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**Abstract**—This book introduces school geometry through a combination of trigonometry and algebra. The content and exercises are based on NCERT textbooks from Class 6-12.

## 1 TRIANGLE

### 1.1 The Right Angled Triangle

1. A right angled triangle looks like Fig. 1.1.1. with angles  $\angle A$ ,  $\angle B$  and  $\angle C$  and sides  $a$ ,  $b$  and  $c$ . The unique feature of this triangle is  $\angle C$  which is defined to be  $90^\circ$ .

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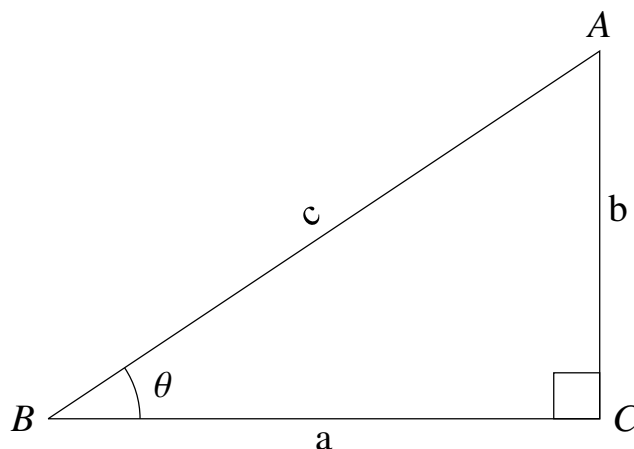


Fig. 1.1.1: Right Angled Triangle

2. For simplicity, let the greek letter  $\theta = \angle B$ . We have the following definitions.

$$\begin{aligned} \sin \theta &= \frac{a}{c} & \cos \theta &= \frac{b}{c} \\ \tan \theta &= \frac{a}{b} & \cot \theta &= \frac{b}{a} \\ \csc \theta &= \frac{c}{a} & \sec \theta &= \frac{c}{b} \end{aligned} \quad (1.1.2.1)$$

### 1.2 Sum of Angles

1. In Fig. 1.2.1, the sum of all the angles on the top or bottom side of the straight line  $XY$  is  $180^\circ$ .
2. In Fig. 1.2.1, the straight line making an angle of  $90^\circ$  to the side  $AC$  is said to be parallel to the side  $BC$ . Note there is an angle at  $A$  that is equal to  $\theta$ . This is one property of parallel lines. Thus,  $\angle YAZ = 90^\circ$ .
3. Show that  $\angle VAZ = 90^\circ - \theta$

**Solution:** Considering the line  $XAZ$ ,

$$\theta + 90^\circ + \angle VAZ = 180^\circ \quad (1.2.3.1)$$

$$\Rightarrow \angle VAZ = 90^\circ - \theta \quad (1.2.3.2)$$

4. Show that  $\angle BAC = 90^\circ - \theta$ .

**Solution:** Consider the line  $VAB$  and use the approach in the previous problem. Note that

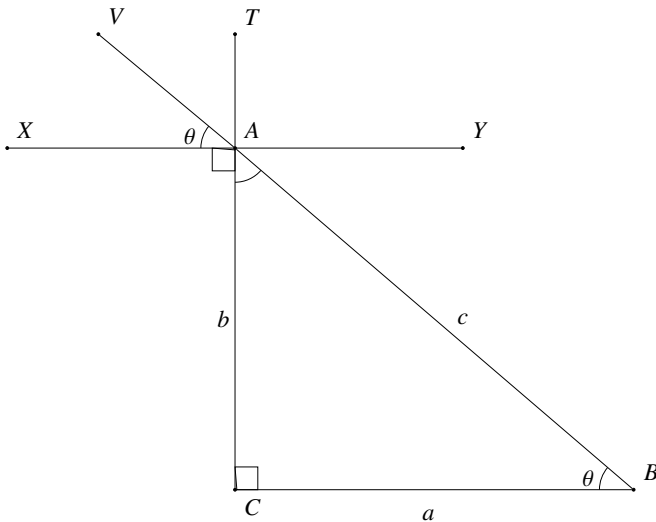


Fig. 1.2.1: Sum of angles of a triangle

this implies that  $\angle VAZ = \angle BAC$ . Such angles are known as vertically opposite angles.

5. Sum of the angles of a triangle is equal to  $180^\circ$

### 1.3 Baudhayana Theorem

1. Using Fig. 1.1.1, show that

$$\cos \theta = \sin(90^\circ - \theta) \quad (1.3.1.1)$$

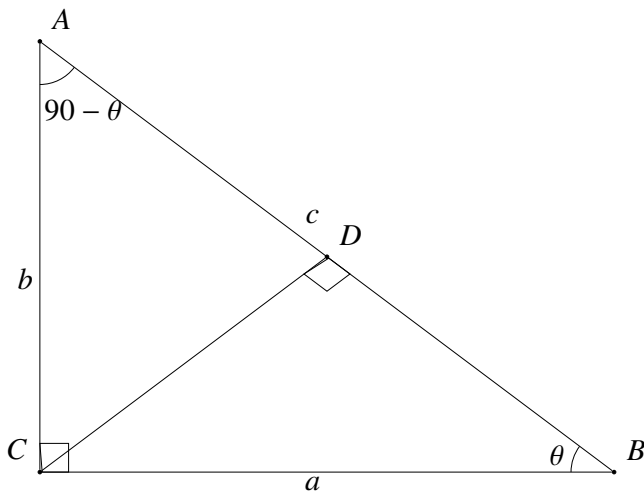


Fig. 1.3.1: Baudhayana Theorem

**Solution:** From Problem 1.2.4 and (1.1.2.1)

$$\cos(90^\circ - \theta) = \frac{b}{c} = \sin \theta \quad (1.3.1.2)$$

2. Using Fig. 1.3.1, show that

$$c = a \cos \theta + b \sin \theta \quad (1.3.2.1)$$

**Solution:** We observe that

$$BD = a \cos \theta \quad (1.3.2.2)$$

$$AD = b \cos(90^\circ - \theta) = b \sin \theta \quad (\text{From } (1.2.4)) \quad (1.3.2.3)$$

Thus,

$$BD + AD = c = a \cos \theta + b \sin \theta \quad (1.3.2.4)$$

3. From (1.3.2.1), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1.3.3.1)$$

**Solution:** Dividing both sides of (1.3.2.1) by  $c$ ,

$$1 = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta \quad (1.3.3.2)$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from } (1.1.2.1)) \quad (1.3.3.3)$$

4. Using (1.3.2.1), show that

$$c^2 = a^2 + b^2 \quad (1.3.4.1)$$

(1.3.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

**Solution:** From (1.3.2.1),

$$c = a \frac{a}{c} + b \frac{b}{c} \quad (\text{from } (1.1.2.1)) \quad (1.3.4.2)$$

$$\Rightarrow c^2 = a^2 + b^2 \quad (1.3.4.3)$$

### 1.4 Area of a Triangle

1. The area of the rectangle  $ACBD$  shown in Fig. 1.4.1 is defined as  $ab$ . Note that all the angles in the rectangles are  $90^\circ$
2. The area of the two triangles constituting the rectangle is the same.
3. The area of the rectangle is the sum of the areas of the two triangles inside.
4. Show that the area of  $\triangle ABC$  is  $\frac{ab}{2}$

**Solution:** From (1.4.3),

$$ar(ABCD) = ar(ACB) + ar(ADB) \quad (1.4.4.1)$$

Also from (1.4.2),

$$ar(ACB) = ar(ADB) \quad (1.4.4.2)$$

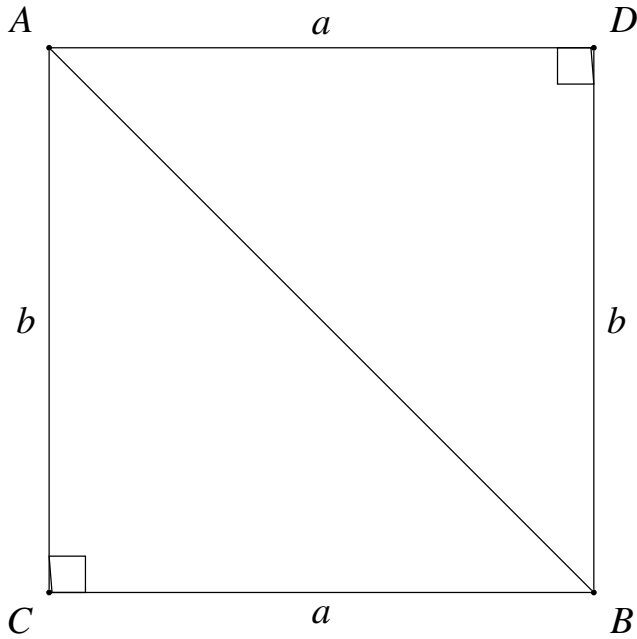


Fig. 1.4.1: Area of a Right Triangle

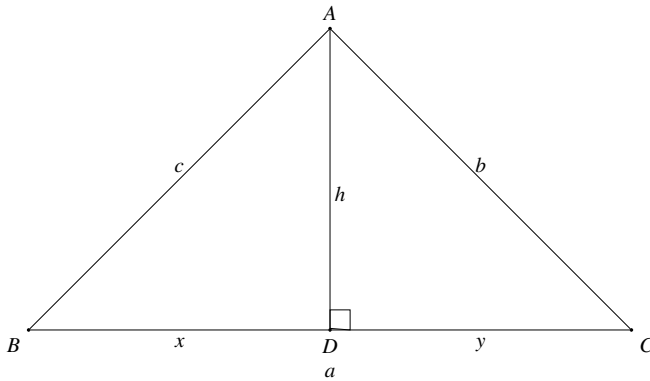


Fig. 1.4.4: Area of a Triangle

From (1.4.4.1) and (1.4.4.2),

$$2ar(\triangle ABC) = ar(ABCD) = ab \text{ (from (1.4.1))} \quad (1.4.4.3)$$

$$\Rightarrow ar(\triangle ABC) = \frac{ab}{2} \quad (1.4.4.4)$$

5. Show that the area of  $\triangle ABC$  in Fig. 1.4.4 is  $\frac{1}{2}ah$ .

**Solution:** In Fig. 1.4.4,

$$ar(\triangle ADC) = \frac{1}{2}hy \quad (1.4.5.1)$$

$$ar(\triangle ADB) = \frac{1}{2}hx \quad (1.4.5.2)$$

Thus,

$$ar(\triangle ABC) = ar(\triangle ADC) + ar(\triangle ADB) \quad (1.4.5.3)$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h(x+y) \quad (1.4.5.4)$$

$$= \frac{1}{2}ah \quad (1.4.5.5)$$

6. Show that the area of  $\triangle ABC$  in Fig. 1.4.4 is  $\frac{1}{2}ab \sin C$ .

**Solution:** We have

$$ar(\triangle ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (1.4.6.1)$$

7. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.4.7.1)$$

**Solution:** Fig. 1.4.4 can be suitably modified to obtain

$$ar(\triangle ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (1.4.7.2)$$

Dividing the above by  $abc$ , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.4.7.3)$$

This is known as the sine formula.

