24-677 Modern Control for Robotics :

Homework 1

Exercise 1

1. Answer : $\boxed{\text{Linear}}$ and $\boxed{\text{Time - invariant .}}$

$y(t) = 0$ for all t

$u_1 \rightarrow y_1$: $y_1(t) = 0$ — (1)

$u_2 \rightarrow y_2$: $y_2(t) = 0$. — (2)

$u_3 \rightarrow y_3$: $y_3(t) = 0$. — (3)

Let $u_3 = \alpha u_1 + \beta u_2$

If system is linear, $y_3 = \alpha y_1 + \beta y_2$.

$y_3(t) = 0$ — (4) [from (1) & (2)]

Since Eq^n (3) = Eq^n (4) , the system is linear

Applying a time shift, $t' = t - \tau$

$y(t-\tau) = 0$ — (5)

The original output $y(t) = 0$.

Eq^n (5) = Eq^n (1)

∴ The output is same after time shift . The system is time invariant

Exercise 1

2. Answer: $\boxed{\text{Non-linear}}$ and $\boxed{\text{Time-invariant}}$

$y(t) = u^3(t)$

$u_1 \rightarrow y_1$: $\quad y_1(t) = u_1^3(t)$ — (1)

$u_2 \rightarrow y_2$: $\quad y_2(t) = u_2^3(t)$ — (2).

$u_3 \rightarrow y_3$: $\quad y_3(t) = u_3^3(t)$ — (3).

$u_3 = \alpha u_1 + \beta u_2$.

If system is linear, $y_3 = \alpha y_1 + \beta y_2$.

$\Rightarrow y_3(t) = \alpha u_1^3(t) + \beta u_2^3(t)$ — (4)

Eqn (3) becomes $\Rightarrow$

$$y_3(t) = [\alpha u_1(t) + \beta u_2(t)]^3$$

$$= \alpha^3 u_1^3(t) + \beta^3 u_2^3(t) + 3\alpha^2\beta u_1^2(t) u_2(t) + 3\alpha\beta^2 u_1(t) u_2^2(t) \text{ — (5)}$$

From Eqn (4) & (5) we can see that this system does not satisfy the superimposition principle.

$\therefore$ The system is Non-linear

Consider $u_2 = u_1(t-\tau)$

$y_2(t) = u_2^3(t) = u_1^3(t-\tau)$ — (6)

If system is time invariant, $y_2(t) = y_1(t-\tau)$

$\Rightarrow y_1(t-\tau) = u_1^3(t-\tau)$ — (7)

From (6) & (7) we see that $y_2(t) = y_1(t-\tau)$ holds true

$\therefore$ The system is time invariant.

Exercise 1

3. Answer: $\boxed{\text{Linear}}$ and $\boxed{\text{Time - varying}}$ (~~invariant~~)

$$y(t) = \cancel{8}\, u(3t)$$

$$u_1 \rightarrow y_1 : \quad y_1(t) = u_1(3t) \quad -(1)$$

$$u_2 \rightarrow y_2 : \quad y_2(t) = u_2(3t) \quad -(2)$$

$$u_3 = \alpha u_1 + \beta u_2 .$$

$$u_3 \rightarrow y_3 : \quad y_3(t) = u_3(3t) \quad -(3).$$

If the system is linear, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

$$y_3(t) = \alpha u_1(3t) + \beta u_2(3t) \quad -(4).$$

Eqn (3) $\Rightarrow$ $\cancel{y_3(t) = \alpha u_1(3t) + \beta u_2(3t)}$

$$y_3(t) = \alpha u_1(3t) + \beta u_2(3t) \quad -(5).$$

Eqn (4) = Eqn (5) $\quad\therefore$ This system is linear as it satisfies the superimposition principle.

Consider $u_2(t) = u_1(t-\tau)$

$$y_2(t) = u_2(3t)$$
$$= u_1(3t-\tau) \quad \text{--- (6)}$$

If system is time invariant, $y_2(t) = y_1(t-\tau)$

$$y_1(t-\tau) = u_1(3t-3\tau) \quad \text{--- (7)}.$$

$\because$ Eq^n (7) $\neq$ Eq^n (6) the system is time varying.

Exercise 1

4. Answer: $\boxed{\text{Linear}}$ & $\boxed{\text{Time-varying}}$

$$y(t) = e^{-t} u(t-T)$$

$u_1(t) \rightarrow y_1(t)$: $y_1(t) = e^{-t} u_1(t-T)$ — (1)

$u_2(t) \rightarrow y_2(t)$: $y_2(t) = e^{-t} u_2(t-T)$ — (2)

$u_3(t) = \alpha u_1(t) + \beta u_2(t)$

$u_3(t) \rightarrow y_3(t)$: $y_3(t) = e^{-t} u_3(t-T)$ — (3).

If system is linear, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

$$\Rightarrow y_3(t) = \alpha e^{-t} u_1(t-T) + \beta e^{-t} u_2(t-T) \quad \text{— (4)}.$$

Eqn (3) becomes $\Rightarrow$

$$y_3(t) = e^{-t}\left[\alpha u_1(t-T) + \beta u_2(t-T)\right] \quad \text{— (5)}.$$

Eqn (4) = Eqn (5) $\therefore$ The system is <u>linear</u> as it satisfies the superimposition principle.

Consider ~~that~~ $u_2(t) = u_1(t-\tau)$

$$y_2(t) = e^{-t} u_2(t-T) = e^{-t} u_1(t-T-\tau) \quad \text{— (6)}.$$

If system is time invariant, $y_2(t) = y_1(t-\tau)$.

$$y_2(t) = e^{-(t-\tau)} u_1(t-\tau-T) \quad \text{— (7)}.$$

Eqn (6) $\neq$ (7) $\therefore$ Our assumption of $y_2(t) = y_1(t-\tau)$ does not hold true.

