

JEE Problems in Linear Algebra: 2D

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Abstract—This manual introduces matrix computations using python and the properties of a triangle.

1 LINE

1.1 Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \quad (1)$$

Draw $\triangle ABC$.

Solution: The following code yields the desired plot in Fig. 1.1

```
#Code by GVV Sharma
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#released under GNU GPL
import numpy as np
import matplotlib.pyplot as plt
#if using termux
import subprocess
import shlex
#end if

A = np.array([-2,-2])
B = np.array([1,3])
C = np.array([4,-1])

len = 10
```

```
lam_1 = np.linspace(0,1,len)

x_AB = np.zeros((2,len))
x_BC = np.zeros((2,len))
x_CA = np.zeros((2,len))
for i in range(len):
    temp1 = A + lam_1[i]*(B-A)
    x_AB[:,i]= temp1.T
    temp2 = B + lam_1[i]*(C-B)
    x_BC[:,i]= temp2.T
    temp3 = C + lam_1[i]*(A-C)
    x_CA[:,i]= temp3.T

#print(x_AB[0,:],x_AB[1,:])
plt.plot(x_AB[0,:],x_AB[1:],label='$AB$')
plt.plot(x_BC[0,:],x_BC[1:],label='$BC$')
plt.plot(x_CA[0,:],x_CA[1:],label='$CA$')

plt.plot(A[0], A[1], 'o')
plt.text(A[0] * (1 + 0.1), A[1] * (1 - 0.1) , '
A')
plt.plot(B[0], B[1], 'o')
plt.text(B[0] * (1 - 0.2), B[1] * (1) , 'B')
plt.plot(C[0], C[1], 'o')
plt.text(C[0] * (1 + 0.03), C[1] * (1 - 0.1) ,
'C')

plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend(loc='best')
plt.grid() # minor

#if using termux
plt.savefig('../figs/triangle.pdf')
plt.savefig('../figs/triangle.eps')
subprocess.run(shlex.split("termux-open ../
figs/triangle.pdf"))
#else
plt.show()
```

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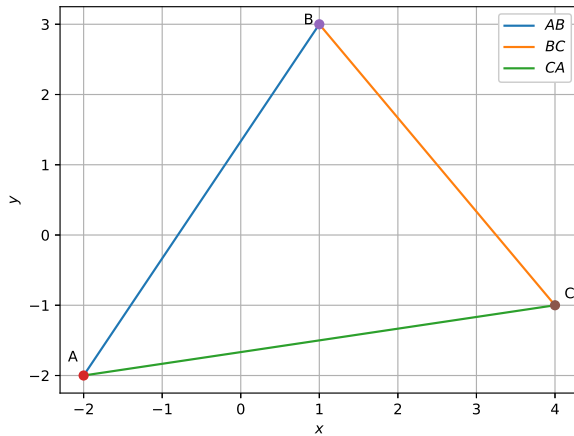


Fig. 1.1

1.2 Find the equation of AB .

Solution: The desired equation is obtained as

$$AB: \mathbf{x} = \mathbf{A} + \lambda_1 (\mathbf{B} - \mathbf{A}) \quad (2)$$

$$= -\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (3)$$

Alternatively, the desired equation is

$$(5 \ -3)(\mathbf{x} - \mathbf{A}) = 0 \quad (4)$$

$$\Rightarrow (5 \ -3)\mathbf{x} = -(5 \ -3)\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -4 \quad (5)$$

1.3 Find the direction vector and the normal vector for AB

Solution: Let

$$T_{AB} = (\mathbf{A} \ \mathbf{B}) = \begin{pmatrix} -2 & 1 \\ -2 & 3 \end{pmatrix} \quad (6)$$

The direction vector of AB is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = T_{AB} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (7)$$

The normal vector \mathbf{n} is defined as

$$\mathbf{n}^T \mathbf{m} = 0 \quad (8)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \quad (9)$$

