Algebra: Maths Olympiad

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***Abstract*—This book provides a collection of the international maths olympiad problems in algebra.**

1. For what real values of x is
$$\sqrt{(x+\sqrt{2x-1}}+\sqrt{(x-\sqrt{2x-1}}=A \tag{1.1}$$
given
a) $A=\sqrt{2}$,
b) $A=1$,
c) $A=2$
where only non-negative real numbers are admitted for square roots?
2. Let a,b,c be real numbers. Consider the quadratic equation in $\cos x$:
$$a\cos^2 x+b\cos x+c=0. \tag{2.1}$$
Using the numbers a,b,c, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.
3. Solve the system of equations:
$$\text{x + y + z = a}$$
$$x^2+y^2+z^2=b^2$$
$$xy=z^2$$
where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.
4. Solve the equation $cos^n x - sin^n x = 1$, where n is a natural number.
5. Determine all real numbers x which satisfy the inequality:
$$\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}$$
6. Solve the equation
$$\cos^2 x+\cos^2 2x+\cos^2 3x=1$$
7. Find all real roots of the equation
$$\sqrt{x^2-p}+2\sqrt{x^2-1}=x \tag{7.1}$$
where p is a real parameter.
8. Find all solutions x_1,x_2,x_3,x_4,x_5 of the system
$$x_5+x_2=yx_1 \tag{8.1}$$
$$x_1+x_3=yx_2 \tag{8.2}$$
$$x_2+x_4=yx_3 \tag{8.3}$$
$$x_3+x_5=yx_4 \tag{8.4}$$
$$x_4+x_1=yx_5 \tag{8.5}$$
where y is a parameter.
9. Suppose a, b, c are the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\le 3abc$$
10. Determine all values x in the interval $0\le x\le 2\pi$ which satisfy the inequality
$$2\cos x\left|\sqrt{(1+\sin 2x)}-\sqrt{(1-\sin 2x)}\right|$$
11. Consider the system of equations
$$a_{11}x_1+a_{12}x_2+a_{13}x_3=0 \tag{11.1}$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=0 \tag{11.2}$$
$$a_{31}x_1+a_{32}x_2+a_{33}x_3=0 \tag{11.3}$$
with unknowns x_1,x_2,x_3. The coefficients satisfy the conditions:

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a) a_{11}, a_{22}, a_{33} are positive numbers;
b) the remaining coefficients are negative numbers;
c) in each equation, the sum of the coefficients is positive.

Prove that the given system has only the solution

$$x_1 = x_2 = x_3 = 0$$

12. Solve the system of equations

$$\begin{aligned} \left|a_1 - a_2\right| x_2 + \left|a_1 - a_3\right| x_3 + \left|a_1 - a_4\right| x_4 &= 1 \\ \left|a_2 - a_1\right| x_1 + \left|a_2 - a_3\right| x_3 + \left|a_2 - a_3\right| x_4 &= 1 \\ \left|a_3 - a_1\right| x_1 + \left|a_3 - a_2\right| x_2 &= 1 \\ \left|a_4 - a_1\right| x_1 + \left|a_4 - a_2\right| x_2 + \left|a_4 - a_3\right| x_3 &= 1 \end{aligned}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

13. Consider the sequence $[c_n]$, where

$$\begin{aligned} c_1 &= a_1 + a_2 + ... + a_8 \\ c_2 &= a_1^2 + a_2^2 + ... + a_8^2 \\ &..... \\ c_2 &= a_1^n + a_2^n + ... + a_8^n \end{aligned}$$

in which $a_1, a_2, ..., a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $[c_n]$ are equal to zero. Find all natural numbers n for which $c_n = 0$.

14. Let k, m, n be natural numbers such that m + k + 1 is a prime greater than n + 1. Let c_s = s(s + 1). Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k).....(c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 .....c_n$.

15. Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

16. Consider the system of equations

$$ax_1^2 + bx_1 + c = x_2$$

$$ax_2^2 + bx_2 + c = x_3$$

....

$$ax_{n-1}^2 + bx_{n-1} + c = x_n$$

$$ax_n^2 + bx_n + c = x_1$$

with unknowns $x_1, x_2, ..., x_n$, where a, b, c are real and a≠0. Let $\Delta = (b - 1)^2 - 4ac$. Prove that for this system

a) if $\Delta < 0$, there is no solution,
b) if $\Delta = 0$, there is exactly one solution,
c) if $\Delta > 0$, there is more than one solution.

17. Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0, x_2 > 0, x_1 y_1 - z_1^2 > 0, x_2 y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \le \frac{1}{x_1 y_1 - z_1^2} + \frac{1}{x_2 y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

18. Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n.

19. Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and B_n are numbers in the system with base b; these are related as follows:

$$A_n = x_n x_{n-1} ....x_0, A_{n-1} = x_{n-1} x_{n-2} ....x_0,$$

$$B_n = x_n x_{n-1} ....x_0, B_{n-1} = x_{n-1} x_{n-2} ....x_0,$$

$$x_n \neq 0, x_{n-1} \neq 0$$

Prove:
$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ if and only if $a > b$

20. The real numbers $a_0, a_1, ..., a_n,......$ satisfy the condition:

$$1 = a_0 \le a_1 \le a_2 \le ..... \le a_n \le ....$$

The numbers $b_1, b_2, ..., b_n, ...$ are defined by

$$b_n = \sum_{k=1}^n \left(1 - \frac{a_{k-1}}{a_k}\right)\left(\frac{1}{\sqrt{a_k}}\right)$$

a) Prove that $0 \le bn < 2$ for all n.
b) Given c with $0 \le c < 2$, prove that there exist numbers $a_0, a_1, ...$ with the above properties such that $b_n > c$ for large enough n.

21. Prove that for every natural number m, there exists a finite set S of points in a plane with the following property: For every point A in S, there are exactly m points in S which are at unit distance from A.

22. Prove that the following assertion is true for n = 3 and n = 5, and that it is false for every other natural number $n > 2$: If $a_1, a_2, ..., a_n$ are arbitrary real numbers, then $(a_1 - a_2)(a_1 - a_3).....(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)......(a_2 - a_n) + ... + (a_n - a_1)(a_n - a_2).......(a_n - a_{n-1} \ge 0$

23. Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \le 0$$

$$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \le 0$$

$$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \le 0$$

$$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \le 0$$

$$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \le 0$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

24. Let a and b be real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

25. Let $a_1, a_2, ..., a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, ..., b_n$ for which

a) $a_k < b_k$ for k = 1, 2,......, n,
b) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for k = 1, 2, ..., n - 1,
c) $b_1 + b_2 + ... + b_n < \frac{1+q}{1-q}(a_1 + a_2 + ... + a_n)$.

26. Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

27. Let P be a non-constant polynomial with integer coefficients. If n(P) is the number of distinct integers k such that $(P(k))^2 = 1$, prove that n(P) - deg(P) $\le$ 2, where deg(P) denotes the degree of the polynomial P.

28. Find all polynomials P, in two variables, with the following properties:
a) for a positive integer n and all real t, x, y
P(tx, ty) = t^nP(x, y)
(that is, P is homogeneous of degree n),
b) for all real a, b, c,
P(b + c, a) + P(c + a, b) + P(a + b, c) = 0,
c) P(1,0) = 1.

29. Consider the system of p equations in q = 2p unknowns $x_1, x_2, ..., x_q$:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1q}x_q = 0$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2q}x_q = 0$$

....

$$a_{p1}x_1 + a_{p2}x_2 + ... + a_{pq}x_q = 0$$

with every coefficient a_{ij} member of the set [-1,0,1]. Prove that the system has a solution $(x_1, x_2, ..., x_q)$ such that
a) all x_j (j = 1,2, ..., q) are integers,
b) there is at least one value of j for which $x_j \ne 0$,
c) $|x_j| \le$q(j = 1,2, ..., q).

30. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j = 2,3, $\cdots$. Show that, for any positive integer n, the roots of the equation $P_n(x)$ = x are real and distinct.

