MTH 598: NPDE
Winter 2020

Homework 1

IIIT-Delhi
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Due: January 19, 2020 (Sunday) by 23.59 hours **Total: 200 points**

Before you start, please read General Submission Guidelines on Page 3.

Problem 1: Modeling and analytical solution (30 points)

A pond initially contains 1,000,000 (one million) liters of water is initiall free of an undesirable chemical. Water containing 0.01 grams of this chemical per liter flows into the pond at the rate of 300 liters per hour, and water also flows out at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

(a) Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

(b) Solve the problem in Part (a) for $Q(t)$. How much chemical is left in the pond after 1 year?

(c) At the end of 1 year, the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

(d) Solve the initial value problem in Part (c). How much chemical remains in the pond after 1 additional year (that is, 2 years from the beginning of the problem)?

(e) How long does it take for $Q(t)$ to be reduced to 10 grams?

(f) Plot $Q(t)$ versus t for 3 years based on your analytical solutions.

Problem 2: Convergence of Euler's method (30 points)

In the forward Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, under suitable conditions on f, it can be shown that the *approximate* (numerical) solution converges to the *true* solution as the step size h decreases.

In this problem, you shall attempt showing this via an example. Consider the initial value problem given by:

$$\frac{dy}{dt} = 1 - t + y, \quad y(t_0) = y_0.$$

(a) Using any previously learned analytical method for solving a first order ODE, show that the exact solution to the above initial value problem is $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b) Assume a uniform discretization of time, that is, $t_n = t_0 + nh$ where $h > 0$ is the time step. Show that using the forward Euler method, the computation of the solution at the n^{th} time step for this ODE is:

$$y_n = (1 + h)^n (y_0 - t_0) + t_n, n \in \mathbb{N}$$

(c) Let $t > t_0$ be fixed such that $t_n = t$. Show that as $h \longrightarrow 0$ (equivalently, $n \longrightarrow \infty$), $y_n \longrightarrow \phi(t)$. Note that you will need to use $h = (t - t_0)/n$ in the forward Euler difference formula for this ODE. You may also need to use that $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

Problem 3: Implementation of single-step methods (50 points)

For each of the following problems, plot the computed solution of the initial value problem $y' = f(t, y), y(t_0) = y_0$ for times $0 \leq t \leq 2$ using:

1. forward Euler method,
2. backward Euler method,
3. Heun's method,
4. fourth-order, four step Runge-Kutta method.

For each of these methods, compute solutions for three different time steps:

(i) $h = 0.05$,

(ii) $h = 0.025$,

(iii) $h = 0.0125$.

In particular, visually compare the solutions of the various methods by highlighting their values in your plots at times $t = \{0.5, 1.0, 1.5, 2.0\}$. (Use your judgement as needed in ensuring the plotting depicts all necessary details in a reasonable manner.)

(a) $y' = (t^2 - y^2)\sin y, \quad y(0) = -1$

(b) $y' = 2t + e^{-ty}, \quad y(0) = 1$

(c) $y' = \dfrac{(y^2 + 2ty)}{(3 + t^2)}, \quad y(0) = 0.5$

Implementation/visualization notes: Make a separate figure for each of the time steps. In each problem's figure, therefore, plot computed solutions for the various methods in that figure. Make sure to clearly specify different plotting styles for various solutions in a given figure. Finally, make sure in each of your figures, you clearly label axes, legends, titles, and have the background grid turned on for ease of visualization.

This implementation problem therefore strives to also set up some minimal standards to be followed in the rest of this course for visualization.

Problem 4: Implementation of multi-step methods (40 points)

In this question, you shall recompute the solution to the ODEs of Problem 3 using multi-step methods. That is, for each of the following problems, plot the computed solution of the initial value problem $y' = f(t, y), y(t_0) = y_0$ for times $0 \le t \le 2$ using:

1. fourth-order Adam-Bashworth method,
2. fourth-order Adam-Moulton method, and
3. fourth-order predictor corrector method with Adam-Bashworth method for prediction step and Adam-Moulton method for the correction step.

For each of these methods, compute solutions for two different time steps:

(i) $h = 0.1$,

(ii) $h = 0.05$,

For bootstrapping each of the methods (that is, for finding y_1, y_2 and y_3, as may be necessary), you should use the fourth-order four-step Runge-Kutta method. Again, visually compare the solutions of the various methods by highlighting their values in your plots at times $t = \{0.5, 1.0, 1.5, 2.0\}$.

Problem 5: Stiff problem (50 points)

Consider the initial value problem:

$$y' = -100y + 100t + 1, \quad y(0) = 1.$$

Verify that the analytical solution to this problem is:

$$\phi(t) = e^{-100t} + t.$$

Plot the graph of this solution from $t = 0$ to $t = 1$. (*Question:* How do you think you should plot this?) Notice that the graph will have a "thin layer" to the right of $t = 0$, and ensure that you use a sufficiently fine discretization of the time to capture this *boundary layer*. Past the boundary layer, the graph of the solution should essentially look like a straight line.

Determine how small a step size h must be chosen (via numercial experimentation) to ensure that the error in the solution, defined to be $e_n = \phi(t_n) - y_n$, at $t = 0.05$ and at $t = 0.1$ is less than 0.0005. Find this step size for forward Euler, backward Euler, and fourth-order four step Runge-Kutta method.

General Submission Guidelines

- **This assignment should be answered individually.**
- **You will be penalized for copying or any plagiarism with an automatic zero.**
- IIIT-Delhi academic policies on honesty and integrity apply to all HWs. This includes not copying from one another, from the internet, a book, or any other online or offline source. A repeat offense will be reported to academic administration.
- **Start working on your solutions early and don't wait until close to due date.**
- If you discuss or read secondary sources (other than class notes), please list all your discussion partners and/or secondary sources in your writeup. Failure to do so will constitute violation of honor code.
- Each problem should have a clear and concise write-up.
- Please clearly show all steps involved in theoretical problems.
- The answer to all theoretical problems can be submitted in a PDF file or turned in handwritten.
- If your code generates an output figure or table, please provide all such results in a single PDF file along with your code submission.
- No late submission will be allowed unless I explicitly permit it.
- All PDF files should be submitted via Google Classroom.
- You will need to write a separate code for each problem and sometimes for each subproblem as well. You should name each such file as `problem_`n`.py` where n is the problem number. For example, your files could be named `problem_1.py`, `problem_4a.py` and `problem_4b.py` in this HW.
- *Python tip:* You can import Python modules as follows:

```python
from __future__ import division
import numpy as np
import scipy as sp
import matplotlib.pyplot as plt
import numpy.linalg as npla
import scipy.linalg as spla
```

Every code you write will have one or more of these import statements.