Python with Linear Algebra: 2D

G V V Sharma*

Contents

***Abstract*—This manual introduces matrix computations using python and the properties of a triangle.**

1 Line

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \tag{1}$$

Find the equation of AB.
Solution: The desired equation is obtained as

$$AB: \quad \mathbf{x} = \mathbf{A} + \lambda_1 (\mathbf{B} - \mathbf{A}) \tag{2}$$

$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{3}$$

1.2 Find the direction vector and the normal vector for AB
Solution: The direction vector of AB is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{4}$$

The normal vector $\mathbf{n}$ is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \tag{5}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{6}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

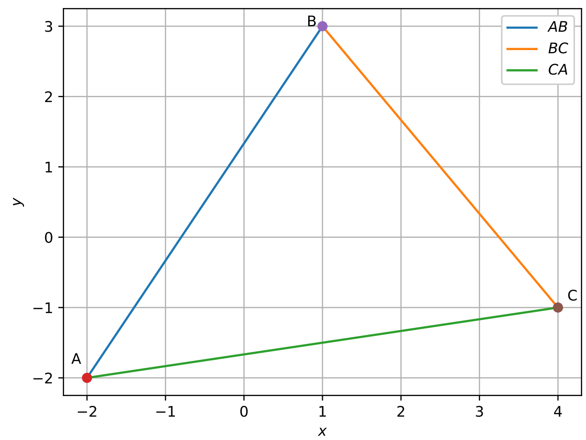

Fig. 1.2

Draw ΔABC.
Solution: The following codes yields the desired plot in Fig. 1.2

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/coeffs.py
```

```
https://raw.githubusercontent.com/gadepall/school/master/linalg/2D/python_2d/codes/draw_triangle.py
```

1.3 Find the equation of the line in terms of the normal vector.
Solution: The desired equation is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = \mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \tag{7}$$

$$\implies \begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = -\begin{pmatrix} 5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -4 \tag{8}$$

1.4 Find the equations of BC and CA.

2 Altitudes of a Triangle

2.1 In ΔABC, Let $\mathbf{P}$ be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of ΔABC.

2.2 Find the equation of AP.
Solution: The normal vector of AP is $\mathbf{B}-\mathbf{C}$. From (7), the equation of AP is

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{x}-\mathbf{A}) = 0 \tag{9}$$

$$\implies \begin{pmatrix}-3 & 4\end{pmatrix}\mathbf{x} = -\begin{pmatrix}-3 & 4\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = -2 \tag{10}$$

2.3 Find the equation of the altitude BQ.
Solution: The desired equation is

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{x}-\mathbf{B}) = 0 \tag{11}$$

$$\implies \begin{pmatrix}6 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}6 & 1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = 9 \tag{12}$$

2.4 Find the equation of the altitude CR.

2.5 Find the point of intersection of AP and BQ.
Solution: (9) and (11) can be stacked together into the matrix equation

$$\begin{pmatrix}-3 & 4\\ 6 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-2\\9\end{pmatrix} \tag{13}$$

The following code computes the point of intersection.

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    orthocentre.py
```

2.6 Find the point of intersection of and BQ and CR. Comment.

2.7 Find $\mathbf{P}$
Solution: The following code finds the required points.

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_foot.py
```

2.8 Find $\mathbf{Q}$ and $\mathbf{R}$.

2.9 Draw AP, BQ and CR and verify that they meet at a point $\mathbf{H}$.
Solution: The following code plots the altitudes in Fig. 2.9

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    alt_draw.py
```

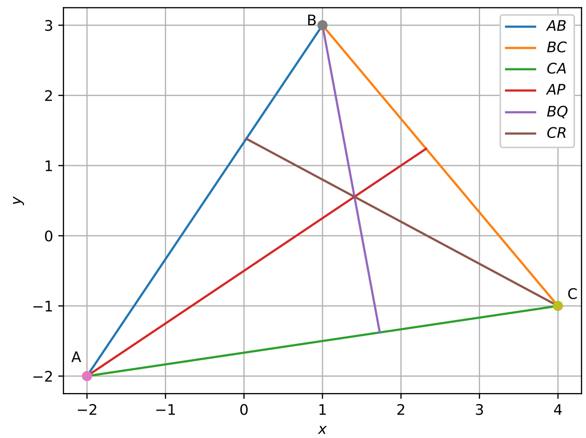

Fig. 2.9

3 Circumcircle

3.1 Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be points on a circle with centre $\mathbf{O}$ and radius r.

