Optimization through School Geometry

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Contents

***Abstract*—This book provides an introduction to optimization based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/optimization/codes
```

1 Constrained Optimization

1. Express the problem of finding the distance of the point $\mathbf{P} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line
$$L: \quad \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{1.1.1}$$
as an optimization problem.
Solution: The given problem can be expressed as
$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \tag{1.1.2}$$
$$\text{s.t.} \quad \mathbf{n}^T \mathbf{x} = c \tag{1.1.3}$$
where
$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.1.4}$$
$$c = 26 \tag{1.1.5}$$
2. Convert (1.1.2) to an *unconstrained* optimization problem.
Solution: L in (1.1.1) can be expressed in terms of the direction vector $\mathbf{m}$ as
$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m}, \tag{1.2.1}$$
where $\mathbf{A}$ is any point on the line and
$$\mathbf{m}^T \mathbf{n} = 0 \tag{1.2.2}$$
Substituting (1.2.1) in (1.1.2), an unconstrained optimization problem
$$\min_{\lambda} f(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\|^2 \tag{1.2.3}$$
is obtained.
3. Solve (1.2.3).
Solution:
$$f(\lambda) = (\lambda \mathbf{m} + \mathbf{A} - \mathbf{P})^T (\lambda \mathbf{m} + \mathbf{A} - \mathbf{P}) \tag{1.3.1}$$
$$= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^T (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \tag{1.3.2}$$
$$\because f^{(2)} \lambda = 2 \|\mathbf{m}\|^2 > 0 \tag{1.3.3}$$

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the minimum value of $f(\lambda)$ is obtained when

$$f^{(1)}(\lambda) = 2\lambda \|\mathbf{m}\|^2 + 2\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0 \tag{1.3.4}$$

$$\implies \lambda_{\min} = -\frac{\mathbf{m}^T (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{1.3.5}$$

Choosing $\mathbf{A}$ such that

$$\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0, \tag{1.3.6}$$

substituting in (1.3.5),

$$\lambda_{\min} = 0 \quad \text{and} \tag{1.3.7}$$

$$\mathbf{A} - \mathbf{P} = \mu\mathbf{n} \tag{1.3.8}$$

for some constant μ. (1.3.8) is a consequence of (1.2.2) and (1.3.6). Also, from (1.3.8),

$$\mathbf{n}^T (\mathbf{A} - \mathbf{P}) = \mu \|\mathbf{n}\|^2 \tag{1.3.9}$$

$$\implies \mu = \frac{\mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} = \frac{c - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} \tag{1.3.10}$$

from (1.1.3). Substituting $\lambda_{\min} = 0$ in (1.2.3),

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} - \mathbf{P}\|^2 = \mu^2 \|\mathbf{n}\|^2 \tag{1.3.11}$$

upon substituting from (1.3.8). The distance between $\mathbf{P}$ and L is then obtained from (1.3.11) as

$$\|\mathbf{A} - \mathbf{P}\| = |\mu| \|\mathbf{n}\| \tag{1.3.12}$$

$$= \frac{\left|\mathbf{n}^T\mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{1.3.13}$$

after substituting for μ from (1.3.10).

2 Lagrange Multipliers

1. A single variable function f is said to be convex if

$$f[\lambda x + (1 - \lambda) y] \le \lambda f(x) + (1 - \lambda) f(y), \tag{2.1.1}$$

for $\quad 0 < \lambda < 1$.

2. Download and execute the following python script. Is $\ln x$ convex or concave?

```
codes/optimization/1.1.py
```

3. Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
codes/optimization/1.2.py
```

4. Execute the following script to obtain Fig. **??**. Comment.

```
codes/optimization/1.3.py
```

5. Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?
6. Let

$$f(\mathbf{x}) = x_1x_2, \quad \mathbf{x} \in \mathbf{R}^2 \tag{2.6.1}$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex.

7. Show that

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} \tag{2.7.1}$$

and find $\mathbf{V}$.

