Exercise 1: Asymptotic & Lyapunov Stability

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(a)

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B} u(k)$$

Finding eigenvalues of A:

$$\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \qquad \left.\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 0.5 \end{matrix}\right\} \text{ upper triangular matrix}$$

For stability of a discrete LTI:

Criteria I: $|\lambda_i| \leq 1 \implies \lambda_1 = 1 \leq 1$ ✓

$\lambda_2 = 0.5 \leq 1$ ✓

Criteria II: $|\lambda_i| = 1 \rightarrow$ non-defective

Here, $\lambda_1 = 1$ is non-defective as its algebraic multiplicity is equal to its geometric multiplicity.

$\therefore$ The given system is Lyapunov stable as is satisfies both the criteria.

For the system to be Asymptotically stable, $\lambda_i < 1$

$\lambda_1 = 1$ is not less than 1.

$\therefore$ The system is not Asymptotically stable.

(b)

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

A B

Finding eigenvalues of A: $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda+7 & 2 & -6 \\ -2 & \lambda+2 & 2 \\ 2 & 2 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda+7)[(\lambda+2)(\lambda-1)-4] - 2[-2(\lambda-1)-4] - 6[-4-2(\lambda+2)] = 0$$

$$\lambda^3 + 9\lambda^2 + 23\lambda + 15 = 0$$

$\lambda_1 = -1$

$\lambda_2 = -3$

$\lambda_3 = -5$

$\lambda_i = Re + Im\, j$

For Asymptotic stability, $\forall \lambda_i$, $Re < 0$

For the above system, λ_1, λ_2 & λ_3 satisfies the above condition.

$\therefore$ This system is asymptotically stable.

$\therefore$ It is also Lyapunov stable

Exercise 2: Stabilizability

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Controllable form:

$P = [B \quad AB \quad A^2B]$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$\text{rank}(P) = 2 < 3$

$\therefore$ The system is not controllable

Using the controllable Kalman decomposition:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$r(M) = 3$

1st two linearly independent columns of P.

arbitrary column to make M full rank.

$$\hat{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

$\therefore$ The controllable form is :

$$\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 3 \end{bmatrix} \hat{x}_c$$

Observability :

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

rank $(Q) = 1 < 2$

$\therefore$ The decomposed system is not observable.

Stabilizability :

The system is stabilizable if its uncontrollable modes are Lyapunov stable.

The decomposed equation is fully controllable, making it automatically stabilizable.

Detectability :

Eigenvalues of A $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ are : $\lambda_1 = 0$

$\lambda_2 = -1$

$\lambda_i = Re + Im\, j$

Finding eigenvectors :

$$A V_1 = \lambda_1 V_1$$

$$(A - \lambda I) V_1 = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} V_1 = 0$$

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A V_2 = \lambda_2 V_2$$

$$(A - \lambda_2 I) V_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} V_2 = 0$$

$$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finding the unobservable mode. ($CV = 0$ means that mode is unobservable)

$\lambda_1 = 0$, $\quad CV_1 = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3 \neq 0$ $\quad \therefore$ Observable

$\lambda_2 = -1$, $\quad CV_2 = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 = 0$ $\quad \therefore$ Not observable

The system is detectable if its unobservable modes are Lyapunov stable.

$\left. \lambda_2 = -1 \qquad Re_2 = -1 \right\}$ unobservable modes

$\therefore \forall \lambda_i$, $Re < 0$ is satisfied $\Rightarrow$ The system is asymptotically stable

Hence, the uncontrollable modes are also Lyapunov stable.

$\therefore$ The decomposed system is detectable.

