MA-109 Information Booklet and Problem Sheets Autumn 2022

INSTRUCTORS:

Madhusudan Manjunath

Sanjoy Pusti

Department of Mathematics

Indian Institute of Technology Bombay

Powai, Mumbai 400076, India

Name:

Roll Number:

Division and Tut Batch:

Contents

Basic Information

Course contents

Sequences of real numbers, Convergence of sequences and series, Review of limit, Continuity and differentiability of functions, Rolle's theorem, Mean value theorems and Taylor's theorem, Maxima, minima and curve sketching, Riemann integral, Fundamental theorem of calculus, Applications to length, area, volume, surface area of revolution.

Functions of several variables, Limit, Continuity and and partial derivatives, Chain rule, Gradient, Directional Derivative and differentiability, Tangent planes and normals, Maxima, minima, saddle points, Lagrange multipliers.

Text/References

[TF] G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley/Narosa, 1998.

[S] James Stewart, *Calculus: Early Transcendentals*, 5th Ed., Thompson Press, 2003 (Second Indian Reprint, 2007).

[A] T. M. Apostol, *Calculus, Volume-I,II*, Wiley Eastern, 1980.

[CJ] Richard Courant and Fritz John, *Introduction to calculus and analysis.* Vol. I. Reprint of the 1989 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1999. xxiv+661 pp.

Course Plan

Sr. No.	Topic	Sections from Text [TF]	No. of Lectures
1.	Real Numbers, Functions	P.1, P.3	1
2.	Sequences	8.1, 8.2	2
3.	Limits and Continuity	1.1-1.5	2
4.	Differentiation	2.1, 2.2, 2.5, 2.6	1
5.	Rolle's, Mean Value, and Taylor's Theorems	3.1, 3.2	2
6.	Maxima/Minima and Curve Sketching	3.1–3.3, 3.5	1
7.	Riemann Integral and the Funda. Theorem of Calculus	4.5-4.8	3
8.	Natural Logarithm and the Exponential Function	6.2-6.4, 6.7	1
9.	Applications of Integrals	5.1-5.6	2
10.	Functions of Several Variables: Limits, Continuity	12.1, 12.2	1
11.	Partial and Total Differentiation	12.3-12.7	2
12.	Maxima, Minima	12.8, 12.9	1

Lectures and Tutorials

Every week we have either three lectures of about one hour duration or two lectures of about one and half hour duration. In addition, there will be a tutorial of one hour duration. The course will be fast paced. Thus it is important that you remain attentive in the class and do not miss lectures.

Consult the text book (and if you wish, the reference books) regularly. Also, be sure to consult the Moodle (`http://moodle.iitb.ac.in/`) page of the course regularly. Slides used to teach in the class will be uploaded there, along with other important resources.

For the purpose of tutorials, each division will be divided into 9 batches. Each batch will be assigned to a "course associate" or a TA. The aim of the tutorials is to clear your doubts and to give you practice for problem solving. Based on the material covered, certain problems from the tutorial sheets in this booklet will be assigned to you each week. You are expected to try the problem before coming to the tutorial class. In case you have doubts, please seek the help of your course associate.

Evaluation Plan

There will be **one quiz** and one **final examination** common for all the four divisions. The quiz will carry 40% weightage and the final examination will carry 60% weightage. The quiz will be held on 23rd November from 8:15-9:15 AM. The end semester will be held some time between 14-20th December 2022.

Instructors and their coordinates

Instructor for Division I and II: Prof. Madhusudan Manjunath, madhu@math.iitb.ac.in
Room No. 204-A, Department of Mathematics.

Instructor for Division III and IV: Prof. Sanjoy Pusti (Instructor-in-charge), spusti@gmail.com
Room No. G3-B. Department of Mathematics.

Timings for Lectures and Tutorials

Lectures of Division I and Division III are scheduled on Mondays 8:30-9:30 AM, Tuesday 9:30-10:30 AM and Thursday 10:30-11:30 AM.

Lecture of Divisions II and IV are scheduled on Monday and Thursday from 2-3:30 PM.

Tutorials of Divisions I,II, III, IV will be held on Wednesdays some time between 2 pm 6 pm. Exact timing and the name of the tutor of each section will be informed via moodle.
Refer the notice boards and Moodle (`http://moodle.iitb.ac.in/`) page of the course for any changes or modifications.

Tutorial Sheets: 0-7

Tutorial sheet No. 0: Revision material on Real numbers

Mark the following statements as True/False:

(1) $+\infty$ and $-\infty$ are both real numbers.

