School Mathematics through Python

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Contents

***Abstract*—This manual shows how to generate figures encountered in high school geometry using python. The process provides simple applications of coordinate geometry.**

1 Line

Problem 1. *Let*

$$A = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, C = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \tag{1}$$

Draw ΔABC.

Solution: The following code yields the desired plot in Fig. 1

```
#This program draws the triangle
    ABC
import numpy as np
import matplotlib.pyplot as plt

def line_coeff(A,B):
        p = np.zeros((2,1))
        p[0] = (A[1]-B[1])/(A[0]-B
            [0])
        p[1] = (A[0]*B[1]-A[1]*B
            [0])/(A[0]-B[0])
        return p

A = np.matrix('-2;-2')
B = np.matrix('1;3')
C = np.matrix('4;-1')

x = np.linspace(np.asscalar(A[0]),
    np.asscalar(B[0]),50)
p = line_coeff(A,B)
y = p[0]*x + p[1]
plt.plot(x,y,label='$5x-3y+4=0$')

plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 + 0.1), A[1] *
    (1 - 0.1) , 'A')

x = np.linspace(np.asscalar(B[0]),
    np.asscalar(C[0]),50)
p = line_coeff(B,C)
y = p[0]*x + p[1]
plt.plot(x,y,label='$4x+3y-13=0$')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 - 0.2), B[1] *
    (1) , 'B')

x = np.linspace(np.asscalar(C[0]),
    np.asscalar(A[0]),50)
p = line_coeff(C,A)
y = p[0]*x + p[1]
plt.plot(x,y,label='$x-6y-10=0$')
plt.plot(C[0], C[1], 'o')
plt.text(C[0] * (1 + 0.03), C[1] *
```

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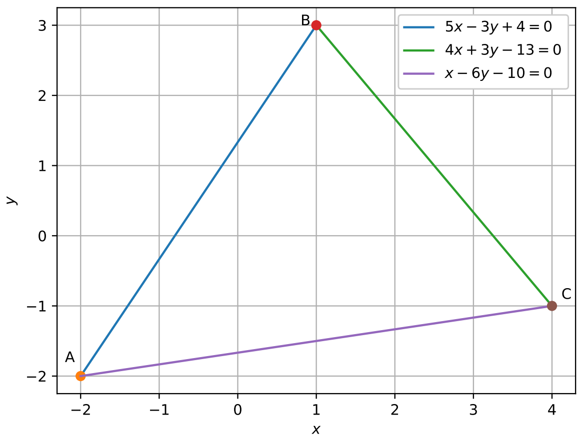

Fig. 1

```
    (1 - 0.1) , 'C')




plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend(loc='best')
#plt.savefig('../figs/triangle.eps
   ')
plt.show()
```

Problem 2. *Consider the line AB with*

$$A = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \tag{2}$$

If AB is expressed by the equation

$$y = p_0 x + p_1, \tag{3}$$

find p_0 and p_1.

Solution: Let

$$A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}, B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}, \tag{4}$$

The equation of AB is given by

$$\frac{y - A_1}{x - A_0} = \frac{A_1 - B_1}{A_0 - B_0} \tag{5}$$

$$\implies y = \frac{A_1 - B_1}{A_0 - B_0} x + A_1 - A_0 \frac{A_1 - B_1}{A_0 - B_0} \tag{6}$$

$$= \frac{A_1 - B_1}{A_0 - B_0} x + \frac{A_0 B_1 - A_1 B_0}{A_0 - B_0} \tag{7}$$

after some algebra. Thus,

$$p_0 = \frac{A_1 - B_1}{A_0 - B_0} \tag{8}$$

$$p_1 = \frac{A_0 B_1 - A_1 B_0}{A_0 - B_0} \tag{9}$$

The following python code computes the numerical values and the equation for AB is

$$y = 1.67x + 1.33 \tag{10}$$

```
import numpy as np
import matplotlib.pyplot as plt


A = np. matrix('-2;-2')
B = np. matrix('1;3')

p = np.zeros((2,1))
p[0] = (A[1]-B[1])/(A[0]-B[0])
p[1] = (A[0]*B[1]- A[1]*B[0])/(A
    [0]-B[0])

print (p)
```

Problem 3. *Let*

$$C = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \tag{11}$$

Find the equations of BC and CA

2 Medians of a Triangle

Problem 4. *Find the coordinates of D, E and F of the mid points of AB, BC and CA respectively for ΔABC.*

