Sets and Relations

Basics of Sets

A set is an unordered collection of "elements".

For example, $\mathbb{Z}, \mathbb{R}, \varnothing$, and $\{1,2\}$ are sets.

We are given (either implicitly or explicitly) a "universal set" from which elements come.
This is a very non-rigorous definition of a set (which is usually defined far more axiomatically), but it will suffice for our requirements.

We write $x \in S$ if the element x is present in the set S and $x \notin S$ otherwise. Eg. $0.5 \in \mathbb{R}$ and $0.5 \notin \mathbb{Z}$

We write $A \subseteq B$ for sets A and B if for all elements x in A, $x \in B$.

The complement of a set A, denoted A^c or $\bar{A}$, is the set of all elements that are not in A.
The union of sets A and B, denoted $A \cup B$, is the set of all elements present in either A or B.
The intersection of sets A and B, denoted $A \cap B$, is the set of all elements present in both A and B.
Given sets A and B, $A \setminus B$ is the set of all elements present in A and not present in B.

Given a predicate p, we can consider the set of all elements for which it holds, denoted as

$$A = \{x \mid p(x)\} \text{ or } \{x : p(x)\}$$

We can also define a "membership predicate" where $p(x)$ iff $x \in A$.

Sets and predicates are essentially the same thing expressed in two different forms.

So we can redefine the earlier operations by

$$x \in A \equiv x \notin \bar{A} \quad \longrightarrow \text{Unary operator}$$

$$\left.\begin{aligned} x \in A \cup B &\equiv x \in A \vee x \in B \\ x \in A \cap B &\equiv x \in A \wedge x \in B \\ x \in A \setminus B &\equiv x \in A \wedge \neg x \in B \\ &\equiv x \in A \not\rightarrow x \in B \\ x \in A \Delta B &\equiv x \in A \oplus x \in B \end{aligned}\right\} \text{Binary operators}$$

$\cup$, $\cap$, and Δ are associative.

Ex. Prove De Morgan's Laws:

$$\overline{S \cup T} = \bar{S} \cap \bar{T}$$

$$\overline{S \cap T} = \bar{S} \cup \bar{T} \quad \text{for sets } S, T.$$

$\cap$ distributes over $\cup$ and $\cup$ distributes over $\cap$.

For sets S, T, $S \subseteq T$ is equivalent to $\forall x \; x \in S \rightarrow x \in T$.
$S \supseteq T$ is equivalent to $\forall x \; x \in S \leftarrow x \in T$.
$S = T$ is equivalent to $\forall x \; x \in S \leftrightarrow x \in T$.

Note that $\emptyset \subseteq X$ is vacuously true for any set X.

If $S \subseteq T$ and $T \subseteq R$, then $S \subseteq R$.
(Just a consequence of $(a \rightarrow b) \rightarrow (b \rightarrow c) \equiv a \rightarrow c$)
If $S \subseteq T$, then $\bar{T} \subseteq \bar{S}$.
(Just the contrapositive)

To show equality of two sets A and B, we usually show $A \subseteq B$ and $B \subseteq A$.

Recall that we did this when showing

$$\{x : \exists u,v \in \mathbb{Z} \ \ x = au+bv\} = \{x : \gcd(a,b) \mid x\}$$

We denote the number of elements in a set S by $|S|$.

The Inclusion-Exclusion Principle states that

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

This can be expanded to three sets as

$$|R \cup S \cup T| = |R| + |S| + |T| - |R \cap S| - |S \cap T| - |T \cap R| + |R \cap S \cap T|$$

This can be extended to any (countable) number of sets using induction on the number of sets.

Def. The Cartesian Product of sets S and T is the set

$$S \times T = \{(s,t) : s \in S \text{ and } t \in T\}$$

$(S = \emptyset \vee T = \emptyset) \longleftrightarrow S \times T = \emptyset$

$|S \times T| = |S| \cdot |T|$

This can be expanded to three sets as

$$R \times S \times T = \{(r,s,t) : r \in R,\ s \in S,\ t \in T\}$$

This is not exactly the same as $((r,s),t)$ but they are essentially the same; there is a bijection between the two.

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

(It also distributes on the other side.)

$$\overline{S \times T} = (\overline{S} \times \overline{T}) \cup (\overline{S} \times T) \cup (S \times \overline{T})$$

Relations

Def. Given sets A and B, a relation is a predicate over $A \times B$.

