Tutorial 2 Solutions

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Note that the square brackets in the question are not meant to be anything like the greatest integer function. They are just brackets.

1. (a) This is false. Take $f(x) = x^2$ and $c = 0$, and $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

   We have $f(x)g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. So $\lim_{x \to 0} f(x)g(x) = 1 \neq 0$

   (b) If $g(x)$ is bounded, let $\max |g(x)| = M$. Since $\lim_{x \to c} f(x) = 0$, we have that

   $$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } |f(x) - 0| < \epsilon \text{ when } 0 < |x - c| < \delta$$

   For a particular epsilon ϵ_0, let δ_0, be the delta that is such that $|f(x) - 0| < \frac{\epsilon_0}{M}$ when $0 < |x - c| < \delta_0$
   Now, whenever $0 < |x - c| < \delta_0$,

   $$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times M < \frac{\epsilon_0}{M} \times M = \epsilon_0$$

   Hence, the limit of $\lim_{x \to c} f(x)g(x)$ exists and is equal to 0.

   (c) Let $\lim_{x \to c} g(x)$ be L. We have that

   $$\exists \delta_0 > 0 \text{ such that } |g(x) - L| < |L| \text{ when } 0 < |x - c| < \delta_0$$
   $$\implies |g(x)| - |L| < |L| \implies |g(x)| < 2|L| \text{ when } 0 < |x - c| < \delta_0$$

   For a particular epsilon ϵ_1, let δ_1, be the delta that is such that $|f(x) - 0| < \frac{\epsilon_1}{2|L|}$ when $0 < |x - c| < \delta_1$
   Now, $\delta_3 = \min(\delta_0, \delta_1)$ is such that whenever $0 < |x - c| < \delta_3$,

   $$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times 2|L| < \frac{\epsilon_1}{2|L|} \times 2|L| = \epsilon_1$$

2. Let $\lim_{x \to \alpha} f(x) = L$

   $$|f(\alpha+h) - f(\alpha-h) - 0| = |f(\alpha+h) - L - (f(\alpha-h) - L)| < |f(\alpha+h) - L| + |f(\alpha-h) - L|$$

Since $\lim_{x\to\alpha} f(x) = L$ $\forall\epsilon, \exists\delta$ such that $|f(x) - L| < \frac{\epsilon}{2}$ when $0 < |x - \alpha| < \delta$ Rewriting $h = x - \alpha$, we get $\forall\epsilon, \exists\delta_0$ such that $|f(\alpha + h) - L| < \frac{\epsilon}{2}$ when $0 < |h| < \delta_0$ Again rewriting with $h = \alpha - x$, we get

$$\forall\epsilon, \exists\delta_1 \text{ such that } |f(\alpha - h) - L| < \frac{\epsilon}{2} \text{ when } 0 < |h| < \delta_1$$

From these two statements, we get that

$$\forall\epsilon, \exists\delta \text{ such that } |f(\alpha-h)-L|+|f(\alpha-h)-L| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2)$$

$$\implies \forall\epsilon, \exists\delta \text{ such that } |f(\alpha-h)-f(\alpha-h)| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2)$$

One could have also argued the same using algebra of limits on functions.

The converse of the result would be that if $\lim_{h\to 0} f(\alpha+h) - f(\alpha-h) = 0$, then $\lim_{x\to\alpha} f(x)$ exists. A counterexample to this would be

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Take $\alpha = 0$, $f(0+h) - f(0-h) = 0$ for all h, since this is an even function, but $\lim_{x\to 0} f(x)$ does not exist.

3. (a)

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In the domain $(0, \infty)$, both $\sin(x)$ and $\frac{1}{x}$ are continuous, hence so is their composition. Same for the domain $(-\infty, 0)$. The only problem is at $x = 0$. The function, in fact, is not continuous at $x = 0$. To show this, we'll use the sequential definition of a limit:

The limit of a function f at a exists and is equal to L if and only if for every sequence $x_n \to a$, $\lim_{n\to\infty} f(x_n) = L$

Take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+\frac{1}{2})\pi}$. We have that $\sin x_n = 0$ and $\sin y_n = 1$ for all n $\in \mathbf{N}$ Thus, we have two sequences tending to 0, such that $f(x_n)$ and $f(y_n)$ tend to different values, hence the limit of $\sin\frac{1}{x}$ does not exist at $x = 0$

(b)

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here again it is easy to argue continuity in the domains $(0, \infty)$ and $(-\infty, 0)$, using the fact the composition and products of continuous functions is continuous. At $x = 0$, we need to find the limit and show that it is equal to $f(0) = 0$.

$|x\sin\frac{1}{x}| = |x||\sin\frac{1}{x}| < |x|$, since the range of sin is $[-1, 1]$

Hence,

$$\forall \epsilon > 0,\ |x \sin \frac{1}{x} - 0| < |x| < \epsilon \text{ when } 0 < |x - 0| < \epsilon$$

Seeing that we can just set $\delta = \epsilon$ always, we get that $\lim_{x \to 0} f(x) = 0 = f(0)$. Hence the function is continuous for all reals.

(c) Note that the pieces of the function in $(1, 2)$ and $(2, 3)$ are continuous. So there can only be a discontinuity at $x = 2$ The right hand and left hand limits at $x = 2$ can both be easily calculated to be 2, however, $f(2) = 1$. Hence, the function is not continuous at $x = 2$

4. We are given that $f(x + y) = f(x) + f(y)$. Functions with this property are also known as linear functions. Set $x = y = 0$ to get $f(0) = 2f(0) \implies f(0) = 0$

$$f(c + h) = f(c) + f(h) \implies f(c + h) - f(c) = f(h) = f(h) - f(0)$$

Since f is continuous at 0

$$\forall \epsilon \exists \delta \text{ such that } |f(h) - 0| < \epsilon \text{ when } 0 < h < \delta$$

$$\implies \forall \epsilon \exists \delta \text{ such that } |f(c + h) - f(c)| = |f(h)| < \epsilon \text{ when } 0 < h < \delta$$

$\implies$ f is continous for all $c \in \mathbf{R}$