Homework 2

Exercise 1 - Caley-Hamilton

Find: A^{10} $\quad e^{At}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using C-H theorem we know that A^{10} can be expressed as a linear combination of eigenvalues

$\because$ A is an upper triangular matrix, its eigenvalues are:

$\lambda_1 = 1 \quad \lambda_2 = 0 \quad \lambda_3 = 1$ [Repeated]

$$\left.\begin{array}{l} f(\lambda) = \lambda^{10} \\ g(\lambda) = \beta_2\lambda^2 + \beta_1\lambda + \beta_0 \end{array}\right\}$$

C-H theorem:

$$f(\lambda) - g(\lambda) = \lambda^{10} - \beta_2\lambda^2 - \beta_1\lambda + \beta_0 \quad — (1)$$

Putting $\lambda = 1$ in Eq^n (1):

$$1 = \beta_2 + \beta_1 + \beta_0 \quad — (2)$$

$\lambda = 0$ in Eq^n (1): $\beta_0 = 0 \quad — (3)$

Taking derivative:

$$\frac{df}{d\lambda} = \frac{dg}{d\lambda}$$

$$10\lambda^9 = 2\beta_2\lambda + \beta_1 \quad — (4)$$

$\lambda = 1 \quad \Rightarrow \quad 10 = 2\beta_2 + \beta_1$

$1 = \beta_2 + \beta_1$

$\Rightarrow \beta_2 = 9$

$\beta_1 = -8$

$$\boxed{\begin{aligned} \therefore\ \beta_0 &= 0 \\ \beta_1 &= -8 \\ \beta_2 &= 9 \end{aligned}}$$

$f(A) = g(A)$

$A^{10} = 9A^2 - 8A$

$$= 9\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 9\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Finding $e^{At} = f(A)$

$f(\lambda) = e^{\lambda t} = g(\lambda) = \beta_2\lambda^2 + \beta_1\lambda + \beta_0$

$\lambda = 1 \Rightarrow e^t = \beta_2 + \beta_1 + \cancel{\beta_0}^{1}$

$\lambda = 0 \Rightarrow 1 = \beta_0$ — 6

Taking derivative:

$\frac{df}{d\lambda} = \frac{dg}{d\lambda} \Rightarrow te^{\lambda t} = 2\beta_2\lambda + \beta_1$

$\lambda = 1$

$te^t = 2\beta_2 + \beta_1$ — 7

$(-)\quad e^t = \beta_2 + \beta_1 + 1$

$e^t(t-1) = \beta_2 + 1$

$\therefore \beta_2 = e^t(t-1) - 1$

$\beta_1 = e^t - \cancel{1} - e^t(t-1) + \cancel{1}$

$= e^t[1 - t + 1] = e^t[2-t]$

$$\boxed{\begin{aligned} \beta_0 &= 1 \\ \beta_1 &= e^t(2-t) \\ \beta_2 &= e^t(t-1)-1 \end{aligned}}$$

$$\therefore e^{At} = \left[e^t(t-1) - 1\right]A^2 + e^t\left[2-t\right]A + I$$

$$= \left[e^t(t-1) - 1\right]\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + e^t\left[2-t\right]\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + I$$

$$= \underbrace{\left[e^t(t-1) - 1\right]}_{A}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + \underbrace{e^t(2-t)}_{B}\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix}e^t & e^t - 1 & e^t(t-1) - 1\\ 0 & 1 & e^t - 1\\ 0 & 0 & e^t\end{bmatrix}}$$

Exercise 2 - Linear dynamics Solution

$$\frac{dx_1}{dt} = -\alpha x_1 + u \qquad \dot{x}_1 = -\alpha x_1 + u$$

$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 \qquad \dot{x}_2 = \alpha x_1 - \beta x_2$$

Modelling the above equations in state space form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{u}_{U} \quad \text{— (1)}$$

$$\dot{X} = AX + BU$$

We need to find $x_1(5)$ & $x_2(5)$.

$\therefore$ Define $y = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} \underbrace{u}_{U} \quad \text{(2)}$$

$$y = CX + DU$$

Given: $\alpha = 0.1$, $\beta = 0.2$, $u = 1$

$x_1(0) = 2$, $x_2(0) = 1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1.$$

We know that,

$$y = Ce^{A(t-t_0)} x(t_0) + Ce^{At} \int_{t_0}^{t} e^{-At} Bu + Du \quad \text{— (3)}$$

where $t_0 = 0$ & $t = 5$.