It is equivalently a subset of $A \times B$.
We restrict ourselves to the case $A = B$, namely homogeneous relations.
We typically write $p(a,b)$ as $a \sqsubset b$, $a \sim b$, $a \leq b$ etc

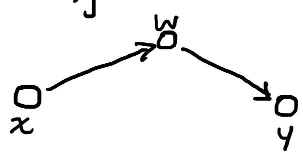
A relation can be represented as
→ A subset of $S \times S$
$$= \{(a,b) : a \sqsubset b\}$$
→ A boolean matrix where $M_{a,b} = T$ iff $a \sqsubset b$.
→ A directed graph where $a \to b$ iff $a \sqsubset b$.
(we will study these later)

Since relations are just sets, we can translate all the set operations into relation operations. (The universal set is just $S \times S$ then)
Given a relation R,
→ The transpose of R, denoted R^T is $\{(x,y) : (y,x) \in R\}$.
(or converse)

$$(M^T_{x,y} = M_{y,x})$$

→ The composition of R and R' is given by
$$R \circ R' = \{(x,y) : \exists w \in S \ (x,w) \in R \text{ and } (w,y) \in R'\}$$

$$(M \circ M')_{x,y} = \exists w \ (M_{x,w} \wedge M_{w,y})$$



↳ Boolean matrix multiplication
$\vee$ instead of $+$ and
$\wedge$ instead of $\times$.

A relation R is said to be
→ reflexive if $\forall x \ R(x,x)$ holds.
all the diagonal entries in the matrix are true.
all the nodes have self-loops.

→ Irreflexive if $\forall x\ \neg R(x,x)$

all the diagonal entries in the matrix are false.

no nodes have self-loops.

→ Symmetric if $\forall x \forall y\ (R(x,y) \leftrightarrow R(y,x))$

the matrix is symmetric.

there are only self loops and bidirectional edges.

→ Anti-Symmetric if $\forall x \forall y((x=y) \vee (R(x,y) \rightarrow \neg R(y,x)))$

(equivalent to $\forall x \forall y\ ((R(x,y) \wedge R(y,x)) \longrightarrow (x=y))$)

the matrix is anti-symmetric.

there are no bidirectional edges.

Note that the equality relation is both symmetric and anti-symmetric.

→ Transitive if $\forall a \forall b \forall c\ ((R(a,b) \wedge R(b,c)) \rightarrow (R(a,c)))$.

$R \circ R \subseteq R \equiv \forall k > 1 (R^k \subseteq R)$ $\quad (R^k = \overbrace{R \circ R \circ \cdots \circ R}^{k \text{ times}})$

if there is a "path" from a to b in the graph, there is an edge (a,b).

→ Intransitive if it is not transitive.

The complete relation $R = S \times S$ is reflexive, symmetric, and transitive.

Def. Given a relation R, we define its reflexive/symmetric/transitive closure as the minimal relation $R' \supseteq R$ s.t. R' is reflexive/symmetric/transitive.

↳ in the sense that we cannot remove any edges.

Def. A relation R is said to be an equivalence relation if it is reflexive, symmetric, and transitive.

e.g. is a relative, is congruent mod 12

Given a relation, we define the equivalence class of x by

$$Eq(x) = \{y : x \sim y\}$$

Note that

- by reflexivity, every element is in its own equivalence class.
- if $Eq(x) \cap Eq(y) \neq \emptyset$, then $Eq(x) = Eq(y)$.

Proof. Let $z \in Eq(x) \cap Eq(y)$. and arbitrary $a \in Eq(x)$.

We have $y \sim z$, $x \sim z$, and $x \sim a$

$\Rightarrow y \sim z$, $z \sim x$, and $x \sim a$. (by symmetry)

$\Rightarrow$ We have $y \sim a$ (by transitivity)

$\Rightarrow a \in Eq(y) \Rightarrow Eq(x) \subseteq Eq(y)$.

Similarly, $Eq(y) \subseteq Eq(x)$ and the two are equal.

The above two imply that the set of equivalence classes partition the domain.

(For $P_1, \dots, P_t \subseteq S$. $\{P_1, \dots, P_t\}$ is said to partition S if $P_1 \cup \dots \cup P_t = S$ and $P_i \cap P_j = \emptyset$ for $i \neq j$.)

These can be visualized as the graph comprising several cliques.

An equivalence relation R is its own symmetric, reflexive, and transitive closure.