Exercise 1 : Canonical Forms

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

on comparing with the standard form.

$$G(s) = \frac{b_n s^n + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

$n = 2$

$b_2 = 0$, $b_1 = 1$, $b_0 = 3$

$a_1 = 3$, $a_0 = 2$

$\therefore$ The state space representation (controllable canonical form)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 - b_2 a_0 & b_1 - b_2 a_1 \end{bmatrix} x + b_2 u$$

$$= \begin{bmatrix} 3-(0)(2) & 1-(0)3 \end{bmatrix} x + 0(u)$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x$$

$\therefore$ The observable canonical form is;

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x$$

Exercise 2 : Realization of MIMO System

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$$

$$G(s=\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$G_{sp} = G - G(\infty) = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

Finding common denominator: $\Delta(s)$

$$\Delta(s) = s(s+1)(s+3) = s(s^2+4s+3)$$
$$= s^3+4s^2+3s$$

$\alpha_1 = 4$
$\alpha_2 = 3$
$\alpha_3 = 0$

$$G_{sp} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

$$G_{sp} = \frac{1}{\Delta(s)} \left\{ \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}}_{C_1} s^2 + \underbrace{\begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix}}_{C_2} s + \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}}_{C_3} \right\}$$

$$A = \begin{bmatrix} -4I_2 & -3I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\therefore$ The state space equation is:

$$\dot{x} = Ax + Bu$$

$$y = Cx + D$$

$$\dot{x} = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

Exercise 3: Minimum Realizations

1st system:

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 2 & 2 \end{bmatrix}}_{C} x$$

we know that, $G_1(s) = C(sI - A)^{-1} B + D$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_1(s) = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-2)(s-1)} \left(2(s-1) \right)$$

$$\boxed{G_1(s) = \frac{2}{(s-2)}}$$

2nd system:

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 2 & 0 \end{bmatrix}}_{C} x$$

$$G_2(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(s-2)(s-1)} \begin{bmatrix} 2(s-1) \end{bmatrix}$$

$$G_2(s) = \frac{2}{(s-2)}$$

Checking controllability & observability for System 1:

$P = [B \;\; AB] = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad r(P) = 1 < 2 \quad \therefore$ It is not controllable

$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ $\quad r(Q) = 1 < 2 \quad \therefore$ It is not observable.

Checking controllability & observability for System 2:

$P = [B \;\; AB] = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ $\quad r(P) = 2 \quad \therefore$ The system is controllable

$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$ $\quad r(Q) = 1 < 2 \quad \therefore$ The system is not observable

The above two systems are equivalent based on their transfer functions.

$\therefore$ They are not minimal realizations because the systems are not both controllable & observable.

Exercise 4: Realization

Transfer function : $g(s) = \dfrac{2s-4}{s^3 - 7s + 6}$

(a) Canonical controllable realisation

here, $n = 3$

$\therefore \; b_3 = 0 \;,\; b_2 = 0 \;,\; b_1 = 2 \;,\; b_0 = -4$

$a_2 = 0 \;,\; a_1 = -7 \;,\; a_0 = 6$

$$\therefore \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} \;,\; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;,\; C = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix}$$

$D = 0$

$\therefore$ The controllable canonical realization is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix} x$$

(b) Canonical observable realization

here, $n = 3$

$\therefore \; b_3 = 0 \;,\; b_2 = 0 \;,\; b_1 = 2 \;,\; b_0 = -4$

$a_2 = 0 \;,\; a_1 = -7 \;,\; a_0 = 6$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$\therefore$ The observable canonical form is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(c) Minimal realization

$$g(s) = \frac{2s-4}{s^3 - 7s + 6}$$

Let's start with the controllable canonical form:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

From this we can find that;

$A_c =$

$$y = \underbrace{\begin{bmatrix} -4 & 2 & 0 \end{bmatrix}}_{C} x$$

Check for observability,

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ -12 & 14 & -4 \end{bmatrix}$$

$\text{rank}(Q) = 2 < 3$ $\therefore$ The above system is not observable.

Using Kalman Decomposition $\Rightarrow$

linearly independent rows of Q

$$M^{-1} = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

arbitrary row to make M full rank

$$\hat{A} = M^{-1} A M = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 7 & 0 \end{bmatrix} \begin{bmatrix} -1/6 & 0 & 1/3 \\ 1/6 & 0 & 2/3 \\ 1/3 & 1/2 & 4/3 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 0 \\ 1/2 & 1/2 & 2 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; \quad \hat{C} = CM = \begin{bmatrix} -4 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1/6 & 0 & 1/3 \\ 1/6 & 0 & 2/3 \\ 1/3 & 1/2 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{x}}_o \\ \dot{\hat{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}\begin{bmatrix} \hat{x}_o \\ \hat{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u, \; y = \begin{bmatrix} C_o & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_o \\ \hat{x}_{\bar{o}} \end{bmatrix} + Du$$

On comparing,

$$\dot{\hat{x}}_o = A_o \hat{x}_o + B_o u \qquad A_o = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \qquad B_o = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$y = C_o \hat{x}_o + Du \qquad C_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$\dot{\hat{x}}_o = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}\hat{x}_o + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\hat{x}_o$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix} \qquad r(P) = 2 \rightarrow \text{Full rank}$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$r(Q) = 2 \rightarrow \text{Full rank.}$$

$\because$ The above system is both controllable & observable, this is the minimal realization in state space form.

