

Tutorial 2 Solutions

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Note that the square brackets in the question are not meant to be anything like the greatest integer function. They are just brackets.

1. (a) This is false. Take $f(x) = x^2$ and $c = 0$, and $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. We have $f(x)g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. So $\lim_{x \rightarrow 0} f(x)g(x) = 1 \neq 0$

- (b) If $g(x)$ is bounded, let $\max |g(x)| = M$. Since $\lim_{x \rightarrow c} f(x) = 0$, we have that

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } |f(x) - 0| < \epsilon \text{ when } 0 < |x - c| < \delta$$

For a particular epsilon ϵ_0 , let δ_0 , be the delta that is such that $|f(x) - 0| < \frac{\epsilon_0}{M}$ when $0 < |x - c| < \delta_0$
Now, whenever $0 < |x - c| < \delta_0$,

$$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times M < \frac{\epsilon_0}{M} \times M = \epsilon_0$$

Hence, the limit of $\lim_{x \rightarrow c} f(x)g(x)$ exists and is equal to 0.

- (c) Let $\lim_{x \rightarrow c} g(x)$ be L . We have that

$$\begin{aligned} &\exists \delta_0 > 0 \text{ such that } |g(x) - L| < |L| \text{ when } 0 < |x - c| < \delta_0 \\ \implies &|g(x)| - |L| < |L| \implies |g(x)| < 2|L| \text{ when } 0 < |x - c| < \delta_0 \end{aligned}$$

For a particular epsilon ϵ , let δ_1 , be the delta that is such that $|f(x) - 0| < \frac{\epsilon}{2|L|}$ when $0 < |x - c| < \delta_1$
Now, $\delta_2 = \min(\delta_0, \delta_1)$ is such that whenever $0 < |x - c| < \delta_2$,

$$|f(x)g(x) - 0| = |f(x)g(x)| \leq |f(x)| \times 2|L| < \frac{\epsilon}{2|L|} \times 2|L| = \epsilon$$

Hence,

$$\forall \epsilon \exists \delta_2 \text{ such that } |f(x)g(x) - 0| < \epsilon \text{ when } 0 < |x - c| < \delta$$

2. Let $\lim_{x \rightarrow \alpha} f(x) = L$

$$|f(\alpha + h) - f(\alpha - h) - 0| = |f(\alpha + h) - L - (f(\alpha - h) - L)| < |f(\alpha + h) - L| + |f(\alpha - h) - L|$$

Since $\lim_{x \rightarrow \alpha} f(x) = L \forall \epsilon, \exists \delta$ such that $|f(x) - L| < \frac{\epsilon}{2}$ when $0 < |x - \alpha| < \delta$ Rewriting $h = x - \alpha$, we get $\forall \epsilon, \exists \delta_0$ such that $|f(\alpha + h) - L| < \frac{\epsilon}{2}$ when $0 < |h| < \delta_0$ Again rewriting with $h = \alpha - x$, we get

$$\forall \epsilon, \exists \delta_1 \text{ such that } |f(\alpha - h) - L| < \frac{\epsilon}{2} \text{ when } 0 < |h| < \delta_1$$

From these two statements, we get that

$$\begin{aligned} &\forall \epsilon, \exists \delta \text{ such that } |f(\alpha + h) - L| + |f(\alpha - h) - L| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2) \\ \implies &\forall \epsilon, \exists \delta \text{ such that } |f(\alpha + h) - f(\alpha - h)| < \epsilon \text{ when } 0 < |h| < \delta = \min(\delta_1, \delta_2) \end{aligned}$$

One could have also argued the same using algebra of limits on functions.

The converse of the result would be that if $\lim_{h \rightarrow 0} f(\alpha + h) - f(\alpha - h) = 0$, then $\lim_{x \rightarrow \alpha} f(x)$ exists. A counterexample to this would be

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Take $\alpha = 0$, $f(0 + h) - f(0 - h) = 0$ for all h , since this is an even function, but $\lim_{x \rightarrow 0} f(x)$ does not exist.

3. (a)

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In the domain $(0, \infty)$, both $\sin(x)$ and $\frac{1}{x}$ are continuous, hence so is their composition. Same for the domain $(-\infty, 0)$. The only problem is at $x = 0$. The function, in fact, is not continuous at $x = 0$. To show this, we'll use the sequential definition of a limit:

The limit of a function f at a exists and is equal to L if and only if for every sequence $x_n \rightarrow a$, $\lim_{n \rightarrow \infty} f(x_n) = L$

Take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+\frac{1}{2})\pi}$. We have that $\sin x_n = 0$ and $\sin y_n = 1$ for all $n \in \mathbf{N}$. Thus, we have two sequences tending to 0, such that $f(x_n)$ and $f(y_n)$ tend to different values, hence the limit of $\sin \frac{1}{x}$ does not exist at $x = 0$

(b)

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here again it is easy to argue continuity in the domains $(0, \infty)$ and $(-\infty, 0)$, using the fact the composition and products of continuous functions is continuous. At $x = 0$, we need to find the limit and show that it is equal to $f(0) = 0$.

$$|x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}| < |x|, \text{ since the range of } \sin \text{ is } [-1, 1]$$

Hence,

$$\forall \epsilon > 0, |x \sin \frac{1}{x} - 0| < |x| < \epsilon \text{ when } 0 < |x - 0| < \epsilon$$

Seeing that we can just set $\delta = \epsilon$ always, we get that $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$. Hence the function is continuous for all reals.

