Exercise 1 : Controllability & Observability

State Equation:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{C} x \qquad D = 0$$

Controllability:

$\because n = 3$ (# of states)

$$P = [B : AB : A^2B]$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A^2B = A.AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$|P| = 1(-1) + 0$$
$$= -1$$

$\because |P| \neq 0$

Rank of P is 3 [Full Rank]

Rank(P) = n = 3 , $\therefore$ The system is controllable

Observability:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix}$$

$$CA^2 = CA.A = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|Q| = 1(-2+2) - 2(-1+1) + 1(-2+2)$$

$$= 0$$

$$\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = -2 + 2 = 0.$$

$\therefore$ Rank $(Q) = 1 \neq 3$, the system is not observable

Exercise 2 : Jordan form test

Jordan form equation :

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{J} x + \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}}_{C} x$$

Controllability :

For $\lambda = 2$, There consists 3 Jordan blocks .

$$\therefore \hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \qquad |\hat{B}^2| = 2(1-2) - 1(1-3) + 1(2-3) = -2 + 2 - 1 = -1$$

$\therefore$ rank $(\hat{B}^2) = 3$

$\because$ $\hat{B}^2$ has full rank, the system is controllable for $\lambda = 2$

For $\lambda = 1$, There consists 2 Jordan blocks

$$\therefore \hat{B}^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{rank } (\hat{B}^1) = 2$$

$\because$ $\hat{B}^1$ has full rank, the system is controllable for $\lambda = 1$

Observability:

for $\lambda = 2$, there are 3 Jordan blocks.

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|\hat{C}^2| = 2(0-2) - 1(1-0) + 3(1-0)$$
$$= -4 - 1 + 3 = -2$$

$\therefore$ rank $(\hat{C}^2) = 3$

$\because$ $\hat{C}^2$ has full rank, the system is observable for $\lambda = 2$

for $\lambda = 1$, there are 2 Jordan blocks

$$\hat{C}^1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

rank $(\hat{C}) = 2$

$\because$ $\hat{C}^1$ has full rank, the system is observable for $\lambda = 1$

$\because$ The system is observable & controllable for all its eigenvalues,
the system is observable & controllable

Exercise 3: Controllability

Previous system:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{u}_{u}$$

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} \underbrace{u}_{u}$$

$$P = [B : AB] = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}$$

$$|P| = \alpha$$

$\therefore$ The system is controllable $\forall\ \alpha \neq 0$

Given in the prev. homework, $\alpha = 0.1$. $\therefore$ The system is controllable.

Modified System:

System dynamics:

$$\frac{dx_1}{dt_1} = -\alpha x_1$$

$$\frac{dx_2}{dt_2} = \alpha x_1 - \beta x_2 + u$$

Modelling the above as state space equations.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$P = [B : AB] = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}$$

$$|P| = 0$$

$\because |P| = 0$, $\text{rank}(P) = 1 \neq 2$.

$\therefore$ The modified system is not controllable

The system is not controllable after the change in inlet pipes, as now the input is only able to influence the level of tank 2. In the first system, the inlet was such that it could influence tank 1 & the level of tank 2 indirectly, therefore being a controllable system.

Exercise 4: Gauss Elimination & LU Decomposition

1. (a)

$$x + y + z = 3$$
$$x + 2y + 3z = 0$$
$$x + 3y + 2z = 3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \qquad Ax = y$$

Augmented matrix:

$$W = [A : y]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right]$$

$R_2 = R_2 - R_1$, $R_3 = R_3 - R_1$

$$W = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$R_3 = R_3 - 2R_2$

$$W = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{array}\right]$$

$z = -2$

$y + 2z = -3 \Rightarrow y = 1$

$x + y + z = 3 \Rightarrow x = 4$

$x = 4$
$y = 1$
$z = -2$

(b)

$$x + 2y - z = 1$$
$$2x + 5y - z = 3$$
$$x + 3y + 2z = 6$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$, $R_3 = R_3 - R_1$

$$W = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$W = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$z = 2$

$y + z = 1 \Rightarrow y = -1$

$x + 2y - z = 1 \Rightarrow x = 1 - 2(-1) + 2$

$= 5$

$x = 5$
$y = -1$
$z = 2$

(c)

$$x_1 + x_2 - x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + x_3 = 4$$
$$3x_1 + 5x_2 + 3x_3 - x_4 = 5$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 3 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}}_{y}$$

$$W = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{bmatrix}$$

$\because r(A) = r(W) = 2 < n(4)$

Infinite solutions exists

There are more variables than equations.