8. In Fig. 1.4.8, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (1.4.8.1)$$

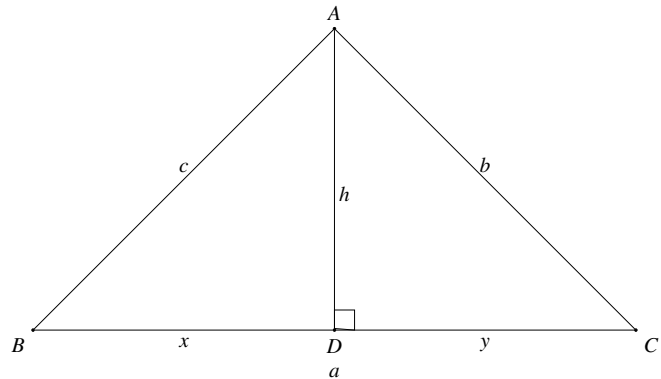


Fig. 1.4.8: The cosine formula

**Solution:** From the figure, the first of the

following equations

$$a = b \cos C + c \cos B \quad (1.4.8.2)$$

$$b = c \cos A + a \cos C \quad (1.4.8.3)$$

$$c = b \cos A + a \cos B \quad (1.4.8.4)$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.4.8.5)$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \quad (1.4.8.6)$$

$$= \frac{b^2 + c^2 - a^2}{2abc} \quad (1.4.8.7)$$

9. Find Hero's formula for the area of a triangle.

**Solution:** From (1.4.6), the area of  $\triangle ABC$  is

$$\frac{1}{2}ab \sin C = \frac{1}{2}ab \sqrt{1 - \cos^2 C} \quad (\text{from (1.3.3.1)}) \quad (1.4.9.1)$$

$$= \frac{1}{2}ab \sqrt{1 - \left( \frac{a^2 + b^2 - c^2}{2ab} \right)^2} \quad (\text{from (1.4.8.1)}) \quad (1.4.9.2)$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2} \quad (1.4.9.3)$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \quad (1.4.9.4)$$

$$= \frac{1}{4} \sqrt{\{(a+b)^2 - c^2\} \{c^2 - (a-b)^2\}} \quad (1.4.9.5)$$

$$= \frac{1}{4} \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (1.4.9.6)$$

Substituting

$$s = \frac{a+b+c}{2} \quad (1.4.9.7)$$

in (1.4.9.6), the area of  $\triangle ABC$  is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (1.4.9.8)$$

This is known as Hero's formula.

### 1.5 Median

1. The line AD in Fig. 1.5.1 that divides the side  $a$  in two equal halves is known as the median.

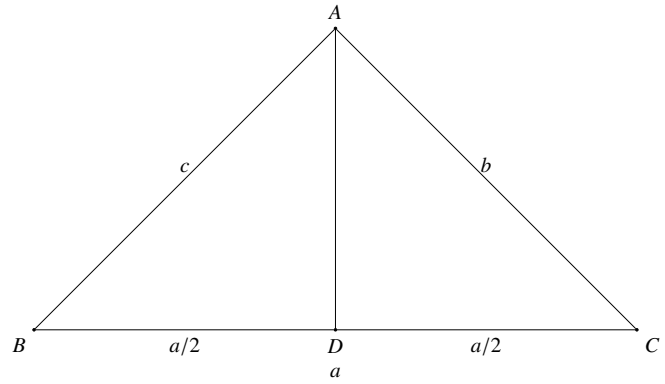


Fig. 1.5.1: Median of a Triangle

2. Show that the median AD in Fig. 1.5.1 divides  $\triangle ABC$  into triangles ADB and ADC that have equal area.

**Solution:** We have

$$ar(\triangle ADB) = \frac{1}{2} \frac{a}{2} c \sin B = \frac{1}{4} ac \sin B \quad (1.5.2.1)$$

$$ar(\triangle ADC) = \frac{1}{2} \frac{a}{2} b \sin C = \frac{1}{4} ab \sin C \quad (1.5.2.2)$$

Using the sine formula,  $b \sin C = c \sin B$ ,

$$ar(\triangle ADB) = ar(\triangle ADC) \quad (1.5.2.3)$$

3. BE and CF are the medians in Fig. 1.5.3. Show that

$$ar(\triangle BFC) = ar(\triangle BEC) \quad (1.5.3.1)$$

**Solution:** Since BE and CF are the medians,

$$ar(\triangle BFC) = \frac{1}{2} ar(\triangle ABC) \quad (1.5.3.2)$$

$$ar(\triangle BEC) = \frac{1}{2} ar(\triangle ABC) \quad (1.5.3.3)$$

From the above, we infer that

$$ar(\triangle BFC) = ar(\triangle BEC) \quad (1.5.3.4)$$

4. We know that the median of a triangle divides it into two triangles with equal area. Using this result along with the sine formula for the area of a triangle in Fig. 1.5.4,

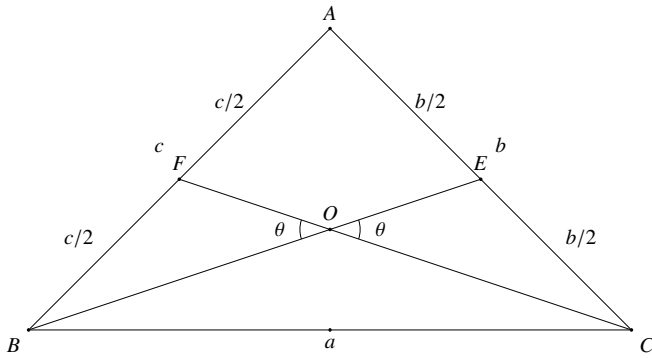


Fig. 1.5.3:  $O$  is the Intersection of Two Medians

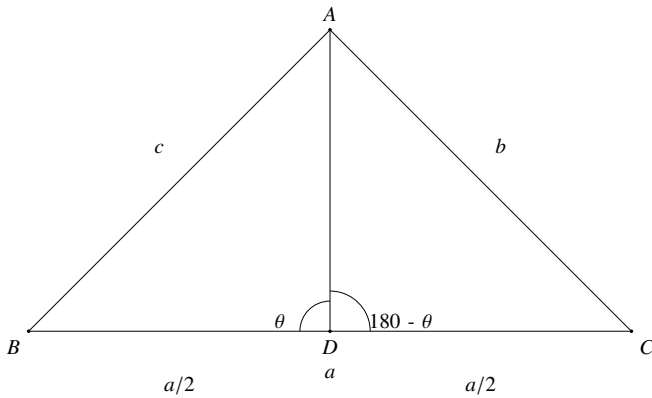


Fig. 1.5.4:  $\sin \theta = \sin(180^\circ - \theta)$

$$\frac{1}{2} \frac{a}{2} AD \sin \theta = \frac{1}{2} \frac{a}{2} AD \sin(180^\circ - \theta) \quad (1.5.4.1)$$

$$\Rightarrow \sin \theta = \sin(180^\circ - \theta). \quad (1.5.4.2)$$

Note that our geometric definition of  $\sin \theta$  holds only for  $\theta < 90^\circ$ . (1.5.4.2) allows us to extend this definition for  $\angle ADC > 90^\circ$ .

5. In Fig. 1.5.5, show that  $EF = \frac{a}{2}$ .

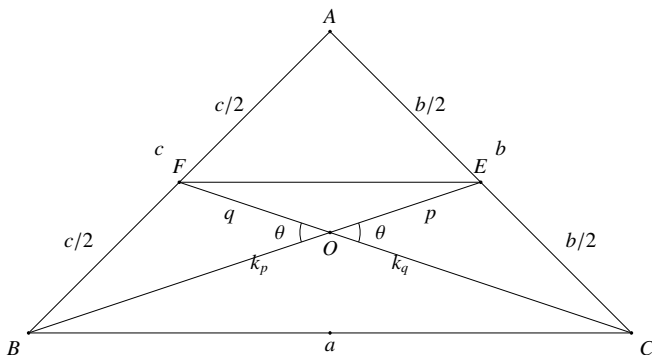


Fig. 1.5.5: Similar Triangles

**Solution:** Using the cosine formula for  $\triangle AEF$ ,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (1.5.5.1)$$

$$= \frac{b^2 + c^2 - 2bc \cos A}{4} \quad (1.5.5.2)$$

$$= \frac{a^2}{4} \quad (1.5.5.3)$$

$$\Rightarrow EF = \frac{a}{2} \quad (1.5.5.4)$$

6. The ratio of sides of triangles  $AEF$  and  $ABC$  is the same. Such triangles are known as similar triangles.

7. Show that similar triangles have the same angles.

**Solution:** Use cosine formula and the proof is trivial.

8. Show that in Fig. 1.5.5,  $EF \parallel BC$ .

**Solution:** Since  $\triangle AEF \sim \triangle ABC$ ,  $\angle AEF = \angle ACB$ . Hence the line  $EF \parallel BC$

9. Show that  $\triangle OEF \sim \triangle OEC$ .

10. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \quad (1.5.10.1)$$

11. Show that the medians of a triangle meet at a point.

## 1.6 Angle Bisectors

1. In Fig. 1.6.1,  $OB$  divides the  $\angle B$  into half, i.e.

$$\angle OBC = \angle OBA \quad (1.6.1.1)$$

$OB$  is known as an angle bisector.

$OB$  and  $OC$  are angle bisectors of angles  $B$  and  $C$ .  $OA$  is joined and  $OD, OF$  and  $OE$  are perpendiculars to sides  $a, b$  and  $c$ .

2. Show that  $OD = OE = OF$ . **Solution:** In  $\triangle ODC$  and  $OEC$ ,

$$OD = OC \sin \frac{C}{2} \quad (1.6.2.1)$$

$$OE = OC \sin \frac{C}{2} \quad (1.6.2.2)$$

$$\Rightarrow OD = OE. \quad (1.6.2.3)$$

Similarly,

$$OD = OF. \quad (1.6.2.4)$$

3. Show that  $OA$  is the angle bisector of  $\angle A$

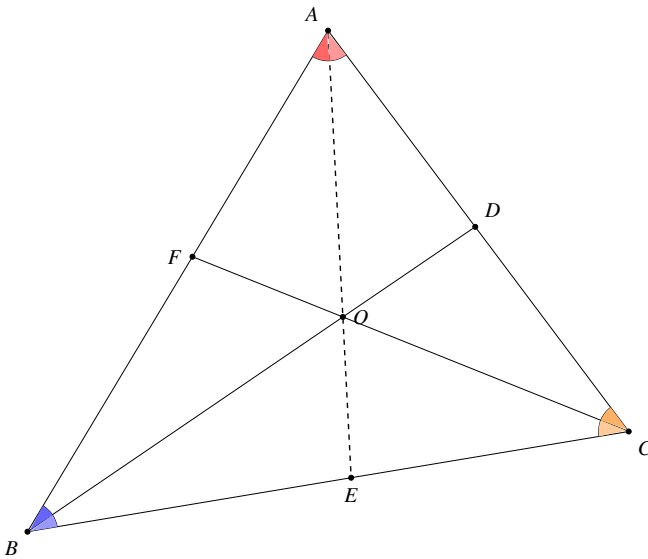


Fig. 1.6.1: Angle bisectors meet at a point

**Solution:** In  $\triangle OFA$  and  $OEA$ ,

$$OF = OE \quad (1.6.3.1)$$

$$\Rightarrow OA \sin OAF = OA \sin OAE \quad (1.6.3.2)$$

$$\Rightarrow \sin OAF = \sin OAE \quad (1.6.3.3)$$

$$\Rightarrow \angle OAF = \angle OAE \quad (1.6.3.4)$$

which proves that  $OA$  bisects  $\angle A$ . **Conclusion:** The angle bisectors of a triangle meet at a point.

### 1.7 Congruent Triangles

1. Show that in  $\triangle ODC$  and  $OEC$ , corresponding sides and angles are equal. Such triangles are known as *congruent* triangles and denoted by  $\triangle ODC \cong \triangle OEC$ .
2. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.
3. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.
4. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.
5. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.
6. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

### 1.8 Perpendicular Bisectors

1. In Fig. 1.8.3,  $OE \perp AC$ ,  $OF \perp AB$ ,  $AE = EC$ ,  $AF = FB$ .  $OE$  and  $OF$  are known as the *perpendicular bisectors* of  $AC$  and  $AB$  respectively. Show that  $OA = OB = OC$ .

**Solution:** Using SAS,  $\triangle OEA \cong \triangle OEC$ . Thus,  $OA = OC$ . Similarly,  $\triangle OFA \cong \triangle OFB \Rightarrow OA = OB$ .

