Tutorial 2 Solutions

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1. (a) This is false. Take $f(x) = x^2$ and $c = 0$, and $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. We have $f(x)g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. So $\lim_{x\to 0} f(x)g(x) = 1 \neq 0$

   (b) If $g(x)$ is bounded, let $\max|g(x)| = M$. Since $\lim_{x\to c} f(x) = 0$, we have that

   $$\forall\epsilon > 0, \exists\delta > 0, \text{ such that } |f(x) - 0| < \epsilon \text{ when } 0 < |x - c| < \delta$$

   For a particular epsilon ϵ_0, let δ_0, be the delta that is such that $|f(x) - 0| < \frac{\epsilon_0}{M}$ when $0 < |x - c| < \delta_0$
   Now, whenever $0 < |x - c| < \delta_0$,

   $$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times M < \frac{\epsilon_0}{M} \times M = \epsilon_0$$

   Hence, the limit of $\lim_{x\to c} f(x)g(x)$ exists and is equal to 0.[1]

   (c) Let $\lim_{x\to c} g(x)$ be L. We have that

   $$\exists\delta_0 > 0 \text{ such that } |g(x) - L| < |L| \text{ when } 0 < |x - c| < \delta_0$$
   $$\implies |g(x)| - |L| < |L| \implies |g(x)| < 2|L| \text{ when } 0 < |x - c| < \delta_0$$

   For a particular epsilon ϵ, let δ_1, be the delta that is such that $|f(x) - 0| < \frac{\epsilon}{2|L|}$ when $0 < |x - c| < \delta_1$
   Now, $\delta_2 = \min(\delta_0, \delta_1)$ is such that whenever $0 < |x - c| < \delta_2$,

   $$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times 2|L| < \frac{\epsilon}{2|L|} \times 2|L| = \epsilon$$

   Hence,
   $$\forall\epsilon > 0, \exists\delta_2 > 0 \text{ such that } |f(x)g(x) - 0| < \epsilon \text{ when } 0 < |x - c| < \delta$$

   which means that the limit exists and is equal to 0, as we wanted to prove.

2. Let $\lim_{x\to\alpha} f(x) = L$

   $$|f(\alpha + h) - f(\alpha - h) - 0| = |f(\alpha + h) - L - (f(\alpha - h) - L)| < |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

   Since $\lim_{x\to\alpha} f(x) = L$ $\forall\epsilon > 0, \exists\delta > 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ when $0 < |x - \alpha| < \delta$
   Rewriting $h = x - \alpha$, we get

   $$\forall\epsilon > 0, \exists\delta_0 > 0 \text{ such that } |f(\alpha + h) - L| < \frac{\epsilon}{2} \text{ when } 0 < |h| < \delta_0$$

   Again rewriting with $h = \alpha - x$, we get

   $$\forall\epsilon > 0, \exists\delta_1 > 0 \text{ such that } |f(\alpha - h) - L| < \frac{\epsilon}{2} \text{ when } 0 < |h| < \delta_1$$

   From these two statements, we get that

   $$\forall\epsilon > 0, \exists\delta > 0 \text{ such that } |f(\alpha - h) - L| + |f(\alpha - h) - L| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2)$$

[1] This remains true if instead ofbeing bounded in the entire domain, g is given to be bounded in a finite punctured neighborhood of c, which serves as a nice motivation for the next subpart

$$\implies \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(\alpha - h) - f(\alpha - h)| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2)$$

Thus the limit exists.[2]

The converse of the result would be that if $\lim_{h \to 0} f(\alpha + h) - f(\alpha - h) = 0$, then $\lim_{x \to \alpha} f(x)$ exists. A counterexample to this is

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Take $\alpha = 0$, $f(0 + h) - f(0 - h) = 0$ for all h, since this is an even function, but $\lim_{x \to 0} f(x)$ does not exist.

3. (a)

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In the domain $(0, \infty)$, both $\sin(x)$ and $\frac{1}{x}$ are continuous, hence so is their composition. Same for the domain $(-\infty, 0)$. The only problem is at $x = 0$. The function, in fact, is not continuous at $x = 0$. To show this, we'll use the sequential definition of a limit:

The limit of a function f at a exists and is equal to L if and only if for every sequence $x_n \to a$, $\lim_{n \to \infty} f(x_n) = L$

Take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n + \frac{1}{2})\pi}$. We have that $\sin x_n = 0$ and $\sin y_n = 1$ for all n $\in \mathbb{N}$ Thus, we have two sequences tending to 0, such that $f(x_n)$ and $f(y_n)$ tend to different values, hence the limit of $\sin \frac{1}{x}$ does not exist at $x = 0$

(b)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here again it is easy to argue continuity in the domains $(0, \infty)$ and $(-\infty, 0)$, using the fact that the composition and products of continuous functions is continuous. At $x = 0$, we need to find the limit and show that it is equal to $f(0) = 0$.