$\therefore$ The system is <u>time variant</u>.

Exercise 1

5. Answer: Linear and time varying

$$y(t) = \begin{cases} 0 & , t \leq 0 \\ u(t) & , t > 0 \end{cases}$$

$u_1(t) \rightarrow y_1(t)$: $y_1(t) = \begin{cases} 0 & ; t \leq 0 \\ u_1(t) ; & t > 0. \end{cases}$ — (1).

$u_2(t) \rightarrow y_2(t)$: $y_2(t) = \begin{cases} 0 & ; t \leq 0 \\ u_2(t) ; & t > 0. \end{cases}$ — (2).

$u_3 = \alpha u_1 + \beta u_2.$

$u_3(t) \rightarrow y_3(t)$: $y_3(t) = \begin{cases} 0 & ; t \leq 0 \\ u_3(t) ; & t > 0 \end{cases}$ — (3)

If system is linear, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

$$y_3(t) = \begin{cases} 0 ; & t \leq 0. \\ \alpha u_1(t) + \beta u_2(t) ; & t > 0 \end{cases} \quad \text{— (4).}$$

Eqⁿ (3) ⇒

$$y_3(t) = \begin{cases} 0. & ; t \leq 0 \\ \alpha u_1(t) + \beta u_2(t) & ; t > 0. \end{cases}$$

$\because$ Eqⁿ (3) = (4) The system is linear. (satisfies superposition)

Consider $u_2(t) = u_1(t-\tau)$

$$y_2(t) = \begin{cases} 0 & , t \le 0 \\ u_2(t) & , t > 0 \end{cases}$$

$$= \begin{cases} 0 & , t \le 0 \\ u_1(t-\tau) & , t > 0. \end{cases} \quad -(6)$$

If system is time invariant, $y_2(t) = y_1(t-\tau)$

$$y_1(t-\tau) = \begin{cases} 0 & , t \le \tau \\ u_1(t-\tau) & , t > \tau. \end{cases} \quad -(7).$$

$\because$ Eqn (7) $\neq$ Eqn (6) the system does not satisfy the law of time invariance.

$\therefore$ The system is time varying.

Exercise 2

Relevant Equations :

signal power : $\boxed{s_i = G_{ii} P_i}$ — ①

Noise + interferance power: $\boxed{q_i = \sigma^2 + \sum_{j \neq i} G_{ij} P_j}$ — ②

SINR : $\boxed{S_i = s_i / q_i}$ — ③

control rule: $\boxed{P_i(k+1) = P_i(k)\left(\alpha\gamma / S_i(k)\right)}$ — ④

Q1. Power control update algorithm :

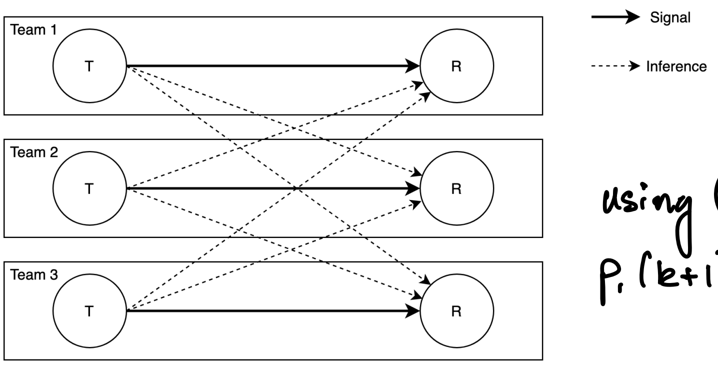
$$p(k+1) = A\,p(k) + B\,\sigma^2 \quad — ⑤$$

A : 3×3 matrix

B : 3×1 matrix

$$p(k) = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$p(k+1) = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{3\times 3} p(k) + \begin{bmatrix} \\ B \\ \\ \end{bmatrix} \sigma^2 \quad — ⑥$$



From Eqⁿ ③ & ④ :

$$P_i(k+1) = P_i(k)\left[\frac{\alpha\gamma}{S(k)} \times q_i(k)\right] \quad — ⑦$$

using ⑦ in ⑥ : for $i = 1$

$$P_1(k+1) = P_1(k)\left[\frac{\alpha\gamma}{S_1(k)} \times q_1(k)\right] =$$

$$a_{11} P_1 + a_{12} P_2 + a_{13} P_3 + b_{11} \sigma^2$$

From Eq (2) :

$$P_1(k+1) = P_1(k)\left[\frac{\alpha\gamma}{G_{11}P_1(k)}\left\{\sigma^2 + G_{12}P_2 + G_{13}P_3\right\}\right] =$$

$$a_{11}p_1 + a_{12}p_2 + a_{13}p_3 + b_{11}\sigma^2$$

On comparing both sides:

$$b_{11} = \frac{\alpha\gamma}{G_{11}} \quad a_{11} = 0 \; ; \quad a_{12} = \alpha\gamma \times \frac{G_{12}}{G_{11}} \; ; \quad a_{13} = \alpha\gamma \times \frac{G_{13}}{G_{11}}$$

Similarly, $P_2(k+1) = P_2(k)\left[\frac{\alpha\gamma}{G_{22}P_2(k)}\left\{\sigma^2 + G_{21}P_1 + G_{23}P_3\right\}\right]$

$$= a_{21}P_1 + a_{22}P_2 + a_{23}P_3 + b_{21}\sigma^2$$

$$b_{21} = \frac{\alpha\gamma}{G_{22}} \; ; \quad a_{21} = \alpha\gamma \times \frac{G_{21}}{G_{22}} \; ; \quad a_{22} = 0 \; ; \quad a_{23} = \alpha\gamma \times \frac{G_{23}}{G_{22}}$$

$$b_{31} = \frac{\alpha\gamma}{G_{33}} \; ; \quad a_{32} = \alpha\gamma \times \frac{G_{32}}{G_{33}} \; ; \quad a_{31} = \alpha\gamma \times \frac{G_{31}}{G_{33}} \; ; \quad a_{33} = 0$$

