Why is the Momentum operator imaginary in Quantum Mechanics?

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In Quantum Mechanics, the momentum operator is given by:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{1}$$

It is evident that this is a complex operator. To understand why this is so, we must first gain a more thorough understanding of Operators in Quantum Mechanics.

In Quantum Mechanics, every observable A is associated with an operator $\hat{A}$. An operator takes a function as an input and another function as an output. In general on measuring the value of an observable for a particle in a state Ψ, we will not get the same value on each measurement.

However for some special wave functions ψ_i, on measuring the value of the observable (say A) we will get the same value (say A_i) each time. These wave functions are said to be eigenfunctions of the operator $\hat{A}$ associated with the observable A with eigenvalue A_i. It can be shown that these eigenfunctions must satisfy the equation:

$$\hat{A}\psi = A_i\psi \tag{2}$$

General wave functions can be expressed as linear combinations of these eigenfunctions.

For example, take an observable A, which can take only two values: A_1 and A_2. Let the corresponding eigenfunctions for these two eigenvalues be ψ_1 and ψ_2. It can be shown that every possible wave function can be expressed as a linear combination of ψ_1 and ψ_2 as:

$$\psi = c_1\psi_1 + c_2\psi_2 \tag{3}$$

Here $|c_1|^2$ and $|c_2|^2$ represent the probabilities of the observable being measured to be A_1 and A_2 respectively (Hence we get the normalization condition $|c_1|^2 + |c_2|^2 = 1$).

The important point to note here is that: The eigenvalues of any eigenfunction must always be real. That is, the observed values (here A_1 and A_2) must always be real. This is as they are the values that we measure when we conduct experiments, and a complex valued observable such as position, energy, or momentum would not make sense.

So let us take a look at two important observables in Quantum Mechanics: Position (x), Momentum (p).

The eigenfunction of position, with eigenvalue x_0, would be:

$$\psi(x) = A\delta(x - x_0) \tag{4}$$

Here $\delta(x)$ represents Dirac's Delta function which satisfies:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0) \tag{5}$$

for all functions $f(x)$, and

$$\delta(x) = 0, \forall x \neq 0 \tag{6}$$

Now the position operator ($\hat{x}$) must be such that:

$$\hat{x}\psi = x_0\psi \tag{7}$$

From the definition of the Delta function, it is clear that:

$$x\delta(x - x_0) = x_0\delta(x - x_0) \tag{8}$$

Therefore:

$$x\psi(x) = x_0\psi(x) \tag{9}$$

Hence, we can conclude that:

$$\boxed{\hat{x} = x} \tag{10}$$

Now let us take a look at the eigenfunctions of Momentum. We know that $p = \hbar k$. Therefore, any eigenfunction of momentum, with eigenvalue p, must be a wave with fixed wave number, i.e. it must be of the form:

$$\psi(x) = Ae^{ikx} = Ae^{\frac{ipx}{\hbar}} \tag{11}$$

Now, note that:

$$\frac{\partial\psi}{\partial x} = \frac{ip}{\hbar}\psi \tag{12}$$

Therefore:

$$-i\hbar\frac{\partial\psi}{\partial x} = p\psi \tag{13}$$

Now the momentum operator must be such that:

$$\hat{p}\psi = p\psi \tag{14}$$

Hence, we can conclude that:

$$\boxed{\hat{p} = -i\hbar\frac{\partial}{\partial x}} \tag{15}$$

Note that, $\hat{p}$ must be complex. This is because, were it not complex, the eigenvalues of $\hat{p}$ would not be real (Note that if $\hat{p}$ were $\hbar\frac{\partial}{\partial x}$, then the eigenvalues of $\hat{p}$ would be ip). The eigenvalues of $\hat{p}$ are the final measured values of momentum. Since, they must be real, it must be the case that the momentum operator $\hat{p}$ is complex.