(Q. 1-A-a.) Let $\{a_n\}$ be a sequence of real numbers. If $\{a_n\}$ is convergent then prove that its limit is unique. [3 marks]

Answer: Let, if possible, the sequence $\{a_n\}$ converge to two different real numbers, say L_1 and L_2, and let $3\delta = |L_1 - L_2|$. [1 mark]

Then for δ we have $N_1, N_2 \in \mathbb{N}$ such that $|L_i - a_n| < \delta \;\; \forall \; n \geq N_i$. [1 mark]

By choosing N to be the maximum of N_1 and N_2, we get a contradiction, thus $L_1 = L_2$. [1 mark]

(Q. 1-B-a.) Let $\{b_n\}$ be a sequence of real numbers. If $\{b_n\}$ is convergent then prove that its limit is unique. [3 marks]

(Q. 1-C-a.) Let $\{c_n\}$ be a sequence of real numbers. If $\{c_n\}$ is convergent then prove that its limit is unique. [3 marks]

(Q. 1-D-a.) Let $\{d_n\}$ be a sequence of real numbers. If $\{d_n\}$ is convergent then prove that its limit is unique. [3 marks]

(Q. 1- $\cdot$ -b.) State the sandwich theorem for sequences. [3 marks]

Answer: Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that [1 mark]

$$a_n \leq b_n \leq c_n \qquad \forall \; n.$$

Assume that $\{a_n\}$ and $\{c_n\}$ are convergent with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \alpha$, say. [1 mark]

Then (the sequence $\{b_n\}$ is also convergent and) $\lim\limits_{n\to\infty} b_n = \alpha$. [1 mark]

(Note: The convergence of the middle sequence, $\{b_n\}$, is a part of the conclusion here. If you assume it in the beginning then your statement is incorrect.)

(Q. 2-A.) Consider the function

$$f(x) = \begin{cases} \frac{7-(16)^{\frac{1}{x}}}{7+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 7 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

Answer: The function is continuous in the interval $(2, 4)$, because the exponential function is continuous on the whole of $\mathbb{R}$ and $1/x$ is continuous for $x \neq 0$. [2 marks]

(Note: If the derivative of the function f is computed correctly and then it is said that f is continuous then 2 marks are to be awarded for the first step.)

Consider $g(x) := f(x) - \frac{1}{2}$, and note that g is also continuous on $(2, 4)$.

Observe that

$$g(2) = f(2) - \frac{1}{2} = \frac{7-4}{7+4} - \frac{1}{2} < 0 \quad \text{and} \quad g(4) = f(4) - \frac{1}{2} = \frac{7-2}{7+2} - \frac{1}{2} > 0.$$ [1 mark each]

Therefore by the intermediate value theorem, there is some $c \in (2, 4)$ such that $g(c) = 0$, or $f(c) = \frac{1}{2}$. [2 marks]

(Q. 2-B.) Consider the function

$$f(x) = \begin{cases} \frac{5-(16)^{\frac{1}{x}}}{5+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{4}$. [6 marks]

(Q. 2-C.) Consider the function

$$f(x) = \begin{cases} \frac{9-(16)^{\frac{1}{x}}}{9+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 9 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 2-D.) Consider the function

$$f(x) = \begin{cases} \frac{11-(16)^{\frac{1}{x}}}{11+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 11 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 2-A.) Consider the function

$$f(x) = \begin{cases} \frac{7-(16)^{\frac{1}{x}}}{7+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 7 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

Answer: Note that $f(c) = \frac{1}{2}$ if and only if $\left(\frac{7}{3}\right)^c = 16.$ [2 marks]

Note that $(\frac{7}{3})^2 < 16 < (\frac{7}{3})^4$ as $2 < \frac{7}{3} < 4.$ [2 marks]

The function $x \mapsto (\frac{7}{3})^x$ is continuous and by intermediate value theorem a required $c \in (2, 4)$ exists. [2 marks]

(Q. 2-B.) Consider the function

$$f(x) = \begin{cases} \frac{5-(16)^{\frac{1}{x}}}{5+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{4}$. [6 marks]

