Problems in Linear Algebra

Contents

1 The Straight Line

1.1 Point

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$
3. The *length* of PQ is defined as
$$\|\mathbf{P} - \mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$
5. The point dividing PQ in the ratio $k:1$ is
$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A} - \mathbf{B})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.7}$$
Solution: Using Baudhayana's theorem,

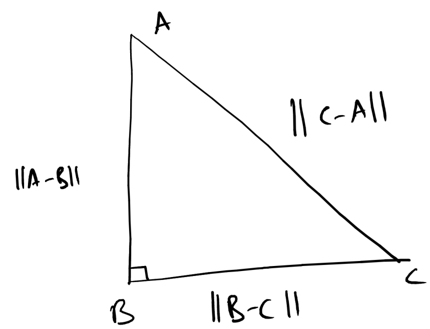

Fig. 1.1.7

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{1.1.7}$$
$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A})$$
$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show that
$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.8}$$
9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that
$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.9}$$

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1.2 Line

1. The points $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ are as shown in Fig. 1.2.1. Find the equation of OA.
Solution: Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be any point on OA. Then, using similar triangles,
$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{1.2.1.1}$$
$$\implies x_2 = mx_1 \tag{1.2.1.2}$$

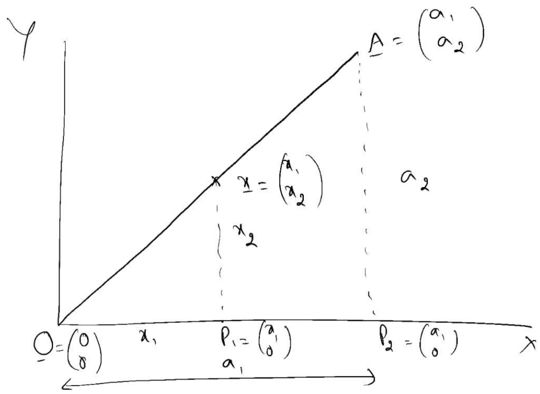

Fig. 1.2.1

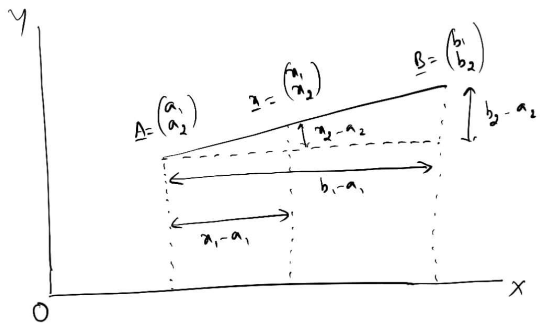

Fig. 1.2.2

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = x_1 \mathbf{m} \tag{1.2.1.3}$$

In general, the above equation is written as

$$\mathbf{x} = \lambda \mathbf{m}, \tag{1.2.1.4}$$

where $\mathbf{m}$ is the direction vector of the line.

2. Find the equation of AB in Fig. 1.2.2
Solution: From Fig. 1.2.2,

$$\frac{x_2 - a_2}{x_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1} = m \tag{1.2.2.1}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{1.2.2.2}$$

From (1.2.2.2),

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ mx_1 + a_2 - ma_1 \end{pmatrix} \tag{1.2.2.3}$$

$$= \mathbf{A} + (x_1 - a_1) \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.2.4}$$

$$= \mathbf{A} + \lambda \mathbf{m} \tag{1.2.2.5}$$

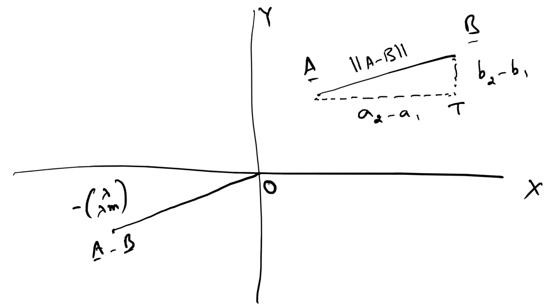

Fig. 1.2.5

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (1.2.2.5) is obtained from (1.2.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 1.2.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T \mathbf{A}}. \tag{1.2.4.1}$$

Also, from (1.2.1.4),

$$\|\mathbf{A}\| = \lambda \sqrt{1 + m^2} \tag{1.2.4.2}$$

Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 1.2.5. From (1.2.2.5), for some λ,

$$\mathbf{B} = \mathbf{A} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.2.5.1}$$

$$\implies \mathbf{A} - \mathbf{B} = -\lambda \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.2.5.2}$$

