CONTROL SYSTEMS ENGINEERING

S. SALIVAHANAN | R. RENGARAJ
G. R. VENKATAKRISHNAN

ALWAYS LEARNING
PEARSON

Control Systems Engineering

This Page is intentionally left blank.

Control Systems Engineering

S. Salivahanan
Principal
SSN College of Engineering
Chennai

R. Rengaraj
Associate Professor
Department of Electrical and Electronics Engineering
SSN College of Engineering
Chennai

G. R. Venkatakrishnan
SSN College of Engineering
Chennai

PEARSON

Chennai • Delhi

Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128, formerly known as TutorVista Global Pvt. Ltd, licensee of Pearson Education in South Asia.

ISBN 978-93-325-3413-1
eISBN 978-93-325-4475-8

Head Office: A-8 (A), 7th Floor, Knowledge Boulevard, Sector 62, Noida 201 309, Uttar Pradesh, India.
Registered Office: Module G4, Ground Floor, Elnet Software City, TS-140, Blocks 2 & 9,
Rajiv Gandhi Salai, Taramani, Chennai 600 113, Tamil Nadu, India.
Fax: 080-30461003, Phone: 080-30461060
www.pearson.co.in, Email: companysecretary.india@pearson.com

About the Authors

S. Salivahanan is the Principal of SSN College of Engineering, Chennai. He obtained his Ph.D. in microwave integrated circuits and has more than three and a half decades of teaching, research, administration and industrial experience both in India and abroad. He is the author of many bestsellers and has had several papers published at national and international journals. He is also the recipient of Bharatiya Vidya Bhavan National Award for Best Engineering College Principal for 2011 from ISTE, and IEEE Outstanding Branch Counselor and Advisor Award in the Asia-Pacific region for 1996–97. He is also a recipient of Tata Rao Gold Medal from the Institution of Engineers (India). He is a senior member of IEEE, Fellow of IETE, Fellow of Institution of Engineers (India), Life Member of ISTE and Life Member of Society for EMC Engineers.

R. Rengaraj is an Associate Professor in the Department of Electrical and Electronics Engineering, SSN College of Engineering, Chennai. He received his Ph.D. in the area of combined heat and power. He has more than ten years of academic experience. Many of his articles have been published in refereed international journals and in the proceedings of international conferences. He is also a recipient of TATA Rao Gold Medal from the Institution of Engineers (India). His areas of research interest include control systems and power systems.

G. R. Venkatakrishnan is a doctoral research scholar in the area of renewable energy sources in the Department of Electrical and Electronics Engineering, SSN College of Engineering, Anna University, Chennai. He has published many research papers in national and international journals and conference proceedings.

This Page is intentionally left blank.

Brief Contents

This Page is intentionally left blank.

Contents

This Page is intentionally left blank.

Preface

The knowledge of control system is essential for engineering graduates in solving many stability design problems arising in electrical, mechanical, aerospace, biomedical and chemical systems. Since control system is an integral part of our modern society, it finds a wide range of applications in aircraft, robots and process control systems. We have developed this book for subjects, such as control system, process instrumentation, dynamics and control, aeronautical engineering and physiological control system, taught in B.E./B.Tech., AMIE and Grade IETE degree programs. The book will also serve as a useful reference for competitive examinations. We are optimistic that this book shall fill the void created by a lack of standard books on the subject of control systems. Ensuring this book available in the libraries of all universities, colleges and polytechnics will certainly help in enriching the readers. The various concepts of the subject have been arranged aesthetically and explained in a simple and reader-friendly language. For the better understanding of the subject, a large number of numerical problems with step-by-step solutions have been provided. The solutions to university questions have also been included. A set of review questions at the end of each chapter will help the readers to test their understanding of the subject.

This book is divided into 14 chapters. Chapter 1 deals with modeling of electrical and mechanical systems. Chapter 2 explains the various components and modeling of physical systems. Chapter 3 concentrates on the block diagram reduction technique in determining the transfer function of a system. Chapter 4 focuses on the signal flow graph in determining the transfer function of a system. Chapter 5 discusses time response analysis of the system. Chapter 6 describes the stability of the system using Routh's array. Chapter 7 elaborates on root locus technique. Chapter 8 deals with frequency response analysis and Bode plot. Chapter 9 concentrates on polar and nyquist plot. Chapter 10 discusses constant *M*- and *N*-circles and Nichols chart. Chapter 11 is devoted to compensators of the system. Chapter 12 deals with physiological control system. Chapter 13 covers the state-space analysis of the system. Chapter 14 is devoted to MATLAB programs related to control systems. All the topics of this book have been illustrated with clear diagrams which are aided by lucid language to facilitate easy understanding of the concepts.

Acknowledgements

We would like to sincerely thank the management of SSN College of Engineering, Chennai, for their constant encouragement and providing necessary facilities for completing this project. We are also thankful to our colleagues, especially V. S. Nagarajan and V. Thiyagarajan, Assistant Professors, and students Mandala Deekshith, B. Akshaya Padma Varshini and G. Deepika for their review and feedback to improve this book.

We are indebted to R. Gopalakrishnan and A. Chakkarabani of SSN College of Engineering, Chennai for word-processing the manuscript of the entire book.

A special thanks to Sojan Jose, R. Pritha and C. Purushothaman of Pearson India Education Services Pvt. Ltd, Chennai for bringing out this book successfully in a short span of time.

Finally, we extend heartfelt thanks to our family members—Kalavathy Salivahanan, S. Santhosh Kanna, S. Subadesh Kanna, Rajalakshmi Rengaraj, R. Harivarshan, R. Devprasath, G. S. Rajan Babu, Sumathi Babu, G. R. Hemalakshmi Prakash and G. R. Jeya Jeyaprakash—for their patience and cheerfulness throughout the whole process.

We will appreciate any constructive suggestions and feedback from the readers for further improvement of this book at salivahanans@hotmail.com.

S. Salivahanan
R. Rengaraj
G. R. Venkatakrishnan

Control Systems Engineering

This Page is intentionally left blank.

1

CONTROL SYSTEM MODELING

1.1 Introduction

A system is a combination of components connected to perform a required action. The control component of a system plays a major role in altering or maintaining the system output based on our desired characteristics. There are two types of control systems: manual control and automatic control. For example, in manual control, a man can switch on or switch off the bore well motor to control the level of water in a tank. On the other hand, in automatic control, level switches and transducers are used to control the level of water in a tank.

Control systems have naturally evolved in our ecosystem. In almost all living things, automatic control regulates the conditions necessary for life by tackling the disturbance through sensing and controlling functionalities. They operate complex systems and processes and achieve control with desired precision. The application of control systems facilitates automated manufacturing processes, accurate positioning and effective control of machine tools. They guide and control space vehicles, aircrafts, ships and high-speed ground transportation systems. Modern automation of a plant involves components such as sensors, instruments, computers and application of techniques that involve data processing and control.

It is essential to understand a system and its characteristics with the help of a model, before creating a control for it. The process of developing a model is known as *modeling*. Physical systems are modeled by applying notable laws that govern their behaviour. For example, mechanical systems are described by Newton's laws and electrical systems are described by Ohm's law, Kirchhoff's law, Faraday's and Lenz's law. These laws form the basis for the *constitutive properties* of the elements in a system.

1.2 Classification of Control System

Control systems are generally classified as (i) open-loop control systems and (ii) closed-loop control systems as shown in Fig. 1.1.

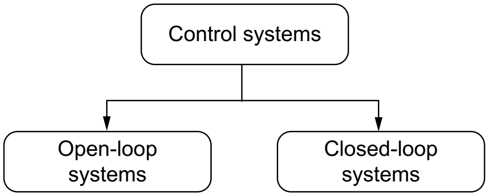


Fig. 1.1 | Classification of control systems

1.2.1 Open-Loop Control System

A control system that cannot adjust itself to the changes is called open-loop control system. In general, manual control systems are open-loop systems. The block diagram of open-loop control system is shown in Fig.1.2.

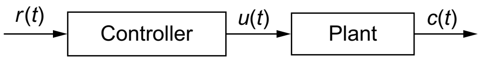


Fig. 1.2 | Block diagram of an open-loop control system

Here, $r(t)$ is the input signal, $u(t)$ is the control signal/actuating signal and $c(t)$ is the output signal.

In this system, the output remains unaltered for a constant input. In case of any discrepancy, the input should be manually changed by an operator. An open-loop control system is suited when there is tolerance for fluctuation in the system and when the system parameter variation can be handled irrespective of the environmental conditions.

In olden days, air conditioners were fitted with regulators that were manually controlled to maintain the desired temperature. This real-time system serves as an open-loop control system as shown in Fig.1.3.

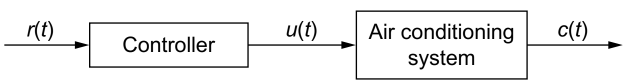


Fig. 1.3 | Open-loop control system of an air conditioner

1.2.2 Closed-Loop Control System

Any system that can respond to the changes and make corrections by itself is known as closed-loop control system. The only difference between open-loop and closed-loop systems is the feedback action. The block diagram of a closed-loop control system is shown in Fig.1.4.

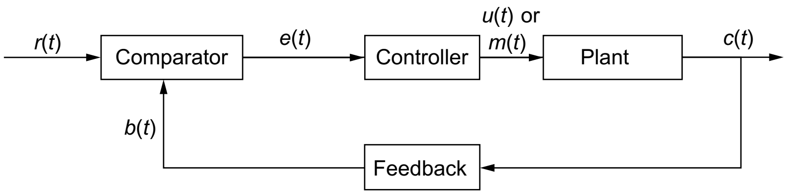


Fig. 1.4 | Block diagram of closed-loop control system

Here, $r(t)$ is the input signal, $e(t)$ is the error signal/actuating signal, $u(t)$ or $m(t)$ is the control signal/manipulated signal, $b(t)$ is the feedback signal and $c(t)$ is the controlled output.

Here, the output of the machine is fed back to a comparator (error detector). The output signal is compared with the reference input $r(t)$ and the error signal $e(t)$ is sent to the controller. Based on the error, the controller adjusts the air conditioners input [control signal $u(t)$]. This process is continued till the error gets nullified. Both manual and automatic controls can be implemented in a closed-loop system. The overall gain of a system is reduced due to the presence of feedback. In order to compensate for the reduction of gain, if an amplifier is introduced to increase the gain of a system, the system may sometimes become unstable.

Present-day air conditioners are designed with temperature sensors, comparators and controller modules. The reference temperature (the desired room temperature) is fed and compared with actual room temperature, which is sensed by a temperature sensor. The difference between these two values, i.e., the error signal is fed to a controller and the controller performs the necessary action to minimize the error. The block diagram for this example is shown in Fig.1.5.

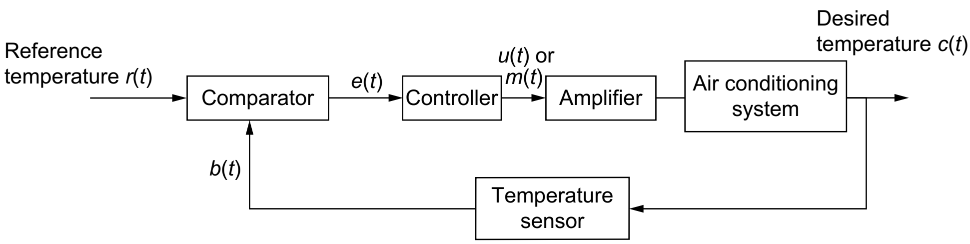


Fig. 1.5 | Closed-loop control system of an air conditioner

A person driving a car is also an example for a closed-loop control system as shown in Fig.1.6. During the ride, the brain controls the hands for steering, gear, horn and the legs for brake and accelerator so as to perform the driving in a perfect manner. The eyes and the ears form the feedback, i.e., eyes for determining the pathway and mirror view and ears for sensing other vehicles nearby. The driver also accelerates or decelerates depending on the terrain and traffic involved.

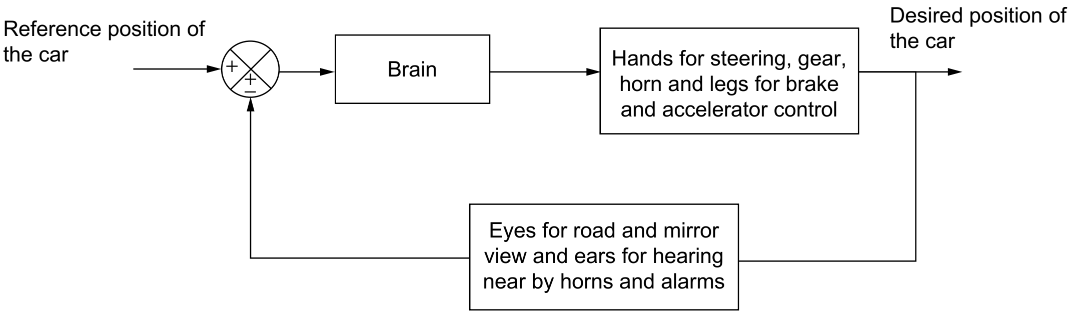


Fig. 1.6 | Closed-loop system of a driver controlled car

1.3 Comparison of Open-Loop and Closed-Loop Control Systems

Table 1.1 discusses the comparison of open-loop and closed-loop control systems.

Table 1.1 | Comparison of open-loop and closed-loop control systems

S. No.	Open-loop control system	Closed-loop control system
1.	The open-loop system is not preferred due to inaccuracy and unreliability.	A closed-loop control system is more accurate, because the error between reference signal and the output is continuously measured through feedback.
2.	System is more economical and simple to construct.	System is expensive and complex to construct.
3.	System is generally stable.	Because of feedback, a system tries to over correct itself, which sometimes leads the system to oscillate. Hence, this is less stable and the overall gain of the system is also reduced.
4.	Stability of a system is affected as it does not adapt to variations of environmental conditions or to external disturbances.	Stability of a system is not affected, due to reduced effects of non-linearities and distortion.

1.4 Differential Equations and Transfer Functions

For the analysis and design of a system, a mathematical model that describes its behaviour is required. The process of obtaining the mathematical description of a system is known as *modeling*. The differential equations for a given system are obtained by applying appropriate laws, for example, Newton's laws for mechanical systems and Kirchhoff's law for electrical systems. These equations may be either linear or non-linear based on the system being modeled. Therefore, deriving appropriate mathematical models is the most important part of the analysis of a system.

A system is said to be *linear* if it obeys the principles of superposition and homogeneity. If a system has responses $c_1(t)$ and $c_2(t)$ for any two inputs $r_1(t)$ and $r_2(t)$ respectively, then the system response to the linear combination of these inputs $a_1r_1(t)+a_2r_2(t)$ is given by the linear combination of the individual outputs $a_1c_1(t)+a_2c_2(t)$, where a_1 and a_2 are constants. The principle of homogeneity and superposition is explained in Fig. 1.7.

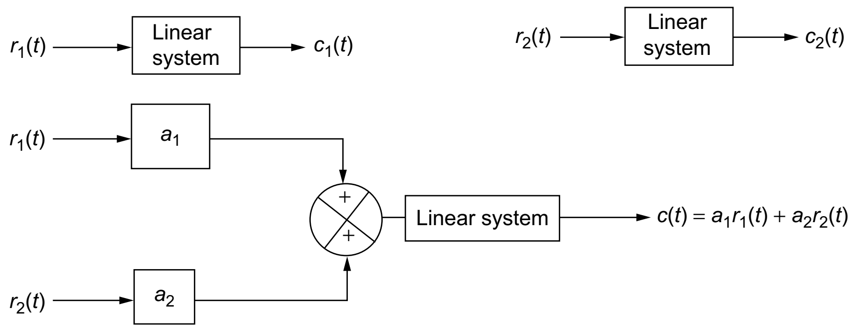


Fig. 1.7 | Principle of homogeneity and superposition

If the coefficients of the differential equations are functions of time (an independent variable), then a system is called *linear time-varying system*. On the other hand, if the coefficients of the differential equations are not a function of time, then a system is called *linear time-invariant system*. For the transient-response or frequency-response analysis of single-input and single-output linear time-invariant system, the transfer function representation will be a convenient tool. But, if a system is subjected to multi-input and multi-output, then state-space analysis will be a convenient tool.

1.4.1 Transfer Function Representation

The transfer function of a linear, time-invariant, differential equation system is given by the ratio of Laplace transform of output variable (response function) to the Laplace transform of the input variable (driving function) under the assumption that all initial conditions are zero.

Transfer function, $G(s)=\dfrac{\mathcal{L}\left[c(t)\right]}{\mathcal{L}\left[r(t)\right]}=\dfrac{C(s)}{R(s)}$ at zero initial conditions.

It is to be noted that non-linear systems and time-varying systems do not have transfer functions because they do not obey the principles of superposition and homogeneity. With the concept of transfer functions, it is possible to represent system dynamics by algebraic equations in s domain. The highest power of s in the denominator of a transfer function is the order of the system.

1.4.2 Features and Advantages of Transfer Function Representation

The following are the features and advantages of transfer function representation:

(i) Using transfer functions, mathematical models can be obtained and analyzed.
(ii) Output response can be obtained for any kind of inputs.
(iii) Stability analysis can be performed.
(iv) The usage of Laplace transform converts complex time domain equations to simple algebraic equations, expressed with complex variable s.
(v) Analysis of a system is simplified due to the use of s-domain variable in the equations, rather than using time-domain variable.

1.4.3 Disadvantages of Transfer Function Representation

The following are the disadvantages of transfer function representation:

(i) It is not applicable to non-linear systems or time-varying systems.
(ii) Initial conditions are neglected.
(iii) Physical nature of a system cannot be found (i.e., whether it is mechanical or electrical or thermal system).

1.4.4 Transfer Function of an Open-Loop System

In the block diagram of a system relating its input and output as shown in Fig.1.8, there exists an input (excitation) which operates through a transfer operator called transfer function and it produces an output (response).

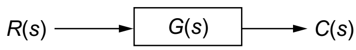


Fig. 1.8 | Block diagram representation of an open-loop transfer function

From the block diagram representation of an open-loop transfer function, we have

$$G(s) = \frac{C(s)}{R(s)}$$

where $R(s)$ is the Laplace transform of the input variable or excitation, $C(s)$ is the Laplace transform of the output variable or response and $G(s)$ is the transfer function.

1.4.5 Transfer Function of a Closed-Loop System

Closed-loop control systems can be classified into two categories, based on the type of feedback signal as (i) Negative feedback systems and (ii) Positive feedback systems.

(i) Negative Feedback Systems

The simplest form of a negative-feedback control system is shown in Fig. 1.9. It has one block in the forward path and one in the feedback path.

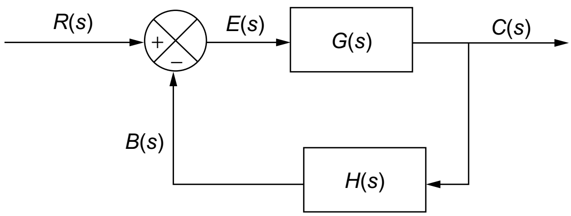


Fig. 1.9 | Block diagram representation of negative feedback control system

Practically, all control systems can be reduced to this form. $G(s)$ represents the overall gain of the blocks in the forward path, $B(s)$ is the feedback signal, $E(s)$ is the error signal and $H(s)$ represents the overall gain of all the blocks in the feedback path.

The objective is to reduce this control system to a single block by determining the closed-loop transfer function (CLTF) for the negative feedback.

The following relationships can be derived from the block diagram:

Feedback signal, $B(s)$ = output × feedback path gain

$$B(s) = C(s)H(s) \tag{1.1}$$

Error signal, $E(s)$ = input signal – feedback signal

$$E(s) = R(s) - B(s)$$

Substituting Eqn. (1.1) in the above equation, we obtain

$$E(s) = R(s) - C(s)H(s) \tag{1.2}$$

Output signal, $C(s) = \text{error signal} \times \text{forward path gain}$

$$C(s) = E(s)G(s) \tag{1.3}$$

Substituting the value of the error signal, we obtain

$$C(s) = R(s)G(s) - C(s)G(s)H(s)$$

$$C(s) + C(s)G(s)H(s) = R(s)G(s)$$

$$C(s)\left(1 + G(s)H(s)\right) = R(s)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Hence, the transfer function of the negative feedback control system is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{1.4}$$

Based on the above expression, Fig. 1.9 is reduced to the simplified form as shown in Fig. 1.10.

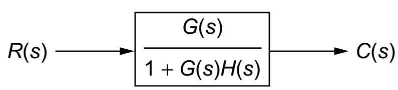


Fig. 1.10 | Reduced form of a negative feedback control system

Thus, the closed-loop transfer function is

$$T(s) = \frac{C(s)}{R(s)} = \frac{\text{forward path gain}}{1 + \text{forward path gain} \times \text{feedback path gain}}$$

Hence, the output, $C(s)$ = closed-loop transfer function $T(s)$ × input $R(s)$.
Observations drawn from this relationship are:

(i) The control system transfer function is a property of the system.
(ii) The transfer function is dependent only on its internal structure and components.
(iii) The transfer function is independent of the input applied to a system.
(iv) When an input signal is applied to a closed-loop control system, an output is generated; i.e., it is dependent on the input as well as the system transfer function.

(ii) Positive Feedback Systems

If the feedback is positive, then the closed-loop transfer function can be similarly derived as

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)} \tag{1.5}$$

The block diagram representing positive feedback control system is shown in Fig. 1.11.

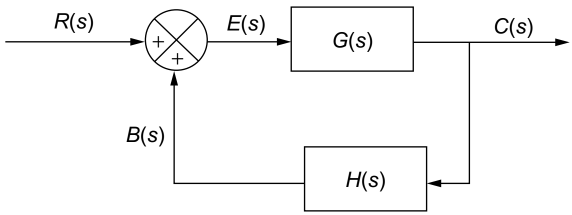


Fig. 1.11 | Block diagram representation of positive feedback control system

1.4.6 Comparison of Positive Feedback and Negative Feedback Systems

Table 1.2 discusses the comparison of characteristics between positive feedback and negative feedback systems.

Table 1.2 | Characteristics of positive feedback and negative feedback systems

Characteristics	Negative feedback	Positive feedback
Stability	High	Low
Magnitude of transfer function	<1	>1
Sensitivity to parameter changes	Low	High
Gain	Low	High
Error signal	Decreased	Increased
Application	Motor control	Oscillators

Example 1.1: A negative feedback system has a forward gain of 10 and feedback gain of 1. Determine the overall gain of the system.

Solution: Given the forward gain $G(s) = 10$ and the feedback gain $H(s) = 1$

The overall gain of a closed-loop system is same as its closed-loop transfer function. For a negative feedback system, the transfer function is given by

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Substituting the values of $G(s)$ and $H(s)$, we obtain overall gain of a closed-loop system,

$$T(s) = \frac{10}{1 + (10 \times 1)} = 0.9091.$$

Example 1.2: A negative feedback system with a forward gain of 2 and a feedback gain of 8 is subjected to an input of 5 V. Determine the output voltage of the system.

Solution: Given the forward gain $G(s) = 2$, feedback gain $H(s) = 8$ and input $R(s) = 5$ V. Output voltage for any system is given by

$$C(s) = \text{Transfer function of the closed-loop system} \times \text{input } R(s)$$

The transfer function of the closed-loop system with negative feedback is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Substituting the values of $G(s)$ and $H(s)$, we obtain

$$\frac{C(s)}{R(s)} = \frac{2}{1 + (2 \times 8)} = 0.1176$$

Therefore, the output voltage of the given system $C(s) = 0.1176 \times 5 = 0.5882$ V.

Example 1.3: A positive feedback system was subjected to an input of 3 V. Determine the output voltage of the following sets of gains:

(i) $G(s) = 1,\ H(s) = 0.75$

(ii) $G(s) = 1,\ H(s) = 0.9$

(iii) $G(s) = 1,\ H(s) = 0.99$

(iv) $G(s) = 1.9,\ H(s) = 0.5$

(v) $G(s) = 1.99,\ H(s) = 0.5$

(vi) $G(s) = 1.999,\ H(s) = 0.5$

Solution: It is known that $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 - G(s)H(s)}$

$$C(s) = \frac{G(s)}{1 - G(s)H(s)} \times R(s)$$

The output voltage $C(s)$ for different cases can be obtained by substituting the different values of $G(s)$, $H(s)$ and $R(s)$.

(i) When $G(s)=1$, $H(s)=0.75$ and $R(s)=3$ V,

$$C(s)=\frac{1}{1-(1\times 0.75)}\times 3=12\ \text{V}$$

(ii) When $G(s)=1$, $H(s)=0.9$ and $R(s)=3$ V,

$$C(s)=\frac{1}{1-(1\times 0.9)}\times 3=30\,\text{V}$$

(iii) When $G(s)=1$, $H(s)=0.99$ and $R(s)=3$ V,

$$C(s)=\frac{1}{1-(1\times 0.99)}\times 3=300\,\text{V}$$

(iv) When $G(s)=1.9$, $H(s)=0.5$ and $R(s)=3$ V,

$$C(s)=\frac{1.9}{1-(1.9\times 0.5)}\times 3=114\,\text{V}$$

(v) When $G(s)=1.99$, $H(s)=0.5$ and $R(s)=3$ V,

$$C(s)=\frac{1.99}{1-(1.99\times 0.5)}\times 3=1194\ \text{V}$$

(vi) When $G(s)=1.999$, $H(s)=0.5$ and $R(s)=3$V,

$$C(s)=\frac{1.999}{1-(1.999\times 0.5)}\times 3=11994\ \text{V}$$

It can be noted that, the output becomes very large and approaches infinity when the loop gain approaches a value of 1, i.e., $G(s)H(s)=1$.

1.5 Mathematical Modeling

The process involved in modeling a system using mathematical equations, formed by the variables and constants of the system is called the mathematical modeling of a system. For example, an electrical network can be modeled using the equations formed by Kirchhoff's laws.

Fig 1.12 shows a flow chart for determining the transfer function of any system.

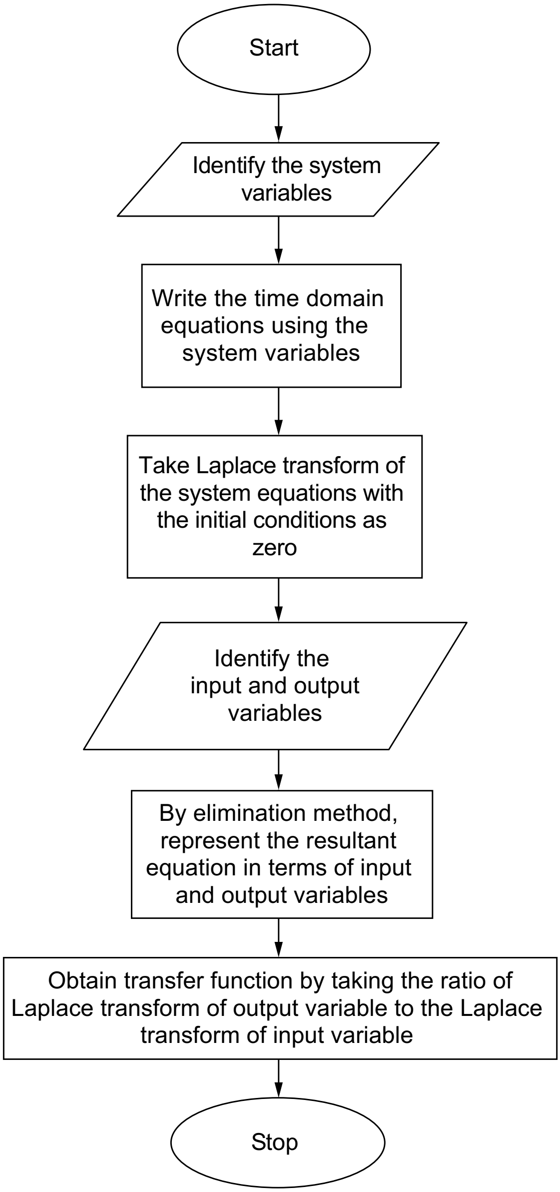


Fig. 1.12 | Flow chart for deriving the transfer function

1.5.1 Mathematical Equations for Problem Solving

Basic Laplace transform of different functions are

$$\mathcal{L}\left(f(t)\right) = F(s)$$

$$\mathcal{L}\left(f'(t)\right) = sF(s) - f(0)$$

$$\mathcal{L}\left(\int f(t)dt\right)=\frac{F(s)}{s}-\frac{f(0)}{s}$$

Cramer's rule:
For a given pair of simultaneous equations, we have

$$a_1x+b_1y=c_1$$

$$a_2x+b_2y=c_2$$

The above equations can be written in matrix form as

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Therefore,

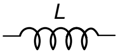
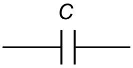
$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=\Delta_x, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=\Delta_y$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\Delta.$

1.6 Modeling of Electrical Systems

An electrical system consists of resistors, capacitors and inductors. The differential equations of electrical systems can be formed by applying Kirchhoff's laws. The transfer function can be obtained by taking Laplace transform of the integro-differential equations and rearranging them as a ratio of output to input. The relationship between voltage and current for different elements in the electrical circuit is given in Table 1.3.

Table 1.3 | Relationship of voltage and current for R, L and C

Element	Voltage drop across the element	Current through the element
R 	$v(t)=Ri(t)$	$i(t)=\frac{v(t)}{R}$
L 	$v(t)=L\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int v(t)dt$
C 	$v(t)=\frac{1}{C}\int i(t)dt$	$i(t)=C\frac{dv(t)}{dt}$

Example 1.4: **A rotary potentiometer is being used as an angular position transducer. It has a rotational range of 0°–300° and the corresponding output voltage is 15 V. The potentiometer is linear over the entire operating range. Determine its transfer function.**

Solution: Given output voltage = 15 V and input rotation = 300°

The transfer function of the potentiometer $T(s)$ is given by

$$T(s) = \frac{\text{output voltage}}{\text{input rotation}}$$

Substituting the given values, we obtain

$$T(s) = \frac{15}{300} = 0.05 \text{ V / degree}$$

Example 1.5: **Determine the transfer function of the electrical network shown in Fig. E1.5.**

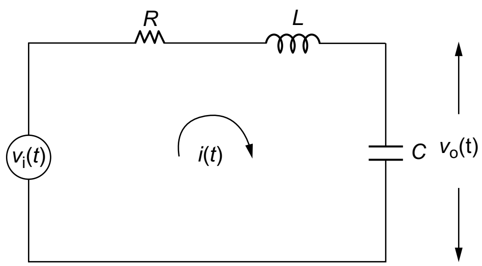


Fig. E1.5

Solution: The output voltage across the capacitor is $v_o(t) = \frac{1}{C}\int i(t)dt$

Applying Kirchhoff's voltage law, the loop equation is given by

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

Taking Laplace transform of the above equations, we obtain

$$V_o(s) = \frac{I(s)}{Cs} \tag{1}$$

$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right)I(s)$$

Simplifying the above equation, we obtain

$$I(s) = \frac{V_i(s)}{\left(R + Ls + \frac{1}{Cs}\right)} \tag{2}$$

Substituting Eqn. (2) in Eqn. (1), we obtain

$$V_o(s) = \frac{V_i(s)}{\left(R + Ls + \frac{1}{Cs}\right)} \times \frac{1}{Cs}$$

Hence, the transfer function of the system is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Example 1.6: Determine the transfer function of the electrical network shown in Fig. E1.6(a).

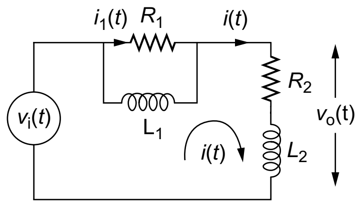


Fig. E1.6(a)

Solution: In the given circuit, there exist two impedances Z_1 and Z_2 as shown in Fig. E1.6(b).

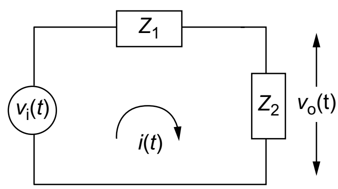


Fig. E1.6(b)

Therefore,

$$Z_1 = R_1 \text{ parallel to } L_1$$

$$= \frac{R_1 \times L_1 s}{R_1 + L_1 s}$$

$$Z_2 = R_2 \text{ series with } L_2$$

$$= R_2 + L_2 s$$

The output voltage across the impedance Z_2 is $v_o(t) = i(t) \times Z_2$

Applying Kirchhoff's voltage law, the loop equation is given by

$$v_i(t) = i(t) \times Z_1 + i(t) \times Z_2$$

Taking Laplace transform of the above equations, we obtain

$$V_o(s) = I(s) \times Z_2 \tag{1}$$

$$V_i(s) = I(s) \times (Z_1 + Z_2)$$

Simplifying, we obtain

$$I(s) = \frac{V_i(s)}{(Z_1 + Z_2)} \tag{2}$$

Substituting Eqn. (2) in Eqn. (1), we obtain

$$V_0(s) = \frac{V_i(s) \times Z_2}{(Z_1 + Z_2)}$$

Hence, the transfer function of the system is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

Therefore,

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2 + L_2 s}{\left(\frac{R_1 \times L_1 s}{R_1 + L_1 s}\right) + (R_2 + L_2 s)} = \frac{L_1 L_2 s^2 + (R_2 L_1 + R_1 L_2)s + R_1 R_2}{L_1 L_2 s^2 + ((R_1 + R_2)L_1 + R_1 L_2)s + R_1 R_2}$$

Example 1.7: Determine the transfer function of the electrical network shown in Fig. E1.7.

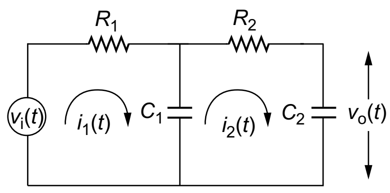


Fig. E1.7

Solution: The output voltage of the given electrical network is

$$v_o(t) = \frac{1}{C_2}\int i_2(t)\,dt$$

Taking Laplace transform, we obtain

$$V_o(s) = I_2(s) \times \frac{1}{C_2 s} \tag{1}$$

The loop equations for each loop can be obtained by applying Kirchhoff's voltage law. Loop equation for loop (1) is

$$v_i(t) = i_1(t) \times R_1 + \frac{1}{C_1}\int \left(i_1(t) - i_2(t)\right)dt$$

Taking Laplace transform on both the sides, we obtain

$$V_i(s) = I_1(s) \times \left(R_1 + \frac{1}{C_1 s}\right) - I_2(s) \times \frac{1}{C_1 s} \tag{2}$$

Loop equation for loop (2) is

$$0 = \frac{1}{C_1}\int \left(i_2(t) - i_1(t)\right)dt + i_2(t) \times R_2 + \frac{1}{C_2}\int i_2(t)\,dt$$

Taking Laplace transform, we obtain

$$0 = -I_1(s) \times \frac{1}{C_1 s} + I_2(s) \times \left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) \tag{3}$$

Representing Eqn. (2) and Eqn. (3) in matrix form,

$$\begin{bmatrix} V_i(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + \dfrac{1}{C_1 s} & \dfrac{-1}{C_1 s} \\ \dfrac{-1}{C_1 s} & R_2 + \dfrac{1}{C_1 s} + \dfrac{1}{C_2 s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

Applying Cramer's rule, we obtain

$$I_2(s) = \frac{\Delta_2}{\Delta}$$

where $$\Delta_2 = \begin{vmatrix} R_1 + \dfrac{1}{C_1 s} & V_i(s) \\ \dfrac{-1}{C_1 s} & 0 \end{vmatrix} = \frac{V_i(s)}{C_1 s}$$

and
$$\Delta = \begin{vmatrix} R_1 + \dfrac{1}{C_1 s} & \dfrac{-1}{C_1 s} \\ \dfrac{-1}{C_1 s} & R_2 + \dfrac{1}{C_1 s} + \dfrac{1}{C_2 s} \end{vmatrix} = \left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \frac{1}{C_1^2 s^2}$$

$$= R_1 R_2 + \frac{R_1}{C_1 s} + \frac{R_1}{C_2 s} + \frac{R_2}{C_1 s} + \frac{1}{C_1 C_2 s^2}$$

$$= \frac{(R_1 R_2 C_1 C_2)s^2 + (R_1 C_2 + R_1 C_1 + R_2 C_2)s + 1}{C_1 C_2 s^2}$$

Therefore,

$$I_2(s) = \frac{\Delta_2}{\Delta} = \frac{\dfrac{V_i(s)}{C_1 s} \times C_1 C_2 s^2}{(R_1 R_2 C_1 C_2)s^2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1} = \frac{V_i(s) \times C_2 s}{(R_1 R_2 C_1 C_2)s^2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1}$$

But, $V_o(s) = \dfrac{I_2(s)}{C_2 s} = \dfrac{V_i(s)}{\left[(R_1 R_2 C_1 C_2)s^2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1\right]}$

Hence, the overall transfer function of the given electrical system is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{(R_1 R_2 C_1 C_2)s^2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1}$$

Example 1.8: Determine the transfer function of the electrical network shown in Fig. E1.8(a).

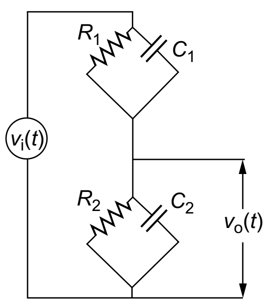


Fig. E1.8(a)

Solution: The above electrical network can be modified as shown in Fig. E1.8(b).

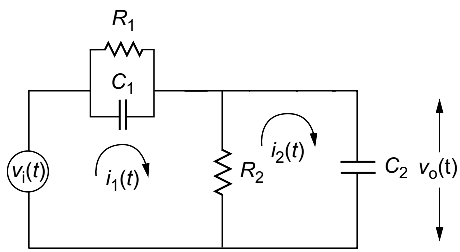


Fig. E1.8(b)

The output voltage of the given electrical circuit is

$$v_o(t) = \frac{1}{C_2}\int i_2(t)\,dt$$

Applying Laplace transform, we obtain

$$V_o(s) = \frac{1}{C_2 s} \times I_2(s) = \frac{I_2(s)}{C_2 s} \qquad (1)$$

Applying Laplace transform, we obtain transformation of the parallel RC circuit as shown in Fig. E1.8(c).

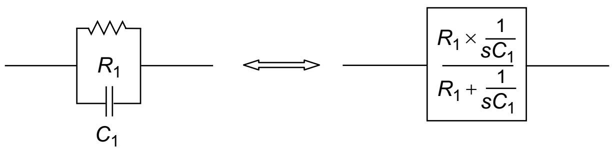


Fig. E1.8(c)

Thus, the given electrical circuit is re-drawn as shown in Fig. E1.8(d).

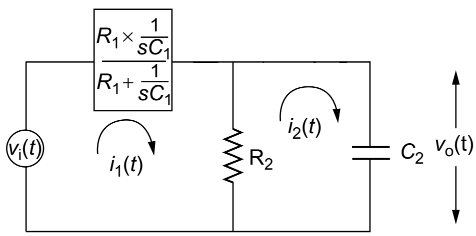


Fig. E1.8(d)

Now, Kirchhoff's voltage law is applied to the individual loops for the above electrical circuit. Then the equation for loop (1) is

$$V_{\mathrm{i}}(s) = I_1(s) \times \left(\frac{\frac{R_1}{C_1 s}}{R_1 + \frac{1}{C_1 s}} \right) + \left(I_1(s) - I_2(s)\right) \times R_2$$

$$= I_1(s) \times \left(\frac{\frac{R_1}{C_1 s}}{\frac{R_1 C_1 s + 1}{C_1 s}} \right) + I_1(s) \times R_2 - I_2(s) \times R_2$$

$$= I_1(s) \times \left(\frac{R_1}{R_1 C_1 s + 1} \right) + I_1(s) \times R_2 - I_2(s) \times R_2$$

$$= I_1(s) \left(\frac{R_1}{s R_1 C_1 + 1} + R_2 \right) - I_2(s) R_2$$

$$= I_1(s) \times \left(\frac{R_1 + R_2 (R_1 C_1 s + 1)}{R_1 C_1 s + 1} \right) - I_2(s) \times R_2 \tag{2}$$

Loop equation for loop (2) is

$$0 = I_2(s) \times \left(\frac{1}{C_2 s} \right) + I_2(s) \times R_2 - I_1(s) \times R_2$$

$$= I_2(s) \times \left(\frac{1}{C_2 s} + R_2 \right) - I_1(s) \times R_2$$

$$= -\left(I_1(s) \times R_2\right) + I_2(s) \times \left(R_2 + \frac{1}{C_2 s} \right) \tag{3}$$

The Eqn. (2) and Eqn. (3) are written in matrix form as

$$\begin{pmatrix} V_{\mathrm{i}}(s) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{R_1 + R_2 (R_1 C_1 s + 1)}{R_1 C_1 s + 1} & -R_2 \\ -R_2 & R_2 + \frac{1}{C_2 s} \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix}$$

Applying Cramer's rule, we obtain

$$I_2(s) = \frac{\Delta_2}{\Delta}$$

$$= \frac{\begin{vmatrix} \dfrac{R_1 + R_2\left(R_1C_1s+1\right)}{R_1C_1s+1} & V_i\left(s\right) \\ -R_2 & 0 \end{vmatrix}}{\begin{vmatrix} \dfrac{R_1 + R_2\left(R_1C_1s+1\right)}{R_1C_1s+1} & -R_2 \\ -R_2 & R_2 + \dfrac{1}{C_2s} \end{vmatrix}}$$

$$= \frac{-\left(-R_2 \times V_i\left(s\right)\right)}{\left[\dfrac{R_1 + R_2\left(R_1C_1s+1\right)}{R_1C_1s+1}\right]\left[R_2 + \dfrac{1}{C_2s}\right] - \left(-R_2 \times -R_2\right)}$$

$$= \frac{R_2 \times \left(R_1C_1s+1\right) \times V_i\left(s\right)}{R_1R_2 + \dfrac{R_1}{C_2s} + \dfrac{R_2\left(R_1C_1s+1\right)}{C_2s}} \tag{4}$$

Substituting Eqn. (4) in Eqn. (1), we obtain

$$V_o\left(s\right) = \frac{\left(\dfrac{R_2 \times \left(R_1C_1s+1\right) \times V_i\left(s\right)}{R_1R_2 + \dfrac{R_1}{C_2s} + \dfrac{R_2\left(R_1C_1s+1\right)}{C_2s}}\right)}{C_2s}$$

$$\frac{V_o\left(s\right)}{V_i\left(s\right)} = \frac{R_2 \times \left(R_1C_1s+1\right)}{C_2s\left[R_1R_2 + \dfrac{R_1}{C_2s} + \dfrac{R_2\left(R_1C_1s+1\right)}{C_2s}\right]}$$

Thus, the overall transfer function is

$$\frac{V_o\left(s\right)}{V_i\left(s\right)} = \frac{R_2 \times \left(R_1C_1s+1\right)}{R_1R_2C_2s + R_1 + R_2\left(R_1C_1s+1\right)}$$

1.7 Modeling of Mechanical Systems

The dimensions in which the movement of a mechanical system can be described are translational, rotational or a combination of both. For modeling of mechanical system, it is necessary to have the equations governing the movement of mechanical systems. The laws that are used directly or indirectly to formulate those equations are obtained from Newton's laws of motion.

The general classification of mechanical system is of two types: (i) translational and (ii) rotational as shown in Fig. 1.13.

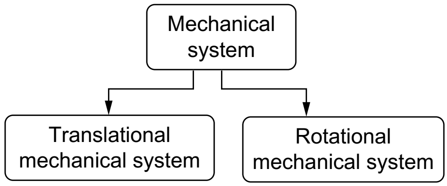


Fig. 1.13 | Classification of mechanical system

1.7.1 Translational Mechanical System

The modeling of linear passive translational mechanical system can be obtained by using three basic elements: mass, spring and damper. The mass that is concentrated at the center of a system is used to represent the weight of a given mechanical system, whereas the spring is used to represent the elastic deformation of the body and the damper is used to represent the friction existing in a mechanical system. An example of such linear passive translational mechanical system is the suspension system existing in the automobiles (such as car, van, etc.). For the analysis of linear passive translational mechanical system, it is assumed that the elements are purely linear.

The opposing forces due to mass, friction and spring act on a system when the system is subjected to a force. Using D'Alembert's principle, for a linear passive translational mechanical system, the sum of forces acting on a body is zero (i.e., the sum of applied forces is equal to the sum of the opposing forces on a body). Displacement, velocity and acceleration are the variables used to describe the linear passive translation mechanical system.

In translational mechanical systems, the energy storage elements are mass and spring and the element that dissipates energy is viscous damper. The analogous elements for energy storage in an electrical circuit are inductors and capacitors, whereas those of the energy dissipating element are resistors.

Inertia Force $f_M(t)$

The element that stores the kinetic energy of translational mechanical system is mass. Mass is considered to be a conservative element, as the energy can be stored and retrieved without loss.

When a force $f(t)$ is applied to a mass M, it experiences an acceleration and it is shown in Fig. 1.14.

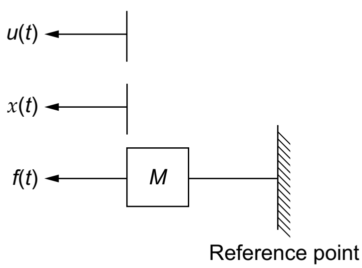


Fig. 1.14 | Mass element

According to Newton's second law, the force experienced by the mass is proportional to the acceleration.

$$f_M(t)\alpha\ a$$

$$f_M(t)\alpha\frac{du(t)}{dt}$$

$$f_M(t)=\text{Mass}\times\text{acceleration}=Ma$$

$$f_M(t)=M\frac{d^2x(t)}{dt^2}=M\frac{du(t)}{dt}$$

where M is the mass (kg), a is the acceleration (m/s^2), $u(t)$ is the velocity (m/s) and $x(t)$ is the displacement (m).

$$f_M(t)=M\frac{d^2x(t)}{dt^2}=M\frac{du(t)}{dt}=Ma$$

Here, $f_M(t)$ is measured in Newton (N) or Kg-m/s^2.

Damper

The damping that is desirable in the motion for a mechanical system is provided by an element called damper. A frictional force that exists between one or two physical systems when there exists a movement or a tendency of movement between them. The characteristics of frictional forces that are non-linear in nature depends on the composition of the surfaces, the pressure between the surfaces, their relative velocity and others that add difficulty in obtaining the mathematical description of the frictional force. The different types of frictions that are commonly used in practical system are viscous friction, static friction and coulomb friction in which the viscous friction is commonly found in a translational mechanical system.

Static Friction or Stiction

The retarding force $f_S(t)$ which tends to prevent the motion is known as static friction or stiction. This type of frictional force exists only when the body is not in motion (stationary), but has the tendency to move. The sign or direction of static friction is opposite to the direction in which the body tends to move or initial direction of the velocity. This frictional force vanishes once the movement of a system is started.

Viscous Friction

The retarding force that is experienced by a system when it is in motion is known as viscous friction. This type of friction exists between a system and a fluid medium. There exists a linear relationship between the applied force and the velocity. Hence, this frictional force $f_B(t)$ is proportional to the velocity of the movement of a system.

$$f_B(t)\ \alpha\ \text{velocity}$$

This frictional force that is more common in a mechanical system is represented by a dashpot or a damper system. When a force is applied to a damping element B, it experiences a velocity and it is shown in Fig. 1.15(a).

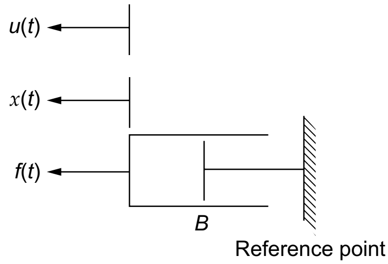


Fig. 1.15 | (a) Dash-pot or Damper element

$$f_B(t) \,\alpha\, u(t)$$

$$f_B(t) = Bu(t) = B\frac{dx(t)}{dt}$$

where B is the viscous friction coefficient (N-s/m), $u(t)$ is the velocity (m/s) and $x(t)$ is the displacement (m).

Dashpot or damper element with two displacements is shown in Fig. 1.15(b).

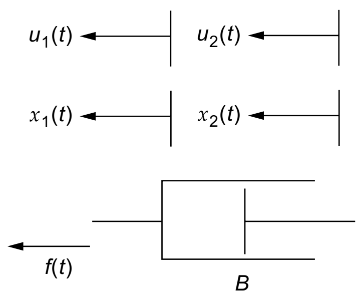


Fig. 1.15 | (b) Dash-pot or Damper element with different displacements

$$f_B(t) = B\left(u_1(t) - u_2(t)\right)$$

$$= B\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right)$$

Here, $f_B(t)$ is measured in Newton (N) or Kg-m/s^2.

Coulomb Friction

The retarding force $f_C(t)$ which is experienced by the system (dry surfaces) when it is in motion is known as coulomb friction. The coulomb friction is similar to the viscous friction, but the coulomb friction has constant amplitude with respect to the change in velocity, but the sign/direction of the frictional force depends on the direction of velocity.

The relationship between the different frictional forces and velocity of the direction of motion is given in Figs. 1.16(a) through (c).

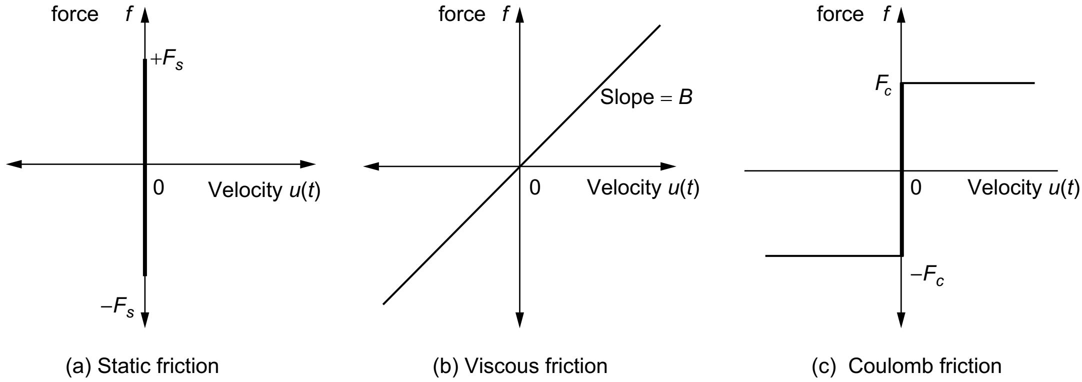


Fig. 1.16 | Frictional force versus velocity

Spring Force $f_K(t)$

In general, an element that stores potential energy in a system is known as spring. The analogous component of spring in an electrical circuit is the capacitor that is used to store voltages. In real time, the springs are non-linear in nature. However, if the spring deformation is small, the behaviour of the spring can be approximated by a linear relationship. The opposing force developed by the spring is directly related to the stiffness of the spring and the total displacement of the spring from its equilibrium position.

When a force $f(t)$ is applied to a spring element K, it experiences a displacement and it is shown in Fig. 1.17.

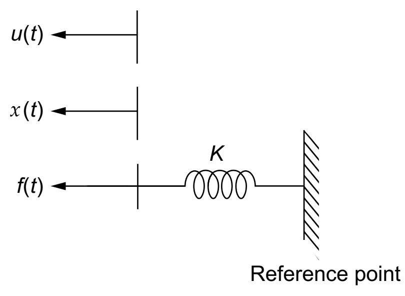


Fig. 1.17 | Spring element

According to Hooke's law, the spring force is directly proportional to the displacement:

$$f_K(t) \; \alpha \; x(t)$$

The spring stores the potential energy. Therefore,

$$f_K(t) = K\,x(t)$$

Velocity, $$u(t) = \frac{dx(t)}{dt}$$

$$dx(t) = u(t)dt$$

Integrating the above equation, we obtain

$$x(t) = \int u(t)dt$$

Therefore, $$f_K(t) = K\int u(t)dt$$

where K is the spring constant (N/m), i.e., $K = \dfrac{1}{\text{compliance}} = \dfrac{1}{k}$, $u(t)$ is the velocity (m/s) and $x(t)$ is the displacement (m).
Spring element with two displacements is shown in Fig. 1.18.