2. In Fig. 1.8.3, show that if  $OD \perp BC$ ,  $BD = DC$ , i.e.,  $OD$  is the perpendicular bisector of  $BC$ .

**Solution:**  $\because OB = OC$ , using RHS congruence,  $\triangle ODC \cong \triangle ODB$ . Hence  $BD = DC$ .

3. In Fig. 1.8.3, show that if  $BD = DC$ ,  $OD \perp BC$ .

**Solution:** Use SSS.

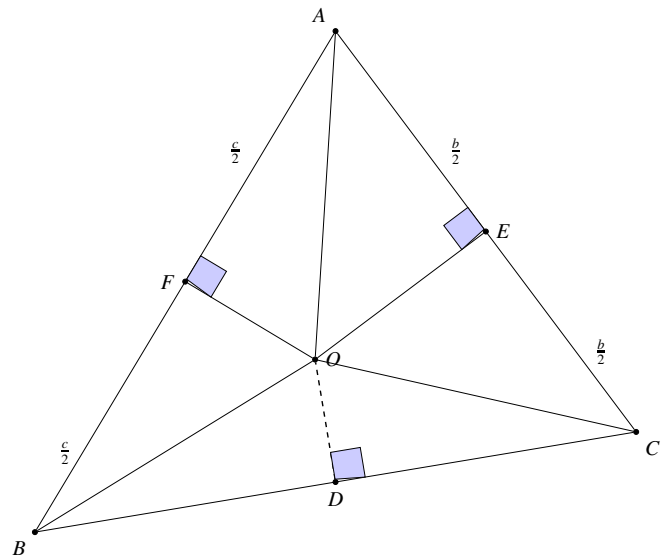


Fig. 1.8.3: Perpendicular bisectors meet at a point

4. In  $\triangle AOB$ ,  $OA = OB$ . Such a triangle is known as an isosceles triangle.

**Conclusion:** The perpendicular bisectors of a triangle meet at a point.

5. In Fig. (1.8.5),  $OA = OB = OC = R$ . Show that  $\angle BOC = 2\angle BAC$ .

**Solution:** Note that  $\alpha$  and  $\beta$  are exterior angles for  $\triangle AOB$  and  $AOC$ , which are isosceles.

### 1.9 Altitudes of a Triangle

1. In Fig. 1.9.1,  $AD \perp BC$  and  $BE \perp AC$ .  $CF$  passes through  $O$  and meets  $AB$  at  $F$ . Show that

$$OE = c \cos A \cot C \quad (1.9.1.1)$$

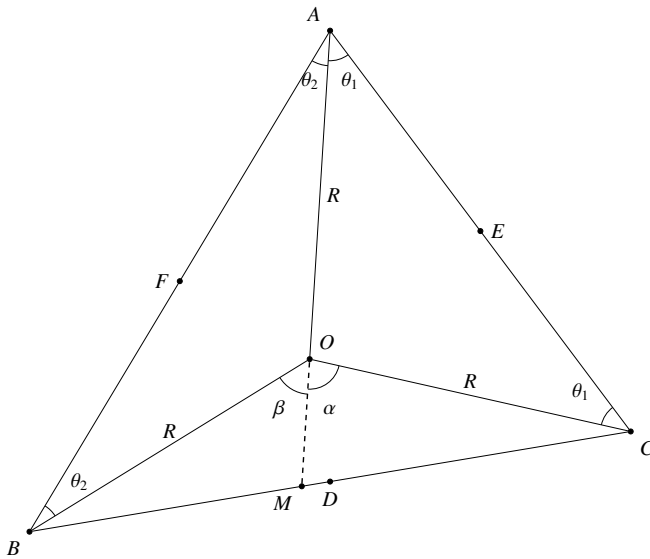


Fig. 1.8.5:  $\angle BOC = 2\angle BAC$

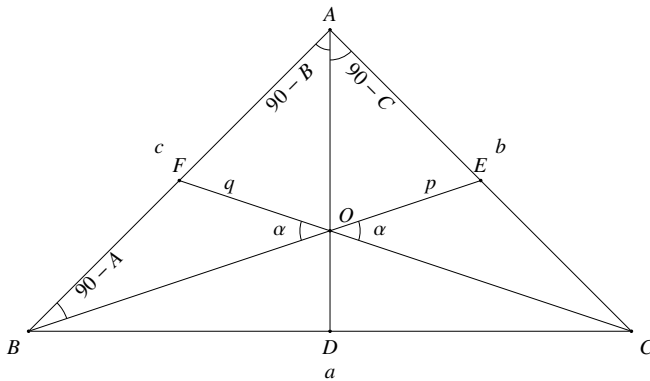


Fig. 1.9.1: Perpendiculars from vertex to opposite side meet at a point

**Solution:** In  $\triangle AEB$  and  $\triangle AEO$ ,

$$AE = c \cos A \quad (1.9.1.2)$$

$$OE = AE \tan(90^\circ - C) \quad (\because ADC \text{ is right angled}) \quad (1.9.1.3)$$

$$= AE \cot C \quad (1.9.1.4)$$

From both the above, we get the desired result.

2. Show that  $\alpha = A$ .

**Solution:** In  $\triangle OEC$ ,

$$CE = a \cos C \quad (\because BEC \text{ is right angled}) \quad (1.9.2.1)$$

Hence,

$$\begin{aligned} \tan \alpha &= \frac{CE}{OE} \\ &= \frac{a \cos C}{c \cos A \cot C} \\ &= \frac{a \cos C \sin C}{c \cos A \cos C} \\ &= \frac{a \sin C}{c \cos A} \\ &= \frac{c \sin A}{c \cos A} \quad \left( \because \frac{a}{\sin A} = \frac{c}{\sin C} \right) \\ &= \tan A \end{aligned}$$

$$\Rightarrow \alpha = A$$

(1.9.2.2)

3. Show that  $CF \perp AB$

**Solution:** Consider triangle OFB and the result of the previous problem.  $\therefore$  the sum of the angles of a triangle is  $180^\circ$ ,  $\angle CFB = 90^\circ$ .  
*Conclusion:* The perpendiculars from the vertex of a triangle to the opposite side meet at a point.

### 1.10 Triangle Inequalities

1. Show that if

$$\theta_1 < \theta_2, \quad \sin \theta_1 < \sin \theta_2. \quad (1.10.1.1)$$

**Solution:** Using Baudhayana's theorem in  $\triangle ABC$  and  $\triangle DBC$

$$l^2 = x^2 + a^2 \quad (1.10.1.2)$$

$$c^2 = b^2 + a^2 \quad (1.10.1.3)$$

$$\Rightarrow c > l \because b > x. \quad (1.10.1.4)$$

Also,

$$a = c \sin \theta_1 = l \sin \theta_2 \quad (1.10.1.5)$$

$$\Rightarrow \frac{\sin \theta_1}{\sin \theta_2} = \frac{l}{c} < 1 \quad \text{from (1.10.1.4)} \quad (1.10.1.6)$$

$$\text{or, } \sin \theta_1 < \sin \theta_2 \quad (1.10.1.7)$$

2. Show that if

$$\theta_1 < \theta_2, \quad \cos \theta_1 > \cos \theta_2. \quad (1.10.2.1)$$

3. Show that in any  $\triangle ABC$ ,  $\angle A > \angle B \Rightarrow a > b$ .

**Solution:** Use (1.4.7.3) and (1.10.1.7)

4. Show that the sum of any two sides of a triangle is greater than the third side.

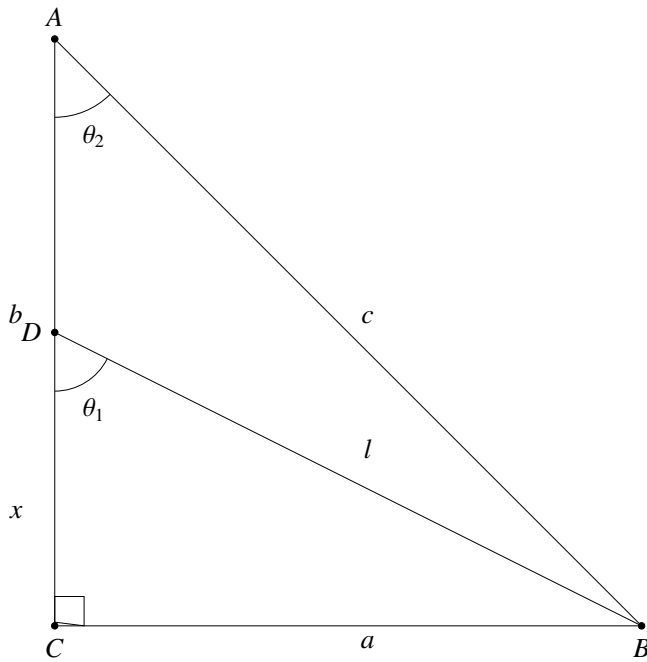


Fig. 1.10.1:  $\theta_1 < \theta_2 \implies \sin \theta_1 < \sin \theta_2$ .

**Solution:** In Hero's formula in (1.4.9.8), all the factors inside the square root should be positive. Thus,

$$(s - a) > 0, (s - b) > 0, (s - c) > 0 \quad (1.10.4.1)$$

$$(1.10.4.2)$$

$$(s - a) > 0 \implies \frac{a + b + c}{2} - a > 0 \quad (1.10.4.3)$$

$$\text{or, } b + c > a \quad (1.10.4.4)$$

Similarly, it can be shown that  $a + b > c, c + a > b$ .

### 1.11 Triangle Exercises

1. Sides opposite to equal angles of a triangle are equal.
2. Each angle of an equilateral triangle is of  $60^\circ$ .
3. Using cosine formula in an equilateral  $\Delta$ , show that  $\cos 60^\circ = \frac{1}{2}$ .
4. Using (1.3.3.1), show that  $\sin 60^\circ = \frac{\sqrt{3}}{2}$ .
5. Find  $\sin 30^\circ$  and  $\cos 30^\circ$  using (1.3.1.2).
6. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
7. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

8. In  $\Delta ABC$ , the bisector  $AD$  of  $\angle A$  is perpendicular to side  $BC$ . Show that  $AB = AC$  and  $\Delta ABC$  is isosceles.
9.  $E$  and  $F$  are respectively the mid-points of equal sides  $AB$  and  $AC$  of  $\Delta ABC$ . Show that  $BF = CE$ .
10. In an isosceles  $\Delta ABC$  with  $AB = AC$ ,  $D$  and  $E$  are points on  $BC$  such that  $BE = CD$ . Show that  $AD = AE$ .
11.  $AB$  is a line-segment.  $P$  and  $Q$  are points on opposite sides of  $AB$  such that each of them is equidistant from the points  $A$  and  $B$ . Show that the line  $PQ$  is the perpendicular bisector of  $AB$ .
12.  $P$  is a point equidistant from two lines  $l$  and  $m$  intersecting at point  $A$ . Show that the line  $AP$  bisects the angle between them.
13.  $D$  is a point on side  $BC$  of  $\Delta ABC$  such that  $AD = AC$ . Show that  $AB > AD$ .
14.  $AB$  is a line segment and line  $l$  is its perpendicular bisector. If a point  $P$  lies on  $l$ , show that  $P$  is equidistant from  $A$  and  $B$ .
15. Line-segment  $AB$  is parallel to another line-segment  $CD$ .  $O$  is the mid-point of  $AD$ . Show that
  - a)  $\Delta AOB \cong \Delta DOC$
  - b)  $O$  is also the mid-point of  $BC$ .
16. In quadrilateral  $ACBD$ ,  $AC = AD$  and  $AB$  bisects  $\angle A$ . Show that  $\Delta ABC \cong \Delta ABD$ . What can you say about  $BC$  and  $BD$ ?
17.  $ABCD$  is a quadrilateral in which  $AD = BC$  and  $\angle DAB = \angle CBA$ . Prove that
  - a)  $\Delta ABD \cong \Delta BAC$
  - b)  $BD = AC$
  - c)  $\angle ABD = \angle BAC$ .
18.  $l$  and  $m$  are two parallel lines intersected by another pair of parallel lines  $p$  and  $q$  to form the quadrilateral  $ABCD$ . Show that  $\Delta ABC \cong \Delta CDA$ .
19. Line  $l$  is the bisector of  $\angle A$  and  $B$  is any point on  $l$ .  $BP$  and  $BQ$  are perpendiculars from  $B$  to the arms of  $\angle A$  (see Fig. 7.20). Show that:
  - a)  $\Delta APB \cong \Delta AQB$
  - b)  $BP = BQ$  or  $B$  is equidistant from the arms of  $\angle A$ .
20.  $ABCE$  is a quadrilateral and  $D$  is a point on  $BC$  such that,  $AC = AE, AB = AD$  and  $\angle BAD = \angle EAC$ . Show that  $BC = DE$ .
21. In right triangle  $ABC$ , right angled at  $C$ ,  $M$  is



the mid-point of hypotenuse  $AB$ .  $C$  is joined to  $M$  and produced to a point  $D$  such that  $DM = CM$ . Point  $D$  is joined to point  $B$ . Show that:

- a)  $\triangle AMC \cong \triangle BMD$
  - b)  $\angle DBC$  is a right angle.
  - c)  $\triangle DBC \cong \triangle ACB$
  - d)  $CM = \frac{1}{2}AB$
22. In an isosceles  $\triangle ABC$ , with  $AB = AC$ , the bisectors of  $\angle B$  and  $\angle C$  intersect each other at  $O$ . Join  $A$  to  $O$ . Show that :
- a)  $OB = OC$
  - b)  $AO$  bisects  $\angle A$
23. In  $\triangle ABC$ ,  $AD$  is the perpendicular bisector of  $BC$ . Show that  $\triangle ABC$  is an isosceles triangle in which  $AB = AC$ .
24.  $ABC$  is an isosceles triangle in which altitudes  $BE$  and  $CF$  are drawn to equal sides  $AC$  and  $AB$  respectively. Show that these altitudes are equal.
25.  $ABC$  is a triangle in which altitudes  $BE$  and  $CF$  to sides  $AC$  and  $AB$  are equal. Show that
- a)  $\triangle ABE \cong \triangle ACF$
  - b)  $AB = AC$ , i.e.,  $ABC$  is an isosceles triangle.
26.  $ABC$  and  $DBC$  are two isosceles triangles on the same base  $BC$ . Show that  $\angle ABD = \angle ACD$ .
27.  $\triangle ABC$  and  $\triangle DBC$  are two isosceles triangles on the same base  $BC$  and vertices  $A$  and  $D$  are on the same side of  $BC$ . If  $AD$  is extended to intersect  $BC$  at  $P$ , show that
- a)  $\triangle ABD \cong \triangle ACD$
  - b)  $\triangle ABP \cong \triangle ACP$
  - c)  $AP$  bisects  $\angle A$  as well as  $\angle D$ .
  - d)  $AP$  is the perpendicular bisector of  $BC$ .
28.  $AD$  is an altitude of an isosceles  $\triangle ABC$  in which  $AB = AC$ . Show that
- a)  $AD$  bisects  $BC$
  - b)  $AD$  bisects  $\angle A$ .
29. Two sides  $AB$  and  $BC$  and median  $AM$  of one triangle  $ABC$  are respectively equal to sides  $PQ$  and  $QR$  and median  $PN$  of  $\triangle PQR$ . Show that:
- a)  $\triangle ABM \cong \triangle PQN$
  - b)  $\triangle ABC \cong \triangle PQR$
30.  $BE$  and  $CF$  are two equal altitudes of a triangle  $ABC$ . Using RHS congruence rule, prove that the triangle  $ABC$  is isosceles.
31.  $ABC$  is an isosceles triangle with  $AB = AC$ . Draw  $AP \perp BC$  to show that  $\angle B = \angle C$ .
32.  $\triangle ABC$  is an isosceles triangle in which  $AB = AC$ . Side  $BA$  is produced to  $D$  such that  $AD = AB$ . Show that  $\angle BCD$  is a right angle.
33.  $ABC$  is a right angled triangle in which  $\angle A = 90^\circ$  and  $AB = AC$ . Find  $\angle B$  and  $\angle C$ .
34. Show that in a right angled triangle, the hypotenuse is the longest side.
35. Sides  $AB$  and  $AC$  of  $\triangle ABC$  are extended to points  $P$  and  $Q$  respectively. Also,  $\angle PBC < \angle QCB$ . Show that  $AC > AB$ .
36. Line segments  $AD$  and  $BC$  intersect at  $O$  and form  $\triangle OAB$  and  $\triangle ODC$ .  $\angle B < \angle A$  and  $\angle C < \angle D$ . Show that  $AD < BC$ .
37.  $AB$  and  $CD$  are respectively the smallest and longest sides of a quadrilateral  $ABCD$ . Show that  $\angle A > \angle C$  and  $\angle B > \angle D$ .
38. In  $\triangle PQR$ ,  $PR > PQ$  and  $PS$  bisects  $\angle QPR$ . Prove that  $\angle PSR > \angle PSQ$ .
39. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.
40.  $ABCD$  is a trapezium with  $AB \parallel DC$ .  $E$  and  $F$  are points on non-parallel sides  $AD$  and  $BC$  respectively such that  $EF$  is parallel to  $AB$ . Show that  $\frac{AE}{ED} = \frac{BF}{FC}$ .
41.  $ST$  is a line joining two points on  $PQ$  and  $PR$  in  $\triangle PQR$ . If  $\frac{PS}{SQ} = \frac{PT}{TR}$  and  $\angle PST = \angle PRQ$ , prove that  $PQR$  is an isosceles triangle.
42. If  $LM \parallel CB$  and  $LN \parallel CD$ , prove that  $\frac{AM}{AB} = \frac{AN}{AD}$ .
43.  $D$  is a point on  $AB$  and  $E, F$  are points on  $BC$  such that  $DE \parallel AC$  and  $DF \parallel AE$ . Prove that  $\frac{BF}{FE} = \frac{BE}{EC}$ .
44.  $O$  is a point in the interior of  $\triangle PQR$ .  $D$  is a point on  $OP$ . If  $DE \parallel OQ$  and  $DF \parallel OR$ . Show that  $EF \parallel QR$ .
45.  $O$  is a point in the interior of  $\triangle PQR$ .  $A, B$  and  $C$  are points on  $OP, OQ$  and  $OR$  respectively such that  $AB \parallel PQ$  and  $AC \parallel PR$ . Show that  $BC \parallel QR$ .
46.  $ABCD$  is a trapezium in which  $AB \parallel DC$  and its diagonals intersect each other at the point  $O$ . Show that  $\frac{AO}{BO} = \frac{CO}{DO}$ .
47. The diagonals of a quadrilateral  $ABCD$  intersect each other at the point  $O$  such that  $\frac{AO}{BO} = \frac{CO}{DO}$ . Show that  $ABCD$  is a trapezium.
48.  $PQ \parallel RS$  and  $PS$  intersects  $QR$  at  $O$ . Show that  $\triangle OPQ \sim \triangle ORS$ .
49.  $CM$  and  $RN$  are respectively the medians of  $\triangle ABC$  and  $\triangle PQR$ . If  $\triangle ABC \sim \triangle PQR$ , prove that

- a)  $\triangle AMC \sim \triangle PNR$   
 b)  $\frac{CM}{RN} = \frac{AB}{PQ}$   
 c)  $\triangle CMB \sim \triangle RNQ$
50. Diagonals  $AC$  and  $BD$  of a trapezium  $ABCD$  with  $AB \parallel DC$  intersect each other at the point  $O$ . Using a similarity criterion for two triangles, show that  $\frac{OA}{OC} = \frac{OB}{OD}$ .
51. In  $\triangle PQR$ ,  $QP$  is extended to  $T$  and  $S$  is a point on  $QR$  such that  $\frac{QR}{QS} = \frac{QT}{PR}$ . If  $\angle PRQ = \angle PQS$ , show that that  $\triangle PQS \sim \triangle TQR$ .
52.  $S$  and  $T$  are points on sides  $PR$  and  $QR$  of  $\triangle PQR$  such that  $\angle P = \angle RTS$ . Show that  $\triangle RPQ \sim \triangle RTS$ .
53. In  $\triangle ABC$ ,  $D$  and  $E$  are points on the sides  $AB$  and  $AC$  respectively. If  $\triangle ABE \cong \triangle ACD$ , show that  $\triangle ADE \sim \triangle ABC$ .
54. Altitudes  $AD$  and  $CE$  of  $\triangle ABC$  intersect each other at the point  $P$ . Show that:  
 a)  $\triangle AEP \sim \triangle CDP$   
 b)  $\triangle ABD \sim \triangle CBE$   
 c)  $\triangle AEP \sim \triangle ADB$   
 d)  $\triangle PDC \sim \triangle BEC$
55.  $E$  is a point on the side  $AD$  produced of a parallelogram  $ABCD$  and  $BE$  intersects  $CD$  at  $F$ . Show that  $\triangle ABE \sim \triangle CFB$ .
56.  $ABC$  and  $AMP$  are two right triangles, right angled at  $B$  and  $M$  respectively.  $M$  lies on  $AC$  and  $AB$  is extended to meet  $P$ . Prove that:  
 a)  $\triangle ABC \sim \triangle AMP$   
 b)  $\frac{CA}{PA} = \frac{BC}{MP}$
57.  $CD$  and  $GH$  are respectively the bisectors of  $\angle ACB$  and  $\angle EGF$  such that  $D$  and  $H$  lie on sides  $AB$  and  $FE$  of  $\triangle ABC$  and  $\triangle EFG$  respectively. If  $\triangle ABC \sim \triangle FEG$ , show that:  
 58.  $\frac{CD}{GH} = \frac{AC}{FG}$   
 59.  $\triangle DCB \sim \triangle HGE$   
 60.  $\triangle DCA \sim \triangle HGF$
61.  $E$  is a point on side  $CB$  produced of an isosceles  $\triangle ABC$  with  $AB = AC$ . If  $AD \perp BC$  and  $EF \perp AC$ , prove that  $\triangle ABD \sim \triangle ECF$ .
62. Sides  $AB$  and  $BC$  and median  $AD$  of a  $\triangle ABC$  are respectively proportional to sides  $PQ$  and  $QR$  and median  $PM$  of  $\triangle PQR$ . Show that  $\triangle ABC \sim \triangle PQR$ .
63.  $D$  is a point on the side  $BC$  of a  $\triangle ABC$  such that  $\angle ADC = \angle BAC$ . Show that  $CA^2 = CB \cdot CD$ .
64. Sides  $AB$  and  $AC$  and median  $AD$  of a  $\triangle ABC$  are respectively proportional to sides  $PQ$  and  $PR$  and median  $PM$  of another  $\triangle PQR$ . Show that  $\triangle ABC \sim \triangle PQR$ .
65. If  $AD$  and  $PM$  are medians of  $\triangle ABC$  and  $\triangle PQR$ , respectively where  $\triangle ABC \sim \triangle PQR$ , prove that  $\frac{AB}{PQ} = \frac{AD}{PM}$ .
66. The line segment  $XY$  is parallel to side  $AC$  of  $\triangle ABC$  and it divides the triangle into two parts of equal areas. Find the ratio  $\frac{AX}{AB}$ .
67. Diagonals of a trapezium  $ABCD$  with  $AB \parallel DC$  intersect each other at the point  $O$ . If  $AB = 2CD$ , find the ratio of the areas of  $\triangle AOB$  and  $\triangle COD$ .
68.  $ABC$  and  $DBC$  are two triangles on the same base  $BC$ . If  $AD$  intersects  $BC$  at  $O$ , show that  $\frac{ar(ABC)}{ar(DBC)} = \frac{AO}{DO}$ .
69. If the areas of two similar triangles are equal, prove that they are congruent.
70.  $D$ ,  $E$  and  $F$  are respectively the mid-points of sides  $AB$ ,  $BC$  and  $CA$  of  $\triangle ABC$ . Find the ratio of the areas of  $\triangle DEF$  and  $\triangle ABC$ .
71. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.
72. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.
73.  $ABC$  and  $BDE$  are two equilateral triangles such that  $D$  is the mid-point of  $BC$ . Find the ratio of the areas of triangles  $ABC$  and  $BDE$ .
74. The sides of two similar triangles are in the ratio  $4 : 9$ . Find the ratio the area of these triangles are in the ratio
75. In  $\triangle ABC$ ,  $\angle ACB = 90^\circ$  and  $CD \perp AB$ . Prove that  $\frac{BC^2}{AC^2} = \frac{BD}{AD}$ .
76. In  $\triangle ABC$ , if  $AD \perp BC$ , prove that  $AB^2 + CD^2 = BD^2 + AC^2$ .
77.  $BL$  and  $CM$  are medians of a  $\triangle ABC$  right angled at  $A$ . Prove that  $4(BL^2 + CM^2) = 5BC^2$ .
78.  $O$  is any point inside a rectangle  $ABCD$ . Prove that  $OB^2 + OD^2 = OA^2 + OC^2$ .
79.  $PQR$  is a triangle right angled at  $P$  and  $M$  is a point on  $QR$  such that  $PM \perp QR$ . Show that  $PM^2 = QM \cdot MR$ .
80.  $ABD$  is a triangle right angled at  $A$  and  $AC \perp BD$ . Show that  
 a)  $AB^2 = BC \cdot BD$   
 b)  $AC^2 = BC \cdot DC$   
 c)  $AD^2 = BD \cdot CD$