$\mathbf{A} - \mathbf{B}$ is marked in Fig. 1.2.5.
6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{1.2.7.1}$$

8. The *normal* to the vector $\mathbf{m}$ is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \tag{1.2.8.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.2.8.2}$$

9. From (1.2.7.1), the equation of a line can also

be expressed as

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B}-\mathbf{A}) \tag{1.2.9.1}$$
$$\implies \mathbf{n}^T\mathbf{x} = c \tag{1.2.9.2}$$

10. The unit vectors on the x and y axis are defined as

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \tag{1.2.10.1}$$
$$\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.2.10.2}$$

11. If a be the *intercept* of the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.11.1}$$

on the $x-$axis, then $\begin{pmatrix}a\\0\end{pmatrix}$ is a point on the line. Thus,

$$\mathbf{n}^T\begin{pmatrix}a\\0\end{pmatrix} = c \tag{1.2.11.2}$$
$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \tag{1.2.11.3}$$

12. The *rotation matrix* is defined as

$$\mathbf{Q} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{1.2.12}$$

where θ is anti-clockwise.

13.

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{1.2.13}$$

where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix*.

14. Find the equation of line L in Fig. 1.2.14.
Solution: The equation of the $x-$axis is

$$\mathbf{x} = \lambda\mathbf{e}_1 \tag{1.2.14.1}$$

Translation by p units along the $y-$axis results in

$$L_0: \quad \mathbf{x} = \lambda\mathbf{e}_1 + p\mathbf{e}_2 \tag{1.2.14.2}$$

Rotation by $90° - \alpha$ in the anti-clockwise direction yields

$$L: \quad \mathbf{x} = \mathbf{Q}\{\lambda\mathbf{e}_1 + p\mathbf{e}_2\} \tag{1.2.14.3}$$
$$= \lambda\mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \tag{1.2.14.4}$$

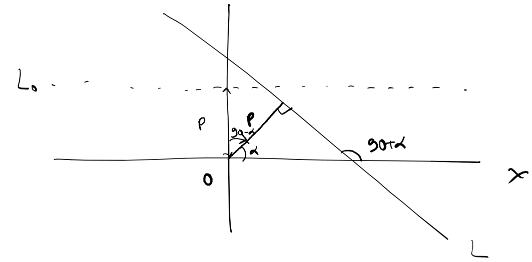

Fig. 1.2.14

where

$$\mathbf{Q} = \begin{pmatrix}\cos(\alpha-90) & -\sin(\alpha-90)\\ \sin(\alpha-90) & \cos(\alpha-90)\end{pmatrix} \tag{1.2.14.5}$$
$$= \begin{pmatrix}\sin\alpha & \cos\alpha\\ -\cos\alpha & \sin\alpha\end{pmatrix} \tag{1.2.14.6}$$

From (1.2.14.4),

$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$
$$= \lambda\mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \tag{1.2.14.7}$$

resulting in

$$L: \quad \begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = p \tag{1.2.14.8}$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T\mathbf{x} = c \tag{1.2.15.1}$$

is

$$p = \frac{c}{\|n\|} \tag{1.2.15.2}$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.16.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.16.2}$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \tag{1.2.16.3}$$

where

$$\mathbf{N} = \begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix} \tag{1.2.16.4}$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{1.2.17.1}$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \tag{1.2.18.1}$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \tag{1.2.18.2}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{1.2.18.3}$$

where

$$\mathbf{n}^T\mathbf{A} = 0, \mathbf{n}^T\mathbf{m} = 0 \tag{1.2.18.4}$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \tag{1.2.18.5}$$

$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \tag{1.2.18.6}$$

From (1.2.15.2), (1.2.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.2.19.1}$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.2.19.2}$$

where

$$\mathbf{V} = \mathbf{V}^T \tag{1.2.19.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{1.2.19.4}$$

20. *Pair of straight lines:* (1.2.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{1.2.20.1}$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \tag{1.2.20.2}$$

21. In Fig. 1.2.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \tag{1.2.21.1}$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k+1} = \mathbf{B}. \tag{1.2.21.2}$$