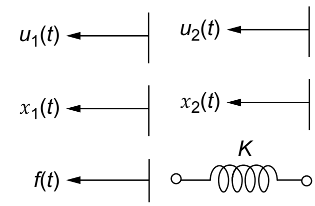


Fig. 1.18 | Spring element with two displacements

$$f_K(t) = K\left(x_1(t) - x_2(t)\right)$$

$$f_K(t) = \frac{1}{k}\left(x_1(t) - x_2(t)\right)$$

$$f_K(t) = \frac{1}{k}\int\left(u_1(t) - u_2(t)\right)dt$$

Here, $f_K(t)$ is measured in Newton (N) or Kg-m/s^2.

1.7.2 A Simple Translational Mechanical System

According to D'Alembert's principle, "The algebraic sum of the externally applied forces to any body is equal to the algebraic sum of the opposing forces restraining motion produced by the elements present in the body."
A simple translational mechanical system with all the basic elements and its free body diagram are shown in Figs. 1.19 (a) and (b) respectively.

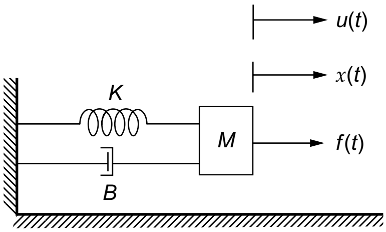


Fig. 1.19 | (a) A simple translational mechanical system

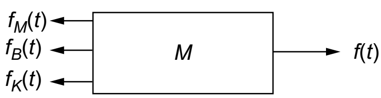


Fig. 1.19 | (b) Free body diagram

Let $f(t)$ be the external force applied to the given system,

$f_M(t) = M\dfrac{du(t)}{dt}$ be the opposing force produced by the element mass M,

$f_B(t) = Bu(t)$ be the opposing force produced by the element damper B and

$f_K(t) = K\int u(t)dt$ be the opposing force produced by the element spring K.

Using D'Alembert's principle,

$$f(t) = f_M(t) + f_B(t) + f_K(t)$$

$$f(t) = M\frac{du(t)}{dt} + Bu(t) + K\int u(t)dt$$

The above equation is referred to as *D'Alembert's basic equation for translational mechanical systems.*

Example 1.9: For the mechanical translational system shown in Fig. E1.9, obtain (i) differential equations and (ii) transfer function of the system.

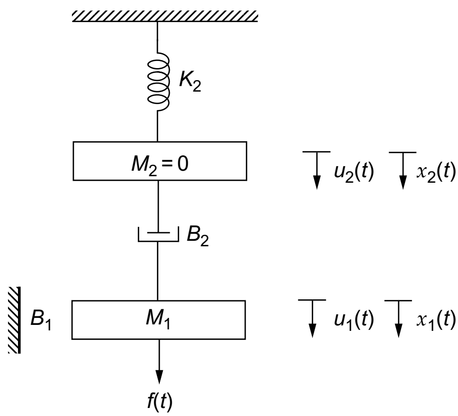


Fig. E1.9

Solution: For the system given in Fig. E1.9, the input is $f(t)$ and the outputs are $x_1(t)$ and $x_2(t)$. Using D' Alembert's principle for the mass M_1, we obtain

$$f(t) = f_{M1}(t) + f_{B1}(t) + f_{B2}(t)$$

$$f(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + B_1 \frac{dx_1(t)}{dt} + B_2 \frac{d(x_1(t) - x_2(t))}{dt} \tag{1}$$

For the mass M_2, we obtain

$$0 = f_{M2}(t) + f_{B2}(t) + f_{K2}(t)$$

$$0 = B_2 \frac{d(x_2(t) - x_1(t))}{dt} + K_2 x_2(t) \tag{2}$$

Taking Laplace transform of Eqn. (1) and Eqn. (2), we obtain

$$\left[M_1 s^2 + (B_1 + B_2)s\right] X_1(s) - B_2 s X_2(s) = F(s)$$

$$-B_2 s X_1(s) + \{B_2 s + K_2\} X_2(s) = 0$$

Representing the above equations in matrix form,

$$\begin{bmatrix} M_1 s^2 + (B_1 + B_2)s & -B_2 s \\ -B_2 s & B_2 s + K_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

Using Cramer's rule, we obtain

$$X_2(s) = \frac{\Delta_2}{\Delta} \text{ and } X_1(s) = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} F(s) & -B_2 s \\ 0 & B_2 s + K_2 \end{vmatrix} = F(s) \times (B_2 s + K_2)$$

$$\Delta_2 = \begin{vmatrix} M_1 s^2 + (B_1 + B_2)s & F(s) \\ -B_2 s & 0 \end{vmatrix} = F(s) \times B_2 s$$

$$\Delta = \begin{vmatrix} M_1 s^2 + (B_1 + B_2)s & -B_2 s \\ -B_2 s & B_2 s + K_2 \end{vmatrix} = \left[M_1 s^2 + (B_1 + B_2)s\right] \times \left[B_2 s + K_2\right] - (B_2 s)^2$$

$$= M_1 B_2 s^3 + K_2 M_1 s^2 + B_1 B_2 s^2 + B_1 K_2 s + B_2 K_2 s$$

Therefore, $$X_1(s) = \frac{\Delta_1}{\Delta} = \frac{F(s)\times(B_2 s + K_2)}{\Delta}$$

and $$X_2(s) = \frac{\Delta_2}{\Delta} = \frac{F(s)\times B_2 s}{\Delta}$$

Thus, the transfer function of the given mechanical system is given by

$$\frac{X_1(s)}{F(s)} = \frac{(B_2 s + K_2)}{\Delta}$$

$$\frac{X_2(s)}{F(s)} = \frac{B_2 s}{\Delta}$$

where $\Delta = M_1 B_2 s^3 + K_2 M_1 s^2 + B_1 B_2 s^2 + B_1 K_2 s + B_2 K_2 s$

Example 1.10: For the mechanical translational system shown in Fig. E1.10, obtain (i) differential equations and (ii) transfer function of the system.

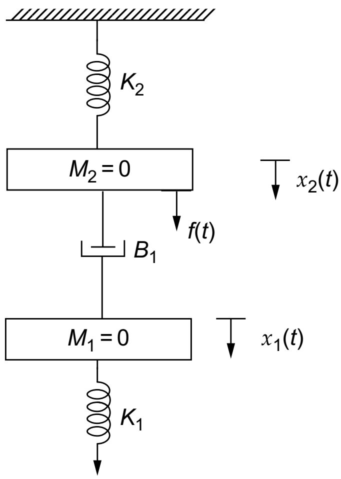


Fig. E1.10

Solution: For the system given in Fig. E1.10, the input is $f(t)$ and the outputs are $x_1(t)$ and $x_2(t)$.
Using D' Alembert's principle for the mass M_1,

$$0 = f_{M1}(t) + f_{B1}(t) + f_{K1}(t)$$

$$0 = B_1 \frac{d\left(x_1(t) - x_2(t)\right)}{dt} + K_1 x_1(t)$$

For the mass M_2,

$$f(t) = f_{M2}(t) + f_{B1}(t) + f_{K2}(t)$$

$$f(t) = B_1 \frac{d\left(x_2(t) - x_1(t)\right)}{dt} + K_2 x_2(t)$$

Taking Laplace transform of the above equations, we obtain

$$\left[B_1 s + K_1\right] X_1(s) - B_1 s \times X_2(s) = 0$$

$$-B_1 s \times X_1(s) + \left\{B_1 s + K_2\right\} X_2(s) = F(s)$$

Representing the above equations in matrix form,

$$\begin{bmatrix} B_1 s + K_1 & -B_1 s \\ -B_1 s & B_1 s + K_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

Using Cramer's rule, we obtain

$$X_2(s) = \frac{\Delta_2}{\Delta} \quad \text{and} \quad X_1(s) = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} 0 & -B_1 s \\ F(s) & B_1 s + K_2 \end{vmatrix} = F(s) \times sB_1$$

$$\Delta_2 = \begin{vmatrix} B_1 s + K_1 & 0 \\ -B_1 s & F(s) \end{vmatrix} = F(s) \times \left(B_1 s + K_1\right)$$

$$\Delta = \begin{vmatrix} B_1 s + K_1 & -B_1 s \\ -B_1 s & B_1 s + K_2 \end{vmatrix} = \left[B_1 s + K_1\right] \times \left[B_1 s + K_2\right] - \left(B_1 s\right)^2$$

$$= \left(B_1 K_2 + B_1 K_1\right) s + K_1 K_2$$

Therefore,

$$X_1(s) = \frac{\Delta_1}{\Delta} = \frac{F(s) \times B_1 s}{\Delta}$$

$$X_2(s) = \frac{\Delta_2}{\Delta} = \frac{F(s)\times(B_1 s + K_1)}{\Delta}$$

Thus, the transfer functions of the given mechanical system is given by

$$\frac{X_1(s)}{F(s)} = \frac{B_1 s}{\Delta}$$

$$\frac{X_2(s)}{F(s)} = \frac{(B_1 s + K_1)}{\Delta}$$

where $\Delta = (B_1K_2 + B_1K_1)s + K_1K_2$

Example 1.11: For the mechanical translational system shown in Fig. E1.11, obtain (i) differential equations and (ii) transfer function of the system.

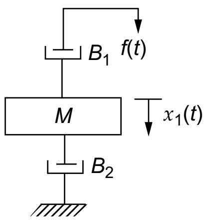


Fig. E1.11

Solution: For the system given in Fig. E1.11, the input is $f(t)$ and the output is $x_1(t)$. Using D' Alembert's principle for the mass M,

$$f(t) = f_{\mathrm{M}}(t) + f_{\mathrm{B1}}(t) + f_{\mathrm{B2}}(t)$$

$$f(t) = M\frac{d^2x_1(t)}{dt^2} + (B_1 + B_2)\frac{dx_1(t)}{dt}$$

Taking Laplace transform on both the sides, we obtain

$$F(s) = \left[Ms^2 + (B_1 + B_2)s\right]X_1(s)$$

Hence, the transfer function of the system is given by

$$\frac{X_1(s)}{F(s)} = \frac{1}{\left[Ms^2 + (B_1 + B_2)s\right]}$$

Example 1.12: For the mechanical translational system shown in Fig. E1.12, obtain (i) differential equations and (ii) transfer function of the system.

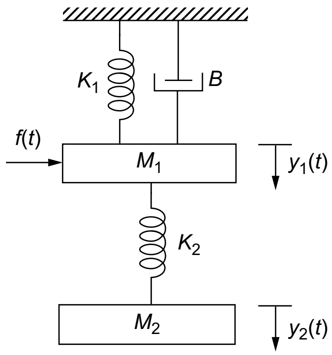


Fig. E1.12

Solution: For the system given in Fig. E1.12, the input is $f(t)$ and the outputs are $y_1(t)$ and $y_2(t)$. Using D' Alembert's principle for the mass M_1,

$$f(t) = f_{M1}(t) + f_{K1}(t) + f_B(t) + f_{K2}(t)$$

$$f(t) = M_1 \frac{d^2 y_1(t)}{dt^2} + K_1 y_1(t) + B_1 \frac{dy_1(t)}{dt} + K_2 \{y_1(t) - y_2(t)\}$$

For the mass M_2,
$$0 = f_{M2}(t) + f_{K2}(t)$$

$$0 = M_2 \frac{d^2 y_2(t)}{dt^2} + K_2 (y_2(t) - y_1(t))$$

Taking Laplace transform of the above equations, we obtain

$$\left[M_1 s^2 + Bs + (K_1 + K_2)\right] Y_1(s) - K_2 Y_2(s) = F(s)$$

$$-K_2 Y_1(s) + \left\{M_2 s^2 + K_2\right\} Y_2(s) = 0$$

Representing the above equations in matrix form,

$$\begin{bmatrix} M_1 s^2 + Bs + (K_1 + K_2) & -K_2 \\ -K_2 & M_2 s^2 + K_2 \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

Using Cramer's rule, we obtain

$$Y_2(s) = \frac{\Delta_2}{\Delta} \text{ and } Y_1(s) = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} F(s) & -K_2 \\ 0 & M_2s^2 + K_2 \end{vmatrix} = F(s) \times (M_2s^2 + K_2)$$

$$\Delta_2 = \begin{vmatrix} M_1s^2 + Bs + (K_1 + K_2) & F(s) \\ -K_2 & 0 \end{vmatrix} = F(s)K_2$$

$$\Delta = \begin{vmatrix} M_1s^2 + Bs + (K_1 + K_2) & -K_2 \\ -K_2 & M_2s^2 + K_2 \end{vmatrix} = \left[M_1s^2 + Bs + (K_1 + K_2)\right] \times \left[M_2s^2 + K_2\right] - K_2{}^2$$

$$= M_1M_2s^4 + K_2M_1s^2 + M_2Bs^3 + K_2Bs + M_2K_1s^2 + M_2K_2s^2 + K_1K_2$$

$$= M_1M_2s^4 + M_2Bs^3 + (K_2M_1 + M_2K_1 + M_2K_2)s^2 + K_2Bs + K_1K_2$$

Therefore,

$$Y_1(s) = \frac{\Delta_1}{\Delta} = \frac{F(s) \times (M_2s^2 + K_2)}{\Delta}$$

and
$$Y_2(s) = \frac{\Delta_2}{\Delta} = \frac{F(s)K_2}{\Delta}$$

Thus, the transfer functions of the given mechanical system are given by

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{\Delta} \text{ and } \frac{Y_1(s)}{F(s)} = \frac{(M_2s^2 + K_2)}{\Delta}$$

where $\Delta = M_1M_2s^4 + M_2Bs^3 + (K_2M_1 + M_2K_1 + M_2K_2)s^2 + K_2Bs + K_1K_2$

Example 1.13: For the mechanical translational system shown in Fig. E1.13, obtain (i) differential equations and (ii) transfer function of the system.

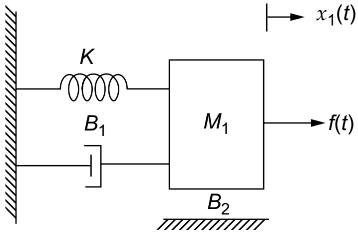


Fig. E1.13

Solution: For the system given in Fig. E1.13, the input is $f(t)$ and the output is $x_1(t)$ Using D' Alembert's principle for the mass M_1,

$$f(t) = f_{M1}(t) + f_K(t) + f_{B1}(t) + f_{B2}(t)$$

$$f(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + Kx_1(t) + (B_1 + B_2)\frac{dx_1(t)}{dt} \tag{1}$$

Taking Laplace transform on both the sides, we obtain

$$F(s) = \left[M_1 s^2 + (B_1 + B_2)s + K\right] X_1(s)$$

Hence, the transfer function of the system is given by

$$\frac{X_1(s)}{F(s)} = \frac{1}{\left[M_1 s^2 + (B_1 + B_2)s + K\right]}$$

Example 1.14: For the mechanical translational system shown in Fig. E1.14, obtain (i) differential equations and (ii) $X_1(s)$

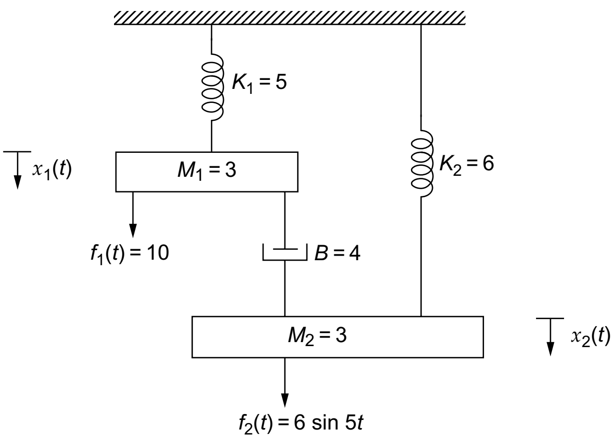


Fig. E1.14

Solution: For the system shown in Fig. E1.14, the inputs are $f_1(t)$ and $f_2(t)$ and the outputs are $x_1(t)$ and $x_2(t)$ respectively.

Using D' Alembert's principle for the mass M_1, $f_1(t) = f_{M1}(t) + f_{K1}(t) + f_B(t)$

$$f_1(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + K_1 x_1(t) + B\frac{d(x_1(t) - x_2(t))}{dt}$$

Using D' Alembert's principle for the mass M_2, $f_2(t) = f_{M2}(t) + f_{K2}(t) + f_B(t)$

$$f_2(t) = M_2 \frac{d^2 x_2(t)}{dt^2} + K_2 x_2(t) + B\frac{d\left(x_2(t) - x_1(t)\right)}{dt}$$

Taking Laplace transform on both the sides, we obtain

$$F_1(s) = \left[M_1 s^2 + K_1 + Bs\right] X_1(s) - Bs \times X_2(s)$$

$$F_2(s) = \left[M_2 s^2 + K_2 + Bs\right] X_2(s) - BsX_1(s)$$

Representing the above equations in matrix form,

$$\begin{bmatrix} M_1 s^2 + Bs + K_1 & -Bs \\ -Bs & M_2 s^2 + K_2 + Bs \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix}$$

Using Cramer's rule, we obtain

$$X_1(s) = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{vmatrix} F_1(s) & -Bs \\ F_2(s) & M_2 s^2 + Bs + K_2 \end{vmatrix} = \left(F_1(s) \times \left(M_2 s^2 + Bs + K_2\right)\right) - \left(F_2(s) \times -Bs\right)$$

$$\Delta = \begin{vmatrix} M_1 s^2 + Bs + K_1 & -Bs \\ -Bs & M_2 s^2 + Bs + K_2 \end{vmatrix} = \left[M_1 s^2 + Bs + K_1\right] \times \left[M_2 s^2 + Bs + K_2\right] - (sB)^2$$

Therefore,

$$X_1(s) = \frac{\Delta_1}{\Delta}$$

$$= \frac{\left(F_1(s) \times \left(M_2 s^2 + Bs + K_2\right)\right) + Bs \times F_2(s)}{\left[M_1 s^2 + Bs + K_1\right] \times \left[M_2 s^2 + Bs + K_2\right] - (Bs)^2}$$

Laplace transform of the given inputs $f_1(t) = 10$ and $f_2(t) = 6 \sin 5t$ are

$$F_1(s) = \frac{10}{s} \text{ and } F_2(s) = \frac{6 \times 5}{s^2 + 25} = \frac{30}{s^2 + 25}$$

Substituting the known values in the above equation, we obtain

$$X_1(s)=\frac{\left(\frac{10}{s}\times\left(3s^2+4s+6\right)\right)-\left(\frac{30}{s^2+25}\times 4s\right)}{\left[3s^2+4s+5\right]\times\left[3s^2+4s+6\right]-(4s)^2}$$

$$X_1(s)=\frac{30s^4+40s^3+930s^2+1000s+1500}{9s^7+24s^6+274s^5+644s^4+1255s^3+1084s^2+750s}$$

Example 1.15: For the mechanical translational system shown in Fig. E1.15, obtain (i) differential equations and (ii) transfer function of the system.

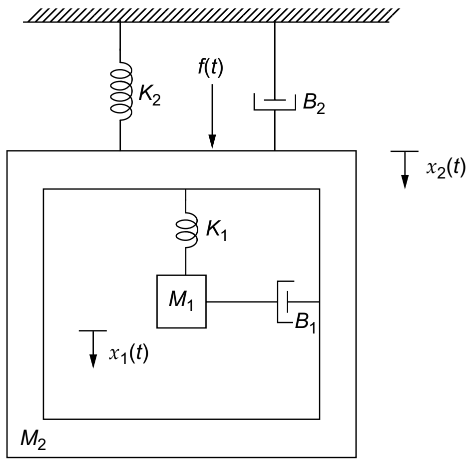


Fig. E1.15

Solution: For the system given in Fig. E1.15, the input is $f(t)$ and the outputs are $x_1(t)$ and $x_2(t)$. Using D' Alembert's principle for the mass M_1, $0=f_{M1}(t)+f_{K1}(t)+f_{B1}(t)$

$$0=M_1\frac{d^2x_1(t)}{dt^2}+B_1\frac{d\{x_1(t)-x_2(t)\}}{dt}+K_1\{x_1(t)-x_2(t)\} \tag{1}$$

For the mass M_2, $f(t)=f_{M2}(t)+f_{K2}(t)+f_{B2}(t)+f_{K1}(t)+f_{B1}(t)$

$$f(t)=M_2\frac{d^2x_2(t)}{dt^2}+B_1\frac{d\{x_2(t)-x_1(t)\}}{dt}+K_1\{x_2(t)-x_1(t)\}+K_2x_2(t)+B_2\frac{dx_2(t)}{dt} \tag{2}$$

Taking Laplace transform of Eqn. (1) and Eqn. (2), we obtain

$$\left[M_1 s^2 + B_1 s + K_1\right] X_1(s) - (K_1 + B_1 s) X_2(s) = 0$$

$$-(K_1 + B_1 s) X_1(s) + \left\{M_2 s^2 + K_2 + K_1 + (B_1 + B_2)s\right\} X_2(s) = F(s)$$

Representing the above equations in matrix form,

$$\begin{bmatrix} M_1 s^2 + B_1 s + K_1 & -(K_1 + B_1 s) \\ -(K_1 + B_1 s) & M_2 s^2 + (B_1 + B_2)s + K_2 + K_1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

Using Cramer's rule, we obtain

$$X_2(s) = \frac{\Delta_2}{\Delta} \text{ and } X_1(s) = \frac{\Delta_1}{\Delta}$$

$$\Delta_1 = \begin{bmatrix} 0 & -(K_1 + B_1 s) \\ F(s) & M_2 s^2 + (B_1 + B_2)s + K_1 + K_2 \end{bmatrix} = F(s) \times (B_1 s + K_1)$$

$$\Delta_2 = \begin{bmatrix} M_1 s^2 + B_1 s + K_1 & 0 \\ -(K_1 + B_1 s) & F(s) \end{bmatrix} = F(s) \times \left(M_1 s^2 + B_1 s + K_1\right)$$

$$\Delta = \begin{vmatrix} M_1 s^2 + B_1 s + K_1 & -(B_1 s + K_1) \\ -(B_1 s + K_1) & M_2 s^2 + (B_1 + B_2)s + (K_1 + K_2) \end{vmatrix}$$

$$= \left[\left(M_1 s^2 + B_1 s + K_1\right) \times \left(M_2 s^2 + (B_1 + B_2)s + K_1 + K_2\right)\right] - (B_1 s + K_1)^2$$

Therefore,

$$X_1(s) = \frac{\Delta_1}{\Delta} = \frac{F(s)(B_1 s + K_1)}{\Delta}$$

$$X_2(s) = \frac{\Delta_2}{\Delta} = \frac{F(s)\left(M_1 s^2 + B_1 s + K_1\right)}{\Delta}$$

Thus, the transfer functions of given mechanical system are given by

$$\frac{X_1(s)}{F(s)} = \frac{(B_1 s + K_1)}{\Delta}$$

$$\frac{X_2(s)}{F(s)} = \frac{M_1 s^2 + B_1 s + K_1}{\Delta}$$

where

$$\Delta = M_1 M_2 s^4 + \left(M_1 B_1 + M_1 B_2 + M_2 B_1\right)s^3 + \left(M_1 K_1 + M_1 K_2 + M_2 K_1 + B_1 B_2\right)s^2 + \left(B_1 K_2 + B_2 K_1\right)s + K_1 K_2$$

1.7.3 Rotational Mechanical System

The modeling of a linear passive rotational mechanical system can be obtained by using three basic elements: inertia, rotational spring and rotational damper. The modeling of a rotational mechanical system is similar to that of a translational mechanical system except that the elements of the system undergo a rotational instead of a translation movement.

The opposing torques due to inertia, rotational spring and rotational damper act on a system when the system is subjected to a torque. Using D'Alembert's principle, for a linear passive rotational mechanical system, the sum of all the torques acting on a body is zero (i.e., the sum of applied torques is equal to the sum of the opposing torques on a body). Angular displacement, angular velocity and angular acceleration are the variables used to describe a linear passive rotational mechanical system.

In rotational mechanical systems, the energy storage elements are inertia and rotational spring and the energy dissipating element is the rotational viscous damper. The analogous for the energy storage elements in an electrical circuit are the inductors and the capacitors and the analogous of energy dissipating element in an electrical circuit is the resistor.

The analogous of linear passive translational and rotational mechanical system is given in Table. 1.4.

Table 1.4 | Analogous of translational and rotational mechanical system

Translational mechanical system	Rotational mechanical system
Force (F)	Torque (T)
Velocity (u)	Angular velocity (ω)
Displacement (x)	Angular displacement (θ)
Mass (M)	Moment of inertia (J)
Damping coefficient (B)	Rotational damping (B)
Spring constant (K)	Rotational spring constant (K)

Inertia Torque $T_J(t)$

The element that stores the kinetic energy of a rotational mechanical system is inertia. Inertia is considered to be a conservative element as the energy can be stored and retrieved without any loss. The factors on which the inertial element depends are geometric composition about the rotational axis and its density.

When a torque $T(t)$ is applied to an inertia element J, it experiences an angular acceleration and it is shown in Fig. 1.20.

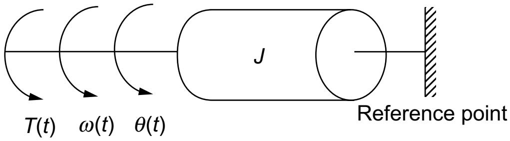


Fig. 1.20 | Inertia element

According to Newton's second law, the inertia torque is proportional to the angular acceleration.

$$T_J(t) \alpha\, a$$

$$T_J(t) \alpha \frac{d\omega(t)}{dt}$$

$$T_J(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

where J is the moment of inertia (kg-m^2/rad), α is the angular acceleration (rad/s^2), $\omega(t)$ is the angular velocity (rad/s), $\theta(t)$ is the angular displacement (rad) and $T_J(t)$ is measured in Newton-meter (N-m).

Damping Torque $T_B(t)$

The description discussed about the damper in the linear translational mechanical system holds good for the linear rotational mechanical system except for the fact that force is replaced by torque. A damper element is represented by a dashpot system.

When a torque $T(t)$ is applied to a damping element B, it experiences an angular velocity and it is shown in Fig. 1.21.

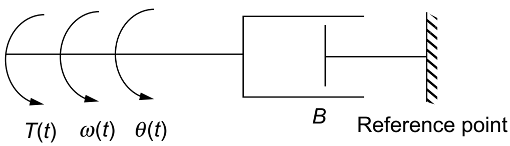


Fig. 1.21 | Damper element

The damping torque is proportional to the angular velocity. Therefore,

$$T_B(t) \alpha\, \omega(t)$$

$$T_B(t) = B\omega(t) = B\frac{d\theta(t)}{dt}$$

where B is viscous friction coefficient (N-s/m), $\omega(t)$ is the angular velocity (rad/s) and $\theta(t)$ is the angular displacement (rad).

Damper element with two angular displacements and a single applied torque is shown in Fig. 1.22.

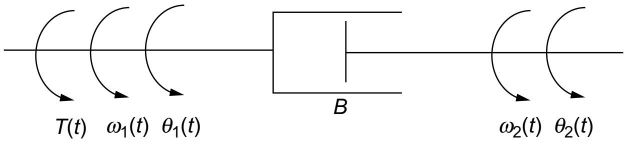


Fig. 1.22 | Damper element with two angular displacements

$$T_B(t) = B(\omega_1(t) - \omega_2(t))$$

$$T_B(t) = B\left(\frac{d\theta_1(t)}{dt} - \frac{d\theta_2(t)}{dt}\right)$$

Here, $T_B(t)$ is measured in Newton-meter.

Torsional/rotational Spring Torque $T_K(t)$

In general, an element that stores potential energy in a rotational mechanical system is known as torsional/rotational spring. The analogous component of torsional/rotational spring in an electrical circuit is the capacitor that is used to store voltages. The opposing force developed by the rotational spring is directly related to the stiffness of the torsional/rotational spring and its total angular displacement from its equilibrium position.

When a torque $T(t)$ is applied to a spring element K, it experiences an angular displacement and it is shown in Fig. 1.23.

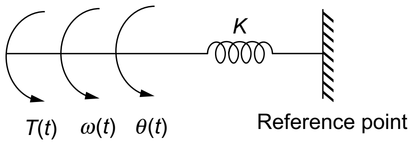


Fig. 1.23 | Torsional spring element

According to Hooke's law, spring torque is proportional to the angular displacement.

$$T_K(t) \alpha \theta(t)$$

$$T_K(t) = K\theta(t) \;= \frac{1}{k}\int \omega(t)\,dt$$

where $\qquad \omega(t) = \dfrac{d\theta(t)}{dt} \quad$ and $\quad d\theta(t) = \omega(t)\,dt$

where K is the spring constant (N-m/rad), i.e., $K = \dfrac{1}{\text{Rotational compliance}} = \dfrac{1}{k}$, $\omega(t)$ is the angular velocity (rad/s) and $\theta(t)$ is the angular displacement (rad).
A spring element with two angular displacements is shown in Fig. 1.24.

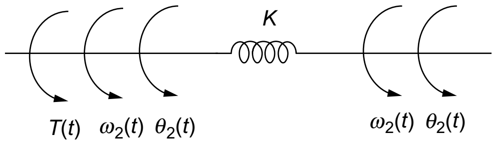


Fig. 1.24 | Torsional spring element with two angular displacements

$$T_K(t) = K\left(\theta_1(t) - \theta_2(t)\right)$$

$$T_K(t) = \frac{1}{k}\left(\theta_1(t) - \theta_2(t)\right)$$

Similarly,

$$T_K(t) = K\int\left(\omega_1(t) - \omega_2(t)\right)dt$$

$$T_K(t) = \frac{1}{k}\int\left(\omega_1(t) - \omega_2(t)\right)dt$$

Here, $T_K(t)$ is measured in Newton-meter.

1.7.4 A Simple Rotational Mechanical System

According to D'Alembert's principle, "The algebraic sum of the externally applied torques to any body is equal to the algebraic sum of opposing torques restraining motion produced by the elements present in the body."

A simple rotational mechanical system with all the basic elements and its free body diagram is shown in Fig.1.25 and Fig.1.26 respectively.

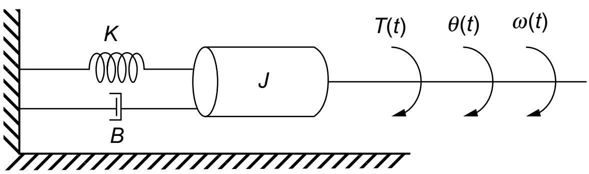


Fig. 1.25 | A simple rotational mechanical system

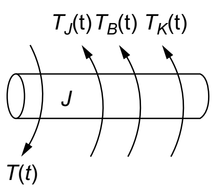


Fig. 1.26 | Free body diagram

Let $T(t)$ be the external torque applied to the given system,

$T_J(t) = J\dfrac{d\omega(t)}{dt}$ be the opposing torque produced by the element mass J,

$T_B(t) = B\omega(t)$ be the opposing torque produced by the element damper B and

$T_K(t) = K\int \omega(t)dt$ be the opposing torque produced by the element spring K.

Using D'Alembert's principle, $T(t) = T_J(t) + T_B(t) + T_K(t)$

$$T(t) = J\frac{d\omega(t)}{dt} + B\omega(t) + \frac{1}{k}\int \omega(t)dt$$

The free body diagram of the system denoting the applied and opposing torques is shown in Fig. 1.26.
Restraining torques are given below:

Inertia torque $T_J(t) = J\frac{d\omega(t)}{dt}$

Damping torque $T_B(t) = B\omega(t)$

Spring torque $T_K(t) = K\int \omega(t)dt$

Example 1.16: **Set up the differential equations for the mechanical system as shown in Fig. E1.16(a). Also, obtain its transfer function $\frac{\theta_1(s)}{T(s)}$.**

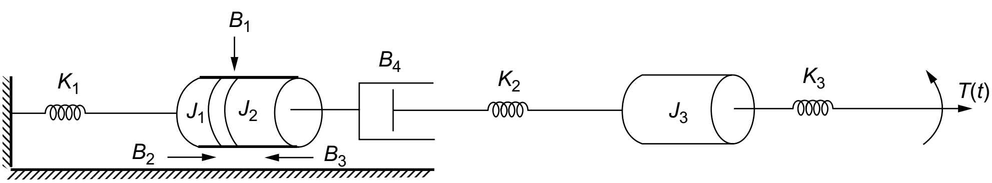


Fig. E1.16(a)

Solution: The given mechanical system is modified indicating the various angular displacements as shown in Fig. E1.16(b).

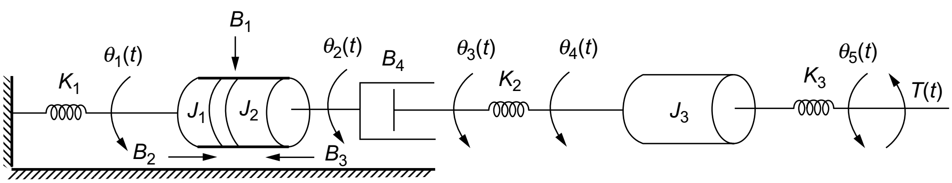


Fig. E1.16(b)

K_3 is between the angular displacements $\theta_4(t)$ and $\theta_5(t)$.

J_3 is under the angular displacement $\theta_4(t)$.

K_2 is between the angular displacements $\theta_4(t)$ and $\theta_3(t)$.

B_4 is between the angular displacements $\theta_3(t)$ and $\theta_2(t)$.

J_2 and B_3 are under the angular displacement $\theta_2(t)$.

B_1 is between the angular displacements $\theta_2(t)$ and $\theta_1(t)$.

J_1, K_1 and B_2 are under the angular displacement $\theta_1(t)$.

The equivalent mechanical system is shown in Fig. E1.16(c).

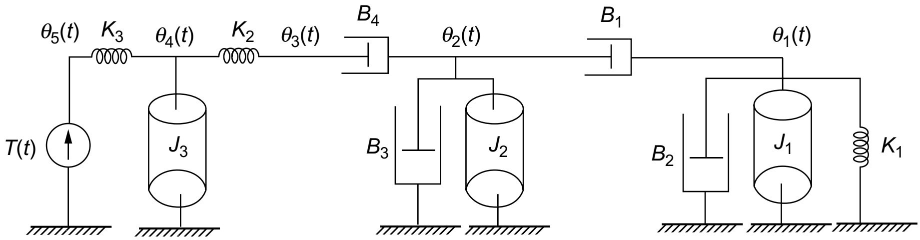


Fig. E1.16(c)

Method 1–Using substitution method

The differential equations are

$$B_1 \frac{d(\theta_1(t)-\theta_2(t))}{dt} + B_2 \frac{d\theta_1(t)}{dt} + J_1 \frac{d^2\theta_1(t)}{dt^2} + K_1\theta_1(t) = 0$$

$$B_4 \frac{d(\theta_2(t)-\theta_3(t))}{dt} + B_3 \frac{d\theta_2(t)}{dt} + J_2 \frac{d^2\theta_2(t)}{dt^2} + B_1 \frac{d(\theta_2(t)-\theta_1(t))}{dt} = 0$$

$$K_2(\theta_3(t)-\theta_4(t)) + B_4 \frac{d(\theta_3(t)-\theta_2(t))}{dt} = 0$$

$$K_3(\theta_4(t)-\theta_5(t)) + J_3 \frac{d^2\theta_4(t)}{dt^2} + K_2(\theta_4(t)-\theta_3(t)) = 0$$

$$K_3(\theta_5(t)-\theta_4(t)) = T(t)$$

Taking Laplace transform of the above five equations, we obtain

$$\theta_1(s)\left[J_1s^2 + (B_1+B_2)s + K_1\right] - B_1s\theta_2(s) = 0 \tag{1}$$

$$\theta_2(s)\left[J_2s^2 + (B_1+B_3+B_4)s\right] - B_4s\theta_3(s) - B_1s\theta_1(s) = 0 \tag{2}$$

$$\theta_3(s)\left[B_4 s+K_2\right]-K_2\theta_4(s)-B_4 s\theta_2(s)=0 \tag{3}$$

$$\theta_4(s)\left[J_3 s^2+\left(K_2+K_3\right)\right]-K_3\theta_5(s)-K_2\theta_3(s)=0 \tag{4}$$

$$K_3\left[\theta_5(s)-\theta_4(s)\right]=T(s) \tag{5}$$

Using Eqn. (1), we obtain

$$\theta_2(s)=\frac{J_1 s^2+\left(B_1+B_2\right)s+K_1}{B_1 s}\theta_1(s) \tag{6}$$

Using Eqn. (2) and Eqn. (6), we obtain

$$\theta_1(s)\left[\frac{J_1 s^2+\left(B_1+B_2\right)s+K_1}{B_1 s}\right]\left[J_2 s^2+\left(B_1+B_3+B_4\right)s\right]-B_4 s\theta_3(s)-B_1 s\theta_1(s)=0$$

Rearranging, we obtain

$$B_4 s\theta_3(s)=\theta_1(s)\left\{\left[\frac{J_1 s^2+\left(B_1+B_2\right)s+K_1}{B_1 s}\right]\left[J_2 s^2+\left(B_1+B_3+B_4\right)s\right]-B_1 s\right\}$$

Therefore, $$\theta_3(s)=\frac{A}{B_1 B_4 s^2}\theta_1(s) \tag{7}$$

where $A=\left[\left(J_1 s^2+\left(B_1+B_2\right)s+K_1\right)\left(J_2 s^2+\left(B_1+B_3+B_4\right)s\right)\right]-B_1^2 s^2$

Using Eqs. (3), (6) and (7), we obtain

$$\frac{\theta_1(s)}{B_1 B_4 s^2}A\left(B_4 s+K_2\right)-K_2\theta_4(s)-B_4 s\left[\frac{J_1 s^2+\left(B_1+B_2\right)s+K_1}{B_1 s}\right]\theta_1(s)=0$$

Therefore, $$\theta_4(s)=\frac{B}{B_1 B_4 K_2 s^2}\theta_1(s) \tag{8}$$

where $B=A\left(B_4 s+K_2\right)-J_1 B_4^2 s^4-B_1 B_4^2 s^3-B_2 B_4^2 s^3-B_4^2 K_1 s^2$

Using Eqs. (4), (7) and (8), we obtain

$$\frac{B}{B_1 B_4 K_2 s^2}\theta_1(s)\left[J_3 s^2+K_2+K_3\right]-K_3\theta_5(s)-K_2\frac{A}{B_1 B_4 s^2}\theta_1(s)=0$$

Therefore, $$\theta_5(s)=\frac{C}{B_1 B_4 K_2 K_3 s^2}\theta_1(s) \tag{9}$$

where $C=B(J_3 s^2+K_2+K_3)-K_2^2 A$

Using Eqs. (9) and (8) in Eqn. (5), we obtain

$$\frac{K_3C}{B_1B_4K_2K_3s^2}\theta_1(s) - \frac{K_3B}{B_1B_4K_2s^2}\theta_1(s) = T(s)$$

The required transfer function of the given system is

$$\frac{\theta_1(s)}{T(s)} = \frac{B_1B_4K_2s^2}{D}$$

where $D = C - K_3B$

$$= B(J_3s^2 + K_2 + K_3) - K_2^2A - K_3B$$

$$= B(J_3s^2 + K_2) - K_2^2A$$

$$= (A(B_4s + K_2) - J_1B_4^2s^4 - B_1B_4^2s^3 - B_2B_4^2s^3 - B_4^2K_1s^2)(J_3s^2 + K_2) - K_2^2A$$

$$= A\left(J_3B_4s^3 + J_3K_2s^2 + B_4K_2s\right) - (J_3s^2 + K_2)(J_1B_4^2s^4 + B_1B_4^2s^3 + B_2B_4^2s^3 + B_4^2K_1s^2)$$

$$= \left[\left[\left(J_1s^2 + (B_1 + B_2)s + K_1\right)\left(J_2s^2 + (B_1 + B_3 + B_4)s\right)\right] - B_1^2s^2\right]\left(J_3B_4s^3 + J_3K_2s^2 + B_4K_2s\right)$$
$$- \left[J_1J_3B_4^2s^6 + (J_3B_1 + J_3B_2)B_4^2s^5 + (J_3K_1 + J_1K_2)B_4^2s^4 + (B_1K_2 + B_2K_2)B_4^2s^3 + B_4^2K_1K_2s^2\right]$$

$$= \left(J_1J_2J_3B_4\right)s^7 + \left(J_1J_3B_1B_4 + J_1J_3B_3B_4 + J_2J_3B_1B_4 + J_2J_3B_2B_4 + J_1J_2J_3K_2 + J_1J_3B_4^2\right)s^6$$
$$+ \begin{pmatrix} J_3B_1B_3B_4 + J_3B_1B_2B_4 + J_3B_2B_3B_4 + J_2J_3B_4K_1 + J_1J_3B_1K_2 + J_1J_3B_3K_2 + J_1J_3B_4K_2 \\ +J_2J_3B_1K_2 + J_2J_3B_2K_2 + J_1J_2B_4K_2 + B_4^2(J_3B_1 + J_3B_2)s^5 \end{pmatrix}$$
$$+ \begin{pmatrix} J_3B_1B_4K_1 + J_3B_3B_4K_1 + J_3B_1B_3K_2 + J_3B_1B_4K_2 + J_3B_2B_3K_2 + J_3B_2B_4K_2 + J_3B_1B_2K_2 \\ +J_2J_3K_1K_2 + J_1B_1B_4K_2 + J_1B_3B_4K_2 + J_2B_1B_4K_2 + J_2B_2B_4K_2 + (J_3K_1 + J_1K_2)B_4^2 \end{pmatrix}s^4$$
$$+ \begin{pmatrix} J_3B_1K_1K_2 + J_3B_3K_1K_2 + J_3B_4K_1K_2 + B_1B_3B_4K_2 + B_1B_2B_4K_2 + B_2B_3B_4K_2 + J_2B_4K_1K_2 \\ +(B_1K_2 + B_2K_2)B_4^2 \end{pmatrix}s^3$$
$$+ \left(B_1B_4K_1K_2 + B_3B_4K_1K_2 + B_4^2K_1K_2\right)s^2 - \begin{pmatrix} J_1J_3B_4^2s^6 + (J_3B_1 + J_3B_2)B_4^2s^5 + (J_3K_1 + J_1K_2)B_4^2s^4 \\ +(B_1K_2 + B_2K_2)B_4^2s^3 + B_4^2K_1K_2s^2 \end{pmatrix}$$

$$= \left(J_1J_2J_3B_4\right)s^7 + \left(J_1J_3B_1B_4 + J_1J_3B_3B_4 + J_2J_3B_1B_4 + J_2J_3B_2B_4 + J_1J_2J_3K_2\right)s^6$$
$$+ \begin{pmatrix} J_3B_1B_3B_4 + J_3B_1B_2B_4 + J_3B_2B_3B_4 + J_2J_3K_1B_4 + J_1J_3K_2B_1 + J_1J_3K_2B_3 + J_1J_3K_2B_4 \\ +J_2J_3K_2B_1 + J_2J_3K_2B_2 + J_1J_2K_2B_4 \end{pmatrix}s^5$$
$$+ \begin{pmatrix} J_3K_1B_1B_4 + J_3K_1B_3B_4 + J_3K_2B_1B_3 + J_3K_2B_1B_4 + J_3K_2B_2B_3 + J_3K_2B_2B_4 + J_3K_2B_1B_2 \\ +J_2J_3K_1K_2 + J_1K_2B_1B_4 + J_1K_2B_3B_4 + J_2K_2B_1B_4 + J_2K_2B_2B_4 \end{pmatrix}s^4$$
$$+ \left(J_3K_1K_2B_1 + J_3K_1K_2B_3 + J_3K_1K_2B_4 + K_2B_1B_3B_4 + K_2B_1B_2B_4 + K_2B_2B_3B_4 + J_2K_1K_2B_4\right)s^3$$
$$+ \left(K_1K_2B_1B_4 + K_1K_2B_3B_4\right)s^2$$

Method 2–Using Cramer's rule
Writing Eqs. (1) through (5) in matrix form, we obtain

$$AX = Y$$

where

$$A = \begin{bmatrix} \left[J_1 s^2 + (B_1 + B_2)s + K_1\right] & -B_1 s & 0 & 0 & 0 \\ -B_1 s & \left[J_2 s^2 + (B_1 + B_3 + B_4)s\right] & -B_4 s & 0 & 0 \\ 0 & -B_4 s & (K_2 + B_4 s) & -K_2 & 0 \\ 0 & 0 & -K_2 & \left[J_3 s^2 + K_3 + K_2\right] & -K_3 \\ 0 & 0 & 0 & -K_3 & K_3 \end{bmatrix}$$

$$X = \left[\theta_1(s) \quad \theta_2(s) \quad \theta_3(s) \quad \theta_4(s) \quad \theta_5(s)\right]^T$$

$$Y = \left[0 \quad 0 \quad 0 \quad 0 \quad T(s)\right]^T$$

Required transfer function of the system,

$$G(s) = \frac{\theta_1(s)}{T(s)}$$

Using Cramer's rule, we obtain

$$\theta_1(s) = \frac{\Delta_1}{\Delta}$$

where $$\Delta_1 = \begin{vmatrix} 0 & -B_1 s & 0 & 0 & 0 \\ 0 & \left[J_2 s^2 + (B_1 + B_3 + B_4)s\right] & -B_4 s & 0 & 0 \\ 0 & -B_4 s & (K_2 + B_4 s) & -K_2 & 0 \\ 0 & 0 & -K_2 & (J_3 s^2 + K_3 + K_2) & -K_3 \\ T(s) & 0 & 0 & -K_3 & K_3 \end{vmatrix}$$

$$\Delta_1 = T(s) \times \begin{vmatrix} -B_1 s & 0 & 0 & 0 \\ \left[J_2 s^2 + (B_1 + B_3 + B_4)s\right] & -B_4 s & 0 & 0 \\ -B_4 s & (K_2 + B_4 s) & -K_2 & 0 \\ 0 & -K_2 & (J_3 s^2 + K_3 + K_2) & -K_3 \end{vmatrix}$$

$$= T(s) \times (-B_1 s) \begin{vmatrix} -B_4 s & 0 & 0 \\ (K_2 + B_4 s) & -K_2 & 0 \\ -K_2 & J_3 s^2 + K_3 + K_2 & -K_3 \end{vmatrix}$$

$$\Delta_1 = B_1 B_4 K_2 K_3 s^2 T(s)$$

$$\Delta = |A|$$

$$|A| = \left[J_1 s^2 + (B_1 + B_2)s + K_1\right] \times \begin{vmatrix} \left[J_2 s^2 + (B_1 + B_3 + B_4)s\right] & -B_4 s & 0 & 0 \\ -B_4 s & (K_2 + B_4 s) & -K_2 & 0 \\ 0 & -K_2 & (K_2 + K_3 + J_3 s^2) & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{vmatrix}$$

$$+ B_1 s \times \begin{vmatrix} -B_1 s & -B_4 s & 0 & 0 \\ 0 & B_4 K_2 s & -K_2 & 0 \\ 0 & -K_2 & K_3 + K_2 + J_3 s^2 & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{vmatrix}$$

$$= \left\{J_1 s^2 + (B_1 + B_2 + K_1)s\right\} L + M B_1 s$$

where $$M = -B_1 s \times \begin{vmatrix} K_2 + B_4 s & -K_2 & 0 \\ -K_2 & K_3 + K_2 J_3 s^2 & -K_3 \\ 0 & -K_3 & K_3 \end{vmatrix}$$

$$M = -B_1 K_3 \left(B_4 J_3 s^2 + K_2 J_3 s + B_4 K_2\right) s^2$$

$$L = \left[J_2 s^2 + (B_1 + B_3 + B_4)s\right] \times \begin{vmatrix} K_2 + B_4 s & -K_2 & 0 \\ -K_2 & K_2 + K_3 J_3 s^2 & -K_3 \\ 0 & -K_3 & K_3 \end{vmatrix} + B_4 s \times \begin{vmatrix} -B_4 s & -K_2 & 0 \\ 0 & K_2 + K_3 + J_3 s^2 & -K_3 \\ 0 & -K_3 & K_3 \end{vmatrix}$$

$$L = (K_3 J_2 J_3 B_4)s^5 + (K_2 K_3 J_2 J_3 + B_4 K_3 J_3 B_1 + B_4 K_3 J_3 B_3)s^4$$
$$+ (K_2 K_3 J_3 B_1 + K_2 K_3 J_3 B_3 + K_2 K_3 J_3 B_4 + K_2 K_3 J_2 B_4)s^3 + (K_2 K_3 B_1 B_4 + K_2 K_3 B_4 B_3)s^2$$

Therefore, $\Delta = |A| = \left\{s^2 J_1 + s(B_1 + B_2) + K_1\right\} L + s B_1 M$

i) $s B_1 M = -s^3 B_1^2 K_3 \left(s^2 B_4 J_3 + s K_2 J_3 + B_4 K_2\right)$

ii) $\left(J_1 s^2 + (B_1 + B_2)s + K_1\right) L$

$$= (J_1 J_2 J_3 K_3 B_4)s^7 + K_3 (J_1 J_3 B_1 B_4 + J_1 J_3 B_3 B_4 + J_2 J_3 B_1 B_4 + J_2 J_3 B_2 B_4 + J_1 J_2 J_3 K_2)s^6$$
$$+ K_3 \begin{pmatrix} J_3 B_1 B_3 B_4 + J_3 B_1 B_2 B_4 + J_3 B_2 B_3 B_4 + J_2 J_3 K_1 B_4 + J_1 J_3 K_2 B_1 + J_1 J_3 K_2 B_3 + J_1 J_3 K_2 B_4 \\ + J_2 J_3 K_2 B_1 + J_2 J_3 K_2 B_2 + J_1 J_2 K_2 B_4 + J_3 B_1^{\,2} B_4 \end{pmatrix} s^5$$

$$+K_3\begin{pmatrix} J_3K_1B_1B_4+J_3K_1B_3B_4+J_3K_2B_1B_3+J_3K_2B_1B_4+J_3K_2B_2B_3+J_3K_2B_2B_4+J_3K_2B_1B_2 \\ +J_2J_3K_1K_2+J_1K_2B_1B_4+J_1K_2B_3B_4+J_2K_2B_1B_4+J_2K_2B_2B_4+J_3{B_1}^2K_2 \end{pmatrix}s^4$$

$$+K_3\begin{pmatrix} J_3K_1K_2B_1+J_3K_1K_2B_3+J_3K_1K_2B_4+K_2B_1B_3B_4+K_2B_1B_2B_4+K_2B_2B_3B_4+J_2K_1K_2B_4 \\ +{B_1}^2B_4K_2 \end{pmatrix}s^3$$

$$+K_3\left(K_1K_2B_1B_4+K_1K_2B_3B_4\right)s^2$$

iii) $\Delta=\left\{J_1s^2+\left(B_1+B_2\right)s+K_1\right\}L+MB_1s$

$$=\left(J_1J_2J_3K_3B_4\right)s^7+K_3\left(J_1J_3B_1B_4+J_1J_3B_3B_4+J_2J_3B_1B_4+J_2J_3B_2B_4+J_1J_2J_3K_2\right)s^6$$

$$+K_3\begin{pmatrix} J_3B_1B_3B_4+J_3B_1B_2B_4+J_3B_2B_3B_4+J_2J_3K_1B_4+J_1J_3K_2B_1+J_1J_3K_2B_3+J_1J_3K_2B_4 \\ +J_2J_3K_2B_1+J_2J_3K_2B_2+J_1J_2K_2B_4 \end{pmatrix}s^5$$

$$+K_3\begin{pmatrix} J_3K_1B_1B_4+J_3K_1B_3B_4+J_3K_2B_1B_3+J_3K_2B_1B_4+J_3K_2B_2B_3+J_3K_2B_2B_4+J_3K_2B_1B_2 \\ +J_2J_3K_1K_2+J_1K_2B_1B_4+J_1K_2B_3B_4+J_2K_2B_1B_4+J_2K_2B_2B_4 \end{pmatrix}s^4$$

$$+K_3\left(J_3K_1K_2B_1+J_3K_1K_2B_3+J_3K_1K_2B_4+K_2B_1B_3B_4+K_2B_1B_2B_4+K_2B_2B_3B_4+J_2K_1K_2B_4\right)s^3$$

$$+K_3\left(K_1K_2B_1B_4+K_1K_2B_3B_4\right)s^2$$

Therefore, the transfer function of the given system is

$$\frac{\theta_1(s)}{T(s)}=\frac{B_1B_4K_2K_3s^2}{\Delta}$$

Example 1.17: For the mechanical rotational system shown in Fig. E1.17(a), obtain (i) differential equations and (ii) transfer function of the system.

