Computational Approach to School Mathematics

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Contents

***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/ncert/codes
```

1 Triangle

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90°, a = 4$ and $b = 3$.
Solution: The vertices of $\triangle ABC$ are
$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1.1}$$
The following code plots Fig. 1.1.1
```
codes/triangle/rt_triangle.py
```

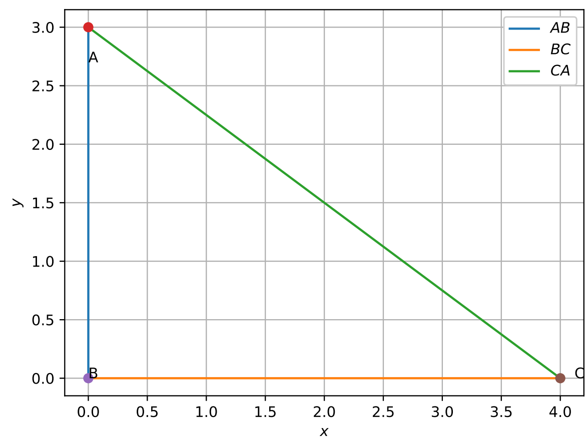

Fig. 1.1.1

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution: Let the vertices of $\triangle ABC$ be
$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.1.2.1}$$
$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \tag{1.1.2.2}$$
$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{1.1.2.3}$$
$$= p \times p + q \times q = p^2 + q^2 \tag{1.1.2.4}$$
Then
$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{1.1.2.5}$$
$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{1.1.2.6}$$
$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \tag{1.1.2.7}$$
From (1.1.2.7),
$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \tag{1.1.2.8}$$
$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{1.1.2.9}$$
$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{1.1.2.10}$$
$$= a^2 + c^2 - 2ap \tag{1.1.2.11}$$
yielding
$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{1.1.2.12}$$

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From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.1.2.14}$$

The following code plots Fig. 1.1.2

codes/triangle/draw_triangle.py

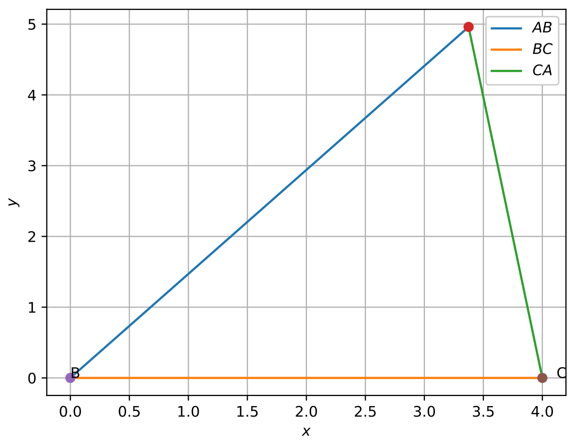

Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.
4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
Solution: Using Baudhayana's theorem,

$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{1.1.4.1}$$

5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^{\circ}$ and $\angle C = 45^{\circ}$, find a, b, c and sketch the triangle.
Solution: From the given information,

$$a + b + c = 11 \tag{1.1.5.1}$$

$$b = c \quad (\because B = C = 45^\circ) \tag{1.1.5.2}$$

$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{1.1.5.3}$$

From (1.1.5.1) and (1.1.5.2),

$$a + 2b = 11 \tag{1.1.5.4}$$

From (1.1.5.2) and (1.1.5.3),

$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{1.1.5.5}$$

(1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{1.1.5.6}$$

which can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{1.1.5.7}$$

$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{1.1.5.8}$$

$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{1.1.5.9}$$

by expanding the determinants. The following code may be used to compute a, b and c.

codes/triangle/triangle_det.py

6. Repeat Problem 1.1.5 using a single matrix equation.
Solution: The equations

$$a + 2b = 11 \tag{1.1.6.1}$$

$$a - b\sqrt{2} = 0 \tag{1.1.6.2}$$

$$b - c = 0 \tag{1.1.6.3}$$

can be expressed as a single matrix equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{1.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{1.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. 1.1.7, $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{1.1.7.1}$$

$$\cos B = \frac{x}{b}, \tag{1.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \tag{1.1.7.3}$$

$$b = c\cos A + a\cos C \tag{1.1.7.4}$$

$$c = b\cos A + a\cos B \tag{1.1.7.5}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.7.6}$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.1.7.7}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1.7.8}$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \tag{1.1.7.9}$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

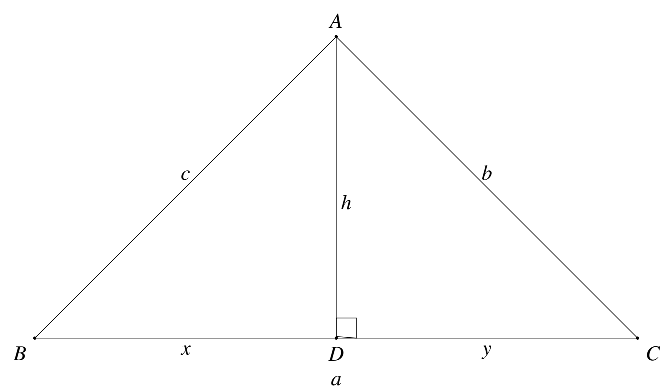

Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
Solution: In Fig. (1.1.7),

$$\sin B = \frac{h}{c} \tag{1.1.8.1}$$

$$\sin C = \frac{h}{b} \tag{1.1.8.2}$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.1.8.3}$$

Thus,

$$c = \frac{a \sin C}{\sin A} \tag{1.1.8.4}$$

where

$$C = 180 - A - B \tag{1.1.8.5}$$

9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30°$.
Solution: From (1.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.1.9.1}$$

which can be expressed as

$$a^2 - 2ab\cos C + b^2 - c^2 = 0. \tag{1.1.9.2}$$

$$\because (a - b\cos C)^2 = a^2 + b^2\cos^2 C - 2ab\cos C, \tag{1.1.9.3}$$

(1.1.9.2) can be expressed as

$$(a - b\cos C)^2 - b^2\cos^2 C + b^2 - c^2 = 0 \tag{1.1.9.4}$$

$$\implies (a - b\cos C)^2 = b^2\left(1 - \cos^2 C\right) - c^2 \tag{1.1.9.5}$$

$$\text{or}, a = b\cos C \pm \sqrt{b^2\left(1 - \cos^2 C\right) - c^2} \tag{1.1.9.6}$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \tag{1.1.10.1}$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{1.1.10.2}$$

Verify (1.1.9.6) using (1.1.10.2).

11. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.
Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \tag{1.1.11.1}$$

$$\implies (18 - c)^2 = 12^2 + c^2 \tag{1.1.11.2}$$

which can be simplified to obtain

$$36c - 180 = 0 \tag{1.1.11.3}$$

$$\implies c = 5 \tag{1.1.11.4}$$

and $b = 13$

12. Find a simpler solution for Problem 1.1.5
Solution: Use cosine formula.
13. In $\triangle ABC$, $a = 7, \angle B = 75^{\circ}$ and $b + c = 13$.
Alternatively,

$$a = b\cos C + c\cos B \tag{1.1.13.1}$$

$$b\sin C = c\sin B \tag{1.1.13.2}$$

$$a + b + c = 11 \tag{1.1.13.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \tag{1.1.13.4}$$

Solving the equivalent matrix equation gives the desired answer.

1.2 Construction Exercises

1. In $\triangle ABC$, $a = 8, \angle B = 45^{\circ}$ and $c - b = 3.5$. Sketch $\triangle ABC$.
2. In $\triangle ABC$, $a = 6, \angle B = 60^{\circ}$ and $b - c = 2$. Sketch $\triangle ABC$.
3. Draw $\triangle ABC$, given that $a+b+c = 11, \angle B = 30^{\circ}$ and $\angle C = 90^{\circ}$.
4. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
5. Draw an equilateral triangle of side 5.5.
6. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
7. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.
8. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60°$.
9. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90°$.
10. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
11. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60°$.
12. Construct $\triangle XYZ$ if $XY = 6, \angle X = 30°$ and $\angle Y = 100°$.
13. If $AC = 7, \angle A = 60°$ and $\angle B = 50°$, can you draw the triangle?

14. Construct $\triangle ABC$ given that $\angle A = 60°$, $\angle B = 30°$ and $AB = 5.8$.
15. Construct $\triangle PQR$ if $PQ = 5, \angle Q = 105°$ and $\angle R = 40°$.
16. Can you construct $\triangle DEF$ such that $EF = 7.2, \angle E = 110°$ and $\angle F = 180°$?
17. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
18. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
19. Construct right angled $\triangle$ whose hypotenuse is 6 and one of the legs is 4.
20. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
21. Construct the triangles in Table 1.2.21.

S.No	Triangle	Given Measurements		
1	△ABC	$\angle A = 85°$	$\angle B = 115°$	AB = 5
2	△PQR	$\angle Q = 30°$	$\angle R = 60°$	QR = 4.7
3	△ABC	$\angle A = 70°$	$\angle B = 50°$	AC = 3
4	△LMN	$\angle L = 60°$	$\angle N = 120°$	LM = 5
5	△ABC	BC = 2	AB = 4	AC = 2
6	△PQR	PQ = 2.5	QR = 4	PR = 3.5
7	△XYZ	XY = 3	YZ = 4	XZ = 5
8	△DEF	DE = 4.5	EF = 5.5	DF = 4

TABLE 1.2.21

1.3 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

Solution: The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{1.3.1.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.3.1.2}$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \tag{1.3.1.3}$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \tag{1.3.1.4}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \tag{1.3.1.5}$$

$$\implies rank(\mathbf{M}) = 2 \tag{1.3.1.6}$$

as the number of non zero rows is 2. The following code plots Fig. 1.3.1

```
codes/triangle/check_tri.py
```

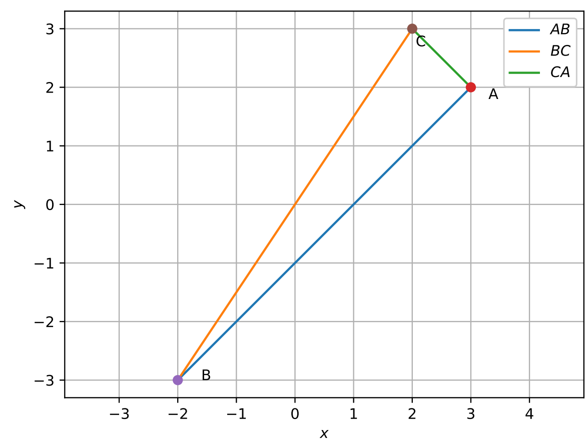

Fig. 1.3.1

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.3.1.7}$$

which, from (1.1.2.10) can be expressed as

$$\|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \tag{1.3.1.8}$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A}) = 0 \tag{1.3.1.9}$$

after simplification. From (1.3.1.1) and (1.3.1.2), it is easy to verify that

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \tag{1.3.1.10}$$

satisfying (1.3.1.9). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.

2. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

3. Draw a triangle whose sides are 8cm and 11cm and the perimeter is 32 cm and find its area.
4. A triangular park ABC has sides 120m, 80m and 50m. A gardener Dhania has to put a fence all around it and also plant grass inside. Draw this park. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.
5. The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Draw the plot and find its area.
6. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.
Solution: The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.3.1.4) as
$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.3.6.1}$$
The computation is done in **area_tri.py**
7. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is
$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \tag{1.3.7.1}$$
8. Find the area of a triangle having the points
$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.3.8.1}$$
as its vertices.
Solution: The area of a triangle using the *vector product* is obtained as
$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})\right\| \tag{1.3.8.2}$$
For any two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.3.8.3}$$
The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
```

9. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point $\mathbf{C}$.
Solution: The centroid of $\triangle ABC$ is given by
$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.3.9.1}$$
Thus,
$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.3.9.2}$$
10. Show that the points
$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.3.10.1}$$
are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.3.10

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since
$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.3.10.2}$$
it is proved that the triangle is indeed right angled.
11. Are the points
$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{1.3.11.1}$$
the vertices of a right angled triangle?
12. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of

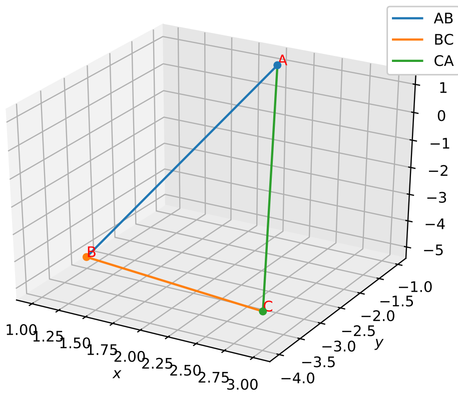

Fig. 1.3.10

the top of the tower is found to be 60°. Find the height of the tower.

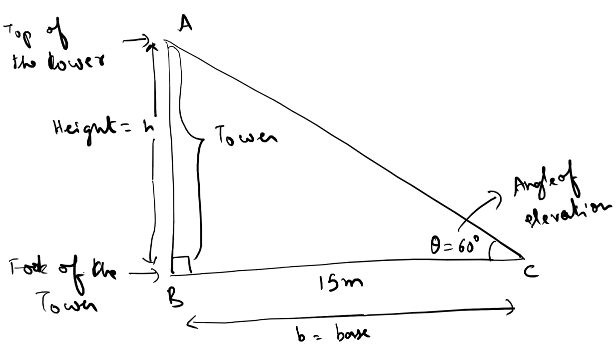

Fig. 1.3.12

Solution: Fig. 1.3.12 summarizes the problem.

$$h = b\tan\theta = 15\tan 60^\circ = 15\sqrt{3} \quad (1.3.12.1)$$

13. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?
Solution: Fig. 1.3.13 summarizes the problem. The objective is to find l and b. From the figure,

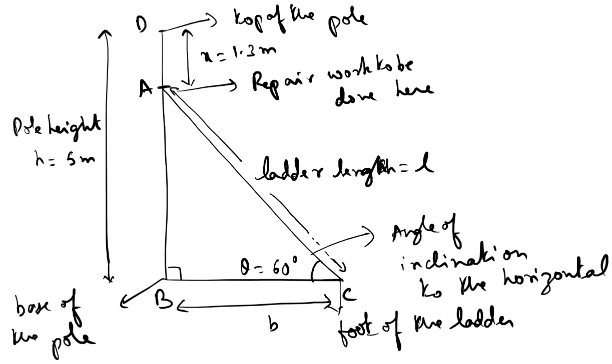

Fig. 1.3.13

if

$$\cot\theta = \frac{1}{\tan\theta}, \quad (1.3.13.1)$$

$$h - x = l\sin\theta = b\tan\theta \quad (1.3.13.2)$$

$$\implies l = (h - x)\csc\theta = 3.7\csc 60^\circ \quad (1.3.13.3)$$

$$\text{and } b = (h - x)\cot\theta = 3.7\cot^\circ \quad (1.3.13.4)$$

14. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

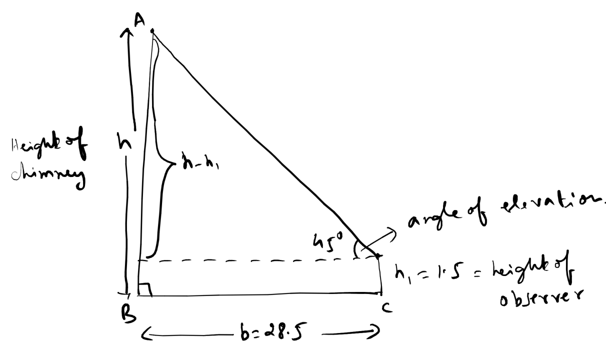

Fig. 1.3.14

Solution: Fig. 1.3.14 summarizes the problem. The objective is to find h. From the figure,

$$h - h_1 = b\tan\theta \quad (1.3.14.1)$$

$$\implies h = h_1 + b\tan\theta \quad (1.3.14.2)$$

$$= 1.5 + 28.5\tan 45^\circ \quad (1.3.14.3)$$

$$= 30m \quad (1.3.14.4)$$

15. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building

and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

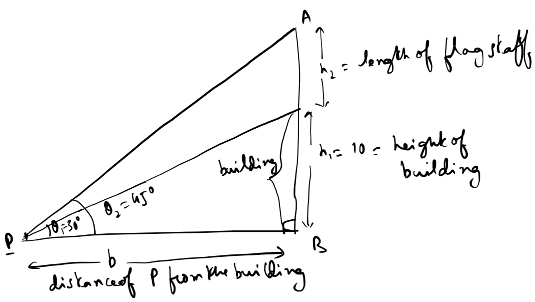

Fig. 1.3.15

Solution: Fig. 1.3.15 summarizes the problem. The objective is to find h_2 and b while h_1 is known. From the figure,

$$h_1 + h_2 = b \tan \theta_1 \tag{1.3.15.1}$$

$$h_1 = b \tan \theta_2 \tag{1.3.15.2}$$

This can be expressed as the matrix equation

$$\begin{pmatrix} \tan \theta_1 & -1 \\ \tan \theta_2 & 0 \end{pmatrix} \begin{pmatrix} b \\ h_2 \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.15.3}$$

and solved.

16. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

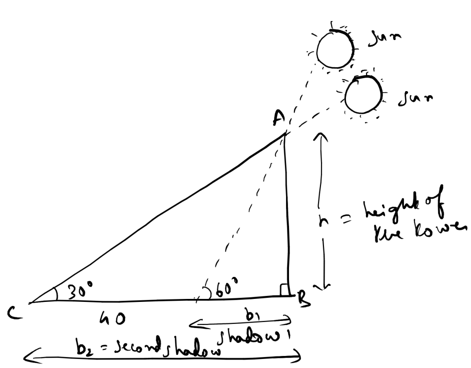

Fig. 1.3.16

Solution: Fig. 1.3.16 summarizes the problem. The objective is to find h. from the figure,

$$b_1 = h \cot 60° \tag{1.3.16.1}$$

$$b_2 = h \cot 30° \tag{1.3.16.2}$$

$$b_2 - b_1 = 40 \tag{1.3.16.3}$$

$$\implies h (\cot 30° - \cot 60°) = 40 \tag{1.3.16.4}$$

$$\text{or } h = \frac{40}{\cot 30° - \cot 60°} \tag{1.3.16.5}$$

17. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

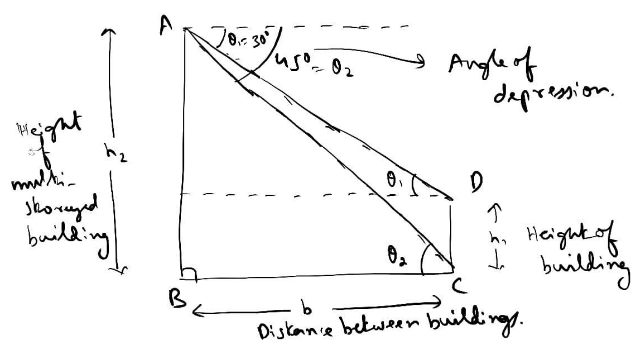

Fig. 1.3.17

Solution: Fig. 1.3.17 summarizes the problem. The objective is to find h_2 and b. From the figure,

$$h_2 = b \tan \theta_2 \tag{1.3.17.1}$$

$$h_2 - h_1 = b \tan \theta_1 \tag{1.3.17.2}$$

which can be expressed as

$$\begin{pmatrix} 1 & -\tan \theta_2 \\ 1 & -\tan \theta_1 \end{pmatrix} \begin{pmatrix} h_2 \\ b \end{pmatrix} = h_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.3.17.3}$$

and solved.

1.4 Triangle Exercises

1. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?

2. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m. The advertisements yield an earning of ₹5000 per m^2 per year. A company hired one o its walls for 3 months. How much rent did it pay?
3. There is a slide in a park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN". If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.
4. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.
5. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.
6. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.
7. Draw the graphs of the equations
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.4.7.1}$$
$$\begin{pmatrix}3 & 2\end{pmatrix} - 12 = 0 \tag{1.4.7.2}$$
Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.
8. In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
9. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
10. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-2\\3\end{pmatrix}, \mathbf{R} = \begin{pmatrix}4\\5\end{pmatrix}$. Find the equation of the median through the vertex $\mathbf{R}$.
11. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\2\end{pmatrix}$, find the equation and length of the altitude from the vertex $\mathbf{A}$.
12. Find the area of the triangle whose vertices are
    a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$
    b) $\begin{pmatrix}-5\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$
13. Find the area of the triangle formed by joining the mid points o the sides of a triangle whose vertices are $\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$.
14. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}4\\-6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\2\end{pmatrix}$ divides it into two triangles of equal areas.
15. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix}4\\6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7\\2\end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.4.15.1}$$
Find
$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{1.4.15.2}$$
16. Let $\mathbf{A} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}1\\4\end{pmatrix}$ be the vertices of $\triangle ABC$.
    a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
    b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
    c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
17. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.4.17.1}$$
18. Show that the points
$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \tag{1.4.18.1}$$
are the vertices of a right angled triangle.
19. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\1\\2\end{pmatrix}$. Find $\angle B$.
20. Show that the vectors $\begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-4\\-4\end{pmatrix}$ form the vertices of a right angled triangle.
21. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix}$ as its vertices.
22. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\3\\5\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}1\\5\\5\end{pmatrix}$

23. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.
24. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$
25. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.
26. Check whether
$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.4.26.1}$$
are the vertices of an isosceles triangle.
27. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.
28. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.
29. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of 30° to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?
30. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is 30°. Find the height of the tower.
31. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.
32. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.
33. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.
34. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.
35. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.
36. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.
37. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.
38. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.
39. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.
40. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of

88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

41. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.
42. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.
43. E and F are points on the sides PQ and PR respectively of a $\triangle PQR$. For each of the following cases, state whether $EF \parallel QR$
    a) $PE = 3.9cm, EQ = 3cm, PF = 3.6cm$ and $FR = 2.4cm$
    b) $PE = 4cm, QE = 4.5cm, PF = 8cm$ and $RF = 9cm$
    c) $PQ = 1.28cm, PR = 2.56cm, PE = 0.18cm$ and $PF = 0.36cm$
44. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.
45. $\triangle ODC \sim \triangle OBA, \angle BOC = 125°$ and $\angle CDO = 70°$. Find $\angle DOC, \angle DCO$ and $\angle OAB$.
46. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?
47. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.
48. Let $\triangle ABC \sim \triangle DEF$ and their areas be, respectively, 64 cm^2 and 121 cm^2. If $EF = 15.4cm$, find BC.
49. A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.
50. Sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse.
    a) 7 cm, 24 cm, 25 cm
    b) 3 cm, 8 cm, 6 cm
    c) 50 cm, 80 cm, 100 cm
    d) 13 cm, 12 cm, 5 cm
51. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.
52. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?
53. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?
54. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.
55. In $\triangle ABC, AB = 63cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.

2 Quadrilateral

2.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
   Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting
   $$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.1.1}$$
   it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x-axis by

an angle θ with

$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.2}$$

and

$$\cos\theta = \frac{a^2 + e^2 - b^2}{2ae} \tag{2.1.1.3}$$

$$\sin\theta = \sqrt{1 - \cos^2\theta} \tag{2.1.1.4}$$

The coordinates of the rotated triangle ACD are

$$\mathbf{D} = \mathbf{P}\begin{pmatrix} h \\ k \end{pmatrix} \tag{2.1.1.5}$$

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.6}$$

$$\mathbf{C} = \mathbf{P}\begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.7}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.1.8}$$

The following code plots quadrilateral $ABCD$

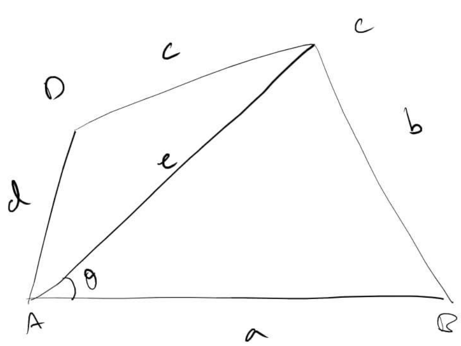

Fig. 2.1.1

in Fig. 2.1.1

codes/quad/draw_quad.py

2. Draw the parallelogram $MORE$ with $OR = 6, RE = 4.5$ and $EO = 7.5$.
   Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel .
3. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.
   Solution: The diagonals of a kite are perpendicular to each other.

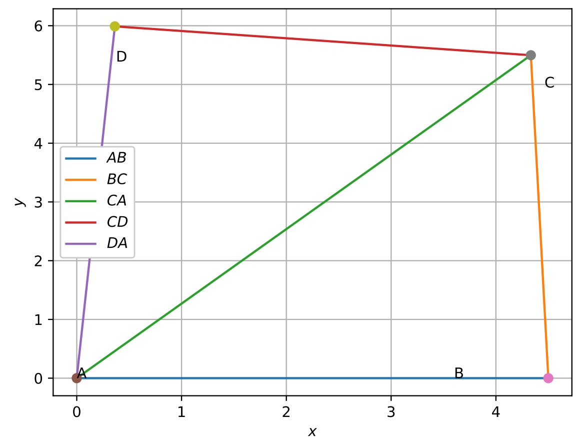

Fig. 2.1.1

4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
   Solution: Diagonals of a rhombus bisect each other at right angles.

2.2 *Construction Exercises*

1. Construct a quadrilateral $ABCD$ such that $AB = 5, \angle A = 50°, AC = 4, BD = 5$ and $AD = 6$.
2. Construct $PQRS$ where $PQ = 4, QR = 6, RS = 5, PS = 5.5$ and $PR = 7$.
3. Draw $JUMP$ with $JU = 3.5, UM = 4, MP = 5, PJ = 4.5$ and $PU = 6.5$
4. Construct a quadrilateral $ABCD$ such that $BC = 4.5, AC = 5.5, CD = 5, BD = 7$ and $AD = 5.5$.
5. Can you construct a quadrilateral $PQRS$ with $PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ = 4$?
6. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.
7. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.
8. DRAW rhombus $BEND$ such that $BN = 5.6$, $DE = 6.5$.
9. construct a quadrilateral MIST where $MI = 3.5, IS = 6.5, \angle M = 75°, \angle I = 105°$ and $\angle S = 120°$.
10. Can you construct the above quadrilateral MIST if $\angle M = 100°$ instead of $75°$.
11. Can you constrcut the quadrilateral PLAN if $PL = 6, LA = 9.5, \angle P = 75°, \angle L = 150°$ and $\angle A = 140°$?

12. Construct $MORE$ where $MO = 6, OR = 4.5, \angle M = 60°, \angle O = 105°, \angle R = 105°$.
13. Construct $PLAN$ where $PL = 4, LA = 6.5, \angle P = 90°, \angle A = 110°$ and $\angle N = 85°$.
14. Constrcut parallelogram $HEAR$ where $HE = 5, EA = 6, \angle R = 85°$.
15. Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.
16. Construct $ABCd$, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105°$ and $\angle C = 80°$.
17. Construct $DEAR$ with $DE = 4, EA = 5, AR = 4.5, \angle E = 60°$ and $\angle A = 90°$.
18. Construct $TRUE$ with $TR = 3.5, RU = 3, UE = 4 \angle R = 75°$ and $\angle U = 120°$.
19. Draw a square of side 4.5.
20. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
21. Draw a square $READ$ with $RE = 5.1$.
22. Draw a rhombus who diagonals are 5.2 and 6.4.
23. Draw a rectangle with adjacent sides 5 and 4.
24. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

2.3 *Quadrilateral Examples*

1. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,
$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.3.1.1}$$
Hence, the diagonals AC and BD bisect each other. Also,
$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.3.1.2}$$
$\implies AC \perp BD$. Hence $ABCD$ is a square.
2. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.
3. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by
$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{2.3.3.1}$$
4. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors
$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{C} \tag{2.3.4.1}$$
$$\mathbf{A}-\mathbf{D} = \mathbf{B}-\mathbf{C} \tag{2.3.4.2}$$
$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \neq 0 \tag{2.3.4.3}$$
Hence, it is not a rectangle. The following code plots Fig. 2.3.4

```
codes/triangle/quad_3d.py
```

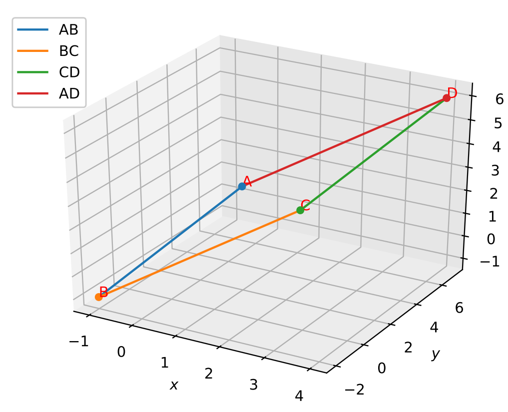

Fig. 2.3.4

5. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and

$\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution: The area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{2.3.5.1}$$

6. Kamla has a triangular field with sides 240 m, 200 m, 360 m, where she grew wheat. In another triangular field with sides 240 m, 320 m, 400 m adjacent to the previous field, she wanted to grow potatoes and onions. She divided the field in two parts by joining the mid-point of the longest side to the opposite vertex and grew patatoes in one part and onions in the other part. Draw the figure for this problem. How much area (in hectares) has been used for wheat, potatoes and onions? (1 hectare = 10000 m^2).
7. Students of a school staged a rally for cleanliness campaign. They walked through the lanes in two groups. One group walked through the lanes AB, BC and CA; while the other through AC, CD and DA. Then they cleaned the area enclosed within their lanes. If AB = 9 m, BC = 40 m, CD = 15 m, DA = 28 m and $\angle B = 90°$, which group cleaned more area and by how much? Draw the corresponding figure. Find the total area cleaned by the students (neglecting the width of the lanes).
8. Sanya has a piece of land which is in the shape of a rhombus. She wants her one daughter and one son to work on the land and produce different crops. She divided the land in two equal parts. If the perimeter of the land is 400 m and one of the diagonals is 160 m, how much area each of them will get for their crops? Draw the rhombus.

2.4 *Quadrilateral Geometry*

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \tag{2.4.2.1}$$
$$\angle B = 3y - 5 \tag{2.4.2.2}$$
$$\angle C = -4x \tag{2.4.2.3}$$
$$\angle D = -7x + 5 \tag{2.4.2.4}$$

Find its angles.

3. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}$, $\begin{pmatrix}0\\7\end{pmatrix}$, $\begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.
4. Find the area of a rhombus if its vertices are $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}4\\5\end{pmatrix}$, $\begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\end{pmatrix}$ taken in order.
5. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.
6. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-5\end{pmatrix}$, $\begin{pmatrix}3\\-2\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$.
7. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
8. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?
9. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
10. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-7\\1\end{pmatrix}$.
11. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix}-1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\-\frac{1}{2}\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-\frac{1}{2}\\4\end{pmatrix}$.
12. The two adjacent sides of a parallelogram are $\begin{pmatrix}2\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. Find the unit vector parallel to

its diagonal. Also, find its area.

13. A park, in the shape of a quadrilateral $ABCD$, has $\angle C = 90°, AB = 9m, BC = 12m, CD = 5m$ and $AD = 8m$. How much area does it occupy? 2. Find the area of a quadrilateral $ABCD$ in which $AB = 3cm, BC = 4cm, CD = 4cm, DA = 5cm$ and $AC = 5cm$.
14. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 26 cm, 28 cm and 30 cm, and the parallelogram stands on the base 28 cm, find the height of the parallelogram.
15. A rhombus shaped field has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48 m, how much area of grass field will each cow be getting?
16. A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The non-parallel sides are 14 m and 13 m. Find the area of the field.
17. $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8$ cm and $CF = 10$ cm, find AD.

3 Line

3.1 Examples: Geometry

1. Check whether –2 and 2 are zeroes of the polynomial $x + 2$.
2. Find a zero of the polynomial $p(x) = 2x + 1$.
3. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = 3x + 1, x = \frac{1}{3}$
   b) $p(x) = 5x - \pi, x = \frac{4}{5}$
   c) $p(x) = 5lx + m, x = -\frac{m}{l}$
   d) $p(x) = 2x + 1, x = \frac{1}{2}$
4. Find the zero of the polynomial in each of the following cases:
   a) $p(x) = x + 5$
   b) $p(x) = x–5$
   c) $p(x) = 2x + 5$
   d) $p(x) = 3x–2$
   e) $p(x) = 3x$
   f) $p(x) = ax, a \neq 0$
   g) $p(x) = cx + d, c \neq 0, c, d$ are real numbers.
5. Find four different solutions of the equation
$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} = 6 \tag{3.1.5.1}$$
6. Find two solutions for each of the following equations:
   a) $\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12$
   b) $\begin{pmatrix} 2 & 5 \end{pmatrix}\mathbf{x} = 0$
   c) $\begin{pmatrix} 0 & 3 \end{pmatrix}\mathbf{x} = 4$
7. Draw the graph of
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 7 \tag{3.1.7.1}$$
8. Draw the graphs of the following equations
   a) $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0$
   b) $\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = 0$
   c) $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0$
   d) $\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -1$
   e) $\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = 4$
   f) $\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 4$
9. Two rails are represented by the equations
$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x}–4 = 0 \text{ and} \tag{3.1.9.1}$$
$$\begin{pmatrix} 2 & 4 \end{pmatrix}\mathbf{x}–12 = 0. \tag{3.1.9.2}$$
   Will the rails cross each other?
10. Check graphically whether the pair of equations
$$\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = 6 \text{ and} \tag{3.1.10.1}$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} = 12 \tag{3.1.10.2}$$
   is consistent. If so, solve them graphically.
11. Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:
$$\begin{pmatrix} 5 & -8 \end{pmatrix}\mathbf{x} = -1 \text{ and} \tag{3.1.11.1}$$
$$\begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix}\mathbf{x} = -\frac{3}{5} \tag{3.1.11.2}$$
12. Solve the following pair of equations
$$\begin{pmatrix} 7 & -15 \end{pmatrix}\mathbf{x} = 2 \tag{3.1.12.1}$$
$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} = 3 \tag{3.1.12.2}$$
13. Find all possibe solutions of
$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 8 \\ \begin{pmatrix} 4 & 6 \end{pmatrix}\mathbf{x} = 7 \end{aligned} \tag{3.1.13.1}$$
14. For which values of p does the pair of equations given below has unique solution?
$$\begin{aligned} \begin{pmatrix} 4 & p \end{pmatrix}\mathbf{x} = -8 \\ \begin{pmatrix} 2 & 2 \end{pmatrix}\mathbf{x} = -2 \end{aligned} \tag{3.1.14.1}$$
15. For what values of k will the following pair of

linear equations have infinitely many solutions?

$$\begin{pmatrix} k & 3 \end{pmatrix}\mathbf{x} = k-3$$
$$\begin{pmatrix} 12 & k \end{pmatrix}\mathbf{x} = k \tag{3.1.15.1}$$

16. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{3.1.16.1}$$

Solution: $x = 2, y = 2, z = 1$.

17. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.1.17.1}$$

verify if
a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
b) $\mathbf{a} = \mathbf{b}$

Solution:
a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

18. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.1.18.1}$$

19. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.
20. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
21. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\| = 7 \tag{3.1.21.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{3.1.21.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{3.1.21.3}$$

22. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \tag{3.1.22.1}$$

23. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{3.1.23.1}$$

24. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \tag{3.1.24.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \tag{3.1.24.2}$$

25. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.

Solution: Refer to Problem 1.3.1.

26. Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix} 7 \\ 1 \end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}\right\|^2 \tag{3.1.26.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix} 7 \\ 1 \end{pmatrix}\right\|^2 - 2\begin{pmatrix} 7 & 1 \end{pmatrix}\mathbf{x}$$
$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix} 3 \\ 5 \end{pmatrix}\right\|^2 - 2\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} \tag{3.1.26.2}$$

which can be simplified to obtain

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 2 \tag{3.1.26.3}$$

which is the desired condition. The following code plots Fig. 3.1.26clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

27. Find a point on the y-axis which is equidistant

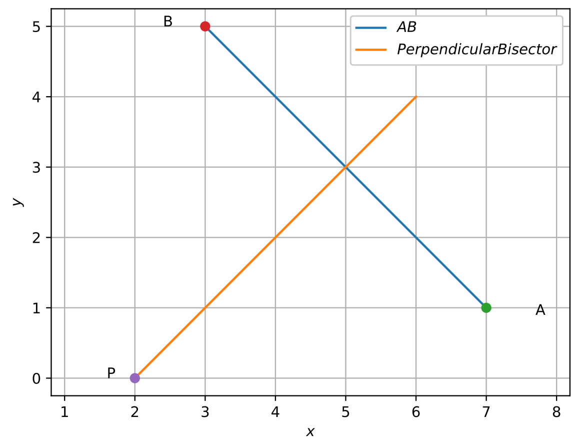

Fig. 3.1.26

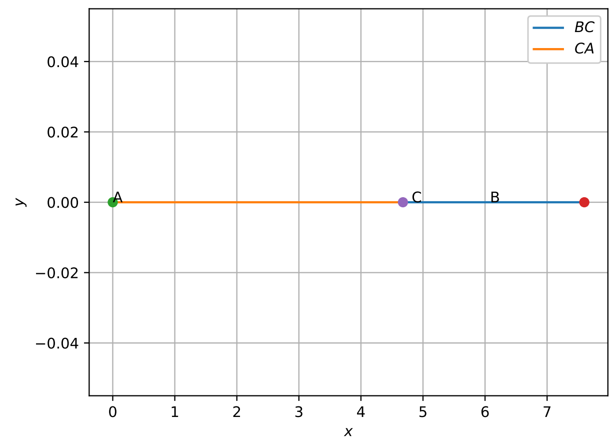

Fig. 3.1.28

from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Solution: Choose $\mathbf{x} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ and follow the approach in Problem (3.1.26). Solve for y.

28. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \tag{3.1.28.1}$$

Then the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{3.1.28.2}$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 3.1.28

```
codes/line/draw_section.py
```

29. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.
Solution: Using (3.1.28.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \tag{3.1.29.1}$$

30. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \tag{3.1.30.1}$$

Solution: Use (3.1.28.2).

31. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \tag{3.1.31.1}$$

Solution: Using (3.1.28.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.1.31.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.1.31.3}$$

32. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.28.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.1.32.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.1.32.2}$$

33. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.3.1.4),

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{B}-\mathbf{A}\end{pmatrix}^T = \begin{pmatrix}2 & k-3\\ 4 & -6\end{pmatrix} \tag{3.1.33.1}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}2 & k-3\\ 2 & -3\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}2 & k-3\\ 0 & -k\end{pmatrix} \tag{3.1.33.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{3.1.33.3}$$

34. Find the direction vectors and slopes of the lines passing through the points
    a) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.
    b) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}7\\-2\end{pmatrix}$.
    c) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
    d) Making an inclination of 60° with the positive direction of the x-axis.

    Solution:

    a) If the direction vector is

    $$\begin{pmatrix}1\\m\end{pmatrix}, \tag{3.1.34.1}$$

    the slope is m. Thus, the direction vector is

    $$\begin{pmatrix}-1\\4\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix} = -\frac{1}{4}\begin{pmatrix}-4\\6\end{pmatrix} \tag{3.1.34.2}$$

    $$= \begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix} \implies m = -\frac{3}{2} \tag{3.1.34.3}$$

    b) The direction vector is

    $$\begin{pmatrix}7\\-2\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} \tag{3.1.34.4}$$

    $$= \begin{pmatrix}1\\0\end{pmatrix} \implies m = 0 \tag{3.1.34.5}$$

    c) The direction vector is

    $$\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix} \tag{3.1.34.6}$$

    $$= \begin{pmatrix}1\\\infty\end{pmatrix} \implies m = \infty \tag{3.1.34.7}$$

    d) The slope is $m = \tan 60° = \sqrt{3}$ and the direction vector is

    $$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{3.1.34.8}$$

35. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.

    Solution: The angle θ between two lines is given by

    $$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{3.1.35.1}$$

    $$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{3.1.35.2}$$

    $$\text{or } m_1 = \frac{5}{3} \tag{3.1.35.3}$$

36. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.

    Solution: Using (1.3.1.9)

    $$\left\{\begin{pmatrix}-2\\6\end{pmatrix} - \begin{pmatrix}4\\8\end{pmatrix}\right\}^T \left\{\begin{pmatrix}8\\12\end{pmatrix} - \begin{pmatrix}x\\24\end{pmatrix}\right\} = 0 \tag{3.1.36.1}$$

    which can be used to obtain x.

37. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

    Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3\\8\end{pmatrix}$ is obtained as

    $$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{3.1.37.1}$$

    $$\implies \begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} - \lambda\begin{pmatrix}-3\\-6\end{pmatrix} \tag{3.1.37.2}$$

    which can be expressed as

    $$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{3.1.37.3}$$

    $$\implies \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = -2 \tag{3.1.37.4}$$

    $$\implies D = 2 + 2T \tag{3.1.37.5}$$

38. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$.

Solution: The line parallel to the x-axis has direction vector $\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.38.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}0\\1\end{pmatrix} \tag{3.1.38.2}$$

The following code plots Fig. 3.1.38

```
codes/line/line_parallel_axes.py
```

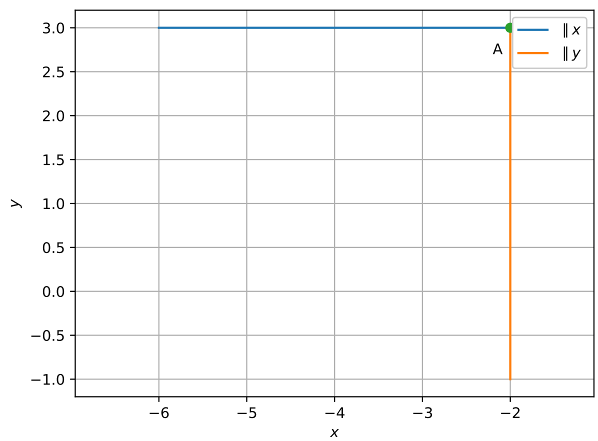

Fig. 3.1.38

39. Find the equation of the line through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$ with slope –4.

    Solution: The direction vector is $\mathbf{m} = \begin{pmatrix}1\\-4\end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{3.1.39.1}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} \tag{3.1.39.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.39.3}$$

$$\implies \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = -5 \tag{3.1.39.4}$$

40. Write the equation of the line through the points $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.

    Solution: Use (3.1.38.1).

41. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and
    a) y-intercept is $-\frac{3}{2}$
    b) x-intercept is 4.

    Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.
    a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$.
    b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix}4\\0\end{pmatrix}$.

    Use the above information get the equations for the lines.

42. Find the equation of a line through the point $\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}3\\2\\-8\end{pmatrix}$.

    Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix}5 & 2\\-4 & \end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-8\end{pmatrix} \tag{3.1.42.1}$$

43. Find the equation of a line passing through the points $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\\6\end{pmatrix}$.

    Solution: Using (3.1.37.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda\begin{pmatrix}4\\4\\4\end{pmatrix} \tag{3.1.43.1}$$

$$= \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{3.1.43.2}$$

44. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{3.1.44.1}$$

find the equation of the line.

Solution: The line can be expressed from

(3.1.44.1) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+2\lambda \\ 5+4\lambda \\ -6+2\lambda \end{pmatrix} \tag{3.1.44.2}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \tag{3.1.44.3}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{3.1.44.4}$$

45. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution: See Problem 3.1.41. The line passes through the points $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
46. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 3.1.46, the foot of the perpendicular P is the intersection of the lines L and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \tag{3.1.46.1}$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.46.2}$$

$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda \left\|\mathbf{n}\right\|^2 = c \tag{3.1.46.3}$$

$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \tag{3.1.46.4}$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{3.1.46.5}$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda| \left\|\mathbf{n}\right\| \tag{3.1.46.6}$$

From (3.1.46.4) and (3.1.46.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \tag{3.1.46.7}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \tag{3.1.46.8}$$

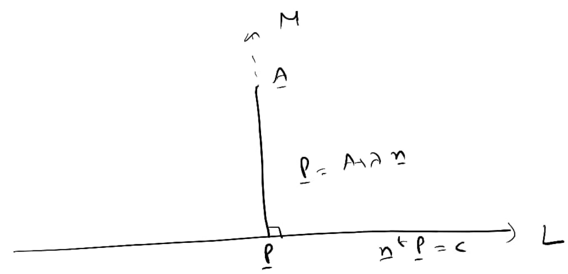

Fig. 3.1.46

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \tag{3.1.46.9}$$

$$= \pm 4 \sec 15^\circ \tag{3.1.46.10}$$

where

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.1.46.11}$$

This follows from (**??**), where

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.1.46.12}$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{3.1.46.13}$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{3.1.46.14}$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \tag{3.1.46.15}$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix} \mathbf{c} = \pm 4 \sec 15^\circ \tag{3.1.46.16}$$

47. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \tag{3.1.47.1}$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.39.3) and (3.1.39.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} 273 \\ 32 \end{pmatrix} \tag{3.1.47.2}$$

$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = -197 \tag{3.1.47.3}$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \tag{3.1.47.4}$$

48. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 10 = 0. \tag{3.1.48.1}$$

Find its
a) slope,
b) x - and y-intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{3.1.48.2}$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \tag{3.1.48.3}$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

49. Find the angle between two vectors **a** and **b** where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T\mathbf{b} = 1. \tag{3.1.49.1}$$

Solution: In Fig. 3.1.49, from the cosine formula in (1.1.7.9)

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\,\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{3.1.49.2}$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\,\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.49.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\,\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.49.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.49.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{3.1.49.6}$$

$$= \frac{1}{2} \tag{3.1.49.7}$$

$$\implies \theta = 60^\circ \tag{3.1.49.8}$$

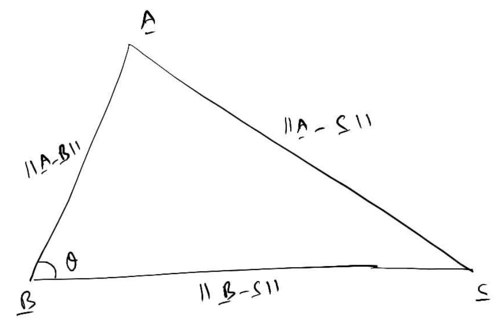

Fig. 3.1.49

50. Find the angle between the lines

$$\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 5 \tag{3.1.50.1}$$

$$\begin{pmatrix} \sqrt{3} & -1 \end{pmatrix} \mathbf{x} = -6. \tag{3.1.50.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{3.1.50.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{3.1.50.4}$$

51. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.51.1}$$

and passes through the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.1.51.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = 0 \tag{3.1.51.3}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.51.4}$$

52. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{3.1.52.1}$$

Solution: Use (3.1.46.7).

53. If the lines

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.53.1}$$

$$\begin{pmatrix} 5 & k \end{pmatrix} \mathbf{x} = 3 \tag{3.1.53.2}$$

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{3.1.53.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & k & 3 \\ 3 & -1 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 5 & k & 3 \end{pmatrix} \tag{3.1.53.4}$$

$$\xleftrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftrightarrow 2R_2 - 3R_1} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & -5 \\ 0 & 2k-5 & -9 \end{pmatrix} \tag{3.1.53.5}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{5}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2k-5 & -9 \end{pmatrix} \tag{3.1.53.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (2k-5) R_2} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2k-4 \end{pmatrix} \tag{3.1.53.7}$$

$$\implies k = -2 \tag{3.1.53.8}$$

54. Find the distance of the line

$$L_1 : \quad \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.54.1}$$

from the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan 135° \end{pmatrix} \tag{3.1.54.2}$$

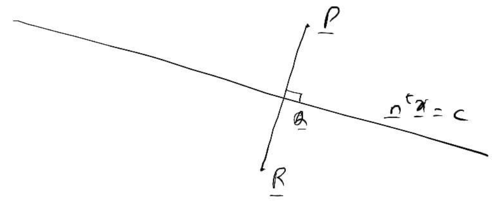

Fig. 3.1.55

Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.54.3}$$

$$\implies \mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.54.4}$$

$$\text{or,} \quad \left\| \mathbf{P} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = d = |\lambda| \, \|\mathbf{m}\| \tag{3.1.54.5}$$

Since $\mathbf{P}$ lies on L_1, from (3.1.54.1),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{P} = 0 \tag{3.1.54.6}$$

Substituting from the above in (3.1.54.3),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{3.1.54.7}$$

$$\implies \lambda = \frac{\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m}}{17} \tag{3.1.54.8}$$

substituting $|\lambda|$ in (3.1.54.5) gives the desired answer.

55. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = -4. \tag{3.1.55.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{3.1.55.2}$$

$$\mathbf{n}^T \mathbf{Q} = c \tag{3.1.55.3}$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \tag{3.1.55.4}$$

where $\mathbf{m}$ is the direction vector of L. From

(3.1.55.2) and (3.1.55.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{3.1.55.5}$$

From (3.1.55.5) and (3.1.55.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.55.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{3.1.55.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{3.1.55.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{3.1.55.9}$$

(3.1.55.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.55.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{3.1.55.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{3.1.55.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{3.1.55.13}$$

The following code plots Fig. 3.1.55 while computing the reflection

codes/line/line_reflect.py

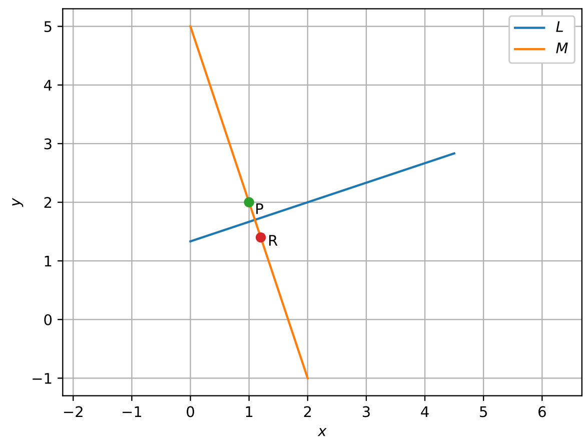

Fig. 3.1.55

56. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix}1\\5\end{pmatrix}$. Obtain its equation.

$$L_1: \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.1.56.1}$$

$$L_2: \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.1.56.2}$$

Solution: Let

$$L: \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.56.3}$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.56.4}$$

$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \tag{3.1.56.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (3.1.56.1), (3.1.56.4) and (3.1.56.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \tag{3.1.56.6}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \tag{3.1.56.7}$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \tag{3.1.56.8}$$

$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \tag{3.1.56.9}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \tag{3.1.56.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{3.1.56.11}$$

57. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.57.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.57.2}$$

are equal is a straight line.