$\Rightarrow$ Converting to Transfer Function.

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}\right]^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -3 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{s(s+2)-3} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$G(s) = \frac{2}{s^2+2s-3}$$

→ This is the minimal realisation of the original transfer function.

Exercise 5: Controllable decomposition

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x$$

Finding $P = \begin{bmatrix} B & AB \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \qquad r(P) = 1 < 2 \quad \therefore \text{ not controllable}$$

Say $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ taking 1 linearly independent column from P & another arbitrary column to make M full rank.

$$\hat{A} = M^{-1} A M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ; \quad \hat{C} = CM = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore$ The controllable form is :

$$\dot{\hat{x}}_c = 3x + u$$

$$y = 2x$$

Checking observability : $Q = [C] = [2]$

Here, $r(Q) = 1$

$\therefore$ The decomposed state is observable.

as Q matrix is full rank.

Exercise 6: Kalman Decomposition

$\lambda_1 = a$

$\lambda_2 = b$

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x$$

A contains 2 Jordan blocks:

$$J_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \quad \& \quad J_2 = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

For J_1: checking observable & controllability.

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \lambda_1 \end{bmatrix} \qquad r(P) = 2$$

$\therefore J_1$ is controllable.

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \lambda_1 \end{bmatrix} \qquad r(Q) = 1 < 2$$

$\therefore J_1$ is not observable.

For J_2: checking observable & controllability.

$$\dot{x} = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & \lambda_2 & \lambda_2^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(P) = 1 < 3 \quad \therefore J_2$ is not controllable.

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_2 & 1 & \lambda_2 \\ \lambda_2^2 & 2\lambda_2 & \lambda_2^2+1 \end{bmatrix}$$

$r(Q) = 3$

$\therefore J_2$ is observable.

J_1 : Controllable but not observable

Using Observable Kalman Decomposition:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$Q = \begin{bmatrix} 0 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

→ Taking 1 row from Q

→ arbitrary row to make M^{-1} full rank

$$\hat{A} = M^{-1} A M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} \overset{A_0}{\lambda_1} & 0 \\ 1 & \lambda_1 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \overset{B_0}{1} \\ 1 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \overset{C_0}{1} & 0 \end{bmatrix}$$

Observable form:

$$\left.\begin{aligned} \dot{\hat{x}}_0 &= \lambda_1 \hat{x}_0 + u \\ y &= \hat{x}_0 \end{aligned}\right\} - (1)$$

J_2: Observable but not controllable

Using Controllable Kalman Decomposition:

$$\dot{x} = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & \lambda_2 & \lambda_2^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking 1 linearly columns from P

Arbitrary columns to make M full rank.

$$\left.\begin{aligned} \hat{A} &= M^{-1}AM = A \\ \hat{B} &= M^{-1}B = B \\ \hat{C} &= CM = C \end{aligned}\right\} \text{ as } M \text{ is identity matrix}$$

$$\dot{\hat{x}}_c = \lambda_2 \hat{x}_c + u$$

$$y_c = \hat{x}_c$$

Combining J_1 & J_2 into original form:

$$J_1: \quad \dot{\hat{x}}_o = \lambda_1 \hat{x}_o + u \qquad y = \hat{x}_o$$

$$J_2: \quad \dot{\hat{x}}_c = \lambda_2 \hat{x}_c + u \qquad y_c = \hat{x}_c$$

$$\left.\begin{aligned} \dot{\hat{x}}_{co} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \hat{x}_{co} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \hat{x}_{co} \end{aligned}\right\} \text{ The full decomposition using Kalman Decomposition.}$$

Exercise 7: Controllable Canonical Form

$$\frac{\theta(s)}{V(s)} = \frac{k_T}{(Ls+R)(Js^2+bs)}$$

$$= \frac{k_T}{\left[LJs^3 + Lbs^2 + RJs^2 + Rbs\right]} = \frac{k_T/LJ}{\left[s^3 + \frac{(Lb+RJ)}{LJ}s^2 + \frac{Rb}{LJ}s\right]}$$

Here, $n = 3$

$b_3 = 0$, $b_2 = 0$, $b_1 = 0$, $b_0 = k_T$

$a_2 = \frac{(Lb+RJ)}{LJ}$, $a_1 = \frac{Rb}{LJ}$, $a_0 = 0$

The controllable canonical form is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Rb/LJ & \frac{(Lb+RJ)}{LJ} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{k_T}{LJ} & 0 & 0 \end{bmatrix} x$$