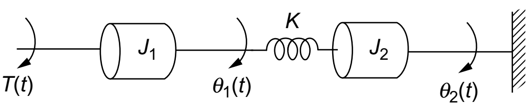


Fig. E1.17(a)

Solution: For the given system, the input and output are applied torque $T(t)$ and angular displacements are $\theta_1(t)$ and $\theta_2(t)$ respectively. Hence, the required transfer function are $\dfrac{\theta_1(s)}{T(s)}$ and $\dfrac{\theta_2(s)}{T(s)}$.

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be $\theta_1(t)$. The Laplace transform of $\theta_1(t)$ is $\theta_1(s)$. The free body diagram of J_1 is shown in Fig. E1.17(b). The opposing torques acting on J_1 are marked as $T_{J1}(t)$ and $T_k(t)$.

$$T_{J1}(t) = J_1 \frac{d^2\theta_1(t)}{dt^2}$$

$$T_K(t) = K_1(\theta_1(t) - \theta(t))$$

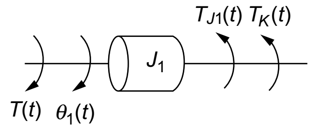


Fig. E1.17(b)

Using Newton's second law, we have

$$T_{J1}(t) + T_K(t) = T(t)$$

$$J_1 \frac{d^2\theta_1(t)}{dt^2} + K(\theta_1(t) - \theta_2(t)) = T(t)$$

Taking Laplace transform, we obtain

$$(J_1 s^2 + K)\theta_1(s) - K\theta_2(s) = T(s) \tag{1}$$

The free body diagram of J_2 is shown in Fig. E1.17(c). The opposing torque acting on J_2 are marked as $T_{J2}(t)$ and $T_K(t)$.

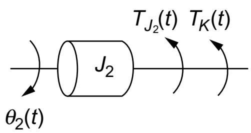


Fig. E1.17(c)

According to D' Alembert's principle, we have

$$T_{J2}(t) + T_K(t) = 0$$

$$J_2 \frac{d^2\theta_2(t)}{dt^2} + K\theta_2(t) - K\theta_1(t) = 0$$

Taking Laplace transform, we obtain

$$J_2 s^2\theta_2(s) + K\theta_2(s) - K\theta_1(s) = 0$$

$$\left(J_2s^2+K\right)\theta_2(s)-K\theta_1(s)=0 \tag{2}$$

The Eqs. (1) and (2) can be written in matrix form as

$$\begin{bmatrix} J_1s^2+K & -K \\ -K & J_2s^2+K \end{bmatrix}\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix}=\begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

Applying Cramer's rule, we obtain

$$\Delta=\begin{vmatrix} J_1s^2+K & -K \\ -K & J_2s^2+K \end{vmatrix}=\left(J_1s^2+K\right)\left(J_2s^2+K\right)-K^2$$

$$=J_1J_2s^4+\left(J_1+J_2\right)Ks^2$$

$$\Delta_1=\begin{bmatrix} T(s) & -K \\ 0 & J_2s^2+K \end{bmatrix}=T(s)\times\left(J_2s^2+K\right)$$

and $$\Delta_2=\begin{bmatrix} J_1s^2+K & T(s) \\ -K & 0 \end{bmatrix}=T(s)\,K$$

Then, $$\theta_1(s)=\frac{\Delta_1}{\Delta}=\frac{T(s)\left(J_2s^2+K\right)}{\left(J_1s^2+K\right)\left(J_2s^2+K\right)-K^2}$$

$$\theta_2(s)=\frac{\Delta_2}{\Delta}=\frac{T(s)K}{\left(J_1s^2+K\right)\left(J_2s^2+K\right)-K^2}$$

Thus, the overall transfer function of the given mechanical system is

$$\frac{\theta_1(s)}{T(s)}=\frac{\left(J_2s^2+K\right)}{\left(J_1s^2+K\right)\left(J_2s^2+K\right)-K^2}$$

$$\frac{\theta_2(s)}{T(s)}=\frac{K}{\left(J_1s^2+K\right)\left(J_2s^2+K\right)-K^2}$$

1.8 Introduction to Analogous System

Two equations of similar form are defined as analogous systems. The advantage of analogous system is that, if the response of one system is known, then the other system is also known without actually solving it.

Generally, mechanical systems are converted into analogous electrical system. There is a similarity between the equilibrium equations (integro-differential equations) of electrical systems and mechanical systems. Due to this, an electrical equivalent can be drawn to a mechanical system.

Example of Analogous Systems

A transformer is analogical to a gear. The transformer adjusts its "voltage-to-current" ratio (V/I) based on the load applied on the transformer that is similar to that of a gear mechanism, by which it adjusts the "torque-to-speed" ratio (T/ω) based on the load requirement.

1.8.1 Advantages of Electrical Analogous System

The following are the advantages of electrical analogous system:

(i) Standard symbols are available in electrical systems such as resistance, inductance, capacitance, etc.

(ii) Standard and simple laws are available in electrical system that makes it easier for calculation.
Example: Kirchhoff's laws, Thévenin's theorem, etc.

(iii) It is easy to analyze a system for different parameter values in an electrical system, because R, L and C cannot be varied and their response can be seen.

In an electrical system, the dual networks are analogous systems and both are in electrical form. The analogous parameters in dual network are listed in Table 1.5.

Table 1.5 | Analogous parameters in dual network

Loop analysis	Nodal analysis
Voltage (V)	Current (I)
Current (I)	Voltage (V)
Charge (q)	Flux (Φ)
Resistance (R)	Conductance (G)
Capacitance (C)	Inductance (L)
Inductance (L)	Capacitance (C)

Types of Analogies

The different types of analogies are discussed as shown in Fig. 1.27.

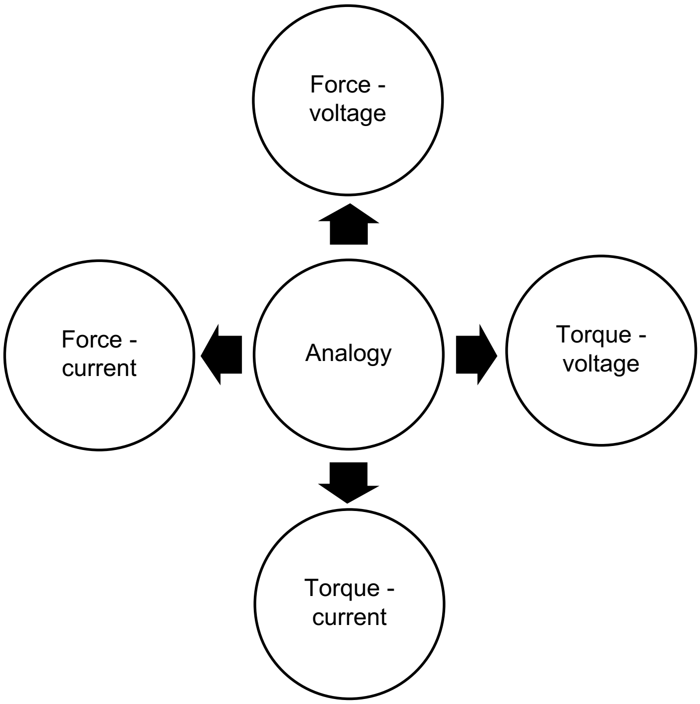


Fig. 1.27 | Types of analogies

1.8.2 Force–Voltage Analogy

Consider a simple translational mechanical system as shown in Fig. 1.28.

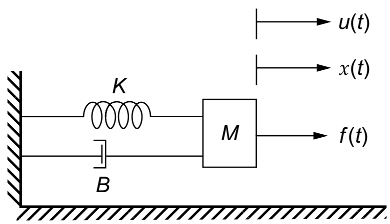


Fig. 1.28 | Translational mechanical system

Using D'Alembert's principle, we have
Sum of the applied forces = sum of the opposing forces

$$f(t) = M\frac{du(t)}{dt} + Bu(t) + K\int u(t)\,dt \tag{1.6}$$

Consider a series RLC circuit as shown in Fig. 1.29.

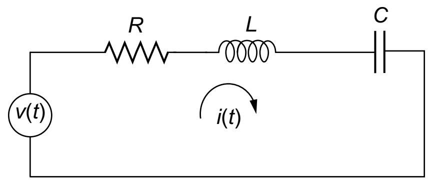


Fig. 1.29 | Series RLC circuit

Using KVL, the integro-differential equations can be written as

$$v(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt \tag{1.7}$$

Rearranging Eqs. (1.6) and (1.7), we obtain

$$f(t) = M\frac{du(t)}{dt} + B\,u(t) + K\int u(t)\,dt$$

$$v(t) = L\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int i(t)\,dt$$

Since the force of mechanical system is made analogous to the voltage of electrical system, the above two equations are represented as analogous system. This analogy is known as force–voltage (or) *F*–*V* analogy. The analogous parameters in *F*–*V* system are given in Table 1.6.

Table 1.6 | *F*–*V* analogous parameters

Translational system	Electrical system
Force (f)	Voltage (v)
Velocity (u)	Current (i)
Displacement (x)	Charge (q)
Mass (M)	Inductance (L)
Damping coefficient (B)	Resistance (R)
Spring constant (K)	1/Capacitance (C)

1.8.3 Force–Current Analogy

Consider a simple parallel RLC Circuit as shown in Fig. 1.30.

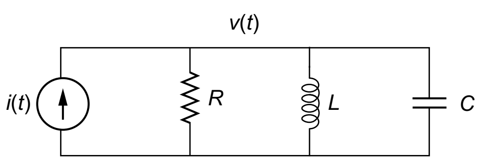


Fig. 1.30 | Parallel RLC circuit

Using KCL, the integro-differential equations can be written as follows:

$$i(t)=\frac{v(t)}{R}+C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)dt$$

$$i(t)=Gv(t)+C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)dt \tag{1.8}$$

The force equation for translational mechanical system is

$$f(t)=M\frac{du(t)}{dt}+Bu(t)+K\int u(t)dt \tag{1.9}$$

Rearranging Eqs. (1.8) and (1.9), we obtain

$$i(t)=C\frac{dv(t)}{dt}+Gv(t)+\frac{1}{L}\int v(t)dt$$

$$f(t)=M\frac{du(t)}{dt}+Bu(t)+K\int u(t)\,dt$$

Since the force of mechanical system is made analogous to the current of electrical system, the above two equations are represented as analogous system. This analogy is known as force–current (or) *F–I* analogy. The analogous parameters in *F–I* system are given in Table 1.7.

Table 1.7 | *F–I* analogous parameters

Translational System	Electrical System
Force (f)	Current (i)
Velocity (u)	Voltage (v)
Displacement (x)	Flux (Φ)
Mass (M)	Capacitance (C)
Damping coefficient (B)	Conductance (G)
Spring constant (K)	1/Inductance (L)

The analogous variables for translational mechanical system and an electrical system are listed in Table 1.8.

Table 1.8 | Analogous of *F–V* and *F–I* to translational mechanical system

Translational Mechanical System	*F–V* Analogy	*F–I* Analogy
Force (f)	Voltage (v)	Current (i)
Velocity (u)	Current (i)	Voltage (v)
Displacement (x)	Charge (q)	Flux (Φ)
Mass (M)	Inductance (L)	Capacitance (C)
Damping coefficient (B)	Resistance (R)	Conductance (G)
Spring constant (K)	1/Capacitance (C)	1/Inductance (L)

Flow chart for problems related to force–voltage and force–current analogy

A flow chart for solving problems related to force–voltage and force–current analogy is shown in Fig. 1.31.

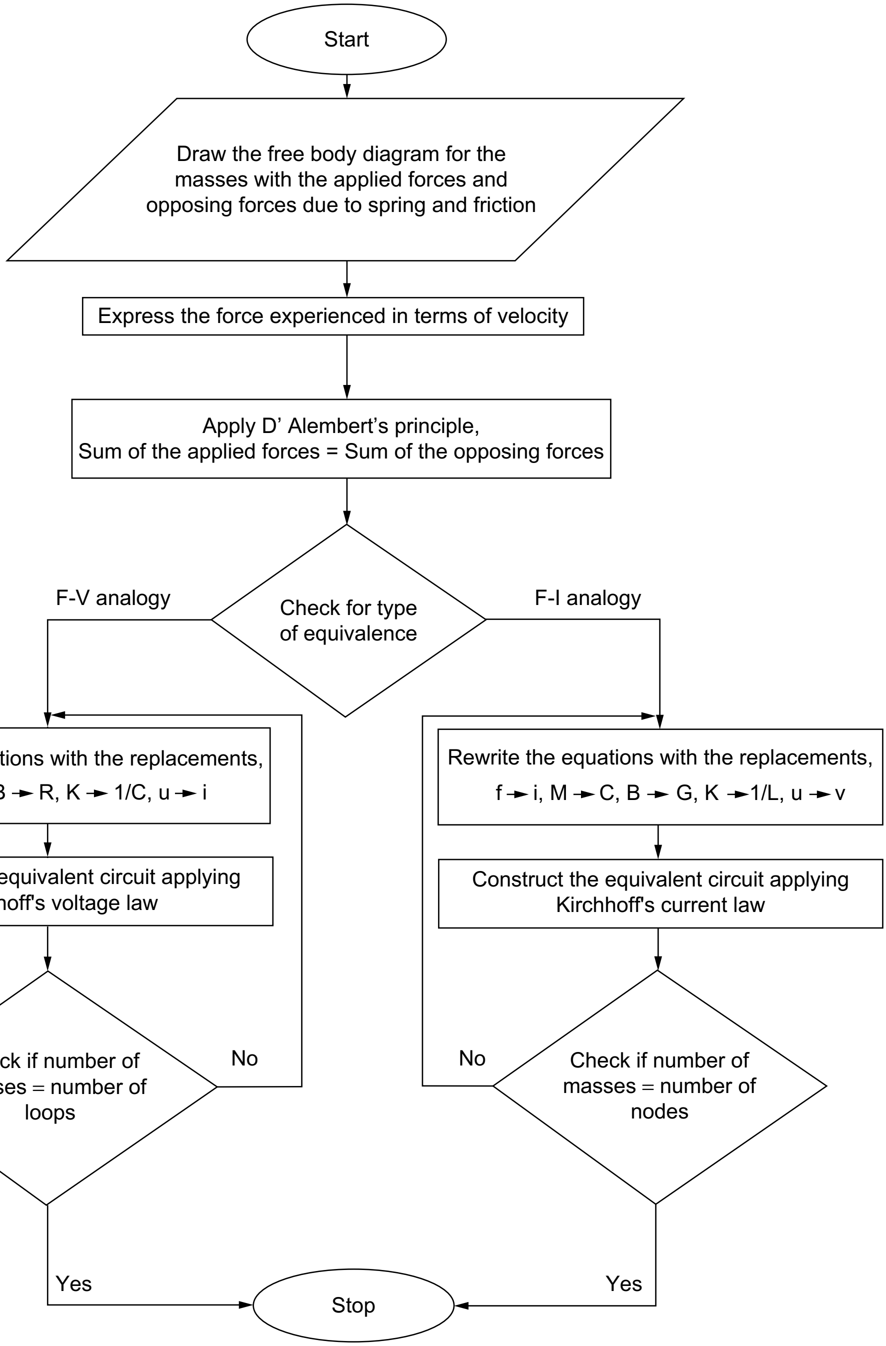


Fig. 1.31 | Flow chart for force-voltage and force-current analogy system

Free Body Diagram

If all the opposing forces and applied forces acting on mass are marked separately, then it is named as *free body diagram*.

Example 1.18: **For the mechanical system shown in Fig. E1.18(a), obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.**

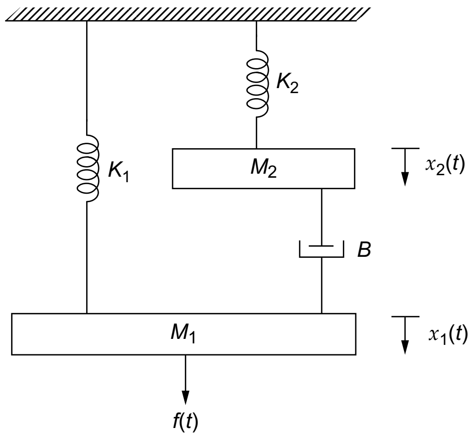


Fig. E1.18(a)

Solution:

Free body diagram of mass M_1:

The mass spring damper system and the free body diagram of mass M_1 indicating the opposing and applied forces acting on it are shown in Figs. E1.18(b) and (c) respectively.

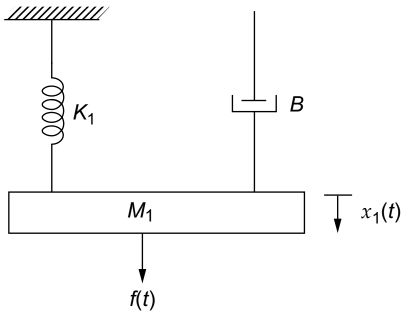


Fig. E1.18(b)

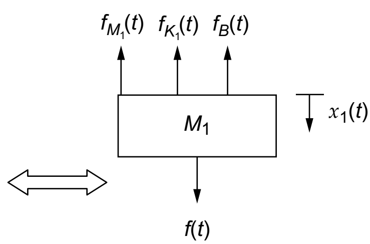


Fig. E1.18(c)

Using D'Alembert's principle, we have

$$f(t) = f_{M1}(t) + f_{K1}(t) + f_B(t)$$

$$f(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + K_1 x_1(t) + B \frac{d(x_1(t) - x_2(t))}{dt} \tag{1}$$

Free body diagram of mass M_2:

The mass spring damper system and the free body diagram of mass M_2 indicating the opposing and applied forces acting on it are shown in Figs. E1.21(d) and (e) respectively.

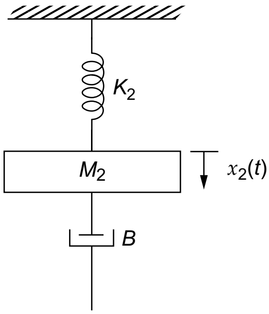


Fig. E1.18(d)

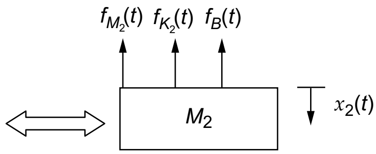


Fig. E1.18(e)

Using D'Alembert's principle, we have

$$0 = f_{M2}(t) + f_{K2}(t) + f_B(t)$$

$$0 = M_2 \frac{d^2 x_2(t)}{dt^2} + K_2 x_2(t) + B \frac{d(x_2(t) - x_1(t))}{dt} \tag{2}$$

Replacing the displacements with velocity in the Eqs. (1) and (2) governing the mechanical translational system, we obtain

$$f(t) = M_1 \frac{du_1(t)}{dt} + K_1 \int u_1(t)dt + B(u_1(t) - u_2(t)) \tag{3}$$

$$0 = M_2 \frac{du_2(t)}{dt} + K_2 \int u_2(t)dt + B(u_2(t) - u_1(t)) \tag{4}$$

(a) *F–V* Electrical Analogy

The *F–V* analogous elements for the mechanical system are given below:

$f(t) \to v(t)$	$M_1 \to L_1$	$B \to R$	$K_1 \to \frac{1}{C_1}$
$u_1(t) \to i_1(t)$	$M_2 \to L_2$		$K_2 \to \frac{1}{C_2}$
$u_2(t) \to i_2(t)$			

The *F–V* analogous equations for Eqs. (3) and (4) are:

$$v(t) = L_1 \frac{di_1(t)}{dt} + \frac{1}{C_1}\int i_1(t)dt + R\left(i_1(t) - i_2(t)\right) \tag{5}$$

$$0 = L_2 \frac{di_2(t)}{dt} + \frac{1}{C_2}\int i_2(t)dt + R\left(i_2(t) - i_1(t)\right) \tag{6}$$

The *F–V* analogous circuit shown in Fig. E1.18(f) is constructed using the above Eqs. (5) and (6). Since the system contains two masses, the electrical circuit is constructed with two loops.

***F–V* Circuit**:

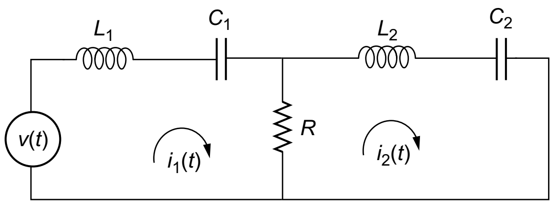


Fig. E1.18(f)

(b) *F–I* Electrical Analogy

The *F–I* analogous elements for the mechanical system are given below:

$f(t) \to i(t)$	$M_1 \to C_1$	$B \to G$	$K_1 \to \frac{1}{L_1}$
$u_1(t) \to v_1(t)$	$M_2 \to C_2$		$K_2 \to \frac{1}{L_2}$
$u_2(t) \to v_2(t)$			

The *F–I* analogous equations for Eqs. (3) and (4) are:

$$i(t) = C_1 \frac{dv_1(t)}{dt} + \frac{1}{L_1}\int v_1(t)dt + G\left(v_1(t) - v_2(t)\right) \tag{7}$$

$$0 = C_2 \frac{dv_2(t)}{dt} + \frac{1}{L_2}\int v_2(t)dt + G\left(v_2(t) - v_1(t)\right) \tag{8}$$

The *F–I* analogous circuit shown in Fig. E1.18(g) is constructed using the above Eqs. (7) and (8). Since the system contains two masses, the electrical circuit is constructed with two nodes.

***F–I* Circuit**:

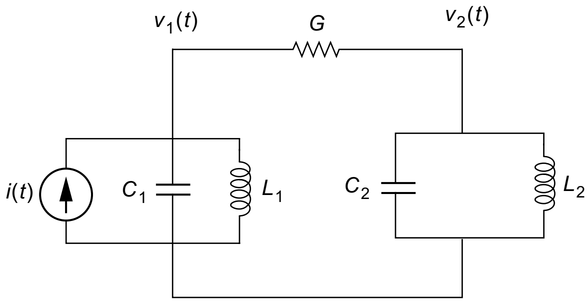


Fig. E1.18(g)

Example 1.19: For the mechanical system shown in Fig. E1.19(a), obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

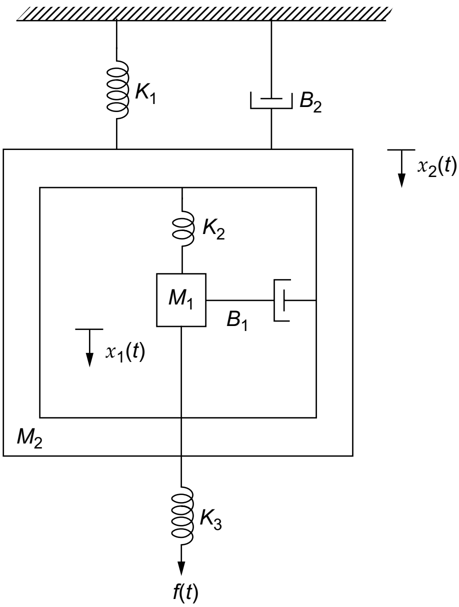


Fig. E1.19(a)

Solution:

Free body diagram of mass M_2:

The mass spring damper system and the free body diagram of mass M_2 indicating the opposing and applied forces acting on it are shown in Figs. E1.19(b) and (c) respectively.

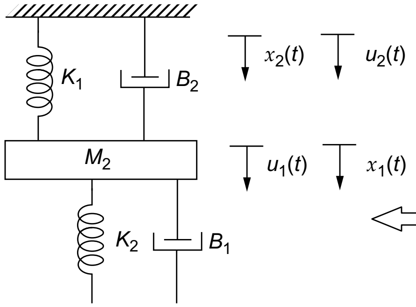


Fig. E1.19(b)

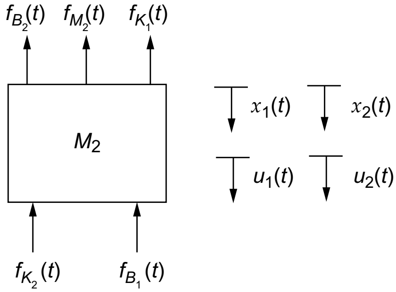


Fig. E1.19(c)

Using D'Alembert's principle, we obtain

$$0 = f_{M2}(t) + f_{K1}(t) + f_{K2}(t) + f_{B1}(t) + f_{B2}(t)$$

$$0 = M_1 \frac{du_2(t)}{dt} + K_1 \int u_2(t)dt + K_2 \int (u_2(t) - u_1(t))dt + B_1(u_2(t) - u_1(t)) + B_2 u_2(t) \tag{1}$$

Free body diagram of mass M_1:

The mass spring damper system and the free body diagram of mass M_1 indicating the opposing and applied forces acting on it are shown in Figs. E1.19(d) and (e) respectively.

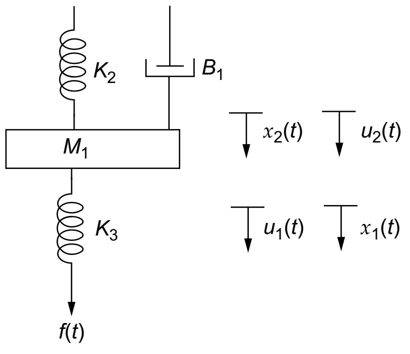


Fig. E1.19(d)

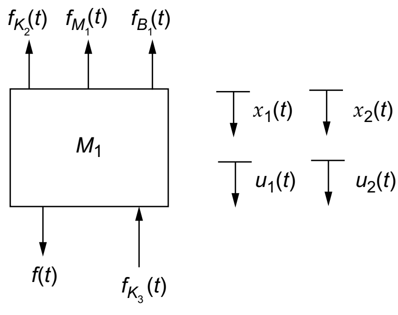


Fig. E1.19(e)

Using D'Alembert's principle, we have

$$f(t) = f_{M1} + f_{K2}(t) + f_{K3}(t) + f_{B1}(t)$$

$$f(t) = M_1 \frac{du_1(t)}{dt} + K_2 \int (u_1(t) - u_2(t))dt + K_3 \int u_1(t)dt + B_1(u_1(t) - u_2(t)) \tag{2}$$

(a) *F–V* Electrical Analogy

The *F–V* electrical analogous elements for the mechanical system are given below:

$f(t) \rightarrow v(t)$ $\quad M_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_3 \rightarrow \frac{1}{C_3}$

$u_1(t) \rightarrow i_1(t)$ $\quad M_2 \rightarrow L_2$ $\quad B_2 \rightarrow R_2$ $\quad K_1 \rightarrow \frac{1}{C_1}$

$u_2(t) \rightarrow i_2(t)$ $\quad K_2 \rightarrow \frac{1}{C_2}$

The *F–V* electrical analogous equations for Eqs. (1) and (2) are:

$$0 = L_2 \frac{di_2(t)}{dt} + \frac{1}{C_1}\int i_2(t)\,dt + \frac{1}{C_2}\int \left(i_2(t) - i_1(t)\right)dt + R_1\left(i_2(t) - i_1(t)\right) + R_2 i_2(t) \tag{3}$$

$$v(t) = L_1 \frac{di_1(t)}{dt} + \frac{1}{C_2}\int \left(i_1(t) - i_2(t)\right)dt + \frac{1}{C_3}\int i_1(t)\,dt + R_1\left(i_1(t) - i_2(t)\right) \tag{4}$$

The *F–V* analogous circuit shown in Fig. E1.19(f) is constructed using the above Eqs. (3) and (4). Since the system contains two masses, the electrical circuit is constructed with two loops.

***F–V* Circuit:**

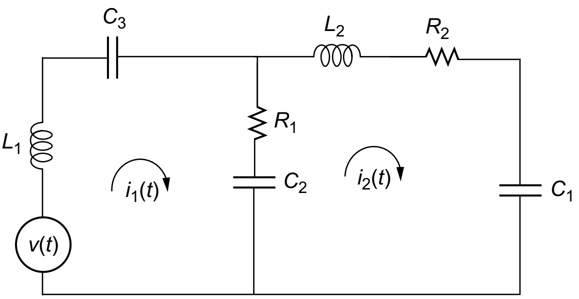


Fig. E1.19(f)

(b) *F–I* Electrical Analogy

The *F–I* electrical analogous elements for the mechanical system are given below:

$f(t) \rightarrow i(t)$ $\quad M_1 \rightarrow C_1$ $\quad B_1 \rightarrow G_1$ $\quad K_1 \rightarrow \frac{1}{L_1}$

$u_1(t) \rightarrow v_1(t)$ $\quad M_2 \rightarrow C_2$ $\quad B_2 \rightarrow G_2$ $\quad K_2 \rightarrow \frac{1}{L_2}$

$u_2(t) \rightarrow v_2(t)$ $\quad K_3 \rightarrow \frac{1}{L_3}$

The *F–I* electrical analogous equations for Eqs. (1) and (2) are:

$$0 = C_2 \frac{dv_2(t)}{dt} + \frac{1}{L_1}\int v_2(t)dt + \frac{1}{L_2}\int \left(v_2(t) - v_1(t)\right)dt + G_1\left(v_2(t) - v_1(t)\right) + G_2 v_2(t) \tag{5}$$

$$i(t) = C_1 \frac{dv_1(t)}{dt} + \frac{1}{L_2}\int \left(v_1(t) - v_2(t)\right)dt + \frac{1}{L_3}\int v_1(t)dt + G_1\left(v_1(t) - v_2(t)\right) \tag{6}$$

The *F–I* analogous circuit shown in Fig. E1.19(g) is constructed using the Eqs. (5) and (6). Since the system contains two masses, the electrical circuit is constructed with two nodes.

***F–I* Circuit**:

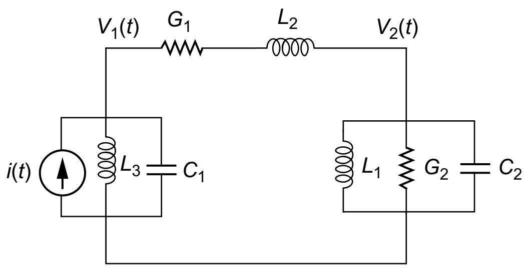


Fig. E1.19(g)

Example 1.20: For the mechanical translational system shown in Fig. E1.20(a), obtain (i) differential equations, (ii) *F–V* analogous circuit and (iii) *F–I* analogous circuit.

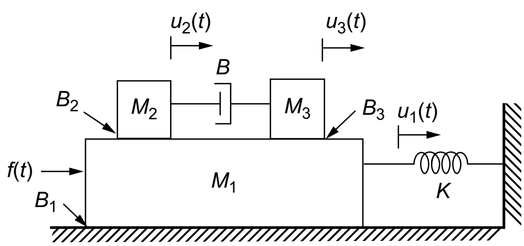


Fig. E1.20(a)

Solution:

Free body diagram of mass M_3:

The mass spring damper system and the free body diagram of mass M_3 indicating the opposing and applied forces acting on it are shown in Figs. E1.20(b) and (c) respectively.

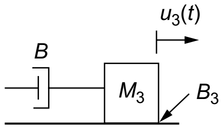


Fig. E1.20(b)

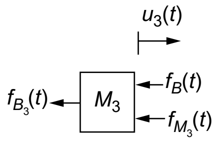


Fig. E1.20(c)

Using D'Alembert's principle, we have

$$0 = f_{M3}(t) + f_{B3}(t) + f_B(t)$$

$$0 = M_3 \frac{dv_3(t)}{dt} + B_3\left(v_3(t) - v_1(t)\right) + B\left(v_3(t) - v_2(t)\right) \tag{1}$$

Free body diagram of mass M_2:
The mass spring damper system and the free body diagram of mass M_2 indicating the opposing and applied forces acting on it are shown in Figs. E1.20(d) and (e) respectively.

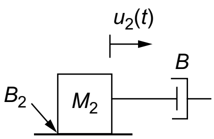


Fig. E1.20(d)

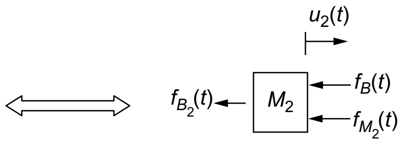


Fig. E1.20(e)

Using D'Alembert's principle, we have

$$0 = f_{M2}(t) + f_{B2}(t) + f_B(t)$$

$$0 = M_2 \frac{dv_2(t)}{dt} + B_2\left(v_2(t) - v_1(t)\right) + B\left(v_2(t) - v_3(t)\right) \tag{2}$$

Free body diagram of mass M_1:
The mass spring damper system and the free body diagram of mass M_1 indicating the opposing and applied forces acting on it are shown in Figs. E1.20(f) and (g) respectively.

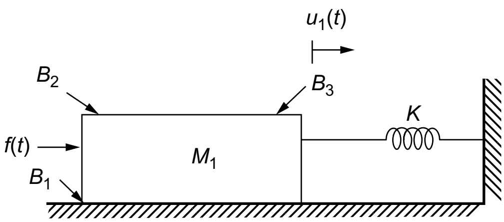


Fig. E1.20(f)

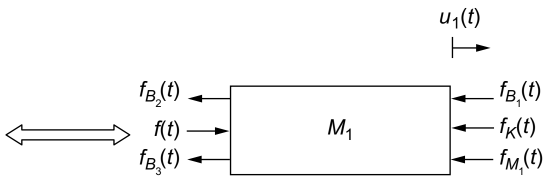


Fig. E1.20(g)

Using D'Alembert's principle, we have

$$f(t) = f_{M1}(t) + f_{B2}(t) + f_{B3}(t) + f_{B1}(t) + f_K(t)$$

$$f(t) = M_1 \frac{dv_1(t)}{dt} + B_1 v_1(t) + B_2\big(v_1(t) - v_2(t)\big) + B_3\big(v_1(t) - v_3(t)\big) + K\int v_1(t)\,dt \tag{3}$$

(a) *F–V* Electrical Analogy

The *F–V* analogous elements for the mechanical system are given by

$f(t) \to v(t)$	$M_1 \to L_1$	$B_1 \to R_1$	$K \to \frac{1}{C}$
$v_1(t) \to i_1(t)$	$M_2 \to L_2$	$B_2 \to R_2$	
$v_2(t) \to i_2(t)$	$M_3 \to L_3$	$B_3 \to R_3$	
$v_3(t) \to i_3(t)$		$B \to R$	

The *F–V* analogous equations for Eqs. (1) through (3) are given by

$$0 = L_3 \frac{di_3(t)}{dt} + R_3\big(i_3(t) - i_1(t)\big) + R\big(i_3(t) - i_2(t)\big) \tag{4}$$

$$0 = L_2 \frac{di_2(t)}{dt} + R_2\big(i_2(t) - i_1(t)\big) + R\big(i_2(t) - i_3(t)\big) \tag{5}$$

$$v(t) = L_1 \frac{di_1(t)}{dt} + R_1 i_1(t) + R_2\big(i_1(t) - i_2(t)\big) + R_3\big(i_1(t) - i_3(t)\big) + \frac{1}{C}\int i_1(t)\,dt \tag{6}$$

The *F–V* analogous circuit shown in Fig. E1.20(h) is constructed using the above Eqs. (4) through (6). Since the system contains three masses, the electrical circuit is constructed with three loops.

***F–V* Circuit**:

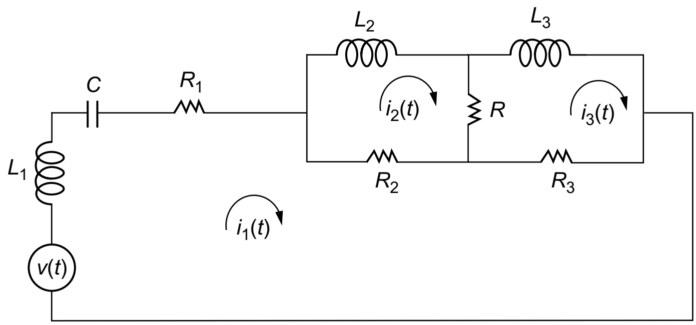


Fig. E1.20(h)

(b) *F–I* Electrical Analogy

The *F–I* analogous elements for the mechanical system are given below:

$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad B_1 \to G_1$ $\quad K_1 \to \dfrac{1}{L_1}$

$u_1(t) \to v_1(t)$ $\quad M_2 \to C_2$ $\quad B_2 \to G_2$ $\quad K_2 \to \dfrac{1}{L_2}$

$u_2(t) \to v_2(t)$ $\quad M_3 \to C_3$

$u_3(t) \to v_3(t)$

The *F–I* analogous equations for Eqs. (1) through (3) are:

$$0 = C_3 \frac{dv_3(t)}{dt} + G_3\left(v_3(t) - v_1(t)\right) + G\left(v_3(t) - v_2(t)\right) \tag{7}$$

$$0 = C_2 \frac{dv_2(t)}{dt} + G_2\left(v_2(t) - v_1(t)\right) + G\left(v_2(t) - v_3(t)\right) \tag{8}$$

$$i(t) = C_1 \frac{dv_1(t)}{dt} + G_1 v_1(t) + G_2\left(v_1(t) - v_2(t)\right) + G_3\left(v_1(t) - v_3(t)\right) + \frac{1}{L}\int v_1(t)\,dt \tag{9}$$

The *F–I* analogous circuit shown in Fig. E1.20(i) is constructed using the Eqs. (7) through (9). Since the system contains three masses, the electrical circuit is constructed with three nodes.

***F–I* Circuit**:

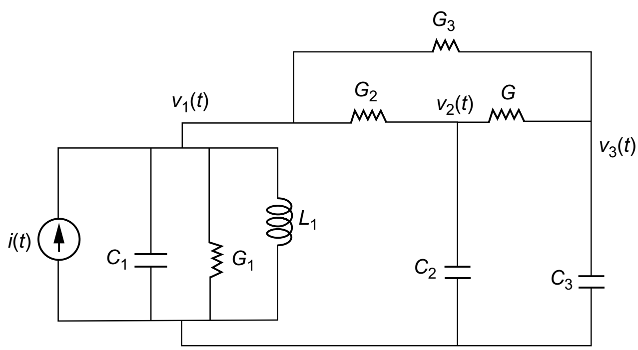


Fig. E1.20(i)

Example 1.21: For the mechanical system shown in Fig. E1.21(a), obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

Solution:

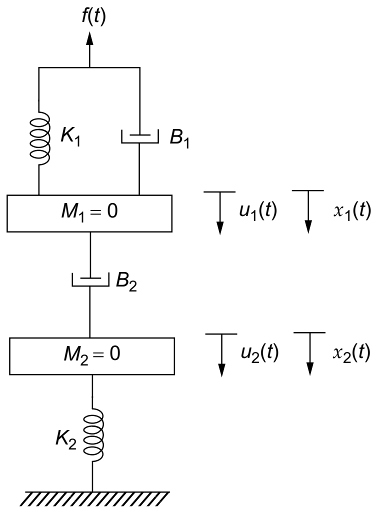


Fig. E1.21(a)

Free body diagram of mass M_1:

The mass spring damper system and the free body diagram of mass M_1 indicating the opposing and applied forces acting on it are shown in Figs. E1.21(b) and (c) respectively.

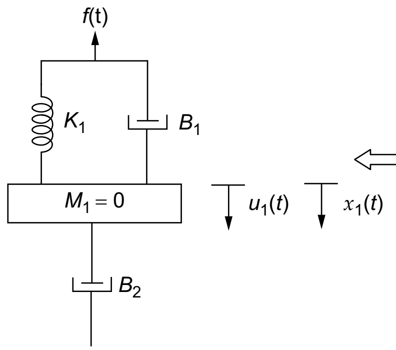


Fig. E1.21(b)

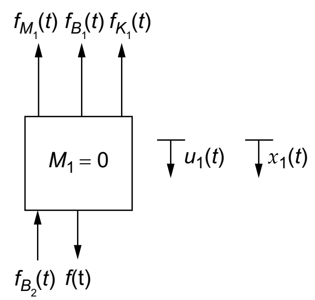


Fig. E1.21(c)

Using D'Alembert's principle, we obtain

$$f(t) = f_{M1}(t) + f_{B1}(t) + f_{K1}(t) + f_{B2}(t)$$

$$f(t) = B_1 u_1(t) + B_2(u_1(t) - u_2(t)) + K_1 \int u_1(t)dt \qquad (1)$$

Free body diagram of mass M_2:

The mass spring damper system and the free body diagram of mass M_2 indicating the opposing and applied forces acting on it are shown in Figs. E1.21(d) and (e) respectively.

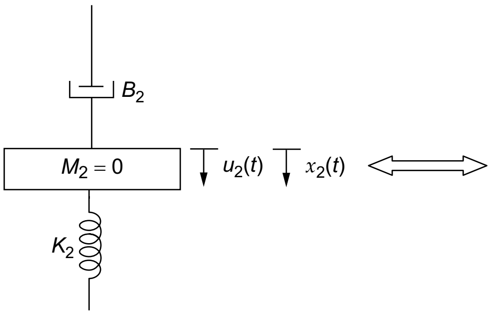


Fig. E1.21(d)

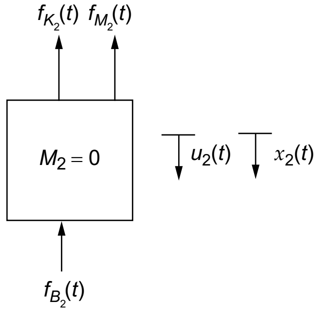


Fig. E1.21(e)

Using D'Alembert's principle, we obtain

$$0 = f_{M2}(t) + f_{B2}(t) + f_{K2}(t)$$

$$0 = B_2(u_2(t) - u_1(t)) + K_2 \int u_2(t)dt \qquad (2)$$

(a) *F–V* Electrical Analogy

The *F–V* electrical analogous elements for the mechanical system are given below:

$f(t) \to v(t)$ $\qquad B_1 \to R_1$ $\qquad K_1 \to \dfrac{1}{C_1}$

$u_1(t) \to i_1(t)$ $\qquad B_2 \to R_2$ $\qquad K_2 \to \dfrac{1}{C_2}$

$u_2(t) \to i_2(t)$

The *F–V* electrical analogous equations for Eqs. (1) and (2) are:

$$v(t) = R_1 i_1(t) + R_2(i_1(t) - i_2(t)) + \frac{1}{C_1}\int i_1(t)dt \qquad (3)$$

$$0 = R_2(i_2(t) - i_1(t)) + \frac{1}{C_2}\int i_2(t)dt \qquad (4)$$

The *F–V* analogous circuit shown in Fig. E1.21(f) is constructed using the above Eqs. (3) and (4). Since the system contains two masses, the electrical circuit is constructed with two loops.

***F–V* Circuit**:

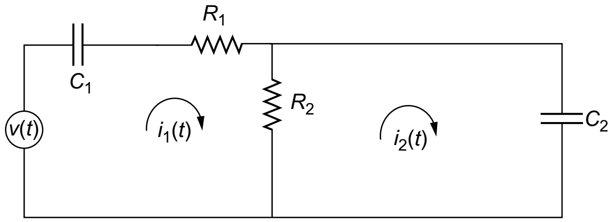


Fig. E1.21(f)

(b) *F–I* Electrical Analogy

The *F–I* electrical analogous elements for the mechanical system are given below:

$f(t) \rightarrow i(t) \qquad B_1 \rightarrow G_1 \qquad K_1 \rightarrow \frac{1}{L_1}$

$u_1(t) \rightarrow v_1(t) \qquad B_2 \rightarrow G_2 \qquad K_2 \rightarrow \frac{1}{L_2}$

$u_2(t) \rightarrow v_2(t)$

The *F–I* electrical analogous equations for Eqs. (1) and (2) are:

$$i(t) = G_1 v_1(t) + G_2(v_1(t) - v_2(t)) + \frac{1}{L_1}\int v_1(t)dt \tag{5}$$

$$0 = G_2(v_2(t) - v_1(t)) + \frac{1}{L_2}\int v_2(t)dt \tag{6}$$

The *F–I* analogous circuit shown in Fig. E1.21(g) is constructed using the above Eqs. (5) and (6). Since the system contains two masses, the electrical circuit is constructed with two nodes.

***F–I* Circuit**:

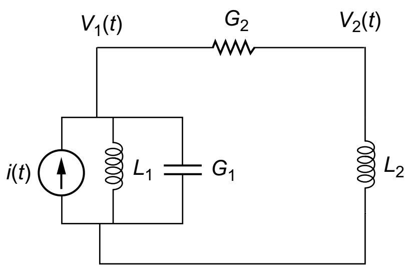


Fig. E1.21(g)

Example 1.22: **For the mechanical system shown in Fig. E1.22(a), obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.**

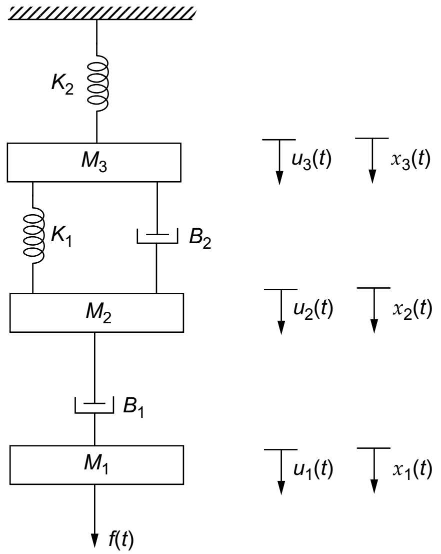


Fig. E1.22(a)

Solution:

Free body diagram of mass M_3:

The mass spring damper system and the free body diagram of mass M_3 indicating the opposing and applied forces acting on it are shown in Figs. E1.22(b) and (c) respectively.

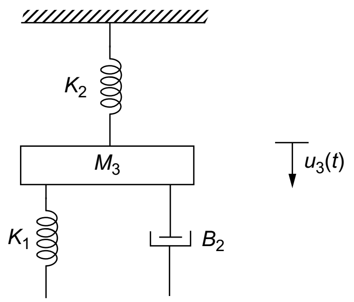


Fig. E1.22(b)

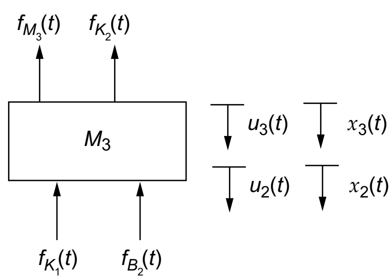


Fig. E1.22(c)

Using D'Alembert's principle, we have

$$0 = f_{M3}(t) + f_{K1}(t) + f_{B2}(t) + f_{K2}(t)$$

$$0 = M_3 \frac{du_3(t)}{dt} + K_2 \int u_3(t)\,dt + B_2\left(u_3(t) - u_2(t)\right) + K_1 \int \left(u_3(t) - u_2(t)\right) dt \tag{1}$$

Free body diagram of mass M_2:
The mass spring damper system and the free body diagram of mass M_2 indicating the opposing and applied forces acting on it are shown in Figs. E1.22(d) and (e) respectively.

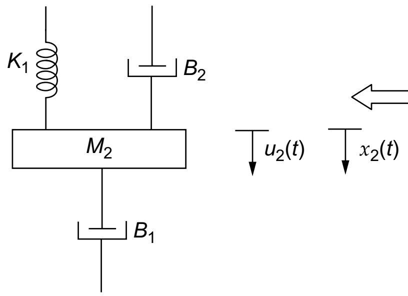


Fig. E1.22(d)

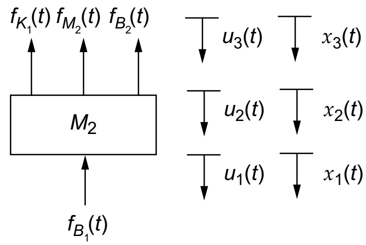


Fig. E1.22(e)

Using D'Alembert's principle, we obtain

$$0 = f_{M2}(t) + f_{K1}(t) + f_{B2}(t) + f_{B1}(t)$$

$$0 = M_2 \frac{du_2(t)}{dt} + K_1 \int \left(u_2(t) - u_3(t)\right) dt + B_2\left(u_2(t) - u_3(t)\right) + B_1\left(u_2(t) - u_1(t)\right) \tag{2}$$

Free body diagram of mass M_1:
The mass spring damper system and the free body diagram of mass M_1 indicating the opposing and applied forces acting on it are shown in Figs. E1.22(f) and (g) respectively.

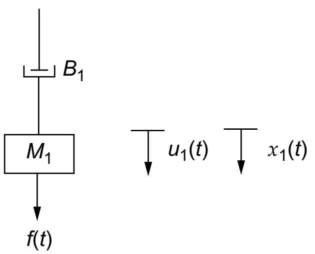


Fig. E1.22(f)

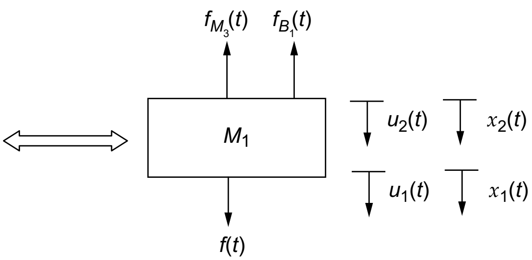


Fig. E1.22(g)

Using D'Alembert's principle, we have

$$f(t) = f_{M1}(t) + f_{B1}(t)$$

$$f(t) = M_1 \frac{du_1(t)}{dt} + B_1\left(u_1(t) - u_2(t)\right) \tag{3}$$

(a) *F–V* Electrical Analogy

The *F–V* analogous elements for the mechanical system are given below:

$f(t) \to v(t)$ $\quad M_1 \to L_1$ $\quad B_1 \to R_1$ $\quad K_1 \to \frac{1}{C_1}$

$u_1(t) \to i_1(t)$ $\quad M_2 \to L_2$ $\quad B_2 \to R_2$ $\quad K_2 \to \frac{1}{C_2}$

$u_2(t) \to i_2(t)$ $\quad M_3 \to L_3$

$u_3(t) \to i_3(t)$

The *F–V* analogous equations for Eqs. (1) through (3) are:

$$0 = L_3 \frac{di_3(t)}{dt} + \frac{1}{C_2}\int i_3(t)dt + R_2\left(i_3(t) - i_2(t)\right) + \frac{1}{C_1}\int \left(i_3(t) - i_2(t)\right)dt \tag{4}$$

$$0 = L_2 \frac{di_2(t)}{dt} + \frac{1}{C_1}\int \left(i_2(t) - i_3(t)\right)dt + R_2\left(i_2(t) - i_3(t)\right) + R_1\left(i_2(t) - i_1(t)\right) \tag{5}$$

$$v(t) = L_1 \frac{di_1(t)}{dt} + R_1\left(i_1(t) - i_2(t)\right) \tag{6}$$

The *F–V* analogous circuit shown in Fig. E1.22(h) is constructed using the above Eqs. (4) through Eqn. (6). Since the system contains three masses, the electrical circuit is constructed with three loops.

