Problems in Linear Algebra

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1 Introduction

1.1 Definitions

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$
3. The *length* of PQ is defined as
$$\|\mathbf{P} - \mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$

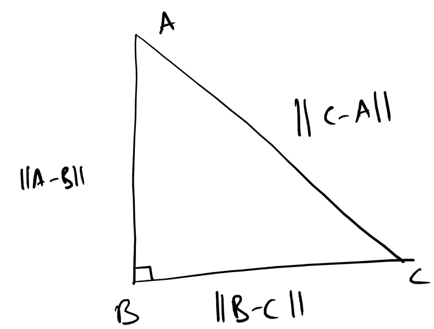

Fig. 1.1.7

5. The point dividing PQ in the ratio $k : 1$ is
$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A} - \mathbf{B})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.7}$$
Solution: Using Baudhayana's theorem,
$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{1.1.7}$$
$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A})$$
$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show that
$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.8}$$
9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that
$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.9}$$
.

1.2 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \qquad (1.2.1)$$

2. Find the length of PQ for
   a) $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$;
   b) $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$;
   c) $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$.
3. Using direction vectors, show that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are the vertices of a parallelogram.
4. Using Baudhayana's theorem, show that the points $\begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ are the vertices of a right-angled traingle. Repeat using orthogonality.
5. Plot the points $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and prove that they are the vertices of a rectangle.
6. Show that $\mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are the vertices of an isosceles triangle.
7. In the last question, find the distance of the vertex $\mathbf{A}$ of the triangle from the middle point of the base BC.
8. Prove that the points $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are the vertices of a square.
9. Prove that the points $\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are the vertices of a parallelogram. Find $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$, the mid points of AB, BC, CD, AD respectively. Show that EG and FH bisect each other.
10. Prove that the points $\begin{pmatrix} 21 \\ -2 \end{pmatrix}, \begin{pmatrix} 15 \\ 10 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -12 \end{pmatrix}$ are the vertices of a rectangle, and find the coordinates of its centre.
11. Find the lengths of the medians of the triangle whose vertices are at the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.
12. Find the coordinates of the points that divide the line joining the points $\begin{pmatrix} -35 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -10 \end{pmatrix}$ into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 25 \\ 10 \end{pmatrix}$.
14. Prove that the middle point of the line joining the points $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ is a point of trisection of the line joining the points $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$.
15. The points $\begin{pmatrix} 8 \\ 5 \end{pmatrix}, \begin{pmatrix} -7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$.
16. The point $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is the intesection of the diagonals of a parallelogram two of whose vertices are at the points $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$. Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.
18. Find the coordinates of points which divide the join of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ externally in the ratio $2 : 3$, and also externally in the ratio $3 : 2$.
19. Prove the centroid of $\triangle ABC$ is

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \qquad (1.2.19)$$

1.3 Loci

1. A point moves so that its distance from the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is double its distance from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the equation of its locus.
2. Find the equation of the perpendicular bisector of the line joining the points $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
3. Find the equation of the circle of radius 5 with centre at $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.
4. A point moves so that its distance from the y-axis is equal to the distance from the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the equation of its locus.

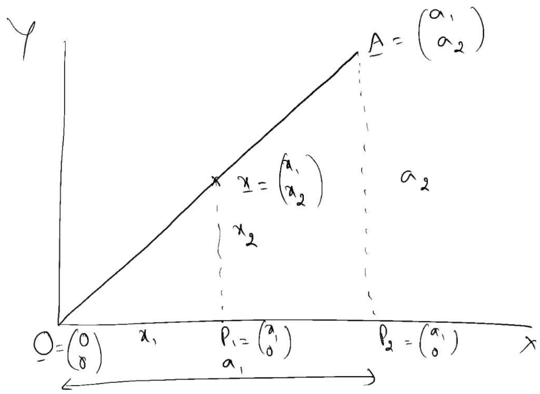

Fig. 2.1.1

5. A point moves so that the sum of the squares of its distance from the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}4\\3\end{pmatrix}$ is constant. Find the equation of the locus.
6. A point moves so that its distance from the axis of x is twice its distance from the point $\begin{pmatrix}0\\1\end{pmatrix}$. Find the equation of the locus.
7. A point moves in such a way that with the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\4\end{pmatrix}$ it forms a triangle of area 8.5. Show that its locus has an equation
$$\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x}\right\}\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x} - 34\right\} = 0 \tag{1.3.7}$$

2 The Straight Line

2.1 Definitions

1. The points $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{A} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$ are as shown in Fig. 2.1.1. Find the equation of OA.
Solution: Let $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ be any point on OA. Then, using similar triangles,
$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{2.1.1.1}$$
$$\implies x_2 = mx_1 \tag{2.1.1.2}$$
where m is known as the slope of the line. Thus, the equation of the line is
$$\mathbf{x} = \begin{pmatrix}x_1\\mx_1\end{pmatrix} = x_1\begin{pmatrix}1\\m\end{pmatrix} = x_1\mathbf{m} \tag{2.1.1.3}$$
In general, the above equation is written as
$$\mathbf{x} = \lambda\mathbf{m}, \tag{2.1.1.4}$$
where $\mathbf{m}$ is the direction vector of the line.