Exercise 3: Stability

$$m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta$$
$$m\ddot{y} = u_1 \cos\theta + \epsilon u_2 \sin\theta - mg$$
$$J\ddot{\theta} = u_2$$

State, $z = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$ Input, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\dot{\theta} = \dot{\theta}$

$$\ddot{x} = -u_1 \frac{\sin\theta}{m} + u_2 \frac{\varepsilon\cos\theta}{m} \quad -(1)$$

$$\ddot{y} = u_1 \frac{\cos\theta}{m} + u_2 \frac{\varepsilon\sin\theta}{m} - g \quad -(2)$$

$$\ddot{\theta} = \frac{u_2}{J} \quad -(3)$$

Equilibrium points : $\tilde{x}(t), \tilde{y}(t), \tilde{\theta}(t) = 0$

$\tilde{u}_1(t) = mg \quad \tilde{u}_2(t) = 0$

State vector: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$

$x_1 = \theta$
$x_2 = \dot{x}$
$x_3 = \dot{y}$
$x_4 = \dot{\theta}$

$\therefore$ The equations become $\Rightarrow$

$$\dot{x}_1 = x_4$$
$$m\dot{x}_2 = -u_1 \sin(x_1) + \varepsilon u_2 \cos(x_1)$$
$$m\dot{x}_3 = u_1 \cos(x_1) + \varepsilon u_2 \sin(x_1) - mg$$
$$\dot{x}_4 = u_2 / J$$

Finding the Jacobian matrix for the above system:

$$J_A = \frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -u_1\cos(x_1)/m \; - \varepsilon u_2 \sin(x_1)/m & 0 & 0 & 0 \\ -u_1 \sin(x_1)/m \; + \varepsilon u_2 \cos(x_1)/m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jacobian @ equilibrium point:

$$\left.\begin{matrix} \tilde{x}(t),\ \tilde{y}(t),\ \tilde{\theta}(t) = 0 \\ \tilde{u}_1(t) = mg \quad \tilde{u}_2(t) = 0 \end{matrix}\right\} \quad \left.\begin{matrix} \dot{\tilde{x}}(t),\ \dot{\tilde{y}}(t),\ \dot{\tilde{\theta}}(t) = 0 \\ \tilde{u}_1(t) = mg, \quad \tilde{u}_2(t) = 0 \end{matrix}\right\} \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{matrix}$$

$$J_A\Big|_{z=z_0} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Doing the same for the input matrix B:

$$J_B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ -\sin(x_1)/m & \varepsilon\cos(x_1)/m \\ \cos(x_1)/m & \varepsilon \sin(x_1)/m \\ 0 & 1/J \end{bmatrix}$$

$$J_B\Big|_{z=z_0} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix}$$

$\therefore$ The linearized system about the equilibrium point is:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \epsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Stability analysis:

Finding eigenvalues: $|\lambda I - A| = 0$

$$\begin{bmatrix} \lambda & 0 & 0 & -1 \\ g & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - (-1) \begin{vmatrix} g & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\lambda \left[\lambda (\lambda^2) \right] + 1 \left[g(0) - \lambda(0) \; 0 \right] = 0$$

$$\lambda^4 = 0$$

$\therefore$ The eigenvalues of the system are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

we see that $\exists \, \lambda_i$, $Re = 0$, $m > 0$

indicating that the system is unstable.

Exercise 4: Lyapunov's direct method

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Given, $V = x_1^2 + x_2^2$

Lyapunov's Direct Method :

Theorem

The origin of $\dot{x} = f(x)$ is stable if

- $V(x) = 0$ *if and only if* $x = 0$
- $V(x) > 0$ *if and only if* $x \neq 0$
- $\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) \leq 0$ *for all values of* $x \neq 0$.

Note: for asymptotic stability, $\dot{V}(x) < 0$ *for* $x \neq 0$ *is required.*

1. $V(x) = 0$ if & only if $x_1 = 0$ & $x_2 = 0$

   $\because$ x_1^2 & x_2^2 are always positive

2. $V(x) > 0$ $\forall$ $x_1, x_2 \neq 0$

Asymptotic stability :

