Problem - 5 Stability of Finite Difference Scheme

$$U_m^{n+1} = \alpha U_{m+1}^n + \beta U_{m-1}^n$$

We want $\sum_{m\in\mathbb{Z}} |U_m^{n+1}|^2$ to be bounded for the scheme to be stable

$$\therefore \sum_{m\in\mathbb{Z}} |U_m^{n+1}|^2 = \sum_{m\in\mathbb{Z}} |\alpha U_{m+1}^n + \beta U_{m-1}^n|^2,$$ Using triangle inequality

$$\leq \sum_{m\in\mathbb{Z}} \left(|\alpha U_{m+1}^n|^2 + |\beta U_{m-1}^n|^2 + 2|\alpha U_{m+1}^n||\beta U_{m-1}^n| \right)$$

$$\leq \sum_{m\in\mathbb{Z}} \left(|\alpha|^2|U_{m+1}^n|^2 + |\beta|^2|U_{m-1}^n|^2 + 2|\alpha||\beta||U_{m+1}^n||U_{m-1}^n| \right)$$

Also, $2|U_{m+1}^n||U_{m-1}^n| \leq |U_{m+1}^n|^2 + |U_{m-1}^n|^2$

$$\Rightarrow \sum_{m\in\mathbb{Z}} |U_m^{n+1}|^2 \leq \sum_{m\in\mathbb{Z}} \left(|\alpha|^2|U_{m+1}^n|^2 + |\beta|^2|U_{m-1}^n|^2 + |\alpha||\beta| \left(|U_{m+1}^n|^2 + |U_{m-1}^n|^2 \right) \right)$$

$$\leq \sum_{m\in\mathbb{Z}} \left(|\alpha|^2|U_{m+1}^n|^2 + |\alpha||\beta||U_{m+1}^n|^2 \right) + \sum_{m\in\mathbb{Z}} \left(|\beta|^2|U_{m-1}^n|^2 + |\alpha||\beta||U_{m-1}^n|^2 \right)$$

since $m \in \mathbb{Z}$, $U_{m+1}^n = U_m^n$, $U_{m-1}^n = U_m^n$

$$\leq \sum_{m\in\mathbb{Z}} \left(|\alpha|^2|U_m^n|^2 + |\beta|^2|U_m^n|^2 + 2|\alpha||\beta||U_m^n|^2 \right)$$

$$\leq \sum_{m\in\mathbb{Z}} (|\alpha| + |\beta|)^2 |U_m^n|^2$$

$$\Rightarrow \sum_{m\in\mathbb{Z}} |U_m^{n+1}|^2 \leq (|\alpha| + |\beta|)^2 \sum_{m\in\mathbb{Z}} |U_m^n|^2$$

$$\leq (|\alpha| + |\beta|)^{2.2} \sum_{m\in\mathbb{Z}} |U_m^{n-1}|^2$$

$$\vdots$$

$$\Rightarrow \sum_{m\in\mathbb{Z}} |U_m^{n+1}|^2 \leq (|\alpha| + |\beta|)^{2n} \sum_{m\in\mathbb{Z}} |U_m^0|^2$$

$\therefore$ the scheme is stable if

$$(|\alpha| + |\beta|)^{2n} < \infty$$

(doesn't explode as $n \to \infty$)

$$\Longleftrightarrow \quad |\alpha| + |\beta| \le 1$$

For Lax Friedrich's scheme

$$\frac{U_m^{n+1} - \frac{1}{2}(U_{m+1}^n + U_{m-1}^n)}{k} + a\frac{(U_{m+1}^n - U_{m-1}^n)}{2h} = 0$$

$$\Rightarrow \frac{U_m^{n+1}}{k} = \frac{1}{2k}(U_{m+1}^n + U_{m-1}^n) - \frac{a}{2h}(U_{m+1}^n - U_{m-1}^n)$$

$$\Rightarrow U_m^{n+1} = \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - \frac{ak}{2h}(U_{m+1}^n - U_{m-1}^n)$$

$$= \frac{1}{2}(U_{m+1}^n + U_{m-1}^n) - \frac{a\lambda}{2}(U_{m+1}^n - U_{m-1}^n) \qquad \because \lambda = k/h$$

$$= U_{m+1}^n\left(\frac{1-a\lambda}{2}\right) + U_{m-1}^n\left(\frac{1+a\lambda}{2}\right)$$

$$\therefore \alpha = \frac{1-a\lambda}{2}, \quad \beta = \frac{1+a\lambda}{2}$$

$$|\alpha| + |\beta| \le 1$$

$$\Rightarrow \left|\frac{1-a\lambda}{2}\right| + \left|\frac{1+a\lambda}{2}\right| \le 1$$

Case (1) $\quad 1 - a\lambda < 0, \quad 1 + a\lambda > 0$

$$\frac{a\lambda - 1}{2} + \frac{1 + a\lambda}{2} \le 1$$

$$\Rightarrow a\lambda \le 1$$

$$\Rightarrow |a\lambda| \le 1$$

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Case 2 $1 - a\lambda > 0$, $1 + a\lambda > 0$

$$\frac{1 - a\lambda}{2} + \frac{1 + a\lambda}{2} = 1 \leq 1 \qquad \text{(trivial)}$$

Case 3 $1 - a\lambda > 0$, $1 + a\lambda < 0$

$$\frac{1 - a\lambda}{2} + \frac{-1 - a\lambda}{2} \leq 1$$

$\Rightarrow \quad -a\lambda \leq 1$ (trivial $\because$ $\lambda > 0$ so $-a\lambda < 0$)

$\Rightarrow \quad -a \leq \frac{1}{\lambda}$

$\Rightarrow \quad \lambda \geq \frac{-1}{a}$ but $\lambda > 0$

$\therefore$ from 1 $|a\lambda| \leq 1$ for stability.