High-Dimensional Convex Geometry

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§1. Introduction

Definition 1.1. A subset S of a Euclidean space is said to be *convex* if for any $u_1, \dots, u_r \in S$ and non-negative $\lambda_1, \dots, \lambda_r$ such that $\lambda_1 + \dots + \lambda_r = 1$, the *affine combination* $\sum_{i=1}^r \lambda_i u_i$ is in S as well.

We primarily consider convex bodies, that is, compact and convex subsets of Euclidean spaces here. To put it more succinctly, a convex body is something that "behaves a bit like a Euclidean ball".
A few simple examples of convex bodies on $\mathbb{R}^n$ are:

- the cube $[-1, 1]^n$. Here, the ratio of the radii of the circumscribed ball to the inscribed ball is $\sqrt{n}$, so it is not much like a Euclidean ball.
- the *n-dimensional regular solid simplex* which is the convex hull of $n+1$ equally spaced points. Here, the ratio of the radii of the circumscribed ball to the inscribed ball is n. This ratio is "maximal" in some sense.
- the n-dimensional "octahedron" or *cross-polytope* which is the convex hull of the $2n$ points $(\pm 1, 0, \dots, 0)$, $(0, \pm 1, 0, \dots, 0)$, $\dots, (0, 0, \dots, 0, \pm 1)$. Note that this is the unit ball on the ℓ_1 norm on $\mathbb{R}^n$ so we denote it as B_1^n. Here, the ratio of the radii of the circumscribed ball to the inscribed ball is $\sqrt{n}$.

Definition 1.2. A *cone* in $\mathbb{R}^n$ is the convex hull of a single point and a convex body of dimension $n-1$. In $\mathbb{R}^n$, the volume of a cone of "height" h over a base of $(n-1)$-dimensional volume B is Bh/n.

Since B_1^n is made up of 2^n pieces similar to the piece with non-negative coordinates, which is a cone of height 1 with base analogous to the similar piece in $\mathbb{R}^{n-1}$, the volume of the non-negative section is $1/n!$. Therefore, the $\mathrm{vol}(B_1^n) = 2^n/n!$.

1.1. The Euclidean Ball

The fourth and final example is the Euclidean ball itself, namely

$$B_2^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \le 1 \right\}.$$

Let us now attempt to calculate $v_n = \mathrm{vol}(B_2^n)$. Note that we can easily get the "surface area" of the ball from the volume by splitting it into "thin" cones from 0 and observing that the volume of each cone is equal to $1/n$ times its base area. Therefore, the surface area of the ball is nv_n.
We perform integration in spherical polar coordinates using two variables - r, which denotes the distance from 0 and θ, which is a point on the unit ball that represents the direction of the point. We obviously have $x = r\theta$. The point θ carries the information of $n-1$ coordinates.
We can then write the integral of a general function on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f = \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta) r^{n-1} \,\mathrm{d}\theta \,\mathrm{d}r \tag{1.1}$$

Here, $\mathrm{d}\theta$ represents the area measure on the sphere. From our earlier observation, its total mass is nv_n. The r^{n-1} factor appears because the sphere of radius r has r^{n-1} times that of the sphere of radius 1.
An important thing to note about the measure corresponding to $\mathrm{d}\theta$ is that it is *rotation-invariant*. If A is a subset of the sphere and U is orthogonal to A, then UA has the same measure as A. Therefore, we often simplify integrals such as 1.1 by pulling out the nv_n factor to get

$$\int_{\mathbb{R}^n} f = nv_n \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta) r^{n-1} \,\mathrm{d}\sigma_{n-1}(\theta) \,\mathrm{d}r \tag{1.2}$$

where σ_{n-1} is the rotation-invariant measure on $\mathbb{R}^{n-1}$ of total mass 1. Now, to evaluate v_n, we choose a suitable f such that the integrals on either side can easily be calculated, namely

$$f : x \mapsto \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right).$$

Then the integral on the left of 1.2 is

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \prod_{i=1}^{n} \exp\left(-\frac{x_i^2}{2}\right) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \exp\left(-\frac{x_i^2}{2}\right) = \left(\sqrt{2\pi}\right)^n$$

and the integral on the right is

$$nv_n \int_0^\infty \int_{S^{n-1}} e^{-r^2/2} r^{n-1} \, \mathrm{d}\sigma_{n-1} \, \mathrm{d}r = nv_n \int_0^\infty e^{-r^2/2} r^{n-1} \, \mathrm{d}r = v_n 2^{n/2} \Gamma\left(\frac{n}{2} + 1\right).$$

