Geometry through Linear Algebra

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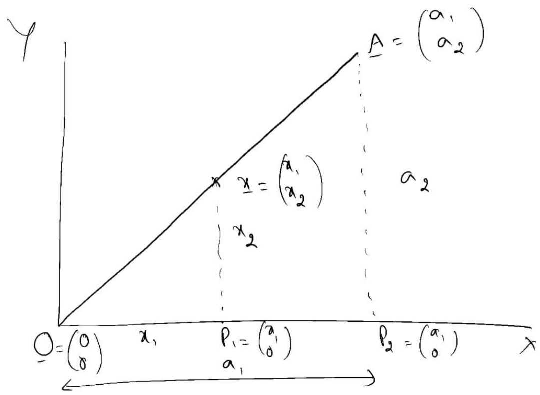

Fig. 1.1

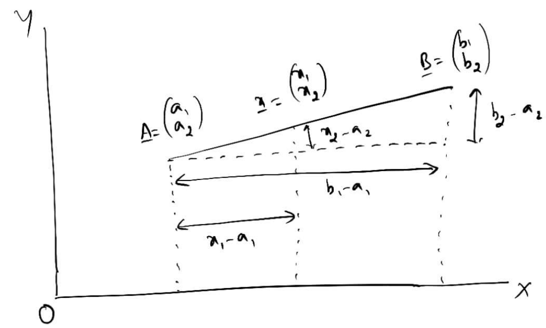

Fig. 1.2

Contents

Abstract—**This textbook introduces linear algebra by exploring Euclidean geometry.**

1 The Straight Line

1.1 The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.1}$$

$$\implies x_2 = mx_1 \tag{1.2}$$

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.3}$$

In general, the above equation is written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.4}$$

1.2 Find the equation of AB in Fig. 1.2

Solution: From Fig. 1.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.5}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.6}$$

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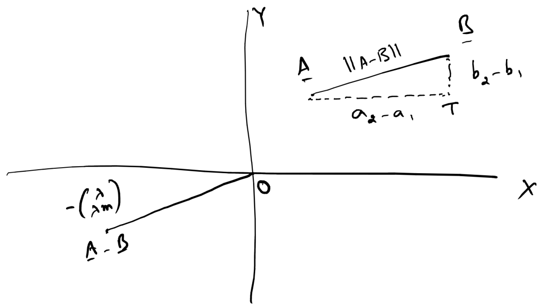

Fig. 1.4

From (1.6),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.7}$$

$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8}$$

$$= \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.9}$$

1.3 Find the length of $\mathbf{A}$ in Fig. 1.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{1.10}$$

Also, from (1.4),

$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{1.11}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.

1.4 Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.4. From (1.9), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.12}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.13}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.4.

1.5 Show that $AB = \|\mathbf{A} - \mathbf{B}\|$

2 Orthogonality

2.1 See Fig. 2.1. In $\triangle ABC, AB \perp BC$. Show that

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.1}$$

Solution: Using Baudhayana's theorem,

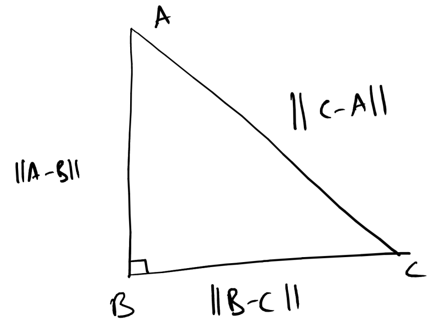

Fig. 2.1

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{2.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})$$

$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{2.3}$$

which can be simplified to obtain (2.1).

2.2 Let $\mathbf{x}$ be any point on AB in Fi.g 2.1. Show that

$$(\mathbf{x} - \mathbf{A})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.4}$$

2.3 If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x} - \mathbf{y})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{2.5}$$

2.4 In Fig. 2.4, $BE \perp AC, CF \perp AB$. Show that $AD \perp BC$.

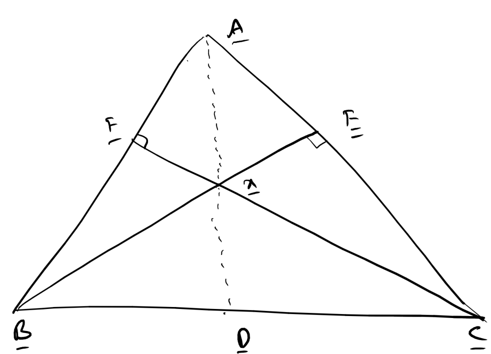

Fig. 2.4

Solution: Let $\mathbf{x}$ be the intersection of BE and

CF. Then, using (2.5),

$$\begin{aligned}(\mathbf{x}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) &= 0\\ (\mathbf{x}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) &= 0\end{aligned} \tag{2.6}$$

$$\implies \mathbf{x}^T(\mathbf{A}-\mathbf{C})-\mathbf{B}^T(\mathbf{A}-\mathbf{C}) = 0 \tag{2.7}$$

$$\text{and } \mathbf{x}^T(\mathbf{A}-\mathbf{B})-\mathbf{C}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{2.8}$$

Subtracting (2.8) from (2.7),

$$\mathbf{x}^T(\mathbf{B}-\mathbf{C})+\mathbf{A}^T(\mathbf{C}-\mathbf{B}) = 0 \tag{2.9}$$

$$\implies \left(\mathbf{x}^T-\mathbf{A}^T\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{2.10}$$

$$\implies (\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.11}$$

which completes the proof.