8. Show that

$$\frac{1}{2}\nabla^2 f(\mathbf{x}) = \mathbf{V} \tag{2.8.1}$$

9. Use (2.1.1) to examine the convexity of $f(\mathbf{x})$.
10. How can you deduce the convexity of $f(\mathbf{x})$ using the eigenvalues of $\mathbf{V}$?
11. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{2.11.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{2.11.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{2.11.3}$$

12. Prove that the point $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 24 \tag{2.12.1}$$

$$\begin{pmatrix} 5 & -3 \end{pmatrix}\mathbf{x} = 15 \tag{2.12.2}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{2.12.3}$$

3 Lagrange Multipliers

1. Find

$$\min_{\mathbf{x}} f(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 = r^2 \tag{3.1.1}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{n}^T\mathbf{x} - c = 0 \tag{3.1.2}$$

by plotting the circles $f(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.
Solution: The following code plots Fig. **??**

```
codes/optimization/2.1.py
```

2. Show that

$$\min r = \frac{5}{\sqrt{2}} \tag{3.2.1}$$

3. Show that
$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.3.1}$$
where
$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{3.3.2}$$
4. Show that
$$\nabla f(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\} \tag{3.4.1}$$
is the direction vector of the normal at $\mathbf{x}$.
5. From Fig. **??**, show that
$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{3.5.1}$$
where $\mathbf{p}$ is the point of contact.
6. Use (3.5.1) and $\mathbf{g}(\mathbf{p}) = 0$ from (3.1.2) to obtain $\mathbf{p}$.
7. Define
$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{3.7.1}$$
and show that $\mathbf{p}$ can also be obtained by solving the equations
$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{3.7.2}$$
What is the sign of λ? L is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.
Solution:

```
codes/optimization/2.3.py
```

4 Linear Programing

1. Solve
$$\max_{\mathbf{x}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \tag{4.1.1}$$
$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{4.1.2}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.1.3}$$
Solution: The given problem can be expressed in general as
$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{4.1.4}$$
$$s.t. \quad \mathbf{A}\mathbf{x} \le \mathbf{b}, \tag{4.1.5}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.1.6}$$
where
$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{4.1.7}$$
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \tag{4.1.8}$$
$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{4.1.9}$$
and can be solved using *cvxpy* through the following code

```
codes/line/lp_exam.py
```

to obtain
$$\mathbf{x} = \begin{pmatrix} 30 \\ 0 \end{pmatrix}, Z = 120 \tag{4.1.10}$$
2. Solve
$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 9 \end{pmatrix} \mathbf{x} \tag{4.2.1}$$
$$s.t. \quad \begin{pmatrix} 1 & 3 \\ -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 60 \\ -10 \\ 0 \end{pmatrix} \tag{4.2.2}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.2.3}$$
Solution: The following code

```
codes/line/lp_exam_mult.py
```

is used to obtain
$$\mathbf{x} = \begin{pmatrix} 15 \\ 15 \end{pmatrix}, Z = 180 \tag{4.2.4}$$
The region in (**??**) is shown in Fig. **??**
3. Solve
$$\min_{\mathbf{x}} Z = \begin{pmatrix} -50 & 20 \end{pmatrix} \mathbf{x} \tag{4.3.1}$$
$$s.t. \quad \begin{pmatrix} -2 & 1 \\ -3 & -1 \\ 2 & -3 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 5 \\ -3 \\ 12 \end{pmatrix} \tag{4.3.2}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.3.3}$$
Solution: The following code

```
codes/line/lp_exam_nosol.py
```

shows that the given problem has no solution.
4. **(Diet problem)**: A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains

1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food ‘I’ and Rs 70 per kg to purchase Food ‘II’. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.

5. **(Allocation problem)** A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

6. **(Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

7. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

8. **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Number of hours required on machines			
Items	**I**	**II**	**III**
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

9. **(Transportation problem)** There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below where A,B,C are cost in ruppes:

From/To	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

4.1 Linear Programing: Exercises

1. Solve
$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{4.1.1.1}$$
$$s.t. \quad \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} -8 \\ 15 \end{pmatrix} \tag{4.1.1.2}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.1.1.3}$$
2. Solve
$$\min_{\mathbf{x}} Z = \begin{pmatrix} 200 & 500 \end{pmatrix} \mathbf{x} \tag{4.1.2.1}$$
$$s.t. \quad \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} -10 \\ 24 \end{pmatrix} \tag{4.1.2.2}$$
$$\mathbf{x} \ge \mathbf{0} \tag{4.1.2.3}$$
3. Maximise Z=3x+4y
subject to the constraints : x+y≤4, x≥0, y≥ 0.
4. Minimise Z=-3x+4y
subject to x+2y≤8, 3x+2y≤12, x≥0, y≥0.
5. Maximise Z=5x+3y subject to 3x+5y≤15, 5x+2y≤10, x≥0, y≥0.
6. Minimise Z=3x+5y such that x+3y≥3, x+y≥2, x,y≥0.
7. Maximise Z=3x+2y subject to x+2y≤10, 3x+y≤15, x,y≥0.
8. Minimise Z=x+2y subject to 2x+y≥3, x+2y≥6, x,y≥0.
Show that the minimum of Z occurs at more than two points.
9. Minimise and Maximise Z=5x+10y subject to x+2y≤120, x+y≥60, x-2y≥0, x,y≥0.
10. Minimise and Maximise Z=x+2y subject to x+2y≥100, 2x-y≤0, 2x+y≤200; x,y≥0.
11. Maximise Z=-x+2y, subject to the constraints: x≥3, x+y≥5, x+2y≥6, y≥0.
12. Maximise Z=x+y, subject to x-y≤-1,-x+y≤0, x,y≥0.
13. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
14. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
15. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
(i) What number of rackets and bats must be made if the factory is to work at full capacity?
(ii)If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
16. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?
17. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it

takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

18. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

19. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

20. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

21. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foodsF_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

22. There are two types of fertilisers F_1 and F_2.F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

23. The corner points of the feasible region determined by the following system of linear inequalities: $2x+y\leq10$, $x+3y\leq15$, $x,y\geq0$ are (0,0), (5,0),(3,4) and (0,5).Let $Z=px+qy$, where $p,q>0$.Condition on p and q so that the maximum of Z occurs at both (3,4) and (0,5) is
(A) $p = q$
(B) $p = 2q$
(C) $p = 3q$
(D) $q = 3p$

24. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

25. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q

costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

26. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	**Vitamin A**	**Vitamin B**	**VitaminC**
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

27. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Machines			
Types of toys	**I**	**II**	**III**
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

29. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per qunital (in Rs)		
From/To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

30. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

31. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phospheric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

32. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

33. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?

5 Circle

5.1 Circle Geometry Examples

1. Find the equation of a circle with centre $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and radius 4.
2. Find the centre and radius of the circle
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 8 \\ 10 \end{pmatrix}\mathbf{x} - 8 = 0 \qquad (5.1.2.1)$$
3. Find the equation of the circle which passes through the points $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$.
4. Find the area enclosed by the circle $\|\mathbf{x}\| = a$
5. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
6. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\| = 2$.
7. Form the differntial equation of the fmaily of circles touching the x-axis at the origin.
8. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

5.2 Circle Geometry Exercises

1. Find the coordinates of a point $\mathbf{A}$, where AB is the diameter of a circle whose centre is $\begin{pmatrix} 2, -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
2. Find the centre Of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
3. Sketch the circles with
   a) centre $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2
   b) centre $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4
   c) centre $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$.
   d) centre $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$.
   e) centre $\begin{pmatrix} -a \\ -b \end{pmatrix}$ and radius $\sqrt{a^2 - b^2}$.
4.
5. Sketch the circles with equation
   a) $\left\|\mathbf{x} - \begin{pmatrix} 5 \\ -3 \end{pmatrix}\right\|^2 = 36$
   b) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 \\ 8 \end{pmatrix}\mathbf{x} - 45 = 0$
   c) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix}\mathbf{x} - 12 = 0$
   d) $2\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{x} = 0$
6. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{x} = 16$.
7. Find the equation of the circle passing through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = 11$.
8. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
9. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.

10. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.
11. Does the point $\begin{pmatrix}-2.5\\3.5\end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T\mathbf{x} = 25$?
12. Find the locus of all the unit vectors in the xy-plane.
13. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
14. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.
15. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.
16. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.
17. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
18. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.
19. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
20. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.
21. If $(x-a)^2+(y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{(1+y_2)^{\frac{3}{2}}}{y_2} \tag{5.2.21.1}$$

is a constant independent of a and b.
22. Form the differential equation of the family of circles touching the y-axis at origin.
23. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

6 Conics

6.1 Examples

1. Find the value of the following polynomial at the indicated value of variables

$$p(x) = 5x^2–3x + 7 \text{ at } x = 1. \tag{6.1.1.1}$$

2. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.
3. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(y) = y^2$.
   b) $p(x) = (x–1)(x + 1)$.
4. Find the roots of the equation $2x^2–5x + 3 = 0$.
5. Find the roots of the quadratic equation $6x^2–x–2 = 0$.
6. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
7. Factorise $6x^2 + 17x + 5$.
8. Factorise $y^2–5y + 6$.
9. Find the zeroes of the quadratic polynomial $x^2+7x+10$ and verify the relationship between the zeroes and the coefficients.
10. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.
11. Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.
12. Find the roots of the equation $5x^2–6x–2 = 0$.
13. Find the roots of $4x^2 + 3x + 5 = 0$.
14. Find the roots of the following quadratic equations, if they exist.
    a) $3x^2 - 5x + 2 = 0$
    b) $x^2 + 4x + 5 = 0$
    c) $2x^2 - 2\sqrt{2}x + 1 = 0$
15. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.
16. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.
17. Solve $x^2 + 2 = 0$.
18. Solve $x^2 + x + 1 = 0$.
19. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.
20. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.
21. Find the equation of the parabola with focus

$\begin{pmatrix}2\\0\end{pmatrix}$ and directrix $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -2$.

22. Find the equation of the parabola with vertex at $\begin{pmatrix}0\\0\end{pmatrix}$ and focus at $\begin{pmatrix}0\\2\end{pmatrix}$.
23. Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $\begin{pmatrix}2\\-3\end{pmatrix}$.
24. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse
$$\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1 \tag{6.1.24.1}$$
25. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse
$$\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 36 \tag{6.1.25.1}$$
26. Find the equation of the ellipse whose vertices are $\begin{pmatrix}\pm 13\\0\end{pmatrix}$ and foci are $\begin{pmatrix}\pm 5\\0\end{pmatrix}$.
27. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix}0\\\pm 5\end{pmatrix}$
28. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.
29. Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas
   a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -16\end{pmatrix}\mathbf{x} = 16$
30. Find the equation of the hyperbola with vertices $\begin{pmatrix}0\\\pm\frac{\sqrt{11}}{2}\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm 3\end{pmatrix}$
31. Find the equation of the hyperbola with foci $\begin{pmatrix}0\\\pm 12\end{pmatrix}$ and length of latus rectum 36.
32. Find the equation of all lines having slope 2 and being tangent to the curve
$$y + \frac{2}{x-3} = 0 \tag{6.1.32.1}$$
33. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.
34. Find the roots of the following equations:
   a) $x + \frac{1}{x} = 3, x \neq= 0$
   b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq= 0, 2$
35. Find points on the curve $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$ at which the tangents are
   a) parallel to x-axis
   b) parallel to y-axis
36. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$.
37. Find the area enclosed by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$
38. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.
39. Find the area bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$.
40. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.
41. Find the area of the region
$$\left\{(x, y) = 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\} \tag{6.1.41.1}$$
42. Find the interals in which the function
$$f(x) = x^2 - 4x + 6 \tag{6.1.42.1}$$
is
   a) increasing
   b) decreasing.
43. Find the shortest distance of the point $\begin{pmatrix}0\\c\end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.
44. An apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $\begin{pmatrix}3\\7\end{pmatrix}$, wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.
45. Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.
46. Discuss the continuity of the function f defined by
$$f(x) = \begin{cases} x & x \ge 0\\ x^2 & x < 0\end{cases} \tag{6.1.46.1}$$