81.  $ABC$  is an isosceles triangle right angled at  $C$ . Prove that  $AB^2 = 2AC^2$ .
82.  $ABC$  is an isosceles triangle with  $AC = BC$ . If  $AB^2 = 2AC^2$ , prove that  $ABC$  is a right triangle.
83.  $ABC$  is an equilateral triangle of side  $2a$ . Find each of its altitudes.
84. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.
85.  $O$  is a point in the interior of a  $\triangle ABC$ ,  $OD \perp BC$ ,  $OE \perp AC$  and  $OF \perp AB$ . Show that
- $OA^2 + OB^2 + BD^2 - OD^2 - OE^2 - OF^2 = AF^2 + BD^2 + CE^2$ .
  - $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$ .
86.  $D$  and  $E$  are points on the sides  $CA$  and  $CB$  respectively of a  $\triangle ABC$  right angled at  $C$ . Prove that  $AE^2 + BD^2 = AB^2 + DE^2$ .
87. The perpendicular from  $A$  on side  $BC$  of a  $\triangle ABC$  intersects  $BC$  at  $D$  such that  $DB = 3CD$ . Prove that  $2AB^2 = 2AC^2 + BC^2$ .
88. In an equilateral  $\triangle ABC$ ,  $D$  is a point on side  $BC$  such that  $BD = \frac{1}{3}BC$ . Prove that  $9AD^2 = 7AB^2$ .
89. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.
90.  $PS$  is the bisector of  $\angle QPR$  of  $\triangle PQR$ . Prove that  $\frac{QS}{SR} = \frac{PQ}{PR}$ .
91.  $D$  is a point on hypotenuse  $AC$  of  $\triangle ABC$ , such that  $BD \perp AC$ ,  $DM \perp BC$  and  $DN \perp AB$ . Prove that :
- $DM^2 = DN \cdot MC$
  - $DN^2 = DM \cdot AN$
92.  $ABC$  is a triangle in which  $\angle ABC > 90^\circ$  and  $AD \perp CB$  produced. Prove that  $AC^2 = AB^2 + BC^2 + 2BC \cdot BD$ .
93.  $ABC$  is a triangle in which  $\angle ABC < 90^\circ$  and  $AD \perp BC$ . Prove that  $AC^2 = AB^2 + BC^2 - 2BC \cdot BD$ .
94.  $AD$  is a median of a  $\triangle ABC$  and  $AM \perp BC$ . Prove that :
- $AC^2 = AD^2 + BC \cdot DM + \left(\frac{BC}{2}\right)^2$
  - $AB^2 = AD^2 - BC \cdot DM + \left(\frac{BC}{2}\right)^2$
  - $AC^2 + AB^2 = 2AD^2 + \frac{1}{2}BC^2$
95. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.
96.  $D$  is a point on side  $BC$  of  $\triangle ABC$  such that

$\frac{BD}{CD} = \frac{AB}{AC}$ . Prove that  $AD$  is the bisector of  $\angle BAC$ .

## 2 QUADRILATERALS

### 2.1 Properties

1. Sum of the angles of a quadrilateral is  $360^\circ$ .

**Solution:** Draw the diagonal and use the fact that sum of the angles of a triangle is  $180^\circ$ .

2. A diagonal of a parallelogram divides it into two congruent triangles.

**Solution:** The alternate angles for the parallel sides are equal. The diagonal is common. Use ASA congruence.

3. In a parallelogram,

- opposite sides are equal
- opposite angles are equal
- diagonals bisect each other

**Solution:** Since the diagonal divides the parallelogram into two congruent triangles, all the above results follow.

4. A quadrilateral is a parallelogram, if

- opposite sides are equal or
- opposite angles are equal or
- diagonals bisect each other or
- a pair of opposite sides is equal and parallel

**Solution:** All the above lead to a quadrilateral that has two parallel sides, by showing that the alternate angles are equal.

5. A rectangle is a parallelogram with one angle that is  $90^\circ$ . Show that all angles of the rectangle are  $90^\circ$ .

**Solution:** Draw a diagonal. Since the diagonal divides the rectangle into two congruent triangles, the angle opposite to the right angle is also  $90^\circ$ . Using congruence, it can be shown that the other two angles are equal. Now use the fact that the sum of the angles of a quadrilateral is  $360^\circ$ .

6. Diagonals of a rectangle bisect each other and are equal and vice-versa.

**Solution:** Use Baudhayana's theorem for equality of diagonals.

7. Diagonals of a rhombus bisect each other at right angles and vice-versa.

**Solution:** The median of an isosceles triangle is also its perpendicular bisector.

8. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.

**Solution:** A square has the properties of a rectangle as well as a rhombus.

9. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.
  10. Two parallel lines  $l$  and  $m$  are intersected by a transversal  $p$ . Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.
  11. Show that the bisectors of angles of a parallelogram form a rectangle.
  12. A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.
  13. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
  14. Area of a parallelogram is the product of its base and the corresponding altitude.
  15. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.
  16. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
  17. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
  18. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
  19. Show that the diagonals of a square are equal and bisect each other at right angles.
  20. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.
3. In  $\triangle ABC$  and  $\triangle DEF$ ,  $AB = DE$ ,  $AB \parallel DE$ ,  $BC = EF$  and  $BC \parallel EF$ . Vertices  $A, B$  and  $C$  are joined to vertices  $D, E$  and  $F$  respectively. Show that
    - a) quadrilateral  $ABED$  is a parallelogram
    - b) quadrilateral  $BEFC$  is a parallelogram
    - c)  $AD \parallel CF$  and  $AD = CF$
    - d) quadrilateral  $ACFD$  is a parallelogram
    - e)  $AC = DF$
    - f)  $\triangle ABC \cong \triangle DEF$ .
  4.  $ABCD$  is a trapezium in which  $AB \parallel CD$  and  $AD = BC$ . Show that
    - a)  $\angle A = \angle B$
    - b)  $\angle C = \angle D$
    - c)  $\triangle ABC \cong \triangle BAD$
    - d) diagonal  $AC =$  diagonal  $BD$
  5.  $ABCD$  is a quadrilateral in which  $P, Q, R$  and  $S$  are mid-points of the sides  $AB, BC, CD$  and  $DA$ .  $AC$  is a diagonal. Show that
    - a)  $SR \parallel AC$  and  $SR = \frac{1}{2}AC$
    - b)  $PQ = SR$
    - c)  $PQRS$  is a parallelogram.
  6.  $ABCD$  is a rhombus and  $P, Q, R$  and  $S$  are the mid-points of the sides  $AB, BC, CD$  and  $DA$  respectively. Show that the quadrilateral  $PQRS$  is a rectangle.
  7.  $ABCD$  is a rectangle and  $P, Q, R$  and  $S$  are mid-points of the sides  $AB, BC, CD$  and  $DA$  respectively. Show that the quadrilateral  $PQRS$  is a rhombus.
  8.  $ABCD$  is a trapezium in which  $AB \parallel DC$ ,  $BD$  is a diagonal and  $E$  is the mid-point of  $AD$ . A line is drawn through  $E \parallel AB$  intersecting  $BC$  at  $F$ . Show that  $F$  is the mid-point of  $BC$ .
  9. In a parallelogram  $ABCD$ ,  $E$  and  $F$  are the mid-points of sides  $AB$  and  $CD$  respectively. Show that the line segments  $AF$  and  $EC$  trisect the diagonal  $BD$ .
  10. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
  11.  $ABCD$  is a parallelogram in which  $P$  and  $Q$  are mid-points of opposite sides  $AB$  and  $CD$ . If  $AQ$  intersects  $DP$  at  $S$  and  $BQ$  intersects  $CP$  at  $R$ , show that:
    - a)  $APCQ$  is a parallelogram.
    - b)  $DPBQ$  is a parallelogram.
    - c)  $PSQR$  is a parallelogram.

## 2.2 Quadrilateral Exercises

1. In parallelogram  $ABCD$ , two points  $P$  and  $Q$  are taken on diagonal  $BD$  such that  $DP = BQ$ . show that
  - a)  $\triangle APD \cong \triangle CQB$
  - b)  $AP = CQ$
  - c)  $\triangle AQB \cong \triangle CPD$
  - d)  $AQ = CP$
  - e)  $APCQ$  is a parallelogram
2.  $ABCD$  is a parallelogram and  $AP$  and  $CQ$  are perpendiculars from vertices  $A$  and  $C$  on diagonal  $BD$ . Show that
  - a)  $\triangle APB \cong \triangle CQD$
  - b)  $AP = CQ$