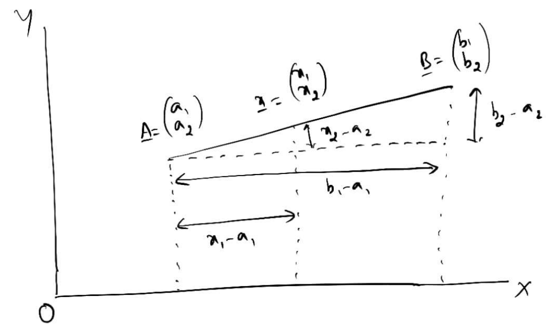

Fig. 2.1.2

2. Find the equation of AB in Fig. 2.1.2
Solution: From Fig. 2.1.2,
$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{2.1.2.1}$$
$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{2.1.2.2}$$
From (2.1.2.2),
$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\mx_1 + a_2 - ma_1\end{pmatrix} \tag{2.1.2.3}$$
$$= \mathbf{A} + (x_1 - a_1)\begin{pmatrix}1\\m\end{pmatrix} \tag{2.1.2.4}$$
$$= \mathbf{A} + \lambda\mathbf{m} \tag{2.1.2.5}$$
3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (2.1.2.5) is obtained from (2.1.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 2.1.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as
$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \tag{2.1.4.1}$$
Also, from (2.1.1.4),
$$\|\mathbf{A}\| = \lambda\sqrt{1 + m^2} \tag{2.1.4.2}$$
Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 2.1.5. From (2.1.2.5), for some λ,
$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \tag{2.1.5.1}$$
$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix}1\\m\end{pmatrix}, \tag{2.1.5.2}$$

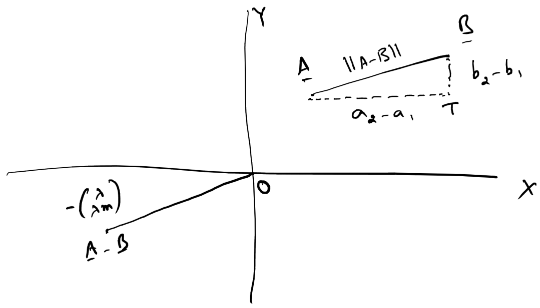

Fig. 2.1.5

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 2.1.5.

6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is
$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{2.1.7.1}$$
8. The *normal* to the vector $\mathbf{m}$ is defined as
$$\mathbf{n}^T\mathbf{m} = 0 \tag{2.1.8.1}$$
$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{m} \tag{2.1.8.2}$$
9. From (2.1.7.1), the equation of a line can also be expressed as
$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T\left(\mathbf{B} - \mathbf{A}\right) \tag{2.1.9.1}$$
$$\implies \mathbf{n}^T\mathbf{x} = c \tag{2.1.9.2}$$
10. The unit vectors on the x and y axis are defined as
$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.1.10.1}$$
$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.1.10.2}$$
11. If a be the *intercept* of the line
$$\mathbf{n}^T\mathbf{x} = c \tag{2.1.11.1}$$
on the $x-$axis, then $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is a point on the line. Thus,
$$\mathbf{n}^T\begin{pmatrix} a \\ 0 \end{pmatrix} = c \tag{2.1.11.2}$$
$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \tag{2.1.11.3}$$
12. The *rotation matrix* is defined as
$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.12}$$
where θ is anti-clockwise.
13. $$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{2.1.13}$$
where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix*.
14. Find the equation of line L in Fig. 2.1.14.
Solution: The equation of the $x-$axis is
$$\mathbf{x} = \lambda\mathbf{e}_1 \tag{2.1.14.1}$$
Translation by p units along the $y-$axis results in
$$L_0: \quad \mathbf{x} = \lambda\mathbf{e}_1 + p\mathbf{e}_2 \tag{2.1.14.2}$$
Rotation by $90° - \alpha$ in the anti-clockwise direction yields
$$L: \quad \mathbf{x} = \mathbf{Q}\left\{\lambda\mathbf{e}_1 + p\mathbf{e}_2\right\} \tag{2.1.14.3}$$
$$= \lambda\mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \tag{2.1.14.4}$$
where
$$\mathbf{Q} = \begin{pmatrix} \cos\left(\alpha - 90\right) & -\sin\left(\alpha - 90\right) \\ \sin\left(\alpha - 90\right) & \cos\left(\alpha - 90\right) \end{pmatrix} \tag{2.1.14.5}$$
$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \tag{2.1.14.6}$$
From (2.1.14.4),
$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$
$$= \lambda\mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \tag{2.1.14.7}$$
resulting in
$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \tag{2.1.14.8}$$
15. Show that the distance from the orgin to the line
$$\mathbf{n}^T\mathbf{x} = c \tag{2.1.15.1}$$
is
$$p = \frac{c}{\|n\|} \tag{2.1.15.2}$$

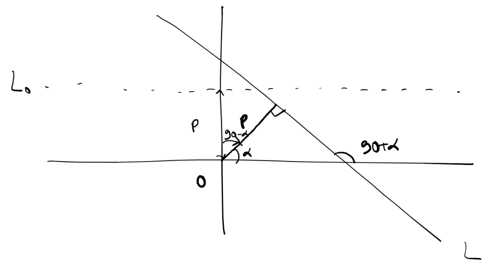

Fig. 2.1.14

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.1.16.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.1.16.2}$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \tag{2.1.16.3}$$

where

$$\mathbf{N} = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix} \tag{2.1.16.4}$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{2.1.17.1}$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{2.1.18.1}$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \tag{2.1.18.2}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{2.1.18.3}$$

where

$$\mathbf{n}^T\mathbf{A} = 0, \mathbf{n}^T\mathbf{m} = 0 \tag{2.1.18.4}$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \tag{2.1.18.5}$$
$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \tag{2.1.18.6}$$

From (2.1.15.2), (2.1.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.1.19.1}$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.1.19.2}$$

where

$$\mathbf{V} = \mathbf{V}^T \tag{2.1.19.3}$$
$$\mathbf{u} = \begin{pmatrix}d & e\end{pmatrix} \tag{2.1.19.4}$$