1.4 Write a python code for computing the direction and normal vectors.

```
import numpy as np
```

```
def dir_vec(AB):
    return np.matmul(AB,dvec)

def norm_vec(AB):
    return np.matmul(omat,np.matmul(AB,dvec
    ))

A = np.array([-2,-2])
B = np.array([1,3])
dvec = np.array([-1,1])
omat = np.array([[0,1],[-1,0]])
AB =np.vstack((A,B)).T

print (dir_vec(AB))
print (norm_vec(AB))
```

1.5 Find the equations of BC and CA

2 MEDIANS OF A TRIANGLE

2.1 Find the coordinates of D, E and F of the mid points of AB, BC and CA respectively for $\triangle ABC$.

Solution: The coordinates of the mid points are given by

$$D = \frac{B+C}{2}, E = \frac{C+A}{2}, F = \frac{A+B}{2} \quad (10)$$

The following code computes the values resulting in

$$D = \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}, E = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}, F = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \quad (11)$$

```
#This program calculates the mid point
between
```

```
#any two coordinates
import numpy as np
import matplotlib.pyplot as plt
```

```
def mid_pt(B,C):
    D = (B+C)/2
    return D
```

```
A = np. matrix(' -2;-2')
B = np. matrix(' 1;3')
C = np. matrix(' 4;-1')
```

```
print(mid_pt(B,C))
print(mid_pt(C,A))
print(mid_pt(A,B))
```

2.2 Find the equations of AD , BE and CF . These lines are the *medians* of $\triangle ABC$

Solution: Use the code in Problem 1.4.

2.3 Find the point of intersection of AD and CF .

Solution: Let the respective equations be

$$\mathbf{n}_1^T \mathbf{x} = p_1 \quad \text{and} \quad (12)$$

$$\mathbf{n}_2^T \mathbf{x} = p_2 \quad (13)$$

This can be written as the matrix equation

$$\begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \end{pmatrix} \mathbf{x} = \mathbf{p} \quad (14)$$

$$\Rightarrow N^T \mathbf{x} = \mathbf{p} \quad (15)$$

where

$$N = (\mathbf{n}_1 \quad \mathbf{n}_2), \quad (16)$$

The point of intersection is then obtained as

$$\mathbf{x} = (N^T)^{-1} \mathbf{p} \quad (17)$$

$$= N^{-T} \mathbf{p} \quad (18)$$

The following code yields the point of intersection

$$\mathbf{G} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (19)$$

```
D = mid_pt(B,C)
```

```
F = mid_pt(A,B)
```

```
AD = np.vstack((A,D)).T
```

```
CF = np.vstack((C,F)).T
```

```
dvec = np.array([-1,1])
```

```
omat = np.array([[0,1],[-1,0]])
```

```
print(line_intersect(AD,CF))
```

2.4 Using the code in Problem 2.3, verify that \mathbf{G} is the point of intersection of BE , CF as well as AD , BE . \mathbf{G} is known as the *centroid* of $\triangle ABC$.

2.5 Graphically show that the medians of $\triangle ABC$ meet at the centroid.

2.6 Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (20)$$

3 ALTITUDES OF A TRIANGLE

3.1 In $\triangle ABC$, Let \mathbf{P} be a point on BC such that $AP \perp BC$. Then AP is defined to be an *altitude* of $\triangle ABC$.

3.2 Find the equation of AP .

3.3 Find the equations of the altitudes BQ and CR .

3.4 Find the point of intersection of AP and BQ .

Solution: Using the code in Problem 2.3, the desired point of intersection is

$$\mathbf{H} = \begin{pmatrix} 1.407 \\ 0.56 \end{pmatrix} \quad (21)$$

Interestingly, BQ and CR also intersect at the same point. Thus, the altitudes of a triangle meet at a single point known as the *orthocentre*

3.5 Find \mathbf{P} , \mathbf{Q} , \mathbf{R} .

Solution: \mathbf{P} is the intersection of AP and BC . Thus, the code in Problem 2.3 can be used to find \mathbf{P} . The desired coordinates are

$$\mathbf{P} = \begin{pmatrix} 2.32 \\ 1.24 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 1.73 \\ -1.38 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0.03 \\ 1.38 \end{pmatrix} \quad (22)$$