(Q. 2-C.) Consider the function

$$f(x) = \begin{cases} \frac{9-(16)^{\frac{1}{x}}}{9+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 9 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 2-D.) Consider the function

$$f(x) = \begin{cases} \frac{11-(16)^{\frac{1}{x}}}{11+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 11 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 2-A.) Consider the function

$$f(x) = \begin{cases} \frac{7-(16)^{\frac{1}{x}}}{7+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 7 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

Answer: Note that $f(c) = \frac{1}{2}$ if and only if $\left(\frac{7}{3}\right)^c = 16$ if and only if $c = \log_{\frac{7}{3}} 16$. [2 marks]

We observe that $\log$ is an increasing function and hence $\log 2 < \log(\frac{7}{3}) < \log 4$. [2 marks]

Therefore, $2 = \log_4 16 < c = \log_{\frac{7}{3}} 16 < 4 = \log_2 16$ and hence $c \in (2, 4)$. [2 marks]

(Q. 2-B.) Consider the function

$$f(x) = \begin{cases} \frac{5-(16)^{\frac{1}{x}}}{5+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{4}$. [6 marks]

(Q. 2-C.) Consider the function

$$f(x) = \begin{cases} \frac{9-(16)^{\frac{1}{x}}}{9+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 9 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 2-D.) Consider the function

$$f(x) = \begin{cases} \frac{11-(16)^{\frac{1}{x}}}{11+(16)^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 11 & \text{if } x = 0. \end{cases}$$

Prove that there is a point c such that $2 < c < 4$ and $f(c) = \frac{1}{2}$. [6 marks]

(Q. 3-A, C.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} \frac{(x^2-y^2)\sin(xy)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) \neq (0,0)$. [2 marks]

Answer: When $(x,y) \neq (0,0)$ we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^4 y\cos(xy) + 4xy^2\sin(xy) - y^5\cos(xy)}{(x^2+y^2)^2} \qquad \text{[1 mark]}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^5\cos(xy) - xy^4\cos(xy) - 4x^2y\sin(xy)}{(x^2+y^2)^2} \qquad \text{[1 mark]}$$

(b) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) = (0,0)$. [2 marks]

Answer:

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0) \qquad \text{[1 mark each]}$$

(Note: A justification is needed for showing this, just writing the above equalities do not guarantee any marks.)

(c) Explain with justification if any of the second order differentials $\frac{\partial^2 f}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at $(0,0)$. [3 marks]

Answer: We compute the second order differentials at $(0,0)$. They are

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1. \qquad \text{[1 mark each]}$$

Since these values are not equal, by the mixed partials theorem, none of the second order differentials $\frac{\partial^2 f}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at $(0,0)$. [1 mark]

(If a student calculates the second order differentials explicitly and concludes, by giving a valid argument, that none of them is continuous at the origin, then the three marks are to be given.

Further, if a student makes an error in computing $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ but writes the final conclusion correctly then we give one mark.)

(Q. 3-B, D.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} \frac{(y^2-x^2)\sin(xy)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) \neq (0,0)$. [2 marks]

Answer: When $(x,y) \neq (0,0)$ we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{-x^4 y\cos(xy) - 4xy^2\sin(xy) + y^5\cos(xy)}{(x^2+y^2)^2} \qquad [1 \text{ mark}]$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-x^5\cos(xy) + xy^4\cos(xy) + 4x^2 y\sin(xy)}{(x^2+y^2)^2} \qquad [1 \text{ mark}]$$

(b) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) = (0,0)$. [2 marks]

Answer:

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0) \qquad [1 \text{ mark each}]$$

(Note: A justification is needed for showing this, just writing the above equalities do not guarantee any marks.)

(c) Explain with justification if any of the second order differentials $\frac{\partial^2 f}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at $(0,0)$. [3 marks]

Answer: We compute the second order differentials at $(0,0)$. They are

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = -1. \qquad [1 \text{ mark each}]$$

Since these values are not equal, by the mixed partials theorem, none of the second order differentials $\frac{\partial^2 f}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at $(0,0)$. [1 mark]

(If a student calculates the second order differentials explicitly and concludes, by giving a valid argument, that none of them is continuous at the origin, then the three marks are to be given.

Further, if a student makes an error in computing $\frac{\partial^2 f}{\partial y \partial x}(0,0)$ and $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ but writes the final conclusion correctly then we give one mark.)

