Python with Linear Algebra: 2D

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Contents

***Abstract*—This manual introduces matrix computations using python and the properties of a triangle.**

1 Line

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \tag{1}$$

Find the direction vector and the normal vector for AB
Solution: The direction vector of AB is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{2}$$
$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{3}$$
$$= \begin{pmatrix} 1-(-2) \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{4}$$

1.2 Find the *normal* vector of AB.
Solution: The normal vector $\mathbf{n}$ is defined as

$$\mathbf{n}^T\mathbf{m} = 0 \tag{5}$$

and can be obtained as

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{6}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{7}$$
$$= \begin{pmatrix} 0\times 3 + 1\times 5 \\ -1\times 3 + 0\times 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{8}$$

1.3 Find the equation of AB.
Solution: The desired equation is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{9}$$
$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{10}$$

1.4 Draw the line AB
Solution: The following code plots AB in Fig. 1.4

```
#Plotting AB
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if


A = np.array([−2,−2])
B = np.array([1,3])
len =10

x_AB = np.zeros((2,len))
lam = np.linspace(0,1,len)
for i in range(len):
    temp1 = A + lam[i]∗(B−A)
    x_AB[:,i]= temp1.T

plt.plot(x_AB[0,:],x_AB[1,:],label='$AB$')
```

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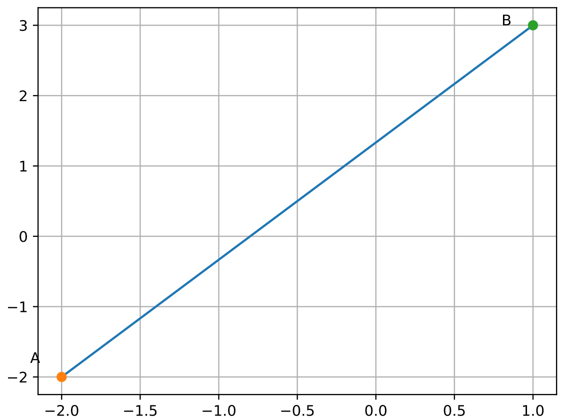

Fig. 1.4

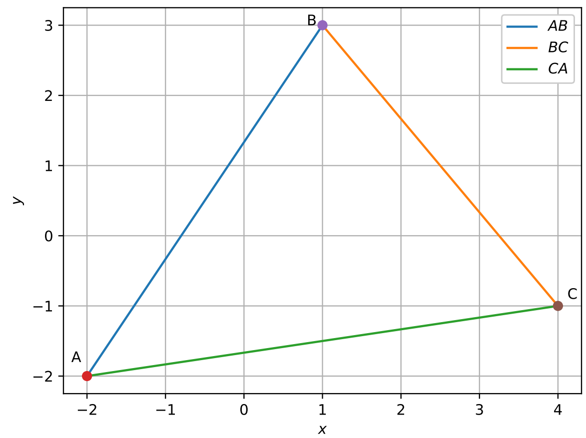

Fig. 1.5

```
plt.grid() # minor

plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 + 0.1), A[1] * (1 − 0.1) , '
    A')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 − 0.2), B[1] * (1) , 'B')

#if using termux
plt.savefig('../figs/draw_line.pdf')
plt.savefig('../figs/draw_line.eps')
subprocess.run(shlex.split("termux−open␣../
    figs/draw_line.pdf"))
#else
#plt.show()
```

1.5 Draw $\triangle ABC$.
Solution: The following codes yields the desired plot in Fig. 1.5

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    coeffs.py
```

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    draw_triangle.py
```

1.6 Find the equation of the line in terms of the normal vector.
Solution: The desired equation is

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = \mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \tag{11}$$

$$\implies \begin{pmatrix} 5 & -3 \end{pmatrix}\mathbf{x} = -\begin{pmatrix} 5 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -4 \tag{12}$$

