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Abstract—This book provides a collection of the international maths olympiad problems in geometry.

1. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.
2. An arbitrary point M is selected in the interior of the segment AB . The squares $AMCD$ and $MBEF$ are constructed on the same side of AB , with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q , intersect at M and also at another point N . Let N_0 denote the point of intersection of the straight lines AF and BC .
 - a) Prove that the points N and N_0 coincide.
 - b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M .
 - c) Find the locus of the midpoints of the segments PQ as M varies between A and B .
3. Two planes, P and Q , intersect along the line p . The point A is given in the plane P , and the point C in the plane Q ; neither of these points lies on the straight line p . Construct an isosceles trapezoid $ABCD$ (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.
4. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P , P_2P , P_3P intersect the opposite sides in points Q_1 , Q_2 , Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2 .

5. Construct triangle ABC if $AC = b$, $AB = c$ and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^\circ$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

6. Consider a plane ε and three non-collinear points A, B, C on the same side of ε ; suppose the plane determined by these three points is not parallel to ε . In plane ε take three arbitrary points A_0, B_0, C_0 . Let L, M, N be the midpoints of segments AA_0, BB_0, CC_0 ; let G be the centroid of triangle LMN . (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ε ?
7. Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A , and the other side intersecting the segment BC .
8. Prove that $\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$
9. Given a triangle ABC , let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that $\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}$.
10. Let ABC be a triangle and P an interior point of ABC . Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to 30° .
11. Consider nine points in space, no four of

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which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

12. In the plane let C be a circle, L a line tangent to the circle C , and M a point on L . Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR .
13. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \frac{\pi}{2}$ and $AC \cdot BD = AD \cdot BC$.
 - a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.
 - b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.
14. ABC is an isosceles triangle with $AB = AC$. Suppose that
 - a) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB ;
 - b) Q is an arbitrary point on the segment BC different from B and C ;
 - c) E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.
15. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y . The line XY meets BC at Z . Let P be a point on the line XY other than Z . The line CP intersects the circle with diameter AC at C and M , and the line BP intersects the circle with diameter BD at B and N . Prove that the lines AM, DN, XY are concurrent.
16. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.
17. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a

grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is \sqrt{r} . The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

- a) Show that the task cannot be done if r is divisible by 2 or 3.
 - b) Prove that the task is possible when $r = 73$.
 - c) Can the task be done when $r = 97$?
18. Let P be a point inside triangle ABC such that
- $$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC , respectively. Show that AP, BD, CE meet at a point.

19. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE , BC is parallel to EF , and CD is parallel to FA . Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF , respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

20. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n , lie along edges of the squares.

Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

- a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.
 - b) Prove that $f(m, n) \leq \frac{1}{2} \max m, n$ for all m and n .
 - c) Show that there is no constant C such that $f(m, n) < C$ for all m and n .
21. The angle at A is the smallest angle of triangle ABC . The points B and C divide the

circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A . The perpendicular bisectors of AB and AC meet the line AU at V and W , respectively. The lines BV and CW meet at T . Show that

$$AU = TB + TC.$$

22. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P , where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.
23. Let I be the incenter of triangle ABC . Let the incircle of ABC touch the sides BC , CA , and AB at K , L , and M , respectively. The line through B parallel to MK meets the lines LM and LK at R and S , respectively. Prove that angle RIS is acute.
24. Two circles G_1 and G_2 are contained inside the circle G , and are tangent to G at the distinct points M and N , respectively. G_1 passes through the center of G_2 . The line passing through the two points of intersection of G_1 and G_2 meets G at A and B . The lines MA and MB meet G_1 at C and D , respectively.
25. AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD , and CD is parallel to AB . The chords NA and CM meet at P ; the chords NB and MD meet at Q . The rays CA and DB meet at E . Prove that $PE = QE$.
26. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i . The line K_1K_2 is reflected in the line L_1L_2 . Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1 . Show that the three new lines form a triangle with vertices on the incircle.
27. Let ABC be an acute-angled triangle with circumcentre O . Let P on BC be the foot of the altitude from A . Suppose that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.
28. In a triangle ABC , let AP bisect $\angle BAC$, with P on BC , and let BQ bisect $\angle ABC$, with Q on CA . It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC ?
29. BC is a diameter of a circle center O . A is any point on the circle with $\angle AOC > 60^\circ$. EF is the chord which is the perpendicular bisector of AO . D is the midpoint of the minor arc AB . The line through O parallel to AD meets AC at J . Show that J is the incenter of triangle CEF .
30. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are O_1, O_2, \dots, O_n . Show that $\sum_{i < j} \frac{1}{O_i O_j} \leq (n-1)\frac{\pi}{4}$.
31. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Show that all the hexagon's angles are equal.
32. $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB , BC , CA are P , Q , R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC if $RP = RQ$.
33. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC . The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R . Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC .
34. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA . The point P lies inside $ABCD$ and satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.
35. A circle is inscribed in $\triangle ABC$ with sides a , b , c . Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from $\triangle ABC$. In each of these triangles, a circle is inscribed. Find the

sum of the areas of all four inscribed circles (in terms of a, b, c).

