

CS 228 (M) - Logic in CS

Tutorial I - Solutions

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Table of Contents

- 1 Question 1
- 2 Question 2
- 3 Question 3
- 4 Question 4
- 5 Question 5
- 6 Question 6
- 7 Question 7
- 8 Question 8

Question 1

We know that all people involved in this story are either *Angels* or *Normals*, since there are no *Vampires* on the island currently. We can therefore ignore the *Vampires*.

Let $Ang(x)$ represent the proposition that person x is an *Angel* and $Norm(x)$ represent the proposition that person x is a *Normal*. For every person x we are concerned with, if they are not an *Angel*, they are a *Normal*, and vice versa. Therefore, we have

$$Norm(x) \equiv \neg Ang(x)$$

Question 1

From the question, we know that A and B are from different tribes, therefore

$$Ang(A) \iff \neg Ang(B)$$

Let P represent A 's answer to our question ("yes" corresponding to **true** and "no" corresponding to **false**). We know that if A is an *Angel* then his answer is truthful, ie we have

$$Ang(A) \implies (P \iff Norm(B))$$

rewriting $Norm(B)$ as $\neg Ang(B)$ and including the previous constraint we get our final constraint as:

$$\boxed{[Ang(A) \iff \neg Ang(B)] \wedge [Ang(A) \implies [P \iff \neg Ang(B)]]}$$

Question 1

The last thing we need to use to solve this puzzle is the fact that after we heard A 's answer, we were able to figure out who is who, ie after we found the value of P , we were also able to find a **unique solution** for $Ang(A)$ and $Ang(B)$.

You can verify that the truth values of $P, Ang(A), Ang(B)$ that satisfy the constraint are:

- ① $(P, Ang(A), Ang(B)) = (T, T, F)$
- ② $(P, Ang(A), Ang(B)) = (T, F, T)$
- ③ $(P, Ang(A), Ang(B)) = (F, F, T)$

Clearly, the solution for $(Ang(A), Ang(B))$ is unique iff P is **false**, and the unique solution is $(P, Ang(A), Ang(B)) = (F, F, T)$.

Therefore, A is a *Normal*, B is an *Angel* and when asked, A (truthfully!) told us that B was not a *Normal*.

Question 2

Let's say the people in the hall are numbered from $0 \dots n-1$, where you are person 0. Let L_i represent the proposition that person i is a liar. L_0 is known to us, and we also know that the total number of liars in the hall is even. This can be written as¹:

$$L_0 = \bigoplus_{i=1}^{n-1} L_i$$

Let Q_i represent the response of the i^{th} person to our question. If we knew both L_i and Q_i for any person, then what we require is just $L_i \oplus Q_i$. Note that $L_i \oplus Q_i$ must be the **same** for every person i . Call this quantity P .

$$P = L_i \oplus Q_i, \forall i \in \{0 \dots n-1\}$$

P is what we want to find.

¹It can be shown that $\bigoplus_{i=1}^n x_i$ is true precisely when an odd number of x_i are true. I am also abusing notation slightly by using $=$ instead of \iff

Question 2

However, we do not know both L_i and Q_i for any given person i . We know L_0 and can choose some $i \in \{1 \dots n-1\}$ and find the corresponding L_i . We can find Q_j for any $j \neq 0, i$ where i was the number chosen earlier.

Our strategy is as follows:

If n is even, we do not ask anyone if they are a liar but instead ask everyone the question, and obtain $Q_i, \forall i \in \{1 \dots n-1\}$. Now, applying \oplus on the last equation for all $i \in \{1 \dots n-1\}$, we get

$$P = \left(\bigoplus_{i=1}^{n-1} L_i \right) \oplus \left(\bigoplus_{i=1}^{n-1} Q_i \right) = L_0 \oplus \left(\bigoplus_{i=1}^{n-1} Q_i \right)$$

and we can hence obtain P .

Question 2

If n is odd, we pick an arbitrary $j \in \{1 \dots n-1\}$ and ask person j if she is a liar. We ask every other person the question. Therefore, we know Q_j $\forall j \in \{1 \dots n-1\} - \{j\}$ and L_j . Now,

$$L_0 \oplus L_j = \bigoplus_{\substack{i=1 \\ i \neq j}}^{n-1} L_i$$

Therefore, on similar lines as before, we get

$$P = L_0 \oplus L_j \oplus \left(\bigoplus_{\substack{i=1 \\ i \neq j}}^{n-1} Q_i \right)$$

Question 3

Note that if B is a knight, then A must be a knave, which makes her statement that B is a knight false, which contradicts the fact that B is a knight. Therefore B is not a knight. This means A 's statement is false, which means A is also not a knight. So both are either normals or knaves, and A 's statement is false. Let us take some cases now:

- ① If A is a knave, then B 's statement is true, which means B must be a normal. This means one of them (B) told the truth but is not a knight.
- ② If B is a knave, then A must be a normal. This means one of them (A) told a lie but is not a knave.
- ③ If both A and B are normals, then both their statements are false, which means one of them (either) told a lie but is not a knave.

