Lyapunov Functions for PID control

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1 Unicycle Bot

In the current scenario, a 2 agent system is considered and the following dynamics describes the nature of the PID control applied.

$$\psi_{12} = ||(r_1 - r_2) - r_{12}^d||^2$$
$$\psi_1 = ||r_1 - r_1^d||^2$$
$$\psi_2 = ||r_2 - r_2^d||^2$$

Constructing the Lyapunov Function,

$$\mathrm{V} = K_{12}\psi_{12} + K_1\psi_1 + K_2\psi_2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r_1} \cdot \frac{dr_1}{dt} + \frac{\partial V}{\partial r_2} \cdot \frac{dr_2}{dt} \tag{1}$$

$$\frac{dV}{dt} = K_{12}[(r_1 - r_2) - r_{12}^d](\dot{r_1} - \dot{r_2}) + K_1(r_1 - r_1^d)\dot{r_1} + K_2(r_2 - r_2^d)\dot{r_2} \tag{2}$$

The above equation gives the control parameter for the unicycle robot.

2 Differential Drive Robot

This section deals with the dynamics of the turtle-bot and how an optimal control can be given by generating the Lyapunov function.

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \omega \end{cases}$$

The Lyapunov function in this case can be written in the matrix form as given below:

$$V = P + A^T K_{12} A + B^T K_1 B + C^T K_2 C \tag{3}$$

Where $P = [1 - \cos(\theta_1 - \theta_2)]$
$A = \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right]$
$B = \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_1^d \\ y_1^d \end{pmatrix}\right]$
$C = \left[\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_2^d \\ y_2^d \end{pmatrix}\right]$

On taking the derivative of V, we will obtain the control equation for the differential drive robot which will consider both the navigation towards goal and reorienting itself in the swarm (if no. of agents is greater than or equal to 2).

$$\frac{dV}{dt} = \frac{\partial V}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial V}{\partial r_2}\frac{dr_2}{dt} + \frac{\partial V}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial V}{\partial \theta_2}\frac{d\theta_2}{dt} \tag{4}$$

$$\frac{dV}{dt} = P + K_{12}Q + K_1 R + K_2 S \tag{5}$$

Where, $P = (\dot{\theta_1} - \dot{\theta_2})\sin(\theta_1 - \theta_2)$
$Q = \begin{bmatrix} (x_1 - x_2 - x_{12}^d)(\dot{x_1} - \dot{x_2}) \\ (y_1 - y_2 - y_{12}^d)(\dot{y_1} - \dot{y_2}) \end{bmatrix}$
$R = \begin{bmatrix} (x_1 - x_1^d)\dot{x_1} \\ (y_1 - y_1^d)\dot{y_1} \end{bmatrix}$
$S = \begin{bmatrix} (x_2 - x_2^d)\dot{x_2} \\ (y_2 - y_2^d)\dot{y_2} \end{bmatrix}$

The above set of equations will be tested in the simulation environment where the setup consists of 2 agents such that the agents are aware of the goal position.