$\dot{x}_1 = a x_1$

$\dot{x}_2 = x_1 - x_2$

$$\dot{V}(x) = 2x_1 \cdot \dot{x}_1 + 2x_2 \cdot \dot{x}_2$$

$$= 2\left[x_1(ax_1) + x_2(x_1 - x_2)\right]$$

$$= 2\left[ax_1^2 + x_1x_2 - x_2^2\right]$$

$\dot{V}(x) < 0$ for $x_1 \neq 0$, $x_2 \neq 0$

PTO

$\dot{V}(x) < 0$

$2\left[ax_1^2 + x_1x_2 - x_2^2\right] < 0$

$ax_1^2 + x_1x_2 - x_2^2 < 0$

$ax_1^2 < x_2^2 - x_1x_2$

$ax_1^2 < x_2 + \frac{1}{4}x_1^2 - \frac{1}{4}x_1^2 - 2\times\frac{1}{2}x_1x_2$

$ax_1^2 < \left[x_2 - \frac{1}{2}x_1\right]^2 - \frac{1}{4}x_1^2$

$ax_1^2 < \left[x_2 - \frac{1}{2}x_1\right]^2 - \left[\frac{x_1}{2}\right]^2$

if $x_1, x_2 \neq 0$, the minimum value of RHS occurs when $x_2 = \frac{x_1}{2}$

The minimum value is $-\frac{x_1^2}{4}$

If ax_1^2 is less than minimum of RHS, then it must satisfy the inequality for all other values of RHS.

$\therefore \; ax_1^2 < -\frac{x_1^2}{4}$

$x_1^2\left(a + \frac{1}{4}\right) < 0 \quad \Rightarrow \quad a + \frac{1}{4} < 0 \quad \Rightarrow \quad a < -\frac{1}{4}$

$\therefore$ $\forall a < -\frac{1}{4}$. the above system is asymptotically stable

Exercise 5: Stability of non-linear systems

$$\dot{x}_1 = x_2 - x_1 x_2^2$$
$$\dot{x}_2 = -x_1^3$$

(a) $$\dot{x} = \begin{bmatrix} x_2 - x_1 x_2^2 \\ -x_1^3 \end{bmatrix}$$

Finding Equilibria: $\begin{cases} x_2 - x_1 x_2^2 = 0 \\ -x_1^3 = 0 \end{cases} \Rightarrow$ $x_2(1 - x_1 x_2) = 0$, $x_1 = 0$, $x_2 = 0$.

$\therefore$ Equilibrium point: $X_0 = (0,0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -x_2^2 & 1 - 2x_1 x_2 \\ -3x_1^2 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{X_0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The linearized system is:

$$\dot{X} = f(X_0) + \left.\frac{\partial f}{\partial x}\right|_{X_0} \delta x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta x$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

Eigenvalues of A: $\lambda_1 = 0$, $\lambda_2 = 0$

$Re(\lambda_i) = 0 \Rightarrow \therefore$ The linearized system is risky

Cannot say if it is stable or unstable using Lyapunov's indirect method. (inconclusive)

(b) $V(x_1, x_2) = x_1^4 + 2x_2^2$

$$\dot{V}(x_1 x_2) = 4x_1^3 \cdot \dot{x}_1 + 4x_2 \cdot \dot{x}_2$$

$$= 4x_1^3 \left[x_2 - x_1 x_2^2 \right] + 4x_2 \left(-x_1^3 \right)$$

$$= \cancel{4x_1^3 x_2} - 4x_1^4 x_2^2 - \cancel{4x_1^3 x_2} \qquad = -4x_1^4 x_2^2$$

condition 1: $V(x) = 0$ only when $x_1 = 0$, $x_2 = 0$

as x_1^4 & $2x_2^2$ is always positive.