47. Verify Rolle's theorem for the function $y = x^2 + 2, a = -2$ and $b = 2$.
48. Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, -4]$.
49. Find the derivative of $f(x) = x^2$.
50. Find the derivative of $x^2 - 2$ at $x = 10$.
51. Find the derivative of $(x-1)(x-2)$.
52. Find

$$\int_0^2 \left(x^2 + 1\right) dx \tag{6.1.52.1}$$

as a limit of a sum.
53. Evaluate the following integral:

$$\int_2^3 x^2\, dx \tag{6.1.53.1}$$

54. Form the differntial equation representing the family of ellipses having foci on x-axis and cenre at the origin.
55. Form the differntial equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.
56. Form a differntial equation representing the following family of curves

$$y^2 = a\left(b^2 - x^2\right) \tag{6.1.56.1}$$

6.2 Exercises

1. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = x^2 - 1, x = 1, -1$
   b) $p(x) = (x+1)(x-2), x = -1, 2$
   c) $p(x) = x^2, x = 0.$
   d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$
2. Find the vaue of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $x^4 - 4x^2 + x + 6, g(x) = x - 3$
3. Factorise :
   a) $12x^2–7x + 1$
   b) $6x^2 + 5x–6$
   c) $2x^2 + 7x + 3$
   d) $3x^2–x–4$
4. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
   a) $x^2–2x–8$
   b) $4u^2 + 8u$
   c) $4s^2–4s + 1$
   d) $t^2–15$
   e) $6x^2–3–7x$
   f) $3x^2–x–4$
5. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.
   a) -1 , $\frac{1}{4}$
   b) 1, 1
   c) 0, $\sqrt{5}$
   d) 4, 1
   e) $\frac{1}{4}, \frac{1}{4}$
   f) $\sqrt{2}, \frac{1}{3}$
6. Find the roots of the following quadratic equations:
   a) $x^2–3x–10 = 0$
   b) $2x^2 + x - 6 = 0$
   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   d) $2x^2–x + \frac{1}{8} = 0$
   e) $100x^2–20x + 1 = 0$
7. Find the roots of the following quadratic equations
   a) $2x^2 - 7x + 3 = 0$
   b) $2x^2 + x - 4 = 0$
   c) $4x^2 + 4\sqrt{3}x + 3 = 0$
   d) $2x^2 + x + 4 = 0$
8. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
   a) $2x^2 - 3x + 5 = 0$
   b) $2x^2 - 6x + 3 = 0$
   c) $3x^2 - 4\sqrt{3}x + 4 = 0$
9. Solve each of the following equations
   a) $x^2 + 3 = 0$
   b) $2x^2 + x + 1 = 0$
   c) $x^2 + 3x + 9 = 0$
   d) $-x^2 + x - 2 = 0$
   e) $x^2 + 3x + 5 = 0$
   f) $x^2 - 3x + 2 = 0$
   g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
   h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
   i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
   j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$
10. Solve each of the following equations
   a) $3x^2 - 4x + \frac{20}{3} = 0$
   b) $x^2 - 2x + \frac{3}{2} = 0$
   c) $27x^2 - 10x + 1 = 0$