Solution: Using (3.1.46.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|}=\frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{3.1.57.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right|=\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right| \tag{3.1.57.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5=\pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right) \tag{3.1.57.5}$$

leading to the possible lines

$$L_1:\quad \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x}=0 \tag{3.1.57.6}$$

$$L_2:\quad \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}=\frac{5}{3} \tag{3.1.57.7}$$

58. Find the distance between the points

$$\mathbf{P}=\begin{pmatrix}1\\-3\\4\end{pmatrix},\mathbf{Q}=\begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{3.1.58.1}$$

Solution: The distance is given by $\|\mathbf{P}-\mathbf{Q}\|$

59. Show that the points $\mathbf{A}=\begin{pmatrix}-2\\3\\5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}7\\0\\-1\end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.3.1.4)

$$\mathbf{M}=\begin{pmatrix}3 & -1 & -2\\9 & -3 & -6\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2-3R_1}\begin{pmatrix}3 & -1 & -2\\0 & 0 & 0\end{pmatrix} \tag{3.1.59.1}$$

$\implies rank(\mathbf{M})=1$. The following code plots Fig. 3.1.59 showing that the points are collinear.

```
codes/line/collinear_3d.py
```

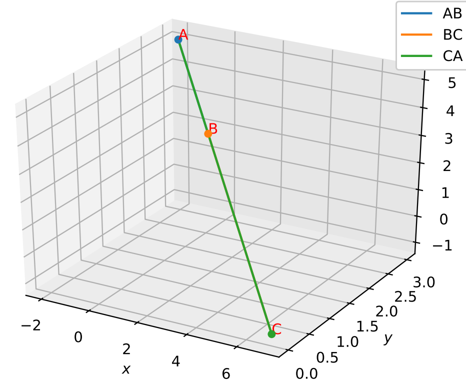

Fig. 3.1.59

60. Show that $\mathbf{A}=\begin{pmatrix}2\\3\\-4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}3\\8\\-11\end{pmatrix}$ are collinear.

Solution: Use the approach in Problem (3.1.59).

61. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2+PB^2=2k^2, \tag{3.1.61.1}$$

$$\mathbf{A}=\begin{pmatrix}3\\4\\5\end{pmatrix},\mathbf{B}=\begin{pmatrix}-1\\3\\-7\end{pmatrix}, \tag{3.1.61.2}$$

respectively.

62. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\begin{pmatrix}3\\4\\-5\end{pmatrix}$ in the ratio 2 : 3
    a) internally, and
    b) externally.

    Solution: Use (3.1.28.2).

63. Prove that the three points $\begin{pmatrix}-4\\6\\10\end{pmatrix}$, $\begin{pmatrix}2\\4\\6\end{pmatrix}$ and $\begin{pmatrix}14\\0\\-2\end{pmatrix}$ are collinear.

    Solution: Use the approach in Problem 3.1.59.

64. Find the ratio in which the line segment joining the points $\begin{pmatrix}4\\8\\10\end{pmatrix}$ and $\begin{pmatrix}6\\10\\-8\end{pmatrix}$ is divided by the YZ-plane.

    Solution: Use (3.1.28.2). The YZ-plane has points $\begin{pmatrix}0\\y\\z\end{pmatrix}$.

65. Find the equation of the set of points $\mathbf{P}$

such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution: Use the approach in Problem 3.1.26.

66. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{3.1.66.1}$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{3.1.66.2}$$

find $\mathbf{R}$, which divides PQ in the ratio $2:1$

a) internally,
b) externally.

Solution: Use (3.1.28.2).

67. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: Use (3.1.49.6)

68. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.1.68.1}$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.1.68.2}$$

Solution: The direction vectors of the lines are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Using(3.1.49.6), the angle between the lines can be obtained.

69. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{3.1.69.1}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{3.1.69.2}$$

Solution: From Problem 3.1.44, the direction vectors of the lines can be expressed as $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The angle between them can then be obtained from (3.1.49.6).

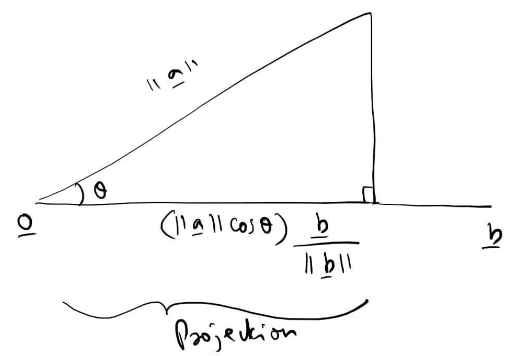

Fig. 3.1.71

70. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Solution: Use (1.3.1.9).

71. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{3.1.71.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{3.1.71.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. 3.1.71. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right) \|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{3.1.71.3}$$

72. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{3.1.72.1}$$

Solution:

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \tag{3.1.72.2}$$

73. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{3.1.73.1}$$

then find $\mathbf{x}$.

Solution:

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{3.1.73.2}$$
$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{3.1.73.3}$$

74. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{3.1.74.1}$$

find $\|\mathbf{a}\times\mathbf{b}\|$.
Solution: Use (1.3.8.3).

75. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{3.1.75.1}$$

Solution: If $\mathbf{x}$ is the desired vector,

$$(\mathbf{a}+\mathbf{b})^T \mathbf{x} = 0 \tag{3.1.75.2}$$
$$(\mathbf{a}-\mathbf{b})^T \mathbf{x} = 0 \tag{3.1.75.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.75.4}$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \tag{3.1.75.5}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.75.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.75.7}$$

76. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are collinear.
Solution: See Problem 3.1.59.

77. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \tag{3.1.77.1}$$
$$\mathbf{C}-\mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \tag{3.1.77.2}$$

$$\because -2(\mathbf{A}-\mathbf{B}) = \mathbf{C}-\mathbf{D}, \tag{3.1.77.3}$$

$\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.

78. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{3.1.78.1}$$

Then,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.1.78.2}$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{3.1.78.3}$$

using (3.1.78.1)

79. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \tag{3.1.79.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{3.1.79.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (3.1.79.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{3.1.79.3}$$
$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{3.1.79.4}$$
$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{3.1.79.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{3.1.79.6}$$

80. Let $\boldsymbol{\alpha} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,
$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.80.1}$$
$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{3.1.80.2}$$
and
$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.80.3}$$
This process is known as *Gram-Schmidth orthogonalization*.
81. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is
$$\mathbf{x} = \begin{pmatrix} \cos 90° \\ \cos 60° \\ \cos 30° \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.1.81.1}$$
$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.
82. Find the distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.2}$$
Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. 3.1.82 From Fig. 3.1.82, the distance is
$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{3.1.82.3}$$
where
$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.4}$$

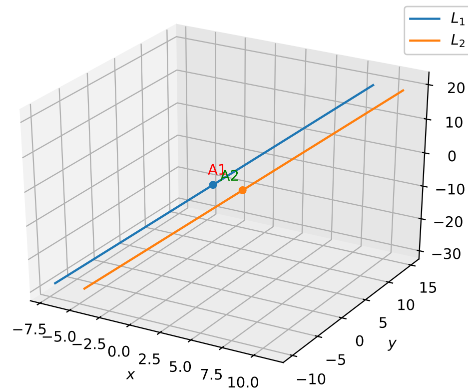

Fig. 3.1.82

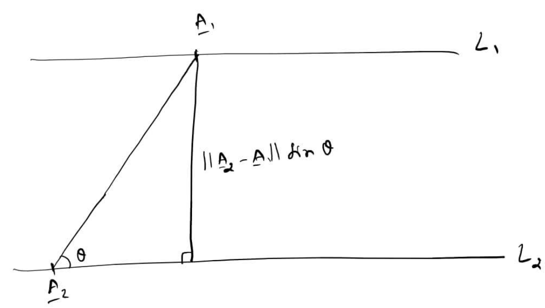

Fig. 3.1.82

83. Find the shortest distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.83.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.83.2}$$
Solution: In the given problem
$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{3.1.83.3}$$

The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{3.1.83.4}$$

$$\implies \lambda_1 \begin{pmatrix}2\\-1\\1\end{pmatrix} - \lambda_2 \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} \tag{3.1.83.5}$$

$$\implies \begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{3.1.83.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}2 & 3 & 1\\-1 & -5 & 0\\1 & 2 & -1\end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix}1 & 2 & -1\\-1 & -5 & 0\\2 & 3 & 1\end{pmatrix} \tag{3.1.83.7}$$

$$\xleftrightarrow[R_3=2R_1-R_3]{R_2=R_1+R_2} \begin{pmatrix}1 & 2 & -1\\0 & -3 & -1\\0 & 1 & -3\end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix}1 & 2 & -1\\0 & 1 & -3\\0 & -3 & -1\end{pmatrix} \tag{3.1.83.8}$$

$$\xleftrightarrow{R_3=3R_2+R_3} \begin{pmatrix}1 & 2 & -1\\0 & 1 & -3\\0 & 0 & -10\end{pmatrix} \tag{3.1.83.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. 3.1.83

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{3.1.83.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \tag{3.1.83.11}$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (3.1.46.7) as

$$\frac{\left|\mathbf{n}^T\left(\mathbf{A}_2 - \mathbf{A}_1\right)\right|}{\|\mathbf{n}\|} = \frac{\left|\left(\mathbf{A}_2 - \mathbf{A}_1\right)^T\left(\mathbf{m}_1 \times \mathbf{m}_2\right)\right|}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \tag{3.1.83.12}$$

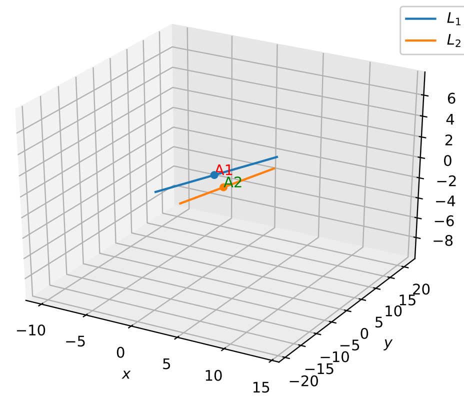

Fig. 3.1.83

84. Find the distance of the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.1.84.1}$$

from the origin.
Solution: From (3.1.46.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{2^2+3^2+4^2}} \tag{3.1.84.2}$$

$$= \frac{6}{\sqrt{29}} \tag{3.1.84.3}$$

85. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix}2\\-3\\4\end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.1.85.1}$$

86. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x} = 1. \tag{3.1.86.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.86.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{3.1.86.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.86.4}$$

87. Find the coordinates of the foot of the perpen-

dicular drawn from the origin to the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{3.1.87.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.87.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda\mathbf{n}$. Substituting in (3.1.87.1),

$$\lambda \|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \tag{3.1.87.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29}\begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.87.4}$$

88. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix}5\\2\\-4\end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (3.1.39.3), the equation of the plane is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.88.1}$$

$$\implies \begin{pmatrix}2\\3\\-1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}5\\2\\-4\end{pmatrix} \tag{3.1.88.2}$$

$$= 20 \tag{3.1.88.3}$$

89. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix}2\\5\\-3\end{pmatrix}, \boldsymbol{S} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix}5\\3\\-3\end{pmatrix}$.
Solution: If the equation of the plane be

$$\mathbf{n}^T\mathbf{x} = c, \tag{3.1.89.1}$$

$$\mathbf{n}^T\mathbf{R} = \mathbf{n}^T\mathbf{S} = \mathbf{n}^T\mathbf{T} = c, \tag{3.1.89.2}$$

$$\implies \begin{pmatrix}\mathbf{R} - \mathbf{S} & \mathbf{S} - \mathbf{T}\end{pmatrix}^T \mathbf{n} = 0 \tag{3.1.89.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix}4 & 8 & -8\\-7 & -6 & 8\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix}1 & 2 & -2\\-7 & -6 & 8\end{pmatrix} \tag{3.1.89.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix}1 & 2 & -2\\0 & 8 & -6\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}1 & 2 & -2\\0 & 4 & -3\end{pmatrix} \tag{3.1.89.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix}2 & 0 & -1\\0 & 4 & -3\end{pmatrix} \tag{3.1.89.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}\frac{1}{2}\\\frac{3}{4}\\1\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} \text{ and} \tag{3.1.89.7}$$

$$c = \mathbf{n}^T\mathbf{T} = 7 \tag{3.1.89.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{n} = 7 \tag{3.1.89.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{3.1.89.10}$$

The equation of the plane is then obtained from (3.1.39.3) as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T(\mathbf{x} - \mathbf{T}) = 0 \tag{3.1.89.11}$$

90. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
Solution: From the given information, the plane passes through the points $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\4\end{pmatrix}$ respectively. The equation can be obtained using Problem 3.1.89.
91. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{3.1.91.1}$$

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = -5 \tag{3.1.91.2}$$

and the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix}1 & 1 & 1 & 6\\ 2 & 3 & 4 & -5\end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & -17\end{pmatrix} \tag{3.1.91.3}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix}1 & 0 & -1 & 23\\ 0 & 1 & 2 & -17\end{pmatrix} \tag{3.1.91.4}$$

$$\implies \mathbf{x} = \begin{pmatrix}23\\ -17\\ 0\end{pmatrix} + \lambda\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \tag{3.1.91.5}$$

Thus, $\begin{pmatrix}23\\ -17\\ 0\end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} - \begin{pmatrix}23\\ -17\\ 0\end{pmatrix}\right) \times \begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \tag{3.1.91.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix}1 & -2 & 1\\ -22 & 18 & 1\end{pmatrix} \xleftrightarrow{R_2=R_2+22R_1} \begin{pmatrix}1 & -2 & 1\\ 0 & -26 & 23\end{pmatrix} \tag{3.1.91.7}$$

$$\xleftrightarrow{R_1=13R_1-R_2} \begin{pmatrix}13 & 0 & -10\\ 0 & -26 & 23\end{pmatrix} \tag{3.1.91.8}$$

$$\implies \mathbf{n} = \begin{pmatrix}\frac{10}{13}\\ \frac{23}{26}\\ 1\end{pmatrix} = \begin{pmatrix}20\\ 23\\ 26\end{pmatrix} \tag{3.1.91.9}$$

Since the plane passes through $\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$, using (3.1.39.3),

$$\begin{pmatrix}20 & 23 & 26\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\right) = 0 \tag{3.1.91.10}$$

$$\implies \begin{pmatrix}20 & 23 & 26\end{pmatrix}\mathbf{x} = 69 \tag{3.1.91.11}$$

Alternatively, the plane passing through the intersection of (3.1.91.1) and (3.1.91.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{3.1.91.12}$$

Substituting $\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = 6 - 5\lambda \tag{3.1.91.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{3.1.91.14}$$

$$\implies \lambda = \frac{3}{14} \tag{3.1.91.15}$$

Substituting this value of λ in (3.1.91.12) yields the equation of the plane.

92. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{3.1.92.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{3.1.92.2}$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.83), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{3.1.92.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix}-3\\ 1\\ 5\end{pmatrix} - \begin{pmatrix}-1\\ 2\\ 5\end{pmatrix} = \begin{pmatrix}-2\\ -1\\ 0\end{pmatrix} \tag{3.1.92.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix}-1 & 2 & 5\\ -3 & 1 & 5\end{pmatrix} \xleftrightarrow{R_2=\frac{R_2-3R_1}{5}} \begin{pmatrix}-1 & 2 & 5\\ 0 & 1 & 2\end{pmatrix} \tag{3.1.92.5}$$

$$\xleftrightarrow{R_1=-R_1+2R_2} \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}-1\\ 2\\ 5\end{pmatrix} \times \begin{pmatrix}-3\\ 1\\ 5\end{pmatrix} = \begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \tag{3.1.92.6}$$

The LHS of (3.1.92.3) is

$$\begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \tag{3.1.92.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left| \mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2 \right| = 0 \tag{3.1.92.8}$$

93. Find the angle between the two planes

$$\begin{pmatrix} 2 & 1 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.93.1}$$

$$\begin{pmatrix} 3 & -6 & -2 \end{pmatrix} \mathbf{x} = 7. \tag{3.1.93.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.49.6).

94. Find the angle between the two planes

$$\begin{pmatrix} 2 & 2 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.94.1}$$

$$\begin{pmatrix} 3 & -6 & 2 \end{pmatrix} \mathbf{x} = 7. \tag{3.1.94.2}$$

Solution: See Problem (3.1.93).

95. Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$\begin{pmatrix} 6 & -3 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{3.1.95.1}$$

Solution: Use (3.1.46.7).

96. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{3.1.96.1}$$

and the plane

$$P: \quad \begin{pmatrix} 10 & 2 & -11 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.96.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left| \begin{pmatrix} 10 & 2 & -11 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \right|}{\sqrt{225} \times \sqrt{49}} = \frac{8}{21} \tag{3.1.96.3}$$

Thus, the desired angle is $90° - \theta$.

97. Find the equation of the plane that contains the point $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \mathbf{x} = 5 \tag{3.1.97.1}$$

$$\begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \mathbf{x} = 8 \tag{3.1.97.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tag{3.1.97.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.97.4}$$

98. Find the distance between the point $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.
Solution: Find the equation of the plane using Problem 3.1.89. Find the distance using (3.1.46.7).

99. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{3.1.99.1}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \tag{3.1.99.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 13 \\ 23 \\ 0 \end{pmatrix} \tag{3.1.99.3}$$

3.2 Complex Numbers

1. Find $\begin{pmatrix} 5 \\ -3 \end{pmatrix}^3$
2. Find $\begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}$.
3. Find the multiplicative inverse of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

4. Find
   a) $\begin{pmatrix}5\\ \sqrt{2}\end{pmatrix}\begin{pmatrix}1\\ -2\sqrt{3}\end{pmatrix}$.
   b) $\begin{pmatrix}0\\1\end{pmatrix}^{-35}$.
   c) Show that the polar representation of $\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}$ is $2\angle 60°$.
5. Convert the complex number $-\frac{16}{\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}}$
6. Find the conjugate of $\frac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$.
7. Find the modulus and argument of the complex numbers
   a) $\frac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}}$.
   b) $\frac{1}{\begin{pmatrix}1\\1\end{pmatrix}}$.
8. Find θ such that

$$\frac{\begin{pmatrix}3\\2\sin\theta\end{pmatrix}}{\begin{pmatrix}1\\-2\sin\theta\end{pmatrix}} \tag{3.2.8.1}$$

   is purely real.
9. Convert the complex number

$$\mathbf{z} = \frac{\begin{pmatrix}-1\\1\end{pmatrix}}{\begin{pmatrix}\cos\frac{\pi}{3}\\ \sin\frac{\pi}{3}\end{pmatrix}} \tag{3.2.9.1}$$

   in the polar form.
10. Simplify

$$\mathbf{z} = \left(\frac{1}{\begin{pmatrix}1\\-4\end{pmatrix}} - \frac{2}{\begin{pmatrix}2\\1\end{pmatrix}}\right)\frac{\begin{pmatrix}3\\-4\end{pmatrix}}{\begin{pmatrix}5\\1\end{pmatrix}} \tag{3.2.10.1}$$

11. Convert the following in the polar form:
    a) $\frac{\begin{pmatrix}1\\7\end{pmatrix}}{\begin{pmatrix}2\\-1\end{pmatrix}^2}$.
    b) $\frac{\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\-2\end{pmatrix}}$.
12. If $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}1\\1\end{pmatrix}$, find $\left\|\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}\right\|$
13. Let $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$. Find
    a) $\mathrm{Re}\left(\frac{\mathbf{z}_1\mathbf{z}_2}{\mathbf{z}_1^*}\right)$.
    b) $\mathrm{Im}\left(\frac{1}{\mathbf{z}_1\mathbf{z}_1^*}\right)$.
14. Find the modulus and argument of the complex number $\frac{\begin{pmatrix}1\\2\end{pmatrix}}{\begin{pmatrix}1\\-3\end{pmatrix}}$.
15. Find the real numbers x, y such that $\begin{pmatrix}x\\-y\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix}$ is the conjugate of $\begin{pmatrix}-6\\-24\end{pmatrix}$.
16. Find the modulus of $\frac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}} - \frac{\begin{pmatrix}1\\-1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}}$.

3.3 Points and Vectors

1. Find the distance between the following pairs of points
   a)
$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{3.3.1.1}$$
   b)
$$\begin{pmatrix}-5\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \tag{3.3.1.2}$$
   c)
$$\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}-1\\b\end{pmatrix} \tag{3.3.1.3}$$
2. Find the distance between the points
$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix} \tag{3.3.2.1}$$
3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.
   a)
$$\begin{pmatrix}-1\\-2\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}-3\\0\end{pmatrix} \tag{3.3.4.1}$$
   b)
$$\begin{pmatrix}-3\\5\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix},\begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.3.4.2}$$
   c)
$$\begin{pmatrix}4\\5\end{pmatrix},\begin{pmatrix}7\\6\end{pmatrix}, \tag{3.3.4.3}$$
$$\begin{pmatrix}4\\3\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix} \tag{3.3.4.4}$$
5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$.
6. Find the point on the x-axis which is equidistant from
$$\begin{pmatrix}2\\-5\end{pmatrix},\begin{pmatrix}-2\\9\end{pmatrix}, \tag{3.3.6.1}$$
7. Find the values of y for which the distance between the points
$$\mathbf{P}=\begin{pmatrix}2\\-3\end{pmatrix},\mathbf{Q}=\begin{pmatrix}10\\y\end{pmatrix} \tag{3.3.7.1}$$
   is 10 units.
8. Show that each of the given three vectors is a unit vector
$$\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix},\frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix},\frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}. \tag{3.3.8.1}$$
   Also, show that they are mutually perpendicular to each other.
9. For
$$\mathbf{a}=\begin{pmatrix}2\\2\\3\end{pmatrix},\mathbf{b}=\begin{pmatrix}-1\\2\\1\end{pmatrix},\mathbf{c}=\begin{pmatrix}3\\1\\0\end{pmatrix}, \tag{3.3.9.1}$$
   $(\mathbf{a}+\lambda\mathbf{b})\perp\mathbf{c}$. Find λ.
10. Find $\mathbf{a}\times\mathbf{b}$ if
$$\mathbf{a}=\begin{pmatrix}1\\-7\\7\end{pmatrix},\mathbf{b}=\begin{pmatrix}3\\-2\\2\end{pmatrix}. \tag{3.3.10.1}$$
11. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where
$$\mathbf{a}=\begin{pmatrix}3\\2\\2\end{pmatrix},\mathbf{b}=\begin{pmatrix}1\\2\\-2\end{pmatrix}. \tag{3.3.11.1}$$
12. If $\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix},\mathbf{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix},\mathbf{c}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$.
13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix},\mathbf{b}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$,
14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.
15. Let $\mathbf{a}=\begin{pmatrix}1\\4\\2\end{pmatrix},\mathbf{b}=\begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d}\perp\mathbf{a},\mathbf{d}\perp\mathbf{b}$ and $\mathbf{d}^T\mathbf{c}=15$.
16. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.
17. The value of
$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+\begin{pmatrix}0\\1\\0\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$
$$+\begin{pmatrix}0\\0\\1\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \tag{3.3.17.1}$$
   is

   a) 0
   b) -1
   c) 1
   d) 3
18. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively.

19. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.

20. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\\6\end{pmatrix}$.

21. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.

22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.

23. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{3.3.23.1}$$

24. Find the projection of the vector

$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{3.3.24.1}$$

on the vector

$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.3.24.2}$$

25. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.

26. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

3.4 Points on a Line

1. Find the coordinates of the point which divides the join of

$$\begin{pmatrix}-1\\7\end{pmatrix}, = \begin{pmatrix}4\\-3\end{pmatrix} \tag{3.4.1.1}$$

in the ratio 2 : 3.

2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}4\\y\end{pmatrix}$, $\begin{pmatrix}x\\6\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.
8. Determine if the points

$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{3.4.8.1}$$

are collinear.

9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix}x\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
    a) $\begin{pmatrix}7\\-2\end{pmatrix}$, $\begin{pmatrix}5\\1\end{pmatrix}$, $\begin{pmatrix}3\\k\end{pmatrix}$
    b) $\begin{pmatrix}8\\1\end{pmatrix}$, $\begin{pmatrix}k\\-4\end{pmatrix}$, $\begin{pmatrix}2\\-5\end{pmatrix}$
12. Find a condition on $\mathbf{x}$ such that the points

$\mathbf{x}, \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}7\\0\end{pmatrix}$ are collinear.

13. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
15. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\8\\7\end{pmatrix}$ are collinear.

3.5 Lines and Planes

1. Sketch the following lines

   a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 9.35$
   b) $\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\mathbf{x} = 10$
   c) $\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 6$
   d) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 0$
   e) $\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0$
   f) $\begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} = -2$
   g) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$
   h) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 5$

2. Write four solutions for each of the following equations
   a) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$
   b) $\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$
   c) $\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$
3. Check which of the following are solutions of the equation

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 4 \tag{3.5.3.1}$$

   a) $\begin{pmatrix}0\\2\end{pmatrix}$
   b) $\begin{pmatrix}2\\0\end{pmatrix}$
   c) $\begin{pmatrix}4\\0\end{pmatrix}$
   d) $\begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix}$
   e) $\begin{pmatrix}1\\1\end{pmatrix}$

4. Find the value of k, if $\begin{pmatrix}2\\1\end{pmatrix}$ is a solution of the equation

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = k \tag{3.5.4.1}$$

5. Draw the graphs of the following equations
   a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$
   b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2$
   c) $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 0$
   d) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3$
   e) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
   f) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
   g) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$
   h) $\begin{pmatrix}7 & -3\end{pmatrix}\mathbf{x} = 2$
   i) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$
   j) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$
   k) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$
   l) $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6$
6. Give the equations of two lines passing through $\begin{pmatrix}2\\14\end{pmatrix}$. How many more such lines are there, and why?
7. If the point $\begin{pmatrix}3\\4\end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a
8. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident
   a)

$$\begin{aligned}\begin{pmatrix}5 & -4\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}7 & 6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.5.8.1}$$

   b)

$$\begin{aligned}\begin{pmatrix}9 & 3\end{pmatrix}\mathbf{x} &= -12\\ \begin{pmatrix}18 & 6\end{pmatrix}\mathbf{x} &= -24\end{aligned} \tag{3.5.8.2}$$

   c)

$$\begin{aligned}\begin{pmatrix}6 & -3\end{pmatrix}\mathbf{x} &= -10\\ \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} &= -9\end{aligned} \tag{3.5.8.3}$$

9. Find out whether the following pair of linear equations are consistent, or inconsistent.
   a)

$$\begin{aligned}\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 7\end{aligned} \tag{3.5.9.1}$$

b)

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 8 \\ \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} = 9 \tag{3.5.9.2}$$

c)

$$\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\end{pmatrix}\mathbf{x} = 7 \\ \begin{pmatrix}9 & -10\end{pmatrix}\mathbf{x} = 14 \tag{3.5.9.3}$$

d)

$$\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = 11 \\ \begin{pmatrix}-10 & 6\end{pmatrix}\mathbf{x} = -22 \tag{3.5.9.4}$$

e)

$$\begin{pmatrix}\frac{4}{3} & 2\end{pmatrix}\mathbf{x} = 8 \\ \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.5.9.5}$$

10. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} = 10 \tag{3.5.10.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 8 \\ \begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} = 16 \tag{3.5.10.2}$$

c)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 6 \\ \begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} = 4 \tag{3.5.10.3}$$

d)

$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \\ \begin{pmatrix}4 & -4\end{pmatrix}\mathbf{x} = 5 \tag{3.5.10.4}$$

11. Given the linear equation $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x}-8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

a) intersecting lines
b) parallel lines
c) coincident lines

12. Find the intersection of the following lines

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 14 \\ \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4 \tag{3.5.12.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3 \\ \begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = 6 \tag{3.5.12.2}$$

c)

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 3 \\ \begin{pmatrix}9 & -3\end{pmatrix}\mathbf{x} = 9 \tag{3.5.12.3}$$

d)

$$\begin{pmatrix}0.2 & 0.3\end{pmatrix}\mathbf{x} = 1.3 \\ \begin{pmatrix}0.4 & 0.5\end{pmatrix}\mathbf{x} = 2.3 \tag{3.5.12.4}$$

e)

$$\begin{pmatrix}\sqrt{2} & \sqrt{3}\end{pmatrix}\mathbf{x} = 0 \\ \begin{pmatrix}\sqrt{3} & \sqrt{8}\end{pmatrix}\mathbf{x} = 0 \tag{3.5.12.5}$$

f)

$$\begin{pmatrix}\frac{3}{2} & -\frac{5}{3}\end{pmatrix}\mathbf{x} = -2 \\ \begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \frac{13}{6} \tag{3.5.12.6}$$

13. Find m if

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 11 \\ \begin{pmatrix}2 & -4\end{pmatrix}\mathbf{x} = -24 \\ \begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = -3 \tag{3.5.13.1}$$

14. Solve the following

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \\ \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.5.14.1}$$

b)

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 10 \\ \begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \tag{3.5.14.2}$$

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 4 \\ \begin{pmatrix}9 & -2\end{pmatrix}\mathbf{x} = 7 \tag{3.5.14.3}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{2}{3}\end{pmatrix}\mathbf{x} = -1 \\ \begin{pmatrix}1 & -\frac{1}{3}\end{pmatrix}\mathbf{x} = 3 \tag{3.5.14.4}$$

15. Which of the following pairs of linear equations has a unique solution, no solution, or

infinitely many solutions?

a)
$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}3 & -9\end{pmatrix}\mathbf{x} = 2 \quad (3.5.15.1)$$

b)
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 8 \quad (3.5.15.2)$$

c)
$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 20$$
$$\begin{pmatrix}6 & -10\end{pmatrix}\mathbf{x} = 40 \quad (3.5.15.3)$$

d)
$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 7$$
$$\begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} = 15 \quad (3.5.15.4)$$

16. For which alues of a and b does the following pair of linear equations have an infinite number of solutions?
$$\begin{aligned}\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 7\\ \begin{pmatrix}a-b & a+b\end{pmatrix}\mathbf{x} &= 3a+b-2\end{aligned} \quad (3.5.16.1)$$
17. For which value of k will the following pair of linear equations have no solution?
$$\begin{aligned}\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} &= 1\\ \begin{pmatrix}2k-1 & k-1\end{pmatrix}\mathbf{x} &= 2k+1\end{aligned} \quad (3.5.17.1)$$
18. Solve the following pair of linear equations
$$\begin{aligned}\begin{pmatrix}8 & 5\end{pmatrix}\mathbf{x} &= 9\\ \begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} &= 4\end{aligned} \quad (3.5.18.1)$$
19. Solve the following pair of linear equations
$$\begin{aligned}\begin{pmatrix}158 & -378\end{pmatrix}\mathbf{x} &= -74\\ \begin{pmatrix}-378 & 152\end{pmatrix}\mathbf{x} &= -604\end{aligned} \quad (3.5.19.1)$$
20. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$.
21. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
22. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.
23. Write the equations for the x and y axes.
24. Find the equation of the line satisfying the following conditions
    a) passing through the point $\begin{pmatrix}-4\\3\end{pmatrix}$ with slope $\frac{1}{2}$.
    b) passing through the point $\begin{pmatrix}0\\0\end{pmatrix}$ with slope m.
    c) passing through the point $\begin{pmatrix}2\\2\sqrt{3}\end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
    d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.
    e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.
    f) passing through the points $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-4\end{pmatrix}$.
    g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.
25. Find the equation of the line passing through $\begin{pmatrix}-3\\5\end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\6\end{pmatrix}$.
26. Find the direction vectors and and y-intercepts of the following lines
    a) $\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 0$.
    b) $\begin{pmatrix}6 & 3\end{pmatrix}\mathbf{x} = 5$.
    c) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$.
27. Find the intercepts of the following lines on the axes.
    a) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 12$.
    b) $\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 6$.
    c) $\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 0$.
28. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
    a) $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = -8$.
    b) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$.
    c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4$.
29. Find the distance of the point $\begin{pmatrix}1\\-1\end{pmatrix}$ from the line $\begin{pmatrix}12 & -5\end{pmatrix}\mathbf{x} = -82$.
30. Find the points on the x-axis, whose distances from the line
$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \quad (3.5.30.1)$$
are 4 units.

31. Find the distance between the parallel lines

$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = 34 \tag{3.5.31.1}$$
$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = -31 \tag{3.5.31.2}$$

32. Find the equation of the line parallel to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = -2 \tag{3.5.32.1}$$

and passing through the point $\begin{pmatrix}-2\\3\end{pmatrix}$.

33. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.5.33.1}$$

and having x intercept 3.

34. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.5.34.1}$$
$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = 1 \tag{3.5.34.2}$$

35. The line through the points $\begin{pmatrix}h\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19 \tag{3.5.35.1}$$

at right angle. Find the value of h.

36. Two lines passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ intersect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.

37. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$.

38. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.5.38.1}$$

39. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.5.39.1}$$

meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.

40. Find θ and p if

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{3.5.40.1}$$

is equivalent to

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{3.5.40.2}$$

41. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

42. Find the equation of the line parallel to the y-axis whose distance from the line

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.5.42.1}$$

4 units.

43. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.5.43.1}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.43.2}$$

44. Find the alue of p so that the three lines

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{3.5.44.1}$$
$$\begin{pmatrix}p & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.5.44.2}$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{3.5.44.3}$$

may intersect at one point.

45. Find the equation of the lines through the point $\begin{pmatrix}3\\2\end{pmatrix}$ which make an angle of 45° with the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3. \tag{3.5.45.1}$$

46. Find the equation of the line passing through the point of intersection of the lines

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = 3 \tag{3.5.46.1}$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -1 \tag{3.5.46.2}$$

that has equal intercepts on the axes.

47. In what ratio is the line joining $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}5\\7\end{pmatrix}$ divided by the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.5.47.1}$$

48. Find the distance of the line

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5 \tag{3.5.48.1}$$

from the point $\begin{pmatrix}1\\2\end{pmatrix}$ along the line

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0. \tag{3.5.48.2}$$

49. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.5.49.1}$$
may be at a distance of 3 units from this point.
50. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Find an equation of the legs of the triangle.
51. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7 \tag{3.5.51.1}$$
assuming the line to be a plane mirror.
52. If the lines
$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.5.52.1}$$
$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.5.52.2}$$
are equally inclined to the line
$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 4, \tag{3.5.52.3}$$
find the value of m.
53. The sum of the perpendicular distances of a variable point $\mathbf{P}$ from the lines
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.53.1}$$
$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.5.53.2}$$
is always 10. Show that $\mathbf{P}$ must move on a line.
54. Find the equation of the line which is equidistant from parallel lines
$$\begin{pmatrix}9 & 7\end{pmatrix}\mathbf{x} = 7 \tag{3.5.54.1}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = -6. \tag{3.5.54.2}$$
55. A ray of light passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\begin{pmatrix}5\\3\end{pmatrix}$. Find the coordinates of $\mathbf{A}$.
56. A person standing at the junction of two straight paths represented by the equations
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.5.56.1}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.5.56.2}$$
wants to reach the path whose equation is
$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{3.5.56.3}$$
in the least time. Find the equation of the path that he should follow.
57. Determine the ratio in which the line
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 4 = 0 \tag{3.5.57.1}$$
divides the line segment joining the points $\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\7\end{pmatrix}$.
58. A line perpendicular to the line segment joining the points $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.
59. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.
60. Find the equation of the line passing through the point $\begin{pmatrix}2\\2\end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.
61. Find the equation of the line through the point $\begin{pmatrix}0\\2\end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.
62. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix}-2\\9\end{pmatrix}$, find the equation of the line.
63. The length L (in cm) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L in terms of C.
64. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
65. Find the equation of a line which passes through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the

vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.

