Algebraic Approach to School Geometry

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Contents

***Abstract*—This book provides an equation based approach to school geometry based on the NCERT textbooks from Class 6-12.**

1 The Right Angled Triangle

1. A right angled triangle looks like Fig. 1.0.1. with angles $\angle A$, $\angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle C$ which is defined to be 90°.

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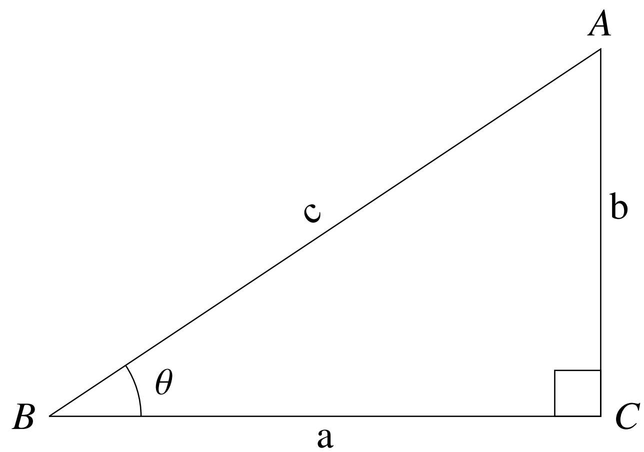

Fig. 1.0.1: Right Angled Triangle

2. For simplicity, let the greek letter $\theta = \angle B$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \frac{a}{c} & \cos\theta &= \frac{b}{c} \\ \tan\theta &= \frac{b}{a} & \cot\theta &= \frac{1}{\tan\theta} \\ \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} \end{aligned} \tag{1.0.2.1}$$

1.1 Sum of Angles

1. In Fig. 1.1.1, the sum of all the angles on the top or bottom side of the straight line XY is 180°.
2. In Fig. 1.1.1, the straight line making an angle of 90° to the side AC is said to be parallel to the side BC. Note there is an angle at A that is equal to θ. This is one property of parallel lines. Thus, $\angle YAZ = 90^{\circ}$.
3. Show that $\angle VAZ = 90^{\circ} - \theta$
Solution: Considering the line XAZ,
$$\theta + 90^{\circ} + \angle VAZ = 180^{\circ} \tag{1.1.3.1}$$
$$\Rightarrow \angle VAZ = 90^{\circ} - \theta \tag{1.1.3.2}$$
4. Show that $\angle BAC = 90^{\circ} - \theta$.
Solution: Consider the line VAB and and use the approach in the previous problem. Note that

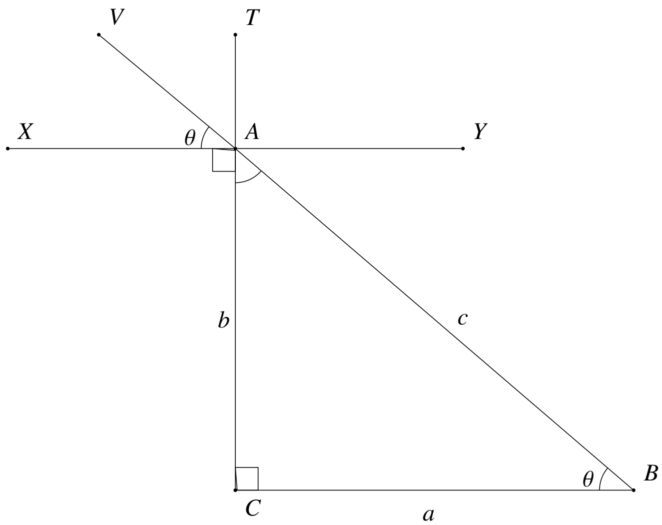

Fig. 1.1.1: Sum of angles of a triangle

this implies that $\angle VAZ = \angle BAC$. Such angles are known as vertically opposite angles.

5. Sum of the angles of a triangle is equal to 180°

1.2 Baudhayana Theorem

1. Using Fig. 1.0.1, show that

$$\cos\theta = \sin\left(90^\circ - \theta\right) \tag{1.2.1.1}$$

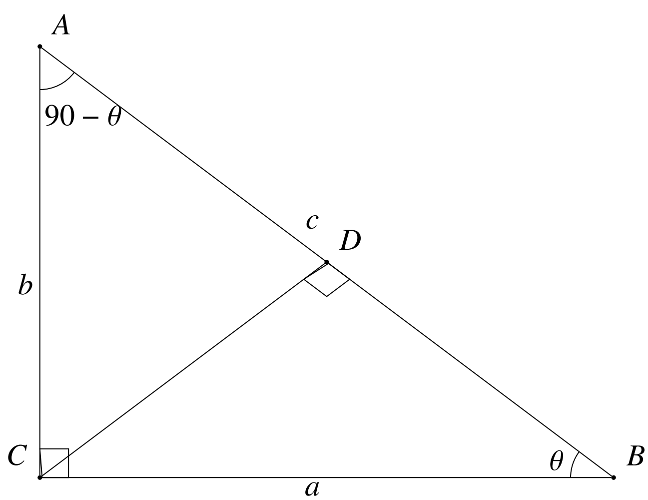

Fig. 1.2.1: Baudhayana Theorem

Solution: From Problem 1.1.4 and (1.0.2.1)

$$\cos\left(90^\circ - \theta\right) = \frac{b}{c} = \sin\theta \tag{1.2.1.2}$$

2. Using Fig. 1.2.1, show that

$$c = a\cos\theta + b\sin\theta \tag{1.2.2.1}$$

Solution: We observe that

$$BD = a\cos\theta \tag{1.2.2.2}$$
$$AD = b\cos(90-\theta) = b\sin\theta \quad \text{(From} \quad (1.1.4)) \tag{1.2.2.3}$$

Thus,

$$BD + AD = c = a\cos\theta + b\sin\theta \tag{1.2.2.4}$$

3. From (1.2.2.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.3.1}$$

Solution: Dividing both sides of (1.2.2.1) by c,

$$1 = \frac{a}{c}\cos\theta + \frac{b}{c}\sin\theta \tag{1.2.3.2}$$
$$\implies \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.0.2.1)) \tag{1.2.3.3}$$

4. Using (1.2.2.1), show that

$$c^2 = a^2 + b^2 \tag{1.2.4.1}$$

(1.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.2.2.1),

$$c = a\frac{a}{c} + b\frac{b}{c} \quad \text{(from} \quad (1.0.2.1)) \tag{1.2.4.2}$$
$$\implies c^2 = a^2 + b^2 \tag{1.2.4.3}$$

