Computational Approach to School Mathematics

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CONTENTS

***Abstract*—This book provides a computational approach to school mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/codes
```

1 Triangle

1.1 Construction Examples

1. Draw $\triangle ABC$ where $\angle B = 90°$, $a = 4$ and $b = 3$.
Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.1.1)$$

The following code plots Fig. 1.1.1

```
codes/triangle/rt_triangle.py
```

2. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution: Let the vertices of $\triangle ABC$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.1.2.1)$$

$$\mathbf{A}^T \triangleq \begin{pmatrix} p & q \end{pmatrix} \quad (1.1.2.2)$$

$$\|\mathbf{A}\|^2 = \mathbf{A}^T\mathbf{A} = \begin{pmatrix} p & q \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \quad (1.1.2.3)$$

$$= p \times p + q \times q = p^2 + q^2 \quad (1.1.2.4)$$

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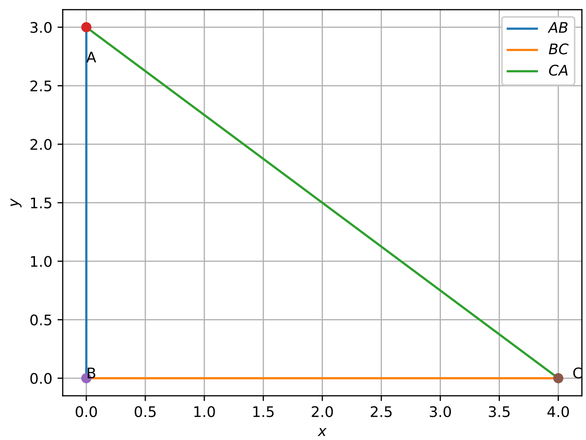

Fig. 1.1.1

Then

$$AB \triangleq \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (1.1.2.5)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (1.1.2.6)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (1.1.2.7)$$

From (1.1.2.7),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (1.1.2.8)$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \quad (1.1.2.9)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \quad (1.1.2.10)$$

$$= a^2 + c^2 - 2ap \quad (1.1.2.11)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (1.1.2.12)$$

From (1.1.2.5),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.1.2.13}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.1.2.14}$$

The following code plots Fig. 1.1.2

```
codes/triangle/draw_triangle.py
```

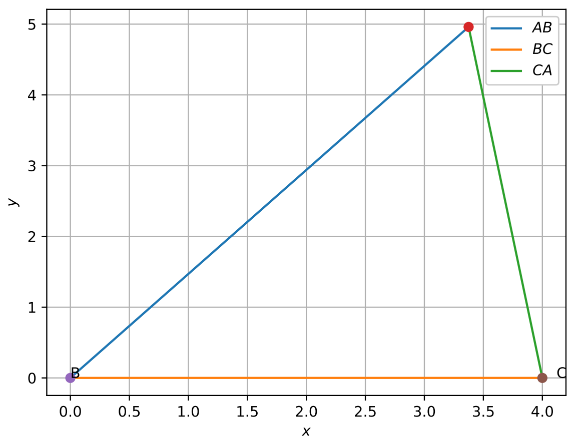

Fig. 1.1.2

3. Construct a triangle of sides $a = 5$, $b = 6$ and $c = 7$. Construct a similar triangle whose sides are $\frac{7}{5}$ times the corresponding sides of the first triangle.
   Solution: The sides of the similar triangle are $\frac{7}{5}a, \frac{7}{5}b$ and $\frac{7}{5}c$.
4. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
   Solution: Using Baudhayana's theorem,
$$b = c = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} \tag{1.1.4.1}$$
5. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^{\circ}$ and $\angle C = 45^{\circ}$, find a, b, c and sketch the triangle.
   Solution: From the given information,
$$a + b + c = 11 \tag{1.1.5.1}$$
$$b = c \quad (\because B = C = 45^\circ) \tag{1.1.5.2}$$
$$a^2 = b^2 + c^2 \quad (\because A = 90^\circ) \tag{1.1.5.3}$$
   From (1.1.5.1) and (1.1.5.2),
$$a + 2b = 11 \tag{1.1.5.4}$$
   From (1.1.5.2) and (1.1.5.3),
$$a^2 = 2b^2 \implies a - b\sqrt{2} = 0 \tag{1.1.5.5}$$
   (1.1.5.4) and (1.1.5.5) can be summarized as the matrix equation
$$\begin{pmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \tag{1.1.5.6}$$
   which can be solved using Cramer's rule as
$$a = \frac{\begin{vmatrix} 11 & 2 \\ 0 & -\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11 \times \left(-\sqrt{2}\right) - 2 \times 0}{1 \times \left(-\sqrt{2}\right) - 2 \times 1} \tag{1.1.5.7}$$
$$= \frac{11\sqrt{2}}{2 + \sqrt{2}} \tag{1.1.5.8}$$
$$b = \frac{\begin{vmatrix} 1 & 11 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -\sqrt{2} \end{vmatrix}} = \frac{11}{2 + \sqrt{2}} \tag{1.1.5.9}$$
   by expanding the determinants. The following code may be used to compute a, b and c.

```
codes/triangle/triangle_det.py
```

6. Repeat Problem 1.1.5 using a single matrix equation.
   Solution: The equations
$$a + 2b = 11 \tag{1.1.6.1}$$
$$a - b\sqrt{2} = 0 \tag{1.1.6.2}$$
$$b - c = 0 \tag{1.1.6.3}$$
   can be expressed as a single matrix equation
$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.1.6.4}$$

and can be solved using Cramer's rule as

$$a = \frac{\begin{vmatrix} 11 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.5}$$

$$b = \frac{\begin{vmatrix} 0 & 11 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.6}$$

$$c = \frac{\begin{vmatrix} 0 & 2 & 11 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix}} \tag{1.1.6.7}$$

The determinant

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 1 & -\sqrt{2} \\ 0 & 1 \end{vmatrix} \tag{1.1.6.8}$$

The determinant can also be expressed as

$$\begin{vmatrix} 0 & 2 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -1 \end{vmatrix} = 0 \times \begin{vmatrix} -\sqrt{2} & 0 \\ 1 & -1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 0 \\ -\sqrt{2} & 0 \end{vmatrix} \tag{1.1.6.9}$$

The determinants of larger matrices can be expressed similarly.

7. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60°$.
Solution: In Fig. 1.1.7, $AD \perp BC$.

$$\cos C = \frac{y}{b}, \tag{1.1.7.1}$$

$$\cos B = \frac{x}{b}, \tag{1.1.7.2}$$

Thus,

$$a = x + y = b\cos C + c\cos B, \tag{1.1.7.3}$$

$$b = c\cos A + a\cos C \tag{1.1.7.4}$$

$$c = b\cos A + a\cos B \tag{1.1.7.5}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.7.6}$$

Using Cramer's rule and determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} \tag{1.1.7.7}$$

$$= \frac{b^2 + c^2 - a^2}{2bc} \tag{1.1.7.8}$$

From (1.1.7.8)

$$b^2 = c^2 + a^2 - 2ca\cos B \tag{1.1.7.9}$$

which is computed by the following code

```
codes/triangle/cos_form.py
```

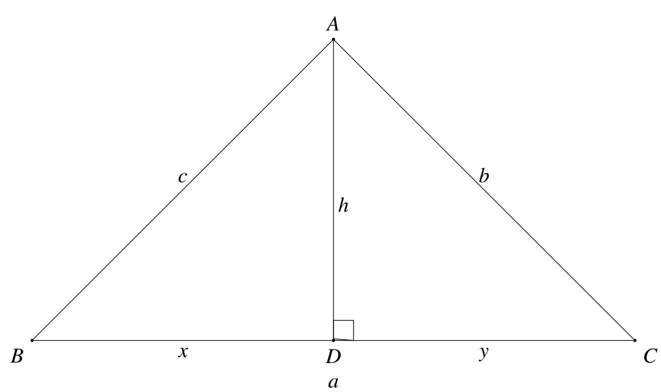

Fig. 1.1.7: The cosine formula

8. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
Solution: In Fig. (1.1.7),

$$\sin B = \frac{h}{c} \tag{1.1.8.1}$$

$$\sin C = \frac{h}{b} \tag{1.1.8.2}$$

which can be used to show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.1.8.3}$$

Thus,

$$c = \frac{a \sin C}{\sin A} \tag{1.1.8.4}$$

where

$$C = 180 - A - B \tag{1.1.8.5}$$

9. Draw $\triangle ABC$ if $AB = 3, AC = 5$ and $\angle C = 30°$.
Solution: From (1.1.7.9),

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.1.9.1}$$

which can be expressed as

$$a^2 - 2ab \cos C + b^2 - c^2 = 0. \tag{1.1.9.2}$$

$$\because (a - b \cos C)^2 = a^2 + b^2 \cos^2 C - 2ab \cos C, \tag{1.1.9.3}$$

(1.1.9.2) can be expressed as

$$(a - b \cos C)^2 - b^2 \cos^2 C + b^2 - c^2 = 0 \tag{1.1.9.4}$$

$$\implies (a - b \cos C)^2 = b^2 \left(1 - \cos^2 C\right) - c^2 \tag{1.1.9.5}$$

$$\text{or}, a = b \cos C \pm \sqrt{b^2 \left(1 - \cos^2 C\right) - c^2} \tag{1.1.9.6}$$

Choose the value(s) for which $a > 0$.

10. The solution of a quadratic equation

$$\alpha x^2 + \beta x + \gamma = 0 \tag{1.1.10.1}$$

is given by

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{1.1.10.2}$$

Verify (1.1.9.6) using (1.1.10.2).

11. $\triangle ABC$ is right angled at $\mathbf{B}$. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.
Solution: From Baudhayana's theorem,

$$b^2 = a^2 + c^2 \tag{1.1.11.1}$$

$$\implies (18 - c)^2 = 12^2 + c^2 \tag{1.1.11.2}$$

which can be simplified to obtain

$$36c - 180 = 0 \tag{1.1.11.3}$$

$$\implies c = 5 \tag{1.1.11.4}$$

and $b = 13$

12. Find a simpler solution for Problem 1.1.5
Solution: Use cosine formula.

13. In $\triangle ABC$, $a = 7, \angle B = 75^\circ$ and $b + c = 13$. Alternatively,

$$a = b \cos C + c \cos B \tag{1.1.13.1}$$

$$b \sin C = c \sin B \tag{1.1.13.2}$$

$$a + b + c = 11 \tag{1.1.13.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & -\cos C & -\cos B \\ 0 & \sin C & -\sin B \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \tag{1.1.13.4}$$

Solving the equivalent matrix equation gives the desired answer.

1.2 Triangle Examples

1. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.
Solution: The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{1.2.1.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.2.1.2}$$

Since

$$\mathbf{B} - \mathbf{A} \neq k \left(\mathbf{C} - \mathbf{A}\right), \tag{1.2.1.3}$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \tag{1.2.1.4}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \tag{1.2.1.5}$$

$$\implies rank(\mathbf{M}) = 2 \tag{1.2.1.6}$$

as the number of non zero rows is 2. The following code plots Fig. 1.2.1

codes/triangle/check_tri.py

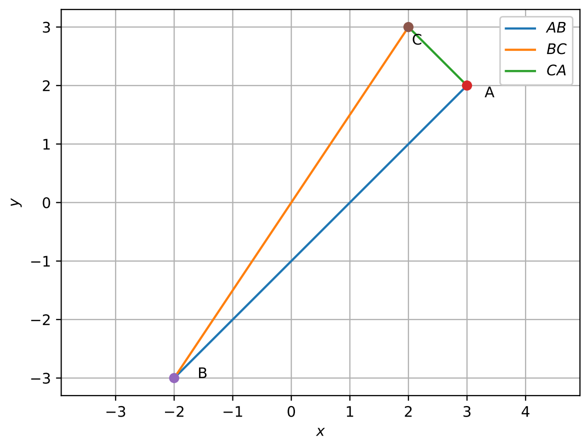

Fig. 1.2.1

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B}-\mathbf{A}\|^2 + \|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{C}\|^2 \tag{1.2.1.7}$$

which, from (1.1.2.10) can be expressed as

$$\|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \tag{1.2.1.8}$$

to obtain

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = 0 \tag{1.2.1.9}$$

after simplification. From (1.2.1.1) and (1.2.1.2), it is easy to verify that

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}-5 & -5\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 0 \tag{1.2.1.10}$$

satisfying (1.2.1.9). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.

2. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\6\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}-3\\-5\end{pmatrix}$.
Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

codes/triangle/area_tri.py

3. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix}5\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\-4\end{pmatrix}$. **Solution:** The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.2.1.4) as

$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.2.3.1}$$

The computation is done in **area_tri.py**

4. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix}-1.5\\3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-2\end{pmatrix}, \mathbf{R} = \begin{pmatrix}-3\\4\end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{vmatrix} \tag{1.2.4.1}$$

5. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.2.5.1}$$

as its vertices.
Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\| \tag{1.2.5.2}$$

For any two vectors $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.2.5.3}$$

The following code computes the area using the vector product.

codes/triangle/area_tri_vec.py

6. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix}3\\-5\\7\end{pmatrix}$ and $\begin{pmatrix}-1\\7\\-6\end{pmatrix}$, respectively, find the coordinates of the point $\mathbf{C}$.

Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.2.6.1}$$

Thus,

$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.2.6.2}$$

7. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.2.7.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.2.7

```
codes/triangle/triangle_3d.py
```

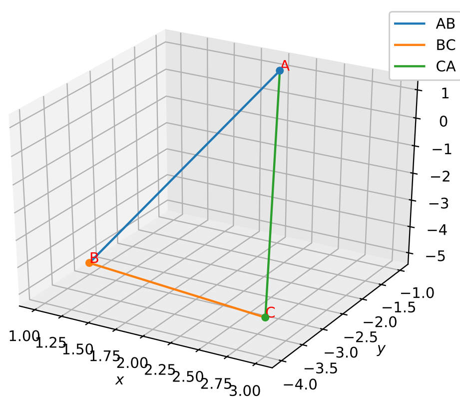

Fig. 1.2.7

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.2.7.2}$$

it is proved that the triangle is indeed right angled.

8. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{1.2.8.1}$$

the vertices of a right angled triangle?

1.3 Triangle Exercises

1. Draw the graphs of the equations

$$\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.3.1.1}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} - 12 = 0 \tag{1.3.1.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

2. In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
3. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
4. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find the equation of the median through the vertex $\mathbf{R}$.
5. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the equation and length of the altitude from the vertex $\mathbf{A}$.
6. Find the area of the triangle whose vertices are
   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
   b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
7. Find the area of the triangle formed by joining the mid points o the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
8. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
9. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.3.9.1}$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{1.3.9.2}$$

10. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the

vertices of $\triangle ABC$.

a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
c) Find the coordinates of the points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

11. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.3.11.1}$$
12. Show that the points
$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.3.12.1}$$
are the vertices of a right angled triangle.
13. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.
14. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.
15. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.
16. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$
17. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$
18. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.
19. Check whether
$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{1.3.19.1}$$
are the vertices of an isosceles triangle.

2 Quadrilateral

2.1 Construction Examples

1. Draw $ABCD$ with $AB = a = 4.5, BC = b = 5.5, CD = c = 4, AD = d = 6$ and $AC = e = 7$.
Solution: Fig. 2.1.1 shows a rough sketch of $ABCD$. Letting
$$\mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.1.1}$$
it is trivial to sketch $\triangle ABC$ from Problem 1.1.2. $\triangle ACD$ is can be obtained by rotating an equivalent triangle with AC on the x-axis by an angle θ with
$$\mathbf{D} = \begin{pmatrix} h \\ k \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.2}$$
and
$$\cos\theta = \frac{a^2+e^2-b^2}{2ae} \tag{2.1.1.3}$$
$$\sin\theta = \sqrt{1-\cos^2\theta} \tag{2.1.1.4}$$
The coordinates of the rotated triangle ACD are
$$\mathbf{D} = \mathbf{P}\begin{pmatrix} h \\ k \end{pmatrix} \tag{2.1.1.5}$$
$$\mathbf{A} = \mathbf{P}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.1.6}$$
$$\mathbf{C} = \mathbf{P}\begin{pmatrix} e \\ 0 \end{pmatrix} \tag{2.1.1.7}$$
where
$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.1.8}$$
The following code plots quadrilateral $ABCD$

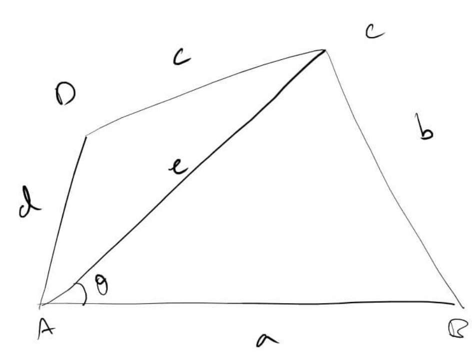

Fig. 2.1.1

in Fig. 2.1.1

```
codes/quad/draw_quad.py
```

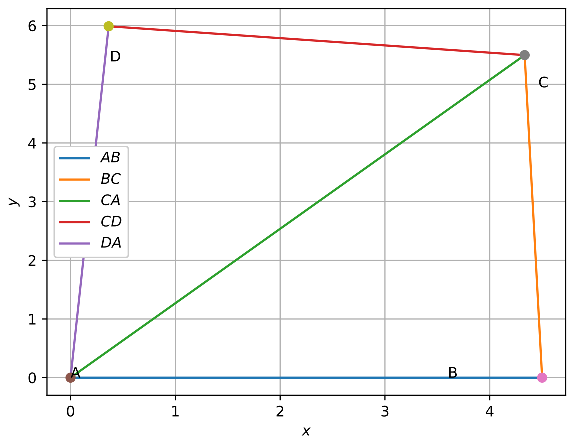

Fig. 2.1.1

2. Draw the parallelogram $MORE$ with $OR = 6, RE = 4.5$ and $EO = 7.5$.
Solution: Diagonals of a parallelogram bisect each other. Opposite sides of a parallelogram are equal and parallel .
3. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.
Solution: The diagonals of a kite are perpendicular to each other.
4. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
Solution: Diagonals of a rhombus bisect each other at right angles.

2.2 *Quadrilateral Examples*

1. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,
$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.2.1.1}$$
Hence, the diagonals AC and BD bisect each other. Also,
$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.2.1.2}$$
$\implies AC \perp BD$. Hence $ABCD$ is a square.
2. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.
3. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by
$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{2.2.3.1}$$
4. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors
$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{C} \tag{2.2.4.1}$$
$$\mathbf{A}-\mathbf{D} = \mathbf{B}-\mathbf{C} \tag{2.2.4.2}$$
$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \neq 0 \tag{2.2.4.3}$$
Hence, it is not a rectangle. The following code plots Fig. 2.2.4

```
codes/triangle/quad_3d.py
```

5. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
Solution: The area is given by
$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{2.2.5.1}$$

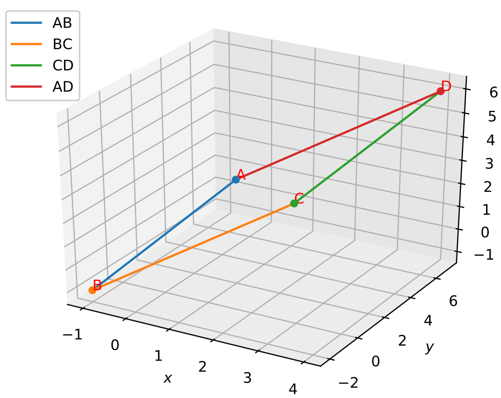

Fig. 2.2.4

2.3 Quadrilateral Geometry

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. $ABCD$ is a cyclic quadrilateral with
$$\angle A = 4y + 20 \tag{2.3.2.1}$$
$$\angle B = 3y - 5 \tag{2.3.2.2}$$
$$\angle C = -4x \tag{2.3.2.3}$$
$$\angle D = -7x + 5 \tag{2.3.2.4}$$
Find its angles.
3. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}$, $\begin{pmatrix}0\\7\end{pmatrix}$, $\begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.
4. Find the area of a rhombus if its vertices are $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}4\\5\end{pmatrix}$, $\begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\end{pmatrix}$ taken in order.
5. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.
6. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}$, $\begin{pmatrix}-3\\-5\end{pmatrix}$, $\begin{pmatrix}3\\-2\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$.
7. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
8. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a
   a) square?
   b) rectangle?
   c) rhombus?
9. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
10. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\-7\\1\end{pmatrix}$.
11. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix}-1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\\frac{1}{2}\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\-\frac{1}{2}\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-\frac{1}{2}\\4\end{pmatrix}$.
12. The two adjacent sides of a parallelogram are $\begin{pmatrix}2\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

3 Line

3.1 Examples: Geometry

1. Check whether –2 and 2 are zeroes of the polynomial $x + 2$.
2. Find a zero of the polynomial $p(x) = 2x + 1$.
3. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = 3x + 1, x = \frac{1}{3}$
   b) $p(x) = 5x - \pi, x = \frac{4}{5}$
   c) $p(x) = 5lx + m, x = -\frac{m}{l}$
   d) $p(x) = 2x + 1, x = \frac{1}{2}$
4. Find the zero of the polynomial in each of the following cases:
   a) $p(x) = x + 5$
   b) $p(x) = x–5$
   c) $p(x) = 2x + 5$
   d) $p(x) = 3x–2$
   e) $p(x) = 3x$
   f) $p(x) = ax, a \neq 0$

g) $p(x) = cx + d, c \neq 0, c, d$ are real numbers.

5. Find four different solutions of the equation

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{3.1.5.1}$$

6. Find two solutions for each of the following equations:

a) $\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12$

b) $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$

c) $\begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 4$

7. Draw the graph of

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7 \tag{3.1.7.1}$$

8. Draw the graphs of the following equations

a) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$

b) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$

c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$

d) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1$

e) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 4$

f) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$

9. Two rails are represented by the equations

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} – 4 = 0 \text{ and} \tag{3.1.9.1}$$

$$\begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} – 12 = 0. \tag{3.1.9.2}$$

Will the rails cross each other?

10. Check graphically whether the pair of equations

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \text{ and} \tag{3.1.10.1}$$

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \tag{3.1.10.2}$$

is consistent. If so, solve them graphically.

11. Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$\begin{pmatrix} 5 & -8 \end{pmatrix} \mathbf{x} = -1 \text{ and} \tag{3.1.11.1}$$

$$\begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\frac{3}{5} \tag{3.1.11.2}$$

12. Solve the following pair of equations

$$\begin{pmatrix} 7 & -15 \end{pmatrix} \mathbf{x} = 2 \tag{3.1.12.1}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 3 \tag{3.1.12.2}$$

13. Find all possibe solutions of

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} &= 8 \\ \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} &= 7 \end{aligned} \tag{3.1.13.1}$$

14. For which values of p does the pair of equations given below has unique solution?

$$\begin{aligned} \begin{pmatrix} 4 & p \end{pmatrix} \mathbf{x} &= -8 \\ \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} &= -2 \end{aligned} \tag{3.1.14.1}$$

15. For what values of k will the following pair of linear equations have infinitely many solutions?

$$\begin{aligned} \begin{pmatrix} k & 3 \end{pmatrix} \mathbf{x} &= k - 3 \\ \begin{pmatrix} 12 & k \end{pmatrix} \mathbf{x} &= k \end{aligned} \tag{3.1.15.1}$$

16. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{3.1.16.1}$$

Solution: $x = 2, y = 2, z = 1$.

17. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{3.1.17.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

18. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.1.18.1}$$

19. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

20. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

21. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such

that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \quad (3.1.21.1)$$

$$\implies |k| = \frac{7}{\sqrt{5}} \quad (3.1.21.2)$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \quad (3.1.21.3)$$

22. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix}2\\2\\-5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\\3\end{pmatrix}. \quad (3.1.22.1)$$

23. Find a unit vector in the direction of

$$\begin{pmatrix}1\\1\\-2\end{pmatrix}. \quad (3.1.23.1)$$

24. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \quad (3.1.24.1)$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix}3\\5\\4\end{pmatrix}, \quad (3.1.24.2)$$

25. Verify if $\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\6\end{pmatrix}$ are points on a line.
Solution: Refer to Problem 1.2.1.
26. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \quad (3.1.26.1)$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$
$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \quad (3.1.26.2)$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \quad (3.1.26.3)$$

which is the desired condition. The following code plots Fig. 3.1.26clearly showing that the above equation is the perpendicular bisector of AB.

codes/line/line_perp_bisect.py

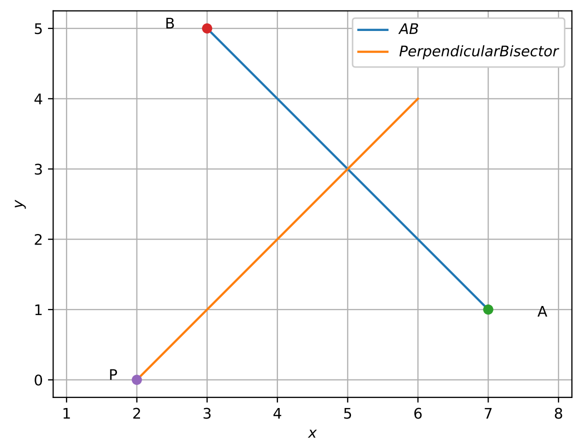

Fig. 3.1.26

27. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution: Choose $\mathbf{x} = \begin{pmatrix}0\\y\end{pmatrix}$ and follow the approach in Problem (3.1.26). Solve for y.
28. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \quad (3.1.28.1)$$

Then the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \quad (3.1.28.2)$$

divides AB in the ratio $k : 1$. For the given problem, $k = \frac{5}{8}$. The following code plots Fig. 3.1.28

codes/line/draw_section.py

29. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix}4\\-3\end{pmatrix}$ and

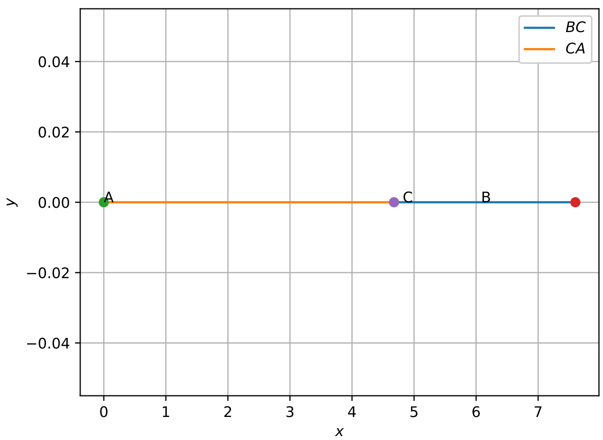

Fig. 3.1.28

$\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.

Solution: Using (3.1.28.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \tag{3.1.29.1}$$

30. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \tag{3.1.30.1}$$

Solution: Use (3.1.28.2).

31. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \tag{3.1.31.1}$$

Solution: Using (3.1.28.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.1.31.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.1.31.3}$$

32. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Solution: Let the corresponding point on the y-axis be$\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.1.28.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.1.32.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.1.32.2}$$

33. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.2.1.4),

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \tag{3.1.33.1}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \tag{3.1.33.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{3.1.33.3}$$

34. Find the direction vectors and slopes of the lines passing through the points
    a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
    b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.
    c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
    d) Making an inclination of 60° with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{3.1.34.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{3.1.34.2}$$

$$= \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \implies m = -\frac{3}{2} \tag{3.1.34.3}$$

b) The direction vector is

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.1.34.4}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies m = 0 \tag{3.1.34.5}$$

c) The direction vector is

$$\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix} \tag{3.1.34.6}$$

$$= \begin{pmatrix}1\\\infty\end{pmatrix} \implies m = \infty \tag{3.1.34.7}$$

d) The slope is $m = \tan 60^\circ = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{3.1.34.8}$$

35. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
Solution: The angle θ between two lines is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{3.1.35.1}$$

$$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{3.1.35.2}$$

$$\text{or } m_1 = \frac{5}{3} \tag{3.1.35.3}$$

36. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
Solution: Using (1.2.1.9)

$$\left\{\begin{pmatrix}-2\\6\end{pmatrix} - \begin{pmatrix}4\\8\end{pmatrix}\right\}^T \left\{\begin{pmatrix}8\\12\end{pmatrix} - \begin{pmatrix}x\\24\end{pmatrix}\right\} = 0 \tag{3.1.36.1}$$

which can be used to obtain x.

37. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3\\8\end{pmatrix}$ is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{3.1.37.1}$$

$$\implies \begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} - \lambda\begin{pmatrix}-3\\-6\end{pmatrix} \tag{3.1.37.2}$$

which can be expressed as

$$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{3.1.37.3}$$

$$\implies \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}T\\D\end{pmatrix} = -2 \tag{3.1.37.4}$$

$$\implies D = 2 + 2T \tag{3.1.37.5}$$

38. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$.
Solution: The line parallel to the x-axis has direction vector $\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.38.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}0\\1\end{pmatrix} \tag{3.1.38.2}$$

The following code plots Fig. 3.1.38

```
codes/line/line_parallel_axes.py
```

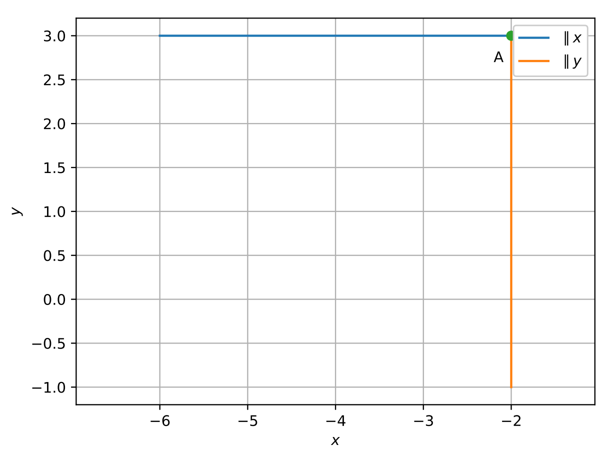

Fig. 3.1.38

39. Find the equation of the line through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$ with slope -4.
Solution: The direction vector is $\mathbf{m} = \begin{pmatrix}1\\-4\end{pmatrix}$.

Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{3.1.39.1}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{3.1.39.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{3.1.39.3}$$

$$\implies \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \tag{3.1.39.4}$$

40. Write the equation of the line through the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.
Solution: Use (3.1.38.1).
41. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and
a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.
Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.
a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$.
b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.
Use the above information get the equations for the lines.
42. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.
Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix} 5 & 2 \\ -4 & \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix} \tag{3.1.42.1}$$

43. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.
Solution: Using (3.1.37.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix} -1 & 0 \\ 2 & \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \tag{3.1.43.1}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.1.43.2}$$

44. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{3.1.44.1}$$

find the equation of the line.
Solution: The line can be expressed from (3.1.44.1) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+2\lambda \\ 5+4\lambda \\ -6+2\lambda \end{pmatrix} \tag{3.1.44.2}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \tag{3.1.44.3}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{3.1.44.4}$$

45. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution: See Problem 3.1.41. The line passes through the points $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
46. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 3.1.46, the foot of the perpendicular P is the intersection of the lines L and M. Thus,

$$\mathbf{n}^T \mathbf{P} = c \tag{3.1.46.1}$$

$$\mathbf{P} = \mathbf{A} + \lambda \mathbf{n} \tag{3.1.46.2}$$

$$\text{or, } \mathbf{n}^T \mathbf{P} = \mathbf{n}^T \mathbf{A} + \lambda \left\|\mathbf{n}\right\|^2 = c \tag{3.1.46.3}$$

$$\implies -\lambda = \frac{\mathbf{n}^T \mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \tag{3.1.46.4}$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda \mathbf{n} \tag{3.1.46.5}$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda| \left\|\mathbf{n}\right\| \tag{3.1.46.6}$$

From (3.1.46.4) and (3.1.46.6)

$$\|\mathbf{P} - \mathbf{A}\| = \frac{\left|\mathbf{n}^T \mathbf{A} - c\right|}{\|\mathbf{n}\|} \tag{3.1.46.7}$$

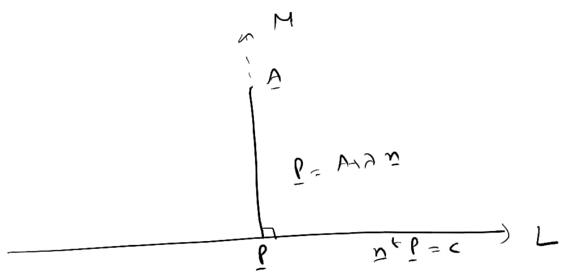

Fig. 3.1.46

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \tag{3.1.46.8}$$

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\|\mathbf{n}\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \tag{3.1.46.9}$$

$$= \pm 4 \sec 15^\circ \tag{3.1.46.10}$$

where

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.1.46.11}$$

This follows from (**??**), where

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.1.46.12}$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{3.1.46.13}$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{3.1.46.14}$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \tag{3.1.46.15}$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix} \mathbf{c} = \pm 4 \sec 15^\circ \tag{3.1.46.16}$$

47. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \tag{3.1.47.1}$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (3.1.39.3) and (3.1.39.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} 273 \\ 32 \end{pmatrix} \tag{3.1.47.2}$$

$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = -197 \tag{3.1.47.3}$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \tag{3.1.47.4}$$

48. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 10 = 0. \tag{3.1.48.1}$$

Find its

a) slope,
b) x - and y-intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{3.1.48.2}$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \tag{3.1.48.3}$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

49. Find the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T \mathbf{b} = 1. \tag{3.1.49.1}$$

Solution: In Fig. 3.1.49, from the cosine formula in (1.1.7.9)

$$\cos\theta = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 - \|\mathbf{A} - \mathbf{C}\|^2}{2\|\mathbf{A} - \mathbf{B}\|\|\mathbf{B} - \mathbf{C}\|} \tag{3.1.49.2}$$

Letting $\mathbf{a} = \mathbf{A} - \mathbf{B}, \mathbf{b} = \mathbf{B} - \mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.49.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.49.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.49.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{3.1.49.6}$$

$$= \frac{1}{2} \tag{3.1.49.7}$$

$$\implies \theta = 60^\circ \tag{3.1.49.8}$$

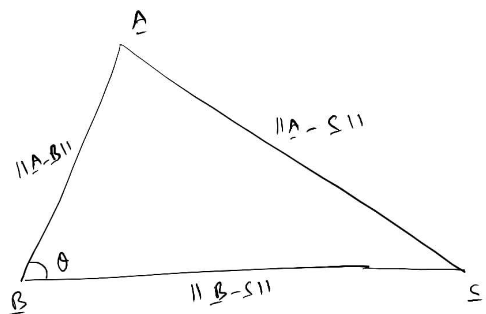

Fig. 3.1.49

50. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \tag{3.1.50.1}$$

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \tag{3.1.50.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{3.1.50.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{3.1.50.4}$$

51. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3 \tag{3.1.51.1}$$

and passes through the point $\begin{pmatrix}1\\-2\end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix}2\\1\end{pmatrix} \tag{3.1.51.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix}2 & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\-2\end{pmatrix}\right) = 0 \tag{3.1.51.3}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.51.4}$$

52. Find the distance of the point $\begin{pmatrix}3\\-5\end{pmatrix}$ from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 26 \tag{3.1.52.1}$$

Solution: Use (3.1.46.7).

53. If the lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{3.1.53.1}$$

$$\begin{pmatrix}5 & k\end{pmatrix}\mathbf{x} = 3 \tag{3.1.53.2}$$

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.1.53.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix}2 & 1 & 3\\5 & k & 3\\3 & -1 & 2\end{pmatrix} \xleftrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix}2 & 1 & 3\\3 & -1 & 2\\5 & k & 3\end{pmatrix} \tag{3.1.53.4}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftrightarrow 2R_2-3R_1} \begin{pmatrix}2 & 1 & 3\\0 & -5 & -5\\0 & 2k-5 & -9\end{pmatrix} \tag{3.1.53.5}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{5}} \begin{pmatrix}2 & 1 & 3\\0 & 1 & 1\\0 & 2k-5 & -9\end{pmatrix} \tag{3.1.53.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(2k-5)R_2} \begin{pmatrix}2 & 1 & 3\\0 & 1 & 1\\0 & 0 & -2k-4\end{pmatrix} \tag{3.1.53.7}$$

$$\implies k = -2 \tag{3.1.53.8}$$

54. Find the distance of the line

$$L_1 : \quad \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.54.1}$$

from the point $\begin{pmatrix}4\\1\end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-

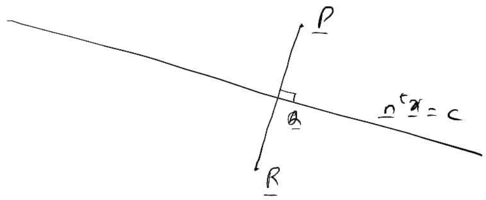

Fig. 3.1.55

axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan 135° \end{pmatrix} \tag{3.1.54.2}$$

Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.54.3}$$

$$\implies \mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{3.1.54.4}$$

$$\text{or, } \left\| \mathbf{P} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = d = |\lambda| \, \|\mathbf{m}\| \tag{3.1.54.5}$$

Since $\mathbf{P}$ lies on L_1, from (3.1.54.1),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{P} = 0 \tag{3.1.54.6}$$

Substituting from the above in (3.1.54.3),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{3.1.54.7}$$

$$\implies \lambda = \frac{\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m}}{17} \tag{3.1.54.8}$$

substituting $|\lambda|$ in (3.1.54.5) gives the desired answer.

55. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the line

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = -4. \tag{3.1.55.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{3.1.55.2}$$

$$\mathbf{n}^T \mathbf{Q} = c \tag{3.1.55.3}$$

$$\mathbf{m}^T \mathbf{R} = \mathbf{m}^T \mathbf{P} \tag{3.1.55.4}$$

where $\mathbf{m}$ is the direction vector of L. From (3.1.55.2) and (3.1.55.3),

$$\mathbf{n}^T \mathbf{R} = 2c - \mathbf{n}^T \mathbf{P} \tag{3.1.55.5}$$

From (3.1.55.5) and (3.1.55.4),

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{R} = \begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix}^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{3.1.55.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{3.1.55.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I} \tag{3.1.55.8}$$

Noting that

$$\begin{pmatrix} \mathbf{m} & -\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.1.55.9}$$

(3.1.55.6) can be expressed as

$$\mathbf{V}^T \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]^T \mathbf{P} + \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{3.1.55.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^{-1} \right]^T \mathbf{P} + \mathbf{V} \begin{pmatrix} 0 \\ 2c \end{pmatrix} \tag{3.1.55.11}$$

$$= \mathbf{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{V}^T \mathbf{P} + 2c\mathbf{n} \tag{3.1.55.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}} \mathbf{P} + c \frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{3.1.55.13}$$

The following code plots Fig. 3.1.55 while computing the reflection

```
codes/line/line_reflect.py
```

56. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Obtain

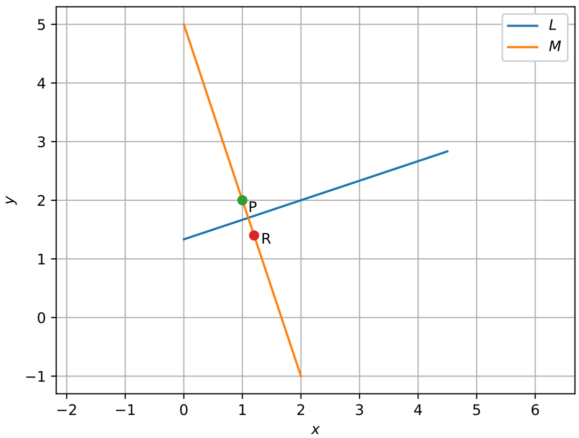

Fig. 3.1.55

its equation.

$$L_1: \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.1.56.1}$$
$$L_2: \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.1.56.2}$$

Solution: Let

$$L: \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.56.3}$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \tag{3.1.56.4}$$
$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \tag{3.1.56.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (3.1.56.1), (3.1.56.4) and (3.1.56.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \tag{3.1.56.6}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \tag{3.1.56.7}$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \tag{3.1.56.8}$$
$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \tag{3.1.56.9}$$
$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \tag{3.1.56.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{3.1.56.11}$$

57. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.57.1}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.57.2}$$

are equal is a straight line.
Solution: Using (3.1.46.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{3.1.57.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right| \tag{3.1.57.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right) \tag{3.1.57.5}$$

leading to the possible lines

$$L_1: \quad \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.1.57.6}$$
$$L_2: \quad \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{3} \tag{3.1.57.7}$$

58. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix}1\\-3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{3.1.58.1}$$

Solution: The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

59. Show that the points $\mathbf{A} = \begin{pmatrix}-2\\3\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7\\0\\-1\end{pmatrix}$ are collinear.
Solution: Forming the matrix in (1.2.1.4)

$$\mathbf{M} = \begin{pmatrix}3 & -1 & -2\\9 & -3 & -6\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix}3 & -1 & -2\\0 & 0 & 0\end{pmatrix} \tag{3.1.59.1}$$

$\implies$ $rank(\mathbf{M}) = 1$. The following code plots Fig. 3.1.59 showing that the points are collinear.

```
codes/line/collinear_3d.py
```

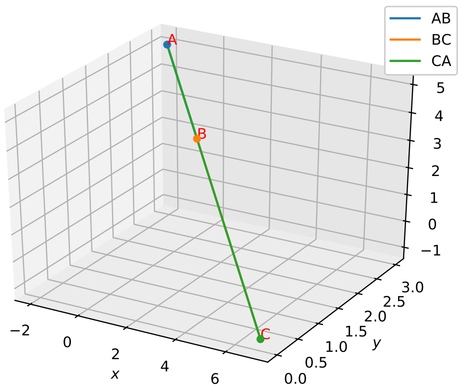

Fig. 3.1.59

60. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.
Solution: Use the approach in Problem (3.1.59).
61. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{3.1.61.1}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \tag{3.1.61.2}$$

respectively.
62. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3
a) internally, and
b) externally.
Solution: Use (3.1.28.2).
63. Prove that the three points $\begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix}$ are collinear.
Solution: Use the approach in Problem 3.1.59.
64. Find the ratio in which the line segment joining the points $\begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$ is divided by the YZ-plane.
Solution: Use (3.1.28.2). The YZ-plane has points $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$.
65. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.
Solution: Use the approach in Problem 3.1.26.
66. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{3.1.66.1}$$
$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{3.1.66.2}$$

find $\mathbf{R}$, which divides PQ in the ratio 2 : 1
a) internally,
b) externally.
Solution: Use (3.1.28.2).
67. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Solution: Use (3.1.49.6)
68. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{3.1.68.1}$$
$$\mathbf{x} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{3.1.68.2}$$

Solution: The direction vectors of the lines are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$. Using(3.1.49.6), the angle between the lines can be obtained.
69. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{3.1.69.1}$$
$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{3.1.69.2}$$

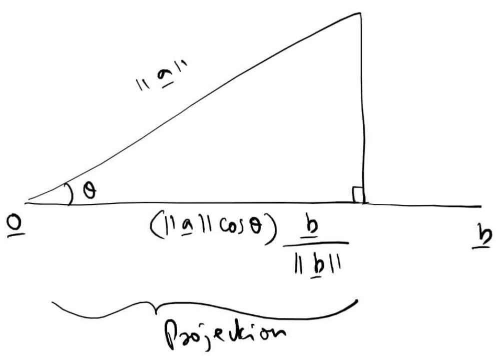

Fig. 3.1.71

Solution: From Problem 3.1.44, the direction vectors of the lines can be expressed as $\begin{pmatrix}3\\5\\4\end{pmatrix}$ and $\begin{pmatrix}1\\1\\2\end{pmatrix}$. The angle between them can then be obtained from (3.1.49.6).

70. If $\mathbf{a} = \begin{pmatrix}5\\-1\\-3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\3\\-5\end{pmatrix}$, then show that the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are perpendicular.
Solution: Use (1.2.1.9).

71. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix}2\\3\\2\end{pmatrix} \tag{3.1.71.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix}1\\2\\1\end{pmatrix}. \tag{3.1.71.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. 3.1.71. It has magnitude $\|\mathbf{a}\|\cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$\left(\|\mathbf{a}\|\cos\theta\right)\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)\|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{3.1.71.3}$$

72. Find $\|\mathbf{a}-\mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{3.1.72.1}$$

Solution:

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \tag{3.1.72.2}$$

73. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = 8, \tag{3.1.73.1}$$

then find $\mathbf{x}$.
Solution:

$$(\mathbf{x}-\mathbf{a})(\mathbf{x}+\mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{3.1.73.2}$$
$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{3.1.73.3}$$

74. Given

$$\mathbf{a} = \begin{pmatrix}2\\1\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\5\\-2\end{pmatrix}, \tag{3.1.74.1}$$

find $\|\mathbf{a}\times\mathbf{b}\|$.
Solution: Use (1.2.5.3).

75. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\2\\3\end{pmatrix}. \tag{3.1.75.1}$$

Solution: If $\mathbf{x}$ is the desired vector,

$$(\mathbf{a}+\mathbf{b})^T\mathbf{x} = 0 \tag{3.1.75.2}$$
$$(\mathbf{a}-\mathbf{b})^T\mathbf{x} = 0 \tag{3.1.75.3}$$

resulting in the matrix equation

$$\begin{pmatrix}2 & 3 & 4\\0 & -1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{3.1.75.4}$$

Performing row operations,

$$\begin{pmatrix}2 & 3 & 4\\0 & -1 & -2\end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_1\leftarrow R_1+3R_2} \begin{pmatrix}2 & 0 & -2\\0 & -1 & -2\end{pmatrix} \tag{3.1.75.5}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1}{2}} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.75.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.75.7}$$

76. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are collinear.
Solution: See Problem 3.1.59.
77. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \tag{3.1.77.1}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \tag{3.1.77.2}$$

$$\because -2\left(\mathbf{A} - \mathbf{B}\right) = \mathbf{C} - \mathbf{D}, \tag{3.1.77.3}$$

$\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.
78. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{3.1.78.1}$$

Then,

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \\ &\quad + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \end{aligned} \tag{3.1.78.2}$$

which reduces to

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{3.1.78.3}$$

using (3.1.78.1)
79. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \tag{3.1.79.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{3.1.79.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (3.1.79.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{3.1.79.3}$$

$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{3.1.79.4}$$

$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{3.1.79.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{3.1.79.6}$$

80. Let $\boldsymbol{\alpha} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.80.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{3.1.80.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.80.3}$$

This process is known as *Gram-Schmidth orthogonalization*.
81. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90° \\ \cos 60° \\ \cos 30° \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.1.81.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.
82. Find the distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. 3.1.82 From Fig. 3.1.82, the distance is

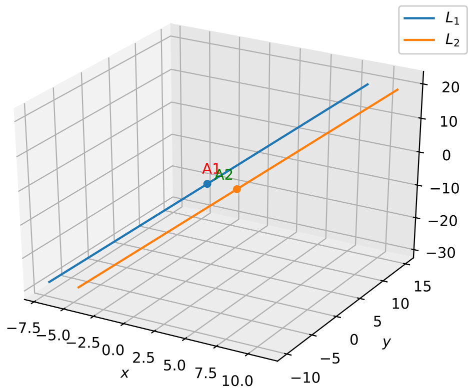

Fig. 3.1.82

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{3.1.82.3}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{3.1.82.4}$$

83. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{3.1.83.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.83.2}$$

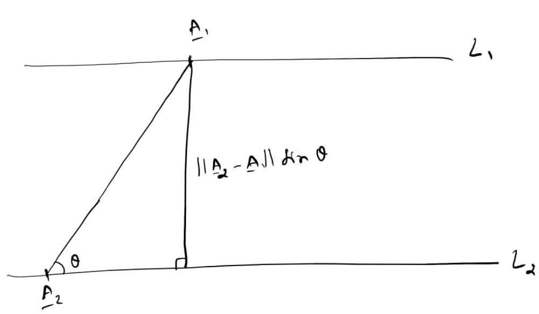

Fig. 3.1.82

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{3.1.83.3}$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{3.1.83.4}$$

$$\implies \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{3.1.83.5}$$

$$\implies \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.1.83.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \tag{3.1.83.7}$$

$$\xleftrightarrow[R_3 = 2R_1 - R_3]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \tag{3.1.83.8}$$

$$\xleftrightarrow{R_3 = 3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \tag{3.1.83.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. 3.1.83

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{3.1.83.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \tag{3.1.83.11}$$

The distance from $\mathbf{A}_1$ to the above line is then

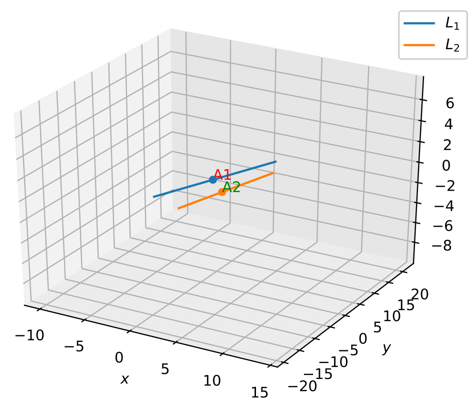

Fig. 3.1.83

obtained using (3.1.46.7) as

$$\frac{\left|\mathbf{n}^T\left(\mathbf{A}_2-\mathbf{A}_1\right)\right|}{\|\mathbf{n}\|}=\frac{\left|\left(\mathbf{A}_2-\mathbf{A}_1\right)^T\left(\mathbf{m}_1\times\mathbf{m}_2\right)\right|}{\|\mathbf{m}_1\times\mathbf{m}_2\|} \tag{3.1.83.12}$$

84. Find the distance of the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x}-6=0 \tag{3.1.84.1}$$

from the origin.
Solution: From (3.1.46.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|}=\frac{6}{\sqrt{2^2+3^2+4^2}} \tag{3.1.84.2}$$

$$=\frac{6}{\sqrt{29}} \tag{3.1.84.3}$$

85. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n}=\begin{pmatrix}2\\-3\\4\end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x}-6=0 \tag{3.1.85.1}$$

86. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x}=1. \tag{3.1.86.1}$$

Solution: The normal vector is

$$\mathbf{n}=\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.86.2}$$

$$\because \|\mathbf{n}\|=7, \tag{3.1.86.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|}=\frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{3.1.86.4}$$

87. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x}-6=0 \tag{3.1.87.1}$$

Solution: The normal vector is

$$\mathbf{n}=\begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.87.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda\mathbf{n}$. Substituting in (3.1.87.1),

$$\lambda\|\mathbf{n}\|^2=6\implies\lambda=\frac{6}{\|\mathbf{n}\|^2}=\frac{6}{29} \tag{3.1.87.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29}\begin{pmatrix}2\\-3\\4\end{pmatrix} \tag{3.1.87.4}$$

88. Find the equation of the plane which passes through the point $\mathbf{A}=\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (3.1.39.3), the equation of the plane is

$$\mathbf{n}^T\left(\mathbf{x}-\mathbf{A}\right)=0 \tag{3.1.88.1}$$

$$\implies\begin{pmatrix}2\\3\\-1\end{pmatrix}\mathbf{x}=\begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}5\\2\\-4\end{pmatrix} \tag{3.1.88.2}$$

$$=20 \tag{3.1.88.3}$$

89. Find the equation of the plane passing through $\boldsymbol{R}=\begin{pmatrix}2\\5\\-3\end{pmatrix}, \boldsymbol{S}=\begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\boldsymbol{T}=\begin{pmatrix}5\\3\\-3\end{pmatrix}$.

Solution: If the equation of the plane be

$$\mathbf{n}^T\mathbf{x} = c, \tag{3.1.89.1}$$

$$\mathbf{n}^T\mathbf{R} = \mathbf{n}^T\mathbf{S} = \mathbf{n}^T\mathbf{T} = c, \tag{3.1.89.2}$$

$$\implies \begin{pmatrix}\mathbf{R}-\mathbf{S} & \mathbf{S}-\mathbf{T}\end{pmatrix}^T \mathbf{n} = 0 \tag{3.1.89.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix} 4 & 8 & -8 \\ -7 & -6 & 8 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 2 & -2 \\ -7 & -6 & 8 \end{pmatrix} \tag{3.1.89.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 8 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.89.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \end{pmatrix} \tag{3.1.89.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and} \tag{3.1.89.7}$$

$$c = \mathbf{n}^T\mathbf{T} = 7 \tag{3.1.89.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{n} = 7 \tag{3.1.89.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R}-\mathbf{S}) \times (\mathbf{S}-\mathbf{T}) \tag{3.1.89.10}$$

The equation of the plane is then obtained from (3.1.39.3) as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{T}) = [(\mathbf{R}-\mathbf{S}) \times (\mathbf{S}-\mathbf{T})]^T(\mathbf{x}-\mathbf{T}) = 0 \tag{3.1.89.11}$$

90. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
Solution: From the given information, the plane passes through the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ respectively. The equation can be obtained using Problem 3.1.89.
91. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{3.1.91.1}$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = -5 \tag{3.1.91.2}$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & -5 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.91.3}$$

$$\xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 23 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{3.1.91.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.91.5}$$

Thus, $\begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.91.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ -22 & 18 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + 22R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.91.7}$$

$$\xleftrightarrow{R_1 = 13R_1 - R_2} \begin{pmatrix} 13 & 0 & -10 \\ 0 & -26 & 23 \end{pmatrix} \tag{3.1.91.8}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{10}{13} \\ \frac{23}{26} \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 23 \\ 26 \end{pmatrix} \tag{3.1.91.9}$$

Since the plane passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using

(3.1.39.3),

$$\begin{pmatrix}20 & 23 & 26\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 0 \tag{3.1.91.10}$$

$$\implies \begin{pmatrix}20 & 23 & 26\end{pmatrix}\mathbf{x} = 69 \tag{3.1.91.11}$$

Alternatively, the plane passing through the intersection of (3.1.91.1) and (3.1.91.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{3.1.91.12}$$

Substituting $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 6 - 5\lambda \tag{3.1.91.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{3.1.91.14}$$

$$\implies \lambda = \frac{3}{14} \tag{3.1.91.15}$$

Substituting this value of λ in (3.1.91.12) yields the equation of the plane.

92. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{3.1.92.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{3.1.92.2}$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (3.1.83), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T(\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{3.1.92.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix}-3\\1\\5\end{pmatrix} - \begin{pmatrix}-1\\2\\5\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\end{pmatrix} \tag{3.1.92.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix}-1 & 2 & 5\\-3 & 1 & 5\end{pmatrix} \xleftrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix}-1 & 2 & 5\\0 & 1 & 2\end{pmatrix} \tag{3.1.92.5}$$

$$\xleftrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}-1\\2\\5\end{pmatrix} \times \begin{pmatrix}-3\\1\\5\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{3.1.92.6}$$

The LHS of (3.1.92.3) is

$$\begin{pmatrix}-2 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0 \tag{3.1.92.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left|\mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2\right| = 0 \tag{3.1.92.8}$$

93. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.93.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.93.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (3.1.49.6).

94. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.1.94.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{3.1.94.2}$$

Solution: See Problem (3.1.93).

95. Find the distance of a point $\begin{pmatrix}2\\5\\-3\end{pmatrix}$ from the plane

$$\begin{pmatrix}6 & -3 & 2\end{pmatrix}\mathbf{x} = 4 \tag{3.1.95.1}$$

Solution: Use (3.1.46.7).

96. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{3.1.96.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{3.1.96.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225}\times\sqrt{49}} = \frac{8}{21} \quad (3.1.96.3)$$

Thus, the desired angle is $90° - \theta$.

97. Find the equation of the plane that contains the point $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix}2 & 3 & -2\end{pmatrix}\mathbf{x} = 5 \quad (3.1.97.1)$$
$$\begin{pmatrix}1 & 2 & -3\end{pmatrix}\mathbf{x} = 8 \quad (3.1.97.2)$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix}2\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\2\\-3\end{pmatrix} \quad (3.1.97.3)$$

The equation of the plane is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \quad (3.1.97.4)$$

98. Find the distance between the point $\mathbf{P} = \begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \boldsymbol{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$.

Solution: Find the equation of the plane using Problem 3.1.89. Find the distance using (3.1.46.7).

99. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.

Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \quad (3.1.99.1)$$
$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \quad (3.1.99.2)$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \quad (3.1.99.3)$$

3.2 Complex Numbers

1. Find $\begin{pmatrix}5\\-3\end{pmatrix}^3$
2. Find $\begin{pmatrix}-\sqrt{3}\\\sqrt{2}\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\-1\end{pmatrix}$.
3. Find the multiplicative inverse of $\begin{pmatrix}2\\-3\end{pmatrix}$.
4. Find
   a) $\begin{pmatrix}5\\\sqrt{2}\end{pmatrix}\begin{pmatrix}1\\-2\sqrt{3}\end{pmatrix}$.
   b) $\begin{pmatrix}0\\1\end{pmatrix}^{-35}$.
   c) Show that the polar representation of $\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$ is $2\angle 60°$.
