

# Geometry: Maths Olympiad

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1. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?
2. In  $\triangle ABC$ , we have  $AC = BC = 7$  and  $AB = 2$ . Suppose that  $D$  is a point on line  $AB$  such that  $B$  lies between  $A$  and  $D$  and  $CD = 8$ . What is the length of the segment  $BD$ ?
3. In rectangle  $ABCD$ ,  $AB = 5$  and  $BC = 3$ . Points  $F$  and  $G$  are on line segment  $CD$  so that  $DF = 1$  and  $GC = 2$ . Lines  $AF$  and  $BG$  intersect  $E$ . What is the area of  $\triangle AEB$ ?
4.  $ABCD$  is a square and  $AB = 1$ . Equilateral triangle  $AYB$  and  $CXD$  are drawn such that  $X$  and  $Y$  are inside the square. What is the length of  $XY$ ?
5.  $O$  and  $I$  are the circumcentre and incentre of  $\triangle ABC$  respectively. Suppose  $O$  lies in the interior of  $\triangle ABC$  and  $I$  lies on the circle passing through  $B$ ,  $O$ , and  $C$ . What is the magnitude of  $\angle BAC$  in degrees?
6.  $PS$  is a line segment of length 4 and  $O$  is the midpoint of  $PS$ . A semicircle arc is drawn with  $PS$  as diameter. Let  $X$  be the midpoint of this arc.  $Q$  and  $R$  are points on the arc  $PXS$  such that  $QR$  is parallel to  $PS$  and the semicircular arc drawn with  $QR$  as diameter is tangent to  $PS$ . What is the area of the region  $QXROQ$  bounded by the two semicircular arcs?
7. Let  $AD$  and  $BC$  be the parallel sides of a trapezium  $ABCD$ . Let  $P$  and  $Q$  be the midpoints of the diagonals  $AC$  and  $BD$ . If  $AD = 16$  and  $BC = 20$ , What is the length of  $PQ$ ?
8. In a triangle  $ABC$ , let  $H$ ,  $I$  and  $O$  be the orthocentre, incentre and circumcentre, respectively. If the points  $B$ ,  $H$ ,  $I$ ,  $C$  lie on a circle, What is magnitude of  $\angle BOC$  in degrees?
9. Let  $ABC$  be an equilateral triangle. Let  $P$  and  $S$  be points on  $AB$  and  $AC$ , respectively, and let  $Q$  and  $R$  be points on  $BC$  such that  $PQRS$  is a triangle. If  $PQ = \sqrt{3}PS$  and the area of  $PQRS$  is  $28\sqrt{3}$ , What is the length of  $PC$ ?
10. Let  $A_1, B_1, C_1, D_1$  be the midpoints of the sides of a convex quadrilateral  $ABCD$  and let  $A_1, B_2, C_2, D_2$  be the midpoints of the quadrilateral  $A_1B_1C_1D_1$ . If  $A_2B_2C_2D_2$  is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of  $ABCD$ ?
11. Let  $S$  be a circle with centre  $O$ . A chord  $AB$ , not a diameter, divides  $S$  into two regions  $R_1$  and  $R_2$  such that  $O$  belongs to  $R_2$ . Let  $S_1$  be a circle with centre in  $R_1$ , touching  $AB$  at  $X$  and  $S$  internally. Let  $S_2$  be a circle with centre in  $R_2$ , touching  $AB$  at  $Y$ , the circle  $S$  internally and passing through the centre of  $S$ . The point  $X$  lies on the diameter passing through the centre of  $S_2$  and  $\angle YXO = 30^\circ$ . If the radius of  $S_2$  is 100 then what is the radius of  $S_1$ ?
12. In a triangle  $ABC$  with  $\angle BCA = 90^\circ$ , the perpendicular bisector of  $AB$  intersects segments  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively. If the ratio of the area of quadrilateral  $BXYC$  to the area of triangle  $ABC$  is  $13 : 18$  and  $BC = 12$  then what is the length of  $AC$ ?
13. Let  $ABCD$  be a convex quadrilateral with perpendicular diagonals. If  $AB = 20$ ,  $BC = 70$  and  $CD = 90$ , then what is the value of  $DA$ ?
14. In a triangle  $ABC$ ,  $X$  and  $Y$  are points on the segments  $AB$  and  $AC$ , respectively, such that  $AX : XB = 1 : 2$  and  $AY : YC = 2 : 1$ . If the area of triangle  $AXY$  is 10 then what is the area of triangle  $ABC$ ?
15. Let  $ABCD$  be a convex quadrilateral with  $\angle DAB = \angle BDC = 90^\circ$ . Let the incircles of triangles  $ABD$  and  $BCD$  touch  $BD$  at  $P$  and  $Q$ , respectively, with  $P$  lying in between  $B$  and  $Q$ . If  $AD = 999$  and  $PQ = 200$  then what is the sum of the radii of the incircles of triangle  $ABD$  and  $BDC$ ?
16. Let  $XOY$  be a triangle with  $\angle XOY = 90^\circ$ . Let  $M$  and  $N$  be the midpoints of legs  $OX$  and  $OY$ , respectively. Suppose that  $XN = 19$  and  $YM =$