(Q. 4-A, C.) Find the radius and the height of the cylindrical can with a lid, which contains 1000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 1000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 1000$. [1 mark]

Then $\nabla f = 2\pi[2r + h, r]$ and $\nabla g = \pi[2rh, r^2]$. [1 mark]

The equation $\nabla f = \lambda \nabla g$ gives us two equalities

$$\lambda rh = 2r + h, \qquad \lambda r^2 = 2r. \qquad \text{[1 mark]}$$

The second equality (and $r \neq 0$ since $g \neq 0$) gives us that $\lambda = 2/r$. [1 mark]

Putting this value in the first equality we get $2r = h$. [1 mark]

Finally, the equality $g(r,h) = 1000$ gives us that $r = \sqrt[3]{\frac{500}{\pi}}$ and $h = \sqrt[3]{\frac{4000}{\pi}}$. [1 mark]

(Q. 4-B, D.) Find the radius and the height of the cylindrical can with a lid, which contains 2000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 1000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 2000$. [1 mark]

Then $\nabla f = 2\pi[2r + h, r]$ and $\nabla g = \pi[2rh, r^2]$. [1 mark]

The equation $\nabla f = \lambda \nabla g$ gives us two equalities

$$\lambda rh = 2r + h, \qquad \lambda r^2 = 2r. \qquad \text{[1 mark]}$$

The second equality (and $r \neq 0$ since $g \neq 0$) gives us that $\lambda = 2/r$. [1 mark]

Putting this value in the first equality we get $2r = h$. [1 mark]

Finally, the equality $g(r,h) = 2000$ gives us that $r = \sqrt[3]{\frac{1000}{\pi}}$ and $h = \sqrt[3]{\frac{8000}{\pi}}$. [1 mark]

If there is an error in writing the function f but the other calculations are consistent with the error then half of the deserving marks are to be given.

(Q. 4-A, C.) Find the radius and the height of the cylindrical can with a lid, which contains 1000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 1000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 1000$. By the volume constraint, we have $h = 1000/(\pi r^2)$ and hence $f(r) = 2\pi(r^2 + 1000/(\pi r)) = 2(\pi r^2 + 1000/r)$. [1 mark]

Then $\dfrac{df}{dr} = 2(2\pi r - 1000/r^2)$. [1 mark]

Hence $r = \sqrt[3]{\frac{500}{\pi}}$ is a critical point for the function f. [1 mark]

Further, $\dfrac{d^2 f}{dr^2} = 4\pi + \dfrac{4000}{r^3}$ [1 mark]

and, $\dfrac{d^2 f}{dr^2}\left(\sqrt[3]{\frac{500}{\pi}}\right) = 12\pi > 0$. [1 mark]

Hence r as above is the required radius and $h = \sqrt[3]{\frac{4000}{\pi}}$ is the required height. [1 mark]

(Q. 4-B, D.) Find the radius and the height of the cylindrical can with a lid, which contains 2000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 2000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 2000$. By the volume constraint, we have $h = 2000/(\pi r^2)$ and hence $f(r) = 2\pi(r^2 + 2000/(\pi r)) = 2(\pi r^2 + 2000/r)$. [1 mark]

Then $\dfrac{df}{dr} = 2(2\pi r - 2000/r^2)$. [1 mark]

Hence $r = \sqrt[3]{\frac{1000}{\pi}}$ is a critical point for the function f. [1 mark]

Further, $\dfrac{d^2 f}{dr^2} = 4\pi + \dfrac{8000}{r^3}$ [1 mark]

and, $\dfrac{d^2 f}{dr^2}\left(\sqrt[3]{\frac{1000}{\pi}}\right) = 12\pi > 0$. [1 mark]

Hence r as above is the required radius and $h = \sqrt[3]{\frac{8000}{\pi}}$ is the required height. [1 mark]

If there is an error in writing the function f but the other calculations are consistent with the error then half of the deserving marks are to be given.

