Problems in Linear Algebra

Contents

1 Introduction

1.1 Properties

1. The *inner product* of $\mathbf{P}$ and $\mathbf{Q}$ is defined as
$$\mathbf{P}^T\mathbf{Q} = p_1q_1 + p_2q_2 \tag{1.1.1}$$
2. The *norm* of a vector
$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1.1.2}$$
is defined as
$$\|\mathbf{P}\| = \sqrt{p_1^2 + p_2^2} \tag{1.1.2}$$
3. The *length* of PQ is defined as
$$\|\mathbf{P} - \mathbf{Q}\| \tag{1.1.3}$$
4. The *direction vector* of the line PQ is defined as
$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{1.1.4}$$
5. The point dividing PQ in the ratio $k : 1$ is
$$\mathbf{R} = \frac{k\mathbf{P} + \mathbf{Q}}{k+1} \tag{1.1.5}$$
6. The *area* of $\triangle PQR$ is the *determinant*
$$\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{vmatrix} \tag{1.1.6}$$
7. *Orthogonality:* See Fig. 1.1.7. In $\triangle ABC, AB \perp BC$. Show that
$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.7}$$
Solution: Using Baudhayana's theorem,
$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{A}\|^2 \tag{1.1.7}$$
$$\implies (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})$$
$$\implies 2\mathbf{A}^T\mathbf{B} - 2\mathbf{B}^T\mathbf{B} + 2\mathbf{B}^T\mathbf{C} - 2\mathbf{A}^T\mathbf{C} = 0 \tag{1.1.7}$$
which can be simplified to obtain (1.1.7).
8. Let $\mathbf{x}$ be any point on AB in Fi.g 1.1.7. Show

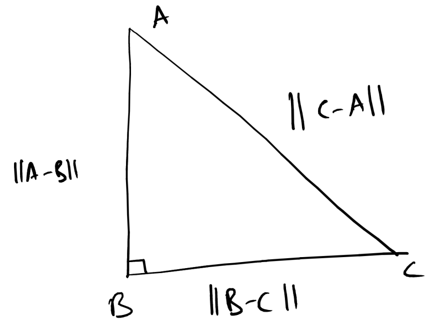

Fig. 1.1.7

that

$$(\mathbf{x} - \mathbf{A})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.8}$$

9. If $\mathbf{x}, \mathbf{y}$ are any two points on AB, show that

$$(\mathbf{x} - \mathbf{y})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.9}$$

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1.2 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.2.1}$$

2. Find the length of PQ for
   a) $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$;
   b) $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$;
   c) $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$.
3. Using direction vectors, show that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are the vertices of a parallelogram.
4. Using Baudhayana's theorem, show that the points $\begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ are the vertices of a right-angled traingle. Repeat using orthogonality.
5. Plot the points $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and prove that they are the vertices of a rectangle.
6. Show that $\mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are the vertices of an isosceles triangle.
7. In the last question, find the distance of the vertex $\mathbf{A}$ of the triangle from the middle point of the base BC.
8. Prove that the points $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are the vertices of a square.
9. Prove that the points $\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are the vertices of a parallelogram. Find $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$, the mid points of AB, BC, CD, AD respectively. Show that EG and FH bisect each other.
10. Prove that the points $\begin{pmatrix} 21 \\ -2 \end{pmatrix}, \begin{pmatrix} 15 \\ 10 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -12 \end{pmatrix}$ are the vertices of a rectangle, and find the coordinates of its centre.
11. Find the lengths of the medians of the triangle whose vertices are at the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.
12. Find the coordinates of the points that divide the line joining the points $\begin{pmatrix} -35 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -10 \end{pmatrix}$ into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 25 \\ 10 \end{pmatrix}$.
14. Prove that the middle point of the line joining the points $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ is a point of trisection of the line joining the points $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$.
15. The points $\begin{pmatrix} 8 \\ 5 \end{pmatrix}, \begin{pmatrix} -7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$.
16. The point $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is the intesection of the diagonals of a parallelogram two of whose vertices are at the points $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$. Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.
18. Find the coordinates of points which divide the

join of $\begin{pmatrix}2\\3\end{pmatrix}$, $\begin{pmatrix}-4\\5\end{pmatrix}$ externally in the ratio 2 : 3, and also externally in the ratio 3 : 2.

19. Prove the centroid of $\triangle ABC$ is

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.2.19}$$

1.3 Loci

1. A point moves so that its distance from the point $\begin{pmatrix}2\\1\end{pmatrix}$ is double its distance from the point $\begin{pmatrix}1\\2\end{pmatrix}$. Find the equation of its locus.
2. Find the equation of the perpendicular bisector of the line joining the points $\begin{pmatrix}3\\-4\end{pmatrix}$ and $\begin{pmatrix}-2\\3\end{pmatrix}$.
3. Find the equation of the circle of radius 5 with centre at $\begin{pmatrix}3\\-4\end{pmatrix}$.
4. A point moves so that its distance from the y-axis is equal to the distance from the point $\begin{pmatrix}2\\1\end{pmatrix}$. Find the equation of its locus.
5. A point moves so that the sum of the squares of its distance from the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}4\\3\end{pmatrix}$ is constant. Find the equation of the locus.
6. A point moves so that its distance from the axis of x is twice its distance from the point $\begin{pmatrix}0\\1\end{pmatrix}$. Find the equation of the locus.
7. A point moves in such a way that with the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\4\end{pmatrix}$ it forms a triangle of area 8.5. Show that its locus has an equation

$$\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x}\right\}\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x} - 34\right\} = 0 \tag{1.3.7}$$

2 The Straight Line

2.1 Properties

1. The points $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{A} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$ are as shown in Fig. 2.1.1. Find the equation of OA.

Solution: Let $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ be any point on OA. Then, using similar triangles,

$$\frac{x_2}{x_1} = \frac{a_2}{a_1} = m \tag{2.1.1.1}$$

$$\implies x_2 = mx_1 \tag{2.1.1.2}$$

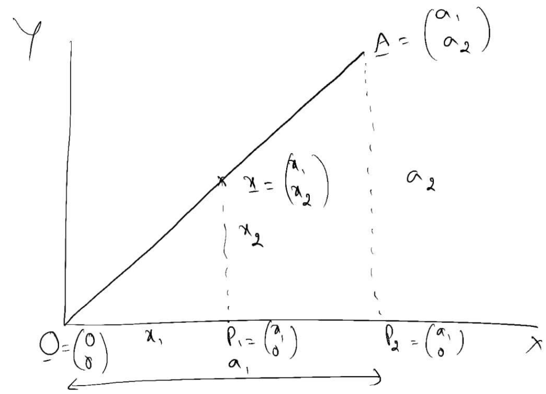

Fig. 2.1.1

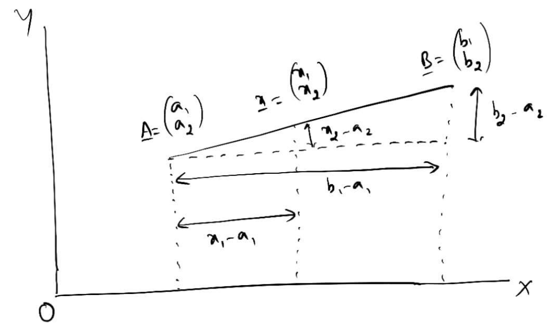

Fig. 2.1.2

where m is known as the slope of the line. Thus, the equation of the line is

$$\mathbf{x} = \begin{pmatrix}x_1\\mx_1\end{pmatrix} = x_1\begin{pmatrix}1\\m\end{pmatrix} = x_1\mathbf{m} \tag{2.1.1.3}$$

In general, the above equation is written as

$$\mathbf{x} = \lambda\mathbf{m}, \tag{2.1.1.4}$$

where $\mathbf{m}$ is the direction vector of the line.

2. Find the equation of AB in Fig. 2.1.2

Solution: From Fig. 2.1.2,

$$\frac{x_2-a_2}{x_1-a_1} = \frac{b_2-a_2}{b_1-a_1} = m \tag{2.1.2.1}$$

$$\implies x_2 = mx_1 + a_2 - ma_1 \tag{2.1.2.2}$$

From (2.1.2.2),

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\mx_1+a_2-ma_1\end{pmatrix} \tag{2.1.2.3}$$

$$= \mathbf{A} + (x_1-a_1)\begin{pmatrix}1\\m\end{pmatrix} \tag{2.1.2.4}$$

$$= \mathbf{A} + \lambda\mathbf{m} \tag{2.1.2.5}$$

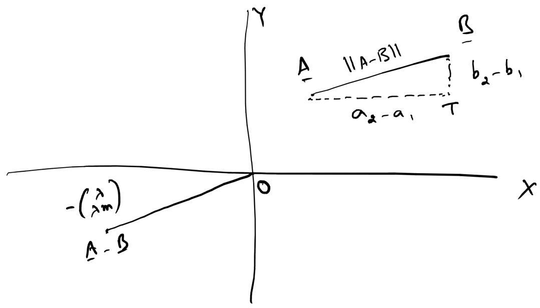

Fig. 2.1.5

3. *Translation:* If the line shifts from the origin by $\mathbf{A}$, (2.1.2.5) is obtained from (2.1.1.4) by adding $\mathbf{A}$.
4. Find the length of $\mathbf{A}$ in Fig. 2.1.1
Solution: Using Baudhayana's theorem, the length of the vector $\mathbf{A}$ is defined as
$$\|\mathbf{A}\| = OA = \sqrt{a_1^2 + a_2^2} = \sqrt{\mathbf{A}^T\mathbf{A}}. \quad (2.1.4.1)$$
Also, from (2.1.1.4),
$$\|\mathbf{A}\| = \lambda\sqrt{1+m^2} \quad (2.1.4.2)$$
Note that λ is the variable that determines the length of $\mathbf{A}$, since m is constant for all points on the line.
5. Find $\mathbf{A} - \mathbf{B}$.
Solution: See Fig. 2.1.5. From (2.1.2.5), for some λ,
$$\mathbf{B} = \mathbf{A} + \lambda\begin{pmatrix}1\\m\end{pmatrix} \quad (2.1.5.1)$$
$$\implies \mathbf{A} - \mathbf{B} = -\lambda\begin{pmatrix}1\\m\end{pmatrix}, \quad (2.1.5.2)$$
$\mathbf{A} - \mathbf{B}$ is marked in Fig. 2.1.5.
6. Show that $AB = \|\mathbf{A} - \mathbf{B}\|$
7. Show that the equation of AB is
$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \quad (2.1.7.1)$$
8. The *normal* to the vector $\mathbf{m}$ is defined as
$$\mathbf{n}^T\mathbf{m} = 0 \quad (2.1.8.1)$$
$$\mathbf{n} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{m} \quad (2.1.8.2)$$
9. From (2.1.7.1), the equation of a line can also be expressed as
$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A} + \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) \quad (2.1.9.1)$$
$$\implies \mathbf{n}^T\mathbf{x} = c \quad (2.1.9.2)$$
10. The unit vectors on the x and y axis are defined as
$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad (2.1.10.1)$$
$$\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix} \quad (2.1.10.2)$$
11. If a be the *intercept* of the line
$$\mathbf{n}^T\mathbf{x} = c \quad (2.1.11.1)$$
on the $x-$axis, then $\begin{pmatrix}a\\0\end{pmatrix}$ is a point on the line. Thus,
$$\mathbf{n}^T\begin{pmatrix}a\\0\end{pmatrix} = c \quad (2.1.11.2)$$
$$\implies a = \frac{c}{\mathbf{n}^T\mathbf{e}_1} \quad (2.1.11.3)$$
12. The *rotation matrix* is defined as
$$\mathbf{Q} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \quad (2.1.12)$$
where θ is anti-clockwise.
13. $$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \quad (2.1.13)$$
where $\mathbf{I}$ is the *identity matrix*. The rotation matrix $\mathbf{Q}$ is also an *orthogonal matrix*.
14. Find the equation of line L in Fig. 2.1.14.
Solution: The equation of the $x-$axis is
$$\mathbf{x} = \lambda\mathbf{e}_1 \quad (2.1.14.1)$$
Translation by p units along the $y-$axis results in
$$L_0: \quad \mathbf{x} = \lambda\mathbf{e}_1 + p\mathbf{e}_2 \quad (2.1.14.2)$$
Rotation by $90° - \alpha$ in the anti-clockwise direction yields
$$L: \quad \mathbf{x} = \mathbf{Q}\{\lambda\mathbf{e}_1 + p\mathbf{e}_2\} \quad (2.1.14.3)$$
$$= \lambda\mathbf{Q}\mathbf{e}_1 + p\mathbf{Q}\mathbf{e}_2 \quad (2.1.14.4)$$

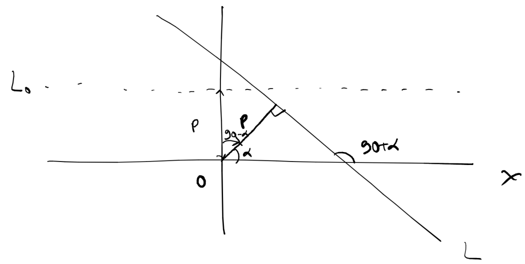

Fig. 2.1.14

where

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha - 90) & -\sin(\alpha - 90) \\ \sin(\alpha - 90) & \cos(\alpha - 90) \end{pmatrix} \quad (2.1.14.5)$$

$$= \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix} \quad (2.1.14.6)$$

From (2.1.14.4),

$$L: \quad \mathbf{e}_2^T\mathbf{Q}^T\mathbf{x} = \lambda\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{Q}^T\mathbf{Q}\mathbf{e}_2$$
$$= \lambda\mathbf{e}_2^T\mathbf{e}_1 + p\mathbf{e}_2^T\mathbf{e}_2 \quad (2.1.14.7)$$

resulting in

$$L: \quad \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \quad (2.1.14.8)$$

15. Show that the distance from the orgin to the line

$$\mathbf{n}^T\mathbf{x} = c \quad (2.1.15.1)$$

is

$$p = \frac{c}{\|n\|} \quad (2.1.15.2)$$

16. Show that the point of intersection of two lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \quad (2.1.16.1)$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \quad (2.1.16.2)$$

is given by

$$\mathbf{x} = \left(\mathbf{N}^T\right)^{-1}\mathbf{c} \quad (2.1.16.3)$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \quad (2.1.16.4)$$

17. The *angle between two lines* is given by

$$\cos^{-1}\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \quad (2.1.17.1)$$

18. Show that the distance of a point $\mathbf{x}_0$ from the line

$$L: \quad \mathbf{n}^T\mathbf{x} = c \quad (2.1.18.1)$$

is

$$\frac{\left|\mathbf{n}^T\mathbf{x}_0 - c\right|}{\|\mathbf{n}\|} \quad (2.1.18.2)$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \quad (2.1.18.3)$$

where

$$\mathbf{n}^T\mathbf{A} = 0, \mathbf{n}^T\mathbf{m} = 0 \quad (2.1.18.4)$$

If $\mathbf{x}_0$ is translated to the origin, the equation of the line L becomes

$$\mathbf{x} = \mathbf{A} - \mathbf{x}_0 + \lambda\mathbf{m} \quad (2.1.18.5)$$
$$\implies \mathbf{n}^T\mathbf{x} = c - \mathbf{n}^T\mathbf{x}_0 \quad (2.1.18.6)$$

From (2.1.15.2), (2.1.18.4) is obtained.

