Linear Algebra through Coordinate Geometry

Contents

1 The Straight Line

1.1 Point

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$
3. The *length* of PQ is defined as
$$\|\mathbf{P} - \mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$
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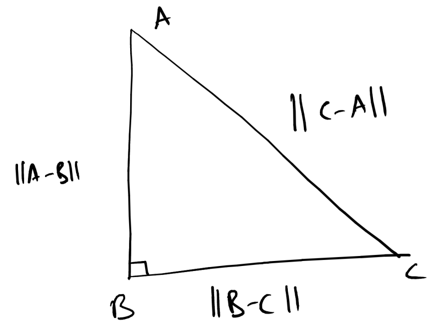

Fig. 1.1.7

5. The point dividing PQ in the ratio $k : 1$ is
$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A} - \mathbf{B})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.7}$$
Solution: Using Baudhayana's theorem,
$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{1.1.7}$$
$$\begin{aligned} \implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) \\ = (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) \\ \implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \end{aligned} \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show that
$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.8}$$
9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that
$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.9}$$

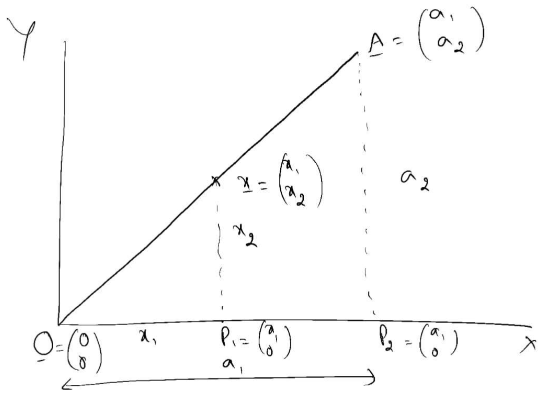

Fig. 1.2.1

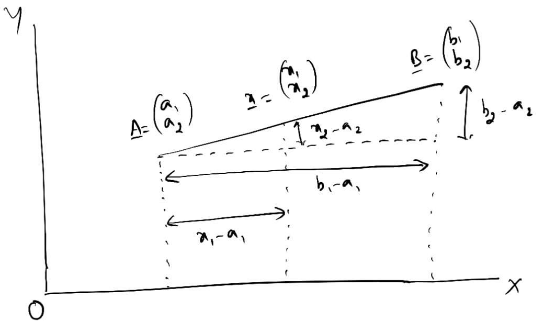

Fig. 1.2.2

1.2 Line

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of OA.

   Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

   $$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.2.1.1}$$

   $$\implies x_2 = mx_1 \tag{1.2.1.2}$$

   where m is known as the slope of the line. Thus, the equation of the line is

   $$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = x_1 \mathbf{m} \tag{1.2.1.3}$$

   In general, the above equation is written as

   $$\mathbf{x} = \lambda \mathbf{m}, \tag{1.2.1.4}$$

   where $\mathbf{m}$ is the direction vector of the line.
2. Find the equation of AB in Fig. 1.2.2

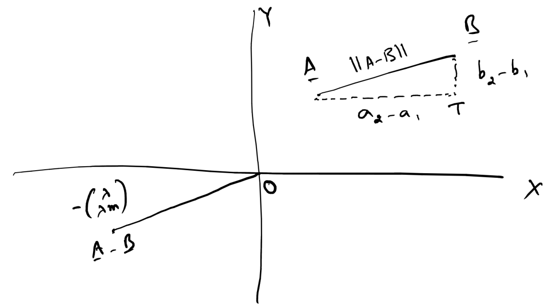

Fig. 1.2.5

   Solution: From Fig. 1.2.2,

   $$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.2.2.1}$$

   $$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.2.2.2}$$

   From (1.2.2.2),

   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.2.2.3}$$

   $$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2.4}$$

   $$= \mathbf{A} + \lambda \mathbf{m} \tag{1.2.2.5}$$

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (1.2.2.5) is obtained from (1.2.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 1.2.1

   Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

   $$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \tag{1.2.4.1}$$

   Also, from (1.2.1.4),

   $$\|\mathbf{A}\| = \lambda \sqrt{1 + m^2} \tag{1.2.4.2}$$

   Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.

   Solution: See Fig. 1.2.5. From (1.2.2.5), for some λ,

   $$\mathbf{B} = \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.5.1}$$

   $$\implies \mathbf{A} - \mathbf{B} = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.2.5.2}$$

   $\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.

6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.2.7.1}$$

8. The *normal* to the vector $\mathbf{m}$ is defined as

$$\mathbf{n}^T\mathbf{m} = 0 \tag{1.2.8.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{1.2.8.2}$$

9. From (1.2.7.1), the equation of a line can also be expressed as

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) \tag{1.2.9.1}$$

$$\implies \mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} = c \tag{1.2.9.2}$$

10. The unit vectors on the x and y axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.2.10.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.10.2}$$

11. If a be the *intercept* of the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.11.1}$$

on the $x-$axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line. Thus,

$$\mathbf{n}^T\begin{pmatrix} a \\ 0 \end{pmatrix} = c \tag{1.2.11.2}$$

$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \tag{1.2.11.3}$$

12. The *rotation matrix* is defined as

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.2.12}$$

where θ is anti-clockwise.

13.

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.13}$$

where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix*.

14. Find the equation of line L in Fig. 1.2.14.
Solution: The equation of the $x-$axis is

$$\mathbf{x} = \lambda\mathbf{e}_1 \tag{1.2.14.1}$$

Translation by p units along the $y-$axis results

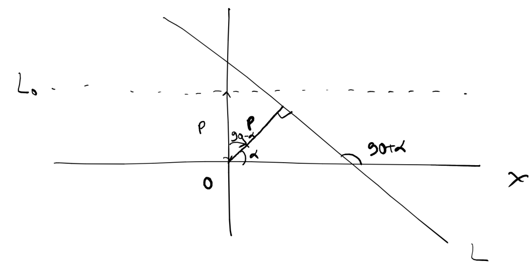

Fig. 1.2.14

in

$$L_0: \quad \mathbf{x} = \lambda\mathbf{e}_1 + p\mathbf{e}_2 \tag{1.2.14.2}$$

Rotation by $90° - \alpha$ in the anti-clockwise direction yields

$$L: \quad \mathbf{x} = \mathbf{Q}\{\lambda\mathbf{e}_1 + p\mathbf{e}_2\} \tag{1.2.14.3}$$

$$= \lambda\mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \tag{1.2.14.4}$$

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \tag{1.2.14.5}$$

$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \tag{1.2.14.6}$$

From (1.2.14.4),

$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$

$$= \lambda\mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \tag{1.2.14.7}$$

resulting in

$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \tag{1.2.14.8}$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.15.1}$$

is

$$p = \frac{c}{\|n\|} \tag{1.2.15.2}$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.16.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.16.2}$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \tag{1.2.16.3}$$

where

$$\mathbf{N} = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix} \tag{1.2.16.4}$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{1.2.17.1}$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.18.1}$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \tag{1.2.18.2}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{1.2.18.3}$$

where

$$\mathbf{n}^T\mathbf{A} = c, \mathbf{n}^T\mathbf{m} = 0 \tag{1.2.18.4}$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \tag{1.2.18.5}$$

$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \tag{1.2.18.6}$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.2.19.1}$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.2.19.2}$$

where

$$\mathbf{V} = \mathbf{V}^T \tag{1.2.19.3}$$

$$\mathbf{u} = \begin{pmatrix}d & e\end{pmatrix} \tag{1.2.19.4}$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = 0 \tag{1.2.20.1}$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \tag{1.2.20.2}$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{1.2.21.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{1.2.21.2}$$

Solution: From (1.2.2.5),

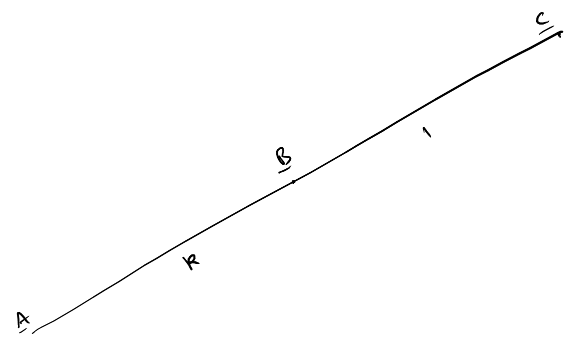

Fig. 1.2.21

$$\begin{aligned}\mathbf{B} &= \mathbf{A} + \lambda_1\mathbf{m}\\ \mathbf{B} &= \mathbf{C} - \lambda_2\mathbf{m}\end{aligned} \tag{1.2.21.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{1.2.21.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{1.2.21.5}$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{1.2.21.6}$$

resulting in (1.2.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{1.2.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{1.2.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{1.2.23.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{1.2.23.3}$$

24. In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude*

of ΔABC.

25. Find the intersection of AP and BQ.
Solution: The normal vector of AP is $\mathbf{B} - \mathbf{C}$. From and , the equation of AP and BQ are

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2.25.1)$$
$$(\mathbf{C} - \mathbf{A})^T (\mathbf{x} - \mathbf{B}) = 0 \quad (1.2.25.2)$$

which can be solved to obtain the intersection point using (1.2.16.3).

26. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.2.26.1)$$
$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.2.26.2)$$

is

$$\frac{\mathbf{n}_1^T \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^T \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \quad (1.2.26.3)$$

27. Find the equation of a line passing through the intersection of the lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (1.2.27.1)$$
$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (1.2.27.2)$$

and passing through the point $\mathbf{p}$.
Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T} \mathbf{c} \quad (1.2.27.3)$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \quad (1.2.27.4)$$
$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (1.2.27.5)$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda \left(\mathbf{N}^{-T} \mathbf{c} - \mathbf{p} \right) \quad (1.2.27.6)$$
$$\implies \mathbf{N}^T \mathbf{x} = \mathbf{N}^T \mathbf{p} + \lambda \left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right) \quad (1.2.27.7)$$

resulting in

$$\left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{x} = \left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{p} \quad (1.2.27.8)$$

28. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L : \quad \mathbf{n}^T \mathbf{x} = c \quad (1.2.28.1)$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

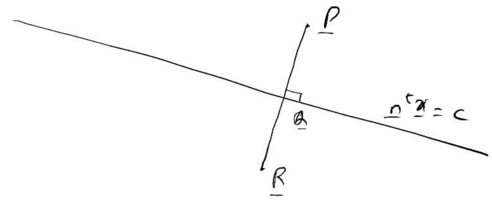

Fig. 1.2.28

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (1.2.28.2)$$
$$\mathbf{n}^T \mathbf{Q} = c \quad (1.2.28.3)$$
$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \quad (1.2.28.4)$$

where $\mathbf{m}$ is the direction vector of L. From (1.2.28.2) and (1.2.28.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \quad (1.2.28.5)$$

From (1.2.28.5) and (1.2.28.4),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.28.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \quad (1.2.28.7)$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \quad (1.2.28.8)$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.2.28.9)$$

(1.2.28.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.28.10)$$
$$\implies \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \quad (1.2.28.11)$$
$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \quad (1.2.28.12)$$

29. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also

given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.2.29.1}$$

1.3 Example

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.3.1.1}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 5 \tag{1.3.1.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 4 \tag{1.3.1.3}$$

Find the area of $\triangle ABC$.
Solution: The centroid $\mathbf{O}$ is the solution of (1.3.1.2),(1.3.1.3) and is obtained as the solution of the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{1.3.1.4}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.3.1.5}$$

Thus,

$$\mathbf{O} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.3.1.6}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{O} \tag{1.3.1.7}$$

$$\implies \mathbf{B} + \mathbf{C} = 3\mathbf{O} - \mathbf{A} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.3.1.8}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{B} + \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{C} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{1.3.1.9}$$

From (1.3.1.3) and (1.3.1.9),

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{B} = 5 \tag{1.3.1.10}$$
$$\implies 5 + \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{C} = 12 \tag{1.3.1.11}$$
$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{C} = 7 \tag{1.3.1.12}$$

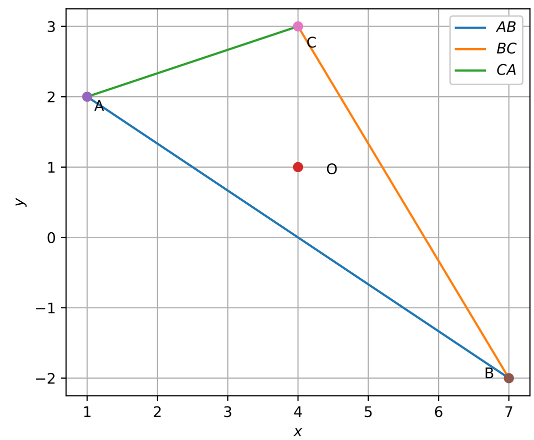

Fig. 1.3.2

From (1.3.1.12) and (1.3.1.3), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{C} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.3.1.13}$$

using the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 7 \\ 1 & 0 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \tag{1.3.1.14}$$

$$\implies \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.3.1.15}$$

From (1.3.1.8),

$$\mathbf{B} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.3.1.16}$$

Thus,

$$\frac{1}{2}\begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 7 & 4 \\ 2 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 9 \tag{1.3.1.17}$$

2. Summarize all the above computations through a Python script and plot $\triangle ABC$.
Solution:

```
https://github.com/gadepall/school/raw/master/
    linalg/2D/manual/codes/triang.py
```

1.4 Programming

1. Find the *orthocentre* of $\triangle ABC$.
Solution: The following code finds the required point using (1.2.25.1) and (1.2.25.2) .

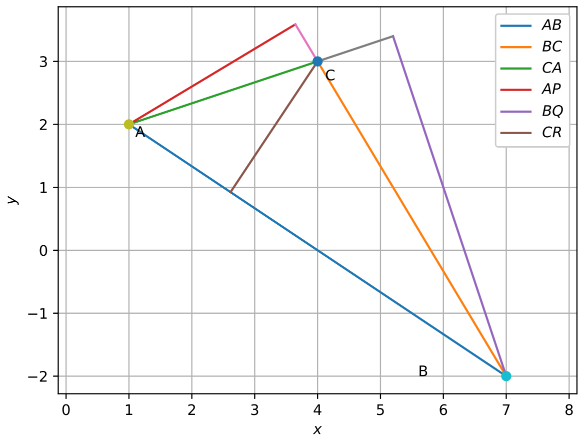

Fig. 1.4.4

https://raw.githubusercontent.com/gadepall/school/master/linalg/book/codes/orthocentre.py

2. Find **P**, the foot of the altitude from **A** upon BC.
Solution:

https://raw.githubusercontent.com/gadepall/school/master/linalg/book/codes/alt_foot.py

3. Find **Q** and **R**.
4. Draw AP, BQ and CR and verify that they meet at a point **H**.
Solution: The following code plots the altitudes in Fig. 1.4.4

https://raw.githubusercontent.com/gadepall/school/master/linalg/book/codes/alt_draw.py

5. Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.
6. Find the equations of AD, BE and CF.
7. Find the point of intersection of AD and CF.
8. Verify that **O** is the point of intersection of BE, CF as well.
9. Graphically show that the medians of ΔABC meet at the centroid.

1.5 Exercises

1. A straight line through the origin **O** meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.5.1}$$
$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.5.1}$$

at **A** and **B** respectively. Find the ratio in which **O** divides AB.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.5.1}$$

Then (1.5.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5.1}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.5.1) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.5.1}$$
$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.5.1}$$

Let **O** divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.5.1}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.5.1}$$
$$\mathbf{A} = -k\mathbf{B} \tag{1.5.1}$$

Substituting in (1.5.1), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{1.5.1}$$
$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{1.5.1}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.5.1}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.5.2}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{1.5.2}$$

by $d = 2\sqrt{3}$ units. If the new point **Q** lies in the third quadrant, then find the equation of the line passing through **Q** and perpendicular to L.

Solution: From (1.5.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.2}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \tag{1.5.2}$$

However,

$$PQ = d \tag{1.5.2}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = d \tag{1.5.2}$$

$$\implies \lambda = \pm \frac{d}{\|\mathbf{m}\|} = \pm \sqrt{6} \tag{1.5.2}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{1.5.2}$$

from (1.5.2). Since $\mathbf{Q}$ lies in the third quadrant, from (1.5.2) and (1.5.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \tag{1.5.2}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \tag{1.5.2}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - 2\sqrt{6} \tag{1.5.2}$$

3. Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.5.3}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \tag{1.5.3}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{1.5.3}$$

find its vertices.
Solution: From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{1.5.3}$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1.5.3}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2}$$
$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.5.3}$$

$$\because AD \parallel BC,$$
$$BC : \begin{pmatrix} 7 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} = -15 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \implies \mathbf{B} = -\frac{1}{3} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.5.3}$$

$$\because AB \parallel CD,$$
$$CD : \begin{pmatrix} 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.5.3}$$

From (1.5.3) and (1.5.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.5.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \implies \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \tag{1.5.3}$$

4. Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{1.5.4}$$

has area 28. Find the orthocentre of this triangle.
Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \tag{1.5.4}$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.5.4}$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T(\mathbf{x}-\mathbf{B}) = 0 \tag{1.5.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2-3k \end{pmatrix}, \tag{1.5.4}$$

The orthocentre is obtained by solving (1.5.4) and (1.5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T\mathbf{A} \\ \mathbf{m}_2^T\mathbf{B} \end{pmatrix} \tag{1.5.4}$$

which can be expressed using (1.5.4), (1.5.4), (1.5.4) and (1.5.4) as

$$\begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix}\mathbf{x} = \begin{pmatrix} k^2+5k+6k-3k^2 \\ 10k-2k-3k^2 \end{pmatrix}$$
$$= k\begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \tag{1.5.4}$$

From (1.5.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$
$$\implies \begin{vmatrix} k & 5-k & -2k \\ -3k & 4k & 2+3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \tag{1.5.4}$$

resulting in

$$(5-k)(2+3k)+8k^2 = 56 \tag{1.5.4}$$
$$\implies 5k^2+13k-46 = 0 \tag{1.5.4}$$
$$\text{or, } k = 2, -\frac{23}{5} \tag{1.5.4}$$

Substituting the above in (1.5.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2, \tag{1.5.5}$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{0} \tag{1.5.5}$$

If AB be the line in (1.5.5), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.5.5}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0 \tag{1.5.5}$$

since CF passes through the origin and $CF \perp AB$. From (1.5.5) and (1.5.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.5.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

From (1.5.5),

$$\mathbf{C} = -(\mathbf{A}+\mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5.5}$$

after substituting from (1.5.5). Thus,

$$CF = \|\mathbf{C}-\mathbf{F}\| = 3\sqrt{2} \tag{1.5.5}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = 2\sqrt{6} \tag{1.5.5}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 6\sqrt{3} \tag{1.5.5}$$

6. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.
Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and

y-axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4\begin{pmatrix}1\\1\end{pmatrix} \tag{1.5.6}$$

Multiplying (1.5.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{1.5.6}$$

$$\begin{aligned}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) &= 4\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &= 4\begin{pmatrix}\cos\theta - \sin\theta\\ \cos\theta + \sin\theta\end{pmatrix}\end{aligned} \tag{1.5.6}$$

$$\begin{aligned}\implies &\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})\\ &= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3} - 1\right)\end{aligned} \tag{1.5.6}$$

for $\theta = 30^{\circ}$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.

2 The Circle

2.1 Definitions

1. The equation of a circle is

$$\|\mathbf{x} - \mathbf{c}\| = r \tag{2.1.1.1}$$

where $\mathbf{c}$ is the centre and r is the radius.

