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CONTENTS

1	Triangle	1
1.1	Construction Examples . . .	1
1.2	Construction Exercises . . .	4
1.3	Triangle Examples	5
1.4	Triangle Exercises	5
2	Quadrilateral	6
2.1	Construction Examples . . .	6
2.2	Construction Exercises . . .	7
2.3	Quadrilateral Examples . . .	8
2.4	Quadrilateral Geometry . . .	8
3	line	9
3.1	Examples	9
3.2	Points and Vectors	13
3.3	Points on a Line	15
3.4	Lines and Planes	16
3.5	Miscellaneous	22
4	Circle	24
4.1	Construction Examples . . .	24
4.2	Construction Exercises . . .	25
4.3	Circle Geometry	25

Abstract—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/ncert/codes
```

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1 TRIANGLE

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90^\circ$, $a = 4$ and $b = 3$.

Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.1.1)$$

The following code plots Fig. 1.1.1

```
codes/triangle/rt_triangle.py
```

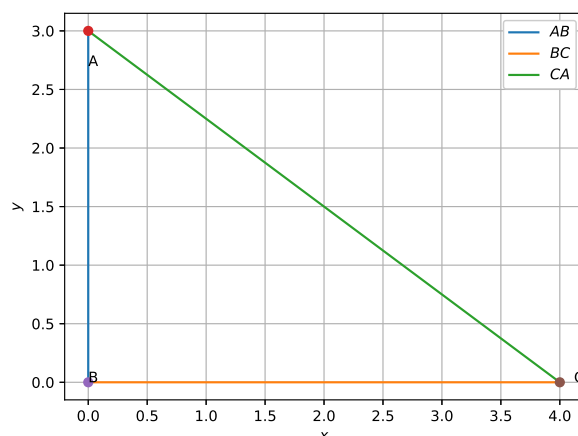


Fig. 1.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.

Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.1.2.1)$$

$$\mathbf{A}^T \triangleq (p \quad q) \quad (1.1.2.2)$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T \mathbf{A} = (p \quad q) \begin{pmatrix} p \\ q \end{pmatrix} \quad (1.1.2.3)$$

$$= p \times p + q \times q = p^2 + q^2 \quad (1.1.2.4)$$

Then

$$AB \triangleq \|A - B\|^2 = \|A\|^2 = c^2 \quad \because B = 0 \quad (1.1.2.5)$$

$$BC = \|C - B\|^2 = \|C\|^2 = a^2 \quad (1.1.2.6)$$

$$AC = \|A - C\|^2 = b^2 \quad (1.1.2.7)$$

From (1.1.2.7),

$$b^2 = \|A - C\|^2 = \|A - C\|^T \|A - C\| \quad (1.1.2.8)$$

$$= A^T A + C^T C - A^T C - C^T A \quad (1.1.2.9)$$

$$= \|A\|^2 + \|C\|^2 - 2A^T C \quad (\because A^T C = C^T A) \quad (1.1.2.10)$$

$$= a^2 + c^2 - 2ap \quad (1.1.2.11)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (1.1.2.12)$$

From (1.1.2.5),

$$\|A\|^2 = c^2 = p^2 + q^2 \quad (1.1.2.13)$$

$$\implies q = \pm \sqrt{c^2 - p^2} \quad (1.1.2.14)$$

The following code plots Fig. 1.1.2

codes/triangle/draw_triangle.py

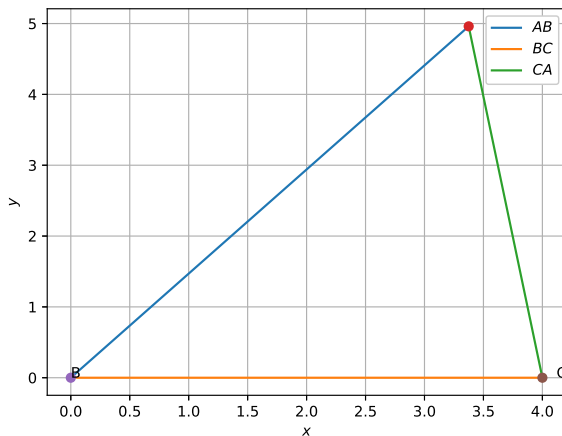


Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.

Solution: The sides of the similar triangle are $\frac{7}{5}a$, $\frac{7}{5}b$ and $\frac{7}{5}c$.

4. Construct an isosceles triangle whose base is $a = 8\text{cm}$ and altitude $AD = h = 4\text{cm}$

Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \quad (1.1.4.1)$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.

Solution: From the given information,

$$a + b + c = 11 \quad (1.1.5.1)$$

$$b = c \quad (\because B = C = 45^\circ) \quad (1.1.5.2)$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \quad (1.1.5.3)$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \quad (1.1.5.4)$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \quad (1.1.5.5)$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \quad (1.1.5.6)$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times (-\sqrt{2}) - 2 \times 0}{1 \times (-\sqrt{2}) - 2 \times 1} \quad (1.1.5.7)$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \quad (1.1.5.8)$$

$$b = \frac{\begin{vmatrix} 11 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \quad (1.1.5.9)$$

by expanding the determinants. The following code may be used to compute a, b and c .

codes/triangle/triangle_det.py

6. Repeat Problem 1.1.5 using a single matrix equation.

Solution: The equations

$$a + 2b = 11 \quad (1.1.6.1)$$

$$a - b\sqrt{2} = 0 \quad (1.1.6.2)$$

$$b - c = 0 \quad (1.1.6.3)$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \quad (1.1.6.4)$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.5)$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.6)$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \quad (1.1.6.7)$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \quad (1.1.6.8)$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \quad (1.1.6.9)$$

The determinants of larger matrices can be

expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60^\circ$.

Solution: In Fig. (1.1.7), $AD \perp BC$.

$$\cos C = \frac{y}{b}, \quad (1.1.7.1)$$

$$\cos B = \frac{x}{a}, \quad (1.1.7.2)$$

Thus,

$$a = x + y = b \cos C + c \cos B, \quad (1.1.7.3)$$

$$b = c \cos A + a \cos C \quad (1.1.7.4)$$

$$c = b \cos A + a \cos B \quad (1.1.7.5)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.1.7.6)$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \quad (1.1.7.7)$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \quad (1.1.7.8)$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca \cos B \quad (1.1.7.9)$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

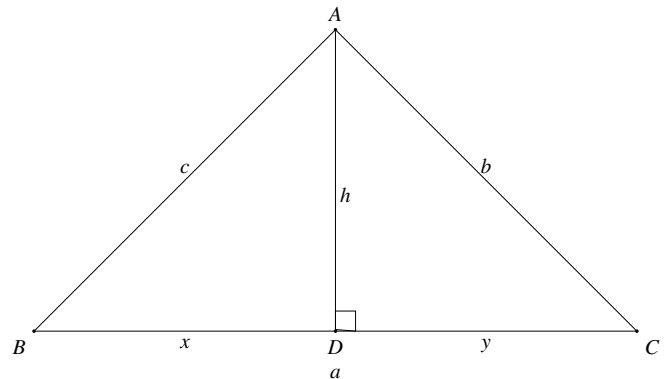


Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7$, $\angle B = 45^\circ$ and $\angle A = 105^\circ$.