$\therefore$ Eq^n (6) $\Rightarrow$

$$\begin{bmatrix} P_1(k+1) \\ P_2(k+1) \\ P_3(k+1) \end{bmatrix} = \alpha\gamma \overbrace{\begin{bmatrix} 0 & G_{12}/G_{11} & G_{13}/G_{11} \\ G_{21}/G_{22} & 0 & G_{23}/G_{22} \\ G_{31}/G_{33} & G_{32}/G_{33} & 0 \end{bmatrix}}^{A} \begin{bmatrix} P_1(k) \\ P_2(k) \\ P_3(k) \end{bmatrix} +$$

$$\underbrace{\alpha\gamma \begin{bmatrix} 1/G_{11} \\ 1/G_{22} \\ 1/G_{33} \end{bmatrix}}_{B} \sigma^2$$

Q2.

$$G = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.1 & 2 & 0.1 \\ 0.3 & 0.1 & 3 \end{bmatrix} \quad ; \quad \begin{matrix} \alpha = 1.2 \\ \sigma = 0.1 \end{matrix}$$

State Space Equation : $x[k+1] = A\,x[k] + B\,u[k]$

In our case: $x[k] = p[k]$

$u(k) = \sigma^2$

$y(k) = p(k)$

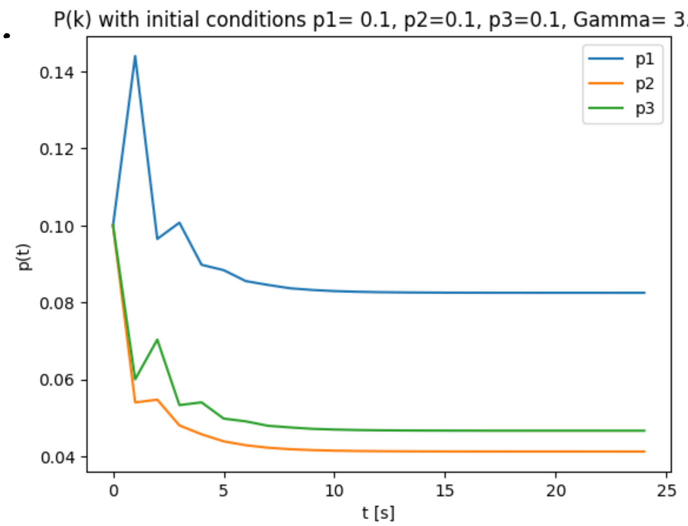
$$P_i(k+1) = A\,p_i(k) + B\sigma^2$$

$$P_i(k) = C\,p_i(k) + D\sigma^2$$

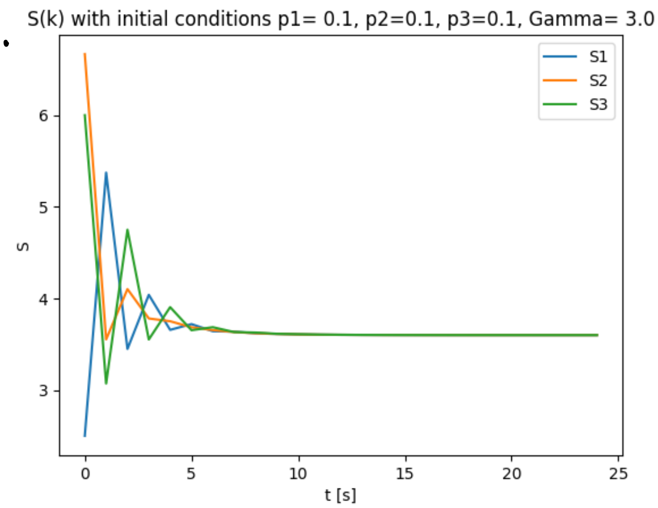
$$\therefore\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

P(t) & S(t) plots from python script output:

1.

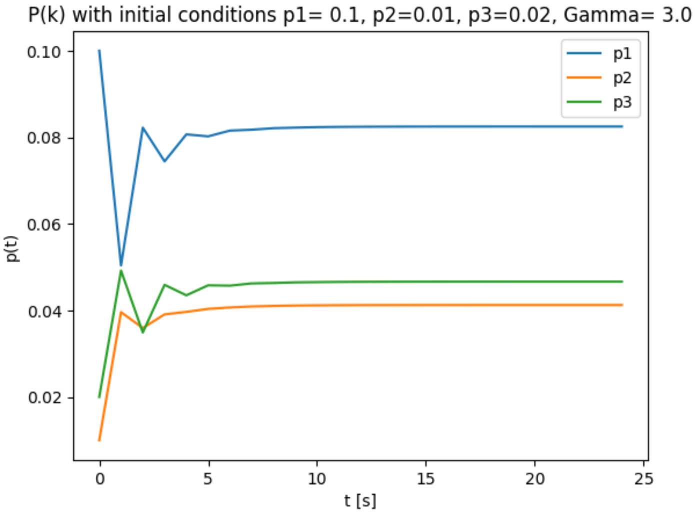


2.