***F–V* Circuit**:

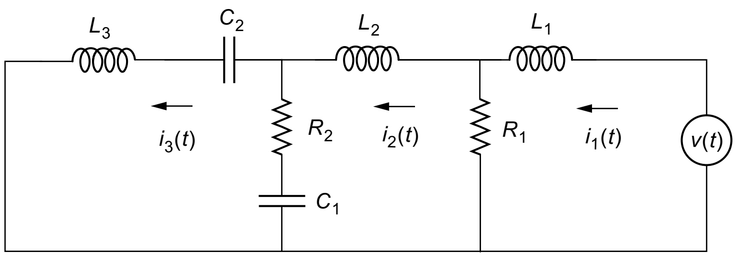


Fig. E1.22(h)

(b) *F–I* Electrical Analogy

The *F–I* analogous elements for the mechanical system are given below:

$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad B_1 \to G_1$ $\quad K_1 \to \frac{1}{L_1}$

$u_1(t) \to v_1(t)$ $\quad M_2 \to C_2$ $\quad B_2 \to G_2$ $\quad K_2 \to \frac{1}{L_2}$

$u_2(t) \to v_2(t) \qquad M_3 \to C_3$

$u_3(t) \to v_3(t)$

The *F–I* analogous equations for Eqs. (1) through (3) are:

$$0 = C_3 \frac{dv_3(t)}{dt} + \frac{1}{L_2}\int v_3(t)\,dt + G_2\left(v_3(t) - v_2(t)\right) + \frac{1}{L_1}\int\left(v_3(t) - v_2(t)\right)dt \tag{7}$$

$$0 = C_2 \frac{dv_2(t)}{dt} + \frac{1}{L_1}\int\left(v_2(t) - v_3(t)\right)dt + G_2\left(v_2(t) - v_3(t)\right) + G_1\left(v_2(t) - v_1(t)\right) \tag{8}$$

$$i(t) = C_1 \frac{dv_1(t)}{dt} + G_1\left(v_1(t) - v_2(t)\right) \tag{9}$$

The *F–I* analogous circuit shown in Fig. E1.22(i) is constructed using the above Eqs. (7) through (9). Since the system contains three masses, the electrical circuit is constructed with three nodes.

***F–I* Circuit**:

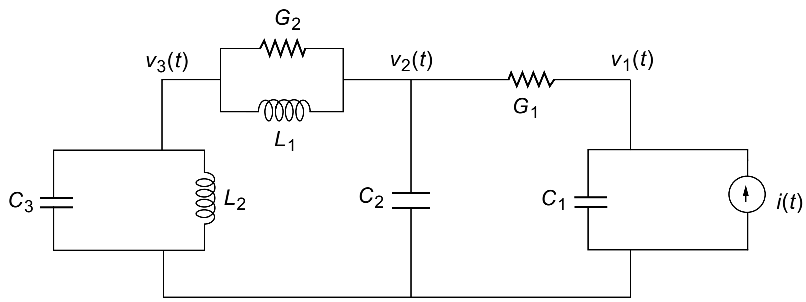


Fig. E1.22(i)

1.8.4 Torque–Voltage Analogy

Consider a simple rotational mechanical system as shown in Fig.1.32.

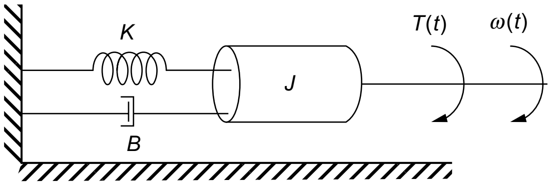


Fig. 1.32 | Rotational mechanical system

Using D'Alembert's principle, we have
Sum of the applied torques = sum of the opposing torques

$$T(t) = J\frac{d\omega(t)}{dt} + B\omega(t) + K\int \omega(t)\,dt \tag{1.10}$$

Consider a series RLC circuit as shown in Fig. 1.33.

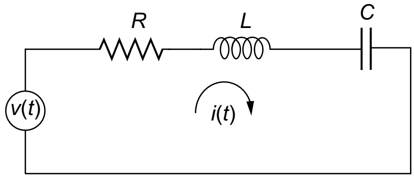


Fig. 1.33 | Series RLC circuit

Applying KVL, we obtain

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)\,dt \tag{1.11}$$

Since the torque of the mechanical system is made analogous to the voltage of the electrical system, Eqs. (1.10) and (1.11) are represented as analogous system. This analogy is known as *torque–voltage* (or) *T–V analogy*. The analogous parameters in *T–V* system are given in Table 1.9.

Table 1.9 | *T–V* analogous parameters

Rotational System	Electrical System
Torque (T)	Voltage (v)
Angular velocity (ω)	Current (i)
Angular displacement (θ)	Charge (q)
Moment of inertia (J)	Inductance (L)
Rotational damping (B)	Resistance (R)
Rotational spring constant (K)	1/Capacitance (C)

1.8.5 Torque–Current Analogy

Consider a simple rotational mechanical system as shown in Fig. 1.34.

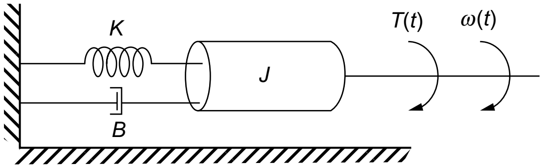


Fig. 1.34 | Rotational mechanical system

Using D'Alembert's principle, we have

$$T(t) = J\frac{d\omega(t)}{dt} + B\omega(t) + K\int \omega(t)\,dt \tag{1.12}$$

Consider parallel RLC Circuit as shown in Fig. 1.35.

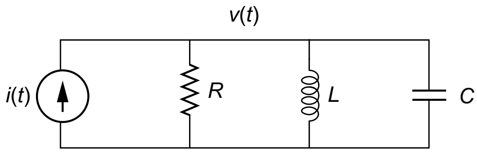


Fig. 1.35 | Parallel RLC circuit

Applying KCL, we obtain

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt$$

$$= C\frac{dv(t)}{dt} + Gv(t) + \frac{1}{L}\int v(t)\,dt \tag{1.13}$$

Since the torque of the mechanical system is made analogous to the current of the electrical system, Eqs. (1.12) and (1.13) are represented as analogous system. This analogy is known as torque–current (or) *T–I* analogy. The analogous parameters in *T–I* system are given in Table 1.10.

Table 1.10 | *T–I* analogous parameters

Rotational Mechanical System	*T-I* Analogous
Torque (T)	Current (i)
Angular velocity (ω)	Voltage (v)
Angular displacement (θ)	Flux (Φ)
Moment of inertia (J)	Capacitance (C)
Rotational spring constant (K)	1/Inductance (L)
Rotational damping (B)	Conductance (G)

The analogous variables for rotational mechanical system and electrical system are listed in Table 1.11.

Table 1.11 | Analogous of *T–V* and *T–I* to rotational mechanical system

Rotational Mechanical System	*T–V* Analogy	*T-I* Analogy
Torque (T)	Voltage (v)	Current (i)
Angular velocity (ω)	Current (i)	Voltage (v)

(*Continued*)

Table 1.11 | (Continued)

Rotational Mechanical System	***T–V* Analogy**	***T-I* Analogy**
Angular displacement (θ)	Charge (q)	Flux (Φ)
Moment of inertia (J)	Inductance (L)	Capacitance (C)
Rotational damping (B)	Resistance (R)	Conductance (G)
Rotational spring constant (K)	1/Capacitance (C)	1/Inductance (L)

A flow chart for solving problems related to torque–voltage and torque–current analogy is shown in Fig. 1.36.

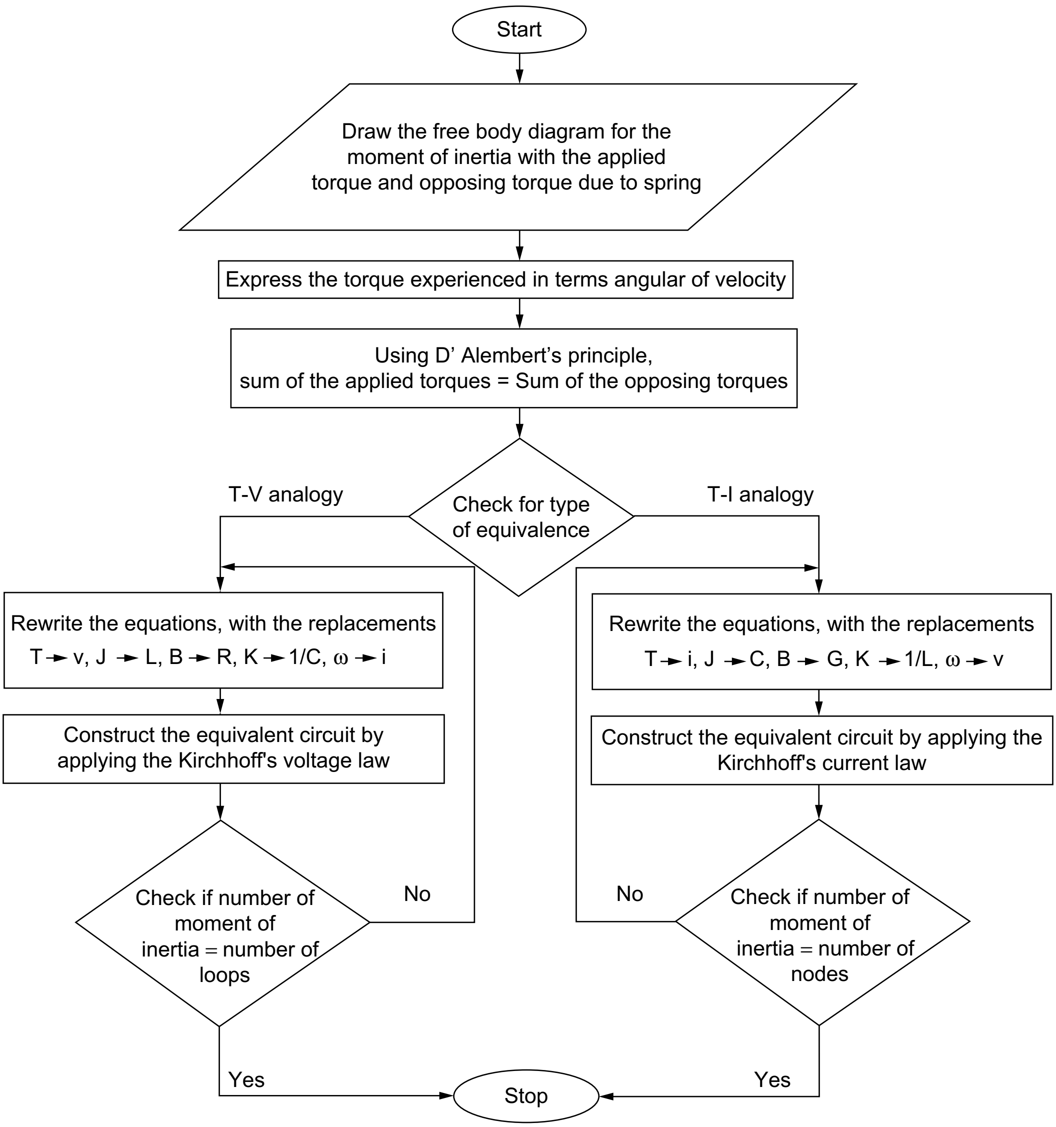


Fig. 1.36 | Flow chart for torque-voltage and torque-current analogy system

Example 1.23: For the mechanical rotational system shown in Fig. E1.23(a), obtain (i) differential equations, (ii) *T–V* analogous circuit and (iii) *T–I* analogous circuit.

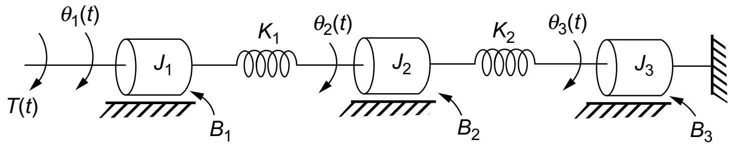


Fig. E1.23(a)

Solution: The given mechanical rotational system has three nodes, i.e., three moment of inertia of masses. The differential equations governing the mechanical rotational systems are given by torque balance equations at these nodes. Let the angular displacements of J_1, J_2 and J_3 be $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ respectively. The corresponding angular velocities are $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$, respectively.

Free body diagram for inertia J_1:

The rotational mechanical system and the free body diagram of J_1 indicating the opposing and applied torques acting on it are shown in Figs. E1.23(b) and (c) respectively.

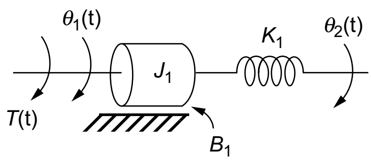


Fig. E1.23(b)

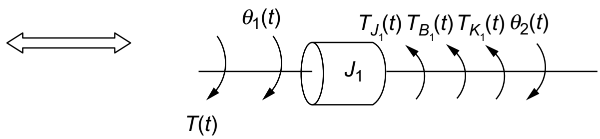


Fig. E1.23(c)

Using D'Alembert's principle, we have

$$T_{J1}(t)+T_{B1}(t)+T_{K1}(t)=T(t)$$

$$J_1\frac{d^2\theta_1(t)}{dt^2}+B_1\frac{d(\theta_1(t))}{dt}+K_1(\theta_1(t)-\theta_2(t))=T(t) \tag{1}$$

Free body diagram for inertia J_2:

The rotational mechanical system and the free body diagram of J_2 indicating the opposing and applied torques acting on it are shown in Figs. E1.23(d) and (e) respectively.

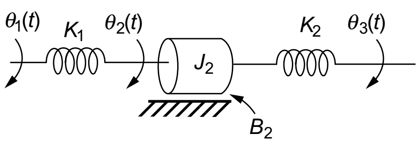


Fig. E1.23(d)

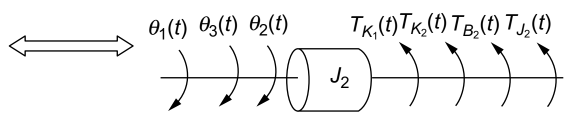


Fig. E1.23(e)

Using D'Alembert's principle, we have

$$T_{J2}(t)+T_{B2}(t)+T_{K2}(t)+T_{K1}(t)=0$$

$$J_2\frac{d^2\theta_2(t)}{dt^2}+B_2\frac{d(\theta_2(t))}{dt}+K_2(\theta_2(t)-\theta_3(t))+K_1(\theta_2(t)-\theta_1(t))=0 \tag{2}$$

Free body diagram for inertia J_3:
The rotational mechanical system and the free body diagram of J_3 indicating the opposing and applied torques acting on it are shown in Figs. E1.23(f) and (g) respectively.

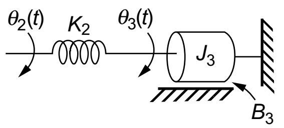


Fig. E1.23(f)

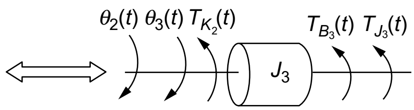


Fig. E1.23(g)

Using D'Alembert's principle, we have

$$T_{J3}(t)+T_{B3}(t)+T_{K2}(t)=0$$

$$J_3\frac{d^2\theta_3(t)}{dt^2}+B_3\frac{d(\theta_3(t))}{dt}+K_2(\theta_3(t)-\theta_2(t))=0 \tag{3}$$

Replacing the angular displacements by angular velocity in the differential Eqs. (1) through (3) governing the mechanical rotational system, we obtain

$$J_1\frac{d\omega_1(t)}{dt}+B_1\omega_1(t)+K_1\int(\omega_1(t)-\omega_2(t))dt=T(t) \tag{4}$$

$$J_2\frac{d\omega_2(t)}{dt}+B_2\omega_2(t)+K_2\int(\omega_2(t)-\omega_3(t))dt+K_1\int(\omega_2(t)-\omega_1(t))dt=0 \tag{5}$$

$$J_3\frac{d\omega_3(t)}{dt}+B_3\omega_3(t)+K_2\int(\omega_3(t)-\omega_2(t))dt=0 \tag{6}$$

***T–V* Electrical Analogy**

The *T–V* analogous elements for the mechanical system are given below:

$T(t)\to v(t)$	$\omega_1(t)\to i_1(t)$	$J_1\to L_1$	$B_1\to R_1$	$K_1\to 1/C_1$
	$\omega_2(t)\to i_2(t)$	$J_2\to L_2$	$B_2\to R_2$	$K_2\to 1/C_2$
	$\omega_3(t)\to i_3(t)$	$J_3\to L_3$	$B_3\to R_3$	

The T–V analogous equations of Eqs. (4) through (6) are:

$$L_1 \frac{di_1(t)}{dt} + R_1 i_1(t) + \frac{1}{C_1}\int \left(i_1(t) - i_2(t)\right)dt = v(t) \tag{7}$$

$$L_2 \frac{di_2(t)}{dt} + R_2 i_2(t) + \frac{1}{C_2}\int \left(i_2(t) - i_3(t)\right)dt + \frac{1}{C_1}\int \left(i_2(t) - i_1(t)\right)dt = 0 \tag{8}$$

$$L_3 \frac{di_3(t)}{dt} + R_3 i_3(t) + \frac{1}{C_3}\int \left(i_3(t) - i_2(t)\right)dt = 0 \tag{9}$$

The T–V analogous circuit shown in Fig. E1.23(h) is constructed using the Eqs. (7) through (9). Since the system contains three inertial elements, the electrical circuit is constructed with three loops.

***T–V* Circuit**:

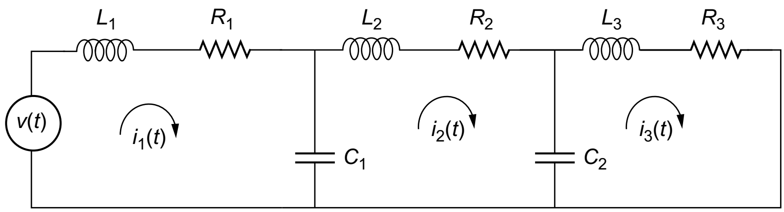


Fig. E1.23(h)

T–I Electrical Analogy

The T–I analogous elements for the mechanical system are given below:

$T(t) \to i(t)$ $\quad \omega_1(t) \to v_1(t) \quad J_1 \to C_1 \quad B_1 \to G_1 \quad K_1 \to 1/L_1$

$\omega_2(t) \to v_2(t) \quad J_2 \to C_2 \quad B_2 \to G_2 \quad K_2 \to 1/L_2$

$\omega_3(t) \to v_3(t) \quad J_3 \to C_3 \quad B_3 \to G_3$

The T–I analogous equations for Eqs. (4) through (6) are:

$$C_1 \frac{dv_1(t)}{dt} + G_1 v_1(t) + \frac{1}{L_1}\int \left(v_1(t) - v_2(t)\right)dt = i(t) \tag{10}$$

$$C_2 \frac{dv_2(t)}{dt} + G_2 v_2(t) + \frac{1}{L_2}\int \left(v_2(t) - v_3(t)\right)dt + \frac{1}{L_1}\int \left(v_2(t) - v_1(t)\right)dt = 0 \tag{11}$$

$$C_3 \frac{dv_3(t)}{dt} + G_3 v_3(t) + \frac{1}{L_2}\int \left(v_3(t) - v_2(t)\right)dt = 0 \tag{12}$$

The *T–I* analogous circuit shown in Fig. E1.23(i) is constructed using the Eqs. (10) through (12). Since the system contains three inertial elements, the electrical circuit is constructed with three nodes.

***T–I* Circuit**:

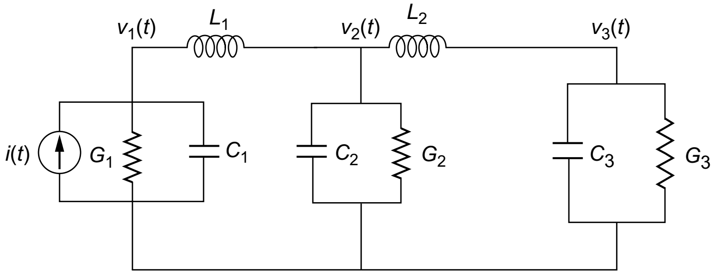


Fig. E1.23(i)

Review Questions

1. What do you mean by a system? Give examples.
2. What do you mean by a control system? Give examples.
3. What are the different types of control system?
4. With the help of block diagram, write a note on open-loop systems.
5. With the help of block diagram, write a note on closed-loop systems.
6. Give a practical example for an open-loop system and explain it.
7. Give a practical example for a closed-loop system and explain it.
8. With the block diagram, illustrate the control of a car by the driver and identify the components of this closed-loop system.
9. Distinguish the open-loop and closed-loop control systems.
10. Explain the concept of principle of homogeneity and superposition.
11. How are the systems classified based on principle of homogeneity and superposition?
12. Define linear and non-linear systems with examples.
13. How are the systems classified based on varying quantity and time?
14. Define time-variant and time-invariant systems with examples.
15. Define transfer function of a system.
16. What are the features and advantages of transfer function?
17. What are the disadvantages of transfer function?
18. Determine the transfer function of the open-loop system.
19. State whether transfer function is applicable to non-linear system.
20. What are the types of closed-loop system?
21. Why is negative feedback preferred in control system?

22. Discuss the concept of negative feedback closed-loop system and derive its transfer function.
23. Discuss the concept of positive feedback closed-loop system and derive its transfer function.
24. What are the features or advantages of negative feedback control system?
25. What is an error detector?
26. What do you mean by modeling of systems?
27. Explain with a neat flowchart, how to determine the transfer function of any physical systems.
28. What are the basic elements used for modeling electrical system?
29. What are the basic laws that are used for modeling electrical systems?
30. Explain the relationship between current and voltage across different elements present in the electrical systems.
31. How are the mechanical systems classified based on the movement of elements?
32. What are the basic laws used for modeling mechanical systems?
33. What do you mean by free body diagram of a mechanical system?
34. What are the basic elements used for modeling translational mechanical system?
35. Explain the mass element in translational mechanical system.
36. Explain the damper element in translational mechanical system.
37. What are the different types of friction in translational mechanical system? Discuss.
38. Explain the spring element in the translational mechanical system.
39. Write a short note on simple translational mechanical system.
40. What are the basic elements used for modeling rotational mechanical system?
41. Explain the inertia element in rotational mechanical system.
42. Explain the damper element in rotational mechanical system.
43. Explain the spring element in rotational mechanical system.
44. Write a short note on simple rotational mechanical system.
45. What do you mean by analogous system? Give examples.
46. Explain the analogous of rotational and translational mechanical system.
47. Explain the analogous existing within the electrical system.
48. What are the advantages of electrical analogous system?
49. What are the different types of analogies existing between the electrical and mechanical system?
50. Explain the force–voltage analogy.
51. Explain the force–current analogy.
52. Tabulate the analogous parameters of the F–V analogy.
53. Tabulate the analogous parameters of the F–I analogy.
54. Explain with a neat flowchart, the procedure for determining the F–V and F–I analogous circuits.
55. Explain the torque–voltage analogy.
56. Explain the torque–current analogy.
57. Tabulate the analogous parameters of the T–V analogy.
58. Tabulate the analogous parameters of the T–I analogy.
59. Explain with a neat flowchart, the procedure for determining the T–V and T–I analogous circuits.

60. For the mechanical rotational system shown in Fig. Q1.60, obtain (i) differential equations, (ii) ***T–V*** analogous circuit and (iii) ***T–I*** analogous circuit.

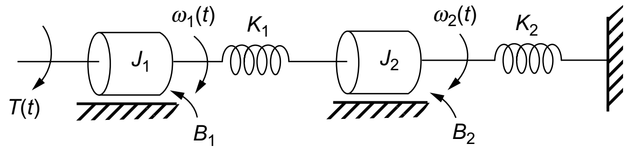


Fig. Q1.60

61. For the mechanical rotational system shown in Fig. Q1.61, obtain (i) differential equations, (ii) T–V analogous circuit and (iii) T–I analogous circuit.

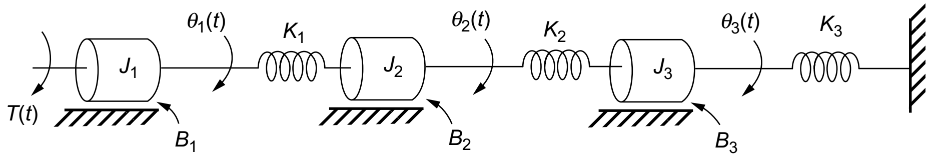


Fig. Q1.61

62. For the mechanical system shown in Fig. Q1.62, obtain (i) differential equations, (ii) transfer function of the system, (iii) force–voltage electrical analogous circuit and (iv) force–current electrical analogous circuit.

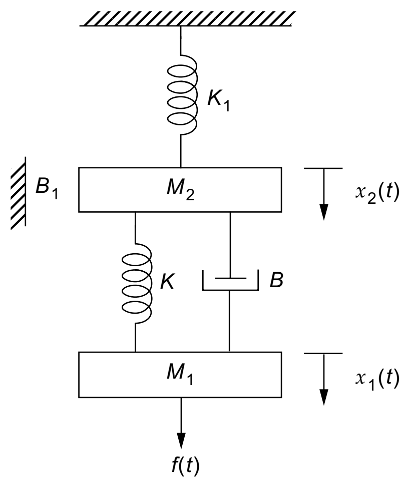


Fig. Q1.62

63. For the mechanical system shown in Fig. Q1.63, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

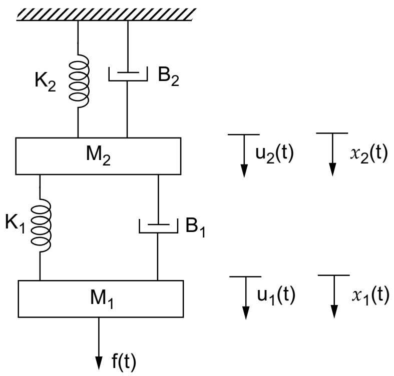


Fig. Q1.63

64. For the mechanical system shown in Fig. Q1.64, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force-current electrical analogous circuit.

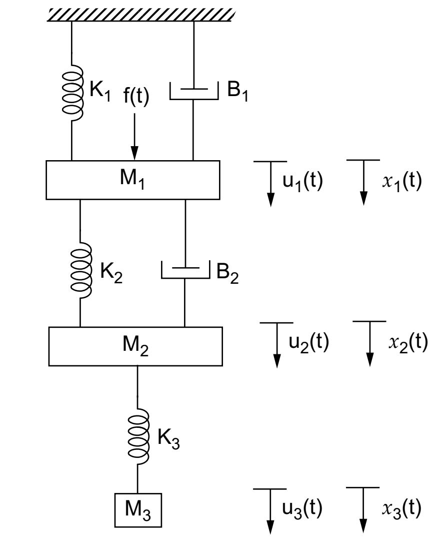


Fig. Q1.64

65. For the mechanical system shown in Fig. Q1.65, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

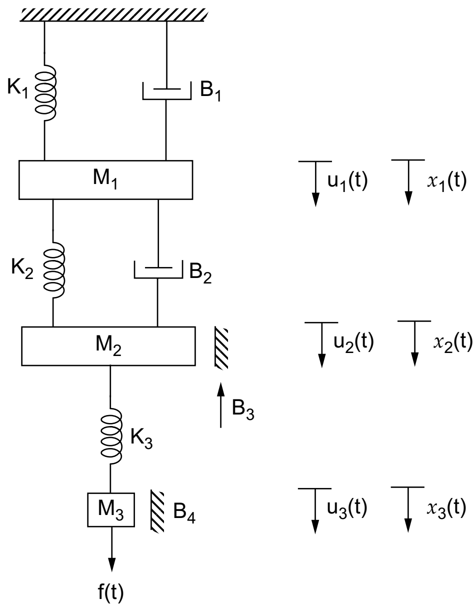


Fig. Q1.65

66. For the mechanical system shown in Fig. Q1.66, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

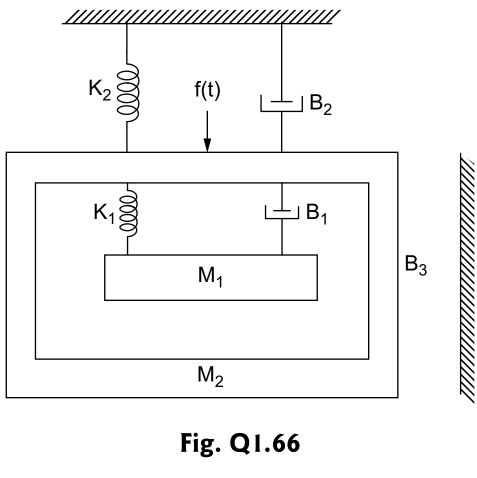


Fig. Q1.66

67. For the mechanical system shown in Fig. Q1.67, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

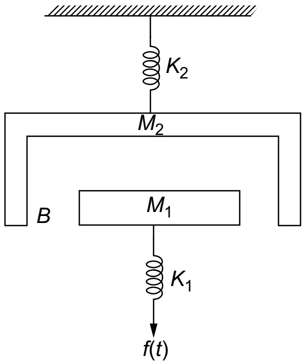


Fig. Q1.67

68. For the mechanical system shown in Fig. Q1.68, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

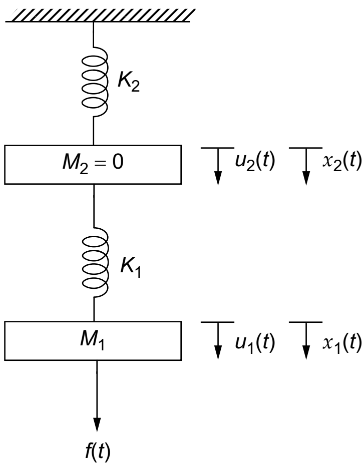


Fig. Q1.68

69. For the mechanical system shown in Fig. Q1.69, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

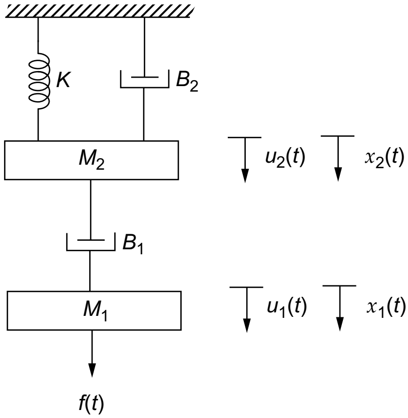


Fig. Q1.69

70. For the mechanical system shown in Fig. Q1.70, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

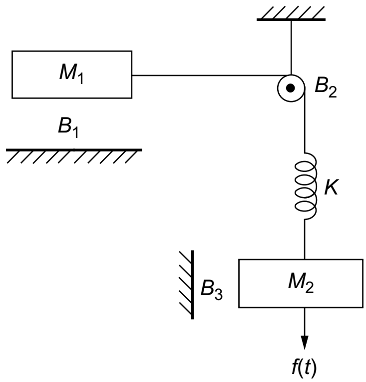


Fig. Q1.70

71. For the mechanical system shown in Fig. Q1.71, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

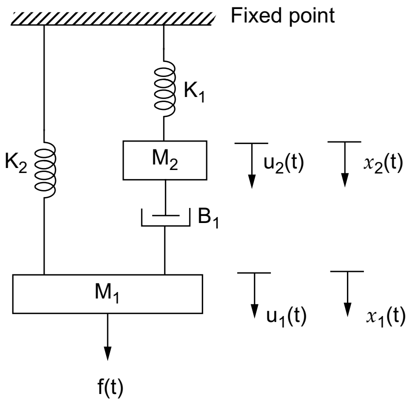


Fig. Q1.71

72. For the mechanical system shown in Fig. Q1.72, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

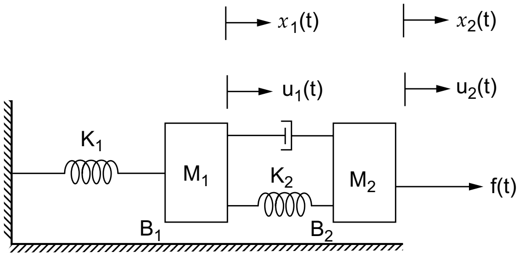


Fig. Q1.72

73. For the mechanical system shown in Fig. Q1.73, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

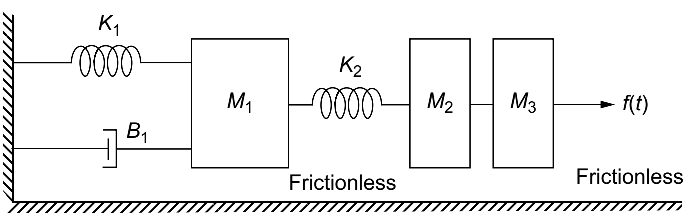


Fig. Q1.73

74. For the mechanical system shown in Fig. Q1.74, obtain (i) differential equations and (ii) F–V electrical analogous circuit.

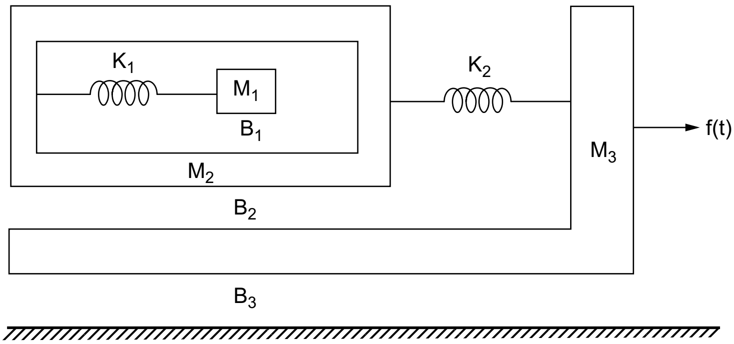


Fig. Q1.74

75. For the mechanical system shown in Fig. Q1.75, obtain (i) differential equations, (ii) F–V electrical analogous circuit and (iii) F–I electrical analogous circuit.

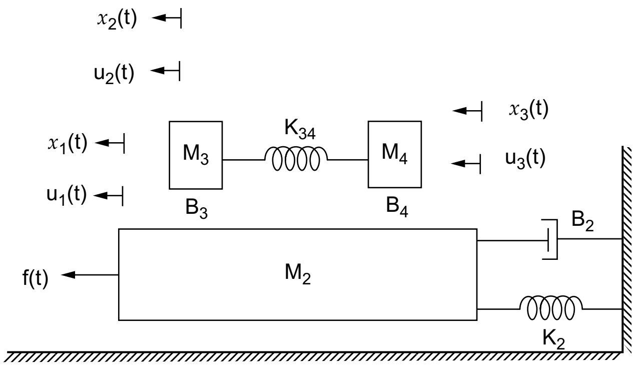


Fig. Q1.75

76. For the mechanical rotational system shown in Fig. Q1.76, obtain (i) differential equations and (ii) transfer function of the system.

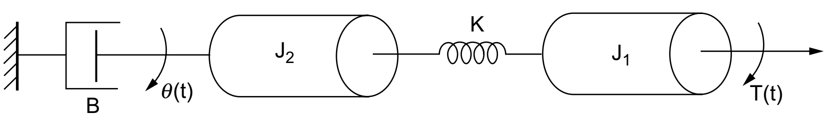


Fig. Q1.76

77. For the mechanical rotational system shown in Fig. Q1.77, obtain (i) differential equations and (ii) transfer function $\theta(s)/T(s)$.

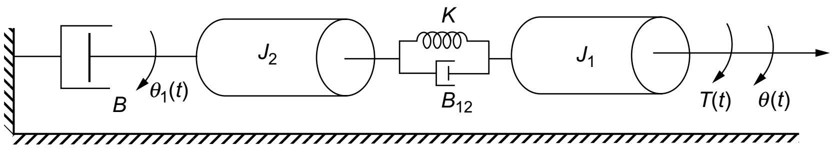


Fig. Q1.77

78. For the mechanical translational system shown in Fig. Q1.78, obtain (i) differential equations and (ii) transfer function of the system.

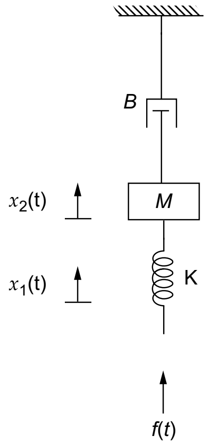


Fig. Q1.78

79. For the mechanical translation system shown in Fig. Q1.79, obtain (i) differential equations and (ii) transfer function of the system.

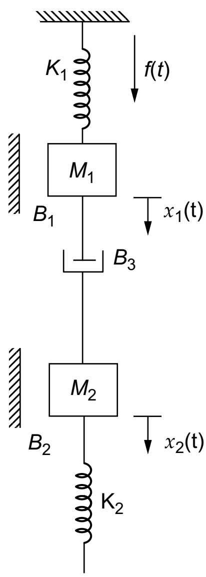


Fig. Q1.79

80. For the mechanical translational system shown in Fig. Q1.80, obtain (i) differential equations and (ii) transfer function $X_2(s)/F(s)$of the system.

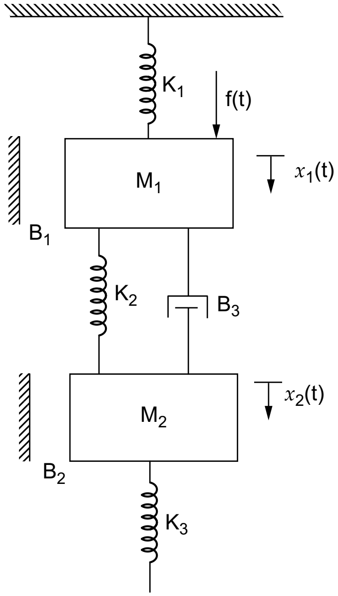
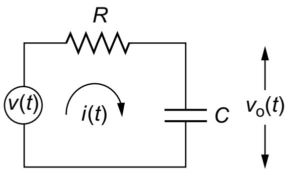


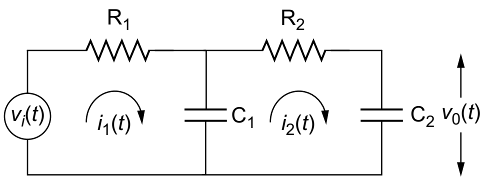
Fig. Q1.80

81. Determine the transfer functions of the electrical networks shown in Fig. Q1.81.

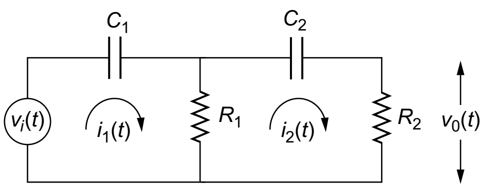
(i)



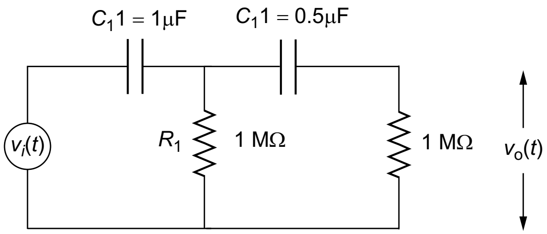
(ii)



(iii)



(iv)



(v)

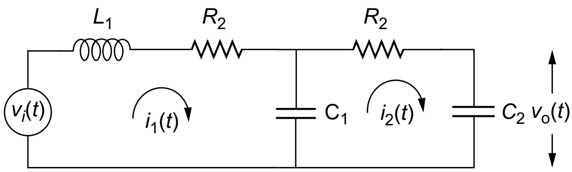


Fig. Q1.81

82. Write the integro-differential equations for the electrical network given in Fig. Q1.82.

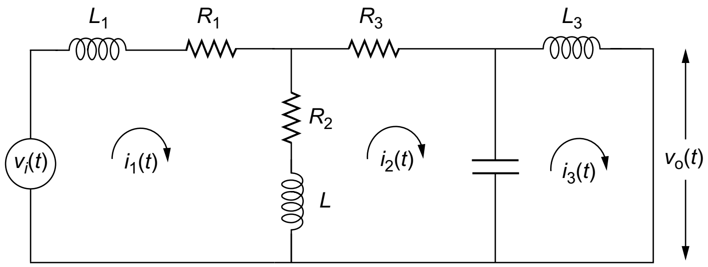


Fig. Q1.82

83. Determine the transfer function for a positive feedback system with a forward gain of 3 and a feedback gain of 0.2.
84. For the mechanical translational system shown in Fig. Q1.84, obtain (i) differential equations and (ii) transfer function of the system.

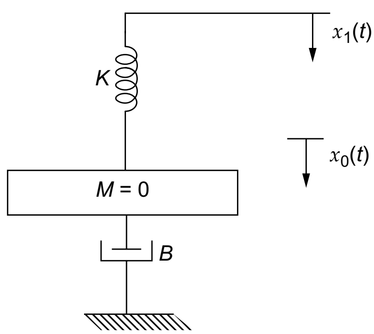


Fig. Q1.84

85. For the mechanical rotational system shown in Fig. Q1.85, obtain transfer function of the system.

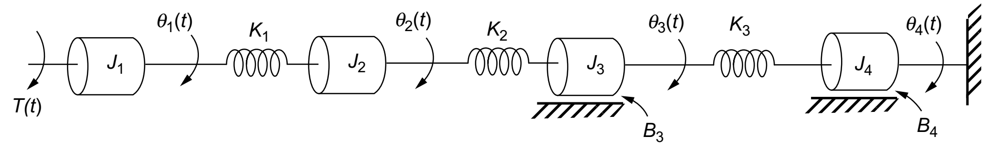


Fig. Q1.85

86. A negative feedback system is subjected to an input of 5 V. Determine the output voltage for the unity feedback system for the given values of forward gain: (i) $G(s) = 1$, (ii) $G(s) = 5$, (iii) $G(s) = 50$ and (iv) $G(s) = 500$

87. By simplifying the block diagram given in Fig. Q1.87, determine the value of the output $C(s)$.

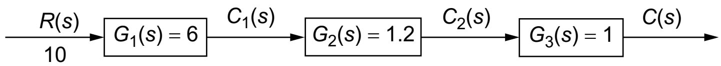


Fig. Q1.87

88. For the mechanical translational system shown in Fig. Q1.88, obtain (i) differential equations and (ii) transfer function of the system.

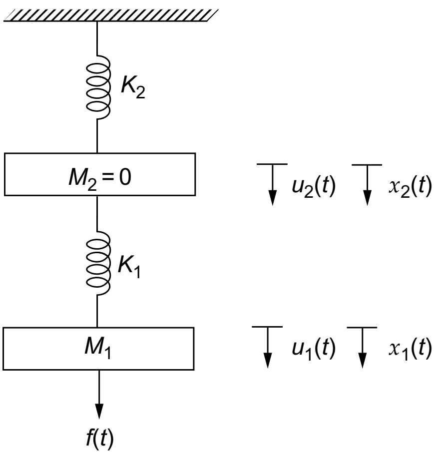


Fig. Q1.88

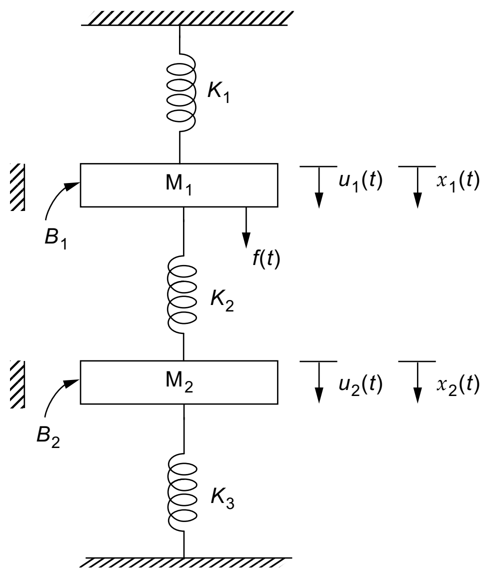


Fig. Q1.89

89. For the mechanical system shown in Fig. Q1.89, obtain (i) differential equations, (ii) force–voltage electrical analogous circuit and (iii) force–current electrical analogous circuit.

2

PHYSICAL SYSTEMS AND COMPONENTS

2.1 Introduction

Modeling is the process of obtaining a desired mathematical description of a physical system. The basic differential equations that may be linear or non-linear for any physical system are obtained by using appropriate laws. The differential equations thus obtained can make the analysis of the system easier. Laplace transformation tool is applied to these differential equations to get the transfer function of the physical system. In this chapter, the differential equations and the transfer functions for physical systems such as electromechanical, hydraulic, pneumatic, liquid level and thermal systems are obtained. The control system components such as controllers, potentiometers, synchros, servomotors, tachogenerators, stepper motor and gear trains are also discussed.

2.2 Electromechanical System

Systems that carry out electrical operations with the help of mechanical elements are known as electromechanical systems. Devices based on electromechanical systems are developed for making the energy conversion easier between electrical and mechanical systems. The subsystems present in the electromechanical systems are shown in Fig. 2.1.

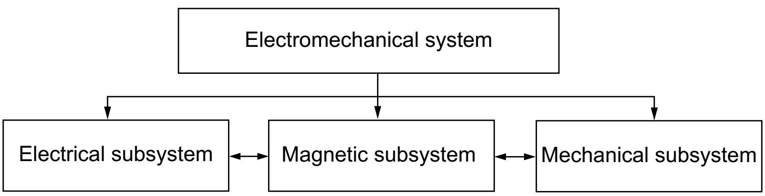


Fig. 2.1 | Subsystems in electromechanical systems

For an electromechanical system, voltages and currents are used to describe the state of its electrical subsystem based on Kirchhoff's laws. Position, velocity and accelerations are used to describe the state of its mechanical subsystems based on Newton's law. The magnetic subsystem or magnetic field acts as a boat in transferring and converting energy between electrical and mechanical subsystems. Magnetic flux, flux density and field strength governed by Maxwell's equations are used to describe the conversion and transformation processes.

Magnetic flux interacting with the electrical subsystem will produce a force or a torque on the mechanical subsystem. In addition, the movement in the mechanical subsystem will produce an induced electromotive force (emf) in the electrical subsystem. Electromotive force is developed due to the variation of the magnetic flux linked with the electrical subsystem. Therefore, the electrical energy and the mechanical energy conversions occur with the help of the magnetic subsystem.

Example 2.1: An electro-mechanical system consisting of a solenoid that drives a lever connected to a mechanical system is shown in Fig. E2.1. Determine the transfer function $\frac{X(s)}{V(s)}$.

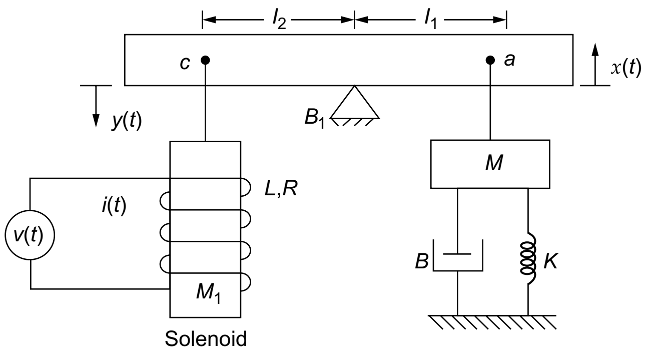


Fig. E2.1

Solution: The exciting current of solenoid is given by

$$I(s) = \frac{V(s)}{(R+Ls)}$$

This produces a force, $F(s) = K_s I(s)$ in the solenoid where K_s = soleniod constant in N/A.

This force,

$$F(s) = \frac{K_s V(s)}{(R+Ls)} = M_1 s^2 Y(s) + F_c \tag{1}$$

where F_c is the reaction force at point *c*.
The force acting at point *a* is

$$F_a = (Ms^2 + Bs + K)\ X(s)$$

Equating momentum at the centre,

$$F_c l_2 = F_a l_1$$

This gives

$$F_c = F_a \frac{l_1}{l_2} = \frac{l_1}{l_2}\left(Ms^2 + Bs + K\right) X(s) \tag{2}$$

The fulcrum will give displacement as

$$y(t) = \frac{l_2}{l_1} x(t)$$

Taking Laplace transform, we obtain

$$Y(s) = \frac{l_2}{l_1} X(s) \tag{3}$$

Substituting Eqn. (3) and Eqn. (2) in Eqn. (1), we obtain

$$\frac{K_s V(s)}{(R + Ls)} = \left[M_1 s^2 \frac{l_2}{l_1} + \frac{l_1}{l_2}\left(Ms^2 + Bs + K\right) \right] X(s)$$

Hence, the transfer function is given by

$$\frac{X(s)}{V(s)} = \frac{K_s}{(R + Ls)\left[M_1 s^2 \frac{l_2}{l_1} + \frac{l_1}{l_2}\left(Ms^2 + Bs + K\right) \right]}$$

Example 2.2: For the electromechanical system shown in Fig. E2.2, obtain the transfer function $X(s)/V(s)$. Assume that coil back emf as $e_b(t) = K_1 \frac{dx(t)}{dt}$ and the force produced by the coil current i_2 on the mass M as $F_c = K_2 i_2(t)$.

Solution:

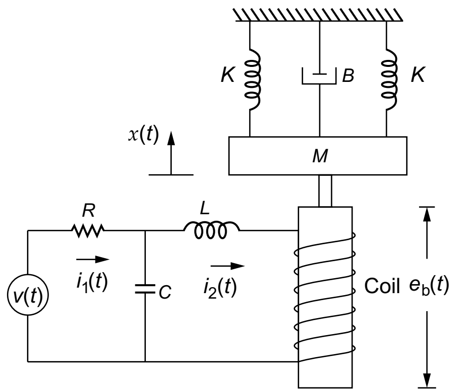


Fig. E2.2

Applying Kirchhoff's voltage law, loop equations are obtained as follows:

$$Ri_1(t) + \frac{1}{C}\int \left[i_1(t) - i_2(t)\right] dt = v(t) \tag{1}$$

$$L\frac{di_2(t)}{dt} + \frac{1}{C}\int\left[i_2(t) - i_1(t)\right] dt = -e_b(t) = -K_1\frac{dx(t)}{dt} \tag{2}$$

where K_1 is a constant.

Taking Laplace transform, we obtain

$$RI_1(s) + \frac{1}{Cs}\left[I_1(s) - I_2(s)\right] = V(s) \tag{3}$$

$$sLI_2(s) + \frac{1}{Cs}\left[I_2(s) - I_1(s)\right] = -E_b(s) = -sK_1X(s) \tag{4}$$

Substituting Eqn. (4) in Eqn. (3), we have

$$RI_1(s) + sLI_2(s) + sK_1X(s) = V(s)$$

From Eqn. (4), we have

$$\frac{1}{Cs}I_1(s) = sLI_2(s) + \frac{1}{Cs}I_2(s) + sK_1\,X(s)$$

$$= \left(sL + \frac{1}{Cs}\right)I_2(s) + sK_1\,X(s)$$

$$I_1(s) = \left(s^2LC + 1\right)I_2(s) + s^2K_1C\,X(s) \tag{5}$$

Substituting Eqn. (5) in Eqn. (3), we obtain

$$\left[R + \frac{1}{Cs}\right]\left[\left(s^2LC + 1\right)I_2(s) + s^2K_1C\,X(s)\right] - \frac{1}{Cs}I_2(s) = V(s)$$

$$\left[s^2RLC + sL + R\right]I_2(s) + \left(s^2K_1RC + sK_1\right)X(s) = V(s) \tag{6}$$

Also, we have

$$F_c(t) = K_2i_2(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + 2K\,x(t)$$

Taking Laplace transform, we obtain

$$K_2I_2(s) = \left[Ms^2 + Bs + 2K\right]X(s)$$

$$I_2(s) = \left[\frac{Ms^2 + Bs + 2K}{K_2}\right] X(s) \tag{7}$$

Substituting Eqn. (7) in Eqn. (6), we obtain

$$\left[\left(s^2 RLC + sL + R\right)\left(\frac{Ms^2 + Bs + 2K}{K_2}\right) + \left(s^2 K_1 RC + sK_1\right)\right] X(s) = V(s)$$

Thus, the transfer function of the given system is

$$\frac{X(s)}{V(s)} = \frac{K_2}{\left[RLCMs^4 + L\left(M + RCB\right)s^3 + \left[RM + LB + RC(2LK + K_1K_2)\right]s^2 + \left(RB + 2LK + K_1K_2\right)s + 2RK\right]}$$

2.3 Hydraulic System

Hydraulics is the study of incompressible liquids and hydraulic devices are the devices that use oil as their working medium. Since the density of the oil does not vary with pressure, it is used as a working medium in the hydraulic systems. Liquid-level systems are a class of hydraulic system. The factors such as positiveness, accuracy, flexibility, high horse power-to-weight ratio, fast starting and stopping, precision, simplicity of operation and smooth reversal of operation make the hydraulic system find wide applications. Hydraulic system that is non-linear in nature is stable and satisfactory under all the operating conditions.