3.2 Find $\mathbf{O}$.
Solution: The equation of the circle is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{14}$$

$$\implies \|\mathbf{x}-\mathbf{O}\|^2 = (\mathbf{x}-\mathbf{O})^T(\mathbf{x}-\mathbf{O}) = R^2 \tag{15}$$

From (14),

$$\|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \tag{16}$$

$$\implies (\mathbf{A}-\mathbf{O})^T(\mathbf{A}-\mathbf{O}) - (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = 0 \tag{17}$$

which can be simplified as

$$(\mathbf{A}-\mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{18}$$

Similarly,

$$(\mathbf{B}-\mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{19}$$

The following code computes $\mathbf{O}$ using the above two equations.

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    circumcentre.py
```

3.3 Find the radius R.

3.4 Plot the *circumcircle* of $\triangle ABC$.
Solution: The following code plots Fig. 3.4

```
https://raw.githubusercontent.com/gadepall/
    school/master/linalg/2D/python_2d/codes/
    circumcircle.py
```

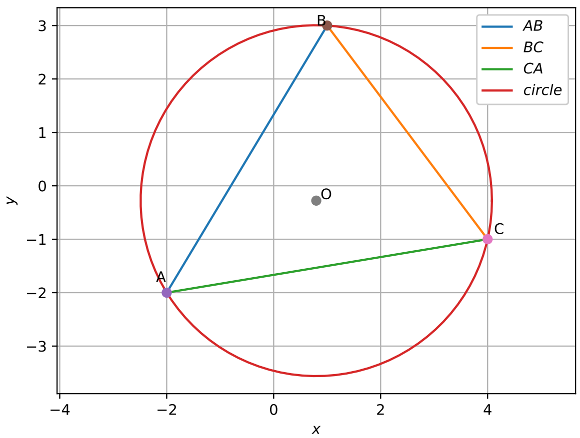

Fig. 3.4

4 Medians of a Triangle

4.1 Find the coordinates of $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ of the mid points of AB, BC and CA respectively for ΔABC.

4.2 Find the equations of AD, BE and CF. These lines are the *medians* of ΔABC

4.3 Find the point of intersection of AD and CF.

4.4 Verify that $\mathbf{G}$ is the point of intersection of BE, CF as well as AD, BE. $\mathbf{G}$ is known as the *centroid* of ΔABC.

4.5 Graphically show that the medians of ΔABC meet at the centroid.

4.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{20}$$

5 Incircle

5.1 Consider a circle with centre $\mathbf{I}$ and radius r that lies within $\triangle ABC$ and touches BC, CA and AB at $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ respectively.

5.2 Show that $IU \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then

$$\|\mathbf{x}_1 - \mathbf{I}\|^2 - \|\mathbf{x}_2 - \mathbf{I}\|^2 = 0 \tag{21}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{22}$$

$$\implies (\mathbf{B} - \mathbf{C})^T \left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} - \mathbf{I}\right) = 0 \tag{23}$$

For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{U}$, x_1x_2 merges into BC and the above equation becomes

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{U} - \mathbf{I}) = 0 \implies OD \perp BC \tag{24}$$

5.3 Find an expression for r if $\mathbf{I}$ is known.
Solution: Let $\mathbf{n}$ be the normal vector of BC. The equation for BC is then given by

$$\mathbf{n}^T (\mathbf{x} - \mathbf{B}) = 0 \tag{25}$$

$$\implies \mathbf{n}^T (\mathbf{U} - \mathbf{B}) = 0 \tag{26}$$

since $\mathbf{U}$ lies on BC. Since $IU \perp BC$,

$$\mathbf{I} = \mathbf{U} + \lambda\mathbf{n} \tag{27}$$

$$\implies \mathbf{I} - \mathbf{U} = \lambda\mathbf{n} \tag{28}$$

$$\text{or} \quad r = \|\mathbf{I} - \mathbf{U}\| = |\lambda| \, \|\mathbf{n}\| \tag{29}$$

From (26) and (27)

$$\mathbf{n}^T\mathbf{I} = \mathbf{n}^T\mathbf{B} + \lambda\mathbf{n}^T\mathbf{n} \tag{30}$$

$$\implies \mathbf{n}^T (\mathbf{I} - \mathbf{B}) = \lambda\|\mathbf{n}\|^2 \tag{31}$$

$$\implies r = |\lambda| \, \|\mathbf{n}\| = \frac{\left|\mathbf{n}^T (\mathbf{I} - \mathbf{B})\right|}{\|\mathbf{n}\|} \tag{32}$$

from (29). Letting

$$\|\mathbf{n}_1\| = \frac{\mathbf{n}}{\|\mathbf{n}\|}, \tag{33}$$

$$r = \left|\mathbf{n}_1^T (\mathbf{I} - \mathbf{B})\right| \tag{34}$$

5.4 Show that

$$r = \frac{\left|(\mathbf{B} - \mathbf{C})^T (\mathbf{I} - \mathbf{B})\right|}{\|\mathbf{B} - \mathbf{C}\|} \tag{35}$$

Solution: Since $\mathbf{U}$ lies on BC,

$$\mathbf{U} = \mathbf{B} + \lambda (\mathbf{B} - \mathbf{C}) \tag{36}$$

$$r = \|\mathbf{I} - \mathbf{U}\| \tag{37}$$

5.5

5.6 Find $\mathbf{U}$
Solution: Since $\mathbf{U}$ lies on BC,

$$\mathbf{n}_{BC}^T (\mathbf{U} - \mathbf{B}) = 0 \tag{38}$$

where $\mathbf{n}_{BC}$ is the normal vector of BC. Sinc $IU \perp BC$,

$$\mathbf{B} - \mathbf{C}^T = \mathbf{B} + \lambda (\mathbf{B} - \mathbf{A}) \tag{39}$$

Since $\mathbf{U}$ lies on the circle,

$$(\mathbf{U} - \mathbf{I})^T (\mathbf{U} - \mathbf{I}) = c^2 \tag{40}$$

5.7 In ΔABC, let U be a point on BC such that $\angle BAU = \angle CAU$. Then AU is known as the *angle bisector*.

Find the length of AB, BC and CA
Solution: The length of CA is given by

$$CA = \|\mathbf{C} - \mathbf{A}\| \tag{41}$$

The following code calculates the respective values as

$$AB = 5.83, BC = 5, CA = 6.08 \tag{42}$$