Therefore, by cases, we can say that one of them told the truth but is not a knight, or that one of them told a lie but is not a knave.

This natural language proof can also be directly translated into a formal proof.

Question 4

Let $P(i, j)$ represent the proposition that the i^{th} pigeon is sitting in the j^{th} hole, where $i \in \{1 \dots n + 1\}$ and $j \in \{1 \dots n\}$.

The Pigeonhole principle states that, if there are $n + 1$ pigeons and n holes, and every pigeon sits in exactly one hole, then there is a hole occupied by more than one pigeon. To convert this into a PL formula, let us convert each side of the implication into PL first.

Every pigeon sits in at least one hole can be expressed in PL as:

$$\bigwedge_{i=1}^{n+1} \bigvee_{j=1}^n P(i, j)$$

Here the inner disjunction refers to the i^{th} pigeon sitting in some hole, and the outer conjunction makes it so that every pigeon must sit in some hole. Call this condition F .

Question 4

We also need no pigeon to sit in multiple holes. Say pigeon i sits in holes j and k with $j < k$. The formula $P(i, j) \wedge P(i, k)$ represents this scenario. There exists a pigeon sitting in multiple holes therefore becomes:

$$\bigvee_{i=1}^{n+1} \bigvee_{\substack{j,k=1 \\ j < k}}^n (P(i, j) \wedge P(i, k))$$

Here, the inner disjunction refers to the i^{th} pigeon sitting in multiple holes and the outer disjunction refers to there existing a pigeon sitting in multiple holes.

Negating this, we get the condition for no pigeon to sit in multiple holes:

$$\bigwedge_{i=1}^{n+1} \bigwedge_{\substack{j,k=1 \\ j < k}}^n (\neg P(i, j) \vee \neg P(i, k))$$

Call this condition G .

Question 4

Now, say hole k is occupied by pigeons i and j with $i < j$. We then have $P(i, k) \wedge P(j, k)$. There exists a hole occupied by more than one pigeon therefore becomes:

$$\bigvee_{k=1}^n \bigvee_{\substack{i,j=1 \\ i < j}}^{n+1} (P(i, k) \wedge P(j, k))$$

Here, the inner disjunction refers to the k^{th} hole being occupied by more than one pigeon and the outer disjunction refers to there existing a hole occupied by multiple pigeons. Call this condition H .

The Pigeonhole Principle therefore becomes:

$$F \wedge G \implies H$$

Question 5

Proof:

- ① $\{(p \implies q) \implies q, q \implies p, \neg p, p\} \vdash p$ (Assumption)
- ② $\{(p \implies q) \implies q, q \implies p, \neg p, p\} \vdash \neg p$ (Assumption)
- ③ $\{(p \implies q) \implies q, q \implies p, \neg p, p\} \vdash \perp$ (\perp introduction on 1, 2)
- ④ $\{(p \implies q) \implies q, q \implies p, \neg p, p\} \vdash q$ (\perp elimination on 3)
- ⑤ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash p \implies q$ (\implies intro on 4)
- ⑥ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash (p \implies q) \implies q$ (Ass.)
- ⑦ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash q$ (Modus Ponens on 5, 6)
- ⑧ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash q \implies p$ (Assumption)
- ⑨ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash p$ (Modus Ponens on 7, 8)
- ⑩ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash \neg p$ (Assumption)
- ⑪ $\{(p \implies q) \implies q, q \implies p, \neg p\} \vdash \perp$ (\perp introduction on 9, 10)
- ⑫ $\{(p \implies q) \implies q, q \implies p\} \vdash \neg \neg p$ (By contradiction on 11)
- ⑬ $\{(p \implies q) \implies q, q \implies p\} \vdash p$ ($\neg \neg$ elimination on 12)
- ⑭ $\{(p \implies q) \implies q\} \vdash (q \implies p) \implies p$ (\implies intro on 13)
- ⑮ $\emptyset \vdash [(p \implies q) \implies q] \implies [(q \implies p) \implies p]$ (\implies intr on 14)



Question 6

We will use the following proof rule (which has often been used implicitly in the slides) along with the proof rules mentioned in the slides (Call it *Monotonicity*). This rule follows as any proof valid for \mathcal{H} will also be valid for any superset of it.