The operating pressure of hydraulic systems is between 1 and 35 MPa. It varies based on the application. The weight and size of the hydraulic system vary inversely with respect to the supply pressure for the same power requirement. A very high driving force for an application can be obtained by using high-pressure hydraulic systems. Due to the advantages of electronic control and hydraulic power, the electronic system in combination with the hydraulic system is widely used.

2.3.1 Advantages of Hydraulic System

The advantages of hydraulic system are:

(i) The fluid (oil) used in the hydraulic system acts as a lubricant.
(ii) Fluid in the system is used to carry the heat generated conveniently to the heat exchanger.
(iii) Hydraulic devices have higher speed of response.
(iv) Hydraulic actuators can be operated at different conditions without damage (continuous, intermittent, reversing and stalled condition).
(v) Flexibility exists in the design, due to the availability of linear and rotary actuators.
(vi) When loads are applied, the speed drop is small due to the presence of low leakage in the actuators.

2.3.2 Disadvantages of Hydraulic System

The disadvantages of hydraulic system are:

(i) Cost of a hydraulic system is higher when compared to an electrical system.
(ii) Occurrence of fire and explosion due to the usage of non-fire-resistant liquids.
(iii) Difficult to maintain the system due to the presence of leakage.
(iv) Failure occurs in the system due to the presence of contaminated oil.
(v) Availability of hydraulic power is less when compared to electrical power.
(vi) Difficult to design the sophisticated hydraulic system due to the involvement of non-linear and complex characteristics.
(vii) Has poor damping characteristics.

2.3.3 Applications of Hydraulic System

Hydraulic system is widely used in the control applications such as power steering and brakes in automobiles, steering mechanisms of large ships, control of large machine tools, etc.

2.3.4 Devices Used in Hydraulic System

Devices such as dashpots, pumps and motors, valves, linear actuator, servo systems, feedback systems and controllers are used in hydraulic systems.

Example 2.3: The hydraulic system shown in Fig. E2.3 is the dashpot (also called as damper) acting as a differentiating element. Obtain the transfer function of the system relating the displacements $x_1(t)$ and $x_2(t)$.

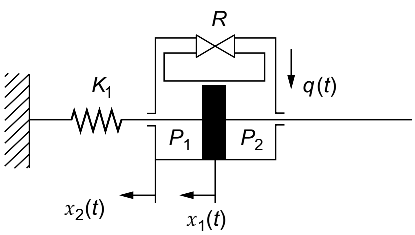


Fig. E2.3

Solution: Let the pressure existing on the left and right sides of the piston be P_1 and P_2 respectively. If the inertia force is taken as a negligible quantity, then the force existing on the piston should balance the spring force. Then,

$$A(P_1 - P_2) = kx_2(t)$$

where A is the piston area in m^2 and k is the spring constant in kg/m.
The flow rate of oil through the resistance R is given by

$$q(t) = \frac{(P_1 - P_2)}{R}$$

where $q(t)$ is the flow rate of oil though resistance in kg/sec and R is the resistance to the flow of oil in $\frac{N/m^2}{m^3/\text{min}}$.

The flow of oil through the resistance during dt seconds must be equal to the change in mass of oil to the right of the piston during the same dt seconds. Therefore,

$$q(t)dt = A\rho\left(dx_1(t) - dx_2(t)\right)$$

where ρ is the density of oil.
Therefore,

$$\frac{q(t)}{A\rho} = \left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right)$$

Substituting the value of $q(t)$ in the above equation and solving, we get

$$\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right) = \frac{P_1 - P_2}{RA\rho} = \frac{kx_2(t)}{RA^2\rho}$$

Taking Laplace transform, we obtain

$$sX_1(s) = sX_2(s) + \frac{kX_2(s)}{RA^2\rho}$$

Thus, the transfer function of the system is given by

$$\frac{X_2(s)}{X_1(s)} = \frac{sT}{sT+1}$$

where $T = \frac{RA^2\rho}{k}$ with the dimension of time.

2.4 Pneumatic Systems

In pneumatic systems, the compressible fluid that is used as a working medium is air or gas. The pneumatic system is most frequently used in industrial control system due to the fact that it provides a greater force than the force obtained from electrical devices. In addition, it provides more safety against fire accidents. The dynamic equations for modeling of pneumatic systems are obtained using conservation of mass. The devices in the pneumatic systems involve with the flow of air or gas through connected pipe lines and pressure vessels. The variables involved in the pneumatic systems are mass flow rate q_{m} and pressure P. The electrical analogous variables to the mass flow rate and pressure are current and voltage respectively.

2.4.1 Gas Flow Resistance and Pneumatic Capacitance

The basic elements of the pneumatic systems are resistance and capacitance. The resistance to gas flow is due to the restrictions present in the pipes and valves. Hence, the gas flow resistance R is defined as the ratio of rate of change in gas pressure to the change in the gas flow rate.

$$R = \frac{\text{Change in gas pressure in N/m}^2}{\text{Change in gas flowrate in kg/sec}}$$

The pneumatic capacitance C of a pressure vessel depends on the type of expansion involved. The pneumatic capacitance of a pressure vessel is defined as the ratio of change in gas stored to the change in gas pressure.

$$C = \frac{\text{Change in gas stored in kg}}{\text{Change in gas pressure in N/m}^2}$$

2.4.2 Advantages of Pneumatic System

The advantages of pneumatic system are:

(i) The working medium used in the pneumatic systems is non-inflammable and offers safety from fire hazards.
(ii) Viscosity of gas or air is negligible.
(iii) Return pipelines are not necessary since the air or gas can be exhausted once the working cycle of device comes to an end.

2.4.3 Disadvantages of Pneumatic System

The disadvantages of pneumatic system are:

(i) Response of the pneumatic system is slower since the working medium is compressible.
(ii) Power output of the pneumatic system is less.
(iii) Accuracy of the actuator in the system is less.

2.4.4 Applications of Pneumatic System

Pneumatic system is used in guided missiles, air craft systems, automation of production machinery and automatic controllers in many fields.

2.4.5 Devices Used in Pneumatic System

Devices such as bellows, flapper valve, relay, actuator, position control system and controllers are used in pneumatic systems.

2.4.6 Comparison between Hydraulic and Pneumatic Systems

The comparison of hydraulic and pneumatic systems based on the common characteristics between the systems is given in Table 2.1.

Table 2.1 | Pneumatic system vs hydraulic system

Characteristics	Pneumatic system	Hydraulic system
Compressibility	Working fluid (air or gas) can be compressed	Working fluid (oil) cannot be compressed
Lubricating property	Air or gas lacks this property	Oil is a good lubricant
Operating pressure	Less	More
Accuracy of actuators	Poor at low velocities	Good at all velocities
Leakage in the system	External leakage is permissible, but internal leakage should be avoided	Internal leakage is permissible, but external leakage should be avoided
Requirement of return lines	Not necessary	Necessary
Operating temperature	5°C–60°C	20°C–70°C
Sensitivity to temperature	Insensitive	Sensitive

Example 2.4: **Consider a pneumatic system as shown in Fig. E2.4. Determine the transfer function relating the change in input pressure $P_i(t)$ and change in output pressure $P_o(t)$ by considering small changes in the pressures from the steady-state value.**

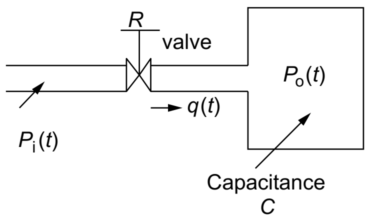


Fig. E2.4

Solution: Consider that $\overline{P}_i$ is the steady-state value of pressure in the system, $P_i(t)$ is the small deviation in the input pressure and $P_o(t)$ is the small deviation in the output pressure.

For small deviation in the input and output pressures, the gas flow resistance R can be obtained as

$$R = \frac{P_i(t) - P_o(t)}{q(t)}$$

and the pneumatic capacitance C can be obtained as

$$C = \frac{dm}{dP}$$

where m is the mass of gas in the device and q is the gas flow rate.

Since the product of pressure change dp_o and the capacitance C is equal to the gas added to the vessel during dt seconds, we obtain

$$CdP_o(t) = q(t)dt$$

$$C\frac{dP_o(t)}{dt} = \frac{P_i(t) - P_o(t)}{R}$$

Therefore, $$RC\frac{dP_o(t)}{dt} + P_o(t) = P_i(t)$$

Taking Laplace transform, we obtain

$$(sRC + 1)P_o(s) = P_i(s)$$

Hence, the transfer function of the system relating the change in input pressure and output pressure is given by

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{sRC + 1}$$

Example 2.5: The pneumatic system shown in Fig. E2.5 is a pneumatic bellow. It consists of a hollow cylinder with thin walls. Determine the transfer function of the system relating the linear displacement $x(t)$ and the input pressure $P_i(t)$.

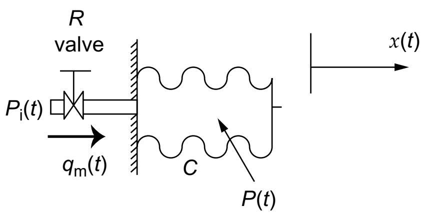


Fig. E2.5

Solution: Let $\overline{P}_i$ be the steady-state value of input pressure,

$P_i(t)$ be the small increment in the input pressure,

$q_m(t)$ be the small increment in the air flow,

$P(t)$ be the small increment in the pressure existing inside the bellow and

$x(t)$ be the displacement of the movable surface due to increase in pressure.

The force on the movable surface is proportional to the pressure inside the bellow and is given by

$$f_{\mathrm{b}} \ \alpha \ P(t)$$

$$f_{\mathrm{b}} = AP(t)$$

where A is the area of the bellow.

Also, the force opposing the movement of the flat surface of bellow walls is proportional to the displacement and is given by

$$f_{\mathrm{o}} \ \alpha \ x(t)$$

$$f_{\mathrm{o}} = Kx(t)$$

where K is the stiffness of the bellow.

At steady state, the above-said forces are in a balanced condition

$$\text{i.e.,} \ f_{\mathrm{b}} = f_{\mathrm{o}}$$

Therefore,
$$AP(t) = Kx(t) \tag{1}$$

The gas flow resistance R is given by

$$R = \frac{P_{\mathrm{i}}(t) - P(t)}{q_{\mathrm{m}}(t)} \tag{2}$$

The pneumatic capacitance C is given by

$$C = \frac{q_{\mathrm{m}}(t)}{\dfrac{dP(t)}{dt}}$$

Therefore,
$$q_{\mathrm{m}}(t) = C\frac{dP(t)}{dt} \tag{3}$$

Using Eqs. (2) and (3), we obtain

$$C\frac{dP(t)}{dt} = \frac{P_{\mathrm{i}}(t) - P(t)}{R}$$

Therefore,
$$RC\frac{dP(t)}{dt} + P(t) = P_{\mathrm{i}}(t) \tag{4}$$

Using Eqn. (1) and Eqn. (4), we obtain

$$RC\left(\frac{K}{A}\frac{dx(t)}{dt}\right)+\frac{K}{A}x(t)=P_{\mathrm{i}}(t)$$

Taking Laplace transform, we obtain

$$sRC\frac{K}{A}X(s)+\frac{K}{A}X(s)=P_{\mathrm{i}}(s)$$

Hence, the transfer function of the system is given by

$$\frac{X(s)}{P_{\mathrm{i}}(s)}=\frac{A/K}{\tau s+1}$$

where $\tau = RC$ is the time constant of the system.

2.5 Thermal Systems

Systems involving the transfer of heat from one substance to another are known as thermal systems. Thermal systems can be modeled by knowing the basic elements involved in the systems. The basic elements in a thermal system are thermal resistance and thermal capacitance that are distributed in nature. But, for the analysis of thermal systems, the nature of the basic elements is assumed to be lumped. The assumption made in the lumped parameter model is that, the substances that are characterized by resistance to heat flow have negligible capacitance and vice versa.

The different ways by which the heat flows from one substance to another substance are conduction, convection and radiation. The heat flow rate for different ways is:

For conduction or convection,

Heat flow rate $q(t)=K\Delta\theta(t)$

where $q(t)$ is the heat flow rate in kcal/sec, K is the coefficient in kcal/sec °C and $\Delta\theta(t)$ is the temperature difference in degree Celsius.

The coefficient K is given by

$$K=\frac{kA}{\Delta X}\text{(for conduction)}$$

$$K=HA\ \text{(for convection)}$$

where k is the thermal conductivity in kcal/m sec °C, A is the area normal to heat flow in m^2, ΔX is the thickness of conductor in m and H is the convection coefficient in kcal/m² sec °C.

For radiation,

Heat flow rate, $q(t)=K_{\mathrm{r}}\left(\theta_1^4(t)-\theta_2^4(t)\right)$

If $\theta_1(t)>>\theta_2(t)$, then $q(t)=K_{\mathrm{r}}\left(\overline{\theta}_4(t)\right)$

where K_r is the radiation coefficient in kcal/sec °C.

Effective temperature difference $\bar{\theta}_4(t) = \left(\theta_1^4(t) - \theta_2^4(t)\right)^{\frac{1}{4}}$

In this section, we consider that the heat flow is through conduction and convection.

2.5.1 Thermal Resistance and Thermal Capacitance

Thermal resistance R for heat transfer between two substances is given by the ratio of change in temperature to the change in heat flow rate.

$$R = \frac{\text{Change in temperature difference in } ^\circ\text{C}}{\text{Change in heat flow rate in kcal/sec}}$$

For conduction and convection,

$$R = \frac{d\left(\Delta\theta(t)\right)}{dq(t)} = \frac{1}{K}$$

Since thermal coefficient K for conduction and convection is almost constant, thermal resistance is also constant for conduction and convection.

Thermal capacitance C is given by the ratio of change in heat stored to change in temperature.

$$C = \frac{\text{Change in heat stored in kcal}}{\text{Change in temperature in degree Celcius}}$$

or $$C = mc$$

where m is the mass of substance considered in kilogram and c is the specific heat of substance in kcal/kg °C.

Example 2.6: The thermal system shown in Fig. E2.6 has an insulated tank to eliminate the heat loss to the surroundings alongwith a heater and a mixer. Determine the transfer function of the system relating the temperature to the input heat flow rate.

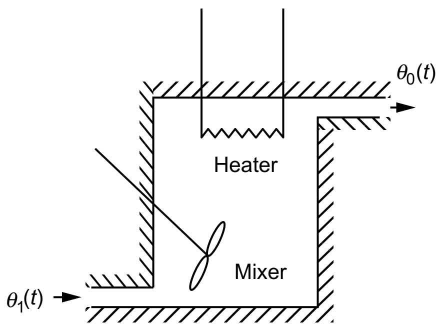


Fig. E2.6

Solution: Let $\bar{\theta}_i$ be the steady-state temperature of input liquid in °C,

$\bar{\theta}_o$ be steady-state temperature of output liquid in °C,

$G(t)$ be liquid flow rate in steady state in kg/sec,

M be the mass of liquid in kg,

c be the specific heat of liquid in kcal/kg °C,

R be the thermal resistance in °C – sec/kcal and

C be the thermal capacitance in kcal/°C.

$Q(t)$ be the input heat rate in steady state in kcal/sec.

If the change in input heat rate is $q_i(t)$, there will be change in the output heat rate and is given by $q_o(t)$. In addition, the temperature of the out-flowing liquid will also be changed and is given by $\theta(t)$.

The equation for the system shown in Fig. E2.6 is obtained as

Change in output heat flow rate $q_0(t)$ = Rate of flow of liquid $G(t) \times$ specific heat of liquid c $\times$ change in temperature $\theta(t)$

i.e., $$q_0(t) = G(t)c\theta(t) \tag{1}$$

Thermal capacitance C = mass $M \times$ specific heat of liquid c

i.e., $$C = Mc \tag{2}$$

Thermal resistance $R = \dfrac{\text{Change in temperature } \theta(t)}{\text{Change in heat flow rate } q_0(t)}$

i.e., $$R = \frac{\theta(t)}{q_0(t)} \tag{3}$$

Substituting Eqn. (1) in Eqn. (3), we obtain

$$R = \frac{1}{Gc} \tag{4}$$

For the given system, the rate of change of temperature is directly proportional to change in input heat rate.

Therefore,

$$\frac{d\theta(t)}{dt} \alpha \left(q_i(t) - q_0(t)\right)$$

$$C\frac{d\theta(t)}{dt} = q_i(t) - q_0(t) \tag{5}$$

Substituting Eqn. (3) in the above equation, we obtain

$$RC\frac{d\theta(t)}{dt} + \theta(t) = Rq_i(t)$$

Taking Laplace transform, we obtain

$$sRC\theta(s)+\theta(s)=RQ_{i}(s)$$

Hence, the transfer function of the given system is

$$\frac{\theta(s)}{Q_{i}(s)}=\frac{R}{sRC+1}$$

Example 2.7: A vessel shown in Fig. E2.7 contains liquid at a constant temperature $Q_{i}(t)$. A thermometer that has a thermal capacitance C to store heat and a thermal resistance R to limit heat flow is dipped in the vessel. Let the temperature indicated by the thermometer be $\theta_{o}(t)$. Determine (i) the transfer function of the system and (ii) discuss the variation in thermometer as a function of time.

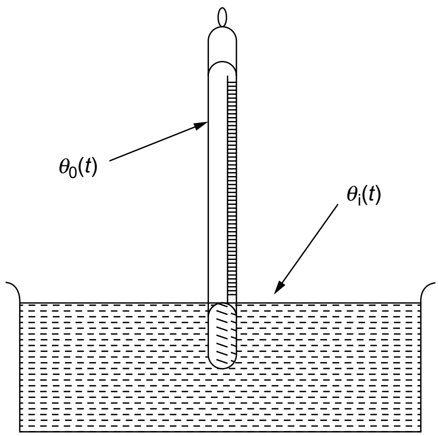


Fig. E2.7

Solution: Assume that, the small change in liquid temperature and change in temperature as indicated by thermometer are $\Delta\theta_{i}(t)$ and $\Delta\theta_{o}(t)$ respectively.

Hence, the heat stored in thermometer is given by

$$H_{s}=C\frac{d\left(\Delta\theta_{o}(t)\right)}{dt}$$

and heat transmitted is given by

$$H_{T}=\frac{\Delta\theta_{o}(t)-\Delta\theta_{i}(t)}{R}$$

Since the net heat equals to zero, the system equation can be written as

$$H_s + H_T = 0$$

$$C\frac{d\left(\Delta\theta_o(t)\right)}{dt} + \frac{\Delta\theta_o(t) - \Delta\theta_i(t)}{R} = 0$$

Taking Laplace transform, we obtain

$$sC\theta_o(s) + \frac{\theta_o(s) - \theta_i(s)}{R} = 0$$

Rearranging the above equation, we obtain the transfer function of the system as

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{1}{1 + sRC}$$

Taking inverse Laplace transform, we obtain

$$\theta_o(t) = \theta_i(t)\left\{1 - e^{-t/RC}\right\}$$

The above equation shows the variation in thermometer as a time function.

2.6 Liquid-Level System

Liquid-level system in any industrial process involves the flow of liquid through connecting pipes and storage tanks. The flow of liquid can be generally classified as laminar and turbulent flow depending on the magnitude of Reynolds number. If the Reynolds number is less than 2,000, then the flow is laminar; else if the Reynolds number is greater than 3,000 and less than 4,000, the flow is turbulent. But, in any industrial process involving the flow of liquid, the flow is often turbulent and not laminar.

In liquid-level control system, the height of the liquid in tanks and the flow rate of the liquid in pipes are the variables that ought to be controlled. Liquid-level system can be represented either by linear or non-linear differential equations depending on the type of flow of liquid. If the flow of liquid is laminar, the system will be represented by linear differential equations and if the flow is turbulent, the system is represented by non-linear differential equations.

2.6.1 Elements of Liquid-Level System

Resistance and capacitance of liquid-level systems describe the dynamic characteristic of the system. The resistance of the liquid is defined by assuming that the flow of liquid is through a short pipe connecting two tanks. The resistance R is defined as the change in liquid-level between two tanks necessary to cause a unit change in flow rate.

$$R = \frac{\text{Change in liquid level in m}}{\text{Change in flow ratein in m}^3\text{/sec}}$$

The relationship between the flow rate and the level difference differs for the laminar and turbulent flow and hence the resistance value also differs between the two.

Resistance for laminar flow $R_l = \dfrac{H}{Q}$

Resistance for turbulent flow $R_t = \dfrac{2H}{Q}$

where H is the steady-state head in m and Q is the steady-state liquid flow rate in m^3/sec.

Capacitance C of a tank is defined to be the change in quantity of stored liquid that is necessary to cause a unit change in potential (head).

$$C = \frac{\text{Change in liquid stored in m}^3}{\text{Change in head in m}}$$

The analogous quantities for different physical system to electrical system are given in Table 2.2.

Table 2.2 | Analogous quantities of different physical system to electrical systems

Electrical systems	Thermal systems	Liquid-level systems	Hydraulic systems	Pneumatic systems
Charge in coulomb (C)	Heat flow in joules (J)	Liquid flow in cubic meter (m^3)	Oil flow in cubic meter (m^3)	Air flow in cubic meter (m^3)
Current in amperes (A)	Heat flow rate in J / min	Liquid flow rate in m^3/min	Oil flow rate in m^3/min	Air flow rate in m^3/min
Voltage in volts (V)	Temperature in degree Celsius (°C)	Head in meter (m)	Piston displacement in metre (m)	Pressure in N/m^2
Resistance R in ohms(Ω)	Resistance R in $\dfrac{°C}{J/min}$	Resistance R in $\dfrac{N/m^2}{m^3/min}$	Resistance R in $\dfrac{N/m^2}{m^3/min}$	Resistance R in $\dfrac{N/m^2}{m^3/min}$
Capacitance C in farad (F)	Capacitance C in J / °C	Capacitance C in m^2	Capacitance C in m^2	Capacitance C in $\dfrac{m^3}{N/m^2}$

Example 2.8: For the liquid-level system shown in Fig. E2.8, obtain (i) mathematical model of the system and (ii) transfer function of the system.

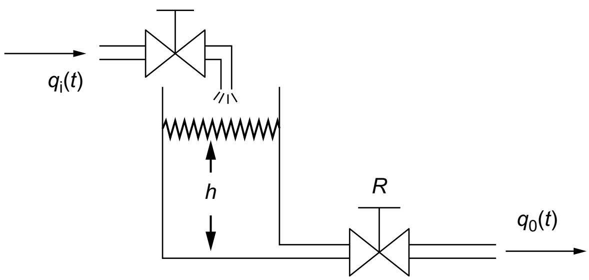


Fig. E2.8

Solution: The variables indicated in the system shown in Fig. E2.8 are defined by

$q_i(t) \rightarrow$ inflow rate of liquid in $m^3/\sec$

$q_o(t) \rightarrow$ outflow rate of liquid in $m^3/\sec$

$h(t) \rightarrow$ height of liquid in m

Under steady-state conditions, we obtain

$q_i(t) = q_o(t) = Q \rightarrow$ steady-state flow rate

$H_o \rightarrow$ steady-state liquid level in tank

If there is a small increase in the inflow rate, there will be two changes occurring in the liquid-level system as follows:

(i) Increase in the liquid level of the tank.
(ii) Increase in the outflow of the liquid.

From the definition of resistance,

$$q_o(t) = \frac{h(t)}{R} \tag{1}$$

Using liquid flow rate balance equation, we obtain

Liquid inflow – liquid outflow = rate of liquid storage in the tank

$$q_i(t) - q_o(t) = C\frac{dh(t)}{dt} \tag{2}$$

Substituting Eqn. (1) in Eqn. (2), we obtain

Therefore,
$$q_i(t) - \frac{h(t)}{R} = C\frac{dh(t)}{dt} \tag{3}$$

where C is the capacitance of tank and R is the resistance of the outlet pipe.

Therefore,
$$RC\frac{d(h(t))}{dt} + h(t) = Rq_i(t)$$

Taking Laplace transform, we obtain

$$(sRC + 1)H(s) = RQ_i(s)$$

Therefore,
$$\frac{H(s)}{Q_i(s)} = \frac{R}{sRC + 1}$$

If $q_i(t)$ is taken as the input and h the output, the transfer function of the system is given by

$$\frac{H(s)}{Q_i(s)} = \frac{R}{sRC + 1}$$

But, if $q_o(t)$ is taken as the output, the transfer function is obtained as follows:
Differentiating Eqn. (1), we obtain

$$\frac{d(h(t))}{dt} = R\frac{d(q_o(t))}{dt} \tag{4}$$

Substituting Eqn. (4) in Eqn. (2), we obtain

$$q_i(t) - q_o(t) = CR\frac{d(q_o(t))}{dt}$$

Taking Laplace transform, we obtain

$$(sRC+1)Q_o(s) = Q_i(s)$$

Thus, the transfer function of the system is given by

$$\frac{Q_o(s)}{Q_i(s)} = \frac{1}{(sRC+1)}$$

2.7 Introduction to Control System Components

The basic components present in a typical closed-loop control system are shown in Fig. 2.2.

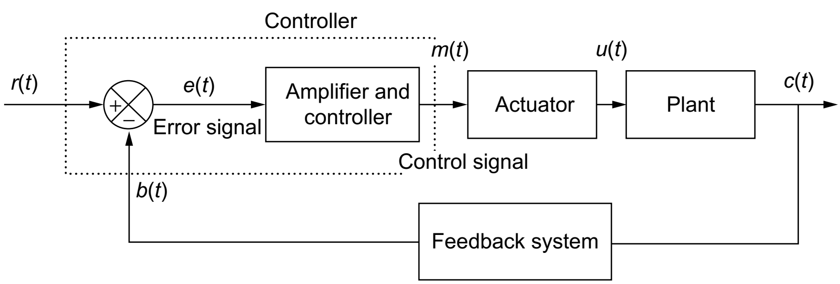


Fig. 2.2 | Block diagram of a typical control system

The basic components present in the control system are the error detector, amplifier, controller, actuator, plant and sensor or feedback system. Controller is the block that combines the error detector, amplifier and controller. The control system shown in Fig. 2.2 will become an open-loop system in the absence of automatic controller unit.

Error detector produces an error signal that is the difference between the reference input $r(t)$ and the feedback signal $b(t)$. The devices that can be used as error-detecting units are potentiometer, linear variable differential transformer (LVDT), synchros, etc.

Amplifier amplifies the error signal so that the controller can modify the error signal to have better control action over the plant or system. In general, in many control systems, controller itself amplifies the error signal.

Controller amplifies the signal produced by the amplifier unit to produce a control signal to stabilize the system. Depending on the nature of error signal/signal from amplifier, the controller used in the control system may be electrical, electronic, hydraulic or pneumatic controllers. The controller may be either proportional (P) controller, proportional integral (PI) controller, proportional derivative (PD) controller and proportional integral derivative (PID) controller. Electrical/electronic controllers are used when the error signal/amplifier output is electrical and the controllers are designed using RC circuit/operational amplifiers. Similarly hydraulic/pneumatic controllers are used when the error signal/amplifier output is mechanical and the controllers are designed using hydraulic servomotors/pneumatic flapper valves.

Actuator is an amplifier that amplifies the controller output and converts the signal into the form that is acceptable by the plant. The devices that can be used in an actuator are pneumatic motor/valve, hydraulic motor/electric motors (DC servomotor and stepper motor)

Feedback system generates a feedback signal proportional to the present output and converts it to an appropriate signal that can be used by the error detector. The devices that can be used for generating the appropriate signal from the present output are transducers, tachogenerators, etc.

2.8 Controllers

A controller is an important component in the control system that produces the control signal by modifying the error signal. In addition, the controller modifies the transient response of the system. The action taken up by the controller in the control system is known as control action.

2.8.1 Controller Output as a Percentage Value

In a closed-loop control system, as a particular controller model can be used for many processes, the measurement variable cannot be anticipated in advance to have controller action. For example, a particular model can be used in temperature -control application as well as in the motor control. Hence, the input and output values of a particular system can be expressed in percentage rather than in its actual unit to determine percentage of controller output to have the desired value as output from the system. The controller output in percentage can be expressed as

$$\text{controller output in \%} = \frac{\text{current output} - \text{minimum output}}{\text{maximum output} - \text{minimum output}} \times 100$$

where 0% is the minimum controller output and 100% is the maximum controller output.

2.8.2 Measured Value as a Percentage Value

In a closed-loop control system, as the unit of measured value and reference input are different, it is convenient to express the measured value and set point as a percentage value to make the error calculation easier. The measured value in percentage can be expressed as follows:

$$\text{measured value in \%} = \frac{\text{original measured value} - \text{minimum measured value}}{\text{maximum measured value} - \text{minimum measured value}} \times 100$$

2.8.3 Set Point as a Percentage Value

In a closed-loop control system, the set point can be expressed as a percentage value to make the error calculation easier. The set point in percentage can be expressed as

$$\text{Set point in \%} = \frac{\text{Set point value} - \text{minimum operating value}}{\text{operating range}} \times 100$$

2.8.4 Error as a Percentage Value

In a closed-loop system, the error in percentage can be expressed if the set point (SP) and measured value (MV) are expressed as percentage values. Therefore,

$$\text{Error in \%} = \%\ \text{SP} - \%\ \text{MV}$$

$$= \frac{\text{Set point value} - \text{measured value}}{\text{operating range}} \times 100$$

Example 2.9: The operating parameters of a speed control for a motor are as follows: maximum and minimum operating speeds are 150 and 1,600 rpm respectively. Set point speed is 900 rpm and actual motor speed is 875 rpm. Determine the set point, measured value and error as a percentage value.

Solution: The operating range of the speed control for a motor = (1,600–150) rpm = 1,450 rpm.

(i) $$\text{Set point in \%} = \frac{\text{Set point value} - \text{minimum operating value}}{\text{operating range}} \times 100$$

$$= \frac{(900-150)}{1450} \times 100 = 51.72\%.$$

(ii) $$\text{Measured value in \%} = \frac{\text{original measured value} - \text{minimum measured value}}{\text{maximum measured value} - \text{minimum measured value}} \times 100$$

$$= \frac{875-150}{1600-150} \times 100 = 50\%$$

Error for the given speed control of a motor = SP – MV = 900 – 875 = 25 rpm

(i) $$\text{Error in \%} = \%\ \text{SP} - \%\ \text{MV} = 51.72\% - 50\% = 1.72\%$$

$$\text{Also, error in \%} = \frac{\text{Set point value} - \text{measured value}}{\text{operating range}} \times 100$$

$$= \frac{900-875}{1450} \times 100 = 1.72\%$$

2.8.5 Types of Controllers

In general, controllers are classified as (i) analog controllers, (ii) digital controllers and (iii) fuzzy controllers.

(i) Analog controllers are further classified as follows:

(a) ON–OFF controller
(b) Proportional controller
(c) Integral controller
(d) Derivative controller
(e) Proportional Derivative controller
(f) Proportional Integral controller
(g) Proportional Integral Derivative controller

(ii) Digital Controller

Initially, digital controller was implemented to perform analog controllers operation by using computers. But nowadays, due to the advancement of digital controller over analog controllers, they have been widely used to control the system. The simple block diagram of a digital controller is shown in Fig. 2.3.

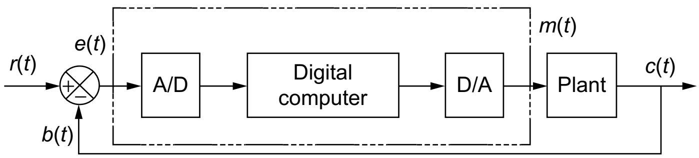


Fig. 2.3 | Block diagram of digital controller

Here, an analog error signal is converted into digital signal using analog-to-digital (A/D) conversion. The converted error signal is processed by a digital algorithm to give a manipulated variable in digital form. The manipulated variable in digital form will be converted to the required form using digital-to-analog (D/A) conversion.

Advantages of digital controller are:

(a) inexpensive
(b) easy to configure and reconfigure through software
(c) scalable
(d) adaptable
(e) less prone to environmental conditions

(iii) Fuzzy Controller

Fuzzy controller is the advanced form of digital controller where the control algorithms are implemented using fuzzy sets. The block diagram of a fuzzy controller is shown in Fig. 2.4.

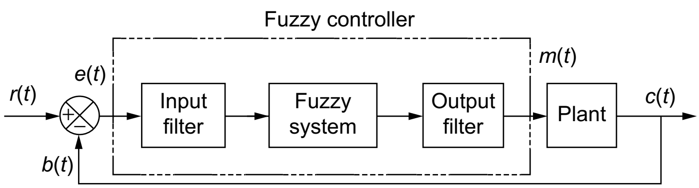


Fig. 2.4 | Block diagram of fuzzy controller

It consists of an input filter, fuzzy system and an output filter. The input filter maps the sensor or error signals to the appropriate signal to be used by the fuzzy system. The fuzzy system contains the control strategy to be implemented to the output of the input filter to produce the manipulated variable. Fuzzy system executes the appropriate rule and generates a result for each. Finally, the output filter converts the fuzzy system output to a specific control output value. Power sources used in the controllers are electricity, compressed air or oil. Hence, the controller used in the control system may be classified as electrical, electronic, hydraulic or pneumatic controllers.

The controller for a particular system is decided based on the type of plant and operating conditions such as cost, availability, reliability, accuracy, weight, size and safety.

2.9 Electronic Controllers

The controllers using electrical signals and digital algorithms which perform the control action are known as electronic controllers. These types of controllers are used where large load changes are encountered and fast response is required. Electronic controllers are classified based on the control action performed by the controllers. The classification of controllers based on the control action is listed in Table 2.3.

Table 2.3 | Control action and its respective controller

Control action	Controller
Two position or ON–OFF control action	Two position ON–OFF controller
Proportional control action	Proportional controller
Integral control action	Integral controller
Derivative control action	Derivative controller
Proportional Integral control action	Proportional Integral controller
Proportional Derivative control action	Proportional Derivative controller
Proportional Integral Derivative control action	Proportional Integral Derivative controller

2.9.1 ON–OFF Controller

ON–OFF control is the simplest form and it drives the manipulated variable from fully closed to fully open or vice versa depending on the error signal. Due to the simplicity of the controller, it is very widely used in both industrial and domestic control system. A common example of ON–OFF control is the temperature control in a domestic heating system. When the temperature is below the thermostat set point, the heating system is switched on and when the temperature is above the set point, the heating system is switched off.

Types of ON–OFF Controller: The different ways in which the ON–OFF controller operated are

(i) Two position ON–OFF controller
(ii) Multi-position control (floating control)

(i) Two Position ON–OFF Controller

In the two position ON-OFF controller, the output of the controller changes when the error value changes from positive to negative or vice versa. The simple electronic device that provides the ON–OFF control action is an electromagnetic relay. It has two contacts: normally open (NO) and normally closed (NC) contacts. The opening of the contact is controlled by the relay coil, which when excited changes the contact from one position to another (i.e., NO→NC or NC→NO).

The output signal from the controller $m(t)$, based on the actuating error signal $e(t)$, may be either at a maximum or minimum value.

$$m(t) = m_1(t) \text{ for } e(t) > 0$$

$$= m_2(t) \text{ for } e(t) < 0$$

The block diagram of a two position ON–OFF controller is shown in Fig. 2.5.

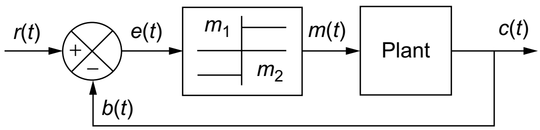


Fig. 2.5 | Block diagram of two position ON–OFF controller

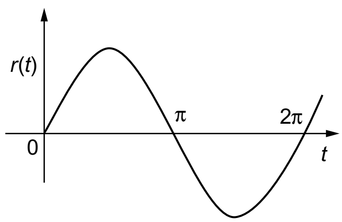


Fig. 2.5 | (a) Input

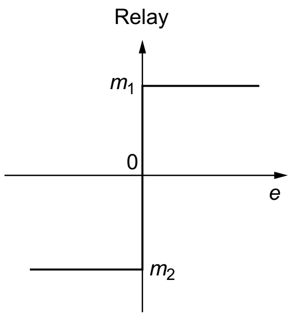


Fig. 2.5 | (b) Relay

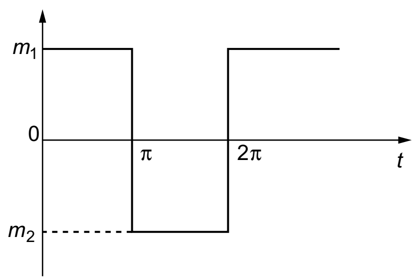


Fig. 2.5 | (c) Output

Example of Electronic Two Position ON–OFF Controller The two-position ON–OFF controller implemented electronically using a comparator circuit is shown in Fig. 2.5(d).

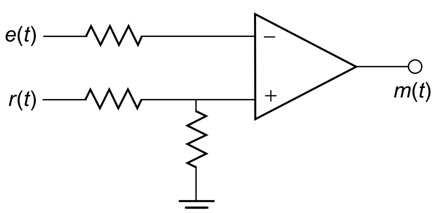


Fig. 2.5 | (d) Electronic ON-OFF controller

Neutral Zone: In practical, a differential gap exists when the controller output changes from one position to another in the two-position ON–OFF control mode of operation. The differential gap existing is known as neutral zone. Since in two-position ON–OFF controller, the controller output often switches between two positions, there exists a chattering effect that results in the premature wearing of the component. Hence, the neutral zone in the two position ON–OFF controller is used to prevent this chattering effect. In the neutral zone, the controller output will remain in its previous value. The two position ON-OFF controller with neutral zone is shown in Fig. 2.6.

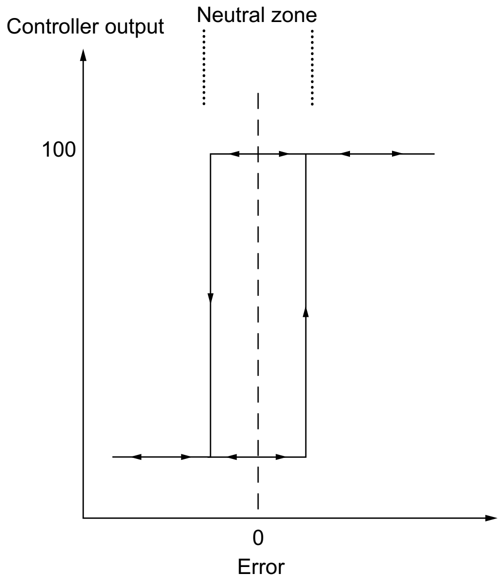


Fig. 2.6 | Two position ON-OFF controller with neutral zone

Application of Two-Position ON–OFF Controller The two position ON–OFF controller is used in

(a) air conditioning/room heating system
(b) refrigerator temperature control system
(c) liquid bath temperature control
(d) liquid-level control in tanks

(ii) Multi-Position Control (Floating Control)

The multi-position/floating control is an extension of two-position ON–OFF control. In this control, the output from the controller can have more than two values. The multi-position controller reduces the controller cycling rate when compared to the two position ON–OFF controller. The most common example of multi-position control is the three position control in which the controller output can have three (0%, 50% and 100%) outputs. The mathematical description of three position controller is as follows:

$$m(t) = \begin{cases} m_1(t)\ for\ e(t) > e_1(t) \\ m_2(t)\ for - e_2(t) < e(t) < e_1(t) \\ m_3(t)\ for\ e(t) < -e_2(t) \end{cases}$$

Fig. 2.7 shows the schematic diagram of the three position controller.

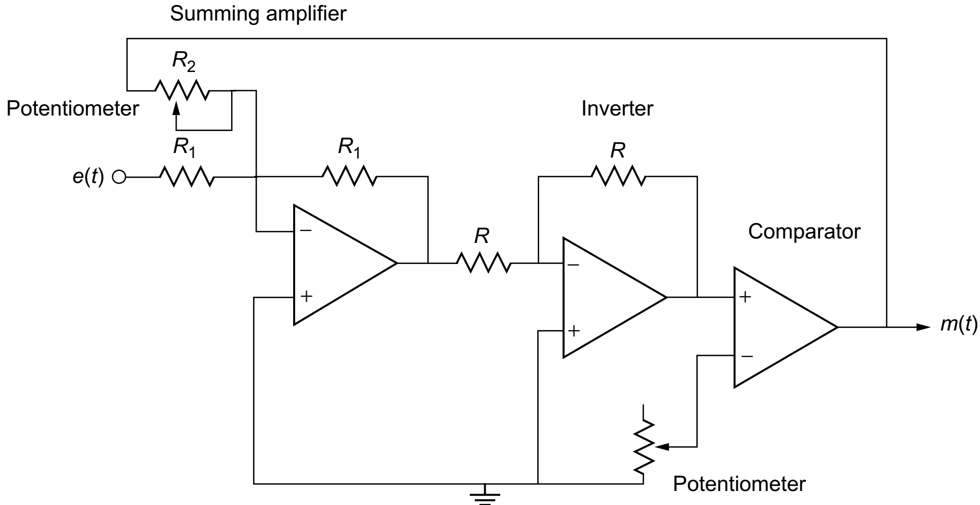


Fig. 2.7 | Three-position controller

The multi-position control is difficult to implement with op-amps, mechanical, pneumatic and hydraulic control elements as it is not as popular as two-position ON–OFF controls. Hence, the multi-position control can be implemented using microprocessor.

2.9.2 Proportional Controller

The proportional (P) controller is simple and the most widely used method of control. The proportional control is more complex than the ON–OFF control system but simpler than the conventional PID-controller. Implementation of proportional controller is simple. In proportional controller, the error signal is amplified to generate the control (output) signal. The output signal of the proportional controller $m(t)$, is proportional to the error signal $e(t)$.

In P-controller, $m(t) \propto e(t)$

$$m(t) = K_{\mathrm{p}} e(t)$$

where K_p is the proportional gain or constant.

Taking Laplace transform, we obtain

$$M(s) = K_{\mathrm{p}} E(s)$$

Therefore, the transfer function of the P-controller is given by

$$\frac{M(s)}{E(s)} = K_{\mathrm{P}}$$

The block diagram of the P-controller is shown in Fig. 2.8.

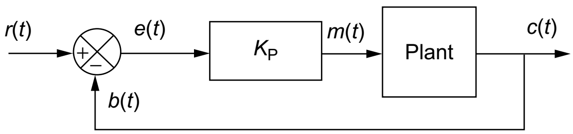


Fig. 2.8 | Block diagram of proportional controller

Proportional Band: The gain in the proportional controller can be referred to the proportional band. The relationship existing between the proportional band and the proportional gain is given below:

$$\text{Proportional Band, PB} = \frac{100}{\text{proportional gain}, K_{\mathrm{p}}}$$

In transfer characteristics, the proportional band is used to represent the maximum percentage of error that will change the controller output from minimum to maximum.

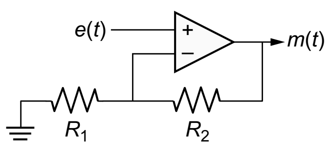
When the controller output is expressed in per cent, the actual controller output can be found using

$$m_a = Km_0 + C$$

where m_{a} is the actual controller output, K is the transfer gain, m_0 is the controller output in per cent and C is the actual output for 0% controller output.

Example of Electronic P-Controller: The proportional controller can be realized either by an inverting amplifier followed by a sign changer or by a non-inverting amplifier with adjustable gain. The analysis of proportional controller using non-inverting amplifier and by using inverting amplifier is given in Table 2.4 and Table 2.5 respectively.

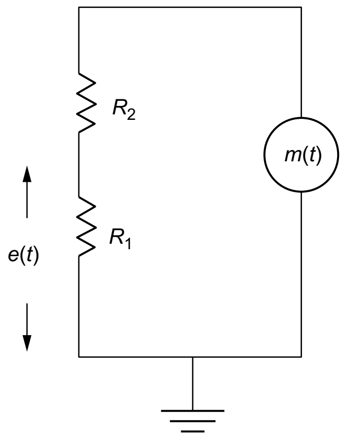
Table 2.4 | P – Controller using non–inverting amplifier

Circuit diagram


(*Continued*)

Table 2.4 | (Continued)

Equivalent circuit



Analysis

Transfer function of the controller is $\dfrac{M(s)}{E(s)} = \dfrac{R_1 + R_2}{R_1}$

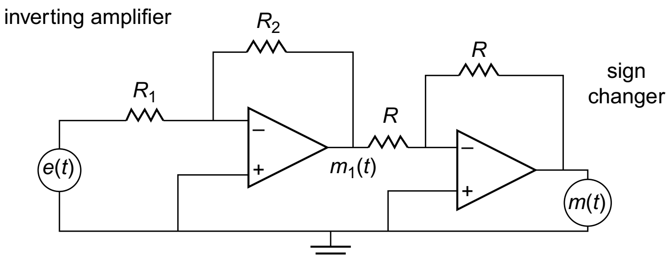
But the transfer function of the P-controller is $\dfrac{M(s)}{E(s)} = K_p$

Comparing the above two equations, we obtain

Proportional gain, $K_p = \dfrac{R_1 + R_2}{R_1}$

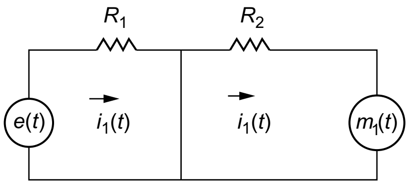
Table 2.5 | P – Controller using inverting amplifier

Circuit diagram



Equivalent circuit

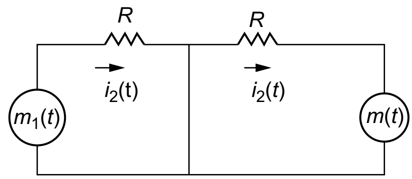
For inverting amplifier



$$e(t) = i_1(t)R_1,$$

$$m_1(t) = -i_1(t)R_2$$

For sign changer



$$m(t) = -i_2(t)R$$

$$m_1(t) = i_2(t)R$$

Analysis

Assumptions

(i) Voltages at both inputs are equal
(ii) Input current is zero

Using the above equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = \frac{R_2}{R_1}$$

But the transfer function of the P-controller is $\dfrac{M(s)}{E(s)} = K_p$

Comparing the above two equations, we obtain

Proportional gain, $K_{\text{p}} = \dfrac{R_2}{R_1}$

Advantages of P-controller

The advantages of P-controller are:

(i) It amplifies the error signal by the gain value K_{p}.
(ii) It increases the loop gain by K_{p}.
(iii) It improves the steady-state accuracy, disturbance signal rejection and relative stability.
(iv) The use of controller makes the system less sensitive to parameter variations.

Disadvantages of P-controller

The disadvantages of P-controller are:

(i) System becomes unstable if the gain of the controller increases by large value.
(ii) P-controller leads to a constant steady-state error.

2.9.3 Integral Controller

The integral (I) controller relates the present error value and the past error value to determine the controller output. The output signal of the integral controller $m(t)$ is proportional to integral of the input error signal $e(t)$.

$$\text{In I-controller, } m(t) \alpha \int e(t)\,dt$$

$$m(t) = K_i \int e(t)\,dt$$

where K_i is the integral gain or constant.

Taking Laplace transform, we obtain

$$M(s) = K_i \frac{E(s)}{s}$$

Therefore, the transfer function of the I-controller is given by

$$\frac{M(s)}{E(s)} = \frac{K_i}{s}$$

The block diagram of the I-controller is shown in Fig. 2.9.

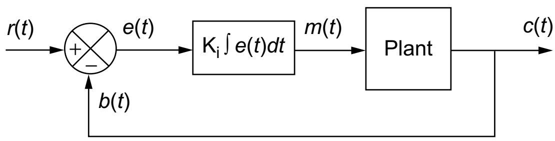


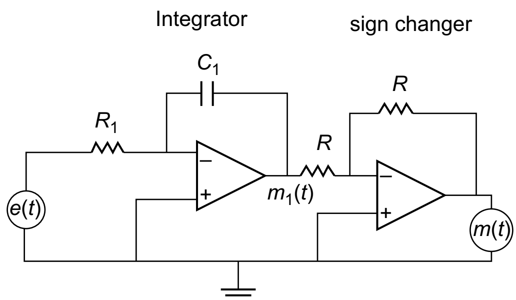
Fig. 2.9 | Block diagram of an integral controller

The integral controller is used alongwith the proportional controller to form the PI-controller or alongwith proportional derivative controller to form the PID controller.

Example of Electronic I-Controller: The integral controller can be realized by an integrator using op-amp followed by a sign changer. The analysis of such an electronic I-controller is given in Table 2.6.

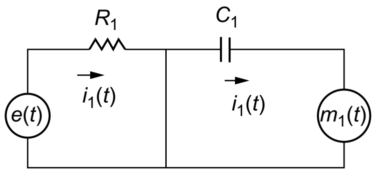
Table 2.6 | I-Controller using op-amp

Circuit diagram



Equivalent circuit

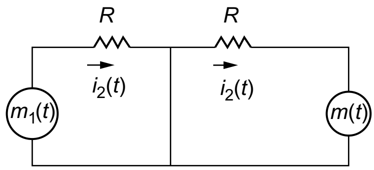
For integrator



$$e(t) = i_1(t) R_1,$$

$$m_1(t) = -\frac{1}{C_1}\int i_1(t)\,dt$$

For sign changer



$$m(t) = -i_2(t) R$$

$$m_1(t) = i_2(t) R$$

Analysis

Assumptions

(i) Voltages at both inputs are equal

(ii) Input current is zero

Using the above equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = \frac{1}{sR_1C_1}$$

But the transfer function of the I-controller is

$$\frac{M(s)}{E(s)} = \frac{K_i}{s}$$

Comparing the above two equations, we obtain

Integral gain, $K_i = \dfrac{1}{R_1C_1}$

Advantages of I-controller

The advantages of I-controller are:

(i) It reduces the steady-state error without the help of manual reset. Hence, the controller is also called as automatic reset.

(ii) It eliminates the error value.

(iii) It eliminates the steady-state error.

Disadvantages of I-controller

The disadvantages of I-controller are:

(i) Action of this controller leads to oscillatory response with increased or decreased amplitude, which is undesirable and the system becomes unstable.

(ii) It involves integral saturation or wind-up effect.

(iii) There is poor transient response.

2.9.4 Derivative Controller

The derivative controller (D-controller) is also known as a rate or anticipatory controller. The controller output of the derivative controller is dependent and is proportional to the differentiation of the error with respect to time. The output signal of the integral controller $m(t)$ is proportional to integral of the input error signal $e(t)$.

In D-controller, $m(t)\ \alpha \dfrac{de(t)}{dt}$

$$m(t) = K_d \frac{de(t)}{dt}$$

where K_d is the derivative gain or constant.

Taking Laplace transform, we obtain

$$M(s) = sK_d E(s)$$

Therefore, the transfer function of the D-controller is given by

$$\frac{M(s)}{E(s)} = sK_d$$

The block diagram of the D-controller is shown in Fig. 2.10.

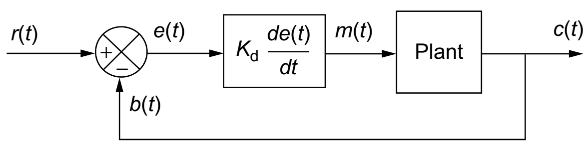
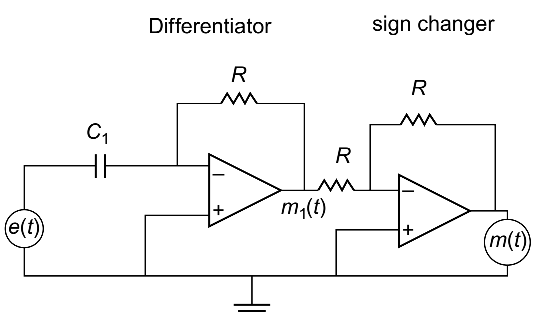


Fig. 2.10 | Block diagram of Derivative controller

Example of Electronic D-Controller: The derivative controller can be realized by a differentiator using op-amp followed by a sign changer. The analysis of such an electronic D-controller is given in Table 2.7.