12.  $l, m$  and  $n$  are three parallel lines intersected by transversals  $p$  and  $q$  such that  $l, m$  and  $n$  cut off equal intercepts  $AB$  and  $BC$  on  $p$ . Show that  $l, m$  and  $n$  cut off equal intercepts  $DE$  and  $EF$  on  $q$  also.
13. Diagonal  $AC$  of a parallelogram  $ABCD$  bisects  $\angle A$ . show that
- it bisects  $\angle C$  also,
  - $ABCD$  is a rhombus.
14.  $ABCD$  is a rhombus. Show that diagonal  $AC$  bisects  $\angle A$  as well as  $\angle C$  and diagonal  $BD$  bisects  $\angle B$  as well as  $\angle D$ .
15.  $ABCD$  is a rectangle in which diagonal  $AC$  bisects  $\angle A$  as well as  $\angle C$ . Show that
- $ABCD$  is a square
  - diagonal  $BD$  bisects  $\angle B$  as well as  $\angle D$ .
16. If  $E, F, G$  and  $H$  are respectively the mid-points of the sides of a parallelogram  $ABCD$ , show that
- $$ar(EFGH) = \frac{1}{2}ar(ABCD). \quad (2.2.16.1)$$
17.  $P$  and  $Q$  are any two points lying on the sides  $DC$  and  $AD$  respectively of a parallelogram  $ABCD$ . Show that  $ar(APB) = ar(BQC)$ .
18.  $P$  is a point in the interior of a parallelogram  $ABCD$ . Show that
- $ar(APB) + ar(PCD) = \frac{1}{2}ar(ABCD)$
  - $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$
19.  $PQRS$  and  $ABRS$  are parallelograms and  $X$  is any point on side  $BR$ . show that
- $ar(PQRS) = ar(ABRS)$
  - $ar(AXS) = \frac{1}{2}ar(PQRS)$
20. A farmer was having a field in the form of a parallelogram  $PQRS$ . She took any point  $A$  on  $RS$  and joined it to points  $P$  and  $Q$ . In how many parts the fields is divided? What are the shapes of these parts? The farmer wants to sow wheat and pulses in equal portions of the field separately. How should she do it?
21.  $ABCD$  is a quadrilateral and  $BE \parallel AC$  and also  $BE$  meets  $DC$  produced at  $E$ . Show that area of  $\triangle ADE$  is equal to the area of the quadrilateral  $ABCD$ .
22.  $E$  is any point on median  $AD$  of a  $\triangle ABC$ . Show that  $ar(ABE) = ar(ACE)$ .
23. In a  $\triangle ABC$ ,  $E$  is the mid-point of median  $AD$ . Show that  $ar(BED) = \frac{1}{4}ar(ABC)$ .
24. Show that the diagonals of a parallelogram divide it into four triangles of equal area.
25.  $ABC$  and  $ABD$  are two triangles on the same base  $AB$ . If line-segment  $CD$  is bisected by  $AB$  at  $O$ , show that  $ar(ABC) = ar(ABD)$ .
26.  $D, E$  and  $F$  are respectively the mid-points of the sides  $BC, CA$  and  $AB$  of a  $\triangle ABC$ . show that
- $BDEF$  is a parallelogram.
  - $ar(BDEF) = \frac{1}{2}ar(ABC)$
27. Diagonals  $AC$  and  $BD$  of quadrilateral  $ABCD$  intersect at  $O$  such that  $OB = OD$ . If  $AB = CD$ , then show that
- $ar(DOC) = ar(AOB)$
  - $ar(DCB) = ar(ACB)$
  - $ar(DEF) = \frac{1}{4}ar(ABC)$
28.  $D$  and  $E$  are points on sides  $AB$  and  $AC$  respectively of  $\triangle ABC$  such that  $ar(DBC) = ar(EBC)$ . Prove that  $DE \parallel BC$ .
29.  $XY$  is a line parallel to side  $BC$  of a  $\triangle ABC$ . If  $BE \parallel AC$  and  $CF \parallel AB$  meet  $XY$  at  $E$  and  $F$  respectively, show that  $ar(ABE) = ar(ACF)$ .
30. The side  $AB$  of a parallelogram  $ABCD$  is produced to any point  $P$ . A line through  $A$  and parallel to  $CP$  meets  $CB$  produced at  $Q$  and then parallelogram  $PBQR$  is completed. Show that  $ar(ABCD) = ar(PBQR)$ .
31. Diagonals  $AC$  and  $BD$  of a trapezium  $ABCD$  with  $AB \parallel DC$  intersect each other at  $O$ . Prove that  $ar(AOD) = ar(BOC)$ .
32.  $ABCDE$  is a pentagon. A line through  $B$  parallel to  $AC$  meets  $DC$  produced at  $F$ . Show that
- $ar(ACB) = ar(ACF)$
  - $ar(AEDF) = ar(ABCDE)$ .
33. A villager Itwaari has a plot of land of the shape of a quadrilateral. The Gram Panchayat of the village decided to take over some portion of his plot from one of the corners to construct a Health Centre. Itwaari agrees to the above proposal with the condition that he should be given equal amount of land in lieu of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented.
34.  $ABCD$  is a trapezium with  $AB \parallel DC$ . A line parallel to  $AC$  intersects  $AB$  at  $X$  and  $BC$  at  $Y$ . Prove that  $ar(ADX) = ar(ACY)$ .
35.  $AP \parallel BQ \parallel CR$ . Prove that  $ar(AQC) = ar(PBR)$ .
36. Diagonals  $AC$  and  $BD$  of a quadrilateral  $ABCD$

intersect at  $O$  in such a way that  $ar(AOD) = ar(BOC)$ . Prove that  $ABCD$  is a trapezium.

37.  $AB \parallel DC \parallel RP$ .  $ar(DRC) = ar(DPC)$  and  $ar(BDP) = ar(ARC)$ . Show that both the quadrilaterals  $ABCD$  and  $DCPR$  are trapeziums.
38. Parallelogram  $ABCD$  and rectangle  $ABEF$  are on the same base  $AB$  and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.
39. In  $\triangle ABC$ ,  $D$  and  $E$  are two points on  $BC$  such that  $BD = DE = EC$ . Show that  $ar(ABD) = ar(ADE) = ar(AEC)$ .
40.  $ABCD$ ,  $DCFE$  and  $ABFE$  are parallelograms. Show that  $ar(ADE) = ar(BCF)$ .
41.  $ABCD$  is a parallelogram and  $BC$  is produced to a point  $Q$  such that  $AD = CQ$ . If  $AQ$  intersect  $DC$  at  $P$ , show that  $ar(BPC) = ar(DPQ)$ .  $ABC$  and  $BDE$  are two equilateral triangles such that  $D$  is the mid-point of  $BC$ . If  $AE$  intersects  $BC$  at  $F$ , show that
- $ar(BDE) = \frac{1}{4}ar(ABC)$
  - $ar(BDE) = \frac{1}{2}ar(BAE)$
  - $ar(ABC) = 2ar(BEC)$
  - $ar(BFE) = ar(AFD)$
  - $ar(BFE) = 2ar(FED)$
  - $ar(FED) = \frac{1}{8}ar(AFC)$
42. Diagonals  $AC$  and  $BD$  of a quadrilateral  $ABCD$  intersect each other at  $P$ . Show that  $ar(APB) \times ar(CPD) = ar(APD) \times ar(BPC)$ .
43.  $P$  and  $Q$  are respectively the mid-points of sides  $AB$  and  $BC$  of a  $\triangle ABC$  and  $R$  is the mid-point of  $AP$ , show that
- $ar(PRQ) = \frac{1}{2}ar(ARC)$
  - $ar(PBQ) = ar(ARC)$
  - $ar(RQC) = \frac{3}{8}ar(ABC)$
44.  $ABC$  is a right triangle right angled at  $A$ .  $BCED$ ,  $ACFG$  and  $ABMN$  are squares on the sides  $BC$ ,  $CA$  and  $AB$  respectively. Line segment  $AX \perp DE$  meets  $BC$  at  $Y$ . Show that
- $\triangle MBC \cong \triangle ABD$
  - $ar(BYXD) = ar(ABMN)$
  - $ar(CYXE) = 2ar(FCB)$
  - $ar(BYXD) = 2ar(MBC)$
  - $\triangle FCB \cong \triangle ACE$
  - $ar(CYXE) = ar(ACFG)$
  - $ar(BCED) = ar(ABMN) + ar(ACFG)$
45.  $L$  is a point on the diagonal  $AC$  of quadrilateral  $ABCD$ . If  $LM \parallel CB$  and  $LN \parallel CD$ ,

prove that  $\frac{AM}{AB} = \frac{AN}{AD}$

### 3 CIRCLE

#### 3.1 Properties

- Fig. 3.1.1 represents a circle, which passes through the vertices  $A, B, C$  of  $\triangle ABC$  in Fig. (1.8.3) The points in the circle are at a distance  $R$  from the centre  $O$ .  $R$  is known as the *radius*. The line joining any two points on a circle is known as a *chord*. Thus, the sides of  $\triangle ABC$  are chords.

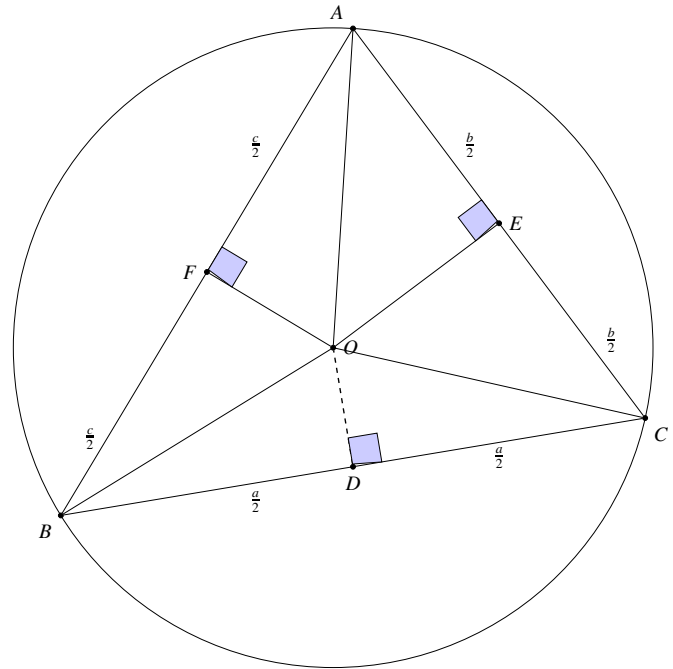


Fig. 3.1.1: Circle Definitions

- The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 3.1.3,  $AB$  is the diameter and passes through the centre  $O$
- In Fig. 3.1.3, show that  $\angle APB = 90^\circ$ .
- In Fig. 3.1.4, show that

$$\begin{aligned}\angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB\end{aligned}\quad (3.1.4.1)$$

**Solution:** Use Problem 1.8.5.

- In Fig. 3.1.4, show that the triangles  $PAB$  and  $PBD$  are similar

**Solution:** Trivial using previous problem

- In Fig. 3.1.4, show that

$$PA.PB = PC.PD \quad (3.1.6.1)$$

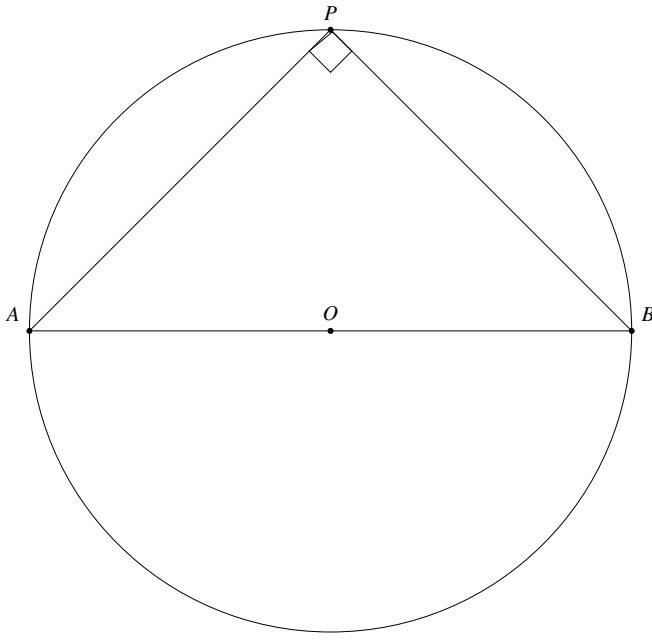
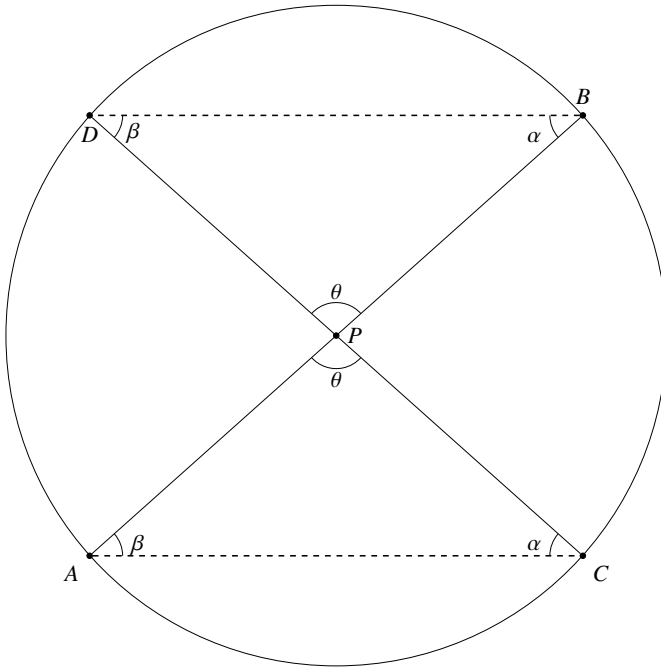


Fig. 3.1.3: Diameter of a circle.