Finding rank of A:

$r(A) = 2$

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 3 \end{vmatrix} = 1(9-5) - 1(6-3) - 1(10-9)$$
$$= 4 - 1(3) - 1$$
$$= 0.$$

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 \neq 0$$

$\therefore r(A) = 2$

Finding rank of W:

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 3 \end{vmatrix} = 1(9-5) - 1(6-3) - 1(10-9)$$
$$= 4 - 3 - 1$$
$$= 0$$

$\therefore r(W) = 2$

2. LU Decomposition

$$x_1 + 2x_2 + 4x_3 = 3$$
$$3x_1 + 8x_2 + 14x_3 = 13$$
$$2x_1 + 6x_2 + 13x_3 = 4$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$L \rightarrow$ lower triangular matrix

$U \rightarrow$ upper triangular matrix

$A = LU$.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

On solving:

$u_{11} = 1$; $u_{12} = 2$; $u_{13} = 4$

$l_{21} = 3$; $u_{22} = 2$; $u_{23} = 2$

$l_{31} = 2$; $l_{32} = \dfrac{6-(2)(2)}{2} = 1$; $u_{33} = 13 - 8 - 2$
$= 3$

$$\therefore U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \& \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 14 \end{bmatrix}$$

$$y_1 = 3$$

$$3y_1 + y_2 = 13 \Rightarrow y_2 = 4$$

$$2y_1 + y_2 + y_3 = 14 \Rightarrow y_3 = 10 - 6 - 4$$
$$= -6$$

$$y = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$Ux = y$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$x_3 = -2$$

$$2x_2 + 2x_3 = 1 \Rightarrow x_2 = \frac{4+4}{2} = 4$$

$$x_1 = 3 - 2(4) - 4(-2)$$
$$= 3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Exercise 5 : SVD

Image : $m \times n$ pixels

$$\underset{\text{image mat.}}{A} = \underset{m \times m}{U} \; \underset{m \times n \text{ (image size)}}{\Sigma} \; \underset{n \times n}{V^T}$$

$A^T A v_i = \sigma_i^2 v_i$ analogous to $A' v_i = \lambda v_i$

σ_i^2 → e-value for $A^T A$

here λ_i is e-value

If compressing image, use only top 'i' singular values.

Original : $m \times n$

New [Compressed] : $i\,[m+n+1]$

$$A_{m \times n} = U_{m \times m} \; \Sigma_{m \times n} \; V_{n \times n}$$

$$A = \underset{m \times i}{\overset{\longrightarrow k}{\begin{bmatrix} U \end{bmatrix}}} \underset{i \times i}{\begin{bmatrix} \Sigma \end{bmatrix}} \underset{i \times n}{\begin{bmatrix} V \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}$$

v_1 —— v_m

v_{21} —— v_{2n}

total storage : $m \times i + i + i \times n$

$\Rightarrow i\,[m+n+1]$

$i\,[m+n+1] = \text{comp}^n \text{ ratio} \times (m \times n)$

$$i \simeq \frac{\text{comp}^n \text{ ratio} \times (m \times n)}{(m+n+1)}$$

Image : 675×1200

HW3

September 27, 2024

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from PIL import Image
import os

image_path = 'C:/Users/ashwi/Desktop/CMU/FALL_2024/Modern_Control_Theory/
 ↪Homework/MCT_HW3/CMU_Grayscale.png'
image = Image.open(image_path)
image_matrix = np.array(image)
# print(image_matrix)
# print(np.shape(image_matrix))
m,n = np.shape(image_matrix)
U, E, V = np.linalg.svd(image_matrix)
#E = np.reshape(E,(-1,1))
print(np.shape(U))
print(np.shape(E))
print(np.shape(V))
```

```
(675, 675)
(675,)
(1200, 1200)
```

[ ]:
```
#Defining some useful functions

def findi(ratio, m, n):
    i = int(ratio*m*n/(m+n+1))
    print(i)
    return i

def gen_image(U, E, V, i):
    E = np.diag(E[:i])
    im = U[:, 0:i] @ E[0:i] @ V[0:i, :]
    return im
```

[ ]:
```
i_50 = findi(0.5, m, n)
i_10 = findi(0.1, m, n)
i_05 = findi(0.05, m, n)
```

```
i_50_image = gen_image(U,E,V,i_50)
i_10_image = gen_image(U,E,V,i_10)
i_05_image = gen_image(U,E,V,i_05)
```

```
215
43
21
```

[ ]:
```
def save_image(image, compression_level, title, output_path):
    output_filename = f'compressed_{compression_level}.png'
    output_path_final = os.path.join(output_path, output_filename)

    plt.imshow(image,cmap='gray')
    plt.title(f"{title}")
    plt.axis('off')
    plt.savefig(output_path_final)
    plt.show()