19. Show that

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.1.19.1)$$

can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.1.19.2)$$

where

$$\mathbf{V} = \mathbf{V}^T \quad (2.1.19.3)$$
$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.1.19.4)$$

20. *Pair of straight lines:* (2.1.19.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.1.20.1)$$

Two intersecting lines are obtained if

$$|\mathbf{V}| < 0 \quad (2.1.20.2)$$

21. In Fig. 2.1.21, let

$$\frac{AB}{BC} = \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = k. \quad (2.1.21.1)$$

Show that

$$\frac{\mathbf{A} + k\mathbf{C}}{k + 1} = \mathbf{B}. \quad (2.1.21.2)$$

Solution: From (2.1.2.5),

$$\mathbf{B} = \mathbf{A} + \lambda_1 \mathbf{m} \tag{2.1.21.3}$$
$$\mathbf{B} = \mathbf{C} - \lambda_2 \mathbf{m}$$

$$\implies \frac{\|\mathbf{A} - \mathbf{B}\|}{\|\mathbf{B} - \mathbf{C}\|} = \frac{\lambda_1}{\lambda_2} = k \tag{2.1.21.4}$$

$$\text{and } \frac{\mathbf{B} - \mathbf{A}}{\lambda_1} = \frac{\mathbf{C} - \mathbf{B}}{\lambda_2} = \mathbf{m}, \tag{2.1.21.5}$$

from (2.1.21.1). Using (2.1.21.4) and (2.1.21.5),

$$\mathbf{A} - \mathbf{B} = k(\mathbf{B} - \mathbf{C}) \tag{2.1.21.6}$$

resulting in (2.1.21.2)

22. If $\mathbf{A}$ and $\mathbf{B}$ are linearly independent,

$$k_1\mathbf{A} + k_2\mathbf{B} = 0 \implies k_1 = k_2 = 0 \tag{2.1.22.1}$$

23. Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1\mathbf{A} + \lambda_2\mathbf{B} + \lambda_3\mathbf{C} \tag{2.1.23.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{2.1.23.2}$$
$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{2.1.23.3}$$

24. Show that the equation of the angle bisectors of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.1.24.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.1.24.2}$$

is

$$\frac{\mathbf{n}_1^T\mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm\frac{\mathbf{n}_2^T\mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{2.1.24.3}$$

25. Find the equation of a line passing through the

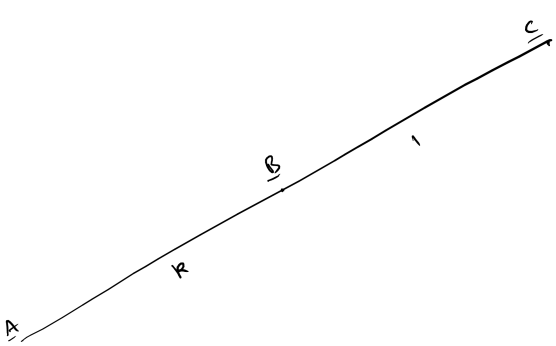

Fig. 2.1.21

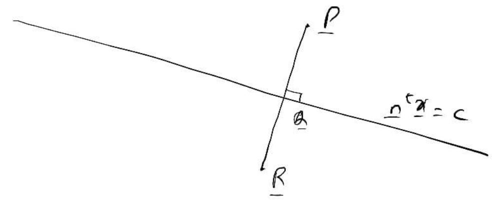

Fig. 2.1.26

intersection of the lines

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.1.25.1}$$
$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.1.25.2}$$

and passing through the point $\mathbf{p}$.

Solution: The intersection of the lines is

$$\mathbf{x} = \mathbf{N}^{-T}\mathbf{c} \tag{2.1.25.3}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \tag{2.1.25.4}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{2.1.25.5}$$

Thus, the equation of the desired line is

$$\mathbf{x} = \mathbf{p} + \lambda\left(\mathbf{N}^{-T}\mathbf{c} - \mathbf{p}\right) \tag{2.1.25.6}$$

$$\implies \mathbf{N}^T\mathbf{x} = \mathbf{N}^T\mathbf{p} + \lambda\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right) \tag{2.1.25.7}$$

resulting in

$$\left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{x} = \left(\mathbf{c} - \mathbf{N}^T\mathbf{p}\right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{N}^T\mathbf{p} \tag{2.1.25.8}$$

26. Find $\mathbf{R}$, the *reflection* of $\mathbf{P}$ about the line

$$L : \quad \mathbf{n}^T\mathbf{x} = c \tag{2.1.26.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{2.1.26.2}$$
$$\mathbf{n}^T\mathbf{Q} = c \tag{2.1.26.3}$$
$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{2.1.26.4}$$

where $\mathbf{m}$ is the direction vector of L. From

(2.1.26.2) and (2.1.26.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{2.1.26.5}$$

From (2.1.26.5) and (2.1.26.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.26.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{2.1.26.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{2.1.26.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{2.1.26.9}$$

(2.1.26.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.26.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.26.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{2.1.26.12}$$

27. Show that, for any $\mathbf{m}, \mathbf{n}$, the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.1.27.1}$$

2.2 *Intercepts*

1. Find the intercepts made on the axes by the straight lines whose equations are

a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 2$
b) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$
c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
d) $\begin{pmatrix}\frac{1}{a+b} & \frac{1}{a-b}\end{pmatrix}\mathbf{x} = \frac{1}{a^2-b^2}$
e) $\begin{pmatrix}1 & -m\end{pmatrix}\mathbf{x} = -c$

2. Write down the equations of straight lines which make the following pairs of intercepts on the axes:

a) 3,-4
b) -5,6
c) $\frac{1}{a}, \frac{1}{b}$
d) $2a, -2a$

3. A straight line passes through a fixed point $\begin{pmatrix}h\\k\end{pmatrix}$ and cuts the axes in $\mathbf{A}, \mathbf{B}$. Parallels to the axes through $\mathbf{A}$ and $\mathbf{B}$ intersect in $\mathbf{P}$. Find the equation of the locus of $\mathbf{P}$.

2.3 *Line Equation*

1. Find the equations of two straight lines at a distance 3 from the origin and making an angle of 120° with OX.
2. Find the equation of a straight line making an angle of 60° with OX and passing through the point $\begin{pmatrix}2\\-2\end{pmatrix}$. Transform the equation to the form

$$\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = p \tag{2.3.2}$$

3. Find the equation of the straight line that passes through the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$. What is its inclination to OX?
4. Find the equation of the straight line through the point $\begin{pmatrix}5\\7\end{pmatrix}$ that makes equal intercepts on the axes.
5. Find the equations of the sides of a triangle whose vertices are $\begin{pmatrix}2\\4\end{pmatrix}$, $\begin{pmatrix}-4\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$.
6. For the same triangle find the equations of the medians
7. Find the equation of a straight line passing through the point $\begin{pmatrix}2\\-3\end{pmatrix}$ parallel to the line $\begin{pmatrix}4 & -1\end{pmatrix}\mathbf{x} + 7 = 0$.
8. Find the intercepts on the axes made by a straight line which passes through the point $\begin{pmatrix}3\\-1\end{pmatrix}$ and makes an angle of 30° with OX.
9. Find the equation of the straight line through the points $\begin{pmatrix}3\\-4\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$ and of the parallel line through $\begin{pmatrix}5\\2\end{pmatrix}$.
10. What is the distance from the origin of the line $\begin{pmatrix}4 & -1\end{pmatrix}\mathbf{x} = 7$? Write down the equation of a parallel line at double the distance.
11. Find the equation of the straight line through the point $\begin{pmatrix}3\\-4\end{pmatrix}$ parallel to the line joining the origin to the point $\begin{pmatrix}2\\-1\end{pmatrix}$.

12. Write down the equation of the straight line which makes intercepts 2 and -7 on the axes, and of the parallel line through the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
13. Find the equations of the straight line joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and of the parallel line through the origin.
14. ABC is a triangle and $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$. Find the equation of the straight line through $\mathbf{A}$ parallel to BC.
15. Find the equation of a line parallel to $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 11$ passing through the middle point of the join of the points $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -11 \end{pmatrix}$.
16. The base of a triangle passes through a fixed point $\begin{pmatrix} f \\ g \end{pmatrix}$ and the sides are bisected at right angles by the axes. Prove that the locus of the vertex is the line
$$\begin{pmatrix} g & f \end{pmatrix} \mathbf{x} = 0 \tag{2.3.16}$$

2.4 Point of Intersection

1. Find the vertices of the triangle whose sides are
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} + 6 = 0, \tag{2.4.1.1}$$
$$\begin{pmatrix} 2 & -5 \end{pmatrix} \mathbf{x} + 4 = 0, \tag{2.4.1.2}$$
$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{2.4.1.3}$$
2. Prove that the lines
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 25 = 0, \tag{2.4.2.1}$$
$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 7 = 0 \tag{2.4.2.2}$$
$$\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.2.3}$$
are concurrent, and find the coordinates of their common point.
3. Find the equation of a line parallel to the line
$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{2.4.3.1}$$
and passing through the intersection of the lines
$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{2.4.3.2}$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 5 \tag{2.4.3.3}$$
4. Find the equation of the line joining the origin to the point of intersection of the lines
$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.4.1}$$
$$\begin{pmatrix} 2 & 7 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{2.4.4.2}$$
5. Find the acute angle between the lines
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -7 \tag{2.4.5.1}$$
$$\begin{pmatrix} 2 + \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 11 \tag{2.4.5.2}$$
6. Find the angle between the lines
$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{2.4.6.1}$$
$$\begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} + 11 = 0 \tag{2.4.6.2}$$
7. Find the equation of a straight line through the point $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ at right angles to the line
$$\begin{pmatrix} 5 & 7 \end{pmatrix} \mathbf{x} + 12 = 0 \tag{2.4.7}$$
and find the point in which the lines intersect.
8. Find the equation of a straight line through the origin and at right angles to the line
$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} + c = 0 \tag{2.4.8}$$
9. Find the equation of a straight line at right angles to the line
$$\begin{pmatrix} 5 & -2 \end{pmatrix} \mathbf{x} + 11 = 0 \tag{2.4.9.1}$$
and passing through the intersecton of the lines
$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} + 1 = 0, \tag{2.4.9.2}$$
$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 7. \tag{2.4.9.3}$$
10. The origin is a corner of a square and two of its sides have equations
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.10.1}$$
$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3. \tag{2.4.10.2}$$
Find the equations of the other two sides.
11. Write down the equations of the perpendiculars from the origin to the lines
$$\begin{pmatrix} 1 & 5 \end{pmatrix} \mathbf{x} = 13, \tag{2.4.11.1}$$
$$\begin{pmatrix} 5 & 1 \end{pmatrix} \mathbf{x} = 13 \tag{2.4.11.2}$$
and find the equation of the line joining the feet of the perpendiculars.

12. Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 11 \tag{2.4.12.1}$$

makes equal angles with the lines

$$\begin{pmatrix}1 & -\left(2-\sqrt{3}\right)\end{pmatrix}\mathbf{x} + 2 = 0, \tag{2.4.12.2}$$

$$\begin{pmatrix}\left(2-\sqrt{3}\right) & -1\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.4.12.3}$$

13. $\mathbf{A}$ is the point $\begin{pmatrix}-4\\0\end{pmatrix}$ and $\mathbf{B}$ is the point $\begin{pmatrix}3\\0\end{pmatrix}$. Find the locus of a point $\mathbf{P}$ such that the angles APO, OPB are equal, where $\mathbf{O}$ is the origin.

2.5 Perpendiculars and Bisectors

1. Find the distance of the point $\begin{pmatrix}4\\2\sqrt{3}\end{pmatrix}$ from the line

$$\begin{pmatrix}\cos 60° & \sin 60°\end{pmatrix}\mathbf{x} = 6 \tag{2.5.1}$$

2. Find the distance of the point $\begin{pmatrix}2\\3\end{pmatrix}$ from each of the straight lines

$$\begin{pmatrix}5 & 12\end{pmatrix}\mathbf{x} - 20 = 0 \tag{2.5.2.1}$$

$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.5.2.2}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} - 28 = 0. \tag{2.5.2.3}$$

3. Find the distance of the pont $\begin{pmatrix}3\\-1\end{pmatrix}$ from the line joining the points $\begin{pmatrix}2\\-3\end{pmatrix}$, $\begin{pmatrix}4\\1\end{pmatrix}$.

4. Are the points $\begin{pmatrix}-2\\3\end{pmatrix}$, $\begin{pmatrix}-2\\4\end{pmatrix}$ on the same or on opposite sides of the line

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 10? \tag{2.5.4}$$

5. Find the equations of the bisectors of the angles between the lines

$$\begin{pmatrix}-4 & 2\end{pmatrix}\mathbf{x} = -9 \tag{2.5.5.1}$$

$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 4 \tag{2.5.5.2}$$

and state which equation refers to the angle which contains the origin.

6. Prove that the bisector of one of the angles between the lines

$$\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{x} - 7 = 0 \tag{2.5.6.1}$$

$$\begin{pmatrix}1 & -5\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2.5.6.2}$$

passes through the origin. What is the equation of the bisector of the other angle?

7. What is the condition that the point $\begin{pmatrix}x\\y\end{pmatrix}$ may be at unit distance from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 10 = 0 \tag{2.5.7}$$

Write down the equations of two straight lines parallel to the given line and at unit distances from it, and state which of the two lies on the same side of the given line as the origin.

8. The sides AB, BC, CA of a triangle have equations

$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 12 \tag{2.5.8.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.5.8.2}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2. \tag{2.5.8.3}$$

Find the coordinates of the centres of the inscribed circle and of the escribed circle opposite to the vertex $\mathbf{A}$.

9. Prove that the point $\begin{pmatrix}4\\4\end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.5.9.1}$$

$$\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = 15 \tag{2.5.9.2}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.5.9.3}$$

10. Find the equation of the line joining the origin to the point of intersection of the lines

$$\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} - 11 = 0 \tag{2.5.10.1}$$

$$\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.5.10.2}$$

11. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.5.11.1}$$

and passing through the intersection of the lines

$$\begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 1 \tag{2.5.11.2}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.5.11.3}$$

12. Find the equation of a line through the intersection of the lines

$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 1 \tag{2.5.12.1}$$

$$\begin{pmatrix}-4 & 1\end{pmatrix}\mathbf{x} = 9 \tag{2.5.12.2}$$

parallel to the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{2.5.12.3}$$

13. The vertices of a triangle are at the points

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \tag{2.5.13}$$

Find the equations of the medians and prove that they meet in a point. What are the coordinates of their point of intersection?
BE and CF are medians of $\triangle ABC$ intersecting at $\mathbf{O}$ as shown in Fig. 2.5.13. We first show that

$$\frac{CO}{OF} = \frac{BO}{OE} = 2 \tag{2.5.13.2}$$

Let

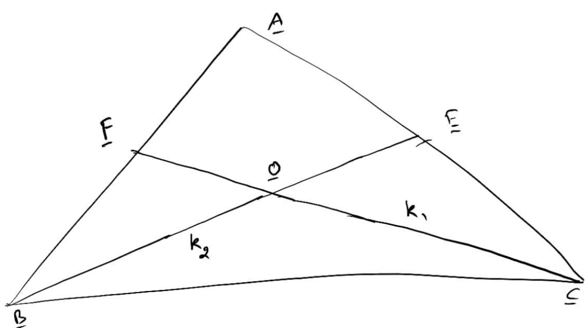

Fig. 2.5.13

$$\frac{CO}{OF} = k_1 \tag{2.5.13.3}$$

$$\frac{BO}{OE} = k_2 \tag{2.5.13.4}$$

Using (2.1.21.2),

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{2.5.13.5}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{2.5.13.6}$$

and

$$\mathbf{O} = \frac{k_1\mathbf{F} + \mathbf{C}}{k_1 + 1} = \frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} \tag{2.5.13.7}$$

$$\mathbf{O} = \frac{k_2\mathbf{E} + \mathbf{B}}{k_2 + 1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \tag{2.5.13.8}$$

From (2.5.13.7) and (2.5.13.8),

$$\frac{k_1\frac{\mathbf{A}+\mathbf{B}}{2} + \mathbf{C}}{k_1 + 1} = \frac{k_2\frac{\mathbf{A}+\mathbf{C}}{2} + \mathbf{B}}{k_2 + 1} \tag{2.5.13.9}$$

$$\begin{aligned} \implies &\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right]\mathbf{A} \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \end{aligned} \tag{2.5.13.10}$$

resulting in $k_1 = k_2$,

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = k_2 = 2, \tag{2.5.13.11}$$

provided $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. Thus, substituting $k_1 = 2$ in (2.5.13.8),

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{2.5.13.12}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly dependent,

$$\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{C} \tag{2.5.13.13}$$

Note that $\mathbf{B}, \mathbf{C}$ are linearly independent. Substituting (2.5.13.13) in (2.5.13.10),

$$\begin{aligned} &\left[\frac{k_1(k_2+1)}{2} - \frac{k_2(k_1+1)}{2}\right][\alpha\mathbf{B} + \beta\mathbf{C}] \\ &+ \left[\frac{k_1(k_2+1)}{2} - (k_1+1)\right]\mathbf{B} \\ &+ \left[(k_2+1) - \frac{k_2(k_1+1)}{2}\right]\mathbf{C} = 0 \end{aligned} \tag{2.5.13.14}$$

$$\implies \begin{aligned} (k_1 - k_2)\alpha + k_1k_2 - k_1 - 2 = 0 \\ (k_1 - k_2)\beta - k_1k_2 + k_2 + 2 = 0 \end{aligned} \tag{2.5.13.15}$$

$$\implies (k_1 - k_2)(\alpha + \beta - 1) = 0 \tag{2.5.13.16}$$

If $\alpha + \beta = 1, \mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear according to (2.1.21.2) resulting in a contradiction. Hence, $k_1 = k_2$, which, upon substitution in (2.5.13.15), yields

$$k_1^2 - k_1 - 2 = 0 \implies k_1 = 2. \tag{2.5.13.17}$$

14. For what multiples k, l, m is the equation

$$\begin{aligned} k\left\{\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 13\right\} + l\left\{\begin{pmatrix}5 & -y\end{pmatrix}\mathbf{x} - 7\right\} \\ +m\left\{\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} + 10\right\} = 0 \end{aligned} \tag{2.5.14.1}$$

an identity? In what point do the lines given by equating the three terms to zero concur?

