

CS 228 (M) - Logic in CS

Tutorial II - Solutions

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Question 1

We abuse notation and use \mathcal{P} to also refer to the set of axioms of \mathcal{P} . For any formulae A, B of \mathcal{P} , we have $(A \implies B) \in \mathcal{P}$.

Let p be a propositional variable of \mathcal{P} , and choose $A = p \vee \neg p$, and let B be \perp . We shall show that $\mathcal{L} \vdash \perp$, ie that \mathcal{L} is inconsistent.

Proof:

- ① $\mathcal{L} \vdash (p \vee \neg p) \implies \perp$ (Premise)
- ② $\mathcal{L} \vdash p \vee \neg p$ (Excluded Middle)
- ③ $\mathcal{L} \vdash \perp$ (Modus Ponens on 1, 2)



Question 2

We assume the set of all connectives is $\{\top, \perp, \neg, \wedge, \vee, \implies\}$.

To show that a subset of these connectives is adequate, we need to prove that for every formula constructed from the original set of connectives there is an equivalent formula constructed from the connectives in the subset. This can be proven via structural induction over the formulae.

- 1 To show that $\{\neg, \wedge\}$ is adequate, we proceed by structural induction.

Base Case:

We have $\top \equiv \neg(p \wedge \neg p)$, $\perp \equiv p \wedge \neg p$, and $p \equiv p$

Inductive Steps:

Say φ and ψ are formulae constructed from the original set of connectives that have equivalent formulae that use only \neg and \wedge . Let these equivalent formulae be φ' and ψ' respectively.

We shall show that every formula constructed from φ and ψ have equivalent formulae that use only \neg and \wedge .

- $\neg\varphi \equiv \neg\varphi'$
- $\varphi \wedge \psi \equiv \varphi' \wedge \psi'$
- $\varphi \vee \psi \equiv \neg(\neg\varphi' \wedge \neg\psi')$
- $\varphi \implies \psi \equiv \neg(\varphi' \wedge \neg\psi')$

Question 2

Since φ' and ψ' contain only \neg and \wedge , the formulae on the RHS also contain only \neg and ψ' , and therefore the induction step is completed. Therefore, every formula can be rewritten into an equivalent formula that uses only \neg and \wedge .

□

To show that $\{\neg, \implies\}$ and $\{\implies, \perp\}$ are also adequate sets we follow a similar method. The equivalences that we will use are:

- $\top \equiv p \implies p$
- $\perp \equiv \neg(p \implies p)$
- $\varphi \wedge \psi \equiv \neg(\varphi \implies \neg\psi)$
- $\varphi \vee \psi \equiv (\neg\varphi \implies \psi)$
- $\neg\varphi \equiv (\varphi \implies \perp)$
- $\varphi \wedge \psi \equiv ([\varphi \implies (\psi \implies \perp)] \implies \perp)$
- $\varphi \vee \psi \equiv ([\varphi \implies \perp] \implies \psi)$

□

Question 2

- ② We shall show that for any $C \subseteq \{\top, \perp, \neg, \wedge, \vee, \implies\}$, if $\perp \notin C$ and $\neg \notin C$, then C cannot be adequate (This is equivalent to proving that if C is adequate then it contains either \neg or \perp).

Before this, notice that if $C \subseteq C' \subseteq \{\top, \perp, \neg, \wedge, \vee, \implies\}$ and if C is adequate, then clearly C' is adequate too. So we shall prove the above statement by showing that $\{\top, \wedge, \vee, \implies\}$ is not adequate. We shall do this by showing that no formula made out of these connectives is equivalent to \perp .

Lemma:

For any formula made out of $\{\top, \wedge, \vee, \implies\}$, setting all the propositional variables to 1 always results in the overall formula having a truth value of 1.

Note that this immediately shows that no formula constructed only out of $\{\wedge, \vee, \implies\}$ can be equivalent to \perp .

We shall prove this lemma via structural induction.

Question 2

Base Case:

If the formula just consists of a single propositional variable p or is just \top , the result clearly follows.

Inductive Step:

Say φ and ψ are formulae constructed only with $\{\top, \wedge, \vee, \implies\}$ and setting all the propositional variables to 1 results in the truth values of both φ and ψ being 1. The formulae we can construct from φ and ψ are $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \implies \psi$. If we set all propositional variables to 1, then by the inductive hypothesis, the truth value of φ and ψ also become 1, and it can be seen that the truth values of the new formulae are also 1. Therefore, by structural induction, the lemma is proven and with it we have proved that $\{\top, \wedge, \vee, \implies\}$ is inadequate.