(Q. 4-A, C.) Find the radius and the height of the cylindrical can with a lid, which contains 1000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 1000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 1000$. By the volume constraint, we have $h = 1000/(\pi r^2)$ and hence $f(r) = 2\pi(r^2 + 1000/(\pi r)) = 2(\pi r^2 + 1000/r)$. [1 mark]

By the AM-GM inequality[1] $f(r) \geq 3\sqrt[3]{2\pi r^2 \times \frac{1000}{r} \times \frac{1000}{r}} = A$, say. [2 marks]

Further, $f(r) = A$, that is, $f(r)$ attains the minimal value exactly when $2\pi r^2 = \frac{1000}{r}$ [2 marks]

or when $r = \frac{10}{\sqrt[3]{2\pi}}$ and then $h = \sqrt[3]{\frac{4000}{\pi}}$. [1 mark]

(Q. 4-B, D.) Find the radius and the height of the cylindrical can with a lid, which contains 2000 cm^3 of water and which has the smallest total surface area. [6 marks]

Answer: We need to design a cylindrical can whose surface area is the smallest among all cylindrical cans of volume 1000 cm^3. Let us denote the base radius of the can by r and the height by h. We will ignore the unit, cm, here by assuming that all the lengths are measured in the centimeters.

We want to minimise $f(r,h) = 2\pi(r^2 + rh)$ subject to $g(r,h) = \pi r^2 h = 2000$. By the volume constraint, we have $h = 2000/(\pi r^2)$ and hence $f(r) = 2\pi(r^2 + 2000/(\pi r)) = 2(\pi r^2 + 2000/r)$. [1 mark]

By the AM-GM inequality[1] $f(r) \geq 3\sqrt[3]{2\pi r^2 \times \frac{2000}{r} \times \frac{2000}{r}} = A$, say. [2 marks]

Further, $f(r) = A$, that is, $f(r)$ attains the minimal value exactly when $2\pi r^2 = \frac{2000}{r}$ [2 marks]

or when $r = \frac{10}{\sqrt[3]{\pi}}$ and then $h = \frac{20}{\sqrt[3]{\pi}}$. [1 mark]

If there is an error in writing the function f but the other calculations are consistent with the error then half of the deserving marks are to be given.

[1] The inequality we are using here is $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ and the equality holds exactly when $a = b = c$.

(Q. 5.) Find the mass M of a metal plate R bounded by $y = x$ and $y = x^2$, with density given by $\delta(x, y) = 1 + xy \ \mathrm{kg/meter^2}$. [6 marks]

(Note that $M = \iint_R \delta(x, y) dA$.)

Answer: We observe that the integrand is continuous, so we can compute this integration by using the iterated integration.

$$M = \iint_R \delta(x, y) dA = \iint_R (1 + xy) dxdy \qquad \text{[1 mark]}$$

$$= \int_0^1 \int_{x^2}^x (1 + xy) dydx \qquad \text{[1 mark]}$$

$$= \int_0^1 (y + xy^2/2)\Big|_{y=x^2}^{y=x} dx \qquad \text{[1 mark]}$$

$$= \int_0^1 (x - x^2 + x^3/2 - x^5/2) dx \qquad \text{[1 mark]}$$

$$= (x^2/2 - x^3/3 + x^4/8 - x^6/12)\Big|_0^1 \qquad \text{[1 mark]}$$

$= 5/24 = 0.208$ kg [1 mark]

(Note: If the limits in the integration are taken in the other way, $\int_x^{x^2}$, leading to the answer $-5/24$ then we have still given the full marks.)

(Q. 6-A.) Consider the vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (3x^2 - y)$ and $Q = (10y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: Note that $\dfrac{\partial P}{\partial y} = -1 = \dfrac{\partial Q}{\partial x}$ on the entire plane which is simply connected. [1/2 + 1/2 marks]
Therefore $\mathbf{F}$ is conservative.

Let $f(x, y)$ be a potential function for $\mathbf{F}$. By definition,

$$f_x = 3x^2 - y, \qquad (A)$$
$$f_y = 10y - x. \qquad (B)$$

[1 mark]

We integrate both sides of equation (A) with respect to x, and obtain

$$f = x^3 - xy + \phi(y), \qquad (C)$$

where $\phi(y)$ is a function of y. [1 mark]

Differentiating equation (C) with respect to y, we get

$$f_y = -x + \phi'(y).$$

Comparing with equaton (B), we get $\phi'(y) = 10y$, [1 mark]

or $\phi(y) = 5y^2 + K$, where K is a constant independent of x and y. [1 mark]

Therefore $f(x, y) = x^3 - xy + 5y^2 + K$ is a required potential function for the given vector field $\mathbf{F}$. [1 mark]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 6-B.) Consider the vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (5x^2 - y)$ and $Q = (8y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: Note that $\dfrac{\partial P}{\partial y} = -1 = \dfrac{\partial Q}{\partial x}$ on the entire plane which is simply connected. [1/2 + 1/2 marks]
Therefore $\mathbf{F}$ is conservative.