condition 2: $V(x) > 0$ only when $x_1 = 0$ & $x_2 = 0$

as $x_1^4 = 0$ only when $x_1 = 0$

$2x_2^2 = 0$ only when $x_2 = 0$

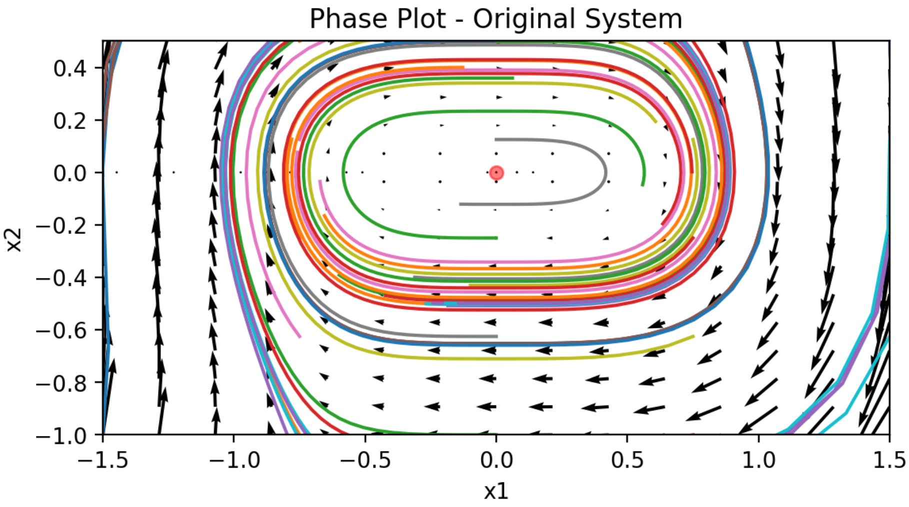
condition 3: $\dot{V}(x) = -4x_1^4 x_2^2$

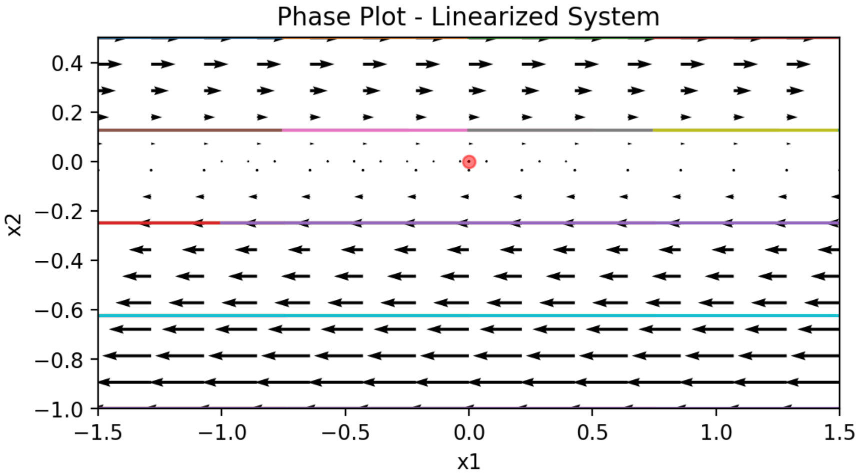
$\dot{V}(x) < 0 \quad \forall \; x_1 x_2 \neq 0$

$\therefore$ The system satisfies the 3 conditions of Lyapunov's direct method.

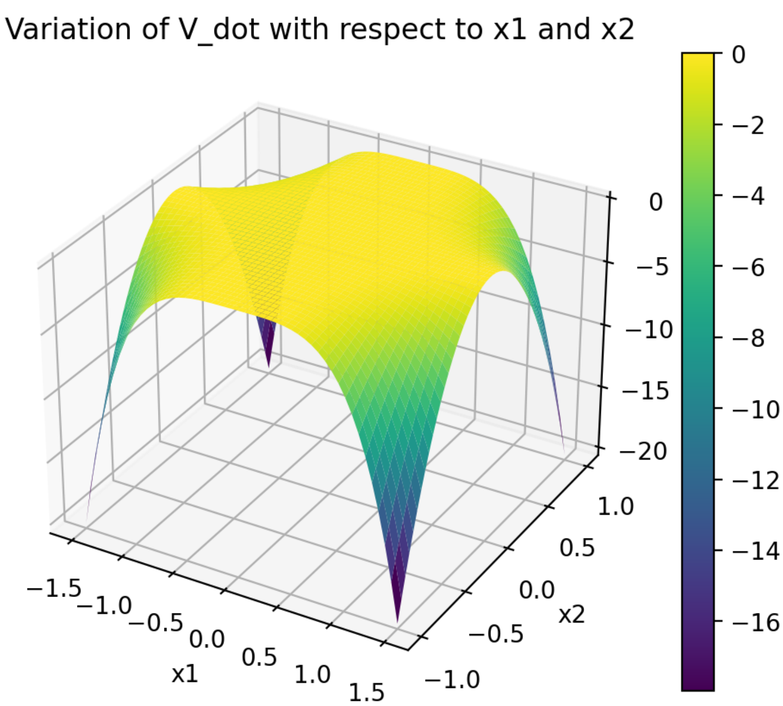
the system is Asymptotically stable.

