Conics and Quadratic Forms

G V V Sharma*

Abstract—**Solved problems from JEE mains papers related to conic sections in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 Two parabolas with a common vertex and with axes along x-axis and y-axis, respectively, intersect each other in the first quadrant. If the length of the latus rectum of each parabola is 3, find the equation of the common tangent to the two parabolas.
Solution: The equation of a conic is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0 \tag{1.1}$$

For the standard parabola,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

$$\mathbf{u} = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.3}$$

$$F = 0 \tag{1.4}$$

The focus

$$\mathbf{F} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.5}$$

The Latus rectum is the line passing through $\mathbf{F}$ with direction vector

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.6}$$

Thus, the equation of the Latus rectum is

$$\mathbf{x} = \mathbf{F} + \lambda\mathbf{m} \tag{1.7}$$

The intersection of the latus rectum and the parabola is obtained from (1.4) , (1.7) and (1.1) as

$$(\mathbf{F} + \lambda\mathbf{m})^T \mathbf{V}(\mathbf{F} + \lambda\mathbf{m}) + 2\mathbf{u}^T(\mathbf{F} + \lambda\mathbf{m}) = 0 \tag{1.8}$$

$$\implies \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\lambda^2 + 2\left(\mathbf{V}\mathbf{F} + \mathbf{u}\right)^T\mathbf{m}\lambda + \left(\mathbf{V}\mathbf{F} + 2\mathbf{u}\right)^T\mathbf{F} = 0 \tag{1.9}$$

From (1.2), (1.3), (1.5) and (1.6),

$$\mathbf{m}^T\mathbf{V}\mathbf{m} = 1 \tag{1.10}$$

$$\left(\mathbf{V}\mathbf{F} + \mathbf{u}\right)^T\mathbf{m} = 0 \tag{1.11}$$

$$\left(\mathbf{V}\mathbf{F} + 2\mathbf{u}\right)^T\mathbf{F} = -4a^2 \tag{1.12}$$

Substituting from (1.10), (1.11) and (1.12) in (1.9),

$$\lambda^2 - 4a^2 = 0 \tag{1.13}$$

$$\implies \lambda_1 = 2a, \lambda_2 = -2a \tag{1.14}$$

Thus, from (1.6), (1.7) and (1.14), the length of the latus rectum is

$$(\lambda_1 - \lambda_2)\|\mathbf{m}\| = 4a \tag{1.15}$$

From the given information, the two parabolas P_1, P_2 have parameters

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2a\begin{pmatrix} 1 \\ 0 \end{pmatrix}, F_1 = 0 \tag{1.16}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -2a\begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_2 = 0 \tag{1.17}$$

$$4a = 3 \tag{1.18}$$

Let L be the common tangent for P_1, P_2 with $\mathbf{c}, \mathbf{d}$ being the respective points of contact. The

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

respectivel normal vectors are

$$\mathbf{n}_1 = \mathbf{V}_1\mathbf{c} + \mathbf{u}_1 = -2a\begin{pmatrix}1\\-\frac{c_2}{2a}\end{pmatrix} \tag{1.19}$$

$$\mathbf{n}_2 = \mathbf{V}_2\mathbf{d} + \mathbf{u}_2 = d_1\begin{pmatrix}1\\-\frac{2a}{d_1}\end{pmatrix} \tag{1.20}$$

From the above equations, since both normals have the same direction vector,

$$\begin{pmatrix}1\\-\frac{c_2}{2a}\end{pmatrix} = \begin{pmatrix}1\\-\frac{2a}{d_1}\end{pmatrix} \implies c_2 d_1 = 4a^2 \tag{1.21}$$

2 Find the product of the perpendiculars drawn from the foci of the ellipse

$$\mathbf{x}^T\begin{pmatrix}25 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 225 \tag{2.1}$$

upon the tangent to it at the point

$$\frac{1}{2}\begin{pmatrix}3\\5\sqrt{3}\end{pmatrix} \tag{2.2}$$

Solution: For the ellipse in (2.1),

$$V = \begin{pmatrix}\frac{1}{9} & 0\\0 & \frac{1}{25}\end{pmatrix}, \mathbf{u} = 0, F = -1 \tag{2.3}$$

The equation of the desired tangent is

$$(\mathbf{VP})^T\mathbf{x} = 1 \tag{2.4}$$

$$\implies \begin{pmatrix}\frac{1}{3} & \frac{\sqrt{3}}{5}\end{pmatrix}\mathbf{x} = 2 \tag{2.5}$$