15. Find the equations of the diagonals of the

parallelogram

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x}+7=0 \tag{2.5.15.1}$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x}-5=0, \tag{2.5.15.2}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5=0 \tag{2.5.15.3}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}+4=0 \tag{2.5.15.4}$$

16. The vertices of a triangle are at the points
$$\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}4\\-3\end{pmatrix},\begin{pmatrix}-2\\1\end{pmatrix} \tag{2.5.16}$$
Find the equations of the perpendiculars to the sides through their middle points.
17. Work the same problem when the vertices of the triangle are at the points
$$\mathbf{A},\mathbf{B},\mathbf{C} \tag{2.5.17.1}$$
and show that the perpendiculars meet in a point.
Solution: In Fig. 2.5.17, $BE \perp AC, CF \perp AB$. We need to show that $AD \perp BC$. Let $\mathbf{x}$ be the

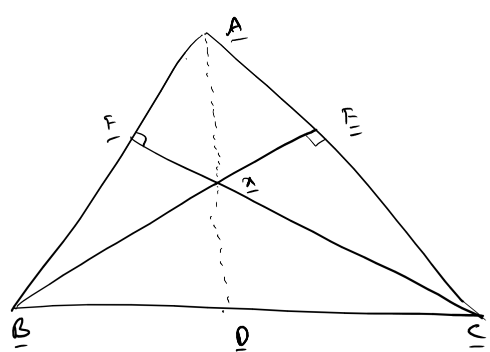

Fig. 2.5.17

intersection of BE and CF. Then, using (1.1.9),
$$\begin{aligned}(\mathbf{x}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})&=0\\(\mathbf{x}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})&=0\end{aligned} \tag{2.5.17.2}$$
$$\implies \mathbf{x}^T(\mathbf{A}-\mathbf{C})-\mathbf{B}^T(\mathbf{A}-\mathbf{C})=0 \tag{2.5.17.3}$$
$$\text{and } \mathbf{x}^T(\mathbf{A}-\mathbf{B})-\mathbf{C}^T(\mathbf{A}-\mathbf{B})=0 \tag{2.5.17.4}$$
Subtracting (2.5.17.4) from (2.5.17.3),
$$\mathbf{x}^T(\mathbf{B}-\mathbf{C})+\mathbf{A}^T(\mathbf{C}-\mathbf{B})=0 \tag{2.5.17.5}$$
$$\implies \left(\mathbf{x}^T-\mathbf{A}^T\right)(\mathbf{B}-\mathbf{C})=0 \tag{2.5.17.6}$$
$$\implies (\mathbf{x}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})=0 \tag{2.5.17.7}$$
which completes the proof.

18. The line
$$\begin{pmatrix}2 & -8\end{pmatrix}\mathbf{x}-4=0 \tag{2.5.18}$$
is the perpendicular bisector of the line AB and $\mathbf{A}$ is the point $\begin{pmatrix}5\\6\end{pmatrix}$. What are the coordinates of $\mathbf{B}$?

2.6 Angle Between Lines

1. What lines are represented by the following equations:
   a) $\mathbf{x}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x}=0$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x}=0$
   c) $\mathbf{x}^T\begin{pmatrix}6 & \frac{1}{2}\\\frac{1}{2} & -1\end{pmatrix}\mathbf{x}=0$
   d) $\mathbf{x}^T\begin{pmatrix}-1 & -\tan\theta\\\tan\theta & 1\end{pmatrix}\mathbf{x}=0$
   e) $x^3+3x^2y-3xy^2-y^3=0$
2. Find the angles between the pairs of straight lines represented by the following equations:
   a) $\mathbf{x}^T\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\mathbf{x}=0$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x}=0$
   c) $\mathbf{x}^T\begin{pmatrix}1 & -\frac{5}{2}\\-\frac{5}{2} & 4\end{pmatrix}\mathbf{x}=0$
   d) $\mathbf{x}^T\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix}\mathbf{x}=0$
   e) $\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\\frac{1}{2} & -6\end{pmatrix}\mathbf{x}=0$
3. Prove that the equations
$$\mathbf{x}^T\begin{pmatrix}1 & -2\\-2 & 1\end{pmatrix}\mathbf{x}=0 \tag{2.6.3.1}$$
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=3 \tag{2.6.3.2}$$
are the sides of an equilateral triangle.

2.7 Miscellaneous

1. Find the locus of a point which is equidistant from the points $\begin{pmatrix}6\\-3\end{pmatrix},\begin{pmatrix}-4\\7\end{pmatrix}$.
2. Find the point on the line
$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x}+7=0 \tag{2.7.2}$$
which is equidistant from the points $\begin{pmatrix}2\\-3\end{pmatrix}$, $\begin{pmatrix}-4\\1\end{pmatrix}$.

3. Find the coordinates of the circumcentre of the triangle whose corners are at the points $\begin{pmatrix}4\\3\end{pmatrix}$, $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}2\\-2\end{pmatrix}$.
4. Find the equations of the lines through $\begin{pmatrix}3\\1\end{pmatrix}$ which are respectively parallel and perpendicular to the line joining the points $\begin{pmatrix}2\\4\end{pmatrix}$, $\begin{pmatrix}5\\-6\end{pmatrix}$.
5. Find the locus of a point at which the join of the points $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-3\\4\end{pmatrix}$ subtends a right angle.
6. Find the orthocentre of a triangle whose corners are at the points $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-4\end{pmatrix}$, $\begin{pmatrix}6\\2\end{pmatrix}$.
7. Prove that the line joining the points $\begin{pmatrix}2\\-1\end{pmatrix}$, $\begin{pmatrix}-3\\5\end{pmatrix}$ makes with the axes a triangle of area $\frac{49}{60}$.
8. $ABCD$ is a parallelogram and the coordinates of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $\begin{pmatrix}2\\4\end{pmatrix}$, $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ Find the coordinates of $\mathbf{D}$.
9. Find the area of the triangle formed by the lines
$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{2.7.9.1}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 7 \tag{2.7.9.2}$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + 2 = 0 \tag{2.7.9.3}$$
10. Find the centre of the inscribed circle of the triangle whose sides are
$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 0 \tag{2.7.10.1}$$
$$\begin{pmatrix}12 & -5\end{pmatrix}\mathbf{x} = 0, \tag{2.7.10.2}$$
$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 8 \tag{2.7.10.3}$$
11. The ends of a diagonal of a square are on the coordinate axes at the points $\begin{pmatrix}2a\\0\end{pmatrix}$, $\begin{pmatrix}0\\a\end{pmatrix}$. Find the equations of the sides.
12. The sides of a triangle ABC are
$$AB = 3, BC = 5, CA = 4 \tag{2.7.12}$$
and $\mathbf{A}, \mathbf{B}$ are on the axes OX, OY respectively, while AC makes an angle θ with OX. Prove that the locus of $\mathbf{C}$, as θ varies, is given by the equation
$$\mathbf{x}^T\begin{pmatrix}16 & -12\\-12 & 25\end{pmatrix}\mathbf{x} = 256 \tag{2.7.12}$$
13. Prove that the locus of a point at which the join of the points $\begin{pmatrix}a\\0\end{pmatrix}$ and $\begin{pmatrix}-a\\0\end{pmatrix}$ subtends an angle of 45° is
$$\mathbf{x}^T\mathbf{x} - 2a\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = a^2 \tag{2.7.13}$$
14. Prove that the line
$$\mathbf{n}^T\mathbf{x} + c = 0 \tag{2.7.14.1}$$
divides the line joining the points $\mathbf{x}_1, \mathbf{x}_2$ in the ratio
$$-\frac{\mathbf{n}^T\mathbf{x}_1 + c}{\mathbf{n}^T\mathbf{x}_2 + c} \tag{2.7.14.2}$$
15. Find the equation of the line joining the point $\mathbf{x}_1$, to the point of intersection of the lines
$$\mathbf{n}^T\mathbf{x} + c = 0 \tag{2.7.15.1}$$
$$\mathbf{n}_1^T\mathbf{x} + c_1 = 0 \tag{2.7.15.2}$$
16. Find the equations of the diagonals of the parallelogram whose sides are
$$\mathbf{n}^T\mathbf{x} + c = 0 \tag{2.7.16.1}$$
$$\mathbf{n}^T\mathbf{x} + d = 0 \tag{2.7.16.2}$$
$$\mathbf{n}_1^T\mathbf{x} + c_1 = 0 \tag{2.7.16.3}$$
$$\mathbf{n}_1^T\mathbf{x} + d_1 = 0 \tag{2.7.16.4}$$
17. Prove that for all values of k the line
$$\begin{pmatrix}2 + k & 1 - 2k\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.7.17}$$
passes through a fixed point, and find its coordinates.
18. Find the angle between the lines
$$\mathbf{x}^T\begin{pmatrix}1 & -\sec\theta\\-\sec\theta & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.18}$$
19. Prove that the pairs of straight lines represented by
$$\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\\frac{1}{2} & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.7.19.1}$$
$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\-\frac{1}{2} & -1\end{pmatrix}\mathbf{x} = 0 \tag{2.7.19.2}$$
are such that the angles between one pair are equal to the angles between the other pair.

20. Find the angles between the lines
$$x^3 - 3x^2y - 3xy^2 + y^3 = 0 \quad (2.7.20)$$
21. Find the area of the triangle whose sides are given by
$$\mathbf{x}^T \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{x} = 0 \quad (2.7.21.1)$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 7 \quad (2.7.21.2)$$
22. Show that the equation
$$\mathbf{x}^T \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -11 & 31 \end{pmatrix} \mathbf{x} - 10 = 0 \quad (2.7.22)$$
represents two straight lines, and find the equations of the bisectors of the angles between them.
23. For what value of k does the equation
$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & k \end{pmatrix} \mathbf{x} + \begin{pmatrix} 13 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (2.7.23)$$
represent two straight lines? What is the angle between them?
24. For what values of k does the equation
$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{k}{2} \\ \frac{k}{2} & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (2.7.24)$$
represent two straight lines?

3 Curves

3.1 Properties

1. The equation of a quadratic curve is given by
$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.1.1)$$
2. Show that
$$\frac{d\left(\mathbf{u}^T \mathbf{x}\right)}{d\mathbf{x}} = \mathbf{u} \quad (3.1.2)$$
3. Show that
$$\frac{d\left(\mathbf{x}^T \mathbf{V} \mathbf{x}\right)}{d\mathbf{x}} = 2\mathbf{V}^T \mathbf{x} \quad (3.1.3)$$
4. Show that
$$\frac{d\mathbf{x}}{dx_1} = \mathbf{m} \quad (3.1.4)$$
5. Find the *normal* vector to the curve in (3.1.1) at point $\mathbf{p}$.

Solution: Differentiating (3.1.1) with respect to x_1,
$$\frac{d\left(\mathbf{x}^T \mathbf{V} \mathbf{x}\right)}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} + \frac{d\left(\mathbf{u}^T \mathbf{x}\right)}{d\mathbf{x}} \frac{d\mathbf{x}}{dx_1} = 0 \quad (3.1.5.1)$$
$$\implies 2\mathbf{x}^T \mathbf{V} \mathbf{m} + 2\mathbf{u}^T \mathbf{m} = 0 \because \left(\frac{d\mathbf{x}}{dx_1} = \mathbf{m}\right) \quad (3.1.5.2)$$
Substituting $\mathbf{x} = \mathbf{p}$ and simplifying
$$(\mathbf{V}\mathbf{p} + \mathbf{u})^T \mathbf{m} = 0 \quad (3.1.5.3)$$
$$\implies \mathbf{n} = \mathbf{V}\mathbf{p} + \mathbf{u} \quad (3.1.5.4)$$
6. The *tangent* to the curve at $\mathbf{p}$ is given by
$$\mathbf{n}^T (\mathbf{x} - \mathbf{p}) = 0 \quad (3.1.6)$$
This results in
$$\left(\mathbf{p}^T \mathbf{V} + \mathbf{u}^T\right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + f = 0 \quad (3.1.6)$$
7. Let $\mathbf{P}$ be a rotation matrix and $\mathbf{c}$ be a vector. Then
$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \quad (3.1.7)$$
is known as an *affine* transformation.
8. Classify the various conic sections based on (3.1.1).
Solution:

Curve	Property
Circle	$V = kI$
Parabola	$\det(V) = 0$
Ellipse	$\det(V) > 0$
Hyperbola	$\det(V) < 0$

TABLE 3.1.8

3.2 Circle

1. Find the centre and radius of the circle
$$C_1 : \mathbf{x}^T \mathbf{x} - \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (3.2.1.1)$$

Solution: let $\mathbf{c}$ be the centre of the circle. Then
$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \quad (3.2.1.2)$$
$$\implies (\mathbf{x} - \mathbf{c})^T (\mathbf{x} - \mathbf{c}) = r^2 \quad (3.2.1.3)$$
$$\implies \mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} = r^2 - \mathbf{c}^T \mathbf{c} \quad (3.2.1.4)$$

Comparing with (3.2.1.1),

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.2.1.5}$$

$$r^2 - \mathbf{c}^T \mathbf{c} = 1 \implies r = \sqrt{2} \tag{3.2.1.6}$$

2. Find the tangent to the circle C_1 at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Solution: From (3.1.6), the tangent T is given by

$$\left[\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right] \mathbf{x} - \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{3.2.2.1}$$

$$\implies T : \mathbf{n}^T \mathbf{x} = 3 \tag{3.2.2.2}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.2.2.3}$$

3. The tangent T in (3.2.2.2) cuts off a chord AB from a circle C_2 whose centre is

$$\mathbf{C} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}. \tag{3.2.3.1}$$

Find $\mathbf{A} + \mathbf{B}$.
Solution: Let the radius of C_2 be r. From the given information,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{C}) = r^2 \tag{3.2.3.2}$$

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = r^2 \tag{3.2.3.3}$$

Subtracting (3.2.3.3) from (3.2.3.2),

$$\mathbf{A}^T \mathbf{A} - \mathbf{B}^T \mathbf{B} - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.2.3.4}$$

$$\implies (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) - 2\mathbf{C}^T (\mathbf{A} - \mathbf{B}) = 0$$

$$\implies (\mathbf{A} + \mathbf{B} - 2\mathbf{C})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{3.2.3.5}$$

$\because$ $\mathbf{A}, \mathbf{B}$ lie on T, from (3.2.2.2),

$$\mathbf{n}^T \mathbf{A} = \mathbf{n}^T \mathbf{B} = 3 \tag{3.2.3.6}$$

$$\implies \mathbf{n}^T (\mathbf{A} - \mathbf{B}) = 0, \tag{3.2.3.7}$$

From (3.2.3.5) and (3.2.3.7)

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = k\mathbf{n} \tag{3.2.3.8}$$

$$\implies \mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C} = k\mathbf{n}^T \mathbf{n} \tag{3.2.3.9}$$

$$\implies \frac{\mathbf{n}^T \mathbf{A} + \mathbf{n}^T \mathbf{B} - 2\mathbf{n}^T \mathbf{C}}{\mathbf{n}^T \mathbf{n}} = k \tag{3.2.3.10}$$

$$\implies k = 2 \tag{3.2.3.11}$$

using (3.2.3.6). Substituting in (3.2.3.8)

$$\mathbf{A} + \mathbf{B} = 2 (\mathbf{n} + \mathbf{C}) \tag{3.2.3.12}$$

4. If $AB = 4$, find $\mathbf{A}^T \mathbf{B}$.
Solution: From the given information,

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{3.2.4.1}$$

resulting in

$$\|\mathbf{A} + \mathbf{B}\|^2 - \|\mathbf{A} - \mathbf{B}\|^2 = 4 \|\mathbf{n} + \mathbf{C}\|^2 - 4^2 \tag{3.2.4.2}$$

$$\implies \mathbf{A}^T \mathbf{B} = \|\mathbf{n} + \mathbf{C}\|^2 - 4 = 17 \tag{3.2.4.3}$$

using (3.2.3.12) and simplifying.

5. Show that

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 8 - r^2 \tag{3.2.5.1}$$

Solution:

$$\|\mathbf{A} - \mathbf{B}\|^2 = 4^2 \tag{3.2.5.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) = 4^2 \tag{3.2.5.3}$$

From (3.2.5.3),

$$[(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})]^T \ [(\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C})] = 4^2 \tag{3.2.5.4}$$

which can be expressed as

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + 2 (\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 4^2 \tag{3.2.5.5}$$

Upon substituting from (3.2.3.3) and (3.2.3.2) and simplifying, (3.2.5.1) is obtained.