$$\mathcal{H} \subseteq \mathcal{H}' \implies \frac{\mathcal{H} \vdash \varphi}{\mathcal{H}' \vdash \varphi}$$

Proof:

- 1 $\mathcal{H} \vdash A \implies B$ (Premise)
- 2 $\mathcal{H} \vdash C \vee A$ (Premise)
- 3 $\mathcal{H} \cup \{C\} \vdash C$ (Assumption)
- 4 $\mathcal{H} \cup \{C\} \vdash B \vee C$ (\vee introduction on 3)
- 5 $\mathcal{H} \cup \{A\} \vdash A$ (Assumption)
- 6 $\mathcal{H} \cup \{A\} \vdash A \implies B$ (Monotonicity on 1)
- 7 $\mathcal{H} \cup \{A\} \vdash B$ (Modus Ponens on 5, 6)
- 8 $\mathcal{H} \cup \{A\} \vdash B \vee C$ (\vee introduction on 7)
- 9 $\mathcal{H} \vdash B \vee C$ (\vee elimination on 2, 4, 8)



Question 7

Proof:

- ① $\mathcal{H} \vdash A \implies C$ (Premise)
- ② $\mathcal{H} \vdash B \implies C$ (Premise)
- ③ $\mathcal{H} \cup \{A \vee B\} \vdash A \vee B$ (Assumption)
- ④ $\mathcal{H} \cup \{A \vee B, A\} \vdash A$ (Assumption)
- ⑤ $\mathcal{H} \cup \{A \vee B, A\} \vdash A \implies C$ (Monotonicity on 1)
- ⑥ $\mathcal{H} \cup \{A \vee B, A\} \vdash C$ (Modus Ponens on 4, 5)
- ⑦ $\mathcal{H} \cup \{A \vee B, B\} \vdash B$ (Assumption)
- ⑧ $\mathcal{H} \cup \{A \vee B, B\} \vdash B \implies C$ (Monotonicity on 2)
- ⑨ $\mathcal{H} \cup \{A \vee B, B\} \vdash C$ (Modus Ponens on 7, 8)
- ⑩ $\mathcal{H} \cup \{A \vee B\} \vdash C$ (\vee elimination on 3, 6, 9)
- ⑪ $\mathcal{H} \vdash (A \vee B) \implies C$ (\implies intro on 10)



Question 8

We abuse notation slightly by using \mathcal{L} to also refer to the axiom set. For any formulae A, B, C , we have $\neg(A \vee B) \vee (B \vee C) \in \mathcal{L}$. We also assume that the proof rules of \perp , \vee and \neg are still present in this system.

We have to show that for any formula F , $\mathcal{L} \vdash F$.

We choose $A = \neg F$, $B = \perp$, $C = F$.

Proof:

- ① $\mathcal{L} \vdash \neg(\neg F \vee \perp) \vee (\perp \vee F)$ (Assumption)
- ② $\mathcal{L} \cup \{\neg(\neg F \vee \perp), \neg F\} \vdash \neg F$ (Assumption)
- ③ $\mathcal{L} \cup \{\neg(\neg F \vee \perp), \neg F\} \vdash \neg F \vee \perp$ (\vee introduction on 3)
- ④ $\mathcal{L} \cup \{\neg(\neg F \vee \perp), \neg F\} \vdash \neg(\neg F \vee \perp)$ (Assumption)
- ⑤ $\mathcal{L} \cup \{\neg(\neg F \vee \perp), \neg F\} \vdash \perp$ (\perp introduction on 4)
- ⑥ $\mathcal{L} \cup \{\neg(\neg F \vee \perp)\} \vdash \neg\neg F$ (By contradiction on 5)
- ⑦ $\mathcal{L} \cup \{\neg(\neg F \vee \perp)\} \vdash F$ ($\neg\neg$ elimination on 6)

Question 8

- 8 $\mathcal{L} \cup \{\perp \vee F\} \vdash \perp \vee F$ (Assumption)
- 9 $\mathcal{L} \cup \{\perp \vee F, \perp\} \vdash \perp$ (Assumption)
- 10 $\mathcal{L} \cup \{\perp \vee F, \perp\} \vdash F$ (\perp elimination on 9)
- 11 $\mathcal{L} \cup \{\perp \vee F, F\} \vdash F$ (Assumption)
- 12 $\mathcal{L} \cup \{\perp \vee F\} \vdash F$ (\vee elimination on 8, 10, 11)
- 13 $\mathcal{L} \vdash F$ (\vee elimination on 1, 7, 12)