Equating the two,

$$v_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

Using Stirling's Formula, we can approximate this slightly better as

$$v_n \approx \frac{\pi^{n/2}}{\sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n+1)/2}} \approx \left(\frac{2\pi e}{n}\right)^{n/2}.$$

This is quite small for large n. The radius of a ball of volume 1 would be approximately $\sqrt{n/2\pi e}$, which is very large!

This is possibly the first hint one should take that following your intuition is probably not a good idea when dealing with high dimensional spaces.

Let us now restrict ourselves to considering the ball of volume 1.
What is the $(n-1)$-dimensional volume of a slice through the center of the ball? Since the slice is an $(n-1)$ dimensional ball, it is equal to

$$v_{n-1} r^{n-1} = v_{n-1} \left(\frac{1}{v_n}\right)^{(n-1)/n}.$$

This is approximately equal to $\sqrt{e}$ (using Stirling's formula once again). More generally, the volume of the slice that is at distance x from the center of the ball is equal to

$$\sqrt{e}\left(\frac{\sqrt{r^2 - x^2}}{r}\right)^{n-1} = \sqrt{e}\left(1 - \frac{x^2}{r^2}\right)^{(n-1)/2} \approx \sqrt{e}\left(1 - \frac{2\pi e x^2}{n}\right)^{(n-1)/2} \approx \sqrt{e}\exp(-\pi e x^2)$$

Note that this is normally distributed but the variance $1/2\pi e$ *does not depend on* n![1] So despite the fact that the radius grows as $\sqrt{n}$, the distribution of the volume stays the same. For example, nearly all the volume (around 96%) is concentrated in the slab with $|x_1| \leq 1/2$.

This might lead us to believe that since the volume is concentrated around any such equator[2] around a subspace, the volume should be concentrated around the intersection of all such equators, which seems to suggest that it should be concentrated around the center. However, for large n, we obviously know that most of the volume should be concentrated on the surface of the sphere[3]. These two points seem to be directly contradictory! However, as might

[1] exclamation mark, not factorial
[2] since we could have equally well taken something other than x_1
[3] the volume of a ball of radius dr $(d < 1)$ is $d^n \ll 1$ times that of a ball of radius r

be expected, this is once again because our intuition fails when dealing with high-dimensional spaces.
The measure of unit ball is "concentrated" both near the surface *and* around the equator, for any equator. To make more sense of this[4], while each x_i is small, the overall distance from 0 is quite large since the small individual coordinates are compensated by the large dimension n. The former leads to the point being close to the equator and the latter leads to the point being close to the surface of the ball.

To conclude, let us write the volume of a general convex body K in spherical polar coordinates. Assume that K has 0 in its interior and for each direction $\theta \in S^{n-1}$, let $r(\theta)$ be the radius of K (in that direction). Then,

$$\mathrm{vol}(K) = nv_n \int_{S^{n-1}} \int_0^{r(\theta)} s^{n-1}\,\mathrm{d}s\,\mathrm{d}\sigma = v_n \int_{S^{n-1}} r(\theta)^n\,\mathrm{d}\sigma.$$

Definition 1.3. A convex body K is said to be *(centrally) symmetric* if $-x \in K$ whenever $x \in K$.

Any symmetric body is the unit ball under some $\|\cdot\|_K$ on $\mathbb{R}^n$ (for example, the octahedron was the unit ball under the ℓ_1 norm). For a general symmetric body with associated norm $\|\cdot\|$, the volume is given by

$$\mathrm{vol}(K) = v_n \int_{S^{n-1}} \|\theta\|^{-n}\,\mathrm{d}\sigma_{n-1}(\theta)$$

1.2. The Cube

So for example. since the volume of the cube $[-1,1]^n$ is 2^n, we can use it to estimate the average radius of the cube[5] as

$$v_n \int_{S^{n-1}} r(\theta)^n = 2^n \implies \int_{S^{n-1}} r(\theta)^n \approx \left(\sqrt{\frac{2n}{\pi e}}\right)^n \text{ so the average radius is approximately } \sqrt{\frac{2n}{\pi e}}$$

That is, the volume of the cube is far more concentrated towards the corners (where the radius is closer to $\sqrt{n}$), rather than the middles of facets (where the radius is closer to 1).
It can actually be shown[6] that the fraction of volume of the intersection of the cube and the ball is less than $\exp(-4n/45)$, which further emphasizes the point that nearly all the volume lies in the corners.