6. Find r.
Solution: (3.2.5.1) can be expressed as

$$\mathbf{A}^T \mathbf{B} - \mathbf{C}^T (\mathbf{A} + \mathbf{B}) + \mathbf{C}^T \mathbf{C} = 8 - r^2 \tag{3.2.6.1}$$

$$\implies 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (\mathbf{A} + \mathbf{B}) - \mathbf{C}^T \mathbf{C} = r^2 \tag{3.2.6.2}$$

$$\implies 8 - \mathbf{A}^T \mathbf{B} + \mathbf{C}^T (2\mathbf{n} + \mathbf{C}) = r^2 \tag{3.2.6.3}$$

$$\implies r = \sqrt{6}. \tag{3.2.6.4}$$

7. Summarize all the above computations through a Python script and plot the tangent and circle.
Solution: The following code generates Fig. 3.2.7.

```
wget
```

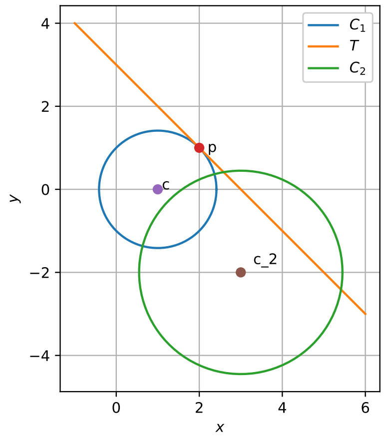

Fig. 3.2.7

```
https://github.com/gadepall/
    school/raw/master/linalg/2D/
    manual/codes/circ.py
```

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3.3 Parabola

1. A standard *parabola* has the equation
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.3.1.1}$$
2. Any point on (3.3.1.1) can be expressed as
$$at\begin{pmatrix}t\\2t\end{pmatrix} \tag{3.3.2.1}$$
3. Find the tangent at $\begin{pmatrix}1\\7\end{pmatrix}$ to the parabola
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} + 6 = 0 \tag{3.3.3.1}$$
Solution: Substituting
$$\mathbf{p} = \begin{pmatrix}1\\7\end{pmatrix}, V = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} \tag{3.3.3.2}$$
in (3.1.6), the desired equation is
$$\left[\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -1\end{pmatrix}\right]\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 7\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} + 6 = 0 \tag{3.3.3.3}$$
resulting in
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -5 \tag{3.3.3.4}$$
4. The line in (3.3.3.4) touches the circle
$$\mathbf{x}^T\mathbf{x} + 4\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + c = 0 \tag{3.3.4.1}$$
Find c.
Solution: Comparing (3.1.1) and (3.3.4.1),
$$\begin{aligned} V &= I, \\ \mathbf{u} &= 2\begin{pmatrix}4\\3\end{pmatrix} \end{aligned} \tag{3.3.4.2}$$
Comparing (3.1.6) and (3.3.3.4),
$$\mathbf{p} + 2\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{3.3.4.3}$$
$$\implies \mathbf{p} = -\begin{pmatrix}6\\7\end{pmatrix} \tag{3.3.4.4}$$
and
$$c + \mathbf{p}^T\mathbf{u} = 5 \tag{3.3.4.5}$$
$$\implies c = 5 + 2\begin{pmatrix}6 & 7\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} \tag{3.3.4.6}$$
$$= 95 \tag{3.3.4.7}$$
5. Plot the parabola, tangent and circle.
Solution: See Fig. 3.3.5.

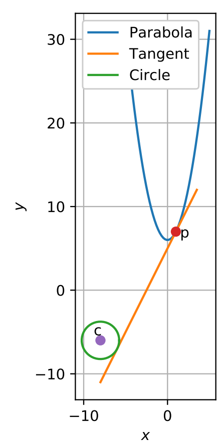

Fig. 3.3.5

3.4 Affine Transformation

1. In general, Fig. 3.3.5 was generated using an *affine transformation*.

2. Express

$$y_2 = y_1^2 \tag{3.4.2.1}$$

as a matrix equation.
Solution: (3.4.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{g}^T\mathbf{y} = 0 \tag{3.4.2.2}$$

where

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{g} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.2.3}$$

3. Given

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + F = 0, \tag{3.4.3.1}$$

where

$$\mathbf{V} = \mathbf{V}^T, \det(\mathbf{V}) = 0, \tag{3.4.3.2}$$

and $\mathbf{P}, \mathbf{c}$ such that

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}. \tag{3.4.3.3}$$

(3.4.3.3) is known as an affine transformation. Show that

$$\begin{aligned} \mathbf{D} &= \mathbf{P}^T\mathbf{V}\mathbf{P} \\ \mathbf{g} &= \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) \\ F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} &= 0 \end{aligned} \tag{3.4.3.4}$$

Solution: Substituting (3.4.3.3) in (3.4.3.1),

$$\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right)^T\mathbf{V}\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + 2\mathbf{u}^T\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + F = 0, \tag{3.4.3.5}$$

which can be expressed as

$$\begin{aligned} \implies \mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right)^T\mathbf{P}\mathbf{y} \\ + F + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} = 0 \end{aligned} \tag{3.4.3.6}$$

Comparing (3.4.3.6) with (3.4.2.2) (3.4.3.4) is obtained.

4. Show that there exists a $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{3.4.4.1}$$

Find $\mathbf{P}$ using

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.4.2}$$

5. Find $\mathbf{c}$ from (3.4.3.4).

Solution:

$$\because \mathbf{g} = \mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right), \tag{3.4.5.1}$$

$$\mathbf{V}\mathbf{c} = \mathbf{P}\mathbf{g} - \mathbf{u} \tag{3.4.5.2}$$

$$\implies \mathbf{c}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\left(\mathbf{P}\mathbf{g} - \mathbf{u}\right) = -F - 2\mathbf{u}^T\mathbf{c} \tag{3.4.5.3}$$

resulting in the matrix equation

$$\begin{pmatrix} \mathbf{V} \\ \left(\mathbf{P}\mathbf{g} + \mathbf{u}\right)^T \end{pmatrix}\mathbf{c} = \begin{pmatrix} \mathbf{P}\mathbf{g} - \mathbf{u} \\ -F \end{pmatrix} \tag{3.4.5.4}$$

for computing $\mathbf{c}$.

.

3.5 Ellipse

1. Express the following equation in the form given in (3.1.1)

$$E: 5x_1^2 - 6x_1x_2 + 5x_2^2 + 22x_1 - 26x_2 + 29 = 0 \tag{3.5.1.1}$$

Solution: (3.5.1.1) can be expressed as

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 29 = 0 \tag{3.5.1.2}$$

where

$$V = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 11 \\ -13 \end{pmatrix} \tag{3.5.1.3}$$

2. Using the affine transformation in (3.1.7), show that (3.5.1.2) can be expressed as

$$\mathbf{y}^T D\mathbf{y} = 1 \tag{3.5.2.1}$$

where

$$\mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.5.2.2}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.5.2.3}$$

for

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I} \tag{3.5.2.4}$$

3. Find $\mathbf{c}$
Solution:

$$\mathbf{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.5.3.1}$$

4. If

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.5.4.1}$$

$$P = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \tag{3.5.4.2}$$

show that

$$Vz = \lambda z \tag{3.5.4.3}$$

where $\lambda \in \{\lambda_1, \lambda_2\}, \mathbf{z} \in \{\mathbf{P}_1, \mathbf{P}_2\}$.

5. Find λ.
Solution: λ is obtained by solving the following equation.

$$|\lambda I - V| = 0 \tag{3.5.5.1}$$

$$\implies \begin{vmatrix} \lambda - 5 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = 0 \tag{3.5.5.2}$$

$$\implies \lambda^2 - 10\lambda + 16 = 0 \tag{3.5.5.3}$$

$$\implies \lambda = 2, 8 \tag{3.5.5.4}$$

6. Sketch 3.5.2.1.
7. Find $\mathbf{P}_1$ and $\mathbf{P}_2$.
Solution: From (3.5.4.3)

$$V\mathbf{P}_1 = \lambda_1 \mathbf{P}_1 \tag{3.5.7.1}$$

$$\implies (V - \lambda I)\,\mathbf{y} = 0 \tag{3.5.7.2}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \tag{3.5.7.3}$$

$$\text{or, } \mathbf{P}_1 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.5.7.4}$$

Similarly,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{P}_2 = 0 \tag{3.5.7.5}$$

$$\text{or, } \mathbf{P}_2 = k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.5.7.6}$$

8. Find $\mathbf{P}$.
Solution: From (3.5.2.4) and (3.5.4.2),

$$k_1 = \frac{1}{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \tag{3.5.8.1}$$

$$k_2 = \frac{1}{\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{2}} \tag{3.5.8.2}$$

Thus,

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.5.8.3}$$

9. Find the equation of the major axis for E.
Solution: The major axis for (3.5.2.1) is the line

$$\mathbf{y} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.5.9.1}$$

Using the affine transformation in (3.1.7)

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.5.9.2}$$

$$\implies \mathbf{x} - \mathbf{c} = \lambda_1 \mathbf{P}_1 \tag{3.5.9.3}$$

$$\text{or, } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.5.9.4}$$

$$= -3 \tag{3.5.9.5}$$

since

$$P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{P}_1 \text{ and } \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{P}_1 = 0 \tag{3.5.9.6}$$

which is the major axis of the ellipse E.

10. Find the minor axis of E.
11. Let $\mathbf{F}_1, \mathbf{F}_2$ be such that

$$\|\mathbf{x} - \mathbf{F}_1\| + \|\mathbf{x} - \mathbf{F}_2\| = 2k \tag{3.5.11.1}$$

Find $\mathbf{F}_1, \mathbf{F}_2$ and k.

12. Plot the ellipses in (3.5.1.1) and (3.5.2.1).
Solution: See Fig. 3.5.12

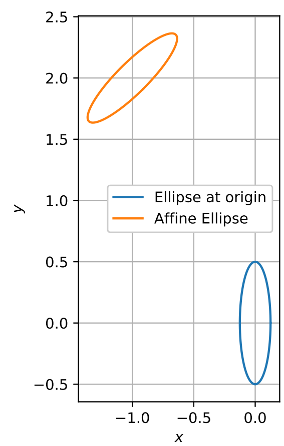

Fig. 3.5.12

3.6 Hyperbola

1. Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \tag{3.6.1.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.6.1.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{3.6.1.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (3.1.6) as

$$\mathbf{P}^T V\mathbf{x} = 36 \quad (3.6.1.4)$$
$$\mathbf{Q}^T V\mathbf{x} = 36. \quad (3.6.1.5)$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (3.6.1.6)$$
$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \quad (3.6.1.7)$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \quad (3.6.1.8)$$
$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \quad (3.6.1.9)$$

which is the equation of PQ.

2. In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad (3.6.2.1)$$

using (2.1.9.2) and (3.6.1.9), the equation of TD is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \quad (3.6.2.2)$$
$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (3.6.2.3)$$

3. Find D.
Solution: From (3.6.1.9) and (3.6.2.3),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (3.6.3.1)$$
$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \quad (3.6.3.2)$$

4. Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \quad (3.6.4.1)$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.6.4.2)$$

5. Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \quad (3.6.5.1)$$

can be expressed as

$$\lambda^2 \mathbf{m}^T V \mathbf{m} + 2\lambda \mathbf{m}^T V \mathbf{D} + \mathbf{D}^T V \mathbf{D} + F = 0 \quad (3.6.5.2)$$

6. Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (3.6.4.1) and (3.6.1.1) (3.6.5.2) is obtained. Substituting from (3.6.4.2), (3.6.1.2) and (3.6.3.2)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \quad (3.6.6.1)$$
$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \quad (3.6.6.2)$$
$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \quad (3.6.6.3)$$

Substituting in (3.6.5.2)

$$4\lambda^2 - 144 = 36 \quad (3.6.6.4)$$
$$\implies \lambda = \pm 3\sqrt{5} \quad (3.6.6.5)$$

Substituting in (3.6.4.1),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3\begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \quad (3.6.6.6)$$
$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3\begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \quad (3.6.6.7)$$

7. Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \quad (3.6.7.1)$$
$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \quad (3.6.7.2)$$

the desired area is

$$\frac{1}{2} PQ \times TD = 45\sqrt{5} \quad (3.6.7.3)$$

8. Repeat the previous exercise using determinants.
9. Plot the hyperbola.
Solution: See Fig. 3.6.9

3.7 Tangent

1. Find the equations of the tangents to the following curves at the points specified:

a) $y = x\left(x^2 - 1\right), x = 2$
b) $y = x^2 + \frac{1}{x^2}, x = 1$
c) $y = x^3 + 2x, x = 0$
d) $y = \left(x + \frac{1}{x}\right)^3, x = 2$
e) $y = \left(x^2 - 1\right)^2, x = 1$
f) $y = x^3 - x + 1, x = 3$
g) $y = (x - a)^3, x = 2a$
h) $y = ax^2 + 2bx + c, (x_1, y_1)$
i) $y = \frac{x^3}{a^3} + \frac{a^3}{x^3}, x = a$
j) $y = \frac{x^2}{a} + \frac{a^2}{x}, x = a$

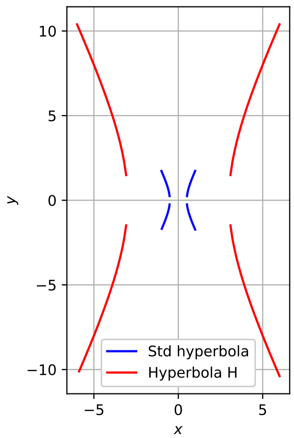

Fig. 3.6.9

2. Find the tangents to the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.2.1}$$

at the points where it is cut by the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{3.7.2.2}$$

and find the point of intersection of the tangents.

3. Prove that the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{3.7.3.1}$$

touches the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + 1 = 0. \tag{3.7.3.2}$$

4. Find the points on the curve

$$3y = x^3 + 3x \tag{3.7.4.1}$$

at which the tangent is parallel to the line

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.4.2}$$

5. Find at what points on the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{3.7.5}$$

the tangents pass through the origin.

6. Show that there are three points on the curve

$$3y = 3x^4 + 8x^3 - 6x^2 \tag{3.7.6.1}$$

at which the tangents are parallel to the line

$$\begin{pmatrix} 8 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.7.6.2}$$

7. Show that the line

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 17 \tag{3.7.7.1}$$

meets the curve

$$y = x^2 + \frac{1}{x^2} \tag{3.7.7.2}$$

in four points and that two of the points of intersection of the tangents at these four points are on the line

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} + 1 = 0, \tag{3.7.7.3}$$

and two are on the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0. \tag{3.7.7.4}$$

3.8 More on Tangents

1. Find the equation of the tangents to the following curves at the points stated:
   a) $\mathbf{x}^T \mathbf{x} = 25, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
   b) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 2, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
   c) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0, \begin{pmatrix} a \\ 2a \end{pmatrix}$
   d) $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2, \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$
   e) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = a^2, \begin{pmatrix} a\sec\theta \\ b\tan\theta \end{pmatrix}$
   f) $\mathbf{x}^T \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{x} = 4, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
   g) $\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} - 20 = 0, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
   h) $x^3 + y^3 - 3xy^2 + a^3 = 0, \begin{pmatrix} a \\ a \end{pmatrix}$
2. Find the equation of the tangent at the point $\mathbf{p}$ on each of the following curves:
   a) $\mathbf{x}^T \mathbf{x} = a^2$
   b) $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2$
   c) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0$
   d) $\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} - c^2 = 0$
   e) $y^2 \left(x^2 - a^2\right) = a^2 \left(x^2 + a^2\right)$

f) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

3.9 Normal

1. Find the equations of the normals to the following curves at the given points
   a) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} = 3, \begin{pmatrix}3\\4\end{pmatrix}$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\mathbf{x} = 13, \begin{pmatrix}1\\2\end{pmatrix}$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0, \begin{pmatrix}a\\2a\end{pmatrix}$
   d) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\b\sin\theta\end{pmatrix}$
   e) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -4\end{pmatrix}\mathbf{x} = 4a^2, \begin{pmatrix}2a\sec\theta\\a\tan\theta\end{pmatrix}$
   f) $x^3 - y^3 - 3xy^2 + a^2 = 0, \begin{pmatrix}a\\-a\end{pmatrix}$
2. Find the equation of the normal at the point $\mathbf{p}$ on each of the following curves:
   a) $\mathbf{x}^T\mathbf{x} = a^2$
   b) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2$
   c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$
   d) $\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\\frac{1}{2} & 0\end{pmatrix}\mathbf{x} = c^2$
3. Prove that for the curve
$$\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \qquad (3.9.3)$$
   the subnormal is of constant length.
4. Prove that the portion of any tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted by the axes is of length a.
5. Prove that for the curve $ay^2 = x^3$ the subnormal varies as the square of the subtangent.
6. Prove that for the curve $y = ae^{\frac{x}{b}}$ the subtangent is of length b.
7. Prove that the area of the triangle formed by the axes and any tangent to the curve
$$\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\\frac{1}{2} & 0\end{pmatrix}\mathbf{x} = c^2 \qquad (3.9.7)$$
   is $2c^2$.
8. Prove that for the curve $x^m y^n = e^{m+n}$ the portion of a tangent intercepted by the axes is divided at the point of contact in the ratio $m : n$.
9. Prove that, if N is the foot of the ordinate and NT is the subtangent at a point on the curve
$$\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\b\sin\theta\end{pmatrix} \qquad (3.9.9)$$
   then $OT.ON = a^2$.
10. Prove that the perpendicular from the foot of the ordinate to the tangent to a curve is of length $\frac{y}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}}$. Show that for the curve $y = c\cosh\frac{x}{c}$, this perpendicular is of length c.
11. Find the equation of the tangent to the curve
$$2x^3 + 2y^3 - 9axy = 0 \qquad (3.9.11.1)$$
    at the point $\begin{pmatrix}2a\\a\end{pmatrix}$; and show that the tangent meets the curve again where
$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 0 \qquad (3.9.11.2)$$

3.10 Affine Transformation: Exercises

1. What does the equation
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} - \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} - 6 = 0 \qquad (3.10.1)$$
   become when the origin is moved to the point $\begin{pmatrix}2\\-3\end{pmatrix}$?
2. To what point must the origin be moved in order that the equation
$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\-\frac{3}{2} & 4\end{pmatrix}\mathbf{x} + \begin{pmatrix}10 & -19\end{pmatrix}\mathbf{x} + 23 = 0 \qquad (3.10.2)$$
   may become
$$\mathbf{x}^T\begin{pmatrix}2 & -\frac{3}{2}\\-\frac{3}{2} & 4\end{pmatrix}\mathbf{x} = 1 \qquad (3.10.2)$$
3. Show that the equation
$$\mathbf{x}^T\mathbf{x} = a^2 \qquad (3.10.3)$$
   remains unaltered by any rotation of the axes.
4. What does the equation
$$\mathbf{x}^T\begin{pmatrix}1 & \sqrt{3}\\\sqrt{3} & -1\end{pmatrix}\mathbf{x} = 2a^2 \qquad (3.10.4)$$
   become when the axes are turned through 30°?

5. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.10.5)$$

become when the axes are turned through 45°?

6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.10.6)$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \quad (3.10.6)$$

and through what angle must the axes be turned in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \quad (3.10.6)$$

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = 1 \quad (3.10.7)$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c \quad (3.10.7)$$

where c is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 13 = 0 \quad (3.10.8)$$

can be transformed to

$$8\mathbf{x}^T\mathbf{x} = 89 \quad (3.10.8)$$

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \quad (3.10.9)$$

can be reduced to

$$\sqrt{85}\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (3.10.9)$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \quad (3.10.10)$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (3.10.10)$$

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \quad (3.10.11)$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \quad (3.10.11)$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \quad (3.10.11)$$

4 Circle

4.1 Properties

1. The equation of a circle is

$$\|\mathbf{x} - \mathbf{c}\| = r \quad (4.1.1)$$

where $\mathbf{c}$ is the centre and r is the radius.

2. By expanding (4.1.1), the equation of a circle can also be expressed as

$$\|\mathbf{x} - \mathbf{c}\|^2 = r^2 \quad (4.1.2.1)$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} - r^2 = 0 \quad (4.1.2.2)$$

3. The direction vector of *normal to the circle* in (4.1.2.2) at point $\mathbf{p}$ is

$$\mathbf{n} = k(\mathbf{p} - \mathbf{c}), \quad (4.1.3.1)$$

where k is a constant.

4. Find the equation of a circle that passes through the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$.
Solution: From (4.1.2.2),

$$\|\mathbf{A} - \mathbf{c}\|^2 = \|\mathbf{B} - \mathbf{c}\|^2 = \|\mathbf{C} - \mathbf{c}\|^2 = r^2 \quad (4.1.4.1)$$

$$\implies \|\mathbf{A} - \mathbf{c}\|^2 - \|\mathbf{B} - \mathbf{c}\|^2 = 0 \quad (4.1.4.2)$$

which can be simplified to obtain

$$(\mathbf{A} - \mathbf{B})^T \mathbf{c} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad \text{and} \quad (4.1.4.3)$$

$$(\mathbf{A} - \mathbf{C})^T \mathbf{c} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{C}\|^2}{2} \quad (4.1.4.4)$$

Solving the two yields $\mathbf{c}$, which can then be used to obtain r.

5. Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be three points on the circle and D be a point on BC such that $OD \perp BC$ as in Fig. 4.1.5. Show that
$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{4.1.5.1}$$

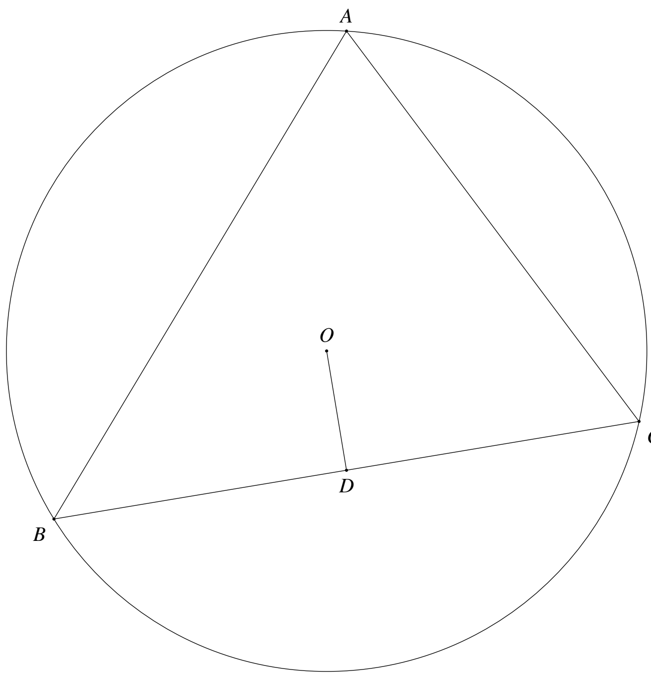

Fig. 4.1.5: Circumcircle.

Solution: From (4.1.1)
$$\|\mathbf{B}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 = r^2 \tag{4.1.5.2}$$
$$\implies (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = (\mathbf{C}-\mathbf{O})^T(\mathbf{C}-\mathbf{O}) \tag{4.1.5.3}$$
$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{B}+\mathbf{C}}{2}-\mathbf{O}\right) = 0 \tag{4.1.5.4}$$
after simplification. Since $OD \perp BC$,
$$(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{O}) = 0 \tag{4.1.5.5}$$
Since D and $\frac{\mathbf{B}+\mathbf{C}}{2}$ lie on BC, using (2.1.7.1),
$$\frac{\mathbf{B}+\mathbf{C}}{2} = \mathbf{B}+\lambda_1(\mathbf{B}-\mathbf{C}) \tag{4.1.5.6}$$
$$\mathbf{D} = \mathbf{B}+\lambda_2(\mathbf{B}-\mathbf{C}) \tag{4.1.5.7}$$
Multiplying (4.1.5.6) and (4.1.5.7) with $(\mathbf{B}-\mathbf{C})^T$ and subtracting, $\lambda_1 = \lambda_2$
$$\implies \mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{4.1.5.8}$$
6. Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.
7. The circle with centre $\mathbf{O}$ and radius r in Fig.4.1.7 is inside $\triangle ABC$ and touches AB, BC and CA at $\mathbf{F}, \mathbf{D}$ and $\mathbf{E}$ respectively. AB, BC and CA are known as *tangents* to the circle.

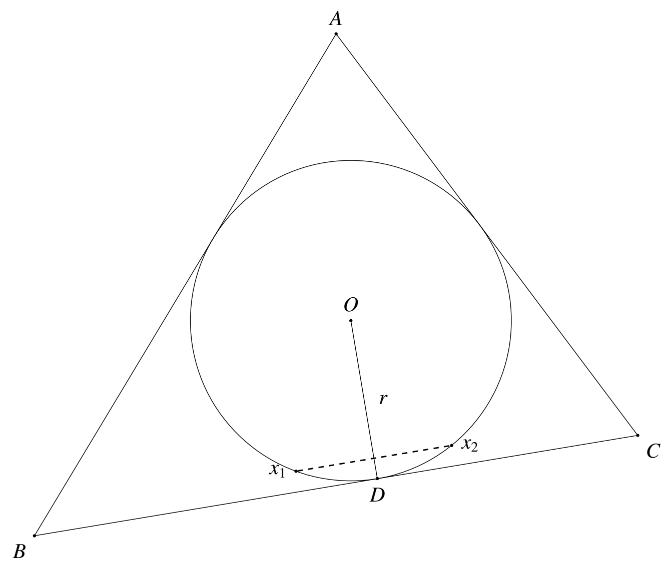

Fig. 4.1.7: Tangent and incircle.

8. Show that $OD \perp BC$.
Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be two points on the circle such that $x_1x_2 \parallel BC$. Then
$$\|\mathbf{x}_1-\mathbf{O}\|^2 - \|\mathbf{x}_2-\mathbf{O}\|^2 = 0 \tag{4.1.8.1}$$
$$\implies (\mathbf{x}_1-\mathbf{x}_2)^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{O}\right) = 0 \tag{4.1.8.2}$$
$$\implies (\mathbf{B}-\mathbf{C})^T\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}-\mathbf{O}\right) = 0 \tag{4.1.8.3}$$
For $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{D}$, x_1x_2 merges into BC and the above equation becomes
$$(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{O}) = 0 \implies OD \perp BC \tag{4.1.8.4}$$
9. Give an alternative proof for the above.
Solution: Let
$$\mathbf{B} = \mathbf{0} \tag{4.1.9.1}$$
$$\mathbf{D} = \lambda\mathbf{m} \tag{4.1.9.2}$$

Then

$$\|\mathbf{D} - \mathbf{O}\|^2 = r^2 \quad (4.1.9.3)$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 - 2\lambda \mathbf{m}^T \mathbf{O} + \|\mathbf{O}\|^2 = r^2 \quad (4.1.9.4)$$

Since the above equation has a single root,

$$\lambda = \frac{\mathbf{m}^T \mathbf{O}}{\|\mathbf{m}\|^2} \quad (4.1.9.5)$$

Thus,

$$(\mathbf{D} - \mathbf{B})^T (\mathbf{D} - \mathbf{O}) = (\lambda \mathbf{m})^T (\lambda \mathbf{m} - \mathbf{O}) \quad (4.1.9.6)$$

$$= \lambda^2 \|\mathbf{m}\|^2 - \lambda \mathbf{m}^T \mathbf{O} \quad (4.1.9.7)$$

$$= \mathbf{O} \text{ (from 4.1.9.5).} \quad (4.1.9.8)$$

$$\implies OD \perp BC \quad (4.1.9.9)$$

10. Find the equation of the tangent at $\mathbf{D}$.
Solution: The equation of the tangent is given by

$$(\mathbf{O} - \mathbf{D})^T (\mathbf{x} - \mathbf{D}) = 0 \quad (4.1.10.1)$$

11. Show that the angle in a semi-circle is a right angle.

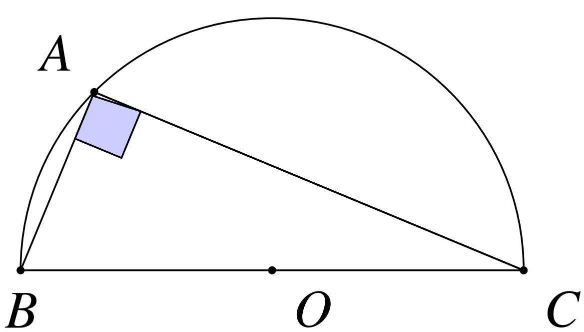

Fig. 4.1.11: Angle in a semi-circle.

Solution: Let

$$\mathbf{O} = 0 \quad (4.1.11.1)$$

From the given information,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = r^2 \quad (4.1.11.2)$$

$$\|\mathbf{B} - \mathbf{C}\|^2 = (2r)^2 \quad (4.1.11.3)$$

$$\mathbf{B} + \mathbf{C} = 0 \quad (4.1.11.4)$$

where r is the radius of the circle. Thus,

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 2\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T (\mathbf{B} + \mathbf{C}) \quad (4.1.11.5)$$

From (4.1.11.4) and (4.1.11.2),

$$\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 = 4r^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (4.1.11.6)$$

Thus, using Baudhayana's theorem, $\triangle ABC$ is right angled.

12. Show that $PA.PB = PC^2$, where PC is the tangent to the circle in Fig. 4.1.12.

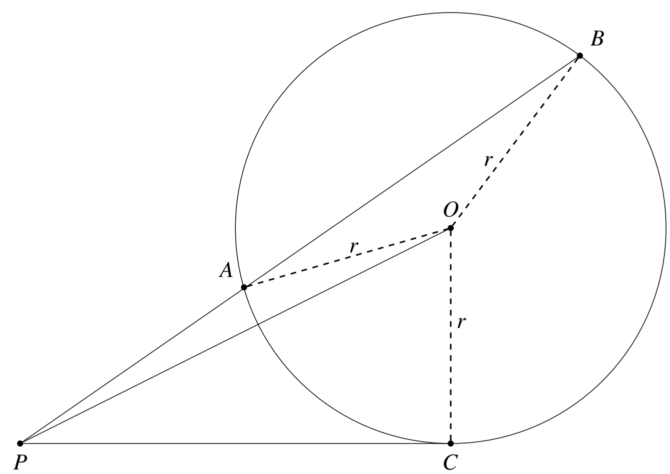

Fig. 4.1.12: $PA.PB = PC^2$.

Solution: Let $\mathbf{P} = \mathbf{0}$. Then, we have the following equations

$$PA.PB = \lambda \|\mathbf{A}\|^2 \quad \because (\mathbf{B} = \lambda \mathbf{A}) \quad (4.1.12.1)$$

$$\|\mathbf{A} - \mathbf{O}\|^2 = \|\mathbf{B} - \mathbf{O}\|^2 = \|\mathbf{C} - \mathbf{O}\|^2 = r^2 \quad (4.1.12.2)$$

$$\|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \quad \triangle PCO \text{ is right angled} \quad (4.1.12.3)$$

$\because$

$$\|\mathbf{B} - \mathbf{O}\|^2 - \|\mathbf{A} - \mathbf{O}\|^2 = 0, \quad (4.1.12.4)$$

$$\left(\lambda^2 - 1\right) \|\mathbf{A}\|^2 - 2(\lambda - 1) \mathbf{A}^T \mathbf{O} = 0 \quad (4.1.12.5)$$

$$\implies PA.PB = \lambda \|\mathbf{A}\|^2 = 2\mathbf{A}^T \mathbf{O} - \|\mathbf{A}\|^2 \quad (4.1.12.6)$$

after substituting from (4.1.12.1) and simplifying. From (4.1.12.3),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{O}\|^2 - \|\mathbf{C}\|^2 = r^2 \tag{4.1.12.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 = PC^2 \tag{4.1.12.8}$$

From (4.1.12.6) and (4.1.12.8),

$$PA.PB = PC^2 \tag{4.1.12.9}$$

13. In Fig. 4.1.13 show that $PA.PB = PC.PD$.

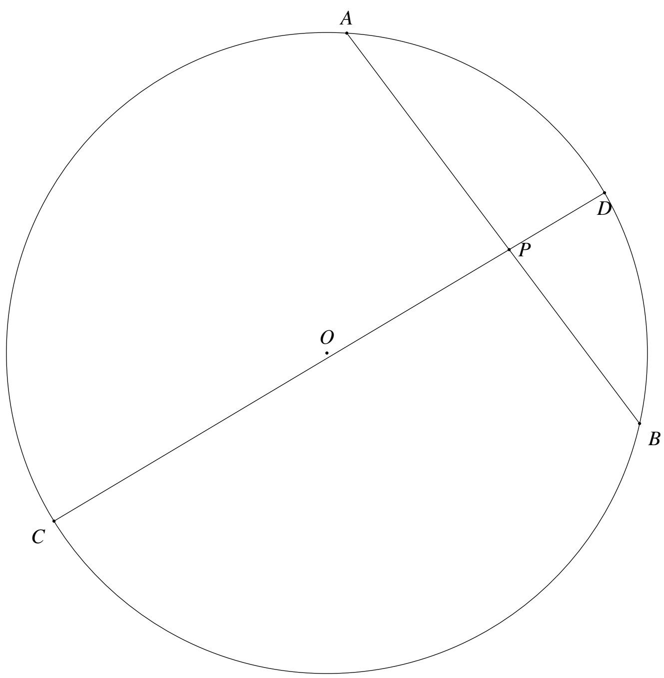

Fig. 4.1.13: Chords of a circle

Solution: Let $\mathbf{P} = \mathbf{0}$. We then have the following equations

$$\begin{aligned}\mathbf{B} &= k_1\mathbf{A}, k_1 = \frac{PB}{PA}\\ \mathbf{D} &= k_2\mathbf{C}, k_2 = \frac{PD}{PC}\end{aligned} \tag{4.1.13.1}$$

$$\begin{aligned}\|\mathbf{A}-\mathbf{O}\|^2 &= \|\mathbf{B}-\mathbf{O}\|^2\\ = \|\mathbf{C}-\mathbf{O}\|^2 &= \|\mathbf{D}-\mathbf{O}\|^2 = r^2\end{aligned} \tag{4.1.13.2}$$

where r is the radius of the circle and $\mathbf{O}$ is the centre. From (4.1.13.2),

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{B}-\mathbf{O}\|^2 \tag{4.1.13.3}$$

$$\implies \|\mathbf{A}-\mathbf{O}\|^2 = \|k\mathbf{A}-\mathbf{O}\|^2 \quad \text{(from (4.1.13.1))} \tag{4.1.13.4}$$

which can be simplified to obtain

$$k_1\|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 \tag{4.1.13.5}$$

Similarly,

$$k_2\|\mathbf{C}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{4.1.13.6}$$

From (4.1.13.2), we also obtain

$$\|\mathbf{A}-\mathbf{O}\|^2 = \|\mathbf{C}-\mathbf{O}\|^2 \tag{4.1.13.7}$$

$$\implies 2\mathbf{A}^T\mathbf{O} - \|\mathbf{A}\|^2 = 2\mathbf{C}^T\mathbf{O} - \|\mathbf{C}\|^2 \tag{4.1.13.8}$$

after simplification. Using this result in (4.1.13.5) and (4.1.13.6),

$$k_1\|\mathbf{A}\|^2 = k_2\|\mathbf{C}\|^2 \tag{4.1.13.9}$$

$$\implies \|\mathbf{A}\|\,\|\mathbf{B}\| = \|\mathbf{C}\|\,\|\mathbf{D}\| \tag{4.1.13.10}$$

which completes the proof.