Fig. 3.1.4:  $PA \cdot PB = PC \cdot PD$ 

**Solution:** Since triangles  $PAC$  and  $PBD$  are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \quad (3.1.6.2)$$

$$\Rightarrow PA \cdot PB = PC \cdot PD \quad (3.1.6.3)$$

7. Show that

$$\sin 0^\circ = 0 \quad (3.1.7.1)$$

**Solution:** From (1.1.2.1),  $\theta \rightarrow 0^\circ \Rightarrow a \rightarrow 0 \Rightarrow \sin \theta$  and

8. Show that

$$\cos 0^\circ = 1 \quad (3.1.8.1)$$

**Solution:** Follows from the fact that  $\sin 0 = 0$  and (1.3.3.1).

9. Show that

$$\cos 90^\circ = 0 \quad (3.1.9.1)$$

**Solution:** Follows from the fact that  $\cos 90^\circ = \sin(90^\circ - 90^\circ) = 0$  using (1.3.1.2).

10. The line  $PX$  in Fig. 3.1.11 touches the circle at exactly one point  $P$ . It is known as the tangent to the circle.

11. Show that  $OP \perp PX$ .

**Solution:** Without loss of generality, let  $0 \leq \theta \leq 90^\circ$ . Using the cosine formula in  $\triangle OPP_n$ ,

$$(r + d_n)^2 > r^2, \quad (3.1.11.1)$$

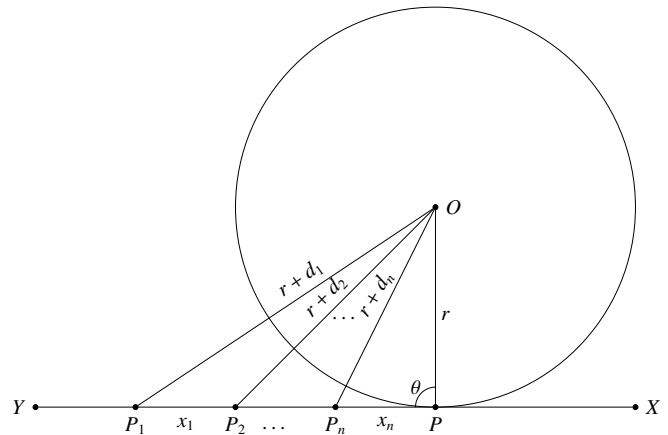


Fig. 3.1.11: Tangent to a Circle.

$$(r + d_n)^2 = r^2 + x_n^2 - 2rx_n \cos \theta > r^2 \quad (3.1.11.2)$$

$$\Rightarrow 0 < \cos \theta < \frac{x_n}{2r}, \quad (3.1.11.3)$$

where  $x_n$  can be made as small as we choose. Thus,

$$\cos \theta = 0 \Rightarrow \theta = 90^\circ. \quad (3.1.11.4)$$

12. In Fig. 3.1.12 show that

$$\angle PCA = \angle PBC \quad (3.1.12.1)$$

$O$  is the centre of the circle and  $PC$  is the tangent.

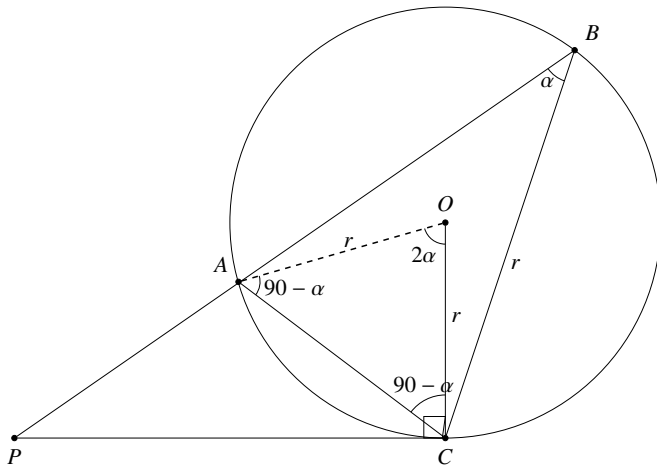


Fig. 3.1.12:  $PA.PB = PC^2$ .

**Solution:** Obvious from the figure once we observe that  $\triangle OAC$  is isosceles.

13. In Fig. 3.1.12, show that the triangles  $PAC$  and  $PBC$  are similar.

**Solution:** From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

14. Show that  $PA.PB = PC^2$

**Solution:** Since  $\triangle PAC \sim \triangle PBC$ , their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \quad (3.1.14.1)$$

$$\Rightarrow PA.PB = PC^2 \quad (3.1.14.2)$$

15. Given that  $PA.PB = PC^2$ , show that  $PC$  is a tangent to the circle.

16. In Fig. 3.1.16, show that

$$PA.PB = PC.PD \quad (3.1.16.1)$$

**Solution:** Draw a tangent and use the previous problem.

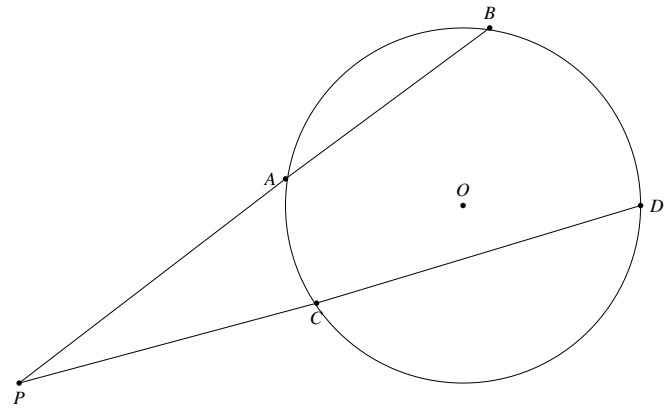


Fig. 3.1.16:  $PA.PB = PC^2$ .

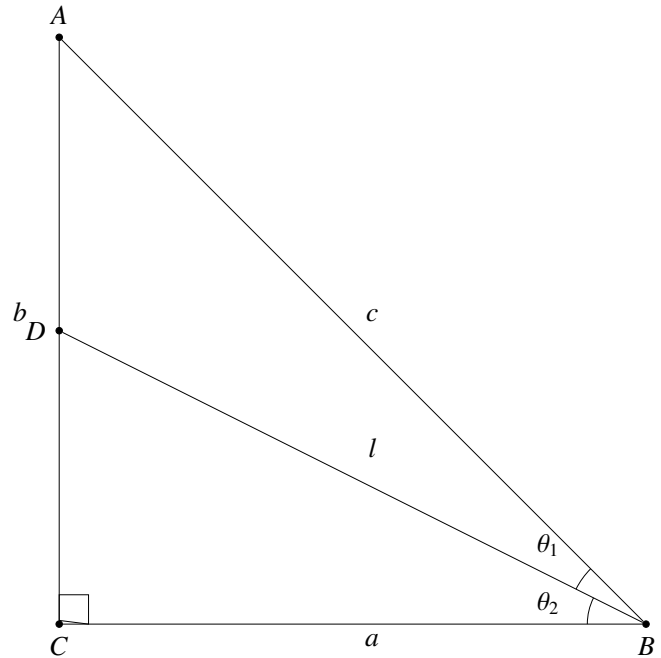


Fig. 3.2.1:  $\sin 2\theta = 2 \sin \theta \cos \theta$

### 3.2 Area of a Circle

1. Using Fig. 3.2.1, show that

$$\sin \theta_1 = \sin (\theta_1 + \theta_2) \cos \theta_2 - \cos (\theta_1 + \theta_2) \sin \theta_2 \quad (3.2.1.1)$$

**Solution:** The following equations can be obtained from the figure using the formula for

the area of a triangle

$$ar(\triangle ABC) = \frac{1}{2}ac \sin (\theta_1 + \theta_2) \quad (3.2.1.2)$$

$$= ar(\triangle BDC) + ar(\triangle ADB) \quad (3.2.1.3)$$

$$= \frac{1}{2}cl \sin \theta_1 + \frac{1}{2}al \sin \theta_2 \quad (3.2.1.4)$$

$$= \frac{1}{2}ac \sin \theta_1 \sec \theta_2 + \frac{1}{2}a^2 \tan \theta_2 \quad (3.2.1.5)$$



( $\because l = a \sec \theta_2$ ). From the above,

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (3.2.1.6)$$

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \cos(\theta_1 + \theta_2) \tan \theta_2 \quad (3.2.1.7)$$

Multiplying both sides by  $\cos \theta_2$ ,

$$\Rightarrow \sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (3.2.1.8)$$

resulting in

$$\Rightarrow \sin \theta_1 = \sin(\theta_1 + \theta_2) \cos \theta_2 - \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (3.2.1.9)$$

2. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (3.2.2.1)$$

b)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (3.2.2.2)$$

**Solution:** In (3.2.1.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \quad (3.2.2.3)$$

This gives (3.2.2.1). In (3.2.2.1), replace  $\alpha$  by  $90^\circ - \alpha$ . This results in

$$\begin{aligned} \sin(90^\circ - \alpha - \beta) \\ = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \\ \Rightarrow \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \end{aligned} \quad (3.2.2.4)$$

3. Using (3.2.1.1) and (3.2.2.2), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (3.2.3.1)$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \quad (3.2.3.2)$$

**Solution:** From (3.2.1.1),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (3.2.3.3)$$

Using (3.2.2.2) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + (\cos \theta_1 \cos \theta_2 \\ &\quad - \sin \theta_1 \sin \theta_2) \sin \theta_2 \end{aligned} \quad (3.2.3.4)$$

which can be expressed as

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + \cos \theta_1 \cos \theta_2 \sin \theta_2 \\ &\quad - \sin \theta_1 \sin^2 \theta_2 \end{aligned} \quad (3.2.3.5)$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (3.2.3.6)$$

we obtain

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \cos \theta_1 \cos \theta_2 \sin \theta_2 \\ &\quad + \sin \theta_1 \cos^2 \theta_2 \end{aligned} \quad (3.2.3.7)$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (3.2.3.8)$$

after factoring out  $\cos \theta_2$ . Using a similar approach, (3.2.3.2) can also be proved.

4. Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (3.2.4.1)$$

5. Show that

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad (3.2.5.1)$$

$$= 1 - \sin^2 \theta \quad (3.2.5.2)$$

$$= 2 \cos^2 \theta - 1 \quad (3.2.5.3)$$

6. The ratio of the perimeter of a circle to its diameter is  $\pi$ .

7. *Radian* is a another unit of the angle defined by

$$\pi \text{ radians} = 180^\circ \quad (3.2.7.1)$$

8. In Fig. 3.2.8, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general,  $n$  number of triangles can be arranged to form a regular polygon.

9. The angle formed by each of the congruent triangles at the centre of a regular polygon of  $n$  sides is  $\frac{2\pi}{n} = \frac{2\pi}{n} \text{ rad}$ .