2. By expanding (2.1.1.1), the equation of a circle can also be expressed as

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.1.2.1}$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} - r^2 = 0 \tag{2.1.2.2}$$

3. Find the equation of the *circumcircle* of $\triangle ABC$ in Fig. 2.1.4.
Solution: Let $\mathbf{O}$ be the centre and R the radius. From (2.1.2.2),

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = R^2 \tag{2.1.3.1}$$

$$\implies \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \tag{2.1.3.2}$$

which can be simplified to obtain

$$(\mathbf{A} - \mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad \text{and} \tag{2.1.3.3}$$

$$(\mathbf{A} - \mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{C}\|^2}{2} \tag{2.1.3.4}$$

Solving the two yields $\mathbf{O}$, which can then be used to obtain R.

4. Given $OD \perp BC$ as in Fig. 2.1.4. Show that

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{2.1.4.1}$$

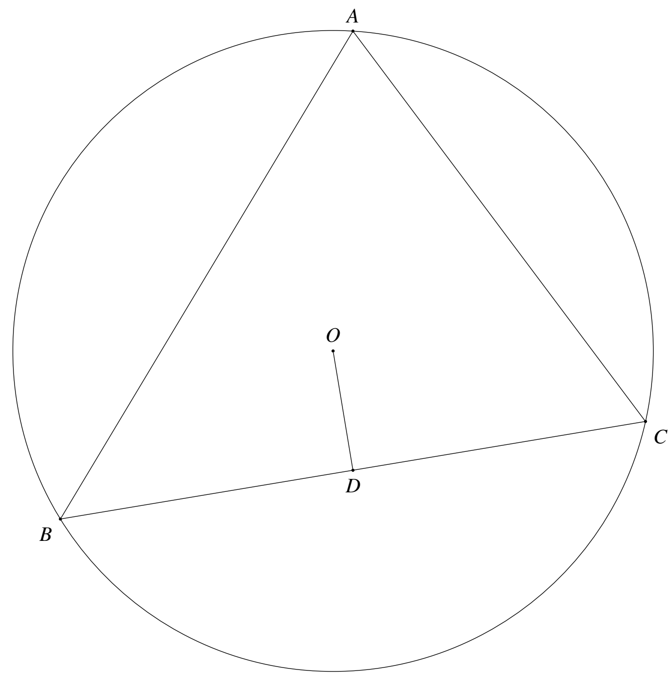

Fig. 2.1.4: Circumcircle.

Solution: From (2.1.1.1)

$$\|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = R^2 \tag{2.1.4.2}$$

$$\implies (\mathbf{B} - \mathbf{O})^T(\mathbf{B} - \mathbf{O}) = (\mathbf{C} - \mathbf{O})^T(\mathbf{C} - \mathbf{O}) \tag{2.1.4.3}$$

$$\implies (\mathbf{B} - \mathbf{C})^T\left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{O}\right) = 0 \tag{2.1.4.4}$$

after simplification. Since $OD \perp BC$,

$$(\mathbf{B} - \mathbf{C})^T(\mathbf{D} - \mathbf{O}) = 0 \tag{2.1.4.5}$$

Since D and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on BC, using (1.2.7.1),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \mathbf{B} + \lambda_1(\mathbf{B} - \mathbf{C}) \tag{2.1.4.6}$$

$$\mathbf{D} = \mathbf{B} + \lambda_2(\mathbf{B} - \mathbf{C}) \tag{2.1.4.7}$$

Multiplying (2.1.4.6) and (2.1.4.7) with

$(\mathbf{B}-\mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$

$$\implies \mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{2.1.4.8}$$

5. Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.
6. The *incircle* with centre $\mathbf{I}$ and radius r in Fig.2.1.6 is inside $\triangle ABC$ and touches AB, BC and CA at $\mathbf{W}, \mathbf{U}$ and $\mathbf{V}$ respectively. AB, BC and CA are known as *tangents* to the circle.

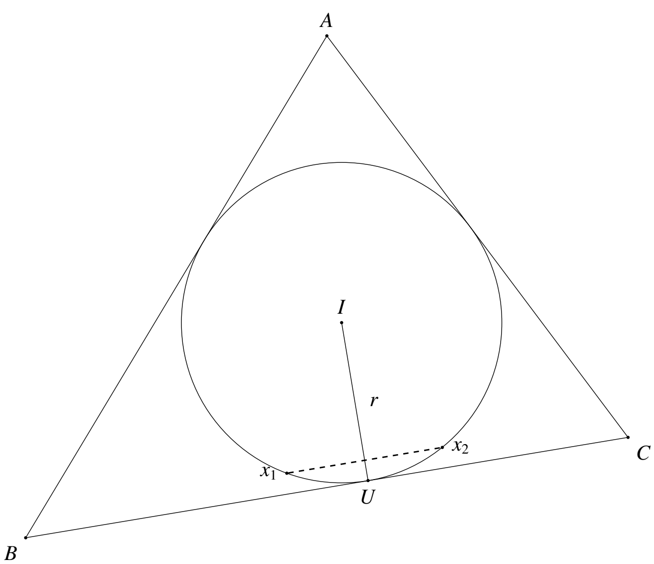

Fig. 2.1.6: Tangent and incircle.

7. Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1-\mathbf{I}\|^2 - \|\mathbf{x}_2-\mathbf{I}\|^2 = 0 \tag{2.1.7.1}$$

$$\implies (\mathbf{x}_1-\mathbf{x}_2)^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{I}\right) = 0 \tag{2.1.7.2}$$

$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{I}\right) = 0 \tag{2.1.7.3}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the above equation becomes

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{U}-\mathbf{I}) = 0 \implies OD \perp BC \tag{2.1.7.4}$$

8. Give an alternative proof for the above.
Solution: Let

$$\mathbf{B} = \mathbf{0} \tag{2.1.8.1}$$

$$\mathbf{U} = \lambda\mathbf{m} \tag{2.1.8.2}$$

Then

$$\|\mathbf{U}-\mathbf{I}\|^2 = r^2 \tag{2.1.8.3}$$

$$\implies \lambda^2\|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{I} + \|\mathbf{I}\|^2 = r^2 \tag{2.1.8.4}$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T\mathbf{I}}{\|\mathbf{m}\|^2} \tag{2.1.8.5}$$

Thus,

$$(\mathbf{U}-\mathbf{B})^T(\mathbf{U}-\mathbf{I}) = (\lambda\mathbf{m})^T(\lambda\mathbf{m}-\mathbf{I}) \tag{2.1.8.6}$$

$$= \lambda^2\|\mathbf{m}\|^2 - \lambda\mathbf{m}^T\mathbf{I} \tag{2.1.8.7}$$

$$= \mathbf{O} \text{ (from 2.1.8.5).} \tag{2.1.8.8}$$

$$\implies OD \perp BC \tag{2.1.8.9}$$

9. Find the equation of the tangent at $\mathbf{U}$.
Solution: The equation of the tangent is given by

$$(\mathbf{I}-\mathbf{U})^T(\mathbf{x}-\mathbf{U}) = 0 \tag{2.1.9.1}$$

10. The direction vector of *normal to the circle* in (2.1.2.2) at point $\mathbf{U}$ is

$$\mathbf{n} = \mathbf{U}-\mathbf{I} \tag{2.1.10.1}$$

11. Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \tag{2.1.11.1}$$

$$\implies \mathbf{n}^T(\mathbf{U}-\mathbf{B}) = 0 \tag{2.1.11.2}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{2.1.11.3}$$

$$\implies \mathbf{I}-\mathbf{U} = \lambda\mathbf{n} \tag{2.1.11.4}$$

$$\text{or} \quad r = \|\mathbf{I}-\mathbf{U}\| = |\lambda|\,\|\mathbf{n}\| \tag{2.1.11.5}$$

From (2.1.11.2) and (2.1.11.3)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{2.1.11.6}$$

$$\implies \mathbf{n}^T(\mathbf{I}-\mathbf{B}) = \lambda\|\mathbf{n}\|^2 \tag{2.1.11.7}$$

$$\implies r = |\lambda|\,\|\mathbf{n}\| = \frac{\left|\mathbf{n}^T(\mathbf{I}-\mathbf{B})\right|}{\|\mathbf{n}\|} \tag{2.1.11.8}$$

from (2.1.11.5). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{2.1.11.9}$$

$$r = \left|\mathbf{n}_1^T(\mathbf{I}-\mathbf{B})\right| \tag{2.1.11.10}$$

12. Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (2.1.11.10),

$$\left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| = \left|\mathbf{n}_2^T (\mathbf{I} - \mathbf{C})\right| = \left|\mathbf{n}_3^T (\mathbf{I} - \mathbf{A})\right| \quad (2.1.12.1)$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (2.1.12.1) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1 \mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \quad (2.1.12.2)$$
$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2 \mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \quad (2.1.12.3)$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3\end{pmatrix}^T \mathbf{I} = \begin{pmatrix}\mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A}\end{pmatrix} \quad (2.1.12.4)$$

13. Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$
14. Let $a = BC, b = CA$, $c = AB$, $x = BU = BW$, $y = CU = CV$, $z = AV = AW$. Find $\mathbf{U}$.
Solution: It is easy to verify that

$$\begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x \\ y \\ z\end{pmatrix} = \begin{pmatrix}a \\ b \\ c\end{pmatrix} \quad (2.1.14.1)$$

which can be used to obtain x and y. Using the section formula,

$$\mathbf{U} = \frac{x\mathbf{B} + \mathbf{A}}{x + y} \quad (2.1.14.2)$$

15. Show that the angle in a semi-circle is a right angle.

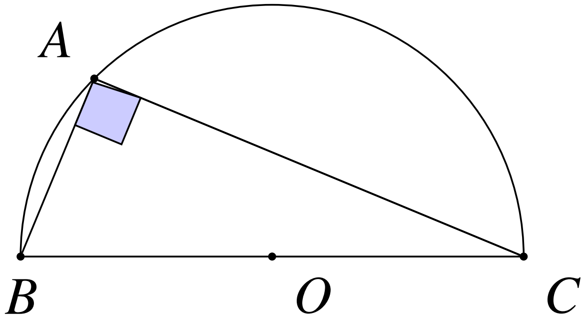

Fig. 2.1.15: Angle in a semi-circle.