Table 2.7 | D-controller using op-amp

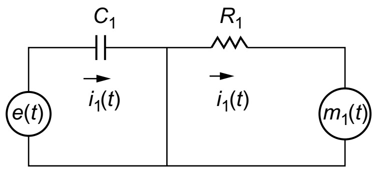
Circuit diagram



<table>
<tr>
<td>

Equivalent circuit

For differentiator

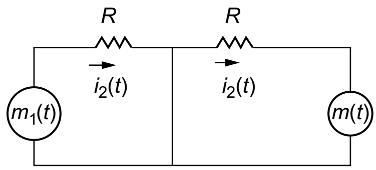


$$e(t) = \frac{1}{C_1}\int i_1(t)\,dt,$$

$$m_1(t) = -i_1(t)R_1$$

</td>
<td>

For sign changer



$$m(t) = -i_2(t)R$$

$$m_1(t) = i_2(t)R$$

</td>
</tr>
<tr>
<td colspan="2">

Analysis

Assumptions

(i) Voltages at both inputs are equal
(ii) Input current is zero

Using the above equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = R_1C_1s$$

But the transfer function of the D-controller is

$$\frac{M(s)}{E(s)} = K_d s$$

Comparing the above two equations, we obtain
Differential gain, $K_d = R_1C_1$

</td>
</tr>
</table>

Advantages of D-controller

The advantages of D-controller are:

(i) Feed forward control
(ii) Resists the change in the system
(iii) Has faster response
(iv) Anticipates the error and initiates an early corrective action that increases the stability of the system
(v) Effective during transient period

Disadvantages of D-controller

The disadvantages of D-Controller are:

(i) Steady-state error is not recognized by the controller even when the error is too large.
(ii) This controller cannot be used separately in the system.
(iii) It is mathematically more complex than P-control.

2.9.5 Proportional Integral Controller

The individual advantages of proportional and integral controller can be used by combining the both in parallel, which results in the PI-controller. The output of the proportional integral controller (PI-controller) consists of two terms: one proportional to error signal and the other proportional to the integral of error signal.

In PI-controller, $m(t)\alpha\left[e(t)+\int e(t)dt\right]$

$$m(t)=K_{\mathrm{p}}e(t)+K_{\mathrm{p}}K_{\mathrm{i}}\int e(t)dt$$

where K_{p} is the proportional gain and K_{i} is the integral gain.

Taking Laplace transform, we obtain

$$M(s)=K_{\mathrm{p}}E(s)+K_{\mathrm{p}}K_{\mathrm{i}}\frac{E(s)}{s}$$

The transfer function of the PI-controller is given by

$$\frac{M(s)}{E(s)}=K_{\mathrm{p}}\left(1+\frac{K_{\mathrm{i}}}{s}\right)$$

The block diagram of PI-controller is shown in Fig. 2.11.

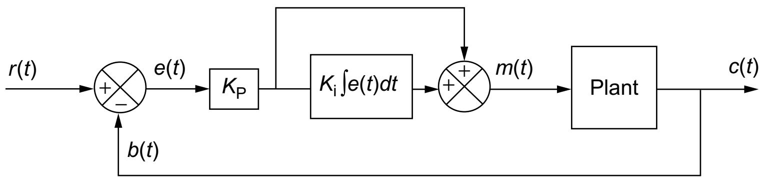


Fig. 2.11 | Block diagram of PI-controller

The equivalent block diagram of the PI controller is shown in Fig. 2.12.

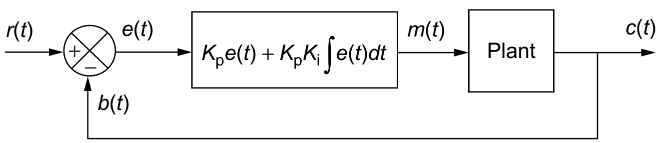
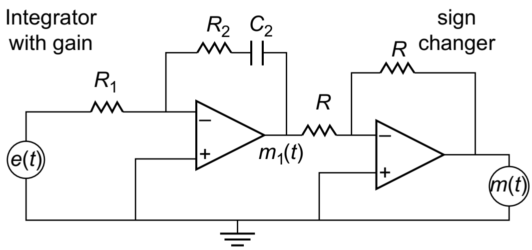


Fig. 2.12 | Equivalent block diagram of PI-controller

Example of Electronic PI-Controller: The PI-controller can be realized by an op-amp integrator with the gain followed by a sign changer. The analysis of such an electronic PI-controller is given in Table 2.8.

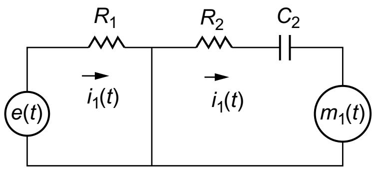
Table 2.8 | PI – controller using op-amp

Circuit diagram



Equivalent circuit

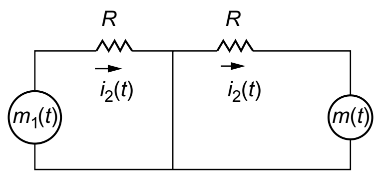
For integrator



$$e(t) = i_1(t)R_1,$$

$$m_1(t) = i_1(t)R_2 - \frac{1}{C_2}\int i_1(t)\,dt$$

For sign changer



$$m(t) = -i_2(t)R$$

$$m_1(t) = i_2(t)R$$

Analysis

Assumptions

(i) Voltages at both inputs are equal

(ii) Input current is zero

Using the above equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{sR_2C_2}\right)$$

But the transfer function of the PI-controller is

$$\frac{M(s)}{E(s)} = K_p\left(1 + \frac{K_i}{s}\right)$$

Comparing the above two equations, we get

Proportional gain, $K_\text{p} = \dfrac{R_2}{R_1}$

Integral gain, $K_\text{i} = \dfrac{1}{R_2C_2}$

Advantages of PI-controller

The advantages of PI-Controller are:

(i) Eliminates the offset present in the proportional controller
(ii) Provides faster response than the integral controller due to the presence of proportional controller also.
(iii) Fluctuation of the system around the set point is minimum
(iv) Has zero steady state error
(v) Form of a feedback control
(vi) Increases the loop gain

Disadvantages of PI-controller

The disadvantages of PI-controller are:

(i) It has maximum overshoot.
(ii) Settling time is more.

2.9.6 Proportional Derivative Controller

The proportional derivative (PD) controller is used in the system to have faster response from the controller. It combines the proportional and derivative controller in parallel. The output of the proportional derivative controller consists of two terms: one proportional to error signal and the other proportional to the derivative of error signal.

In PD controller, $m(t)\alpha\left[e(t)+\frac{de(t)}{dt}\right]$

Therefore, $$m(t)=K_p e(t)+K_p K_d \frac{de(t)}{dt}$$

where K_p is the proportional gain and K_d is the derivative gain.
Taking Laplace transform, we obtain

$$M(s)=K_p E(s)+K_p K_d sE(s)$$

Therefore, the transfer function of the PD-controller becomes

$$\frac{M(s)}{E(s)}=K_p\left(1+K_d s\right)$$

The block diagram of PD-controller is shown in Fig. 2.13.

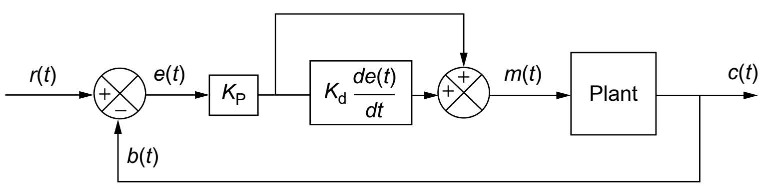


Fig. 2.13 | Block diagram of PD-controller

The equivalent block diagram of PD controller is shown in Fig. 2.14.

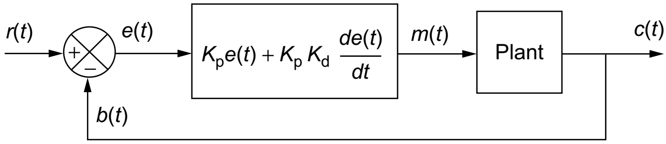
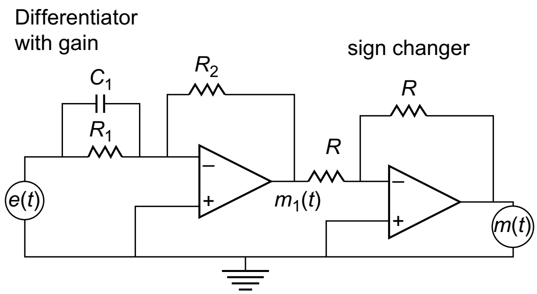
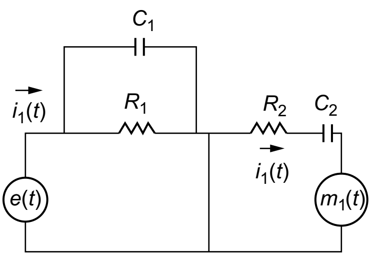
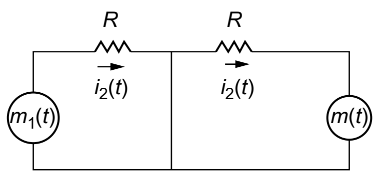


Fig. 2.14 | Equivalent block diagram of PD-controller

Example of Electronic PD-Controller The PD-controller can be realized by an op-amp differentiator with the gain followed by a sign changer. The analysis of such an electronic PD-controller is given in Table 2.9.

Table 2.9 | PD – controller using op-amp

Circuit diagram	
	
Equivalent circuit For differentiator  $i_1(t) = \dfrac{e(t)}{R_1} + C_1 \dfrac{de(t)}{dt}$ $i_1(t) R_2 = -m_1(t)$	For sign changer  $m(t) = -i_2(t) R$ $m_1(t) = i_2(t) R$

(*Continued*)

Table 2.9 | (Continued)

Analysis

Assumptions

(i) Voltages at both inputs are equal
(ii) Input current is zero

Using the equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + R_1 C_1 s\right)$$

But the transfer function of the PD-controller is

$$\frac{M(s)}{E(s)} = K_p\left(1 + K_d s\right)$$

Comparing the above two equations, we get

Proportional gain, $K_p = \dfrac{R_2}{R_1}$

Derivative gain, $K_d = R_1 C_1$

Advantages of PD-controller

The advantages of PD-controller are:

(i) It has smaller maximum overshoot due to the faster derivative action.
(ii) It eliminates excessive oscillations.
(iii) Damping is increased.
(iv) Rise time in the transient response of the system is lower

Disadvantages of PD-controller

The disadvantages of PD-Controller are:

(i) It does not eliminate the offset.
(ii) It is used in slow systems

2.9.7 Proportional Integral Derivative Controller

The universally used controller in the control system is the proportional integral derivative (PID) controller. The PID controller combines the advantages of PI- and PD-controllers. It is the parallel combination of P I and D-controllers. By tuning the parameters in the PID-controller, the control action for specific process could be obtained. The output of the PID-controller consists of three terms: first one proportional to error signal, second one proportional to the integral of error signal and the third one proportional to the derivative of error signal.

In PID-controller, $m(t)\alpha\left[e(t)+\int e(t)dt+\frac{de(t)}{dt}\right]$

Therefore, $$m(t)=K_p e(t)+K_p K_i \int e(t)dt+K_p K_d \frac{de(t)}{dt}$$

where K_p is the proportional gain, K_i is the integral gain and K_d is the derivative gain. Taking Laplace transform, we obtain

$$M(s)=K_p E(s)+K_p K_i \frac{E(s)}{s}+K_p K_d sE(s).$$

Therefore, the transfer function of the PID-controller is given by

$$\frac{M(s)}{E(s)}=K_p\left(1+\frac{K_i}{s}+K_d s\right)$$

The block diagram of PID-controller is shown in Fig. 2.15.

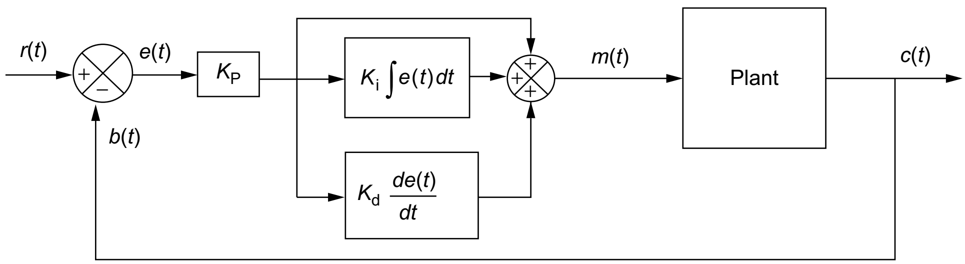


Fig. 2.15 | Block diagram of PID-controller

The equivalent block diagram of PID-controller is shown in Fig. 2.16.

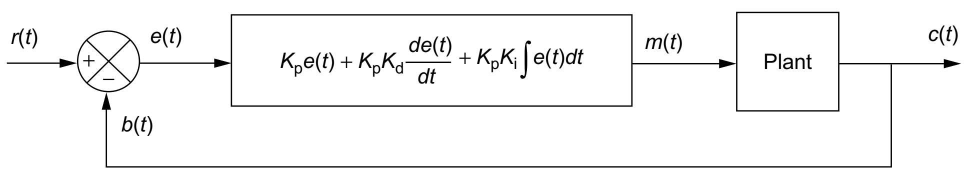
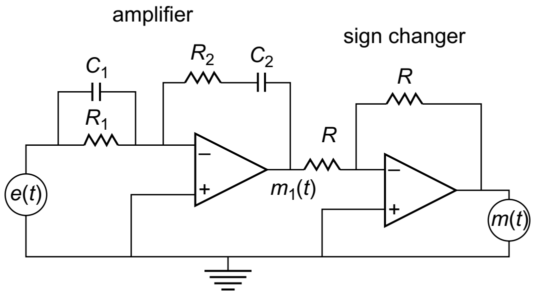


Fig. 2.16 | Equivalent block diagram of PID and controller

Example of Electronic PID-Controller: The analysis of electronic PID controller is given in Table 2.10.

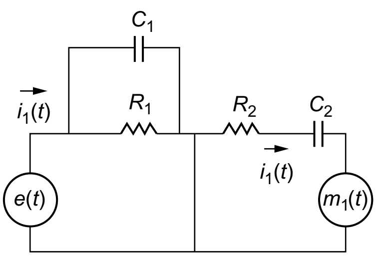
Table 2.10 | PID controller using op – amp

Circuit diagram



Equivalent circuit

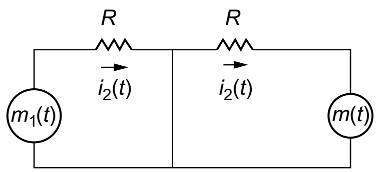
For amplifier



$$i_1(t) = \frac{e(t)}{R_1} + C_1 \frac{de(t)}{dt}$$

$$i_1(t) R_2 + \frac{1}{C_2} \int i_1(t)dt = -m_1(t)$$

For sign changer



$$m(t) = -i_2(t) R$$

$$m_1(t) = i_2(t) R$$

Analysis

Assumptions

(i) Voltages at both inputs are equal

(ii) Input current is zero

Using the above equations, the transfer function of the system is given by

$$\frac{M(s)}{E(s)} = \frac{R_2}{R_1}\left(\frac{R_1 C_1 + R_2 C_2}{R_2 C_2} + \frac{1}{s R_1 C_2} + s R_1 C_1\right)$$

But the transfer function of the PID-controller is

$$\frac{M(s)}{E(s)} = K_p \left(1 + \frac{1}{K_i s} + K_d s \right)$$

Comparing the above two equations, we get

Proportional gain $K_p = \frac{R_2}{R_1}$

Derivative gain $K_d = R_1 C_1$

Integral gain $K_i = \frac{1}{R_1 C_2}$

Also, $\frac{R_1 C_1 + R_2 C_2}{R_2 C_2} = 1$

Advantages of PID-controller

The advantages of PID-controller are:

(i) It reduces maximum overshoot
(ii) Steady-state error is zero.
(iii) It increases the stability of the system.
(iv) It improves the transient response of the system.
(v) It is possible to tune the parameters in the controllers.

Disadvantages of PID-controller

The disadvantages of PID-controller are:

(i) It is difficult to use in non-linear systems.
(ii) It is difficult to implement in large industries where complex calculations are required.

2.10 Potentiometers

A potentiometer is a variable resistor that is used to convert a linear or angular displacement into voltage. Since the resistance varies according to the linear/angular displacement of sliding/wiper contact, it is used to measure the mechanical displacement. There are two types of potentiometers: DC potentiometer and AC potentiometer. Three terminals are provided: two of them are connected to the ends of the variable resistance track and the third one is connected to the sliding contact. When an input voltage $v_i(t)$ is applied across the fixed end terminals, the output voltage $v_o(t)$ at the sliding terminal is proportional to the displacement. The schematic arrangement of the rotary DC potentiometer is shown in Fig. 2.17. If the sliding contact is connected to a shaft, the output gives the measure of the shaft position. Hence, a potentiometer will be useful in comparing the position of two shafts.

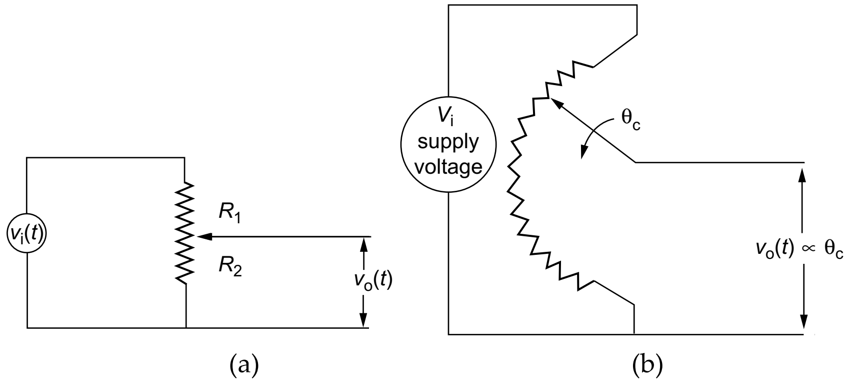


Fig. 2.17 | DC potentiometer

$$v_o(t) = \frac{R_2}{R_1 + R_2} v_i(t)$$

A variable reference voltage is obtained by using two potentiometers. Here, one potentiometer will produce an output voltage corresponding to reference shaft position and the other will produce an output voltage corresponding to the shaft that has to be controlled. The arrangement using two rotary potentiometers is shown in Fig. 2.18. When a constant voltage (AC/DC) is applied to the fixed end terminals of potentiometer, the output voltage will be directly proportional to the difference between the positions of the two shafts.

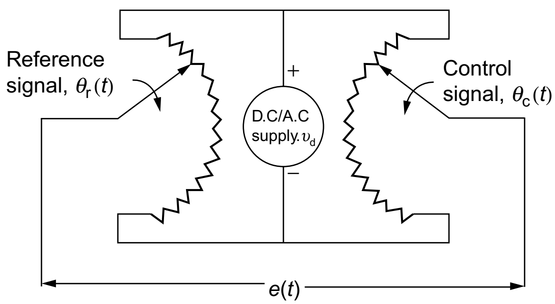


Fig. 2.18 | Error sensor having variable reference voltage

If DC voltage is applied to the potentiometers, the polarity of the error determines the relative position of the two shafts. If AC voltage is applied, then their phase determines the relative position of two shafts. But the error signal $e(t)$ is the difference between the reference signal $\theta_r(t)$ and the controlled signal $\theta_c(t)$. Thus, the error signal is given by

$$e(t) = K\left[\theta_r(t) - \theta_c(t)\right]$$

where K is the sensitivity of error detector in volts/radian. The value of K depends on the voltage and displacement of the potentiometer and is given by

$$K = \frac{v_i}{n \times 2\pi} \text{ volts/radian}$$

where v_i is the supply voltage of the potentiometer and n is the number of revolutions made by the shaft (i.e., displacement). If the applied voltage is 100 volts and the potentiometer makes 10 revolutions, then the value of K is given by

$$K = \frac{100}{10 \times 2\pi} = 5 \text{ volts/radian}$$

2.10.1 Characteristics of Potentiometers

(i) **Linearity**: It produces the output voltage that is proportional to the displacement or change in the shaft position. The deviation in the linearity occurs due to inequality in the turns (size, space) and the diameter of the resistance wire may not be uniform along its length.

(ii) **Normal Linearity**: It is the deviation in percentage of total measured resistance of the actual resistance value at any point from the best straight line drawn to the resistance versus shaft position curve.

(iii) **Resolution**: It is the smallest incremental change in resistance that can be measured between the slider and one end of fixed end of potentiometer.

(iv) **Loading Error**: There is a variation in resistance of the potentiometer due to non-linearity when load resistance is connected across it. It may be minimized by adding a non-linear resistance in the potentiometer or by using a very large load resistance.

(v) **Zero-Based Linearity**: It is the voltage between the brush and the bottom point that is zero. The linearity curve passes through zero.

2.10.2 Power-Handling Capacity

The potentiometer can dissipate the power (P) in watts continuously. Hence, the maximum voltage that can be applied across the potentiometer is $V_{i(\max)} = \sqrt{PR_p}$, where R_p is the total resistance of the potentiometer.

2.10.3 Applications of Potentiometer

The potentiometer is used to measure the mechanical displacement and to determine an error signal by using two identical potentiometers. In addition, it is used for position feedback in position control systems such as robots. It is also used for volume control in radio and brightness control in television.

2.11 Synchros

The practical problem in the potentiometer is that a physical contact exists between the wiper and the variable resistance in order to produce an output. This problem could be overcome by the usage of a synchro. A synchro is an electromagnetic rotary transducer that converts an angular shaft position into an electric signal (AC voltage) or vice versa. In control systems, the synchros are widely used as error detectors and encoders due to their rugged construction and good reliability. The trade names for the synchros are Selsyn, Autosyn, Telesyn, Dichloyn, Teletorque, etc. The elements in the synchros are synchro transmitter and synchro control transformer. Interconnection of synchro transmitter and synchro control transformer forms the synchro pair.

Reason for Trade Name Selsyn

In wound rotor induction motor, the starting resistance is placed in series with the rotor windings to limit the starting current. In addition, the starting resistance is used for printing and draw bridges where two motors are to be synchronized. Synchronizing torque during the starting process is proportional to the value of the resistance during the start. Once the motors are started using starting resistance, the rotors are short circuited and no synchronizing torque will be available. But a substantial synchronizing torque will be present in the motor even after the removal of starting resistance. This is the reason why synchros are called "Selsyn" that is known as "self-synchronous".

2.11.1 Synchro Transmitter

Synchro transmitter is the basic element of the synchro system or synchro pair. The construction of the synchro transmitter and three phase alternator are very similar. It has a stationary stator and rotor. The stationary stator is made up of laminated silicon steel and is slotted on the inner periphery to accommodate a balanced, concentric, identical three-phase star connected winding with their axis 120° apart. The rotor is made up of dumbbell construction wound with a concentric coil. An AC voltage is applied to the rotor winding through slip rings. The synchro transmitter resembles a single-phase transformer in which the rotor coil acts as the primary and the stator coils form the three secondary coils.

The constructional features and schematic diagram of a synchro transmitter are shown in Fig. 2.19 and Fig. 2.20 respectively.

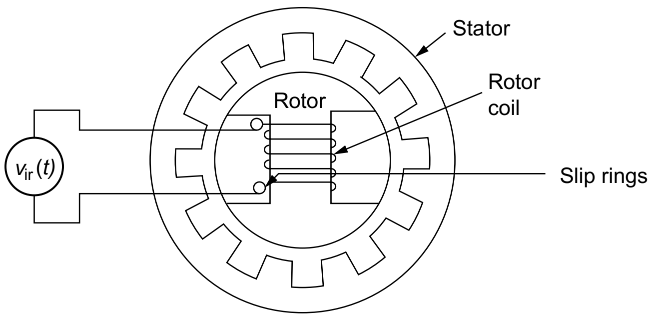


Fig. 2.19 | Constructional Features of Synchro Transmitter

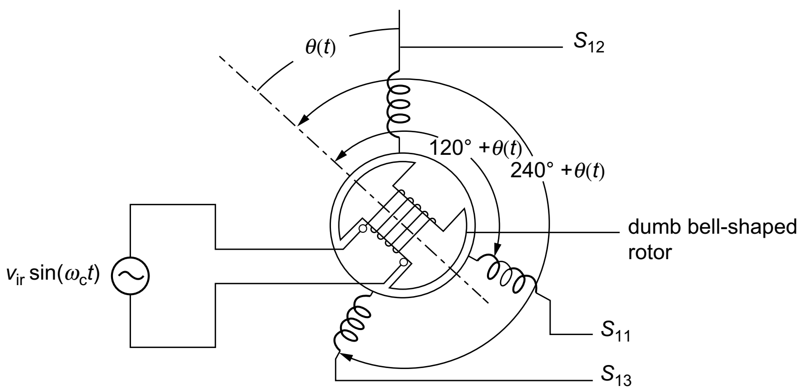


Fig. 2.20 | Schematic Diagram of Synchro Transmitter

Working of Synchro Transmitter: The schematic diagram of synchro transmitter is shown in Fig. 2.20. When an AC voltage $v_{\mathrm{ir}}(t) = V_{\mathrm{ir}} \sin \omega_c t$, is applied to the rotor of the synchro transmitter via slip rings as shown in Fig. 2.19 and Fig. 2.20, a magnetizing current that is induced in the rotor coil produces a sinusoidally time varying flux that is distributed in the air gap along the stator periphery that induces the voltage in each of the stator coils. The flux linking any stator coil and the induced voltage in the stator coil is proportional to the cosine of the angle between the rotor and stator coil axes as the flux in the air gap is sinusoidally distributed.

Let v_{s1}, v_{s2} and v_{s3} be the voltages induced in the stator coils and S_1, S_2 and $S_3$with respect to the neutral respectively. Then, when the angle between the rotor axis and stator coil S_2 is $\theta(t)$ as shown in Fig. 2.20, the voltage induced in the stator coils are given by:

$$v_{s1}(t) = K_s V_{ir} \sin(\omega_c t) \cos\left(\theta(t) + 120^\circ\right) \tag{2.1}$$

$$v_{s2}(t) = K_s V_{ir} \sin(\omega_c t) \cos\theta(t) \tag{2.2}$$

$$v_{s3}(t) = K_s V_{ir} \sin(\omega_c t) \cos\left(\theta(t) + 240^\circ\right) \tag{2.3}$$

Also, the terminal voltages of the stator are given by:

$$v_{s1s2}(t) = v_{s1}(t) - v_{s2}(t) = \sqrt{3}\, K_s\, V_{\mathrm{ir}} \sin\left(\theta(t) + 240^\circ\right) \sin(\omega_{\mathrm{c}}\, t) \tag{2.4}$$

$$v_{s2s3}(t) = v_{s2}(t) - v_{s3}(t) = \sqrt{3}\, K_s\, V_{\mathrm{ir}} \sin\left(\theta(t) + 120^\circ\right) \sin(\omega_{\mathrm{c}}\, t) \tag{2.5}$$

$$v_{s3s1}(t) = v_{s3}(t) - v_{s1}(t) = \sqrt{3}\, K_s\, V_{\mathrm{ir}} \sin\theta(t) \sin(\omega_{\mathrm{c}}\, t) \tag{2.6}$$

The terminal voltages of the stator that are given in Eqs. (2.4) through (2.6) can be plotted as shown in Fig. 2.21.

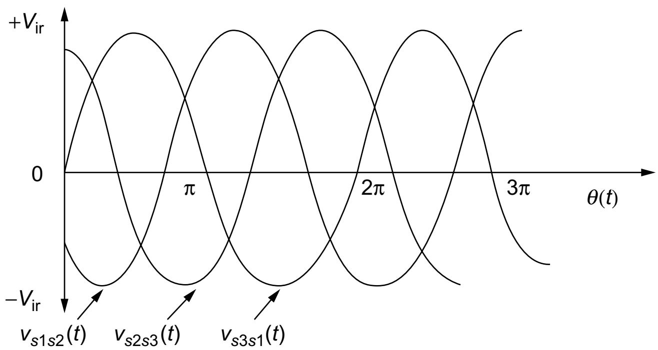


Fig. 2.21 | Waveform for the terminal voltages of the stator

Electrical Zero Position of Transmitter: When the rotor axis and stator coil are in parallel, i.e., when $\theta(t) = 0$, a maximum voltage will be induced in the corresponding stator coil while the terminal voltage across the other two terminals is zero. That corresponding position of the rotor is defined as the "electrical zero" position of the transmitter. For the shown rotor position in Fig. 2.19, if the angle $\theta(t) = 0$, then from Eqs. (2.1) through (2.3), it is clear that the maximum voltage is induced in the stator coil S_2 and also, the terminal voltage across the other two terminals, i.e., v_{S3S1} is zero. This electrical zero position is used as reference to specify the angular position of the rotor.

2.11.2 Synchro Control Transformer

The synchro control transformer is similar to the synchro transmitter with the exception in the shape of the rotor. The shape of the rotor in the synchro control transformer is made cylindrical to have uniform air gap. The generated emf of the synchro transmitter is applied as an input to the stator coils of the synchro control transformer. The load whose position is to be maintained at a desired value is connected with the rotor shaft. The constructional features and schematic diagram of a synchro transmitter are shown in Fig. 2.22(a) and Fig. 2.22(b) respectively.

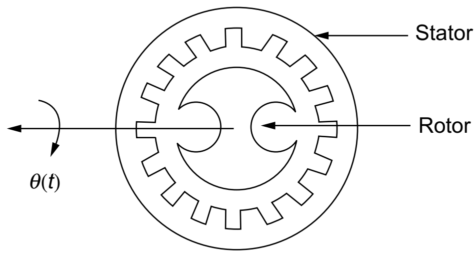


Fig. 2.22 | (a) Constructional features of Synchro control transformer

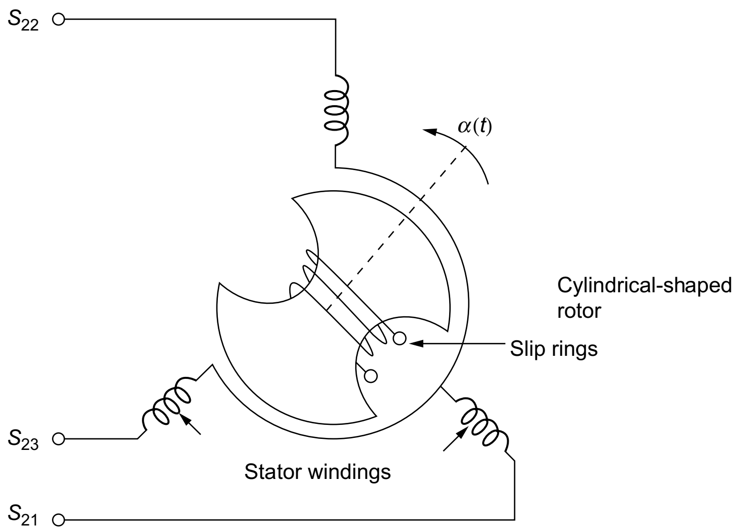


Fig. 2.22 | (b) Schematic diagram of Synchro control transformer

Working of Synchro Control Transformer: The terminal voltages obtained across the stator windings of transmitter is applied across the stator windings of control transformer. Due to the presence of uniform air gap and the position of rotor, a voltage will be induced in the rotor winding. This voltage is known as error voltage that is proportional to the difference in the rotor positions of transmitter and control transformer. In addition, this induced voltage in the rotor is used to drive the motor for correcting the position of load.

The constructional details of the synchro transmitter and the synchro control transformer are given in Table. 2.11.

Table. 2.11 | Synchro transmitter and synchro control transformer

Characteristics	Synchro transmitter	Synchro control transformer
Rotor	Dumb-bell shape	Cylindrical shape
Air gap	Non-uniform	Uniform
Input voltage	AC voltage	EMF generated by the transmitter
Input voltage applied	Across the rotor coils	Across the stator coils
Output	EMF across the stator coils	Voltage across the rotor coils

2.11.3 Synchro Error Detector

The system (synchro transmitter–synchro control transformer pair) acts as an error detector. The schematic diagram of the synchro error detector is shown in Fig. 2.23.

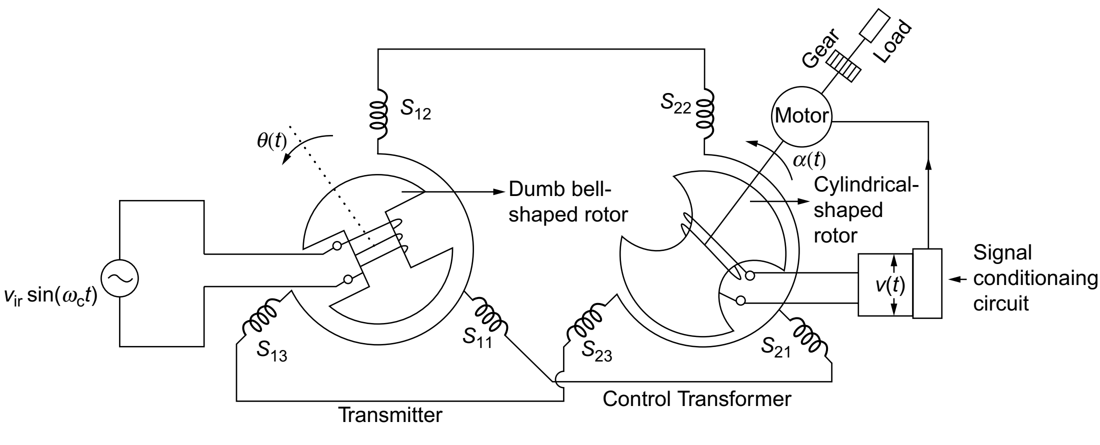


Fig. 2.23 | Synchro error detector

Let $\theta(t)$ and $\alpha(t)$ be the angles through which the rotor of synchro transmitter and synchro control transformer rotates in the same direction. When an AC supply is applied to the rotor of the synchro transmitter an induced voltage produced in the stator will be applied to the stator of the synchro control transformer. Due to the voltage induced in the stator of control transformer, an induced voltage will be produced in the rotor of control transformer, which is proportional to cosine of the angle difference between the rotor positions, i.e., cos $(\theta(t) - \alpha(t))$.

The voltage induced in the rotor of the control transformer is known as error voltage and is given by

$$v(t) = K_s\, V_{ir} \cos\left[\theta(t) - \alpha(t)\right] \sin \omega_c\, t \tag{2.7}$$

where the constant of proportionality K_s is called gain of the error detector.

When the error voltage is modulated and amplified, it can be used to drive the servomotor that is connected to load.

Electrical Zero Position of Synchro Control Transformer: When the two rotors are at right angles, the voltage induced in the rotor of the control transformer is zero. This position is known as the "electrical zero" position of the control transformer.

2.12 Servomotors

The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft. They are normally coupled to the controlled device by a gear train or some mechanical linkage. Their ratings vary from a fractional kilowatt to few kilowatts. The main objective of the servomotor system is to control the position of an object and hence the system is known as servomechanism.

2.12.1 Classification of Servomotor

The general classification of servomotors is shown in Fig. 2.24.

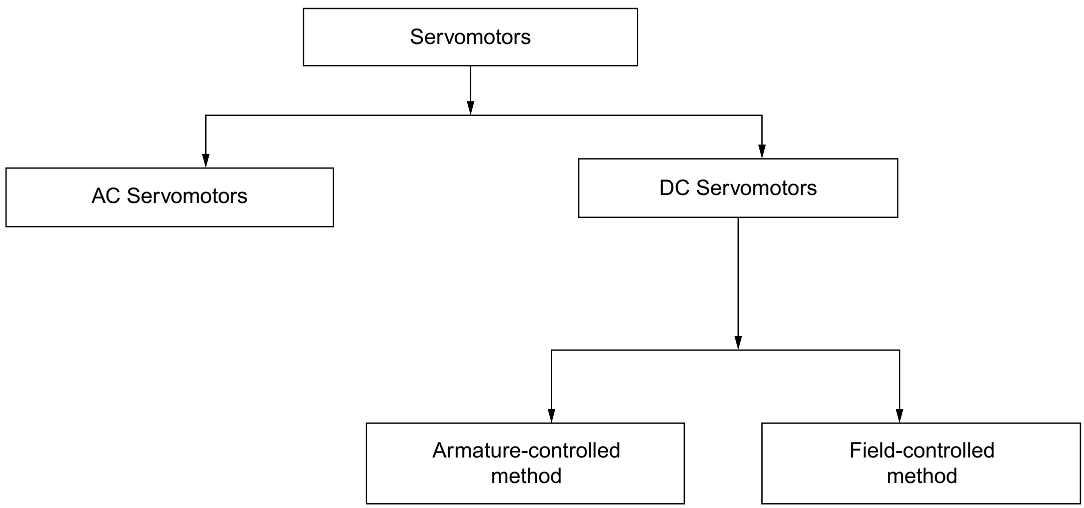


Fig. 2.24 | Classification of servomotor

2.12.2 Features of Servomotor

The maximum acceleration is the most important characteristics of servo motor. For control system application, there exists a variety of servomotors. But selection of a servomotor for a particular application depends on system characteristics and its operating conditions. In general, there are some features that a servomotor should have to suit for any application.

(i) Relationship between the control signal and speed should be linear.
(ii) Speed control should have a wider range.
(iii) Response should be faster.
(iv) There should be steady-state stability.
(v) Mechanical and electrical inertia should be low.
(vi) Mechanical characteristics should be linear throughout the speed range.

2.12.3 DC Servomotor

DC servomotor is used in applications such as machine tools and robotics, where there is an appreciable amount of shaft power is required. DC servomotors are expensive and more efficient than AC servomotors. While deriving transfer functions, the system is approximated by a linearly lumped constant parameters model by making suitable assumptions. The modes in which it could be operated are (a) field control mode and (b) armature control mode.

Transfer Function of a Field-Controlled DC Motor: In the case of field-controlled DC motor, the armature voltage and current are kept constant. The torque developed and hence the speed of the motor will be directly proportional to the field flux, generated by the field current flowing in the field winding. When the error signal is zero, there will be no field current and hence no torque is developed. The direction of rotation of the motor depends on the polarity of the field and hence on the nature of the error signal. If the field polarity is reserved, the motor will develop torque in the opposite direction. A field-controlled DC motor is shown in Fig. 2.25

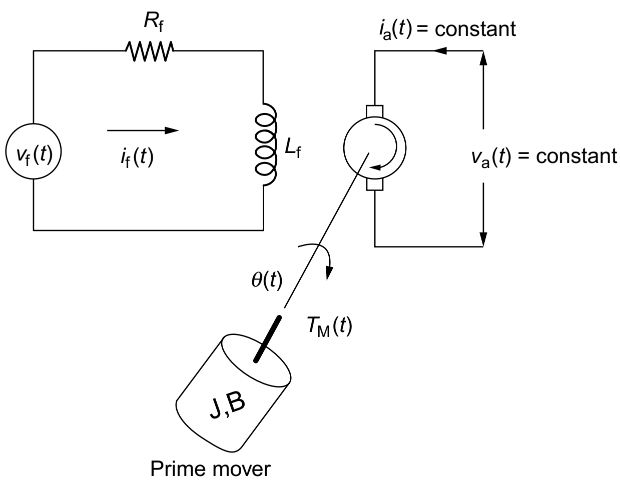


Fig. 2.25 | Field controlled DC motor

Let R_f and L_f be the resistance and inductance of the field circuit in Ω and H respectively,
$v_f(t)$ be the excitation voltage/control input of the field circuit in V,
$i_f(t)$ be the current flowing through the field circuit in A,

$T_M(t)$ be the torque developed by the motor in N-m,
J be the moment of inertia in kg-m^2,
B be the coefficient of friction in N-m/(rad/sec) and
$\theta(t)$ be the angular displacement of motor shaft in radians.

Here, the armature current is maintained constant by applying a constant voltage source to the armature and having a very large resistance in series with armature. Although the efficiency of the field-controlled DC motor is low, it is used for speed control.

The equivalent electrical field circuit and equivalent mechanical circuit of the load are shown in Fig. 2.26 and Fig. 2.27 respectively.

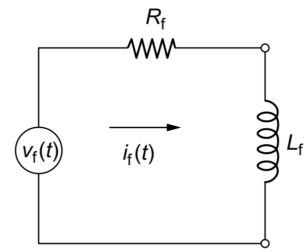


Fig. 2.26 | Equivalent electrical field circuit

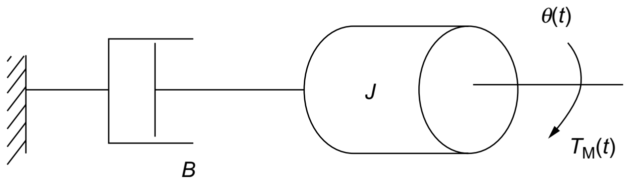


Fig. 2.27 | Equivalent mechanical circuit of load

Applying Kirchhoff's voltage law to the equivalent electrical field circuit shown in Fig. 2.26, we obtain

$$v_f(t) = R_f\, i_f(t) + L_f \frac{di_f(t)}{dt} \tag{2.8}$$

Taking Laplace transform, we obtain

$$v_f(s) = R_f I_f(s) + sL_f I_f(s)$$

$$= \left(R_f + sL_f\right) I_f(s)$$

$$I_f(s) = \frac{v_f(s)}{\left(R_f + sL_f\right)} \tag{2.9}$$

The torque developed is proportional to the field current since the armature current is constant.

$$T_M(t) = K_t\, i_f(t)$$

where K_t is a constant.

Taking Laplace transform of the above equation, we obtain

$$T_M(s) = K_t \cdot I_f(s) \tag{2.10}$$

The torque equation for the load from its equivalent mechanical circuit shown in Fig. 2.27 is given by

$$T_M(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} \tag{2.11}$$

Taking Laplace transform of the above equation, we obtain

$$T_M(s) = \left(s^2J + sB\right)\theta(s) \tag{2.12}$$

Comparing Eqn. (2.10) and Eqn. (2.12), we obtain

$$\left(s^2J + sB\right)\theta(s) = K_t \cdot I_f(s)$$

Substituting Eqn. (2.9) in the above equation, we obtain

$$s\left(sJ + B\right)\theta(s) = K_t \cdot \frac{V_f(s)}{\left(R_f + sL_f\right)}$$

Hence, the transfer function is given by

$$G(s) = \frac{\theta(s)}{V_f(s)}$$

$$= \frac{K_t}{s\left(sJ + B\right)\left(R_f + sL_f\right)}$$

$$= \frac{K_t'}{sB\left[1 + s\left(\frac{J}{B}\right)\right] R_f\left[1 + s\left(\frac{L_f}{R_f}\right)\right]}$$

$$G(s) = \frac{K_t'}{s\left(1 + s\tau_{me}\right)\left(1 + s\tau_f\right)}$$

where $K_t' = \dfrac{K_t}{R_f B}$, mechanical time constant $\tau_{\text{me}} = \dfrac{J}{B}$ and field time constant $\tau_f = \dfrac{L_f}{R_f}$.

The resultant transfer function of field-controlled DC motor is of higher order (third order) due to the field inductance that is not negligible. Hence, the stability of the system is poor when compared to the armature-controlled DC motor.

Transfer Function of an Armature-Controlled DC Servo Motor: A DC armature controlled motor is shown in Fig. 2.28.

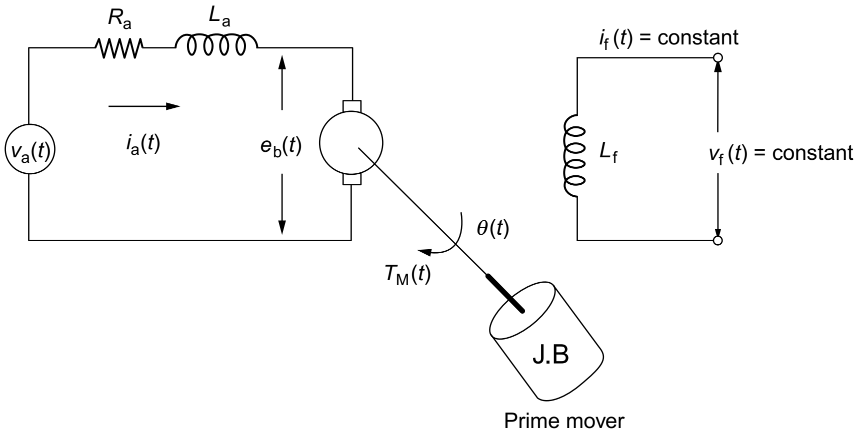


Fig. 2.28 | Armature controlled DC motor

Let R_a and L_a be the resistance and inductance of the armature circuit in Ω and H respectively,

$v_a(t)$ be the applied voltage in the armature circuit in V,

$i_a(t)$ be the current flowing through the armature circuit in A,

$e_b(t)$ be the back emf in the armature circuit in V,

$i_f(t)$ be the constant field current flowing through the field windings in A,

$T_M(t)$ be the torque developed in the motor in N-m,

J be the moment of inertia in kg-m^2,

B be the coefficient of friction in N-m/(rad/sec) and

$\theta(t)$ be the angular displacement of motor shaft in radians.

Control signals are applied to the armature current and the field current is maintained at a constant value. Maintaining a constant field current is easier than maintaining a constant armature current due to the presence of back emf in the armature circuit.

The equivalent electrical armature circuit and equivalent mechanical circuit of the load are shown in Fig. 2.29 and Fig. 2.30 respectively.

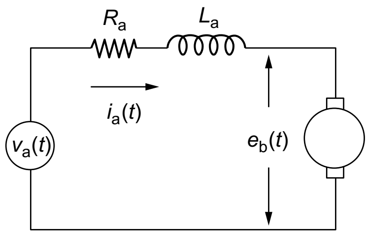


Fig. 2.29 | Equivalent electrical armature circuit

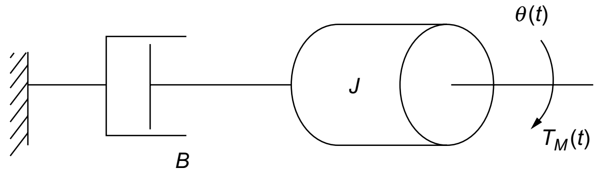


Fig. 2.30 | Equivalent mechanical circuit of load

Applying Kirchhoff's voltage law to the armature circuit shown in Fig. 2.29, we obtain

$$v_a(t) = R_a i_a(t) + L_a \cdot \frac{di_a(t)}{dt} + e_b(t) \tag{2.13}$$

Taking Laplace transform of the above equation, we have

$$V_a(s) = [R_a + sL_a] I_a(s) + E_b(s)$$

$$I_a(s) = \frac{V_a(s) - E_b(s)}{(R_a + sL_a)} \tag{2.14}$$

Since the field current is kept constant, the torque developed is proportional to the armature current. Therefore,

$$T_M(t) = K_t\, i_a(t) \tag{2.15}$$

Taking Laplace transform of the above equation, we have

$$T_M(s) = K_t\, I_a(s) \tag{2.16}$$

The torque equation for the load from its equivalent mechanical circuit shown in Fig. 2.30 is given by

$$T_M(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} \tag{2.17}$$

Taking Laplace transform of the above equation, we have

$$T_M(s) = s^2 J\, \theta(s) + sB\, \theta(s) \tag{2.18}$$

Substituting Eqn. (2.16) in Eqn. (2.18), we have

$$\theta(s)(sB + s^2 J) = K_t\, I_a(s)$$

Substituting Eqn. (2.14) in the above equation, we obtain

$$s\theta(s)\left[B+sJ\right]=K_{\mathrm{t}}\frac{\left[V_{\mathrm{a}}(s)-E_{\mathrm{b}}(s)\right]}{\left(R_{\mathrm{a}}+sL_{\mathrm{a}}\right)} \tag{2.19}$$

Motor back emf is proportional to speed and therefore,

$$e_{\mathrm{b}}(t)=K_{\mathrm{b}}\,.\,\frac{d\theta(t)}{dt}$$

where K_{b} is the back emf constant.

Taking Laplace transform of the above equation, we have

$$E_{\mathrm{b}}(s)=sK_{\mathrm{b}}\,\theta(s) \tag{2.20}$$

Substituting Eqn. (2.20) in Eqn. (2.19), we have

$$s\,\theta(s)\quad(B+sJ)\quad=\quad\frac{K_{\mathrm{t}}\left[V_{\mathrm{a}}(s)-sK_{\mathrm{b}}\,.\,\theta(s)\right]}{\left(R_{\mathrm{a}}+sL_{\mathrm{a}}\right)}$$

$$s\theta(s)(B+sJ)(R_a+sL_a)+sK_t\,.\,K_b\,\theta(s)=K_tV_a(s)$$

Therefore, the transfer function is given by

$$G(s)=\frac{\theta(s)}{V_{\mathrm{a}}(s)}$$

$$=\frac{K_t}{sK_{\mathrm{t}}\,K_{\mathrm{b}}+s(B+sJ)(R_{\mathrm{a}}+sL_{\mathrm{a}})}$$

$$=\frac{K_t}{sK_{\mathrm{t}}\,K_{\mathrm{b}}+sB\left(1+s\frac{J}{B}\right)R_{\mathrm{a}}\left(1+s\frac{L_{\mathrm{a}}}{R_{\mathrm{a}}}\right)}$$

$$=\frac{K_t}{s\left[K_{\mathrm{t}}\,K_{\mathrm{b}}+B(1+s\tau_{\mathrm{me}})\,R_{\mathrm{a}}\,(1+s\tau_{\mathrm{a}})\right]}$$

where $\tau_{\mathrm{me}}=\frac{J}{B}$ is the Mechanical time constant and $\tau_{\mathrm{a}}=\frac{L_{\mathrm{a}}}{R_{\mathrm{a}}}$ is the armature time constant.

Generally, for an armature controlled machine, $\tau_{\mathrm{a}}=0$. Hence,

$$G(s)=\frac{K_t}{s\left[K_{\mathrm{t}}\,K_{\mathrm{b}}+B\,R_{\mathrm{a}}\,(1+s\tau_{\mathrm{me}})\right]}$$

Advantages of DC Servomotor

The advantages of DC servomotor are:

(i) It has a higher power output than a 50Hz motor of the same size and weight.

(ii) Linear characteristics can easily be achieved.
(iii) Speed control can be easily achieved (from zero speed to full speed).
(iv) Quick acceleration of loads is possible due to high torque to inertia ratio.
(v) Time constants in the transfer function have low values.
(vi) It involves less acoustic noise.
(vii) Encoder sets resolution and accuracy.
(viii) It has higher efficiency since it can reach up to 90% of light loads.
(ix) It is free from vibration and resonance.

Disadvantages of DC Servomotor

The disadvantages of DC servomotor are:

(i) It involves higher cost due to its complex architecture (encoders, servo drives).
(ii) It requires larger motor or gear box.
(iii) There is necessity of safety circuits.
(iv) It needs to be tuned so as to get steady feedback loop.
(v) Motor generates peak power only at higher speeds and requires frequent gearing.
(vi) Inefficient cooling mechanism. If motors are ventilated, they get easily contaminated.

2.12.4 AC Servomotor

The AC servomotor is a two-phase induction motor with some special design features. The simple constructional feature of AC servomotor is shown in Fig. 2.31. It comprises of a stator winding and a rotor, which may take one of several forms: (i) squirrel cage, (ii) drag cup and (iii) solid iron.