Solution: The coordinates of the mid points are given by

$$D = \frac{B + C}{2}, E = \frac{C + A}{2}, F = \frac{A + B}{2} \tag{12}$$

The following code computes the values resulting in

$$D = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}, E = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}, F = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \tag{13}$$

```
#This program calculates the mid
   point between
#any two coordinates
import numpy as np
import matplotlib.pyplot as plt

def mid_pt(B,C):
                D = (B+C)/2
                return D

A = np. matrix('-2;-2')
B = np. matrix('1;3')
C = np. matrix('4;-1')

print(mid_pt(B,C))
print(mid_pt(C,A))
print(mid_pt(A,B))
```

Problem 5. *Find the equations of AD, BE and CF. These lines are the* medians *of ΔABC*

Solution: Use the code in Problem 2.

Problem 6. *Find the point of intersection of AD and CF.*

Solution: Let the respective equations be

$$y = p_0 x + p_1 \text{ and} \tag{14}$$

$$y = q_0 x + q_1 \tag{15}$$

This can be written as the matrix equation

$$\begin{pmatrix} p_0 & -1 \\ q_0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \tag{16}$$

The following code yields the point of intersection

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{17}$$

```
#This program calculates the
#intersection of AD and CF
import numpy as np
import matplotlib.pyplot as plt

def mid_pt(B,C):
                D = (B+C)/2
                return D

def line_intersect(p,q):
        P = np.matrix([[p
           [0][0],-1],[q
           [0][0],-1]])
        c = -np.matrix([[p
           [1][0]],[q[1][0]]])
        return np.linalg.inv(P)*c



def line_coeff(A,B):
        p = np.zeros((2,1))
        p[0] = (A[1]-B[1])/(A[0]-B
           [0])
        p[1] = (A[0]*B[1]-A[1]*B
           [0])/(A[0]-B[0])
        return p

A = np.matrix('-2;-2')
B = np.matrix('1;3')
C = np.matrix('4;-1')

D = mid_pt(B,C)
F = mid_pt(A,B)

p = line_coeff(A,D)
q = line_coeff(C,F)

print(line_intersect(p,q))
```

Problem 7. *Using the code in Problem 6, verify that G is the point of intersection of BE, CF as well as AD, BE. G is known as the* centroid *of ΔABC.*

Problem 8. *Graphically show that the medians of ΔABC meet at the centroid.*

Problem 9. *Verify that*

$$G = \frac{A + B + C}{3} \tag{18}$$

3 Altitudes of a Triangle

Definition 10. *In ΔABC, Let P be a point on BC such that $AP \perp BC$. Then AP is defined to be an* altitude *of ΔABC.*

Problem 11. *Find the equation of AP.*

Solution: Let the equation for BC and AP be

$$y = p_0 x + p_1 \tag{19}$$
$$y = q_0 x + q_1 \tag{20}$$

respectively. Since, $AP \perp BC$,

$$p_0 q_0 = -1 \tag{21}$$

The equation for AP is then obtained as

$$y - A_1 = q_0 (x - A_0) \tag{22}$$
$$\implies y = q_0 x + A_1 - q_0 A_0 \tag{23}$$

From the following python code, AP can be expressed as

$$y = 0.75x - 0.5 \tag{24}$$

```
#This program calculates the
#equation of the altitude
import numpy as np
import matplotlib.pyplot as plt

def line_coeff(A,B):
        p = np.zeros((2,1))
        p[0] = (A[1]-B[1])/(A[0]-B
            [0])
        p[1] = (A[0]*B[1]-A[1]*B
            [0])/(A[0]-B[0])
        return p

def alt_coeff(p,A):
        q = np.zeros((2,1))
        q[0] = -1/p[0]
        q[1] = A[1] - q[0]*A[0]
        return q

A = np. matrix('-2;-2')
B = np. matrix('1;3')
C = np. matrix('4;-1')

p = line_coeff(B,C)
q = alt_coeff(p,A)
print (q)
```

Problem 12. *Find the equations of the altitudes BQ and CR.*

Solution: Using the code in Problem 11, the respective equations are

$$y = -6x + 9 \tag{25}$$
$$y = -0.6x + 1.4 \tag{26}$$

Problem 13. *Find the point of intersection of AP and BQ.*

Solution: Using the code in Problem 6, the desired point of intersection is

$$H = \begin{pmatrix} 1.407 \\ 0.56 \end{pmatrix} \tag{27}$$

Interestingly, BQ and CR also intersect at the same point. Thus, the altitudes of a triangle meet at a single point known as the *orthocentre*