3 Medians of a triangle

3.1 In Fig. 3.1,

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{3.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{3.2}$$

Solution: From (1.9),

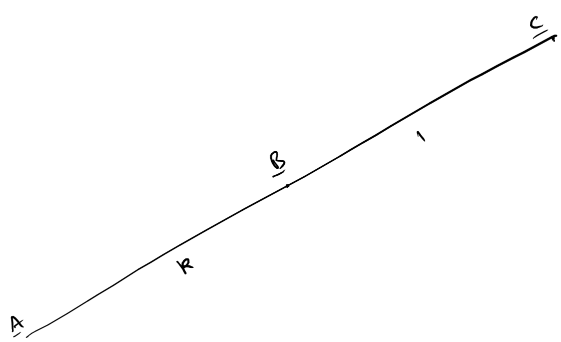

Fig. 3.1

$$\begin{aligned}\mathbf{B} &= \mathbf{A}+\lambda_1\begin{pmatrix}1\\m\end{pmatrix},\\ \mathbf{B} &= \mathbf{C}-\lambda_2\begin{pmatrix}1\\m\end{pmatrix}.\end{aligned} \tag{3.3}$$

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{3.4}$$

$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \begin{pmatrix}1\\m\end{pmatrix}, \tag{3.5}$$

from (3.1). Using (3.4) and (3.4),

$$\mathbf{A}-\mathbf{B} = k(\mathbf{B}-\mathbf{C}) \tag{3.6}$$

resulting in (3.2).

3.2 If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A}+k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{3.7}$$

3.3 BE and CF are medians of $\triangle ABC$ intersecting at O as shown in Fig. 3.3. Show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \tag{3.8}$$

Solution: Let

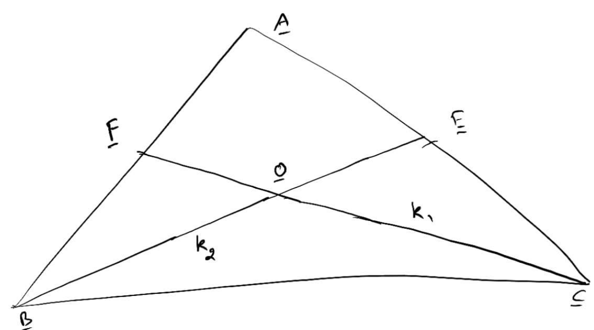

Fig. 3.3

$$\frac{CO}{OF} = k_1 \tag{3.9}$$

$$\frac{BO}{OE} = k_2 \tag{3.10}$$

Using (3.2),

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{3.11}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{3.12}$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F}+\mathbf{C}}{k_1+1} = \frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} \tag{3.13}$$

$$\mathbf{O} = \frac{k_2\mathbf{E}+\mathbf{B}}{k_2+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{3.14}$$

From (3.13) and (3.14),

$$\frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{3.15}$$

$$\implies \left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right]\mathbf{A} + \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} + \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \quad (3.16)$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = k_2 = 2, \quad (3.17)$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (3.14),

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (3.18)$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \quad (3.19)$$

Note that $\mathbf{B}, \mathbf{C}$ are linearly independent. Substituting (3.19) in (3.16),

$$\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right][\alpha\mathbf{B} + \beta\mathbf{C}] + \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} + \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \quad (3.20)$$

$$\implies \begin{aligned} (k_1 - k_2)\alpha + k_1k_2 - k_1 - 2 &= 0 \\ (k_1 - k_2)\beta - k_1k_2 + k_2 + 2 &= 0 \end{aligned} \quad (3.21)$$

$$\implies (k_1 - k_2)(\alpha + \beta - 1) = 0 \quad (3.22)$$

If $\alpha + \beta = 1, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (3.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in (3.21), yields

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = 2. \quad (3.23)$$

4 Matrix Transformations

4.1 Find $\mathbf{R}$, the reflection of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \quad (4.1)$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \quad (4.2)$$

$$\mathbf{n}^T\mathbf{Q} = c \quad (4.3)$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \quad (4.4)$$

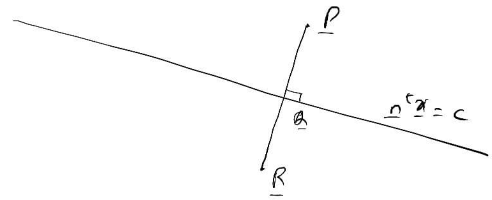

Fig. 4.1

where $\mathbf{m}$ is the direction vector of L. From (4.2) and (4.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \quad (4.5)$$

From (4.5) and (4.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0 \\ 2c\end{pmatrix} \quad (4.6)$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \quad (4.7)$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal,

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \quad (4.8)$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \quad (4.9)$$

(4.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0 \\ 2c\end{pmatrix} \quad (4.10)$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0 \\ 2c\end{pmatrix} \quad (4.11)$$

$$= \mathbf{V}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \quad (4.12)$$

4.2 Rotate $\mathbf{P}$ through an angle of θ about the origin in the counter clockwise direction to obtain $\mathbf{S}$.
Solution:

$$\mathbf{S} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\mathbf{P} \quad (4.13)$$

5 Circle

5.1 A circle with centre at origin $\mathbf{O}$ and radius r has the equation

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{5.1}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be two points on the circle. Show that

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} + \mathbf{B}) = 0 \tag{5.2}$$

5.2 Let $\mathbf{D}$ be the mid point of AB. Show that $OD \perp AB$.

5.3 Show that the direction vector of the normal to the tangent at $\mathbf{P}$ is $\mathbf{P}$.

5.4 Let $\mathbf{A}$ be any point on the tangent. Show that $OP \perp AP$.

5.5 Show that the angle in a semi-circle is a right angle.