

# Topology

Lecture 1 - 06/01/21 Introduction and examples of topologies

Def.

A **topology** on a set  $X$  is a collection  $\tau$  of subsets of  $X$  such that

i)  $\emptyset \in \tau$  and  $X \in \tau$ .

ii) If  $U_i \in \tau$  for all  $i \in I$ , where  $I$  is some indexing set, then

$$\bigcup_{i \in I} U_i \in \tau.$$

iii) If  $U_j \in \tau$  for all  $j \in J$ , where  $J$  is some finite indexing set, then

$$\bigcap_{j \in J} U_j \in \tau.$$

Recall the definition of a metric space and an open set. (from Real Analysis)

Since the set of open sets is closed under arbitrary unions and finite intersections, observe that the set of open subsets of a metric space  $(X, d)$  is a topology. That is,

$$\tau = \{ U \subseteq X : U \text{ is open in } (X, d) \}$$

is a topology. ( $\emptyset$  and  $X$  are trivially open)

Topologies essentially extend the idea of open sets. How?

Def.

A **topological space**  $(X, \tau)$  is a set  $X$  along with a topology  $\tau$  on  $X$ .

Topological Space

For a topological space, we call the elements of  $\tau$  **open**.

Open Set

$(X, \{\emptyset, X\})$  is a trivial topological space on a set  $X$ .

We now introduce the analogues of interior points, closed sets, etc. Since we don't have "balls" in topological spaces, we have to define everything in an alternate way that remains consistent.

Metric  
Topology

For a metric space  $(X, d)$ , the topology  $\mathcal{T} = \{U \subseteq X : U \text{ is open}\}$  is called the **metric topology** induced by the metric  $d$ .

Discrete  
Topology

For a set  $X$ , the topology  $\mathcal{P}(X)$  is called the **discrete topology** on  $X$ .

Observe that this is the metric topology induced by the discrete metric. (for  $x, y \in X$ ,  $d(x, y) = 0$  if  $x = y$  and 1 otherwise)

Indiscrete  
Topology

For a set  $X$ , the topology  $\{\emptyset, X\}$  is called the **indiscrete topology** on  $X$ .

Let  $X$  be a set and

Finite  
Complement  
Topology

$\mathcal{T}_f = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}$ .  
 $\mathcal{T}_f$  is a topology on  $X$  and is called the **finite complement topology** or the **co-finite topology**.

- Clearly,  $\emptyset$  and  $X$  are in  $\mathcal{T}_f$ .

- For  $(U_i)_{i \in I}$  in  $\mathcal{T}_f$ ,

$$\left(\bigcup_{i \in I} U_i\right)^c = \bigcap_{i \in I} U_i^c \text{ is finite (since each } U_i^c \text{ is finite)}$$

- For  $(U_i)_{i=1}^n$  in  $\mathcal{T}_f$ ,

$$\left(\bigcap_{i=1}^n U_i\right)^c = \bigcup_{i=1}^n U_i^c \text{ is finite (a finite union of finite sets)}$$

We have seen that any metric defines a topology. Is the converse true?

No!

Topologies that are induced by a metric are said to be **metrizable**.

→ Consider the indiscrete topology  $\{\emptyset, X\}$ .

Use the fact that distinct points are separable by neighbourhoods.



If  $X$  is a finite set, the finite complement topology is the discrete topology.

Similar to the co-finite topology  $\tau_f$ , we can define  $\tau_c$ , the **co-countable topology**.

$$\left( \{\emptyset\} \cup \{ U \subseteq X : X \setminus U \text{ is countable} \} \right)$$