \angle B$ (ii) $\angle C = \angle D$ (iii) $\triangle ABC \cong \triangle BAD$ (iv) diagonal $AC =$ diagonal BD
30. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA AC is a diagonal. Show that : (i) $SR \parallel AC$ and $SR = \frac{1}{2}AC$ (ii) $PQ = SR$ (iii) $PQRS$ is a parallelogram.
31. $ABCD$ is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rectangle.
32. $ABCD$ is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.
33. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and E is the mid-point of AD. A line is drawn through $E \parallel AB$ intersecting BC at F. Show that F is the mid-point of BC.

34. In a parallelogram $ABCD$, E and F are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.
35. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

7 Circle

7.1 Chord of a Circle

1. Fig. 7.1.1 represents a circle. The points in the circle are at a distance r from the centre O. r is known as the radius.

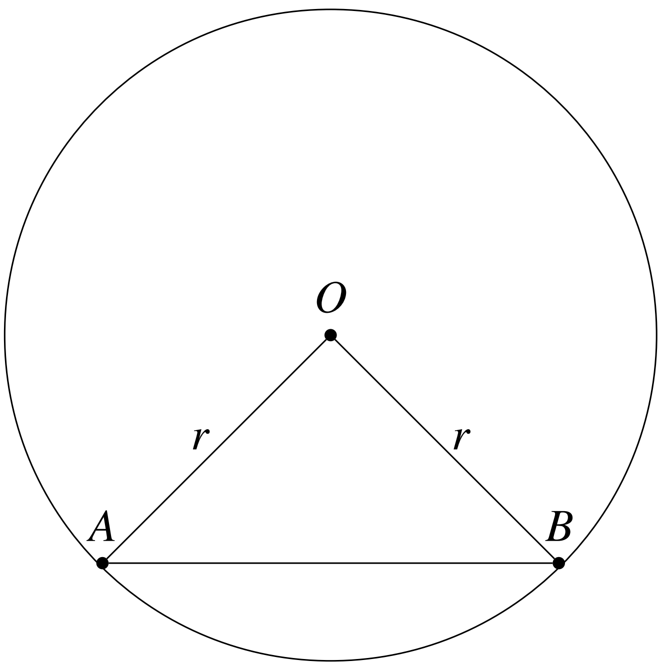

Fig. 7.1.1: Circle Definitions

7.2 Chords of a circle

1. In Fig. 7.1.1, A and B are points on the circle. The line AB is known as a chord of the circle.
2. In Fig. 7.2.2 Show that $\angle AOB = 2\angle ACB$.
Solution: In Fig. 7.2.2, the triangeles OPA and OPB are isosceles. Hence,

$$\angle OCA = \angle OAC = \theta_1 \quad (7.2.2.1)$$
$$\angle OCB = \angle OBC = \theta_2 \quad (7.2.2.2)$$

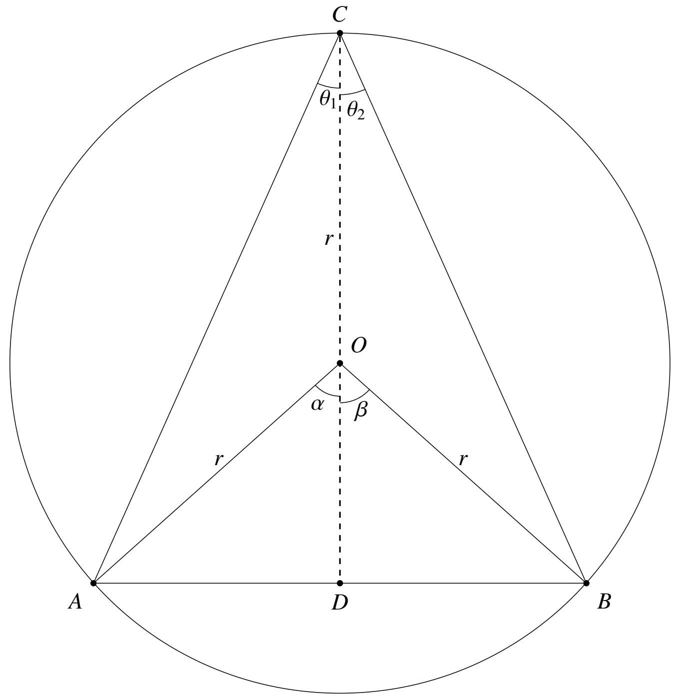

Fig. 7.2.2: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Also, α and β are exterior angles corresponding to the triangle AOC and BOC respectively. Hence