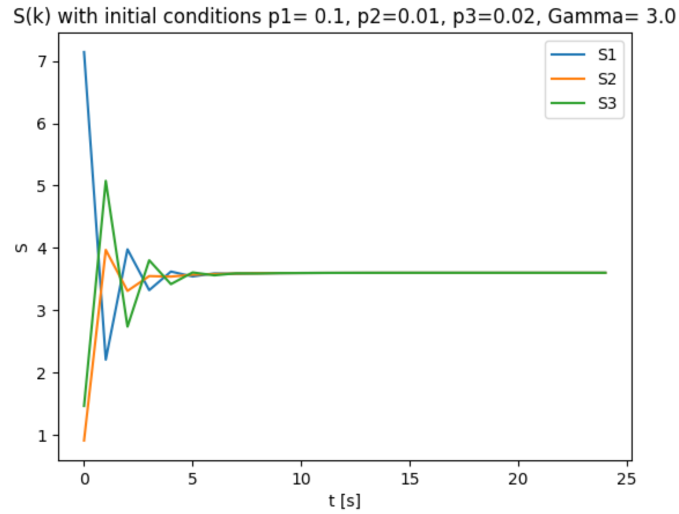


$$\boxed{\begin{matrix} S_i(t) \longrightarrow \alpha\gamma \\ S_i(t) \longrightarrow 3.6 \end{matrix}}$$

3.



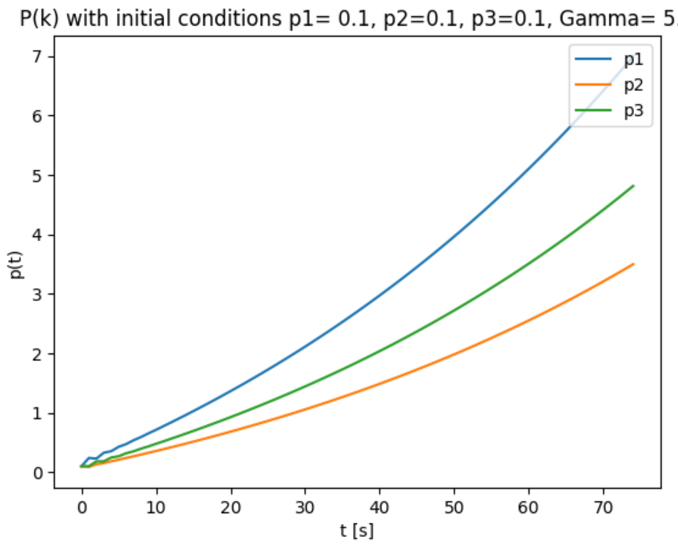
4.



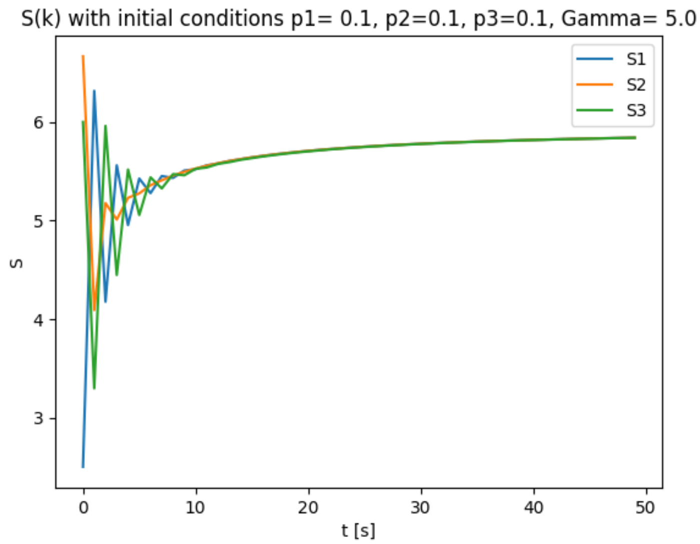
$S_i(t) \longrightarrow \alpha\gamma$

$S_i(t) \longrightarrow 3.6$

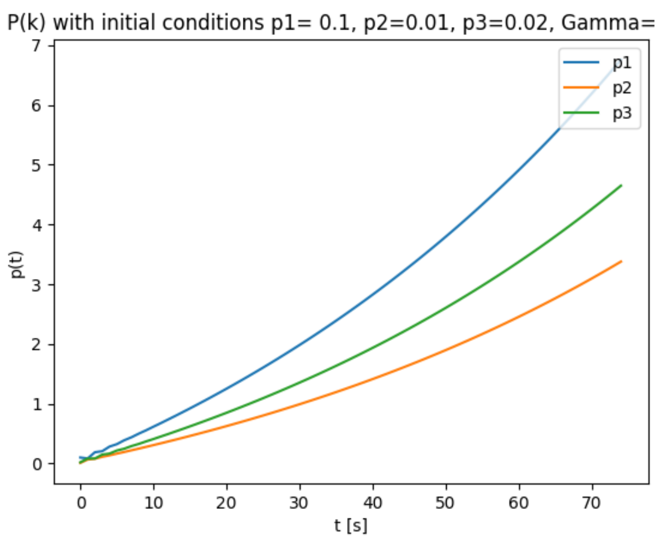
5.



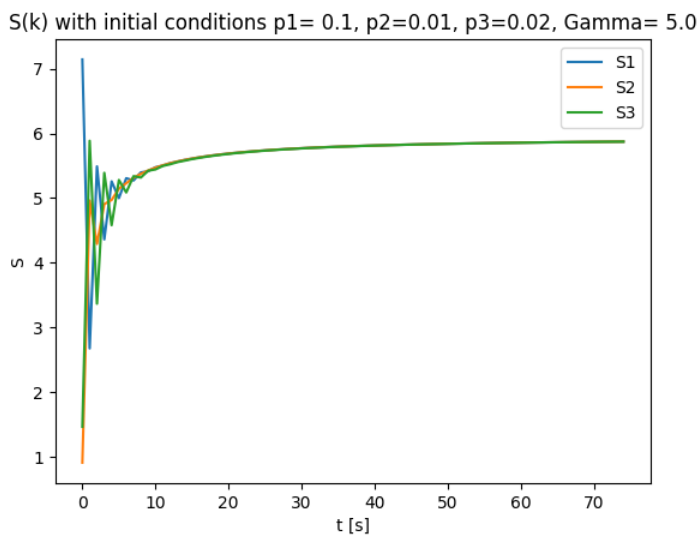
6.