Finding e^{At}:

$$A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}$$

Eigen values : $\left.\begin{array}{l} \lambda_1 = -0.1 \\ \lambda_2 = -0.2 \end{array}\right\}$ upper Δ^{le} matrix

$f(\lambda) = e^{\lambda t}$

$g(\lambda) = \beta_1 \lambda + \beta_0$

$f(\lambda) = g(\lambda)$

$\lambda_1 = -0.1$

$$f(\lambda_1) = e^{-0.1t} = \beta_1(-0.1) + \beta_0. \quad — 4$$

$\lambda_2 = -0.2$

$$e^{-0.2t} = \beta_1(-0.2) + \beta_0 \quad — 5$$

$5 - 4 \Rightarrow$

$$e^{-0.2t} - e^{-0.1t} = -0.1\,\beta_1$$

$$\beta_1 = -\left(\frac{e^{-0.2t} - e^{-0.1t}}{0.1}\right)$$

$$\beta_0 = e^{-0.2t} - 2\left[e^{-0.2t} - e^{-0.1t}\right] = 2e^{-0.1t} - e^{-0.2t}$$

$$\boxed{\begin{aligned} \beta_1 &= \frac{e^{-0.1t} - e^{-0.2t}}{0.1} \\ \beta_0 &= 2e^{-0.1t} - e^{-0.2t} \end{aligned}}$$

$$e^{At} = \beta_1 A + \beta_0 I \Rightarrow \frac{e^{-0.1t} - e^{-0.2t}}{0.1}\begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}$$

$$+ \left(2e^{-0.1t} - e^{-0.2t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cancel{e^{-0.2t}} - e^{-0.1t} + 2e^{-0.1t} - \cancel{e^{-0.2t}} & 0 \\ e^{-0.1t} - e^{-0.2t} & -2e^{-0.1t} + 2e^{-0.2t} + 2e^{-0.1t} - e^{-0.2t} \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}}$$

Using equation 3 :

$$y(t) = C e^{A(t-t_0)} x(t_0) + C \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

$t_0 = 0$

$x_2(0) = 1$

$x_1(0) = 2$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \int_0^5 \cdot \begin{bmatrix} e^{-0.1(t-\tau)} & 0 \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} & e^{-0.2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \cdot d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \int_0^5 \begin{bmatrix} e^{-0.1(t-\tau)} \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} \int e^{-0.1(t-\tau)} \cdot d\tau \\ \int e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} . d\tau \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{0.1} e^{-0.1(t-\tau)} \Big|_0^5 \\ \frac{1}{0.1} e^{-0.1(t-\tau)} - \frac{1}{0.2} e^{-0.2(t-\tau)} \Big|_0^5 \end{bmatrix}$$

$$\begin{bmatrix} 10\left[e^{-0.1(t-5)} - e^{-0.1t}\right] \\ 10\left[e^{-0.1(t-5)} - e^{-0.1t}\right] - 5\left[e^{-0.2(t-5)} - e^{-0.2t}\right] \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 10e^{-0.1(t-5)} - 8e^{-0.1t} \\ 10e^{-0.1(t-5)} - 5e^{-0.2(t-5)} - 8e^{-0.1t} + 4e^{-0.2t} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1(5) \\ x_2(5) \end{bmatrix} = \begin{bmatrix} 10 - 8e^{-0.5} \\ 10 - 5 - 8e^{-0.5} + 4e^{-1} \end{bmatrix}$$

$$\begin{bmatrix} x_1(5) \\ x_2(5) \end{bmatrix} = \begin{bmatrix} 5.1477 \\ 1.619 \end{bmatrix}$$

Exercise 3 - Jordan form, decomposition

#1: $A = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Upper Δ^{le} matrix: Eigenvalues $\rightarrow \lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

for $\lambda_1 = 1$,

$$[A - \lambda_1 I] v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 4 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v_1 = 0$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies the above eq^n

for $\lambda_2 = 2$,

$$[A - \lambda_2 I] v_2 = 0 \Rightarrow \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_2 = 0$$

$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ satisfies the above eq^n

for $\lambda_3 = 3$,

$$[A - \lambda_3 I] v_3 = 0 \Rightarrow \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0$$

$v_3 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ satisfies the above eqⁿ

$\therefore A = MJM^{-1}$

$$= \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#2: $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$