Let $f(x, y)$ be a potential function for $\mathbf{F}$. By definition,

$$f_x = 5x^2 - y, \qquad (A)$$
$$f_y = 8y - x. \qquad (B)$$

[1 mark]

We integrate both sides of equation (A) with respect to x, and obtain

$$f = \frac{5}{3}x^3 - xy + \phi(y), \qquad (C)$$

where $\phi(y)$ is a function of y. [1 mark]

Differentiating equation (C) with respect to y, we get

$$f_y = -x + \phi'(y).$$

Comparing with equaton (B), we get $\phi'(y) = 8y$, [1 mark]

or $\phi(y) = 4y^2 + K$, where K is a constant independent of x and y. [1 mark]

Therefore $f(x, y) = \frac{5}{3}x^3 - xy + 4y^2 + K$ is a required potential function for the given vector field $\mathbf{F}$. [1 mark]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 6-C.) Consider the vector field $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (6x^2 - y)$ and $Q = (9y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: Note that $\dfrac{\partial P}{\partial y} = -1 = \dfrac{\partial Q}{\partial x}$ on the entire plane which is simply connected. [1/2 + 1/2 marks]
Therefore $\mathbf{F}$ is conservative.

Let $f(x,y)$ be a potential function for $\mathbf{F}$. By definition,

$$f_x = 6x^2 - y, \qquad (A)$$
$$f_y = 9y - x. \qquad (B)$$

[1 mark]

We integrate both sides of equation (A) with respect to x, and obtain

$$f = 2x^3 - xy + \phi(y), \qquad (C)$$

where $\phi(y)$ is a function of y. [1 mark]

Differentiating equation (C) with respect to y, we get

$$f_y = -x + \phi'(y).$$

Comparing with equaton (B), we get $\phi'(y) = 9y$, [1 mark]

or $\phi(y) = \frac{9}{2}y^2 + K$, where K is a constant independent of x and y. [1 mark]

Therefore $f(x,y) = 2x^3 - xy + \frac{9}{2}y^2 + K$ is a required potential function for the given vector field $\mathbf{F}$. [1 mark]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 6-D.) Consider the vector field $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (10x^2 - y)$ and $Q = (3y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: Note that $\dfrac{\partial P}{\partial y} = -1 = \dfrac{\partial Q}{\partial x}$ on the entire plane which is simply connected. [1/2 + 1/2 marks]
Therefore $\mathbf{F}$ is conservative.

Let $f(x,y)$ be a potential function for $\mathbf{F}$. By definition,

$$f_x = 10x^2 - y, \qquad (A)$$
$$f_y = 3y - x. \qquad (B)$$

[1 mark]

We integrate both sides of equation (A) with respect to x, and obtain

$$f = \frac{10}{3}x^3 - xy + \phi(y), \qquad (C)$$

where $\phi(y)$ is a function of y. [1 mark]

Differentiating equation (C) with respect to y, we get

$$f_y = -x + \phi'(y).$$

Comparing with equaton (B), we get $\phi'(y) = 3y$, [1 mark]

or $\phi(y) = \frac{3}{2}y^2 + K$, where K is a constant independent of x and y. [1 mark]

Therefore $f(x,y) = \frac{10}{3}x^3 - xy + \frac{3}{2}y^2 + K$ is a required potential function for the given vector field $\mathbf{F}$. [1 mark]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 6-A.) Consider the vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (3x^2 - y)$ and $Q = (10y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: Note that $\dfrac{\partial P}{\partial y} = -1 = \dfrac{\partial Q}{\partial x}$ on the entire plane which is simply connected. [1/2 + 1/2 marks]
Therefore $\mathbf{F}$ is conservative.