Problem 14. *Find P, Q, R.*

Solution: P is the intersection of AP and BC. Thus, the code in Problem 6 can be used to find P. The desired coordinates are

$$P = \begin{pmatrix} 2.32 \\ 1.24 \end{pmatrix}, Q = \begin{pmatrix} 1.73 \\ -1.38 \end{pmatrix}, R = \begin{pmatrix} 0.03 \\ 1.38 \end{pmatrix} \tag{28}$$

Problem 15. *Draw AP, BQ and CR and verify that they meet at H.*

4 Angle Bisectors of a Triangle

Definition 16. *In ΔABC, let U be a point on BC such that $\angle BAU = \angle CAU$. Then AU is known as the* angle bisector.

Problem 17. *Find the length of AB, BC and CA*

Solution: The length of CA is given by

$$CA = \sqrt{(C_0 - A_0)^2 + (C_1 - A_1)^2} \tag{29}$$

The following code calculates the respective values as

$$AB = 5.83, BC = 5, CA = 6.08 \tag{30}$$

```
#This program calculates the
   distance between
#two points
import numpy as np
import matplotlib.pyplot as plt


def side_length(A,B):
        return np.sqrt((A[0]-B[0])
            **2 + (A[1]-B[1])**2)
```

```
A = np.  matrix('-2;-2')
B = np.  matrix('1;3')
C = np.  matrix('4;-1')

 print (side_length(A,B))
 print (side_length(B,C))
 print (side_length(C,A))
```

Problem 18. *If AU, BV and CW are the angle bisectors, find the coordinates of U, V and W.*

Solution: Using the section formula,

$$W = \frac{AW.B + WB.A}{AW + WB} = \frac{\frac{AW}{WB}.B + A}{\frac{AW}{WB} + 1} \tag{31}$$

$$= \frac{\frac{CA}{BC}.B + A}{\frac{CA}{BC} + 1} \tag{32}$$

$$= \frac{CA \times B + BC \times A}{BC + CA} \tag{33}$$

$$= \frac{a \times A + b \times B}{a + b} \tag{34}$$

where $a = BC, b = CA$, since the angle bisector has the property that

$$\frac{AW}{WB} = \frac{CA}{AB} \tag{35}$$

The following code computes the coordinates as

$$U = \begin{pmatrix} 2.47 \\ 1.04 \end{pmatrix}, V = \begin{pmatrix} 1.23 \\ -1.46 \end{pmatrix} \approx \begin{pmatrix} -0.35 \\ 0.75 \end{pmatrix} \tag{36}$$

```
#This program calculates point
#where the angle bisector meets
   the
#opposite side

import numpy as np
import matplotlib.pyplot as plt


def angle_bisect_coord(b,c,B,C):
        return np.multiply((np.
           multiply(b,B)+np.
           multiply(c,C)),1/(b+c))

A = np.  matrix('-2;-2')
B = np.  matrix('1;3')
C = np.  matrix('4;-1')

a = side_length(B,C)
b = side_length(C,A)
c = side_length(A,B)

U = angle_bisect_coord(b,c,B,C)
V = angle_bisect_coord(c,a,C,A)
W = angle_bisect_coord(a,b,A,B)

print (U)
print (V)
print (W)
```

Problem 19. *Find the intersection of AU and BV.*

Solution: Using the code in Problem 6, the desired point of intersection is

$$I = \begin{pmatrix} 1.15 \\ 0.14 \end{pmatrix} \tag{37}$$

It is easy to verify that even BV and CW meet at the same point. I is known as the *incentre* of ΔABC.

Problem 20. *Draw AU, BV and CW and verify that they meet at a point I.*

Problem 21. *Verify that*

$$I = \frac{BC.A + CA.B + AB.C}{AB + BC + CA} \tag{38}$$

Problem 22. *Let the perpendiculars from I to AB, BC and CA be IX, IY, IZ. Verify that*

$$IX = IY = IZ = r \tag{39}$$

r is known as the inradius *of ΔABC.*

Solution: The distance of a point (a, b) from the line $y = p_0x + p_1$ is given by

