Indian Institute of Technology Bombay

MA 105 Calculus

Autumn 2019 SRG/MM/MM

Solutions and Marking Scheme for Mid-Semester Examination

Date: September 16, 2019 Weightage: 30 %
Time: 4.00 PM – 6.00 PM Max. Marks: 60

1. (i) Define when a sequence (a_n) of real numbers is said to be convergent. [2]

   (ii) Prove that if a sequence (a_n) of real numbers is convergent, then it is bounded. Is the converse true? Justify your answer. [3 + 2]

   Solution: (i) A sequence (a_n) of real numbers is said to be convergent if there is $L \in \mathbb{R}$ satisfying the following: For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq n_0$.
   (ii) Suppose $a_n \to L$. Then choosing $\varepsilon = 1$, one can find $n_0 \in \mathbb{N}$ such that $L - 1 < a_n < L + 1$ for all $n \geq n_0$. Then $\min\{a_1, ..., a_{n_0-1}, L-1\}$ and $\max\{a_1, ..., a_{n_0-1}, L+1\}$ serve as lower and upper bounds of (a_n) respectively.

   The converse is not true. The sequence $((-1)^n)$ is bounded, but not convergent.

2. Consider $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \cos x & \text{if } x \text{ is rational,} \\ \sin x & \text{if } x \text{ is irrational.} \end{cases}$$

   Determine the points $c \in [0, 2\pi]$ at which f is continuous. Justify your answer. [5]

   Solution: Suppose f is continuous at a point $x \in [0, 2\pi]$. Now, consider a sequence (a_n) of rational numbers and a sequence (b_n) of irrational numbers such that $a_n \to x$ and $b_n \to x$. (Such sequences can be found, by picking a rational as well as an irrational between x and $x + (1/n)$ for each $n \in \mathbb{N}$.) [1]

   Now $f(a_n) = \cos a_n \to \cos x$ and $f(b_n) = \sin b_n \to \sin x$. But since f is continuous at x, $f(a_n) \to f(x)$ and $f(b_n) \to f(x)$. Hence $\sin x = \cos x$ and since $x \in [0, 2\pi]$, we must have $x = \pi/4$ or $5\pi/4$. [1 + 1]

On the other hand if $c = \pi/4$ or $c = 5\pi/4$, then $\sin c = \cos c = d$, say (in fact, $d = \pm 1/\sqrt{2}$), and from the continuity of cosine and sine functions, we see that for any $\epsilon > 0$, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x - c| < \delta_1 \Longrightarrow |\cos x - d| < \epsilon$$

and

$$|x - c| < \delta_2 \Longrightarrow |\sin x - d| < \epsilon.$$

Thus, if we take $\delta := \min\{\delta_1, \delta_2\}$, then we obtain

$$|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

This shows that f is continuous at $\pi/4, 5\pi/4$. [2]

3. Find the number of real roots of the equation $8x - \sin 2x = 2\pi$. [6]

Solution: Let $f(x) := 8x - 2\pi - \sin 2x$ for $x \in \mathbb{R}$. Then $f'(x) = 8 - 2\cos 2x \geq 8 - 2 = 6 > 0$. Hence by Rolle's theorem, the number of roots can at most be one. [3]
Choosing $a = -10$, $b = 400$, for example, we see that $f(-10) < 0$ and $f(400) > 0$. [1]
Since f is clearly continuous, it has the IVP, which implies that f has at least one root. So the equation has exactly one real root. [2]

Aliter: For the first part, one can also argue that since $f'(x) > 0$ for all $x \in \mathbb{R}$, f is strictly increasing, and so f can have at most one root.
Note: There will be no universal agreement regarding the choice of a and b, and any right choice is fine. However, "graphical proofs" are not really proofs, so cannot be awarded a total of at most 3 marks.

4. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that $f'(0) = 0$. Further, show that $f''(0) = 0$. [4 + 2]

Solution: Since f is differentiable, it is continuous. Now since $\frac{1}{n} \to 0$, continuity of f at 0 shows that $f(0) = 0$. [2]

Next, note that f is differentiable at a point c if and only if there exists $L \in \mathbb{R}$ such that for every sequence (x_n) in $(-1, 1)$ such that $x_n \to c$ with $x_n \neq c$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} \dfrac{f(x_n) - f(c)}{x_n - c} = L$. Since f is differentiable, taking $c = 0$ and $x_n = 1/n$ for $n \in \mathbb{N}$, we see that $f'(0) = 0$.

[2]

Applying Rolle's theorem to f in the interval $[\frac{1}{n+1}, \frac{1}{n}]$, we obtain $y_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $f'(y_n) = 0$. The same argument as above now applies to f' and the sequence (y_n) thus obtained, which yields $f''(0) = 0$. [2]

Aliter (for the first step): By Rolle's theorem applied to f' on $[\frac{1}{n+1}, \frac{1}{n}]$, there exist $c_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $f'(c_n) = 0$. Now since f' is differentiable, it is continuous and moreover by Sandwich theorem, $c_n \to 0$. Thus it follows that $f'(0) = 0$. [4]

Note: It is not correct to assume that f'' is continuous.

5. Sketch the graph of the function $f(x) = \dfrac{3x^2+2}{x^2+1}$ after finding the following:

   (i) intervals where the function is increasing/decreasing [2]

   (ii) points of local minima/maxima [1]

   (iii) intervals of concavity/convexity [2]

   (iv) points of inflection [1]

   (v) asymptotes [2]

   Use this information to sketch a graph of f. [2]

   Solution: Since $f'(x) = \frac{2x}{(x^2+1)^2}$, we see that $f'(x) \geq 0$ when $x \geq 0$ and $f'(x) \leq 0$ when $x \leq 0$. This implies the following.

   (i) Thus f is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. . [2]

   (ii) There is no local maximum and only one point of local minimum: $x = 0$. . [1]

   Next, we observe that

$$f''(x) = \frac{-6x^4 - 4x^2 + 2}{(1+x^2)^4} = \frac{2(1-3x^2)(1+x^2)}{(1+x^2)^4} = \frac{2(1-3x^2)}{(1+x^2)^3}.$$

   This gives the following.

   (iii) f is convex on $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ and concave on $(-\infty, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, \infty)$. . [2]

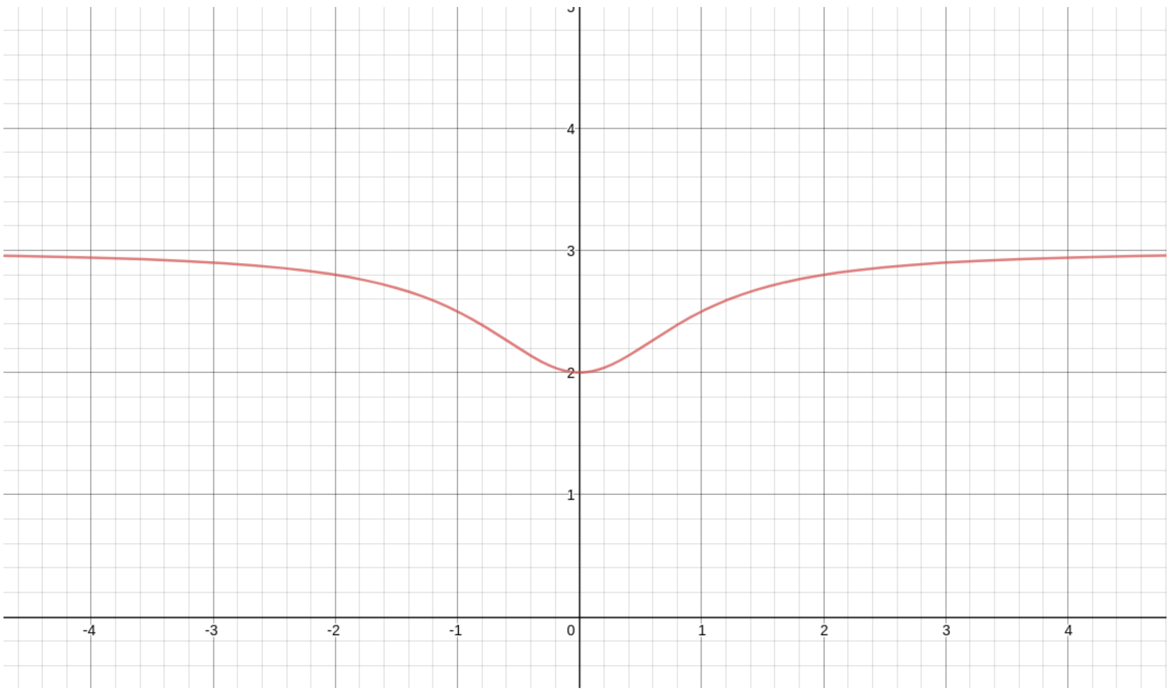
   (iv) The points of inflection are $x = \pm\frac{1}{\sqrt{3}}$. . [1]

