The Quantum Harmonic Oscillator

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1 Normalizing the Ground State Wave Function of the Quantum Harmonic Oscillator

We know that the Wave Function of the Ground State of the Harmonic Oscillator is given by

$$\psi(x) = A\exp{(-\frac{m\omega x^2}{2\hbar})} \tag{1}$$

To normalize this, we must impose the condition that

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1 \tag{2}$$

In this case, that condition becomes

$$|A|^2 \int_{-\infty}^{\infty} \exp{(-\frac{m\omega x^2}{\hbar})} dx = 1 \tag{3}$$

From the Standard Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{4}$$

it can be deduced that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \tag{5}$$

This implies that

$$|A|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} \tag{6}$$

Assuming, without loss of Generality, that A is a positive Real Number, we get $A = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}$. Therefore, the Normalized Wave Function of the Ground State[1] becomes:

$$\psi(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \exp{(-\frac{m\omega x^2}{2\hbar})} \tag{7}$$

[1] The normalized Wave Functions for the nth Stationary State of the Harmonic Oscillator are given by $\psi_n(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{\frac{m\omega}{\hbar}} x) \exp{(-\frac{m\omega x^2}{2\hbar})}$, where H_n represents the nth Hermite Polynomial.

2 The Uncertainty in Position and Momentum for a Particle in the Ground State of the Quantum Harmonic Oscillator

The Wave Function of the Ground State of the Quantum Harmonic Oscillator is given by:

$$\psi(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \exp{(-\frac{m\omega x^2}{2\hbar})} \tag{8}$$

To calculate σ_x and σ_p, we shall use the equations:

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \tag{9}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \tag{10}$$

Note, that since the Ground State Wave Function is an even function

$$\langle x\rangle = \langle\psi|x\psi\rangle = 0 \tag{11}$$

From Ehrenfest's Theorem, we can then conclude that:

$$\langle p\rangle = m\frac{d\,\langle x\rangle}{dt} = 0 \tag{12}$$

To calculate, $\langle x^2\rangle$, however, we must integrate.

$$\langle x^2\rangle = \langle\psi|x^2\psi\rangle = \int_{-\infty}^{\infty} x^2|\psi(x)|^2 dx = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 \exp{(-\frac{m\omega x^2}{\hbar})} dx \tag{13}$$

On integrating, we find that

$$\langle x^2\rangle = \frac{\hbar}{2m\omega} \tag{14}$$

And therefore,

$$\sigma_x = \sqrt{\frac{\hbar}{2m\omega}} \tag{15}$$

To find $\langle p^2\rangle$, notice that

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \tag{16}$$

Since, $\langle p^2\rangle = \langle\psi|\hat{p}^2\psi\rangle$, we can say that:

$$\langle p^2\rangle = \langle\psi|2m(\hat{H} - v(x))\psi\rangle = 2m(\langle E\rangle - \langle V(x)\rangle) \tag{17}$$

Since, ψ is a Stationary State, $\langle E\rangle = E$, and since the Ground State of the Harmonic Oscillator has Energy $\frac{\hbar\omega}{2}$

$$\langle p^2\rangle = m\omega\hbar - 2m\,\langle V(x)\rangle = m\omega\hbar - m^2\omega^2\,\langle x^2\rangle \tag{18}$$

Using the previously calculated value of $\langle x^2 \rangle$, we get

$$\langle p^2 \rangle = \frac{m\omega\hbar}{2} \tag{19}$$

Therefore,

$$\sigma_p = \sqrt{\frac{m\omega\hbar}{2}} \tag{20}$$

And hence, we get

$$\boxed{\sigma_x \sigma_p = \frac{\hbar}{2}} \tag{21}$$

Note that here equality occurs in Heisenberg's Uncertainty Principle ($\sigma_x \sigma_p \geq \frac{\hbar}{2}$). This is as the Ground State Wave Function of the Quantum Harmonic Oscillator is a Gaussian Function.
In fact, it can be proven that for equality to occur in Heisenberg's Uncertainty Principle, the Wave Function must be a Gaussian Wave Function.
The Generalized Uncertainty Principle states that

$$\sigma_A \sigma_B \geq \frac{|\langle [A, B] \rangle|}{2} \tag{22}$$

Applying this to $\hat{x}$ and $\hat{p}$ and using the relation $[\hat{x}, \hat{p}] = i\hbar$, we get Heisenberg's Uncertainty Principle:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{23}$$

In the Generalized Uncertainty Principle equality occurs only when

$$(\hat{A} - \langle A \rangle)\psi = i\lambda(\hat{B} - \langle B \rangle)\psi \tag{24}$$

for some real number λ. Putting $\hat{A} = \hat{p} = -i\hbar\frac{d}{dx}$ and $\hat{B} = \hat{x} = x$, we get:

$$\frac{d\psi}{dx} = \frac{i\,\langle p \rangle}{\hbar}\psi - \frac{\lambda}{\hbar}(x - \langle x \rangle)\psi \tag{25}$$

On solving this Differential Equation, we get:

$$\psi(x) = Ae^{\frac{i\langle p \rangle x}{\hbar}} e^{-\frac{\lambda}{2\hbar}(x - \langle x \rangle)^2} \tag{26}$$

Noting that $\lambda > 0$, as otherwise the function would not be normalizable, we conclude that the Minimum Uncertainty Wave Function must be a Gaussian Function.