The Spin Statistics Theorem

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21st February, 2022

The Spin of a Particle is a type of Angular Momentum intrinsic to the particle itself. Since Spin is a type of Angular Momentum, its operators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ must satisfy the same commutations relations as $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$. That is:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \tag{1}$$

$$[\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \tag{2}$$

$$[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \tag{3}$$

$$[\hat{S}^2, \hat{S}_x] = [\hat{S}^2, \hat{S}_y] = [\hat{S}^2, \hat{S}_z] = 0 \tag{4}$$

These commutation relations ultimately arise from imposing Rotational Invariance in three Dimensions.
From these commutation relations we can find the eigenfunctions of the Spin operators. Since, $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ do not commute, we can try to find only the simultaneous eigenfunctions of $\hat{S}^2$ and S_z. On doing this we see that the eigenvalues of $\hat{S}^2$ and S_z are such that:

$$\hat{S}^2 f_{sm} = s(s+1)\hbar^2 f_{sm} \tag{5}$$

$$\hat{S}_z f_{sm} = m_s\hbar f_{sm} \tag{6}$$

Where s is known as the Spin of the particle and it can be any non-negative integer or half-integer.

$$s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots\} \tag{7}$$

and m_s is known as the Spin Quantum number, and it can be:

$$m_s \in \{-s, -s+1 \ldots s-1, s\} \tag{8}$$

The Spin Statistics Theorem states that every elementary particle (such as electrons, photons, etc) have a specific value of s instrinsic to them. Since, as shown before, s must be either a integer or a half integer, there cannot be any particles with fractional spin (other than the ones with half integer spins).