Definition 1.4. A body which is bounded by a finite number of flat facets is called a *polytope*.

A polytope is essentially the intersection of a finite number of half-spaces (we shall later look at the Hahn-Banach separation theorem, which is a huge generalization of this to convex bodies in general).
Note that the cube is a polytope with $2n$ facets.

Earlier, we remarked that the cube is not much like a Euclidean ball. So a question that might come to mind is: If K is a polytope with m facets, how close can K be to the Euclidean ball?

Let us define this "closeness" more concretely.

Definition 1.5. The distance $d(K, L)$ between symmetric convex bodies K and L is the least positive d for which there is a linear image $\tilde{L}$ of L such that $\tilde{L} \subseteq K \subseteq d\tilde{L}$.

This corresponds to the how we thought of inscribing/circumscribing a ball earlier, since the ratio of the two radii we considered is just this distance.
If we wanted to make this distance a metric, then we should consider $\log d$ instead of d (this metric is multiplicative and for any K, $d(K, K) = 1$).

[4] The answers to this mathoverflow question might further aid understanding

[5] more concretely, the expectation of a random variable that corresponds to the distance of a point uniformly randomly chosen from the unit cube from the origin

[6] Consider the random variable $z_i = x_i^2$ where x_i is drawn uniformly randomly from $[-1, 1]$. Show that $\mathbf{E}[z_i] = 1/3$ and $\mathbf{Var}[z_i] = 4/45$ and use the Chernoff bound to get a bound on $\Pr[\sum_i z_i \le 1]$.

From what we mentioned earlier, we know that the distance between the cube and the Euclidean ball in $\mathbb{R}^n$ is at most $\sqrt{n}$. We shall prove later that it is indeed equal to $\sqrt{n}$.

As might be expected, if we want a polytope that approximates the ball very well, we would need a very large number of facets.

Definition 1.6. For a fixed unit vector v and some $\varepsilon \in [0,1)$, the set

$$C(\varepsilon, v) = \{\theta \in S^{n-1} : \langle \theta, v\rangle \geq \varepsilon\}$$

is called the *ε-cap about v* or more generally, a *spherical cap* (or just *cap*).
It is often better to write a cap in terms of its radius rather than in terms of ε. The *cap of radius r about v* is

$$\{\theta \in S^{n-1} : |\theta - v| \leq r\}$$

It is easy to see that a cap of radius r is a $(1 - \frac{r^2}{2})$-cap.
As we shall see in the proof of Theorem 1.3, it is useful to know some upper and lower bounds on the area of an ε-cap.

Lemma 1.1 (Upper bound on the area of spherical caps)**.** For $\varepsilon \in [0,1)$, the cap $C(\varepsilon, u)$ on S^{n-1} has measure (under σ_{n-1}) at most $e^{-n\varepsilon^2/2}$.

Proof. Suppose $n \geq 2$ and let $\alpha = \cos^{-1}(\varepsilon)$. We can assume that $\alpha \in [0, \frac{\pi}{2}]$ since we can prove the other case similarly. Then the measure of the cap is given by

$$\begin{aligned}
A(n,\alpha) &= \int_0^\alpha \frac{v_{n-1}}{n v_n} (\sin\theta)^{n-1}\,\mathrm{d}\theta \\
&= \frac{\Gamma\left(\frac{n}{2}+1\right)}{n\Gamma\left(\frac{n-1}{2}+1\right)\sqrt{\pi}} \int_0^\alpha \sin^{n-1}(\theta)\,\mathrm{d}\theta \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}} \int_0^\alpha \sin^{n-1}(\theta)\,\mathrm{d}\theta \\
&\leq \frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}} \int_0^\alpha \sin^{n-1}(\alpha)\,\mathrm{d}\theta \\
&\leq \frac{1}{2\sqrt{\pi}} \alpha \sin^{n-1}(\alpha) \\
&= \frac{\sqrt{\pi}}{4}\left(\frac{2\alpha}{\pi}\right)\sin^{n-1}(\alpha) \\
&\leq \sin^n(\alpha) && (2\alpha/\pi \leq \sin(\alpha) \text{ for } \alpha \text{ in } [0,\pi/2]) \\
&= (1-\varepsilon^2)^{n/2} \\
&\leq \exp(-n\varepsilon^2/2).
\end{aligned}$$