7.



8.



For 6 & 8 $\quad S_i(t) \not\longrightarrow \alpha\gamma$; $\quad S_i(t) \not\longrightarrow 6$

Python Script :*

* The initial conditions for the following code snippet is $p_1 = p_2 = p_3 = 0.1$ & $\gamma = 3$

In [1]:
```
import numpy as np
from scipy.signal import StateSpace, dlsim
import matplotlib.pyplot as plt
```

In [2]:
```
# Assigning value to constants

g = np.asarray([[1., 0.2, 0.1],
             [0.1, 2., 0.1],
             [0.3, 0.1, 3.]])
alpha = 1.2
sigma = 0.1  #self noise

p1 = float(input("Enter the initial value of p1:",))
p2 = float(input("Enter the initial value of p2:",))
p3 = float(input("Enter the initial value of p3:",))
gamma = float(input("Enter the value of threshold:",))
```

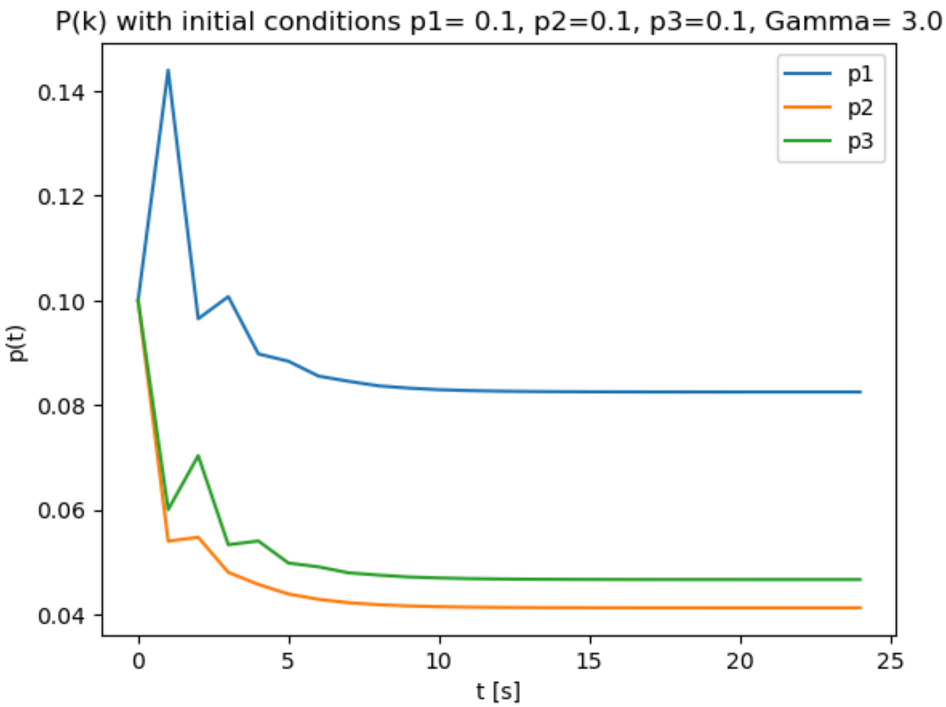
```
Enter the initial value of p1:0.1
Enter the initial value of p2:0.1
Enter the initial value of p3:0.1
Enter the value of threshold:3
```

In [3]:
```
#Calculating the value of A, B, C & D
A = np.asarray(np.zeros((3,3), dtype=float))
B = np.asarray(np.zeros((3,1), dtype=float))

#Using the formula calculated in Ex2 Q1
for i in range(0,3):
    for j in range(0,3):
        if i==j:
            A[i,j] = 0.
        else:
            A[i,j] = (alpha*gamma)*(g[i,j]/g[i,i])    #Using the formula calculated in Ex2 Q1

for i in range(0,3):
    for j in range(0,1):
            B[i,j] = alpha*gamma*1/g[i,i]      #Using the formula calculated in Ex2 Q1

C = np.identity(3, dtype=float)
D = np.zeros((3,1), dtype=float)

drone_sys = StateSpace(A,B,C,D,dt=1)
```

In [4]:
```
T= int(input("Enter Time(s) upto which the system runs"))
t = np.arange(0, T, 1)
u = np.ones((1,len(t)))
u = sigma**2 * u
u = np.transpose(u)      # taking transpose to make sure number of rows in 'u' is same as elements in 't'
# u  = sigma**2 * np.ones_like(t)
_, y, x = dlsim(drone_sys, u, t, x0=[p1, p2, p3])  # Using the dlsim function for discrete systems
```

