Algebra: Pre Regional Maths Olympiad

G V V Sharma*

***Abstract*—This book provides a collection of the international maths olympiad problems in algebra.**

1. For how many pairs of positive integers (x, y) is x + 3y = 100?
2. Let

$$S_n = n^2 + 20n + 12$$

n is a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

3. Suppose that

$$4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, ..., 126^{X_{123}} = 127,$$

$$127^{X_{124}} = 128$$

What is the value of the product $X_1 X_2 ..... X_{124}$?

4. Let

$$P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$$

What is the largest integer that is a divisor of P(n) for all positive even integers n ?

5. If

$$a = b - c, b = c - d, c = d - a$$

and $abcd \neq 0$ then what is the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

6. Let x_1, x_2, x_3 be the roots of the equation

$$x^3 + 3x + 5 = 0$$

What is the value of the expression

$$(x_1 + \frac{1}{x_1})(x_2 + \frac{1}{x_2})(x_3 + \frac{1}{x_3})$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

7. How many integer pairs (x, y) satisfy

$$x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0 \tag{7.1}$$

8. What is the sum of the squares of the roots of the equation

$$x^2 - 7[x] + 5 = 0 \tag{8.1}$$

(Here [x] denotes the greatest integer less than or equal to x. For example [3.4] = 3 and [-2.3] = -3.)

9. What is the smallest positive integer k such that

$$k(3^3 + 4^3 + 5^3) = a^n \tag{9.1}$$

for some positive integers a and n, with $n > 1$?

10. Let

$$S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

What is the value of

$$\sum_{n=1}^{90} \frac{1}{S_n + S_{n-1}}$$

11. It is given that the equation

$$x^2 + ax + 20 = 0 \tag{11.1}$$

has integer roots. What is the sum of all possible values of a?

12. Three real numbers x, y, z are such that

$$x^2 + 6y = -17 \tag{12.1}$$

$$y^2 + 4z = 1 \tag{12.2}$$

$$z^2 + 2x = 2 \tag{12.3}$$

What is the value of $x^2 + y^2 + z^2$?

13. Let

$$f(x) = x^3 - 3x + b$$

$$g(x) = x^2 + bx - 3$$

where b is a real number. What is the sum of the all possible values of b for which the equations $f(x) = 0$ and $g(x) = 0$ have a common root?

14. A natural number k is such that

$$k^2 < 2014 < (k+1)^2$$

What is the largest prime factor of k?

15. If real numbers a, b, c, d, e satisfy

$$a + 1 = b + 2 = c + 3 = d + 4$$
$$= e + 5 = a + b + c + d + e + 3,$$

What is the value of

$$a^2 + b^2 + c^2 + d^2 + e^2$$

16. What is the smallest possible natural number n for which the equation

$$x^2 - nx + 2014 = 0$$

has integer roots?

17. If $x^{x^4} = 4$, what is the value of$x^{x^2} + x^{x^8}$?

18. Natural numbers k, l, p and q are such that if a and b are roots of

$$x^2 - kx + 1 = 0$$

then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of

$$x^2 - px + q = 0$$

What is the sum of all possible values of q?

19. For natural numbers x and y, let(x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \leq y$ satisfy the equation xy = x + y + (x, y)?

20. For a natural number b, let N(b) denote the number of natural numbers a for which the equation

$$x^2 + ax + b = 0$$

has integer roots. What is the smallest value of b for which N(b) = 20?

21. Positive integers a and b are such that a + b = a/b + b/a. What is the value of $a^2 + b^2$?

22. The equations

$$x^2 - 4x + k = 0$$

$$x^2 + kx - 4 = 0$$

where k is a real number, have exactly one common root. What is the value of k?

23. Let P(x) be a non-zero polynomial with integer coefficients. If P(n) is divisible by n for each positive integer n, what is the value of P(0)?

24. Let a, b and c be real numbers such that

$$a - 7b + 8c = 4$$

$$8a + 4b - c = 7$$

What is the value of $a^2 - b^2 + c^2$?

25. If

$$3^x + 2y = 985$$

$$3^x - 2^y = 473$$

what is the value of xy?

26. Let a, b and c be such that a + b + c = 0 and

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of P?

27. Suppose a, b are positive real numbers such that

$$a\sqrt{a} + b\sqrt{b} = 183$$

$$a\sqrt{b} + b\sqrt{a} = 182$$

Find $\frac{9}{5}(a + b)$.