(c)


Phase Plot - Original System
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
x2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x1


Phase Plot - Linearized System
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
x2
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x1

(d)


Variation of V_dot with respect to x1 and x2
0
−5
−10
−15
−20
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x1
1.0
0.5
0.0
−0.5
−1.0
x2
0
−2
−4
−6
−8
−10
−12
−14
−16

Exercise 6: BIBO Stability

(a)

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 5 & 5 \end{bmatrix} x(k)$$

For the discrete system above:

$$G_D(z) = C\,(zI - A)^{-1} B + D$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} * \frac{1}{(z-1)(z-0.5)} * \begin{bmatrix} (z-0.5) & 0 \\ -0.5 & (z-1) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 0$$

Since $G(s) = 0$, it means that the input has no change to the output.

Output is always zero (bounded) for any input signal as it does not affect the output.

$\therefore$ The system is BIBO stable.

(b)

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

Converting to transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s^2+2s-7}{(s+5)(s+1)(s+3)} & \frac{-2}{(s+1)(s+3)} & \frac{2(3s+11)}{(s+5)(s^2+4s+3)} \\ \frac{2}{(s+5)(s+3)} & \frac{1}{(s+3)} & \frac{-2}{(s+5)(s+3)} \\ \frac{-2}{s^2+4s+3} & \frac{-2}{s^2+4s+3} & \frac{s+5}{s^2+4s+3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

On solving:

$$G(s) = \begin{bmatrix} 0 & 0 \\ \frac{1}{s+3} & 0 \end{bmatrix}$$

Poles of $G(s)$: $s = -3$ (G_{11}, G_{12}, G_{21} are constants and do not contribute any poles)

$\because$ Every pole of $G(s)$ has negative real part.

$\therefore$ The above system is BIBO stable.

Exercise 7: BIBO Stability

$$V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + \beta(T_H - T_C)$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + \beta(T_C - T_H)$$

$f_c = f_H = 0.1$ $\quad y_1 = T_c$ $\quad u_1 = T_{ci}$

$\beta = 0.2$ $\quad y_2 = T_H$ $\quad u_2 = T_{Hi}$

$V_H , V_c = 1$

1.
$$\begin{cases} \dfrac{dT_c}{dt} = 0.1 \left[T_{ci} - T_c \right] + 0.2 \left[T_H - T_c \right] \\ \dfrac{dT_H}{dt} = 0.1 \left[T_{Hi} - T_H \right] + 0.2 \left[T_c - T_H \right] \end{cases}$$

$$\begin{cases} \dot{y}_1 = 0.1 \left[u_1 - y_1 \right] + 0.2 \left[y_2 - y_1 \right] \\ \dot{y}_2 = 0.1 \left[u_2 - y_2 \right] + 0.2 \left[y_1 - y_2 \right] \end{cases}$$

$$\begin{cases} \dot{y}_1 = 0.1 u_1 + 0.2 y_2 - 0.3 y_1 \\ \dot{y}_2 = 0.1 u_2 - 0.3 y_2 + 0.2 y_1 \end{cases}$$

Considering state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $\begin{matrix} x_1 = T_c \\ x_2 = T_H \end{matrix} \Rightarrow \begin{matrix} x_1 = y_1 \\ x_2 = y_2 \end{matrix}$

$$\begin{cases} \dot{x}_1 = -0.3x_1 + 0.2x_2 + 0.1u_1 \\ \dot{x}_2 = 0.2x_1 - 0.3x_2 + 0.1u_2 \end{cases}$$

$\therefore$ The state space equation form is as follows:

$$\boxed{\begin{aligned} \dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}}$$

2. If input is absent, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Finding eigenvalues of the system:

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda + 0.3 & -0.2 \\ -0.2 & \lambda + 0.3 \end{vmatrix} = 0 \Rightarrow \begin{aligned} &(\lambda+0.3)^2 - 0.04 = 0 \\ &\lambda^2 + 0.09 + 0.6\lambda - 0.04 = 0 \\ &\lambda^2 + 0.6\lambda + 0.05 = 0 \\ &(\lambda+0.5)(\lambda+0.1) = 0 \end{aligned}$$

