MTH 598, NPDE, Winter 2020 | **Midterm (50 + 50 = 100 points)**

Date: Feb. 16, 2020 | **In-class exam: 15:00 to 16:00 hours**

Theory Exam (50 points, 1 hour)

Problem 1. True or False 3 x 2 = 6 points, 6 minutes

For each of the following statements, please state whether it is *True* or *False*. You do not have to provide a justification.

(a) In approximating a solutions of an ordinary differential equation (ODE) numerically, the global error grows only if the solution to the ODE is unstable.

(b) In numerically approximating a stable solution of an ODE, one can take arbitrarily large time steps using an unconditionally stable method, and still achieve any required accuracy.

(c) For solving a time dependent partial differential equation (PDE), a finite difference method that is both consistent and stable converges to the true solution as the spatial and time step sizes go to zero.

Problem 2. Forward Euler Method 7.5 + 7.5 + 5 = 20 points, 25 minutes

Consider the initial value problem $y' = f(t,y)$ with $y(t_0) = y_0$. When the functions $f(t,y)$ and $\dfrac{\partial f}{\partial y}(t,y)$ are continuous in a closed, bounded region R of the t-$y-$plane that includes the point (t_0, y_0), then there exists a constant L such that: $|f(t,y) - f(t,\widetilde{y})| \leq L|y - \widetilde{y}|$,where (t,y) and $(t,\widetilde{y})$ are any two points in R with the same t coordinate.

Let $\phi(t)$ be the unique solution of the above initial value problem, that is, $\phi'(t) = f(t,\phi(t))$. And, let $\phi(t_n + h) := \phi(t_{n+1})$. Then, the Taylor expansion of ϕ around t_n is given by:

$$\phi(t_{n+1}) = \phi(t_n) + hf(t_n, \phi(t_n)) + \frac{1}{2}\phi''(\overline{t})h^2,$$

where $t_n < \overline{t} < t_{n+1}$. Let E_n denote the global truncation error, that is, $E_n := \phi(t_n) - y_n$.

(a) Show that:

$$|E_{n+1}| \leq |E_n| + h|f(t,\phi(t_n) - f(t_n, y_n))| + \frac{1}{2}h^2|\phi''(\overline{t_n})| \leq \alpha|E_n| + \beta h^2,$$

where $\alpha := 1 + hL$ and $\beta := \max \dfrac{1}{2}|\phi''(t)|$ on $t_0 \leq t \leq t_n$.

(b) Assume that $E_0 = 0$ and E_n satisfies the above equation (in part (a)). Then show that:

$$|E_n| \leq \beta h^2 \frac{(\alpha^n - 1)}{(\alpha - 1)},$$

for $\alpha \neq 1$, and α and β are as defined in part (a).

(c) Using $(1 + hL)^n \leq e^{nhL}$, and the result in part (b), show that:

$$|E_n| \leq \frac{e^{nhL} - 1}{L}\beta h.$$

Note: (read at home!) A logical conclusion for showing part (c) is that if we select an ending time $T > t_0$ and choose the step size h so that n steps are required to traverse the interval $[t_0, T]$, then $nh = Tt_0$, and:

$$|E_n| \leq \frac{e^{(T-t_0)L} - 1}{L}\beta h = Kh,$$

where K depends only on the length $T - t_0$ of the interval, and the constants L and β are determined only by f. Thus, the global error for the forward Euler method is $\mathcal{O}(h)$.

Problem 3. FTBS Scheme — 4 points, 5 minutes

Consider the nonlinear advection equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

Write down a finite difference discretization of this PDE using a first-order forward difference scheme in time, and a first-order backward difference scheme in space.

