The Straight Line

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***Abstract*—Solved problems from JEE mains papers related to 2D lines in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1. A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1}$$
$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{3}$$

Then (1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{4}$$
$$2\mathbf{n}^T \mathbf{x} = -5 \tag{5}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{6}$$
$$2\mathbf{n}^T \mathbf{B} = -5 \tag{7}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{8}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{9}$$
$$\mathbf{A} = -k\mathbf{B} \tag{10}$$

Substituting in (6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{11}$$
$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{12}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{13}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{14}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{15}$$

by $2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (15), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \tag{17}$$

However,

$$PQ = 2\sqrt{3} \tag{18}$$
$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \, \|\mathbf{m}\| = 2\sqrt{3} \tag{19}$$
$$\implies \lambda = \pm \frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{20}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{21}$$

from (16). Since $\mathbf{Q}$ lies in the third quadrant,

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from (17) and (20),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{6} \\ 1-\sqrt{6} \end{pmatrix} \tag{22}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x}-\mathbf{Q}) = 0 \tag{23}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3-\sqrt{6} \tag{24}$$

3. A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{25}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 12 \tag{26}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.
Solution: The intersection of the lines in (25) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{27}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{28}$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{29}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} \tag{30}$$

$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R} \tag{31}$$

Let the equation of AB be

$$\mathbf{n}^T(\mathbf{x}-\mathbf{C}) = 0 \tag{32}$$

Since **R** lies on AB,

$$\mathbf{n}^T(\mathbf{R}-\mathbf{C}) = 0 \tag{33}$$

Also,

$$\mathbf{n}^T(\mathbf{A}-\mathbf{B}) = 0 \tag{34}$$

Substituting from (30) in (34),

$$\mathbf{n}^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} = 0 \tag{35}$$

From (33) and (35),

$$(\mathbf{R}-\mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} \tag{36}$$

for some constant k. Multiplying both sides of (36) by

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{37}$$

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R}-\mathbf{C}) = k\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R}$$
$$= k\mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \tag{38}$$

$$\because \mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \tag{39}$$

which can be easily verified for any **R**. from (38),

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R}-\mathbf{C}) = 0$$
$$\implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{C} = 0$$
$$\implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{C}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \tag{40}$$

which is the desired locus.

4. Two sides of a rhombus are along the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{41}$$

$$\begin{pmatrix} 7 & -1 \end{pmatrix}\mathbf{x} - 5 = 0. \tag{42}$$

If its diagonals intersect at

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{43}$$

find its vertices.

5. Let k be an integer such that the triangle with vertices

$$\begin{pmatrix} k \\ -3k \end{pmatrix}, \begin{pmatrix} 5 \\ k \end{pmatrix}, \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{44}$$

has area 28. Find the orthocentre of this triangle.

6. If an equlateral triangle, having centroid at the

origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{45}$$

then find the area of this triangle.

7. Find the locus of the point of intersection of the straight lines

$$\begin{pmatrix} t & -2 \end{pmatrix} \mathbf{x} - 3t = 0 \tag{46}$$

$$\begin{pmatrix} 1 & -2t \end{pmatrix} \mathbf{x} + 3 = 0 \tag{47}$$

8. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.
9. Find the locus of the point of intersection of the lines

$$\begin{pmatrix} \sqrt{2} & -1 \end{pmatrix} \mathbf{x} + 4\sqrt{2}k = 0 \tag{48}$$

$$\begin{pmatrix} \sqrt{2}k & k \end{pmatrix} \mathbf{x} - 4\sqrt{2} = 0 \tag{49}$$

10. The sides of a rhombus ABC are parallel to the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{50}$$

$$\begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} + 3 = 0. \tag{51}$$

If the diagonals of the rhombus intersect at

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{52}$$

and the vertex **A** (different) from the origin is on the y-axis, then find the ordinate of A.