Let $f(x, y)$ be a potential function for $\mathbf{F}$. By definition,

$$f_x = 3x^2 - y, \qquad f_y = 10y - x.$$

[1 mark]

From the above two equations we get

$$f(x, y) = x^3 - xy + g(y), \qquad f(x, y) = 5y^2 - xy + h(x)$$

[1 + 1 marks]

This further implies that

$$x^3 - h(x) = 5y^2 - g(y) = c \quad \text{and hence} \quad h(x) = x^3 + c, \qquad g(y) = 5y^2 + c.$$

[1 mark]

Hence

$$f(x, y) = x^3 + 5y^2 - xy + c.$$

[1 mark]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 6-A.) Consider the vector field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ on the entire plane, where $P = (3x^2 - y)$ and $Q = (10y - x)$. Prove that $\mathbf{F}$ is conservative and find a scalar potential function for $\mathbf{F}$.

[6 marks]

Answer: We fix a point $P_0 = (0, 0)$ and let $P_1 = (x, y)$ be any point in the plane.

Note that curl$(\mathbf{F}) = 0$ and $\mathbf{F}$ is defined on the whole plane (which is simply connected) so for any two paths C_1 and C_2 joining P_0 to P_1 we get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

[1 + 1 marks]

We define

$$f(P_1) = \int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{r}$$

[1 mark]

Then

$$f(P_1) = \int_{P_0}^{P_1} Pdx + Qdy$$

[1 mark]

$$= \int_{P_0}^{P_1} (3x^2 - y)dx + (10y - x)dy = x^3 + 5y^2 - xy.$$

[1 + 1 marks]

[Note: Since the question asks for *a* potential function, marks should not be deducted for a final answer without the constant K. If a potential function f is computed and then it is said that "$\nabla f = \mathbf{F}$ and hence $\mathbf{F}$ is conservative" then full 6 marks are to be awarded.]

(Q. 7-A.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-B.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2z^2\mathbf{i} + y^2\mathbf{j} + 2x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-C.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3z^2\mathbf{i} + y^2\mathbf{j} + 3x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-D.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 4z^2\mathbf{i} + y^2\mathbf{j} + 4x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = mz^2\mathbf{i} + y^2\mathbf{j} + mx\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

Answer: Note that $\operatorname{curl} \mathbf{F} = m(2z-1)\mathbf{j}$. [1 mark]

The unit normals to the surface S bounded the traingle are given by $\pm\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. The orientation given by the upward normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ is compatible with the counter-clockwise orientation of the boundary curve. [1 mark]

Therefore by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS$$

[1 mark]

If D is the projection of S on the xy-plane, then D is the triangle cut by the positive x and y axes and $y = -x+1$. Since S is given by $z = g(x,y) = 1-x-y$, we know that $dS = \sqrt{g_x^2 + g_y^2 + 1}\, dA = \sqrt{3}dA$, where dA is the area element of the plane D. Therefore, the required integral is equal to

$$\iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\sqrt{3}dA = \iint_D m(2z-1)\mathbf{j} \cdot (\mathbf{i}+\mathbf{j}+\mathbf{k})dA.$$

[2 marks]

$$= \int_0^1 \int_0^{1-x} m(1-2x-2y)dy\, dx$$

[1 mark]

$$= \int_0^1 m(y-2xy-y^2)\big|_0^{1-x}\, dx = \int_0^1 m(x^2-x)dx = -m/6.$$

[1 mark]

(Q. 7-A.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-B.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2z^2\mathbf{i} + y^2\mathbf{j} + 2x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-C.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3z^2\mathbf{i} + y^2\mathbf{j} + 3x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7-D.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 4z^2\mathbf{i} + y^2\mathbf{j} + 4x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

(Q. 7.) Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = mz^2\mathbf{i} + y^2\mathbf{j} + mx\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise orientation, viewed from above. [7 marks]

Answer: Note that $\operatorname{curl} \mathbf{F} = m(2z-1)\mathbf{j}$. [1 mark]

Let $O = (0,0,0)$, $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. We consider the surface S to be the union of three surfaces S_1, S_2 and S_3 bound, respectively, by the triangles OAB, OBC and OCA. The orientation given by the inward normals is compatible with the counter-clockwise orientation of the boundary curve.