1.3 Area of a Triangle

1. The area of the rectangle $ACBD$ shown in Fig. 1.3.1 is defined as ab. Note that all the angles in the rectangles are 90°
2. The area of the two triangles constituting the rectangle is the same.
3. The area of the rectangle is the sum of the areas of the two triangles inside.
4. Show that the area of ΔABC is $\frac{ab}{2}$
Solution: From(1.3.3),

$$ar(ABCD) = ar(ACB) + ar(ADB) \tag{1.3.4.1}$$

Also from (1.3.2),

$$ar(ACB) = ar(ADB) \tag{1.3.4.2}$$

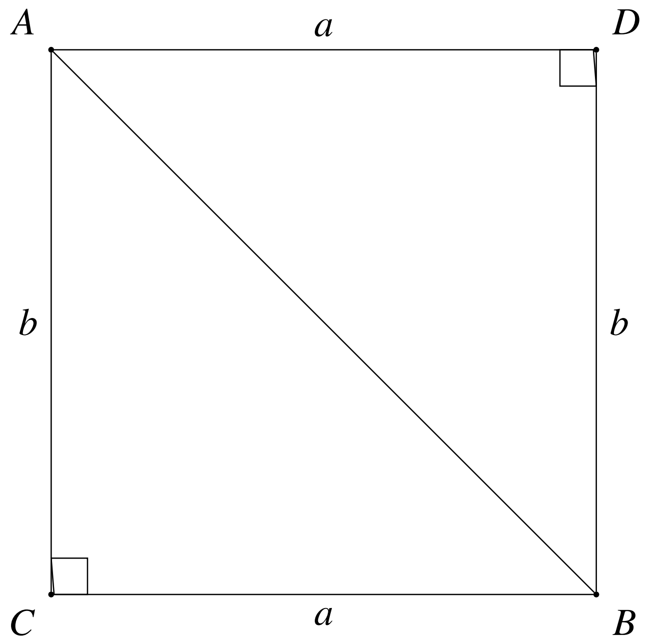

Fig. 1.3.1: Area of a Right Triangle

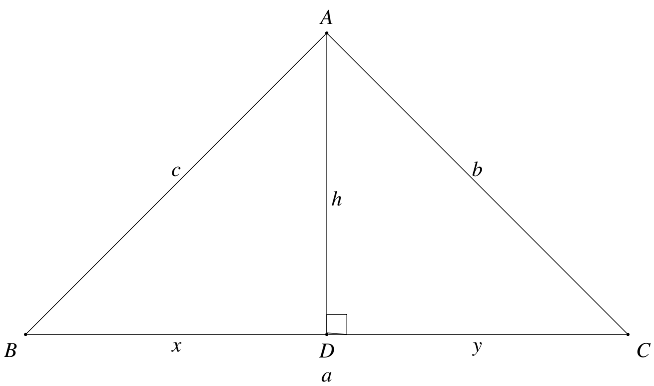

Fig. 1.3.4: Area of a Triangle

From (1.3.4.1) and (1.3.4.2),

$$2ar\,(ACB) = ar\,(ABCD) = ab\,(\text{from} \quad (1.3.1)) \tag{1.3.4.3}$$

$$\Rightarrow ar\,(ACB) = \frac{ab}{2} \tag{1.3.4.4}$$

5. Show that the area of ΔABC in Fig. 1.3.4 is $\frac{1}{2}ah$.
Solution: In Fig. 1.3.4,

$$ar\,(\Delta ADC) = \frac{1}{2}hy \tag{1.3.5.1}$$

$$ar\,(\Delta ADB) = \frac{1}{2}hx \tag{1.3.5.2}$$

Thus,

$$ar\,(\Delta ABC) = ar\,(\Delta ADC) + ar\,(\Delta ADB) \tag{1.3.5.3}$$

$$= \frac{1}{2}hy + \frac{1}{2}hx = \frac{1}{2}h\,(x+y) \tag{1.3.5.4}$$

$$= \frac{1}{2}ah \tag{1.3.5.5}$$

6. Show that the area of ΔABC in Fig. 1.3.4 is $\frac{1}{2}ab\sin C$.
Solution: We have

$$ar\,(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C)\,. \tag{1.3.6.1}$$

7. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.3.7.1}$$

Solution: Fig. 1.3.4 can be suitably modified to obtain

$$ar\,(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{s}bc\sin A = \frac{1}{2}ca\sin B \tag{1.3.7.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.3.7.3}$$

This is known as the sine formula.

8. In Fig. 1.3.8, show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.3.8.1}$$

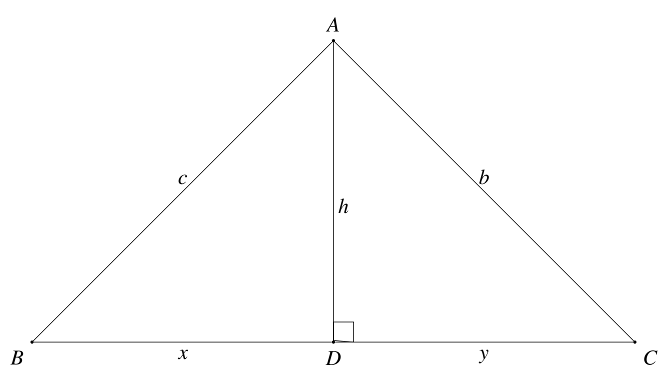

Fig. 1.3.8: The cosine formula

Solution: From the figure, the first of the following equations

$$a = b\cos C + c\cos B \tag{1.3.8.2}$$
$$b = c\cos A + a\cos C \tag{1.3.8.3}$$
$$c = b\cos A + a\cos B \tag{1.3.8.4}$$

is obvious and the other two can be similarly obtained. The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.3.8.5}$$

Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.3.8.6}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} \tag{1.3.8.7}$$

9. Find Hero's formula for the area of a triangle.
Solution: From (1.3.6), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.2.3.1))} \tag{1.3.9.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.3.8.1))} \tag{1.3.9.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2 - (a^2+b^2-c^2)} \tag{1.3.9.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.3.9.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.3.9.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.3.9.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.3.9.7}$$

in (1.3.9.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.3.9.8}$$

This is known as Hero's formula.