Question 3

The following equivalences (which can be verified via truth tables) imply that \downarrow (NAND) is complete, ie it can express all binary connectives.

$$\textcircled{1} \quad p \wedge q \equiv (p \downarrow q) \downarrow (p \downarrow q)$$

$$\textcircled{2} \quad p \vee q \equiv (p \downarrow p) \downarrow (q \downarrow q)$$

$$\textcircled{3} \quad p \implies q \equiv p \downarrow (q \downarrow q)$$

\downarrow can also express the unary and nullary connectives (\neg and \perp) respectively:

$$\textcircled{1} \quad \neg p \equiv p \downarrow p$$

$$\textcircled{2} \quad \top \equiv p \downarrow (p \downarrow p)$$

$$\textcircled{3} \quad \perp \equiv (p \downarrow (p \downarrow p)) \downarrow (p \downarrow (p \downarrow p))$$

We can show, via structural induction, that a subset of connectives can express all connectives (not just binary ones) iff it is adequate.

Question 4

The truth table for \oplus is as follows:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

We will show by structural induction that any formula φ constructed from two propositional variables (say p, q) will have an even number of 1s in its truth table. This means a formula like $p \wedge q$ that has an odd number of 1s in its truth table cannot be expressed via \oplus , ie \oplus is not complete¹.

Base Case:

\top , \perp , p and q all have an even number of 1s in their truth tables.

¹We will prove something stronger, as our proof will also allow \top , \perp to be used - we prove $\{\oplus, \top, \perp\}$ is neither complete nor adequate

Question 4

Inductive Step:

Say φ and ψ are formulae formed with \oplus . By the inductive hypothesis, they have an even number of 1s in their truth tables. Let's say φ and ψ are both 0 in i places, are both 1 in j places, φ is 0 and ψ is 1 in k places and φ is 1 while ψ is 0 in l places. By the inductive hypothesis, $j + k$ and $j + l$ are even. Therefore, their sum, $2j + k + l$ is even, which means $k + l$ is also even. By truth table, $\varphi \oplus \psi$ is 1 in $k + l$ places, and therefore its truth table also has an even number of 1s.

Therefore, any formula formed this way has an even number of 1s, and hence \oplus is not complete².



²As homework, you can show that all formulae whose truth tables contain an even number of 1s can be expressed with $\{\top, \perp, \oplus\}$

Question 5

First, we will show that satisfiability implies consistency, ie if \mathcal{F} is satisfiable ($\models \mathcal{F}$), then it is consistent ($\mathcal{F} \not\vdash \perp$).

Assume, this was not the case and there exists an assignment α such that $\alpha \models \mathcal{F}$ and $\mathcal{F} \vdash \perp$. Since our Formal Proof System is sound³, we have $\mathcal{F} \models \perp$, and therefore, we have an assignment α such that $\alpha \models \perp$, which is not possible! Therefore, satisfiability implies consistency.

Now, we will show the reverse, ie if \mathcal{F} is consistent ($\mathcal{F} \not\vdash \perp$) then it must be satisfiable ($\models \mathcal{F}$). Since our Formal Proof System is complete⁴, we have $\mathcal{F} \not\vdash \perp$, ie there exists an assignment α such that $\alpha \models \mathcal{F}$ and $\alpha \not\models \perp$. The latter is always true, but the former shows that \mathcal{F} is satisfiable ($\models \mathcal{F}$).

□

³This was proven in class by structural induction

⁴The proof of this is quite complex

Question 6

We know that \mathcal{F} is inconsistent, ie $\mathcal{F} \vdash \perp$

- ① By the previous result, \mathcal{F} must be unsatisfiable, ie for all assignments α , $\alpha \not\models \mathcal{F}$. Now, \mathcal{F} can be written as $\mathcal{F}_G \cup G$, ie $\forall \alpha, \alpha \not\models \mathcal{F}_G \cup G$, which can be rewritten as $\forall \alpha, \neg(\alpha \models \mathcal{F}_G) \vee \alpha \not\models G$. This can be further rewritten as $\forall \alpha, \alpha \models \mathcal{F}_G \implies \alpha \models \neg G$, which is the definition of $\mathcal{F}_G \models \neg G$. Now, since the Formal Proof System is complete, this also means that $\mathcal{F}_G \vdash \neg G$.

□

- ②
 - ① $\mathcal{F}_G \cup \{G\} \vdash \perp$ (Premise)
 - ② $\mathcal{F}_G \vdash \neg G$ (\neg introduction on 1)

□