Linear Algebra through Coordinate Geometry

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Contents

***Abstract*—This manual introduces linear algebra through coordinate geometry using a problem solving approach.**

1 The Straight Line

1.1 The equation of the line between two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{A} - \mathbf{B}) \tag{1.1}$$

Alternatively, it can be expressed as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{1.2}$$

where $\mathbf{m}$ is the solution of

$$(\mathbf{A} - \mathbf{B})^T \mathbf{m} = 0 \tag{1.3}$$

2 Locus

2.1 The line through

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1}$$

intersects the coordinate axes at $\mathbf{P}$ and $\mathbf{Q}$. $\mathbf{O}$ is the origin and rectangle $OPRQ$ is completed as shown i Fig. (2.1),

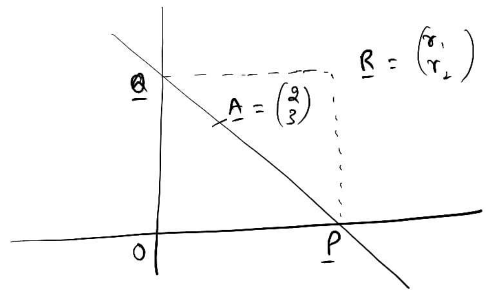

Fig. 2.1

2.2 Show that

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R} \tag{2.2}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R} \tag{2.3}$$

$$\mathbf{P} + \mathbf{Q} = \mathbf{R} \tag{2.4}$$

2.3 Show that

$$\begin{aligned} (\mathbf{A} - \mathbf{P})^T \mathbf{m} &= 0 \\ (\mathbf{A} - \mathbf{Q})^T \mathbf{m} &= 0 \\ (\mathbf{P} - \mathbf{Q})^T \mathbf{m} &= 0 \end{aligned} \tag{2.5}$$

Solution: Trivial using (1.2) and (1.3).

2.4 Show that

$$(2\mathbf{A} - \mathbf{R})^T \mathbf{m} = 0 \tag{2.6}$$

$$\mathbf{R}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{m} = 0 \tag{2.7}$$

Solution: From (2.5) and (2.4)

$$[2\mathbf{A} - (\mathbf{P} + \mathbf{Q})]^T \mathbf{m} = 0 \tag{2.8}$$

resulting in (2.6). From (2.5) and (2.2),(2.3), (2.7) is obtained.

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2.5 Show that

$$\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0. \tag{2.9}$$

2.6 Find the locus of $\mathbf{R}$.
Solution: For $\mathbf{m}$ to be unique in (2.6),(2.7),

$$(\mathbf{2A} - \mathbf{R}) = k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}$$

$$\implies \mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\mathbf{2A} - \mathbf{R})$$

$$= k\mathbf{R}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{R}$$

$$= k\mathbf{R}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{R} = 0 \tag{2.10}$$

where k is some constant.

3 Conics

3.1 The equation of quadratic curve is given by

$$Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0 \tag{3.1}$$

Show that (3.1) can be expressed as

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + F = 0 \tag{3.2}$$

Find the matrix V and vector $\mathbf{u}$.

3.2 The tangent to (3.1) at a point $\mathbf{p}$ on the curve is given by

$$\begin{pmatrix} \mathbf{p}^T & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{u} \\ \mathbf{u}^T & F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = 0 \tag{3.3}$$

Show that (3.3) can be expressed as

$$\left(\mathbf{p}^T V + \mathbf{u}^T\right) \mathbf{x} + \mathbf{p}^T \mathbf{u} + F = 0 \tag{3.4}$$

4 Parabola

4.1 Find the tangent at $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ to the parabola

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 6 = 0 \tag{4.1}$$

Solution: Substituting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4.2}$$

in (3.4), the desired equation is

$$\left[\begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \end{pmatrix}\right] \mathbf{x}$$

$$+ \frac{1}{2} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 6 = 0 \tag{4.3}$$

resulting in

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 5 \tag{4.4}$$

4.2 The line in (4.4) touches the circle

$$\mathbf{x}^T \mathbf{x} + 4 \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} + c = 0 \tag{4.5}$$

Find c.
Solution: Comparing (3.2) and (4.5),

$$V = I,$$
$$\mathbf{u} = 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{4.6}$$