(2) The set of all even natural numbers is bounded.

(3) The set $\{x\}$ is an open interval for every $x \in \mathbb{R}$.

(4) The set $\{2/m | m \in \mathbb{N}\}$ is bounded above.

(5) The set $\{2/m | m \in \mathbb{N}\}$ is bounded below.

(6) Union of intervals is also an interval.

(7) Nonempty intersection of intervals is also an interval.

(8) Nonempty intersection of open intervals is also an open interval.

(9) Nonempty intersection of closed intervals is also a closed interval.

(10) Nonempty finite intersection of closed intervals is also a closed interval.

(11) For every $x \in \mathbb{R}$, there exists a rational $r \in \mathbb{Q}$, such that $r > x$.

(12) Between any two rational numbers there lies an irrational number.

Tutorial Sheet No.1:
Sequences

1. Using (ϵ-n_0) definition prove the following:
   (i) $\lim_{n\to\infty} \frac{10}{n} = 0$
   (ii) $\lim_{n\to\infty} \frac{5}{3n+1} = 0$
   (iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$
   (iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$
2. Show that the following limits exist and find them :
   (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$
   (ii) $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)$
   (iii) $\lim_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right)$
   (iv) $\lim_{n\to\infty} (n)^{1/n}$
   (v) $\lim_{n\to\infty} \left(\frac{\cos \pi\sqrt{n}}{n^2}\right)$
   (vi) $\lim_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$
3. Show that the following sequences are not convergent :
   (i) $\{\frac{n^2}{n+1}\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n\geq 1}$
4. Determine whether the sequences are increasing or decreasing :
   (i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$
   (ii) $\left\{\frac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$
   (iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}$
5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits :
   (i) $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall\, n \geq 1$
   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}\ \forall\ n \geq 1$
   (iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2}\ \forall\, n \geq 1$
6. If $\lim_{n\to\infty} a_n = L$, find the following : $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \qquad \text{for all } n \geq n_0.$$

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.
   Optional: State and prove a corresponding result if $a_n \to L > 0$.

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following :
   (i) $\{a_nb_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   (ii) $\{a_nb_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Supplement

1. A sequence $\{a_n\}_{n\geq 1}$ is said to be **Cauchy** if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \geq n_0$.
   In other words, the elements of a Cauchy sequence come arbitrarily close to each other after some stage. One can show that *every convergent sequence is also Cauchy and conversely, every Cauchy sequence in* $\mathbb{R}$ *is also convergent.* This is an equivalent way of stating the **Completeness property of real numbers**.)

2. To prove that a sequence $\{a_n\}_{n\geq 1}$ is convergent to L, one needs to find a real number L (not given by the sequences) and verify the required property. However the concept of 'Cauchyness' of a sequence is purely an 'intrinsic' property of the given sequence. Nonetheless a sequence of real numbers is Cauchy if and only if it is convergent.

3. In problem 5(i) we defined

$$a_0 = 1, \;\; a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \;\; \forall\, n \geq 1.$$

   The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find rational approximation (in computing machines) of $\sqrt{2}$.

Tutorial Sheet No. 2: Limits, Continuity and Differentiability

1. Let $a, b, c \in \mathbb{R}$ with $a < c < b$ and let $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements.
   (i) $\lim_{x\to c}[f(x)g(x)] = 0$.
   (ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.
   (iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that
$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$
   Analyze the converse.
3. Discuss the continuity of the following functions :
   (i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if} \quad 1 \leq x < 2 \\ 1 & \text{if} \quad x = 2 \\ \sqrt{6-x} & \text{if} \quad 2 \leq x \leq 3 \end{cases}$
4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.
   (Optional) Show that the function f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.
5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?
6. Let $f : (a, b) \to \mathbb{R}$ be a function such that
$$|f(x + h) - f(x)| \leq C|h|^\alpha$$
   for all $x, x + h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.
7. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that
$$\lim_{h\to 0^+} \frac{f(c + h) - f(c - h)}{2h}$$
   exists and equals $f^{'}(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]
8. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy
$$f(x + y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.
(Optional) Show that f has a derivative of every order on $\mathbb{R}$.

9. Using the Theorem on derivative of inverse function. Compute the derivative of
(i) $\cos^{-1} x, \ -1 < x < 1$. (ii) $\mathrm{cosec}^{-1} x, \ |x| > 1$.