$\therefore \lambda_1 = -0.5$ & $\lambda_2 = -0.1$

Finding eigenvectors:

$$A v_1 = \lambda_1 v_1$$

$$(A - \lambda I) v_1 = 0 \quad \Rightarrow \quad \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix} v_1 = 0$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - \lambda_2 I) v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} -0.2 & 0.2 \\ 0.2 & -0.2 \end{bmatrix} v_2 = 0$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-0.1t}$

$\because$ $y_1(t) = x_1(t)$

$y_2(t) = x_2(t)$

$\therefore$ The values of $y_1(t)$ & $y_2(t)$ are:

$$\boxed{\begin{aligned} y_1(t) &= c_1 e^{-0.5t} + c_2 e^{-0.1t} \\ y_2(t) &= -c_1 e^{-0.5t} + c_2 e^{-0.1t} \end{aligned}}$$

c_1 & c_2 are constants that can be determined from the initial condition of the system.

3. To check if the system is BIBO stable, we need to convert it to a transfer function from state space.

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+0.3 & -0.2 \\ -0.2 & s+0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$= \frac{1}{s^2+0.6s+0.05} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+0.3 & 0.2 \\ 0.2 & s+0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$G(s) = \frac{1}{s^2+0.6s+0.05} \begin{bmatrix} 0.1(s+0.3) & 0.02 \\ 0.02 & 0.1(s+0.3) \end{bmatrix}$$

Poles of the above transfer function: $s^2+0.6s+0.05 = 0$

$(s+0.1)(s+0.5) = 0$

$\therefore$ The poles are -0.1 & -0.5

$\because$ Every pole of the transfer function has a negative real part, the system is BIBO stable.