Solution: From (1.2.2.5),

$$\begin{aligned} \mathbf{B} &= \mathbf{A} + \lambda_1\mathbf{m} \\ \mathbf{B} &= \mathbf{C} - \lambda_2\mathbf{m} \end{aligned} \tag{1.2.21.3}$$

$$\implies \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{1.2.21.4}$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{1.2.21.5}$$

from (1.2.21.1). Using (1.2.21.4) and (1.2.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{1.2.21.6}$$

resulting in (1.2.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{1.2.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{1.2.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{1.2.23.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{1.2.23.3}$$

24. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{1.2.24.1}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{1.2.24.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{1.2.24.3}$$

25. Find the equation of a line passing through the

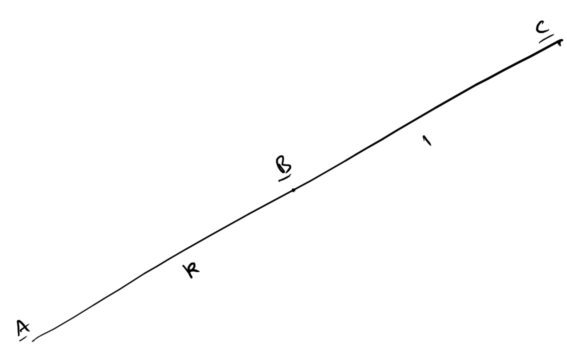

Fig. 1.2.21

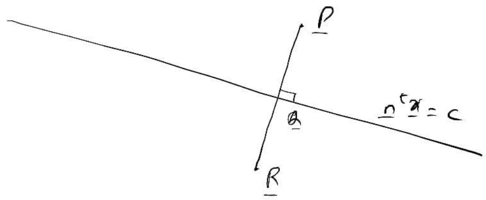

Fig. 1.2.26

intersection of the lines

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{1.2.25.1}$$
$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{1.2.25.2}$$

and passing through the point $\mathbf{p}$.

Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T} \mathbf{c} \tag{1.2.25.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{1.2.25.4}$$
$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{1.2.25.5}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda \left(\mathbf{N}^{-T} \mathbf{c} - \mathbf{p} \right) \tag{1.2.25.6}$$
$$\implies \mathbf{N}^T \mathbf{x} = \mathbf{N}^T \mathbf{p} + \lambda \left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right) \tag{1.2.25.7}$$

resulting in

$$\left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{x} = \left(\mathbf{c} - \mathbf{N}^T \mathbf{p} \right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T \mathbf{p} \tag{1.2.25.8}$$

26. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L: \quad \mathbf{n}^T \mathbf{x} = c \tag{1.2.26.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.2.26.2}$$
$$\mathbf{n}^T \mathbf{Q} = c \tag{1.2.26.3}$$
$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \tag{1.2.26.4}$$

where $\mathbf{m}$ is the direction vector of L. From (1.2.26.2) and (1.2.26.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \tag{1.2.26.5}$$

From (1.2.26.5) and (1.2.26.4),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.2.26.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{1.2.26.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \tag{1.2.26.8}$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2.26.9}$$

(1.2.26.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.2.26.10}$$
$$\implies \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{1.2.26.11}$$
$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \tag{1.2.26.12}$$

27. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.2.27.1}$$

1.3 Example

1. In $\triangle ABC$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.3.1}$$

and the equations of the medians through $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{1.3.1}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \tag{1.3.1}$$

Find the area of $\triangle ABC$.

Solution: The centroid $\mathbf{O}$ is the solution of (1.3.1),(1.3.1) and is obtained as the solution of the matrix equation

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\ 4\end{pmatrix} \tag{1.3.1}$$

which can be solved using the augmented matrix as follows.

$$\begin{pmatrix}1 & 1 & 5\\ 1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 5\\ 0 & 1 & 1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & 1\end{pmatrix} \tag{1.3.1}$$

Thus,

$$\mathbf{O} = \begin{pmatrix}4\\ 1\end{pmatrix} \tag{1.3.1}$$

Let AD be the median through $\mathbf{A}$. Then,

$$\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \mathbf{O} \tag{1.3.1}$$

$$\implies \mathbf{B}+\mathbf{C} = 3\mathbf{O}-\mathbf{A} = \begin{pmatrix}11\\ 1\end{pmatrix} \tag{1.3.1}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}11\\ 1\end{pmatrix} \tag{1.3.1}$$

From (1.3.1) and (1.3.1),

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{B} = 5 \tag{1.3.1}$$

$$\implies 5 + \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 12 \tag{1.3.1}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{C} = 7 \tag{1.3.1}$$