14. (Pole and Polar:) The polar of a point $\mathbf{x}$ with respect to the curve

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1.14.1}$$

is the line

$$\mathbf{n}^T\mathbf{x} = c \tag{4.1.14.2}$$

where

$$\begin{pmatrix}\mathbf{n}^T\\ -c\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ 1\end{pmatrix} \tag{4.1.14.3}$$

The pole of the line in (4.1.14.2) is obtained as $\frac{1}{x_3}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, where

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{n}^T\\ -c\end{pmatrix} \tag{4.1.14.4}$$

15. $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be conjugate points for (4.1.14.1) if $\mathbf{x}_2$ lies on the polar of $\mathbf{x}_1$ and vice-versa. A similar definition holds for conjugate lines as well.
16. Let $\mathbf{p}$ be a point of intersection of two circles with centres $\mathbf{c}_1$ and $\mathbf{c}_2$. The circles are said to be orthogonal if their tangents at $\mathbf{p}$ are perpendicular to each other. Show that if r_1 and r_2 are their respective radii,

$$\|\mathbf{c_1} - \mathbf{c}_2\|^2 = r_1^2 + r_2^2 \tag{4.1.16.1}$$

17. Show that the length of the tangent from a point $\mathbf{p}$ to the circle

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.17.1}$$

is

$$\mathbf{p}^T\mathbf{p} - 2\mathbf{c}^T\mathbf{p} + f \tag{4.1.17.2}$$

This length is also known as the *power* of the point $\mathbf{p}$ with respect to the circle.

18. The *radical axis* of the circles

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 = 0 \tag{4.1.18.1}$$
$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{4.1.18.2}$$

is the locus of the points from which lengths of the tangents to the circles are equal. From (4.1.21), this locus is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}_1^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} - 2\mathbf{c}_2^T\mathbf{x} + f_2 = 0 \tag{4.1.18.3}$$
$$\implies 2\left(\mathbf{c}_1 - \mathbf{c}_2\right)^T\mathbf{x} + f_2 - f_1 = 0 \tag{4.1.18.4}$$

19. Show that the radical axis of the circles is perpendicular to the line joining their centres.
20. *Coaxal circles* have the same radical axis.
21. Obtain a family of coaxal circles from and find their *limit points*.

Solution: The family of circles is obtained as

$$\mathbf{x}^T\mathbf{x} - 2\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\mathbf{x} + f_1 + \lambda f_2 = 0 \tag{4.1.21.1}$$

The limit points are the centres of those circles whose radii are 0. From (4.1.21.1) and (4.1.2.2), this results in

$$f_1 + \lambda f_2 = \left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right)^T\left(\mathbf{c}_1 + \lambda\mathbf{c}_2\right) \tag{4.1.21.2}$$
$$\implies \lambda^2\left\|\mathbf{c}_2\right\|^2 + \lambda\left(2\mathbf{c}_1^T\mathbf{c}_2 - f_2\right) + \left\|\mathbf{c}_1\right\|^2 - f_1 = 0 \tag{4.1.21.3}$$

Solving for λ, the limit points are given by

$$\mathbf{c}_1 + \lambda\mathbf{c}_2 \tag{4.1.21.4}$$

4.2 Equation

1. Find the radius and the coordinates of the centre of each of the following circles:
   a) $3\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-12 & 6\end{pmatrix}\mathbf{x} + 11 = 0$
   b) $\mathbf{x}^T\mathbf{x} = a^2 + b^2$
   c) $2\mathbf{x}^T\mathbf{x} + \begin{pmatrix}16 & -4\end{pmatrix} + 88 = 0$
   d) $36\mathbf{x}^T\mathbf{x} - \begin{pmatrix}36 & 24\end{pmatrix}\mathbf{x} - 131 = 0$
2. Find the equation of the circle that passes through the points $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}$.
3. Find the equation of the circle that passes through the points $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}5\\1\end{pmatrix}$.
4. Find the equation of the circle that passes through the points $\begin{pmatrix}2a\\0\end{pmatrix}, \begin{pmatrix}0\\2b\end{pmatrix}, \begin{pmatrix}a+b\\a+b\end{pmatrix}$.
5. A circle has its centre on the line $x = 2y$ and passes through the points $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}$. Find the coordinates of the centre and the equation of the circle.
6. Find the locus of the centre of a circle which touches the line

$$\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = p \tag{4.2.6.1}$$

and the circle

$$\left\|\mathbf{x} - \mathbf{c}\right\| = r \tag{4.2.6.2}$$

4.3 Tangent and Normal

1. Without drawing a figure, determine whether the points $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\-4\end{pmatrix}$ lie outside, on the circumference, or inside the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-5 & 2\end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.3.1}$$

2. Find the points of intersection of the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 12 \tag{4.3.2.1}$$

and the circle

$$\left\|\mathbf{x}\right\|^2 = 13 \tag{4.3.2.2}$$

and find for what values of c the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = c \tag{4.3.2.3}$$

touches the circle.

3. Prove that the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 30 \tag{4.3.3.1}$$

touches the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}10 & 2\end{pmatrix}\mathbf{x} + 13 = 0 \tag{4.3.3.2}$$

and find the coordinates of the point of contact.

4. For what values of m does the line

$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = 0 \tag{4.3.4.1}$$

touch the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 2\end{pmatrix}\mathbf{x} + 8 = 0 \tag{4.3.4.2}$$

5. Prove that the circle

$$\mathbf{x}^T\mathbf{x} - 2a\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} + a^2 = 0 \tag{4.3.5}$$

touches the coordinate axes.

6. Show that two circles can be drawn to pass through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touch the coordinate axes, and find their equations.

7. Find the length of the tangent from the point $\begin{pmatrix}7\\4\end{pmatrix}$ to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} + 12 = 0 \tag{4.3.7}$$

8. Find the equations of the tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} + 5 = 0 \tag{4.3.8.1}$$

that are parallel to the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{4.3.8.2}$$

9. Find the equations of the tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}7 & 5\end{pmatrix}\mathbf{x} + 18 = 0 \tag{4.3.9}$$

at the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$, showing that they are parallel.

10. Find the equations of the tangent and normal to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-6 & 4\end{pmatrix}\mathbf{x} - 12 = 0 \tag{4.3.10}$$

at the point $\begin{pmatrix}6\\2\end{pmatrix}$.

11. Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{4.3.11.1}$$

touches the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 7 = 0 \tag{4.3.11.2}$$

and find the equations of the parallel and perpendicular tangents.

12. Find the equation of the tangent at the origin to the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x} = 0 \tag{4.3.12}$$

13. Prove that the line

$$\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = p \tag{4.3.13.1}$$

touches the circle

$$\left\|\mathbf{x} - \begin{pmatrix}a\\b\end{pmatrix}\right\| = r \tag{4.3.13.2}$$

if

$$r = \pm(p - a\cos\alpha - b\sin\alpha) \tag{4.3.13.3}$$

14. Find the points of contact of the tangents to the circle

$$\|\mathbf{x}\| = 5 \tag{4.3.14}$$

that pass through the point $\begin{pmatrix}7\\1\end{pmatrix}$ and write down the equations of the tangents.

15. Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \tag{4.3.15.1}$$

at the point $\begin{pmatrix}1\\-2\end{pmatrix}$ also touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-8 & 6\end{pmatrix}\mathbf{x} + 20 = 0 \tag{4.3.15.2}$$

and find the coordinates of the point of contact.

16. Find the equations of the circles that touch the lines

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{4.3.16.1}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 4 \tag{4.3.16.2}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 2 \tag{4.3.16.3}$$

17. Find the coordinates of the middle point of the chord

$$\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 25 \tag{4.3.17.1}$$

of the circle

$$\|\mathbf{x}\| = 5 \tag{4.3.17.2}$$

18. Find the equation of the chord of the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} - 23 = 0 \tag{4.3.18}$$

which has the point $\begin{pmatrix}4\\1\end{pmatrix}$ as its middle point.

19. Prove that the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 9 = 0 \tag{4.3.19}$$

subtends an angle $\tan^{-1}\frac{12}{5}$ at the origin.

20. Find the condition that the line

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} + n = 0 \tag{4.3.20.1}$$

should touch the circle

$$\left\|\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right\| = r \tag{4.3.20.2}$$

21. Verify that the perpendicular bisector of the chord joining two points $\mathbf{x}_1, \mathbf{x}_2$ on the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \tag{4.3.21}$$

passes through the centre.

4.4 Pole and Polar

1. Write down the equations of the polars of the following points with regard to the circle

$$\|\mathbf{x}\|^2 = 6 \tag{4.4.1}$$

a) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
b) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2. Find the poles of the following lines with regard to the circle

$$\|\mathbf{x}\| = 3 \tag{4.4.2}$$

a) $\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 7$
b) $\begin{pmatrix} 5 & -1 \end{pmatrix}\mathbf{x} + 6 = 0$

and verify that the polar of their point of intersection is the line joining their poles.

3. Show that the points $\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}$ are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 24 \tag{4.4.3}$$

4. Prove that the lines

$$\begin{pmatrix} 5 & 3 \end{pmatrix}\mathbf{x} = 40 \tag{4.4.4.1}$$

$$\begin{pmatrix} 7 & -5 \end{pmatrix}\mathbf{x} = 10 \tag{4.4.4.2}$$

are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 20 \tag{4.4.4.3}$$

5. Find the polar of the point $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ with regard to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} - 8 = 0 \tag{4.4.5}$$

6. Find the pole of

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} + n = 0 \tag{4.4.6}$$

with regard to

a) $\|\mathbf{x}\| = a$
b) $\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0$

7. Prove that if two lines at right angles are conjugate with regard to a circle one of them must pass through the centre.
8. Prove that, if the chords of contact of pairs of tangents to a circle from **P** and **Q** intersect in in **R**, then the line joining **R** to the centre is perpendicular to PQ.

4.5 Systems of Circles

1. Find the equation of a circle which passes through the points $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and cuts orthogonally the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 3 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.5.1}$$

2. Find the equation of a circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 4 & -5 \end{pmatrix}\mathbf{x} + 6 = 0 \tag{4.5.2.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 5 & -6 \end{pmatrix}\mathbf{x} + 7 = 0 \tag{4.5.2.2}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{4.5.2.3}$$

3. Find the equation of a circle which cuts orthogonally the two circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 2 & 2 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{4.5.3.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -3 & 6 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{4.5.3.2}$$

and passes through the point $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

4. Write down the equations of the radical axes of the following pairs of circles:

a)

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & -5 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{4.5.4.1}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 5 & -6 \end{pmatrix}\mathbf{x} = 0 \tag{4.5.4.2}$$

b)

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 3 & -2 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{4.5.4.3}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 3 & -5 \end{pmatrix}\mathbf{x} + 2 = 0 \tag{4.5.4.4}$$

c)

$$\mathbf{x}^T\mathbf{x} + 2g\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} + c = 0 \tag{4.5.4.5}$$

$$\mathbf{x}^T\mathbf{x} + 2f\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} + c = 0 \tag{4.5.4.6}$$

5. Find the equation of a circle coaxal with

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} - 1 = 0 \tag{4.5.5.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 1 = 0 \tag{4.5.5.2}$$

and passing through the point $\begin{pmatrix}2\\1\end{pmatrix}$.

6. Find the coordinates of the point from which the tanges to the three circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & 4\end{pmatrix}\mathbf{x} + 7 = 0 \tag{4.5.6.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} + 3 = 0 \tag{4.5.6.2}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} = 0 \tag{4.5.6.3}$$

are of equal length.

7. Find the limiting points of the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} - 4 = 0 \tag{4.5.7.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} - 10 = 0 \tag{4.5.7.2}$$

8. Prove that if a point moves so that the tangent from it to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} + 6 = 0 \tag{4.5.8.1}$$

is double the length of the tangent to the circle

$$\|\mathbf{x}\| = 2, \tag{4.5.8.2}$$

its locus is the circle

$$3\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} - 22 = 0 \tag{4.5.8.3}$$

9. Prove that the locus of a point such that the lengths of the tangents from it to two given circles are in a constant ratio is a circle coaxal with the given circles.

10. Find the equations of the two circles coaxal with

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8 & -10\end{pmatrix}\mathbf{x} + 2 = 0 \tag{4.5.10.1}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} - 1 = 0 \tag{4.5.10.2}$$

that touch the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 3 = 0 \tag{4.5.10.3}$$

11. Find the centre and radius of the circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 12 = 0 \tag{4.5.11.1}$$

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{4.5.11.2}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \tag{4.5.11.3}$$

12. The line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} + 2 = 0 \tag{4.5.12.1}$$

is the radical axis of a family of coaxal circles of which one circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} - 1 = 0. \tag{4.5.12.2}$$

Find the equation of the member of the family that passes through the point $\begin{pmatrix}-3\\1\end{pmatrix}$.

4.6 Miscellaneous

1. Find the locus of a point which moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant.
2. Find the locus of a point which moves so that the sum of the squares of its distances from n fixed points is constant.
3. Find the locus of a point at which two given circles subtend equal angles.
4. A circle passes through the four points $\begin{pmatrix}a\\0\end{pmatrix}$, $\begin{pmatrix}b\\0\end{pmatrix}$, $\begin{pmatrix}0\\c\end{pmatrix}$, $\begin{pmatrix}0\\d\end{pmatrix}$. By what relation are a, b, c, d connected? Find the equation of the circle and show that the tangent at the point $\begin{pmatrix}a\\c+d\end{pmatrix}$ is

$$\begin{pmatrix}a-b & c+d\end{pmatrix}\mathbf{x} - a(a-b) - (c+d)^2 = 0 \tag{4.6.4}$$

5. Write down the equations of the tangents to the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2a & 0\end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.6.5.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & -2b\end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.6.5.2}$$

at their points of intersection and verify that they cut at right angles.

6. Find the equation of the tangent to the circle

$$\|\mathbf{x}\| = a \tag{4.6.6.1}$$

at the point $\begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}$ and show that the length of the tangent intercepted by the lines

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{4.6.6.2}$$

is $\pm 2a\sec\theta$.

