Do there exist other pairs of Conjugate Variables?

Ashwin Abraham

22nd December, 2021

Conjugate variables are those variables which cannot both be known precisely at the same time. That is to say, they follow reciprocity relations similar to the well known Heisenberg-Uncertainty principle, which is the reciprocity relation between the position and the momentum of an object. It is known that any observable quantity of a particle can be found from its wave function, by composing it with an operator. For example the momentum operator is given by:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x} \tag{1}$$

Note that, in general, for operators $\hat{A}$ and $\hat{B}$, multiplication is not commutative ($\hat{A}\hat{B} \neq \hat{B}\hat{A}$). In fact, it is the case that, if two operators commute, then their corresponding observables can both be measured to any desired level of precision simultaneously. However, if two operators, do not commute, there exist reciprocity relations, that constrain the accuracy with which both observables can be measured. The commutator of 2 operators $\hat{A}$ and $\hat{B}$ (denoted by $[\hat{A}, \hat{B}]$ is defined as $\hat{A}\hat{B} - \hat{B}\hat{A}$. If this commutator is non-zero, then the two variables would be Conjugate variables.

Some standard (non-zero) commutators are [1]:

$$[\hat{x}, \hat{p}_x] = i\hbar \tag{2}$$

$$[\hat{y}, \hat{p}_y] = i\hbar \tag{3}$$

$$[\hat{z}, \hat{p}_z] = i\hbar \tag{4}$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \tag{5}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \tag{6}$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{7}$$

$$[\hat{L}_z, \hat{x}] = i\hbar\hat{y} \tag{8}$$

$$[\hat{L}_z, \hat{y}] = -i\hbar\hat{x} \tag{9}$$

$$[\hat{L}_z, \hat{p}_x] = i\hbar\hat{p}_y \tag{10}$$

$$[\hat{L}_z, \hat{p}_y] = -i\hbar\hat{p}_x \tag{11}$$

1

[1] Taken from: An Introduction to Quantum Mechanics (Third Edition) - David J. Griffiths, Darryl F. Schroeter

The existence of all these non-zero commutators leads to the existence of reciprocity relations for their respective variables. For example, equation (2) implies the famous Heisenberg Uncertainty Principle (in the x direction).

$$\boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}} \tag{12}$$

From equations (3) and (4) come similar relations in the y and z directions.

$$\Delta y \Delta p_y \geq \frac{\hbar}{2} \tag{13}$$

$$\Delta z \Delta p_z \geq \frac{\hbar}{2} \tag{14}$$

From equations (5), (6) and (7) we see that:

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle| \tag{15}$$

$$\Delta L_y \Delta L_z \geq \frac{\hbar}{2} |\langle L_x \rangle| \tag{16}$$

$$\Delta L_z \Delta L_x \geq \frac{\hbar}{2} |\langle L_y \rangle| \tag{17}$$

It is also known that:

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{18}$$

Hence there are indeed many Conjugate Variables other than the famous pair of Position and Momentum.