Quantum Tunnelling

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1 Normalisation of the Stationary States of the Step Potential

Let our step potential be $V(x)$, such that:

$$V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \tag{1}$$

We know that the Stationary States of Energy E satisfying $E > V_0$ are given by:

$$\psi(x) = \begin{cases} A(e^{ik_1x} + \frac{k_1-k_2}{k_1+k_2}e^{-ik_1x}) & x \leq 0 \\ A(\frac{2k_1}{k_1+k_2})e^{ik_2x} & x > 0 \end{cases} \tag{2}$$

Therefore,

$$|\psi(x)|^2 = \begin{cases} |A|^2(1 + (\frac{k_1-k_2}{k_1+k_2})^2 + 2\frac{k_1-k_2}{k_1+k_2}\cos{(k_1x)}) & x \leq 0 \\ |A|^2(\frac{2k_1}{k_1+k_2})^2 & x > 0 \end{cases} \tag{3}$$

Clearly, $\int_{-\infty}^{\infty}|\psi(x)|^2dx$ diverges (unless $A = 0$, in which case $\psi(x) = 0$, which is not an acceptable Wave Function), and hence this Wave Function is not Normalizable.

2 Bound States of the Step Potential

A Stationary state is said to be a Bound State when it satisfies $E < V(\infty)$ and $E < V(-\infty)$. Note that all stationary states satisfy $E \geq V_{min}$. In the step potential $V_{min} = V(-\infty) = 0$. Therefore $E \geq V(-\infty)$ always. Therefore, a step potential can have no bound states, and all it's Stationary States are Scattering States. Any state with $E > V_0$ would be non-square Integrable on $x < 0$ and $x > 0$ and therefore, would also be a Scattering State.
Note that, Scattering States have a continuous Energy Spectrum and are non-normalizable. Hence we could have found the answer for the previous Question without finding $|\psi|^2$!