Problem - 3 Advection Eqn Analytical Soln (system)

$$u_t + u_x + v_x = 0 \qquad u(x,0) = u_0(x)$$
$$v_t + u_x - v_x = 0 \qquad v(x,0) = v_0(x)$$

The system of eqn s can be written as

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} u_x \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we diagonalize $A = P \Lambda P^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}+1/2\sqrt{2} & 1/2\sqrt{2} \\ \sqrt{2}-1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix}$$

Writing the system as

$$U_t + AU_x = 0$$

$$\Rightarrow \quad PP^{-1}U_t + P\Lambda P^{-1}U_x = 0$$

Suppose $W = P^{-1}U \Rightarrow W_t = P^{-1}U_t, \; W_x = P^{-1}U_x$

$$\Rightarrow \quad PW_t + P\Lambda W_x = 0$$
$$\Rightarrow \quad P(W_t + \Lambda W_x) = 0$$

$\therefore$ we need to solve

$$W_t + \Lambda W_x = 0 \qquad \because P \neq 0$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t + \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we get

$$(w_1)_t + \sqrt{2}\,(w_1)_x = 0$$
$$(w_2)_t - \sqrt{2}\,(w_2)_x = 0$$

$$\therefore \quad w_1(t,x) = w_{1,0}(x - \sqrt{2}t)$$
$$w_2(t,x) = w_{2,0}(x + \sqrt{2}t)$$

$$\begin{bmatrix} (w_1)_0 \\ (w_2)_0 \end{bmatrix} = W_0 = P^{-1}U = \begin{bmatrix} \sqrt{2}+1/2\sqrt{2} & 1/2\sqrt{2} \\ \sqrt{2}-1/2\sqrt{2} & -1/2\sqrt{2} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

$$\Rightarrow (w_1)_0 = \frac{(\sqrt{2}+1)u_0 + v_0}{2\sqrt{2}} = w_{1,0}$$

$$\Rightarrow (w_2)_0 = \frac{(\sqrt{2}-1)u_0 - v_0}{2\sqrt{2}} = w_{2,0}$$

And $\quad U = PW \Rightarrow \begin{bmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\therefore$ we get

$$u(t,x) = W_1 + W_2$$
$$= W_{1,0} + W_{2,0}$$
$$= \frac{(\sqrt{2}+1)\,u_0(x-\sqrt{2}t) + v_0(x-\sqrt{2}t) + (\sqrt{2}-1)\,u_0(x+\sqrt{2}t) - v_0(x+\sqrt{2}t)}{2\sqrt{2}}$$

$$\Rightarrow u(t,x) = \frac{(\sqrt{2}+1)\,u_0(x-\sqrt{2}t) + v_0(x-\sqrt{2}t) + (\sqrt{2}-1)\,u_0(x+\sqrt{2}t) - v_0(x+\sqrt{2}t)}{2\sqrt{2}}$$

and
$$v(t,x) = W_1(\sqrt{2}-1) - W_2(\sqrt{2}+1)$$
$$= W_{1,0}(\sqrt{2}-1) - W_{2,0}(\sqrt{2}+1)$$
$$= \frac{u_0(x-\sqrt{2}t) + (\sqrt{2}-1)\,v_0(x-\sqrt{2}t) - \left(u_0(x+\sqrt{2}t) - (\sqrt{2}+1)\,v_0(x+\sqrt{2}t)\right)}{2\sqrt{2}}$$

$$v(t,x) = \frac{u_0(x-\sqrt{2}t) + (\sqrt{2}-1)\,v_0(x-\sqrt{2}t) - u_0(x+\sqrt{2}t) - (\sqrt{2}+1)\,v_0(x+\sqrt{2}t)}{2\sqrt{2}}$$

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