   Finally, since

$$f(x) = 3 - \frac{1}{1+x^2} \to 3 \text{ as } x \to \pm\infty,$$

we see that $y = 3$ is a horizontal asymptote. It is clear that there are no other (vertical or oblique) asymptotes. . [1]

Finally, using the above properties and noting that the graph of f is symmetric about the y-axis, we can draw it as follows. [2]



Note: In grading this question, particular care must be taken that the intervals (closed/half-open/open etc.) are mentioned precisely.
Also, if in a certain step the calculation is wrong (let's say the calculation of f'' is wrong), but the rest of the solution shows proper line of thinking (even if based on wrong calculation), some marks might be awarded

6. Let $f : [0, 2] \to \mathbb{R}$ be defined as follows: $f(x) = \begin{cases} x, & x \in [0,1] \\ 1, & x \in [1,2] \end{cases}$ Show that f is Riemann integrable from first principles and evaluate the integral $\int_0^2 f(x)dx$. [6]

Solution: Consider the partition $P_n = \{t_0, t_1, \ldots, t_n, t_{n+1}, \ldots, t_{2n}\}$ of $[0, 2]$ given by $P_n = \{0, 1/n, 2/n, \ldots, 1, 1 + 1/n, 1 + 2/n, \ldots, 2\}$. We compute the upper and lower Riemann sums $U(P_n, f)$ and $L(P_n, f)$ to obtain:

$$U(P_n, f) = \sum_{i=1}^{2n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n}(i/n)(1/n) + \sum_{i=1}^{n}(1/n) = \frac{n(n+1)}{2n^2} + 1$$

and

$$L(P_n, f) = \sum_{i=1}^{2n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n}((i-1)/n)(1/n) + \sum_{i=1}^{n}(1/n) = \frac{(n-1)n}{2n^2} + 1.$$

In this calculation, we used the fact that f is continuous and monotonically increasing (in particular in $[0, 1]$) and hence, $M_i = f(t_i)$ and $m_i = f(t_{i-1})$. [2]
Note that both the sequences $(U(P_n, f))$ and $(L(P_n, f))$ are convergent and converge to $3/2$. In particular, $U(P_n, f) - L(P_n, f) \to 0$. [2]
Hence, by the Riemann condition, f is Riemann integrable and $\int_0^2 f(x)dx = 3/2$. [2]

7. Let $F(x) = \int_0^{x^3} \cos t \sin t \, dt$. If F differentiable? Justify your answer. If F is differentiable, determine $\dfrac{dF}{dx}$. [6]

Solution: We let $y(x) = x^3$ and define $G(y) = \int_0^y \cos t \sin t dt$. [1]
Note that the Fundamental theorem of calculus (Part 1) applies here because $\cos t \sin t$ is a continuous function. [1]
Note also that $F(x) = G(y(x))$, and since G and y are differentiable functions, by Chain rule, their composite is differentiable. [1]
Moreover, by the Fundamental theorem of calculus (Part 1), we note that $\frac{dG}{dy} = \cos y \sin y$. Also, by the chain rule, we obtain $\frac{dF}{dx} = \frac{dG}{dy}\frac{dy}{dx}$. Since $\frac{dy}{dx} = 3x^2$, we can conclude that $\frac{dF}{dx} = 3x^2 \cos x^3 \sin x^3$. [3]
Aliter: It is possible to calculate directly the integral $F(x)$ and differentiate it directly, using the chain rule of differentiation. If written correctly in all the details, this would get full points.
Note: It is not correct to use FTC on the integral given, and conclude immediately without justification that it is differentiable with respect to x. That is not what the FTC says.