    # if os.path.exists(output_path_final):
    #     print(f"Image successfully saved at: {output_path_final}")
    # else:
    #     print("Failed to save the image.")
```

[ ]:
```
plt.imshow(image,cmap='gray')
plt.title("Original Image")
plt.axis('off')
plt.show()

save_image(i_50_image, 50, f"Compressed to 50%, No: of singular values={i_50}",␣
 ↪"C:/Users/ashwi/Desktop/CMU/FALL_2024/Modern_Control_Theory/Homework/MCT_HW3/
 ↪Compressed_Images")
save_image(i_10_image, 10, f"Compressed to 10%, No: of singular values={i_10}",␣
 ↪"C:/Users/ashwi/Desktop/CMU/FALL_2024/Modern_Control_Theory/Homework/MCT_HW3/
 ↪Compressed_Images")
save_image(i_05_image, 5,f"Compressed to 5%, No: of singular values={i_05}", "C:
 ↪/Users/ashwi/Desktop/CMU/FALL_2024/Modern_Control_Theory/Homework/MCT_HW3/
 ↪Compressed_Images")
```

Original Image

Compressed to 50%, No: of singular values=215

Compressed to 10%, No: of singular values=43

Compressed to 5%, No: of singular values=21

[ ]:
```
################
NOTEBOOK_NAME = "HW3.ipynb"
!apt-get install -y pandoc > /dev/null 2>&1
!sudo apt-get install texlive-xetex texlive-fonts-recommended␣
↪texlive-plain-generic > /dev/null 2>&1
!pip install nbconvert > /dev/null 2>&1
```

```python
from google.colab import drive, files
drive.mount('/content/drive')
notebook_path = rf'/content/drive/MyDrive/Colab Notebooks/{NOTEBOOK_NAME}'

import os
if os.path.exists(notebook_path):
  !jupyter nbconvert --to pdf "{notebook_path}" > /dev/null 2>&1
  pdf_filename = notebook_path.replace('.ipynb', '.pdf')
  files.download(pdf_filename)
else:
  print("Notebook not found. Please check the path.")
################
```

Exercise 6: Design for Controllability & Observability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. controllable but not observable

$$P = [B : AB] = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

This system is controllable if rank $(P) = 2$.

$$\Rightarrow |P| \neq 0$$

$$\begin{vmatrix} 1 & -3 \\ 0 & \gamma \end{vmatrix} \neq 0$$

$$\gamma \neq 0$$

$\therefore$ The system is controllable $\forall\ \gamma \neq 0$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \gamma-3 & -1 \end{bmatrix}$$

The system is not observable if rank $(Q) < 2$

$$\therefore |Q| = 0$$

$$\begin{vmatrix} 1 & 1 \\ \gamma-3 & -1 \end{vmatrix} = 0 \Rightarrow -1 - 1(\gamma-3) = 0$$

$$-1 - \gamma + 3 = 0$$

$$\gamma = 2.$$

$\therefore$ The system is controllable but not observable for $\gamma = 2$

2. Observable but not controllable.

The system is observable if rank $(Q) = 2$

$|Q| \neq 0$

$$\begin{vmatrix} 1 & 1 \\ \gamma - 3 & -1 \end{vmatrix} \neq 0 \quad \Rightarrow \quad \gamma \neq 2$$

The system is not controllable if rank $(P) < 2$

$|P| = 0$

$$\begin{vmatrix} 1 & -3 \\ 0 & \gamma \end{vmatrix} = 0 \quad \Rightarrow \quad \gamma = 0.$$

$\therefore$ The system is observable & not controllable for $\gamma = 0$

Exercise 7: State Space Representation, Controllability

1. $u(t) \rightarrow$ brightness command
   $x(t) \rightarrow$ brightness of LED
   $y(t) \rightarrow$ state itself

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

k	1	2	3	4	5
k	u_1	–	–	–	–
$k+T$	u_2	u_1	–	–	–
$k+2T$	u_3	u_2	u_1	–	–
$k+3T$	u_4	u_3	u_2	u_1	–
$k+4T$	u_5	u_4	u_3	u_2	u_1
$k+5T$	u_6	u_5	u_4	u_3	u_2

Repeat.

We find a pattern here as follows:

so $x_1(k+1) = u(k) \longrightarrow$ new command.

$$x_2(k+1) = x_1(k)$$

$$x_3(k+1) = x_2(k)$$

$$x_4(k+1) = x_3(k)$$

$$x_5(k+1) = x_4(k)$$

$\therefore$ Modelling the above as a state-space eq^n:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$X[k+1] = A_d X[k] + B_d u(k)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t)$$

$$y(k) = C_d x[k] + D u[k]$$

2. Controllability:

$$P = [B : AB : A^2B : A^3B : A^4B]$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} ; \quad A^2B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^4B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_{5\times 5}$$

$\therefore$ rank $(P) = 5 \rightarrow$ full rank. Hence, the system is controllable.

In the above system, we can see that the input signal directly controls the state 1. We can also see that each of the consecutive states are dependent on the previous one. Therefore, with the current arrangement we can influence the state of all the LEDs as each one is dependent on the previous & ultimately the input signal. Therefore, making it controllable.