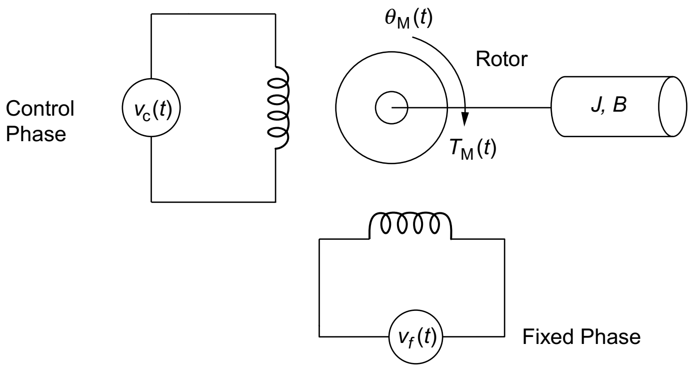


Fig. 2.31 | AC servomotor

The stator consists of two pole-pairs mounted on the inner periphery of the stator, such that their axes are at an angle of 90° in space. The rotor bars are placed on the slots and short-circuited at both ends by end rings. The diameter of the rotor is kept small in order to reduce inertia and to obtain good accelerating characteristics.

Each pole-pair carries a winding. The exciting current in the windings should have a phase displacement of 90° as shown in Fig. 2.32. The voltages applied to these windings are not balanced.

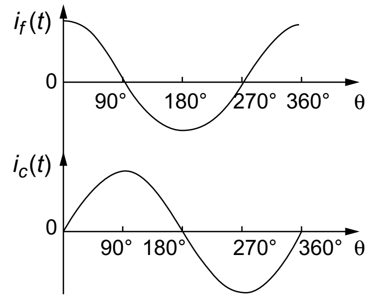


Fig. 2.32 | Exciting currents in the winding

Under normal operating conditions, a fixed voltage form a constant voltage source is applied to one phase, which is called the reference phase. The other phase called the control phase is energized by a voltage of variable magnitude and polarity, which is at 90° out of phase with respect to the fixed phase. The control phase is usually applied from a servo amplifier. The direction of rotation reverses if the phase sequence is reversed.

In feedback control system, the AC servomotor resembles two-phase induction motor. The differences that exist between normal induction motor and servomotor are:

(i) To achieve linear speed torque characteristics, the rotor resistance of the servomotor is made high so that the ratio X/R is made small.

(ii) Phase difference between the excitation voltages applied to two stator windings is 90°.

The torque-speed characteristics for the induction motor and servomotor are shown in Fig. 2.33.

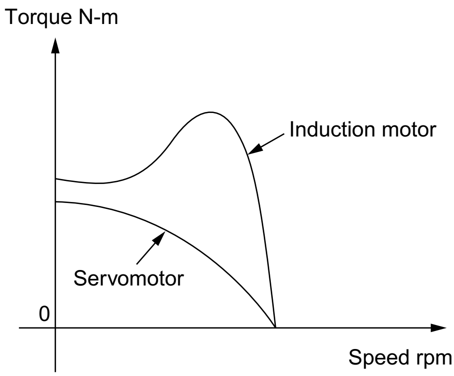


Fig. 2.33 | Torque-speed characteristics

In the two-phase servomotor, the polarity of the control voltage determines the direction of rotation. Then the instantaneous control voltage $v_c(t)$ is of the form

$$v_c(t) = V_c(t)\sin\omega t \quad \text{for } V_c(t) > 0$$

$$= |V_c(t)| \sin(\omega t + \pi) \quad \text{for } V_c(t) < 0$$

Since the reference voltage is constant, the torque $T(t)$ and the angular speed $\theta(t)$ are also functions of the control voltage $V_c(t)$. If variations in $V_c(t)$ are also compared with the ac supply frequency, the torque developed by the motor is proportional to $V_c(t)$. Figure 2.34 shows the curves $\theta_c(t)$ versus t, $V_c(t)$ versus t and torque $T(t)$ versus t. The angular speed at steady state is proportional to the control voltage $V_c(t)$.

Transfer Function of AC Servomotor: For higher values of rotor resistance, the torque–speed characteristics are linear. For servo application, the motor characteristics should be linear with negative slope. Therefore, ordinary two-phase induction motor with low rotor resistance is not suitable for servo applications. Figure 2.35 shows the speed–torque characteristics of servomotor for different values of control voltage.

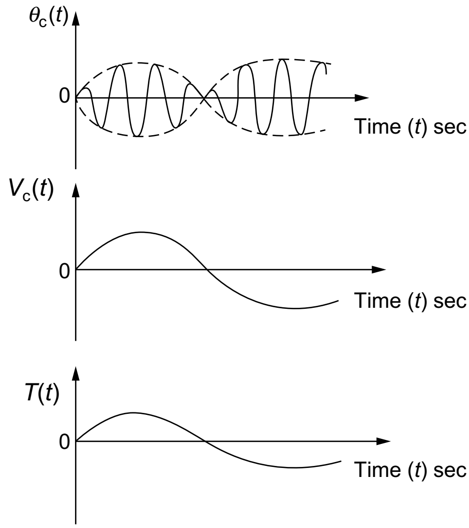


Fig. 2.34 | $V_c(t)$, $T(t)$ and $\theta_c(t)$ versus time

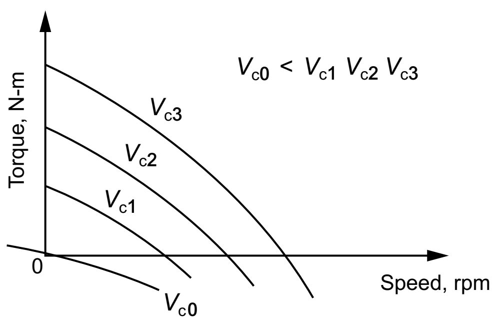


Fig. 2.35 | Torque – speed characteristics

The following assumptions are to be made to derive the transfer function of the motor:

(i) Linear portion of the torque–speed curve is extended up to the high-speed region since the servomotor operates normally at high speed.

(ii) Speed–torque characteristics for different voltages are assumed as straight line parallel to the characteristics at rated control voltage.

The blocked rotor torque at rated voltage per unit control voltage is represented as follows:

$$K = \frac{\text{Blocked rotor torque at } V_c(t) = V_f(t)}{\text{Rated control voltage} V_f(t)} = \frac{T_0(t)}{V_f(t)}$$

The slope of the linearized torque–speed curve is given by

$$m = \frac{\text{Blocked rotor torque}}{\text{No load speed}} = \frac{T_0(t)}{n_0}$$

Then for any torque $T_m(t)$, the family of straight line torque–speed curve is represented as follows:

$$T_m(t) = KV_c(t) + m\left(\frac{d\theta_m(t)}{dt}\right)$$

If we consider that the moment of inertia of the motor and shaft is J_m and the viscous friction is B_m, we have

$$T_m(t) = KV_c(t) + m\left(\frac{d\theta_m(t)}{dt}\right) = J_m\left(\frac{d^2\theta_m(t)}{dt^2}\right) + B_m\left(\frac{d\theta_m(t)}{dt}\right)$$

Taking the Laplace transform, we obtain

$$KV_c(s) + ms\theta_m(s) = \left(s^2 J_m + sB_m\right)\theta_m(s)$$

Therefore, the transfer function of the two-phase motor is obtained as

$$\frac{\text{Motor shaft displacement}}{\text{Control phase voltage}} = \frac{\theta_m(s)}{V_c(s)} = \frac{K}{(B_m - m)s\left(1 + sJ_m/(B_m - m)\right)}$$

or
$$\frac{\theta_m(s)}{V_c(s)} = \frac{K_m}{s(1 + sT_m)}$$

where $K_m = \dfrac{K}{(B_m - m)}$ is the motor gain constant and $T_m = \dfrac{J_m}{(B_m - m)}$ is the motor time constant.

Advantages of AC Servomotors

(i) AC servomotors involve low cost and low maintenance since there is no commutator and brushes.
(ii) These are of high efficiency.

Applications of AC Servomotors

(i) These are used in X–Y recorders.
(ii) These are also used in disk drives, tape drives printers, etc.

2.12.5 Comparison between AC Servomotor and DC Servomotor

The comparison between the AC servomotor and DC servomotor based on different characteristics is given in Table 2.12.

Table 2.12 | Comparison between AC and DC servomotors

Characteristics	AC servomotor	DC servomotor
Life (in terms of hours)	20,000 or up	3000–5000
Maintenance	Less (absence of brushes and commutators)	More (needs replacement of brushes periodically)
Noise	Less	More
Efficiency	More	Less
Speed and Torque	More	Less due to brush flashover
Response	Very quick	Quick
Cleanliness	Good	Bad

2.13 Tachogenerators

Tachogenerator is an electromechanical device that converts mechanical energy (shaft speed) to a proportional electrical energy (voltage). It is an ordinary generator that generates a voltage proportional to the magnitude of the angular velocity of the shaft. It is used as speed indicator, velocity feedback device and signal integrator. Tachogenerators are classified as AC tachogenerator and DC tachogenerator.

2.13.1 DC Tachogenerator

The schematic representation of a DC tachogenerator is shown in Fig. 2.36. The iron core rotor is the most common part in all DC tachogenerator. A DC tachogenerator is an ordinary DC generator with linear characteristics. The parts of the DC tachogenerator are stator with a permanent magnet to produce necessary magnetic field, rotating armature circuit connected to the commutator and brush assembly. The output voltage is measured across the pair of brushes that are connected with commutators. If required, modulated output of DC tachogenerator can be used in controlling the AC system components.

The output voltage of the tachogenerator is proportional to the angular velocity of the shaft. The polarity of the output voltage depends on the direction of rotation of the shaft.

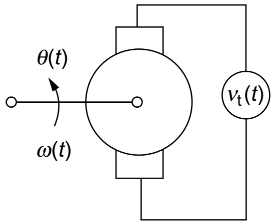


Fig. 2.36 | Schematic diagram of DC Tachogenerator

Let $v_t(t)$ be the output voltage of tachogenerator in volts,

$\theta(t)$ be the angular displacements in rad,

$\omega(t) = \dfrac{d\theta(t)}{dt}$ be the angular velocity in rad/sec and

K_t be the sensitivity of tachogenerator in volts–sec/rad.

The output voltage of tachogenerator,

$$v_t(t)\,\alpha\,\frac{d\theta(t)}{dt}$$

or,
$$v_t(t) = K_t\frac{d\theta(t)}{dt}$$

Taking Laplace transform of the above equation, we obtain

$$V_t(s) = K_t s\theta(s)$$

Hence, the transfer function of DC tachogenerator is

$$\frac{V_t(s)}{\theta(s)} = sK_t$$

Advantages of DC Tachogenerator

The advantages of DC tachogenerator are:

(i) It is used alongwith high pass output filters to reduce the servo velocity lags.
(ii) It generates very high-voltage gradients in small sizes.
(iii) There is no presence of residual voltage at zero speed.

Disadvantages of DC Tachogenerator

The disadvantages of DC tachogenerator are:

(i) inherent ripples in the output waveform
(ii) poor commutation
(iii) requirement of additional filter to avoid the interference

2.13.2 AC Tachogenerator

The AC tachogenerator and a two-phase induction motor resemble each other. The parts of the AC tachogenerator are two-stator windings (reference and quadrature windings) with the angle between their axes as 90° are fixed on the inner periphery of stator arranged in space quadrature and the squirrel cage rotor. Stator windings are referred to as reference winding and output winding. The reference winding is excited by a sinusoidal voltage with frequency, ω_c. When the rotor rotates, a voltage will be induced in the quadrature winding. The schematic diagram of AC tachogenerator is shown in Fig. 2.37. An AC tachogenerator alongwith a phase-sensitive demodulator circuit can be used in control system to convert an AC output to a DC output.

If the reference winding is excited by the voltage $v_r(t) = V_r \sin\omega_c t$, a reference flux $\varphi_r \sin\omega_c t$ will be produced in the reference winding due to the presence of negligible resistance and reactance in the reference coil. The rotor speed equation is given by

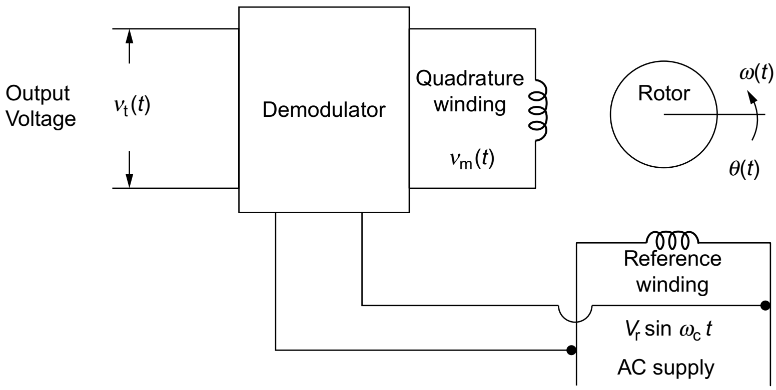


Fig. 2.37 | Schematic diagram of AC Tachogenerator

$$\dot{\theta}(t) = \dot{\theta}_m(t)\ \cos(a\omega_c t)$$

The induced voltage produced in the quadrature winding, which is in phase with the voltage applied in the reference coil, is directly proportional to the rotor speed. The modulated signal available at the output terminal is given by

$$v_t(t) = K_t \frac{d\theta(t)}{dt}$$

Taking Laplace transform, we obtain

$$\frac{V_t(s)}{\theta(s)} = sK_t$$

The above equation gives the transfer function of the AC tachogenerator, which is same as the transfer function of DC tachogenerator.

2.14 Stepper Motor

A special type of synchronous motor designed to rotate through a specific angle for each electrical pulse applied is known as stepper motor. The specific angle through which the stepper motor rotates is called step. Electrical pulses are received from the control unit of stepper motor. Stepper motor is used alongwith electronic switching devices whose function is to switch the control windings of stepper motor according to the command received.

The stepper motor has gained importance in recent years because of the ease with which it can be interfaced with digital circuits. The stepper motor completes a full rotation by sequencing through a series of discrete rotational steps (stepwise rotation). Each step position is an equilibrium position, so that without further excitation the rotor position stays at the latest step. Thus, continuous rotation is achieved by a train of input pulses, each of which causes an advance of one step. The stepper motor can be variable reluctance, permanent magnet or hybrid stepper motor depending on the type of rotor. The stepper motor can also be classified into two-phase, three-phase or four-phase stepper motor depending on the number of windings on the stator.

2.14.1 Permanent Magnet Stepper Motor

In this type of stepper motor, the stator has salient poles that carry control windings. A phase is created when the two control windings are connected in series. The rotor of this type of stepper motor is made in the form of a spider cast integral permanent magnet or assembled permanent magnets. The schematic diagram showing the different types of rotor structure is shown in Fig. 2.38.

Fig. 2.38 | Rotor structure types

The permanent magnet stepper motor operates at a larger steps upto 90° at a maximum response rate of 300 pps.

2.14.2 Variable Reluctance Stepper Motor

In this type of stepper motor, it has a single or several stacks of stator and rotor. The stator has a common frame, whereas the rotor has a common shaft. The stator and rotor of this type of stepper motor has toothed structure. The longitudinal cross sectional view of three-stack variable reluctance stepper motor is shown in Fig. 2.39.

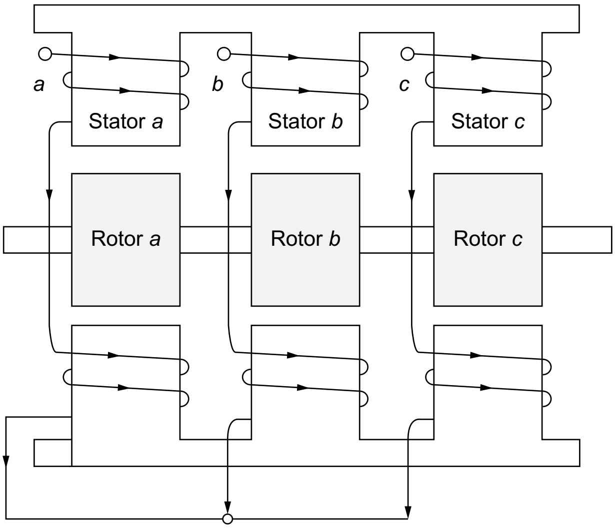


Fig. 2.39 | 3 Stack variable reluctance motor

The difference in angular displacement of stator and rotor when the teeth of the rotors are perfectly aligned is given by

$$\alpha = \frac{360°}{nT}$$

where n is the number of stacks and T is the number of rotor teeth.

The schematic diagram explaining the concept of the separation between the stator and rotor is shown in Fig. 2.40.

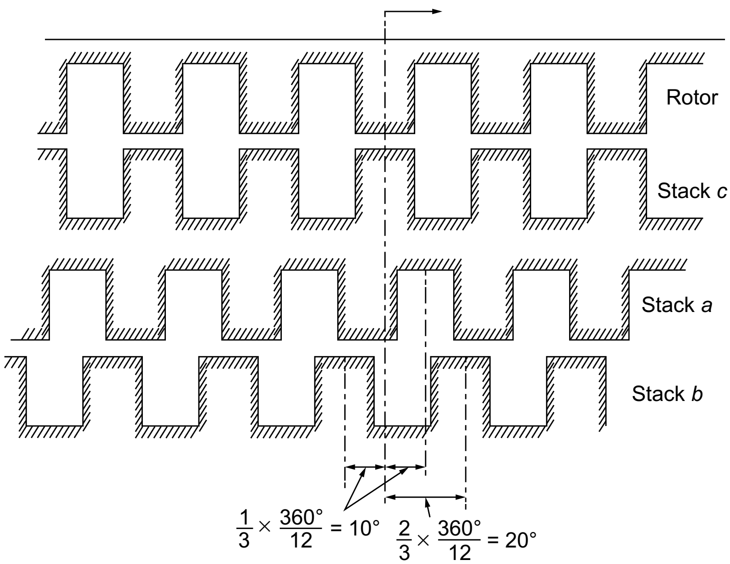


Fig. 2.40 | Separation between stator and rotor

2.14.3 Hybrid Stepper Motor

This is similar to the permanent magnet stepper motor with the constructional features rotor adopted from variable reluctance stepper motor. The polarities of the teeth on the stack of the rotor at both the ends are of different polarities. Thus, the two sets of teeth in the rotor are displaced from each other by one-half of the tooth pitch (pole pitch). The constructional features of this type of stepper motor are shown in Fig. 2.41(a) and (b).

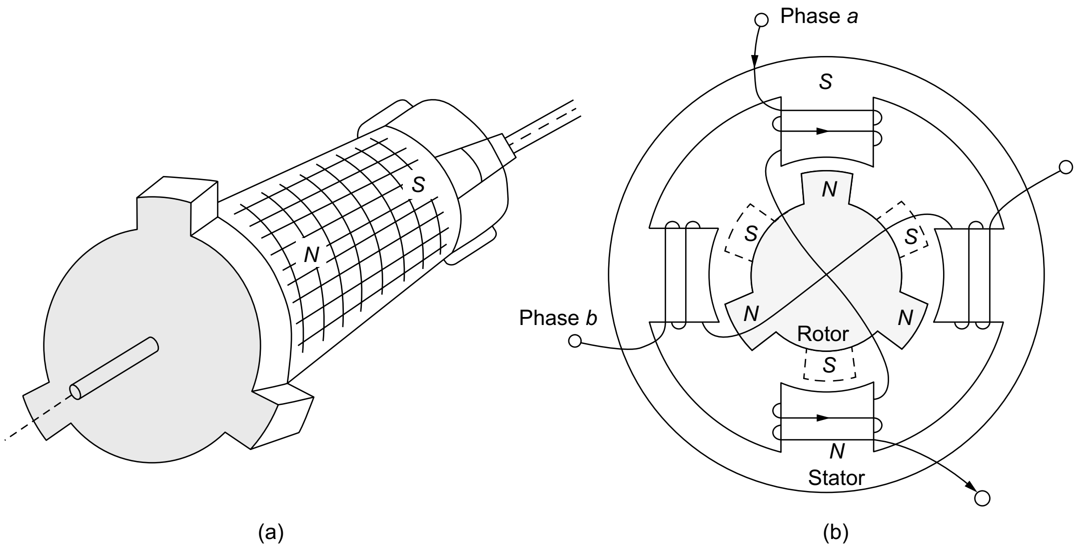


Fig. 2.41 | Constructional features of hybrid stepper motor

2.14.4 Operation of Stepper Motor

Depending on the step angle by which the stepper motor rotates, the operation of stepper motor can be classified as

(i) full-step operation
(ii) half-step operation
(iii) micro-step operation

Full-Step Operation: In this type of operation, the stepper motor moves one full step for each of the input pulse applied.

$$\text{Full step} = \frac{360°}{N_R N_S}$$

where N_R is the number of rotor poles and N_S the number of stator pole pairs.

Half Step Operation: In this type of operation, the stepper motor moves one-half of the full step for each of the input pulse applied. If in the full-step operation of stepper motor, for each input pulse the motor rotates at $X°$, then in the half-step operation of stepper motor, the motor will rotate at $X°/2$ for each input pulse.

Micro Step Operation: In this type of operation, the stepper motor moves through an angle of 1/10, 1/16, 1/32 and 1/125 of a full step. The major advantage of this type of operation is that it provides much finer resolution.

2.14.5 Advantages of Stepper Motor

The advantages of stepper motor are:

(i) It is compatible with digital systems.
(ii) No sensors are required to sense the speed and position.
(iii) Motors are available in larger power ratings with reduced cost.
(iv) Motors are available with the torque range of 0.5 micro N-m to 100 N-m.
(v) It is used to upgrade mechanical systems to give greater precision and production rate.

2.14.6 Applications of Stepper Motor

The applications of stepper motor are:

(i) It is used in computer peripherals (printers, disk drives, etc.), X–Y plotters, scientific instruments and machine tools.
(ii) It is used in quartz crystal watches.
(iii) Many supporting roles in the manufacture of packaged foodstuffs, commercial end products and even in the production of science fiction movies.

2.15 Gear Trains

The conventional servomotor usually operates at high speeds with low torque. Gear trains are used in servo positioning applications to operate the servomotor at low speed with low torque. In addition, to alter the speed-to-torque ratio of the rotational power transmitted from motor-to-load gear trains are used. Gear trains are necessary to match the torque requirement of the load to that of the motor. Hence, the knowledge of the actions of a gear train is important to the control systems designer. The analogous component of gear train in mechanical rotational system is the transformer in electrical system that is shown in Fig. 2.42 and Fig. 2.43.

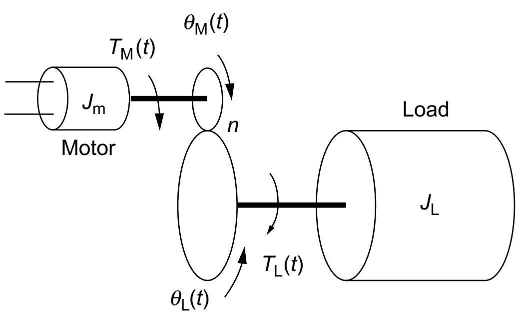


Fig. 2.42 | Schematic diagram of gear trains

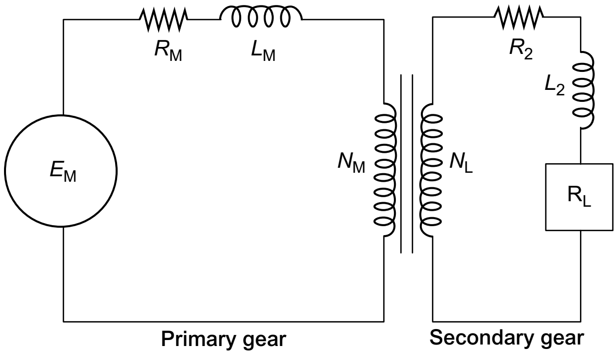


Fig. 2.43 | Equivalent circuit of gear trains

2.15.1 Single Gear Train

The schematic diagram representing the gear train driving the load is shown in Fig. 2.44. The gear train shown in Fig. 2.44 consists of two gears with teeth N_M and N_L.The gear 1 (primary gear) is connected to motor shaft and gear 2 (secondary gear) is connected to load shaft. The subscript M in the following section denotes the gear 1 (primary gear) and subscript L in the following section denotes the gear 2 (secondary gear).

Let N_M and N_L be the number of teeth's in gears,
r_M and r_L be the radius of gears,
θ_M and θ_L be the angular displacement of shafts,
J_M and J_L be the moment of inertia,
B_M and B_L be the viscous friction coefficient,
T_1 be the load torque developed by motor,
T_M be the torque developed by motor,
T_2 be the torque transmitted to gear 2 from gear 1 and
T_L be the load torque.

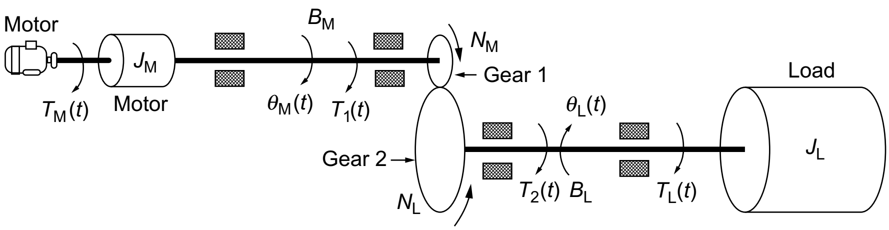


Fig. 2.44 | Schematic diagram for gear trains

The torque developed by motor is balanced by the sum of load torque requirement on gear 1 and opposing torque due to J_M and B_M. Hence, the torque balance equation for motor shaft is given by

$$J_M \frac{d^2\theta_M(t)}{dt^2} + B_M \frac{d\theta_M(t)}{dt} + T_1(t) = T_M(t) \tag{2.21}$$

The torque transmitted to gear 2 is balanced by the sum of load torque and opposing torque balance equation for load shaft is given by

$$J_L \frac{d^2\theta_L(t)}{dt^2} + B_L \frac{d\theta_L(t)}{dt} + T_L(t) = T_2(t) \tag{2.22}$$

When the motor drives the load, the linear distance travelled by each gear is same. The linear distance travelled by a gear is given by the product of the radius and angular displacement.

Linear distance travelled by the gear $= \theta_M(t) r_M = \theta_L(t) r_L$

Therefore,
$$\frac{\theta_L(t)}{\theta_M(t)} = \frac{r_M}{r_L} \tag{2.23}$$

The number of teeth in each gear is proportional to its radius.

Therefore, $\quad N_M \alpha\ r_M$ and $N_L \alpha\ r_L$

Hence,
$$\frac{N_M}{N_L} = \frac{r_M}{r_L} \tag{2.24}$$

From Eqn. (2.23) and Eqn. (2.24), we obtain

$$\frac{\theta_L(t)}{\theta_M(t)} = \frac{N_M}{N_L} \tag{2.25}$$

In ideal gear train system, there is no power loss in transmission. Hence, the work done by both gears are equal. The work done by a gear is given by the product of torque acting on it and its angular displacement.

Work done by the gear $= T_1(t)\theta_M(t) = T_2(t)\theta_L(t)$ (2.26)

Therefore,
$$\frac{T_1(t)}{T_2(t)} = \frac{\theta_L(t)}{\theta_M(t)} \tag{2.27}$$

Cross-multiplying and differentiating the above equation, we obtain

$$T_1(t)\frac{d\theta_M(t)}{dt} = T_2 \frac{d\theta_L(t)}{dt} \tag{2.28}$$

Therefore, $\quad T_1(t)\alpha_M(t) = T_2(t)\alpha_L(t)$

and
$$\frac{T_1(t)}{T_2(t)} = \frac{\alpha_L(t)}{\alpha_M(t)} \tag{2.29}$$

When $\omega_{\mathrm{M}}(t)$ and $\omega_{\mathrm{L}}(t)$ are angular speeds of motor shaft and load shaft respectively. Differentiating Eqn. (2.28) again, we obtain

$$T_1(t)\frac{d^2\theta_{\mathrm{M}}(t)}{dt^2} = T_2(t)\frac{d^2\theta_{\mathrm{L}}(t)}{dt^2}$$

Therefore,

$$T_1(t)\alpha_{\mathrm{M}}(t) = T_2(t)\alpha_{\mathrm{L}}(t)$$

and

$$\frac{T_1(t)}{T_2(t)} = \frac{\alpha_{\mathrm{L}}(t)}{\alpha_{\mathrm{M}}(t)} \tag{2.30}$$

where $\alpha_{\mathrm{M}}(t)$ and $\alpha_{\mathrm{L}}(t)$ are angular acceleration of motor shaft and load shaft respectively.

From Eqs. (2.23), (2.24), (2.27), (2.29) and (2.30), we obtain

$$\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}} = \frac{r_{\mathrm{M}}}{r_{\mathrm{L}}} = \frac{T_1(t)}{T_2(t)} = \frac{\theta_{\mathrm{L}}(t)}{\theta_{\mathrm{M}}(t)} = \frac{\omega_{\mathrm{L}}(t)}{\omega_{\mathrm{M}}(t)} = \frac{\alpha_{\mathrm{L}}(t)}{\alpha_{\mathrm{M}}(t)} \tag{2.31}$$

Using the above equation, it can be said that

When $N_{\mathrm{M}} > N_{\mathrm{L}}$, the gear train increases the speed and reduces the torque.

When $N_{\mathrm{M}} = N_{\mathrm{L}}$,there is no change in speed and torque.

When $N_{\mathrm{M}} < N_{\mathrm{L}}$,the gear train decrease the speed and increase the torque.

Torque Equation Referred to Motor Shaft: In systems with gear trains, it will be easier for analysis if the torques acting on the system is transferred to one of the sides.

Equation (2.31) can be rewritten as follows:

$$\alpha_{\mathrm{L}}(t) = \alpha_{\mathrm{M}}(t)\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}, \qquad \frac{d^2\theta_{\mathrm{L}}(t)}{dt^2} = \frac{d^2\theta_{\mathrm{M}}(t)}{dt^2}\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}} \tag{2.32}$$

$$\omega_{\mathrm{L}}(t) = \omega_{\mathrm{M}}(t)\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}, \qquad \frac{d\theta_{\mathrm{L}}(t)}{dt} = \frac{d\theta_{\mathrm{M}}(t)}{dt}\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}} \tag{2.33}$$

$$T_2(t) = T_1(t)\frac{N_{\mathrm{L}}}{N_{\mathrm{M}}} \tag{2.34}$$

Substituting the values for $\frac{d^2\theta_{\mathrm{L}}(t)}{dt^2}, \frac{d\theta_{\mathrm{L}}(t)}{dt}, T_2\left(t\right)$ from the above equations in Eqn. (2.22), we obtain

$$J_{\mathrm{L}}\frac{d^2\theta_{\mathrm{M}}(t)}{dt^2}\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}} + B_{\mathrm{L}}\frac{d\theta_{\mathrm{M}}(t)}{dt}\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}} + T_{\mathrm{L}}(t) = T_1(t)\frac{N_{\mathrm{L}}}{N_{\mathrm{M}}}$$

Multiplying the above equation by $N_{\mathrm{M}} / N_{\mathrm{L}}$, we obtain

$$J_{\mathrm{L}}\frac{d^2\theta_{\mathrm{M}}(t)}{dt^2}\left(\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}\right)^2 + B_{\mathrm{L}}\frac{d\theta_{\mathrm{M}}(t)}{dt}\left(\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}\right)^2 + T_{\mathrm{L}}(t)\left(\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}\right) = T_1(t) \tag{2.35}$$

Substituting the value for $T_1(t)$ from Eqn. (2.35) in Eqn. (2.21), we obtain

$$J_M \frac{d^2\theta_M(t)}{dt^2} + B_M \frac{d\theta_M(t)}{dt} + J_L \frac{d^2\theta_M(t)}{dt^2}\left(\frac{N_M}{N_L}\right)^2 + B_L \frac{d\theta_M(t)}{dt}\left(\frac{N_M}{N_L}\right)^2 + T_L(t)\left(\frac{N_M}{N_L}\right) = T_M(t)$$

$$\left(J_M + J_L\left(\frac{N_M}{N_L}\right)^2\right)\frac{d^2\theta_M(t)}{dt^2} + \left[B_M + B_L\left(\frac{N_M}{N_L}\right)^2\right]\frac{d\theta_M(t)}{dt} + T_L(t)\left(\frac{N_M}{N_L}\right) = T_M(t)$$

$$J_{eqM}\frac{d^2\theta_M(t)}{dt^2} + B_{eqM}\frac{d\theta_M(t)}{dt} + T_L(t)\left(\frac{N_M}{N_L}\right) = T_M(t) \tag{2.36}$$

where $J_{eqM} = J_M + J_L\left(\frac{N_M}{N_L}\right)^2$ = total equivalent moment of inertia referred to motor shaft

and $B_{eqM} = B_M + B_L\left(\frac{N_M}{N_L}\right)^2$ = total equivalent viscous friction coefficient referred to motor shaft.

Equation (2.36) is the torque equation of the gear train system referred to motor shaft.

Torque Equation Referred to Load Shaft:

From Eqn. (2.31), we can write

$$\alpha_M(t) = \alpha_L(t)\frac{N_L}{N_M}, \qquad \frac{d^2\theta_M(t)}{dt^2} = \frac{d^2\theta_L(t)}{dt^2}\frac{N_L}{N_M} \tag{2.37}$$

$$\omega_M(t) = \omega_L(t)\frac{N_L}{N_M}, \qquad \frac{d\theta_M(t)}{dt} = \frac{d\theta_L(t)}{dt}\frac{N_L}{N_M} \tag{2.38}$$

$$T_1(t) = T_2(t)\frac{N_M}{N_L} \tag{2.39}$$

Substituting the values of $\frac{d^2\theta_M(t)}{dt^2}, \frac{d\theta_M(t)}{dt}, T_1(t)$ from the above equations in Eqn. (2.21), we obtain

$$J_M \frac{d^2\theta_L(t)}{dt_2}\frac{N_L}{N_M} + B_M \frac{d\theta_L(t)}{dt}\frac{N_L}{N_M} + T_2(t)\frac{N_M}{N_L} = T_M(t) \tag{2.40}$$

Substituting the value of $T_2(t)$ from Eqn. (2.22) in Eqn. (2.40), we obtain

$$J_M \frac{d^2\theta_L(t)}{dt^2}\frac{N_L}{N_M} + \mathrm{B}_M \frac{d\theta_L(t)}{dt}\frac{N_L}{N_M} + \left(J_L \frac{d^2\theta_L(t)}{dt^2} + B_L \frac{d\theta_L(t)}{dt} + T_L(t)\right)\frac{N_M}{N_L} = T_M(t)$$

Multiplying the above equation by N_L / N_M, we obtain

$$J_M \frac{d^2\theta_L(t)}{dt^2}\left(\frac{N_L}{N_M}\right)^2 + B_M \frac{d\theta_L(t)}{dt}\left(\frac{N_L}{N_M}\right)^2 + J_L \frac{d^2\theta_L(t)}{dt^2} + B_L \frac{d\theta_L(t)}{dt} + T_L(t) = T_M(t)\left(\frac{N_L}{N_M}\right)$$

$$\left(J_M\left(\frac{N_L}{N_M}\right)^2 + J_L\right)\frac{d^2\theta_L(t)}{dt^2} + \left[B_M\left(\frac{N_L}{N_M}\right)^2 + B_L\right]\frac{d\theta_L(t)}{dt} + T_L(t) = T_M(t)\left(\frac{N_L}{N_M}\right)$$

$$J_{eqL}\frac{d^2\theta_L(t)}{dt^2} + B_{eqL}\frac{d\theta_L(t)}{dt} + T_L(t) = T_M(t)\left(\frac{N_L}{N_M}\right) \tag{2.41}$$

where $J_{eqL} = J_M\left(\frac{N_L}{N_M}\right)^2 + J_L$ = total equivalent moment of inertia referred to load shaft and

$B_{eqL} = B_M\left(\frac{N_L}{N_M}\right)^2 + B_L$ = Total equivalent viscous friction coefficient referred to load shaft.

Equation (2.41) is the torque equation of gear train systems referred to load shaft.

2.15.2 Multiple Gear Trains

The concept of equivalent inertia, friction and torque just described can also be extended to a multiple gear train system as shown in Fig. 2.45.

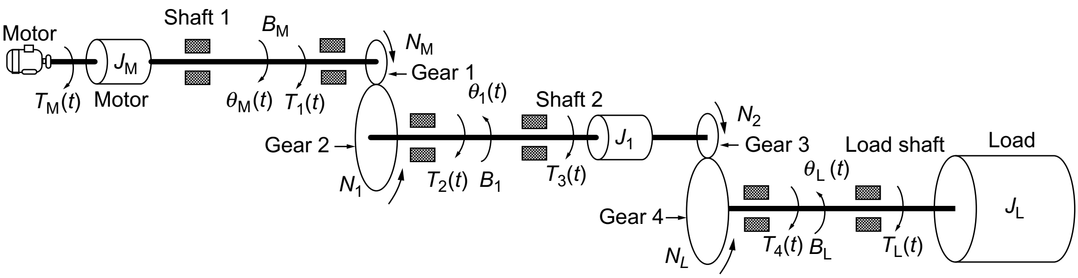


Fig. 2.45 | Multiple gear train system

The inertia J_L and B_L at the load shaft are referred to shaft 2 and the equivalent quantities are as follows:

$$J_{eq1} = J_1 + J_L\left(\frac{N_2}{N_L}\right)^2$$

$$B_{eq1} = B_1 + B_L\left(\frac{N_2}{N_L}\right)^2$$

The equivalent inertia and friction at shaft 2 are now referred to shaft 1 resulting

$$J_{\text{eqM}} = J_1 + J_{\text{eq1}}\left(\frac{N_{\text{M}}}{N_1}\right)^2$$

$$= J_{\text{M}} + J_1\left(\frac{N_{\text{M}}}{N_1}\right)^2 + J_2\left(\frac{N_{\text{M}}N_2}{N_1N_{\text{L}}}\right)^2$$

$$B_{\text{eqM}} = B_{\text{M}} + B_{\text{eq1}}\left(\frac{N_{\text{M}}}{N_1}\right)^2$$

$$= B_{\text{M}} + B_1\left(\frac{N_{\text{M}}}{N_1}\right)^2 + B_2\left(\frac{N_{\text{M}}N_2}{N_1N_{\text{L}}}\right)^2$$

Similarly, the inertia and friction at shaft 1 can be referred to the load shaft. Therefore, we have

$$J_{\text{eqL}} = J_{\text{L}} + J_1 + J_{\text{eq1}}\left(\frac{N_{\text{L}}}{N_2}\right)^2 + J_{\text{M}}\left(\frac{N_1N_{\text{L}}}{N_{\text{M}}N_2}\right)^2$$

$$B_{\text{eqL}} = B_{\text{L}} + B_1 + B_{\text{eq1}}\left(\frac{N_{\text{L}}}{N_2}\right)^2 + B_{\text{M}}\left(\frac{N_1N_{\text{L}}}{N_{\text{M}}N_2}\right)^2$$

It is noted that direction of each gear is opposite to the direction of previous gear in the given gear train system.

Example 2.10: A simple gear system is shown in Fig. E2.10. For the given gear system obtain (i) the ratio of diameters D_1 / D_2. of the gears, (ii) displacement of gear 2 if gear 1 is displaced by 80 rad, (iii) angular velocity of gear 2 given the angular velocity of gear 1 as 60 rad/sec, (iv) angular acceleration of gear 1 given the angular acceleration of gear2 as 8 rad/sec^2 and (v) if the torque of gear 1 is 10 N-m, determine the torque of gear 2.

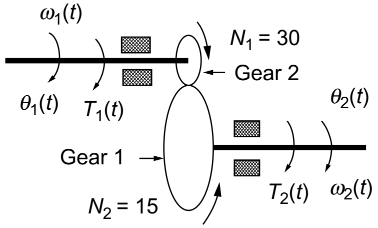


Fig. E2.10 | A simple gear system

Solution: Using Eqn. 2.31, we obtain the gear ratio as

$$\frac{N_{\mathrm{M}}}{N_{\mathrm{L}}}=\frac{r_{\mathrm{M}}}{r_{\mathrm{L}}}=\frac{T_1(t)}{T_2(t)}=\frac{\theta_{\mathrm{L}}(t)}{\theta_{\mathrm{M}}(t)}=\frac{\omega_{\mathrm{L}}(t)}{\omega_{\mathrm{M}}(t)}=\frac{\alpha_{\mathrm{L}}(t)}{\alpha_{\mathrm{M}}(t)}$$

For the given gear system, the subscripts M and L in the above equation can be changed into 1 and 2 respectively. Hence, the above equation can be modified as

$$\frac{N_1}{N_2}=\frac{r_1}{r_2}=\frac{T_1(t)}{T_2(t)}=\frac{\theta_2(t)}{\theta_1(t)}=\frac{\omega_2(t)}{\omega_1(t)}=\frac{\alpha_2(t)}{\alpha_1(t)}$$

where $N_1$and N_2 are number of turns in the gear 1and gear 2 respectively.

r_1 and r_2 are the radius of gear 1 and gear 2 respectively in m.

$T_1(t)$ and $T_2(t)$ are the torques of gear 1 and gear 2 respectively in N-m.

$\theta_1(t)$ and $\theta_2(t)$ are the angular displacements of gear 1 and gear 2 respectively in rad.

$\omega_1(t)$ and $\omega_2(t)$ are the angular velocities of gear 1 and gear 2 respectively in rad/sec.

$\alpha_1(t)$ and $\alpha_2(t)$ are the angular accelerations of gear 1 and gear 2 respectively in rad/sec^2.

From the given data, we have

$$\frac{N_1}{N_2}=\frac{30}{15}=2$$

(i) **Ratio of diameters of gears**
Using the above equation, we obtain

$$\frac{N_1}{N_2}=\frac{r_1}{r_2}=\frac{D_1}{D_2}$$

$$\frac{D_1}{D_2}=2$$

(ii) **Angular displacement of gear 2 $\theta_2(t)$**
Using the above equation, we obtain

$$\frac{N_1}{N_2}=\frac{\theta_2(t)}{\theta_1(t)}$$

$$\theta_2(t)=2\times\theta_1(t)=160 \text{ rad}$$

(iii) **Angular velocity of gear 2 $\omega_2(t)$**
Using the above equation, we obtain

$$\frac{N_1}{N_2}=\frac{\omega_2(t)}{\omega_1(t)}$$

$$\omega_2(t)=2\times\omega_1(t)=120 \text{ rad/sec}$$

(iv) **Angular acceleration of gear 1** $\alpha_1(t)$

Using the above equation, we obtain

$$\frac{N_1}{N_2} = \frac{\alpha_2(t)}{\alpha_1(t)}$$

$$\alpha_1(t) = \frac{\alpha_2(t)}{2} = 4 \text{ rad / sec}^2$$

(v) **Torque of gear 2** $T_2(t)$

Using the above equation, we obtain

$$\frac{N_1}{N_2} = \frac{T_1(t)}{T_2(t)}$$

$$T_2(t) = \frac{T_1(t)}{2} = 5 \text{ N-m}$$

Review Questions

1. What do you mean by an electromechanical system and why is it required?
2. What are the subsystems available under electromechanical systems?
3. What are the different laws used for governing an electromechanical system?
4. Explain hydraulic systems.
5. What are the advantages and disadvantages of hydraulic systems?
6. What are devices used in the hydraulic system?
7. Name some of the applications of hydraulic systems.
8. Explain pneumatic systems.
9. What are the basic elements of the pneumatic systems? Explain.
10. What are the advantages and disadvantages of pneumatic systems?
11. What are devices used in the pneumatic system?
12. Compare pneumatic systems with hydraulic systems.
13. Explain thermal systems.
14. What are the different ways by which a heat flows from one substance to another substance? Explain.
15. Explain the basic elements of thermal systems.
16. What do you mean by liquid-level system?
17. What are the basic elements of liquid-level system. Explain.
18. Tabulate the analogous quantities of different physical systems (hydraulic, thermal, pneumatic and liquid-level systems) with electrical systems.
19. With a neat block diagram, explain the control system components.
20. Explain the following terms:
    (i) Controller output as a percentage value
    (ii) Measured value as a percentage value
    (iii) Set point as a percentage value
    (iv) Error as a percentage value

21. What are the general classifications of a controller?
22. List the types of analog controllers.
23. Explain the digital controller with a neat block diagram.
24. With a neat block diagram, explain the fuzzy controller.
25. What are the advantages of digital controller?
26. Define electronic controllers.
27. How is the controller classified based on the control action performed by it?
28. Define ON–OFF controller.
29. Explain the different types of ON–OFF controller.
30. What do you mean by neutral zone in the ON–OFF controller?
31. Draw the electronic version of two position ON–OFF controller.
32. What are the applications of two position ON–OFF controller?
33. Draw the electronic version of three position ON–OFF controller.
34. With a neat block diagram, explain the proportional controller or P-controller.
35. Define proportional band of P-controller.
36. Explain the electronic P-controller using inverting and non-inverting amplifier.
37. What are the advantages and disadvantages of P-controller?
38. With a neat block diagram, explain the following controllers:
    (i) Integral controller or I-controller
    (ii) Derivative controller or D-controller
    (iii) Proportional–integral controller or PI-controller
    (iv) Proportional–derivative controller or PD-controller
    (v) Proportional–integral–derivative controller or PID-controller.
39. Explain the following electronic controllers:
    (i) I-controller using inverting amplifier
    (ii) D-controller using inverting amplifier
    (iii) PI-controller using inverting amplifier
    (iv) PD-controller using inverting amplifier
    (v) PID controller using inverting amplifier
40. What are the advantages and disadvantages of the following controllers?
    (i) I-controller
    (ii) D-controller
    (iii) PI-controller
    (iv) PD-controller
    (v) PID-controller
41. With a neat diagram, explain DC and AC potentiometer.
42. What are the characteristics of potentiometers?
43. Define power-handling capacity in potentiometer.
44. What are the applications of potentiometers?
45. Explain the concept of synchro.
46. Differentiate the characteristics of synchro transmitter and synchro control transformer.
47. With a neat diagram, explain synchro transmitter.

48. Explain synchro control transformer with a neat diagram.
49. Define electrical zero position of synchro control transformer and synchro transmitter.
50. Explain the application of synchro as synchro error detector.
51. Explain the concept, classification and features of servomotor.
52. Derive the transfer function of field- and armature-controlled DC motor.
53. What are the advantages and disadvantages of DC servomotor?
54. Explain the concept of AC servomotor and derive its transfer function.
55. Compare different characteristics of AC servomotor and DC servomotor.
56. Define tachogenerator. What are the different types of tachogenerator?
57. Derive the transfer function of DC tachogenerator. Also, mention its advantages and disadvantages.
58. Explain the concept of AC tachogenerator.
59. Explain the concept of stepper motor. Also, explain briefly about the different types of stepper motor.
60. Explain the different concepts of stepper motor operation.
61. What are the advantages and applications of stepper motor.
62. Explain gear train.
63. With a neat schematic diagram, explain the concept of single gear train and derive all the possible relations.
64. With a neat schematic diagram, explain the concept of multiple gear trains.

65. Consider two identical dashpots connected in series as shown in Fig. Q2.65. Obtain the transfer functions $\dfrac{X_3(s)}{X_1(s)}$ of the system.

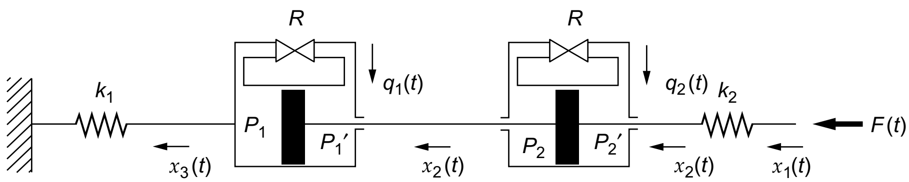


Fig. Q2.65

66. The pneumatic system in Fig. Q2.66 has two identical bellows. The outputs of the bellows (mechanical ends) are connected by a link from which the mechanical linear movement $x(t)$ is taken. The linear movement is proportional to the differential force of the bellows. The linear movement is zero if there exists a steady input pressure. Determine (i) the transfer function of the given system relating the input pressure $P_1(t)$ to linear displacement output $x(t)$ if the bellows stiffness is k_s and (ii) determine the steady-state value of $x(t)$ for a unit step change in input pressure.

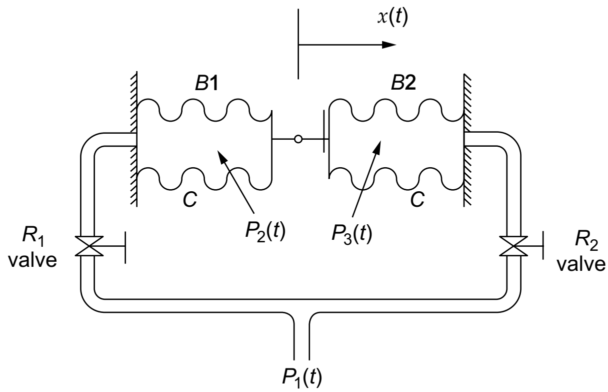


Fig. Q2.66

67. For the non-interacting liquid-level system shown in Fig. Q2.67, determine the transfer function of the system relating the inflow rate into the tank 1 and the liquid level in the tank 2.

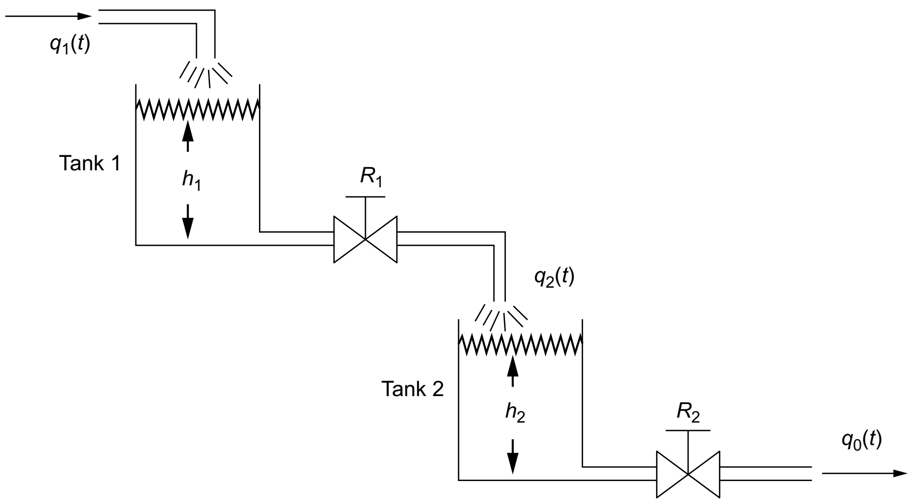


Fig. Q2.67

68. For the interacting liquid-level system shown in Fig. Q2.68(a), determine the transfer function of the system relating (i) the inflow rate $q(t)$ and the height of liquid in tank 2 $h_2(t)$, (ii) the inflow rate $q(t)$ and the outflow rate $q_2(t)$ and (iii) the inflow rate $q(t)$ and the height of liquid in tank 1 $h_1(t)$ when the inflow of liquid is at (a) first tank and (b) second tank.