20. *Pair of straight lines:* (2.1.19.2) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = 0 \tag{2.1.20.1}$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \tag{2.1.20.2}$$

21. In Fig. 2.1.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = k. \tag{2.1.21.1}$$

Show that

$$\frac{\mathbf{A}+k\mathbf{C}}{k+1} = \mathbf{B}. \tag{2.1.21.2}$$

Solution: From (2.1.2.5),

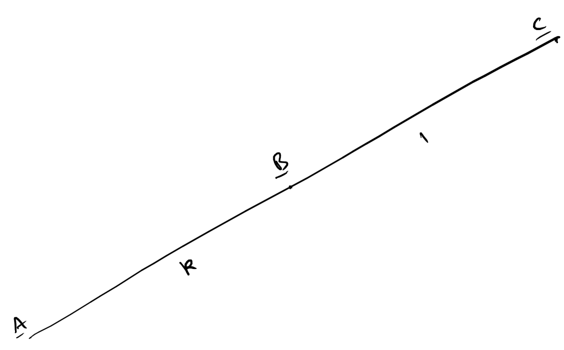

Fig. 2.1.21

$$\begin{aligned}\mathbf{B} &= \mathbf{A} + \lambda_1\mathbf{m}\\ \mathbf{B} &= \mathbf{C} - \lambda_2\mathbf{m}\end{aligned} \tag{2.1.21.3}$$
$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{2.1.21.4}$$
$$\text{and } \frac{\mathbf{B}-\mathbf{A}}{\lambda_1} = \frac{\mathbf{C}-\mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{2.1.21.5}$$

from (2.1.21.1). Using (2.1.21.4) and (2.1.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{2.1.21.6}$$

resulting in (2.1.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{2.1.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \mathbf{A} + \lambda_1(\mathbf{B} - \mathbf{A}) + \lambda_2(\mathbf{C} - \mathbf{A}) \tag{2.1.23.1}$$

such that

$$0 \le \lambda_1 \le 1, 0 \le \lambda_2 \le 1, 0 \le \lambda_1 + \lambda_2 \le 1, \tag{2.1.23.2}$$

24. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.1.24.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.1.24.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{2.1.24.3}$$

25. Find the equation of a line passing through the intersection of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.1.25.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.1.25.2}$$

and passing through the point $\mathbf{p}$.
Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T}\mathbf{c} \tag{2.1.25.3}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda\left(\mathbf{N}^{-T}\mathbf{c} - \mathbf{p}\right) \tag{2.1.25.4}$$

$$\implies (\mathbf{c} - \mathbf{N}\mathbf{p})^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}\mathbf{x} = (\mathbf{c} - \mathbf{N}\mathbf{p})^T \mathbf{N}\mathbf{p} \tag{2.1.25.5}$$

2.2 Intercepts

1. Find the intercepts made on the axes by the straight lines whose equations are

   a) $\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 2$
   b) $\begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = -5$
   c) $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$
   d) $\begin{pmatrix} \frac{1}{a+b} & \frac{1}{a-b} \end{pmatrix}\mathbf{x} = \frac{1}{a^2-b^2}$
   e) $\begin{pmatrix} 1 & -m \end{pmatrix}\mathbf{x} = -c$

2. Write down the equations of straight lines which make the following pairs of intercepts on the axes:

   a) 3,-4
   b) -5,6
   c) $\frac{1}{a}, \frac{1}{b}$
   d) $2a, -2a$

3. A straight line passes through a fixed point $\begin{pmatrix} h \\ k \end{pmatrix}$ and cuts the axes in $\mathbf{A}, \mathbf{B}$. Parallels to the axes through $\mathbf{A}$ and $\mathbf{B}$ intersect in $\mathbf{P}$. Find the equation of the locus of $\mathbf{P}$.

2.3 Line Equation

1. Find the equations of two straight lines at a distance 3 from the origin and making an angle of 120° with OX.
2. Find the equation of a straight line making an angle of 60° with OX and passing through the point $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$. Transform the equation to the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \tag{2.3.2}$$

3. Find the equation of the straight line that passes through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. What is its inclination to OX?
4. Find the equation of the straight line through the point $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ that makes equal intercepts on the axes.
5. Find the equations of the sides of a triangle whose vertices are $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
6. For the same triangle find the equations of the medians
7. Find the equation of a straight line passing through the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ parallel to the line $\begin{pmatrix} 4 & -1 \end{pmatrix}\mathbf{x} + 7 = 0$.
8. Find the intercepts on the axes made by a straight line which passes through the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and makes an angle of 30° with OX.
9. Find the equation of the straight line through the points $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and of the parallel line through $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

10. What is the distance from the origin of the line $\begin{pmatrix} 4 & -1 \end{pmatrix} \mathbf{x} = 7$? Write down the equation of a parallel line at double the distance.
11. Find the equation of the straight line through the point $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ parallel to the line joining the origin to the point $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
12. Write down the equation of the straight line which makes intercepts 2 and -7 on the axes, and of the parallel line through the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
13. Find the equations of the straight line joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and of the parallel line through the origin.
14. ABC is a triangle and $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$. Find the equation of the straight line through $\mathbf{A}$ parallel to BC.
15. Find the equation of a line parallel to $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 11$ passing through the middle point of the join of the points $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -11 \end{pmatrix}$.
16. The base of a triangle passes through a fixed point $\begin{pmatrix} f \\ g \end{pmatrix}$ and the sides are bisected at right angles by the axes. Prove that the locus of the vertex is the line
$$\begin{pmatrix} g & f \end{pmatrix} \mathbf{x} = 0 \tag{2.3.16}$$