66. Find the equaion off the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.
67. Find the equation of the line which passes through the point $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and parallel to the line given by
$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{3.5.67.1}$$
68. Find the equation of the line given by
$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \tag{3.5.68.1}$$
69. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.
70. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.
71. Find the angle between the following pair of lines:

a)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.5.71.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.5.71.2}$$

b)
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{3.5.71.3}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{3.5.71.4}$$

72. Find the angle between the following pair of lines

a)
$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{3.5.72.1}$$
$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{3.5.72.2}$$

b)
$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{3.5.72.3}$$
$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{3.5.72.4}$$

73. Find the values of p so that the lines
$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{3.5.73.1}$$
$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{3.5.73.2}$$
are at right angles.
74. Show that the lines
$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{3.5.74.1}$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{3.5.74.2}$$
are perpendicular to each other.
75. Find the shortest distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.5.75.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{3.5.75.2}$$
76. Find the shortest distance between the lines
$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \tag{3.5.76.1}$$
$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{3.5.76.2}$$
77. Find the shortest distance between the lines
$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{3.5.77.1}$$
$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.5.77.2}$$

78. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \tag{3.5.78.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \tag{3.5.78.2}$$

79. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\mathbf{x} = 2$ c) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix}\mathbf{x} = -8$
b) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 1$ d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\mathbf{x} = 5$

80. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

81. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}\mathbf{x} = 12$ c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 1$
b) $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix}\mathbf{x} = 0$ d) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix}\mathbf{x} = -8$

82. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

83. Find the equation of the planes that passes through three points

a) $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$.

84. Find the intercepts cut off by the plane $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}\mathbf{x} = 5$.

85. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

86. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 3 & -1 & 2 \end{pmatrix}\mathbf{x} = 4$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = -2$ and the pont $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

87. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\mathbf{x} = 9$ and the pont $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

88. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\mathbf{x} = 0$.

89. Find the angle between the planes whose equations are $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix} 3 & -3 & 5 \end{pmatrix}\mathbf{x} = 3$

90. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix} 7 & 5 & 6 \end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix} 3 & -1 & -10 \end{pmatrix}\mathbf{x} = -4$
b) $\begin{pmatrix} 2 & 1 & 3 \end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix} 1 & -2 & 5 \end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix} 2 & -2 & 4 \end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix} 3 & -3 & 6 \end{pmatrix}\mathbf{x} = 1$
d) $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix}\mathbf{x} = -3$
e) $\begin{pmatrix} 4 & 8 & 1 \end{pmatrix}\mathbf{x} = 8$ and $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\mathbf{x} = 4$

91. In the following cases, find the distance of each of the given points from the corresponding plane.

92. Show that the line joining the origin to the point $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$.

93. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, then find the angle between the lines AB and CD.

94. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{3.5.94.1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{3.5.94.2}$$

Item	Point	Plane
a)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}3 & -4 & 12\end{pmatrix}\boldsymbol{x} = 3$
b)	$\begin{pmatrix}3\\-2\\1\end{pmatrix}$	$\begin{pmatrix}2 & -1 & 2\end{pmatrix}\boldsymbol{x} = -3$
c)	$\begin{pmatrix}2\\3\\-5\end{pmatrix}$	$\begin{pmatrix}1 & 2 & -2\end{pmatrix}\boldsymbol{x} = 9$
d)	$\begin{pmatrix}-6\\0\\0\end{pmatrix}$	$\begin{pmatrix}2 & -3 & 6\end{pmatrix}\boldsymbol{x} = 2$

TABLE 3.5.91

find the value of k.

95. Find the equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\mathbf{x} = -9 \tag{3.5.95.1}$$

96. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}6\\2\\2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-2\\2\end{pmatrix} \text{ and} \tag{3.5.96.1}$$

$$\mathbf{x} = \begin{pmatrix}-4\\0\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-2\\-2\end{pmatrix} \tag{3.5.96.2}$$

97. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the YZ-plane.

98. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the ZX-plane.

99. Find the coordinates of the point where the line through $\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}\\-3\\1\end{pmatrix}$ crosses the plane

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.5.99.1}$$

100. Find the equation of the plane passing through the point $\begin{pmatrix}-1\\3\\2\end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 5 \tag{3.5.100.1}$$

$$\begin{pmatrix}3 & 3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.100.2}$$

101. If the points $\begin{pmatrix}1\\1\\p\end{pmatrix}$ and $\begin{pmatrix}-3\\0\\1\end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix}3 & 4 & -12\end{pmatrix}\mathbf{x} = -13, \tag{3.5.101.1}$$

then find the value of p.

102. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1 \text{ and} \tag{3.5.102.1}$$

$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.5.102.2}$$

and parallel to the x-axis.

103. If **O** be the origin and the coordinates of **P** be $\begin{pmatrix}1\\2\\3\end{pmatrix}$, then find the equation of the plane passing through **P** and perpendicular to OP.

104. Find the equation of the plane which contains the line of intersection of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 4 \tag{3.5.104.1}$$

$$\begin{pmatrix}2 & 1 & -1\end{pmatrix}\mathbf{x} = -5 \tag{3.5.104.2}$$

and which is perpendicular to the plane

$$\begin{pmatrix}5 & 3 & -6\end{pmatrix}\mathbf{x} = -8 \tag{3.5.104.3}$$

105. Find the distance of the point $\begin{pmatrix}-1\\-5\\-10\end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\2\end{pmatrix} \tag{3.5.105.1}$$

and the plane

$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{3.5.105.2}$$

106. Find the vector equation of the line passing

through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.5.106.1}$$

$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{3.5.106.2}$$

107. Find the vector equation of the line passing through the point $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \tag{3.5.107.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{3.5.107.2}$$

108. Distance between the two planes

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.5.108.1}$$

$$\begin{pmatrix}4 & 6 & 8\end{pmatrix}\mathbf{x} = 12 \tag{3.5.108.2}$$

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

109. The planes

$$\begin{pmatrix}2 & -1 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.5.109.1}$$

$$\begin{pmatrix}5 & -\frac{5}{2} & 10\end{pmatrix}\mathbf{x} = 6 \tag{3.5.109.2}$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix}0\\0\\\frac{5}{4}\end{pmatrix}$

3.6 Examples: Applications

1. The cost of a notebook is twice the cost of a pen. Write a linear equation in two variables to represent this statement.
2. The taxi fare in a city is as follows: For the first kilometre, the fare is ₹8 and for the subsequent distance it is ₹5 per km. Taking the distance covered as x km and total fare as ₹y, write a linear equation for this information, and draw its graph.
3. Yamini and Fatima, two students of Class IX of a school, together contributed ₹100 towards the Prime Minister's Relief Fund to help the earthquake victims. Write a linear equation which satisfies this data. (You may take their contributions as ₹x and ₹y.) Draw the graph of the same.
4. In countries like USA and Canada, temperature is measured in Fahrenheit, whereas in countries like India, it is measured in Celsius. Here is a linear equation that converts Fahrenheit to Celsius:

$$F = \frac{9}{5}C + 32 \tag{3.6.4.1}$$

   a) Draw the graph of the linear equation above using Celsius for x-axis and Fahrenheit for y-axis.
   b) If the temperature is 30°C, what is the temperature in Fahrenheit?
   c) If the temperature is 95°F, what is the temperature in Celsius?
   d) If the temperature is 0°C, what is the temperature in Fahrenheit and if the temperature is 0°F, what is the temperature in Celsius?
   e) Is there a temperature which is numerically the same in both Fahrenheit and Celsius? If yes, find it.
5. Romila went to a stationery shop and purchased 2 pencils and 3 erasers for ₹9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹18. Represent this situation algebraically and graphically. Find the cost of each pencil and eraser.
6. Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically. Find their respective ages.
7. The coach of a cricket team buys 3 bats and 6 balls for ₹3900. Later, she buys another bat and 3 more balls of the same kind for ₹1300. Represent this situation algebraically and geometrically.Find the cost of each bat and ball.
8. The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹300. Represent the situation algebraically and geometrically. Find the cost of apples and grape.
9. Form the pair of linear equations in the follow-

ing problems, and find their solutions.

10. 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
11. 5 pencils and 7 pens together cost ₹50, whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.
12. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.
13. Form the pair of linear equations for the following problems and find their solution
14. The difference between two numbers is 26 and one number is three times the other. Find them.
15. The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.
16. The coach of a cricket team buys 7 bats and 6 balls for ₹3800. Later, she buys 3 bats and 5 balls for ₹1750. Find the cost of each bat and each ball.
17. The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹105 and for a journey of 15 km, the charge paid is ₹155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?
18. A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. . Find the fraction.
19. Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages
20. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly incomes.
21. The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?
22. If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$. if we only add 1 to the denominator. What is the fraction?
23. Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?
24. The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.
25. Meena went to a bank to withdraw ₹2000. She asked the cashier to give her ₹50 and ₹100 notes only. Meena got 25 notes in all. Find how many notes of ₹50 and ₹100 she received.
26. A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹27 for a book kept for seven days, while Susy paid ₹21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.
27. The cost of 5 oranges and 3 apples is ₹35 and the cost of 2 oranges and 4 apples is ₹28. Let us find the cost of an orange and an apple.
28. From a bus stand in Bangalore , if we buy 2 tickets to Malleswaram and 3 tickets to Yeshwanthpur, the total cost is ₹46; but if we buy 3 tickets to Malleswaram and 5 tickets to Yeshwanthpur the total cost is ₹74. Find the fares from the bus stand to Malleswaram, and to Yeshwanthpur.
29. A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹1000 as hostel charges whereas a student B, who takes food for 26 days, pays ₹1180 as hostel charges. Find the fixed charges and the cost of food per day.
30. A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes when 8 is added to its denominator. Find the fraction.
31. Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?
32. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two

cars?

33. The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.
34. Solve the pair of equations:

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} &= 13 \\ \begin{pmatrix} 5 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} &= -2 \end{aligned} \tag{3.6.34.1}$$

35. Solve the pair of equations by reducing them to a pair of linear equations

$$\begin{aligned} \begin{pmatrix} 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{x-1} \\ \frac{1}{y-2} \end{pmatrix} &= 2 \\ \begin{pmatrix} 6 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{x-1} \\ \frac{1}{y-2} \end{pmatrix} &= 1 \end{aligned} \tag{3.6.35.1}$$

36. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.
37. Solve the following pairs of equations

a)

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} &= 2 \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} &= \frac{13}{6} \end{aligned} \tag{3.6.37.1}$$

b)

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{x}} \\ \frac{1}{\sqrt{y}} \end{pmatrix} &= 2 \\ \begin{pmatrix} 4 & -9 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{x}} \\ \frac{1}{\sqrt{y}} \end{pmatrix} &= -1 \end{aligned} \tag{3.6.37.2}$$

c)

$$\begin{aligned} \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ y \end{pmatrix} &= 14 \\ \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{x} \\ y \end{pmatrix} &= 23 \end{aligned} \tag{3.6.37.3}$$

d)

$$\begin{aligned} \begin{pmatrix} 10 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{x+y} \\ \frac{1}{x-y} \end{pmatrix} &= 4 \\ \begin{pmatrix} 15 & -5 \end{pmatrix} \begin{pmatrix} \frac{1}{x+y} \\ \frac{1}{x-y} \end{pmatrix} &= -2 \end{aligned} \tag{3.6.37.4}$$

e)

$$\begin{aligned} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3x+y} \\ \frac{1}{3x-y} \end{pmatrix} &= \frac{3}{4} \\ \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3x+y} \\ \frac{1}{3x-y} \end{pmatrix} &= -\frac{1}{8} \end{aligned} \tag{3.6.37.5}$$

38. Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.
39. 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.
40. Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.
41. The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju is twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju.
42. One says, "Give me a hundred, friend! I shall then become twice as rich as you". The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their (respective) capital? [From the Bijaganita of Bhaskara II].
43. A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.
44. The students of a class are made to stand in rows. If 3 students are extra in a row, there

would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.

3.7 Miscellaneous

1. Solve the following pair of linear equations

a)

$$\begin{pmatrix} p & q \end{pmatrix}\mathbf{x} = p - q$$
$$\begin{pmatrix} q & -p \end{pmatrix}\mathbf{x} = p + q \tag{3.7.1.1}$$

b)

$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = c$$
$$\begin{pmatrix} b & a \end{pmatrix}\mathbf{x} = 1 + c \tag{3.7.1.2}$$

c)

$$\begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \end{pmatrix}\mathbf{x} = 0$$
$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = a^2 + b^2 \tag{3.7.1.3}$$

2. Solve the following pair of equations

$$\begin{aligned}\begin{pmatrix} a-b & a+b \end{pmatrix}\mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix} a+b & a+b \end{pmatrix}\mathbf{x} &= a^2 + b^2\end{aligned} \tag{3.7.2.1}$$

3. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.7.3.1}$$

4. (Cauchy-Schwarz Inequality:) Show that

$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\| \, \|\mathbf{b}\| \tag{3.7.4.1}$$

5. (Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{3.7.5.1}$$

6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

7. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.

8. If three points $\begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

9. $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix}\mathbf{x} = 2 \tag{3.7.9.1}$$

10. Point $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

11. Show that two lines

$$\begin{pmatrix} a_1 & b_1 \end{pmatrix}\mathbf{x} + c_1 = 0 \tag{3.7.11.1}$$
$$\begin{pmatrix} a_2 & b_2 \end{pmatrix}\mathbf{x} + c_2 = 0 \tag{3.7.11.2}$$

   are
   a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
   b) perpendicular if $a_1a_2 - b_1b_2 = 0$.

12. Find the distance between the parallel lines

$$l\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = -p \tag{3.7.12.1}$$
$$l\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = r \tag{3.7.12.2}$$

13. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line

$$\begin{pmatrix} A & B \end{pmatrix}\mathbf{x} = -C \tag{3.7.13.1}$$

14. If p and q are the lengths of perpendiculars from the origin to the lines

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\mathbf{x} = k\cos 2\theta \tag{3.7.14.1}$$
$$\begin{pmatrix} \sec\theta & \operatorname{cosec}\theta \end{pmatrix}\mathbf{x} = k \tag{3.7.14.2}$$

   respectively, prove that $p^2 + 4q^2 = k^2$.

15. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{3.7.15.1}$$

16. Show that the area of the triangle formed by the lines

$$\begin{pmatrix} -m_1 & 1 \end{pmatrix}\mathbf{x} = c_1 \tag{3.7.16.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix}\mathbf{x} = c_2 \tag{3.7.16.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.7.16.3}$$

   is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

17. Find the values of k for which the line
$$\begin{pmatrix} k-3 & -\left(4-k^2\right) \end{pmatrix}\mathbf{x} + k^2 - 7k + 6 = 0 \tag{3.7.17.1}$$
is
    a) parallel to the x-axis
    b) parallel to the y-axis
    c) passing through the origin.
18. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$.
19. Find the area of the triangle formed by the lines
$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 0 \tag{3.7.19.1}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{3.7.19.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = k \tag{3.7.19.3}$$
20. If three lines whose equations are
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix}\mathbf{x} = c_1 \tag{3.7.20.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix}\mathbf{x} = c_2 \tag{3.7.20.2}$$
$$\begin{pmatrix} -m_3 & 1 \end{pmatrix}\mathbf{x} = c_3 \tag{3.7.20.3}$$
are concurrent, show that
$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \tag{3.7.20.4}$$
21. Find the equation of the line passing through the origin and making an angle θ with the line
$$\begin{pmatrix} -m & 1 \end{pmatrix}\mathbf{x} = c \tag{3.7.21.1}$$
22. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{a^2-b^2} \\ 0 \end{pmatrix}$ to the line
$$\begin{pmatrix} \frac{\cos\theta}{a} & \frac{\sin\theta}{b} \end{pmatrix}\mathbf{x} = 1 \tag{3.7.22.1}$$
is b^2.
23. If $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1n_2 - m_2n_1 \\ n_1l_2 - n_2l_1 \\ l_1m_2 - l_2m_1 \end{pmatrix}$.
24. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{3.7.24.1}$$
25. Show that the lines
$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \tag{3.7.25.1}$$
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{3.7.25.2}$$
are coplanar.
26. Find $\mathbf{R}$ which divides the line joining the points
$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{3.7.26.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{3.7.26.2}$$
externally in the ratio 1 : 2.
27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if
$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}-\mathbf{b}) = 8 \tag{3.7.27.1}$$
$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \tag{3.7.27.2}$$
28. Evaluate the product
$$(3\mathbf{a} - 5\mathbf{b})^T(2\mathbf{a} + 7\mathbf{b}) \tag{3.7.28.1}$$
29. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if
$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{3.7.29.1}$$
$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{3.7.29.2}$$
and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.
30. Show that
$$(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a}) \perp (\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a}) \tag{3.7.30.1}$$
31. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?
32. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that
$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \tag{3.7.32.1}$$
find the value of
$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.7.32.2}$$
33. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if
    a) $\lambda = 1$
    b) $\lambda = -1$
    c) $\|\mathbf{a}\| = |\lambda|$
    d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$
34. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-

axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

35. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\,(\mathbf{a} \times \mathbf{b}) \qquad (3.7.35.1)$$

36. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?
37. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \qquad (3.7.37.1)$$

38. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a) $\frac{\pi}{6}$ c) $\frac{\pi}{3}$
b) $\frac{\pi}{4}$ d) $\frac{\pi}{2}$

39. Prove that

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \qquad (3.7.39.1)$$
$$\iff \mathbf{a} \perp \mathbf{b}. \qquad (3.7.39.2)$$

40. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

a) $0 < \theta < \frac{\pi}{2}$ c) $0 < \theta < \pi$
b) $0 \leq \theta \leq \frac{\pi}{2}$ d) $0 \leq \theta \leq \pi$

41. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$ c) $\theta = \frac{\pi}{2}$
b) $\theta = \frac{\pi}{3}$ d) $\theta = \frac{2\pi}{3}$

42. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a} \times \mathbf{b}\|$ when θ is equal to

a) 0 c) $\frac{\pi}{2}$
b) $\frac{\pi}{4}$ d) π.

43. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.
44. Find the equation of a line parallel to the x-axis and passing through the origin.
45. Find the equation of a plane passing through $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x}x = 2 \qquad (3.7.45.1)$$

46. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \qquad (3.7.46.1)$$

4 Circle

4.1 Construction Examples

1. ABC is a triangle. Locate a point in the interior of $\triangle ABC$ which is equidistant from all the vertices of $\triangle ABC$.
Solution: Let $\mathbf{O}$ be the desired point. Then,

$$\|\mathbf{A} - \mathbf{O}\| = \|\mathbf{B} - \mathbf{O}\| = \|\mathbf{C} - \mathbf{O}\| = R \quad (4.1.1.1)$$

From (4.1.1.1),

$$\|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \qquad (4.1.1.2)$$

$$\implies (\mathbf{A} - \mathbf{O})^T (\mathbf{A} - \mathbf{O}) - (\mathbf{B} - \mathbf{O})^T (\mathbf{B} - \mathbf{O}) = 0 \quad (4.1.1.3)$$

which can be simplified as

$$(\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \qquad (4.1.1.4)$$

Similarly,

$$(\mathbf{B} - \mathbf{C})^T \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \qquad (4.1.1.5)$$

From and , $\mathbf{O}$ can be computed. A circle with centre $\mathbf{O}$ can be drawn through $\mathbf{A}, \mathbf{B}, \mathbf{C}$. This circle is known as the *circumcircle*. The following code plots Fig. 4.1.1

```
codes/circle/circle_const_ccircle.py
```

2. In a triangle locate a point in its interior which is equidistant from all the sides of the triangle.
3. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.
Solution: The tangent is perpendicular to the radius. From the given information, in $\triangle ABC, AC \perp AB, a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \qquad (4.1.3.1)$$

The following code plots Fig. 4.1.3

```
codes/circle/draw_circle_eg.py
```

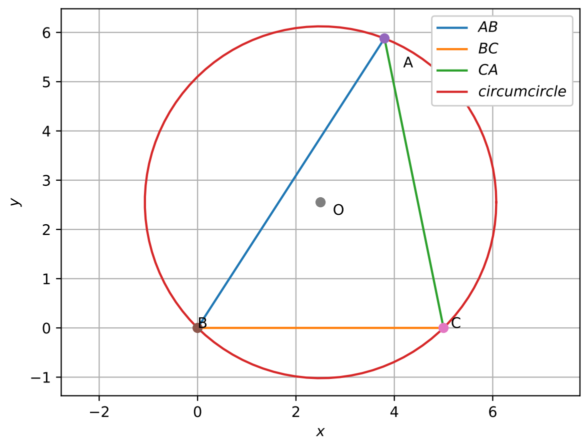

Fig. 4.1.1

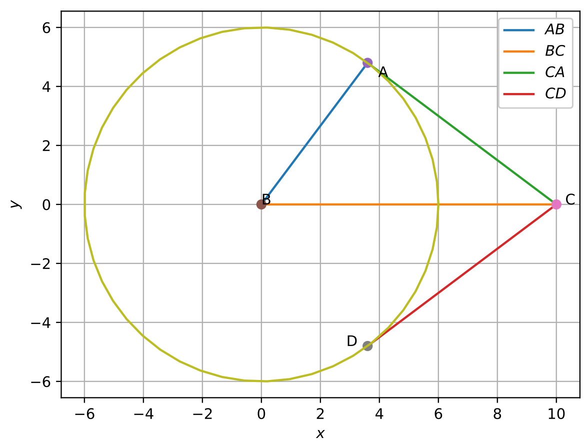

Fig. 4.1.3

4. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.
Solution: For any angle θ, a point on the circle with radius 3 has coordinates

$$3\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \tag{4.1.4.1}$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
Solution: Take the centre of both circles to be at the origin.
6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
Solution: Take the diameter to be on the x-axis.
7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.
Solution: The tangent is perpendicular to the radius.
8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8 \\ 0\end{pmatrix}. \tag{4.2.4.1}$$

9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^{\circ}$. BD is the perpendicular from **B** on AC (altitude). The circle through $\mathbf{B}, \mathbf{C}, \mathbf{D}$ (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry Examples

1. Find the equation of a circle with centre $\begin{pmatrix}-3 \\ 2\end{pmatrix}$ and radius 4.
2. Find the centre and radius of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}8 \\ 10\end{pmatrix}\mathbf{x} - 8 = 0 \tag{4.3.2.1}$$

3. Find the equation of the circle which passes through the points $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and whose centre lies on the line $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$.
4. Find the area enclosed by the circle $\|\mathbf{x}\| = a$
5. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
6. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\| = 2$.

4.4 Circle Geometry Exercises

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\begin{pmatrix} 2, -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.
2. Find the centre Of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
3. Sketch the circles with
   a) centre $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2
   b) centre $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4
   c) centre $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$.
   d) centre $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and radius $\sqrt{2}$.
   e) centre $\begin{pmatrix} -a \\ -b \end{pmatrix}$ and radius $\sqrt{a^2 - b^2}$.
4.
5. Sketch the circles with equation
   a) $\left\|\mathbf{x} - \begin{pmatrix} 5 \\ -3 \end{pmatrix}\right\|^2 = 36$
   b) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 \\ 8 \end{pmatrix} \mathbf{x} - 45 = 0$
   c) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix} \mathbf{x} - 12 = 0$
   d) $2\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{x} = 0$
6. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16$.
7. Find the equation of the circle passing through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 11$.
8. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
9. Find the equation of the circle passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and making intercepts a and b on the coordinate axes.
10. Find the equation of a circle with centre $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and passes through the point $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$.
11. Does the point $\begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T\mathbf{x} = 25$?
12. Find the locus of all the unit vectors in the xy-plane.
13. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$
14. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
15. A,B and C are three points on a circle with centre O such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.
16. $\angle PQR = 100°$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$.
17. A, B, C, D are points on a circle such that $\angle ABC = 69°$, $\angle ACB = 31°$, find $\angle BDC$.
18. A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130°$ and $\angle ECD = 20°$. Find $\angle BAC$.
19. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70°$, $\angle BAC$ is 30°, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.
20. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.
21. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?
22. A tangent PQ at a point P of a circle of radius

5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Find the length of PQ.

23. PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.
24. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. Find the radius of the circle is
25. If TP and TQ are the two tangents to a circle with centre O so that $\angle POQ = 110°$, then find $\angle PTQ$
26. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80 ° , then find $\angle POA$
27. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.
28. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.
29. A $\triangle ABC$ is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively. Find the sides AB and AC.
30. The cost of fencing a circular field at the rate of ₹24 per metre is ₹5280. The field is to be ploughed at the rate of ₹0.50 per m^2. Find the cost of ploughing the field.
31. The radii of two circles are 19 cm and 9 cm respectively. Find the radius of the circle which has circumference equal to the sum of the circumferences of the two circles.
32. The radii of two circles are 8 cm and 6 cm respectively. Find the radius of the circle having area equal to the sum of the areas of the two circles.
33. A circular archery target is marked with its five scoring regions from the centre outwards as Gold, Red, Blue, Black and White. The diameter of the region representing Gold score is 21 cm and each of the other bands is 10.5 cm wide. Find the area of each of the five scoring regions.
34. The wheels of a car are of diameter 80 cm each. How many complete revolutions does each wheel make in 10 minutes when the car is travelling at a speed of 66 km per hour?
35. Find the area of the sector of a circle with radius 4 cm and of angle 30 ° . Also, find the area of the corresponding major sector.
36. Find the area of the segment AYB, if radius of the circle is 21 cm and $\angle AOB = 120°$.
37. Find the area of a sector of a circle with radius 6 cm if angle of the sector is 60 ° .
38. Find the area of a quadrant of a circle whose circumference is 22 cm. 3. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes.
39. A chord of a circle of radius 10 cm subtends a right angle at the centre. Find the area of the corresponding :
    a) minor segment
    b) major sector.
40. In a circle of radius 21 cm, an arc subtends an angle of 60 ° at the centre. Find:
    a) the length of the arc
    b) area of the sector formed by the arc
    c) area of the segment formed by the corresponding chord
41. A chord of a circle of radius 15 cm subtends an angle of 60 ° at the centre. Find the areas of the corresponding minor and major segments of the circle.
42. A chord of a circle of radius 12 cm subtends an angle of 120 ° at the centre. Find the area of the corresponding segment of the circle.
43. A horse is tied to a peg at one corner of a square shaped grass field of side 15 m by means of a 5 m long rope. Find
    a) the area of that part of the field in which the horse can graze.
    b) the increase in the grazing area if the rope were 10 m long instead of 5 m.
44. A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire is also used in making 5 diameters which divide the circle into 10 equal sectors. Find :
    a) the total length of the silver wire required.
    b) the area of each sector of the brooch
45. An umbrella has 8 ribs which are equally spaced. Assuming umbrella to be a flat circle of radius 45 cm, find the area between the two consecutive ribs of the umbrella.
46. A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115 ° . Find the total

area cleaned at each sweep of the blades.

47. To warn ships for underwater rocks, a lighthouse spreads a red coloured light over a sector of angle 80 ° to a distance of 16.5 km. Find the area of the sea over which the ships are warned.
48. A round table cover has six equal designs. If the radius of the cover is 28 cm, find the cost of making the designs at the rate of ₹0.35 per cm^2 .
49. Two circular flower beds are located on opposite sides of a square lawn $ABCD$ of side 56 m. If the centre Of each circular flower bed is the point of intersection O of the diagonals of the square lawn, find the sum of the areas of the lawn and the flower beds.
50. Four circles are inscribed inside a square $ABCD$ of side 14 cm such that each one touches exernally two adjacent sides of the square and two other circles. Find the region between the circles and the square.
51. $ABCD$ is a square of side 10 cm and semicircles are drawn with each side of the square as diameter. Find the area enclosed by the circular arcs.
52. P is a point on the semi-circle formed with diameter QR. Find the area between the semicircle and $\triangle PQR$ if PQ = 24 cm, PR = 7 cm and O is the centre Of the circle.
53. AC and BD are two arcs on concentric circles with radii 14 cm and 7 cm respectively, such that $\angle AOC = 40°$. Find the area of the region $ABDC$.
54. Find the area between a square $ABCD$ of side 14cm and the semi circles APD and BPC.
55. Find the area of the region enclosed by a circular arc of radius 6 cm drawn with vertex O of an equilateral triangle OAB of side 12 cm as centre.
56. From each corner of a square of side 4 cm a quadrant of a circle of radius 1 cm is cut and also a circle of diameter 2 cm is cut. Find the area of the remaining portion of the square.
57. In a circular table cover of radius 32 cm, a design is formed leaving an equilateral $\triangle ABC$ in the middle. Find the area of the design.
58. $ABCD$ is a square of side 14 cm. With centres A, B, C and D, four circles are drawn such that each circle touches externally two of the remaining three circles. Find the area within the square that lies outside the circles.
59. The left and right ends of a racing track are semicircular. The distance between the two inner parallel line segments is 60 m and they are each 106 m long. If the track is 10 m wide, find :
    a) the distance around the track along its inner edge
    b) the area of the track.
60. AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OD is the diameter of a smaller circle inside. If OA = 7 cm, find the area of the smaller circle.
61. The area of an equilateral $\triangle ABC$ is 17320.5 cm^2 . With each vertex of the triangle as centre, a circle is drawn with radius equal to half the length of the side of the triangle. Find the area of region within the triangle but outside the circles.
62. On a square handkerchief, nine circular designs are inscribed touching each other, each of radius 7 cm. Find the area of the remaining portion of the handkerchief.
63. $OACB$ is a quadrant of a circle with centre O and radius 3.5 cm. D is a point on OA. If OD = 2 cm, find the area of the
    a) quadrant $OACB$,
    b) the region between the quadrant and $\triangle OBD$.
64. A square $OABC$ is inscribed in a quadrant $OPBQ$. If OA = 20 cm, find the area between the square and the quadrant.
65. AB and CD are respectively arcs of two concentric circles of radii 21 cm and 7 cm and centre O. If $\angle AOB = 30°$, find the area of the region $ABCD$.
66. ABC is a quadrant of a circle of radius 14 cm and a semicircle is drawn with BC as diameter. Find the area of the crescent formed.
67. Find the area common between the two quadrants of circles of radius 8 cm each if the centres of the circles lie on opposite sides of a square.
68. Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector.
69. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
70. Find the area of the region in the first quadrant

enclosed by x-axis, line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^{\mathbf{x}} = 4$.

71. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^{\mathbf{x}} = 4$ and the lines $x = 0$ and $x = 2$.
72. Find the area of the circle $4\mathbf{x}^{\mathbf{x}} = 9$.
73. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
74. Find the smaller area enclosed by the circle $\mathbf{x}^{\mathbf{x}} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

4.5 Circle Applications

1. Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?
2. A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

5 Conics

5.1 Examples

1. Find the value of the following polynomial at the indicated value of variables
$$p(x) = 5x^2 - 3x + 7 \text{ at } x = 1. \tag{5.1.1.1}$$
2. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.
3. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(y) = y^2$.
   b) $p(x) = (x-1)(x+1)$.
4. Find the roots of the equation $2x^2 - 5x + 3 = 0$.
5. Find the roots of the quadratic equation $6x^2 - x - 2 = 0$.
6. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
7. Factorise $6x^2 + 17x + 5$.
8. Factorise $y^2 - 5y + 6$.
9. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$ and verify the relationship between the zeroes and the coefficients.
10. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.
11. Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.
12. Find the roots of the equation $5x^2 - 6x - 2 = 0$.
13. Find the roots of $4x^2 + 3x + 5 = 0$.
14. Find the roots of the following quadratic equations, if they exist.
   a) $3x^2 - 5x + 2 = 0$
   b) $x^2 + 4x + 5 = 0$
   c) $2x^2 - 2\sqrt{2}x + 1 = 0$
15. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.
16. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.
17. Solve $x^2 + 2 = 0$.
18. Solve $x^2 + x + 1 = 0$.
19. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.
20. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.
21. Find the equation of the parabola with focus $\begin{pmatrix}2\\0\end{pmatrix}$ and directrix $\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -2$.
22. Find the equation of the parabola with vertex at $\begin{pmatrix}0\\0\end{pmatrix}$ and focus at $\begin{pmatrix}0\\2\end{pmatrix}$.
23. Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $\begin{pmatrix}2\\-3\end{pmatrix}$.
24. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse
$$\mathbf{x}^T \begin{pmatrix}\frac{1}{25} & 0\\ 0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1 \tag{5.1.24.1}$$
25. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse
$$\mathbf{x}^T \begin{pmatrix}9 & 0\\ 0 & 4\end{pmatrix}\mathbf{x} = 36 \tag{5.1.25.1}$$
26. Find the equation of the ellipse whose vertices

are $\begin{pmatrix}\pm 13\\0\end{pmatrix}$ and foci are $\begin{pmatrix}\pm 5\\0\end{pmatrix}$.

27. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix}0\\\pm 5\end{pmatrix}$
28. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.
29. Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas
    a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -16\end{pmatrix}\mathbf{x} = 16$
30. Find the equation of the hyperbola with vertices $\begin{pmatrix}0\\\pm\frac{\sqrt{11}}{2}\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm 3\end{pmatrix}$
31. Find the equation of the hyperbola with foci $\begin{pmatrix}0\\\pm 12\end{pmatrix}$ and length of latus rectum 36.
32. Find the equation of all lines having slope 2 and being tangent to the curve
$$y + \frac{2}{x-3} = 0 \tag{5.1.32.1}$$
33. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.
34. Find the roots of the following equations:
    a) $x + \frac{1}{x} = 3, x \neq= 0$
    b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq= 0, 2$
35. Find points on the curve $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$ at which the tangents are
    a) parallel to x-axis
    b) parallel to y-axis
36. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$.
37. Find the area enclosed by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$
38. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.
39. Find the area bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{a^2} & 0\\0 & \frac{1}{b^2}\end{pmatrix}\mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1–e^2)$ and $e < 1$.
40. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.
41. Find the area of the region
$$\left\{(x, y) = 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\} \tag{5.1.41.1}$$

5.2 Exercises

1. Verify whether the following are zeroes of the polynomial, indicated against them.
    a) $p(x) = x^2 - 1, x = 1, -1$
    b) $p(x) = (x+1)(x-2), x = -1, 2$
    c) $p(x) = x^2, x = 0$.
    d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$.
2. Find the vaue of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:
    a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
    b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
    c) $x^4 - 4x^2 + x + 6, g(x) = x - 3$
3. Factorise :
    a) $12x^2–7x + 1$
    b) $6x^2 + 5x–6$
    c) $2x^2 + 7x + 3$
    d) $3x^2–x–4$
4. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
    a) $x^2–2x–8$
    b) $4u^2 + 8u$
    c) $4s^2–4s + 1$
    d) $t^2–15$
    e) $6x^2–3–7x$
    f) $3x^2–x–4$
5. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.
    a) -1 , $\frac{1}{4}$
    b) 1, 1
    c) 0, $\sqrt{5}$
    d) 4, 1
    e) $\frac{1}{4}, \frac{1}{4}$
    f) $\sqrt{2}, \frac{1}{3}$
6. Find the roots of the following quadratic equations:
    a) $x^2–3x–10 = 0$
    b) $2x^2 + x - 6 = 0$
    c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
    d) $2x^2–x + \frac{1}{8} = 0$

e) $100x^2–20x + 1 = 0$

7. Find the roots of the following quadratic equations
   a) $2x^2 - 7x + 3 = 0$
   b) $2x^2 + x - 4 = 0$
   c) $4x^2 + 4\sqrt{3}x + 3 = 0$
   d) $2x^2 + x + 4 = 0$
8. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
   a) $2x^2 - 3x + 5 = 0$
   b) $2x^2 - 6x + 3 = 0$
   c) $3x^2 - 4\sqrt{3}x + 4 = 0$
9. Solve each of the following equations
   a) $x^2 + 3 = 0$
   b) $2x^2 + x + 1 = 0$
   c) $x^2 + 3x + 9 = 0$
   d) $-x^2 + x - 2 = 0$
   e) $x^2 + 3x + 5 = 0$
   f) $x^2 - 3x + 2 = 0$
   g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
   h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
   i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
   j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$
10. Solve each of the following equations
    a) $3x^2 - 4x + \frac{20}{3} = 0$
    b) $x^2 - 2x + \frac{3}{2} = 0$
    c) $27x^2 - 10x + 1 = 0$
    d) $21x^2 - 28x + 10 = 0$
11. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum
    a) $y^2 = 12x$
    b) $x^2 = 6y$
    c) $y^2 = -8x$
    d) $x^2 = -16y$
    e) $y^2 = 10x$
    f) $x^2 = -9y$
12. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:
    a) Focus $\begin{pmatrix}6\\0\end{pmatrix}$, directrix $\begin{pmatrix}1 & 0\end{pmatrix} = -6$.
    b) Focus $\begin{pmatrix}0\\-3\end{pmatrix}$, directrix $\begin{pmatrix}0 & 1\end{pmatrix} = 3$.
    c) Focus $\begin{pmatrix}3\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.
    d) Focus $\begin{pmatrix}-2\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.
    e) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}2\\2\end{pmatrix}$ and axis is along the x-axis
    f) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}5\\2\end{pmatrix}$ and symmetric with respect to the y-axis.
13. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.
    a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{36} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$
    c) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$
    d) $\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{100}\end{pmatrix}\mathbf{x} = 1$
    e) $\mathbf{x}^T\begin{pmatrix}\frac{1}{49} & 0\\0 & \frac{1}{36}\end{pmatrix}\mathbf{x} = 1$
    f) $\mathbf{x}^T\begin{pmatrix}\frac{1}{100} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$
    g) $\mathbf{x}^T\begin{pmatrix}36 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 144$
    h) $\mathbf{x}^T\begin{pmatrix}16 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 16$
    i) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 36$
14. In each of the following, find the equation for the ellipse that satisfies the given conditions:
    a) Vertices $\begin{pmatrix}\pm5\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$
    b) Vertices $\begin{pmatrix}0\\\pm13\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$
    c) Vertices $\begin{pmatrix}\pm6\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$
    d) Ends of major axis $\begin{pmatrix}\pm3\\0\end{pmatrix}$, ends of minor axis $\begin{pmatrix}0\\\pm2\end{pmatrix}$
    e) Ends of major axis $\begin{pmatrix}0\\\pm5\end{pmatrix}$, ends of minor axis $\begin{pmatrix}\pm1\\0\end{pmatrix}$
    f) Length of major axis 26, foci $\begin{pmatrix}\pm5\\0\end{pmatrix}$

g) Length of minor axis 16, foci $\begin{pmatrix} 0 \\ \pm 6 \end{pmatrix}$.

h) Foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, a = 4

i) b = 3, c = 4, centre at the origin; foci on the x axis.

j) Centre at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

k) Major axis on the x-axis and passes through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

15. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & -\frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{27} \end{pmatrix} \mathbf{x} = 1$

c) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} = 36$

d) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 576$

e) $\mathbf{x}^T \begin{pmatrix} 5 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x} = 36$

f) $\mathbf{x}^T \begin{pmatrix} 49 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 784$

16. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$

b) Vertices $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 8 \end{pmatrix}$

c) Vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

d) Transverse axis length 8, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.

e) Conjugate axis length 24, foci $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$.

f) Latus rectum length 8, foci $\begin{pmatrix} \pm 3\sqrt{5} \\ 0 \end{pmatrix}$.

g) Latus rectum lenght 12, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$.

h) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$

i) Vertices $\begin{pmatrix} \pm 7 \\ 0 \end{pmatrix}$, $e = \frac{4}{3}$

j) Foci $\begin{pmatrix} 0 \\ \pm\sqrt{10} \end{pmatrix}$, passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

17. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

18. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

19. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

20. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.

21. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are

a) parallel to x-axis

b) parallel to y-axis.

22. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

23. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

24. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.

25. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

26. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

27. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

28. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

29. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

30. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

31. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

32. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
33. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
34. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
35. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
36. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
37. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
38. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
39. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
40. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
41. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.
42. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
43. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
44. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
45. Find the area between $y = x^2$ and $y = x$.
46. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
47. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
48. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
49. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.
50. Find the area bounded by the curves
$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{5.2.50.1}$$
51. Find the area of the region
$$\left\{(x, y) : y^2 \le 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{5.2.51.1}$$
52. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

6 Curves

6.1 Examples

1. Find the value of each of the following polynomials at the indicated value of variables:
   a) $q(y) = 3y^3 - 4y + 11$ at $y = 2$.
   b) $p(t) = 4t^4 + 5t^3 - t2 + 6$ at $t = a$.
2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(t) = 2 + t + 2t^2 - t^3$
   b) $p(x) = x^3$
3. Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.
4. Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.
5. Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.
6. Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.
7. Factorize $x^3 - 23x^2 + 142x - 120$.
8. Verify that $3, -1, \frac{1}{3}$, are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.
9. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$
10. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$
11. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
12. Find the equation of the tangent to the curve $\begin{pmatrix} a\sin^3 t \\ b\cos^3 t \end{pmatrix}$ at $t = \frac{\pi}{2}$.
13. Find the equation of tangents to the curve $y = cos(x + y), -2\pi \le x \le 2\pi$ that are parallel to the line $\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 0$.
14. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.
15. Sketch the graph of $y = |x + 3|$ and evaluate its area for $-6 \le x \le 0$.
16. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

6.2 Exercises

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by
   a) $x + 1$
   b) $x - \frac{1}{2}$
   c) x
   d) $x + \pi$
   e) $5 + 2x$
2. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.
3. Determine which of the following polynomials has $(x + 1)$ as a factor:
   a) $x^3 + x^2 + x + 1$
   b) $x^4 + x^3 + x^2 + x + 1$
   c) $x^4 + 3x^3 + 3x^2 + x + 1$
   d) $x^3 - x^2 - \left(2 + \sqrt{2}\right) + \sqrt{2}$.
4. Determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $p(x) = x^4 - 4x^2 + x + 6, g(x) = x - 3$
5. Factorise :
   a) $x^3–2x^2–x + 2$
   b) $x^3–3x^2–9x–5$
   c) $x^3 + 13x^2 + 32x + 20$
   d) $2y^3 + y^2–2y–1$
6. Find the roots of the following equations:
   a) $x - \frac{1}{x} = 3, x \neq= 0$
   b) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq= -4, 7$
7. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
8. Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 2.
9. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
10. Find the slope of the normal to the curve $\mathbf{x} = a\begin{pmatrix}\cos^3\theta \\ \sin^3\theta\end{pmatrix}$ at $\theta = \frac{\pi}{4}$.
11. Find the slope of the normal to the curve $\mathbf{x} = \begin{pmatrix}1 - a\sin\theta \\ b\cos^2\theta\end{pmatrix}$ at $\theta = \frac{\pi}{2}$.
12. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to th x-axis.
13. Find the point on the curve $y = x^3–11x + 5$ at which the tangent is $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 11$.
14. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$.
15. Find the equations of the tangent and normal to the given curves at the indicated points:
    a) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}0 \\ 5\end{pmatrix}$.
    b) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}1 \\ 3\end{pmatrix}$.
    c) $y = x^3$ at $\begin{pmatrix}1 \\ 1\end{pmatrix}$.
16. Show that the tangents to the curve $y = 7x^3+11$ at the points where $x = 2$ and $x = -2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3–2x^5$ find all the points at which the tangent passes through the origin.
19. Find the equation of the normal at the point $\begin{pmatrix}am^2 \\ am^3\end{pmatrix}$ for the curve $ay^2 = x^3$
20. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $\begin{pmatrix}1 & 14\end{pmatrix} + 4 = 0$.
21. Find the slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$. Show that the normal at any point θ to the curve $\mathbf{x} = \begin{pmatrix}acos\theta + a\theta sin\theta \\ asin\theta–a\theta cos\theta\end{pmatrix}$ is at a constant distance from the origin.
22. Find the slope of the tangent to the curve $\mathbf{x} = \begin{pmatrix}t^2 + 3t - 8 \\ 2t^2 - 2t - 5\end{pmatrix}$ at the point $\begin{pmatrix}2 \\ -1\end{pmatrix}$.
23. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes.
24.
25. Find the area under $y = x^4, x = 1, x = 5$ and x-axis.
26. Using integration find the area of region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).
27. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$, $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -1$ and x = 4.
28. Find the area of the region bounded by the line $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -2$, the x-axis and the ordinates $x = -1, x = 1$.
29. Find the area bounded by the curve $|x|+|y| = 1$.
30. Using the method of integration find the area of $\triangle ABC$, whose vertices are $\mathbf{A} = \begin{pmatrix}2 \\ 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4 \\ 5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6 \\ 3\end{pmatrix}$.

31. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 4$, $\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 6$ and $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$.
32. Find the area bounded by the curve $y = x^3$, $x = -2$, $x = 1$ and the x-axis.
33. Find the area bounded by the curve $y = x|x|$, $x = -1$, $x = 1$ and the x-axis.
34. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$.

7 Miscellaneous Examples

1. Divide $p(x)$ by $g(x)$, where $p(x) = x + 3x^2 - 1$ and $g(x) = 1 + x$.
2. Divide the polynomial $p(x) = 3x^4 - 4x^3 - 3x - 1$ by $x - 1$.
3. Find the value of k, if x–1 is a factor of $p(x) = 4x^3 + 3x^2 - 4x + k$.
4. Divide $2x^2 + 3x + 1$ by $x + 2$.
5. Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.
6. Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.
7. John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. We would like to find out how many marbles they had to start with.
8. A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹750. We would like to find out the number of toys produced on that day.
9. The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?
10. Find two consecutive odd positive integers, sum of whose squares is 290.
11. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m. Find its length and breadth.
12. A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.
13. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?
14. The focus of a parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB .
15. A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?
16. 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point **A** lies on x-axis and end point **B** lies on y-axis. A point **P** is taken on the rod in such a way that $AP = 6$ cm. Show that the locus of P is an ellipse
17. Using integration find the area of region bounded by the triangle whose vertices are $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$.
18. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.
19.

8 Miscellaneous Exercises

1. Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.
2. Find the value of k, if x – 1 is a factor of p(x) in each of the following cases:
   a) $p(x) = x^2 + x + k$
   b) $p(x) = kx^2 - \sqrt{2}x + 1$
   c) $p(x) = 2x^2 + kx + \sqrt{2}$
   d) $p(x) = kx^2 - 3x + k$
3. Divide the polnyomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following:
   a) $p(x) = x^3 - 3x^2 + 5x - 3$, $g(x) = x^2 - 2$.
   b) $p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$.
   c) $p(x) = x^4 - 5x + 6$, $g(x) = 2 - x^2$.

4. Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:
   a) $t^2 - x, 2t^4 + 3t^3 - 2t^2 - 9t - 12$.
   b) $x^2 + 3x + 1, 3x^4 + 5x^3 - 7x^2 + 2x + 2$.
   c) $x^3 - 3x + 1, x^5 - 4x^3 + x^2 + 3x + 1$.
5. Obtain all the other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$and $-\sqrt{\frac{5}{3}}$.
6. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x-2$ and $-2x + 4$respectively. Find $g(x)$.
7. Verify that the numbers given alongside the cubic polynomials below are their zeroes. Also verify if the relationship between the zeroes and the coefficients in each case:
   a) $2x^3 + x^2 - 5x + 2; \frac{1}{2}, 1, -2$
   b) $x^3 - 4x^2 + 5x - 2; 2, 1, 1$
8. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, -7, -4 respectively.
9. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find the other zeroes.
10. If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.
11. The area of a rectangular plot is 528 m^2 . The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.
12. The product of two consecutive positive integers is 306. We need to find the integers.
13. Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Rohan's present age.
14. A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.
15. Find two numbers whose sum is 27 and product is 182.
16. Find two consecutive positive integers, sum of whose squares is 365.
17. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.
18. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹90, find the number of articles produced and the cost of each article.
19. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.
20. In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.
21. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.
22. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.
23. A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less for the same journey. Find the speed of the train.
24. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.
25. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11km/h more than that of the passenger train, find the average speed of the two trains.
26. Sum of the areas of two squares is 468 $m^2$2 find the sides of the two squares.
27. Find the values of k for each of the following quadratic equations, so that they have two equal roots:
   a) $2x^2 + kx + 3 = 0$
   b) $kx(x - 2) + 6 = 0$
28. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2 ? If so, find its length and breadth.

29. Is the following situation possible? If so, determine their present ages. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48.
30. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2 its length and breadth
31. If
$$\mathbf{x} = \sqrt{\frac{\begin{pmatrix} a \\ -b \end{pmatrix}}{\begin{pmatrix} c \\ -d \end{pmatrix}}} \tag{8.0.31.1}$$
prove that
$$\|\mathbf{x}\|^2 = \frac{\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\|}{\left\|\begin{pmatrix} c \\ d \end{pmatrix}\right\|} \tag{8.0.31.2}$$
32. For any two complex numbers $\mathbf{z}_1, \mathbf{z}_2$, prove that
$$\Re \mathbf{z}_1 \mathbf{z}_2 = \Re \mathbf{z}_1 \Re \mathbf{z}_2 - \Im \mathbf{z}_1 \Im \mathbf{z}_2 \tag{8.0.32.1}$$
33. If $\mathbf{x} = \frac{\begin{pmatrix} a \\ b \end{pmatrix}}{\begin{pmatrix} a \\ -b \end{pmatrix}}$, show that $\|\mathbf{x}\| = 1$
34. If $\mathbf{x} = \frac{\begin{pmatrix} x \\ 1 \end{pmatrix}^2}{2x^2+1}$, prove that $\|\mathbf{x}\|^2 = \frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$.
35. If $\begin{pmatrix} x \\ y \end{pmatrix}^3 = \mathbf{uv}$, then show that $\frac{u}{x} + \frac{v}{y} = 4\left(x^2 - y^2\right)$.
36. If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are different complex numbers with $\|\boldsymbol{\beta}\| = 1$, then find $\left\|\frac{\beta-\alpha}{1-\alpha^*\beta}\right\|$.
37. Find the number of non-zero integral solutions of the equation $\|1 - 1\|^x = 2^x$.
38. If $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$, then show that $\left(a^2 + b^2\right)\left(c^2 + d^2\right)\left(e^2 + f^2\right)\left(g^2 + h^2\right) = A^2 + B^2$.
39. If $\left(\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}\right) = 1$, the find hte least positive integral value of m.
40. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
41. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?
42. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
43. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.
44. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.
45. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.
46. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
47. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.
48. Prove that the curves $x = y^2$ and $kx = y$ cut at right angles if $8k^2 = 1$
49. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $\begin{pmatrix} at^2 \\ 2at \end{pmatrix}$.
50. Find the equations of the tangent and normal to the hyperbola $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ at the point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.
51. Find the area of the smaller part of the circle $\mathbf{x}^{\mathbf{x}} = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
52. Find the area enclosed between the parabola $y^2 = 4ax$ ad the line $y = mx$.