Geometry: Maths Olympiad

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1. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.
2. In a convex quadrilateral PQRS, PQ = RS, $(\sqrt{3}+1)$QR = SP and $\angle$RSP - $\angle$SPQ = 30^o. Prove that
$$\angle PQR - \angle QRS = 90^o.$$
3. Let ABC be a triangle in which no angle is 90^o. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:
   a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$
   b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$
   c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.
4. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle$ DAB = $\angle$ BCA and $\angle$ DAC = 15^o. Show that $\angle$ ADC is obtuse. Further, if O is the circumcentre of ADC, Prove that triangle AOD is equilateral.
5. For a convex hexagon ABCDEF in which each pair of opposite sides is unequal, consider the following statements:
   (a_1) AB is parallel DE (a_2) AE = BD
   (a_1) BC is parallel EF (a_2) BF = CE
   (a_1) CD is parallel FA (a_2) CA = DF
   a) Show that if the all the six statements are true, then the hexagon is cyclic.
   b) Prove that in fact, any five of these six statements also imply that the hexagon is cyclic.
6. Consider an acute triangle ABC and let P be an interior point of ABC. Suppose the lines BP and CP, when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC. Show that DK bisects $\angle$ EKF.
7. Let ABC be a triangle with sides a,b,c. Consider a triangle $A_1B_1C_1$ with sides equal to $a+\frac{b}{2}$, $b+\frac{c}{2}$, $c+\frac{a}{2}$. Show that
$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$
where [XYZ] denotes the area of the triangle XYZ.
8. Consider a convex quadrilateral ABCD, in which K,L,M,N are the midpoints of the BC, CD, DA respectively. Suppose
   a) BD bisects KM at Q;
   b) QA = QB = QC = QD; and
   c) LK/LM = CD/CB

   Prove that ABCD is a square.
9. Let R denotes the circum radius of a triangle ABC; a,b,c its sides BC, CA, AB; and r_a exradii opposite A,B,C. If $2R \leq r_a$, Prove that

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a) $a > b$ and $a > c$
b) $2R > r_b$ and $2R > r_c$

10. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect ABC at K and L,K being nearer to A than L. If AK = KL = LM, Prove that the sides of triangle ABC are in the ratio 5:10:13 in some order.
11. In a non-equilateral triangle ABC, the sides a,b,c form an arithmetic progression. Let I and O denote the incentre and circum centre of the triangle respectively.
    a) Prove that IO is perpendicular to BI.
    b) Suppose BI extended meets AC in K, and D,E are the midpoints of BC, BA respectively. Prove that I is the circumcentre of the triangle DKE.
12. In a cyclic quadrilateral ABCD, AB = a, BC = b, CD = c, $\angle$ ABC = 120^o, and $\angle$ ABD = 30^o, Prove that
    a) c $\geq$ a + b;
    b) $\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c-a-b}$.
13. In a triangle ABC right-angles at C, the median through B bisects the angle between BA and the bisector of $\angle$B. Prove that
$$\frac{5}{2} < \frac{AB}{BC} < 3.$$
14. Let ABC be a triangle in which AB = AC. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. If $\frac{AP}{PD} = \frac{BP}{PE} = \lambda$, $\frac{BD}{AD} = m$ and z = $m^2(1+\lambda)$, Prove that
$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$
Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.
15. Let ABC be a triangle and let P be an interior point such that $angle$BPC = 90^O, $angle$BAP = $angle$BCP. Let M,N be the mid points of AC, BC respectively. Suppose BP = 2PM. Prove that A,P,N are collinear.
16. Let ABC be an acute-angles triangle and let H be its ortho-centre. Let h_{max} denote the largest altitude of the triangle ABC. Prove that
$$AH + BH + CH \leq 2h_{max}$$
17. Let ABCD be a quadrilateral inscribed in a circle. Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA of the circle. Suppose AC.BD = EG.FH. Prove that AC, BD, EG, FH are concurrent.
18. Let D,E,F be points on the sides BC, CA, AB respectively of a triangle ABC such that BD = CE = AF and $\angle$BDF = $\angle$CED = $\angle$AFE. Prove that ABC is equilateral.
19. Let ABC an acute-angled triangle and let D,E,F be points on BC, CA, AB respectively such that AD is the median, BE is the internal angle bisector and CF is the altitude. Suppose $\angle$FDE = $\angle$C, $\angle$DEF = $\angle$A and $\angle$EFD = $\angle$B.Prove that ABC is equilateral.
20. Let ABC be a triangle. An interior point P of ABC is said to be good if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 smaller triangles of equal area. Determine the number of good points for a given triangle ABC.
21. Let ABCD be a quadrilateral inscribed in a circle. Suppose AB = $\sqrt{2+\sqrt{2}}$ and AB subtends 135^o at the centre of the circle. Find the maximum possible area of ABCD.
22. Let T_1 and T_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to T_2 at P and passing through the centre O_1 of T_1. Similarly, let l_2 be a line which is tangent to T_2 at Q and passing through the centre O_2 of T_2. Suppose l_1 and l_2 are not parallel and intersect at K.

If KP = KQ, Prove that the triangle PQR is equilateral.

23. In an acute triangle ABC, O is the circumcentre, H is the orthocentre and G is the centroid. Let OD be perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the midpoint of AB. Suppose the areas of triangles ODC, HEA and GFB are equal. Find all the possible values of C.
24. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD, respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.
25. In a triangle ABC, let D be a point on the segment BC such that AB + BD = AC + CD. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that AB = AC.
26. Let ABC be a right-angled triangle with $angle$B = 90^o. Let BD be the altitude from B on to AC. Let P, Q and I be the incentres of triangles ABD, CBD and ABC respectively. Show that the circumcentre of of the triangle PIQ lies on the hypotenuse AC.
27. Let ABCD be a convex quadrilateral. Let the diagonals AC and BD intersect in P . Let PE, PF, PG and PH be the altitudes from P on to the sides AB, BC, CD and DA respectively. Show that ABCD has an incircle if and only if
$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}$$
28. Let ABC be triangle in which AB = AC. Suppose the orthocentre of the triangle lies on the incircle. Find the ratio $\frac{AB}{BC}$.
29. Let ABC be a right-angled triangle with ∠B = 90^o. Let D be a point on AC such that the inradii of the triangles ABD and CBD are equal. If this common value is r_0 and if r is the inradius of triangle ABC, prove that
$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}$$
30. ABCD is a square sheet of paper. It is folded along EF such that A goes to a point A' different from B and C, on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles GD'F and A'BE.
31. Let ABCDE be a convex pentagon in which ∠A = ∠B = ∠C = ∠D = 120^o and side lengths are five consecutive integers in some order. Find all possible values of AB + BC + CD.
32. Let ABC be a triangle with ∠A = 90^o and $AB < AC$. Let AD be the altitude from A on to BC. Let P, Q and I denote respectively the incentres of triangles ABD, ACD and ABC. Prove that AI is perpendicular to PQ and AI = PQ.