Comparing (3.4) and (4.4),

$$\mathbf{p} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{4.7}$$

$$\implies \mathbf{p} = -\begin{pmatrix} 6 \\ 7 \end{pmatrix} \tag{4.8}$$

and

$$c + \mathbf{p}^T \mathbf{u} = 5 \tag{4.9}$$

$$\implies c = 5 + 2 \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{4.10}$$

$$= 95 \tag{4.11}$$

5 Hyperbola

5.1 Tangents are drawn to the hyperbola

$$\mathbf{x}^T V \mathbf{x} = 36 \tag{5.1}$$

where

$$V = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$. If these tangents intersect at

$$\mathbf{T} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \tag{5.3}$$

find the equation of PQ.
Solution: The equations of the two tangents are obtained using (3.4) as

$$\mathbf{P}^T V \mathbf{x} = 36 \tag{5.4}$$

$$\mathbf{Q}^T V \mathbf{x} = 36. \tag{5.5}$$

Since both pass through $\mathbf{T}$

$$\mathbf{P}^T V\mathbf{T} = 36 \implies \mathbf{P}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{5.6}$$

$$\mathbf{Q}^T V\mathbf{T} = 36 \implies \mathbf{Q}^T \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 36 \tag{5.7}$$

Thus, $\mathbf{P}, \mathbf{Q}$ satisfy

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} = -36 \tag{5.8}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{5.9}$$

5.2 In $\triangle PTQ$, find the equation of the altitude $TD \perp PQ$.
Solution: Since

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{5.10}$$

using (1.2) and (5.9), the equation of TD is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{T}) = 0 \tag{5.11}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.12}$$

5.3 Find D.
Solution: From (5.9) and (5.12),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{5.13}$$

$$\implies \mathbf{D} = \begin{pmatrix} 0 \\ -12 \end{pmatrix} \tag{5.14}$$

5.4 Show that the equation of PQ can also be expressed as

$$\mathbf{x} = \mathbf{D} + \lambda \mathbf{m} \tag{5.15}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.16}$$

5.5 Show that for $\mathbf{V}^T = \mathbf{V}$,

$$(\mathbf{D} + \lambda \mathbf{m})^T V (\mathbf{D} + \lambda \mathbf{m}) + F = 0 \tag{5.17}$$

can be expressd as

$$\lambda^2 \mathbf{m}^T V\mathbf{m} + 2\lambda \mathbf{m}^T V\mathbf{D} + \mathbf{D}^T V\mathbf{D} + F = 0 \tag{5.18}$$

5.6 Find $\mathbf{P}$ and $\mathbf{Q}$.
Solution: From (5.15) and (5.1) (5.18) is obtained. Substituting from (5.16), (5.2) and (5.14)

$$\mathbf{m}^T V\mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \tag{5.19}$$

$$\mathbf{m}^T V\mathbf{D} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = 0 \tag{5.20}$$

$$\mathbf{D}^T V\mathbf{D} = \begin{pmatrix} 0 & -12 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -12 \end{pmatrix} = -144 \tag{5.21}$$

Substituting in (5.18)

$$4\lambda^2 - 144 = 36 \tag{5.22}$$

$$\implies \lambda = \pm 3\sqrt{5} \tag{5.23}$$

Substituting in (5.15),

$$\mathbf{P} = \mathbf{D} + 3\sqrt{5}\mathbf{m} = 3 \begin{pmatrix} \sqrt{5} \\ -4 \end{pmatrix} \tag{5.24}$$

$$\mathbf{Q} = \mathbf{D} - 3\sqrt{5}\mathbf{m} = -3 \begin{pmatrix} \sqrt{5} \\ 4 \end{pmatrix} \tag{5.25}$$

5.7 Find the area of $\triangle PTQ$.
Solution: Since

$$PQ = \|\mathbf{P} - \mathbf{Q}\| = 6\sqrt{5} \tag{5.26}$$

$$TD = \|\mathbf{T} - \mathbf{D}\| = 15, \tag{5.27}$$

the desired area is

$$\frac{1}{2} PQ \times TD = 45\sqrt{5} \tag{5.28}$$

5.8 Repeat the previous exercise using determinants.

6 JEE

6.1 Tangent and normal are drawn at

$$\mathbf{P} = \begin{pmatrix} 16 \\ 16 \end{pmatrix} \tag{6.1}$$

on the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 16 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{6.2}$$

which intersect the axis of the parabola at $\mathbf{A}$ and $\mathbf{B}$ respectively. If $\mathbf{C}$ is the centre of the circle through the ponts $\mathbf{P}$ $\mathbf{A}$ and $\mathbf{B}$, find $\tan CPB$.