The Straight Line

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Abstract—**Solved problems from JEE mains papers related to 2D lines in coordinate geometry are available in this document. These problems are solved using linear algebra/matrix analysis.**

1 A straight line through the origin **O** meets the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 10 \tag{1.1}$$

$$\begin{pmatrix} 8 & 6 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.2}$$

at **A** and **B** respectively. Find the ratio in which **O** divides *AB*.
Solution: Let

$$\mathbf{n} = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{1.3}$$

Then (1.1) can be expressed as

$$\mathbf{n}^T \mathbf{x} = 10 \tag{1.4}$$

$$2\mathbf{n}^T \mathbf{x} = -5 \tag{1.5}$$

and since **A**, **B** satisfy (1.4) respectively,

$$\mathbf{n}^T \mathbf{A} = 10 \tag{1.6}$$

$$2\mathbf{n}^T \mathbf{B} = -5 \tag{1.7}$$

Let **O** divide the segment *AB* in the ratio $k : 1$. Then

$$\mathbf{O} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \tag{1.8}$$

$$\because \mathbf{O} = \mathbf{0}, \tag{1.9}$$

$$\mathbf{A} = -k\mathbf{B} \tag{1.10}$$

Substituting in (1.6), and simplifying,

$$\mathbf{n}^T \mathbf{B} = \frac{10}{-k} \tag{1.11}$$

$$\mathbf{n}^T \mathbf{B} = \frac{-5}{2} \tag{1.12}$$

resulting in

$$\frac{10}{-k} = \frac{-5}{2} \implies k = 4 \tag{1.13}$$

2 The point

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1}$$

is translated parallel to the line

$$L : \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{2.2}$$

by $2\sqrt{3}$ units. If the new point **Q** lies in the third quadrant, then find the equation of the line passing through **Q** and perpendicular to *L*.
Solution: From (2.2), the direction vector of *L* is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.3}$$

Thus,

$$\mathbf{Q} = \mathbf{P} + \lambda\mathbf{m} \tag{2.4}$$

However,

$$PQ = 2\sqrt{3} \tag{2.5}$$

$$\implies \|\mathbf{P} - \mathbf{Q}\| = |\lambda| \|\mathbf{m}\| = 2\sqrt{3} \tag{2.6}$$

$$\implies \lambda = \pm\frac{2\sqrt{3}}{\|\mathbf{m}\|} = \pm\sqrt{6} \tag{2.7}$$

$$\because \|\mathbf{m}\| = \sqrt{\mathbf{m}^T \mathbf{m}} = \sqrt{2} \tag{2.8}$$

from (2.3). Since **Q** lies in the third quadrant,

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from (2.4) and (2.7),

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sqrt{6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - \sqrt{6} \\ 1 - \sqrt{6} \end{pmatrix} \qquad (2.9)$$

The equation of the desired line is then obtained as

$$\mathbf{m}^T(\mathbf{x} - \mathbf{Q}) = 0 \qquad (2.10)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 - \sqrt{6} \qquad (2.11)$$

3 A variable line drawn through the intersection of the lines

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \qquad (3.1)$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 12 \qquad (3.2)$$

meets the coordinate axes at **A** and **B**, then find the locus of the midpoint of *AB*.

Solution: The intersection of the lines in (3.1) is obtained through the matrix equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \qquad (3.3)$$

by forming the augmented matrix and row reduction as

$$\begin{pmatrix} 4 & 3 & 12 \\ 3 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 4 & 3 & 12 \\ 0 & 7 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 28 & 0 & 48 \\ 0 & 7 & 12 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 7 & 0 & 12 \\ 0 & 7 & 12 \end{pmatrix} \qquad (3.4)$$

resulting in

$$\mathbf{C} = \frac{1}{7}\begin{pmatrix} 12 \\ 12 \end{pmatrix} \qquad (3.5)$$

Let the **R** be the mid point of *AB*. Then,

$$\mathbf{A} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{R} \qquad (3.6)$$

$$\mathbf{B} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{R} \qquad (3.7)$$