2.4 Point of Intersection

1. Find the vertices of the triangle whose sides are
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} + 6 = 0, \tag{2.4.1.1}$$
$$\begin{pmatrix} 2 & -5 \end{pmatrix} \mathbf{x} + 4 = 0, \tag{2.4.1.2}$$
$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{2.4.1.3}$$
2. Prove that the lines
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 25 = 0, \tag{2.4.2.1}$$
$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 7 = 0 \tag{2.4.2.2}$$
$$\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.2.3}$$
are concurrent, and find the coordinates of their common point.
3. Find the equation of a line parallel to the line
$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{2.4.3.1}$$
and passing through the intersection of the lines
$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{2.4.3.2}$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 5 \tag{2.4.3.3}$$
4. Find the equation of the line joining the origin to the point of intersection of the lines
$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.4.1}$$
$$\begin{pmatrix} 2 & 7 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{2.4.4.2}$$
5. Find the acute angle between the lines
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -7 \tag{2.4.5.1}$$
$$\begin{pmatrix} 2 + \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.5.2}$$
6. Find the angle between the lines
$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{2.4.6.1}$$
$$\begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} + 11 = 0 \tag{2.4.6.2}$$
7. Find the equation of a straight line through the point $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ at right angles to the line
$$\begin{pmatrix} 5 & 7 \end{pmatrix} \mathbf{x} + 12 = 0 \tag{2.4.7}$$
and find the point in which the lines intersect.
8. Find the equation of a straight line through the origin and at right angles to the line
$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} + c = 0 \tag{2.4.8}$$
9. Find the equation of a straight line at right angles to the line
$$\begin{pmatrix} 5 & -2 \end{pmatrix} \mathbf{x} + 11 = 0 \tag{2.4.9.1}$$
and passing through the intersecton of the lines
$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} + 1 = 0, \tag{2.4.9.2}$$
$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 7. \tag{2.4.9.3}$$
10. The origin is a corner of a square and two of its sides have equations
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.10.1}$$
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3. \tag{2.4.10.2}$$
Find the equations of the other two sides.

11. Write down the equations of the perpendiculars from the origin to the lines

$$\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x} = 13, \tag{2.4.11.1}$$
$$\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{x} = 13 \tag{2.4.11.2}$$

and find the equation of the line joining the feet of the perpendiculars.

12. Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 11 \tag{2.4.12.1}$$

makes equal angles with the lines

$$\begin{pmatrix}1 & -\left(2-\sqrt{3}\right)\end{pmatrix}\mathbf{x} + 2 = 0, \tag{2.4.12.2}$$
$$\begin{pmatrix}\left(2-\sqrt{3}\right) & -1\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.4.12.3}$$

13. **A** is the point $\begin{pmatrix}-4\\0\end{pmatrix}$ and **B** is the point $\begin{pmatrix}3\\0\end{pmatrix}$. Find the locus of a point **P** such that the angles APO, OPB are equal, where **O** is the origin.

2.5 Perpendiculars and Bisectors

1. Find the distance of the point $\begin{pmatrix}4\\2\sqrt{3}\end{pmatrix}$ from the line

$$\begin{pmatrix}\cos 60^\circ & \sin 60^\circ\end{pmatrix}\mathbf{x} = 6 \tag{2.5.1}$$

2. Find the distance of the point $\begin{pmatrix}2\\3\end{pmatrix}$ from each of the straight lines

$$\begin{pmatrix}5 & 12\end{pmatrix}\mathbf{x} - 20 = 0 \tag{2.5.2.1}$$
$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.5.2.2}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} - 28 = 0. \tag{2.5.2.3}$$

3. Find the distance of the pont $\begin{pmatrix}3\\-1\end{pmatrix}$ from the line joining the points $\begin{pmatrix}2\\-3\end{pmatrix}$, $\begin{pmatrix}4\\1\end{pmatrix}$.

4. Are the points $\begin{pmatrix}-2\\3\end{pmatrix}$, $\begin{pmatrix}-2\\4\end{pmatrix}$ on the same or on opposite sides of the line

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 10? \tag{2.5.4}$$

5. Find the equations of the bisectors of the angles between the lines

$$\begin{pmatrix}-4 & 2\end{pmatrix}\mathbf{x} = -9 \tag{2.5.5.1}$$
$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 4 \tag{2.5.5.2}$$

and state which equation refers to the angle which contains the origin.

6. Prove that the bisector of one of the angles between the lines

$$\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{x} - 7 = 0 \tag{2.5.6.1}$$
$$\begin{pmatrix}1 & -5\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2.5.6.2}$$

passes through the origin. What is the equation of the bisector of the other angle?

7. What is the condition that the point $\begin{pmatrix}x\\y\end{pmatrix}$ may be at unit distance from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 10 = 0 \tag{2.5.7}$$

Write down the equations of two straight lines parallel to the given line and at unit distances from it, and state which of the two lies on the same side of the given line as the origin.

8. The sides AB, BC, CA of a triangle have equations

$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 12 \tag{2.5.8.1}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.5.8.2}$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2. \tag{2.5.8.3}$$

Find the coordinates of the centres of the inscribed circle and of the escribed circle opposite to the vertex **A**.

