Mixed Yee-Crank-Nicolson Finite-Difference Time-Domain Scheme for 1D Maxwell's Equations

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(Dated: February 9, 2026)

Abstract

Maxwell's equations form the foundation of classical electromagnetism and the basis for a wide range of modern technologies such as wireless communications and precision sensing. Accurate and efficient numerical solutions of these equations are therefore of significant importance. In this paper, a one-dimensional formulation of Maxwell's equations is solved using a mixed finite-difference time-domain (FDTD) approach. The magnetic field is advanced explicitly using the standard Yee leapfrog scheme, while the electric field is evolved using the Crank–Nicolson (CN) method to achieve unconditional stability. This hybrid approach combines the simplicity and efficiency of explicit FDTD with the improved stability properties of implicit time integration. Reflection coefficients at material interfaces can be computed using Fourier analysis and compared against analytical results to validate accuracy. Additional benchmarks, including convergence studies and energy conservation tests, can be used to confirm second-order accuracy and physical consistency of the scheme.

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I. PHYSICAL PROBLEM

Maxwell's equations were an important development in nineteenth-century physics, demonstrating that electricity and magnetism are intrinsically linked, unified as one fundamental force, electromagnetism. This unified theory stated that electromagnetic waves propagate at the constant speed of light, which later influenced Einstein while he was working on his theory of relativity. This was an essential step in modern physics and engineering[1][2].

Accurately modelling Maxwell's equations is critical for current technologies such as mobile phones, GPS, and the internet. It plays an important role in optimization and predicting electromagnetic phenomena. The first step towards developing a numerical solution of these equations is starting with the basics and studying a 1D system. An algorithm can be built to accurately describe the coupling between the electric and magnetic fields as they evolve through time. Then more complexities can be added into the system with multiple mediums, interfaces, and various initial conditions[1][2].

This is motivated by prior research into eddy current dampers, where a deep understanding and accurate handling of electromagnetism is crucial in modelling the interaction between a moving magnetic field and a conducting medium to optimize the geometry and achieve the required damping. Similar techniques are used to develop damping mechanisms for ultralight dark-matter and gravitational wave detectors, automotive suspension, braking systems, tuned-mass dampers for buildings in earthquake prone areas, and aerospace applications where long-lasting dampers need to handle extreme conditions. These cases are few of many that require robust numerical solutions of Maxwell's equations.

II. COMPUTATIONAL METHODS

A. Standard Yee FDTD

The finite difference time domain (FDTD) method, or Yee's method, is a widely used technique to solve complex time-dependent differential equations, such as Maxwell's equations. The spatial domain is a discretized grid with the spatial and time derivatives replaced with difference quotients, calculating a finite difference to evaluate instead[3][4].

In the typical FDTD setup, a staggered grid is used, evaluating different functions at different spatial locations. In 1D, the electric field is evaluated at the traditional, main points as without a staggered grid, and the magnetic field at the half-integer points. This is known as the leapfrog method, where the arrangement reduces the numerical artifacts and improves the stability of the solution while remaining accurate to second-order in both space and time when using centred differences[3][4].

An important condition that needs to be met is the Courant–Friedrichs–Lewy (CFL) condition, where the numerical scheme is limited in speed to ensure it doesn't travel faster than the physical wave speed[4][5]. For the 1D Maxwell equations, this is written as:

$$\alpha = \frac{c\Delta t}{\Delta x} \leq 1$$

where c is the wave speed in a specific medium, Δx is the spatial step, and Δt is the time step. Typically, α is chosen to be ≤ 0.9 to provide a safety buffer[4][3]. This explicit method is what is used to calculate the magnetic field component B_y, coming from Maxwell's curl equation in 1D:

$$\frac{\partial B_y}{\partial t} = \frac{-1}{\mu} \frac{\partial E_z}{\partial x}$$

For second-order accuracy, centred differences in time and space is used to generate the update equation:

$$\frac{B^{n+1/2}_{i+1/2} - B^{n-1/2}i + 1/2}{\Delta t} = \frac{-1}{\mu} \frac{E^n i + 1 - E^n_i}{\Delta x}$$

The FDTD method is incredibly useful for its simplicity, computational efficiency, and straightforward explicit time-stepping, allowing it to be used in a variety of cases. However, the stability is conditional on the CFL constraint, restricting how fine the grid mesh can be. This also means that fine geometrical features, such as a wire, and long propagation distances are difficult to model without the fine grid, leading to expensive computational time.