Solution: Let

$$\mathbf{O} = 0 \quad (2.1.15.1)$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \quad (2.1.15.2)$$
$$\|\mathbf{B} - \mathbf{C}\|^2 = (2r)^2 \quad (2.1.15.3)$$
$$\mathbf{B} + \mathbf{C} = 0 \quad (2.1.15.4)$$

where r is the radius of the circle. Thus,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T(\mathbf{B} + \mathbf{C}) \quad (2.1.15.5)$$

From (2.1.15.4) and (2.1.15.2),

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 4r^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (2.1.15.6)$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

16. Show that $PA.PB = PC^2$, where PC is the tangent to the circle in Fig. 2.1.16.

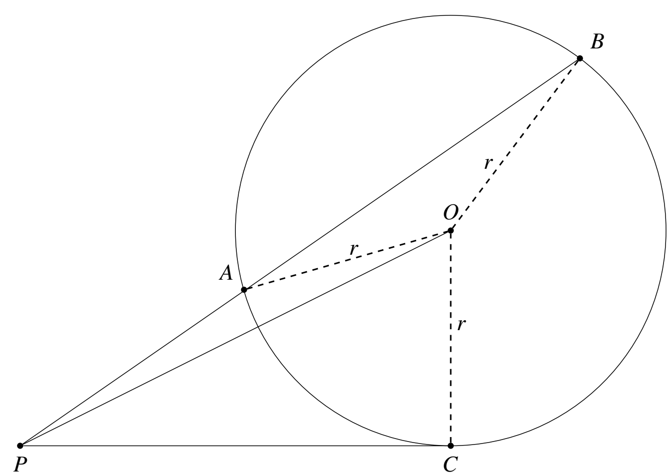

Fig. 2.1.16: $PA.PB = PC^2$.

Solution: Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda\|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda\mathbf{A}) \quad (2.1.16.1)$$

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \quad (2.1.16.2)$$

$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \quad (2.1.16.3)$$

$\because$

$$\|\mathbf{B}-\mathbf{O}\|^2-\|\mathbf{A}-\mathbf{O}\|^2=0, \tag{2.1.16.4}$$

$$\left(\lambda^2-1\right)\|\mathbf{A}\|^2-2\left(\lambda-1\right)\mathbf{A}^T\mathbf{O}=0 \tag{2.1.16.5}$$

$$\implies PA.PB=\lambda\|\mathbf{A}\|^2=2\mathbf{A}^T\mathbf{O}-\|\mathbf{A}\|^2 \tag{2.1.16.6}$$

after substituting from (2.1.16.1) and simplifying. From (2.1.16.3),

$$\|\mathbf{A}-\mathbf{O}\|^2=\|\mathbf{O}\|^2-\|\mathbf{C}\|^2=r^2 \tag{2.1.16.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O}-\|\mathbf{A}\|^2=\|\mathbf{C}\|^2=PC^2 \tag{2.1.16.8}$$

From (2.1.16.6) and (2.1.16.8),

$$PA.PB=PC^2 \tag{2.1.16.9}$$

17. In Fig. 2.1.17 show that $PA.PB=PC.PD$.

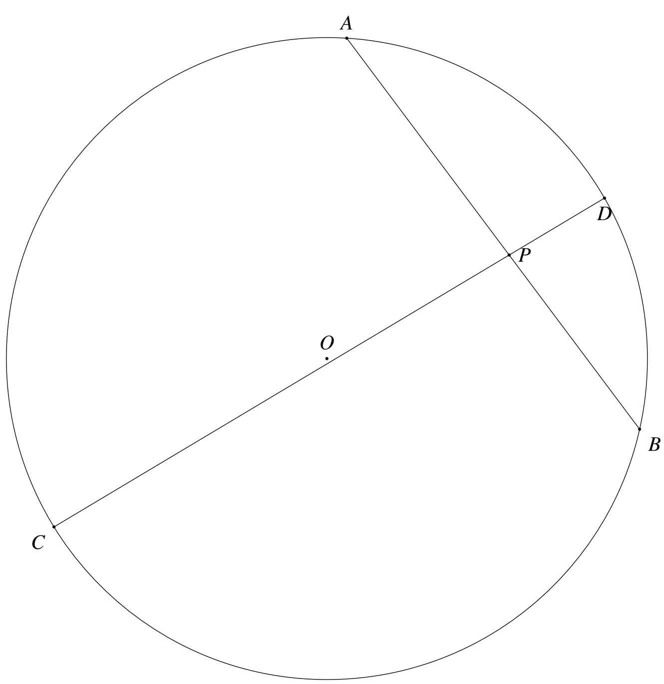

Fig. 2.1.17: Chords of a circle

Solution: Let $\mathbf{P}=\mathbf{0}$. We then have the following equations

$$\mathbf{B}=k_1\mathbf{A}, k_1=\frac{PB}{PA}$$
$$\mathbf{D}=k_2\mathbf{C}, k_2=\frac{PD}{PC} \tag{2.1.17.1}$$

$$\|\mathbf{A}-\mathbf{O}\|^2=\|\mathbf{B}-\mathbf{O}\|^2$$
$$=\|\mathbf{C}-\mathbf{O}\|^2=\|\mathbf{D}-\mathbf{O}\|^2=r^2 \tag{2.1.17.2}$$

where r is the radius of the circle and $\mathbf{O}$ is the centre. From (2.1.17.2),

$$\|\mathbf{A}-\mathbf{O}\|^2=\|\mathbf{B}-\mathbf{O}\|^2 \tag{2.1.17.3}$$

$$\implies \|\mathbf{A}-\mathbf{O}\|^2=\|k\mathbf{A}-\mathbf{O}\|^2 \quad (\text{from } (2.1.17.1)) \tag{2.1.17.4}$$

which can be simplified to obtain

$$k_1\|\mathbf{A}\|^2=2\mathbf{A}^T\mathbf{O}-\|\mathbf{A}\|^2 \tag{2.1.17.5}$$

Similarly,

$$k_2\|\mathbf{C}\|^2=2\mathbf{C}^T\mathbf{O}-\|\mathbf{C}\|^2 \tag{2.1.17.6}$$

From (2.1.17.2), we also obtain

$$\|\mathbf{A}-\mathbf{O}\|^2=\|\mathbf{C}-\mathbf{O}\|^2 \tag{2.1.17.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O}-\|\mathbf{A}\|^2=2\mathbf{C}^T\mathbf{O}-\|\mathbf{C}\|^2 \tag{2.1.17.8}$$

after simplification. Using this result in (2.1.17.5) and (2.1.17.6),

$$k_1\|\mathbf{A}\|^2=k_2\|\mathbf{C}\|^2 \tag{2.1.17.9}$$

$$\implies \|\mathbf{A}\|\,\|\mathbf{B}\|=\|\mathbf{C}\|\,\|\mathbf{D}\| \tag{2.1.17.10}$$

which completes the proof.

18. (Pole and Polar:) The polar of a point $\mathbf{x}$ with respect to the curve

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}+f=0 \tag{2.1.18.1}$$

is the line

$$\mathbf{n}^T\mathbf{x}=c \tag{2.1.18.2}$$

where

$$\begin{pmatrix}\mathbf{n}^T\\-c\end{pmatrix}=\begin{pmatrix}\mathbf{V}&\mathbf{u}\\\mathbf{u}^T&f\end{pmatrix}\begin{pmatrix}\mathbf{x}\\1\end{pmatrix} \tag{2.1.18.3}$$

The pole of the line in (2.1.18.2) is obtained as $\frac{1}{x_3}\begin{pmatrix}x_1\\x_2\end{pmatrix}$, where

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{n}^T\\ -c\end{pmatrix} \tag{2.1.18.4}$$

19. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be conjugate points for (2.1.18.1) if $\mathbf{x}_2$ lies on the polar of $\mathbf{x}_1$ and vice-versa. A similar definition holds for conjugate lines as well.
20. Let $\mathbf{p}$ be a point of intersection of two circles with centres $\mathbf{c}_1$ and $\mathbf{c}_2$. The circles are said to be orthogonal if their tangents at $\mathbf{p}$ are perpendicular to each other. Show that if r_1 and r_2 are their respective radii,

$$\left\|\mathbf{c_1} - \mathbf{c}_2\right\|^2 = r_1^2 + r_2^2 \tag{2.1.20.1}$$

21. Show that the length of the tangent from a point $\mathbf{p}$ to the circle

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{2.1.21.1}$$

is

$$\mathbf{p}^T\mathbf{p} - 2\mathbf{c}^T\mathbf{p} + f \tag{2.1.21.2}$$

This length is also known as the *power* of the point $\mathbf{p}$ with respect to the circle.
22. The *radical axis* of the circles

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 = 0 \tag{2.1.22.1}$$
$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.2}$$

is the locus of the points from which lengths of the tangents to the circles are equal. From (2.1.25), this locus is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{2.1.22.3}$$
$$\implies 2\left(\mathbf{c}_1 - \mathbf{c}_2\right)^T\mathbf{x} + f_2 - f_1 = 0 \tag{2.1.22.4}$$

23. Show that the radical axis of the circles is perpendicular to the line joining their centres.
24. *Coaxal circles* have the same radical axis.
25. Obtain a family of coaxal circles from and find their *limit points*.
Solution: The family of circles is obtained as

$$\mathbf{x}^T\mathbf{x} - 2\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\mathbf{x} + f_1 + \lambda f_2 = 0 \tag{2.1.25.1}$$

The limit points are the centres of those circles whose radii are 0. From (2.1.25.1) and (2.1.2.2), this results in

$$f_1 + \lambda f_2 = \left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right) \tag{2.1.25.2}$$
$$\implies \lambda^2\left\|\mathbf{c}_2\right\|^2 + \lambda\left(2\mathbf{c}_1^T\mathbf{c}_2 - f_2\right) + \left\|\mathbf{c}_1\right\|^2 - f_1 = 0 \tag{2.1.25.3}$$