$$\alpha = 2\theta_1 \quad (7.2.2.3)$$
$$\beta = 2\theta_2 \quad (7.2.2.4)$$

Thus,

$$\angle AOB = \alpha + \beta \quad (7.2.2.5)$$
$$= 2(\theta_1 + \theta_2) \quad (7.2.2.6)$$
$$= 2\angle ACB \quad (7.2.2.7)$$

3. The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 7.2.4, AB is the diameter and passes through the centre O
4. In Fig. 7.2.4, show that $\angle APB = 90^\circ$.
5. In Fig. 7.2.5, show that

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \quad (7.2.5.1)$$

Solution: Use Problem 7.2.2.
6. In Fig. 7.2.5, show that the triangles PAB and PBD are similar
Solution: Trivial using previous problem

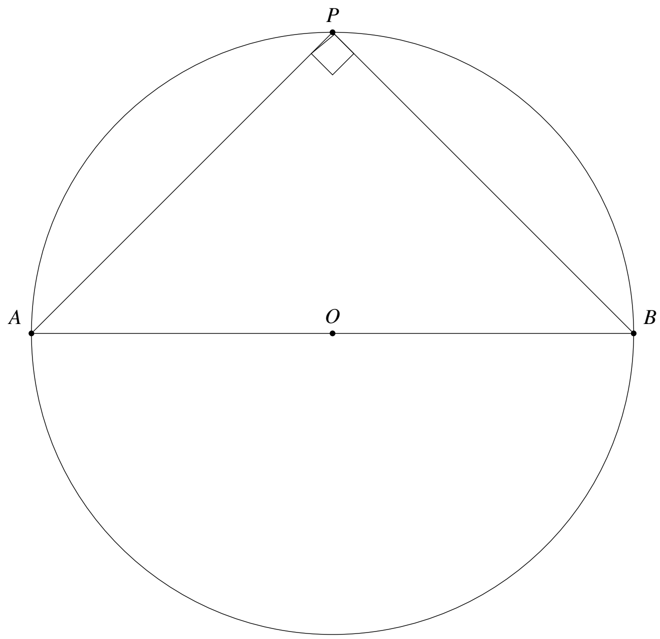

Fig. 7.2.4: Diameter of a circle.

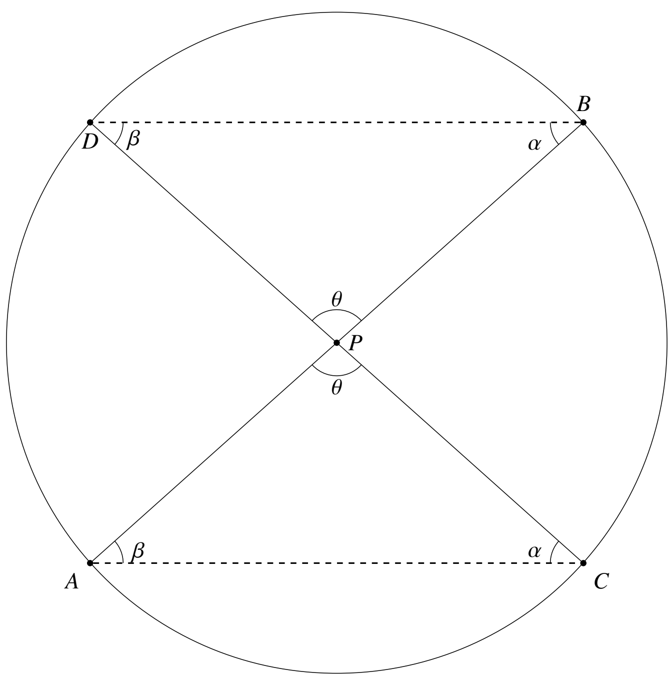

Fig. 7.2.5: $PA.PB = PC.PD$

7. In Fig. 7.2.5, show that

$$PA.PB = PC.PD \tag{7.2.7.1}$$

Solution: Since triangles PAC and PBD are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{7.2.7.2}$$

$$\Rightarrow PA.PB = PC.PD \tag{7.2.7.3}$$

8. Show that

$$\sin 0^{\circ} = 0 \tag{7.2.8.1}$$

Solution: Follows from (4.0.1.1).