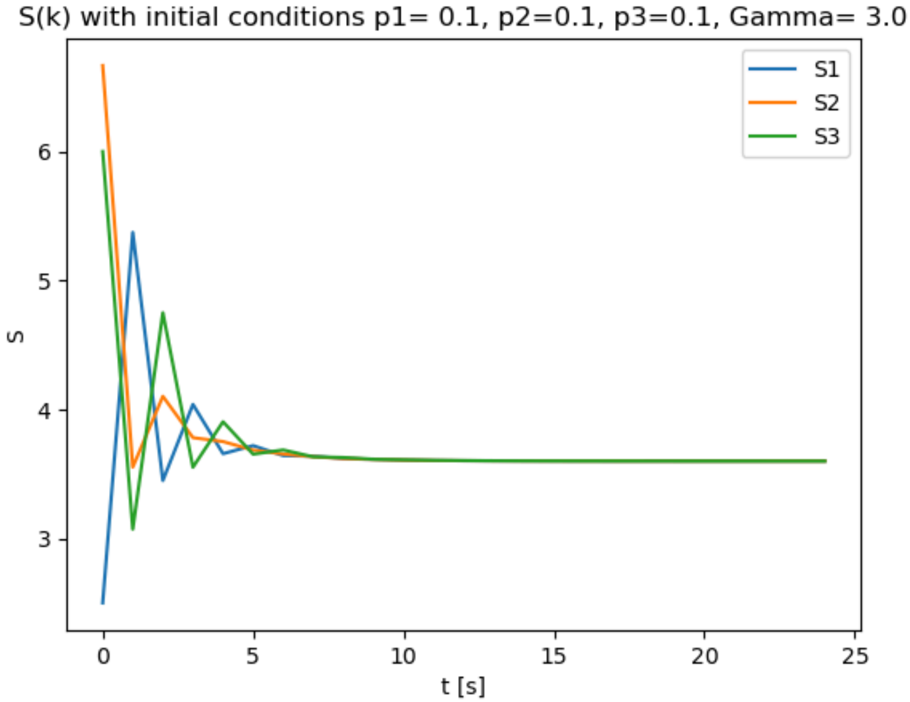
```
Enter Time(s) upto which the system runs25
```

In [5]:
```
#Plotting 'P' as a function of time
plt.figure(dpi=100)
plt.plot(t, y[:,0], label= 'p1')
plt.plot(t, y[:,1], label= 'p2')
plt.plot(t, y[:,2], label= 'p3')
plt.legend(loc='upper right')
plt.ylabel('p(t)')
plt.xlabel('t [s]')
plt.title('P(k) with initial conditions p1= {}, p2={}, p3={}, Gamma= {}'.format(p1,p2,p3,gamma))
plt.show()
```



In [6]:
```
#Computing the value of SINR
S = np.ones_like(y)    #creating S with the same dimensions as 'y'
for i in t:
    for j in range(0,3):
        s_small = g[j,j]*y[i,j]
        q = sigma**2
        for k in range(0,3):
            if j!=k:
                q += g[j,k]*y[i,k]
        S[i,j] = s_small/q
```

In [7]:
```
#Plotting 'S' as a function of time
plt.figure(dpi=100)
plt.plot(t, S[:,0], label= 'S1')
plt.plot(t, S[:,1], label= 'S2')
plt.plot(t, S[:,2], label= 'S3')
plt.legend(loc='upper right')
plt.ylabel('S')
plt.xlabel('t [s]')
plt.title('S(k) with initial conditions p1= {}, p2={}, p3={}, Gamma= {}'.format(p1,p2,p3,gamma))
plt.show()
```



In [8]:
```
#Displaying the S(t) matrix
S
```

Out[8]:
```
array([[2.5       , 6.66666667, 6.        ],
       [5.37313433, 3.55263158, 3.07167235],
       [3.44867029, 4.10194903, 4.74963977],
       [4.03909087, 3.78122677, 3.55197919],
       [3.65694569, 3.75103862, 3.90488145],
       [3.71971169, 3.68488022, 3.65311752],
       [3.6420946 , 3.65475263, 3.68600349],
       [3.63763772, 3.63304051, 3.63281054],
       [3.61870166, 3.6203637 , 3.62756671],
       [3.61299761, 3.61239746, 3.61409893],
       [3.60735214, 3.60756856, 3.60960327],
       [3.60468701, 3.60460901, 3.6054883 ],
       [3.60277917, 3.60280726, 3.60347471],
       [3.60171897, 3.60170885, 3.60206782],
       [3.6010365 , 3.60104015, 3.60127591],
       [3.60063433, 3.60063302, 3.60077032],
       [3.60038475, 3.60038523, 3.60047103],
       [3.60023459, 3.60023442, 3.60028583],
       [3.60014259, 3.60014265, 3.60017422],
       [3.60008682, 3.6000868 , 3.60010591],
       [3.60005281, 3.60005282, 3.60006448],
       [3.60003214, 3.60003214, 3.60003922],
       [3.60001956, 3.60001956, 3.60002387],
       [3.6000119 , 3.6000119 , 3.60001452],
       [3.60000724, 3.60000724, 3.60000884]])
```

Exercise 3

$$\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0 \quad — (1)$$

Defining state as $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

$\therefore$ The state space equation can be written as ;

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -(1+y)\dot{y} + 2y - 0.5y^3 \end{bmatrix}$$

Finding Equilibrium points , $\dot{x} = 0$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} \dot{y} = 0 \\ -\cancel{(1+y)\dot{y}}^{\,0} + 2y - 0.5y^3 = 0 \Rightarrow y(2 - y^2/2) = 0 \end{array}$$

$y = 0 \; , \; \pm 2$

Equilibrium points : $(0,0)$, $(2,0)$, $(-2,0)$

$f_1 = \dot{y}$ $\qquad x_1 = y$
$f_2 = \ddot{y}$ $\qquad x_2 = \dot{y}$

Jacobian matrix :

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dot{y} + 2 - 1.5y^2 & -(1+y) \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \partial f_1 / \partial u \\ \partial f_2 / \partial u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\because$ For our system there exists no input .

