HOMEWORK - 2

Problem - I Advection Eqn Analytical soln
(constant coefficient)

$$u_t + a u_x = f(t,x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$u(0,x) = u_0(x) = 0$$

Comparing it with general advection eqn

$$u_t + a u_x + bu = f(t,x),$$

we have $b = 0$

we know its soln is given by

$$u(t,x) = u_0(x-at)e^{-bt} + \int_0^t f(s, x-a(t-s))\, e^{-b(t-s)}\, ds$$

for the given eqn

$$u(0,x) = u_0(x) = 0 \quad , \quad b = 0$$

$$\therefore u(t,x) = \int_0^t f(s, x-a(t-s))\, ds = \int_0^t f(s, x-at+as)\, ds$$

This can be broken down into 3 cases

① $x < 0 \rightarrow f(s, x-a(t-s)) = 0$ $0 \leq s \leq t$, $x < 0$

② $x - at \leq$ and $x - at + as \geq 0$

$\Rightarrow s \geq t - \frac{x}{a}$

$\rightarrow f(s, x-a(t-s)) = 1$ for $0 \leq x \leq at$, $t - \frac{x}{a} \leq s \leq t$

③ $x - at \geq 0$ and $x - at + as \geq 0$

$\rightarrow f(s, x-a(t-s)) = 1$ for $0 \leq s \leq t$, $x \geq at$

$$\therefore \quad u(t,x) = \begin{cases} \int_0^t 0 \, ds & x < 0 \\ \int_{t-x/a}^t 1 \, ds & 0 \le x \le at \\ \int_0^t 1 \, ds & x \ge at \end{cases}$$

$$\Rightarrow u(t,x) = \begin{cases} 0 & x < 0 \\ t - (t - x/a) & 0 \le x \le at \\ t & x \ge at \end{cases}$$

$$u(t,x) = \begin{cases} 0 & x < 0 \\ x/a & 0 \le x \le at \\ t & x \ge at \end{cases}$$