Problem - 2 Advection Eqn analytical soln (variable coeff)

$$u_t + \frac{1}{1 + \frac{1}{2}\cos x} u_x = 0$$

Comparing with a general PDE of the form $u_t + a u_x = 0$ with $u(0,x) = u_0(x)$ which can be recast into a system of 2 ODE's

$$\frac{du}{dt} = 0 \quad \text{on} \quad (t, x(t)) \;, \quad \frac{dx}{dt} = a$$

$$u(x,0) = u_0(x)$$

Comparing, we get $a = \frac{1}{1 + \frac{1}{2}\cos x}$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{1 + \frac{1}{2}\cos x}$$

$$\Rightarrow \quad dx = \frac{dt}{1 + \frac{1}{2}\cos x}$$

$$\Rightarrow \quad \left(1 + \frac{1}{2}\cos x\right) dx = dt$$

Integrating both sides,

$$\Rightarrow \int_{\xi}^{x} \left(1 + \frac{1}{2}\cos x\right) dx = \int_{0}^{t} dt$$

if $u(x,0) = u_0(\xi)$

$$\Rightarrow \quad \left[x + \frac{\sin x}{2}\right]_{\xi}^{x} = [t]_0^t$$

$$\Rightarrow \quad x + \frac{\sin x}{2} - \xi - \frac{\sin \xi}{2} = t$$

$$\Rightarrow \quad \xi + \frac{\sin \xi}{2} = x + \frac{\sin x}{2} - t$$

$\Rightarrow$ if $u(x,0) = u(\xi,0) = u_0(\xi)$, ξ is the soln for

$$\xi + \frac{\sin \xi}{2} = x + \frac{\sin x}{2} - t$$

soln for the PDE $u(x,t)$ can be written as

$$u(x,t) = u_0(x-at)\, e^{-bt} + \int f(x-a(t-s), s)\, e^{-b(t-s)}\, ds$$

(e^{-bt} : $t \to 0$, $e^{-b(t-s)}$: $t \to 1$)

$$\Rightarrow \quad u(x,t) = u_0(x-at)$$

substitute $\xi = x - at$

$\tau = t$

$$\Rightarrow \quad u(t,x) = u_0(\xi)$$

where ξ is the soln for

$$\xi + \frac{\sin \xi}{2} = x + \frac{\sin x}{2} - t$$