$$\frac{|ap_0 - b + p_1|}{\sqrt{p_0^2 + 1}} \tag{40}$$

The following code computes IX.

```
#This program calculates the
   inradius

import numpy as np
import matplotlib.pyplot as plt

def line_coeff(A,B):
        p = np.zeros((2,1))
        p[0] = (A[1]-B[1])/(A[0]-B
           [0])
```

```
        p[1] = (A[0]*B[1]-A[1]*B
            [0])/(A[0]-B[0])
        return p

def line_dist(I,p):
        return np.abs((I[0]*p[0]-I
            [1]+p[1])/(np.sqrt(p
            [0]**2+1)))

I = np. matrix('1.15;0.14')
A = np. matrix('-2;-2')
B = np. matrix('1;3')
C = np. matrix('4;-1')

AB = line_coeff(A,B)
BC = line_coeff(B,C)
CA = line_coeff(C,A)


print(line_dist(I,AB))
print(line_dist(I,BC))
print(line_dist(I,CA))
```

5 Circle

Definition 23. *From Problem 22, it is obvious that a circle with centre at I and radius r passing through X, Y, Z can be drawn. The* incircle *is defined as the circle with centre at the incentre I and radius equal to the inradius.*

Problem 24. *Obtain the equation of the incircle of ΔABC and draw it.*

Solution: Letting $I = (a, b)$, the equation of the incircle is given by

$$(x-a)^2 + (y-b)^2 = r^2, \tag{41}$$

where r is the inradius. The following code plots this circle in Fig. 24

```
#This program plots the incircle
import numpy as np
import matplotlib.pyplot as plt

r=1.6
a=1.15
b=0.14
x=np.linspace(a-r,a+r,1000)
y1=b+np.sqrt((r)**2-((x-a)**2))
y2=b-np.sqrt((r)**2-((x-a)**2))
plt.plot(x,y1)
plt.plot(x,y2)
plt.grid()
plt.axis("equal")
#plt.savefig('../figs/incircle.eps
    ')
plt.show()
```

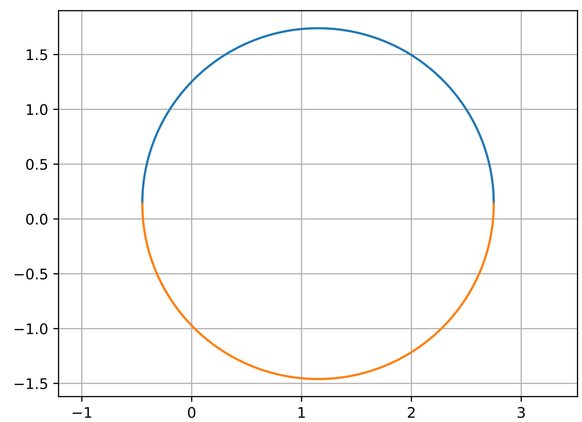

Fig. 24

6 Tangent and Derivative

Definition 25. *A line that meets the circle at exactly one point is known as a* tangent *to the circle.*

Problem 26. *Draw ΔABC and its incircle in the same graph and verify that the lines AB, BC, CA are tangents to the incircle*

Solution: Fig. 26 can be drawn using the codes in Problems 1 and 24. It is obvious from the figure that AB, BC and CA are tangents to the incircle.

Problem 27. *Let the equation of AB be*

$$p_0 x + p_1 y + p_2 = 0 \tag{42}$$

and the incircle be

$$(x-a)^2 + (y-b)^2 = r^2, \tag{43}$$

Verify that

$$\left[p_0\left(p_2 + bp_1\right) - ap_1^2\right]^2 = \left(p_0^2 + p_1^2\right)\left[p_1^2 a^2 + \left(p_2 + bp_1\right)^2 - p_1^2 r^2\right] \tag{44}$$

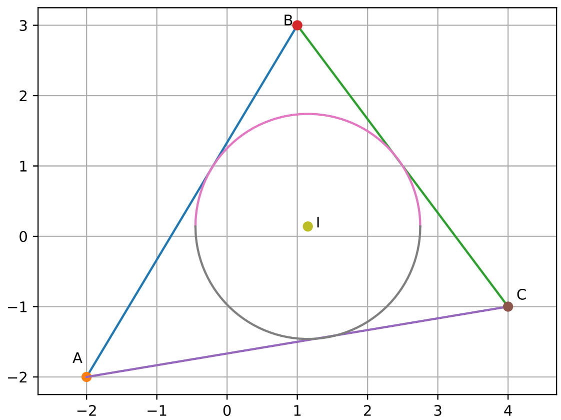

Fig. 26

and the point of contact

$$X = \begin{pmatrix} \frac{ap_1^2 - p_0(p_2+bp_1)}{p_0^2+p_1^2} \\ -\frac{p_2}{p_1} + \frac{p_0}{p_1}\frac{ap_1^2 - p_0(p_2+bp_1)}{p_0^2+p_1^2} \end{pmatrix} \tag{45}$$

