The following is a model answer for the midsem question for Code A. The same guideline is to be followed for other Codes.

1.(a) If $p > 1$, prove that $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$.

Sol. By definition $p^{\frac{1}{n}} = e^{\frac{1}{n}\log p}$. [1]
Therefore,

$$\lim_{n\to\infty} p^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log p} = e^{\left\{\lim\limits_{n\to\infty} \frac{1}{n}\log p\right\}},$$

by continuity of the exponential function

[1]

$$= e^0 = 1.$$

[1]

Alternative Solution. Let $x_n = p^{\frac{1}{n}} - 1$. Then $x_n > 0$ for all $n \in \mathbb{N}$. [1]

By the Binomial theorem we have

$$1 + nx_n \leq (1 + x_n)^n = p,$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

[1]

Since $\lim\limits_{n\to\infty} \dfrac{p-1}{n} = 0$, by Sandwich theorem $\lim\limits_{n\to\infty} x_n = 0$ which implies that $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = 1$. [1]

(b) If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that $|a_n| \geq \frac{|L|}{2}$, for all $n \geq n_0$.
Sol. Choose $\epsilon = |L|/2$. [1]
By the definition of limit of a sequence, there is n_0 such that for $n \geq n_0$, $|L - a_n| < |L|/2$. Therefore, if $n \geq n_0$, then by triangle inequality, $|L|/2 > |L| - |a_n|$, or $|a_n| \geq |L|/2$. [1]

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$, which is bijective (i.e. one-one and onto), but is discontinuous at every $c \in \mathbb{R}$.
Sol. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows : $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = x - 1$ for $x \notin \mathbb{Q}$. [1]

For x rational (irrational), $f(x)$ is rational (irrational). Therefore, the images of $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ under f are disjoint, f is one-one on restrictions to these two sets, hence one-one. [1/2]

Every rational $x = f(x)$, whereas for every irrational y, $x = y - 1$ is also irrational and $y = f(x)$. Therefore f is onto. [1/2]

Both $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ are dense in $\mathbb{R}$. Suppose f is continuous at $a \in \mathbb{R}$. Choose any $\epsilon < 1/2$. If $0 < \delta < \epsilon$, then there is $x \in (a-\delta, a+\delta) \cap \mathbb{Q}$ and $y \in (a-\delta, a+\delta) - \mathbb{Q}$. But $f(x) = x$ and $f(y) = y - 1$, and

$$f(x) = x > a - \delta > a - 1 + \delta > y - 1 = f(y).$$

If $a \in \mathbb{Q}$, this means that $|f(y) - f(a)| \geq 1 - \delta > 1 - \epsilon > \epsilon$, and if $a \notin \mathbb{Q}$, we have $|f(x) - f(a)| \geq 1 - \delta > 1 - \epsilon > \epsilon$. [3]

[Note: Many students used sesquential criterion of continuity to prove that the function is discontinuous at every point. Let $c \in \mathbb{R}$. Since both $\mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ are dense in $\mathbb{R}$, there are sequences, x_n of rational numbers, and y_n of irrational number such that both $x_n \to c$ and $y_n \to c$. Therefore $\lim_n f(x_n) = \lim_n x_n = c \neq c - 1 = \lim_n y_n - 1 = \lim_n f(y_n)$.]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose there is $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$. Let $a_1 \in \mathbb{R}$, and set $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$. Show that the sequence $\{a_n\}$ converges.

Sol. For every $x \neq y$, by the mean value theorem there is ξ in between x and y such that $|f(x) - f(y)| = |x - y||f'(\xi)| \leq \alpha|x - y|$, since f is differentiable. [1]

Therefore by induction,

$$|f(a_n) - f(a_{n-1})| = |a_{n+1} - a_n| \leq \alpha^{n-1}|a_2 - a_1|.$$

[1]

Now, by triangle inequality, if $m > n$,

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + \cdots |a_{n+1}, a_n| \\
&\leq |a_2 - a_1|\alpha^n(1 + \alpha + \cdots \alpha^{m-n-1}) \\
&\leq |a_2 - a_1|\alpha^n(\sum_{i=1}^{\infty} \alpha^i) \\
&= \alpha^n \frac{|a_2 - a_1|}{1 - \alpha}
\end{aligned}$$

[2]

Therefore a_n is Cauchy. Since every Cauchy sequence converges, a_n is convergent. [1]

4. Using the definition of Riemann integrability in terms of Riemann sum $S(f,\dot{\mathcal{P}})$, prove that the following function is integrable over $[1,3]$ and compute its integral:

$$f(x) = \left\{ \begin{array}{ll} 5, & \text{if } 1 \leq x < 2 \\ -4, & \text{if } 2 \leq x \leq 3. \end{array} \right\}$$