Solution: In Fig. (1.1.7),

$$\sin B = \frac{h}{c} \quad (1.1.8.1)$$

$$\sin C = \frac{h}{b} \quad (1.1.8.2)$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.1.8.3)$$

Thus,

$$c = \frac{a \sin C}{\sin A} \quad (1.1.8.4)$$

where

$$C = 180 - A - B \quad (1.1.8.5)$$

9. Draw $\triangle ABC$ if $AB = 3$, $AC = 5$ and $\angle C = 30^\circ$.

Solution: From (1.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (1.1.9.1)$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \quad (1.1.9.2)$$

$$\therefore (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \quad (1.1.9.3)$$

(1.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \quad (1.1.9.4)$$

$$\Rightarrow (a - b \cos C)^2 = b^2 (1 - \cos^2 C) - c^2 \quad (1.1.9.5)$$

$$\text{or, } a = b \cos C \pm \sqrt{b^2 (1 - \cos^2 C) - c^2} \quad (1.1.9.6)$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \quad (1.1.10.1)$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (1.1.10.2)$$

Verify (1.1.9.6) using (1.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.

Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \quad (1.1.11.1)$$

$$\Rightarrow (18 - c)^2 = 12^2 + c^2 \quad (1.1.11.2)$$

which can be simplified to obtain

$$36c - 180 = 0 \quad (1.1.11.3)$$

$$\Rightarrow c = 5 \quad (1.1.11.4)$$

and $b = 13$

12. Find a simpler solution for Problem 1.1.5

Solution: Use cosine formula.

13. In $\triangle ABC$, $a = 7$, $\angle B = 75^\circ$ and $b + c = 13$. Alternatively,

$$a = b \cos C + c \cos B \quad (1.1.13.1)$$

$$b \sin C = c \sin B \quad (1.1.13.2)$$

$$a + b + c = 11 \quad (1.1.13.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (1.1.13.4)$$

Solving the equivalent matrix equation gives the desired answer.

1.2 Construction Exercises

1. In $\triangle ABC$, $a = 8$, $\angle B = 45^\circ$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6$, $\angle B = 60^\circ$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a + b + c = 11$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$.
4. Construct $\triangle xyz$ where $xy = 4.5$, $yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4$, $QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5$, $BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3$, $QR = 5.5$ and $\angle PQR = 60^\circ$.
9. Construct $\triangle DEF$ such that $DE = 5$, $DF = 3$ and $\angle D = 90^\circ$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110° .
11. Construct $\triangle ABC$ with $BC = 7.5$, $AC = 5$ and $\angle C = 60^\circ$.

12. Construct $\triangle XYZ$ if $XY = 6$, $\angle X = 30^\circ$ and $\angle Y = 100^\circ$.
13. If $AC = 7$, $\angle A = 60^\circ$ and $\angle B = 50^\circ$, can you draw the triangle?
14. Construct $\triangle ABC$ given that $\angle A = 60^\circ$, $\angle B = 30^\circ$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5$, $\angle Q = 105^\circ$ and $\angle R = 40^\circ$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2$, $\angle E = 110^\circ$ and $\angle F = 180^\circ$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled \triangle whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 1.2.21.

S.No	Triangle	Given Measurements		
1	$\triangle ABC$	$\angle A = 85^\circ$	$\angle B = 115^\circ$	$AB = 5$
2	$\triangle PQR$	$\angle Q = 30^\circ$	$\angle R = 60^\circ$	$QR = 4.7$
3	$\triangle ABC$	$\angle A = 70^\circ$	$\angle B = 50^\circ$	$AC = 3$
4	$\triangle LMN$	$\angle L = 60^\circ$	$\angle N = 120^\circ$	$LM = 5$
5	$\triangle ABC$	$BC = 2$	$AB = 4$	$AC = 2$
6	$\triangle PQR$	$PQ = 2.5$	$QR = 4$	$PR = 3.5$
7	$\triangle XYZ$	$XY = 3$	$YZ = 4$	$XZ = 5$
8	$\triangle DEF$	$DE = 4.5$	$EF = 5.5$	$DF = 4$

TABLE 1.2.21

1.3 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

Solution: The direction vectors of AB and BC are

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad (1.3.1.1)$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (1.3.1.2)$$

Since

$$\mathbf{A} - \mathbf{B} \neq k(\mathbf{B} - \mathbf{C}), \quad (1.3.1.3)$$

the points are not collinear and form a triangle.
The following code plots Fig. ??

```
codes/triangle/check_tri.py
```

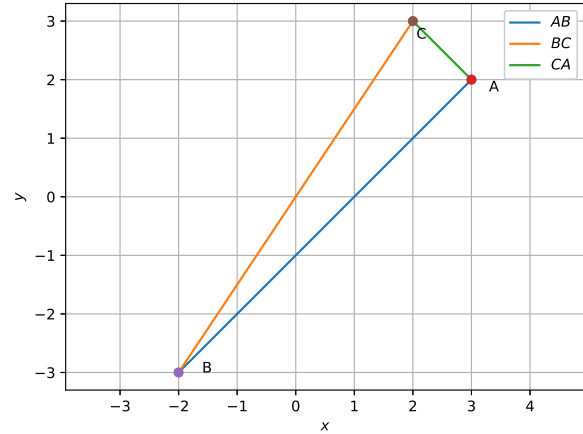


Fig. 1.3.1

2. Find the area of a triangle whose vertices are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
3. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.
4. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
5. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \quad (1.3.5.1)$$

the vertices of a right angled triangle?

6. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.3.6.1)$$

7. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of \mathbf{A} and \mathbf{B} are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point \mathbf{C} .

8. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.3.8.1)$$

are the vertices of a right angled triangle.

9. (Cauchy-Schwarz Inequality:) Show that

$$|\mathbf{a}^T \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad (1.3.9.1)$$

10. (Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \quad (1.3.10.1)$$

11. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.3.11.1)$$

as its vertices.

1.4 Triangle Exercises

1. The vertices of $\triangle PQR$ are

$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find the equation of the median through the vertex \mathbf{R} .

2. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the equation and length of the altitude from the vertex \mathbf{A} .

3. Find the area of the triangle whose vertices are

a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

4. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

5. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.

6. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (1.4.6.1)$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (1.4.6.2)$$

7. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the vertices of $\triangle ABC$.

a) The median from \mathbf{A} meets BC at \mathbf{D} . Find the coordinates of the point \mathbf{D} .