Finding eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{vmatrix} = 0$

$$-\lambda(3\lambda + \lambda^2 + 4) - 1(0+2) = 0$$

$$-3\lambda^2 - \lambda^3 - 4\lambda - 2 = 0$$

$$\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

$\therefore$ Eigenvalues: $\lambda_1 = -1$

$\lambda_2 = -1+i$

$\lambda_3 = -1-i$

for $\lambda_1 = -1$,

$[A - \lambda_1 I] v_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} v_1 = 0$

$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{matrix}$

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ satisfies the above eq^n

for $\lambda_2 = -1 + i$

$[A - \lambda_2 I] v_2 = 0 \Rightarrow \begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & -2-i \end{bmatrix} v_2 = 0$.

~~$-i + h$~~

$(1-i)$

$v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ solving for x, y & z:

$$(1-i)x + y = 0$$
$$(1-i)y + z = 0$$
$$-2x - 4y - z(2+i) = 0$$

$y = x(i-1)$

$z = y(i-1) = x(i^2 + 1 - 2i) = x(-2i)$

$\therefore v_2 = \begin{bmatrix} 1 \\ i-1 \\ -2i \end{bmatrix}$ satisfies this eq^n

for $\lambda_3 = -1-i$

$$[A - \lambda_3 I] V_3 = 0 \Rightarrow \begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix} V_3 = 0.$$

$$V_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

solving for x, y & z:

$$(1+i)x + y = 0$$

$$+ (1+i)y + z = 0$$

$$-2x - 4y + (-2+i)z = 0$$

$$y = -x(1+i)$$

$$z = -(1+i)y$$

$$z = -(1+i) \cdot -x(1+i)$$

$$= (1+i^2+2i)x \Rightarrow z = (2i)x$$

$$\therefore V_3 = \begin{bmatrix} 1 \\ -1-i \\ 2i \end{bmatrix}$$

satisfies the above eqn

$$\therefore A = MJM^{-1}$$

$$= [V_1 V_2 V_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [V_1 V_2 V_3]^{-1}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & i-1 & -1-i \\ 1 & -2i & 2i \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & i-1 & 0 \\ 0 & 0 & -i-1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & i-1 & -1-i \\ 1 & -2i & 2i \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & i-1 & -1-i \\ 1 & -2i & 2i \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & i-1 & 0 \\ 0 & 0 & -i-1 \end{bmatrix} \begin{bmatrix} 1 & -i/2 & i/2 \\ -1 & (-1+i)/2 & (-1-i)/2 \\ 1 & 1 & 1 \end{bmatrix}$$

#3: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Upper Δ^{le} matrix : $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$.

$\lambda_1 = 1$

$[A - \lambda_1 I] V_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V_1 = 0.$

$\therefore V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies the eq^n

$\lambda_2 = 1 \Rightarrow \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\because$ V_1 & V_2 are linearly independent, these can be 2 eigenvectors

$\lambda_3 = 2$

$[A - \lambda_3 I] V_3 = 0 \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_3 = 0.$

$\therefore V_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ satisfies the eq^n

$A = MJM^{-1}$

$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

#4:

$$A = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Finding eigenvalues: $|A - \lambda I| = 0.$

$$\begin{vmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{vmatrix} = 0$$

$$-\lambda\{(20-\lambda)(-20-\lambda) + 400\}$$

$$-\lambda[\lambda^2 - 400 + 400] = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

$\lambda_1 = 0$

$$[A - \lambda_1 I]v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} v_1 = 0$$

$\therefore v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies the eq^n

$\because$ There are no linearly independent eigenvalues, we use the Jordan form.

Finding V_2 :

$$[A - \lambda I] V_2 = V_1 \Rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore V_2 = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$ satisfies the eqn

Finding V_3 :

$$[A - \lambda I] V_3 = V_2 \Rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} V_3 = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$$

$\therefore V_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ satisfies the eqn

$\therefore A = MJM^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & -5 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & -5 & -1 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & -5 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -4 & -3 \\ 0 & -1 & -1 \\ 0 & 5 & 4 \end{bmatrix}$$

