

3D Geometry through Linear Algebra

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Abstract—This manual introduces linear algebra by exploring 3D geometry through a problem solving approach.

1 LINES AND PLANES

1.1 L_1 is the intersection of planes

$$\begin{aligned} \begin{pmatrix} 2 & -2 & 3 \end{pmatrix} \mathbf{x} &= 2 \\ \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} &= -1 \end{aligned} \quad (1.1)$$

Find its equation.

Solution: (1.1) can be written in matrix form as

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad (1.2)$$

and solved using the augmented matrix as follows

$$\begin{pmatrix} 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & 2 \end{pmatrix} \quad (1.3)$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 4 \end{pmatrix} \quad (1.4)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - 5 \\ x_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (1.5)$$

which is the desired equation.

1.2 L_2 is the intersection of the planes

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \mathbf{x} = 3 \quad (1.6)$$

$$\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 1 \quad (1.7)$$

Show that its equation is

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \quad (1.8)$$

1.3 Do L_1 and L_2 intersect? If so, find their point of intersection.

Solution: From (1.5), (1.8), the point of intersection is given by

$$\mathbf{x} = \frac{1}{7} \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (1.9)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \\ 1 & -5 \\ 0 & -7 \end{pmatrix} \Lambda = \frac{1}{7} \begin{pmatrix} 40 \\ 8 \\ -28 \end{pmatrix} \quad (1.10)$$

This matrix equation can be solved as

$$\begin{pmatrix} 1 & 3 & \frac{40}{7} \\ 1 & -5 & \frac{8}{7} \\ 0 & -7 & -4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 8 & 0 & \frac{224}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 1 & \frac{4}{7} \end{pmatrix} \quad (1.11)$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & \frac{4}{7} \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 4 \\ 4 \\ \frac{4}{7} \end{pmatrix} \quad (1.12)$$

Substituting $\lambda_1 = 4$ in (1.9)

$$\mathbf{x} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \quad (1.13)$$

2 NORMAL TO A PLANE

2.1 The cross product of \mathbf{a}, \mathbf{b} is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (2.1)$$

From (1.5), (1.8), the direction vectors of L_1 and L_2 are

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \quad (2.2)$$

respectively. Find the direction vector of the normal to the plane spanned by L_1 and L_2 .

Solution: The desired vector is obtained as

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ 8 \end{pmatrix} = \mathbf{n} \quad (2.3)$$

2.2 Find the equation of the plane spanned by L_1 and L_2 .

Solution: Let \mathbf{x}_0 be the intersection of L_1 and L_2 . Then the equation of the plane is

$$(\mathbf{x} - \mathbf{x}_0)^T \mathbf{n} = 0 \quad (2.4)$$

$$\Rightarrow \mathbf{x}^T \mathbf{n} = \mathbf{x}_0^T \mathbf{n} \quad (2.5)$$

$$\Rightarrow \mathbf{x}^T \begin{pmatrix} 7 \\ -7 \\ 8 \end{pmatrix} = (-1 \ 4 \ 4) \begin{pmatrix} 7 \\ -7 \\ 8 \end{pmatrix} = -3 \quad (2.6)$$

2.3 Find the distance of the origin from the plane containing the lines L_1 and L_2 .

Solution: The distance from the origin to the plane is given by

$$\frac{|\mathbf{x}_0^T \mathbf{n}|}{\|\mathbf{n}\|} = \frac{1}{3\sqrt{2}} \quad (2.7)$$

3 PROJECTION ON A PLANE

3.1 Find the equation of the line L joining the points

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \end{pmatrix}^T \quad (3.1)$$

$$\mathbf{B} = \begin{pmatrix} 4 & -1 & 3 \end{pmatrix}^T \quad (3.2)$$

Solution: The desired equation is

$$\mathbf{x} = \mathbf{B} + \lambda (\mathbf{A} - \mathbf{B}) \quad (3.3)$$

$$= \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad (3.4)$$

3.2 Find the intersection of L and the plane P given by

$$(1 \ 1 \ 1)\mathbf{x} = 7 \quad (3.5)$$

Solution: From (3.4) and (3.5),

$$(1 \ 1 \ 1) \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \lambda (1 \ 1 \ 1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 7 \quad (3.6)$$