b) Find the coordinates of the point \mathbf{P} on AD such that $AP : PD = 2 : 1$.

c) Find the coordinates of the points \mathbf{Q} and \mathbf{R} on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

8. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.4.8.1)$$

9. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.4.9.1)$$

are the vertices of a right angled triangle.

10. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.

11. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.

12. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ as its vertices.}$$

13. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

14. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

15. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} =$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$$

16. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.
17. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.4.17.1)$$

are the vertices of an isosceles triangle.

2 QUADRILATERAL

2.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5$, $BC = b = 5.5$, $CD = c = 4$, $AD = d = 6$ and $AC = e = 7$.
Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting

$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.1.1.1)$$

it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ can be obtained by rotating an equivalent triangle with AC on the x -axis by an angle θ with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (2.1.1.2)$$

and

$$\cos \theta = \frac{a^2 + e^2 - b^2}{2ae} \quad (2.1.1.3)$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} \quad (2.1.1.4)$$

The coordinates of the rotated triangle ACD are

$$\mathbf{D} = \mathbf{P} \begin{pmatrix} h \\ k \end{pmatrix} \quad (2.1.1.5)$$

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.1.6)$$

$$\mathbf{C} = \mathbf{P} \begin{pmatrix} e \\ 0 \end{pmatrix} \quad (2.1.1.7)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (2.1.1.8)$$

The following code plots quadrilateral $ABCD$ in Fig. 2.1.1

```
codes/quad/draw_quad.py
```

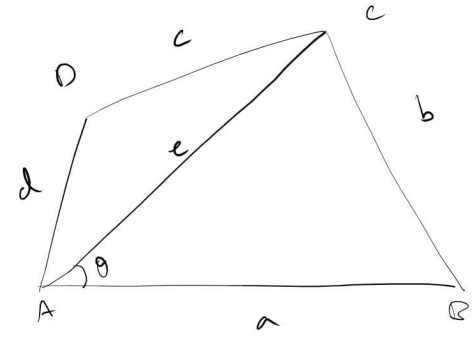


Fig. 2.1.1

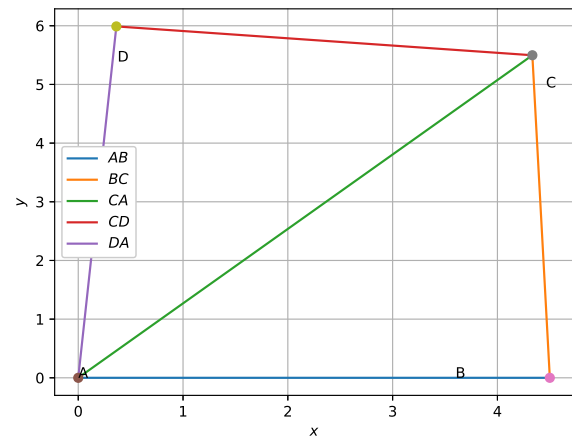


Fig. 2.1.1

2. Draw the parallelogram $MORE$ with $OR = 6$, $RE = 4.5$ and $EO = 7.5$.

Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel.

3. Construct a kite $EASY$ if $AY = 8$, $EY = 4$ and $SY = 6$.

Solution: The diagonals of a kite are perpendicular to each other.

4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.

Solution: Diagonals of a rhombus bisect each other at right angles.

2.2 Construction Exercises

- Construct a quadrilateral $ABCD$ such that $AB = 5$, $\angle A = 50^\circ$, $AC = 4$, $BD = 5$ and $AD = 6$.
- Construct $PQRS$ where $PQ = 4$, $QR = 6$, $RS = 5$, $PS = 5.5$ and $PR = 7$.

3. Draw *JUMP* with $JU = 3.5$, $UM = 4$, $MP = 5$, $PJ = 4.5$ and $PU = 6.5$
4. Construct a quadrilateral *ABCD* such that $BC = 4.5$, $AC = 5.5$, $CD = 5$, $BD = 7$ and $AD = 5.5$.
5. Can you construct a quadrilateral *PQRS* with $PQ = 3$, $RS = 3$, $PS = 7.5$, $PR = 8$ and $SQ = 4$?
6. Construct *LIFT* such that $LI = 4$, $IF = 3$, $TL = 2.5$, $LF = 4.5$, $IT = 4$.
7. Draw *GOLD* such that $OL = 7.5$, $GL = 6$, $GD = 6$, $LD = 5$, $OD = 10$.
8. DRAW rhombus *BEND* such that $BN = 5.6$, $DE = 6.5$.
9. construct a quadrilateral *MIST* where $MI = 3.5$, $IS = 6.5$, $\angle M = 75^\circ$, $\angle I = 105^\circ$ and $\angle S = 120^\circ$.
10. Can you construct the above quadrilateral *MIST* if $\angle M = 100^\circ$ instead of 75° .
11. Can you construct the quadrilateral *PLAN* if $PL = 6$, $LA = 9.5$, $\angle P = 75^\circ$, $\angle L = 150^\circ$ and $\angle A = 140^\circ$?
12. Construct *MORE* where $MO = 6$, $OR = 4.5$, $\angle M = 60^\circ$, $\angle O = 105^\circ$, $\angle R = 105^\circ$.
13. Construct *PLAN* where $PL = 4$, $LA = 6.5$, $\angle P = 90^\circ$, $\angle A = 110^\circ$ and $\angle N = 85^\circ$.
14. Construct parallelogram *HEAR* where $HE = 5$, $EA = 6$, $\angle R = 85^\circ$.
15. Draw rectangle *OKAY* with $OK = 7$ and $KA = 5$.
16. Construct *ABCD*, where $AB = 4$, $BC = 5$, $CD = 6.5$, $\angle B = 105^\circ$ and $\angle C = 80^\circ$.
17. Construct *DEAR* with $DE = 4$, $EA = 5$, $AR = 4.5$, $\angle E = 60^\circ$ and $\angle A = 90^\circ$.
18. Construct *TRUE* with $TR = 3.5$, $RU = 3$, $UE = 4$, $\angle R = 75^\circ$ and $\angle U = 120^\circ$.
19. Draw a square of side 4.5.
20. Can you construct a rhombus *ABCD* with $AC = 6$ and $BD = 7$?
21. Draw a square *READ* with $RE = 5.1$.
22. Draw a rhombus whose diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram *OKAY* with $OK = 5.5$ and $KA = 4.2$.

2.3 Quadrilateral Examples

1. Show that the points $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.

2. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p .
3. If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral *ABCD*.
4. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$ are the vertices of a parallelogram *ABCD* but it is not a rectangle.
5. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

2.4 Quadrilateral Geometry

1. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$. Also, find its area.
2. Find the area of a rhombus if its vertices are $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ taken in order.
3. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.
4. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
5. The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.
6. *ABCD* is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of *AB, BC, CD, DA* respectively. Is the quadrilateral *PQRS* a
 - a) square?
 - b) rectangle?
 - c) rhombus?

7. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
8. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.
9. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$.
10. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

3 LINE

3.1 Examples

- Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.
- Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.
- Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.
- Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 3.

Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \quad (3.1.4.1)$$

Then the point \mathbf{C}

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k + 1} \quad (3.1.4.2)$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{3}$. The following code plots Fig. 3.1.4

codes/line/draw_section.py

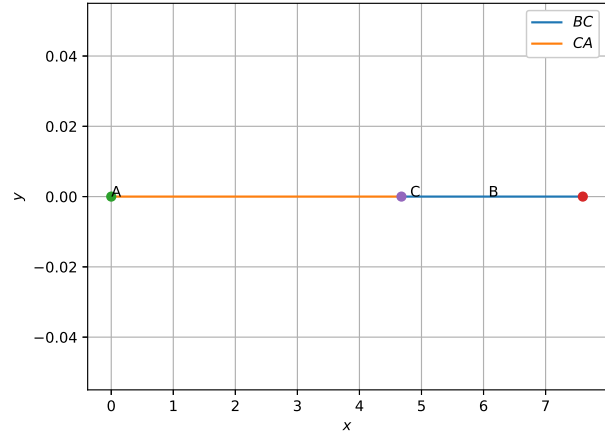


Fig. 3.1.4

- Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.
- In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points $\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$ (3.1.6.1)
- Find the coordinates of the points of trisection of the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$ (3.1.7.1)
- Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
- Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ k \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.
- Find the direction vectors and slopes of the lines passing through the points
 - $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
 - $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

- c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
- d) Making an inclination of 60° with the positive direction of the x-axis.
11. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
12. The line through the points $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} x \\ 24 \end{pmatrix}$. Find the value of x .
13. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
14. Find the equations of the lines parallel to the axes and passing through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
15. Find the equation of the line through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope -4 .
16. Find the equations of the lines parallel to axes and passing through $(-2, 3)$.
17. Write the equation of the line through the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.
18. Write the equation of the lines for which $\tan \theta = \frac{1}{2}$, where θ is the inclination of the line and
- a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.
19. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
20. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15° .
21. The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F , when $K = 0$.
22. Equation of a line is
- $$(3 \ -4) \cdot \mathbf{x} + 10 = 0. \quad (3.1.22.1)$$
- Find its
- a) slope,
- b) x - and y-intercepts.
23. Find the angle between the lines
- $$(1 \ -\sqrt{3}) \cdot \mathbf{x} = 5 \quad (3.1.23.1)$$
- $$(\sqrt{3} \ -1) \cdot \mathbf{x} = -6. \quad (3.1.23.2)$$
24. Find the equation of a line perpendicular to the line
- $$(1 \ -2) \cdot \mathbf{x} = 3 \quad (3.1.24.1)$$
- and passes through the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
25. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line
- $$(3 \ -4) \cdot \mathbf{x} = 26 \quad (3.1.25.1)$$
26. If the lines
- $$(2 \ 1) \cdot \mathbf{x} = 3 \quad (3.1.26.1)$$
- $$(5 \ k) \cdot \mathbf{x} = 3 \quad (3.1.26.2)$$
- $$(3 \ 1) \cdot \mathbf{x} = 2 \quad (3.1.26.3)$$
- are concurrent, find the value of k .
27. Find the distance of the line
- $$(4 \ 1) \cdot \mathbf{x} = 0 \quad (3.1.27.1)$$
- from the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ measured along the line making an angle of 135° with the positive x-axis.
28. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line
- $$(1 \ -3) \cdot \mathbf{x} = -4. \quad (3.1.28.1)$$
29. A line is such that its segment between the lines
- $$(5 \ -1) \cdot \mathbf{x} = -4 \quad (3.1.29.1)$$
- $$(3 \ 4) \cdot \mathbf{x} = 4 \quad (3.1.29.2)$$
- is bisected at the point $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Obtain its equation.
30. Show that the path of a moving point such that

its distances from two lines

$$(3 \ -2)\mathbf{x} = 5 \quad (3.1.30.1)$$

$$(3 \ 2)\mathbf{x} = 5 \quad (3.1.30.2)$$

are equal is a straight line.

31. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \quad (3.1.31.1)$$

32. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and

$$\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix} \text{ are collinear.}$$

33. Find the equation of set of points \mathbf{P} such that

$$PA^2 + PB^2 = 2k^2, \quad (3.1.33.1)$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \quad (3.1.33.2)$$

respectively.

34. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and

$$\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \text{ in the ratio } 2 : 3$$

- a) internally, and
b) externally.

35. Using section formula, prove that the three

$$\text{points } \begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix} \text{ are collinear.}$$

36. Find the ratio in which the line segment joining

$$\text{the points } \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix} \text{ and } \begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix} \text{ is divided by the YZ-}$$

plane.

37. Find the equation of the set of points \mathbf{P} such that its distances from the points $\mathbf{A} =$

$$\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

38. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \quad (3.1.38.1)$$

39. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (3.1.39.1)$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

40. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

41. Find a vector \mathbf{x} in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$.

42. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \quad (3.1.42.1)$$

43. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \quad (3.1.43.1)$$

44. Find the direction vector of PQ , where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \quad (3.1.44.1)$$

45. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (3.1.45.1)$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (3.1.45.2)$$

find \mathbf{R} , which divides PQ

a) internally,

b) externally.

46. Find the angle between two vectors \mathbf{a} and \mathbf{b} where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T \mathbf{b} = 1. \quad (3.1.46.1)$$

47. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

48. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

49. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad (3.1.49.1)$$

on the vector

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \quad (3.1.49.2)$$

50. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T \mathbf{b} = 4. \quad (3.1.50.1)$$

51. If \mathbf{a} is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \quad (3.1.51.1)$$

then find \mathbf{x} .

52. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (3.1.52.1)$$

find $\|\mathbf{a} \times \mathbf{b}\|$.

53. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (3.1.53.1)$$

54. Show that $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and

$$\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}, \text{ are collinear.}$$

55. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.

56. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (3.1.56.1)$$

evaluate

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}, \quad (3.1.56.2)$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

57. Let $\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find β_1, β_2 such that

$$\beta = \beta_1 + \beta_2, \beta_1 \parallel \alpha \text{ and } \beta_2 \perp \alpha.$$

58. Find a unit vector that makes an angle of $90^\circ, 60^\circ$ and 30° with the positive x, y and z axis respectively.

59. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

60. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

61. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} =$

$$\begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix} \text{ are collinear.}$$

62. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.

63. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.

64. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}, \quad (3.1.64.1)$$

find the equation of the line.

65. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (3.1.65.1)$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (3.1.65.2)$$

66. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad (3.1.66.1)$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad (3.1.66.2)$$

67. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.67.1)$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (3.1.67.2)$$

68. Find the distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.68.1)$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad (3.1.68.2)$$

69. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

70. Find the unit normal vector of the plane

$$(6 \ -3 \ -2)\mathbf{x} = 1. \quad (3.1.70.1)$$

71. Find the distance of the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (3.1.71.1)$$

from the origin.

72. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$(2 \ -3 \ 4)\mathbf{x} - 6 = 0 \quad (3.1.72.1)$$

73. Find the equation of the plane which passes through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the line with direction vector $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

74. Find the equation of the plane passing through $\mathbf{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, $\mathbf{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$.

75. Find the equation of the plane with intercepts

2, 3 and 4 on the x, y and z axis respectively.

76. Find the equation of the plane passing through the intersection of the planes

$$(1 \ 1 \ 1)\mathbf{x} = 6 \quad (3.1.76.1)$$

$$(2 \ 3 \ 4)\mathbf{x} = -5 \quad (3.1.76.2)$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

77. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \quad (3.1.77.1)$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \quad (3.1.77.2)$$

are coplanar.

78. Find the angle between the two planes

$$(2 \ 1 \ -2)\mathbf{x} = 5 \quad (3.1.78.1)$$

$$(3 \ -6 \ -2)\mathbf{x} = 7. \quad (3.1.78.2)$$

79. Find the angle between the two planes

$$(2 \ 2 \ -2)\mathbf{x} = 5 \quad (3.1.79.1)$$

$$(3 \ -6 \ 2)\mathbf{x} = 7. \quad (3.1.79.2)$$

Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$(6 \ -3 \ 2)\mathbf{x} = 4 \quad (3.1.79.3)$$

Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \quad (3.1.79.4)$$

and the plane

$$(10 \ 2 \ -11)\mathbf{x} = 3 \quad (3.1.79.5)$$

80. Find the equation of the plane that contains the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and is perpendicular to each of the planes

$$(2 \ 3 \ -2)\mathbf{x} = 5 \quad (3.1.80.1)$$

$$(1 \ 2 \ -3)\mathbf{x} = 8 \quad (3.1.80.2)$$

81. Find the distance between the point $P = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane determined by the points $A = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $C = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.
82. Find the coordinates of the point where the lines through the points $A = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ crosses the XY plane.

3.2 Points and Vectors

1. Find the distance between the following pairs of points

a)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.2.1.1)$$

b)

$$\begin{pmatrix} -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (3.2.1.2)$$

c)

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -1 \\ b \end{pmatrix} \quad (3.2.1.3)$$

2. Find the distance between the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (3.2.2.1)$$

3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (3.2.4.1)$$

b)

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (3.2.4.2)$$

c)

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \quad (3.2.4.3)$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.4.4)$$

5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.
6. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \quad (3.2.6.1)$$

7. Find the values of y for which the distance between the points

$$P = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, Q = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (3.2.7.1)$$

is 10 units.

8. Show that each of the given three vectors is a unit vector

$$\frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \quad (3.2.8.1)$$

Also, show that they are mutually perpendicular to each other.

9. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (3.2.9.1)$$

$(\mathbf{a} + \lambda \mathbf{b}) \perp \mathbf{c}$. Find λ .

10. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \quad (3.2.10.1)$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad (3.2.11.1)$$

12. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

13. Find a vector of magnitude 5 units, and parallel

to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

15. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Find a vector \mathbf{d} such that $\mathbf{d} \perp \mathbf{a}$, $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T \mathbf{c} = 15$.

16. The scalar product of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$ is unity. Find the value of λ .

17. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (3.2.17.1)$$

is

- a) 0 c) 1
b) -1 d) 3

18. Find a unit vector that makes an angle of 90° , 135° and 45° with the positive x, y and z axis respectively.

19. Show that the lines with direction vectors $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$,

$$\begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix} \text{ are mutually perpendicular.}$$

20. Show that the line through the points $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$,

$$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \text{ is parallel to the line through the points}$$

21. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

23. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (3.2.23.1)$$

24. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \quad (3.2.24.1)$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (3.2.24.2)$$

25. Write down a unit vector in the xy-plane, making an angle of 30° with the positive direction of the x-axis.

26. Find the value of x for which $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.

3.3 Points on a Line

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (3.3.1.1)$$

in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

3. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

4. Find the ratio in which the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x -axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y .
6. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of \mathbf{P} such that $AP = \frac{3}{7}AB$ and \mathbf{P} lies on the line segment AB .
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.
8. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \quad (3.3.8.1)$$

are collinear.

9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix} x \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
- $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ k \end{pmatrix}$
 - $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$, $\begin{pmatrix} k \\ -4 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$
12. Find a condition on \mathbf{x} such that the points \mathbf{x} , $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.

13. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and

$$\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix} \text{ are collinear.}$$

14. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and

$$\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix} \text{ are collinear, and find the ratio in}$$

which \mathbf{B} divides AC .

15. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

3.4 Lines and Planes

- Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$.
- The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
- Find the slope of the line, which makes an angle of 30° of y -axis measured anticlockwise.
- Write the equations for the x and y axes.
- Find the equation of the line satisfying the following conditions
 - passing through the point $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ with slope $\frac{1}{2}$.
 - passing through the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m .
 - passing through the point $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x -axis at an angle of 75° .
 - Intersecting the x -axis at a distance of 3 units to the left of the origin with slope -2 .
 - intersecting the y -axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x -axis.
 - passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.
 - perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x -axis is 30° .
- Find the equation of the line passing through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$.
- Find the direction vectors and y -intercepts of the following lines
 - $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0$.
 - $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5$.
 - $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$.
- Find the intercepts of the following lines on the axes.
 - $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12$.

- b) $(4 \ -3)\mathbf{x} = 6$.
 c) $(3 \ 2)\mathbf{x} = 0$.
9. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
 a) $(1 \ -\sqrt{3})\mathbf{x} = -8$.
 b) $(0 \ 1)\mathbf{x} = 2$.
 c) $(1 \ -1)\mathbf{x} = 4$.
10. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $(12 \ -5)\mathbf{x} = -82$.
11. Find the points on the x-axis, whose distances from the line
 $(4 \ 3)\mathbf{x} = 12$ (3.4.11.1)
 are 4 units.
12. Find the distance between the parallel lines
 $(15 \ 8)\mathbf{x} = 34$ (3.4.12.1)
 $(15 \ 8)\mathbf{x} = -31$ (3.4.12.2)
13. Find the equation of the line parallel to the line
 $(3 \ -4)\mathbf{x} = -2$ (3.4.13.1)
 and passing through the point $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
14. Find the equation of a line perpendicular to the line
 $(1 \ -7)\mathbf{x} = -5$ (3.4.14.1)
 and having x intercept 3.