d) $21x^2 - 28x + 10 = 0$

11. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum
    a) $y^2 = 12x$
    b) $x^2 = 6y$
    c) $y^2 = -8x$
    d) $x^2 = -16y$
    e) $y^2 = 10x$
    f) $x^2 = -9y$
12. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:
    a) Focus $\begin{pmatrix}6\\0\end{pmatrix}$, directrix $\begin{pmatrix}1 & 0\end{pmatrix} = -6$.
    b) Focus $\begin{pmatrix}0\\-3\end{pmatrix}$, directrix $\begin{pmatrix}0 & 1\end{pmatrix} = 3$.
    c) Focus $\begin{pmatrix}3\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.
    d) Focus $\begin{pmatrix}-2\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.
    e) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}2\\2\end{pmatrix}$ and axis is along the x-axis
    f) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}5\\2\end{pmatrix}$ and symmetric with respect to the y-axis.
13. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.
    a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{36} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$
    c) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$
    d) $\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{100}\end{pmatrix}\mathbf{x} = 1$
    e) $\mathbf{x}^T\begin{pmatrix}\frac{1}{49} & 0\\0 & \frac{1}{36}\end{pmatrix}\mathbf{x} = 1$
    f) $\mathbf{x}^T\begin{pmatrix}\frac{1}{100} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    g) $\mathbf{x}^T\begin{pmatrix}36 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 144$
    h) $\mathbf{x}^T\begin{pmatrix}16 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 16$
    i) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 36$
14. In each of the following, find the equation for the ellipse that satisfies the given conditions:
    a) Vertices $\begin{pmatrix}\pm5\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$
    b) Vertices $\begin{pmatrix}0\\\pm13\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$
    c) Vertices $\begin{pmatrix}\pm6\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$
    d) Ends of major axis $\begin{pmatrix}\pm3\\0\end{pmatrix}$, ends of minor axis $\begin{pmatrix}0\\\pm2\end{pmatrix}$
    e) Ends of major axis $\begin{pmatrix}0\\\pm5\end{pmatrix}$, ends of minor axis $\begin{pmatrix}\pm1\\0\end{pmatrix}$
    f) Length of major axis 26, foci $\begin{pmatrix}\pm5\\0\end{pmatrix}$
    g) Length of minor axis 16, foci $\begin{pmatrix}0\\\pm6\end{pmatrix}$.
    h) Foci $\begin{pmatrix}\pm3\\0\end{pmatrix}$, a = 4
    i) b = 3, c = 4, centre at the origin; foci on the x axis.
    j) Centre at $\begin{pmatrix}0\\0\end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\6\end{pmatrix}$.
    k) Major axis on the x-axis and passes through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$.
15. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.
    a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & -\frac{1}{9}\end{pmatrix}\mathbf{x} = 1$
    b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{27}\end{pmatrix}\mathbf{x} = 1$
    c) $\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & -4\end{pmatrix}\mathbf{x} = 36$
    d) $\mathbf{x}^T\begin{pmatrix}16 & 0\\0 & -9\end{pmatrix}\mathbf{x} = 576$
    e) $\mathbf{x}^T\begin{pmatrix}5 & 0\\0 & -9\end{pmatrix}\mathbf{x} = 36$
    f) $\mathbf{x}^T\begin{pmatrix}49 & 0\\0 & -16\end{pmatrix}\mathbf{x} = 784$

16. In each of the following, find the equation for the ellipse that satisfies the given conditions:
   a) Vertices $\begin{pmatrix}\pm 2\\ 0\end{pmatrix}$, foci $\begin{pmatrix}\pm 3\\ 0\end{pmatrix}$
   b) Vertices $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$, foci $\begin{pmatrix}0\\ \pm 8\end{pmatrix}$
   c) Vertices $\begin{pmatrix}0\\ \pm 3\end{pmatrix}$, foci $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$
   d) Transverse axis length 8, foci $\begin{pmatrix}\pm 5\\ 0\end{pmatrix}$.
   e) Conjugate axis length 24, foci $\begin{pmatrix}0\\ \pm 13\end{pmatrix}$.
   f) Latus rectum length 8, foci $\begin{pmatrix}\pm 3\sqrt{5}\\ 0\end{pmatrix}$.
   g) Latus rectum lenght 12, foci $\begin{pmatrix}\pm 4\\ 0\end{pmatrix}$.
   h) Ends of major axis $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$, ends of minor axis $\begin{pmatrix}\pm 1\\ 0\end{pmatrix}$
   i) Vertices $\begin{pmatrix}\pm 7\\ 0\end{pmatrix}$, $e = \frac{4}{3}$
   j) Foci $\begin{pmatrix}0\\ \pm\sqrt{10}\end{pmatrix}$, passing through $\begin{pmatrix}2\\ 3\end{pmatrix}$.
17. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
18. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix}2\\ 0\end{pmatrix}$ and $\begin{pmatrix}4\\ 4\end{pmatrix}$.
19. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$
20. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
21. Find points on the curve $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$ at which tangents are
   a) parallel to x-axis
   b) parallel to y-axis.
22. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix}0\\ 0\end{pmatrix}$.
23. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
   a) parallel to the line $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -9$
   b) perpendicular to the line $\begin{pmatrix}-15 & 5\end{pmatrix}\mathbf{x} = 13$.
24. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix}4 & 2\end{pmatrix}\mathbf{x} + 5 = 0$.
25. Find the point at which the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
26. The line $\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.
27. Find the normal at the point $\begin{pmatrix}1\\ 1\end{pmatrix}$ on the curve $2y + x^2 = 3$
28. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix}1\\ 2\end{pmatrix}$.
29. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
30. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
31. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
32. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\ 0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$
33. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$
34. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
35. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
36. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$.
37. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
38. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
39. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
40. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix}1 & 0\end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
41. AOBA is the part of the ellipse $\mathbf{x}^T\begin{pmatrix}9 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.
42. Find the area lying between the curves $y^2 = 4x$

and $y = 2x$.

43. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
44. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
45. Find the area between $y = x^2$ and $y = x$.
46. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
47. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
48. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
49. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
50. Find the area bounded by the curves
$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{6.2.50.1}$$
51. Find the area of the region
$$\left\{(x, y) : y^2 \leq 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{6.2.51.1}$$
52. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.
53. Find the intervals in which the function given by
$$f(x) = 2x^2 - 3x \tag{6.2.53.1}$$
is
    a) increasing
    b) decreasing.
54. Find the intervals in which the following functions are strictly increasing or decreasing
    a) $x^2 + 2x - 5$
    b) $10 - 6x - 2x^2$
    c) $6 - 9x - x^2$
55. Prove that the function f given by $f(x) = x^2 - x+1$ is neither strictly increasing nor decreasing on $(1, -1)$.
56. Find the maximum and minimum values, if any, of the following functions given by
    a) $f(x) = (2x - 1)^2 + 3$
    b) $f(x) = 9x^2 + 12x + 2$
    c) $f(x) = -(x - 1)^2 + 10$
    d) $f(x) = x^2$.
57. Find the absolute maximum and absoute minimum value of the following functions in the given intervals
    a) $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$
    b) $f(x) = (x - 1)^2 + 3, x \in (-3, 1)$
58. Find the maximum profit that a company can make, if the profit function is given by
$$p(x) = 41 - 72x - 18x^2 \tag{6.2.58.1}$$
59. Find the point on the curve $x^2 = 2y$ which is nearest to the point $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.
60. Find the maximum area of an isosceles triangle inscribed in the ellipse
$$\mathbf{x}^T \begin{pmatrix} a^20 & 0 & b^2 \end{pmatrix} \mathbf{x} = a^2b^2 \tag{6.2.60.1}$$
with its vertex at one end of the major axis.
61. Examine the continuity of the function $f(x) = 2x^2–1$ at $x = 3$.
62. Find all points of discontinuity of f, where $f$is defined by
$$f(x) = \begin{cases} x + 1, & x \geq 1, \\ x^2 + 1, & x < 1, \end{cases} \tag{6.2.62.1}$$
63. For what value of $\lambda$is the function defined by
$$f(x) = \begin{cases} \lambda\left(x^2 - 2x\right), & x \leq 0, \\ 4x + 1, & x > 0 \end{cases} \tag{6.2.63.1}$$
continuous at $x = 0$? What about continuity at $x = 1$?
64. For what value of k is the following function continuous at the given point.
$$f(x) = \begin{cases} kx^2, & x \leq 2, \\ 3, & x > 2, \end{cases} \quad x = 2 \tag{6.2.64.1}$$
65. Find $\frac{dy}{dx}$ in the following
$$x^2 + xy + y^2 = 100 \tag{6.2.65.1}$$
66. Verify Rolle's theorem for the function . $f(x) = x^2 + 2x - 8, x \in [-4, 2]$
67. Examine if Rolle's theorem is applicable to the following function $f(x) = x^2 - 1, x \in [1, 2]$. Can you say some thing about the converse of Rolle's theorem from this example?
68. Examine the applicability of the mean value theorem for the function in Problem 6.2.66.
69. Find $\lim_{x\to 1} \pi r^2$.

70. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases} \tag{6.2.70.1}$$

71. For some constans a and b, find the derivative of

$$(x-a)(x-b) \tag{6.2.71.1}$$

72. Integrate the following as limit of sums:
    (i) $\int_2^3 x^2\, dx$
    (ii) $\int_1^4 \left(x^2 - x\right)\, dx$
73. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.
74. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.
75. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.