Homework 1

IIIT-Delhi
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Due: February 9, 2020 (Sunday) by 23.59 hours **Total: 200 points**

Before you start, please read General Submission Guidelines on Page 3.

Problem 1. Advection eqn. analytical soln. (constant coeff.) (20 points)

Consider the initial value problem for the equation: $u_t + a\,u_x = f(t,x)$ where u_t denotes $\dfrac{\partial u}{\partial t}$ and u_x denotes $\dfrac{\partial u}{\partial x}$. Let $u(0,x) = 0$ and let $f(t,x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$
Assume that $a > 0$, and show that the solution is given by:

$$u(t,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0, \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0. \end{cases}$$

Problem 2. Advection eqn. analytical soln. (variable coeff.) (20 points)

Solve the initial value problem for:

$$u_t + \frac{1}{1 + \frac{1}{2}\cos x} u_x = 0.$$

Show that the solution is given by $u(t,x) = u_0(\xi)$ where ξ is the unique solution of:

$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

Problem 3. Advection eqn. analytical soln. (system) (10 points)

Obtain the solution of the system:

$$\begin{aligned} u_t + u_x + v_x &= 0, \quad & u(x,0) &= u_0(x), \\ v_t + u_x - v_x &= 0, \quad & v(x,0) &= v_0(x). \end{aligned}$$

Problem 4. FTCS consistency (10 points)

Show that the forward time central space (FTCS) scheme for the constant coefficient advection equation is consistent with the partial differential equation.

Problem 5. Stability of finite difference scheme (10 points)

Show that schemes of the form: $U_m^{n+1} = \alpha U_{m+1}^n + \beta U_{m-1}^n$ are stable if $|\alpha| + |\beta| \leq 1$. Conclude therefore that Lax-Friedrichs scheme is stable if $|a\lambda| \leq 1$ where $\lambda = k/h$.

Problem 6. Advection equation finite difference solution (100 points)

For $x \in [-3, 3]$ and $t \in [0, 2.4]$, solve the advection equation $u_t + u_x = 0$ with initial data:

$$u(0, x) = \begin{cases} \cos^2(\pi x) & \text{if } |x| \leq \dfrac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and boundary data $u(t, -3) = 0$.

Use the following four schemes with spatial steps $h = [1/10, 1/20, 1/40, 1/80]$.

(a) FTBS with $\lambda = 0.8$,

(b) FTCS with $\lambda = 0.8$,

(c) Lax-Friedrichs scheme with $\lambda = 0.8$ and $\lambda = 1.6$,

(d) Leapfrog Scheme with $\lambda = 0.8$,

where $\lambda = k/h$ and k is the time step.

For schemes (b), (c) and (d), at the right boundary use condition $U^n_{M+1} = U^n_M$ where M is such that $x_{M+1} = 3$. For scheme (d), use scheme (b) to compute the solution at $n = 1$.

For each scheme, determine whether the scheme is a *useful* or *useless* scheme. For this purpose, we will define these terms as follows. A scheme is *useless* if $|U^n_m| \geq 5$ for any value of (m, n). Otherwise, it is *useful*.

In each case, plot the solution only for the last time that they were computed. However, you will need to check *useful*-ness or *useless*-ness by keeping track of the computed solution for all j, k as you have been doing so for the heat equation.

Problem 7. Advection equation system solution (30 points)

Solve the system:

$$\begin{aligned} u_t + \frac{1}{3}(t-2)u_x + \frac{2}{3}(t+1)w_x + \frac{1}{3}u &= 0, \\ w_t + \frac{1}{3}(t+1)u_x + \frac{1}{3}(2t-1)w_x - \frac{1}{3}w &= 0, \end{aligned}$$

by Lax-Friedrichs scheme. (That is approximate each time derivative as in the Lax-Friedrichs scheme for the one-dimensional advection equation, and each spatial derivative by central differences again as in the the Lax-Friedrichs scheme for the one-dimensional advection equation.) The initial data are:

$$\begin{aligned} u(0, x) &= \max\{0, 1 - |x|\}, \\ w(0, x) &= \max\{0, 1 - 2|x|\}, \end{aligned}$$

Take $x \in [-3, 3]$ and $t \in [0, 2]$. Also, take $h = 1/20$ and $\lambda = 1/2$. At each boundary, set $u = 0$ and w to equal the newly computed value one grid point inside from the boundary. Plot the solutions for u and w as a function of $t \in [0, 2]$.

General Submission Guidelines

- **This assignment should be answered individually.**
- **You will be penalized for copying or any plagiarism with an automatic zero.**
- IIIT-Delhi academic policies on honesty and integrity apply to all HWs. This includes not copying from one another, from the internet, a book, or any other online or offline source. A repeat offense will be reported to academic administration.
- **Start working on your solutions early and don't wait until close to due date.**
- If you discuss or read secondary sources (other than class notes), please list all your discussion partners and/or secondary sources in your writeup. Failure to do so will constitute violation of honor code.
- Each problem should have a clear and concise write-up.
- Please clearly show all steps involved in theoretical problems.
- The answer to all theoretical problems can be submitted in a PDF file or turned in handwritten.
- If your code generates an output figure or table, please provide all such results in a single PDF file along with your code submission.
- No late submission will be allowed unless I explicitly permit it.
- All PDF files should be submitted via Google Classroom.
- You will need to write a separate code for each problem and sometimes for each subproblem as well. You should name each such file as `problem_`n`.py` where n is the problem number. For example, your files could be named `problem_1.py`, `problem_4a.py` and `problem_4b.py` in this HW.
- *Python tip:* You can import Python modules as follows:

```
from __future__ import division
import numpy as np
import scipy as sp
import matplotlib.pyplot as plt
import numpy.linalg as npla
import scipy.linalg as spla
```

  Every code you write will have one or more of these import statements.