The foci of the ellipse are located at

$$\mathbf{F}_1 = \begin{pmatrix}0\\4\end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix}0\\-4\end{pmatrix} \tag{2.6}$$

The product of the perpendiculars is

$$\frac{\left|\begin{pmatrix}\frac{1}{3} & \frac{\sqrt{3}}{5}\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix}-2\right|\left|\begin{pmatrix}\frac{1}{3} & \frac{\sqrt{3}}{5}\end{pmatrix}\begin{pmatrix}0\\-4\end{pmatrix}-2\right|}{\left\|\begin{pmatrix}\frac{1}{3} & \frac{\sqrt{3}}{5}\end{pmatrix}\right\|^2} = 9 \tag{2.7}$$

3 Let $\mathbf{P}$ be the point on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}8 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.1}$$

which is at a minimum distance from the centre $\mathbf{C}$ of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 12\end{pmatrix}\mathbf{x} = 1 \tag{3.2}$$

Find the equation of the circle passing through $\mathbf{C}$ and having its centre at $\mathbf{P}$.

Solution: From (3.2),

$$\mathbf{C} = \begin{pmatrix}0\\-6\end{pmatrix} \tag{3.3}$$

Thus, the given problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x}-\mathbf{C}\|^2 \tag{3.4}$$

$$s.t. \quad \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} = 0 \tag{3.5}$$

Using Lagrange multipliers,

$$2(\mathbf{x}-\mathbf{C}) - 2\lambda(\mathbf{V}\mathbf{x}+\mathbf{u}) = 0 \tag{3.6}$$

$$\implies (\mathbf{I}-\lambda\mathbf{V})\mathbf{x} = \mathbf{C}+\lambda\mathbf{u} \tag{3.7}$$

$$\implies \mathbf{x} = (\mathbf{I}-\lambda\mathbf{V})^{-1}(\mathbf{C}+\lambda\mathbf{u}) \tag{3.8}$$

$$= \begin{pmatrix}1 & 0\\0 & -\frac{1}{\lambda}\end{pmatrix}\begin{pmatrix}-4\lambda\\-6\end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix}-4\lambda\\\frac{6}{\lambda}\end{pmatrix} \tag{3.10}$$

Substituting in (3.1),

$$\left(\frac{6}{\lambda}\right)^2 = 8(-4\lambda) \tag{3.11}$$

$$\implies \lambda^3 = -\frac{3}{16} \tag{3.12}$$

Thus,

$$\mathbf{P} = \mathbf{C} + \lambda\mathbf{u}, \tag{3.13}$$

and the equation of the circle is

$$\|\mathbf{x}-\mathbf{P}\| = \|\mathbf{P}-\mathbf{C}\| \tag{3.14}$$

4 Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then find the area of the quadrilateral inscribed in the ellipse, with the vertices as the vertices of the ellipse.

5 Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation

$$9e^2 - 18e + 5 = 0 \tag{5.1}$$

If

$$\mathbf{S} = \begin{pmatrix}5\\0\end{pmatrix} \tag{5.2}$$

is a focus and

$$\begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} = 9 \tag{5.3}$$

is the corresponding directrix of this hyperbola, then find $a^2 - b^2$.

6 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{6.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \tag{6.2}$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of AB.

Solution: The intersection of the lines in (6.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{6.3}$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \tag{6.4}$$

resulting in

$$\mathbf{C} = \frac{1}{7} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{6.5}$$

Let the **R** be the mid point of AB. Then,

$$\mathbf{A} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{6.6}$$

$$\mathbf{B} = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{6.7}$$

Let the equation of AB be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \tag{6.8}$$

Since **R** lies on AB,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \tag{6.9}$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{6.10}$$

Substituting from (6.6) in (6.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \tag{6.11}$$

From (6.9) and (6.11),

$$(\mathbf{R} - \mathbf{C}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \tag{6.12}$$

for some constant k. Multiplying both sides of (6.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.13}$$

$$\begin{aligned} \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) &= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ &= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{6.14}$$

$$\because \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{6.15}$$

which can be easily verified for any **R**. from (6.14),

$$\begin{aligned} &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0 \\ \implies &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0 \\ \implies &\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \end{aligned} \tag{6.16}$$

which is the desired locus.