15. Find angles between the lines
 $(\sqrt{3} \ 1)\mathbf{x} = 1$ (3.4.15.1)
 $(1 \ \sqrt{3})\mathbf{x} = 1$ (3.4.15.2)
16. The line through the points $\begin{pmatrix} h \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ intersects the line
 $(7 \ -9)\mathbf{x} = 19$ (3.4.16.1)
 at right angle. Find the value of h .
17. Two lines passing through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ intersect each other at angle of 60° . If the slope of one line is 2, find the equation of the other line.
18. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.
19. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ to the line
 $(3 \ -4)\mathbf{x} = 16$. (3.4.19.1)
20. The perpendicular from the origin to the line
 $(-m \ 1)\mathbf{x} = c$ (3.4.20.1)
 meets it at the point $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Find the values of m and c .
21. Find θ and p if
 $(\sqrt{3} \ 1)\mathbf{x} = -2$ (3.4.21.1)
 is equivalent to
 $(\cos \theta \ \sin \theta)\mathbf{x} = p$ (3.4.21.2)
22. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.
23. Find the equation of the line parallel to the y-axis whose distance from the line
 $(4 \ 3)\mathbf{x} = 12$ (3.4.23.1)
 is 4 units.
24. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines
 $(1 \ -7)\mathbf{x} = -5$ (3.4.24.1)
 $(3 \ 1)\mathbf{x} = 0$ (3.4.24.2)
25. Find the value of p so that the three lines
 $(3 \ 1)\mathbf{x} = 2$ (3.4.25.1)
 $(p \ 2)\mathbf{x} = 3$ (3.4.25.2)
 $(2 \ -1)\mathbf{x} = 3$ (3.4.25.3)
 may intersect at one point.
26. Find the equation of the lines through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which make an angle of 45° with the line
 $(1 \ -2)\mathbf{x} = 3$. (3.4.26.1)
27. Find the equation of the line passing through

the point of intersection of the lines

$$(4 \ 7)\mathbf{x} = 3 \quad (3.4.27.1)$$

$$(2 \ -3)\mathbf{x} = -1 \quad (3.4.27.2)$$

that has equal intercepts on the axes.

28. In what ratio is the line joining $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ divided by the line

$$(1 \ 1)\mathbf{x} = 4 \quad (3.4.28.1)$$

29. Find the distance of the line

$$(4 \ 7)\mathbf{x} = -5 \quad (3.4.29.1)$$

from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ along the line

$$(2 \ -1)\mathbf{x} = 0. \quad (3.4.29.2)$$

30. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ so that its point of intersection with the line

$$(1 \ 1)\mathbf{x} = 4 \quad (3.4.30.1)$$

may be at a distance of 3 units from this point.

31. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Find an equation of the legs of the triangle.

32. Find the image of the point $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ with respect to the line

$$(1 \ 3)\mathbf{x} = 7 \quad (3.4.32.1)$$

assuming the line to be a plane mirror.

33. If the lines

$$(-3 \ 1)\mathbf{x} = 1 \quad (3.4.33.1)$$

$$(-1 \ 2)\mathbf{x} = 3 \quad (3.4.33.2)$$

are equally inclined to the line

$$(-m \ 1)\mathbf{x} = 4, \quad (3.4.33.3)$$

find the value of m .

34. The sum of the perpendicular distances of a variable point \mathbf{P} from the lines

$$(1 \ 1)\mathbf{x} = 0 \quad (3.4.34.1)$$

$$(3 \ -2)\mathbf{x} = -7 \quad (3.4.34.2)$$

is always 10. Show that \mathbf{P} must move on a line.

35. Find the equation of the line which is equidistant from parallel lines

$$(9 \ 7)\mathbf{x} = 7 \quad (3.4.35.1)$$

$$(3 \ 2)\mathbf{x} = -6. \quad (3.4.35.2)$$

36. A ray of light passing through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ reflects on the x-axis at point \mathbf{A} and the reflected ray passes through the point $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Find the coordinates of \mathbf{A} .

37. A person standing at the junction of two straight paths represented by the equations

$$(2 \ -3)\mathbf{x} = 4 \quad (3.4.37.1)$$

$$(3 \ 4)\mathbf{x} = 5 \quad (3.4.37.2)$$

wants to reach the path whose equation is

$$(6 \ -7)\mathbf{x} = -8 \quad (3.4.37.3)$$

in the least time. Find the equation of the path that he should follow.

38. Determine the ratio in which the line

$$(2 \ 1)\mathbf{x} - 4 = 0 \quad (3.4.38.1)$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

39. A line perpendicular to the line segment joining the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.

40. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

41. Find the equation of the line passing through the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.

42. Find the equation of the line through the point $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

43. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix} -2 \\ 9 \end{pmatrix}$, find the equation of

the line.

44. The length L (in cm) of a copper rod is a linear function of its Celsius temperature C . In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L in terms of C .

45. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?

46. Find the equation of a line which passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and is parallel to the

vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

47. Find the equation of the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

48. Find the equation of the line which passes through the point $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \quad (3.4.48.1)$$

49. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \quad (3.4.49.1)$$

50. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

51. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

52. Find the angle between the following pair of lines:

a)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad (3.4.52.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad (3.4.52.2)$$

b)

$$L_1 : \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad (3.4.52.3)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \quad (3.4.52.4)$$

53. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \quad (3.4.53.1)$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \quad (3.4.53.2)$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \quad (3.4.53.3)$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \quad (3.4.53.4)$$

54. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \quad (3.4.54.1)$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \quad (3.4.54.2)$$

are at right angles.

55. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \quad (3.4.55.1)$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad (3.4.55.2)$$

are perpendicular to each other.

56. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.4.56.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad (3.4.56.2)$$

57. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad (3.4.57.1)$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad (3.4.57.2)$$

58. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (3.4.58.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (3.4.58.2)$$

59. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \quad (3.4.59.1)$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \quad (3.4.59.2)$$

60. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $(0 \ 0 \ 1)\mathbf{x} = 2$ c) $(0 \ 5 \ 0)\mathbf{x} = -8$

b) $(1 \ 1 \ 1)\mathbf{x} = 1$ d) $(2 \ 3 \ -1)\mathbf{x} = 5$

61. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

62. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $(2 \ 3 \ 4)\mathbf{x} = 12$ c) $(1 \ 1 \ 1)\mathbf{x} = 1$

b) $(3 \ 4 \ -6)\mathbf{x} = 0$ d) $(0 \ 5 \ 0)\mathbf{x} = -8$

63. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vector to the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

64. Find the equation of the planes that pass through three points

a) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

65. Find the intercepts cut off by the plane $(2 \ 1 \ 1)\mathbf{x} = 5$.

66. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOY plane.