B. Crank-Nicolson

The Crank-Nicolson (CN) method, like Yee's method, is another FD scheme to solve PDEs. It is an implicit FD scheme, relying on the trapezoidal rule, allowing it to be second-order accurate for both space and time. Using both the explicit and implicit Euler formulas, a tridiagonal linear system representing the physical system can be solved at each time step[6][7].

$$\frac{\partial E_z}{\partial t} = \frac{-1}{\epsilon}\frac{\partial B_y}{\partial x}$$

From the curl Maxwell equation for the magnetic field and using the equation above, one can isolate for the electric field. The system to be solved is $AE_{interior} = b$, where A is a tridiagonal matrix, b is the known values of the system at the previous time step, and $E_{interior}$ is a matrix of the E values, excluding the boundaries which are enforced separately.

Equation for unknown E^{n+1}, with $\alpha = \frac{c^2 \Delta t^2}{2\Delta x^2}$:

$$E_i^{n+1} - \alpha\left(E_{i+1}^{n+1} - 2E_i^{n+1} + E_{i-1}^{n+1}\right) = 2E_i^n - E_i^{n-1} + \alpha\left(E_{i+1}^{n-1} - 2E_i^{n-1} + E_{i-1}^{n-1}\right)$$

The known vector b from previous time steps and enforced boundary conditions:

$$b_i = (1+2\alpha)E_i^{n+1} - \alpha E_{i+1}^{n+1} - \alpha E_{i-1}^{n+1}$$

The elements in tridiagonal matrix A are:

$$a_{i,i} = 1 + \alpha_{i-1} + \alpha_i,\ a_{i,i-1} = -\alpha_{i-1},\ a_{i,i+1} = -\alpha_i$$

The CN method offers unconditional stability and high accuracy even with larger time steps and when running long time simulations, this is much more computationally efficient compared to an explicit method[7].

However, it also has its issues, one being that spurious oscillations can occur if $\Delta t/\Delta x^2$ is too large. In addition, solving a sparse linear system can save on memory, but when dealing with highly complex systems such as introducing higher dimensions or sharp edges, the computational cost is generally not worth it[7].

C. Mixed Explicit-Implicit Scheme

Combining the explicit Yee update and the CN update for the magnetic and electric field respectively, allows the use of the benefits of both methods. The electric field has improved stability while the magnetic field is computationally cheap. This is particularly useful when the simulation time is long or larger steps are needed.

D. Fourier Analysis

In order to compute the reflection coefficient at an interface, the incident and reflected wave need to be studied against the wave in a homogeneous cavity. By choosing a specific spatial position, the time-domain data for both simulations can be extracted. The homogeneous run can be subtracted from the interface run to isolate the reflected wave[8].

Then the time-domain data can then be transformed using an FFT to be in the frequency domain. There are various signal processing techniques that can be used to improve the result. Hann windowing tapers the ends of the signal to zero, which avoids sharp cuts from false high frequencies and any spectral leakage, improving the frequency resolution[9]. Zero-padding increases the frequency resolution without changing the signal by appending zeros to the time domain signal before performing a discrete Fourier transform (DFT). The spectrum looks much smoother and can be easier to visually analyze without changing the physics. However, this method assumes that the signal outside of the window is actually zero, and if not it can introduce artifacts[8].

Peaks in the frequency spectrum can be identified and, if necessary, refined using cross-correlation or fitting techniques

III. VALIDATION AND ANALYTICAL BENCHMARKS

A. Reflection Coefficient

The reflection and transmission coefficients, R and T respectively, are specific to an interface between two mediums. Briefly, the coefficients represent how much of an incident wave on an interface gets reflected and how much gets transmitted, following $R + T = 1$. The below would be Fresnel's equations for the reflection and transmission coefficients for the 1D case, 2D would require choosing the normal component of the plane wave to the interface[1].

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Where η is the impedance of a medium, $\eta_i = sqrt(\frac{\mu_i}{\epsilon_i})$. This is a clean and straightforward way to test the validity of a simulation. The computational value from the FFT should closely match the analytical value from the above equation.