28. Let a, b be integers such that all the roots of the equation

$$(x^2 + ax + 20)(x^2 + 17x + b) = 0$$

are negative integers. What is the smallest possible value of a + b?

29. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^4 0 = v^n = w^5 0$. Find the value of n.

30. Let the sum

$$\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$$

written in its lowest terms be $\frac{p}{q}$. Find the value of q - p.

31. Find the number of positive integers n, such

that

$$\sqrt{n} + \sqrt{n+1} < 11$$

32. Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here [x] denotes the integer part of x and $\{x\} = x - [x]$.)
33. Integers 1, 2, 3,.......n, where $n > 2$, are written on a board. Two numbers m, k such that $1 < m < n$, $1 < k < n$ are removed and the average of the remaining numbers is found be 17. What is the maximum sum of the two removed numbers?
34. If the real numbers x, y, z are such that

$$x^2 + 4y^2 + 16z^2 = 48 \tag{34.1}$$

$$xy + 4yz + 2zx = 24 \tag{34.2}$$

What is the value of $x^2 + y^2 + z^2$?
35. Suppose 1, 2, 3 are the roots of the equation

$$x^2 + ax^2 + bx = c \tag{35.1}$$

Find the value of c.
36. What is the number of triples(a, b, c) of positive integers such that
   a) $a < b < c < 10$
   b) a, b, c, 10 froms the sides of a quadrilateral?
37. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.
38. Suppose an integer x, a natural number n and a prime number p satisfy the equation

$$7x^2 - 44x + 12 = p^n \tag{38.1}$$

Find the largest value of p.
39. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for all positive integers n. Find the least possible value of $p + q$?
40. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$(that is 1, 6, 5, 8, 9, 0 are digits in base b in the above eqaution). Find the sum of all possible values of $b \geq 10$ satisfying the equation.
41. Integers a, b, c satisfy

$$a + b - c = 1 \tag{41.1}$$

$$a^2 + b^2 - c^2 = -1 \tag{41.2}$$

What is the sum of all possible values of $a^2 + b^2 + c^2$?
42. Suppose a, b are integers and a + b is a root of

$$x^2 + ax + b = 0 \tag{42.1}$$

What is the maximum possible value of b^2?
43. If

$$x = \cos 1^o \cos 2^o \cos 3^o .... \cos 89^o$$

$$y = \cos 2^o \cos 6^o \cos 10^o .... \cos 86^o$$

then what is the integer nearest $\frac{2}{7} log_2(y/x)$?
44. Let a and b be natural numbers such that 2z - b, a - 2b amd a + b are all distinct squares. What is the smallest possible value of b?
45. What is the value of

$$\sum_{1 \leq i < j \leq 10 (odd)} (i + j) - \sum_{1 \leq i < j \leq 10 (even)} (i + j)$$

46. If $a, b, c \geq 4$ are integers, not all equal and

$$4abc = (a + 3)(b + 3)(c + 3)$$

then what is the value of $a + b + c$?
47. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$?
48. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29} + \frac{c}{31}} + \frac{b^2}{\frac{c}{29} + \frac{a}{31}} + \frac{c^2}{\frac{a}{29} + \frac{b}{31}} \geq n(a + b + c)$$

49. Let

$$P(x) = a_0 + a_1 x + a_2 x + a_2 x^2 + ...a_n x^n$$

be a polynomial in which a_i is a non-negative integer for each $i \in \{0, 1, , 2, 3, ...\}$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$?
50. Let

$$f(x) = x^2 + ax + b$$

If for all non-zero real x

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

and the roots of $f(x) = 0$ are integers, What is the value of $a^2 + b^2$?
51. Let $\overline{abc}$ be a three digit numbers with non-zero digits such that $a^2 + b^2 = c^2$. What is the largest

possible prime factor $\overline{abc}$?

52. How many positive integers n are there such that $3 \leq n \leq 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?
53. A natutal number $k > 1$ iscalled good if there exist natural numbers

$$a_1 < a_2 < ....a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + ... + \frac{1}{\sqrt{a_k}} = 1$$

Let $f(n)$ be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which $f(n+5)/f(n)$ is an integer.

54. Find the number of ordered triples (a, b, c) of positive integers such that

$$30a + 50b + 70c \leq 343$$

55. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$?