31. Let a and b be positive integers. When $a^2 + b^2$ is divided by a+b, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

32. Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real θ, then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

33. m and n are natural numbers with $1 \leq m \leq n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that m + n has its least value.
34. Let a_k(k = 1, 2, 3,....., n,.....) be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n}(\frac{a_k}{k^2}) \geq \sum_{k=1}^{n}(\frac{1}{k})$$

35. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5}(kx_k) = a, \sum_{k=1}^{5}(k^3 x_k) = a^2, \sum_{k=1}^{5}(k^5 x_k) = a^3,$$

36. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{1318} + \frac{1}{1319}.$$

37. a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining n - 1 numbers?
    b) For which values of $n > 2$ is there exactly one set having the stated property?
38. Determine the maximum value of $m^3 + n^3$, where m and n are integers satisfying m, n ∈ (1, 2,...,1981) and

$$(n^2 - mn - m^2)^2 = 1.$$

39. Consider the infinite sequences x_n of positive real numbers with the following properties: x_0 = 1, and for all $i \geq 0, x_{i+1} \leq x_i$.
    a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ...\frac{x_{n-1}^2}{x_n} \geq 3.999.$$

    b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ...\frac{x_{n-1}^2}{x_n} < 4$$

40. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n \qquad (40.1)$$

has a solution in integers (x, y), then it has at least three such solutions.Show that the equation has no solution in integers when n=2891.

41. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that 2abc - ab - bc - ca is the largest integer which cannot be expressed in the form xbc + yca + zab, where x, y and z are non-negative integers.
42. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

43. Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers for which x + y + z = 1.
44. Find one pair of positive integers a and b such that:
    a) ab(a + b) is not divisible by 7;
    b) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7.

    Justify your answer.
45. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and ad = bc. Prove that if a + d = 2^k and b + c = 2^m for some integers k and m, then a = 1.
46. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set M = (1,2,.., $n-1$) is colored either blue or white. It is given that
    a) for each i ∈ M, both i and n - i have the same color;
    b) for each i ∈ M, i ≠ k, both i and $|i-k|$ have the same color. Prove that all numbers in M must have the same color.
47. For any polynomial $P(x) = a_0 + a_1 + ... + a_k x^k$ with integer coefficients, the number of coefficients which are odd is denoted by w(P). For i = 0, 1,....., let $Q_i(x) = (1 + x)^i$. Prove that if $i_1 i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < ..... < i_n$, then $w(Q_{i1} + Q_{i2}, +Q_{in}) \geq w(Q_{i1})$.
48. For every real number x_1, construct the sequence $x_1, x_2,....$ by setting

$$x_{n+1} = x_n(x_n + \frac{1}{n})$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.

49. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set 2, 5, 13, d such that ab - 1 is not a perfect square.
50. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \frac{\sqrt{n}}{3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.
51. Let n be a positive integer and let $A_1, A_2, ..., A_{2n+1}$ be subsets of a set B. Suppose that
    a) Each A_i has exactly 2n elements,
    b) Each $A_i \cap A_j (1 \leq i < j \leq 2n + 1)$ contains exactly one element, and
    c) Every element of B belongs to at least two of the A_i.

    Foe which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has zero assign to exactly n of its elements.
52. Let a and b be positive integers such that ab + 1 divides $a^2 + b^2$. Show that
    $$\frac{a^2 + b^2}{ab + 1}$$
    is the square of an integer.
53. Let n and k be positive integers and let S be a set of n points in the plane such that
    a) No three points of S are collinear, and
    b) For any point P of S there are at least k points of S equidistant from P .

    Prove that $k < \frac{1}{2} + \sqrt{2n}$
54. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.
55. Given an initial integer $n_0 > 1$, two players, A and B, choose integers $n_1, n_2, n_3,.......$ alternately according to the following rules: Knowing n_{2k}, A chooses any integer n_{2k+1} such that
    $n_{2k} \geq n_{2k+1} \geq n_{2k}^2$.Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $\frac{n_{2k+1}}{n_{2k+2}}$ is a prime raised to a positive integer power.
    Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which n_0 does:
    a) A have a winning strategy?
    b) B have a winning strategy?
    c) Neither player have a winning strategy?
56. Determine all integers $n > 1$ such that $\frac{2^n+1}{n^2}$ is an integer.
57. Let $n > 6$ be an integer and $a_1, a_2, ...a_k$ be all the natural numbers less than n and relatively prime to n. If
    $a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0$
    prove that n must be either a prime number or a power of 2.

58. An infinite sequence $x_0, x_1, x_2,.......$ of real numbers is said to be bounded if there is a constant C such that $\left|x_i\right| \leq C$ for every i ≥ 0. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2,.......$such that
    $$\left|x_i - x_j\right| \left|i - j\right|^a \geq 1$$
    for every pair of distinct non negative integers i, j.