7. $\mathbf{A}$ and $\mathbf{B}$ are two fixed points $\begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} -c \\ 0 \end{pmatrix}$, and $\mathbf{P}$ moves so that $PA = k.PB$. Find the locus of $\mathbf{P}$ and prove that it is cut orthogonally by any circle through $\mathbf{A}$ and $\mathbf{B}$.
8. Show that the common chord of the circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 6 & 4 \end{pmatrix}\mathbf{x} + 9 = 0 \tag{4.6.8.1}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 8 & 6 \end{pmatrix}\mathbf{x} + 23 = 0 \tag{4.6.8.2}$$

is a diameter of the latter circle and find the angle at which the circles cut.
9. Prove analytically that the tangents to a circle at the ends of a chord are equally inclined to the chord.
10. Prove that for different values of a the equation

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2a\operatorname{cosec}\alpha & 0 \end{pmatrix}\mathbf{x} + a^2\cot^2\alpha = 0 \tag{4.6.10.1}$$

represents a family of circles touching the lines

$$\begin{pmatrix} \pm\tan\alpha & 1 \end{pmatrix}\mathbf{x} = 0 \tag{4.6.10.2}$$

Prove also that the locus of the poles of the line

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} = 0 \tag{4.6.10.3}$$

with regard to the circles is the line

$$\begin{pmatrix} m\sin^2\alpha & l\cos^2\alpha \end{pmatrix}\mathbf{x} = 0 \tag{4.6.10.4}$$

11. Find the coordinates of the middle point of the chord

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} = 1 \tag{4.6.11.1}$$

of the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \tag{4.6.11.2}$$

12. Prove that the points of intersection of the line

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} = 1 \tag{4.6.12.1}$$

and the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \tag{4.6.12.2}$$

subtend a right angle at the origin if

$$c\left(l^2 + m^2\right) + 2gl + 2fm + 2 = 0 \tag{4.6.12.3}$$

13. Prove that the equation of the circle having for diameter the portion of the line

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \tag{4.6.13.1}$$

intercepted by the circle

$$\|\mathbf{x}\| = a \tag{4.6.13.2}$$

is

$$\mathbf{x}^T\mathbf{x} - 2p\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} + 2p^2 - a^2 = 0 \tag{4.6.13.3}$$

14. Prove that if a chord of the circle

$$\|\mathbf{x}\| = a \tag{4.6.14.1}$$

subtends a right angle at a fixed point $\mathbf{x}_1$, the locus of the middle point of the chord is

$$2\mathbf{x}^T\mathbf{x} - 2\mathbf{x}_1^T\mathbf{x} + \|\mathbf{x}_1\|^2 - a^2 = 0 \tag{4.6.14.2}$$

15. Prove that the equation of any tangent to the circle

$$\left\|\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right\| = r \tag{4.6.15.1}$$

may be written in the form

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right) = r \tag{4.6.15.2}$$

Deduce that the equation of the tangents from $\mathbf{x}_1$ to the circle is

$$r^2\|\mathbf{x} - \mathbf{x}_1\|^2 = \left[\left\{\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right\}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left\{\mathbf{x}_1 - \begin{pmatrix} a \\ b \end{pmatrix}\right\}\right]^2 \tag{4.6.15.3}$$

16. Prove that the distances of two points from the centre of a circle are proportional to the distance of each point from the polar of the other.
17. Prove that the tangents to the circles of a coaxal system drawn from a limiting point are bisected by the radical axis.
18. Show that a common tangent to the two circles is bisected by their radical axis and subtends a right angle at either limiting point.
19. Prove that if a point moves so that the dif-

ference of the squares of the tangents from it to two given circles is constant its locus is a straight line parallel to the radical axis of the circles.

20. Prove that the polars of a fixed point with regard to a family of coaxal circles all pass through another fixed point.
21. The circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2a\sec\alpha & 0\end{pmatrix}\mathbf{x} - a^2 = 0 \quad (4.6.21.1)$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & -2a\operatorname{cosec}\alpha\end{pmatrix}\mathbf{x} - a^2 = 0 \quad (4.6.21.2)$$

where α is a given angle, both cut orthogonally every member of a coaxal family of circles. Find the radical axis and the limiting points of the family.

22. Prove that, if two points $\mathbf{P}$, $\mathbf{Q}$ are conjugate with regard to a circle, the circle on PQ as diameter cuts the first circle orthogonally.
23. Prove that if $\mathbf{P}$, $\mathbf{Q}$ are conjugate points with regard to a circle, the circles with $\mathbf{P}$, $\mathbf{Q}$ as centres which cut the given circle orthogonally are orthogonal to one another.
24. Prove that, if PQ is a diameter of a circle, then $\mathbf{P}$, $\mathbf{Q}$ are conjugate points with regard to any circle which cuts the given circle orthogonally.
25. Prove that if $\mathbf{P}$, $\mathbf{Q}$ are conjugate points with regard to a circle, the square on PQ is equal to the sum of the squares on the tangents from $\mathbf{P}$, $\mathbf{Q}$ to the circle.
26. The equation

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2g & 0\end{pmatrix}\mathbf{x} + 2g - 5 = 0 \quad (4.6.26)$$

where g is a variable parameter, represents a family of coaxial circles. Show that the radius of the smallest circle of the family is 2.

27. Prove that, if perpendiculars are drawn from a fixed point $\mathbf{P}$ to the polars of $\mathbf{P}$ with regard to a family of coaxial circles, then the locus of the feet of these perpendiculars is a circle whose centre lies on the radical axis of the family.
28. Prove that, if the points in which the line

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} + n = 0 \quad (4.6.28.1)$$

meets the circle,

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x} + c = 0 \quad (4.6.28.2)$$

and those in which the line

$$\begin{pmatrix}l_1 & m_1\end{pmatrix}\mathbf{x} + n_1 = 0 \quad (4.6.28.3)$$

meets

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}g_1 & f_1\end{pmatrix}\mathbf{x} + c_1 = 0 \quad (4.6.28.4)$$

lie on a circle, then

$$2(g-g_1)(mn_1 - m_1n) + 2(f-f_1)(nl_1 - n_1l) + (c-c_1)(lm_1 - l_1m) = 0 \quad (4.6.28.5)$$

29. Show that, if a diameter of a circle is the portion of the line

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} = 1 \quad (4.6.29.1)$$

intercepted by the lines

$$\mathbf{x}^T\begin{pmatrix}a & h\\ h & b\end{pmatrix}\mathbf{x} = 0 \quad (4.6.29.2)$$

then the equation of the circle is

$$\left(am^2 - 2hlm + bl^2\right)\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}(hm-bl) & (hl-am)\end{pmatrix}\mathbf{x} + a + b = 0 \quad (4.6.29.3)$$

30. Prove that, as k varies, the equation

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} + c + 2k\left\{\begin{pmatrix}a & -b\end{pmatrix}\mathbf{x} + 1\right\} = 0 \quad (4.6.30.1)$$

represents a system of coaxial circles. Also prove that the orthogonal system is given by

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}\frac{c+2}{2a} & \frac{c-2}{2b}\end{pmatrix}\mathbf{x} + h\left\{\begin{pmatrix}\frac{1}{2a} & \frac{1}{2b}\end{pmatrix}\mathbf{x} + 1\right\} = 0 \quad (4.6.30.2)$$

where h is a variable parameter.

5 Parabola

5.1 Tangent and Normal

1. Show that the line

$$\begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} + a = 0 \quad (5.1.1.1)$$

touches the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (5.1.1.2)$$

and find the coordinates of the point of contact.

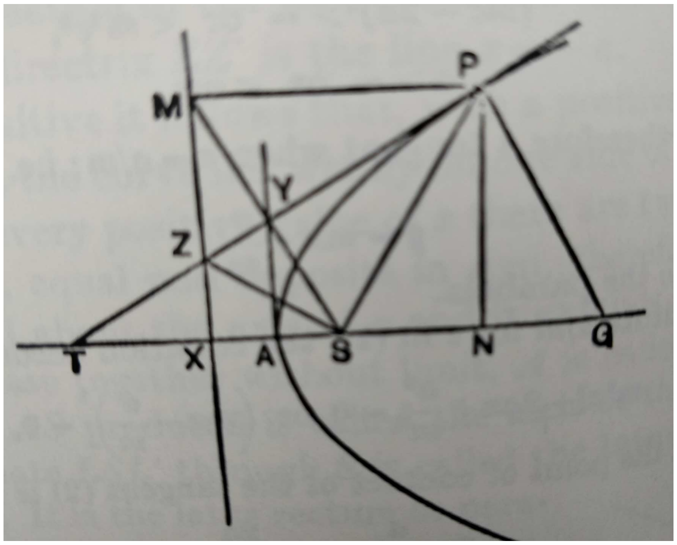

Fig. 5.1.0

2. Find the point of intersection of the parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.2.1)$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.2.2)$$

other than the origin, and prove that the tangents at this point are inclined at an angle $\tan^{-1} \frac{3}{4}$.

3. Find the points in which the line

$$\begin{pmatrix} 8 & -1 \end{pmatrix} \mathbf{x} = a \qquad (5.1.3.1)$$

cuts the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.3.2)$$

and find the point where the tangents at these points intersect.

4. A line through the vertex $\mathbf{A}$ of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.4)$$

makes an angle of 60° with the axis and cuts the curve again in $\mathbf{P}$. Find the equation of the tangent at $\mathbf{P}$, and show that the area of the triangle this tangent makes with the axes is $\frac{4a^2}{3\sqrt{3}}$.

5. Prove that in Fig. 5.1.0

$$SG = SP \qquad (5.1.5)$$

6. Prove that $t_1 t_2 = -1$ is the condition that the chord joining the points with parameters t_1, t_2 on a parabola shall pass through the focus.
7. Prove that the tangents drawn to a parabola from a point on the directrix are at right angles and that their chord of contact passes through the focus.
8. If in Fig. 5.1.0 PS cuts the curve again in $\mathbf{Q}$, prove that QA passes through $\mathbf{M}$.
9. Through the vertex $\mathbf{A}$ of a parabola chords AP, AP at right angles to one another are drawn. Prove that PA cuts the axis in a fixed point.
10. Find the coordinates of the other point in which the normal at $\begin{pmatrix} at^2 \\ 2at \end{pmatrix}$ meets the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.10)$$

; and prove that two normal chords that cut at right angles divide one another in the ratio 1 : 3.

11. Three normals are drawn to a parabola from the point $\begin{pmatrix} h \\ k \end{pmatrix}$. Prove that the centroid of the triangle formed by their feet is the point $\begin{pmatrix} \frac{2}{3}(h-2a) \\ 0 \end{pmatrix}$.
12. Find the equation of the tangent to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.12.1)$$

that is parallel to the normal at $\mathbf{P} = \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$; and prove that, if this tangent meets the axis in T and PN is the ordinate of P and A is the vertex, then

$$TA.AN = a^2 \qquad (5.1.12.2)$$

13. Prove that the circle

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} g & f \end{pmatrix} \mathbf{x} + c = 0 \qquad (5.1.13.1)$$

cuts the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.13.2)$$

in four points the sum of whose ordinates is zero; and conversely that if four ponits on a parabola be such that the sum of their ordinates is zero then the four points lie on a circle.

14. Prove that the orthocentre of a triangle whose sides all touch a parabola lies on the directrix.

15. A chord POQ of a parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.15.1)$$

cuts the axis in a fixed point $\mathbf{O}$. PN, QM are the ordinates of $\mathbf{P}$ and $\mathbf{Q}$, and $\mathbf{A}$ is the vertex. Prove that

$$NP.MQ + 4a.AO = 0 \qquad (5.1.15.2)$$

16. From a point $\mathbf{P} = \begin{pmatrix} at_1^2 \\ 2at_1 \end{pmatrix}$ on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.1.16.1)$$

two chords PQ, PR are drawn normal to the curve at $\mathbf{Q}$ and $\mathbf{R}$. Prove that, if $\mathbf{Q}$, $\mathbf{R}$ are the points with parameters t_2, t_3 on the curve, then $t_2 t_3 = 2$, and the equation of QR is

$$\begin{pmatrix} 2 & t_1 \end{pmatrix} \mathbf{x} + 4a = 0 \qquad (5.1.16.2)$$

17. Prove that the normals to a parabola at the ends of a chord whose inclination to the axis is θ meet on the normal whose inclination is $\tan^{-1}(2\cot\theta)$.
18. Prove that, if two parabolas are on the same side of the same directrix and have their axes in the same line, then they intersect at a distance from the directrix equal to one-quarter of the sum of their latera recta.

5.2 Miscellaneous

In the following problems, the equation of a parabola is assumed to be (3.3.1.1) and capital letters refer to Fig. 5.1.0 unless the contrary is stated.

1. Prove that as $\mathbf{P}$ moves along the curve GP^2 varies as SG.
2. Prove that, if PP_1 is a double ordinate and PX meets the curve in $\mathbf{Q}$, then P_1Q passes through $\mathbf{S}$.
3. Prove that, if PSP_1 is a focal chord and AP, AP_1 meet the latus rectum in $\mathbf{Q}$, $\mathbf{Q}_1$, the SQ, SQ_1 are equal to the ordinates of $\mathbf{P}_1$ and $\mathbf{P}$.
4. Prove that, if the tangents at $\mathbf{P}$, $\mathbf{Q}$ intersect in $\mathbf{T}$, then

$$ST^2 = SP.SQ. \qquad (5.2.4)$$

5. Prove that, if the tangent at the end $\mathbf{Q}$ of a focal chord PSQ meets the latus rectum in $\mathbf{R}$, then PGR is a right angle.
6. Tangents at $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ on a parabola form a triangle UVW. Show that the centroids of the triangles PQR and UVW lie on the same diameter.
7. Prove that, if the difference of the ordinates of two points on a parabola is constant, then the locus of the point of intersection of the tangents at these points is an equal parabola.
8. Prove that, if two tangents intercept a fixed length on the tangent at the vertex, the locus of their intersections is an equal parabola.
9. The chord of contact of tangents from any point $\mathbf{Q}$ meets the tangent at the vertex in $\mathbf{R}$. Prove that the tangent of the angle which AQ makes with the axis is $\frac{2a}{AR}$.
10. The parameters, t, t_1 of two points on a parabola are connected by the relation $t = k^2 t_1$, prove that the tangents at the points intersect on the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \left(k + \frac{1}{k}\right)^2 a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (5.2.10)$$

11. Show that the length of the normal chord at the point of parameter t is

$$\frac{4a}{t^2}\left(1 + t^2\right)^{\frac{3}{2}} \qquad (5.2.11)$$

12. Prove that the locus of intersection of tangents at the ends of a normal chord is

$$(x + 2a)\, y^2 + 4a^2 = 0. \qquad (5.2.12)$$

13. Prove that the locus of the point of intersection of perpendicular normals is the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 3a^2 = 0 \quad (5.2.13)$$

14. Prove that if the tangents at two points on the parabola intersect in the point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, the corresponding normals intersect in the point

$$\begin{pmatrix} 2a - x_1 + \frac{y_1^2}{a} \\ -\frac{x_1 y_1}{a} \end{pmatrix}. \qquad (5.2.14)$$

15. Show that, if the tangent at $\mathbf{P}$ meets the latus rectum in $\mathbf{K}$, then SK is a mean proportional between the segments of the focal chord

through **P**.