From (1.3.1) and (1.3.1), $\mathbf{C}$ can be obtained by solving

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}7\\ 4\end{pmatrix} \tag{1.3.1}$$

using the augmented matrix as

$$\begin{pmatrix}1 & 1 & 7\\ 1 & 0 & 4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 7\\ 0 & 1 & 3\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & 4\\ 0 & 1 & 3\end{pmatrix} \tag{1.3.1}$$

$$\implies \mathbf{C} = \begin{pmatrix}4\\ 3\end{pmatrix} \tag{1.3.1}$$

From (1.3.1),

$$\mathbf{B} = \begin{pmatrix}11\\ 1\end{pmatrix} - \begin{pmatrix}4\\ 3\end{pmatrix} = \begin{pmatrix}7\\ -2\end{pmatrix} \tag{1.3.1}$$

Thus,

$$\frac{1}{2}\begin{vmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\\ 1 & 1 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}1 & 7 & 4\\ 2 & -2 & 3\\ 1 & 1 & 1\end{vmatrix} = 9 \tag{1.3.1}$$

2. Summarize all the above computations through a Python script and plot $\triangle ABC$.

Solution:

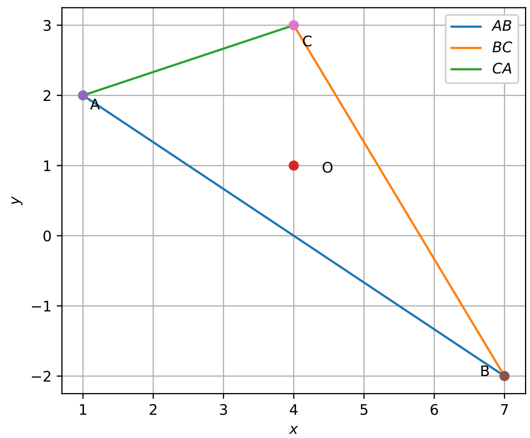

Fig. 1.3.2

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https://github.com/gadepall/school/raw/master/linalg/2D/manual/codes/triang.py
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1.4 Problems

1. A straight line through the origin $\mathbf{O}$ meets the lines

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 10 \tag{1.4.1}$$

$$\begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 5 = 0 \tag{1.4.1}$$

at $\mathbf{A}$ and $\mathbf{B}$ respectively. Find the ratio in which $\mathbf{O}$ divides AB.

Solution: Let

$$\mathbf{n} = \begin{pmatrix}4\\ 3\end{pmatrix} \tag{1.4.1}$$

Then (1.4.1) can be expressed as

$$\mathbf{n}^T\mathbf{x} = 10 \tag{1.4.1}$$

$$2\mathbf{n}^T\mathbf{x} = -5 \tag{1.4.1}$$

and since $\mathbf{A}, \mathbf{B}$ satisfy (1.4.1) respectively,

$$\mathbf{n}^T\mathbf{A} = 10 \tag{1.4.1}$$

$$2\mathbf{n}^T\mathbf{B} = -5 \tag{1.4.1}$$

Let $\mathbf{O}$ divide the segment AB in the ratio $k:1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.4.1}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.4.1}$$

$$\mathbf{A} = -k\mathbf{B} \tag{1.4.1}$$

Substituting in (1.4.1), and simplifying,

$$\mathbf{n}^T\mathbf{B} = \frac{10}{-k} \tag{1.4.1}$$

$$\mathbf{n}^T\mathbf{B} = \frac{-5}{2} \tag{1.4.1}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.4.1}$$

2. The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.4.2}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 4 \tag{1.4.2}$$

by $d = 2\sqrt{3}$ units. If the new point $\mathbf{Q}$ lies in the third quadrant, then find the equation of the line passing through $\mathbf{Q}$ and perpendicular to L.
Solution: From (1.4.2), the direction vector of L is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.4.2}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \tag{1.4.2}$$

However,

$$PQ = d \tag{1.4.2}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda|\,\|\mathbf{m}\| = d \tag{1.4.2}$$

$$\implies \lambda = \pm\frac{d}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{1.4.2}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T\mathbf{m}} = \sqrt{2} \tag{1.4.2}$$

from (1.4.2). Since $\mathbf{Q}$ lies in the third quadrant, from (1.4.2) and (1.4.2),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{6} \\ 1-\sqrt{6} \end{pmatrix} \tag{1.4.2}$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{Q}) = 0 \tag{1.4.2}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 - 2\sqrt{6} \tag{1.4.2}$$

3. Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.4.3}$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix}\mathbf{x} - 5 = 0. \tag{1.4.3}$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \tag{1.4.3}$$

find its vertices.
Solution: From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix}\mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \tag{1.4.3}$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{1.4.3}$$

Since diagonals of a rhombus bisect each other,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{C}}{2}$$

$$\mathbf{C} = 2\mathbf{P} - \mathbf{A} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.4.3}$$

$$\because AD \parallel BC,$$

$$BC : \begin{pmatrix} 7 & -1 \end{pmatrix}(\mathbf{x} - \mathbf{C}) = 0$$

$$\implies \begin{pmatrix} 7 & -1 \end{pmatrix}\mathbf{x} = -15 \tag{1.4.3}$$

From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{B} = \begin{pmatrix} -15 \\ -1 \end{pmatrix} \tag{1.4.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & -15 \\ 1 & -1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & -15 \\ 0 & 3 & -4 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 3 & 0 & -7 \\ 0 & 3 & -4 \end{pmatrix} \implies \mathbf{B} = -\frac{1}{3}\begin{pmatrix} 7 \\ 4 \end{pmatrix} \tag{1.4.3}$$

$$\because AB \parallel CD,$$
$$CD : \begin{pmatrix} 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{C}) = 0$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.4.3}$$

From (1.4.3) and (1.4.3),

$$\begin{pmatrix} 7 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.4.3}$$

resulting in the augmented matrix

$$\begin{pmatrix} 7 & -1 & 5 \\ 1 & -1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 7 & -1 & 5 \\ 0 & 3 & -8 \end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -8 \end{pmatrix} \implies \mathbf{D} = \frac{1}{3} \begin{pmatrix} 1 \\ -8 \end{pmatrix} \tag{1.4.3}$$

4. Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \tag{1.4.4}$$

has area 28. Find the orthocentre of this triangle.
Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \tag{1.4.4}$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.4.4}$$

Similarly, considering the side AC the equation of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \tag{1.4.4}$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \tag{1.4.4}$$

The orthocentre is obtained by solving (1.4.4) and (1.4.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \tag{1.4.4}$$

which can be expressed using (1.4.4), (1.4.4), (1.4.4) and (1.4.4) as

$$\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix}$$
$$= k \begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix} \tag{1.4.4}$$

From (1.4.4), using the expression for the area of triangle,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$
$$\implies \begin{vmatrix} k & 5 - k & -2k \\ -3k & 4k & 2 + 3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \tag{1.4.4}$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \tag{1.4.4}$$
$$\implies 5k^2 + 13k - 46 = 0 \tag{1.4.4}$$
$$\text{or, } k = 2, -\frac{23}{5} \tag{1.4.4}$$

Substituting the above in (1.4.4) and solving yields the orthocentre.

5. If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.4.5}$$

then find the area of this triangle. Also draw the equilateral triangle and two medians to verify your results.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{0}$$
$$\implies \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0} \tag{1.4.5}$$

If AB be the line in (1.4.5), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.4.5}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.4.5}$$

since CF passes through the origin and $CF \perp AB$. From (1.4.5) and (1.4.5),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.4.5}$$

Forming the augmented matrix,

$$\begin{pmatrix}1 & 1 & 2\\ 1 & -1 & 0\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 2\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\end{pmatrix}$$
$$\leftrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} \implies \mathbf{F} = \begin{pmatrix}1\\ 1\end{pmatrix} \tag{1.4.5}$$

From (1.4.5),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1.4.5}$$

after substituting from (1.4.5). Thus,

$$CF = \|\mathbf{C} - \mathbf{F}\| = 3\sqrt{2} \tag{1.4.5}$$
$$\implies AB = CF\frac{2}{\sqrt{3}} = 2\sqrt{6} \tag{1.4.5}$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 6\sqrt{3} \tag{1.4.5}$$

6. A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.

Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that $\mathbf{B}$ and $\mathbf{D}$ lie on the x and y-axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4\begin{pmatrix}1\\ 1\end{pmatrix} \tag{1.4.6}$$

Multiplying (1.4.6) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{1.4.6}$$

$$\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 4\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$
$$= 4\begin{pmatrix}\cos\theta - \sin\theta\\ \cos\theta + \sin\theta\end{pmatrix} \tag{1.4.6}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})$$
$$= 4(\cos\theta - \sin\theta) = 2\left(\sqrt{3} - 1\right) \tag{1.4.6}$$

for $\theta = 30^\circ$. Draw the square with sides on the axis as well as the rotated square in the same graph to verify your result.