Why is Schrödinger's Equation first order in time but second order in position?

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The Schrödinger Equation in one dimension is:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi} \tag{1}$$

It can be clearly seen that this equation is second order in space, but only first order in time.

The reason why Schrödinger's Equation is linear in time, but second order is because it is a non-relativistic equation.

In the Special Theory of Relativity, position and time are treated on an equal footing, i.e. Physical Equations should be of the same order in both position and time. In the Special Theory of Relativity, Energy and Momentum are related by:

$$E^2 = (pc)^2 + (mc^2)^2 \tag{2}$$

Note that this equation has E and p raised to the same power.

However, in Non-Relativistic Physics, there is no obligation to treat position and time on the same footing. In Non-Relativistic Mechanics:

$$E = \frac{p^2}{2m} + V \tag{3}$$

Note that this equation does not have E and p raised to the same power. This is the reason why Schrödinger's equation is of different orders in time and position.

Now, according to Planck's Law:

$$E = \hbar\omega \tag{4}$$

and according to De Broglie's Law:

$$p = \hbar k \tag{5}$$

So for a wave $\Psi = Ae^{i(kx-\omega t)}$, $\hat{E}$ and $\hat{p}$ operators should be such that:

$$\hat{E}\Psi = \omega\Psi \tag{6}$$

and

$$\hat{p}\Psi = k\Psi \tag{7}$$

Notice that $i\hbar\frac{\partial\Psi}{\partial t} = \omega\Psi$ and $-i\hbar\frac{\partial\Psi}{\partial x} = k\Psi$. So it is evident that:

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \tag{8}$$

and

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{9}$$

Now, in applying the formula for Energy in Non-Relativistic Physics, we get:

$$\hat{E} = \frac{\hat{p}^2}{2m} + V \tag{10}$$

Using the expressions for $\hat{E}$ and $\hat{p}$ we now obtain:

$$i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \tag{11}$$

Applying this operator equation to the Wave Function Ψ, we get:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \tag{12}$$

This is nothing but Schrödinger's Equation!

Now we see that the reason that it is first order in time but second order in position is because in the non-relativistic formula for Energy, Energy and Momentum have different powers.

If we use instead the Relativistic Formula for Energy ($E^2 = (pc)^2 + (mc^2)^2$), we get:

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2 c^2\frac{\partial^2}{\partial x^2} + m^2c^4 \tag{13}$$

On rearranging some terms we get the operator equation:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m^2c^2}{\hbar^2} = 0 \tag{14}$$

Applying this operator equation to the wave function Ψ, we obtain:

$$\boxed{\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} + \frac{m^2c^2\Psi}{\hbar^2} = 0} \tag{15}$$

This equation is known as the Klein-Gordon Equation, and it is a relativistic equivalent to Schrödinger's Equation in Quantum Mechanics. As expected, it has the same order (two) in position and time.