10. Compute $\dfrac{dy}{dx}$, given
$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f^{'}(x) = \sin(x^2).$$

Optional Exercises:

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

12. Let $f(x) = \begin{cases} 1, & \text{if } \ x \text{ is rational,} \\ 0, & \text{if } \ x \text{ is irrational.} \end{cases}$
Show that f is discontinuous at every $c \in \mathbb{R}$.

13. **(Optional)**
Let $g(x) = \begin{cases} x, & \text{if } \ x \text{ is rational,} \\ 1-x, & \text{if } \ x \text{ is irrational.} \end{cases}$
Show that g is continuous only at $c = 1/2$.

14. **(Optional)**
Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ be such that $\lim\limits_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c + h) > \alpha \ \text{ for all } \ 0 < |h| < \delta.$$
(See also question 7 of Tutorial Sheet 1.

15. **(Optional)** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent :
(i) f is differentiable at c.
(ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$
(iii) There exists $\alpha \in \mathbb{R}$ such that
$$\lim_{h \to 0} \left(\frac{|f(c + h) - f(c) - \alpha h|}{|h|} \right) = 0.$$

Tutorial Sheet No. 3:
Rolle's and Mean Value Theorems, Maximum/Minimum

1. Show that the cubic $x^3 - 6x + 3$ has all roots real.
2. Let p and q be two real numbers with $p > 0$. Show that the cubic x^3+px+q has exactly one real root.
3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.
4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:
   (i) $p < 0$.
   (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.
   (iii) The maximum/minimum values are of opposite signs.
5. Use the MVT to prove $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f^{'}(c_1) + f^{'}(c_2) = 2$.
7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.
   Optional: Show that under the given conditions, in fact $f(x) = x$ for every x.
8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.
   (i) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 1$
   (ii) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 2$
   (iii) $f^{''}(x) \geq 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 100$ for all $x > 0$
   (iv) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 1$ for all $x < 0$
9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.
10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

Tutorial Sheet No. 4: Curve Sketching, Riemann Integration

1. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?
   (i) $y = 2x^3 + 2x^2 - 2x - 1$
   (ii) $y = \dfrac{x^2}{x^2+1}$
   (iii) $y = 1 + 12|x| - 3x^2,\ x \in [-2, 5]$

2. Sketch a continuous curve $y = f(x)$ having all the following properties:
   $f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ \ f^{'}(2) = f^{'}(-2) = 0;$
   $f^{'}(x) > 0$ for $|x| > 2,\ f^{'}(x) < 0$ for $|x| < 2;$
   $f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0.$

3. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is
   (i) strictly increasing and convex.
   (ii) strictly increasing and concave.
   (iii) strictly decreasing and convex.
   (iv) strictly decreasing and concave.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?
   (i) If f and g have a local maximum at $x = c$, then so does h.
   (ii) If f and g have a point of inflection at $x = c$, then so does h.

5. Let $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 2$ if $x \in (1, 2]$. Show from the first principles that f is Riemann integrable on $[0, 2]$ and find $\displaystyle\int_0^2 f(x)dx$.

6. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.
   (b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

7. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:
   (i) $S_n = \dfrac{1}{n^{5/2}} \displaystyle\sum_{i=1}^{n} i^{3/2}$

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

(iii) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

(iv) $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{i\pi}{n}$

(v) $S_n = \frac{1}{n} \left\{ \sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^2 \right\}$

8. Compute

(a) $\frac{d^2y}{dx^2}$, if $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \int_0^{x^2} \cos(t)dt$.

9. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t)dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Tutorial Sheet No. 5: Applications of Integration

(1) Find the area of the region bounded by the given curves in each of the following cases.
(i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$
(ii) $y = x^4 - 2x^2$ and $y = 2x^2$.
(iii) $x = 3y - y^2$ and $x + y = 3$

(2) Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5

(3) Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1 + \cos\theta)$.

(4) Find the arc length of the each of the curves described below.
(i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$
(ii) $y = \displaystyle\int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4$.

(5) For the following curve, find the arc length as well as the the area of the surface generated by revolving it about the line $y = -1$.

$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$$

(6) The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

(8) A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

(9) Find the volume of the solid generated when the region bounded by the curves $y = 3 - x^2$ and $y = -1$ is revolved about the line $y = -1$, by both the Washer Method and the Shell Method.

