JEE Problems in Discrete Mathematics

Abstract—**A collection of problems from JEE papers related to matrices are available in this document. Verify your soluions using C.**

1 Signal Processing: Z Transform

1.1 Let

$$a(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n) \tag{1}$$

$$b(n) = a(n-1) + a(n+1) - \delta(n) \tag{2}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{3}$$

and

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{4}$$

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{5}$$

1.2 Verify your results through a C program.

1.3 Show that the Z transform of $u(n)$

$$U(z) \triangleq \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{6}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{7}$$

1.4 Show that

$$A(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \tag{8}$$

1.5 Let

$$y(n) = a(n) * u(n) \triangleq \sum_{k=-\infty}^{\infty} a(k) u(n-k) \tag{9}$$

Show that

$$y(n) = \sum_{k=0}^{n} a(k) \tag{10}$$

1.6 Show that

$$Y(z) = A(z) U(z) \tag{11}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{12}$$

1.7 Show that

$$w(n) = \left[a(n+2) - 1\right] u(n-1) \tag{13}$$

$$= a(n+2) - u(n+1) + 2\delta(n) \tag{14}$$

1.8 Is $W(z) = Y(z)$?

1.9 Verify if

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} = \frac{10}{89} \tag{15}$$

1.10 Verify if

$$\sum_{n=1}^{\infty} \frac{b(n)}{10^n} = \frac{8}{89} \tag{16}$$

2 Algebra: Modular Arithmetic

Let $AP\,(a; d)$ denote an A.P. with $d > 0$

2.1 Express $AP\,(a; d)$ in modulo arithmetic.
Solution:

$$A \equiv a \pmod{d} \tag{17}$$

2.2 Express the intersection of $AP\,(1; 3), AP\,(2; 5)$ and $AP\,(3; 7)$ using modulo arithmetic.
Solution: The desired AP can be expressed as

$$A \equiv 1 \pmod{3} \tag{18}$$

$$\equiv 2 \pmod{5} \tag{19}$$

$$\equiv 3 \pmod{7} \tag{20}$$

2.3 Two numbers are said to be coprime if their greatest common divisor (gcd) is 1. Verify if (3,5), (5,7) and (3,7) are pairwise coprime.

2.4 Does a solution for (18) exist?
Solution: The Chinese remainder theorem

guarantees that the system in (18) has a solution since 3,5,7 are pairwise coprime.

2.5 Simplify

$$(7 \times 5) \pmod 3 \tag{21}$$

Solution: (21) can be expressed as

$$\begin{aligned}(7 \times 5) \pmod 3 &= 35 \pmod 3 \\ &= 2 \pmod 3\end{aligned} \tag{22}$$

2.6 Find x in

$$2x = 1 \pmod 3 \tag{23}$$

Solution: By inspection, for $x = 2$,

$$2x = 2 \times 2 = 4 = 3 + 1 = 1 \pmod 3 \tag{24}$$

Thus $x = 2$ is a solution of (23).

2.7 In general, x in

$$ax = 1 \pmod d \tag{25}$$

is defined to be the modular multiplicative inverse of (17).

2.8 Show that the multiplicative inverse of

$$(3 \times 5) \pmod 7 = y = 1 \tag{26}$$

2.9 Show that the multiplicative inverse of

$$(3 \times 7) \pmod 5 = z = 1 \tag{27}$$

2.10 Find $a + d$.
Solution:

$$\begin{aligned}&(5 \times 7 \times 1 \times x) + (3 \times 5 \times 3 \times y) \\ &\quad + (3 \times 7 \times 2 \times z) = 157\end{aligned} \tag{28}$$

2.11 Find a and d.
Solution:

$$d = LCM\,(3, 5, 7) = 105 \tag{29}$$

$$\begin{aligned}A &= 157 \pmod{105} \\ &= 52 \pmod{105} \\ \implies a &= 52\end{aligned} \tag{30}$$

2.12 Given the APs

$$a_1 \pmod{d_1} \tag{31}$$

$$a_2 \pmod{d_2} \tag{32}$$

$$a_3 \pmod{d_3}, \tag{33}$$

such that

$$gcd(d_1, d_2) = gcd(d_2, d_3) = gcd(d_3, d_1) = 1, \tag{34}$$

show that their intersection

$$a \pmod d \tag{35}$$

is obtained through

$$\begin{aligned}&a + d = \\ &\quad (d_1 \times d_2 \times a_3 \times x) + (d_2 \times d_3 \times a_1 \times y) \\ &\qquad + (d_3 \times d_1 \times a_2 \times z)\end{aligned} \tag{36}$$

$$d = LCM\,(d_1, d_2, d_3)\,, \tag{37}$$

where x, y, z are the modular multiplicative inverses given by

$$x = [(d_1 \times d_2) \pmod{d_3}]^{-1} \tag{38}$$

$$y = [(d_2 \times d_3) \pmod{d_1}]^{-1} \tag{39}$$

$$z = [(d_3 \times d_1) \pmod{d_2}]^{-1} \tag{40}$$

respectively.