59. Find all integers a, b, c with $1 < a < b < c$ such that

    (a - 1)(b - 1)(c - 1) is a divisor of abc - 1.
60. For each positive integer n, S(n) is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.
    a) Prove that $(n) \leq n^2 - 14$ for each $n \geq 4$.

    b) Find an integer n such that $(n) = n^2 - 14$.

    c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.
61. Let f(x) = $x^n + 5x^{n-1}$+3, where $n > 1$ is an integer. Prove that f(x) cannot be expressed as the product of two nonconstant polynomials with integer coefficients.
62. There are n lamps L_0, ... , L_{n-1} in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps s_0, s_1, ... as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are

on. Show that:

a) There is a positive integer M(n) such that after M(n) steps all the lamps are on again;
b) If n = 2^k, we can take M(n) = n^2 - 1;
c) If n = 2^k + 1, we can take M(n) = n^2-n+1.

63. Let m and n be positive integers. Let a_1, a_2, ... ,a_m be distinct elements of 1, 2, ... , n such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i \leq j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1+a_2+...+a_m}{m} \geq \frac{n+1}{2}.$$

64. . Determine all ordered pairs (m, n) of positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

65. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin$ A each of which is a product of k distinct elements of S for some $k \geq 2$.

66. Let a, b, c be positive real numbers such that abc = 1. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(b+a)} \geq \frac{3}{2}$$

67. Find the maximum value of x_0 for which there exists a sequence x_0, x_1 ... , x_{1995} of positive reals with $x_0 = x_{1995}$, such that for i = 1, ... , 1995,

$$x_{i-1}+\frac{2}{x_{i-1}}=2x_i+\frac{1}{x_i}$$

68. The positive integers a and b are such that the numbers 15a + 16b and 16a - 15b are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

69. Let $x_1, x_2, ..., x_n$ be real numbers satisfying the conditions

$$\left|x_1 + x_2 + ... + x_n\right| = 1$$

and

$$\left|x_i\right| \leq \frac{n+1}{2} \quad i = 1, 2, . . . , n$$

Show that there exists a permutation $y_1, y_2, ..., y_n$ of $x_1, x_2, ..., x_n$ such that

$$\left|y_1 + 2y_2 + ... + ny_n\right| \leq \frac{n+1}{2}$$

70. Find all pairs (a, b) of integers a, $b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

71. For each positive integer n , let f(n) denote the number of ways of representing n as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, f(4) = 4, because the number 4 can be represented in the following four ways:

4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.

Prove that, for any integer $n \geq 3$,

$$2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$$

72. For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n)$ = k for some n.

73. Determine all pairs (a, b) of positive integers such that ab^2 + b + 7 divides $a^2b + a + b$.

74. Consider an n x n square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

75. Determine all pairs (n, p) of positive integers such that

p is a prime,

n not exceeded 2p, and

$(p-1)^n + 1$ is divisible by n^{p-1}.

76. A, B, C are positive reals with product 1. Prove that $(A-1+\frac{1}{B})(B-1+\frac{1}{C})(C-1+\frac{1}{A}) \le 1$.

77. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A move is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B0 to the right of A such that $AB^{'} = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

78. Prove that

$$\frac{a}{\sqrt{(a^2+8bc)}}+\frac{b}{\sqrt{(b^2+8ac)}}+\frac{c}{\sqrt{(c^2+8ba)}} \ge 1$$

for all positive real numbers a,b and c.

79. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

ac + bd = (b + d a - c) (b + d - a + c).

Prove that ab + cd is not prime.

80. S is the set of all (h, k) with h, k non-negative integers such that h + $k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h^{'} \le h, k^{'} \le k$, then $(h^{'}, k^{'})$ is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

81. Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which k^n+k^2-1 divides k^m+k+1.

82. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < ... < d_k$, so that $d_1 = 1$, $d_k = n$. Let $d = d_1d_2 + d_2d_3 + ... + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

83. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2-n^3+1}$ is a positive integer.

84. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

85. Find all polynomials f with real coefficients such that for all reals a,b,c such that ab + bc + ca = 0 we have the following relations

f(a - b)+f(b - c)+f(c - a) = 2f(a+b+c).

86. Let $n \ge 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that

$n^2 + 1 > (t_1 + t_2 + ... + t_n)(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n})$.

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \le i < j < k \le n$.

87. Let x, y, z be three positive reals such that xyz $\ge$ 1. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} \ge 0$$

88. Let a_1, a_2,....... be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, ..., a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence a_1, a_2,.......

89. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, n \ge 1.$$

90. Real numbers $a_1, a_2, ..., a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = max\{a_j : 1 \le j \le i\} - min\{a_j : i \le j \le n\}$$

and let

$$d = max\{d_i : 1 \le i \le n\}.$$

a) Prove that, for any real numbers $x_1 \le x_2 \le .... \le x_n$, $\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}$.
b) Show that there are real numbers $x_1 \le x_2 \le .... \le x_n$such that equality holds in.

91. Let a and b be positive integers. Show that if 4ab-1 divides $(4a^2-1)^2$, then a = b.

92. Let n be a positive integer. Consider

$S = \{(x, y, z) : x, y, z \in \{0, 1, ..., n\}, x + y + z > 0\}$

as a set of $(n + 1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include (0, 0, 0).

93. a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying xyz = 1.

b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying xyz = 1.

94. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

95. Let n and k be positive integers with k ≥ n and k - n an even number. Let 2n lamps labelled 1, 2,....., 2n be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps n + 1 through 2n are all off. Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps n + 1 through 2n are all off, but where none of the lamps n + 1 through 2n is ever switched on. Determine the ratio $\frac{N}{M}$.

96. Let $a_1, a_2, ..., a_n$be distinct positive integers and let M be a set of n - 1 positive integers not containing $s = a_1+a_2+........+a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, ..., a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

97. Let $a_1, a_2, a_3, ...$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = max\{a_k + a_{n-k} | 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers l and N, with $l \leq s$ and such that $a_n = a_l + a_{n-l}$ for all $n \leq N$.

98. Given any set A = $\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with 1 ≤ i ≤ j ≤ 4 for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

99. Find all positive integers n for which there exist non-negative integers $a_1, a_2, ..., a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + ... + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + ... + \frac{n}{3^{a_n}} = 1$$

100. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, . . . , m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = (1 + \frac{1}{m_1})(1 + \frac{1}{m_2})\Delta\Delta\Delta(1 + \frac{1}{m_k}).$$

101. Let Q > 0 be the set of positive rational numbers. Let f : Q>0 → R be a function satisfying the following three conditions:

a) for all x, y ∈ Q>0, we have f(x)f(y) ≥ f(xy);

b) for all x, y ∈ Q>0, we have f(x + y) ≥ f(x) + f(y);

c) there exists a rational number a > 1 such that f(a) = a. Prove that f(x) = x for all x ∈ Q > 0.

102. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer n ≤ 1 such that

$$a_n < \frac{a_0 + a_1 + ... + a_n}{n} \leq a_{n+1}$$

103. Determine all triple (a,b,c) of positive integers such that each of the numbers.
ab - c, bc - a , ca - b is a power of 2.
(A power of 2 is an integer of the form 2^n,where n is a non-negative integer)

104. The sequence a_1,a_2,....of integers satisfies the following conditions:

a) $1 \leq a_j \leq 2015$ for all j ≥ 1;

b) $k + a_k \neq l + a_l$ for all 1 ≤ k < l.

Prove that there exist two positive integers b and N such that

$$\sum_{j=m+1}^{n} (a_j - b) \leq (1007)^2$$

for all integers m and n satisfying n > m ≥ N.

105. A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements.Let P(n) =

$n^2 + n + 1$ what is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set
$\{P(a+1), P(a+2), ..., P(a+b)\}$
is fragrant?

106. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, ...$ by:
$a_{n+1} =$
$$\begin{cases} \sqrt{a_n} if \sqrt{a_n} \ is \ an \ integer, \\ a_{n+3} \ otherwise \end{cases}$$
for all each n ≥ 0
Determine all values of a_0 for which there is a number A such that a_n = A for infinitely many values of n.
107. An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, . . . , a_n$ such that, for each (x, y) in S, we have:
$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n = 1.$$
108. Find all integers n ≥ 3 for which there exist real numbers $a_1, a_2, . . . , a_n + 2$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and $a_ia_{i+1} + 1 = a_{i+2}$ for i = 1, 2, . . . , n.
109. Let $a_1, a_2, . . .$ be an infinite sequence of positive integers. Suppose that there is an integer N > 1 such that, for each n ≥ N, the number
$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$
is an integer. Prove that there is a positive integer M such that$a_m = a_{m+1}$ for all m ≥ M.