Sol. Let $\dot{\mathcal{P}}$ be a tagged partition of $[1,3]$ with $\|\dot{\mathcal{P}}\| < \delta$. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ consisting of tags in $[1,2)$ and $\dot{\mathcal{P}}_2$ be the remaining subset consisting of tags in $[2,3]$. The union of all subintervals in $\dot{\mathcal{P}}_1$ contains $[1,2-\delta)$ and is contained in $[1,2+\delta)$. [1]

So the Riemann sum is bounded by

$$5(2-1-\delta) \leq S(f,\dot{\mathcal{P}}_1) \leq 5(2-1+\delta).$$

[1]

Similarly, [1]

$$-4(3-2+\delta) \leq S(f,\dot{\mathcal{P}}_2) \leq -4(3-2-\delta).$$ [1]

Thus we get

$$5(2-1)-4(3-2)-(5+4)\delta \leq S(f,\dot{\mathcal{P}}) \leq 5(2-1)-4(3-2)+(5+4)\delta$$

that is

$$1-9\delta \leq S(f,\dot{\mathcal{P}}) \leq 1+9\delta.$$

and hence

$$\left|S(f,\dot{\mathcal{P}})-1\right| < \epsilon$$

whenever $\dot{\mathcal{P}}$ is a tagged partition of $[a,b]$ with $\|\dot{\mathcal{P}}\| < \dfrac{\epsilon}{9}$.

For the calculation of the integral [1]

5. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \left\{ \begin{array}{ll} x\cos(1/y), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{array} \right\}.$$

Find out the points in $[0,1] \times [0,1]$ at which f is continuous.

Sol. The given function is obviously continuous at every point of the form (a,b), with $b \neq 0$, in its domain of definition by being composition and product of continuous functions. [1]

At $(0,0)$

$$\lim_{(x,y)\to(0,0)} x\cos\frac{1}{y} = 0,$$

because of the fact that $\cos\frac{1}{y}$ is bounded in a neighborhood of $(0,0)$ and $\lim\limits_{(x,y)\to(0,0)} x = 0$. [2]

At the points $(a,0)$, with $a \neq 0$, the function is discontinuous. Consider two different sequences $\{(a, \frac{1}{(2n+1)\pi/2})\}$ and $\{(a, \frac{1}{2n\pi})\}$, both of them tend to $(a,0)$ as $n\to\infty$ but $\{f(a, \frac{1}{(2n+1)\pi/2})\}$ and $\{f(a, \frac{1}{2n\pi})\}$ do not converge to the same limits. [2]

Note. The word **similarly** indicates that you have thought about the containment of the union of subintervals in the partition $\dot{P}_2$ between suitable subsets of $[1,3]$. This is important because otherwise we do not get the following inequality. Note that if the reasoning in the first part about this containment is incorrect then you do not get any marks for writing **similarly**.

6. Let $f(0,0)=0$ and

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right), \text{ for } (x,y)\neq(0,0).$$

Show that the partial derivatives of f exist but are not bounded in any disc around $(0,0)$.

Sol. To show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$.
Now

$$\frac{\partial f}{\partial x}|_{(0,0)} = \lim_{h\to o}\frac{f((0,0)+(h,0))-f(0,0)}{h} = \lim_{h\to o}\frac{(h)^2\sin\frac{1}{h^2}}{h}$$
$$= \lim_{h\to o} h\sin\frac{1}{h^2} = 0.$$

This is because $\sin\frac{1}{h^2}$ is bounded in a neighborhood of 0 and $\lim\limits_{h\to o} h = 0$. [1]

The proof of existence of $\partial f/\partial y$ at $(0,0)$ is similar because the given function is a symmetric function. [1]

The function f is differentiable everywhere except $(0,0)$, by being a product and composition of differentiable functions.The partial derivatives are the following functions

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cos\frac{1}{x^2+y^2}.(\frac{-2x}{(x^2+y^2)^2})\\ &= 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}.\end{aligned}$$

[1/2]

Now let $N(\bar{0};r)$ be a neighborhood of $\bar{0}=(0,0)$. Since $2x\sin\dfrac{1}{x^2+y^2}$ is a product of two functions $2x$ and $\sin\dfrac{1}{x^2+y^2}$, each of which is bounded in $N(\bar{0};r)$, it is bounded in $N(\bar{0};r)$. [1/2]

The function $\cos\dfrac{1}{x^2+y^2}$ is bounded in $N(\bar{0};r)$ but $\dfrac{2x}{x^2+y^2}$ is unbounded in $N(\bar{0};r)$. This is because for any $M>1$, we can choose a real number $R>M$ such that $1/R<r$. Now for $(x_1,y_1)=(1/R,1/R)$, $\dfrac{2x_1}{x_1^2+y_1^2}>M$. If $0<M\le 1$, then also we are done. [1]

Similarly one can show that $\dfrac{\partial f}{\partial y}$ is unbounded around the origin. [1]

Note. If there is a mistake in a part then the marks are deducted from the similar part as well if one uses the term **similarly**.