Solving for λ, the limit points are given by

$$\mathbf{c}_1 + \lambda\mathbf{c}_2 \tag{2.1.25.4}$$

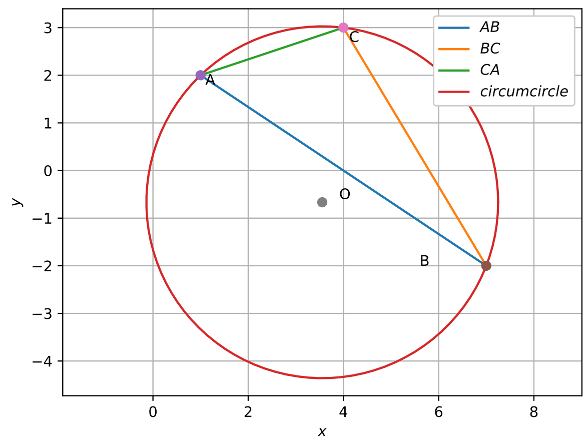

Fig. 2.2.2

2.2 *Programming*

1. Find the circumcentre $\mathbf{O}$ and radius R of $\triangle ABC$ in Fig. 1.3.2
Solution: $\mathbf{O}$ can be obtained from The following code computes $\mathbf{O}$ using (2.1.3.3) and (2.1.3.4)

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/book/codes/
    circumcentre.py
```

2. Plot the circumcircle of $\triangle ABC$.
Solution: The following code plots Fig. 2.2.2

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/book/codes/
    circumcircle.py
```

3. Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.
4. Compute $\mathbf{I}$ and r.
Solution: The following code uses (2.1.12.4) and (2.1.11.10) to compute $\mathbf{I}$ and r respectively.

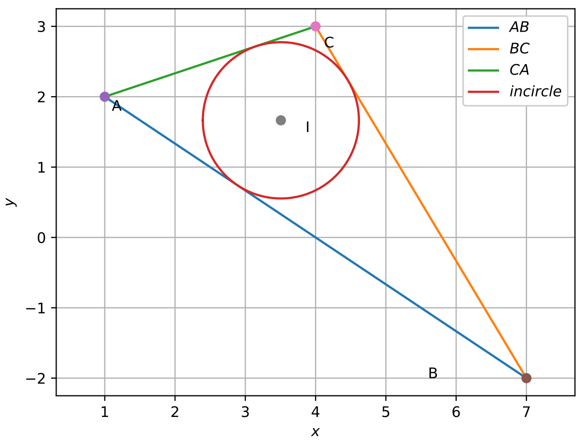

Fig. 2.2.5

https://raw.githubusercontent.com/gadepall/school/master/linalg/book/codes/incentre.py

5. Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 2.2.5

https://raw.githubusercontent.com/gadepall/school/master/linalg/book/codes/incircle.py

2.3 Example

1. Find the centre and radius of the circle
$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{2.3.1.1}$$
Solution: let $\mathbf{c}$ be the centre of the circle. Then
$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \tag{2.3.1.2}$$
$$\implies (\mathbf{x} - \mathbf{c})^T(\mathbf{x} - \mathbf{c}) = r^2 \tag{2.3.1.3}$$
$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} = r^2 - \mathbf{c}^T\mathbf{c} \tag{2.3.1.4}$$
Comparing with (2.3.1.1),
$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.3.1.5}$$
$$r^2 - \mathbf{c}^T\mathbf{c} = 1 \implies r = \sqrt{2} \tag{2.3.1.6}$$
2. Find the tangent to the circle C_1 at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Solution: From (3.1.6), the tangent T is given by
$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right]\mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{2.3.2.1}$$
$$\implies T : \mathbf{n}^T\mathbf{x} = 3 \tag{2.3.2.2}$$
where
$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3.2.3}$$
3. The tangent T in (2.3.2.2) cuts off a chord AB from a circle C_2 whose centre is
$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{2.3.3.1}$$
Find $\mathbf{A} + \mathbf{B}$.
Solution: Let the radius of C_2 be r. From the given information,
$$(\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{C}) = r^2 \tag{2.3.3.2}$$
$$(\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = r^2 \tag{2.3.3.3}$$
Subtracting (2.3.3.3) from (2.3.3.2),
$$\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} - 2\mathbf{C}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.4}$$
$$\implies (\mathbf{A} + \mathbf{B})^T(\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T(\mathbf{A} - \mathbf{B}) = 0$$
$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T(\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.5}$$
$\because$ $\mathbf{A}, \mathbf{B}$ lie on T, from (2.3.2.2),
$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = 3 \tag{2.3.3.6}$$
$$\implies \mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0, \tag{2.3.3.7}$$
From (2.3.3.5) and (2.3.3.7)
$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{2.3.3.8}$$
$$\implies \mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C} = k\mathbf{n}^T\mathbf{n} \tag{2.3.3.9}$$
$$\implies \frac{\mathbf{n}^T\mathbf{A} + \mathbf{n}^T\mathbf{B} - 2\mathbf{n}^T\mathbf{C}}{\mathbf{n}^T\mathbf{n}} = k \tag{2.3.3.10}$$
$$\implies k = 2 \tag{2.3.3.11}$$
using (2.3.3.6). Substituting in (2.3.3.8)
$$\mathbf{A} + \mathbf{B} = 2(\mathbf{n} + \mathbf{C}) \tag{2.3.3.12}$$
4. If $AB = 4$, find $\mathbf{A}^T\mathbf{B}$.
Solution: From the given information,
$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{2.3.4.1}$$

resulting in

$$\|\mathbf{A}+\mathbf{B}\|^2 - \|\mathbf{A}-\mathbf{B}\|^2 = 4\,\|\mathbf{n}+\mathbf{C}\|^2 - 4^2 \tag{2.3.4.2}$$

$$\implies \mathbf{A}^T\mathbf{B} = \|\mathbf{n}+\mathbf{C}\|^2 - 4 = 17 \tag{2.3.4.3}$$

using (2.3.3.12) and simplifying.

5. Show that

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 8 - r^2 \tag{2.3.5.1}$$

Solution:

$$\|\mathbf{A}-\mathbf{B}\|^2 = 4^2 \tag{2.3.5.2}$$

$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) = 4^2 \tag{2.3.5.3}$$

From (2.3.5.3),

$$[(\mathbf{A}-\mathbf{C})-(\mathbf{B}-\mathbf{C})]^T\;[(\mathbf{A}-\mathbf{C})-(\mathbf{B}-\mathbf{C})] = 4^2 \tag{2.3.5.4}$$

which can be expressed as

$$\|\mathbf{A}-\mathbf{C}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 + 2\,(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 4^2 \tag{2.3.5.5}$$

Upon substituting from (2.3.3.3) and (2.3.3.2) and simplifying, (2.3.5.1) is obtained.

6. Find r.
Solution: (2.3.5.1) can be expressed as

$$\mathbf{A}^T\mathbf{B} - \mathbf{C}^T(\mathbf{A}+\mathbf{B}) + \mathbf{C}^T\mathbf{C} = 8 - r^2 \tag{2.3.6.1}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T(\mathbf{A}+\mathbf{B}) - \mathbf{C}^T\mathbf{C} = r^2 \tag{2.3.6.2}$$

$$\implies 8 - \mathbf{A}^T\mathbf{B} + \mathbf{C}^T(2\mathbf{n}+\mathbf{C}) = r^2 \tag{2.3.6.3}$$