Let the equation of *AB* be

$$\mathbf{n}^T(\mathbf{x} - \mathbf{C}) = 0 \qquad (3.8)$$

Since **R** lies on *AB*,

$$\mathbf{n}^T(\mathbf{R} - \mathbf{C}) = 0 \qquad (3.9)$$

Also,

$$\mathbf{n}^T(\mathbf{A} - \mathbf{B}) = 0 \qquad (3.10)$$

Substituting from (3.6) in (3.10),

$$\mathbf{n}^T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} = 0 \qquad (3.11)$$

From (3.9) and (3.11),

$$(\mathbf{R} - \mathbf{C}) = k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R} \qquad (3.12)$$

for some constant k. Multiplying both sides of (3.12) by

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (3.13)$$

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R} - \mathbf{C}) = k\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{R}$$

$$= k\mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0$$

$$\qquad (3.14)$$

$$\because \mathbf{R}^T\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \qquad (3.15)$$

which can be easily verified for any **R**. from (3.14),

$$\mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\mathbf{R} - \mathbf{C}) = 0$$

$$\implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{C} = 0$$

$$\implies \mathbf{R}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} - \mathbf{C}^T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{R} = 0 \quad (3.16)$$

which is the desired locus.

4 Two sides of a rhombus are along the lines

$$AB : \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \qquad (4.1)$$

$$AD : \begin{pmatrix} 7 & -1 \end{pmatrix}\mathbf{x} - 5 = 0. \qquad (4.2)$$

If its diagonals intersect at

$$\mathbf{P} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \qquad (4.3)$$

find its vertices.

Solution: From (4.1) and (4.2),

$$\begin{pmatrix} 1 & -1 \\ 7 & -1 \end{pmatrix}\mathbf{A} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \qquad (4.4)$$

By row reducing the augmented matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 7 & -1 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 6 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$
(4.5)

From (4.3) and (4.5)

$$AP = \|\mathbf{A} - \mathbf{P}\| = 2\sqrt{5} = d(say) \qquad (4.6)$$

The direction vector of AP is

$$\mathbf{m} = \mathbf{A} - \mathbf{P} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (4.7)$$

$$\implies \|\mathbf{m}\| = 2\sqrt{5} \qquad (4.8)$$

Since the direction of AP is the same as AC,

$$\mathbf{C} = \mathbf{P} - d\frac{\mathbf{m}}{\|\mathbf{m}\|}$$

$$= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} \qquad (4.9)$$

Let $\mathbf{n} \perp \mathbf{m}$. Then

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (4.10)$$

and

$$\mathbf{B}, \mathbf{D} = \mathbf{P} \pm d\frac{\mathbf{n}}{\|\mathbf{n}\|}$$

$$= -\begin{pmatrix} 1 \\ 2 \end{pmatrix} \pm 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \qquad (4.11)$$

5 Let k be an integer such that the triangle with vertices

$$\mathbf{A} = \begin{pmatrix} k \\ -3k \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -k \\ 2 \end{pmatrix} \qquad (5.1)$$

has area 28. Find the orthocentre of this triangle.

Solution: Let $\mathbf{m}_1$ be the direction vector of BC. Then,

$$\mathbf{m}_1 = \begin{pmatrix} 5 + k \\ k - 2 \end{pmatrix}, \qquad (5.2)$$

If AD be an altitude, its equation can be obtained as

$$\mathbf{m}_1^T (\mathbf{x} - \mathbf{A}) = 0 \qquad (5.3)$$

Similarly, considering the side AC the equation

of the altitude BE is

$$\mathbf{m}_2^T (\mathbf{x} - \mathbf{B}) = 0 \qquad (5.4)$$

where

$$\mathbf{m}_2 = \begin{pmatrix} 2k \\ -2 - 3k \end{pmatrix}, \qquad (5.5)$$

The orthocentre is obtained by solving (5.3) and (5.4) using the matrix equation

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} \mathbf{m}_1^T \mathbf{A} \\ \mathbf{m}_2^T \mathbf{B} \end{pmatrix} \qquad (5.6)$$