$$\Rightarrow 6 + 2\lambda = 7 \quad (3.7)$$

$$\Rightarrow \lambda = \frac{1}{2} \quad (3.8)$$

Substituting in (3.4),

$$\mathbf{x} = \frac{1}{2} (9 \ -1 \ 7) \quad (3.9)$$

3.3 Find $\mathbf{C} \in P$ such that $AC \perp P$.

Solution: From (3.5), the direction vector of AC is $(1 \ 1 \ 1)^T$. Hence, the equation of AC is

$$\mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.10)$$

Substituting in (3.5)

$$(1 \ 1 \ 1) \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} + \lambda_1 (1 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 7 \quad (3.11)$$

$$\Rightarrow 8 + 3\lambda_1 = 7 \quad (3.12)$$

$$\Rightarrow \lambda_1 = -\frac{1}{3} \quad (3.13)$$

Thus,

$$\mathbf{C} = \frac{1}{3} \begin{pmatrix} 14 \\ -4 \\ 11 \end{pmatrix} \quad (3.14)$$

3.4 Show that if $BD \perp P$ such that $\mathbf{D} \in P$,

$$\mathbf{D} = \frac{1}{3} \begin{pmatrix} 13 \\ -2 \\ 10 \end{pmatrix} \quad (3.15)$$

3.5 Find the projection of AB on the plane P .

Solution: The projection is given by

$$CD = \|\mathbf{C} - \mathbf{D}\| = \sqrt{\frac{2}{3}} \quad (3.16)$$

4 COPLANAR VECTORS

4.1 If $\mathbf{u}, \mathbf{A}, \mathbf{B}$ are coplanar, show that

$$\mathbf{u}^T (\mathbf{A} \times \mathbf{B}) = 0 \quad (4.1)$$

4.2 Find $\mathbf{A} \times \mathbf{B}$ given

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}^T \quad (4.2)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T \quad (4.3)$$

Solution: From (2.1),

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (4.4)$$

$$= \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \quad (4.5)$$

4.3 Let \mathbf{u} be coplanar with such that $\mathbf{u} \perp \mathbf{A}$ and

$$\mathbf{u}^T \mathbf{B} = 24. \quad (4.6)$$

Find $\|\mathbf{u}\|^2$.

Solution: From (4.5) and the given informa-

tion,

$$\mathbf{u}^T \begin{pmatrix} 4 & -2 & 2 \end{pmatrix} = 0 \quad (4.7)$$

$$\mathbf{u}^T \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = 0 \quad (4.8)$$

$$\mathbf{u}^T \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 24 \quad (4.9)$$

$$\Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 24 \end{pmatrix} \quad (4.10)$$

$$\Rightarrow \mathbf{u} = 4 \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \quad (4.11)$$

$$\Rightarrow \|\mathbf{u}\|^2 = 336 \quad (4.12)$$

5 LEAST SQUARES

5.1 Find the equation of the plane P containing the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad (5.1)$$

5.2 Show that the vector

$$\mathbf{y} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \quad (5.2)$$

lies outside P .

5.3 Find the point $\mathbf{w} \in P$ closest to \mathbf{y} .

5.4 Show that

$$\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 = \|\mathbf{y}\|^2 - \mathbf{w}^T \mathbf{X}^T \mathbf{y} \quad (5.3)$$

$$- \mathbf{y}^T \mathbf{A}\mathbf{w} + \mathbf{w}^T \mathbf{X}^T \mathbf{X}\mathbf{w} \quad (5.4)$$

5.5 Assuming 2×2 matrices and 2×1 vectors, show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{y} = \frac{\partial}{\partial \mathbf{w}} \mathbf{y}^T \mathbf{X}\mathbf{w} = \mathbf{y}^T \mathbf{X} \quad (5.5)$$

5.6 Show that

$$\frac{\partial}{\partial \mathbf{w}} \mathbf{w}^T \mathbf{X}^T \mathbf{X}\mathbf{w} = 2\mathbf{w}^T (\mathbf{X}^T \mathbf{X}) \quad (5.6)$$

5.7 Show that

$$\hat{\mathbf{w}} = \min_{\mathbf{w}} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 \quad (5.7)$$

$$= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \quad (5.8)$$

5.8 Let

$$\mathbf{X} = (\mathbf{x}_1 \quad \mathbf{x}_2). \quad (5.9)$$

from (5.1). Verify (5.7).