The state space Equation :

$$\dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} \delta u$$

$$\Rightarrow \quad \dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \delta x$$

I: The State Space equation @ $(0,0)$ $\Rightarrow$

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \delta x \quad , \quad \text{where} \quad \delta x = x - \bar{x}$$

$$\delta x = \begin{bmatrix} y - 0 \\ \dot{y} - 0 \end{bmatrix} \quad , \quad \dot{\delta x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + 0$$

$$\Rightarrow \ddot{y} + \dot{y} - 2y = 0$$

II: The State space equation @ $(2,0)$ $\Rightarrow$

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \delta x$$

$$\delta x = \begin{bmatrix} y - 2 \\ \dot{y} \end{bmatrix} \quad ; \quad \dot{\delta x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} y - 2 \\ \dot{y} \end{bmatrix}$$

$$\Rightarrow \ddot{y} = -4y + 8 - 3\dot{y}$$

III : The state space equation @ (-2,0)

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \delta x$$

$$\delta x = \begin{bmatrix} y+2 \\ \dot{y} \end{bmatrix} ; \qquad \dot{\delta x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y+2 \\ \dot{y} \end{bmatrix}$$

$$\Rightarrow \ddot{y} = -4y + \dot{y} - 8$$

Exercise 4

Vertical ascent:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g\left[\dfrac{D}{x_1(t)+D}\right]^2 + \dfrac{\ln(u)}{m} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \text{state of the system}$$

$$\dot{x} = 0 \Rightarrow \quad x_2(t) = 0 \quad \& \quad -g\left[\frac{D}{x_1(t)+D}\right]^2 + \frac{\ln(u)}{m} = 0$$

$$\frac{D}{x_1(t)+D} = \pm\sqrt{\frac{\ln(u)}{mg}}$$

$$x_1(t) = \pm D\sqrt{\frac{mg}{\ln(u)}} - D$$

$$x_1^* = D\left[\sqrt{\frac{mg}{\ln(u)}} - 1\right] \quad \& \quad x_1^* = -D\left[\sqrt{\frac{mg}{\ln(u)}} + 1\right]$$

$\therefore$ The equilibrium points (x_1^*, x_2^*):

$$\left(D\left[\sqrt{\frac{mg}{\ln(u)}} - 1\right], 0\right) \quad \& \quad \left(-D\left(\sqrt{\frac{mg}{\ln(u)}} + 1\right), 0\right)$$

Jacobian:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} ; \quad \frac{\partial f}{\partial u} = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{\partial f_2}{\partial x_1} & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ $\because$ our system has no input.

Finding $\frac{\partial f_2}{\partial x_1}$:

$$\frac{\partial}{\partial x_1}\left[-g\left[\frac{D}{x_1+D}\right]^2 + \frac{\ln(u)}{m} \right]$$

$$\Rightarrow \frac{\partial}{\partial x_1}\left[-g\left(\frac{D}{x_1+D}\right)^2 \right] + 0$$

$$\Rightarrow \frac{\partial}{\partial x_1}\left[-g\left(\frac{x_1+D}{D}\right)^{-2} \right] \Rightarrow \frac{\partial}{\partial x_1}\left[-gD^2\,(x_1-D)^{-2}\right]$$

$$\Rightarrow -gD^2\;(-2)\;(x_1+D)^{-3}$$

$$\therefore \frac{\partial f_2}{\partial x_1} = \frac{2gD^2}{(x_1+D)^3}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{2gD^2}{(x_1+D)^3} & 0 \end{bmatrix}$$

The state space Equation:

$$\dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}} \delta u$$

$$\Rightarrow \dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}} \delta x$$

For Point $\left\{ D\left[\sqrt{\frac{mg}{\ln(u)}} - 1\right], 0 \right\} \Rightarrow$

$$\delta x = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}, \quad \dot{\delta x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1^* x_2^*} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{D\left[mg/\ln(u)\right]^{3/2}} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{D\left[mg/\ln(u)\right]^{3/2}} & 0 \end{bmatrix} \begin{bmatrix} x_1 - D\sqrt{mg/\ln(u)} + D \\ x_2 - 0 \end{bmatrix}$$

$$\boxed{\dot{x}_1 = x_2}$$

$$\dot{x}_2 = \frac{2g}{D\left[mg/\ln(u)\right]^{3/2}} \left[x_1 - D\sqrt{mg/\ln(u)} + D \right]$$

$$\boxed{\dot{x}_2 = \frac{2g}{D\left[mg/\ln(u)\right]^{3/2}} x_1 - \frac{2\ln(u)}{m} + \frac{2g}{\left[mg/\ln(u)\right]^{3/2}}}$$

For Point $\left\{ -D\left[\sqrt{\frac{mg}{\ln(u)}} + 1\right], 0 \right\} \Rightarrow$

$$\delta x = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}, \qquad \delta \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1^* x_2^*} = \begin{bmatrix} 0 & 1 \\ \dfrac{-2g}{D\left[mg/\ln(u)\right]^{3/2}} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-2g}{D\left[mg/\ln(u)\right]^{3/2}} & 0 \end{bmatrix} \begin{bmatrix} x_1 + D\sqrt{mg/\ln(u)} + D \\ x_2 - 0 \end{bmatrix}$$

$$\boxed{\dot{x}_1 = x_2}$$

$$\dot{x}_2 = \frac{-2g}{D\left[mg/\ln(u)\right]^{3/2}} x_1 + \frac{-2g}{\cancel{D}\left[mg/\ln(u)\right]^{3/2}} \times \cancel{D}\left[mg/\ln(u)\right]^{1/2} - \frac{2g}{\left[mg/\ln(u)\right]^{3/2}}$$

$$\dot{x}_2 = \frac{-2g}{D\left[mg/\ln(u)\right]^{3/2}} x_1 - \frac{2g}{\left[mg/\ln(u)\right]} - \frac{2g}{\left[mg/\ln(u)\right]^{3/2}}$$

$$\boxed{\dot{x}_2 = \frac{-2g}{D\left[mg/\ln(u)\right]^{3/2}} x_1 - \frac{2}{\left[m/\ln(u)\right]} \left[1 + \frac{1}{\sqrt{mg/\ln(u)}}\right]}$$