(10) A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Tutorial Sheet No. 6:
Functions of two variables, Limits, Continuity

(1) Find the natural domains of the following functions of two variables:

$$(i)\ \frac{xy}{x^2-y^2} \qquad (ii)\ \ln(x^2+y^2)$$

(2) Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x,y) = x - y$ (ii) $f(x,y) = x^2 + y^2$ (iii) $f(x,y) = xy$

(3) Using definition, examine the following functions for continuity at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \frac{x^3y}{x^6+y^2} \quad (ii)\ xy\frac{x^2-y^2}{x^2+y^2} \quad (iii)\ ||x|-|y||-|x|-|y|.$$

(4) Suppose $f,\ g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x,y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\ g(y)\}$
(iv) $\min\{f(x),\ g(y)\}$.

(5) Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2} \ \text{ for } (x,y) \neq (0,0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \text{ and } \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

exist and both are equal to 0, but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(6) Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.
(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$
(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

(7) Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \ \text{ for } \ (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

(8) Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y\sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

(9) Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \ xy\frac{x^2 - y^2}{x^2 + y^2} \qquad (ii)\ \ \frac{x^3}{x^2 + y^2} \qquad (iii)\ \ (x^2 + y^2)\sin\frac{1}{x^2 + y^2}$$

(10) Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2 + y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Tutorial Sheet No. 7:
Maxima, Minima, Saddle Points

(1) Let $F(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $F(x,y,z) = 7$ at $(1,-1,3)$.

(2) Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x-5y+2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2+y^2+z^2 = 9$ at $(2,2,1)$.

(3) Given $\sin(x+y) + \sin(y+z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

(4) If $f(0,0) = 0$ and

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2} \text{ for } (x,y) \neq (0,0),$$

show that both f_{xy} and f_{yx} exist at $(0,0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0,0)$?

(5) Show that the following functions have local minima at the indicated points.

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0,y_0) = (-1,2)$

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0,y_0) = (0,0)$

(6) Analyze the following functions for local maxima, local minima and saddle points :

(i) $f(x,y) = (x^2-y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x,y) = x^3 - 3xy^2$

(7) Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2-4x)\cos y \text{ for } 1 \le x \le 3,\ -\pi/4 \le y \le \pi/4.$$

(8) The temperature at a point (x,y,z) in 3-space is given by $T(x,y,z) = 400xyz$. Find the highest temperature on the unit sphere $x^2+y^2+z^2 = 1$.

(9) Maximize the $f(x,y,z) = xyz$ subject to the constraints

$$x+y+z = 40 \text{ and } x+y = z.$$

(10) Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraints

$$x+2y+3z = 6 \text{ and } x+3y+4z = 9.$$

Answers: Tutorial Sheets 0-7

Tutorial sheet No. 0:

(1) False

(2) False

(3) False

(4) True

(5) True.

(6) False

(7) True

(8) False

(9) True

(10) True.

(11) True

(12) True

Tutorial sheet No. 1

(1) For a given $\epsilon > 0$, select $n_0 \in \mathbb{N}$ satisfying:

$$\text{(i) } n_0 > \frac{10}{\epsilon}, \quad \text{(ii) } n_0 > \frac{5-\epsilon}{3\epsilon}, \quad \text{(iii) } n_0 > \frac{1}{\epsilon^3}\,, \quad \text{(iv) } n_0 > \frac{2}{\epsilon}.$$

(2) The limits of the sequences are as follows:

(i) 1, (ii) 0, (iii) 0, (iv) 1, (v) 0, (vi) 1/2.

(3) (i) Not convergent, (ii) Not convergent.

(4) (i) Decreasing, (ii) Increasing, (iii) Increasing.

(5) Hint: In each case, use induction on n to show that $\{a_n\}$ is bounded and monotonic. The limits are:
(i) $\sqrt{2}$, (ii) 2, (iii) 6.

(7) Hint: Consider $\epsilon = L/2$.

(9) Both the statements are **False**.

Tutorial sheet No. 2

(1) (i) False, (ii) True, (iii) True.

(2) Yes. The converse is **False**.

(3) (i) Not continuous at $\alpha = 0$
(ii) Continuous
(iii) Continuous everywhere except at $x = 2$

(5) Continuous for $x \neq 0$, not continuous at $x = 0$.

(7) The converse is **False** .

(9) (i) $\dfrac{-1}{\sqrt{1-\cos^2(x)}} = \dfrac{-1}{\sqrt{1-y^2}}$, (ii) $\dfrac{1}{\sqrt{(1-\frac{1}{x^2})}}\left(\dfrac{-1}{x^2}\right)$, $|x| > 1$.

