Topology

Lecture 1 - 06/01/21 Introduction and examples of topologies

Topology

Def. A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii) If $U_i \in \mathcal{T}$ for all $i \in I$, where I is some indexing set, then
$$\bigcup_{i \in I} U_i \in \mathcal{T}.$$

iii) If $U_j \in \mathcal{T}$ for all $j \in J$, where J is some finite indexing set, then
$$\bigcap_{j \in J} U_j \in \mathcal{T}.$$

Unless mentioned otherwise, assume $X \neq \emptyset$.

Recall the definition of a metric space and an open set. (from Real Analysis)

Since the set of open sets is closed under arbitrary unions and finite intersections, observe that the set of open subsets of a metric space (X,d) is a topology. That is,
$$\mathcal{T} = \{ U \subseteq X : U \text{ is open in } (X,d) \}$$
is a topology. ($\emptyset$ and X are trivially open)

Topologies essentially extend the idea of open sets. How?

Topological Space

Def. A topological space $(X, \mathcal{T})$ is a set X along with a topology $\mathcal{T}$ on X.

Open Set

For a topological space, we call the elements of $\mathcal{T}$ open.

$(X, \{\emptyset, X\})$ is a trivial topological space on a set X.

We now introduce the analogues of interior points, closed sets, etc. Since we don't have "balls" in topological spaces, we have to define everything in an alternate way that remains consistent.

Metric Topology

For a metric space (X,d), the topology
$$\mathcal{T} = \{ U \subseteq X : U \text{ is open} \}$$
is called the metric topology induced by the metric d.

Discrete Topology

For a set X, the topology $\mathcal{P}(X)$ is called the discrete topology on X.
Observe that this is the metric topology induced by the discrete metric. (for $x,y \in X$, $d(x,y) = 0$ if $x=y$ and 1 otherwise)

Indiscrete Topology

For a set X, the topology $\{\emptyset, X\}$ is called the indiscrete topology on X.

Finite Complement Topology

Let X be a set and
$$\mathcal{T}_f = \{\emptyset\} \cup \{ U \subseteq X : X \setminus U \text{ is finite} \}.$$
$\mathcal{T}_f$ is a topology on X and is called the finite complement topology or the co-finite topology.

- Clearly, $\emptyset$ and X are in $\mathcal{T}_f$.
- For $(U_i)_{i \in I}$ in $\mathcal{T}_f$,
$$\left(\bigcup_{i \in I} U_i \right)^c = \bigcap_{i \in I} U_i^c \text{ is finite (since each } U_i^c \text{ is finite)}$$
- For $(U_i)_{i=1}^n$ in $\mathcal{T}_f$,
$$\left(\bigcap_{i=1}^n U_i \right)^c = \bigcup_{i=1}^n U_i^c \text{ is finite (a finite union of finite sets)}$$

We have seen that any metric defines a topology. Is the converse true?
No!
Topologies that are induced by a metric are said to be metrizable.
Consider the indiscrete topology $\{\emptyset, X\}$. (for $|X| > 1$)
Use the fact that distinct points are separable by neighbourhoods.

If X is a finite set, the finite complement topology is the discrete topology.

Similar to the co-finite topology τ_f, we can define τ_c, the co-countable topology.

$$\left(\{\emptyset\} \cup \{ U \subseteq X : X \setminus U \text{ is countable} \} \right)$$