Solution: The following code computes the point of contact

```
#This program computes the point
    of contact between a circle
#and its tangent

import numpy as np
import matplotlib.pyplot as plt

def line_coeff(A,B):
        p = np.zeros((3,1))
        p[0] = A[1]-B[1]
        p[1] = B[0]-A[0]
        p[2] = A[0]*B[1]- A[1]*B
            [0]
        return p

A = np.matrix('-2;-2')
B = np.matrix('1;3')
p = line_coeff(A,B)
r=1.6
a=1.15
b=0.14

print((p[0]*(p[2]+b*p[1])-a*p
    [1]**2)**2)
print((p[0]**2+p[1]**2)*(p[1]**2*a
    **2+(p[2]+b*p[1])**2-p[1]**2*r
    **2))

X = np.zeros((2,1))
X[0] = -(p[0]*(p[2]+b*p[1])-a*p
    [1]**2)/(p[0]**2+p[1]**2)
X[1] = -(p[2]+p[0]*X[0])/p[1]
print(X)
```

Problem 28. *Verify that*

$$AX = AZ \tag{46}$$
$$BX = BY \tag{47}$$
$$CY = CZ \tag{48}$$

Problem 29. *Devise a method for calculating the slope of the tangent at X, given the equation of the circle and the point X.*

Solution: In Fig. 29, it can be seen that the tangent at X has the same slope as the chord BC. From the equation of the circle,

$$(p_1 - a)^2 + (q_1 - b)^2 = r^2 \tag{49}$$
$$(p_2 - a)^2 + (q_2 - b)^2 = r^2 \tag{50}$$

which, after simplification, leads to the slope of the chord as

$$\frac{q_1 - q_2}{p_1 - p_2} = -\frac{p_1 + p_2 - 2a}{q_1 + q_2 - 2b} \tag{51}$$
$$\Rightarrow \frac{\Delta y}{\Delta x} = -\frac{p_1 + p_2 - 2a}{q_1 + q_2 - 2b} \tag{52}$$

If we keep choosing smaller chords parallel to BC, p_1 and p_2 come closer while q_1 and q_2 come closer, without any change in the slope on the LHS. The limiting behaviour results in $p_1 = p_2 = p$ and $q_1 = q_2 = q$. This results in an expression for the slope of the tangent at X

$$\frac{dy}{dx} = -\frac{p - a}{q - b} \tag{53}$$

where

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}. \tag{54}$$

Problem 30. *Verify that the derivative of the circle at X is actually the slope of AB.*

Solution: The verificaton is done by the following program.

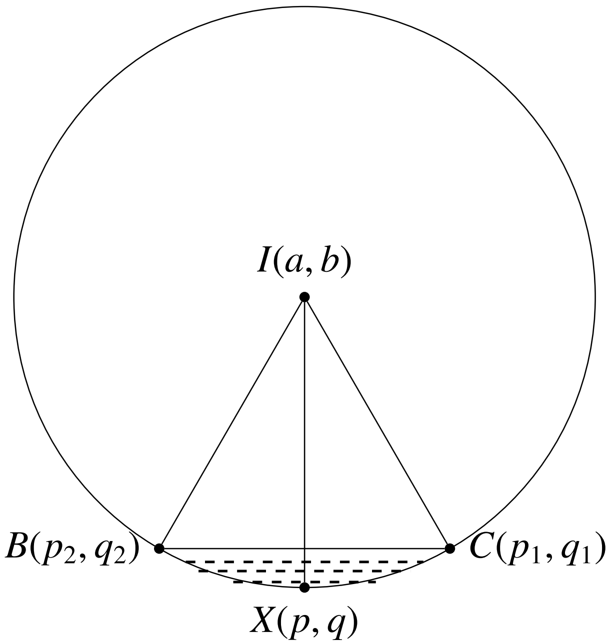

Fig. 29: Notion of the derivative.

```
import numpy as np

def slope_coeff(A,B):
        p = np.zeros((2,1))
        p[0] = (A[1]-B[1])/(A[0]-B
            [0])
        p[1] = (A[0]*B[1]-A[1]*B
            [0])/(A[0]-B[0])
        return p

A = np.matrix('-2;-2')
B = np.matrix('1;3')
p = slope_coeff(A,B)
print(p[0])
X=np.matrix('-0.22;0.96')
I=np.matrix('1.15;0.14')
print(-(X[0]-I[0])/(X[1]-I[1]))




#C=np.matrix('2.43;1.09')
```

