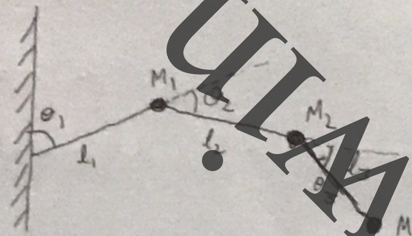
3DOF Robotic arm



$x_1 = l_1 \sin\theta_1$
$x_2 = x_1 + l_2 \sin(\theta_1+\theta_2)$
$x_3 = x_2 + l_3 \sin(\theta_1+\theta_2+\theta_3)$

$y_1 = l_1 \cos\theta_1$
$y_2 = y_1 + l_2 \cos(\theta_1+\theta_2)$
$y_3 = y_2 + l_3 \cos(\theta_1+\theta_2+\theta_3)$

$$KE = \frac{1}{2}M_1\dot{x}_1^2 + \frac{1}{2}M_2\dot{x}_2^2 + \frac{1}{2}M_3\dot{x}_3^2 + \frac{1}{2}M_1\dot{y}_1^2 + \frac{1}{2}M_2\dot{y}_2^2 + \frac{1}{2}M_3\dot{y}_2^2$$

$\dot{x}_1 = l_1 c\theta_1 \dot{\theta}_1$
$\dot{x}_2 = \dot{x}_1 + l_2 c(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$
$\dot{x}_3 = \dot{x}_2 + l_3 c(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)$

$\dot{y}_1 = -l_1 s\theta_1 \dot{\theta}_1$
$\dot{y}_2 = \dot{y}_1 - l_2 s(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)$
$\dot{y}_3 = \dot{y}_2 - l_3 s(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)$

$\dot{x}_1^2+\dot{y}_1^2 = l_1^2\dot{\theta}_1^2$

$\dot{x}_2^2+\dot{y}_2^2 = \dot{x}_1^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2\dot{x}_1 l_2 c(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) - 2\dot{y}_1 l_2 s(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + \dot{y}_1^2 +$

$\dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2(\dot{\theta}_1+\dot{\theta}_2)l_2 l_1\dot{\theta}_1 c\theta_2$

$\dot{x}_3^2+\dot{y}_3^2 = \dot{x}_2^2+\dot{y}_2^2 + l_3^2(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 + 2\dot{x}_2 l_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)c(\theta_1+\theta_2+\theta_3) - 2\dot{y}_2 l_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)s(\theta_1+\theta_2+\theta_3)$

simplifying δ,

$\cancel{\dot{\theta}}\ l_1\dot{\theta}_1 c(\theta_2+\theta_3) + l_2(\dot{\theta}_1+\dot{\theta}_2)c\theta_3$

$\rightarrow \mu + 2l_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\left[l_1\dot{\theta}_1 c(\theta_2+\theta_3) + l_2(\dot{\theta}_1+\dot{\theta}_2)c\theta_3\right]$

$$\therefore KE = \frac{1}{2}M_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}M_2\left[l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2(\dot{\theta}_1+\dot{\theta}_2)l_1 l_2\dot{\theta}_1 c\theta_2\right] + \frac{1}{2}M_3\left[l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2 + 2(\dot{\theta}_1+\dot{\theta}_2)l_1 l_2\dot{\theta}_1 c\theta_2 + l_3^2(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 + 2l_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\left\{l_1\dot{\theta}_1 c(\theta_2+\theta_3) + l_2(\dot{\theta}_1+\dot{\theta}_2)c\theta_3\right\}\right]$$

$$PE = M_1 g l_1 c\theta_1 + M_2 g(l_1 c\theta_1 + l_2 c(\theta_1+\theta_2)) + M_3 g(l_1 c\theta_1 + l_2 c(\theta_1+\theta_2) + l_3 c(\theta_1+\theta_2+\theta_3))$$

$L = KE - PE$ {Lagrangian}

$\tau_{\theta_1} = \frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}_1}\right] - \frac{\partial L}{\partial\theta_1}$

$\tau_{\theta_2} = \frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}_2}\right] - \frac{\partial L}{\partial\theta_2}$

$\tau_{\theta_3} = \frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}_3}\right] - \frac{\partial L}{\partial\theta_3}$

