# CS 228 (M) - Logic in CS
## Tutorial III - Solutions

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# Question 1

This statement is **False**. An easy counterexample to this would be $\mathcal{F} = \{p, \neg p\}$ and $\mathcal{G} = \{q, \neg q\}$.

# Question 2

## Theorem

*A set of formulae $\Sigma$ is satisfiable iff every finite subset of it is satisfiable.*

This theorem is known as the **Compactness Theorem**.

## Proof.

Proving the backward direction is trivial, as clearly if $\Sigma$ is satisfiable then every finite subset of $\Sigma$ is satisfiable (indeed, every subset is satisfiable). Let us show that if $\Sigma$ is not satisfiable, then there exists a finite subset of it that is unsatisfiable (this suffices to show the forward direction). By the Completeness[a] of our Formal Proof System, if $\Sigma$ is unsatisfiable, then it is inconsistent, ie $\Sigma \vdash \bot$. The proof of this statement can use only a finite number of formulae in $\Sigma$ (since all proofs are finite). Call this finite subset $\Sigma'$. Our proof of $\Sigma \vdash \bot$ will also show that $\Sigma' \vdash \bot$, and so this $\Sigma'$ is a finite subset of $\Sigma$ that is unsatisfiable. $\square$

[a]For this proof to be airtight, our proof of completeness should not depend on the Compactness Theorem, even in the infinite case. Such proofs do exist.

# Question 2

Since $\mathcal{F}$ is inconsistent (and therefore also unsatisfiable), by the Compactness Theorem there exists a finite subset of $\mathcal{F}$ (say $\mathcal{F}'$) that is unsatisfiable (and therefore inconsistent). Since $\mathcal{F}$ is closed under conjunction $\left(\bigwedge_{f \in \mathcal{F}'} f\right) \in \mathcal{F}$. Call this $F$. Clearly $\{F\} \equiv \mathcal{F}'$, and therefore $\{F\} \vdash \bot$. By $\bot$ elimination, for any formula $G$, we have $\{F\} \vdash \neg G$. Therefore, we have shown that there exists $F \in \mathcal{F}$ such that for any $G \in \mathcal{F}$, $\{F\} \vdash \neg G$. This is in fact a stronger statement than what we set out to prove!

$\square$

# Question 3

We have to show that if $F$ is not a contradiction and $G$ is not a tautology, and $\models (F \implies G)$, then there exists a formula $H$ such that $\models (F \implies H)$, $\models (H \implies G)$ and $Vars(H) \subseteq Vars(F) \cap Vars(G)$.
Firstly, note that we do not need the statement that $F$ is not a contradiction and $G$ is not a tautology. If $F$ is a contradiction, then we can take $H = \bot$ and if $G$ is a tautology we can take $H = \top$.
Removing this clause from the question statement, we shall prove the rest via induction on $|Vars(F) - Vars(G)|$. Our inductive hypothesis will be if $|Vars(F) - Vars(G)| = k$ and $\models (F \implies G)$, then there exists $H$ such that $\models (F \implies H)$, $\models (H \implies G)$ and $Vars(H) \subseteq Vars(F) \cap Vars(G)$.
**Base Case**:
When $k = 0$, we have $Vars(F) \subseteq Vars(G)$, and therefore we can choose $H = F$, which satisfies all the conditions.

# Question 3

Before we proceed to the inductive step,

**Lemma**:

Say $q \in Vars(F) - Vars(G)$ and $\models (F \implies G)$.
Let $H = F[q/\bot] \vee F[q/\top]$. Then we have $\models (F \implies H)$ and $\models (H \implies G)$.
Note that for any formula $F$, $F[p/G]$ denotes the formula obtained by replacing all instances of $p$ in $F$ by $G$.

**Proof**:

Say an assignment $\alpha$ has $\alpha \models F$. If $\alpha(q) = 0$, then we have $\alpha \models F[q/\bot]$ and therefore $\alpha \models H$. On the other hand, if $\alpha(q) = 1$, then $\alpha \models F[q/\top]$ and we still have $\alpha \models H$. Therefore, we have $\alpha \models F \implies \alpha \models H$ for all $\alpha$, ie $F \implies H$ is valid, ie $\models (F \implies H)$.
Now, let us show the other part. Some notation first: For an assignment $\alpha$, $\alpha[q \to b]$ is an assignment identical to $\alpha$ except at $q$, where it is $b$. We have $\alpha[q \to 0] \models F \iff \alpha \models F[q/\bot]$, $\alpha[q \to 1] \models F \iff \alpha \models F[q/\top]$.

