Commutators in Quantum Mechanics

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The commutator of two Operators $\hat{A}$ and $\hat{B}$ is defined as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{1}$$

When the commutator acts on a function f, $[\hat{A}, \hat{B}]f = \hat{A}\hat{B}f - \hat{B}\hat{A}f$. Since Operators are, in general not commutative, $[\hat{A}, \hat{B}]$ is not always 0.
For example, the momentum operator is $\hat{p} = -i\hbar\frac{d}{dx}$ and the position operator is given by $\hat{x} = x$. Therefore,

$$[\hat{x}, \hat{p}]f = \hat{x}\hat{p}f - \hat{p}\hat{x}f = x(-i\hbar\frac{df}{dx}) - (-i\hbar\frac{d}{dx}(xf(x))) \tag{2}$$

Which means,

$$[\hat{x}, \hat{p}]f = -i\hbar x\frac{df}{dx} + i\hbar f + i\hbar x\frac{df}{dx} = i\hbar f \tag{3}$$

Therefore,

$$\boxed{[\hat{x}, \hat{p}] = i\hbar} \tag{4}$$

This means that,

$$\hat{x}\hat{p} = \hat{p}\hat{x} + i\hbar \tag{5}$$

Multiplying by $\hat{x}$ on both sides, we get:

$$\hat{x}^2\hat{p} = \hat{x}\hat{p}\hat{x} + i\hbar\hat{x} = (\hat{p}\hat{x} + i\hbar)\hat{x} + i\hbar\hat{x} = \hat{p}\hat{x}^2 + 2i\hbar\hat{x} \tag{6}$$

Doing this n times, we get:

$$\hat{x}^n\hat{p} = \hat{p}\hat{x}^n + ni\hbar\hat{x}^{n-1} \tag{7}$$

Therefore, we get:

$$\boxed{[\hat{x}^n, \hat{p}] = ni\hbar x^{n-1}} \tag{8}$$

We could have also multiplied by $\hat{p}$ instead of $\hat{x}$ (after rewriting the commutator relation as $\hat{p}\hat{x} = \hat{x}\hat{p} - i\hbar$). Doing this would give:

$$\hat{p}^2\hat{x} = \hat{p}\hat{x}\hat{p} - i\hbar\hat{p} = (\hat{x}\hat{p} - i\hbar)\hat{p} - i\hbar\hat{p} = \hat{x}\hat{p}^2 - 2i\hbar\hat{p} \tag{9}$$

Repeating this n times would give:

$$\hat{p}^n\hat{x} = \hat{x}\hat{p}^n - ni\hbar\hat{p}^{n-1} \tag{10}$$

Therefore, we get:

$$\boxed{[\hat{x}, \hat{p}^n] = ni\hbar\hat{p}^{n-1} = (-1)^{n-1}n(i\hbar)^n\frac{d^{n-1}}{dx^{n-1}}} \tag{11}$$