8. Consider the region bounded by the graphs of the curve $y^2 = x^3$ and the line $y = x$ in the first quadrant. Let S be the solid obtained by revolving this region about the y-axis. Compute its volume using the washer method and the shell method. [8]

Solution: We note that the points of intersection between the two curves in the first quadrant are $(0, 0)$ and $(1, 1)$. [2]

Next, we compute the integral using the two methods.
Washer method: Since, we are revolving about the y-axis the variable of integration

will be y and it will vary in $[0,1]$. We take $g_1(y) = y$ and $g_2(y) = y^{2/3}$ and note that $g_2(y) \geq g_1(y)$ for all $y \in [0,1]$. By the formula for the volume by the washer method, the volume is given by

$$V_{\text{washer}} = \int_0^1 \pi(g_2^2(y) - g_1^2(y))dy = \pi(\int_0^1 (y^{4/3} - y^2)dy = \pi\left(\frac{3}{7} - \frac{1}{3}\right) = \frac{2\pi}{21}. \quad [3]$$

Shell Method: Since, we are revolving about the y-axis the variable of integration will be x and will vary in $[0,1]$. We take $f_2(x) = x$ and $f_1(x) = x^{3/2}$, and note that $f_2(x) \geq f_1(x)$ for all $x \in [0,1]$. By the formula for the volume by the shell method, the volume is given by

$$\begin{aligned} V_{\text{shell}} &= \int_0^1 2\pi x(f_2(x) - f_1(x))dx \\ &= \int_0^1 2\pi x(x - x^{3/2})dx = 2\int_0^1 (x^2 - x^{5/2})dx = 2\pi\left(\frac{1}{3} - \frac{2}{7}\right) = \frac{2\pi}{21}. \quad [3] \end{aligned}$$

This confirms that the volumes V_{washer} and V_{shell} obtained by the two methods are equal.

9. Prove that there exists a unique $x \in [0,10]$ such that

$$\int_0^x e^{t^2}\,dt = \int_x^{10} e^{t^2}\,dt. \quad [6]$$

Solution: Observe that one can work with the function $F(x) = \int_0^x e^{t^2}dt - \int_x^{10} e^{t^2}dt$.[1]
By the Fundamental theorem of calculus (Part 1), we know that $F(x)$ is continuous (since e^{t^2} is integrable in $[0,10]$, being continuous). Next, note that

$$F(0) = -\int_0^{10} e^{t^2}dt < 0 \quad \text{and} \quad F(10) = \int_0^{10} e^{t^2}dt > 0.$$

This is because $e^{t^2} > 0$ in $[0,10]$ and is continuous, hence $\int_0^{10} e^{t^2}dt > 0$. [Alternatively, $\int_0^{10} e^{t^2}dt \geq (10-0)e^0 = 10 > 0$.] [1]
By the intermediate value theorem, we conclude that there exists an $x_0 \in [0,10]$ such that $F(x_0) = 0$. [1]
For the uniqueness, we prove that that $F(x)$ is strictly increasing by the First derivative test. First, note that e^{t^2} is continuous in $[0,10]$ and $F(x) = 2\int_0^x e^{t^2}dt - \int_0^{10} e^{t^2}dt$ (by the domain additivity property of Riemann integrals). [1]
Hence, by the fundamental theorem of calculus (Part 1), $F(x)$ is differentiable and $F'(x) = 2 \cdot e^{x^2} > 0$ in $[0,10]$. [1]
Hence, $F(x)$ is strictly increasing in $[0,10]$ and therefore it has a unique root. [1]