The inward unit normal to the surface S_1 is $\mathbf{k}$, that to the surface S_2 is $\mathbf{i}$ and that to the surface S_3 is $\mathbf{j}$. [1/2 + 1/2 + 1/2 marks]

Then by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS = \sum_i \iint_{S_i} \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS.$$

[1 mark]

We observe that the integrals over the surfaces S_1 and S_2 are zero as the curl$(\mathbf{F})$ is perpendicular to the outward normals. [1 mark]

So the only surviving integral is

$$\iint_{S_3} \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dS = \iint_{S_3} m(2z-1)\mathbf{j} \cdot (\mathbf{j}) dS.$$

[1/2 marks]

$$= m \int_0^1 \int_0^{1-x} (2z-1) dz dx$$

[1 mark]

$$= -m/6.$$

[1 mark]

(Q. 8.) Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Given any point $\mathbf{r}$ on the ellipsoid, let $p(\mathbf{r})$ denote the perpendicular distance of the origin from the tangent plane to the ellipsoid at $\mathbf{r}$. Show that

$$\iint_S \frac{1}{p}\, dS = \frac{4}{3}\pi abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right).$$

[6 marks]

Answer: Note that $p = p(\mathbf{r}) = \mathbf{n} \cdot \mathbf{r}$, where $\mathbf{n}$ is the outward unit normal. [1 mark]

Let $\psi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$. Then $\psi \equiv 1$ on the ellipsoid S.

$$\iint_S \frac{1}{p}\, dS = \iint_S \left(\frac{\mathbf{n}}{\mathbf{r} \cdot \mathbf{n}} \right) \cdot \mathbf{n}\, dS.$$

Note that since the surface S is a level surface of ψ, $\mathbf{n} = \nabla\psi / |\nabla\psi|$. [1 mark]

Therefore the above surface integral is given by

$$\iint_S \frac{\nabla\psi}{\mathbf{r} \cdot \nabla\psi} \cdot \mathbf{n}\, dS = \iint_S \frac{\nabla\psi}{2} \cdot \mathbf{n}\, dS.$$

[1 mark]

By the Divergence Theorem, this is equal to

$$\iiint_V \frac{\nabla^2\psi}{2}\, dV$$

[1 mark]

$$= \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \iiint_V dV = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) vol(V).$$

[1 mark]

Now, we know that the volume of the ellipsoid is given by $\frac{4}{3}\pi abc$. Therefore the integral is given by

$$\frac{4}{3}\pi abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right).$$

[1 mark]

(Q. 8.) Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Given any point $\mathbf{r}$ on the ellipsoid, let $p(\mathbf{r})$ denote the perpendicular distance of the origin from the tangent plane to the ellipsoid at $\mathbf{r}$. Show that

$$\iint_S \frac{1}{p}\, dS = \frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

[6 marks]

Answer: Note that $p = p(\mathbf{r}) = \mathbf{n}\cdot\mathbf{r}$, where $\mathbf{n}$ is the outward unit normal. [1 mark]

If we take $\mathbf{F} = \dfrac{x}{a^2}\mathbf{i} + \dfrac{y}{b^2}\mathbf{j} + \dfrac{z}{c^2}\mathbf{k}$ then $\mathbf{F}\cdot\mathbf{n} = \dfrac{1}{p}$. [1 mark]

By Gauss' divergence theorem

$$\iint_S \frac{1}{p} dS = \iiint_W (\nabla\cdot\mathbf{F}) dW$$

where W is the region enclosed by the ellipsoid S. [1 mark]

Now

$$\nabla\cdot\mathbf{F} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

[1 mark]

Hence

$$\iint_S \frac{1}{p} dS = \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\iiint_W dW.$$

[1 mark]

This gives

$$\iint_S \frac{1}{p} dS = \frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

[1 mark]

(Q. 8.) Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Given any point $\mathbf{r}$ on the ellipsoid, let $p(\mathbf{r})$ denote the perpendicular distance of the origin from the tangent plane to the ellipsoid at $\mathbf{r}$. Show that

$$\iint_S \frac{1}{p}\,dS = \frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

[6 marks]

Answer: Note that $p = p(\mathbf{r}) = \dfrac{1}{\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}}}$. [1 mark]