$$\implies r = \sqrt{6}. \tag{2.3.6.4}$$

7. Summarize all the above computations through a Python script and plot the tangent and circle.
Solution: The following code generates Fig. 2.3.7.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/circle/codes/circ.py
```

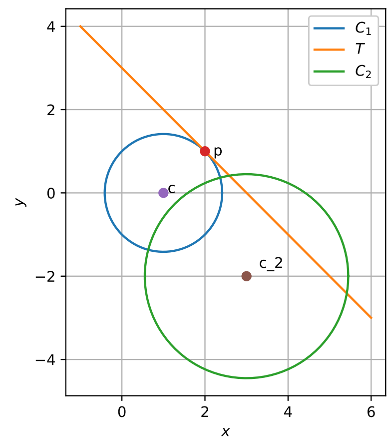

Fig. 2.3.7

2.4 Exercises

1. A circle passes through the points $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}4\\5\end{pmatrix}$. If its centre $\mathbf{O}$ lies on the line

$$\begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.4.1.1}$$

find its radius.
Solution: Let

$$\mathbf{C} = \frac{\mathbf{A}+\mathbf{B}}{2} \implies \mathbf{C} = \begin{pmatrix}3\\4\end{pmatrix} \tag{2.4.1.2}$$

The direction vector of AB is

$$\mathbf{m} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{2.4.1.3}$$

$$\because OC \perp AB,$$

$$OC : \mathbf{m}^T(\mathbf{x}-\mathbf{C}) = 0$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{2.4.1.4}$$

Thus, $\mathbf{O}$ is the intersection of (2.4.1.1) and (2.4.1.4) and is the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\-1 & 4\end{pmatrix}\mathbf{x} = \begin{pmatrix}7\\3\end{pmatrix} \tag{2.4.1.5}$$

From the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 7 \\ -1 & 4 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 7 \\ 0 & 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\implies \mathbf{O} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.4.1.6}$$

Thus the radius of the circle

$$OA = \|\mathbf{O} - \mathbf{A}\| = \sqrt{10} \tag{2.4.1.7}$$

2. If a circle C_1, whose radius is 3, touches externally the circle

$$C_2 : \mathbf{x}^T\mathbf{x} + \begin{pmatrix} 2 & -4 \end{pmatrix}\mathbf{x} = 4 \tag{2.4.2.1}$$

at the point $\mathbf{P} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, then find the length of the intercept cut by this circle C on the x-axis.
Solution: From (2.4.2.1), the centre of C_2 is

$$\mathbf{O}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.4.2.2}$$

The radius of the circle is given by

$$r_2^2 - \mathbf{O}_2^T\mathbf{O}_2 = 4 \implies r_2 = 3 \tag{2.4.2.3}$$

Since the radius of C_1 is $r_1 = r_2 = 3$ and $\mathbf{O}_1, \mathbf{P}, \mathbf{O}_2$ are collinear,

$$\frac{\mathbf{O}_1 + \mathbf{O}_2}{2} = \mathbf{P}$$

$$\implies \mathbf{O}_1 = 2\mathbf{P} - \mathbf{O}_2$$

$$\implies \mathbf{O}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{2.4.2.4}$$

The intercepts of C_1 on the $x-$axis can be expressed as

$$\mathbf{x} = \lambda\mathbf{m} \tag{2.4.2.5}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4.2.6}$$

Susbtituting in the equation for C_1,

$$\|\lambda\mathbf{m} - \mathbf{O}_1\|^2 = r_1^2 \tag{2.4.2.7}$$

which can be expressed as

$$\lambda^2 \|\mathbf{m}\|^2 - 2\lambda\mathbf{m}^T\mathbf{O}_1 + \|\mathbf{O}_1\|^2 - r_1^2 = 0$$

$$\implies \lambda^2 - 10\lambda + 20 = 0 \tag{2.4.2.8}$$

resulting in

$$\lambda = 5 \pm \sqrt{5} \tag{2.4.2.9}$$

after substituting from (2.4.2.6) and (2.4.2.4).

3. A line drawn through the point

$$\mathbf{P} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{2.4.3.1}$$

cuts the circle

$$C : \mathbf{x}^T\mathbf{x} = 9 \tag{2.4.3.2}$$

at the points $\mathbf{A}$ and $\mathbf{B}$. Find $PA.PB$. Draw PAB for any two points $\mathbf{A}, \mathbf{B}$ on the circle.
Solution: Since the points $\mathbf{P}, \mathbf{A}, \mathbf{B}$ are collinear, the line PAB can be expressed as

$$L : \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{2.4.3.3}$$

for $\|\mathbf{m}\| = 1$. The intersection of L and C yields

$$(\mathbf{P} + \lambda\mathbf{m})^T (\mathbf{P} + \lambda\mathbf{m}) = 9$$

$$\implies \lambda^2 + 2\lambda\mathbf{m}^T\mathbf{P} + \|\mathbf{P}\|^2 - 9 = 0 \tag{2.4.3.4}$$

The product of the roots in (2.4.3.4) is

$$PA.PB = \|\mathbf{P}\|^2 - 9 = 56 \tag{2.4.3.5}$$

4. Find the equation of the circle C_2, which is the mirror image of the circle

$$C_1 : \mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{2.4.4.1}$$

in the line

$$L : \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3. \tag{2.4.4.2}$$

Solution: From (2.4.4.1), circle C_1 has centre at

$$\mathbf{O}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4.4.3}$$

and radius

$$r_1 = \mathbf{O}_1^T\mathbf{O}_1 = 1 \tag{2.4.4.4}$$

The centre of C_2 is the reflection of $\mathbf{O}_1$ about L and is obtained as

$$\frac{\mathbf{O}_2}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{O}_1 + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.4.4.5}$$

where the relevant parameters are obtained from (2.4.4.2) as

$$\mathbf{n} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 & -1 \end{pmatrix}, c = 3. \tag{2.4.4.6}$$

Substituting the above in (2.4.4.5),

$$\frac{\mathbf{O}_2}{2} = \frac{\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}{4}\mathbf{O}_1 + c\frac{\mathbf{n}}{2}$$

$$\implies \mathbf{O}_2 = \begin{pmatrix}3\\ 4\end{pmatrix} \tag{2.4.4.7}$$

Thus

$$C_2 : \left\|\mathbf{x} - \begin{pmatrix}3\\ 4\end{pmatrix}\right\| = 1 \tag{2.4.4.8}$$

5. One of the diameters of the circle, given by

$$C : \mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 12 \tag{2.4.5.1}$$

is a chord of a circle S, whose centre is at

$$\mathbf{O}_2 = \begin{pmatrix}-3\\ 2\end{pmatrix}. \tag{2.4.5.2}$$

Find the radius of S.
Solution: From (2.4.5.1), the centre of C is

$$\mathbf{O}_1 = \begin{pmatrix}2\\ -3\end{pmatrix} \tag{2.4.5.3}$$

and the radius is

$$r_1 = \sqrt{\mathbf{O}_1^T\mathbf{O}_1 - 12} = 5 \tag{2.4.5.4}$$

From (2.4.5.3) and (2.4.5.2),

$$O_1O_2 = \|\mathbf{O}_1 - \mathbf{O}_2\| = 5\sqrt{2}$$

$$\implies r_2 = \sqrt{O_1O_2^2 - r_1^2} = 5 \tag{2.4.5.5}$$

6. A circle C passes through

$$\mathbf{P} = \begin{pmatrix}-2\\ 4\end{pmatrix} \tag{2.4.6.1}$$

and touches the y-axis at

$$\mathbf{Q} = \begin{pmatrix}0\\ 2\end{pmatrix}. \tag{2.4.6.2}$$

Which one of the following equations can represent a diameter of this circle?

(i) $\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 6$ (iii) $\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 3$

(ii) $\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} + 10 = 0$(iv) $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} + 4 = 0$

Solution: Let $\mathbf{O}$ be the centre of C. Then the equation of the normal, OQ is

$$\begin{pmatrix}0 & 1\end{pmatrix}(\mathbf{O} - \mathbf{Q}) = 0$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{O} = 2 \tag{2.4.6.3}$$

Also,

$$\|\mathbf{O} - \mathbf{P}\|^2 = \|\mathbf{O} - \mathbf{Q}\|^2$$

$$\implies 2(\mathbf{P} - \mathbf{Q})^T\mathbf{O} = \|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2$$

$$\text{or, } \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{O} = -4 \tag{2.4.6.4}$$

(2.4.6.3) and (2.4.6.4) result in the matrix equation

$$\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\mathbf{O} = \begin{pmatrix}-4\\ 2\end{pmatrix} \tag{2.4.6.5}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & -4\\ 0 & 1 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 2\end{pmatrix} \implies \mathbf{O} = \begin{pmatrix}-2\\ 2\end{pmatrix} \tag{2.4.6.6}$$

Hence, option ii) is correct.

7. Find the equation of the tangent to the circle, at the point

$$\mathbf{P} = \begin{pmatrix}1\\ -1\end{pmatrix}, \tag{2.4.7.1}$$

whose centre $\mathbf{O}$ is the point of intersection of the straight lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.4.7.2}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{2.4.7.3}$$

Solution: From (2.4.7.2) and (2.4.7.3), we obtain the matrix equation

$$\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\mathbf{O} = \begin{pmatrix}3\\ 1\end{pmatrix} \tag{2.4.7.4}$$

yielding the augmented matrix

$$\begin{pmatrix}1 & -1 & 1\\ 2 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 1\\ 0 & 3 & 1\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}3 & 0 & 4\\ 0 & 3 & 1\end{pmatrix} \implies \mathbf{O} = \frac{1}{3}\begin{pmatrix}4\\ 1\end{pmatrix} \tag{2.4.7.5}$$

Thus, the equation of the desired tangent is

$$(\mathbf{O} - \mathbf{P})^T(\mathbf{x} - \mathbf{P}) = 0$$

$$\implies \begin{pmatrix}1 & 4\end{pmatrix}\mathbf{x} = -3 \tag{2.4.7.6}$$

8. The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.4.8.1}$$

intersects the circle

$$\Omega : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{2.4.8.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{R} = -\frac{3}{5} \tag{2.4.8.3}$$

Find m.

Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.4.8.4}$$

The intersection of (2.4.8.1) and (2.4.8.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{2.4.8.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T(\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \tag{2.4.8.6}$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1\mathbf{m} \tag{2.4.8.7}$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2\mathbf{m} \tag{2.4.8.8}$$

$$\implies \frac{\mathbf{P} + \mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2}\mathbf{m} \tag{2.4.8.9}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P} + \mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{m} \tag{2.4.8.10}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{c} - \frac{\mathbf{m}^T(\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \tag{2.4.8.11}$$

using the sum of roots in (2.4.8.6). From (2.4.8.3) and (2.4.8.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5}\left(1 + m^2\right) \tag{2.4.8.12}$$

$$\implies m^2 - 5m + 6 = 0 \tag{2.4.8.13}$$

$$\implies m = 2 \text{ or } 3 \tag{2.4.8.14}$$

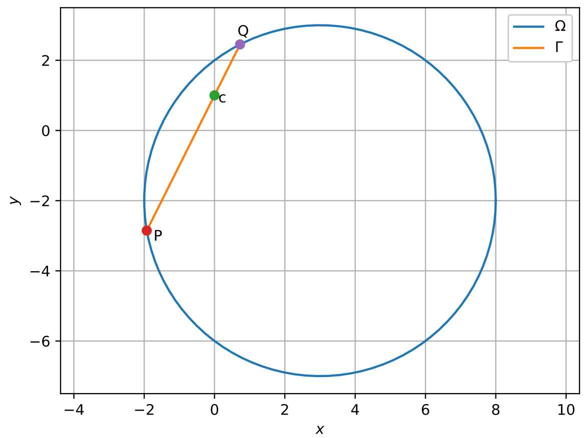

Fig. 2.4.8

From (2.4.8.6),

$$\lambda = \frac{-\mathbf{m}^T(\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{\left(\mathbf{m}^T(\mathbf{c} - \mathbf{O})\right)^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{2.4.8.15}$$

Fig. 2.4.8 summarizes the solution for $m = 2$.

3 Conics

3.1 Definitions

1. From (1.2.19.2), the equation of a conic section is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.1.1.1}$$

for

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} \neq 0 \tag{3.1.1.2}$$

2. Show that

$$\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \tag{3.1.2.1}$$

3. Show that

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T\mathbf{x} \tag{3.1.3.1}$$

4. Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \tag{3.1.4.1}$$

5. Find the *normal* vector to the curve in (1.2.19.2) at point $\mathbf{p}$.
Solution: Differentiating (1.2.19.2) with respect to x_1,
$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} + \frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} = 0 \tag{3.1.5.1}$$
$$\implies 2\mathbf{x}^T\mathbf{V}\mathbf{m} + 2\mathbf{u}^T\mathbf{m} = 0 \because \left(\frac{d\mathbf{x}}{dx_1} = \mathbf{m}\right) \tag{3.1.5.2}$$
Substituting $\mathbf{x} = \mathbf{p}$ and simplifying
$$\left(\mathbf{V}\mathbf{p} + \mathbf{u}\right)^T \mathbf{m} = 0 \tag{3.1.5.3}$$
$$\implies \mathbf{n} = \mathbf{V}\mathbf{p} + \mathbf{u} \tag{3.1.5.4}$$
6. The *tangent* to the curve at $\mathbf{p}$ is given by
$$\mathbf{n}^T\left(\mathbf{x} - \mathbf{p}\right) = 0 \tag{3.1.6}$$
This results in
$$\left(\mathbf{p}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{x} + \mathbf{p}^T\mathbf{u} + f = 0 \tag{3.1.6}$$
7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then
$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.1.7}$$
is known as an *affine* transformation.
8. Classify the various conic sections based on (1.2.19.2).
Solution:

Curve	**Property**
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 3.1.8

3.2 Parabola

1. Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} + 6 = 0 \tag{3.2.1.1}$$
Solution: Substituting
$$\mathbf{p} = \begin{pmatrix}1\\7\end{pmatrix}, V = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \tag{3.2.1.2}$$
in (3.1.6), the desired equation is
$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} + 6 = 0 \tag{3.2.1.3}$$
resulting in
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -5 \tag{3.2.1.4}$$
2. The line in (3.2.1.4) touches the circle
$$\mathbf{x}^T\mathbf{x} + 4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + c = 0 \tag{3.2.2.1}$$
Find c.
Solution: Comparing (1.2.19.2) and (3.2.2.1),
$$V = I, \quad \mathbf{u} = 2\begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.2.2}$$
Comparing (3.1.6) and (3.2.1.4),
$$\mathbf{p} + 2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{3.2.2.3}$$
$$\implies \mathbf{p} = -\begin{pmatrix}6\\7\end{pmatrix} \tag{3.2.2.4}$$
and
$$c + \mathbf{p}^T\mathbf{u} = 5 \tag{3.2.2.5}$$
$$\implies c = 5 + 2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.2.6}$$
$$= 95 \tag{3.2.2.7}$$
3. Summarize all the above computations through a Python script and plot the parabola, tangent and circle.
Solution: The following code generates Fig. 3.2.3.

