

The Straight Line

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Abstract—Solved problems from JEE mains papers related to 2D lines in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.

- 1 A straight line through the origin **O** meets the lines

$$(4 \ 3)\mathbf{x} = 10 \quad (1.1)$$

$$(8 \ 6)\mathbf{x} + 5 = 0 \quad (1.2)$$

at **A** and **B** respectively. Find the ratio in which **O** divides **AB**.

Solution: Let

$$\mathbf{n} = (4 \ 3) \quad (1.3)$$

Then (1.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \quad (1.4)$$

$$2\mathbf{n}^T \mathbf{x} = -5 \quad (1.5)$$

and since **A**, **B** satisfy (1.4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \quad (1.6)$$

$$2\mathbf{n}^T \mathbf{B} = -5 \quad (1.7)$$

Let **O** divide the segment **AB** in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (1.8)$$

$$\therefore \mathbf{O} = \mathbf{0}, \quad (1.9)$$

$$\mathbf{A} = -k\mathbf{B} \quad (1.10)$$

Substituting in (1.6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \quad (1.11)$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \quad (1.12)$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \quad (1.13)$$

- 2 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.1)$$

is translated parallel to the line

$$L : (1 \ -1)\mathbf{x} = 4 \quad (2.2)$$

by $2\sqrt{3}$ units. If the new point **Q** lies in the third quadrant, then find the equation of the line passing through **Q** and perpendicular to **L**.

Solution: From (2.2), the direction vector of **L** is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.3)$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda \mathbf{m} \quad (2.4)$$

However,

$$PQ = 2\sqrt{3} \quad (2.5)$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = 2\sqrt{3} \quad (2.6)$$

$$\implies \lambda = \pm \frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm \sqrt{6} \quad (2.7)$$

$$\therefore \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \quad (2.8)$$

from (2.3). Since **Q** lies in the third quadrant,

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from (2.4) and (2.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \quad (2.9)$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{Q}) = 0 \quad (2.10)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 - \sqrt{6} \quad (2.11)$$

- 3 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (3.1)$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 12 \quad (3.2)$$

meets the coordinate axes at \mathbf{A} and \mathbf{B} , then find the locus of the midpoint of AB .

Solution: The intersection of the lines in (3.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (3.3)$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \quad (3.4)$$

resulting in

$$\mathbf{C} = \frac{1}{7} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \quad (3.5)$$

Let the \mathbf{R} be the mid point of AB . Then,

$$\mathbf{A} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \quad (3.6)$$

$$\mathbf{B} = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \quad (3.7)$$

Let the equation of AB be

$$\mathbf{n}^T (\mathbf{x} - \mathbf{C}) = 0 \quad (3.8)$$

Since \mathbf{R} lies on AB ,

$$\mathbf{n}^T (\mathbf{R} - \mathbf{C}) = 0 \quad (3.9)$$

Also,

$$\mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0 \quad (3.10)$$

Substituting from (3.6) in (3.10),

$$\mathbf{n}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} = 0 \quad (3.11)$$

From (3.9) and (3.11),

$$(\mathbf{R} - \mathbf{C}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \quad (3.12)$$

for some constant k . Multiplying both sides of (3.12) by

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.13)$$

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = k \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R} \\ = k \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (3.14)$$

$$\therefore \mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (3.15)$$

which can be easily verified for any \mathbf{R} . from (3.14),

$$\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{R} - \mathbf{C}) = 0 \\ \Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = 0 \\ \Rightarrow \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} - \mathbf{C}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \quad (3.16)$$

which is the desired locus.

- 4 Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (4.1)$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} - 5 = 0. \quad (4.2)$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad (4.3)$$

find its vertices.

Solution: From (4.1) and (4.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad (4.4)$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (4.5)$$

From (4.3) and (4.5)

$$AP = \|\mathbf{A} - \mathbf{P}\| = 2\sqrt{5} = d(\text{say}) \quad (4.6)$$

The direction vector of AP is

$$\mathbf{m} = \mathbf{A} - \mathbf{P} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (4.7)$$

$$\Rightarrow \|\mathbf{m}\| = 2\sqrt{5} \quad (4.8)$$

Since the direction of AP is the same as AC ,

$$\mathbf{C} = \mathbf{P} - d \frac{\mathbf{m}}{\|\mathbf{m}\|} \\ = - \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \quad (4.9)$$

Let $\mathbf{n} \perp \mathbf{m}$. Then

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (4.10)$$

and

$$\mathbf{B}, \mathbf{D} = \mathbf{P} \pm d \frac{\mathbf{n}}{\|\mathbf{n}\|} \\ = - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \pm 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (4.11)$$

5 Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \quad (5.1)$$

has area 28. Find the orthocentre of this triangle.