9. Show that

$$\cos 0^{\circ} = 1 \tag{7.2.9.1}$$

10. The line PX in Fig. 7.2.11 touches the circle at exactly one point P. It is known as the tangent to the circle.
11. Show that $OP \perp PX$.
    Solution: Without loss of generality, let $0 \le \theta \le 90^{\circ}$. Using the cosine formula in $\triangle OPP_n$,

$$(r + d_n)^2 > r^2, \tag{7.2.11.1}$$

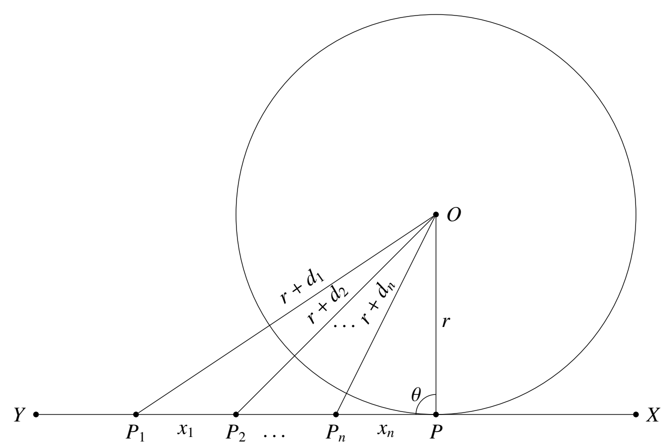

Fig. 7.2.11: Tangent to a Circle.

$$(r + d_n)^2 = r^2 + x_n^2 - 2rx_n \cos\theta > r^2 \tag{7.2.11.2}$$

$$\implies 0 < \cos\theta < \frac{x_n}{2r}, \tag{7.2.11.3}$$

where x_n can be made as small as we choose. Thus,

$$\cos\theta = 0 \implies \theta = 90^{\circ}. \tag{7.2.11.4}$$

12. In Fig. 7.2.12 show that

$$\angle PCA = \angle PBC \tag{7.2.12.1}$$

O is the centre of the circle and PC is the tangent.

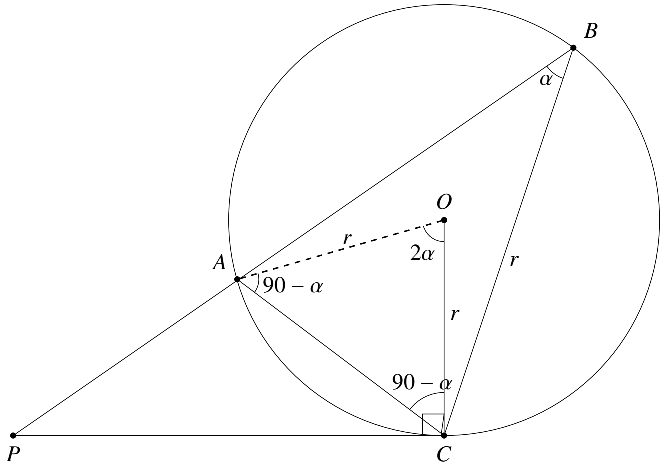

Fig. 7.2.12: $PA.PB = PC^2$.

Solution: Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

13. In Fig. 7.2.12, show that the triangles PAC and PBC are similar.

    Solution: From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

14. Show that $PA.PB = PC^2$

    Solution: Since $\Delta PAC \sim \Delta PBC$, their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{7.2.14.1}$$

$$\Rightarrow PA.PB = PC^2 \tag{7.2.14.2}$$

15. Given that $PA.PB = PC^2$, show that PC is a tangent to the circle.
16. In Fig. 7.2.16, show that

$$PA.PB = PC.PD \tag{7.2.16.1}$$

Solution: Draw a tangent and use the previous problem.