```
wget
https://github.com/gadepall/school/raw/master/
    linalg/book/codes/parab.py
```

3.3 Affine Transformation

1. In general, Fig. 3.2.3 was generated using an *affine transformation*.
2. Express
$$y_2 = y_1^2 \tag{3.3.2.1}$$

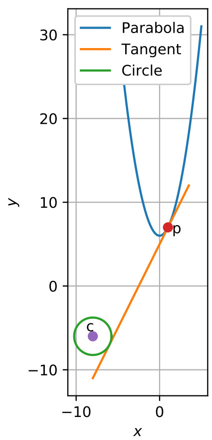

Fig. 3.2.3

as a matrix equation.
Solution: (3.3.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{g}^T\mathbf{y} = 0 \tag{3.3.2.2}$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.2.3}$$

3. Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0, \tag{3.3.3.1}$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \tag{3.3.3.2}$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.3.3.3}$$

(3.3.3.3) is known as an affine transformation. Show that

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P}$$
$$\mathbf{g} = \mathbf{P}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) \tag{3.3.3.4}$$
$$F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0$$

Solution: Substituting (3.3.3.3) in (3.3.3.1),

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + F = 0, \tag{3.3.3.5}$$

which can be expressed as

$$\implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \tag{3.3.3.6}$$

Comparing (3.3.3.6) with (3.3.2.2) (3.3.3.4) is obtained.

4. Show that there exists a $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{3.3.4.1}$$

Find $\mathbf{P}$ using

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.3.4.2}$$

5. Find $\mathbf{c}$ from (3.3.3.4).
Solution:

$$\because \mathbf{g} = \mathbf{P}^T(\mathbf{V}\mathbf{c} + \mathbf{u}), \tag{3.3.5.1}$$
$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{3.3.5.2}$$
$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T(\mathbf{P}\mathbf{g} - \mathbf{u}) = -F - 2\mathbf{u}^T\mathbf{c} \tag{3.3.5.3}$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ (\mathbf{P}\mathbf{g} + \mathbf{u})^T \end{pmatrix}\mathbf{c} = \begin{pmatrix} \mathbf{P}\mathbf{g} - \mathbf{u} \\ -F \end{pmatrix} \tag{3.3.5.4}$$

for computing $\mathbf{c}$.

3.4 Ellipse: Eigenvalues and Eigenvectors

1. Express the following equation in the form given in (1.2.19.2)

$$E: 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{3.4.1.1}$$

Solution: (3.4.1.1) can be expressed as

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{3.4.1.2}$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \tag{3.4.1.3}$$

2. Using the affine transformation in (3.3.3.3), show that (3.4.1.2) can be expressed as

$$\mathbf{y}^T D\mathbf{y} = 1 \tag{3.4.2.1}$$

where

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.2.2}$$
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.2.3}$$

for

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I} \tag{3.4.2.4}$$

3. Find $\mathbf{c}$

Solution:

$$\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.3.1}$$

4. If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.4.4.1}$$

$$P = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{3.4.4.2}$$

show that

$$V\mathbf{z} = \lambda\mathbf{z} \tag{3.4.4.3}$$

where $\lambda \in \{\lambda_1, \lambda_2\}, \mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$.

5. Find λ.
Solution: λ is obtained by solving the following equation.

$$|\lambda I - V| = 0 \tag{3.4.5.1}$$

$$\implies \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \tag{3.4.5.2}$$

$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{3.4.5.3}$$

$$\implies \lambda = 2, 8 \tag{3.4.5.4}$$

6. Sketch 3.4.2.1.
7. Find $\mathbf{P}_1$ and $\mathbf{P}_2$.
Solution: From (3.4.4.3)

$$V\mathbf{P}_1 = \lambda_1\mathbf{P}_1 \tag{3.4.7.1}$$

$$\implies (V - \lambda I)\,\mathbf{y} = 0 \tag{3.4.7.2}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.7.3}$$

$$\text{or, } \mathbf{P}_1 = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.4.7.4}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{P}_2 = 0 \tag{3.4.7.5}$$

$$\text{or, } \mathbf{P}_2 = k_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.4.7.6}$$

8. Find $\mathbf{P}$.
Solution: From (3.4.2.4) and (3.4.4.2),

$$k_1 = \frac{1}{\left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.1}$$

$$k_2 = \frac{1}{\left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{2}} \tag{3.4.8.2}$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.4.8.3}$$

9. Find the equation of the major axis for E.
Solution: The major axis for (3.4.2.1) is the line

$$\mathbf{y} = \lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.4.9.1}$$

Using the affine transformation in (3.3.3.3)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.9.2}$$

$$\implies \mathbf{x} - \mathbf{c} = \lambda_1\mathbf{P}_1 \tag{3.4.9.3}$$

$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.4.9.4}$$

$$= -3 \tag{3.4.9.5}$$

since

$$P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{P}_1 = 0 \tag{3.4.9.6}$$

which is the major axis of the ellipse E.

10. Find the minor axis of E.
11. Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{3.4.11.1}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

12. Summarize all the above computations through a Python script and plot the ellipses in (3.4.1.1) and (3.4.2.1).
Solution: The following script plots Fig. 3.4.12 using the principles of an affine transformation.