9. Prove that the point $\begin{pmatrix}4\\4\end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.5.9.1}$$
$$\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = 15 \tag{2.5.9.2}$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.5.9.3}$$

10. Find the equation of the line joining the origin to the point of intersection of the lines

$$\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} - 11 = 0 \tag{2.5.10.1}$$
$$\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.5.10.2}$$

11. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.5.11.1}$$

and passing through the intersection of the lines

$$\begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 1 \tag{2.5.11.2}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.5.11.3}$$

12. Find the equation of a line through the intersection of the lines

$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 1 \tag{2.5.12.1}$$

$$\begin{pmatrix}-4 & 1\end{pmatrix}\mathbf{x} = 9 \tag{2.5.12.2}$$

parallel to the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{2.5.12.3}$$

13. The vertices of a triangle are at the points

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \tag{2.5.13}$$

Find the equations of the medians and prove that they meet in a point. What are the coordinates of their point of intersection?
BE and CF are medians of $\triangle ABC$ intersecting at $\mathbf{O}$ as shown in Fig. 2.5.13. We first show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \tag{2.5.13.2}$$

Let

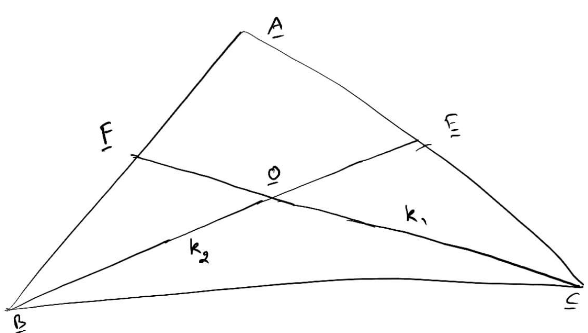

Fig. 2.5.13

$$\frac{CO}{OF} = k_1 \tag{2.5.13.3}$$

$$\frac{BO}{OE} = k_2 \tag{2.5.13.4}$$

Using (2.1.21.2),

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{2.5.13.5}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{2.5.13.6}$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F}+\mathbf{C}}{k_1+1} = \frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} \tag{2.5.13.7}$$

$$\mathbf{O} = \frac{k_2\mathbf{E}+\mathbf{B}}{k_2+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{2.5.13.8}$$

From (2.5.13.7) and (2.5.13.8),

$$\frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2}+\mathbf{C}}{k_1+1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2}+\mathbf{B}}{k_2+1} \tag{2.5.13.9}$$

$$\begin{aligned}\implies &\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right]\mathbf{A} \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0\end{aligned} \tag{2.5.13.10}$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = k_2 = 2, \tag{2.5.13.11}$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (2.5.13.8),

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{2.5.13.12}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \tag{2.5.13.13}$$

Note that $\mathbf{B}, \mathbf{C}$ are linearly independent. Substituting (2.5.13.13) in (2.5.13.10),

$$\begin{aligned}&\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right][\alpha\mathbf{B} + \beta\mathbf{C}] \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0\end{aligned} \tag{2.5.13.14}$$

$$\implies \begin{aligned}(k_1-k_2)\alpha + k_1k_2 - k_1 - 2 = 0 \\ (k_1-k_2)\beta - k_1k_2 + k_2 + 2 = 0\end{aligned} \tag{2.5.13.15}$$

$$\implies (k_1-k_2)(\alpha+\beta-1) = 0 \tag{2.5.13.16}$$

If $\alpha + \beta = 1, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (2.1.21.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in

(2.5.13.15), yields

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = 2. \tag{2.5.13.17}$$

14. For what multiples k, l, m is the equation

$$k\left\{\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 13\right\} + l\left\{\begin{pmatrix}5 & -y\end{pmatrix}\mathbf{x} - 7\right\} \\ +m\left\{\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} + 10\right\} = 0 \tag{2.5.14.1}$$

an identity? In what point do the lines given by equating the three terms to zero concur?

15. Find the equations of the diagonals of the parallelogram

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2.5.15.1}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} - 5 = 0, \tag{2.5.15.2}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5 = 0 \tag{2.5.15.3}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{2.5.15.4}$$

16. The vertices of a triangle are at the points

$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\-3\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix} \tag{2.5.16}$$

Find the equations of the perpendiculars to the sides through their middle points.

17. Work the same problem when the vertices of the triangle are at the points

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \tag{2.5.17.1}$$

and show that the perpendiculars meet in a point.
Solution: In Fig. 2.5.17, $BE \perp AC, CF \perp AB$. We need to show that $AD \perp BC$. Let $\mathbf{x}$ be the intersection of BE and CF. Then, using (1.1.9),

$$\begin{aligned}(\mathbf{x} - \mathbf{B})^T (\mathbf{A} - \mathbf{C}) = 0 \\ (\mathbf{x} - \mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0\end{aligned} \tag{2.5.17.2}$$

$$\implies \mathbf{x}^T (\mathbf{A} - \mathbf{C}) - \mathbf{B}^T (\mathbf{A} - \mathbf{C}) = 0 \tag{2.5.17.3}$$

$$\text{and } \mathbf{x}^T (\mathbf{A} - \mathbf{B}) - \mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.5.17.4}$$

Subtracting (2.5.17.4) from (2.5.17.3),

$$\mathbf{x}^T (\mathbf{B} - \mathbf{C}) + \mathbf{A}^T (\mathbf{C} - \mathbf{B}) = 0 \tag{2.5.17.5}$$

$$\implies \left(\mathbf{x}^T - \mathbf{A}^T\right)(\mathbf{B} - \mathbf{C}) = 0 \tag{2.5.17.6}$$

$$\implies (\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{2.5.17.7}$$

which completes the proof.