Solution: Let \mathbf{m}_1 be the direction vector of BC . Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5+k \\ k-2 \end{pmatrix}, \quad (5.2)$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \quad (5.3)$$

Similarly, considering the side AC the equation

of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \quad (5.4)$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2-3k \end{pmatrix}, \quad (5.5)$$

The orthocentre is obtained by solving (5.3) and (5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \quad (5.6)$$

which can be expressed using (5.2), (5.5), (5.3) and (5.4) as

$$\begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2+5k+6k-3k^2 \\ 10k-2k-3k^2 \end{pmatrix} \\ = k \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \quad (5.7)$$

The solution to the above is

$$\mathbf{x} = k \begin{pmatrix} 5+k & k-2 \\ 2k & -2-3k \end{pmatrix}^{-1} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\ = \frac{k}{-3k^2-17k-10-2k^2+4k} \\ \times \begin{pmatrix} -2-3k & -k+2 \\ -2k & 5+k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\ = \frac{k}{5k^2+13k+10} \\ \times \begin{pmatrix} 2+3k & k-2 \\ 2k & -5-k \end{pmatrix} \begin{pmatrix} 11-4k \\ 8-3k \end{pmatrix} \\ = \frac{k}{5k^2+13k+10} \\ \times \begin{pmatrix} 22-12k^2-8k+33k-3k^2+6k+8k-16 \\ 22k-8k^2-40+15k-8k+3k^2 \end{pmatrix} \\ = \frac{k}{5k^2+13k+10} \begin{pmatrix} 6+39k-15k^2 \\ -40+29k-5k^2 \end{pmatrix} \quad (5.8)$$

From the given information,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56 \\ \Rightarrow \begin{vmatrix} k & 5-k & -2k \\ -3k & 4k & 2+3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \quad (5.9)$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \quad (5.10)$$

$$\implies 5k^2 + 13k - 46 = 0 \quad (5.11)$$

$$\text{or, } k = 2, -\frac{23}{5} \quad (5.12)$$

- 6 If an equilateral triangle, having centroid at the origin, has a side along the line

$$(1 \ 1)\mathbf{x} = 2, \quad (6.1)$$

then find the area of this triangle.

Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\begin{aligned} \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} &= \mathbf{0} \\ \implies \mathbf{A} + \mathbf{B} + \mathbf{C} &= \mathbf{0} \end{aligned} \quad (6.2)$$

If AB be the line in (6.1), the equation of CF , where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (6.3)$$

is

$$(1 \ -1)\mathbf{x} = 0 \quad (6.4)$$

since CF passes through the origin and $CF \perp AB$. From (6.1) and (6.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (6.5)$$

Forming the augmented matrix,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} &\leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \quad (6.6)$$

From (6.2),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6.7)$$

after substituting from (6.6). Thus,

$$CF = \|\mathbf{C} - \mathbf{F}\| = 3\sqrt{2} \quad (6.8)$$

$$\implies AB = CF \frac{2}{\sqrt{3}} = \sqrt{6} \quad (6.9)$$

and the area of the triangle is

$$\frac{1}{2}AB \times CF = 3\sqrt{3} \quad (6.10)$$

- 7 A square, of each side 2, lies above the x -axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x -axis, then find the sum of the x -coordinates of the vertices of the square.

Solution: Consider the square $ABCD$ with $\mathbf{A} = \mathbf{0}, AB = 2$ such that \mathbf{B} and \mathbf{D} lie on the x and y -axis respectively. Then

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (7.1)$$

Multiplying (7.1) with the rotation matrix

$$\mathbf{T} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (7.2)$$

$$\begin{aligned} \mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) &= 4 \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 4 \begin{pmatrix} \cos \theta - \sin \theta \\ \cos \theta + \sin \theta \end{pmatrix} \end{aligned} \quad (7.3)$$

$$\begin{aligned} \implies (1 \ 0)\mathbf{T}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) \\ = 4(\cos \theta - \sin \theta) = 2(\sqrt{3} - 1) \end{aligned} \quad (7.4)$$

for $\theta = 30^\circ$.