(10) $\dfrac{3}{(x+1)^2}\sin\left(\dfrac{2x-1}{x+1}\right)^2$.

Tutorial sheet No. 3

(8) (i) Not possible
(ii) Possible
(iii) Not possible
(iv) Possible

(9) The absolute maximum is 13 which is attained at $x = \pm 2$ and the absolute minimum is -14 which is attained at $x = 5$.

(10) $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$.

Tutorial sheet No. 4

(4) (i) **True** .
(ii) **False**; consider $f(x) = g(x) = 1 + \sin(x), c = 0$.

(7) (i) $\dfrac{2}{5}$, (ii) $\dfrac{\pi}{4}$, (iii) $2(\sqrt{2}-1)$, (iv) 0, (v) $\dfrac{1}{2} + \dfrac{2}{5}(4\sqrt{2}-1) + \dfrac{19}{3}$.

(8) (a) $\dfrac{y}{\sqrt{1+y^2}}\dfrac{dy}{dx} = y$
(b) (i) $F'(x) = 2\cos(4x^2)$, (ii) $F'(x) = 2x\cos(x^2)$.

Tutorial Sheet No. 5

(1) (i) $\frac{1}{6}$, (ii) $\frac{128}{15}$, (iii) $\frac{4}{3}$.

(2) $a = -2$

(3) $4\pi a^2$

(4) (i) 8, (ii) 1.

(5) $\frac{dy}{dx} = x^2 + \left(-\frac{1}{4x^2}\right)$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}$$

Therefore $L = \frac{53}{6}$. The surface area is $\left(101\frac{5}{18}\right)\pi$.

(6) $\frac{512\pi}{15}$

(7) Volume is $\frac{16a^3}{3}$

(8) Volume is r^2h

(9) $2^9\pi/15$

(10) $\frac{28\pi}{3}$.

Tutorial Sheet No. 6

(1) (i) $\{(x, y) \in \mathbb{R}^2 \mid x \neq \pm y\}$.
(ii) $\mathbb{R}^2 - \{(0, 0)\}$

(2) (i) Level curves are parallel lines $x - y = c$. Contours are the same lines shifted to $z = c$. (same c)

(ii) Level curves do not exist for $c \leq -1$. It is just a point for $c = 0$ and are concentric circles for $c = 1, 2, 3, 4$. Contours are the sections of paraboloid of revolution $z = x^2 + y^2$ by $z = c$, i.e., concentric circles in the plane $z = c$.

(iii) Level curves are rectangular hyperbolas. Branches are in first and third quadrant for for $c > 0$ and in second and fourth quadrant for $c < 0$. For $c = 0$ it is the union of x-axis and y-axis.

(3) (i) Discontinuous at $(0, 0)$
(ii) Continuous at $(0, 0)$
(iii) Continuous at $(0, 0)$

(6) (i) $f_x(0, 0) = 0 = f_y(0, 0)$.
(ii) f is continuous at $(0, 0)$. Both $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

(9) (i) $(D_{\vec{v}}f)(0,0)$ exists and equals 0 for every $\vec{v} \in \mathbb{R}^2$;
f is also differentiable at $(0,0)$.
(ii) It is not differentiable, but for every vector $\vec{v} = (a,b)$, $D_{\vec{v}}f(0,0)$ exists.
(iii) $(D_{\vec{v}}f)(0,0) = 0$; f is differentiable at $(0,0)$.

Tutotial Sheet No. 7

(1) Tangent plane:

$$0.(x-1) + 4(y+1) + 6(z-3) = 0, \text{ i.e., } 2y + 3z = 7$$

Normal line: $x = 1,\ 3y - 2z + 9 = 0$

(2) $-\dfrac{2}{3}$

(3) $\dfrac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\dfrac{\cos^2(x+y)}{\cos^2(y+z)}$.

(6) (i) $(0,0)$ is a saddle point;
$(\pm\sqrt{2},0)$ are local maxima;
$((0,\pm\sqrt{2})$ are local minima.
(ii) $(0,0)$ is a saddle point.

(7) $f_{\min} = -4$ at $(2,0)$ and $f_{\max} = -3/\sqrt{2}$ at $(3,\pm\pi/4)$

(8) $T_{\max} = \dfrac{400}{3\sqrt{3}}$ at

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),$$
$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

(9) $F(10,10,20) = 2000$ is the maximum value.

(10) At $(x,y,z) = (-1,2,1)$, $f(-1,2,1) = 6$ is the minimum value.
