Indian Institute of Technology Bombay

MA 105 Calculus

Autumn 2019 SRG/MM/MM

Solutions and Marking Scheme for Quiz 1

Date: Seetember 11, 2019 Weightage: 10 %

Time: 8.35 AM – 9.15 AM Max. Marks: 30

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1. `Let` (a_n) `be a sequence of real numbers satisfying` $3 < a_1 < 4$ `and` $a_{n+1} = \sqrt{12 + a_n}$ `for` $n \in \mathbb{N}$`. Determine if` $\{a_n\}$ `is convergent. Justify your answer.`

   Solution: We will first show using induction on n that $3 < a_n < 4$ for all $n \in \mathbb{N}$. [1]
   This is given to be true for $n = 1$. Further if $3 < a_n < 4$, then

   $$3 < \sqrt{12 + 3} < a_{n+1} < \sqrt{12 + 4} = 4$$

   and so $3 < a_{n+1} < 4$. This proves that (a_n) is bounded. [2]

   Next, for any $n \in \mathbb{N}$,

   $$a_{n+1} - a_n = \sqrt{12 + a_n} - a_n = \frac{12 + a_n - a_n^2}{\sqrt{12 + a_n} + a_n} = \frac{-(a_n - 4)(a_n + 3)}{\sqrt{12 + a_n} + a_n}.$$

   Since $3 < a_n < 4$ for all $n \in \mathbb{N}$, it follows that $a_{n+1} - a_n > 0$, that is, $a_{n+1} > a_n$ for all $n \in \mathbb{N}$, Thus the sequence $(a_n)$is monotonically increasing. [2]

   Since (a_n) is monotonic and bounded, it follows that (a_n) is convergent. [1]

2. `Let` $f : [0, 1] \to \mathbb{R}$ `be a continuous function such that` $f(0) = 0$ `and` $f(x)$ `is a rational number for every` $x \in [0, 1]$`. Determine` $f(1)$`. Justify your answer.`

   Solution: We claim that $f(1) = 0$. [1]

   To see this, assume the contrary, i.e., suppose $f(1) \neq 0 = f(0)$. Then between $f(0)$ and $f(1)$ there must be an irrational number, say t. [2]

   Since f is continuous on $[0, 1]$, it has the IVP on $[0, 1]$. [2]

Hence $t = f(x)$ for some $x \in (0,1)$. But this is a contradiction since f only takes rational values. [1]

Thus the Claim is proved and $f(1) = 0$.

3. `Let` $f : [0,1] \to \mathbb{R}$ `be continuous on` $[0,1]$ `and differentiable on` $(0,1)$`. Suppose` $\lim_{x \to 0^+} f'(x) = L$ `for some real number` L`. Show that` $f'(0)$ `exists and` $f'(0) = L$`.`

Solution: Since f is defined on $[0,1]$, by definition, $f'(0)$ is the right hand limit $\lim_{x\to 0^+} \big(f(x) - f(0)\big)/(x-0)$. For each $x \in (0,1]$, by the MVT, there is $c_x \in (0,x)$ satisfying

$$\frac{f(x) - f(0)}{x} = f'(c_x). \qquad [2]$$

Since c_x is between 0 and x, we see that $c_x \to 0^+$ as $x \to 0^+$. [2]

Thus, taking limits as $x \to 0^+$, and using the fact that $\lim_{x\to 0} f'(x) = L$, we obtain

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = L. \qquad [2]$$

Thus $f'(0)$ exists and $f'(0) = L$.

Aliter: Let $\epsilon > 0$ be given. Since $\lim_{x \to 0^+} f'(x) = L$, there is $\delta > 0$ such that

$$0 < x < \delta \Longrightarrow |f'(x) - L| < \epsilon. \qquad [1]$$

Now, for each $x \in (0, \delta)$, by the MVT, there is $c_x \in (0,x)$ satisfying

$$\frac{f(x) - f(0)}{x} = f'(c_x). \qquad [2]$$

Now $0 < c_x < x < \delta$ and so $|f'(c_x) - L| < \epsilon$. [1]

It follows that

$$0 < x < \delta \Longrightarrow \left| \frac{f(x) - f(0)}{x} - L \right| < \epsilon. \qquad [1]$$

This proves that

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = L.$$

Thus $f'(0)$ exists and $f'(0) = L$. [1]

[**Note:** A solution that uses interchange of limits such as $\lim\limits_{h\to 0} \lim\limits_{x\to 0} = \lim\limits_{x\to 0} \lim\limits_{h\to 0}$ without proper justification may be given at most 2 points. A solution using L'Hôpital's rule, but without a proof of that rule, may be given at most 4 points.]

4. `Consider points` $x_1, ..., x_n \in \mathbb{R}$, `and the function` $f : \mathbb{R} \to \mathbb{R}$ `defined by`

$$f(x) = \sqrt{(x - x_1)^2 + ... + (x - x_n)^2} \quad \texttt{for } x \in \mathbb{R}.$$

`Determine if the absolute minimum of` f `exists, and if yes, find the value of` x `at which it is attained.`

Solution: If $x_1 = \cdots = x_n$, then $f(x) = \sqrt{n}\,|x - x_1|$ for all $x \in \mathbb{R}$ and it is clear that the absolute minimum of f is 0, which is attained at $x = x_1$. [1]

Now suppose not all $x_1, \ldots, x_n$ are equal. Then f is differentiable on $\mathbb{R}$. [1]

Moreover, for any $x \in \mathbb{R}$,

$$f'(x) = \frac{n(x - \overline{x})}{\sqrt{(x - x_1)^2 + \cdots + (x - x_n)^2}} \quad \text{where} \quad \overline{x} = \frac{1}{n}(x_1 + \cdots + x_n). \qquad [1]$$

Hence we see that

$$f'(x) \le 0 \text{ if } x \le \overline{x} \quad \text{and} \quad f'(x) \ge 0 \text{ if } x \ge \overline{x}. \qquad [1]$$

Thus f is decreasing on $(-\infty, \overline{x}]$ and increasing on $[\overline{x}, \infty)$. Hence

$$f(x) \ge f(\overline{x}) \quad \text{for all } x \in \mathbb{R}. \qquad [1]$$

This proves that the absolute minimum of f exists and the value at which it is attained is $x = \overline{x}$. [1]

5. `Determine if`

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cos^2\left(\frac{i\pi}{n}\right)$$

`exists and if yes, find the limit.`

Solution: Consider $f : [0, 1] \to \mathbb{R}$ defined by $f(x) := \cos^2 \pi x$ for $x \in [0, 1]$. Given any $n \in \mathbb{N}$, if P_n denotes the partition of $[0, 1]$ into n equal parts, i.e., if $P_n = \{0,\ 1/n,\ 2/n, \ldots, (n-1)/n,\ 1\}$, then

$$\sum_{i=1}^{n} \frac{1}{n} \cos^2\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = S(P_n, f), \qquad [2]$$

i.e., the sum on the left is (an approximate) Riemann sum for f w.r.t. P_n.

Since f is continuous, it is integrable on $[0, 1]$, and hence $S(P_n, f)$ converges to $\int_0^1 f(x)dx$ as $n \to \infty$. [2]

Consequently, the given limit exists and it can be obtained as follows.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cos^2\left(\frac{i\pi}{n}\right) = \int_0^1 \cos^2 \pi x dx = \int_0^1 \frac{1 + \cos 2\pi x}{2} dx = \frac{1}{2}. \qquad [2]$$