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Indian Institute of Technology Bombay

MA 105 Calculus

Autumn 2019 SRG/MM/MM

Solutions and Marking Scheme for the End-Semester Examination

Date: November 11 , 2019 Weightage: 40%

Time: 2:00 PM - 5:00 PM Max. Marks: 80

Note: *This is a question paper cum answer book. Answers to the first six objective questions are to be provided here by encircling the correct choice(s) or writing down the requisite quantity. Answers to the last six subjective questions (including relevant intermediate steps) are to be written in the space provided. In case you require additional space to write your answers, use the blank sheets attached to the question paper cum answer book. A blank answerbook provided separately can be used for rough work. Write your Roll number, division and tutorial batch on the question paper as well as the answerbooks and any supplements you may use for your rough work. Failure to do so can result in a penalty of 2 marks.*

As usual, $\mathbb{N}$ *denotes the set of all positive integers and* $\mathbb{R}$ *denotes the set of all real numbers.*

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1. Consider the sequences (a_n), (b_n) and (c_n) defined by

$$a_n := \frac{(-1)^n}{2^n}, \quad b_n := \frac{2^n}{n!} \quad \text{and} \quad c_n := 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n} \quad \text{for } n \in \mathbb{N}.$$

Then which of the following statements are true? [4 marks]

A Each of (a_n), (b_n) and (c_n) is convergent.

B (a_n) is divergent, but (b_n) and (c_n) are convergent.

C (b_n) is divergent, but (a_n) and (c_n) are convergent.

D (c_n) is divergent, but (a_n) and (b_n) are convergent.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := xy^2 - 8y$ for $(x, y) \in \mathbb{R}^2$. Find the directional derivative $D_{\mathbf{u}} f(2, 3)$ of f at $(2, 3)$ in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{5}}(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$. Also, write the equation for the tangent plane to the graph of f through the point $(2, 3, -6)$. [4 marks]

$$D_{\mathbf{u}} f(2, 3) = \frac{22}{\sqrt{5}}$$

Equation for the tangent plane: $9x + 4y - z = 36$

3. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy + |xy|$ for $(x, y) \in \mathbb{R}^2$. Then which of the following statements are true? [4 marks]

   A f is continuous at $(0, 0)$, but $\nabla f(0, 0)$ does not exist.

   B $\nabla f(0, 0)$ exists, but $D_{\mathbf{u}} f(0, 0)$ does not exist for some unit vector $\mathbf{u}$.

   C $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u}$, but f is not differentiable at $(0, 0)$.

   D f is differentiable at $(0, 0)$.

4. Compute the surface area of the parametrized surface Φ defined by

$$\Phi(u, v) := (3u + 2v,\ 4u + v,\ 5v) \quad \text{where } (u, v) \in [0, 1] \times [0, 1].$$

[4 marks]

Area$(\Phi) = \sqrt{650}$

5. Let S be the unit sphere in $\mathbb{R}^3$ with the normal $\mathbf{n}$ at each point of S in the outward direction, and let D be an open set in $\mathbb{R}^3$ containing S. Suppose $\mathbf{F}$ is a smooth vector field on D with the property that $\mathbf{F}(\mathbf{x})$ is of unit length for each $\mathbf{x}) \in D$, and moreover, the surface integral $\iint_S \boldsymbol{F} \cdot d\mathbf{S}$ has the maximum possible value among the surface integrals over S of all smooth vector fields on D with this property. Then determine $\mathbf{F}(1/2,\ 1/2,\ 1/\sqrt{2})$. [4 marks]

$$\mathbf{F}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$$

6. Let S be the surface in $\mathbb{R}^3$ formed by the part of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane (that is, where $z \geq 0$) and let $\boldsymbol{F} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + 2(1 - z)\hat{\boldsymbol{k}}$. Taking the upward normal $\boldsymbol{n}$ on S for positive orientation, calculate $\iint_S \boldsymbol{F} \cdot d\mathbf{S}$.

[4 marks]

$$\iint_S \boldsymbol{F} \cdot d\mathbf{S} = 2\pi.$$

7. Let $f : (0, 1) \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 1/q & \text{if } x = p/q, \text{ where } p, q \text{ are relatively prime positive integers,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is discontinuous at each rational in $(0, 1)$ and it is continuous at each irrational in $(0, 1)$. [10 marks]

Answer: Let $c \in (0, 1)$ be a rational number. Then $c = p/q$ for unique $p, q \in \mathbb{N}$ such that p, q are relatively prime, and so $f(c) = 1/q \neq 0$. [1]

On the other hand, there exists a sequence (x_n) of irrational numbers such that $x_n \to c$. [2]

But then $f(x_n) = 0$ for all $n \in \mathbb{N}$. Hence $f(x_n) \to 0 \neq f(c)$. This proves that f is not continuous at c. [1]

