Problem-4 FTCS Consistency

Given any PDE $Pu = f$: $\frac{\partial u}{\partial t} + \frac{a \partial u}{\partial x} = 0 \quad t > 0$
(advection)

we express the finite-difference discretization as

$$P_{h,k} v = f : \frac{v_m^{n+1} - v_m^n}{k} + \frac{a\left(v_{m+1}^n - v_{m-1}^n\right)}{2h}$$

(forward time centered space scheme)

let $\phi(t,x)$ be a sufficiently smooth function in $\mathbb{R}^+ \times \mathbb{R}$. we write taylor series expansion for the following

$$\phi_m^{n+1} = \phi(t_n + k, x_m) = \phi(t_n, x_m) + k\partial_t \phi_m^n + \frac{k^2}{2}\partial_{tt}\phi_m^n + O(k^3)$$

$$\Rightarrow \phi_m^{n+1} = \phi_m^n + k\partial_t\phi_m^n + \frac{k^2}{2}\partial_{tt}\phi_m^n + O(k^3) \quad - ①$$

$$\phi_{m+1}^n = \phi(t_n, x_m + h) = \phi(t_n, x_m) + h\partial_x\phi_m^n + \frac{h^2}{2}\partial_{xx}\phi_m^n + O(h^3)$$

$$\Rightarrow \phi_{m+1}^n = \phi_m^n + h\partial_x\phi_m^n + \frac{h^2}{2}\partial_{xx}\phi_m^n + O(h^3) \quad - ②$$

Similarly $\phi_{m-1}^n = \phi_m^n - h\partial_x\phi_m^n + \frac{h^2}{2}\partial_{xx}\phi_m^n - O(h^3) \quad - ③$

$$\therefore P_{h,k}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + \frac{a\left(\phi_{m+1}^n - \phi_{m-1}^n\right)}{2h}$$

Plugging values from ①, ②, ③, we get

$$\Rightarrow P_{h,k}\phi = \partial_t\phi_m^n + \frac{k}{2}\partial_{tt}\phi_m^n + O(k^2) + a\left(\partial_x\phi_m^n + O(h^2)\right)$$

$$P_{h,k}\phi = \partial_t\phi_m^n + a\partial_x\phi_m^n + \frac{k}{2}\partial_{tt}\phi_m^n + O(h^2 + k^2)$$

Now we write

$$P\phi - P_{h,k}\phi = (\partial_t\phi + a\partial_x\phi) - \left(\partial_t\phi + a\partial_x\phi + \tfrac{k}{2}\partial_{tt}\phi + O(h^2+k^2)\right)$$

$$\Rightarrow |P\phi - P_{h,k}\phi| = \left|\frac{k}{2}\partial_{tt}\phi + O(h^2+k^2)\right|$$

we use triangle inequality

$$\Rightarrow |P\phi - P_{h,k}\phi| \le \left|\tfrac{k}{2}\partial_{tt}\phi\right| + |O(h^2+k^2)|$$

$$\le \underbrace{\left|\frac{k}{2}\frac{\partial^2(\phi)}{\partial t^2}\right|}_{} + \underbrace{|c(h^2+k^2)|}_{\text{tends to } 0 \text{ as } h,k \to 0}, \quad c>0$$

is a bounded constant at (t,x) due to smoothness of $\phi(t,x)$ and $\to 0$ as $k \to 0$

$\therefore$ $|P\phi - P_{h,k}\phi| \longrightarrow 0$ as $h,k \longrightarrow 0$

Thus, FTCS is consistent.