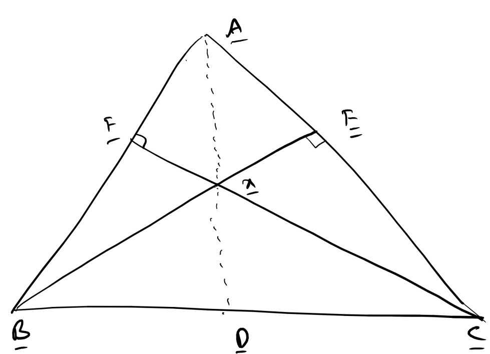

Fig. 2.5.17

18. The line

$$\begin{pmatrix}2 & -8\end{pmatrix}\mathbf{x} - 4 = 0 \tag{2.5.18}$$

is the perpendicular bisector of the line AB and $\mathbf{A}$ is the point $\begin{pmatrix}5\\6\end{pmatrix}$. What are the coordinates of $\mathbf{B}$?

2.6 Angle Between Lines

1. What lines are represented by the following equations:

a) $\mathbf{x}^T \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x} = 0$

b) $\mathbf{x}^T \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 0$

c) $\mathbf{x}^T \begin{pmatrix}6 & \frac{1}{2}\\\frac{1}{2} & -1\end{pmatrix}\mathbf{x} = 0$

d) $\mathbf{x}^T \begin{pmatrix}-1 & -\tan\theta\\\tan\theta & 1\end{pmatrix}\mathbf{x} = 0$

e) $x^3 + 3x^2y - 3xy^2 - y^3 = 0$

2. Find the angles between the pairs of straight lines represented by the following equations:

a) $\mathbf{x}^T \begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\mathbf{x} = 0$

b) $\mathbf{x}^T \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 0$

c) $\mathbf{x}^T \begin{pmatrix}1 & -\frac{5}{2}\\-\frac{5}{2} & 4\end{pmatrix}\mathbf{x} = 0$

d) $\mathbf{x}^T \begin{pmatrix}1 & 2\\2 & 0\end{pmatrix}\mathbf{x} = 0$

e) $\mathbf{x}^T \begin{pmatrix}1 & \frac{1}{2}\\\frac{1}{2} & -6\end{pmatrix}\mathbf{x} = 0$

3. Prove that the equations

$$\mathbf{x}^T \begin{pmatrix}1 & -2\\-2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.6.3.1}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.6.3.2}$$

are the sides of an equilateral triangle.

2.7 Miscellaneous

1. Find the locus of a point which is equidistant from the points $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$.
2. Find the point on the line
$$\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} + 7 = 0 \tag{2.7.2}$$
which is equidistant from the points $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.
3. Find the coordinates of the circumcentre of the triangle whose corners are at the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$.
4. Find the equations of the lines through $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ which are respectively parallel and perpendicular to the line joining the points $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$.
5. Find the locus of a point at which the join of the points $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ subtends a right angle.
6. Find the orthocentre of a triangle whose corners are at the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.
7. Prove that the line joining the points $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ makes with the axes a triangle of area $\frac{49}{60}$.
8. $ABCD$ is a parallelogram and the coordinates of **A**, **B** and **C** are $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ Find the coordinates of **D**.
9. Find the area of the triangle formed by the lines
$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{2.7.9.1}$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 7 \tag{2.7.9.2}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{2.7.9.3}$$
10. Find the centre of the inscribed circle of the triangle whose sides are
$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 0 \tag{2.7.10.1}$$
$$\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = 0, \tag{2.7.10.2}$$
$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 8 \tag{2.7.10.3}$$
11. The ends of a diagonal of a square are on the coordinate axes at the points $\begin{pmatrix} 2a \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ a \end{pmatrix}$. Find the equations of the sides.
12. The sides of a triangle ABC are
$$AB = 3, BC = 5, CA = 4 \tag{2.7.12}$$
and $\mathbf{A}, \mathbf{B}$ are on the axes OX, OY respectively, while AC makes an angle θ with OX. Prove that the locus of **C**, as θ varies, is given by the equation
$$\mathbf{x}^T \begin{pmatrix} 16 & -12 \\ -12 & 25 \end{pmatrix} \mathbf{x} = 256 \tag{2.7.12}$$
13. Prove that the locus of a point at which the join of the points $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ subtends an angle of 45° is
$$\mathbf{x}^T \mathbf{x} - 2a \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = a^2 \tag{2.7.13}$$
14. Prove that the line
$$\mathbf{n}^T \mathbf{x} + c = 0 \tag{2.7.14.1}$$
divides the line joining the points $\mathbf{x}_1, \mathbf{x}_2$ in the ratio
$$-\frac{\mathbf{n}^T \mathbf{x}_1 + c}{\mathbf{n}^T \mathbf{x}_2 + c} \tag{2.7.14.2}$$
15. Find the equation of the line joining the point $\mathbf{x}_1$, to the point of intersection of the lines
$$\mathbf{n}^T \mathbf{x} + c = 0 \tag{2.7.15.1}$$
$$\mathbf{n}_1^T \mathbf{x} + c_1 = 0 \tag{2.7.15.2}$$
16. Find the equations of the diagonals of the parallelogram whose sides are
$$\mathbf{n}^T \mathbf{x} + c = 0 \tag{2.7.16.1}$$
$$\mathbf{n}^T \mathbf{x} + d = 0 \tag{2.7.16.2}$$
$$\mathbf{n}_1^T \mathbf{x} + c_1 = 0 \tag{2.7.16.3}$$
$$\mathbf{n}_1^T \mathbf{x} + d_1 = 0 \tag{2.7.16.4}$$
17. Prove that for all values of k the line
$$\begin{pmatrix} 2 + k & 1 - 2k \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.7.17}$$
passes through a fixed point, and find its coordinates.
18. Find the angle between the lines
$$\mathbf{x}^T \begin{pmatrix} 1 & -\sec\theta \\ -\sec\theta & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.7.18}$$

19. Prove that the pairs of straight lines represented by

$$\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.7.19.1}$$

$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\ -\frac{1}{2} & -1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.19.2}$$

are such that the angles between one pair are equal to the angles between the other pair.