3.6 Draw AP , BQ and CR and verify that they meet at \mathbf{H} .

```
#This program calculates the
#intersection of AD and CF
import numpy as np

def mid_pt(B,C):
    D = (B+C)/2
    return D

def norm_vec(AB):
    return np.matmul(omat,np.matmul(AB,dvec))

def line_intersect(AD,CF):
    n1=norm_vec(AD)
    n2=norm_vec(CF)
    N =np.vstack((n1,n2))
    p = np.zeros(2)
    p[0] = np.matmul(n1,AD[:,0])
    p[1] = np.matmul(n2,CF[:,0])
    return np.matmul(np.linalg.inv(N),p)

A = np.array([-2,-2])
B = np.array([1,3])
C = np.array([4,-1])
```

4 ANGLE BISECTORS OF A TRIANGLE

4.1 In $\triangle ABC$, let U be a point on BC such that $\angle BAU = \angle CAU$. Then AU is known as the *angle bisector*.

4.2 Find the length of AB , BC and CA

Solution: The length of CA is given by

$$CA = \|\mathbf{C} - \mathbf{A}\| \quad (23)$$

The following code calculates the respective values as

$$AB = 5.83, BC = 5, CA = 6.08 \quad (24)$$

```
#This program calculates the distance
between
#two points
import numpy as np
import matplotlib.pyplot as plt

A = np.array([-2,-2])
B = np.array([1,3])
C = np.array([4,-1])

print (np.linalg.norm(A-B))
```

4.3 If AU , BV and CW are the angle bisectors, find the coordinates of \mathbf{U} , \mathbf{V} and \mathbf{W} .

Solution: Using the section formula,

$$\mathbf{W} = \frac{AW \cdot \mathbf{B} + WB \cdot \mathbf{A}}{AW + WB} = \frac{\frac{AW}{WB} \cdot \mathbf{B} + \mathbf{A}}{\frac{AW}{WB} + 1} \quad (25)$$

$$= \frac{\frac{CA}{BC} \cdot \mathbf{B} + \mathbf{A}}{\frac{CA}{BC} + 1} \quad (26)$$

$$= \frac{CA \times \mathbf{B} + BC \times \mathbf{A}}{BC + CA} \quad (27)$$

$$= \frac{a \times \mathbf{A} + b \times \mathbf{B}}{a + b} \quad (28)$$

where $a = BC$, $b = CA$, since the angle bisector has the property that

$$\frac{AW}{WB} = \frac{CA}{AB} \quad (29)$$

4.4 Write a program to find \mathbf{U} , \mathbf{V} , \mathbf{W} .

4.5 Find the intersection of AU and BV .

Solution: Using the code in Problem 2.3, the desired point of intersection is

$$\mathbf{I} = \begin{pmatrix} 1.15 \\ 0.14 \end{pmatrix} \quad (30)$$

It is easy to verify that even BV and CW meet at the same point. \mathbf{I} is known as the *incentre* of $\triangle ABC$.

4.6 Draw AU , BV and CW and verify that they meet at a point \mathbf{I} .

4.7 Verify that

$$\mathbf{I} = \frac{BC \cdot \mathbf{A} + CA \cdot \mathbf{B} + AB \cdot \mathbf{C}}{AB + BC + CA} \quad (31)$$

4.8 Let the perpendicular from \mathbf{I} to AB be IX . If the equation of AB is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \quad (32)$$

show that

$$IX = \frac{|\mathbf{n}^T (\mathbf{I} - \mathbf{A})|}{\|\mathbf{n}\|} \quad (33)$$

Verify through a Python script.

4.9 If $IY \perp BC$ and $IZ \perp CA$, verify that

$$IX = IY = IZ = r \quad (34)$$

r is known as the *inradius* of $\triangle ABC$.

4.10 Draw the incircle of $\triangle ABC$

4.11 Draw the circumcircle of $\triangle ABC$