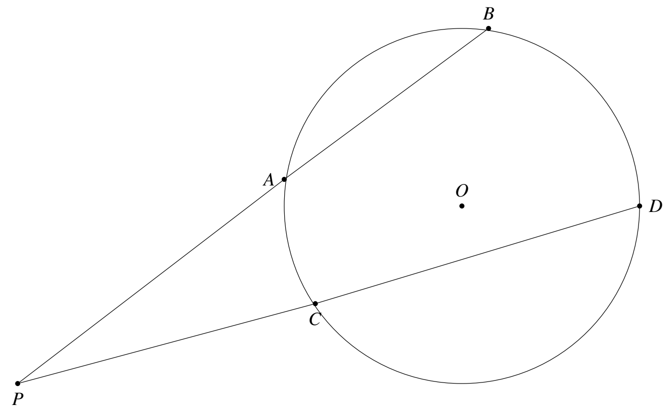

Fig. 7.2.16: $PA.PB = PC^2$.

8 Area of a Circle

8.1 The Regular Polygon

1. In Fig. 8.1.1, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.

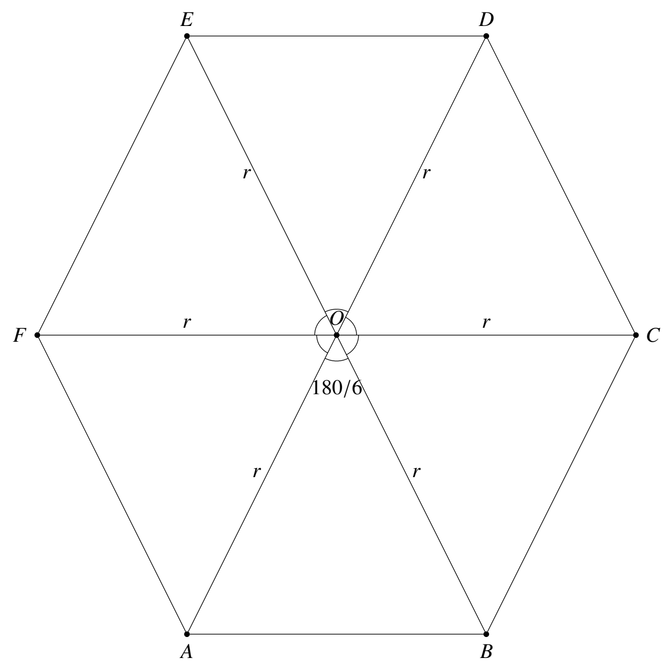

Fig. 8.1.1: Polygon Definition

2. The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{360^\circ}{n}$.
3. Show that the area of a regular polygon is given by

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} \tag{8.1.3.1}$$

Solution: The triangle that forms the polygon of n sides is given in Fig. 8.1.3. Thus,

$$\begin{aligned} ar\,(polygon) &= nar\,(\Delta ABC) \\ &= \frac{n}{2}r^2 \sin\frac{360^\circ}{n} \end{aligned} \tag{8.1.3.2}$$

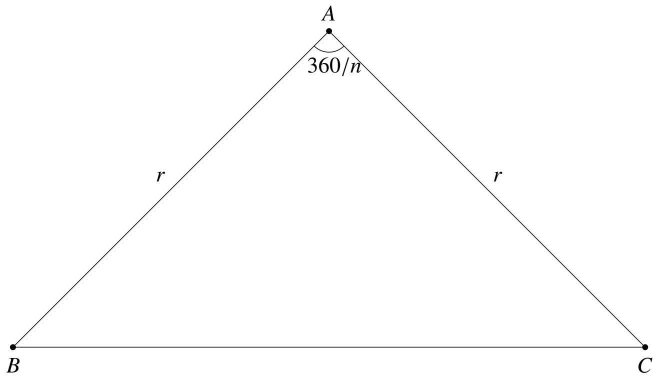

Fig. 8.1.3: Polygon Area

4. Using Fig. 8.1.4, show that

$$\frac{n}{2}r^2 \sin \frac{360^\circ}{n} < \text{ area of circle } < nr^2 \tan \frac{180^\circ}{n} \quad (8.1.4.1)$$

The portion of the circle visible in Fig. 8.1.4 is defined to be a sector of the circle.