2 Medians of a Triangle

2.1 Median

1. The line AD in Fig. 2.1.1 that divides the side a in two equal halfs is known as the median.

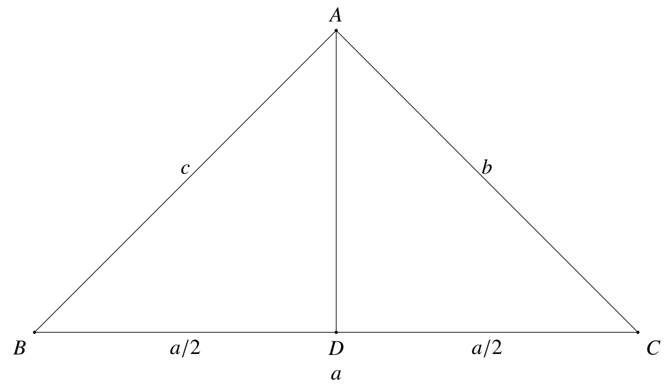

Fig. 2.1.1: Median of a Triangle

2. Show that the median AD in Fig. 2.1.1 divides ΔABC into triangles ADB and ADC that have equal area.
Solution: We have

$$ar(\Delta ADB) = \frac{1}{2}\frac{a}{2}c\sin B = \frac{1}{4}ac\sin B \tag{2.1.2.1}$$

$$ar(\Delta ADC) = \frac{1}{2}\frac{a}{2}b\sin C = \frac{1}{4}ab\sin C \tag{2.1.2.2}$$

Using the sine formula, $b\sin C = c\sin B$,

$$ar(\Delta ADB) = ar(\Delta ADC) \tag{2.1.2.3}$$

3. BE and CF are the medians in Fig. 2.1.3. Show that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.1.3.1}$$

Solution: Since BE and CF are the medians,

$$ar(\Delta BFC) = \frac{1}{2}ar(\Delta ABC) \tag{2.1.3.2}$$

$$ar(\Delta BEC) = \frac{1}{2}ar(\Delta ABC) \tag{2.1.3.3}$$

From the above, we infer that

$$ar(\Delta BFC) = ar(\Delta BEC) \tag{2.1.3.4}$$

4. We know that the median of a triangle divides it into two triangles with equal area. Using this

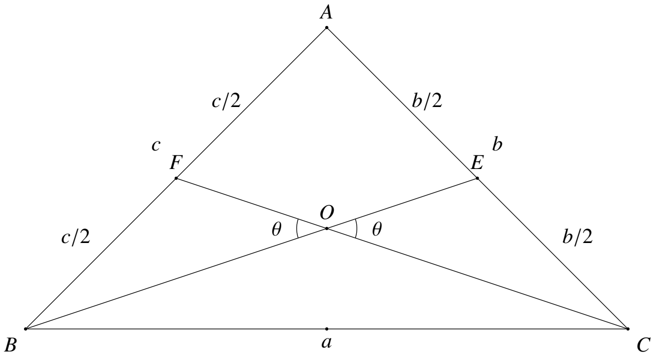

Fig. 2.1.3: O is the Intersection of Two Medians

result along with the sine formula for the area of a triangle in Fig. 2.1.4,

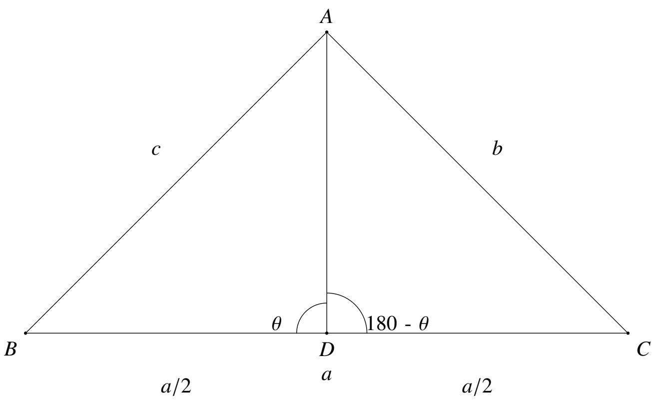

Fig. 2.1.4: $\sin\theta = \sin\left(180^{\circ} - \theta\right)$

$$\frac{1}{2}\frac{a}{2}AD\sin\theta = \frac{1}{2}\frac{a}{2}AD\sin\left(180^{\circ} - \theta\right) \quad (2.1.4.1)$$

$$\implies \sin\theta = \sin\left(180^{\circ} - \theta\right). \quad (2.1.4.2)$$

Note that our geometric definition of $\sin\theta$ holds only for $\theta < 90^{\circ}$. (2.1.4.2) allows us to extend this definition for $\angle ADC > 90^{\circ}$.

5. In Fig. 2.1.5, show that $EF = \frac{a}{2}$.
Solution: Using the cosine formula for ΔAEF,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (2.1.5.1)$$

$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \quad (2.1.5.2)$$

$$= \frac{a^2}{4} \quad (2.1.5.3)$$

$$\implies EF = \frac{a}{2} \quad (2.1.5.4)$$

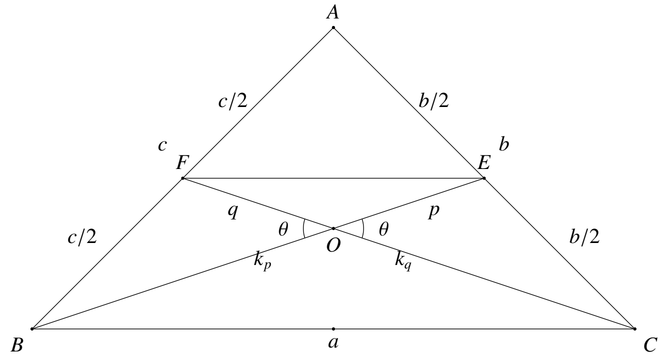

Fig. 2.1.5: Similar Triangles

6. The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.
7. Show that similar triangles have the same angles.
Solution: Use cosine formula and the proof is trivial.
8. Show that in Fig. 2.1.5, $EF \| BC$.
Solution: Since $\Delta AEF \sim \Delta ABC$, $\angle AEF = \angle ACB$. Hence the line $EF \| BC$
9. Show that $\triangle OEF \sim \triangle OEC$.
10. Show that

$$\frac{OB}{OE} = \frac{OC}{OF} = 2 \quad (2.1.10.1)$$

11. Show that the medians of a triangle meet at a point.

3 Angle and Perpendicular Bisectors

3.1 Angle Bisectors

1. In Fig. 3.1.1, OB divides the $\angle B$ into half, i.e.

$$\angle OBC = \angle OBA \quad (3.1.1.1)$$

OB is known as an angle bisector.
OB and OC are angle bisectors of angles B and C. OA is joined and OD, OF and OE are perpendiculars to sides a, b and c.
2. Show that $OD = OE = OF$. **Solution:** In Δs ODC and OEC,

$$OD = OC\sin\frac{C}{2} \quad (3.1.2.1)$$

$$OE = OC\sin\frac{C}{2} \quad (3.1.2.2)$$

$$\implies OD = OE. \quad (3.1.2.3)$$

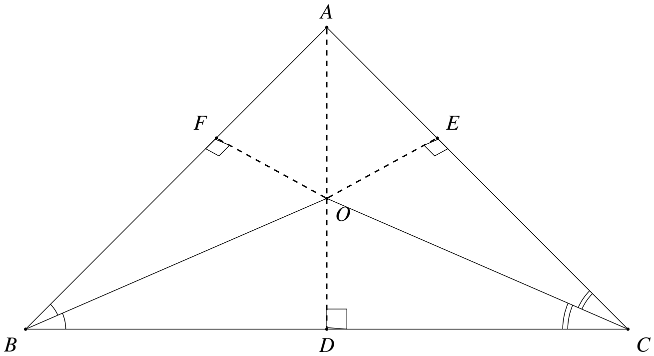

Fig. 3.1.1: Angle bisectors meet at a point

Similarly,

$$OD = OF. \tag{3.1.2.4}$$

3. Show that OA is the angle bisector of $\angle A$
Solution: In Δs OFA and OEA,

$$OF = OE \tag{3.1.3.1}$$
$$\Rightarrow OA \sin OAF = OA \sin OAE \tag{3.1.3.2}$$
$$\Rightarrow \sin OAF = \sin OAE \tag{3.1.3.3}$$
$$\Rightarrow \angle OAF = \angle OAE \tag{3.1.3.4}$$

which proves that OA bisects $\angle A$. *Conclusion:* The angle bisectors of a triangle meet at a point.