Exercise 5

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \quad — (1)$$

$$\ddot{\theta} = -2\frac{\dot{\theta}}{r}\dot{r} + \frac{u_2}{r} \quad — (2)$$

Reference orbit : $u_1 = u_2 = 0$

$r(t) \equiv p \qquad \theta(t) = \omega t$

Q1. $r(t) = p \Rightarrow \dot{r}(t) = 0 = \ddot{r}(t)$

$\theta(t) = \omega t \Rightarrow \dot{\theta}(t) = \omega$

Eqn (1) $\Rightarrow \cancel{\ddot{r}}^{\,0} = r\dot{\theta}^2 - k/r^2 + \cancel{u_1}^{\,0}$

$$0 = r\omega^2 - \frac{k}{r^2} + 0$$

$$k = r^3\omega^2$$

$$\boxed{k = p^3\omega^2}$$

Q2. $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$

u_1 & u_2 are control input

$\ddot{\theta} = -2\frac{\dot{\theta}}{r}\dot{r} + \frac{u_2}{r}$

Defining the states as

$$x = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

We need these 4 states to represent the system.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \begin{array}{ll} x_1 = r & x_3 = \theta \\ x_2 = \dot{r} & x_4 = \dot{\theta} \\ \Rightarrow \ddot{r} = \dot{x}_2 & \Rightarrow \ddot{\theta} = \dot{x}_4 \end{array}$$

$$\dot{x}_2 = x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \qquad \Rightarrow \text{Putting } k = p^3 w^2$$

$$\Rightarrow \dot{x}_2 = x_1 x_4^2 - \frac{p^3 w^2}{x_1^2} + u_1$$

$$\dot{x}_4 = -2\frac{x_4}{x_1} x_2 + \frac{u_2}{x_1}$$

State space equation:

$$\dot{x} = Ax + Bu \qquad \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

We need to linearize the above system.
The resulting form of the equation should be as follows:

$$\dot{\delta}_x = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}\,\bar{u}} \delta_x + \left.\frac{\partial f}{\partial u}\right|_{\bar{x}\,\bar{u}} \delta_u$$

$\therefore$ Defining the variables as:

$$\delta_{x_1} = x_1 - \bar{x}_1 \qquad \delta_{x_2} = x_2 - \bar{x}_2 \quad \ldots \text{ and so on}$$

$$\delta_{u_1} = u_1 - \bar{u}_1 \qquad \delta_{u_2} = u_2 - \bar{u}_2$$

Since we are linearizing on trajectory & not at equilibrium point, we do not use $\dot{x} = 0$. Instead, we put $\bar{x}$ or $\bar{u}$ as the reference orbit conditions.

$$\delta_{x_1} = x_1 - p \;\Big|\; \delta_{x_2} = x_2 - 0 \;\Big|\; \delta_{x_3} = x_3 - wt \;\Big|\; \delta_{x_4} = x_4 - w$$

$$\dot{\delta}_{x_1} = \dot{x}_1 \;\Big|\; \dot{\delta}_{x_2} = \dot{x}_2 \;\Big|\; \dot{\delta}_{x_3} = \dot{x}_3 - w \;\Big|\; \dot{\delta}_{x_4} = \dot{x}_4$$

$\delta_{u_1} = u_1 - 0$ $\qquad$ $\delta_{u_2} = u_2 - 0$

$\dot{\delta}_{u_1} = \dot{u}_1$ $\qquad$ $\dot{\delta}_{u_2} = \dot{u}_2$

Jacobian :

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots\cdots & \frac{\partial f_4}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_4}{\partial x_1} & & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

$f_1 = \dot{x}_1 = x_2$

$f_2 = x_1 x_4^2 - \frac{p^3 w^2}{x_1^2} + u_1$

$f_3 = \dot{x}_3 = x_4$

$f_4 = -2 \frac{x_4}{x_1} x_2 + \frac{u_2}{x_1}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2p^3w^2}{x_1^3} + x_4^2 & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \frac{2x_4x_2}{x_1^2} - \frac{u_2}{x_2^2} & \frac{-2x_4}{x_1} & 0 & -\frac{2x_2}{x_1} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \vdots \\ \vdots & \vdots \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{x_1} \end{bmatrix}$$

$\therefore$ The state space form becomes $\Rightarrow$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}\bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{2p^3w^2}{p^3} + w^2 & 0 & 0 & 2pw \\ 0 & 0 & 0 & 1 \\ 2w(0) - 0 & -\frac{2w}{p} & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}\bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2pw \\ 0 & 0 & 0 & 1 \\ 0 & -2w/p & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_{\bar{x}\bar{u}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/p \end{bmatrix}$$

$$\begin{bmatrix} \dot{\delta}_{x_1} \\ \dot{\delta}_{x_2} \\ \dot{\delta}_{x_3} \\ \dot{\delta}_{x_4} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2pw \\ 0 & 0 & 0 & 1 \\ 0 & -2w/p & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 - p \\ x_2 \\ x_3 - wt \\ x_4 - w \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/p \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 - w \\ \dot{x}_4 \end{bmatrix}$$

$\therefore$ The Linearized system is as follows:

$x_1 = r$ $\quad$ $x_3 = \theta$

$x_2 = \dot{r}$ $\quad$ $x_4 = \dot{\theta}$

$\Rightarrow \ddot{r} = \dot{x}_2$ $\quad$ $\Rightarrow \ddot{\theta} = \dot{x}_4$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 3w^2(x_1 - p) + 2pw(x_4 - w) + u_1$$

$$\dot{x}_2 = 3w^2 x_1 + 2pw x_4 - 5w^2 p$$

$$\dot{x}_3 - w = x_4 - w$$

$$\dot{x}_4 = -\frac{2w}{p} x_2 + \frac{u_2}{p}$$

Converting the state variables to its original values

$\therefore$ The linearized trajectory equation is:

$$\boxed{\begin{aligned} \ddot{r} &= 3w^2 r + 2pw\dot{\theta} - 5w^2 p + u_1 \\ \ddot{\theta} &= -\frac{2w}{p}\dot{r} + \frac{u_2}{p} \end{aligned}}$$