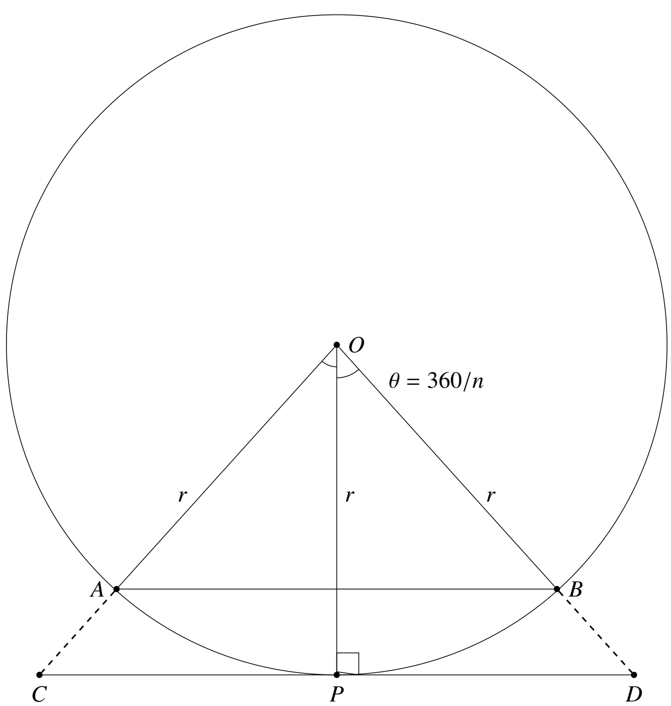

Fig. 8.1.4: Circle Area in between Area of Two Polygons

Solution: Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. 8.1.4, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$ar(\Delta OAB) = \frac{1}{2}r^2 \sin \frac{360^\circ}{n} \quad (8.1.4.2)$$

$$ar(\Delta OPQ) = 2 \times \frac{1}{2} \times r \tan \frac{360/n}{2} \times r \quad (8.1.4.3)$$

$$= r^2 \tan \frac{180^\circ}{n}, \quad (8.1.4.4)$$

we obtain (8.1.4.1).

5. Using Fig. 8.1.5, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \quad (8.1.5.1)$$

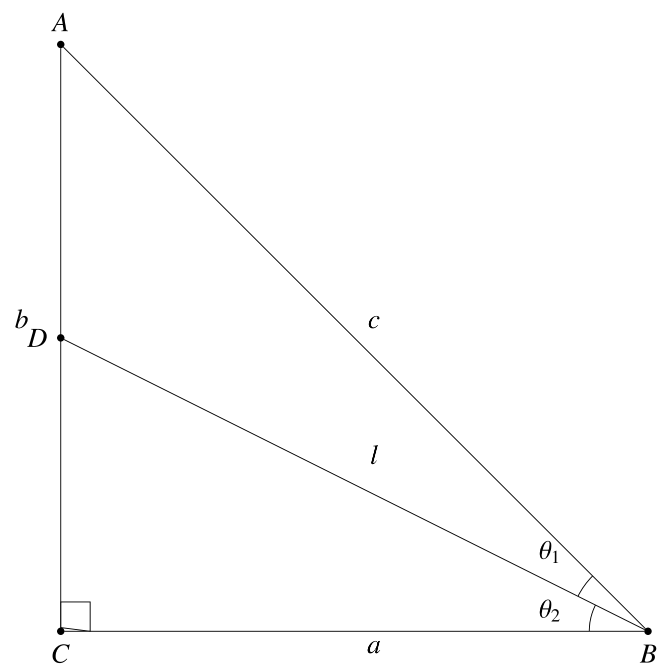

Fig. 8.1.5: $\sin 2\theta = 2\sin\theta\cos\theta$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1+\theta_2) \quad (8.1.5.2)$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \quad (8.1.5.3)$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \quad (8.1.5.4)$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \quad (8.1.5.5)$$