3.2 Congruent Triangles

1. Show that in Δs ODC and OEC, corresponding sides and angles are equal.
2. Note that Δs ODC and OEC are known as congruent triangles. To show that two triangles are congruent, it is sufficient to show that some angles and sides are equal.
3. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.
4. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.
5. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.
6. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

3.3 Perpendicular Bisectors

1. In Fig. 3.3.2, OD $\perp$ BC and $BD = DC$. OD is defined as the perpendicular bisector of BC.
2. In Fig. 3.3.2, show that $OA = OB = OC$.

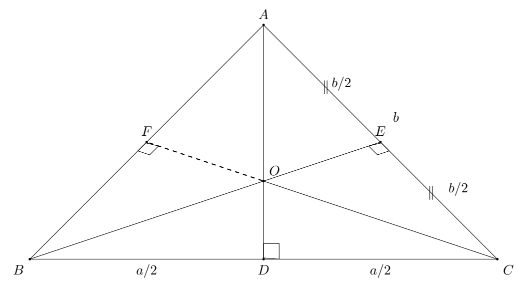

Fig. 3.3.2: Perpendicular bisectors meet at a point

Solution: In Δs ODB and ODC, using Budhayana's theorem,

$$\begin{aligned} OB^2 &= OD^2 + BD^2 \\ OC^2 &= OD^2 + DC^2 \end{aligned} \tag{3.3.2.1}$$

Since $BD = DC = \frac{a}{2}$, $OB = OC$. Similarly, it can be shown that $OA = OC$. Thus, $OA = OB = OC$.

3. In ΔAOB, $OA = OB$. Such a triangle is known as an isoceles triangle.
4. Show that $AF = BF$.
Solution: Trivial using Budhayana's theorem. This shows that OF is a perpendicular bisector of AB. *Conclusion:* The perpendicular bisectors of a triangle meet at a point.

3.4 Perpendiculars from Vertex to Opposite Side

1. In Fig. 3.4.1, $AD \perp BC$ and $BE \perp AC$. CF passes through O and meets AB at F. Show that

$$OE = c \cos A \cot C \tag{3.4.1.1}$$

Solution: In Δ s AEB and AEO,

$$AE = c \cos A \tag{3.4.1.2}$$
$$OE = AE \tan\left(90^{\circ} - C\right) (\because ADC \text{ is right angled}) \tag{3.4.1.3}$$
$$= AE \cot C \tag{3.4.1.4}$$

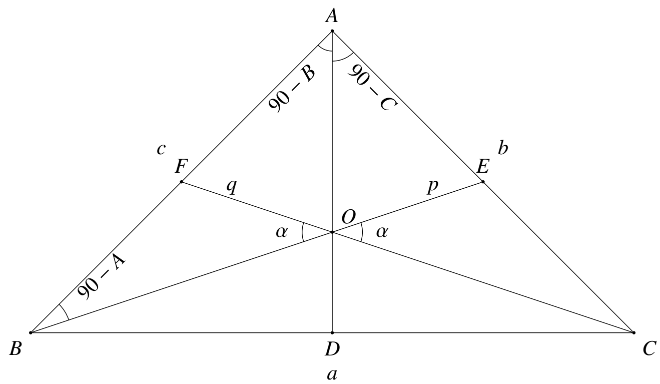

Fig. 3.4.1: Perpendiculars from vertex to opposite side meet at a point

From both the above, we get the desired result.

2. Show that $\alpha = A$.
Solution: In ΔOEC,

$$CE = a\cos C \, (\because BEC \text{ is right angled}) \tag{3.4.2.1}$$

Hence,

$$\begin{aligned}\tan\alpha &= \frac{CE}{OE} \\ &= \frac{a\cos C}{c\cos A\cot C} \\ &= \frac{a\cos C\sin C}{c\cos A\cos C} \\ &= \frac{a\sin C}{c\cos A} \\ &= \frac{c\sin A}{c\cos A}\left(\because \frac{a}{\sin A} = \frac{c}{\sin C}\right) \\ &= \tan A\end{aligned}$$

$$\Rightarrow \alpha = A \tag{3.4.2.2}$$

3. Show that $CF \perp AB$
Solution: Consider triangle OFB and the result of the previous problem. $\because$ the sum of the angles of a triangle is 180°, $\angle CFB = 90^{\circ}$. *Conclusion: The perperdiculars from the vertex of a triangle to the opposite side meet at a point.*

4 Triangle Inequalities

1. Show that if

$$\theta_1 < \theta_2, \quad \sin\theta_1 < \sin\theta_2. \tag{4.0.1.1}$$

Solution: Using Baudhayana's theorem in

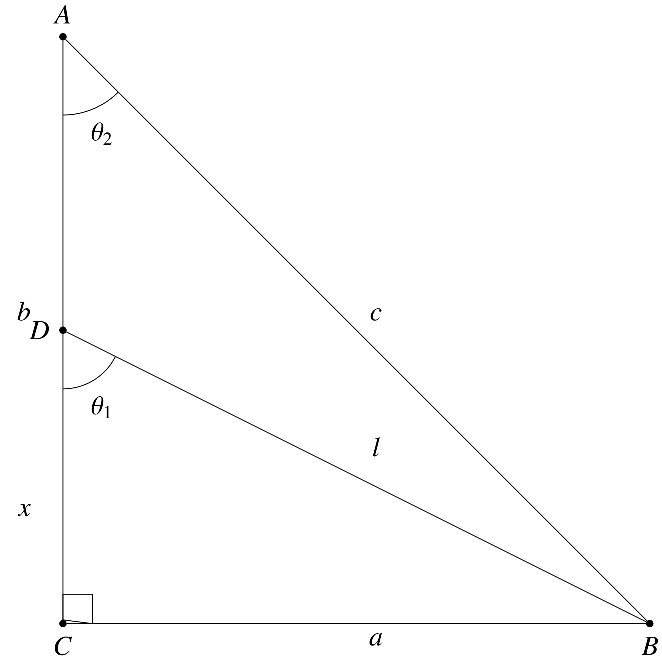

Fig. 4.0.1: $\theta_1 < \theta_2 \implies \sin\theta_1 < \sin\theta_2$.