$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}_1}\right] =$

$$\frac{\partial L}{\partial\dot{\theta}_1} = M_1 l_1^2\dot{\theta}_1 + \frac{1}{2}M_2\left[2l_1^2\dot{\theta}_1 + 2l_2^2(\dot{\theta}_1+\dot{\theta}_2) + 2l_1 l_2 c\theta_2(2\dot{\theta}_1+\dot{\theta}_2)\right] + \frac{1}{2}M_3\left[2l_1^2\dot{\theta}_1 + 2l_2^2(\dot{\theta}_1+\dot{\theta}_2) + 2l_1 l_2 c\theta_2(2\dot{\theta}_1+\dot{\theta}_2) + 2l_3(l_1\dot{\theta}_1 c(\theta_2+\theta_3) + l_2 c\theta_3(\dot{\theta}_1+\dot{\theta}_2)) + 2l_3\dot{\theta}_1(l_1 c(\theta_2+\theta_3) + l_2 c\theta_3) + 2l_3(\dot{\theta}_2+\dot{\theta}_3)\left[l_1 c(\theta_2+\theta_3) + l_2 c\theta_3\right] + 2l_3^2(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \ddot\theta_1 + \frac{1}{2}M_2\left[2l_1^2\ddot\theta_1 + 2l_2^2(\ddot\theta_1+\ddot\theta_2) + 2l_1 l_2 c\theta_2(2\ddot\theta_1+\ddot\theta_2) - 2l_1 l_2 s\theta_2(2\dot\theta_1+\dot\theta_2)\right] + \frac{1}{2}M_3\Big[2l_1^2\ddot\theta_1 + 2l_2^2(\ddot\theta_1+\ddot\theta_2) + 2l_1 l_2 c\theta_2(2\ddot\theta_1+\ddot\theta_2) - 2l_1 l_2 s\theta_2(\dot\theta_1+\dot\theta_2) +$$

$$2l_3^2(\ddot\theta_1+\ddot\theta_2+\ddot\theta_3) + 2l_3\Big\{l_1\ddot\theta_1 c(\theta_2+\theta_3) - l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) + l_2 c\theta_3(\ddot\theta_1+\ddot\theta_2) - l_2 s\theta_3(\dot\theta_1+\dot\theta_2) + \ddot\theta_1\left(l_1 c(\theta_2+\theta_3) + l_2 c\theta_3\right) + \dot\theta_1\left(-l_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) - l_2 s\theta_3\right) +$$

$$(\ddot\theta_1+\ddot\theta_3)\left[l_1 c(\theta_2+\theta_3) + l_2 c\theta_3\right] + (\dot\theta_1+\dot\theta_2)\left[-l_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) - l_2 s\theta_3\dot\theta_2\right]\Big\}\Big] - \left\{M_1 g l_1 s_1 + M_2 g\left[l_1 s\theta_1 + l_2 s(\theta_1+\theta_2)\right] + M_3 g\left[l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) + l_3 s(\theta_1+\theta_2+\theta_3)\right]\right\}$$

$$\frac{\partial L}{\partial \dot\theta_2} = M_2 l_2^2(\dot\theta_1+\dot\theta_2) + M_2 l_1 l_2 c\theta_2\dot\theta_1 + M_3 l_2^2(\dot\theta_1+\dot\theta_2) + M_3 l_1 l_2 c\theta_2\dot\theta_1 + M_3 l_2^2(\dot\theta_1+\dot\theta_2+\dot\theta_3) + \cancel{2l_3 l_2 c\theta_3}\ 2l_3(\dot\theta_1+\dot\theta_3) l_2 c\theta_3 + 2l_3\dot\theta_2 c\theta_3 l_2 + 2l_3\left(l_1\dot\theta_1 c(\theta_2+\theta_3) + l_2(\dot\theta_1+\dot\theta_2)c\theta_3\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = M_2 l_2^2(\ddot\theta_1+\ddot\theta_2) + M_2 l_1 l_2 c\theta_2\ddot\theta_1 - M_2 l_1 l_2 s\theta_2\dot\theta_2\dot\theta_1 + M_3 l_2^2(\ddot\theta_1+\ddot\theta_2) + M_3 l_1 l_2\ddot\theta_1 - M_3 l_1 l_2 s\theta_2\dot\theta_1\dot\theta_2 + M_3 l_3^2(\ddot\theta_1+\ddot\theta_2+\ddot\theta_3) + 2l_3(\ddot\theta_1+\ddot\theta_3) l_2 c\theta_3 - 2l_3(\dot\theta_1+\dot\theta_3) l_2 s\theta_3\dot\theta_3 + 2l_3\ddot\theta_2 l_2 c\theta_3$$
$$-2l_3\dot\theta_2 s\theta_3\dot\theta_3 l_2 + 2l_3\left(l_1\ddot\theta_1 c(\theta_2+\theta_3) - l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) + l_2(\ddot\theta_1+\ddot\theta_2)c\theta_3 - l_2(\dot\theta_1+\dot\theta_2)s\theta_3\dot\theta_3\right)$$

