Now suppose f is continuous at x, that is, for $\varepsilon > 0$, $\exists\, \delta > 0$

$$\underset{t \in [a,b]}{|t-x| < \delta} \Rightarrow |f(t) - f(x)| < \varepsilon$$

For $x - \delta < t < x + \delta$,

$$\left| \frac{F(t) - F(x)}{t-x} - f(x) \right| = \left| \frac{1}{t-x} \int_x^t f(y)\,dy - f(x) \right|$$

$$= \left| \frac{1}{t-x} \left[\int_x^t (f(y) - f(x))\,dy \right] \right|$$

$$\leq \left| \frac{1}{t-x} \int_x^t \varepsilon\, dy \right| \quad (\text{as } |y-x| \leq |t-x| < \delta)$$

$$= \varepsilon$$

$\Rightarrow$ F is differentiable at x and $F'(x) = f(x)$.

Theo.

Let f be Riemann integrable on $[a,b]$. Let there be a differentiable function F on $[a,b]$ s.t. $F'(x) = f(x)$ $\forall x \in [a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Fix $\varepsilon > 0$. Let there be a partition $P = \{a = x_0, x_1, \cdots, x_n = b\}$ of $[a,b]$ s.t. $U(P;f) - L(P;f) < \varepsilon$.

Now, by the Mean Value Theorem, for $i = 1, 2, \cdots, n$

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) f(t_i) \text{ for some } t_i \in (x_{i-1}, x_i)$$

As we have shown earlier,

$$\left| \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \left| \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \left| (F(b) - F(a)) - \int_a^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$