Next, suppose $c \in (0, 1)$ is irrational. Then $f(c) = 0$. Thus to prove continuity of f at c it suffices to show that for any $\epsilon > 0$, there is $\delta > 0$ such that

$$(*) \quad x \in (0, 1) \text{ and } |x - c| < \delta \Longrightarrow |f(x)| < \epsilon. \qquad [1]$$

Now (*) clearly holds if $x \in (0, 1)$ is irrational. To prove it for rational $x \in (0, 1)$, note that for any given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. [1]

Now there are only finitely many $q \in \mathbb{N}$ such that $q \leq n_0$. Also if $p, q \in \mathbb{N}$ are such that $p/q \in (0, 1)$, then we must have $1 \leq p < q$. It follows that there are only finitely many rational numbers in $(0, 1)$ of the form p/q, where $p, q \in \mathbb{N}$ are relatively prime and $q \leq n_0$. [2]

Hence we can choose $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq (0, 1)$ and $(c - \delta, c + \delta)$ contains none among these finitely many rationals. [1]

Now if $x \in (c - \delta, c + \delta)$ is a rational number, then $x = p/q$ for some $p, q \in \mathbb{N}$ such that p, q are relatively prime and $q > n_0$. Hence $f(x) = \frac{1}{q} < \frac{1}{n_0} < \epsilon$. [1]

This proves that (*) holds and thus f is continuous at c.

Note: An incomplete argument for continuity at irrationals using decimal representation may be given at most 2 points.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) = e^{x^2}$ and $f(0) = 6$. Prove that $6.7 \le f(1) \le 9.2$. [9 marks]

Answer: By the Mean Value Theorem, there exists $c \in (0,1)$ such that

$$f(1) - f(0) = f'(c) = e^{c^2} \qquad [4]$$

Since $f(0) = 6$ and $1 < e^{c^2} < e$, this implies that

$$7 \le f(1) \le 6 + e. \qquad [4]$$

Since $2 < e < 3$, it follows that $6.7 \le f(1) \le 9.2$. [1]

Aliter: Since $f'(x) = e^{x^2}$, we see that $f'(0) = 1$ and $f'(1) = e < 3$. Also f' is differentiable on $[0,1]$ and

$$f''(x) = 2xe^{x^2} \ge 0 \quad \text{for all } x \in [0,1]. \qquad [2]$$

Hence f' is (monotonically) increasing and so

$$1 = f'(0) \le f'(x) \le f'(1) < 3 \quad \text{for all } x \in [0,1]. \qquad [2]$$

This implies that

$$1 = \int_0^1 1dx = \int_0^1 e^{x^2} dx \le \int_0^1 3dx \quad \text{and hence by FTC,} \quad 1 \le f(1) - f(0) \le 6. \qquad [3]$$

Since $f(0) = 6$, we obtain $7 \le f(1) \le 9$, which yields $6.7 \le f(1) \le 9.2$. [2]

9. Let $a, b \in \mathbb{R}$ with $a < b$. Define when a function $F : [a, b] \to \mathbb{R}$ is said to be convex. Use your definition to show that if $f : [a, b] \to \mathbb{R}$ is a monotonically increasing function, then $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) := \int_a^x f(t)dt \quad \text{for } x \in [a, b]$$

is a convex function. [10 marks]

Answer: A function $F : [a, b] \to \mathbb{R}$ is said to be convex (on $[a, b]$) if for any $x_1, x, , x_2 \in [a, b]$ with $x_1 < x < x_2$, we have

$$F(x) \le F(x_1) + \frac{F(x_2) - F(x_1)}{x_2 - x_1}(x - x_1). \qquad [2]$$

Aliter: A function $F : [a, b] \to \mathbb{R}$ is said to be convex if for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x + (1 - \lambda)y) \le \lambda F(x) + (1 - \lambda)F(y). \qquad [2]$$

Now suppse $f : [a, b] \to \mathbb{R}$ is a monotonically increasing function, and let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) := \int_a^x f(t)dt \quad \text{for } x \in [a, b]$$

Fix any $x_1, x, , x_2 \in [a, b]$ with $x_1 < x < x_2$. Then

$$\frac{F(x) - F(x_1)}{x - x_1} = \frac{1}{x - x_1}\int_{x_1}^x f(t)dt \le \frac{1}{x - x_1}\int_{x_1}^x f(x)dt = f(x),$$

where the inequality above follows since f is monotonically increasing. [2]

Similarly, since f is monotonically increasing,

$$f(x) \le \frac{1}{x_2 - x}\int_x^{x_2} f(t)dt = \frac{F(x_2) - F(x)}{x_2 - x}. \qquad [2]$$

It follows that

$$\frac{F(x) - F(x_1)}{x - x_1} \le \frac{F(x_2) - F(x)}{x_2 - x} \quad \text{and so} \quad \left(\frac{1}{x - x_1} + \frac{1}{x_2 - x}\right) F(x) \le \frac{F(x_1)}{x - x_1} + \frac{F(x_2)}{x_2 - x}.$$

[2]

This implies that

$$F(x) \le \frac{x_2 - x}{x_2 - x_1}F(x_1) + \frac{x - x_1}{x_2 - x_1}F(x_2) = F(x_1) + \frac{F(x_2) - F(x_1)}{x_2 - x_1}(x - x_1),$$

where the last step follows by writing $x_2 - x = (x_2 - x_1) - (x - x_1)$. [2]

Thus F is convex on $[a, b]$.