Exercise 4: CT & DT Dynamics

Given: $x(0) = 0$

$u \rightarrow$ Unit step input

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$$

where, $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

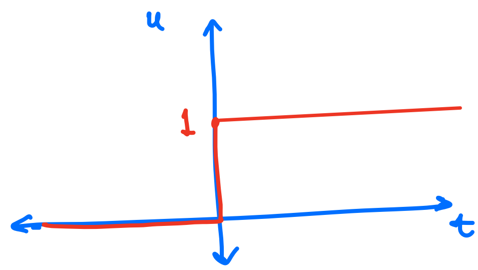
$C = \begin{bmatrix} 2 & 3 \end{bmatrix}$ $\quad D = 0$

(i) Find $y(5)$ for CT system

for a CT system: $\boxed{y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}$

here, $t_0 = 0$

$$u(t) = \begin{cases} 0 , & x < 0 \\ 1 , & x \geq 0 \end{cases}$$



for $t=5$,

$$y(5) = \cancelto{0}{Ce^{A(5-0)}x(0)} + C\int_0^5 e^{A(5-\tau)} B u(\tau) d\tau + \cancelto{0}{D}\,\cancelto{1}{u(5)}$$

$$y(5) = C \int_0^5 e^{A(5-\tau)} \underbrace{B\, u(\tau)\, d\tau}$$

integrating from $0 \rightarrow 5$, during which $u(\tau)$ remains const (value $= 1$)

$$y(s) = C \int_0^5 e^{A(s-\tau)} \cdot d\tau \cdot B$$

Finding e^{At} :

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

Eigenvalues of A: $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-2-\lambda) + 2 = 0$$

$$2\lambda + \lambda^2 + 2 = 0$$

$\therefore \lambda_1 = -1 + i$; $\lambda_2 = -1 - i$

Jordan form: Distinct eigenvalues :

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix}$$

Finding eigenvectors:

for $\lambda_1 = -1+i$, $[A - \lambda_1 I] v_1 = 0$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} v_1 = 0$$

$1 - i - 1 + i$

$-2 + 1 - i^2$

$-2 + 2$

$\therefore v_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$ is the eigenvector

for $\lambda_2 = -1-i$, $[A - \lambda_2 I] v_2 = 0$

$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} v_2 = 0$$

$\therefore \; v_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$ is the eigenvector.

$\therefore \; A = MJM^{-1}$

Here, $M = [v_1 \; v_2] = \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix}$

$$e^{At} = M e^{Jt} M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix} \begin{bmatrix} e^{(-1+i)t} & 0 \\ 0 & e^{(-1-i)t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix}^{-1}$$

$$\begin{bmatrix} (1-i)/2 & -i/2 \\ (1+i)/2 & i/2 \end{bmatrix}$$

we need to evaluate $\int_0^5 e^{A(s-\tau)} \, d\tau$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{(-1+i)t} = e^{-t} \cdot e^{ti}$$

$$= e^{-t} [\cos t + i \sin t]$$

$$e^{(-1-i)t} = e^{-t} [\cos t - i \sin t]$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix} \begin{bmatrix} e^{-t}[\cos t + i\sin t] & 0 \\ 0 & e^{-t}[\cos t - i\sin t] \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix}^{-1}$$

writing $\cos t = c$ & $\sin t = s$ for simplicity :

$$e^{At} = \begin{bmatrix} e^{-t}(c+is) & e^{-t}(c-is) \\ e^{-t}\left[-c-s+i(c+s)\right] & -e^{-t}\left[c-s+i(c-s)\right] \end{bmatrix} \begin{bmatrix} (1-i)/2 & -i/2 \\ (1+i)/2 & i/2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t}(c+s) & e^{-t}s \\ -2e^{-t}s & e^{-t}(c-s) \end{bmatrix}$$

$$\therefore \int_0^s e^{A(s-\tau)} . d\tau = \begin{bmatrix} \int_0^s e^{-(s-\tau)}(\cos(s-\tau)+\sin(s-\tau)) d\tau & \int_0^s e^{-(s-\tau)} . \sin(s-\tau) . d\tau \\ -2\int_0^s e^{-(s-\tau)} \sin(s-\tau) . d\tau & \int_0^s e^{-(s-\tau)}(\cos(s-\tau) - \sin(s-\tau)) . d\tau \end{bmatrix}$$

Solving the integral

$$\int_0^s e^{A(s-\tau)} . d\tau = \begin{bmatrix} 1 - e^{-s}\cos s & \frac{1}{2} - \frac{e^{-s}}{2}\left[\sin(s) + \cos(s)\right] \\ e^{-s}\left[\sin(s) + \cos(s)\right] - 1 & e^{-s}\sin(s) \end{bmatrix}$$