7 Conic Sections

Problem 31. *Plot the circle*

$$x^2 + y^2 = 1 \tag{55}$$

Solution:

```
#This program draws the unit
   circle
import numpy as np
import matplotlib.pyplot as plt


r = 1
theta = np.linspace(-np.pi,np.pi
   ,50)
x = r*np.cos(theta)
y = r*np.sin(theta)

plt.plot(x,y)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
plt.show()
```

Problem 32. *Show that* (55) *can be expressed as*

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \tag{56}$$

Problem 33. *Show that*

$$\begin{pmatrix} x & y & 1 \end{pmatrix} M^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \tag{57}$$

for

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \tag{58}$$

can be expressed as

$$x^2 + 2xy + y^2 - 4x - 2y - 1 = 0 \tag{59}$$

Problem 34. *Show that* (59) *results in the curve in Fig. 34. This is known as a parabola.*

Problem 35. *Show that using*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{60}$$

in Problem 33 results in

$$y^2 - x^2 = 1 \tag{61}$$

Problem 36. *Sketch* (61) *to obtain Fig. 36. This curve is known as a* hyperbola

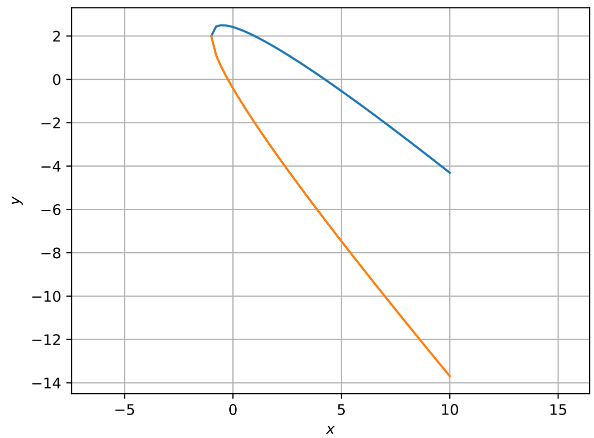

Fig. 34

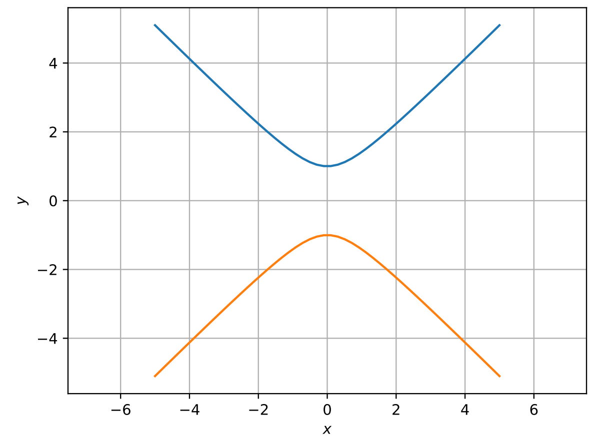

Fig. 36

Solution:

```
#This program draws a hyperbola
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-5,5,50)
y1 = np.sqrt(1+x**2)
y2 = -np.sqrt(1+x**2)

plt.plot(x,y1,x,y2)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
plt.savefig('../figs/circle_
    hyperbola.eps')
plt.show()
```

Problem 37. *Generate the points*

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \tag{62}$$

where x, y are the points generated in Problem 31. Plot y_1 with respect to x_1. The figure that you obtain in Fig. 37 is known as an ellipse.

Solution:

```
#This program draws the triangle
    ABC
import numpy as np
import matplotlib.pyplot as plt

r = 1
theta = np.linspace(-np.pi,np.pi
    ,50)
x = r*np.cos(theta)
y = r*np.sin(theta)
X = np.row_stack((x,y))
M = np.matrix('3,0;0,2')
Y = M*X
x1 = np.array(Y)[0]
y1 = np.array(Y)[1]

plt.plot(x1,y1)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$y$')
plt.axis('equal')
#plt.savefig('../figs/ellipse_
    transform.eps')
plt.show()
```

Problem 38. *Draw the curve*

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \tag{63}$$

Comment.