(c) Note that the pieces of the function in $(1, 2)$ and $(2, 3)$ are continuous. So there can only be a discontinuity at $x = 2$. The right hand and left hand limits at $x = 2$ can both be easily calculated to be 2, however, $f(2) = 1$. Hence, the function is not continuous at $x = 2$

4. We are given that $f(x + y) = f(x) + f(y)$. Functions with this property are also known as additive functions.

$$\text{Set } x = y = 0 \text{ to get } f(0) = 2f(0) \implies f(0) = 0$$

$$f(c + h) = f(c) + f(h) \implies f(c + h) - f(c) = f(h) = f(h) - f(0)$$

Since f is continuous at 0

$$\forall \epsilon \exists \delta \text{ such that } |f(h) - 0| < \epsilon \text{ when } 0 < h < \delta$$

$$\implies \forall \epsilon \exists \delta \text{ such that } |f(c + h) - f(c)| = |f(h)| < \epsilon \text{ when } 0 < h < \delta$$

$$\implies f \text{ is continuous for all } c \in \mathbf{R}$$

For more information about this see [this wikipedia entry](#)

5. For $x \neq 0$, $\frac{1}{x}$ is differentiable while $\sin x$ and x are differentiable at all points. Hence, a function formed by their compositions and products would also be differentiable at $x \neq 0$. At $x = 0$, we'll need to work by first principles. We need to see if $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} x \sin \frac{1}{x}$ exists. Deja vu to q3 ii), the limit exists and is equal to 0. Hence f is differentiable on \mathbf{R} . However, the derivative is not continuous. Using chain rule for $x \neq 0$, we get that

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} + \cos \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

We can show that $\lim_{x \rightarrow 0} f'(x)$ does not exist, using a method similar to q3 i)

6. We have that $\forall x, x + h \in (a, b)$

$$\frac{|f(x + h) - f(x)|}{|h|} < C|h|^{\alpha-1}$$

$$\implies -C|h|^{\alpha-1} < \frac{f(x+h) - f(x)}{h} < C|h|^{\alpha-1}$$

Since $\alpha - 1 > 0$, $\lim_{h \rightarrow 0} C|h|^{\alpha-1} = \lim_{h \rightarrow 0} -C|h|^{\alpha-1} = 0$

Hence, by sandwich theorem we can conclude that $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exists and equals 0. By definition, this means that f is differentiable over (a, b) and $f'(x) = 0 \forall x \in (a, b)$

7.

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h \rightarrow 0^+} \frac{\frac{f(c+h) - f(c)}{h} + \frac{f(c) - f(c-h)}{h}}{2} \\ &= \frac{\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} + \lim_{h \rightarrow 0^+} \frac{f(c) - f(c-h)}{h}}{2} \end{aligned}$$

Since f is differentiable, $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} = \text{Right hand derivative} = \text{Left hand derivative} = \lim_{h \rightarrow 0^+} \frac{f(c) - f(c-h)}{h} = f'(c)$

Hence, we are done.

The converse is false, consider $f(x) = |x|$ and $c = 0$. Then $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c-h)}{2h} = \lim_{h \rightarrow 0^+} \frac{h-h}{2} = 0$, but the function is not differentiable at 0.

8. First, note that if $\exists c$ such that $f(c) = 0$, then $f(x) = f(x - c + c) = f(x - c)f(c) = 0 \forall x \in \mathbf{R}$. In this case f is obviously differentiable, with $f'(c) = 0 \forall c = f'(0)f(c)$ (trivially).

The only other possibility is that f is always non-zero. Set $x = y = 0$ to get

$$f(0) = f(0)^2 \implies f(0) = 0 \text{ or } 1$$

. If f is not identically 0 then $f(0) = 1$.

$$\frac{f(c+h) - f(c)}{h} = \frac{f(c)f(h) - f(c)}{h} = f(c) \times \frac{f(h) - 1}{h}$$

Since $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - 1}{h}$ exists, we have that $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0} f(c) \times \frac{f(h) - 1}{h}$ exists and is equal to $f(c) \times \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f(c)f'(0)$. Thus, by definition, f is differentiable at every $c \in \mathbf{R}$ and $f'(c) = f(c)f'(0)$

9. Calculating derivative of inverse of a function. Let $f : (a, b) \rightarrow \mathbf{R}$ be a differentiable and invertible function. Let the inverse be denoted by $g(x)$

$$f(g(x)) = x$$

Differentiating $f(g(x))$ using the chain rule, we get that

$$f'(g(x)) \times g'(x) = \frac{d}{dx} x = 1 \implies g'(x) = \frac{1}{f'(g(x))}$$

Using this both the parts can be solved, solving is left to the reader.

10. Apply chain rule and simplify. Left to the reader.