B. Convergence Study

The second-order accuracy can be checked via a convergence study, by comparing the result from a simulation in a homogeneous medium and the same scenario analytically, such as with a propagating Gaussian pulse[7]:

$$E(x,t) = exp\left(-\frac{(x - x_0 - ct)^2}{2\sigma^2}\right)$$

For each time step, the electric field can be computed for all x and c, with c being spatially dependent[7]. The root mean squared error can then be computed for each time:

$$L_2(\Delta x) = \sqrt{\sum E_{num} - E_{exact}^2}$$

This can be computed for various finer spatial step values (Δx, $\Delta x/2$, $\Delta x/4$), and plotted on a log-log plot. As it is second-order accurate, the slope of the plot should be roughly equal to 2[7]. This would be an excellent validation for the accuracy of the algorithm.

C. Energy Conservation

Assuming a lossless media, the electromagnetic energy throughout should be conserved. The total energy as a function of time is a great diagnostic:

$$U(t) = \int (\epsilon E^2 + \mu B^2)dx$$

A constant value of $U(t)$ shows strong evidence that the numerical scheme adheres to the physical conservation laws[10][4].

IV. DISCUSSION

The figures 1 and figure 2 both showcase that the FDTD framework accurately describes the physics of electromagnetic wave propagation in both dimensions. When an interface between two mediums is present, the resulting reflected and transmitted wave look consistent with the impedance mismatch without distortion. Comparing the Gaussian, Ricker,

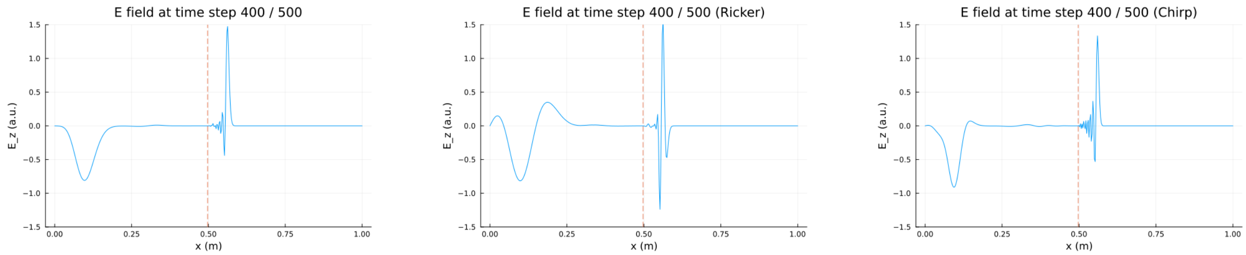

FIG. 1. The z component of the electric field in the 1D case with three different initial wavelets. From left to right, the Gaussian, Ricker, and chirp wavelets were used.

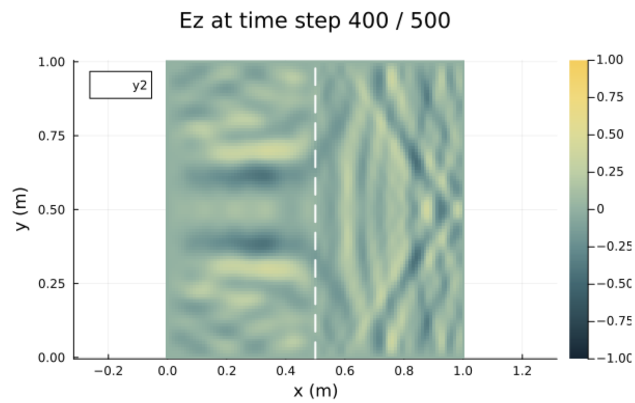

FIG. 2. The z component of the electric field in the 2D case with a Gaussian pulse as the initial condition.

and chirped sources show the increased numerical dispersion, expected from broader-band excitations.

Boundary conditions (BCs), such as a perfect electric conductor (PEC) further test the scheme. When PEC BCs are present, the tangential electric field gets set to zero while the normal component gets inversely reflected, and the normal magnetic field also goes to zero. If perfect magnetic conductor (PMC) BCs were implemented, the normal electric field gets reflected, but unlike before, the sign stays preserved[4]. The PEC result is as expected from the analytical case for electromagnetic waves, giving confidence in the update equations.