which can be expressed using (5.2), (5.5), (5.3) and (5.4) as

$$\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix} \mathbf{x} = \begin{pmatrix} k^2 + 5k + 6k - 3k^2 \\ 10k - 2k - 3k^2 \end{pmatrix}$$

$$= k\begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix} \qquad (5.7)$$

The solution to the above is

$$\mathbf{x} = k\begin{pmatrix} 5 + k & k - 2 \\ 2k & -2 - 3k \end{pmatrix}^{-1} \begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix}$$

$$= \frac{k}{-3k^2 - 17k - 10 - 2k^2 + 4k}$$

$$\times \begin{pmatrix} -2 - 3k & -k + 2 \\ -2k & 5 + k \end{pmatrix}\begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix}$$

$$= \frac{k}{5k^2 + 13k + 10}$$

$$\times \begin{pmatrix} 2 + 3k & k - 2 \\ 2k & -5 - k \end{pmatrix}\begin{pmatrix} 11 - 4k \\ 8 - 3k \end{pmatrix}$$

$$= \frac{k}{5k^2 + 13k + 10}$$

$$\times \begin{pmatrix} 22 - 12k^2 - 8k + 33k - 3k^2 + 6k + 8k - 16 \\ 22k - 8k^2 - 40 + 15k - 8k + 3k^2 \end{pmatrix}$$

$$= \frac{k}{5k^2 + 13k + 10}\begin{pmatrix} 6 + 39k - 15k^2 \\ -40 + 29k - 5k^2 \end{pmatrix} \qquad (5.8)$$

From the given information,

$$\begin{vmatrix} k & 5 & -k \\ -3k & k & 2 \\ 1 & 1 & 1 \end{vmatrix} = 56$$

$$\implies \begin{vmatrix} k & 5 - k & -2k \\ -3k & 4k & 2 + 3k \\ 1 & 0 & 0 \end{vmatrix} = 56 \qquad (5.9)$$

resulting in

$$(5 - k)(2 + 3k) + 8k^2 = 56 \tag{5.10}$$

$$\implies 5k^2 + 13k - 46 = 0 \tag{5.11}$$

$$\text{or, } k = 2, -\frac{23}{5} \tag{5.12}$$

6 If an equilateral triangle, having centroid at the origin, has a side along the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{6.1}$$

then find the area of this triangle.
Solution: Let the vertices be $\mathbf{A}, \mathbf{B}, \mathbf{C}$. From the given information,

$$\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \mathbf{0}$$

$$\implies \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0} \tag{6.2}$$

If AB be the line in (6.1), the equation of CF, where

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{6.3}$$

is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{6.4}$$

since CF passes through the origin and $CF \perp AB$. From (6.1) and (6.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{6.5}$$

Forming the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \implies \mathbf{F} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.6}$$

From (6.2),

$$\mathbf{C} = -(\mathbf{A} + \mathbf{B}) = -2\mathbf{F} = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.7}$$

after substituting from (6.6).

7 Find the locus of the point of intersection of the straight lines

$$\begin{pmatrix} t & -2 \end{pmatrix} \mathbf{x} - 3t = 0 \tag{7.1}$$

$$\begin{pmatrix} 1 & -2t \end{pmatrix} \mathbf{x} + 3 = 0 \tag{7.2}$$

8 A square, of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle $30°$ with the positive direction of the x-axis, then find the sum of the x-coordinates of the vertices of the square.

9 Find the locus of the point of intersection of the lines

$$\begin{pmatrix} \sqrt{2} & -1 \end{pmatrix} \mathbf{x} + 4\sqrt{2}k = 0 \tag{9.1}$$

$$\begin{pmatrix} \sqrt{2}k & k \end{pmatrix} \mathbf{x} - 4\sqrt{2} = 0 \tag{9.2}$$

10 The sides of a rhombus ABC are parallel to the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{10.1}$$

$$\begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} + 3 = 0. \tag{10.2}$$

If the diagonals of the rhombus intersect at

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{10.3}$$

and the vertex $\mathbf{A}$ (different) from the origin is on the y-axis, then find the ordinate of A.