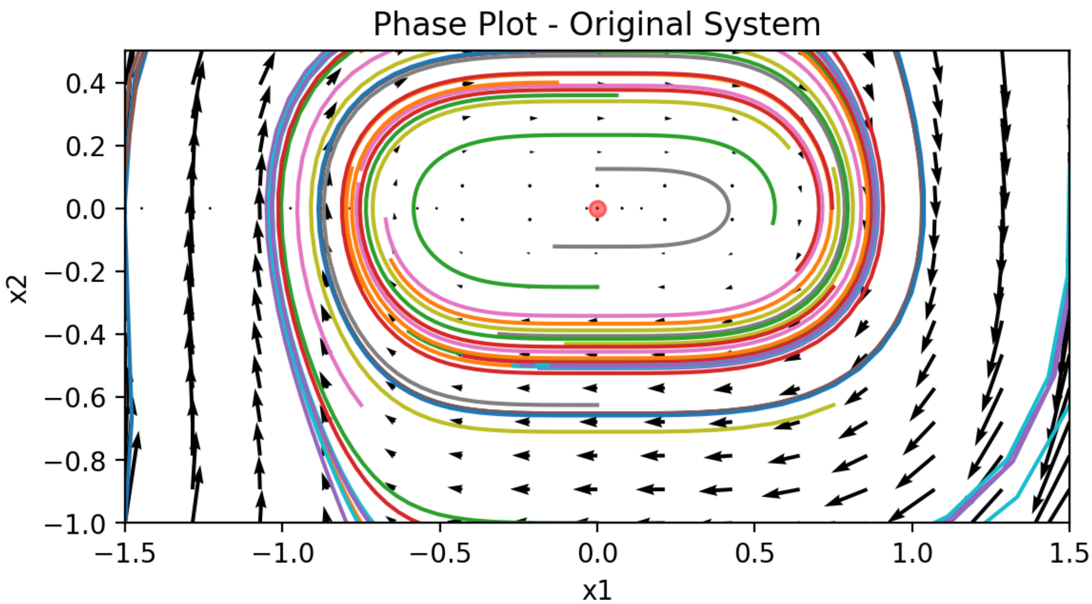
hw5_abijunai

October 12, 2024

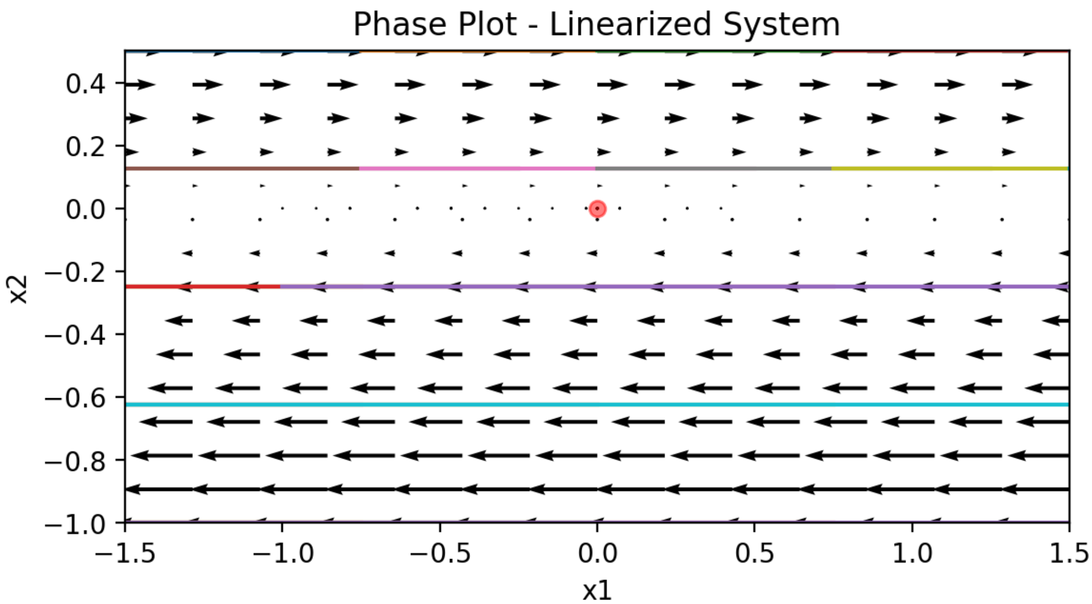
[ ]:
```
import os
import numpy as np
import matplotlib.pyplot as plt
from control.phaseplot import phase_plot
from numpy import pi
from mpl_toolkits.mplot3d import Axes3D
plt.close('all')
```

[ ]:
```
def org_system(x, t):
    # x1, x2 = X
    dx1dt = x[1] - x[0] * x[1]**2
    dx2dt = -x[0]**3
    return [dx1dt, dx2dt]

def lin_system(x,t):
    dx1dt = x[1]
    dx2dt = x[0]*0
    return np.array([dx1dt, dx2dt])
```

[ ]:
```
plt.figure(dpi = 200)
plt.clf()
plt.gca().set_aspect('equal', adjustable='box')
plt.axis([-1.5, 1.5, -1, 0.5])

plt.title('Phase Plot - Original System')
x1 = np.linspace(-1.5, 1.5, 5)
x2 = np.linspace(-1, 0.5, 5)
X1, X2 = np.meshgrid(x1, x2)
X0 = np.hstack((X1.reshape(-1,1), X2.reshape(-1,1)))

# Outer trajectories
phase_plot(org_system, X0 = X0, X = (-1.5, 1.5, 15), Y = (-1, 0.5, 15), T = 10)

plt.scatter(x = [0], y = [0], c='r', alpha=0.5)

plt.show()
```