8 Area Within a Parabola

Problem 39. *Sketch the parabola*

$$y^2 = x \tag{64}$$

Problem 40. *Using n rectangles of equal width as shown in Fig. 40, find the limting area of the parabola in $x \in (0, 1)$.*

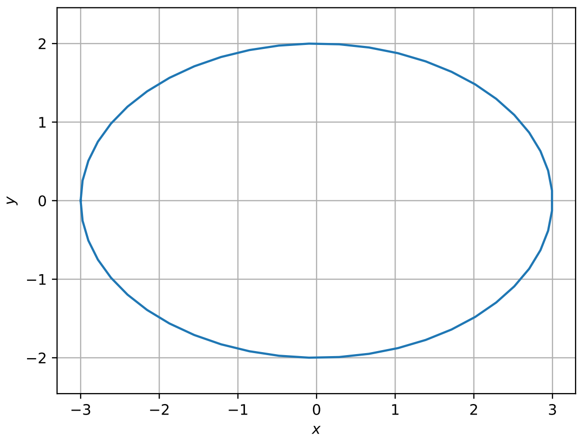

Fig. 37

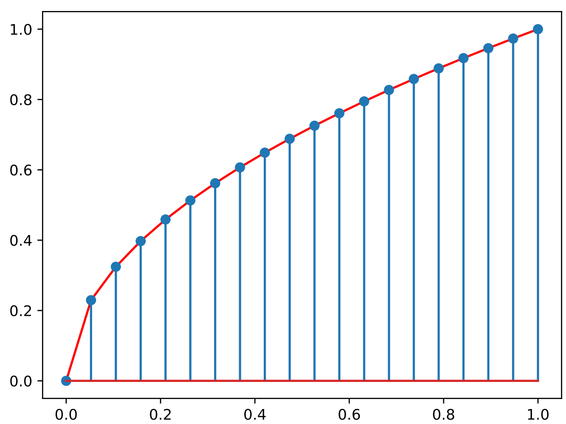

Fig. 40

Solution: Considering the width of the rectangle as $h = \frac{1}{n}, n = 100$, the approximate area of the parabola can be computed as

$$A = h * \left(\sqrt{h} + \sqrt{2h} + \cdots + \sqrt{100h}\right) \tag{65}$$
$$\approx 0.67 \tag{66}$$

using the following program

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt

n = 100
h = 1/n
x = np.linspace(1,n,n)
y = np.sqrt(h*x)
A = h*np.sum(y)
print(A)
```

8.1 Arithmetic Progression

Problem 41. *Plot*

$$y = x^2 \tag{67}$$

and verify that the area under this parabola for $x \in (0, 1)$ *is* $A_0 = 1 - A$.

Solution: The following code

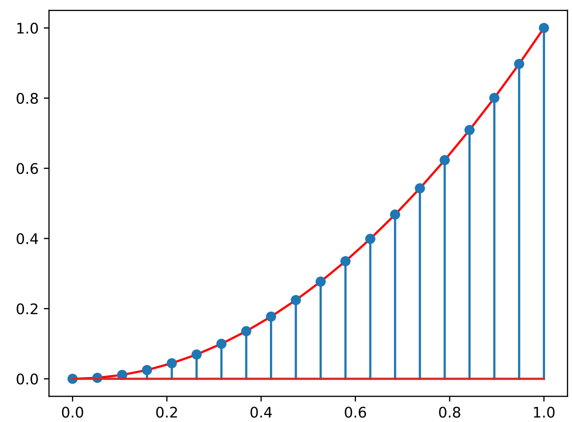

Fig. 41

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt

n = 100
h = 1/n
x = np.linspace(1,n,n)
y = (h*x)**2
A_1 = h*np.sum(y)
print(A_1)
```

yields the area in Fig.41 as $A_0 \approx 0.33 = 1 - A = 1 - 0.67$.