20. Find the angles between the lines

$$x^3 - 3x^2y - 3xy^2 + y^3 = 0 \tag{2.7.20}$$

21. Find the area of the triangle whose sides are given by

$$\mathbf{x}^T\begin{pmatrix}1 & -2\\ -2 & 3\end{pmatrix}\mathbf{x} = 0 \tag{2.7.21.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 7 \tag{2.7.21.2}$$

22. Show that the equation

$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\ -\frac{1}{2} & -15\end{pmatrix}\mathbf{x} + \begin{pmatrix}-11 & 31\end{pmatrix}\mathbf{x} - 10 = 0 \tag{2.7.22}$$

represents two straight lines, and find the equations of the bisectors of the angles between them.

23. For what value of k does the equation

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & k\end{pmatrix}\mathbf{x} + \begin{pmatrix}13 & -1\end{pmatrix}\mathbf{x} + 3 = 0 \tag{2.7.23}$$

represent two straight lines? What is the angle between them?

24. For what values of k does the equation

$$\mathbf{x}^T\begin{pmatrix}6 & \frac{k}{2}\\ \frac{k}{2} & -3\end{pmatrix}\mathbf{x} + \begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} - 2 = 0 \tag{2.7.24}$$

represent two straight lines?

3 Curves

3.1 Definitions

1. The equation of a quadratic curve is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.1.1}$$

2. Show that

$$\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \tag{3.1.2}$$

3. Show that

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T\mathbf{x} \tag{3.1.3}$$

4. Show that

$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \tag{3.1.4}$$

5. Find the *normal* vector to the curve in (3.1.1) at point $\mathbf{p}$.
Solution: Differentiating (3.1.1) with respect to x_1,

$$\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} + \frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}}\frac{d\mathbf{x}}{dx_1} = 0 \tag{3.1.5.1}$$

$$\implies 2\mathbf{x}^T\mathbf{V}\mathbf{m} + 2\mathbf{u}^T\mathbf{m} = 0 \because \left(\frac{d\mathbf{x}}{dx_1} = \mathbf{m}\right) \tag{3.1.5.2}$$

Substituting $\mathbf{x} = \mathbf{p}$ and simplifying

$$(\mathbf{V}\mathbf{p} + \mathbf{u})^T\,\mathbf{m} = 0 \tag{3.1.5.3}$$

$$\implies \mathbf{n} = \mathbf{V}\mathbf{p} + \mathbf{u} \tag{3.1.5.4}$$

6. The *tangent* to the curve at $\mathbf{p}$ is given by

$$\mathbf{n}^T(\mathbf{x} - \mathbf{p}) = 0 \tag{3.1.6}$$

7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.1.7}$$

is known as an *affine* transformation.

3.2 Tangent

1. Find the equations of the tangents to the following curves at the points specified:

a) $y = x\left(x^2 - 1\right), x = 2$
b) $y = x^2 + \frac{1}{x^2}, x = 1$
c) $y = x^3 + 2x, x = 0$
d) $y = \left(x + \frac{1}{x}\right)^3, x = 2$
e) $y = \left(x^2 - 1\right)^2, x = 1$
f) $y = x^3 - x + 1, x = 3$
g) $y = (x - a)^3, x = 2a$
h) $y = ax^2 + 2bx + c, (x_1, y_1)$
i) $y = \frac{x^3}{a^3} + \frac{a^3}{x^3}, x = a$
j) $y = \frac{x^2}{a} + \frac{a^2}{x}, x = a$

2. Find the tangents to the curve

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.2.2.1}$$

at the points where it is cut by the line

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 4 = 0 \tag{3.2.2.2}$$

and find the point of intersection of the tangents.

3. Prove that the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 4 = 0 \tag{3.2.3.1}$$

touches the curve

$$\mathbf{x}^T\begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{2} & 0\end{pmatrix}\mathbf{x} + 1 = 0. \tag{3.2.3.2}$$

4. Find the points on the curve

$$3y = x^3 + 3x \tag{3.2.4.1}$$

at which the tangent is parallel to the line

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.2.4.2}$$

5. Find at what points on the curve

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} + 9 = 0 \tag{3.2.5}$$

the tangents pass through the origin.

6. Show that there are three points on the curve

$$3y = 3x^4 + 8x^3 - 6x^2 \tag{3.2.6.1}$$

at which the tangents are parallel to the line

$$\begin{pmatrix}8 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.2.6.2}$$