Notice that $|x \sin \frac{1}{x}| = |x||\sin \frac{1}{x}| < |x|$, since the range of sin is $[-1, 1]$

Hence,

$$\forall \epsilon > 0, \exists \delta, \text{ such that } |x \sin \frac{1}{x} - 0| < |x| < \epsilon \text{ when } 0 < |x - 0| < \delta$$

namely, one such delta would be $\delta = \epsilon$ always. We get that $\lim_{x \to 0} f(x) = 0 = f(0)$. Hence the function is continuous for all reals.

(c) Note that the pieces of the function in $(1, 2)$ and $(2, 3)$ are continuous. So there can only be a discontinuity at $x = 2$ The right hand and left hand limits at $x = 2$ can both be easily calculated to be 2, however, $f(2) = 1$. Hence, the function is not continuous at $x = 2$

4. We are given that $f(x + y) = f(x) + f(y)$. Functions with this property are also known as additive functions.[3]

Set $x = y = 0$ to get $f(0) = 2f(0) \implies f(0) = 0$

$$f(c + h) = f(c) + f(h) \implies f(c + h) - f(c) = f(h) = f(h) - f(0)$$

Since f is continuous at 0

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(h) - 0| < \epsilon \text{ when } 0 < h < \delta$$

$$\implies \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(c + h) - f(c)| = |f(h)| < \epsilon \text{ when } 0 < h < \delta$$

$$\implies \text{f is continous for all } c \in \mathbb{R}$$

5. For $x \neq 0$, $\frac{1}{x}$ is differentiable while $\sin x$ and x are differentiable at all points. Hence, a function formed by their compositions and products would also be differentiable at $x \neq 0$. At $x = 0$, we'll need to work by first principles. We need to see if $\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \sin \frac{1}{x}$ exists. Deja vu to q3 ii), the limit exists and is equal to 0. Hence f is differentiable on $\mathbb{R}$

[2] One could have also argued the same using algebra of limits on functions.

[3] For more information about this, see this wikipedia entry

However, the derivative is not continuous. Using chain rule for $x \neq 0$, we get that

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} + \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

We can show that $\lim\limits_{x\to 0} f'(x)$ does not exist, using a method similar to q3 i)

6. We have that $\forall x, x+h \in (a,b)$

$$\frac{|f(x+h)-f(x)|}{|h|} < C|h|^{\alpha-1}$$

$$\implies -C|h|^{\alpha-1} < \frac{f(x+h)-f(x)}{h} < C|h|^{\alpha-1}$$

Since $\alpha - 1 > 0$, $\lim\limits_{h\to 0} C|h|^{\alpha-1} = \lim\limits_{h\to 0} -C|h|^{\alpha-1} = 0$

Hence, by sandwich theorem we can conclude that $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and equals 0. By definition, this means that f is differentiable over (a,b) and $f'(x) = 0\, \forall x \in (a,b)$

7.

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h} &= \lim_{h\to 0^+} \frac{\frac{f(c+h)-f(c)}{h} + \frac{f(c)-f(c-h)}{h}}{2} \\ &= \frac{\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c)}{h} + \lim\limits_{h\to 0^+} \frac{f(c)-f(c-h)}{h}}{2} \end{aligned}$$

Since f is differentiable, $\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c)}{h} =$ Right hand derivative $=$ Left hand derivative $= \lim\limits_{h\to 0^+} \frac{f(c)-f(c-h)}{h} = f'(c)$ and hence, we are done.

The converse is false, consider $f(x) = |x|$ and $c = 0$. Then $\lim\limits_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h} = \lim\limits_{h\to 0^+} \frac{h-h}{2} = 0$, but the function is not differentiable at 0.

8. First, note that if $\exists c$ such that $f(c) = 0$, then $f(x) = f(x-c+c) = f(x-c)f(c) = 0\, \forall x \in \mathbb{R}$. In this case f is obviously differentiable, with $f'(c) = 0 \forall c = f'(0)f(c)$ (trivially).

The only other possibility is that f is always non-zero. Set $x = y = 0$ to get

$$f(0) = f(0)^2 \implies f(0) = 0 \text{ or } 1$$

. If f is not identically 0 then $f(0) = 1$.

$$\frac{f(c+h)-f(c)}{h} = \frac{f(c)f(h)-f(c)}{h} = f(c) \times \frac{f(h)-1}{h}$$

Since $f'(0) = \lim\limits_{h\to 0} \frac{f(h)-1}{h}$ exists, we have that $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim\limits_{h\to 0} f(c) \times \frac{f(h)-1}{h}$ exists and is equal to $f(c) \times \lim\limits_{h\to 0} \frac{f(h)-f(0)}{h} = f(c)f'(0)$. Thus, by definition, f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f(c)f'(0)$

9. Calculating derivative of inverse of a function. Let $f : (a,b) \to \mathbb{R}$ be a differentiable and invertible function. Let the inverse be denoted by $g(x)$, which means that

$$f(g(x)) = x$$

Differentating $f(g(x))$ using the chain rule, we get that

$$f'(g(x)) \times g'(x) = \frac{d}{dx}x = 1 \implies g'(x) = \frac{1}{f'(g(x))}$$

Using this both the parts can be solved, solving is left to the reader.

10. Apply chain rule and simplify. Left to the reader.