There is a clear benefit to using the mixed Yee–Crank–Nicolson approach as compared to one of either. In the magnetic field, the explicit method is simple and computationally efficient. The CFL constraint doesn't weigh as heavy due to using the CN method for the electric field, providing the stability from the implicit evolution through time and allowing for larger time steps and longer simulations. This does mean that if the spatial and time

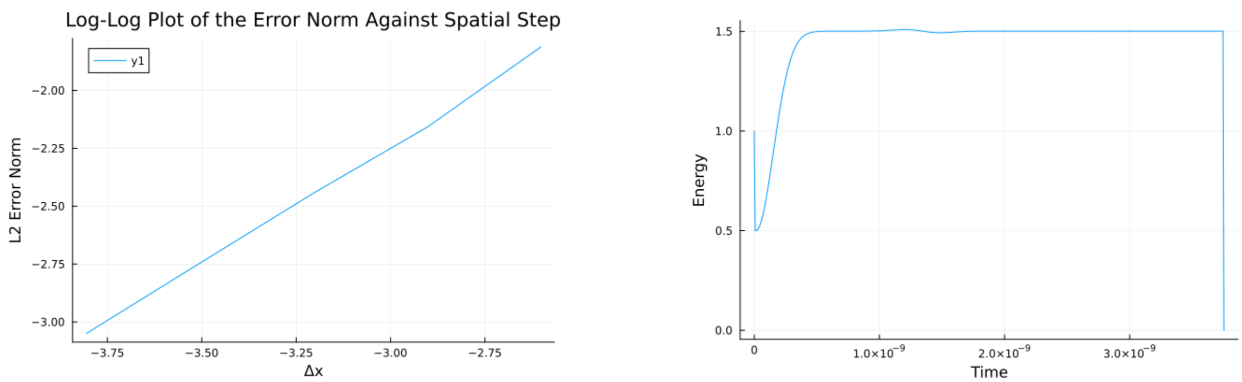

FIG. 3. The result from the convergence and energy validations. The slope from the convergence test is roughly p = 1.02, which show it is only first-order accurate. The total energy is divided by the initial value, starting at 1 but later holds steady through time at 1.5 times the initial value. This shows that once the source added additional energy to the bare cavity, the energy held constant and so was conserved.

steps aren't chosen wisely, as mentioned in an earlier section, spurious oscillations can occur if $\Delta t/\Delta x^2$ is too large.

The validation of the scheme by testing the convergence and energy conservation was observed, though not as strongly as expected (figure 3).

Ideally, the trends would show that the second-ordere convergence would lead to a slope of two, however the slope was numerically calculated to be 1.02. However, the trend can still provide some information as to what went wrong. Deviations may have arisen due to finite domain size, boundary reflections, errors in the phase or time, and other numerical dispersion issues. This result emphasizes the practical challenges of numerical validation and the need for incredible care when it comes to setting up these frameworks.

The reflection coefficient and energy conservation diagnostics proved much more successful. The initial energy would be that of the empty cavity, then the addition of the source adds more energy to the system but it holds constant, showing that the energy remains conserved after the initial introduction of the wave pulse. The reflection coefficient also showed promise, where the analytical and numerical results agreed to four significant figures, 0.7988. These qualitative validation tools are helpful even when awry, as information about the system can still be gleaned, as the case was with the convergence study.

The extension into a 2D simulation showcases a good foundation for flexibility within the framework, where in the future it can be used to model more complex, realistic geometries. Various other boundary conditions, such as perfectly matched layer (PML) BCs can be used to model absorption, dispersive, and lossy media. Within my own research investigating electromagnetic wave propagation around superconductors, modelling the flow of field around a superconducting torus, including complexities like skin effects and screening currents due to the Meissner effect, would have been incredibly useful. The mixing is a more stable method than Yee's method alone, and more computationally efficient than the CN method alone, providing a good structure to investigate the subtleties present in electromagnetic effects in real physical systems.

ACKNOWLEDGMENTS

I'd like to thank my peers Keiran and Tyler, who were immensely helpful in discussing the topics covered in class, and our TAs who were incredibly helpful during the help sessions and over email. Additionally, I would like to thank my professor, Dr. Sacchi, who taught an incredibly thorough course with interesting assignments, there was a lot I learned that I will be taking with me later on.

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