$\triangle ABC$ and $\triangle DBC$

$$l^2 = x^2 + a^2 \tag{4.0.1.2}$$

$$c^2 = b^2 + a^2 \tag{4.0.1.3}$$

$$\implies c > l \because b > x. \tag{4.0.1.4}$$

Also,

$$a = c\sin\theta_1 = l\sin\theta_2 \tag{4.0.1.5}$$

$$\implies \frac{\sin\theta_1}{\sin\theta_2} = \frac{l}{c} < 1 \quad \text{from (4.0.1.4)} \tag{4.0.1.6}$$

$$\text{or, } \sin\theta_1 < \sin\theta_2 \tag{4.0.1.7}$$

2. Show that if

$$\theta_1 < \theta_2, \cos\theta_1 > \cos\theta_2. \tag{4.0.2.1}$$

3. Show that in any $\triangle ABC$, $\angle A > \angle B \implies a > b$.
Solution: Use (1.3.7.3) and (4.0.1.7)
4. Show that the sum of any two sides of a triangle is greater than the third side.
Solution: In Hero's formula in (1.3.9.8), all the factors inside the square root should be positive. Thus,

$$(s-a) > 0, (s-b) > 0\,(s-c) > 0 \tag{4.0.4.1}$$

$$(4.0.4.2)$$

$$(s-a) > 0 \implies \frac{a+b+c}{2} - a > 0 \quad (4.0.4.3)$$

$$\text{or, } b+c > a \quad (4.0.4.4)$$

Similarly, it can be shown that $a+b > c, c+a > b$.

5 Triangle Exercises

1. Angles opposite to equal sides of a triangle are equal.
   Solution: Using the sine formula in (1.3.7.3),
   $$\frac{\sin A}{a} = \frac{\sin B}{b} \quad (5.0.1.1)$$
   Thus, if $A = B$, $\sin A = \sin B \implies a = b$.
2. Sides opposite to equal angles of a triangle are equal.
3. Each angle of an equilateral triangle is of 60°.
4. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
5. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.
6. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.
7. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.
8. In an isosceles $\triangle ABC$ with $AB =$ AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.
9. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.
10. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.
11. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$
12. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.
13. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that
    a) $\triangle AOB \cong \triangle DOC$
    b) O is also the mid-point of BC.
14. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?
15. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that
    a) $\triangle ABD \cong \triangle BAC$
    b) $BD = AC$
    c) $\angle ABD = \angle BAC$.
16. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.
17. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$ (see Fig. 7.20). Show that:
    a) $\triangle APB \cong \triangle AQB$
    b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.
18. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.
19. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:
    a) $\triangle AMC \cong \triangle BMD$
    b) $\angle DBC$ is a right angle.
    c) $\triangle DBC \cong \triangle ACB$
    d) $CM = \frac{1}{2}AB$
20. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
    a) $OB = OC$
    b) AO bisects $\angle A$
21. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.
22. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.
23. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that
    a) $\triangle ABE \cong \triangle ACF$
    b) $AB = AC$, i.e., ABC is an isosceles triangle.
24. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$.

25. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
    a) $\triangle ABD \cong \triangle ACD$
    b) $\triangle ABP \cong \triangle ACP$
    c) AP bisects $\angle A$ as well as $\angle D$.
    d) AP is the perpendicular bisector of BC.
26. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that
    a) AD bisects BC
    b) AD bisects $\angle A$.
27. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:
    a) $\triangle ABM \cong \triangle PQN$
    b) $\triangle ABC \cong \triangle PQR$
28. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.
29. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.
30. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.
31. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.
32. Show that in a right angled triangle, the hypotenuse is the longest side.
33. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.
34. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.
35. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.
36. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.
37. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.

6 Circle

6.1 Chord of a Circle

1. Fig. 5.1.1 represents a circle. The points in the circle are at a distance r from the centre O. r is known as the radius.

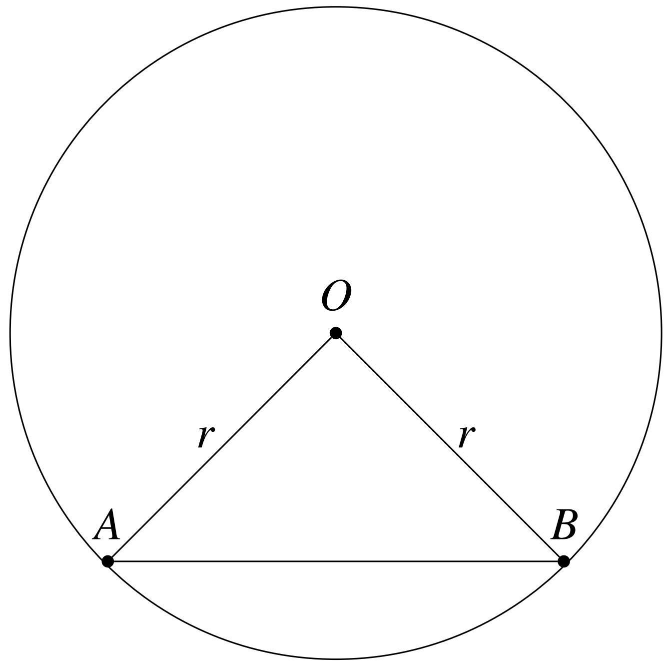

Fig. 6.1.1: Circle Definitions

6.2 Chords of a circle

1. In Fig. 5.1.1, A and B are points on the circle. The line AB is known as a chord of the circle.
2. In Fig. 5.2.2 Show that $\angle AOB = 2\angle ACB$.

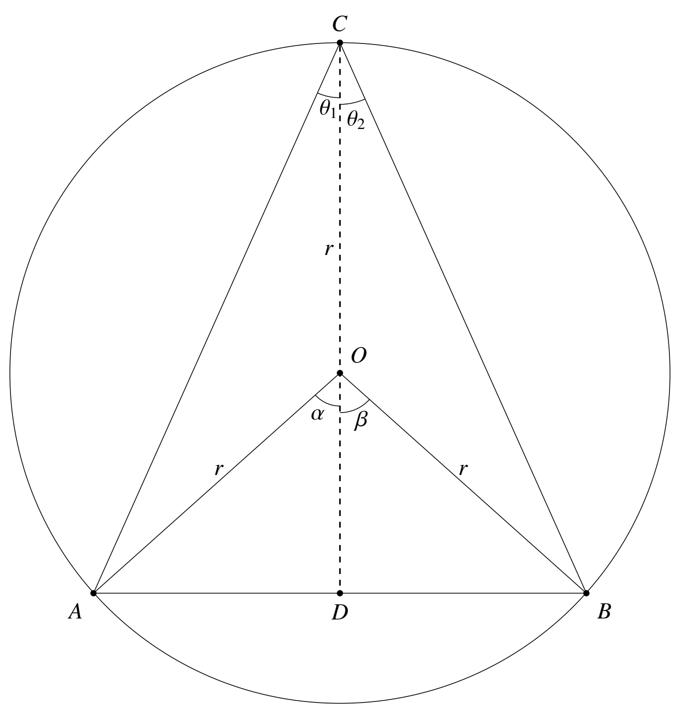

Fig. 6.2.2: Angle subtended by chord AB at the centre O is twice the angle subtended at P.