Solving =>

$$\int_0^s e^{A(s-\tau)} . d\tau = \begin{bmatrix} 0.9981 & 0.50227 \\ -1.004549 & -0.00646 \end{bmatrix}$$

$$y(s) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0.9981 & 0.50227 \\ -1.004549 & -0.00646 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(s) = \begin{bmatrix} -1.017487 & 0.958516 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(s) = -0.0323$$

(ii)
$$\left.\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= C_d x(k) + D_d u(k) \end{aligned}\right\} \text{Equation form of DT-LTI}$$

where $A_d = e^{AT}$, $B_d = A^{-1}\left(e^{At} - I\right)B$, $T = 1s$

$C_d = C$ $D_d = D$ $T = 57.296$ deg

$$A_d = e^{AT} = \begin{bmatrix} e^{-T}(\cos T + \sin T) & e^{-T}\sin T \\ -2e^{-T}\sin T & e^{-T}(\cos T - \sin T) \end{bmatrix} = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -0.4917 & 0.3096 \\ -0.6191 & -1.1108 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -0.1821 \\ -1.7299 \end{bmatrix} \Rightarrow B_d = \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$$

$C_d = \begin{bmatrix} 2 & 3 \end{bmatrix}$

$D_d = 0$

$$\therefore \quad x(k+1) = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} x(k) + \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(k)$$

(iii) $y(5)$ for DT-LTI system

$$y(5) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(5)$$

$$\begin{cases} x(k) = A_d^k x(0) + \sum_{m=0}^{k-1} A_d^{k-m-1} B_d u(m) \\ y(k) = C A_d^k x(0) + \sum_{m=0}^{k-1} C A_d^{k-m-1} B_d u(m) + D u(k) \end{cases}$$

$$x(5) = \cancel{A_d^5 x(0)}^{\,0} + \sum_{m=0}^{4} A_d^{4-m} \cdot B_d \cdot u(m)$$

$$= \left[A_d^4 u(0) + A_d^3 u(1) + A_d^2 u(2) + A_d^1 u(3) + A_d^0 u(4) \right] B_d$$

$$A_d^2 = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix}$$

$$A_d^2 = \begin{bmatrix} 0.06669 & 0.12306 \\ -0.24609 & -0.17939 \end{bmatrix}$$

$$A_d^3 = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} \begin{bmatrix} 0.06669 & 0.12306 \\ -0.24609 & -0.17939 \end{bmatrix}$$

$$A_d^3 = \begin{bmatrix} -0.04228 & 0.00701 \\ -0.01402 & -0.05631 \end{bmatrix}$$

$$A_d^4 = \begin{bmatrix} 0.06669 & 0.12306 \\ -0.24609 & -0.17939 \end{bmatrix} \begin{bmatrix} 0.06669 & 0.12306 \\ -0.24609 & -0.17939 \end{bmatrix}$$

$$= \begin{bmatrix} -0.02583 & -0.01386 \\ 0.02773 & 0.00189 \end{bmatrix}$$

$$\therefore x(5) = \begin{bmatrix} 1.50687 & 0.4258 \\ -0.8511 & 0.65539 \end{bmatrix} \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5003 \\ -1.01053 \end{bmatrix}$$

$$\therefore y(5) = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1.5003 \\ -1.01053 \end{bmatrix}$$

$$y(5) = -0.03099$$

Exercise 5 : Diagonalization

Fibonacci sequence: 0, 1, 1, 2 . . . .

Equation : $F_{k+2} = F_{k+1} + F_k$

$k = 0, 1, 2 \ldots n$

Find F_{20}

Construct a discrete system :

$$\left.\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ F_{k+2} &= F_{k+1} + F_k \end{aligned}\right\} \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} + B[0]$$

$x[k+1] = A\,x[k] + Bu$

→ Discrete time LTI

$F_{20} = ?$

$$\begin{bmatrix} F_{20} \\ F_{19} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{19} \\ F_{18} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{18} \\ F_{17} \end{bmatrix} \ldots\ldots$$

$$\begin{bmatrix} F_{20} \\ F_{19} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{19} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$\begin{bmatrix} F_{20} \\ F_{19} \end{bmatrix} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore F_{20} = 6765$

→ 20^{th} term of Fibonacci sequence.