16. Show that, if the tangents from **Q** to the parabola form with the tangent at the vertex, a triangle of constant area c^2, then the locus of **Q** is the curve
$$x^2\left(y^2-4ax\right)=4c^4. \tag{5.2.16}$$
17. Show that the normals at the ends of each of a series of parallel chords of a parabola intersect on a fixed straight line, itself a normal to the parabola.
18. **P**, **Q** are points on the parabola subtending a constant angle α at the vertex. Show that the locus of the intersection of the tangents at **P**, **Q** is the curve
$$\mathbf{x}^T\begin{pmatrix}0&0\\0&1\end{pmatrix}\mathbf{x}-\begin{pmatrix}4a+\frac{\tan^2\alpha}{4}&0\end{pmatrix}\mathbf{x}=a^2\tan^2\alpha \tag{5.2.18}$$
19. Prove that the exterior angle between two tangents to a parabola is equal to the angle which either of them subtends at the focus.
20. Two perpendicular focal chords of a parabola meet the directrix in **T** and $\mathbf{T}_1$. Show that the tangents to the parabola which are parallel to these chords intersect in the middle point of TT_1.
21. Prove that, if the tangents at the points **Q**, **R** intersect at **P**, then
$$PQ^2:PR^2=SQ:SR \tag{5.2.21}$$
22. The tangents at any two points **P**, **Q** meet at **T** and the normals meet at **N**. Prove that the projection of TN on the axis is equal to the sum of the distances of **P** and **Q** from the directrix.
23. Prove that the circumscribing circle of the triangle formed by three tangents to a parabola passes through the focus.
24. PQ is a chord of a parabola normal at **P**; the circle on PQ as diameter cuts the parabola again in **R**. Prove that the projection of QR on the axis is twice the latus rectum.
25. Prove that the distance between a tangent and the parallel normal is $a\operatorname{cosec}\theta\sec^2\theta$, where θ is the angle which either makes with the axis.
26. Prove that, if the normals at **P** and **Q** intersect on the curve, then PQ cuts the axis in a fixed point.
27. Prove that, if the normals at **P** and **Q** meet at the point $\mathbf{R}\left(x_1,y_1\right)$ on the parabola, and the tangents at **P** and **Q** meet at **T**, then
$$TP.TQ=\frac{1}{2}\left(x_1-8a\right)\sqrt{y_1^2+4a^2}. \tag{5.2.27}$$
28. Show that, in the last problem, as **R** moves along the parabola, the middle point of PQ always lies on the parabola
$$\mathbf{x}^T\begin{pmatrix}0&0\\0&1\end{pmatrix}\mathbf{x}-2a\begin{pmatrix}1&0\end{pmatrix}\mathbf{x}=4a^2 \tag{5.2.28}$$
29. Prove that the area of the triangle formed by the tangents at the points t_1,t_2, and their chord of contact is
$$\frac{1}{2}a^2\left(t_1-t_2\right)^2 \tag{5.2.29}$$
30. Prove that the area of the triangle formed by three points t_1, t_2, t_3 on the parabola is
$$a^2\left(t_2-t_3\right)\left(t_3-t_1\right)\left(t_1-t_2\right) \tag{5.2.30}$$
and that this is double the area of the triangle formed by the tangents at these points.
31. Prove that, if a line through any point $P\begin{pmatrix}x_1\\y_1\end{pmatrix}$ making an angle θ with the axis meets the parabola at **Q** and **R**, then
$$PQ.PR=\left(y_1^2-4ax_1\right)\operatorname{cosec}^2\theta. \tag{5.2.31}$$
32. Two chords QR, Q_1R_1 of a parabola meet at **O**, and the diameters bisecting them meet the curve at **P** and P_1. Prove that
$$QO.OR:Q_1O.OR_1=SP:SP_1 \tag{5.2.32}$$
33. Show that, if **P** is on the parabola, the length of the chord through **P** that makes an angle θ with the axis is
$$4a\sin\left(\alpha-\theta\right)\operatorname{cosec}^2\theta\operatorname{cosec}\alpha \tag{5.2.33}$$
where α is the inclination of the tangent at **P** to the axis.
34. Show that the locus of the middle point of a chord which passes through the fixed point $\begin{pmatrix}h\\k\end{pmatrix}$ is the parabola
$$\mathbf{x}^T\begin{pmatrix}0&0\\0&1\end{pmatrix}\mathbf{x}-\begin{pmatrix}2a&k\end{pmatrix}\mathbf{x}+2ah=0 \tag{5.2.34}$$

35. A tangent to the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4b \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.2.35)$$

meets the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.2.35)$$

at **P**, **Q**. Prove that the locus of the middle point of PQ is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 2a+b \end{pmatrix} \mathbf{x} - 4a^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.2.35)$$

36. Prove that the polar of the focus of a parabola is the directrix.
37. Prove that, if a chord of the parabola subtends a right angle at the vertex, the locus of its pole is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 4a = 0 \qquad (5.2.37)$$

38. Show that, if parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (5.2.38)$$

are drawn corresponding to different values of a, the feet of the perpendiculars from a fixed point on its polar lines all lie on a circle passing through the point.
39. Prove that, if from a point $\mathbf{Q} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ a perpendicular be drawn to the polar of **Q** with regard to the parabola cutting it in **R** and the axis in G, then

$$SG = SR = x_1 + a \qquad (5.2.39)$$

40. Prove that, if the diameter through a point **P** of a parabola meets any chord in **O** and the tangents at the ends of the chord in $\mathbf{T}, \mathbf{T}_1$, then

$$PO^2 = PT.PT_1 \qquad (5.2.40)$$

41. QQ_1 is a chord of a parabola and TOR is a diameter which meets the tangent at **Q** in **T**, the curve in **O** and QQ_1 in **R**. Prove that

$$TO : OR = QR : RQ_1 \qquad (5.2.41)$$

42. Prove that if the normals at three points **P**, **Q**, **R** on a parabola concur, then the points **P**, **Q**, **R** and the vertex of the parabola are concyclic.
43. Prove that in general, two members of the family of parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4a^2 \qquad (5.2.43)$$

, where a is the parameter specifying members of the family, pass through any assigned point of the plane, and that these two parabolas cut orthogonally at **P**.

6 The Ellipse

6.1 Properties

1. Find the length of the latus rectum, the eccentricity and the coordinates of the foci of the ellipses
   a) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$
   b) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36$
   c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 8$
2. Find the equation of the ellipse whose foci are the points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and eccentricity $\frac{1}{2}$. What are the equations of the directrices?
3. Find the equation of the ellipse of eccentricity $\frac{3}{4}$, which has its centre at the point $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and touches the axis of y at the origin. What is the length of its latus rectum?
4. An ellipse has the axis of y for directrix and its centre at the point $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Find its equation if its eccentricity is $\frac{3}{4}$.
5. Find the equation of the ellipse which has a focus at $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, corresponding directrix the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 6 \qquad (6.1.5.1)$$

and eccentricity $\frac{1}{2}$. What are the lengths of its axes?
6. An ellipse has its centre at $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and a focus at $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and its eccentricity is $\frac{1}{2}$. Find its equation.
7. Are the points $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ inside or outside the

ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} = 20 \tag{6.1.7.1}$$

8. Prove that the line $y = x - 5$ touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \mathbf{x} = 144 \tag{6.1.8.1}$$

and find the coordinates of the point of contact.

9. The line

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = c \tag{6.1.9.1}$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 1 \tag{6.1.9.2}$$

Find the values of c.

10. Find the equation of the tangent to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 8 \tag{6.1.10.1}$$

at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and also the equations of the two tangents perpendicular to this.

11. Find the equations of the tangents to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{6.1.11.1}$$

which makes an angle of 60° with the major axis.

12. Find at what point the line

$$\begin{pmatrix} e & 1 \end{pmatrix} \mathbf{x} = a \tag{6.1.12.1}$$

touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2 \left(1 - e^2\right) \tag{6.1.12.2}$$

13. Prove that the normal at an end of the latus rectum meets the major axis at a distance ae^3 from the centre.

14. Prove that the normal at an end L of the latus rectum meets the minor axis in g, and l is the projection of L on the minor axis, then

$$gl = a \tag{6.1.14.1}$$

15. Prove that the normal at $\begin{pmatrix} x \\ y \end{pmatrix}$ divides the major axis into segments of lengths $a - e^2x$ and $a + e^2x$.

16. Prove that if the normal at $\mathbf{P}$ meets the major axis in $\mathbf{G}$, and the minor axis in $\mathbf{G}_1$, then
    a) $SG = eSP$;
    b) $PG : PG_1 = b^2 : a^2$

6.2 *Pole and Polar*

1. Find the polar of the point $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 6 \tag{6.2.1.1}$$

2. Find the pole of the line

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5 \tag{6.2.2.1}$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 5 \tag{6.2.2.2}$$

3. Find the poles of the lines

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 1 \tag{6.2.3.1}$$

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 4 \tag{6.2.3.2}$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 6, \tag{6.2.3.3}$$

and verify that the line joining the poles is the polar of the point of intersection of the lines.

4. Prove that the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are conjugate with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 5 \tag{6.2.4.1}$$

5. Find the equatioin of a line through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and conjugate to the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{6.2.5.1}$$

with regard to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{6.2.5.2}$$

6. Prove that, if the polar of a point is at right angles to the line joining the point to the centre of an ellipse, the point must lie on one of the axes.

7. Prove that, if $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is the pole of the normal at $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, then
$$\frac{x_1 x_2}{a^4} = -\frac{y_1 y_2}{b^4} = \frac{1}{a^2 - b^2} \qquad (6.2.7.1)$$
8. Prove that a directrix of an ellipse is the polar of the corresponding focus.

6.3 Eccentric Angles

The following problems refer to the ellipse whose equation is
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2$$

1. Show that the eccentric angle of one end of a latus rectum is $\cos^{-1} e$.
2. If α is the eccentric angle of a point $\mathbf{P}$ on an ellipse, what are the coordinates of the corresponding point $\mathbf{Q}$ on the auxilary circle? Write down the equations of the tangents at $\mathbf{P}$ and $\mathbf{Q}$ and show that they intersect on the major axis.
3. Prove that, if CP, CD are conjugate radii of an ellipse and ω the angle between them, then $\sin^2 \omega$ varies as $CP^{-2} + CD^{-2}$.
4. Prove that if a parallelogram is inscribed in an ellipse, its sides are parallel to the conjugate diameters.
5. Prove that if QQ_1 is a chord of an ellipse parallel to the tangent at $\mathbf{P}$ the eccentric angles of $\mathbf{Q}$ and $\mathbf{Q}_1$ differ from the eccentric angle at $\mathbf{P}$ by equal amounts.
6. Prove that the equation of the perpendicular bisector of the chord joining the points $\mathbf{Q}$, $\mathbf{Q}_1$ whose eccentric angles are $\alpha + \beta$, $\alpha - \beta$ is
$$\begin{pmatrix} a \sec \alpha & -b \operatorname{cosec} \alpha \end{pmatrix} \mathbf{x} = \left(a^2 - b^2\right) \cos \beta \qquad (6.3.6.1)$$
and deduce the equation of the normal at the point whose eccentric angle is α.
7. Prove that, if the same chord passes through a focus, then
$$\cos \beta = \pm e \cos \alpha \qquad (6.3.7.1)$$
8. Prove that the equation of a focal chord parallel to the tangent at the point whose eccentric angle is α is
$$\begin{pmatrix} \frac{\cos \alpha}{a} & \frac{\sin \alpha}{b} \end{pmatrix} \mathbf{x} = \pm e \cos \alpha \qquad (6.3.8.1)$$
9. Prove that, if α is variable and β constant, the chord joining the points whose eccentric angles are $\alpha + \beta$ and $\alpha - \beta$ touches the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \cos^2 \beta \qquad (6.3.9.1)$$
and that the locus of the poles of the chord is
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \sec^2 \beta \qquad (6.3.9.2)$$
10. Prove that the tangents at the points whose eccentric angles are α, $\alpha + \frac{\pi}{2}$ intersect on the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = 2a^2 b^2 \qquad (6.3.10.1)$$
11. Prove that, if $\mathbf{P}$, $\mathbf{Q}$ are corresponding points on an ellipse and its auxiliary circle and the normals at $\mathbf{P}$, $\mathbf{Q}$ intersect in R, then
$$CR = a + b \qquad (6.3.11.1)$$
12. Prove that, if the line joining the ends of two equal conjugate radii of an ellipse passes through a focus the eccentricity is $\frac{1}{\sqrt{2}}$.
13. Prove that the chords that join the ends of conjugate radii all touch the ellipse
$$2\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \qquad (6.3.13.1)$$

6.4 Miscellaneous

The following problems refer to the ellipse whose equation is
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2$$
and that C is its centre and S, S_1 its foci.

1. Prove that, if the tangent and normal at a point P on an ellipse meet the major axis in T, G, then the tangent from either end of the minor axis to the circle TPG is equal in length to half the major axis.
2. Show that if (x_1, y_1) are the coordinates of a point of intersection of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

and $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$, the equations of their common tangents are $\pm\dfrac{xx_1}{aa_1}\pm\dfrac{yy_1}{bb_1}=1$.

3. The normal at any point P of the ellipse meets the axis in G; a point Q is taken in the tangent so that $PQ=\lambda.PG$, where λ is constant; prove that the locus of Q is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^2+\lambda b^2}{a^2}$$

4. Prove that the line $\dfrac{ax}{k^2-1}+\dfrac{by}{2k}+\dfrac{a^2-b^2}{k^2+1}=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of k.
5. Prove that the foot of the focal perpendicular on the normal at any point of an ellipse is at a distance from the centre equal to the difference between the semi-major axis and the focal radius vector to the point at which the normal is drawn.
6. Prove that the locus of the poles of normal chords of the ellipse is the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=\left(a^2-b^2\right)^2.$$

7. P is a point (x_1,y_1) on the ellipse and PS, PS_1 meet the curve again in Q, R. Prove that the equation of QR is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\frac{1+e^2}{1-e^2}+1=0,$$

where e is the eccentricity.

8. Show that two parallel tangents to an ellipse are met by any other tangent in points which lie on conjugate diameters.
9. Prove that if CP, CD be any two conjugate semi-diameters of an ellipse and PF be drawn perpendicular to CD and produced both ways to E, E_1 so that $PE=PE_1=CD$, then $CE.CE_1=CS^2$, where S is a focus.
10. Tangents are drawn from points on the ellipse to the circle $x^2+y^2=r^2$, show that the chords of contact touch the ellipse

$$a^2x^2+b^2y^2=r^4$$

11. Tangents TP, TQ_1 are drawn to an ellipse so that SP, S_1Q are parallel. Prove that CT is parallel to SP or S_1Q.
12. the normal to an ellipse at a point P passes through one end of the minor axis, and CD is the semidiameter conjugate to CP. The perpendicular from C to CD meets the auxiliary circle in E. Prove that DE is equal to half the distance betwen the directrices.
13. Prove that, if (x_1,y_1) be the middle point of a chord of the ellipse, the equation of the chord is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

14. If Q be the pole of the chord of the ellipse which is normal at a point P, and if CR drawn through the centre C perpendicular to CQ meet the normal at P in R, prove that the locus of R is

$$\frac{x^2}{b^6}+\frac{y^2}{a^6}=\frac{1}{a^2b^2}$$

15. Prove that, if the point P lies on the ellipse $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$, its polar with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the ellipse

$$\frac{a_1^2x^2}{a^4}+\frac{b_1^2y^2}{b^4}=1$$

16. Show that the polar with regard to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of a point on the circle $x^2+y^2=c^2$ touches the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.
17. Prove that, if the tangents to an ellipse at (x_1,y_1) and (x_2,y_2) meet at (x,y) and the normals at (ξ,η), then $a^2\xi=e^2xx_1x_2$ and $b^2\eta=-e^2yy_1y_2$, where e is the eccentricity.
18. Show that, if (x,y) is the middle point of a chord of the ellipse, and the tangents at the ends of the chord intersect in (x_1,y_1) and the normals in (x_2,y_2), then

$$\frac{a^2x_2}{x_1}+\frac{b^2y_2}{y_1}=\left(a^2-b^2\right)\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}\right).$$

19. Prove that, if the normals at two points P, Q on an ellipse intersect on the diameter that bisects PQ, then the two normals are at right angles.
20. Prove that if a chord of the ellipse subtends a right angle at the centre then it touches the circle

$$\left(x^2+y^2\right)\left(a^2+b^2\right)=a^2b^2$$

21. The locus of middle points of chords of the ellipse which subtend a right angle at its centre

is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{a^2 + b^2}{a^2 b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

22. Show that the tangents at the extremties of all chords of the ellipse which subtend a right angle at the centre, intersect on the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$$

23. If P be any point on the ellipse whose axes are AA_1, BB_1 and if the parallel lines AP, BQ be drawn, Q being on the ellipse; Q will be one extremity of the diamter conjugate to that drawn from P.
24. From a any point T on one of the equiconjugate diamters of a conic whose centre is O, tangents TP, TQ are drawn to the conic. Show that O, P, Q, T are concyclic.
25. Prove that, if two conjugate radii of an ellipse cut the director circle in T, T_1 then TT_1 touches the ellipse.
26. If PSP_1, QSQ_1 be two focal chords and if PQ be parallel to the major axis, show that P_1Q_1 bisects the distance between S and the nearer directrix.
27. Tangents are drawn at the extremities of conjugate diameters of an ellipse, and meet in O. Prove that the perpendicular from O on the focal radius to a point of contact is half the minor axis.
28. Show that the area of the rectangle fromed by two parallel tangents and the corresponding normals to an ellipse is never greater than half the square on the line joining the foci.
29. Prove that the angle btween the normal and the central radius at a point on an ellipse is greatest when the point is the end of one of the equiconjugate diameters.
30. P, Q are two points on the ellipse, and PS, QS_1 intersect on the curve, prove that the locus of the pole of PQ is

$$\frac{x^2}{a^2} + \frac{b^2 y^2}{(2a^2 - b^2)^2} = 1$$

31. Two conjugate diameters of an ellipse meet a fixed straight line $ls + my = 1$ in P, Q, and the straight lines through P, Q perpendicular to these diameters intersect in R; prove that the locus of R is the straight line

$$a^2 lx + b^2 my = a^2 + b^2$$

32. Prove that if α, β are the eccentric angles of two points P, P_1 on an ellipse such that the focal distances SP, S_1P_1 are parallel, then

$$\tan\frac{\alpha}{2} : \tan\frac{\beta}{2} = 1 \pm e : 1 \mp e,$$

and PP_1 touches the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2}$$

33. Prove that the square of the sum of two conjugate radii is greatest when the radii are equal.
34. Prove that, if on the inward normal to an ellipse at P a length PQ be taken equal to the conjugate radius CD, the locus of Q is a circle of radius $a - b$.
35. Prove that, if P, Q are corresponding points on an ellipse and its auxiliary circle, and the normal at P to the ellipse meets the normal at Q to the circle in R, then the locus of R is a circle of radius $a + b$.
36. Prove that, if lines drawn from any poin on an ellipse to the ends of a diameter PCP_1 meet the conjugate diameter DCD_1 in M, M_1, then $CM.CM_1 = CD^2$.
37. Prove that, if an ellipse slides between two straight lines at right angles to one another, the locus of its centre is a circle.