Solution: In Fig. 5.2.2, the triangeles OPA and

OPB are isosceles. Hence,

$$\angle OCA = \angle OAC = \theta_1 \tag{6.2.2.1}$$
$$\angle OCB = \angle OBC = \theta_2 \tag{6.2.2.2}$$

Also, α and β are exterior angles corresponding to the triangle AOC and BOC respectively. Hence

$$\alpha = 2\theta_1 \tag{6.2.2.3}$$
$$\beta = 2\theta_2 \tag{6.2.2.4}$$

Thus,

$$\angle AOB = \alpha + \beta \tag{6.2.2.5}$$
$$= 2(\theta_1 + \theta_2) \tag{6.2.2.6}$$
$$= 2\angle ACB \tag{6.2.2.7}$$

3. The diameter of a circle is the chord that divides the circle into two equal parts. In Fig. 5.2.4, AB is the diameter and passes through the centre O
4. In Fig. 5.2.4, show that $\angle APB = 90^\circ$.

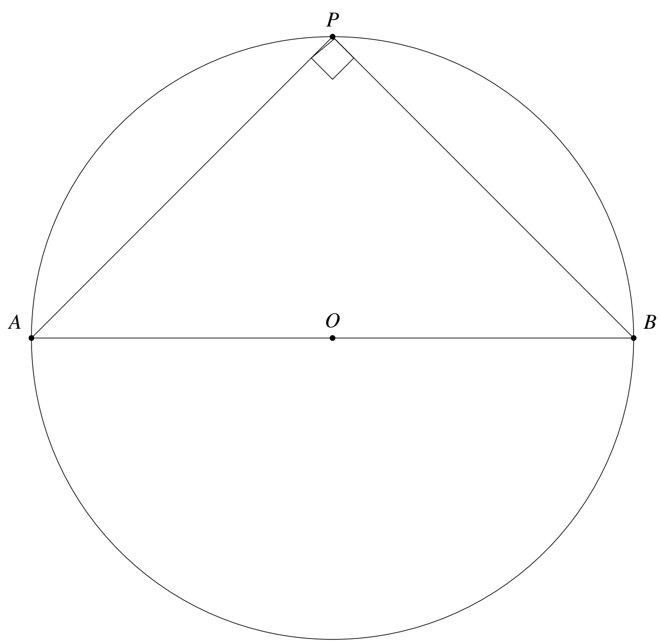

Fig. 6.2.4: Diameter of a circle.

5. In Fig. 5.2.5, show that

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \tag{6.2.5.1}$$

Solution: Use Problem 5.2.2.

6. In Fig. 5.2.5, show that the triangles PAB and PBD are similar

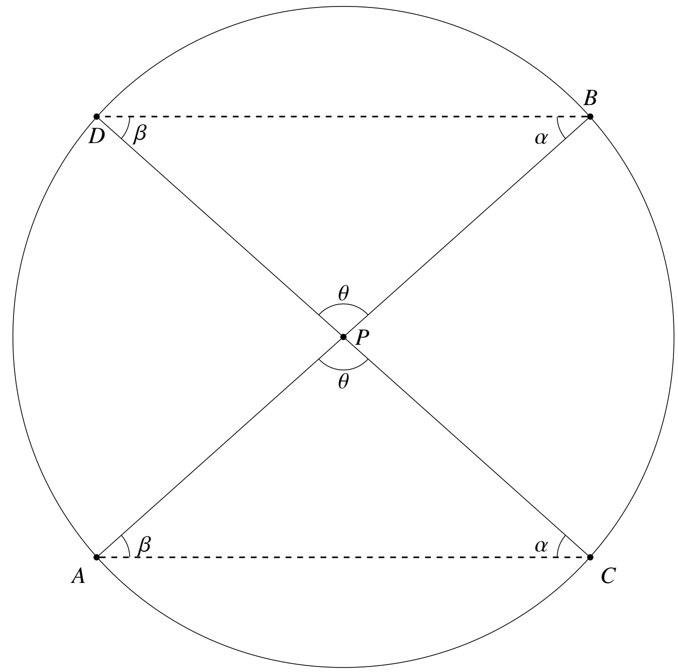

Fig. 6.2.5: $PA.PB = PC.PD$

Solution: Trivial using previous problem

7. In Fig. 5.2.5, show that

$$PA.PB = PC.PD \tag{6.2.7.1}$$

Solution: Since triangles PAC and PBD are similar,

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{6.2.7.2}$$
$$\implies PA.PB = PC.PD \tag{6.2.7.3}$$

8. Show that

$$\sin 0^\circ = 0 \tag{6.2.8.1}$$

Solution: Follows from (4.0.2.1).

9. Show that

$$\cos 0^\circ = 1 \tag{6.2.9.1}$$

10. The line PX in Fig. 5.2.11 touches the circle at exactly one point P. It is known as the tangent to the circle.
11. Show that $OP \perp PX$.
Solution: Without loss of generality, let $0 \leq \theta \leq 90^\circ$. Using the cosine formula in $\triangle OPP_n$,

$$(r + d_n)^2 > r^2, \tag{6.2.11.1}$$

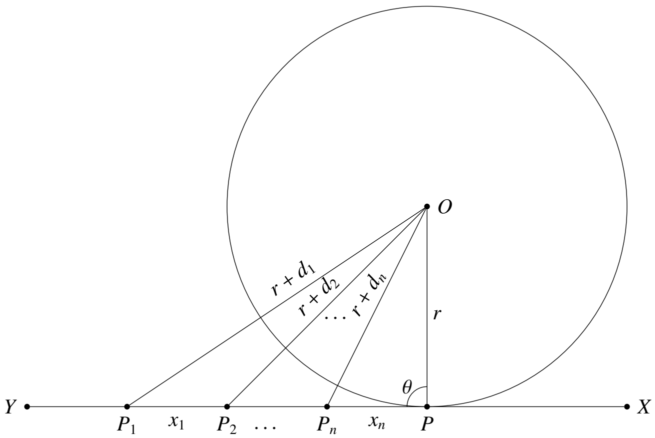

Fig. 6.2.11: Tangent to a Circle.

$$(r+d_n)^2 = r^2 + x_n^2 - 2rx_n\cos\theta > r^2 \tag{6.2.11.2}$$

$$\implies 0 < \cos\theta < \frac{x_n}{2r}, \tag{6.2.11.3}$$

where x_n can be made as small as we choose. Thus,

$$\cos\theta = 0 \implies \theta = 90^\circ. \tag{6.2.11.4}$$

12. In Fig. 5.2.12 show that

$$\angle PCA = \angle PBC \tag{6.2.12.1}$$

O is the centre of the circle and PC is the tangent.