1.7 Find the equations of BC and CA.

2 Altitudes of a Triangle

2.1 In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of ΔABC.
2.2 Find the equation of AP.
Solution: The normal vector of AP is $\mathbf{B}-\mathbf{C}$. From (11), the equation of AP is

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{x}-\mathbf{A}) = 0 \tag{13}$$

$$\implies \begin{pmatrix} -3 & 4 \end{pmatrix}\mathbf{x} = -\begin{pmatrix} -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -2 \tag{14}$$

2.3 Find the equation of the altitude BQ.
Solution: The desired equation is

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{x}-\mathbf{B}) = 0 \tag{15}$$

$$\implies \begin{pmatrix} 6 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9 \tag{16}$$

2.4 Find the equation of the altitude CR.
2.5 Find the point of intersection of AP and BQ.
Solution: (13) and (15) can be stacked together into the matrix equation

$$\begin{pmatrix} -3 & 4 \\ 6 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \tag{17}$$

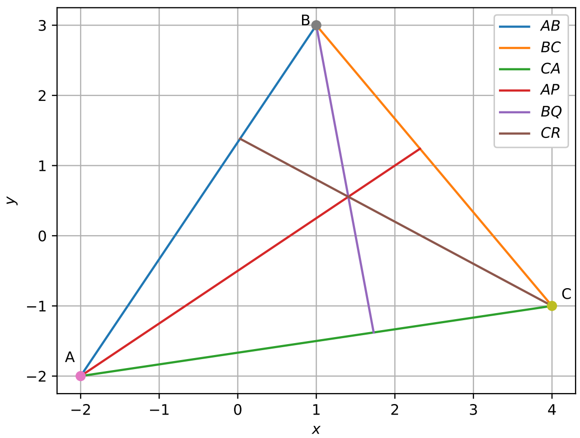

Fig. 2.9

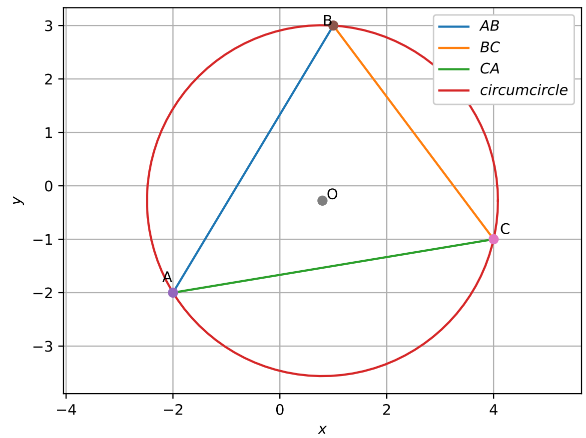

Fig. 3.4

The following code computes the point of intersection.

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/orthocentre.py

2.6 Find the point of intersection of and BQ and CR. Comment.

2.7 Find $\mathbf{P}$
Solution: The following code finds the required points.

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/alt_foot.py

2.8 Find $\mathbf{Q}$ and $\mathbf{R}$.

2.9 Draw AP, BQ and CR and verify that they meet at a point $\mathbf{H}$.
Solution: The following code plots the altitudes in Fig. 2.9

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/alt_draw.py

3 Circumcircle

3.1 Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be points on a circle with centre $\mathbf{O}$ and radius r.

3.2 Find $\mathbf{O}$.
Solution: The equation of the circle is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{18}$$

$$\implies \|\mathbf{x}-\mathbf{O}\|^2 = (\mathbf{x}-\mathbf{O})^T(\mathbf{x}-\mathbf{O}) = R^2 \tag{19}$$

From (18),

$$\|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \tag{20}$$

$$\implies (\mathbf{A}-\mathbf{O})^T(\mathbf{A}-\mathbf{O}) - (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = 0 \tag{21}$$

which can be simplified as

$$(\mathbf{A}-\mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{22}$$

Similarly,

$$(\mathbf{B}-\mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{23}$$

The following code computes $\mathbf{O}$ using the above two equations.