5. Convert the complex number $-\frac{16}{\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}$
6. Find the conjugate of $\frac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$.
7. Find the modulus and argument of the complex numbers
   a) $\frac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}}$.
   b) $\frac{1}{\begin{pmatrix}1\\1\end{pmatrix}}$.
8. Find θ such that

$$\frac{\begin{pmatrix}3\\2\sin\theta\end{pmatrix}}{\begin{pmatrix}1\\-2\sin\theta\end{pmatrix}} \quad (3.2.8.1)$$

is purely real.

9. Convert the complex number
$$\mathbf{z} = \frac{\begin{pmatrix}-1\\1\end{pmatrix}}{\begin{pmatrix}\cos\frac{\pi}{3}\\ \sin\frac{\pi}{3}\end{pmatrix}} \tag{3.2.9.1}$$
in the polar form.
10. Simplify
$$\mathbf{z} = \left(\frac{1}{\begin{pmatrix}1\\-4\end{pmatrix}} - \frac{2}{\begin{pmatrix}2\\1\end{pmatrix}}\right)\frac{\begin{pmatrix}3\\-4\end{pmatrix}}{\begin{pmatrix}5\\1\end{pmatrix}} \tag{3.2.10.1}$$
11. Convert the following in the polar form:
   a) $\dfrac{\begin{pmatrix}1\\7\end{pmatrix}}{\begin{pmatrix}2\\-1\end{pmatrix}^2}$.
   b) $\dfrac{\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\-2\end{pmatrix}}$.
12. If $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}1\\1\end{pmatrix}$, find $\left\|\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}\right\|$
13. Let $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$. Find
   a) $\text{Re}\left(\frac{\mathbf{z}_1\mathbf{z}_2}{\mathbf{z}_1^*}\right)$.
   b) $\text{Im}\left(\frac{1}{\mathbf{z}_1\mathbf{z}_1^*}\right)$.
14. Find the modulus and argument of the complex number $\dfrac{\begin{pmatrix}1\\2\end{pmatrix}}{\begin{pmatrix}1\\-3\end{pmatrix}}$.
15. Find the real numbers x, y such that $\begin{pmatrix}x\\-y\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix}$ is the conjugate of $\begin{pmatrix}-6\\-24\end{pmatrix}$.
16. Find the modulus of $\dfrac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}} - \dfrac{\begin{pmatrix}1\\-1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}}$.

3.3 Points and Vectors

1. Find the distance between the following pairs of points
   a)
$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{3.3.1.1}$$
   b)
$$\begin{pmatrix}-5\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \tag{3.3.1.2}$$
   c)
$$\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}-1\\b\end{pmatrix} \tag{3.3.1.3}$$
2. Find the distance between the points
$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix} \tag{3.3.2.1}$$
3. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
4. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.
   a)
$$\begin{pmatrix}-1\\-2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-3\\0\end{pmatrix} \tag{3.3.4.1}$$
   b)
$$\begin{pmatrix}-3\\5\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.3.4.2}$$
   c)
$$\begin{pmatrix}4\\5\end{pmatrix}, \begin{pmatrix}7\\6\end{pmatrix}, \tag{3.3.4.3}$$
$$\begin{pmatrix}4\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \tag{3.3.4.4}$$
5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$.
6. Find the point on the x-axis which is equidistant from
$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{3.3.6.1}$$
7. Find the values of y for which the distance between the points
$$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{3.3.7.1}$$

is 10 units.

8. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix}, \frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix}, \frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}. \tag{3.3.8.1}$$

Also, show that they are mutually perpendicular to each other.

9. For

$$\mathbf{a} = \begin{pmatrix}2\\2\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \mathbf{c} = \begin{pmatrix}3\\1\\0\end{pmatrix}, \tag{3.3.9.1}$$

$(\mathbf{a} + \lambda\mathbf{b}) \perp \mathbf{c}$. Find λ.

10. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix}1\\-7\\7\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\2\end{pmatrix}. \tag{3.3.10.1}$$

11. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix}3\\2\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\2\\-2\end{pmatrix}. \tag{3.3.11.1}$$

12. If $\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\-1\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

13. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$,

14. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.

15. Let $\mathbf{a} = \begin{pmatrix}1\\4\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.

16. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.

17. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T \left(\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\1\\0\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\0\\1\end{pmatrix}^T \left(\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix}\right) \tag{3.3.17.1}$$

is

a) 0
b) -1
c) 1
d) 3

18. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively.

19. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.

20. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}3\\5\\6\end{pmatrix}$.

21. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.

22. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.

23. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{3.3.23.1}$$

24. Find the projection of the vector
$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{3.3.24.1}$$
on the vector
$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{3.3.24.2}$$
25. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.
26. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

3.4 Points on a Line

1. Find the coordinates of the point which divides the join of
$$\begin{pmatrix}-1\\7\end{pmatrix}, = \begin{pmatrix}4\\-3\end{pmatrix} \tag{3.4.1.1}$$
in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.
3. Find the ratio in which the line segment joining the points $\begin{pmatrix}-3\\10\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$ is divided by $\begin{pmatrix}-1\\6\end{pmatrix}$.
4. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix}1\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\5\end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
5. If $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}4\\y\end{pmatrix}$, $\begin{pmatrix}x\\6\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
6. If $\mathbf{A} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\-4\end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
7. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.
8. Determine if the points
$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{3.4.8.1}$$
are collinear.
9. By using the concept of equation of a line, prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.
10. Find the value of x for which the points $\begin{pmatrix}x\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ are collinear.
11. In each of the following, find the value of k for which the points are collinear
    a) $\begin{pmatrix}7\\-2\end{pmatrix}$, $\begin{pmatrix}5\\1\end{pmatrix}$, $\begin{pmatrix}3\\k\end{pmatrix}$
    b) $\begin{pmatrix}8\\1\end{pmatrix}$, $\begin{pmatrix}k\\-4\end{pmatrix}$, $\begin{pmatrix}2\\-5\end{pmatrix}$
12. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}7\\0\end{pmatrix}$ are collinear.
13. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
14. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
15. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\8\\7\end{pmatrix}$ are collinear.

3.5 Lines and Planes

1. Sketch the following lines

    a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 9.35$
    b) $\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\mathbf{x} = 10$
    c) $\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 6$
    d) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 0$
    e) $\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0$
    f) $\begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} = -2$
    g) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$
    h) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 5$

2. Write four solutions for each of the following equations
    a) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$
    b) $\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$

c) $\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$

3. Check which of the following are solutions of the equation

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 4 \tag{3.5.3.1}$$

a) $\begin{pmatrix}0\\2\end{pmatrix}$

b) $\begin{pmatrix}2\\0\end{pmatrix}$

c) $\begin{pmatrix}4\\0\end{pmatrix}$

d) $\begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix}$

e) $\begin{pmatrix}1\\1\end{pmatrix}$

4. Find the value of k, if $\begin{pmatrix}2\\1\end{pmatrix}$ is a solution of the equation

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = k \tag{3.5.4.1}$$

5. Draw the graphs of the following equations

a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$

b) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2$

c) $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 0$

d) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3$

e) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$

f) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$

g) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$

h) $\begin{pmatrix}7 & -3\end{pmatrix}\mathbf{x} = 2$

i) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$

j) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$

k) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$

l) $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6$

6. Give the equations of two lines passing through $\begin{pmatrix}2\\14\end{pmatrix}$. How many more such lines are there, and why?

7. If the point $\begin{pmatrix}3\\4\end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a

8. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{aligned}\begin{pmatrix}5 & -4\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}7 & 6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.5.8.1}$$

b)

$$\begin{aligned}\begin{pmatrix}9 & 3\end{pmatrix}\mathbf{x} &= -12\\ \begin{pmatrix}18 & 6\end{pmatrix}\mathbf{x} &= -24\end{aligned} \tag{3.5.8.2}$$

c)

$$\begin{aligned}\begin{pmatrix}6 & -3\end{pmatrix}\mathbf{x} &= -10\\ \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} &= -9\end{aligned} \tag{3.5.8.3}$$

9. Find out whether the following pair of linear equations are consistent, or inconsistent.

a)

$$\begin{aligned}\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 7\end{aligned} \tag{3.5.9.1}$$

b)

$$\begin{aligned}\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{3.5.9.2}$$

c)

$$\begin{aligned}\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\end{pmatrix}\mathbf{x} &= 7\\ \begin{pmatrix}9 & -10\end{pmatrix}\mathbf{x} &= 14\end{aligned} \tag{3.5.9.3}$$

d)

$$\begin{aligned}\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} &= 11\\ \begin{pmatrix}-10 & 6\end{pmatrix}\mathbf{x} &= -22\end{aligned} \tag{3.5.9.4}$$

e)

$$\begin{aligned}\begin{pmatrix}\frac{4}{3} & 2\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 12\end{aligned} \tag{3.5.9.5}$$

10. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{aligned}\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} &= 10\end{aligned} \tag{3.5.10.1}$$

b)

$$\begin{aligned}\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} &= 16\end{aligned} \tag{3.5.10.2}$$

c)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6$$
$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.10.3}$$

d)

$$\begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 2$$
$$\begin{pmatrix} 4 & -4 \end{pmatrix} \mathbf{x} = 5 \tag{3.5.10.4}$$

11. Given the linear equation $\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

a) intersecting lines
c) coincident lines
b) parallel lines

12. Find the intersection of the following lines

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 14$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.12.1}$$

b)

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = 6 \tag{3.5.12.2}$$

c)

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} 9 & -3 \end{pmatrix} \mathbf{x} = 9 \tag{3.5.12.3}$$

d)

$$\begin{pmatrix} 0.2 & 0.3 \end{pmatrix} \mathbf{x} = 1.3$$
$$\begin{pmatrix} 0.4 & 0.5 \end{pmatrix} \mathbf{x} = 2.3 \tag{3.5.12.4}$$

e)

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \end{pmatrix} \mathbf{x} = 0$$
$$\begin{pmatrix} \sqrt{3} & \sqrt{8} \end{pmatrix} \mathbf{x} = 0 \tag{3.5.12.5}$$

f)

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{3} \end{pmatrix} \mathbf{x} = -2$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{13}{6} \tag{3.5.12.6}$$

13. Find m if

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 11$$
$$\begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} = -24 \tag{3.5.13.1}$$
$$\begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = -3$$

14. Solve the following

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.14.1}$$

b)

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 10$$
$$\begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 2 \tag{3.5.14.2}$$

c)

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 9 & -2 \end{pmatrix} \mathbf{x} = 7 \tag{3.5.14.3}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \end{pmatrix} \mathbf{x} = -1$$
$$\begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \mathbf{x} = 3 \tag{3.5.14.4}$$

15. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} 3 & -9 \end{pmatrix} \mathbf{x} = 2 \tag{3.5.15.1}$$

b)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 8 \tag{3.5.15.2}$$

c)

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 20$$
$$\begin{pmatrix} 6 & -10 \end{pmatrix} \mathbf{x} = 40 \tag{3.5.15.3}$$

d)

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 7$$
$$\begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{3.5.15.4}$$

16. For which alues of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 7$$
$$\begin{pmatrix} a-b & a+b \end{pmatrix} \mathbf{x} = 3a+b-2 \tag{3.5.16.1}$$

17. For which value of k will the following pair of linear equations have no solution?

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 1$$
$$\begin{pmatrix} 2k-1 & k-1 \end{pmatrix} \mathbf{x} = 2k+1 \tag{3.5.17.1}$$

18. Solve the following pair of linear equations
$$\begin{pmatrix} 8 & 5 \end{pmatrix} \mathbf{x} = 9$$
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.18.1}$$
19. Solve the following pair of linear equations
$$\begin{pmatrix} 158 & -378 \end{pmatrix} \mathbf{x} = -74$$
$$\begin{pmatrix} -378 & 152 \end{pmatrix} \mathbf{x} = -604 \tag{3.5.19.1}$$
20. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$.
21. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
22. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.
23. Write the equations for the x and y axes.
24. Find the equation of the line satisfying the following conditions
    a) passing through the point $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ with slope $\frac{1}{2}$.
    b) passing through the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.
    c) passing through the point $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
    d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.
    e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.
    f) passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.
    g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.
25. Find the equation of the line passing through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$.
26. Find the direction vectors and and y-intercepts of the following lines
    a) $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0$.
    b) $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5$.
    c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$.
27. Find the intercepts of the following lines on the axes.
    a) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12$.
    b) $\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 6$.
    c) $\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 0$.
28. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
    a) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8$.
    b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2$.
    c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$.
29. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = -82$.
30. Find the points on the x-axis, whose distances from the line
$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{3.5.30.1}$$
are 4 units.
31. Find the distance between the parallel lines
$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = 34 \tag{3.5.31.1}$$
$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = -31 \tag{3.5.31.2}$$
32. Find the equation of the line parallel to the line
$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -2 \tag{3.5.32.1}$$
and passing through the point $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
33. Find the equation of a line perpendicular to the line
$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \tag{3.5.33.1}$$
and having x intercept 3.
34. Find angles between the lines
$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{3.5.34.1}$$
$$\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 1 \tag{3.5.34.2}$$
35. The line through the points $\begin{pmatrix} h \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ intersects the line
$$\begin{pmatrix} 7 & -9 \end{pmatrix} \mathbf{x} = 19 \tag{3.5.35.1}$$
at right angle. Find the value of h.
36. Two lines passing through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ inter-

sect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.

37. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$.

38. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line
$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.5.38.1}$$

39. The perpendicular from the origin to the line
$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.5.39.1}$$
meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.

40. Find θ and p if
$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{3.5.40.1}$$
is equivalent to
$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{3.5.40.2}$$

41. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

42. Find the equation of the line parallel to the y-axis whose distance from the line
$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{3.5.42.1}$$
4 units.

43. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines
$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{3.5.43.1}$$
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.43.2}$$

44. Find the alue of p so that the three lines
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{3.5.44.1}$$
$$\begin{pmatrix}p & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.5.44.2}$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{3.5.44.3}$$
may intersect at one point.

45. Find the equation of the lines through the point $\begin{pmatrix}3\\2\end{pmatrix}$ which make an angle of 45° with the line
$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 3. \tag{3.5.45.1}$$

46. Find the equation of the line passing through the point of intersection of the lines
$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = 3 \tag{3.5.46.1}$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -1 \tag{3.5.46.2}$$
that has equal intercepts on the axes.