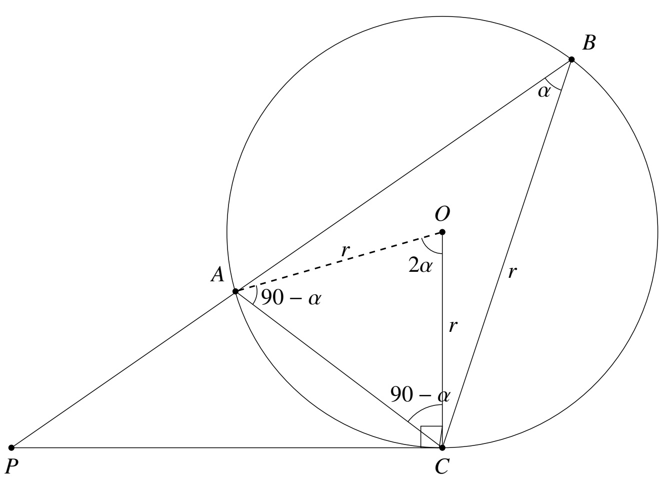

Fig. 6.2.12: $PA.PB = PC^2$.

Solution: Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

13. In Fig. 5.2.12, show that the triangles PAC and PBC are similar.

Solution: From the previous problem, it is obvious that corresponding angles of both triangles are equal. Hence they are similar.

14. Show that $PA.PB = PC^2$

Solution: Since $\Delta PAC \sim \Delta PBC$, their sides are in the same ratio. Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{6.2.14.1}$$

$$\Rightarrow PA.PB = PC^2 \tag{6.2.14.2}$$

15. Given that $PA.PB = PC^2$, show that PC is a tangent to the circle.
16. In Fig. 5.2.16, show that

$$PA.PB = PC.PD \tag{6.2.16.1}$$

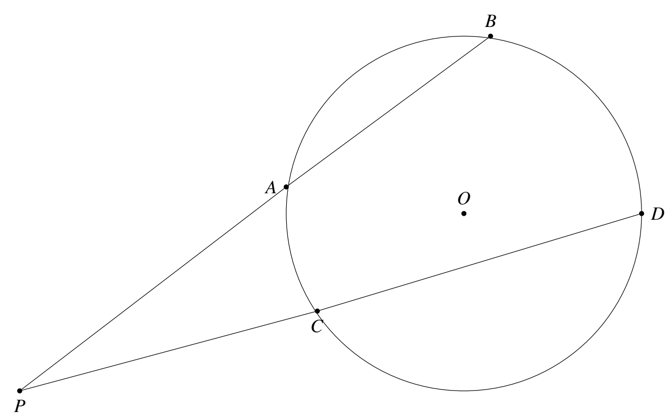

Fig. 6.2.16: $PA.PB = PC^2$.

Solution: Draw a tangent and use the previous problem.

7 Area of a Circle

7.1 The Regular Polygon

1. In Fig. 6.1.1, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.
2. The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{360^\circ}{n}$.
3. Show that the area of a regular polygon is given by

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} \tag{7.1.3.1}$$

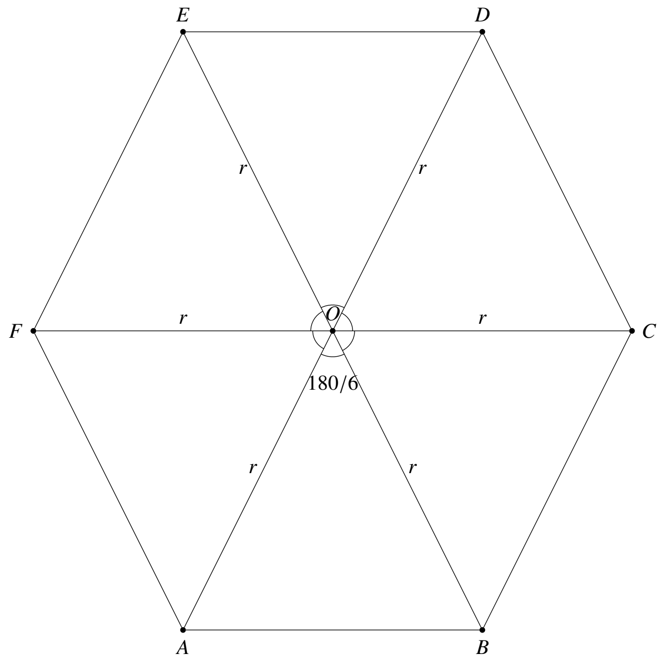

Fig. 7.1.1: Polygon Definition

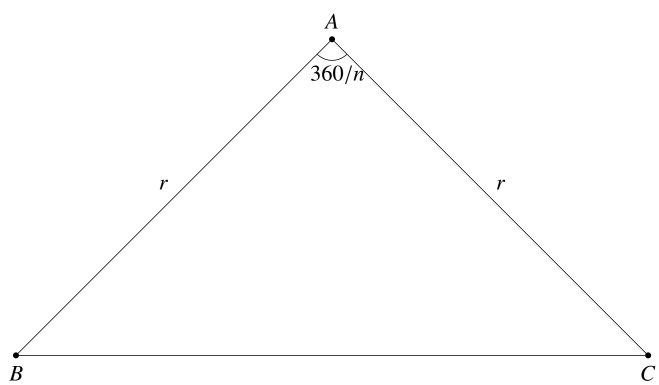

Fig. 7.1.3: Polygon Area

Solution: The triangle that forms the polygon of n sides is given in Fig. 6.1.3. Thus,

$$ar\,(polygon) = nar\,(\Delta ABC)$$
$$= \frac{n}{2}r^2 \sin\frac{360^\circ}{n} \tag{7.1.3.2}$$

4. Using Fig. 6.1.4, show that

$$\frac{n}{2}r^2 \sin\frac{360^\circ}{n} < \text{ area of circle } < nr^2 \tan\frac{180^\circ}{n} \tag{7.1.4.1}$$

The portion of the circle visible in Fig. 6.1.4 is defined to be a sector of the circle.

Solution: Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. 6.1.4, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$ar\,(\Delta OAB) = \frac{1}{2}r^2 \sin\frac{360^\circ}{n} \tag{7.1.4.2}$$
$$ar\,(\Delta OPQ) = 2 \times \frac{1}{2} \times r\tan\frac{360/n}{2} \times r \tag{7.1.4.3}$$
$$= r^2 \tan\frac{180^\circ}{n}, \tag{7.1.4.4}$$

we obtain (6.1.4.1).

