Lyapunov Functions for PID control

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1 Unicycle Bot

In the current scenario, a 2 agent system is considered and the following dynamics describes the nature of the PID control applied.

$$\psi_{12} = ||(r_1 - r_2) - r_{12}^d||^2$$
$$\psi_1 = ||r_1 - r_1^d||^2$$
$$\psi_2 = ||r_2 - r_2^d||^2$$

Constructing the Lyapunov Function,

$$\text{V} = K_{12}\psi_{12} + K_1\psi_1 + K_2\psi_2$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r_1} \cdot \frac{dr_1}{dt} + \frac{\partial V}{\partial r_2} \cdot \frac{dr_2}{dt} \tag{1}$$

$$\frac{dV}{dt} = K_{12}[(r_1 - r_2) - r_{12}^d](\dot{r_1} - \dot{r_2}) + K_1(r_1 - r_1^d)\dot{r_1} + K_2(r_2 - r_2^d)\dot{r_2} \tag{2}$$

The above equation gives the control parameter for the unicycle robot.

2 Differential Drive Robot

This section deals with the dynamics of the turtle-bot and how an optimal control can be given by generating the Lyapunov function.

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \omega \end{cases}$$

The Lyapunov function in this case can be written in the matrix form as given below:

$$V = P + A^T K_{12} A + B^T K_1 B + C^T K_2 C \tag{3}$$

Where $P = [1 - \cos(\theta_1 - \theta_2)]$
$A = \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right]$
$B = \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_1^d \\ y_1^d \end{pmatrix}\right]$
$C = \left[\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_2^d \\ y_2^d \end{pmatrix}\right]$

On taking the derivative of V, we will obtain the control equation for the differential drive robot which will consider both the navigation towards goal and reorienting itself in the swarm (if no. of agents is greater than or equal to 2).

$$\frac{dV}{dt} = \frac{\partial V}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial V}{\partial r_2}\frac{dr_2}{dt} + \frac{\partial V}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial V}{\partial \theta_2}\frac{d\theta_2}{dt} \tag{4}$$

$$\frac{dV}{dt} = P + K_{12} Q + K_1 R + K_2 S \tag{5}$$

Where, $P = (\dot{\theta_1} - \dot{\theta_2})\sin(\theta_1 - \theta_2)$
$Q = \begin{bmatrix} (x_1 - x_2 - x_{12}^d)(\dot{x_1} - \dot{x_2}) \\ (y_1 - y_2 - y_{12}^d)(\dot{y_1} - \dot{y_2}) \end{bmatrix}$
$R = \begin{bmatrix} (x_1 - x_1^d)\dot{x_1} \\ (y_1 - y_1^d)\dot{y_1} \end{bmatrix}$
$S = \begin{bmatrix} (x_2 - x_2^d)\dot{x_2} \\ (y_2 - y_2^d)\dot{y_2} \end{bmatrix}$

Now, the above Lyapunov function will be used for checking the asymptotic stability of the dynamical system. The following control law will guarantee the global exponential stability upon the substitution of time derivative.

$$\frac{dV}{dt} = -kV \tag{6}$$

Hence, applying the above condition in our dynamical system, we will get the expressions of control parameters $\dot{r1}, \dot{r2}, \dot{\theta_1}$ and $\dot{\theta_2}$.

$$\dot{r_1} = \frac{-\alpha[K_{12}(r_1 - r_2 - r_{12}^d)^2 + K_1(r_1 - r_1^d)^2]}{K_{12}(r_1 - r_2 - r_1^d) + K_2(r_2 - r_2^d)} \tag{7}$$

$$\dot{r_2} = \frac{-\alpha[K_{12}(r_1 - r_2 - r_{12}^d)^2 + K_2(r_2 - r_2^d)^2]}{K_{12}(r_1 - r_2 - r_1^d) + K_2(r_2 - r_2^d)} \tag{8}$$

$$\dot{\theta_1} = K_t \sin(\theta_{1err} - \theta_1) \tag{9}$$

$$\dot{\theta_2} = K_t \sin(\theta_{2err} - \theta_2) \tag{10}$$

Where $\theta_{1err} = \tan^{-1}(\frac{y_1^d - y_1}{x_1^d - x_1})$ and
$\theta_{2err} = \tan^{-1}(\frac{y_2^d - y_2}{x_2^d - x_2})$

The above set of inputs gave the optimal control inputs for the systems and were used in the simulation.