2.13 Write a C program to find x, y and z.

3 Discrete Fourier Transform

3.1 Show that

$$\sum_{k=0}^{n-1} e^{\jmath \frac{2\pi k}{n}} = \begin{cases} 1 & n = 1, \\ 0 & n > 1 \end{cases} \tag{41}$$

3.2 Show that

$$\sum_{k=0}^{n} \cos\left(\frac{2k + r}{n + 2}\pi\right) = -\cos\left(\frac{r - 2}{n + 2}\pi\right) \tag{42}$$

Solution: From (41),

$$\begin{aligned}&\sum_{k=0}^{n+1} e^{\jmath \frac{2k+r}{n+2}\pi} = 0 \\ \implies &\sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} + e^{\jmath \frac{2(n+1)+r}{n+2}\pi} = 0 \\ \implies &\sum_{k=0}^{n} e^{\jmath \frac{2k+r}{n+2}\pi} = -e^{\jmath \frac{2(n+2)+r-2}{n+2}\pi} \\ &\qquad = -e^{\jmath \frac{r-2}{n+2}\pi}\end{aligned} \tag{43}$$

Taking the real part on both sides yields (42).

3.3 Show that

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right) \sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+2}{n+2}\pi\right)} \tag{44}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right)}{n + \cos\left(\frac{2\pi}{n+2}\right)} \tag{45}$$

Solution: Let

$$\theta_n = \frac{\pi}{n+2} \tag{46}$$

$$\because \sin\{(k+1)\theta_n\}\sin\{(k+2)\theta_n\},$$
$$= \frac{1}{2}\left[\cos\theta_n - \cos\{(2k+3)\theta_n\}\right] \tag{47}$$

from (44) and (42),

$$f(n) = \frac{n\cos\theta_n - \sum_{k=0}^{n}\cos\{(2k+3)\theta_n\}}{n - \sum_{k=0}^{n}\cos\{(2k+4)\theta_n\}}$$
$$= \frac{n\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{n + \cos\left(\frac{2\pi}{n+2}\right)} \tag{48}$$

resulting in (45). Verify if

3.4

$$f(4) = \frac{\sqrt{3}}{2} \tag{49}$$

3.5

$$\lim_{n\to\infty} f(n) = \frac{1}{2} \tag{50}$$

3.6

$$\sin\left(7\cos^{-1} f(5)\right) = 0 \tag{51}$$

3.7 If

$$\alpha = \tan\left(\cos^{-1} f(6)\right) \tag{52}$$

verify if

$$\alpha^2 + 2\alpha - 1 = 0 \tag{53}$$

4 Combinatorics

4.1 Find

$$\sum_{k=0}^{n} k \tag{54}$$

Solution: (54) can be expressed as

$$\frac{n(n+1)}{2} \tag{55}$$

4.2 Find

$$\sum_{k=0}^{n} {}^nC_k k^2 \tag{56}$$

Solution:

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k \tag{57}$$

$$\implies n(1+x)^{n-1} = \sum_{k=0}^{n} k\, {}^nC_k x^{k-1} \tag{58}$$

upon differentiation. Multiplying (58) by x and differentiating,

$$\frac{d}{dx}\left[nx(1+x)^{n-1}\right] = \sum_{k=0}^{n} k^2\, {}^nC_k x^{k-1} \tag{59}$$

$$\implies n(n-1)x(1+x)^{n-2} + n(1+x)^{n-1}$$
$$= \sum_{k=0}^{n} k^2\, {}^nC_k x^{k-1} \tag{60}$$

Substituting $x = 1$ in (60),

$$\sum_{k=0}^{n} {}^nC_k k^2 = n(n-1)2^{n-2} + n2^{n-1}$$
$$= n(n+1)2^{n-2} \tag{61}$$

4.3 Find

$$\sum_{k=0}^{n} {}^nC_k k \tag{62}$$

Solution: Substituting $x = 1$ in (58),

$$\sum_{k=0}^{n} {}^nC_k k = n2^{n-1} \tag{63}$$

4.4 Find

$$\sum_{k=0}^{n} {}^nC_k 3^k \tag{64}$$

Solution: Substituting $x = 2$ in (57),

$$\sum_{k=0}^{n} {}^nC_k 3^k = 4^n \tag{65}$$

4.5 If

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(n+1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0 \tag{66}$$

for some n, find

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{67}$$

Solution: (66) can be expressed as

$$n(n+1)2^{2n-3} \begin{vmatrix} 1 & 1 \\ n & 4 \end{vmatrix} = 0 \tag{68}$$

$$\implies n = 4 \tag{69}$$

Integrating (57) from 0 to 1,

$$\frac{2^{n+1}}{n+1} = \sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} \tag{70}$$

Substituting $n = 4$ in the above,

$$\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1} = \frac{2^5 - 1}{5} = \frac{31}{5} \tag{71}$$

5 Exercises

1.