[**Note:** Cut 4 marks if it is argued that F is convex because $F' = f$ is mononically increasing or because $F'' \ge 0$. Since f is not given to be continuous, we can not assume that F is differentiable.]

10. Find the global extrema of the function $f : [0, 2] \times [0, 2] \to \mathbb{R}$ defined by

$$f(x, y) := 3x + y - x^2 - y^2 \quad \text{for } (x, y) \in [0, 2] \times [0, 2].$$

Also determine the points in $[0, 2] \times [0, 2]$ where the global extrema are attained.

[9 marks]

Answer: We calculate that the only critical point of f on the interior of the square is $(x_0, y_0) = (3 - 2x_0, 1 - 2y_0) = (0, 0)$. Hence, $(x_0, y_0) = (3/2, 1/2)$. The value at $(3/2, 1/2)$ is $10/4$. [2]

We now turn to the boundary of the square:

(a) On the edge $x = 0$, the function specialises to $f(0, y) = y - y^2$. It has a critical point at $y_0 = 1/2$ with value $1/4$. [1]

(b) On the edge $x = 2$, the function specialises to $f(2, y) = 2 + y - y^2$. It has a critical point at $y_0 = 1/2$ with value $9/4$. [1]

(c) On the edge $y = 0$, the function specialises to $f(x, 0) = 3x - x^2$. It has a critical point at $x_0 = 3/2$ with value $9/4$. [1]

(d) On the edge $y = 2$, the function specialises to $f(x, 2) = 3x - x^2 - 2$. It has a critical point at $x_0 = 3/2$ with value $1/4$. [1]

The value of f at the end points $(0, 0)$, $(0, 2)$, $(2, 0)$ and $(2, 2)$ are $0, -2, 2, 0$ respecitvely. [2]

Hence, the global maximum is attained at the point $(3/2, 1/2)$ with value $10/4$ and the global minimum is attained at the point $(0, 2)$ with value -2. [1]

11. Find the absolute value of the line integral of the vector field $\mathbf{F}$ in $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) := (ye^z, xe^z, xye^z)$$

over the curve C cut out by the intersection of the plane $x+y+z=1$ with the portion of the cylinder $x^2+y^2=1$ where $x \geq 3/5$. [9 marks]

Answer: We note that the given vector field $\mathbf{F}$ is the gradient field ∇f of the function $f(x, y, z) = xye^z$. [3]

By the FTC for line integrals (part II), we obtain

$$\left| \int_C \mathbf{F}\mathbf{ds} \right| = \left| \int_C \nabla f \mathbf{ds} \right| = |f(b) - f(a)|$$

where a and b are the two endpoints of the path C. [3]

We calculate the two endpoints of the path to be: $a = (3/5, 4/5, -2/5)$ and $b = (3/5, -4/5, 6/5)$. [2]

Hence,

$$\left| \int_C \mathbf{F} \cdot d\mathbf{s} \right| = |f(b) - f(a)| = \frac{12}{25}\left(e^{6/5} + e^{-2/5}\right). \qquad [1]$$

Note: Those who have computed the curl($\mathbf{F}$) to be zero and concluded that the line integral is zero may be given at most 4 marks.

12. Evaluate the line integral

$$I = \int_C z dx + (x + e^{y^2}) dy + (y + e^{z^2}) dz,$$

where C is the curve which is the intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. Orient C counterclockwise as viewed from above.

[9 marks]

Answer: Let $\boldsymbol{F} := (z, x + e^{y^2}, y + e^{z^2})$. Then $\mathrm{curl}(\boldsymbol{F}) = (1, 1, 1)$. [2]

If D represents the union of C and the region bounded by C (which is the inside of an ellipse), then by Stokes' theorem,

$$I = \iint_S \mathrm{curl}(\boldsymbol{F}) \cdot \boldsymbol{n} dS,$$

where S is the surface in $\mathbb{R}^3$ given the parametrization $\Phi(x, y) = (x, y, 3 - y)$, where $x^2 + y^2 \leq 4$. [4]

Now since, $\Phi_x = (1, 0, 0), \Phi_y = (0, 1, -1)$, we see that

$$I = \iint_{x^2+y^2 \leq 4} 2d(x, y) = 8\pi. \quad [3]$$

Note: Some other variations as well as alternative solutions for Q. 12 are provided in a separate note by a grader.