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcentre.py

3.3 Find the radius R.

3.4 Plot the *circumcircle* of $\triangle ABC$.
Solution: The following code plots Fig. 3.4

https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/circumcircle.py

4 Medians of a Triangle

4.1 Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.

4.2 Find the equations of AD, BE and CF. These lines are the *medians* of ΔABC
4.3 Find the point of intersection of AD and CF.
4.4 Verify that $\mathbf{G}$ is the point of intersection of BE, CF as well as AD, BE. $\mathbf{G}$ is known as the *centroid* of ΔABC.
4.5 Graphically show that the medians of ΔABC meet at the centroid.
4.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{24}$$

5 Incircle

5.1 Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.
5.2 Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \tag{25}$$
$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{26}$$
$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{27}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{U} - \mathbf{I}) = 0 \implies OD \perp BC \tag{28}$$

5.3 Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \tag{29}$$
$$\implies \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \tag{30}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{31}$$
$$\implies \mathbf{I} - \mathbf{U} = \lambda\mathbf{n} \tag{32}$$
$$\text{or} \quad r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \, \|\mathbf{n}\| \tag{33}$$

From (30) and (31)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{34}$$
$$\implies \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda\|\mathbf{n}\|^2 \tag{35}$$
$$\implies r = |\lambda| \, \|\mathbf{n}\| = \frac{\left|\mathbf{n}^T (\mathbf{I} - \mathbf{B})\right|}{\|\mathbf{n}\|} \tag{36}$$

from (33). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{37}$$
$$r = \left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| \tag{38}$$

5.4 Find $\mathbf{I}$.
Solution: Since $r = IU = IV = IW$, from (38),

$$\left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| = \left|\mathbf{n}_2^T (\mathbf{I} - \mathbf{C})\right| = \left|\mathbf{n}_3^T (\mathbf{I} - \mathbf{A})\right| \tag{39}$$

where $\mathbf{n}_2, \mathbf{n}_3$ are unit normals of CA, AB respectively. (39) can be expressed as

$$\mathbf{n}_1^T (\mathbf{I} - \mathbf{B}) = k_1\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) \tag{40}$$
$$\mathbf{n}_2^T (\mathbf{I} - \mathbf{C}) = k_2\mathbf{n}_3^T (\mathbf{I} - \mathbf{A}) \tag{41}$$

where $k_1, k_2 = \pm 1$. The above equations can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{n}_1 - k_1\mathbf{n}_2 & \mathbf{n}_2 - k_2\mathbf{n}_3 \end{pmatrix}^T \mathbf{I} = \begin{pmatrix} \mathbf{n}_1^T\mathbf{B} - k_1\mathbf{n}_2^T\mathbf{C} \\ \mathbf{n}_2^T\mathbf{C} - k_2\mathbf{n}_3^T\mathbf{A} \end{pmatrix} \tag{42}$$

5.5 Show that $\mathbf{I}$ lies inside $\triangle ABC$ for $k_1 = k_2 = 1$
5.6 Compute $\mathbf{I}$ and r.
Solution:

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    incentre.py
```

5.7 Plot the incircle of $\triangle ABC$
Solution: The following code plots the incircle in Fig. 5.7

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    incircle.py
```

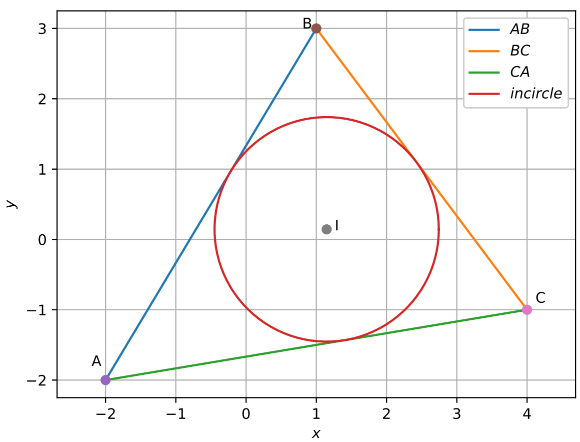

Fig. 5.7