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22. What is  $XY$ ?
17. In a triangle  $ABC$ , let  $I$  denote the incenter. Let the lines  $AI$ ,  $BI$  and  $CI$  intersect the incircle at  $P$ ,  $Q$  and  $R$ , respectively. If  $\angle BAC = 40^\circ$ , what is the value of  $\angle QPR$  in degrees?
18. In acute-angled triangle  $ABC$ , let  $D$  be the foot of the altitude from  $A$ , and  $E$  be the midpoint of  $BC$ . Let  $F$  be the midpoint of  $AC$ . Suppose  $\angle BAE = 40^\circ$ . If  $\angle DAE = \angle DFE$ , What is the magnitude of  $\angle ADF$  in degrees?
19. The circle  $\Omega$  touches the circle  $\Omega$  internally at  $P$ . The centre  $O$  of  $\Omega$  is outside  $\Omega$ . Let  $XY$  be a diameter of  $\Omega$  which is also tangent to  $\Omega$ . Assume  $PY > PX$ . Let  $PY$  intersect  $\Omega$  at  $Z$ . If  $YZ = 2PZ$ , What is the magnitude of  $\angle PYX$  in degrees?
20. In rectangle  $ABCD$ ,  $AB = 8$  and  $BC = 20$ . Let  $P$  be a point on  $AD$  such that  $\angle BPC = 90^\circ$ . If  $r_1, r_2, r_3$  are the radii of the incircles of triangles  $APB$ ,  $BPC$  and  $CPD$ , What is the value of  $r_1 + r_2 + r_3$ ?
21. In a rectangle  $ABCD$ ,  $E$  is the midpoint of  $AB$ ;  $F$  is a point on  $AC$  such that  $BF$  is perpendicular to  $AC$ ; and  $FE$  perpendicular to  $BD$ . Suppose  $BC = 8\sqrt{3}$ . Find  $AB$ ?
22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?
23. Let  $P$  be an interior point of a triangle  $ABC$  whose side lengths are 26, 65, 78. The line through  $P$  parallel to  $BC$  meets  $AB$  in  $K$  and  $AC$  in  $L$ . The line through  $P$  parallel to  $CA$  meets  $BC$  in  $M$  and  $BA$  in  $N$ . The line through  $P$  parallel to  $AB$  meets  $CA$  in  $S$  and  $CB$  in  $T$ . If  $KL$ ,  $MN$ ,  $ST$  are of equal lengths, find this common length.
24. Let  $ABCD$  be a rectangle and let  $E$  and  $F$  be points on  $CD$  and  $BC$  respectively such that  $\text{area}(ADE) = 16$ ,  $\text{area}(CEF) = 9$  and  $\text{area}(ABF) = 25$ . What is the area of triangle  $AEF$ ?
25. Let  $AB$  and  $CD$  be two parallel chords in a circle with radius 5 such that the centre  $O$  lies between these chords. Suppose  $AB = 6$ ,  $CD = 8$ . Suppose further that the area of the part of the circle lying between the chords  $AB$  and  $CD$  is  $\frac{m\pi+n}{k}$ , where  $m, n, k$  are positive integers with  $\gcd(m, n, k) = 1$ . What is the value of  $m + n$ ?
26. Consider the areas of the four triangles obtained by drawing the diagonals  $AC$  and  $BD$  of a trapezium  $ABCD$ . The product of these areas, taken two at a time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.
27. In a quadrilateral  $ABCD$ , it is given that  $AB = AD = 13$ ,  $BC = CD = 20$ ,  $BD = 24$ . If  $r$  is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to  $r$ ?
28. Let  $ABCD$  be a trapezium in which  $AB \parallel CD$  and  $AD \perp AB$ . Suppose  $ABCD$  has an incircle which touches  $AB$  at  $Q$  and  $CD$  at  $P$ . Given that  $PC = 36$  and  $QB = 49$ , find  $PQ$ ?
29. Let  $AB$  be a chord of a circle with centre  $O$ . Let  $C$  be a point on the circle such that  $\angle ABC = 30^\circ$  and  $O$  lies inside the triangle  $ABC$ . Let  $D$  be a point on  $AB$  such that  $\angle DCO = \angle OCB = 20^\circ$ . Find the measure of  $\angle CDO$  in degrees.
30. In a triangle  $ABC$ , right-angled at  $A$ , the altitude through  $A$  and the internal bisector of  $\angle A$  have lengths 3 and 4, respectively. Find the length of the median through  $A$ .
31. Triangles  $ABC$  and  $DEF$  are such that  $\angle A = \angle D$ ,  $AB = DE = 17$ ,  $BC = EF = 10$  and  $AC - DF = 12$ . What is  $AC + DF$ ?
32. Let  $ABC$  be an acute-angled triangle and let  $H$  be its orthocentre. Let  $G_1, G_2$  and  $G_3$  be the centroids of the triangles  $HBC$ ,  $HCA$  and  $HAB$ , respectively. If the area of triangle  $G_1G_2G_3$  is 7 units, what is the area of triangle  $ABC$ ?
33. Let  $D$  be an interior point of the side  $BC$  of a triangle  $ABC$ . Let  $I_1$  and  $I_2$  be the incentres of triangles  $ABD$  and  $ACD$  respectively. Let  $AI_1$  and  $AI_2$  meet  $BC$  in  $E$  and  $F$  respectively. If  $\angle BI_1E = 60^\circ$ , what is the measure of  $\angle CI_2F$  in degrees?
34. In a triangle  $ABC$ , the median from  $B$  to  $CA$  is perpendicular to the median from  $C$  to  $AB$ . If the median from  $A$  to  $BC$  is 30, determine  $(BC^2 + CA^2 + AB^2)/100$ .
35. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?
36. Let  $ABC$  be a triangle and let  $\Omega$  be its circum-