67. Find the equation of the plane through the intersection of the planes $(3 \ -1 \ 2)\mathbf{x} = 4$ and $(1 \ 1 \ 1)\mathbf{x} = -2$ and the point $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

68. Find the equation of the plane passing through the intersection of the planes $(2 \ 2 \ -3)\mathbf{x} = 7$ and $(2 \ 5 \ 3)\mathbf{x} = 9$ and the point $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

69. Find the equation of the plane through the intersection of the planes $(1 \ 1 \ 1)\mathbf{x} = 1$ and $(2 \ 3 \ 4)\mathbf{x} = 5$ which is perpendicular to the plane $(1 \ -1 \ 1)\mathbf{x} = 0$.

70. Find the angle between the planes whose equations are $(2 \ 2 \ -3)\mathbf{x} = 5$ and $(3 \ -3 \ 5)\mathbf{x} = 3$

71. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $(7 \ 5 \ 6)\mathbf{x} = -30$ and $(3 \ -1 \ -10)\mathbf{x} = -4$

b) $(2 \ 1 \ 3)\mathbf{x} = 2$ and $(1 \ -2 \ 5)\mathbf{x} = 0$

c) $(2 \ -2 \ 4)\mathbf{x} = -5$ and $(3 \ -3 \ 6)\mathbf{x} = 1$

d) $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \mathbf{x} = 1$ and $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \mathbf{x} = -3$

e) $\begin{pmatrix} 4 & 8 & 1 \end{pmatrix} \mathbf{x} = 8$ and $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 4$

72. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -4 & 12 \end{pmatrix} \mathbf{x} = 3$
b)	$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} = -3$
c)	$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \mathbf{x} = 9$
d)	$\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 & 6 \end{pmatrix} \mathbf{x} = 2$

TABLE 3.4.72

73. Show that the line joining the origin to the point $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the line deter-

mined by the points $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$.

74. If the coordinates of the points A, B, C, D be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, then find the angle between the lines AB and CD .

75. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \quad (3.4.75.1)$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \quad (3.4.75.2)$$

find the value of k .

76. Find the equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix} 1 & 2 & -5 \end{pmatrix} \mathbf{x} = -9 \quad (3.4.76.1)$$

77. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad \text{and} \quad (3.4.77.1)$$

$$\mathbf{x} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad (3.4.77.2)$$

78. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the YZ -plane.

79. Find the coordinates of the point where the line through $\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ crosses the ZX -plane.

80. Find the coordinates of the point where the line through $\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ crosses the plane

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \mathbf{x} = 7 \quad (3.4.80.1)$$

81. Find the equation of the plane passing through the point $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \mathbf{x} = 5 \quad (3.4.81.1)$$

$$\begin{pmatrix} 3 & 3 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.81.2)$$

82. If the points $\begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix} 3 & 4 & -12 \end{pmatrix} \mathbf{x} = -13, \quad (3.4.82.1)$$

then find the value of p .

83. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1 \quad \text{and} \quad (3.4.83.1)$$

$$\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = -4 \quad (3.4.83.2)$$

and parallel to the x -axis.

84. If \mathbf{O} be the origin and the coordinates of \mathbf{P} be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then find the equation of the plane passing through \mathbf{P} and perpendicular to OP .

85. Find the equation of the plane which contains

the line of intersection of the planes

$$(1 \ 2 \ 3)\mathbf{x} = 4 \quad (3.4.85.1)$$

$$(2 \ 1 \ -1)\mathbf{x} = -5 \quad (3.4.85.2)$$

and which is perpendicular to the plane

$$(5 \ 3 \ -6)\mathbf{x} = -8 \quad (3.4.85.3)$$

86. Find the distance of the point $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \quad (3.4.86.1)$$

and the plane

$$(1 \ -1 \ 1)\mathbf{x} = 5 \quad (3.4.86.2)$$

87. Find the vector equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and parallel to the planes

$$(1 \ -1 \ 2)\mathbf{x} = 5 \quad (3.4.87.1)$$

$$(3 \ 1 \ 1)\mathbf{x} = 6 \quad (3.4.87.2)$$

88. Find the vector equation of the line passing through the point $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \quad (3.4.88.1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad (3.4.88.2)$$

89. Distance between the two planes

$$(2 \ 3 \ 4)\mathbf{x} = 4 \quad (3.4.89.1)$$

$$(4 \ 6 \ 8)\mathbf{x} = 12 \quad (3.4.89.2)$$

- a) 2 c) 8
b) 4 d) $\frac{2}{\sqrt{29}}$

90. The planes

$$(2 \ -1 \ 4)\mathbf{x} = 5 \quad (3.4.90.1)$$

$$(5 \ -\frac{5}{2} \ 10)\mathbf{x} = 6 \quad (3.4.90.2)$$

are

a) Perpendicular

b) Parallel

c) intersect y-axis

d) passes through $\begin{pmatrix} 0 \\ 0 \\ \frac{5}{4} \end{pmatrix}$

3.5 Miscellaneous

1. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

2. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when

a) PQ is parallel to the y-axis.

b) PQ is parallel to the x-axis.

3. If three points $\begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

4. $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\left(\frac{1}{a} \ \frac{1}{b}\right)\mathbf{x} = 2 \quad (3.5.4.1)$$

5. Point $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

6. Show that two lines

$$(a_1 \ b_1)\mathbf{x} + c_1 = 0 \quad (3.5.6.1)$$

$$(a_2 \ b_2)\mathbf{x} + c_2 = 0 \quad (3.5.6.2)$$

are

a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and

b) perpendicular if $a_1a_2 - b_1b_2 = 0$.

7. Find the distance between the parallel lines

$$l(1 \ 1)\mathbf{x} = -p \quad (3.5.7.1)$$

$$l(1 \ 1)\mathbf{x} = r \quad (3.5.7.2)$$

8. Find the equation of the line through the point \mathbf{x}_1 and parallel to the line

$$(A \ B)\mathbf{x} = -C \quad (3.5.8.1)$$

9. If p and q are the lengths of perpendiculars

from the origin to the lines

$$(\cos \theta \quad \sin \theta) \mathbf{x} = k \cos 2\theta \quad (3.5.9.1)$$

$$(\sec \theta \quad \operatorname{cosec} \theta) \mathbf{x} = k \quad (3.5.9.2)$$

respectively, prove that $p^2 + 4q^2 = k^2$.

10. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b , then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \quad (3.5.10.1)$$

11. Show that the area of the triangle formed by the lines

$$(-m_1 \quad 1) \mathbf{x} = c_1 \quad (3.5.11.1)$$

$$(-m_2 \quad 1) \mathbf{x} = c_2 \quad (3.5.11.2)$$

$$(1 \quad 0) \mathbf{x} = 0 \quad (3.5.11.3)$$

is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

12. Find the values of k for which the line

$$(k - 3 \quad -(4 - k^2)) \mathbf{x} + k^2 - 7k + 6 = 0 \quad (3.5.12.1)$$

is

- a) parallel to the x-axis
- b) parallel to the y-axis
- c) passing through the origin.

13. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and $\begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$.

14. Find the area of the triangle formed by the lines

$$(1 \quad -1) \mathbf{x} = 0 \quad (3.5.14.1)$$

$$(1 \quad 1) \mathbf{x} = 0 \quad (3.5.14.2)$$

$$(1 \quad 0) \mathbf{x} = k \quad (3.5.14.3)$$

15. If three lines whose equations are

$$(-m_1 \quad 1) \mathbf{x} = c_1 \quad (3.5.15.1)$$

$$(-m_2 \quad 1) \mathbf{x} = c_2 \quad (3.5.15.2)$$

$$(-m_3 \quad 1) \mathbf{x} = c_3 \quad (3.5.15.3)$$

are concurrent, show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \quad (3.5.15.4)$$

16. Find the equation of the line passing through the origin and making an angle θ with the line

$$(-m \quad 1) \mathbf{x} = c \quad (3.5.16.1)$$

17. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$ to the line

$$\left(\frac{\cos \theta}{a} \quad \frac{\sin \theta}{b} \right) \mathbf{x} = 1 \quad (3.5.17.1)$$

is b^2 .

18. If $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, then show that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1 n_2 - m_2 n_1 \\ n_1 l_2 - n_2 l_1 \\ l_1 m_2 - l_2 m_1 \end{pmatrix}$.

19. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}. \quad (3.5.19.1)$$

20. Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}, \quad (3.5.20.1)$$

$$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma} \quad (3.5.20.2)$$

are coplanar.

21. Find \mathbf{R} which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \quad (3.5.21.1)$$

$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \quad (3.5.21.2)$$

externally in the ratio 1 : 2.

22. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \quad (3.5.22.1)$$

$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| \quad (3.5.22.2)$$

23. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \quad (3.5.23.1)$$

24. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \quad (3.5.24.1)$$

$$\mathbf{a}^T \mathbf{b} = \frac{1}{2} \quad (3.5.24.2)$$

and the angle between \mathbf{a} and \mathbf{b} is 60° .

25. Show that

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \perp (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a}) \quad (3.5.25.1)$$

26. If $\mathbf{a}^T \mathbf{a} = 0$ and $\mathbf{a} \mathbf{b} = 0$, what can be concluded about the vector \mathbf{b} ?

27. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \quad (3.5.27.1)$$

find the value of

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \quad (3.5.27.2)$$

28. If $\mathbf{a} \neq \mathbf{0}$, $\lambda \neq 0$, then $\|\lambda \mathbf{a}\| = 1$ if

- a) $\lambda = 1$
- b) $\lambda = -1$
- c) $\|\mathbf{a}\| = |\lambda|$
- d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

29. If a unit vector \mathbf{a} makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and \mathbf{a} .

30. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) \quad (3.5.30.1)$$

31. If $\mathbf{a}^T \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = 0$, what can you conclude about \mathbf{a} and \mathbf{b} ?

32. Find \mathbf{x} if \mathbf{a} is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \quad (3.5.32.1)$$

33. If $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between \mathbf{a} and \mathbf{b} is

- a) $\frac{\pi}{6}$
- b) $\frac{\pi}{4}$
- c) $\frac{\pi}{3}$
- d) $\frac{\pi}{2}$

34. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (3.5.34.1)$$

$$\iff \mathbf{a} \perp \mathbf{b}. \quad (3.5.34.2)$$

35. If θ is the angle between two vectors \mathbf{a} and \mathbf{b} , then $\mathbf{a}^T \mathbf{b} \geq 0$ only when

$$\text{a) } 0 < \theta < \frac{\pi}{2}$$

$$\text{b) } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\text{c) } 0 < \theta < \pi$$

$$\text{d) } 0 \leq \theta \leq \pi$$

36. Let \mathbf{a} and \mathbf{b} be two unit vectors and θ be the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

$$\text{a) } \theta = \frac{\pi}{4}$$

$$\text{b) } \theta = \frac{\pi}{3}$$

$$\text{c) } \theta = \frac{\pi}{2}$$

$$\text{d) } \theta = \frac{2\pi}{3}$$

37. If θ is the angle between any two vectors \mathbf{a} and \mathbf{b} , then $\|\mathbf{a}^T \mathbf{b}\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

$$\text{a) } 0$$

$$\text{b) } \frac{\pi}{4}$$

$$\text{c) } \frac{\pi}{2}$$

$$\text{d) } \pi.$$

38. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.

39. Find the equation of a line parallel to the x-axis and passing through the origin.

40. Find the equation of a plane passing through $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane

$$(1 \ 1 \ 1)\mathbf{x} = 2 \quad (3.5.40.1)$$

41. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \quad (3.5.41.1)$$

4 CIRCLE

4.1 Construction Examples

1. Draw a circle with centre \mathbf{B} and radius 6. If \mathbf{C} be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC$, $AC \perp AB$, $a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \quad (4.1.1.1)$$

The following code plots Fig. 4.1.1

```
codes/circle/draw_circle_eg.py
```

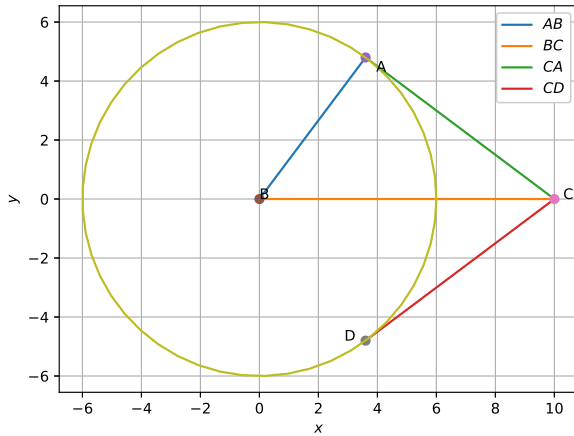



Fig. 4.1.1

2. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.

Solution: For any angle θ , a point on the circle with radius 3 has coordinates

$$3 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (4.1.2.1)$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
Solution: Take the centre of both circles to be at the origin.
6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
Solution: Take the diameter to be on the x -axis.

7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60° .

Solution: The tangent is perpendicular to the radius.

8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (4.2.2.1)$$

9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through **B, C, D** (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $(2, -3)$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
2. Find the centre of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
3. Find the locus of all the unit vectors in the xy -plane.