36. Seventeen people correspond by mail with one another each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.
37. Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.
38. In tetrahedron $ABCD$, vertex D is connected with D_0 the centroid of $\triangle ABC$. Lines parallel to DD_0 are drawn through A, B and C . These lines intersect the planes BCD, CAD and ABD in points A_1, B_1 and C_1 , respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point D_0 is selected anywhere within $\triangle ABC$?
39. Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d , and the angle between them is ω . Tetrahedron $ABCD$ is divided into two solids by plane ε , parallel to lines AB and CD . The ratio of the distances of ε from AB and CD is equal to k . Compute the ratio of the volumes of the two solids obtained.
40. Consider a triangle OAB with acute angle AOB . Through a point $M \neq O$ perpendiculars are drawn to OA and OB , the feet of which are P and Q respectively. The point of intersection of the altitudes of $\triangle OPQ$ is H . What is the locus of H if M is permitted to range over
 - a) the side AB ,
 - b) the interior of $\triangle OAB$?
41. In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d . Prove that the number of diameters of the given set is at most n .
42. In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A , the number who solved B was twice the number who solved C . The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A . How many students solved only problem B ?
43. Let a, b, c be the lengths of the sides of a triangle, and α, β, γ , respectively, the angles opposite these sides. Prove that if

$$a + b = \tan \frac{\gamma}{2}(a \tan \alpha + b \tan \beta)$$
 the triangle is isosceles.
44. Prove the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.
45. In the interior of sides BC, CA, AB of triangle ABC , any points K, L, M , respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC .
46. Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\angle BAD = \alpha$. If $\triangle ABD$ is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if

$$a \geq \cos \alpha + \sqrt{3} \sin \alpha.$$
47. Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq \frac{1}{8}$.
48. Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\triangle A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C , respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA , and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.
49. Prove that there is one and only one triangle whose side lengths are consecutive integers,

and one of whose angles is twice as large as another.

50. Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.
51. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.
52. A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B, and D is the foot of the perpendicular from C to AB. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line AB. Of these, γ_1 is inscribed in $\triangle ABC$, while γ_2 and γ_3 are both tangent to CD and to γ , one on each side of CD. Prove that γ_1, γ_2 and γ_3 have a second tangent in common.
53. Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the inscribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}$$

54. In the tetrahedron ABCD, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of $\triangle ABC$. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

55. Consider a convex polyhedron P_1 with nine vertices A_1, A_2, \dots, A_9 ; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to A_i ($i = 2, 3, \dots, 9$). Prove that at least two of the polyhedra P_1, P_2, \dots, P_9 have an interior point in common.
56. All the faces of tetrahedron ABCD are acute-angled triangles. We consider all closed polygonal paths of the form XY ZTX defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges BCCD, DA, respectively. Prove:
- a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.

- b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2AC \sin \frac{\alpha}{2}$ where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

57. Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.
58. Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.
59. Point O lies on the line $g: \overrightarrow{OP_1}, \overrightarrow{OP_2}, \dots, \overrightarrow{OP_n}$ are unit vectors such that points P_1, P_2, \dots, P_n all lie in a plane containing g and on one side of g . Prove that if n is odd, $|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \dots + \overrightarrow{OP_n}|$ Here $|\overrightarrow{OM}|$ denotes the length of vector \overrightarrow{OM} .
60. Determine whether or not there exists a finite set M of points in space not lying in the same plane such that, for any two points A and B of M, one can select two other points C and D of M so that lines AB and CD are parallel and not coincident.
61. In the triangle ABC, prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin \frac{C}{2}$$

62. Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.
63. On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45^\circ, \angle BCP = \angle ACQ = 30^\circ, \angle ABR = \angle BAR = 15^\circ$. Prove that $\angle QRP = 90^\circ$ and $QR = RP$.
64. In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.
65. A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.
66. Equilateral triangles ABK, BCL, CDM, DAN

are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL , LM , MN , NK and the midpoints of the eight segments $AKBK$, BL , CL , CM , DM , DN , AN are the twelve vertices of a regular dodecagon.