circle. The internal bisectors of angles  $A$ ,  $B$  and  $C$  intersect  $\Omega$  at  $A_1, B_1$ , and  $C_1$ , respectively, and the internal bisectors of angles  $A_1, B_1$ , and  $C_1$ , of the triangle  $A_1B_1C_1$ , intersect  $\Omega$  at  $A_2, B_2$ , and  $C_2$ , respectively. If the smallest angle of triangle  $ABC$  is  $40^\circ$ , what is the magnitude of the smallest angle of triangle  $A_2B_2C_2$ , in degrees?

37. Let  $AB$  be a diameter of a circle and let  $C$  be a point on the segment  $AB$  such that  $AC : CB = 6 : 7$ . Let  $D$  be a point on the circle such that  $DC$  is perpendicular to  $AB$ . Let  $DE$  be the diameter through  $D$ . If  $[XYZ]$  denotes the area of the triangle  $XYZ$ , find  $[ABD]/[CDE]$  to the nearest integer.
38. Let  $ABCD$  be a convex cyclic quadrilateral. Suppose  $P$  is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of  $ABCD$  is the least. If  $(PA, PB, PC, PD) = (3, 4, 6, 8)$ , What is the maximum possible area of  $ABCD$ ?
39. Let  $ABC$  be a triangle with sides 51, 52, 53. Let  $\Omega$  denote the incircle of  $\triangle ABC$ . Draw tangents to  $\Omega$  which are parallel to the sides of  $ABC$ . Let  $r_1, r_2, r_3$  be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed  $r_1 + r_2 + r_3$ .
40. In a triangle  $ABC$ , the median  $AD$  (with  $D$  on  $BC$ ) and the angle bisector  $BE$  (with  $E$  on  $AC$ ) are perpendicular to each other. If  $AD = 7$  and  $BE = 9$ , find the integer nearest to the area of triangle  $ABC$ .