```
#This program calculates the distance
    between
#two points
import numpy as np
import matplotlib.pyplot as plt


A = np.array([-2,-2])
B = np.array([1,3])
C = np.array([4,-1])

print (np.linalg.norm(A-B))
```

5.8 If AU, BV and CW are the angle bisectors, find the coordinates of $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$.
Solution: Using the section formula,

$$\mathbf{W} = \frac{AW.\mathbf{B} + WB.\mathbf{A}}{AW + WB} = \frac{\frac{AW}{WB}.\mathbf{B} + \mathbf{A}}{\frac{AW}{WB} + 1} \tag{43}$$

$$= \frac{\frac{CA}{BC}.\mathbf{B} + \mathbf{A}}{\frac{CA}{BC} + 1} \tag{44}$$

$$= \frac{CA \times \mathbf{B} + BC \times \mathbf{A}}{BC + CA} \tag{45}$$

$$= \frac{a \times \mathbf{A} + b \times \mathbf{B}}{a + b} \tag{46}$$

where $a = BC, b = CA$, since the angle bisector has the property that

$$\frac{AW}{WB} = \frac{CA}{AB} \tag{47}$$

5.9 Write a program to find $\mathbf{U}, \mathbf{V}, \mathbf{W}$.
5.10 Find the intersection of AU and BV.
Solution: Using the code in Problem 4.3, the desired point of intersection is

$$\mathbf{I} = \begin{pmatrix} 1.15 \\ 0.14 \end{pmatrix} \tag{48}$$

It is easy to verify that even BV and CW meet at the same point. $\mathbf{I}$ is known as the *incentre* of ΔABC.
5.11 Draw AU, BV and CW and verify that they meet at a point $\mathbf{I}$.
5.12 Verify that

$$\mathbf{I} = \frac{BC.\mathbf{A} + CA.\mathbf{B} + AB.\mathbf{C}}{AB + BC + CA} \tag{49}$$

5.13 Let the perpendicular from $\mathbf{I}$ to AB be IX. If the equation of AB is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{50}$$

show that

$$IX = \frac{\left|\mathbf{n}^T (\mathbf{I} - \mathbf{A})\right|}{\|\mathbf{n}\|} \tag{51}$$

Verify through a Python script.
5.14 If $IY \perp BC$ and $IZ \perp CA$, verify that

$$IX = IY = IZ = r \tag{52}$$

r is known as the *inradius* of ΔABC.
5.15 Draw the incircle of $\triangle ABC$
5.16 Draw the circumcircle of $\triangle ABC$