7. Show that the line

$$\begin{pmatrix}0 & 4\end{pmatrix}\mathbf{x} = 17 \tag{3.2.7.1}$$

meets the curve

$$y = x^2 + \frac{1}{x^2} \tag{3.2.7.2}$$

in four points and that two of the points of intersection of the tangents at these four points are on the line

$$\begin{pmatrix}0 & 4\end{pmatrix}\mathbf{x} + 1 = 0, \tag{3.2.7.3}$$

and two are on the line

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0. \tag{3.2.7.4}$$

3.3 More on Tangents

1. Find the equation of the tangents to the following curves at the points stated:
   a) $\mathbf{x}^T\mathbf{x} = 25, \begin{pmatrix}3\\4\end{pmatrix}$
   b) $\mathbf{x}^T\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}\mathbf{x} = 2, \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0, \begin{pmatrix}a\\2a\end{pmatrix}$
   d) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\ b\sin\theta\end{pmatrix}$
   e) $\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} = a^2, \begin{pmatrix}a\sec\theta\\ b\tan\theta\end{pmatrix}$
   f) $\mathbf{x}^T\begin{pmatrix}1 & -\frac{1}{2}\\ -\frac{1}{2} & 1\end{pmatrix}\mathbf{x} = 4, \begin{pmatrix}2\\2\end{pmatrix}$
   g) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} - 20 = 0, \begin{pmatrix}3\\1\end{pmatrix}$
   h) $x^3 + y^3 - 3xy^2 + a^3 = 0, \begin{pmatrix}a\\a\end{pmatrix}$
2. Find the equation of the tangent at the point $\mathbf{p}$ on each of the following curves:
   a) $\mathbf{x}^T\mathbf{x} = a^2$
   b) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$
   d) $\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & \end{pmatrix}\mathbf{x} - c^2 = 0$
   e) $y^2\left(x^2 - a^2\right) = a^2\left(x^2 + a^2\right)$
   f) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

3.4 Normal

1. Find the equations of the normals to the following curves at the given points
   a) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} = 3, \begin{pmatrix}3\\4\end{pmatrix}$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\mathbf{x} = 13, \begin{pmatrix}1\\2\end{pmatrix}$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0, \begin{pmatrix}a\\2a\end{pmatrix}$
   d) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\ b\sin\theta\end{pmatrix}$
   e) $\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -4\end{pmatrix}\mathbf{x} = 4a^2, \begin{pmatrix}2a\sec\theta\\ a\tan\theta\end{pmatrix}$
   f) $x^3 - y^3 - 3xy^2 + a^2 = 0, \begin{pmatrix}a\\-a\end{pmatrix}$
2. Find the equation of the normal at the point $\mathbf{p}$ on each of the following curves:
   a) $\mathbf{x}^T\mathbf{x} = a^2$
   b) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$
   d) $\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{x} = c^2$

3. Prove that for the curve

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.4.3}$$

the subnormal is of constant length.
4. Prove that the portion of any tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted by the axes is of length a.
5. Prove that for the curve $ay^2 = x^3$ the subnormal varies as the square of the subtangent.
6. Prove that for the curve $y = ae^{\frac{x}{b}}$ the subtangent is of length b.
7. Prove that the area of the triangle formed by the axes and any tangent to the curve

$$\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{x} = c^2 \tag{3.4.7}$$

is $2c^2$.
8. Prove that for the curve $x^m y^n = e^{m+n}$ the portion of a tangent intercepted by the axes is divided at the point of contact in the ratio $m : n$.
9. Prove that, if N is the foot of the ordinate and NT is the subtangent at a point on the curve

$$\mathbf{x}^T\begin{pmatrix}b^2 & 0\\ 0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\ b\sin\theta\end{pmatrix} \tag{3.4.9}$$

then $OT.ON = a^2$.
10. Prove that the perpendicular from the foot of the ordinate to the tangent to a curve is of length $\frac{y}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}}$. Show that for the curve $y = c\cosh\frac{x}{c}$, this perpendicular is of length c.
11. Find the equation of the tangent to the curve

$$2x^3 + 2y^3 - 9axy = 0 \tag{3.4.11.1}$$

at the point $\begin{pmatrix}2a\\ a\end{pmatrix}$; and show that the tangent meets the curve again where

$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.4.11.2}$$

3.5 Affine Transformation

1. What does the equation

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} - \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.5.1}$$

become when the origin is moved to the point $\begin{pmatrix}2\\ -3\end{pmatrix}$?
2. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\ -\frac{3}{2} & 4\end{pmatrix}\mathbf{x} + \begin{pmatrix}10 & -19\end{pmatrix}\mathbf{x} + 23 = 0 \tag{3.5.2}$$

may become

$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\ -\frac{3}{2} & 4\end{pmatrix}\mathbf{x} = 1 \tag{3.5.2}$$

3. Show that the equation

$$\mathbf{x}^T\mathbf{x} = a^2 \tag{3.5.3}$$

remains unaltered by any rotation of the axes.
4. What does the equation

$$\mathbf{x}^T\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\mathbf{x} = 2a^2 \tag{3.5.4}$$

become when the axes are turned through 30°?
5. What does the equation

$$\mathbf{x}^T\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\mathbf{x} - 4\sqrt{2}a\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.5}$$

become when the axes are turned through 45°?
6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix}\mathbf{x} + \begin{pmatrix}10 & -4\end{pmatrix}\mathbf{x} = 0 \tag{3.5.6}$$

may become

$$\mathbf{x}^T\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix}\mathbf{x} = 1 \tag{3.5.6}$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T\begin{pmatrix}p & 0\\ 0 & q\end{pmatrix}\mathbf{x} = 1 \tag{3.5.6}$$

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\mathbf{x} = 1 \tag{3.5.7}$$

to the form

$$\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\mathbf{x} = c \tag{3.5.7}$$

where c is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix}\mathbf{x} - 13 = 0 \qquad (3.5.8)$$

can be transformed to

$$8\mathbf{x}^T\mathbf{x} = 89 \qquad (3.5.8)$$

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \qquad (3.5.9)$$

can be reduced to

$$\sqrt{85}\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \qquad (3.5.9)$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \qquad (3.5.10)$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \qquad (3.5.10)$$

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix}\mathbf{x} + 27 = 0 \qquad (3.5.11)$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \qquad (3.5.11)$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \qquad (3.5.11)$$