Problem 4. von Neumann Stability — 8 + 12 = 20 points, 24 minutes

For the advection equation: $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$ with $a \in \mathbb{R}$, $t > 0$ and $x \in \mathbb{R}$, and $u(0, x) = u_0(x)$, consider the following forward time, centered space (FTCS) discretization:

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0,$$

where h, k are positive numbers such that $(t_n, x_m) = (nk, mh)$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

The amplification factor $g(h\xi)$ relates the Fourier transformed solution at time step $n+1$ with the Fourier transformed solution in the previous time step: $\widehat{U}^{n+1}(\xi) = g(h\xi)\widehat{U}^n$. A simple procedure for computation of this amplification factor is to replace U_m^n in the numerical scheme by $g^n e^{imh\xi}$, and so on (depending on the subscript label)!

Recall that in von Neumann analysis, a scheme is stable if and only if there is a constant K (independent of $h\xi$, k, h) such that $|g(h\xi, k, h)| \leq 1 + Kk$ or if $g(h\xi, k, h)$ is independent of h and k, then $|g(h\xi)| \leq 1$.

With this background, please answer the following two questions.

(a) Replace each U_m^n by $g^n e^{imh\xi}$ and compute the amplification factor g for the FTCS scheme.

(b) Show that if one takes $k = h^2$, then the FTCS scheme is stable (under the requirement of von Neumann stability).

Take-home Exam (50 points)

Deadline: Feb. 18, 2020, Tuesday by 23:59 hours.

Note: This is an individual take-home examination. You are not allowed to consult with any sources – online or offline. All necessary information for solving this problem is contained within this page. You should work on your solution on your own. Please email me if you have any questions.

Consider the following variable-coefficient advection equation:

$$\frac{\partial u}{\partial t} + (1+t)\frac{\partial u}{\partial x} = 0, \quad \text{in } (x,t) \in (0,1) \times \mathbb{R}^+,$$

with initial condition $u(x,0) = x^2$ on $[0,1]$, and boundary condition $u(0,t) = 0$. (Note that the initial and boundary conditions agree at $x = 0$.) The analytical solution for this PDE is the following finite difference scheme:

$$u(x,t) = \begin{cases} \left(x - \frac{3}{2}t^2\right)^2, & \text{if } x \geq \frac{3}{2}t^2, \\ 0, & \text{if } x < \frac{3}{2}t^2. \end{cases}$$

The second-order accurate (in both time and space) *Wendoff box scheme* for this advection equation is given by:

$$\frac{U_m^{n+1} + U_{m+1}^{n+1} - U_m^n - U_{m+1}^n}{2k} + (1 + t_{n+1})\frac{U_{m+1}^{n+1} + U_{m+1}^n - U_m^{n+1} - U_m^n}{2h} = 0,$$

with the initial and boundary conditions discretized as:

$$U_0^n = g(t_n), \quad \text{and} \quad U_m^0 = u(mh, 0).$$

Computationally solve the given advection equation using the Wendoff box scheme for the following four choices of (h,k):

(a) $h = k = 1/10$,

(b) $h = k = 1/20$,

(c) $h = k = 1/40$,

(d) $h = k = 1/80$,

and calculate the error $|U_m^n - u(mh, nk)|$ at $(x,t) = (1, 1/2)$ for each of these choices. For each choice of (h,k), plot the solution $u(x,t)$ for all $x \in (0,1)$ and at $t = \{0, 1/4, 1/2\}$. Plot also the computed error in a $\log - \log$ plot with the x-axis being $h = k$, and verify that the convergence is indeed $O(h^2)$. (On the $\log - \log$ plot, show also the curve (h, h^2) to demonstrate this visually.)

Submission Checklist Please submite the following via Google Classroom.

(1) Your solution code (in Python for example) named `midterm.py` (or similar with appropriate extension).

(2) A PDF named `midterm.pdf` containing the following clearly labeled figures.

  (a) Three plots of the analytical solution at times $t = 0$, $t = 1/4$ and $t = 1/2$.

  (b) Three plots of the discrete solution at times $t = 0$, $t = 1/4$, $t = 1/2$ for each of the choices of the spatial and time steps.

  (c) A single $\log - \log$ plot showing the second-order convergence of the error for each of the choices of spatial and time steps computed at $x = 1$, $t = 1/2$.