47. In what ratio is the line joining $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}5\\7\end{pmatrix}$ divided by the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.5.47.1}$$

48. Find the distance of the line
$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5 \tag{3.5.48.1}$$
from the point $\begin{pmatrix}1\\2\end{pmatrix}$ along the line
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0. \tag{3.5.48.2}$$

49. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.5.49.1}$$
may be at a distance of 3 units from this point.

50. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\1\end{pmatrix}$. Find an equation of the legs of the triangle.

51. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7 \tag{3.5.51.1}$$
assuming the line to be a plane mirror.

52. If the lines
$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 1 \tag{3.5.52.1}$$
$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{3.5.52.2}$$
are equally inclined to the line
$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 4, \tag{3.5.52.3}$$
find the value of m.

53. The sum of the perpendicular distances of a

variable point **P** from the lines

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.53.1}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{3.5.53.2}$$

is always 10. Show that **P** must move on a line.

54. Find the equation of the line which is equidistant from parallel lines

$$\begin{pmatrix}9 & 7\end{pmatrix}\mathbf{x} = 7 \tag{3.5.54.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = -6. \tag{3.5.54.2}$$

55. A ray of light passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$ reflects on the x-axis at point **A** and the reflected ray passes through the point $\begin{pmatrix}5\\3\end{pmatrix}$. Find the coordinates of **A**.
56. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{3.5.56.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.5.56.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{3.5.56.3}$$

in the least time. Find the equation of the path that he should follow.

57. Determine the ratio in which the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 4 = 0 \tag{3.5.57.1}$$

divides the line segment joining the points $\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\7\end{pmatrix}$.

58. A line perpendicular to the line segment joining the points $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ divides it in the ratio $1 : n$. Find the equation of the line.
59. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.
60. Find the equation of the line passing through the point $\begin{pmatrix}2\\2\end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.
61. Find the equation of the line through the point $\begin{pmatrix}0\\2\end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.
62. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix}-2\\9\end{pmatrix}$, find the equation of the line.
63. Find the equation of a line which passes through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}3\\2\\-2\end{pmatrix}$.
64. Find the equaion off the line that passes through $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and is in the direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.
65. Find the equation of the line which passes through the point $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{3.5.65.1}$$

66. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \tag{3.5.66.1}$$

67. Find the equation of the line passing through the origin and the point $\begin{pmatrix}5\\-2\\3\end{pmatrix}$.
68. Find the equation of the line passing through the points $\begin{pmatrix}3\\-2\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\\6\end{pmatrix}$.
69. Find the angle between the following pair of lines:

a)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}2\\-5\\1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{3.5.69.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix}7\\-6\\0\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{3.5.69.2}$$

b)

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \tag{3.5.69.3}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \tag{3.5.69.4}$$

70. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{3.5.70.1}$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{3.5.70.2}$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{3.5.70.3}$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{3.5.70.4}$$

71. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{3.5.71.1}$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{3.5.71.2}$$

are at right angles.

72. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{3.5.72.1}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{3.5.72.2}$$

are perpendicular to each other.

73. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.5.73.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{3.5.73.2}$$

74. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \tag{3.5.74.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{3.5.74.2}$$

75. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{3.5.75.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.5.75.2}$$

76. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} \tag{3.5.76.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} \tag{3.5.76.2}$$

77. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 2$ c) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \mathbf{x} = -8$

b) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1$ d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = 5$

78. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}$.

79. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 12$ c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1$

b) $\begin{pmatrix} 3 & 4 & -6 \end{pmatrix} \mathbf{x} = 0$ d) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \mathbf{x} = -8$

80. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

81. Find the equation of the planes that passes through three points

a) $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}6\\4\\-5\end{pmatrix}, \begin{pmatrix}-4\\-2\\3\end{pmatrix}$

b) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\-1\end{pmatrix}$.

82. Find the intercepts cut off by the plane $\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 5$.
83. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.
84. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}3 & -1 & 2\end{pmatrix}\mathbf{x} = 4$ and $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = -2$ and the pont $\begin{pmatrix}2\\2\\1\end{pmatrix}$.
85. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix}2 & 5 & 3\end{pmatrix}\mathbf{x} = 9$ and the pont $\begin{pmatrix}2\\1\\3\end{pmatrix}$.
86. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 0$.
87. Find the angle between the planes whose equations are $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix}3 & -3 & 5\end{pmatrix}\mathbf{x} = 3$
88. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.
    a) $\begin{pmatrix}7 & 5 & 6\end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix}3 & -1 & -10\end{pmatrix}\mathbf{x} = -4$
    b) $\begin{pmatrix}2 & 1 & 3\end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix}1 & -2 & 5\end{pmatrix}\mathbf{x} = 0$
    c) $\begin{pmatrix}2 & -2 & 4\end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix}3 & -3 & 6\end{pmatrix}\mathbf{x} = 1$
    d) $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = -3$
    e) $\begin{pmatrix}4 & 8 & 1\end{pmatrix}\mathbf{x} = 8$ and $\begin{pmatrix}0 & 1 & 1\end{pmatrix}\mathbf{x} = 4$
89. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}3 & -4 & 12\end{pmatrix}\boldsymbol{x} = 3$
b)	$\begin{pmatrix}3\\-2\\1\end{pmatrix}$	$\begin{pmatrix}2 & -1 & 2\end{pmatrix}\boldsymbol{x} = -3$
c)	$\begin{pmatrix}2\\3\\-5\end{pmatrix}$	$\begin{pmatrix}1 & 2 & -2\end{pmatrix}\boldsymbol{x} = 9$
d)	$\begin{pmatrix}-6\\0\\0\end{pmatrix}$	$\begin{pmatrix}2 & -3 & 6\end{pmatrix}\boldsymbol{x} = 2$

TABLE 3.5.89

90. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.
91. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}, \begin{pmatrix}-4\\3\\-6\end{pmatrix}, \begin{pmatrix}2\\9\\2\end{pmatrix}$, then find the angle between the lines AB and CD.
92. If the lines
$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{3.5.92.1}$$
$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{3.5.92.2}$$
find the value of k.
93. Find the equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane
$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\mathbf{x} = -9 \tag{3.5.93.1}$$
94. Find the shortest distance between the lines
$$\mathbf{x} = \begin{pmatrix}6\\2\\2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-2\\2\end{pmatrix} \text{ and} \tag{3.5.94.1}$$
$$\mathbf{x} = \begin{pmatrix}-4\\0\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-2\\-2\end{pmatrix} \tag{3.5.94.2}$$
95. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the YZ-plane.
96. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the ZX-plane.

97. Find the coordinates of the point where the line through $\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}-3\\1\end{pmatrix}$ crosses the plane

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.5.97.1}$$

98. Find the equation of the plane passing through the point $\begin{pmatrix}-1\\3\\2\end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 5 \tag{3.5.98.1}$$
$$\begin{pmatrix}3 & 3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.5.98.2}$$

99. If the points $\begin{pmatrix}1\\1\\p\end{pmatrix}$ and $\begin{pmatrix}-3\\0\\1\end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix}3 & 4 & -12\end{pmatrix}\mathbf{x} = -13, \tag{3.5.99.1}$$

then find the value of p.

100. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1 \text{ and} \tag{3.5.100.1}$$
$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = -4 \tag{3.5.100.2}$$

and parallel to the x-axis.

101. If **O** be the origin and the coordinates of **P** be $\begin{pmatrix}1\\2\\3\end{pmatrix}$, then find the equation of the plane passing through **P** and perpendicular to OP.

102. Find the equation of the plane which contains the line of intersection of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 4 \tag{3.5.102.1}$$
$$\begin{pmatrix}2 & 1 & -1\end{pmatrix}\mathbf{x} = -5 \tag{3.5.102.2}$$

and which is perpendicular to the plane

$$\begin{pmatrix}5 & 3 & -6\end{pmatrix}\mathbf{x} = -8 \tag{3.5.102.3}$$

103. Find the distance of the point $\begin{pmatrix}-1\\-5\\-10\end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\2\end{pmatrix} \tag{3.5.103.1}$$

and the plane

$$\begin{pmatrix}1 & -1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{3.5.103.2}$$

104. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.5.104.1}$$
$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{3.5.104.2}$$

105. Find the vector equation of the line passing through the point $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \tag{3.5.105.1}$$
$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{3.5.105.2}$$

106. Distance between the two planes

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{3.5.106.1}$$
$$\begin{pmatrix}4 & 6 & 8\end{pmatrix}\mathbf{x} = 12 \tag{3.5.106.2}$$

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

107. The planes

$$\begin{pmatrix}2 & -1 & 4\end{pmatrix}\mathbf{x} = 5 \tag{3.5.107.1}$$
$$\begin{pmatrix}5 & -\frac{5}{2} & 10\end{pmatrix}\mathbf{x} = 6 \tag{3.5.107.2}$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix}0\\0\\\frac{5}{4}\end{pmatrix}$

3.6 Miscellaneous

1. Solve the following pair of linear equations

a)

$$\begin{pmatrix} p & q \end{pmatrix}\mathbf{x} = p - q$$
$$\begin{pmatrix} q & -p \end{pmatrix}\mathbf{x} = p + q \tag{3.6.1.1}$$

b)

$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = c$$
$$\begin{pmatrix} b & a \end{pmatrix}\mathbf{x} = 1 + c \tag{3.6.1.2}$$

c)

$$\begin{pmatrix} \frac{1}{a} & -\frac{1}{b} \end{pmatrix}\mathbf{x} = 0$$
$$\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = a^2 + b^2 \tag{3.6.1.3}$$

2. Solve the following pair of equations
$$\begin{aligned} \begin{pmatrix} a-b & a+b \end{pmatrix}\mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix} a+b & a+b \end{pmatrix}\mathbf{x} &= a^2 + b^2 \end{aligned} \tag{3.6.2.1}$$
3. In $\triangle ABC$, Show that the centroid
$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.6.3.1}$$
4. (Cauchy-Schwarz Inequality:) Show that
$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\| \, \|\mathbf{b}\| \tag{3.6.4.1}$$
5. (Triangle Inequality:) Show that
$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{3.6.5.1}$$
6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
7. Find the distance between $\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ when
   a) PQ is parallel to the y-axis.
   b) PQ is parallel to the x-axis.
8. If three points $\begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
9. $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is
$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix}\mathbf{x} = 2 \tag{3.6.9.1}$$
10. Point $\mathbf{R} = \begin{pmatrix} h \\ k \end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
11. Show that two lines
$$\begin{pmatrix} a_1 & b_1 \end{pmatrix}\mathbf{x} + c_1 = 0 \tag{3.6.11.1}$$
$$\begin{pmatrix} a_2 & b_2 \end{pmatrix}\mathbf{x} + c_2 = 0 \tag{3.6.11.2}$$
are
   a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
   b) perpendicular if $a_1a_2 - b_1b_2 = 0$.
12. Find the distance between the parallel lines
$$l\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = -p \tag{3.6.12.1}$$
$$l\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = r \tag{3.6.12.2}$$
13. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line
$$\begin{pmatrix} A & B \end{pmatrix}\mathbf{x} = -C \tag{3.6.13.1}$$
14. If p and q are the lengths of perpendiculars from the origin to the lines
$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\mathbf{x} = k\cos 2\theta \tag{3.6.14.1}$$
$$\begin{pmatrix} \sec\theta & \operatorname{cosec}\theta \end{pmatrix}\mathbf{x} = k \tag{3.6.14.2}$$
respectively, prove that $p^2 + 4q^2 = k^2$.
15. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{3.6.15.1}$$
16. Show that the area of the triangle formed by the lines
$$\begin{pmatrix} -m_1 & 1 \end{pmatrix}\mathbf{x} = c_1 \tag{3.6.16.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix}\mathbf{x} = c_2 \tag{3.6.16.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.6.16.3}$$
is $\frac{(c_1-c_2)^2}{2|m_1-m_2|}$.
17. Find the values of k for which the line
$$\begin{pmatrix} k-3 & -\left(4-k^2\right) \end{pmatrix}\mathbf{x} + k^2 - 7k + 6 = 0 \tag{3.6.17.1}$$
is
   a) parallel to the x-axis
   b) parallel to the y-axis
   c) passing through the origin.
18. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and

$\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$.

19. Find the area of the triangle formed by the lines

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{3.6.19.1}$$
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.6.19.2}$$
$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = k \tag{3.6.19.3}$$

20. If three lines whose equations are

$$\begin{pmatrix}-m_1 & 1\end{pmatrix}\mathbf{x} = c_1 \tag{3.6.20.1}$$
$$\begin{pmatrix}-m_2 & 1\end{pmatrix}\mathbf{x} = c_2 \tag{3.6.20.2}$$
$$\begin{pmatrix}-m_3 & 1\end{pmatrix}\mathbf{x} = c_3 \tag{3.6.20.3}$$

are concurrent, show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0 \tag{3.6.20.4}$$

21. Find the equation of the line passing through the origin and making an angle θ with the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.6.21.1}$$

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix}\sqrt{a^2-b^2}\\ 0\end{pmatrix}$ and $\begin{pmatrix}\sqrt{a^2-b^2}\\ 0\end{pmatrix}$ to the line

$$\begin{pmatrix}\frac{\cos\theta}{a} & \frac{\sin\theta}{b}\end{pmatrix}\mathbf{x} = 1 \tag{3.6.22.1}$$

is b^2.

23. If $\begin{pmatrix}l_1\\ m_1\\ n_1\end{pmatrix}$ and $\begin{pmatrix}l_2\\ m_2\\ n_2\end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix}m_1n_2 - m_2n_1\\ n_1l_2 - n_2l_1\\ l_1m_2 - l_2m_1\end{pmatrix}$.

24. A line makes angles $\alpha,\beta,\gamma,\delta$ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{3.6.24.1}$$

25. Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}, \tag{3.6.25.1}$$
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{3.6.25.2}$$

are coplanar.

26. Find $\mathbf{R}$ which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{3.6.26.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{3.6.26.2}$$

externally in the ratio 1 : 2.

27. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}-\mathbf{b}) = 8 \tag{3.6.27.1}$$
$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \tag{3.6.27.2}$$

28. Evaluate the product

$$(3\mathbf{a}-5\mathbf{b})^T(2\mathbf{a}+7\mathbf{b}) \tag{3.6.28.1}$$

29. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{3.6.29.1}$$
$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \tag{3.6.29.2}$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

30. Show that

$$(\|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a}) \perp (\|\mathbf{a}\|\mathbf{b} - \|\mathbf{b}\|\mathbf{a}) \tag{3.6.30.1}$$

31. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

32. If $\mathbf{a},\mathbf{b},\mathbf{c}$ are unit vectors such that

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0, \tag{3.6.32.1}$$

find the value of

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{3.6.32.2}$$

33. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if
   a) $\lambda = 1$
   b) $\lambda = -1$
   c) $\|\mathbf{a}\| = |\lambda|$
   d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

34. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

35. Show that

$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b}) \tag{3.6.35.1}$$

36. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

37. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x}-\mathbf{a})^T(\mathbf{x}+\mathbf{a}) = 12. \tag{3.6.37.1}$$

38. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a}\times\mathbf{b}$ is a unit vector

if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

39. Prove that

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \quad (3.6.39.1)$$
$$\iff \mathbf{a} \perp \mathbf{b}. \quad (3.6.39.2)$$

40. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

a) $0 < \theta < \frac{\pi}{2}$
b) $0 \leq \theta \leq \frac{\pi}{2}$
c) $0 < \theta < \pi$
d) $0 \leq \theta \leq \pi$

41. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

42. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a}\times\mathbf{b}\|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π.

43. Find the angle between the lines whose direction vectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix}$.
44. Find the equation of a line parallel to the x-axis and passing through the origin.
45. Find the equation of a plane passing through $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x}x = 2 \quad (3.6.45.1)$$

46. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \quad (3.6.46.1)$$

4 Circle

4.1 Construction Examples

1. ABC is a triangle. Locate a point in the interior of $\triangle ABC$ which is equidistant from all the vertices of $\triangle ABC$.

Solution: Let $\mathbf{O}$ be the desired point. Then,

$$\|\mathbf{A}-\mathbf{O}\| = \|\mathbf{B}-\mathbf{O}\| = \|\mathbf{C}-\mathbf{O}\| = R \quad (4.1.1.1)$$

From (4.1.1.1),

$$\|\mathbf{A}-\mathbf{O}\|^2 - \|\mathbf{B}-\mathbf{O}\|^2 = 0 \quad (4.1.1.2)$$

$$\implies (\mathbf{A}-\mathbf{O})^T(\mathbf{A}-\mathbf{O}) - (\mathbf{B}-\mathbf{O})^T(\mathbf{B}-\mathbf{O}) = 0 \quad (4.1.1.3)$$

which can be simplified as

$$(\mathbf{A}-\mathbf{B})^T\mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (4.1.1.4)$$

Similarly,

$$(\mathbf{B}-\mathbf{C})^T\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (4.1.1.5)$$

From and , $\mathbf{O}$ can be computed. A circle with centre $\mathbf{O}$ can be drawn through $\mathbf{A}, \mathbf{B}, \mathbf{C}$. This circle is known as the *circumcircle*. The following code plots Fig. 4.1.1

```
codes/circle/circle_const_ccircle.py
```

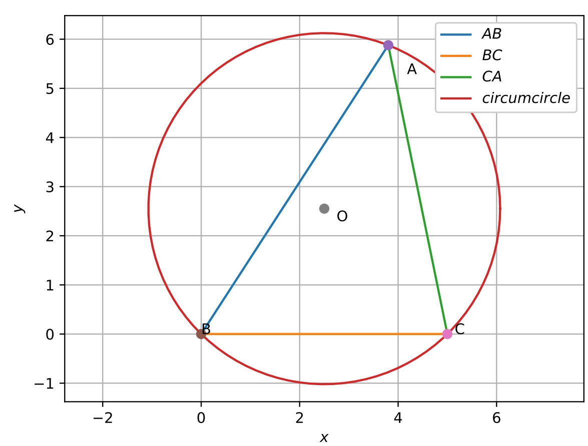

Fig. 4.1.1

2. In a triangle locate a point in its interior which is equidistant from all the sides of the triangle.
3. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

Solution: The tangent is perpendicular to the radius. From the given information, in

$\triangle ABC, AC \perp AB, a = 10$ and $c = 6$.

$$b = \sqrt{a^2 - c^2} \tag{4.1.3.1}$$

The following code plots Fig. 4.1.3

codes/circle/draw_circle_eg.py

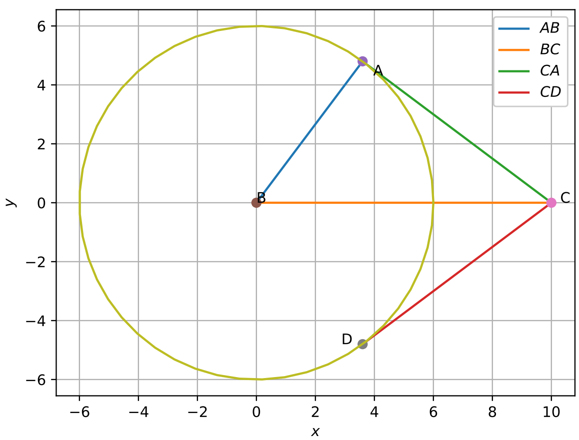

Fig. 4.1.3

4. Draw a circle of radius 3. Mark any point **A** on the circle, point **B** inside the circle and point **C** outside the circle.
Solution: For any angle θ, a point on the circle with radius 3 has coordinates

$$3\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \tag{4.1.4.1}$$

4.2 Construction Exercises

1. Draw a circle of diameter 6.1
2. With the same centre **O**, draw two circles of radii 4 and 2.5
3. Draw a circle of radius 3 and any two of its diameters. draw the ends of these diameters. What figure do you get?
4. Let **A** and **B** be two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at **C** and **D**. Is $AB \perp CD$?
5. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
Solution: Take the centre of both circles to be at the origin.
6. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.
Solution: Take the diameter to be on the x-axis.
7. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.
Solution: The tangent is perpendicular to the radius.
8. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8 \\ 0\end{pmatrix}. \tag{4.2.4.1}$$

9. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through $\mathbf{B}, \mathbf{C}, \mathbf{D}$ (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.
10. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**

4.3 Circle Geometry Examples

1. Find the equation of a circle with centre $\begin{pmatrix}-3 \\ 2\end{pmatrix}$ and radius 4.
2. Find the centre and radius of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}8 \\ 10\end{pmatrix}\mathbf{x} - 8 = 0 \tag{4.3.2.1}$$

3. Find the equation of the circle which passes through the points $\begin{pmatrix}2 \\ -2\end{pmatrix}$ and $\begin{pmatrix}3 \\ 4\end{pmatrix}$ and whose centre lies on the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.
4. Find the area enclosed by the circle $\|\mathbf{x}\| = a$
5. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
6. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix}2 \\ 0\end{pmatrix}\right\| = 2$.

4.4 Circle Geometry Exercises

1. Find the coordinates of a point **A**, where AB is the diameter of a circle whose centre is $\begin{pmatrix}2, -3\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}1\\4\end{pmatrix}$.
2. Find the centre Of a circle passing through the points $\begin{pmatrix}6\\-6\end{pmatrix}$, $\begin{pmatrix}3\\-7\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$.
3. Sketch the circles with
   a) centre $\begin{pmatrix}0\\2\end{pmatrix}$ and radius 2
   b) centre $\begin{pmatrix}-2\\32\end{pmatrix}$ and radius 4
   c) centre $\begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}$ and radius $\frac{1}{12}$.
   d) centre $\begin{pmatrix}1\\1\end{pmatrix}$ and radius $\sqrt{2}$.
   e) centre $\begin{pmatrix}-a\\-b\end{pmatrix}$ and radius $\sqrt{a^2 - b^2}$.
4.
5. Sketch the circles with equation
   a) $\left\|\mathbf{x} - \begin{pmatrix}5\\-3\end{pmatrix}\right\|^2 = 36$
   b) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4\\8\end{pmatrix}\mathbf{x} - 45 = 0$
   c) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8\\-10\end{pmatrix}\mathbf{x} - 12 = 0$
   d) $2\mathbf{x}^T\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\mathbf{x} = 0$
6. Find the equation of the circle passing through the points $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}6\\5\end{pmatrix}$ and whose centre is on the line $\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 16$.
7. Find the equation of the circle passing through the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ and whose centre is on the line $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 11$.
8. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.
9. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.
10. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.
11. Does the point $\begin{pmatrix}-2.5\\3.5\end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T\mathbf{x} = 25$?
12. Find the locus of all the unit vectors in the xy-plane.
13. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
14. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^{\mathbf{x}} = 4$.
15. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^{\mathbf{x}} = 4$ and the lines $x = 0$ and $x = 2$.
16. Find the area of the circle $4\mathbf{x}^{\mathbf{x}} = 9$.
17. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$
18. Find the smaller area enclosed by the circle $\mathbf{x}^{\mathbf{x}} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

5 Conics

5.1 Examples

1. Find the value of the following polynomial at the indicated value of variables
$$p(x) = 5x^2 – 3x + 7 \text{ at } x = 1. \tag{5.1.1.1}$$
2. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.
3. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(y) = y^2$.
   b) $p(x) = (x–1)(x + 1)$.
4. Find the roots of the equation $2x^2–5x + 3 = 0$.
5. Find the roots of the quadratic equation $6x^2–x–2 = 0$.
6. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
7. Factorise $6x^2 + 17x + 5$.
8. Factorise $y^2–5y + 6$.
9. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$ and verify the relationship between the zeroes and the coefficients.
10. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.
11. Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.
12. Find the roots of the equation $5x^2–6x–2 = 0$.

13. Find the roots of $4x^2 + 3x + 5 = 0$.
14. Find the roots of the following quadratic equations, if they exist.
    a) $3x^2 - 5x + 2 = 0$
    b) $x^2 + 4x + 5 = 0$
    c) $2x^2 - 2\sqrt{2}x + 1 = 0$
15. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.
16. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.
17. Solve $x^2 + 2 = 0$.
18. Solve $x^2 + x + 1 = 0$.
19. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.
20. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.
21. Find the equation of the parabola with focus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and directrix $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -2$.
22. Find the equation of the parabola with vertex at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and focus at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.
23. Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
24. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse
$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1 \tag{5.1.24.1}$$
25. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse
$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36 \tag{5.1.25.1}$$
26. Find the equation of the ellipse whose vertices are $\begin{pmatrix} \pm 13 \\ 0 \end{pmatrix}$ and foci are $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.
27. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$
28. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
29. Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas
    a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
    b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 16$
30. Find the equation of the hyperbola with vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$
31. Find the equation of the hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of latus rectum 36.
32. Find the equation of all lines having slope 2 and being tangent to the curve
$$y + \frac{2}{x - 3} = 0 \tag{5.1.32.1}$$
33. Find the point at which the tangent to the curve $y = \sqrt{4x - 3} - 1$ has its slope $\frac{2}{3}$.
34. Find the roots of the following equations:
    a) $x + \frac{1}{x} = 3, x \neq= 0$
    b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq= 0, 2$
35. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$ at which the tangents are
    a) parallel to x-axis
    b) parallel to y-axis
36. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
37. Find the area enclosed by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$
38. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.
39. Find the area bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$.
40. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.
41. Find the area of the region
$$\left\{(x, y) = 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\} \tag{5.1.41.1}$$

42. Find the interals in which the function
$$f(x) = x^2 - 4x + 6 \tag{5.1.42.1}$$
is
   a) increasing
   b) decreasing.

5.2 Exercises

1. Verify whether the following are zeroes of the polynomial, indicated against them.
   a) $p(x) = x^2 - 1, x = 1, -1$
   b) $p(x) = (x+1)(x-2), x = -1, 2$
   c) $p(x) = x^2, x = 0.$
   d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$
2. Find the vaue of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $x^4 - 4x^2 + x + 6, g(x) = x - 3$
3. Factorise :
   a) $12x^2–7x + 1$
   b) $6x^2 + 5x–6$
   c) $2x^2 + 7x + 3$
   d) $3x^2–x–4$
4. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.
   a) $x^2–2x–8$
   b) $4u^2 + 8u$
   c) $4s^2–4s + 1$
   d) $t^2–15$
   e) $6x^2–3–7x$
   f) $3x^2–x–4$
5. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.
   a) -1 , $\frac{1}{4}$
   b) 1, 1
   c) 0, $\sqrt{5}$
   d) 4, 1
   e) $\frac{1}{4}, \frac{1}{4}$
   f) $\sqrt{2}, \frac{1}{3}$
6. Find the roots of the following quadratic equations:
   a) $x^2–3x–10 = 0$
   b) $2x^2 + x - 6 = 0$
   c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
   d) $2x^2–x + \frac{1}{8} = 0$
   e) $100x^2–20x + 1 = 0$
7. Find the roots of the following quadratic equations
   a) $2x^2 - 7x + 3 = 0$
   b) $2x^2 + x - 4 = 0$
   c) $4x^2 + 4\sqrt{3}x + 3 = 0$
   d) $2x^2 + x + 4 = 0$
8. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:
   a) $2x^2 - 3x + 5 = 0$
   b) $2x^2 - 6x + 3 = 0$
   c) $3x^2 - 4\sqrt{3}x + 4 = 0$
9. Solve each of the following equations
   a) $x^2 + 3 = 0$
   b) $2x^2 + x + 1 = 0$
   c) $x^2 + 3x + 9 = 0$
   d) $-x^2 + x - 2 = 0$
   e) $x^2 + 3x + 5 = 0$
   f) $x^2 - 3x + 2 = 0$
   g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
   h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
   i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
   j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$
10. Solve each of the following equations
   a) $3x^2 - 4x + \frac{20}{3} = 0$
   b) $x^2 - 2x + \frac{3}{2} = 0$
   c) $27x^2 - 10x + 1 = 0$
   d) $21x^2 - 28x + 10 = 0$
11. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum
   a) $y^2 = 12x$
   b) $x^2 = 6y$
   c) $y^2 = -8x$
   d) $x^2 = -16y$
   e) $y^2 = 10x$
   f) $x^2 = -9y$
12. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:
   a) Focus $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, directrix $\begin{pmatrix} 1 & 0 \end{pmatrix} = -6.$
   b) Focus $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, directrix $\begin{pmatrix} 0 & 1 \end{pmatrix} = 3.$

c) Focus $\begin{pmatrix}3\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.

d) Focus $\begin{pmatrix}-2\\0\end{pmatrix}$, vertex $\begin{pmatrix}0 & 0\end{pmatrix}$.

e) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}2\\2\end{pmatrix}$ and axis is along the x-axis

f) vertex $\begin{pmatrix}0 & 0\end{pmatrix}$ passing through $\begin{pmatrix}5\\2\end{pmatrix}$ and symmetric with respect to the y-axis.

13. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{36} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$

b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{25}\end{pmatrix}\mathbf{x} = 1$

c) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$

d) $\mathbf{x}^T\begin{pmatrix}\frac{1}{25} & 0\\0 & \frac{1}{100}\end{pmatrix}\mathbf{x} = 1$

e) $\mathbf{x}^T\begin{pmatrix}\frac{1}{49} & 0\\0 & \frac{1}{36}\end{pmatrix}\mathbf{x} = 1$

f) $\mathbf{x}^T\begin{pmatrix}\frac{1}{100} & 0\\0 & \frac{1}{16}\end{pmatrix}\mathbf{x} = 1$

g) $\mathbf{x}^T\begin{pmatrix}36 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 144$

h) $\mathbf{x}^T\begin{pmatrix}16 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 16$

i) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 36$

14. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix}\pm5\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$

b) Vertices $\begin{pmatrix}0\\\pm13\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$

c) Vertices $\begin{pmatrix}\pm6\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$

d) Ends of major axis $\begin{pmatrix}\pm3\\0\end{pmatrix}$, ends of minor axis $\begin{pmatrix}0\\\pm2\end{pmatrix}$

e) Ends of major axis $\begin{pmatrix}0\\\pm5\end{pmatrix}$, ends of minor axis $\begin{pmatrix}\pm1\\0\end{pmatrix}$

f) Length of major axis 26, foci $\begin{pmatrix}\pm5\\0\end{pmatrix}$

g) Length of minor axis 16, foci $\begin{pmatrix}0\\\pm6\end{pmatrix}$.

h) Foci $\begin{pmatrix}\pm3\\0\end{pmatrix}$, $a = 4$

i) $b = 3$, $c = 4$, centre at the origin; foci on the x axis.

j) Centre at $\begin{pmatrix}0\\0\end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}1\\6\end{pmatrix}$.

k) Major axis on the x-axis and passes through the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$.

15. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & -\frac{1}{9}\end{pmatrix}\mathbf{x} = 1$

b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\0 & -\frac{1}{27}\end{pmatrix}\mathbf{x} = 1$

c) $\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & -4\end{pmatrix}\mathbf{x} = 36$

d) $\mathbf{x}^T\begin{pmatrix}16 & 0\\0 & -9\end{pmatrix}\mathbf{x} = 576$

e) $\mathbf{x}^T\begin{pmatrix}5 & 0\\0 & -9\end{pmatrix}\mathbf{x} = 36$

f) $\mathbf{x}^T\begin{pmatrix}49 & 0\\0 & -16\end{pmatrix}\mathbf{x} = 784$

16. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix}\pm2\\0\end{pmatrix}$, foci $\begin{pmatrix}\pm3\\0\end{pmatrix}$

b) Vertices $\begin{pmatrix}0\\\pm5\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm8\end{pmatrix}$

c) Vertices $\begin{pmatrix}0\\\pm3\end{pmatrix}$, foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$

d) Transverse axis length 8, foci $\begin{pmatrix}\pm5\\0\end{pmatrix}$.

e) Conjugate axis length 24, foci $\begin{pmatrix}0\\\pm13\end{pmatrix}$.

f) Latus rectum length 8, foci $\begin{pmatrix}\pm3\sqrt{5}\\0\end{pmatrix}$.

g) Latus rectum lenght 12, foci $\begin{pmatrix}\pm4\\0\end{pmatrix}$.

h) Ends of major axis $\begin{pmatrix}0\\\pm5\end{pmatrix}$, ends of minor axis

$\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$

i) Vertices $\begin{pmatrix} \pm 7 \\ 0 \end{pmatrix}$, $e = \frac{4}{3}$

j) Foci $\begin{pmatrix} 0 \\ \pm\sqrt{10} \end{pmatrix}$, passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

17. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
18. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.
19. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$
20. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
21. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$ at which tangents are
    a) parallel to x-axis
    b) parallel to y-axis.
22. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
23. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$
    a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$
    b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.
24. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & 2 \end{pmatrix} \mathbf{x} + 5 = 0$.
25. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.
26. The line $\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.
27. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$
28. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
29. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.
30. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.
31. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.
32. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
33. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
34. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
35. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
36. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.
37. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
38. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
39. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
40. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix} 1 & 0 \end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
41. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.
42. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
43. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
44. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
45. Find the area between $y = x^2$ and $y = x$.
46. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
47. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.
48. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1$ and the line $\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1$
49. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2$ and the x-axis.

50. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{5.2.50.1}$$

51. Find the area of the region

$$\left\{(x, y) : y^2 \le 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{5.2.51.1}$$

52. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.
53. Find the intervals in which the function given by

$$f(x) = 2x^2 - 3x \tag{5.2.53.1}$$

is
   a) increasing
   b) decreasing.
54. Find the intervals in which the following functions are strictly increasing or decreasing
   a) $x^2 + 2x - 5$
   b) $10 - 6x - 2x^2$
   c) $6 - 9x - x^2$
55. Prove that the function f given by $f(x) = x^2 - x+1$ is neither strictly increasing nor decreasing on $(1, -1)$.

6 Curves

6.1 Examples

1. Find the value of each of the following polynomials at the indicated value of variables:
   a) $q(y) = 3y^3 - 4y + 11$ at $y = 2$.
   b) $p(t) = 4t^4 + 5t^3 - t2 + 6$ at $t = a$.
2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(t) = 2 + t + 2t^2 - t^3$
   b) $p(x) = x^3$
3. Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.
4. Check whether the polynomial $q(t) = 4t^3+4t^2-t - 1$ is a multiple of $2t + 1$.
5. Examine whether $x+2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.
6. Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.
7. Factorize $x^3 - 23x^2 + 142x - 120$.
8. Verify that $3, -1, \frac{1}{3}$, are the zeroes of the cubic polynomial $p(x) = 3x^3-5x^2-11x-3$, and then verify the relationship between the zeroes and the coefficients.
9. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$
10. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$
11. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
12. Find the equation of the tangent to the curve $\begin{pmatrix} a\sin^3 t \\ b\cos^3 t \end{pmatrix}$ at $t = \frac{\pi}{2}$.
13. Find the equation of tangents to the curve $y = cos(x+y), -2\pi \le x \le 2\pi$ that are parallel to the line $\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} = 0$.
14. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.
15. Sketch the graph of $y = |x + 3|$ and evaluate its area for $-6 \le x \le 0$.
16. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.
17. Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R} \tag{6.1.17.1}$$

is increasing on $\mathbf{R}$.
18. Prove that the function given by $f(x) = \cos x$ is
   a) decreasing in $(0, \pi)$.
   b) increasing in $(\pi, 2\pi)$ and
19. Find the intervals in which the function

$$f(x) = 4x^3 - 6x^2 - 72x + 30 \tag{6.1.19.1}$$

is
   a) increasing
   b) decreasing.
20. Find the intervals in which the function given by

$$f(x) = \sin x, x \in \left[0, \frac{\pi}{2}\right] \tag{6.1.20.1}$$

is
   a) increasing
   b) decreasing.
21. Find the intervals in which the function given by

$$f(x) = \sin x + \cos x, x \in [0, 2\pi] \tag{6.1.21.1}$$

is increasing or decreasing.

6.2 Exercises

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by
   a) $x + 1$
   b) $x - \frac{1}{2}$
   c) x
   d) $x + \pi$
   e) $5 + 2x$
2. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.
3. Determine which of the following polynomials has $(x + 1)$ as a factor:
   a) $x^3 + x^2 + x + 1$
   b) $x^4 + x^3 + x^2 + x + 1$
   c) $x^4 + 3x^3 + 3x^2 + x + 1$
   d) $x^3 - x^2 - \left(2 + \sqrt{2}\right) + \sqrt{2}$.
4. Determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $p(x) = x^4 - 4x^2 + x + 6, g(x) = x - 3$
5. Factorise :
   a) $x^3 – 2x^2 – x + 2$
   b) $x^3 – 3x^2 – 9x – 5$
   c) $x^3 + 13x^2 + 32x + 20$
   d) $2y^3 + y^2 – 2y – 1$
6. Find the roots of the following equations:
   a) $x - \frac{1}{x} = 3, x \neq= 0$
   b) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq= -4, 7$
7. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
8. Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 2.
9. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
10. Find the slope of the normal to the curve $\mathbf{x} = a\begin{pmatrix}\cos^3\theta \\ \sin^3\theta\end{pmatrix}$ at $\theta = \frac{\pi}{4}$.
11. Find the slope of the normal to the curve $\mathbf{x} = \begin{pmatrix}1 - a\sin\theta \\ b\cos^2\theta\end{pmatrix}$ at $\theta = \frac{\pi}{2}$.
12. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to th x-axis.
13. Find the point on the curve $y = x^3 – 11x + 5$ at which the tangent is $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 11$.
14. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$.
15. Find the equations of the tangent and normal to the given curves at the indicated points:
   a) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}0 \\ 5\end{pmatrix}$.
   b) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}1 \\ 3\end{pmatrix}$.
   c) $y = x^3$ at $\begin{pmatrix}1 \\ 1\end{pmatrix}$.
16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3 – 2x^5$ find all the points at which the tangent passes through the origin.
19. Find the equation of the normal at the point $\begin{pmatrix}am^2 \\ am^3\end{pmatrix}$ for the curve $ay^2 = x^3$
20. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $\begin{pmatrix}1 & 14\end{pmatrix} + 4 = 0$.
21. Find the slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$. Show that the normal at any point θ to the curve $\mathbf{x} = \begin{pmatrix}acos\theta + a\theta sin\theta \\ asin\theta – a\theta cos\theta\end{pmatrix}$ is at a constant distance from the origin.
22. Find the slope of the tangent to the curve $\mathbf{x} = \begin{pmatrix}t^2 + 3t - 8 \\ 2t^2 - 2t - 5\end{pmatrix}$ at the point $\begin{pmatrix}2 \\ -1\end{pmatrix}$.
23. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes.
24.
25. Find the area under $y = x^4, x = 1, x = 5$ and x-axis.
26. Using integration find the area of region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).
27. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$, $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -1$ and x = 4.
28. Find the area of the region bounded by the line $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -2$, the x-axis and the ordinates $x = -1, x = 1$.
29. Find the area bounded by the curve $|x| + |y| = 1$.
30. Using the method of integration find the area of $\triangle ABC$, whose vertices are $\mathbf{A} = \begin{pmatrix}2 \\ 0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4 \\ 5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6 \\ 3\end{pmatrix}$.

31. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 4$, $\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 6$ and $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$.
32. Find the area bounded by the curve $y = x^3, x = -2, x = 1$ and the x-axis.
33. Find the area bounded by the curve $y = x|x|, x = -1, x = 1$ and the x-axis.
34. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$.
35. Show that the function given by $f(x) = 3x+17$ is increasing on $\mathbf{R}$.
36. Show that the function given by $f(x) = e^{2x}$ is increasing on $\mathbf{R}$.
37. Show that the function given by

$$f(x) = \sin x \tag{6.2.37.1}$$

is

a) increasing in $\left(0, \frac{\pi}{2}\right)$
b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

38. Find the intervals in which the function given by

$$f(x) = 2x^3 - 3x^2 - 36x + 7 \tag{6.2.38.1}$$

is

a) increasing
b) decreasing.

39. Find the intervals in which the following functions are strictly increasing or decreasing

a) $(x+1)^3(x-3)^3$
b) $-2x^3 - 9x^2 - 12x + 1$

40. Show that

$$y = \log(1+x) - \frac{2x}{2+x}, x > -1, \tag{6.2.40.1}$$

is an increasing function of x throughout its domain.

41. Find the values of x for which $y = x(x-2)^2$ is an increasing function.
42. Prove that

$$y = \frac{4\sin\theta}{2+\cos\theta} - \theta \tag{6.2.42.1}$$

is an incresing function of θ in $\left[0, \frac{\pi}{2}\right]$.

43. Prove that the logarithmic function is increasing on $(0, \infty)$.
44. Which of the following functions are decreasing on $\left[0, \frac{\pi}{2}\right]$?

a) $\cos x$
b) $\cos 2x$
c) $\cos 3x$
d) $\tan x$

45. Find the intervals on which

$$f(x) = x^100 + \sin x - 1 \tag{6.2.45.1}$$

is decreasing.

46. Let I be any interval disjoint from $[1, -1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.
47. Prove that the function f given by $f(x) = \log\sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.
48. Prove that the function f given by $f(x) = \log|\cos x|$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.
49. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in $\mathbf{R}$.
50. Find the interval(s) in which $f(x) = x^2e^{-x}$ is increasing.

7 Miscellaneous Exercises

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?
3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.
5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.
9. Prove that the curves $x = y^2$ and $kx = y$ cut at right angles if $8k^2 = 1$
10. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $\begin{pmatrix} at^2 \\ 2at \end{pmatrix}$.
11. Find the equations of the tangent and normal to the hyperbola $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ at the point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.
12. Find the area of the smaller part of the circle $\mathbf{x}^{\mathbf{x}} = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
13. Find the area enclosed between the parabola $y^2 = 4ax$ ad the line $y = mx$.
14. The focus of a parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB .
15. A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?
16. 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point **A** lies on x-axis and end point **B** lies on y-axis. A point **P** is taken on the rod in such a way that $AP = 6$ cm. Show that the locus of P is an ellipse
17. Using integration find the area of region bounded by the triangle whose vertices are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
18. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.
19. Find the rate of change of the area of a circle per second with respect to its radius when $r =$ 5cm.
20. The volume of a cube is increasing at a rate of 9 cu cm per second. How fast is the surface area increasing when the length of an edge is 10 cm?
21. A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
22. The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y =6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.
23. The total cost $C(x)$ in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005x^3–0.02x^2 + 30x + 5000$ Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
24. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.
25. Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm
26. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
27. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
28. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
29. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?
30. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
31. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When

x = 8cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

32. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
33. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
34. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
35. A particle moves along the curve $6y = x3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
36. The radius of an air bubble is increasing at the rate of 12cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
37. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}2x + 1$. Find the rate of change of its volume with respect to x.
38. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
39. The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 – 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.
40. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x2 + 26x + 15$. Find the marginal revenue when x = 7.
41. Find the rate of change of the area of a circle with respect to its radius r at r = 6 cm.
42. The total revenue in | received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when x = 15.
43. For what vaues of a the function given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?