# Question 3

Assume $\alpha \models H$. We have:

1. $\alpha \models F[q/\bot] \vee F[q/\top]$
2. $\alpha[q \to 0] \models F \vee \alpha[q \to 1] \models F$
3. $\alpha[q \to 0] \models G \vee \alpha[q \to 1] \models G$ (Since $\forall \alpha, \alpha \models F \implies \alpha \models G$)

Now, since $q \notin Vars(G)$, $\alpha[q \to b] \models G \iff \alpha \models G$, $b \in \{0, 1\}$.
Therefore,

4. $\alpha \models G \vee \alpha \models G$
5. $\alpha \models G$

Therefore, $\forall \alpha, \alpha \models H \implies \alpha \models G$, ie $\models (H \implies G)$
$\square$

# Question 3

Now, back to the main proof. Our inductive hypothesis is that for any formulae $F$ and $G$ if $|Vars(F) - Vars(G)| = k$ and $\models (F \implies G)$, then there exists $H$ such that $\models (F \implies H)$, $\models (H \implies G)$, and $Vars(H) \subseteq Vars(F) \cap Vars(G)$.
Assuming this, we have to prove the hypothesis for the case where $|Vars(F) - Vars(G)| = k + 1$. Let $q \in Vars(F) - Vars(G)$, and let $H = F[q/\top] \vee F[q/\bot]$. By the previous lemma, we have $\models (F \implies H)$ and $\models (H \implies G)$. Note that $|Vars(H) - Vars(G)| = k$. Applying the inductive hypothesis, there exists $H'$ such that $\models (H \implies H')$, $\models (H' \implies G)$ and $Vars(H') \subseteq Vars(H) \cap Vars(G)$. Using $\models (F \implies H)$ and the fact that $Vars(H) \subseteq Vars(F)$, we get $\models (F \implies H')$, $\models (H' \implies G)$, and $Vars(H') \subseteq Vars(F) \cap Vars(G)$. Therefore, the inductive hypothesis is proven for $k + 1$, and thus the statement in the question is also proven.

$\square$

# Question 4

Firstly, note that the empty set $\emptyset$ is satisfiable (in fact, it is valid)[1]. Now, it can be easily shown that the set

$$\Sigma_n = \{p_1, \ldots p_n, \bigvee_{i=1}^{n} \neg p_i\}$$

is an example of a minimal unsatisfiable set for $n \geq 1$.

[1] This is because all universally quantified propositions over the empty set are true - these are known as vacuous truths.

# Question 5

(a) Mechanically keep calculating $Res^n(\psi)$ by resolution, until you find that $\emptyset \in Res^*(\psi) = Res^3(\psi)$. This correctly tells us that $\psi$ is unsatisfiable due to the soundness of the resolution proof system.
(b) Let us do resolution in a slightly different way.
Our algorithm is as follows:

1. If $Vars(\psi)$ is empty, then we can immediately conclude the satisfiability of $\psi$ by checking if $\emptyset \in \psi$.
2. If not, pick a variable $p \in Vars(\psi)$ such that resolution[2] can be done with pairs of clauses in $\psi$ with $p$ as pivot.
3. If no such variable exists, then we are done with resolution, and we can check satisfiability by checking if $\emptyset \in \psi$.
4. If such a variable exists, replace $\psi$ with $R_p(\psi)$, where $R_p(\psi)$ is formed by removing all clauses that were involved in resolution from $\psi$ and replacing them with the newly generated resolved clauses.
5. Go to step 1

[2] We do not consider resolutions that lead to tautologies

# Question 5

To show that this algorithm works, we show that $\psi$ and $R_p(\psi)$ are equisatisfiable, ie $\psi \vdash \bot \iff R_p(\psi) \vdash \bot$.

The reverse direction is easy to prove here, the clauses of $R(\psi)$ are either members of $\psi$ or are formed from $\psi$ by resolution, ie any proof that $R_p(\psi) \vdash \bot$ can easily be converted into a proof that $\psi \vdash \bot$ by replacing the steps assuming the resolved clauses with their resolutions.

For the forward direction, let us prove the contrapositive, ie $R_p(\psi)$ is satisfiable $\implies \psi$ is satisfiable.

Let $\psi = \{\{p\} \cup A_i : i \in \{1 \dots m\}\} \cup \{\{\neg p\} \cup B_j : j \in \{1 \dots n\}\} \cup C$ where $A_i, B_j$ and $C$ do not contain $p$.

We have $R_p(\psi) = \{A_i \cup B_j : (i, j) \in [m] \times [n], A_i \cup B_j \textit{ not a tautology}\} \cup C$

Let's say some assignment $\alpha$ has $\alpha \vDash R_p(\psi)$. Firstly, clearly $\alpha \vDash C$. If $\alpha \vDash A_i$ for all $i \in [m]$, then $\alpha[p \to 0] \vDash \psi$. If there is some $k \in [m]$ such that $\alpha \nvDash A_k$, then for all $j \in [n]$, we have $\alpha \vDash A_k \cup B_j$ (this follows from the membership of the clause in $R_p(\psi)$ for non-tautological clauses and by definition for the tautologies). Since $\alpha \nvDash A_k$, we must have $\alpha \vDash B_j$, for all $j \in [n]$. Therefore, $\alpha[p \to 1] \vDash \psi$. Therefore, $\psi$ is satisfiable. $\square$