```
https://github.com/gadepall/school/raw/master/
    linalg/book/codes/ellipse.py
```

3.5 Hyperbola

1. Tangents are drawn to the hyperbola

$$\mathbf{x}^T V\mathbf{x} = 36 \tag{3.5.1.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5.1.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{3.5.1.3}$$

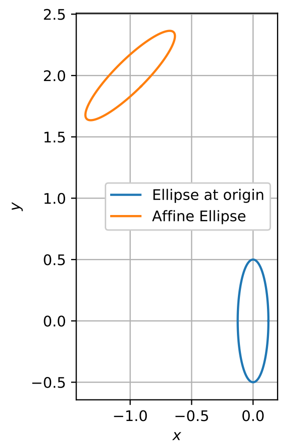

Fig. 3.4.12

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (3.1.6) as

$$\mathbf{P}^T V\mathbf{x} = 36 \tag{3.5.1.4}$$
$$\mathbf{Q}^T V\mathbf{x} = 36. \tag{3.5.1.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.6}$$
$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{3.5.1.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \tag{3.5.1.8}$$
$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{3.5.1.9}$$

which is the equation of PQ.

2. In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{3.5.2.1}$$

using (1.2.25) and (3.5.1.9), the equation of TD is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \tag{3.5.2.2}$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.2.3}$$

3. Find D.
Solution: From (3.5.1.9) and (3.5.2.3),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.1}$$
$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{3.5.3.2}$$

4. Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \tag{3.5.4.1}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.4.2}$$

5. Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \tag{3.5.5.1}$$

can be expressed as

$$\lambda^2 \mathbf{m}^T V \mathbf{m} + 2\lambda \mathbf{m}^T V \mathbf{D} + \mathbf{D}^T V \mathbf{D} + F = 0 \tag{3.5.5.2}$$

6. Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (3.5.4.1) and (3.5.1.1) (3.5.5.2) is obtained. Substituting from (3.5.4.2), (3.5.1.2) and (3.5.3.2)

$$\mathbf{m}^T V \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \tag{3.5.6.1}$$
$$\mathbf{m}^T V \mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \tag{3.5.6.2}$$
$$\mathbf{D}^T V \mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \tag{3.5.6.3}$$

Substituting in (3.5.5.2)

$$4\lambda^2 - 144 = 36 \tag{3.5.6.4}$$
$$\implies \lambda = \pm 3\sqrt{5} \tag{3.5.6.5}$$

Substituting in (3.5.4.1),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \tag{3.5.6.6}$$
$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \tag{3.5.6.7}$$

7. Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{3.5.7.1}$$
$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{3.5.7.2}$$

the desired area is

$$\frac{1}{2}PQ \times TD = 45\sqrt{5} \tag{3.5.7.3}$$

8. Repeat the previous exercise using determinants.
9. Summarize all the above computations through a Python script and plot the hyperbola.

3.6 Exercises

1. Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.
Solution: The equation of a conic is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0 \tag{3.6.1.1}$$

For the standard parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6.1.2}$$

$$\mathbf{u} = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.1.3}$$

$$F = 0 \tag{3.6.1.4}$$

The focus

$$\mathbf{F} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.1.5}$$

The Latus rectum is the line passing through $\mathbf{F}$ with direction vector

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.6.1.6}$$

Thus, the equation of the Latus rectum is

$$\mathbf{x} = \mathbf{F} + \lambda\mathbf{m} \tag{3.6.1.7}$$

The intersection of the latus rectum and the parabola is obtained from (3.6.1.4) , (3.6.1.7) and (3.6.1.1) as

$$(\mathbf{F} + \lambda\mathbf{m})^T \mathbf{V}(\mathbf{F} + \lambda\mathbf{m}) + 2\mathbf{u}^T(\mathbf{F} + \lambda\mathbf{m}) = 0 \tag{3.6.1.8}$$

$$\implies \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\lambda^2 + 2(\mathbf{V}\mathbf{F} + \mathbf{u})^T\mathbf{m}\lambda + (\mathbf{V}\mathbf{F} + 2\mathbf{u})^T\mathbf{F} = 0 \tag{3.6.1.9}$$

From (3.6.1.2), (3.6.1.3), (3.6.1.5) and (3.6.1.6),

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = 1 \tag{3.6.1.10}$$

$$(\mathbf{V}\mathbf{F} + \mathbf{u})^T\mathbf{m} = 0 \tag{3.6.1.11}$$

$$(\mathbf{V}\mathbf{F} + 2\mathbf{u})^T\mathbf{F} = -4a^2 \tag{3.6.1.12}$$

Substituting from (3.6.1.10), (3.6.1.11) and (3.6.1.12) in (3.6.1.9),

$$\lambda^2 - 4a^2 = 0 \tag{3.6.1.13}$$

$$\implies \lambda_1 = 2a, \lambda_2 = -2a \tag{3.6.1.14}$$

Thus, from (3.6.1.6), (3.6.1.7) and (3.6.1.14), the length of the latus rectum is

$$(\lambda_1 - \lambda_2)\|\mathbf{m}\| = 4a \tag{3.6.1.15}$$

From the given information, the two parabolas P_1, P_2 have parameters

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \tag{3.6.1.16}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a\begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \tag{3.6.1.17}$$

$$4a = 3 \tag{3.6.1.18}$$

Let L be the common tangent for P_1, P_2 with $\mathbf{c}, \mathbf{d}$ being the respective points of contact. The respectivel normal vectors are

$$\mathbf{n}_1 = \mathbf{V}_1\mathbf{c} + \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} \tag{3.6.1.19}$$

$$\mathbf{n}_2 = \mathbf{V}_2\mathbf{d} + \mathbf{u}_2 = d_1\begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \tag{3.6.1.20}$$

From the above equations, since both normals have the same direction vector,

$$\begin{pmatrix} 1 \\ -\frac{c_2}{2a} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2a}{d_1} \end{pmatrix} \implies c_2d_1 = 4a^2 \tag{3.6.1.21}$$

2. Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}\mathbf{x} = 225 \tag{3.6.2.1}$$

upon the tangent to it at the point

$$\frac{1}{2}\begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix} \tag{3.6.2.2}$$

Solution: For the ellipse in (3.6.2.1),

$$V = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}, \mathbf{u} = 0, F = -1 \tag{3.6.2.3}$$

The equation of the desired tangent is

$$(\mathbf{VP})^T \mathbf{x} = 1 \tag{3.6.2.4}$$

$$\implies \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix} \mathbf{x} = 2 \tag{3.6.2.5}$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{3.6.2.6}$$

The product of the perpendiculars is

$$\frac{\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} - 2\right|\left|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\begin{pmatrix} 0 \\ -4 \end{pmatrix} - 2\right|}{\left\|\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{3}}{5} \end{pmatrix}\right\|^2} = 9 \tag{3.6.2.7}$$

3. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.
Solution: If a and b be the semi-major and minor-axis respectively, the foci of the ellipse are

$$\mathbf{F}_1 = ae\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{F}_2 = -ae\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.3.1}$$

From the given information,

$$e = \frac{3}{5}, 2ae = 6 \tag{3.6.3.2}$$

$$\implies a = 5, b = a\sqrt{1 - e^2} = 4 \tag{3.6.3.3}$$

Thus, the vertices of the ellipse are

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ -b \end{pmatrix} \tag{3.6.3.4}$$

and the area of the quadrilateral is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & b \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & -a & 0 \\ 0 & 0 & -b \end{vmatrix} = 2ab = 40 \tag{3.6.3.5}$$

4. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{3.6.4.1}$$

If

$$\mathbf{S} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.6.4.2}$$

is a focus and

$$\begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{3.6.4.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.
Solution: From (3.6.4.1),

$$(3e - 1)(3e - 5) = 0 \tag{3.6.4.4}$$

$$\implies e = \frac{5}{3}, \because e > 1 \tag{3.6.4.5}$$

for a hyperbola. Let $\mathbf{x}$ be a point on the hyperbola. From (3.6.4.3), its distance from the directrix is

$$\frac{\left|\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} - 9\right|}{5} \tag{3.6.4.6}$$

and from the focus is

$$\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right\| \tag{3.6.4.7}$$

From the definition of a hyperbola, the eccentricity is the ratio of these distances and (3.6.4.5) , (3.6.4.6) and (3.6.4.7),

$$\frac{5\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 0 \end{pmatrix}\right\|}{\left|\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} - 9\right|} = \frac{5}{3} \tag{3.6.4.8}$$

$$\implies 9\left\{(x_1 - 5)^2 + x_2^2\right\} = (5x_1 - 9)^2 \tag{3.6.4.9}$$

$$\text{or, } \mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 225 \tag{3.6.4.10}$$

which is the equation of the hyperbola. Thus,

$$a^2 = \frac{225}{16}, b^2 = \frac{225}{9} \tag{3.6.4.11}$$

$$\implies a^2 - b^2 = -\frac{175}{16} \tag{3.6.4.12}$$

5. A variable line drawn through the intersection

of the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.6.5.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 12 \tag{3.6.5.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (3.6.5.1) is obtained through the matrix equation

$$\begin{pmatrix}4 & 3\\3 & 4\end{pmatrix}\mathbf{x} = \begin{pmatrix}12\\12\end{pmatrix} \tag{3.6.5.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix}4 & 3 & 12\\3 & 4 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}4 & 3 & 12\\0 & 7 & 12\end{pmatrix} \leftrightarrow \begin{pmatrix}28 & 0 & 48\\0 & 7 & 12\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}7 & 0 & 12\\0 & 7 & 12\end{pmatrix} \tag{3.6.5.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix}12\\12\end{pmatrix} \tag{3.6.5.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{R} \tag{3.6.5.6}$$

$$\mathbf{B} = 2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{R} \tag{3.6.5.7}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x} - \mathbf{C}) = 0 \tag{3.6.5.8}$$

Since **R** lies on AB,

$$\mathbf{n}^T(\mathbf{R} - \mathbf{C}) = 0 \tag{3.6.5.9}$$

Also,

$$\mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0 \tag{3.6.5.10}$$

Substituting from (3.6.5.6) in (3.6.5.10),

$$\mathbf{n}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R} = 0 \tag{3.6.5.11}$$

From (3.6.5.9) and (3.6.5.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R} \tag{3.6.5.12}$$

for some constant k. Multiplying both sides of (3.6.5.12) by

$$\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \tag{3.6.5.13}$$

$$\begin{aligned}\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{R}\\ &= k\mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R} = 0\end{aligned} \tag{3.6.5.14}$$

$$\because \mathbf{R}^T\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{R} = 0 \tag{3.6.5.15}$$

which can be easily verified for any **R**. from (3.6.5.14),

$$\begin{aligned}&\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(\mathbf{R} - \mathbf{C}) = 0\\ \implies &\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} - \mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{C} = 0\\ \implies &\mathbf{R}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} - \mathbf{C}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{R} = 0\end{aligned} \tag{3.6.5.16}$$

which is the desired locus.

3.7 Miscellaneous

1. What does the equation

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} - \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.7.1.1}$$

become when the origin is moved to the point $\begin{pmatrix}2\\-3\end{pmatrix}$?

2. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\-\frac{3}{2} & 4\end{pmatrix}\mathbf{x} + \begin{pmatrix}10 & -19\end{pmatrix}\mathbf{x} + 23 = 0 \tag{3.7.2.1}$$

may become

$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\-\frac{3}{2} & 4\end{pmatrix}\mathbf{x} = 1 \tag{3.7.2.2}$$

3. Show that the equation

$$\mathbf{x}^T\mathbf{x} = a^2 \tag{3.7.3.1}$$

remains unaltered by any rotation of the axes.

4. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x} = 2a^2 \tag{3.7.4.1}$$

become when the axes are turned through 30°?

5. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.5.1}$$

become when the axes are turned through 45°?

6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.6.1}$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.6.2}$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \tag{3.7.6.3}$$

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.7.1}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c \tag{3.7.7.2}$$

where c is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 13 = 0 \tag{3.7.8.1}$$

can be transformed to

$$8\mathbf{x}^T\mathbf{x} = 89 \tag{3.7.8.2}$$

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.9.1}$$

can be reduced to

$$\sqrt{85}\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{3.7.9.2}$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \tag{3.7.10.1}$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.7.10.2}$$

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \tag{3.7.11.1}$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.11.2}$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{3.7.11.3}$$