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# Indian Institute of Technology Bombay

## MA 105 Calculus

Autumn 2019 SRG/MM/MM

**Solution and Marking Scheme for Quiz 2**

Date: October 25, 2019 Weightage: 10 %

Time: 8.35 AM – 9.15 AM Max. Marks: 30

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Note for the Graders: *Give a penalty of 2 marks to those who did not follow the instructions and turned in their papers on an answerbook instead of the question paper cum answer book. There are no partial marks for the first four objective questions irrespective of the explanations given. Encircling by pencils should be circled again with a red pen. Failure to write roll number and division/tutorial batch should result in a penalty of 2 marks.*

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) := 0$ and $f(x,y) := (x^4y + x^3y^2)/(x^6 + y^2)$ for $(x,y) \neq (0,0)$. Then which of the following statements are true?

   A. $f$ is continuous at $(0,0)$, but $D_{\mathbf{u}}f(0,0)$ does not exist for some unit vector $\mathbf{u}$.

   B. $f$ is continuous at $(0,0)$ and $D_{\mathbf{u}}f(0,0)$ exists for every unit vector $\mathbf{u}$.

   C. $f$ is not continuous at $(0,0)$, but $D_{\mathbf{u}}f(0,0)$ exists for every unit vector $\mathbf{u}$.

   D. $f$ is not continuous at $(0,0)$ and $D_{\mathbf{u}}f(0,0)$ does not exist for some unit vector $\mathbf{u}$.

   [4 marks]

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := x^2y$. Then which of the following statements are true?

   A. $f$ has a local minimum at $(0,1)$.

   B. $f$ has a local maximum at $(0,1)$.

   C. $f$ has a saddle point at $(0,1)$.

   D. None of the above. [4 marks]

3. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^{-x} \sin y$. Let $\mathbf{u}$ be the unit vector such that $f$ has steepest ascent at $(0, 0)$ in the direction of $\mathbf{u}$. Also let $\mathbf{n}$ be a unit normal vector to the surface $z = f(x, y)$ at $(0, 0, 0)$. Then $\mathbf{u}$ and $\mathbf{n}$ are given by

$$\mathbf{u} = (0, 1) \qquad \text{and} \qquad \mathbf{n} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

(Note $\mathbf{n} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is also acceptable,] [4 = 2 + 2 marks]

4. Let $\mathbf{F}$ be the vector field in $\mathbb{R}^2$ defined by $\mathbf{F}(x, y) := (y, x)$, and let $C$ be the parametrized curve $x = 1 + y^5(1 - y)$, $0 \le y \le 1$. Then the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 1.$$

[4 marks]

5. Find the absolute minimum and absolute maximum values of $f(x, y) = (x^2 - 3x) \cos y$ over the region $1 \le x \le 3$, $\frac{-\pi}{4} \le y \le \frac{\pi}{4}$. [7 marks]

**Answer:** Since $f$ is continuous and the given region is a closed and bounded rectangle, $f$ does have an absolute minimum and absolute maximum. Moreover these are attained either at a critical point or at a boundary point. [1]

We calculate that

$$f_x = (2x - 3) \cos y \quad \text{and} \quad f_y = (x^2 - 3x)(-\sin y). \qquad [1]$$

Hence the only critical point of $f$ in the given region is the interior point $(3/2, 0)$, the value of $f$ is $-9/4$. [1]

The boundary consists of the four sides of the rectangle $[1, 3] \times [\frac{-\pi}{4}, \frac{\pi}{4}]$.

Along $x = 1$, $f(1, y) = -2 \cos y$, $y \in [\frac{-\pi}{4}, \frac{\pi}{4}]$. The max and min here are respectively $-\sqrt{2}$ and $-2$.. [1]

Along $x = 3$, $f(3, y) = 0$, so the max and min are both 0. [1]

Along $y = \pm\frac{\pi}{4}$, $f(x, \pm\pi/4)) = x^2 - 3x$, $x \in [1, 3]$. So we only need to calculate $f$ at $(\frac{3}{2}, \pm\frac{\pi}{4})$, $(1, \pm\frac{\pi}{4})$ and $(3, \pm\frac{\pi}{4})$. The values there are respectively $-\frac{9}{4\sqrt{2}}$, $-\sqrt{2}$ and 0. [1]

By comparing all these values, we conclude that the absolute maximum of $f$ is 0 and the absolute minimum is $-\frac{9}{4}$. [1]

6. Evaluate the following double integral:

$$I = \int_0^{\pi/2} \int_y^{\pi/2} x^4 \sin(x^2 y) dx dy.$$

[7 marks]

**Answer:** We start by noting that the domain of integration is the triangle given by

$$T = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \pi/2,\ y \leq x \leq \pi/2\}. \qquad [2]$$

(A pictorial description of $T$ is also acceptable.)

Note that $T$ is an elementary region and the integrand is continuous. [1]

Hence, we can use the Fubini's theorem for elementary regions to interchange the order of integration and obtain the following double integral:

$$I = \int_0^{\pi/2} \int_0^{x} x^4 \sin(x^2 y) dy dx. \qquad [2]$$

This can be computed as follows:

$$I = \int_0^{\pi/2} x^2(1 - \cos(x^3)) dx = \frac{1}{3}\left[x^3 - \sin x^3\right]_0^{\pi/2} = \frac{\pi^3}{24} - \frac{1}{3}\sin(\pi^3/8). \qquad [2]$$