The ellipsoid is given by $\Phi(x,y) = (x, y, \pm c\sqrt{1-(x^2/a^2)-(y^2/b^2)})$ defined on a subset D of $\mathbb{R}^2$. Then

$$\Phi_x \times \Phi_y = \pm\left(\frac{cx/a^2}{\sqrt{1-(x^2/a^2)-(y^2/b^2)}}, \frac{cy/b^2}{\sqrt{1-(x^2/a^2)-(y^2/b^2)}}, 1\right)$$

[1 mark]

and

$$\|\Phi_x \times \Phi_y\| = \sqrt{\frac{\frac{c^2x^2}{a^4}+\frac{c^2y^2}{b^4}+\frac{z^2}{c^2}}{\frac{z^2}{c^2}}}.$$

[1 mark]

Hence

$$\iint_S \frac{1}{p}dS = \iint_D \frac{c^2}{z}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)dxdy.$$

We now use the following change of co-ordinates to compute the integral:

$$x = a\sin\phi\cos\theta, y = b\sin\phi\sin\theta, z = c\cos\phi.$$

Then

$$\iint_S \frac{1}{p}dS = \int_0^{2\pi}\int_0^{\pi}\frac{c}{\cos\phi}\left(\frac{\sin^2\phi\cos^2\theta}{a^2}+\frac{\sin^2\phi\sin^2\theta}{b^2}+\frac{\cos^2\phi}{c^2}\right)|J|d\phi d\theta.$$

[1 mark]

Here $|J(\theta,\phi)| = ab\sin\phi\cos\phi$ [1 mark]

and hence the required integral is

$$\int_0^{2\pi}\int_0^{\pi}\frac{bc}{a}(\sin^3\phi\cos^2\theta)d\phi d\theta + \int_0^{2\pi}\int_0^{\pi}\frac{ac}{b}(\sin^3\phi\sin^2\theta)d\phi d\theta + \int_0^{2\pi}\int_0^{\pi}\frac{ab}{c}(\cos^2\phi\sin\phi)d\phi d\theta$$

$$= \frac{4\pi}{3}\frac{bc}{a}+\frac{4\pi}{3}\frac{ac}{b}+\frac{4\pi}{3}\frac{ab}{c} = \frac{4\pi}{3}abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

[1 mark]

(Q. 8.) Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Given any point $\mathbf{r}$ on the ellipsoid, let $p(\mathbf{r})$ denote the perpendicular distance of the origin from the tangent plane to the ellipsoid at $\mathbf{r}$. Show that

$$\iint_S \frac{1}{p}\, dS = \frac{4}{3}\pi abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

[6 marks]

Answer: Let $\psi = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$, $\mathbf{n} = \dfrac{\nabla\psi}{\|\nabla\psi\|}$. Then

$$\frac{1}{p} = \left(\frac{\mathbf{r}}{p^2}\right) \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{n}.$$

[1 mark]

Then by Gauss divergence theorem

$$\iint_S \frac{1}{p} dS = \iiint_W \nabla \cdot \left(\frac{\mathbf{r}}{p^2}\right) dV = 5 \iiint_W \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) dV.$$

[1 + 1 marks]

We now use the following change of co-ordinates to compute this integral:

$$x = ar\sin\theta\cos\phi, \quad y = br\sin\theta\sin\phi, \quad z = cr\cos\theta, \qquad 0 \le r \le 1, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$$

where the Jacobian is given by

$$|J| = abcr^2 \sin\theta dr d\theta d\phi.$$

[1 mark]

Then the required integral is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 5\left(\frac{a^2r^2\sin^2\theta\cos^2\phi}{a^4} + \frac{b^2r^2\sin^2\theta\sin^2\phi}{b^4} + \frac{c^2r^2\cos^2\theta}{c^4}\right) abcr^2\sin\theta dr d\theta d\phi.$$

$$= \int_0^{2\pi}\int_0^{\pi} abc\left(\frac{\sin^3\theta\cos^2\phi}{a^2} + \frac{\sin^3\theta\sin^2\phi}{b^2} + \frac{\cos^2\theta\sin\theta}{c^2}\right) d\theta d\phi$$

$$= \frac{4\pi}{3} abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

[2 marks]