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Lemma 1.2 (Lower bound on the area of spherical caps)**.** For $r \in [0,2]$, a cap of radius r on S^{n-1} has measure (under σ_{n-1}) at least $\frac{1}{2}(r/2)^{n-1}$.

Theorem 1.3. Let K be a symmetric polytope in $\mathbb{R}^n$ with $d(K, B_2^n) = d$. Then K has at least $\exp(n/2d^2)$ facets. On the other hand, for each n, there is a polytope with 4^n facets whose distance from the ball is at most 2.

Before proving the above theorem, let us reformulate what a symmetric polytope is in another way. Suppose you have a symmetric polytope K with m pairs of facets. Then it is basically the intersection of m slabs in $\mathbb{R}^n$ each of the form $\{x : |\langle x, v_i\rangle| \leq 1\}$ for some $v_i \in \mathbb{R}^n$. That is,

$$K = \{x : |\langle x, v_i\rangle| \leq 1 \text{ for } 1 \leq i \leq m\} \tag{1.3}$$

We can then consider a linear map from $K \to \mathbb{R}^m$ given by

$$T : x \mapsto (\langle x, v_1\rangle, \ldots, \langle x, v_m\rangle)$$

This maps $\mathbb{R}^n$ to a subspace of $\mathbb{R}^m$. By the formulation of K given in 1.3 , the intersection of this subspace with the unit cube is just the image of K under T! This is just an n-dimensional slice of $[-1, 1]^m$. Even conversely, any n-dimensional slice of $[-1, 1]^m$ is a convex body with at most m pairs of facets.

Proof. For the proof, let us write what it means for each v_i (following the above notation) if $B_2^n \subseteq K \subseteq dB_2^n$.

- The first inclusion just says that each v_i is of length at most 1 (otherwise, one could consider $v_i/|v_i|$, which would be in B_2^n but not in K).
- The latter says that if $|x| > d$, then there is some i for which $\langle x, v_i\rangle > 1$. That is, for any unit vector θ, there is some i such that
$$\langle \theta, v_i\rangle > \frac{1}{d}.$$

Since we want to minimize m while satisfying the above two conditions, we can clearly do no better than have $|v_i| = 1$ for each i. We want that every $\theta \in S^{n-1}$ is in one of the m $(1/d)$-caps about the (v_i).

- Obviously, to do this, we should attempt to estimate the area of a general ε-cap ($\varepsilon = 1/d$ here). Given Lemma 1.1, we get that
$$m \geq \frac{1}{\exp(-n\varepsilon^2/2)} = \exp\left(\frac{n}{2d^2}\right).$$
- To show that there exists a polytope with the given number of facets, it is enough to find $2 \cdot 4^{n-1}$ points $v_1, \ldots, v_m$ such that the caps of radius 1 centered at these points covers the sphere. Such a set is called a 1-*net*. Now, suppose we choose a set of points on the sphere such that any two of them are at least distance 1 apart. Such a set is called a 1-*separated set*.

  Note that the caps of radius 1/2 centered at each of the points in a 1-separated set are disjoint. Since the measure of a cap of radius 1/2 is at least $1/2 \cdot 4^{n-1}$ (by Lemma 1.2), the number of points in a 1-separated set is at most $2 \cdot 4^{n-1}$.
  It is then enough to choose a "maximal" 1-separated set (a 1-separated set S such that $S \cup x$ is not 1-separated for any $x \in S^{n-1}$) since it is then automatically a 1-net!

  Therefore, there is a 1-net with at most $2 \cdot 4^{n-1}$ points.

  The corresponding polytope then has twice as many, that is, 4^n facets (since each v_i corresponds to a pair of facets).

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