5. Using Fig. 6.1.5, show that

$$\sin\theta_1 = \sin(\theta_1 + \theta_2)\cos\theta_2 - \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{7.1.5.1}$$

Solution: The following equations can be obtained from the figure using the forumula for

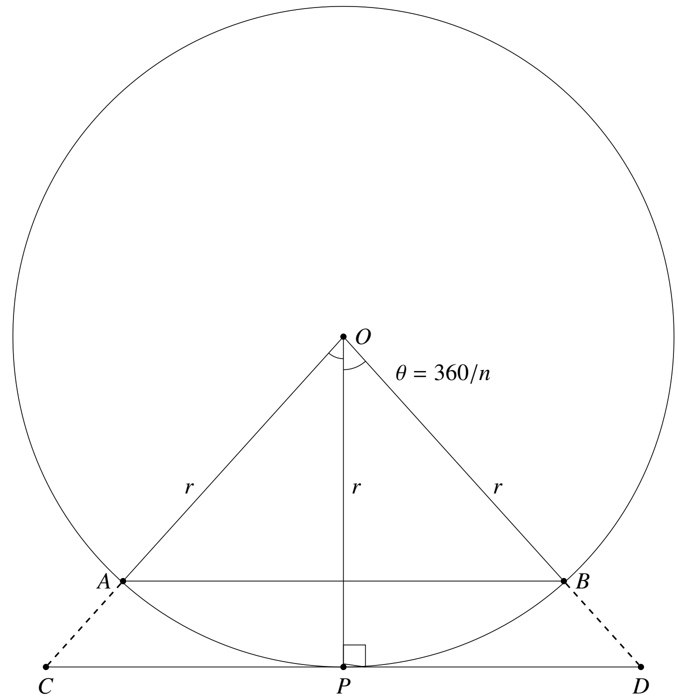

Fig. 7.1.4: Circle Area in between Area of Two Polygons

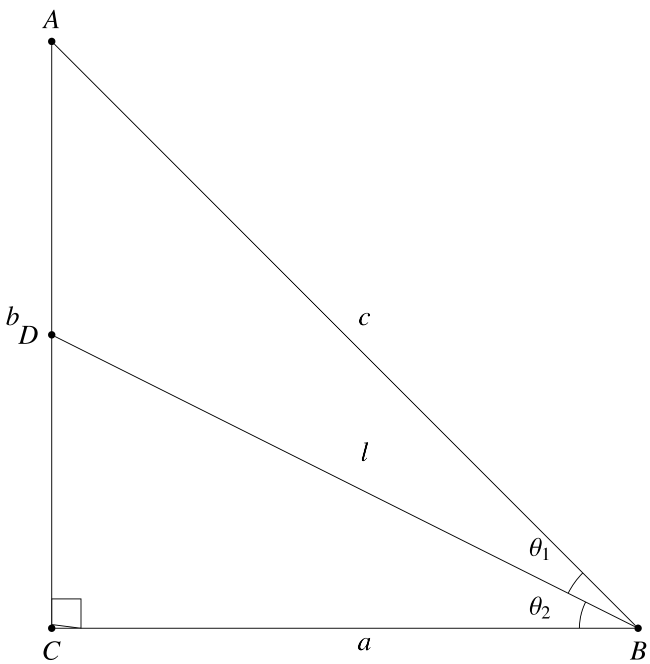

Fig. 7.1.5: $\sin 2\theta = 2\sin\theta\cos\theta$

the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1+\theta_2) \tag{7.1.5.2}$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{7.1.5.3}$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{7.1.5.4}$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{7.1.5.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{7.1.5.6}$$

$$\Rightarrow \sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \tag{7.1.5.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\Rightarrow \sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{7.1.5.8}$$

resulting in

$$\Rightarrow \sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{7.1.5.9}$$

6. Prove the following identities

a)

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{7.1.6.1}$$

b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{7.1.6.2}$$

Solution: In (6.1.5.1), let

$$\begin{aligned}\theta_1+\theta_2 &= \alpha\\ \theta_2 &= \beta\end{aligned} \tag{7.1.6.3}$$

This gives (6.1.6.1). In (6.1.6.1), replace α by $90^\circ - \alpha$. This results in

$$\begin{aligned}&\sin\left(90^\circ - \alpha - \beta\right)\\ &= \sin\left(90^\circ-\alpha\right)\cos\beta - \cos\left(90^\circ-\alpha\right)\sin\beta\\ &\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned} \tag{7.1.6.4}$$

7. Using (6.1.5.1) and (6.1.6.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{7.1.7.1}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{7.1.7.2}$$

Solution: From (6.1.5.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{7.1.7.3}$$

Using (6.1.6.2) in the above,

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (&\cos\theta_1\cos\theta_2\\ &- \sin\theta_1\sin\theta_2)\sin\theta_2\end{aligned} \tag{7.1.7.4}$$

which can be expressed as

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1&+\cos\theta_1\cos\theta_2\sin\theta_2\\ &- \sin\theta_1\sin^2\theta_2\end{aligned} \tag{7.1.7.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{7.1.7.6}$$

we obtain

$$\begin{aligned}\sin(\theta_1+\theta_2)\cos\theta_2 = &\cos\theta_1\cos\theta_2\sin\theta_2\\ &+ \sin\theta_1\cos^2\theta_2\end{aligned} \tag{7.1.7.7}$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \tag{7.1.7.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (6.1.7.2) can also be proved.

8. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{7.1.8.1}$$

9. Show that

$$\cos^2\frac{180^\circ}{n} < \frac{\text{area of circle}}{nr^2\tan\frac{180^\circ}{n}} < 1 \tag{7.1.9.1}$$

Solution: From (6.1.4.1) and (6.1.8.1),

$$\begin{aligned} \frac{n}{2}r^2\sin\frac{360^\circ}{n} &< \text{ area of circle} \\ &< nr^2\tan\frac{180^\circ}{n} \\ \Rightarrow nr^2\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} &< \text{ area of circle} \\ &< nr^2\tan\frac{180^\circ}{n} \end{aligned} \tag{7.1.9.2}$$

10. Show that for large values of n

$$\cos^2\frac{180^\circ}{n} = 1 \tag{7.1.10.1}$$

Solution: Follows from previous problem.

11. The previous result can be expressed as

$$\lim_{n\to\infty}\cos^2\frac{180^\circ}{n} = 1 \tag{7.1.11.1}$$

12. Show that

$$\text{area of circle } = r^2\lim_{n\to\infty} n\tan\frac{180^\circ}{n} \tag{7.1.12.1}$$

13.

$$\pi = \lim_{n\to\infty} n\tan\frac{180^\circ}{n} \tag{7.1.13.1}$$

Thus, the area of a circle is πr^2.

14. The radian is a unit of angle defined by

$$1 \text{ radian} = \frac{360^\circ}{2\pi} \tag{7.1.14.1}$$

15. Show that the circumference of a circle is $2\pi r$.
16. Show that the area of a sector with angle θ in radians is $\frac{1}{2}r^2\theta$.
17. Show that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{7.1.17.1}$$