$$\frac{\partial L}{\partial \theta_2} = -M_2 l_1 l_2\dot\theta_1 2(\dot\theta_1+\dot\theta_2)s\theta_2 - M_3(\dot\theta_1+\dot\theta_2)l_1 l_2\dot\theta_1 s\theta_2 - M_3 l_3(\dot\theta_1+\dot\theta_2+\dot\theta_3)\left\{l_1\dot\theta_1 s(\theta_2+\theta_3)\right\} + M_2 g l_2 s(\theta_1+\theta_2) + M_3 g\left(l_2 s(\theta_1+\theta_2) + l_3 s(\theta_1+\theta_2+\theta_3)\right)$$

$$k = -M_2 l_1 l_2 s\theta_2\dot\theta_1\dot\theta_2 - M_3 l_1 l_2 s\theta_2\dot\theta_1\dot\theta_2 - M_3 l_3 l_2 s\theta_3\dot\theta_3(\dot\theta_1+\dot\theta_3)$$
$$- M_3 l_3\dot\theta_2 s\theta_3\dot\theta_3 l_2 + M_3 l_3 l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3)$$
$$- M_3 l_3 l_2(\dot\theta_1+\dot\theta_2)s\theta_3\dot\theta_3 + M_2 l_1 l_2\dot\theta_1(\dot\theta_1+\dot\theta_2)s\theta_2$$
$$+ M_3(\dot\theta_1+\dot\theta_2)l_1 l_2\dot\theta_1 s\theta_2 - M_3 l_3(\dot\theta_1+\dot\theta_2+\dot\theta_3)\left\{l_1\dot\theta_1 s(\theta_2+\theta_3)\right\}$$

$$\frac{\partial L}{\partial \dot\theta_3} = M_3 l_3^2(\dot\theta_1+\dot\theta_2+\dot\theta_3) + M_3 l_3\left(l_1\dot\theta_1 c(\theta_2+\theta_3) + l_2(\dot\theta_1+\dot\theta_2)c\theta_3\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_3}\right) = M_3 l_3^2(\ddot\theta_1+\ddot\theta_2+\ddot\theta_3) + M_3 l_3\left\{l_1\ddot\theta_1 c(\theta_2+\theta_3) - l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) + l_2(\ddot\theta_1+\ddot\theta_2)c\theta_3 - l_2(\dot\theta_1+\dot\theta_2)s\theta_3\right\}$$

$$\frac{\partial L}{\partial \theta_3} = M_3 l_3(\dot\theta_1+\dot\theta_2+\dot\theta_3)\left\{-l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) - l_2(\dot\theta_1+\dot\theta_2)s\theta_3\right\} + M_3 g l_3 s(\theta_1+\theta_2+\theta_3)$$

The manipulator motor eq^n is given as $B(q)\ddot q + C(q,\dot q) + g(q) = F$

$B(q)$ = inertia matrix

$C(q)$ =

$g(q)$ = gravity matrix

here the physical inputs are the torques

$$l = -M_3 l_1 l_3\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) - M_3 l_2 l_3(\dot\theta_1+\dot\theta_2)s\theta_3$$
$$+ M_3 l_3(\dot\theta_1+\dot\theta_2+\dot\theta_3)\left\{l_1\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) + l_2(\dot\theta_1+\dot\theta_2)s\theta_3\right\}$$

$$e = M_2 l_2^2 + M_3 l_2^2 + M_3 l_3^2 + M_3 l_2 l_3 c\theta_3 + 2M_3 l_2 l_3 c\theta_3$$
$$f = M_3 l_3^2 + M_3 l_2 l_3 c\theta_3$$
$$g = M_3 l_3^2 + M_3 l_1 l_3 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3$$
$$h = M_3 l_3^2 + M_3 l_2 l_3 c\theta_3$$
$$i = M_3 l_3^2$$

$$B(\theta) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3} \qquad C(\theta,\dot\theta) = \begin{bmatrix} j \\ k \\ l \end{bmatrix}_{3\times1} \qquad g(\theta) = \begin{bmatrix} m \\ n \\ o \end{bmatrix}_{3\times1}$$

$$j = -M_2 l_1 l_2 s\theta_2(2\dot\theta_1+\dot\theta_2) - M_3 l_1 l_2 s\theta_2(\dot\theta_1+\dot\theta_2)$$
$$- M_3 l_1 l_3\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3) - M_3 l_3 l_2 s\theta_3(\dot\theta_1+\dot\theta_2)$$
$$- M_3 l_1 l_3\dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_1+\dot\theta_3)$$
$$- M_3 l_3 l_1(\dot\theta_1+\dot\theta_3) s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3)$$
$$- M_3 l_1 l_3 \dot\theta_1 s\theta \ldots$$

