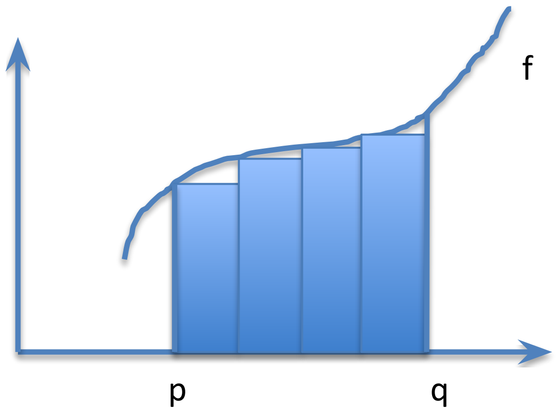
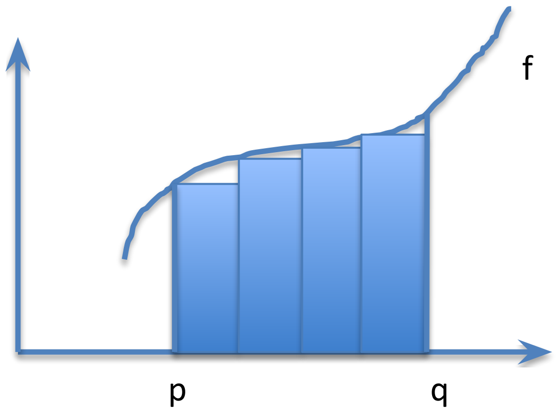
Numerical integration

- In many scientific calculations we need to integrate.
  - Closed form may not be available.
- We can integrate numerically
  - Integral = area under the curve
  - Approximate area by rectangles.
  - The more rectangles we use, more accurate is the answer



Plan for writing a program

- Read p, q.
- Read n = number of rectangles.
- Calculate w = width of rectangle
  = (q-p)/n.
- Consider i^{th} rectangle, i=0,1,..,n-1
  - Begins at x = p+iw.
  - Height = f(x) = f(p+iw)
  - Area = w * f(p+iw)
- Integral = sum over all i.



The generic program

- Read p, q.
- Read n = number of rectangles.
- Calculate w = width of rectangle = (q-p)/n.
- Consider i^{th} rectangle, i=0,1,..,n-1
  - Begins at x = p+iw.
  - Height = f(x) = f(p+iw)
  - Area = (p+iw) * f(p+iw)
- Integral = sum over all i.

Should put code to evaluate f(p+iw)
Next

```
main_program{
  double p, q; cin >>p>>q;
  int n; cin >> n;
  double w = (q-p)/n;
  double area = 0;
  for(int i=0; i<n; i++){
    area = area + w*f(p+iw);
  }
  cout << area << endl;
}
```

Numerical integration to calculate ln(x)

ln(x) = natural logarithm
=
area under curve f(x)=1/x from 1 to x.

```
main_program{
  double p, q; cin >>p>>q;
  int n; cin >> n;
  double w = (q-p)/n;
  double area = 0;
  for(int i=0; i<n; i++){
    area = area +
        w*1/(p+iw);
  }
  cout << area << endl;
  cout << log(x) << endl;
}
```

Analysis of the error

- Error 1: due to the gap between the rectangles and the curve.
  - Can be reduced by increasing the number of rectangles.
- Error 2: in area of each rectangle
  - Each number is expressed to precision of few digits: 7-8 for float, 16-17 for double.
  - So error of 10^{-8} or 10^{-17} per rectangle.
  - If you add up n such areas error increases to $n*10^{-8}$ or $n*10^{-17}$.
  - So be careful in increasing n too much.
- Ways of decreasing errors:
  - Use trapeziums instead of rectangles, hug curve better
  - Set rectangle height = function value at the midpoint of its width. (See text)

Exercise

- In the picture earlier it appears that the answer we calculate will be smaller than the actual integral.
  - Show that the answer could be larger than the integral for a different curve for f.
- Will the error be less if we take the height to be the function value in the middle of the rectangle rather than at the beginning?
  - How does this work for $f(x) = x$?

What we discussed

- Integration is needed in many places in scientific computing.
  – Numerical integration can be used if closed form solutions are not available.
- In order to perform numerical integration we just need to be able to evaluate the given function at an arbitrary point.
- Error will depend upon how many and what kind of rectangles you use.

Next: Bisection method for finding roots

♞