Problem 42. *Show that the limiting area in Problem 41 is*

$$A_0 = \lim_{n\to\infty} \left(\frac{1}{n}\right)^3 \sum_{k=1}^{n} k^2 = \frac{1}{3} \tag{68}$$

Solution: We have

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1 \tag{69}$$
$$\implies n^3 = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + n \tag{70}$$
$$\implies \sum_{k=1}^{n} k^2 = \frac{1}{3}\left[n^3 + 3\sum_{k=1}^{n} k - n\right] \tag{71}$$

Letting

$$S_n = 1 + 2 + \cdots + n \tag{72}$$
$$S_n = n + n - 1 + \cdots + 1 \tag{73}$$
$$\implies 2S_n = n(n+1) \tag{74}$$
$$\implies S_n = \frac{n(n+1)}{2} \tag{75}$$

Thus,

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[n^3 + 3\frac{n(n+1)}{2} - n\right] \tag{76}$$
$$= \frac{n}{6}\left[2n^2 + 3n + 1\right] \tag{77}$$
$$= \frac{n(n+1)(2n+1)}{6} \tag{78}$$

and

$$A_0 = \lim_{n\to\infty}\left(\frac{1}{n}\right)^3 \sum_{k=1}^{n} k^2 \tag{79}$$
$$= \lim_{n\to\infty}\frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \tag{80}$$
$$= \frac{1}{3} \tag{81}$$

This process of finding the area under a curve is known as *integration*. Integration is the opposite of differentiation. The sequence that is summed in S_n is known as an *Arithmetic Progression*.

8.2 Geometric Progression

Problem 43. *Plot the parabola*

$$y = x^2 \tag{82}$$

for $x \in (0,1)$ *with points* $\left(r^k, 0\right), k = 0, 1, \ldots, n$ *for* $r = 0.8, n = 10$.

Solution: The following code

```
import numpy as np
import matplotlib.pyplot as plt
n = 10
r = 0.8
temp = [ r ** (k - 1) for k in
    range(1, n + 1)]
x = np.array(temp)
y = x**2
plt.plot(x,y,'r')
plt.stem(x,y)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('../figs/parabola_area
    _gp.eps')
plt.show()
```

plots Fig. 43

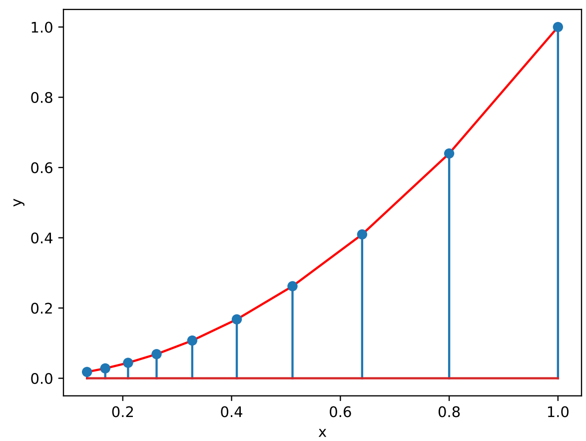

Fig. 43

Problem 44. *Calculate the area of the parabola using Fig. 43 with* $n = 100, r = 0.98$.

Solution: The intervals are of width $r^{k-1}(1-r), k = 1, \ldots, n$. The corresponding heights are r^{2k-2}. Thus, the area is

$$A_0 = \sum_{k=1}^{n} r^{k-1}(1-r)\, r^{2k-2} \tag{83}$$
$$= (1-r)\sum_{k=1}^{n} r^{3k-3} \tag{84}$$

The following code calculates the desired area as 0.33

```
#Area under the parabola
import numpy as np
import matplotlib.pyplot as plt
```

```
n = 100
r = 0.98
k = np.linspace(1,n,n)
y = r**(3*k-3)
A = (1-r)*np.sum(y)
print(A)
```

Problem 45. *Obtain the area in the previous problem as the limit of a sum.*

Solution: Let

$$A_0 = \lim_{\substack{r \to 1 \\ n \to \infty}} (1-r) \sum_{k=1}^{n} r^{3k-3}, \quad r < 1 \tag{85}$$

If $p = r^3$,

$$S_n = \sum_{k=1}^{n} p^{k-1} \tag{86}$$

$$\Rightarrow pS_n = \sum_{k=1}^{n} p^k \tag{87}$$

$$\Rightarrow (1-p) S_n = 1 - p^n \tag{88}$$

$$\Rightarrow S_n = \frac{1-p^n}{1-p} \tag{89}$$

The sequence $p^{k-1}, k = 1, \dots, n$ is known as a *Geometric Progression*. Substituting in (85),

$$A_0 = \lim_{\substack{r \to 1 \\ n \to \infty}} (1-r) \frac{1-r^{3n}}{1-r^3} \tag{90}$$

$$= \lim_{\substack{r \to 1 \\ n \to \infty}} \frac{1-r^{3n}}{1+r+r^2} \tag{91}$$

$$= \frac{1}{3} \quad \left(\because r^{3n} \to 0\right) \tag{92}$$