Linear Algebra through Coordinate Geometry

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Contents

***Abstract*—This manual introduces linear algebra through coordinate geometry using a problem solving approach.**

1 The Straight Line

1.1 The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{A} - \mathbf{B}) \tag{1.1}$$

Alternatively, it can be expressed as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.2}$$

where $\mathbf{m}$ is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{m} = 0 \tag{1.3}$$

2 Locus

2.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1}$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed as shown i Fig. (2.1),

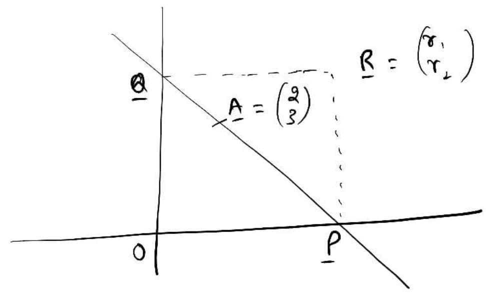

Fig. 2.1

2.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{2.2}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{2.3}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{2.4}$$

2.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{m} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{m} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{m} &= 0 \end{aligned} \tag{2.5}$$

Solution: Trivial using (1.2) and (1.3).

2.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{m} = 0 \tag{2.6}$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{m} = 0 \tag{2.7}$$

Solution: From (2.5) and (2.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{m} = 0 \tag{2.8}$$

resulting in (2.6). From (2.5) and (2.2),(2.3), (2.7) is obtained.

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2.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \tag{2.9}$$

2.6 Find the locus of $\mathbf{R}$.

Solution: For $\mathbf{m}$ to be unique in (2.6),(2.7),

$$\begin{aligned}(\mathbf{2A} - \mathbf{R}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ \implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{2A} - \mathbf{R}) \\ = k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ = k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{2.10}$$

where k is some constant.

3 Conics

3.1 The equation of quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{3.1}$$

Show that (3.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{3.2}$$

Find the matrix V and vector $\mathbf{u}$.

3.2 The tangent to (3.1) at a point $\mathbf{p}$ on the curve is given by

$$\begin{pmatrix} \mathbf{p}^T & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{3.3}$$

Show that (3.3) can be expressed as

$$\left(\mathbf{p}^T V + \mathbf{u}^T\right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \tag{3.4}$$

3.3 Find the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \tag{3.5}$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{3.6}$$

in (3.4), the desired equation is

$$\begin{aligned}\left[\begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\right] \mathbf{x} \\ + \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \end{aligned} \tag{3.7}$$

touches the circle

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \tag{3.8}$$

Find c.