($\because l = a\sec\theta_2$). From the above,

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{8.1.5.6}$$

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \tag{8.1.5.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\Rightarrow \sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{8.1.5.8}$$

resulting in

$$\Rightarrow \sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{8.1.5.9}$$

6. Prove the following identities

a)

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{8.1.6.1}$$

b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{8.1.6.2}$$

Solution: In (8.1.5.1), let

$$\begin{aligned} \theta_1+\theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{8.1.6.3}$$

This gives (8.1.6.1). In (8.1.6.1), replace α by $90^\circ - \alpha$. This results in

$$\begin{aligned} &\sin\left(90^\circ - \alpha - \beta\right) \\ &= \sin\left(90^\circ-\alpha\right)\cos\beta - \cos\left(90^\circ-\alpha\right)\sin\beta \\ &\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned} \tag{8.1.6.4}$$

7. Using (8.1.5.1) and (8.1.6.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{8.1.7.1}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{8.1.7.2}$$

Solution: From (8.1.5.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{8.1.7.3}$$

Using (8.1.6.2) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{8.1.7.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{8.1.7.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{8.1.7.6}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{8.1.7.7}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{8.1.7.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (8.1.7.2) can also be proved.

8. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{8.1.8.1}$$

9. Show that

$$\cos^2\frac{180^\circ}{n} < \frac{\text{area of circle}}{nr^2\tan\frac{180^\circ}{n}} < 1 \tag{8.1.9.1}$$

Solution: From (8.1.4.1) and (8.1.8.1),

$$\begin{aligned} &\frac{n}{2}r^2\sin\frac{360^\circ}{n} < \text{ area of circle} \\ &\qquad < nr^2\tan\frac{180^\circ}{n} \\ &\Rightarrow nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} < \text{ area of circle} \\ &\qquad < nr^2\tan\frac{180^\circ}{n} \end{aligned} \tag{8.1.9.2}$$

10. Show that for large values of n

$$\cos^2\frac{180^\circ}{n} = 1 \tag{8.1.10.1}$$

Solution: Follows from previous problem.

11. The previous result can be expressed as

$$\lim_{n\to\infty}\cos^2\frac{180^\circ}{n} = 1 \tag{8.1.11.1}$$

12. Show that

$$\text{area of circle } = r^2\lim_{n\to\infty} n\tan\frac{180^\circ}{n} \tag{8.1.12.1}$$

13. $$\pi = \lim_{n\to\infty} n \tan \frac{180^\circ}{n} \qquad (8.1.13.1)$$

Thus, the area of a circle is πr^2.

14. The radian is a unit of angle defined by

$$1 \text{ radian} = \frac{360^\circ}{2\pi} \qquad (8.1.14.1)$$

15. Show that the circumference of a circle is $2\pi r$.
16. Show that the area of a sector with angle θ in radians is $\frac{1}{2}r^2\theta$.
17. Show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (8.1.17.1)$$

9 Circle Exercises

1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3. The perpendicular from the centre of a circle to a chord bisects the chord.
4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
5. There is one and only one circle passing through three non-collinear points.
6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
9. Congruent arcs of a circle subtend equal angles at the centre.
10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
11. Angles in the same segment of a circle are equal.
12. Angle in a semicircle is a right angle.
13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
15. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.
16. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.
17. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$
18. Two circles intersect at two points $AandB$. AD and AC are diameters to the two circles (see Fig.10.34). Prove that B lies on the line segment DC.
19. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
20. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
21. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
22. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
23. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.
24. A,B and C are three points on a circle with centre O such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.
25. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
26. $\angle PQR = 100°$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$. Fig. 10.37
27. A, B, C, D are points on a circle such that $\angle ABC = 69°, \angle ACB = 31°$, find $\angle BDC$.
28. A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130°$ and $\angle ECD = 20°$. Find $\angle BAC$.

29. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70°, \angle BACis30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.
30. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
31. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
32. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.
33. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
34. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.
35. Prove that a cyclic parallelogram is a rectangle.
36. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
37. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.
38. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?
39. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.
40. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.
41. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.
42. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.
43. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90°-\frac{A}{2}, 90°-\frac{B}{2}$ and $90°-\frac{C}{2}$.
44. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.
45. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.