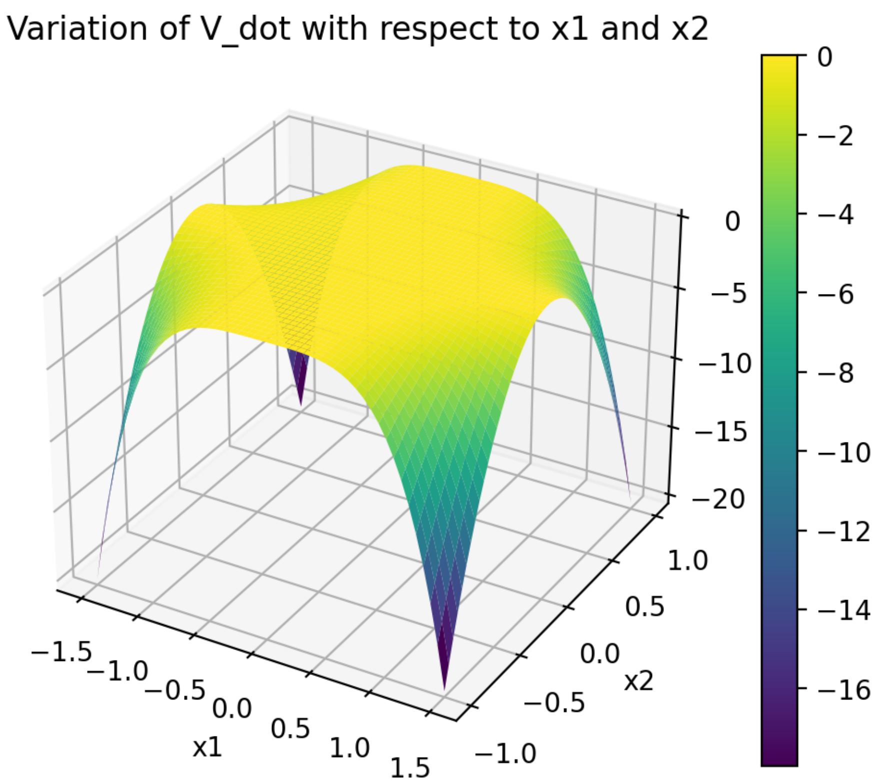


[ ]:
```
plt.figure(dpi = 200)
plt.clf()
plt.gca().set_aspect('equal', adjustable='box')
plt.axis([-1.5, 1.5, -1, 0.5])

plt.title('Phase Plot - Linearized System')
x1 = np.linspace(-1.5, 1.5, 5)
x2 = np.linspace(-1, 0.5, 5)
X1, X2 = np.meshgrid(x1, x2)
X0 = np.hstack((X1.reshape(-1,1), X2.reshape(-1,1)))

# Outer trajectories
phase_plot(lin_system, X0 = X0, X = (-1.5, 1.5, 15), Y = (-1, 0.5, 15), T = 10)

plt.scatter(x = [0], y = [0], c='r', alpha=0.5)

plt.show()
```



[ ]:
```
def V(x1, x2):
    return x1**4 + 2*x2**2


def V_dot(x1, x2):
    dV_dx1 = 4 * x1**3
    dV_dx2 = 4 * x2
    x1_dot = x2 - x1 * (x2**2)
    x2_dot = -x1**3
    return dV_dx1 * x1_dot + dV_dx2 * x2_dot

x1 = np.linspace(-1.5, 1.5, 50)
x2 = np.linspace(-1, 1, 50)
X1, X2 = np.meshgrid(x1, x2)

vdot = V_dot(X1, X2)

# Create the 3D plot
fig = plt.figure(dpi=200)
ax = fig.add_subplot(111, projection='3d')

# Plot the surface
surf = ax.plot_surface(X1, X2, vdot, cmap='viridis')

# Set labels and title
ax.set_xlabel('x1')
```

```
ax.set_ylabel('x2')
ax.set_zlabel('V_dot')
ax.set_title('Variation of V_dot with respect to x1 and x2')

# Add a color bar
fig.colorbar(surf)

# Show the plot
plt.show()
```




```
[ ]: ################
NOTEBOOK_NAME = "hw5_abijunai.ipynb"
!apt-get install -y pandoc > /dev/null 2>&1
!sudo apt-get install texlive-xetex texlive-fonts-recommended␣
  ↪texlive-plain-generic > /dev/null 2>&1
!pip install nbconvert > /dev/null 2>&1

from google.colab import drive, files
drive.mount('/content/drive')
notebook_path = rf'/content/drive/MyDrive/Colab Notebooks/{NOTEBOOK_NAME}'
```

```
import os
if os.path.exists(notebook_path):
  !jupyter nbconvert --to pdf "{notebook_path}" > /dev/null 2>&1
  pdf_filename = notebook_path.replace('.ipynb', '.pdf')
  files.download(pdf_filename)
else:
  print("Notebook not found. Please check the path.")
################
```