10. The triangle that forms a polygon of  $n$  sides is given in Fig. 3.2.10. Show that

$$BC = 2r \sin \frac{\pi}{n} \quad (3.2.10.1)$$

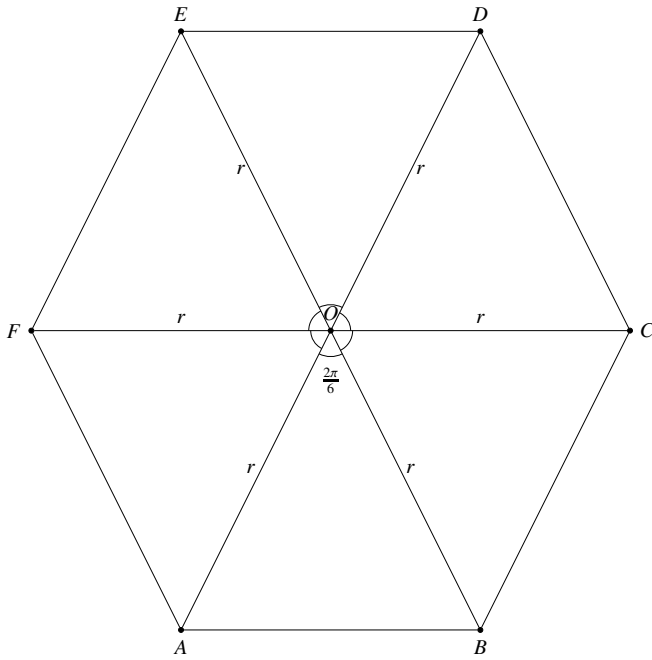


Fig. 3.2.8: Polygon Definition

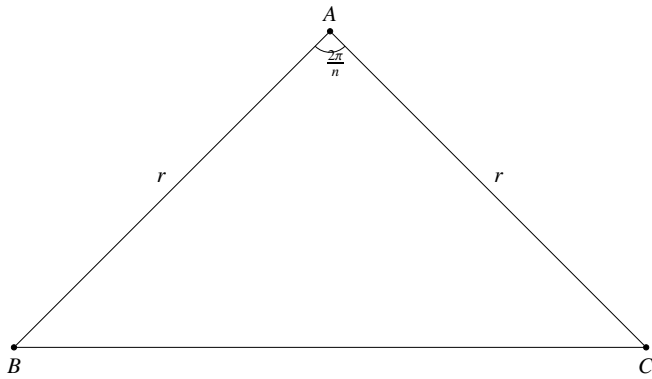


Fig. 3.2.10: Triangle that forms a polygon

**Solution:** Using cosine formula,

$$BC^2 = 2r^2 - 2r^2 \cos \frac{2\pi}{n} \quad (3.2.10.2)$$

$$\Rightarrow BC^2 = 2r^2 \left(1 - \cos \frac{2\pi}{n}\right) = 4r^2 \sin^2 \frac{\pi}{n} \quad (3.2.10.3)$$

upon substituting from (3.2.5.2). Taking the square root results in (3.2.10.1)

11. Show that the perimeter of a regular polygon is given by

$$2rn \sin \frac{\pi}{n} \quad (3.2.11.1)$$

12. Show that the area of a regular polygon is given

by

$$\frac{n}{2} r^2 \sin \frac{2\pi}{n} \quad (3.2.12.1)$$

**Solution:** From Fig. 3.2.10

$$\begin{aligned} ar(\text{polygon}) &= n \times ar(\triangle ABC) \\ &= \frac{n}{2} r^2 \sin \frac{2\pi}{n} \end{aligned} \quad (3.2.12.2)$$

13. Using Fig. 3.2.13, show that

$$\frac{n}{2} r^2 \sin \frac{2\pi}{n} < \text{area of circle} < nr^2 \tan \frac{\pi}{n} \quad (3.2.13.1)$$

The portion of the circle visible in Fig. 3.2.13 is defined to be a sector of the circle.

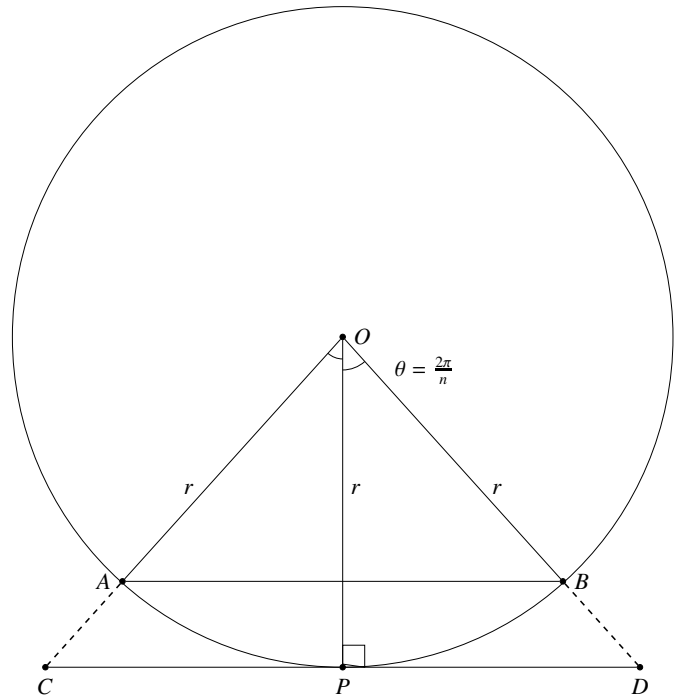


Fig. 3.2.13: Circle Area in between Area of Two Polygons

**Solution:** Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. 3.2.13, the area of the circle should be in between the areas of

the inner and outer polygons. Since

$$ar(\Delta OAB) = \frac{1}{2}r^2 \sin \frac{2\pi}{n} \quad (3.2.13.2)$$

$$ar(\Delta OPQ) = 2 \times \frac{1}{2} \times r \tan \frac{2\pi/n}{2} \times r \quad (3.2.13.3)$$

$$= r^2 \tan \frac{\pi}{n}, \quad (3.2.13.4)$$

we obtain (3.2.13.1).

14. Show that

$$\cos^2 \frac{\pi}{n} < \frac{\text{area of circle}}{nr^2 \tan \frac{\pi}{n}} < 1 \quad (3.2.14.1)$$

**Solution:** From (3.2.13.1) and (3.2.4.1),

$$\frac{n}{2}r^2 \sin \frac{2\pi}{n} < \text{area of circle} < nr^2 \tan \frac{\pi}{n} \quad (3.2.14.2)$$

$$\Rightarrow nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} < \text{area of circle} < nr^2 \tan \frac{\pi}{n} \quad (3.2.14.3)$$

which yields (3.2.14.1) upon making use of the fact that

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \quad (3.2.14.4)$$

15. Show that for large values of  $n$

$$\cos^2 \frac{\pi}{n} = 1 \quad (3.2.15.1)$$

**Solution:** As  $n \rightarrow \infty$ ,  $\frac{\pi}{n} \rightarrow 0$ . From (3.1.8.1), this yields (3.2.15.1).

16. (3.2.15.1) is a *limit* and expressed as

$$\lim_{n \rightarrow \infty} \cos^2 \frac{\pi}{n} = 1 \quad (3.2.16.1)$$

17. Show that

$$\text{area of circle} = r^2 \lim_{n \rightarrow \infty} n \tan \frac{\pi}{n} \quad (3.2.17.1)$$

**Solution:** From (3.2.14.1) and (3.2.16.1),

$$\lim_{n \rightarrow \infty} \cos^2 \frac{\pi}{n} < \lim_{n \rightarrow \infty} \frac{\text{area of circle}}{nr^2 \tan \frac{\pi}{n}} < 1 \quad (3.2.17.2)$$

$$1 = \lim_{n \rightarrow \infty} \frac{\text{area of circle}}{nr^2 \tan \frac{\pi}{n}} < 1 \quad (3.2.17.3)$$

resulting in (3.2.17.1).

18. Show that

$$\pi = \lim_{n \rightarrow \infty} n \tan \frac{\pi}{n} \quad (3.2.18.1)$$

**Solution:** From (3.2.6) and (3.2.11.1), the perimeter of the circle is

$$\lim_{n \rightarrow \infty} 2rn \sin \frac{\pi}{n} = 2\pi r \implies \lim_{n \rightarrow \infty} n \sin \frac{\pi}{n} = \pi \quad (3.2.18.2)$$

Also, from Fig. (3.2.13), using the fact that the inner and outer polygons converge into a circle for large  $n$ ,

$$\lim_{n \rightarrow \infty} nCD - nAB = 0 \quad (3.2.18.3)$$

$$\implies \lim_{n \rightarrow \infty} 2rn \tan \frac{\pi}{n} - 2rn \sin \frac{\pi}{n} = 0 \quad (3.2.18.4)$$

from which, we obtain (3.2.18.1) by substituting from (3.2.18.2).

19. Show that the area of a circle is  $\pi r^2$ .

**Solution:** Use (3.2.18.1) in (3.2.17.1).

20. Show that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \quad (3.2.20.1)$$

21. Show that the area of a sector with angle  $\theta$  in radians is  $\frac{1}{2}r^2\theta$ .

### 3.3 Circle Exercises

1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3. The perpendicular from the centre of a circle to a chord bisects the chord.
4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
5. There is one and only one circle passing through three non-collinear points.
6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.

9. Congruent arcs of a circle subtend equal angles at the centre.
10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
11. Angles in the same segment of a circle are equal.
12. Angle in a semicircle is a right angle.
13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
14. The sum of either pair of opposite angles of a cyclic quadrilateral is  $180^\circ$ .
15. If sum of a pair of opposite angles of a quadrilateral is  $180^\circ$ , the quadrilateral is cyclic.
16.  $AB$  is a diameter of the circle,  $CD$  is a chord equal to the radius of the circle.  $AC$  and  $BD$  when extended intersect at a point  $E$ . Prove that  $\angle AEB = 60^\circ$ .
17. Two circles intersect at two points  $A$  and  $B$ .  $AD$  and  $AC$  are diameters to the two circles. Prove that  $B$  lies on the line segment  $DC$ .
18. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
19. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
20. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
21. If a line intersects two concentric circles (circles with the same centre) with centre  $O$  at  $A$ ,  $B$ ,  $C$  and  $D$ , prove that  $AB = CD$ .
22. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
23. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
24. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
25. Two circles intersect at two points  $B$  and  $C$ . Through  $B$ , two line segments  $ABD$  and  $PBQ$  are drawn to intersect the circles at  $A, D$  and  $P, Q$  respectively. Prove that  $\angle ACP = \angle QCD$ .
26. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
27.  $ABC$  and  $ADC$  are two right triangles with common hypotenuse  $AC$ . Prove that  $\angle CAD = \angle CBD$ .
28. Prove that a cyclic parallelogram is a rectangle.
29. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
30. Let the vertex of an angle  $ABC$  be located outside a circle and let the sides of the angle intersect equal chords  $AD$  and  $CE$  with the circle. Prove that  $\angle ABC$  is equal to half the difference of the angles subtended by the chords  $AC$  and  $DE$  at the centre.
31. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.
32.  $ABCD$  is a parallelogram. The circle through  $A, B$  and  $C$  intersect  $CD$  (produced if necessary) at  $E$ . Prove that  $AE = AD$ .
33.  $AC$  and  $BD$  are chords of a circle which bisect each other. Prove that (i)  $AC$  and  $BD$  are diameters, (ii)  $ABCD$  is a rectangle.
34. Bisectors of angles  $A, B$  and  $C$  of a  $\triangle ABC$  intersect its circumcircle at  $D, E$  and  $F$  respectively. Prove that the angles of the  $\triangle DEF$  are  $90^\circ - \frac{A}{2}$ ,  $90^\circ - \frac{B}{2}$  and  $90^\circ - \frac{C}{2}$ .
35. Two congruent circles intersect each other at points  $A$  and  $B$ . Through  $A$  any line segment  $PAQ$  is drawn so that  $P, Q$  lie on the two circles. Prove that  $BP = BQ$ .
36. In any  $\triangle ABC$ , if the angle bisector of  $\angle A$  and perpendicular bisector of  $BC$  intersect, prove that they intersect on the circumcircle of the  $\triangle ABC$ .
37. The lengths of tangents drawn from an external point to a circle are equal.
38. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.
39. Two tangents  $TP$  and  $TQ$  are drawn to a circle with centre  $O$  from an external point  $T$ . Prove that  $\angle PTQ = 2\angle OPQ$ .
40. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.
41. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

42. A quadrilateral  $ABCD$  is drawn to circumscribe a circle. Prove that  $AB + CD = AD + BC$ .
43.  $XY$  and  $X'Y'$  are two parallel tangents to a circle with centre  $O$  and another tangent  $AB$  with point of contact  $C$  intersecting  $XY$  at  $A$  and  $X'Y'$  at  $B$ . Prove that  $\angle AOB = 90^\circ$
44. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.
45. Prove that the parallelogram circumscribing a circle is a rhombus.
46. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
47. Find the area of a sector of angle  $p$  (in degrees) of a circle with radius  $R$ .
48. Two chords  $AB$  and  $CD$  intersect each other at the point  $P$ . Prove that :
  - a)  $\triangle APC \sim \triangle DPB$
  - b)  $AP.PB = CP.DP$
49. Two chords  $AB$  and  $CD$  of a circle intersect each other at the point  $P$  (when produced) outside the circle. Prove that
  - a)  $\triangle PAC \sim \triangle PDB$
  - b)  $PA.PB = PC.PD$