where

$$a = M_1 l_1^2 + M_2 l_2^2 + 2M_1 l_1 l_2 c\theta_2 + M_3 l_1^2 + M_3 l_2^2 + 2M_3 l_1 l_2 c\theta_2 + M_3 l_3^2 + M_3 l_1 l_3 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3 + M_3 l_3\left(l_1 c(\theta_2+\theta_3) + l_2 c\theta_3\right)$$
$$b = M_2 l_2^2 + M_2 l_1 l_2 c\theta_2 + M_3 l_2^2 + M_3 l_1 l_2 c\theta_2 + M_3 l_3^2 + M_3 l_2 l_3 c\theta_3 + M_3 l_3\left[l_1 c(\theta_2+\theta_3) + l_2 c\theta_3\right]$$
$$c = M_3 l_3^2 + M_3 l_3 l_1 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3$$
$$d = M_2 l_2^2 + M_2 l_1 l_2 c\theta_2 + M_3 l_2^2 + M_3 l_1 l_2 + M_3 l_3^2 + M_3 l_2 l_3 c\theta_3 + M_3 l_1 l_3 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3$$

$b = M_2 l_2^2 + M_2 l_1 l_2 c\theta_2 + M_3 l_2^2 + M_3 l_1 l_2 c\theta_2 + M_3 l_3^2 + M_3 l_2 l_3 c\theta_3 + M_3 l_3 [l_1 c(\theta_2+\theta_3) + l_2 c\theta_3]$

$c = M_3 l_3^2 + M_3 l_3 l_1 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3$

$d = M_2 l_2^2 + M_2 l_1 l_2 c\theta_2 + M_3 l_2^2 + M_3 l_1 l_2 + M_3 l_3^2 + M_3 l_2 l_3 c\theta_3 + M_3 l_1 l_3 c(\theta_2+\theta_3) + M_3 l_2 l_3 c\theta_3$

$- M_3 l_1 l_3 \dot\theta_1 s(\theta_2+\theta_3)(\dot\theta_2+\dot\theta_3)$

$- M_3 l_3 l_1 (\dot\theta_1+\dot\theta_3)\, s(\theta_2+\theta_3)(\dot\theta_2$

$- M_3 l_2 l_3 \dot\theta_2\, s\theta_3$

$m = M_1 g l_1 s_1 + M_2 g(l_1 s\theta_1 + l_2 s(\theta_1+\theta_2)) + M_3 g[l_1 s\theta_1 + l_2 s(\theta_1+\theta_2) + l_3 s(\theta_1+\theta_2+\theta_3)]$

$n = M_2 g l_2 s(\theta_1+\theta_2) + M_3 g(l_2 s(\theta_1+\theta_2) + l_3 s(\theta_1+\theta_2+\theta_3))$

$o = M_3 g l_3 s(\theta_1+\theta_2+\theta_3)$

$\dot x_1 = \theta_{1f} - \theta_1$
$\dot x_2 = \theta_{2f} - \theta_2$
$\dot x_3 = \theta_{3f} - \theta_3$

$$B(\theta)\ddot\theta + C(\theta,\dot\theta) + g(\theta) = F$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{bmatrix} + \begin{bmatrix} j \\ k \\ l \end{bmatrix} + \begin{bmatrix} m \\ n \\ o \end{bmatrix} = F = \begin{bmatrix} f_{\theta_1} \\ f_{\theta_2} \\ f_{\theta_3} \end{bmatrix}$$

$$\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{bmatrix} = B(\theta)^{-1}\left[-C(\theta,\dot\theta) - g(\theta)\right] + \begin{bmatrix} K_{p_1}(\theta_{1f}-\theta_1) - K_{D_1}\dot\theta_1 + K_{i_1}x_1 \\ K_{p_2}(\theta_{2f}-\theta_2) - K_{D_2}\dot\theta_2 + K_{i_2}x_2 \\ K_{p_3}(\theta_{3f}-\theta_3) - K_{D_3}\dot\theta_3 + K_{i_3}x_3 \end{bmatrix}$$

$$\ddot\theta = B^{-1}[-C-g] + \hat F$$

$$\hat F = B^{-1}F$$

$$B\hat F = F$$

set $\hat F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

error signals

$e(\theta_1) = \theta_{1f} - \theta_1$
$e(\theta_2) = \theta_{2f} - \theta_2$
$e(\theta_3) = \theta_{3f} - \theta_3$

θ_{1f}, θ_{2f} & θ_{3f} are defined

PID controller for joint torque inputs

$$f = K_p e + K_d \dot e + K_i \int e\,dt$$

$f_1 = K_{p_1}(\theta_{1f} - \theta_1) - K_{d_1}\dot\theta_1 + K_{i_1}\int e(\theta_1)\,dt$

$f_2 = K_{p_2}(\theta_{2f} - \theta_2) - K_{d_2}\dot\theta_2 + K_{i_2}\int e(\theta_2)\,dt$

$f_3 = K_{p_3}(\theta_{3f} - \theta_3) - K_{d_3}\dot\theta_3 + K_{i_3}\int e(\theta_3)\,dt$