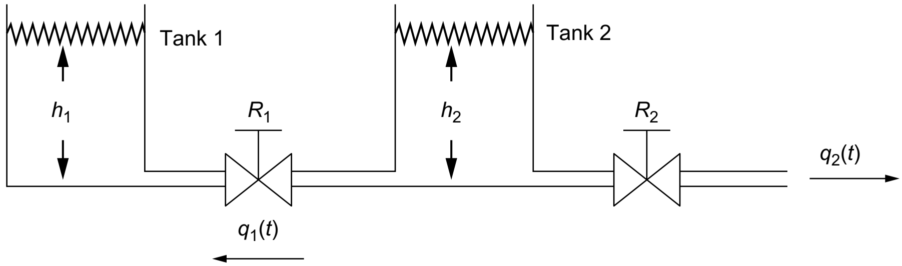


Fig. Q2.68

69. The composite gear system that consists of many gears is shown in Fig. Q2.69. Obtain the (i) angular displacement $\theta(t)$ of gear 4 and gear 6 if the angular displacement of gear 2 $\theta_2(t)$ is 10 **rad**, (ii) angular velocity $\omega(t)$ of gear 1 and gear 2 if the angular velocity of gear 6 $\omega_6(t)$ is 15 **rad / sec** and (iii) torque $T(t)$ of gear 3 and gear 7 if the torque of gear 1 $T_1(t)$ is 20 N-m

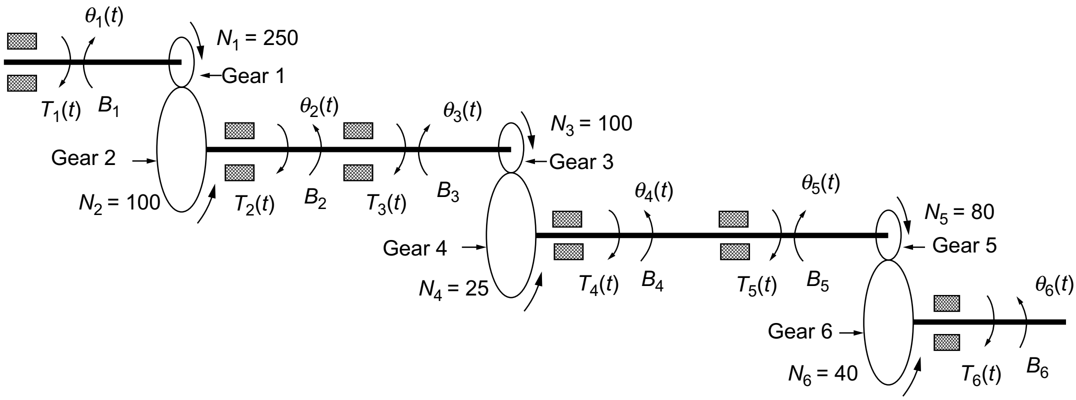


Fig. Q2.69

This Page is intentionally left blank.

3

BLOCK DIAGRAM REDUCTION TECHNIQUES

3.1 Introduction to Block Diagram

A block diagram is a diagrammatic representation of the cause-and-effect relationship between the input and output of a physical system represented by the flow of signals. It gives the relationship that exists between various components of a system. The open-loop and the closed-loop systems discussed in Chapters 1 and 2 can be represented by using the block diagram reduction techniques.

3.2 Open-Loop and Closed-Loop Systems Using Block Diagram

A system that cannot change its output in accordance to the change in input is known as a open-loop system. Such an open-loop system can be represented by using a block diagram as shown in Fig. 3.1.

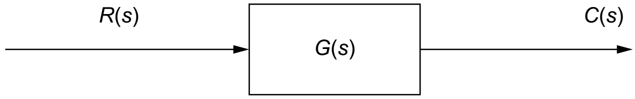


Fig. 3.1 | Open-loop system

A system that can change its output in accordance with change in input is known as a closed-loop system. This can be implemented by introducing a feedback path in a open-loop system and manipulating the input that is applied to the system. Such a closed-loop system can be represented by using a block diagram shown in Fig. 3.2.

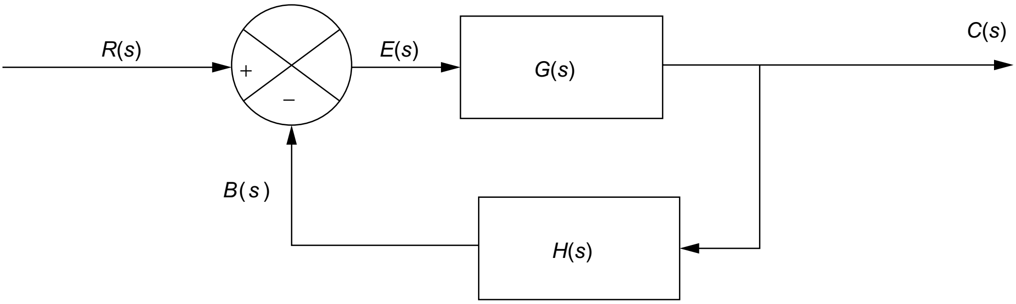


Fig. 3.2 | Closed-loop system

3.2.1 Advantages of Block Diagram Representation

The advantages of block diagram representation are:

(i) It facilitates easier representation of complex systems.
(ii) Calculation of transfer function by block diagram reduction techniques is easy.
(iii) Performance analysis of a complex system is simplified by determining its transfer function.
(iv) It facilitates easier access of individual elements in a system that is represented by a block diagram.
(v) It facilitates visualization of operation of the whole system by the flow of signals.

3.2.2 Disadvantages of Block Diagram Representation

The disadvantages of block diagram representation are:

(i) It is difficult to determine the actual composition of individual elements in a system.
(ii) Representation of a system using block diagram is not unique.
(iii) The main source of signal flow cannot be represented definitely in a block diagram.

3.3 Block Diagram Representation of Electrical System

The flow chart to represent any electrical system by means of a block diagram is shown in Fig. 3.3.

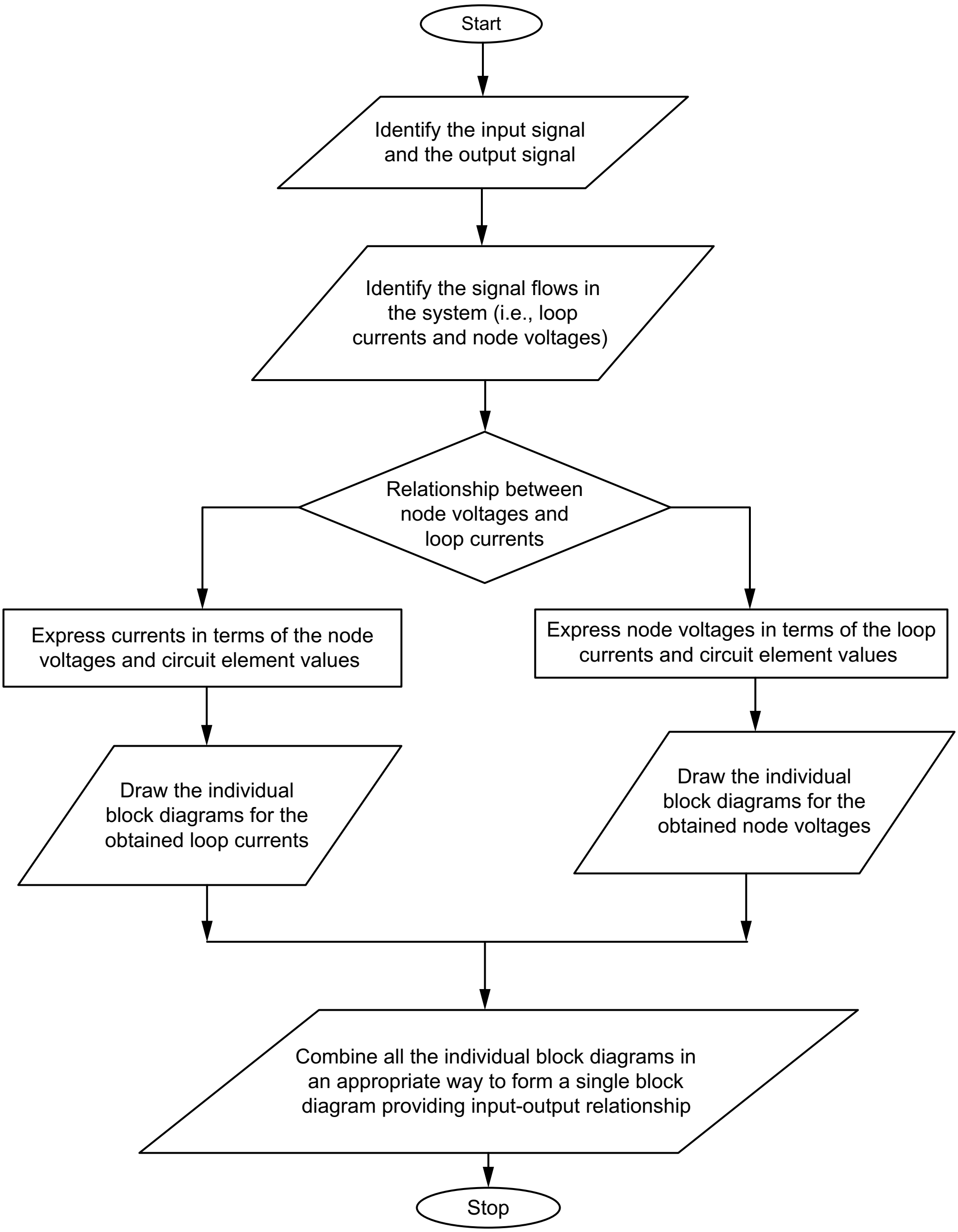


Fig. 3.3 | Flow chart for representing an electrical system by a block diagram

Example 3.1: Represent the electrical circuit shown in Fig. E3.1(a) by a block diagram.

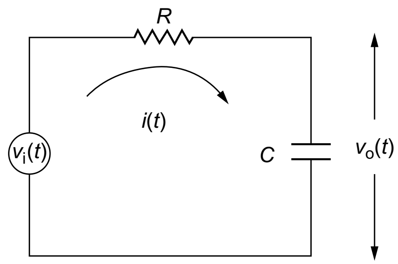


Fig. E3.1(a)

Solution: The current flowing through the electrical circuit shown in Fig. E3.1(a) is given by

$$i(t) = \frac{v_i(t) - v_o(t)}{R} \tag{1}$$

The output voltage across the capacitor C is determined by

$$v_o(t) = \frac{1}{C}\int i(t)dt \tag{2}$$

Individual block diagrams representing each of the above equations can be determined by examining the input and output of the respective equations. Then, these individual block diagrams can be combined in an appropriate way to represent the given electrical circuit.

To Determine the Individual Block Diagrams:

For Eqn. (1):

Taking Laplace transform of Eqn. (1), we obtain

$$I(s) = \frac{V_i(s) - V_o(s)}{R} \tag{3}$$

where the inputs are $V_i(s)$ and $V_o(s)$ and the output is $I(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.1(b).

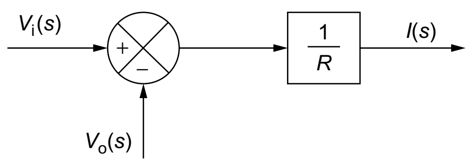


Fig. E3.1(b)

For Eqn. (2):

Taking Laplace transform of Eqn. (2), we obtain

$$V_o(s) = \frac{I(s)}{sC} \tag{4}$$

where the input is $I(s)$ and the output is $V_o(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.1(c).

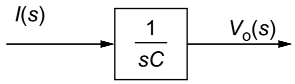


Fig. E3.1(c)

Thus, the overall block diagram for the given electrical circuit is obtained by interconnecting the block diagrams shown in Figs. E3.1(b) and (c). The overall block diagram of the given electrical circuit is shown in Fig. E3.1(d).

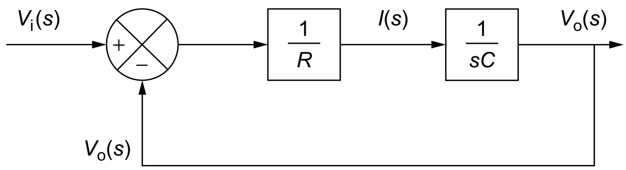


Fig. E3.1(d)

Example 3.2: Represent the electrical circuit shown in Fig. E3.2(a) by a block diagram.

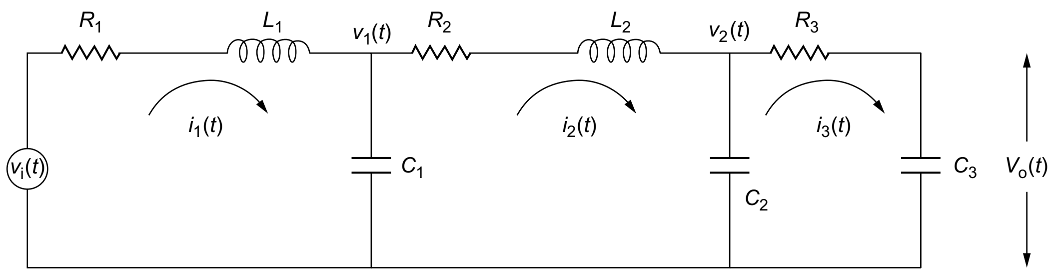


Fig. E3.2(a)

Solution: Resistors R_1, R_2, R_3 and inductances L_1, L_2 are in a series path. Capacitors C_1, C_2 and C_3 are in a shunt path. The loop currents indicated in the given electrical circuit are $i_1(t)$, $i_2(t)$ and $i_3(t)$. The node voltages for the given electrical circuit are $v_1(t)$ and $v_2(t)$. The input voltage is $v_i(t)$ and the output voltage across the capacitor C_3 is $v_o(t)$.

The current flowing through each element in series path is determined by using the node voltages.

The same current flows through resistor R_1 and inductor L_1. Hence, using the node voltages $v_i(t)$ and $v_1(t)$, the current $i_i(t)$ can be obtained as

$$R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} = v_i(t) - v_1(t) \tag{1}$$

Similarly, the current flowing through resistor R_2 and inductor L_2 can be obtained by using the node voltages $v_1(t)$ and $v_2(t)$ as

$$R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} = v_1(t) - v_2(t) \tag{2}$$

The current flowing through resistor R_3 can be obtained by using the node voltages $v_2(t)$ and $v_o(t)$ as

$$R_3 i_3(t) = v_2(t) - v_o(t) \tag{3}$$

The voltages across each element present in the shunt path can be determined by using the loop currents.

The voltage $v_1(t)$ across the capacitor C_1 is given by

$$v_1(t) = \frac{1}{C_1}\int (i_1(t) - i_2(t))\,dt \tag{4}$$

The voltage $v_2(t)$ across the capacitor C_2 is given by

$$v_2(t) = \frac{1}{C_2}\int (i_2(t) - i_3(t))\,dt \tag{5}$$

The voltage $v_o(t)$ across the capacitor C_3 is given by

$$v_o(t) = \frac{1}{C_3}\int i_3(t)\,dt \tag{6}$$

Individual block diagrams representing each of the above equations can be determined by examining the input and output of the respective equations. Then, these individual block diagrams can be combined in an appropriate way to represent the given electrical circuit.

To determine the individual block diagrams:

For Eqn. (1):

Taking Laplace transform of Eqn. (1), we obtain

$$(R_1 + sL_1)I_1(s) = V_i(s) - V_1(s)$$

$$I_1(s) = \frac{V_i(s) - V_1(s)}{(R_1 + sL_1)} \tag{7}$$

where the inputs are $V_i(s)$ and $V_1(s)$ and the output is $I_1(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(b).

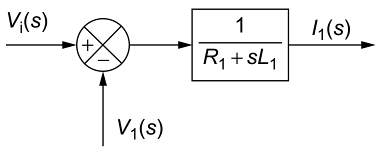


Fig. E3.2(b)

For Eqn. (2):

Taking Laplace transform of Eqn. (2), we obtain

$$(R_2 + sL_2)I_2(s) = V_1(s) - V_2(s)$$

$$I_2(s) = \frac{V_1(s) - V_2(s)}{(R_2 + sL_2)} \tag{8}$$

where the inputs are $V_1(s)$ and $V_2(s)$ and the output is $I_2(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(c).

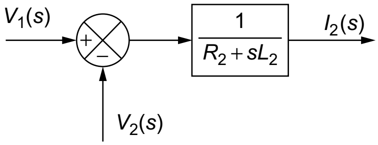


Fig. E3.2(c)

For Eqn. (3):

Taking Laplace transform of Eqn. (3), we obtain

$$R_3 I_3(s) = V_2(s) - V_o(s)$$

Therefore,
$$I_3(s) = \frac{V_2(s) - V_o(s)}{R_3} \tag{9}$$

where the inputs are $V_2(s)$ and $V_o(s)$ and the output is $I_3(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(d).

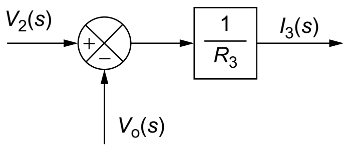


Fig. E3.2(d)

For Eqn. (4):

Taking Laplace transform of Eqn. (4), we obtain

$$V_1(s) = \frac{I_1(s) - I_2(s)}{sC_1} \tag{10}$$

where the inputs are $I_1(s)$ and $I_2(s)$ and the output is $V_1(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(e).

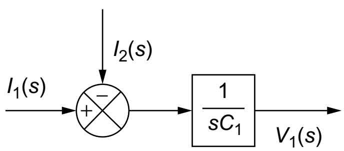


Fig. E3.2(e)

For Eqn. (5):
Taking Laplace transform of Eqn. (5), we obtain

$$V_2(s) = \frac{I_2(s) - I_3(s)}{sC_2} \tag{11}$$

where the inputs are $I_2(s)$ and $I_3(s)$ and the output is $V_2(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(f).

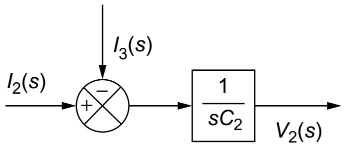


Fig. E3.2(f)

For Eqn. (6):
Taking Laplace transform of Eqn. (6), we obtain

$$V_o(s) = \frac{I_3(s)}{sC_3} \tag{12}$$

where the input is $I_3(s)$ and the output is $V_o(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.2(g).

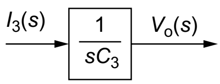


Fig. E3.2(g)

Hence, the overall block diagram for the given electrical circuit is obtained by interconnecting the block diagrams shown in Figs. E3.2(b) through (g). The overall block diagram of the given electrical circuit is shown in Fig. E3.2(h).

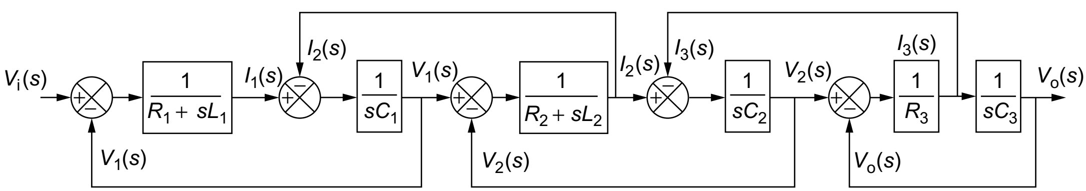


Fig. E3.2(h)

Example 3.3: Represent the armature-controlled DC motor shown in Fig. E3.3(a) by a block diagram.

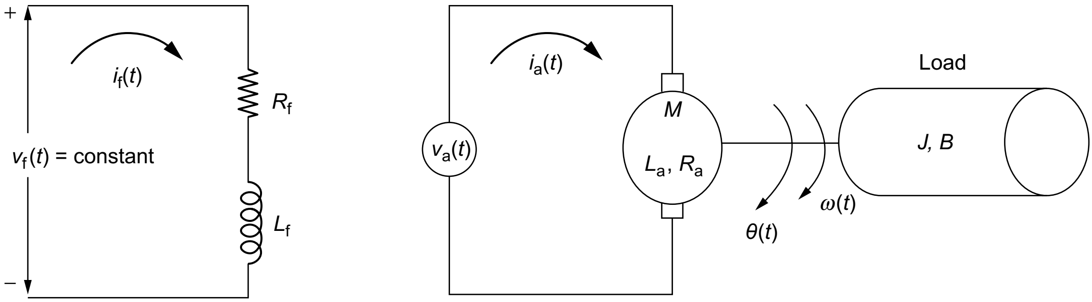


Fig. E3.3(a)

Solution: The equivalent electrical circuit of armature circuit is shown in Fig. E3.3(b).

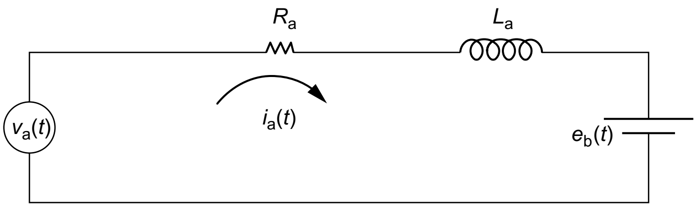


Fig. E3.3(b)

Applying Kirchhoff's voltage law, the differential equation can be obtained as

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_b(t) \tag{1}$$

The relationship between the torque applied to the load and armature current is given by

$$T(t) = K_t i_a(t) \tag{2}$$

The equivalent rotational mechanical system of the load connected to DC motor is shown in Fig. E3.3(c).

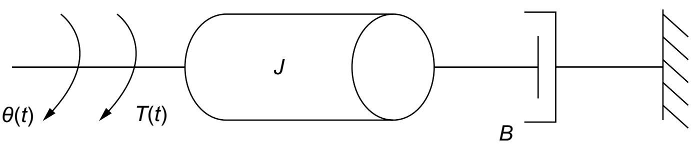


Fig. E3.3(c)

Using D'Alembert's principle, we have

$$T(t) = J\frac{d\omega(t)}{dt} + B\omega(t) \tag{3}$$

where $\omega(t)$ is the angular velocity.
The relation between back emf $e_b(t)$ and angular velocity $\omega(t)$ is given by

$$e_b(t) = K_b\omega(t) \tag{4}$$

The equation relating the angular velocity and angular displacement $\theta(t)$ is given by

$$\omega(t) = \frac{d\theta(t)}{dt} \tag{5}$$

Individual block diagrams representing each of the above equations can be determined by examining the input and output of the respective equations. Then, these individual block diagrams can be combined in an appropriate way to represent the armature-controlled DC motor.

To determine the individual block diagrams:
For Eqn. (1):
Taking Laplace transform of Eqn. (1), we obtain

$$V_a(s) = R_aI_a(s) + sL_aI_a(s) + E_b(s) \tag{6}$$

where the inputs are $V_a(s)$ and $E_b(s)$ and the output is $I_a(s)$. Hence, Eqn. (6) can be rewritten as

$$I_a(s) = \frac{V_a(s) - E_b(s)}{\left[R_a + sL_a\right]} \tag{7}$$

Hence, the block diagram for the above equation is shown in Fig. E3.3(d).

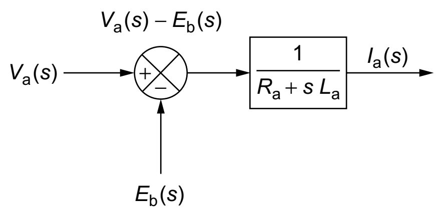


Fig. E3.3(d)

For Eqn. (2):
Taking Laplace transform of Eqn. (2), we obtain

$$T(s) = K_tI_a(s) \tag{8}$$

where the input is $I_a(s)$ and the output is $T(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.3(e).

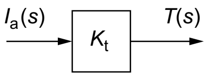


Fig. E3.3(e)

For Eqn. (3):
Taking Laplace transform of Eqn. (3), we obtain

$$T(s) = sJ\omega(s) + B\omega(s) \tag{9}$$

where the input is $T(s)$ and the output is $\omega(s)$. Hence Eqn. (9) can be rewritten as

$$\omega(s) = \frac{1}{sJ + B} T(s) \tag{10}$$

Thus, the block diagram for Eqn. (10) is shown in Fig. E3.3(f).

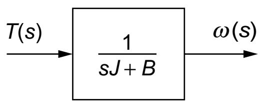


Fig. E3.3(f)

For Eqn. (4):
Taking Laplace transform of Eqn. (4), we obtain

$$E_b(s) = K_b\omega(s) \tag{11}$$

where the input is $\omega(s)$ and the output is $E_b(s)$. Hence, the block diagram for the above equation is shown in Fig. E3.3(g).

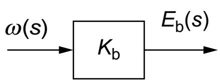


Fig. E3.3(g)

For Eqn. (5):
Taking Laplace transform of Eqn. (5), we obtain

$$\omega(s) = s\theta(s) \tag{12}$$

where the input is $\omega(s)$ and the output is $\theta(s)$. Hence, Eqn. (12) can be rewritten as

$$\theta(s) = \frac{1}{s}\omega(s) \tag{13}$$

Hence, the block diagram for the above equation is shown in Fig. E3.3(h).

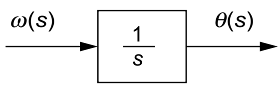


Fig. E3.3(h)

Hence, the overall block diagram of armature-controlled DC motor is obtained by interconnecting the block diagrams shown in Fig. E3.3(d) through (h). The overall block diagram of the armature-controlled DC motor is shown in Fig. E3.3(i).

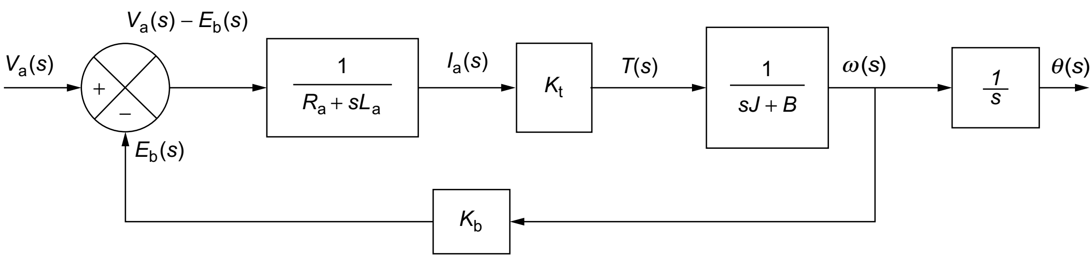


Fig. E3.3(i)

Example 3.4: Represent the field-controlled DC motor shown in Fig. E3.4(a) by a block diagram.

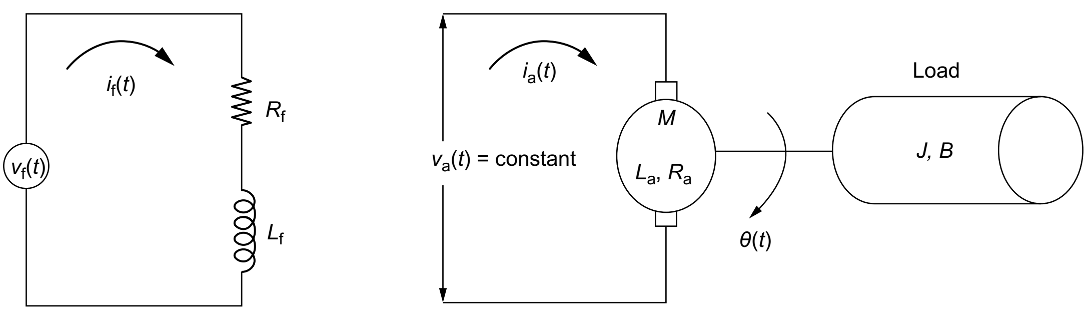


Fig. E3.4(a)

Solution: The equivalent circuit of electrical circuit is shown in Fig. E3.4(b).

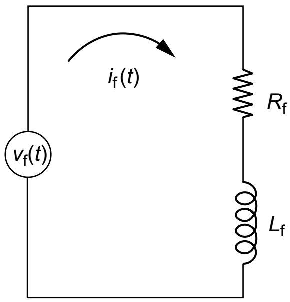


Fig. E3.4(b)

Applying Kirchhoff's voltage law, we have

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt} \tag{1}$$

The relationship between the torque applied to the load and the field current is given by

$$T(t) = K_{tf} i_f(t) \tag{2}$$

The equivalent rotational mechanical system of the load connected to a DC motor is shown in Fig. E3.4(c).

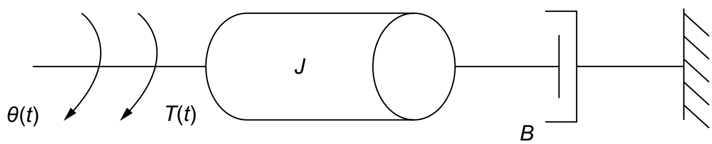


Fig. E3.4(c)

Using D'Alembert's principle, we have

$$T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} \tag{3}$$

Individual block diagrams representing each of the above equations can be determined by examining the input and output of the respective equations. Then, these individual block diagrams can be combined in an appropriate way to represent the field-controlled DC motor.

To determine the individual block diagrams:

For Eqn. (1):

Taking Laplace transform of Eqn. (1), we obtain

$$V_f(s) = R_f I_f(s) + sL_f I_f(s) \tag{4}$$

In the above equation, the input is $V_f(s)$ and the output is $I_f(s)$. Hence, the above equation can be rewritten as

$$I_f(s) = \frac{1}{R_f + sL_f} V_f(s) \tag{5}$$

The block diagram representing the above equation is shown in Fig. E3.4(d).

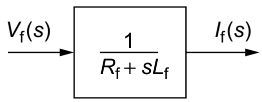


Fig. E3.4(d)

For Eqn. (2):
Taking Laplace transform of Eqn. (2), we obtain

$$T(s) = K_{tf} I_f(s) \tag{6}$$

In the above equation, $I_f(s)$ is the input and $T(s)$ is the output. Hence, the block diagram for the above equation is shown in Fig. E3.4(e).

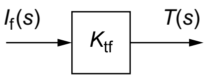


Fig. E3.4(e)

For Eqn. (3):
Taking Laplace transform of Eqn. (3), we obtain

$$T(s) = s^2 J \theta(s) + sB\theta(s) \tag{7}$$

where $T(s)$ is the input and $\theta(s)$ is the output. Hence, the above equation can be rewritten as

$$\theta(s) = \frac{1}{s^2 J + sB} T(s) \tag{8}$$

The block diagram representing the above equation is shown in Fig. E3.4(f).

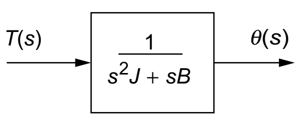


Fig. E3.4(f)

Hence, the overall block diagram of field-controlled DC motor is obtained by interconnecting the block diagrams shown in Figs. E3.4(d) through (f). The overall block diagram of the field-controlled DC motor is shown in Fig. E3.4(g).

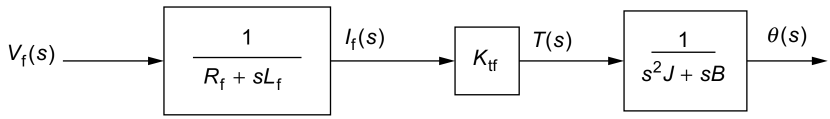


Fig. E3.4(g) | Block diagram of field-controlled DC motor

3.4 Block Diagram Reduction

3.4.1 Need for Block Diagram Reduction

In any complex real-time system, many individual sub-systems exist which can be represented by means of individual blocks. The individual blocks of each sub-system shown in Fig. 3.4 are combined to form the block diagram for the system. These individual blocks contain the transfer function of each sub-system.

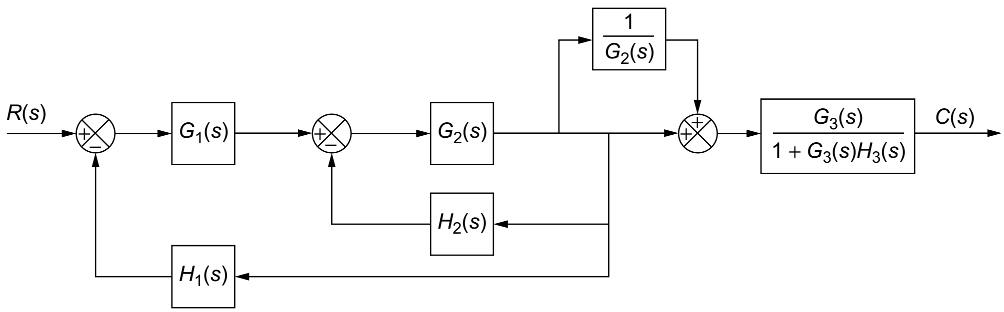


Fig. 3.4 | Block diagram of complex system

After applying certain rules to the block diagram shown in Fig. 3.4, the complex block diagram can be reduced to a simpler one as shown in Fig. 3.5. Thus, from the reduced form of the block diagram, the transfer function of the complex system can be determined.

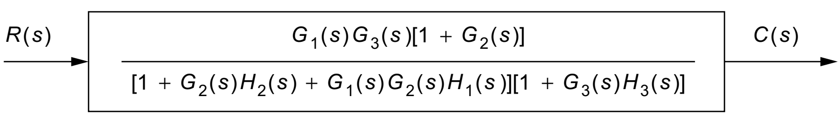


Fig. 3.5 | Simplified block diagram

3.4.2 Block Diagram Algebra

Before knowing the rules in reducing the complex block diagram to a simple block diagram, it is necessary to know some terminologies related to the block diagram. Consider a simple block diagram as shown in Fig. 3.6.

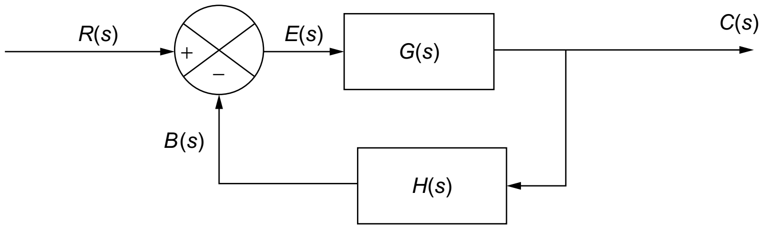


Fig. 3.6 | Block diagram

Block

The individual elements of a system can be represented by using blocks, which processes the input signal applied to it and produces an output. The blocks can either be in the forward path or in the feedback path. The blocks for the given block diagram are represented by dotted lines as shown in Fig. 3.7.

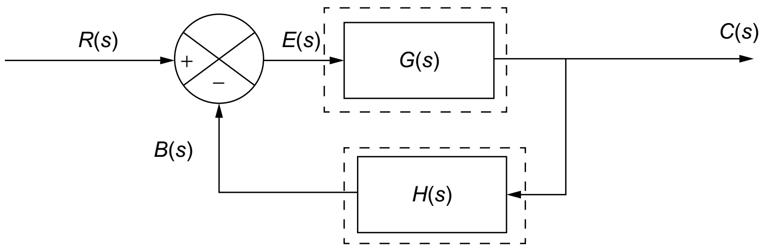


Fig. 3.7 | Blocks in the block diagram

Input Signal

A solid line with an arrow at the end pointing towards any block/summing point in the block diagram representation of a system is known as an input signal. A schematic diagram representing one of the input signals is represented by dotted lines in Fig. 3.8.

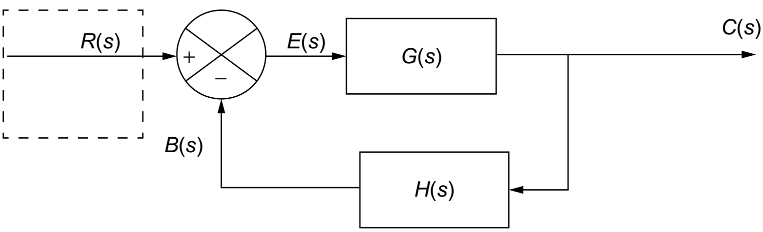


Fig. 3.8 | Input signal

Forward Path

The flow of signal from the input to output represents the forward path. The representation of the forward path for the block diagram shown in Fig. 3.6 is represented by dotted lines in Fig. 3.9.

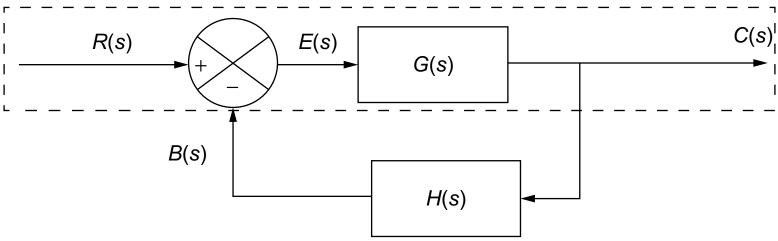


Fig. 3.9 | Forward path

Feedback Path

The flow of signal from the output to input represents the feedback path. The representation of the feedback path for the block diagram shown in Fig. 3.6 is indicated by dotted lines in Fig. 3.10.

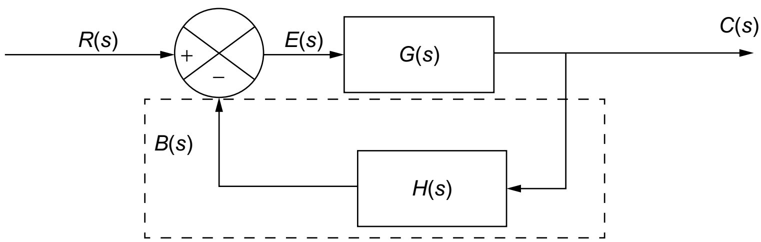


Fig. 3.10 | Feedback path

Feedback Signal

The signal in the feedback path is known as the feedback signal. For the block diagram given in Fig. 3.6, the feedback signal is denoted by $B(s)$.

Error Signal

The signal obtained by the algebraic sum/difference of any two signals is known as the error signal. In the block diagram shown in Fig. 3.6, the error signal is represented as E(s), which is the algebraic difference of a source input signal and the feedback signal. The error signal is mathematically expressed as

$$E(s) = R(s) - B(s)$$

Summing Point

A block/an element at which two or more signals are either added or subtracted is known as the summing point. The diagram denoting the summing point for the block diagram given in Fig. 3.6 is represented by dotted lines in Fig. 3.11.

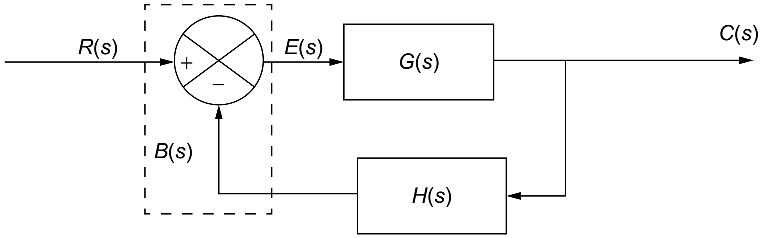


Fig. 3.11 | Summing point

Output Signal

The signal that comes out from each block/summing point in any block diagram representing a system is known as the output signal. In the block diagram shown in Fig. 3.6, the output signals are C(s) and $B(s)$.

Take-Off Point

The point at which a signal is applied to two or more blocks is known as the take-off point. The signal might either be an input signal or an output signal. The take-off point for the block diagram shown in Fig. 3.6 is represented by dotted lines in Fig. 3.12. In this case, the output signal is taken as the regulated output and also given as the input to the block in the feedback path.

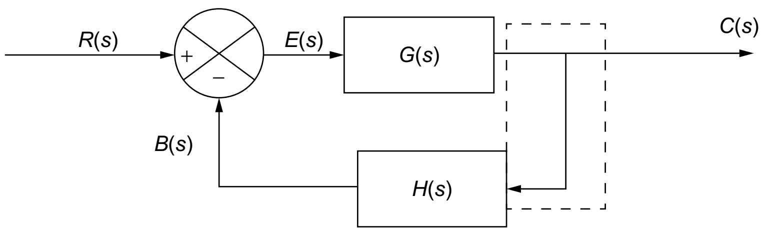


Fig. 3.12 | Take-off point

3.4.3 Rules for Block Diagram Reduction

Any complex block diagram representing a system can be reduced to a simple block diagram for determining the transfer function by applying certain rules. The rules that can be used in simplifying the given complex block diagram are as explained below. Proof for the rules is obtained by determining the output of the original block diagram and its equivalent block diagram.

(i) Cascading the blocks in series:

When there are two or more blocks in series without any take-off/summing point in between them, then the individual transfer function of each block is multiplied to form a single block. The original block diagram and its equivalent block diagram are shown in Figs. 3.13(a) and (b) respectively.

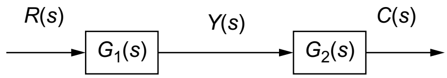


Fig. 3.13(a)

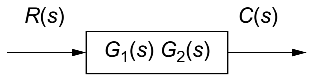


Fig. 3.13(b)

Proof:

The intermediate output $Y(s)$ is given by

$Y(s) = G_1(s)R(s)$

The total output $C(s)$ is given by

$C(s) = G_2(s)Y(s)$

Using the above two equations, we obtain

$C(s) = G_2(s)G_1(s)R(s)$ (3.1)

The output $C(s)$ is given by

$C(s) = G_2(s)G_1(s)R(s)$ (3.2)

From Eqs. (3.1) and (3.2), it is clear that the output obtained after cascading the blocks in series is similar to the original block diagram output.

(ii) Cascading the blocks in parallel:

When there are two or more blocks in parallel in the forward path, then the individual transfer function of each block is added to form a single block. The original block diagram and its equivalent block diagram are shown in Figs. 3.14(a) and (b).

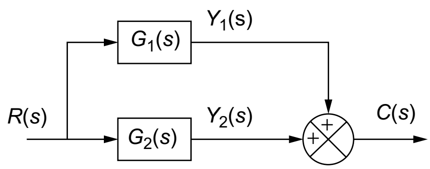


Fig. 3.14(a)

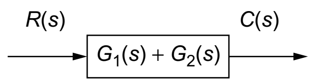


Fig. 3.14(b)

Proof:

The intermediate outputs $Y_1(s)$ and $Y_2(s)$ are given by

$Y_1(s) = G_1(s)R(s)$

$Y_2(s) = G_2(s)R(s)$

The total output $C(s)$ is given by

$C(s) = Y_1(s) + Y_2(s)$

Using the above equations, we obtain

$C(s) = G_2(s)\ R(s) + G_1(s)R(s)$

$C(s) = (G_2(s) + G_1(s))R(s)$ (3.3)

The output $C(s)$ is given by

$C(s) = (G_2(s) + G_1(s))\ R(s)$ (3.4)

From Eqs. (3.3) and (3.4), it is clear that the output obtained after cascading the blocks in parallel in the forward path is similar to the original block diagram output.

(iii) Reduction of feedback loop:

When there exists a feedback loop in a system, the feedback loop can be reduced to a single block. The feedback loop can either be negative or positive. The blocks representing the feedback loop and its equivalent block diagram are shown in Figs. 3.15(a) and (b).

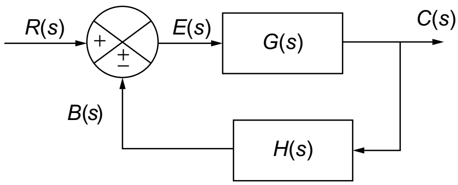


Fig. 3.15(a)

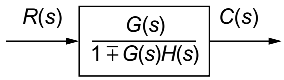


Fig. 3.15(b)

Proof:

The intermediate outputs E(s) and $B(s)$ are given by

$E(s) = R(s) \pm B(s)$

$B(s) \ = H(s)C(s)$

The total output $C(s)$ is given by

$C(s) = G(s)E(s)$

Substituting $E(s)$ in the above equation, we obtain

$C(s) = G(s)(R(s) \pm B(s))$

$C(s) = G(s)\ R(s) \pm G(s)\ \ B(s)$

Substituting $B(s)$ in the above equation, we obtain

$C(s) = G(s)\ R(s) \pm G(s)(H(s)C(s))$

$C(s) \mp G(s)H(s)C(s) = G(s)R(s)$

$C(s)(1 \mp G(s)H(s)) = G(s)R(s)$

Hence, the output $C(s)$ is given by

$$C(s) = \frac{G(s)\ R(s)}{1 \mp G(s)H(s)} \qquad (3.5)$$

The output $C(s)$ is given by

$$C(s) = \frac{G(s)\ R(s)}{1 \mp G(s)H(s)} \qquad (3.6)$$

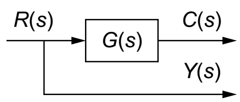
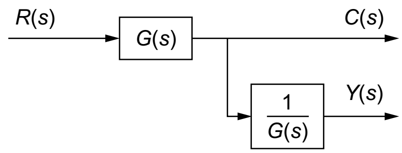
From Eqs. (3.5) and (3.6), it is clear that the output obtained after reducing the feedback loop is similar to the original feedback loop output.

(iv) Moving the take-off point after/before a block:

The take-off point that exists between the blocks can be moved either before the block or after the block.

(a) After a block

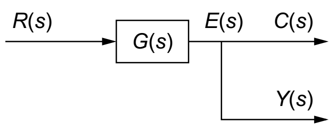
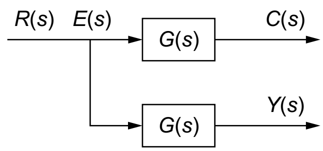
The changes to be made in the block diagram in moving the take-off point after the block is shown in Fig. 3.16(b). The original block diagram is shown in Fig. 3.16(a).

 **Fig. 3.16(a)**	 **Fig. 3.16(b)**
Proof: The first output $C(s) = R(s)G(s)$ The second output $Y(s) = R(s)$ (3.7)	The first output $C(s) = R(s)G(s)$ The second output $Y(s) = (G(s)R(s))\dfrac{1}{G(s)} = R(s)$ (3.8)

From Eqs. (3.7) and (3.8), it is clear that the output obtained after shifting the take-off point after a block is similar to the original block diagram output.

(b) Before a block

The changes to be made in the block diagram in moving the take-off point before the block is shown in Fig. 3.17(b). The original block diagram is shown in Fig. 3.17(a).

 **Fig. 3.17(a)**	 **Fig. 3.17(b)**
Proof: The output at node $E(s) = G(s)R(s)$ The first output $C(s) = E(s) = G(s)\,R(s)$ The second output $Y(s) = E(s) = G(s)\,R(s)$	The output at node $E(s) = R(s)$ The first output $C(s) = G(s)E(s) = G(s)\,R(s)$ The second output $Y(s) = G(s)E(s) = G(s)\,R(s)$

From the above proof, it is clear that the output obtained after shifting the take-off point before a block is similar to the original block diagram output.

(v) Moving the summing point after/before a block:
The summing point that adds/subtracts two or more signals can be moved either after or before a block.

(a) After a block
The changes to be made in the block diagram in moving the summing point after the block is shown in Fig. 3.18(b). The original block diagram is shown in Fig. 3.18(a).

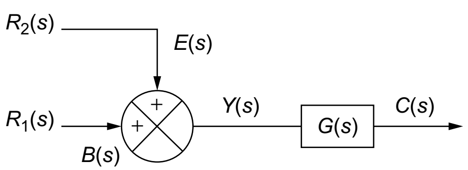


Fig. 3.18(a)

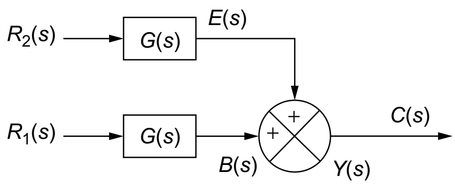


Fig. 3.18(b)

Proof:

Inputs to summing point are:

$E(s) = R_2(s)$

$B(s) = R_1(s)$

The output of the summing point

$$Y(s) = E(s) + B(s)$$
$$= R_2(s) + R_1(s)$$

The output $C(s) = G(s)\ Y(s)$

$$= G(s)(R_1(s) + R_2(s)) \quad (3.9)$$

Inputs to summing point are:

$E(s) = R_2(s)\ G(s)$

$B(s) = R_1(s)\ G(s)$

The output of the summing point

$$Y(s) = E(s) + B(s)$$
$$= (R_2(s) + R_1(s))\ G(s)$$

The output $C(s) = Y(s)$

$$= G(s)(R_1(s) + R_2(s)) \quad (3.10)$$

From Eqs. (3.9) and (3.10), it is clear that the output obtained after shifting the summing point after a block is similar to the original block diagram output.

(b) Before a block
The changes to be made in the block diagram in moving the summing point before the block is shown in Fig. 3.19(b). The original block diagram is shown in Fig. 3.19(a).

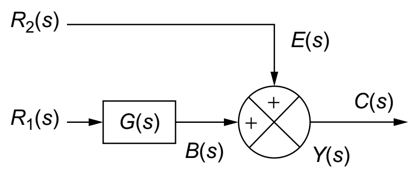


Fig. 3.19(a)

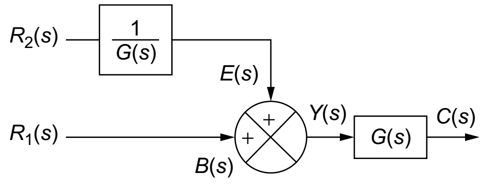


Fig. 3.19(b)

Proof:

Inputs to summing point are:

$E(s) = R_2(s)$

$B(s) = R_1(s)\ G(s)$

The output of the summing point

$Y(s) = E(s) + B(s)$

$\quad = R_2(s) + R_1(s)\ G(s)$

The output $C(s) = Y(s)$

$$= G(s) R_1(s) + R_2(s) \qquad (3.11)$$

Inputs to summing point are:

$E(s) = \dfrac{R_2(s)}{G(s)}$

$B(s) = R_1(s)$

The output of the summing point

$Y(s) = E(s) + B(s)$

$\quad = \dfrac{R_2(s)}{G(s)} + R_1(s)$

The output $C(s) = Y(s)\ G(s)$

$$= G(s)\ R_1(s) + R_2(s) \qquad (3.12)$$

From Eqs. (3.11) and (3.12), it is clear that the output obtained after shifting the summing point before a block is similar to the original block diagram output.

(vi) Splitting/joining a summing point:

The summing point that adds/subtracts two or more signals can be split into two summing points, or two summing points can be joined together to have a single summing point.

(a) Splitting the summing points

A summing point that adds/subtracts three signals can be split into two summing points. The block diagram explaining the concept is shown in Figs. 3.20 (a) and (b).

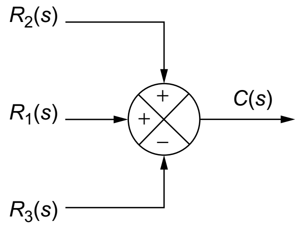


Fig. 3.20(a)

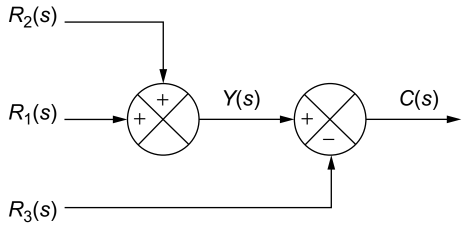


Fig. 3.20(b)

Proof: The output of the summing point: $C(s) = R_1(s) + R_2(s) - R_3(s)$ (3.13)	The intermediate output $Y(s) = R_1(s) + R_2(s)$ The output of the summing point: $C(s) = Y(s) - R_3(s) = R_1(s) + R_2(s) - R_3(s)$ (3.14)

From Eqs. (3.13) and (3.14), it is clear that the output obtained after splitting the summing point is same as the output of the summing point without splitting.

(b) Joining the summing points

Two or more summing points present in the block diagram can be joined together to form a single summing point. The block diagram explaining the concept is shown in Figs. 3.21(a) and (b).

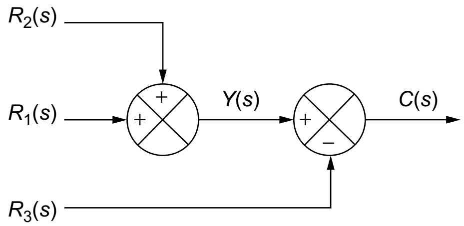


Fig. 3.21(a)

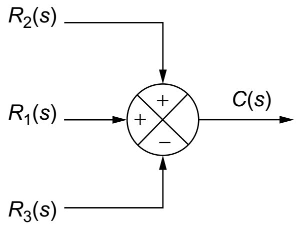


Fig. 3.21(b)

Proof: The intermediate output $Y(s) = R_1(s) + R_2(s)$ The output of the summing point $C(s) = Y(s) - R_3(s)$ $= R_1(s) + R_2(s) - R_3(s)$ (3.14)	The output of the summing point $C(s) = R_1(s) + R_2(s) - R_3(s)$ (3.15)

From Eqs. (3.14) and (3.15), it is clear that the output obtained after joining the summing point is same as the output of the summing point without joining.

(vii) Moving a take-off point after/before a summing point:

The take-off point that exists in the block diagram can be moved after or before a summing point without changing the output of the system. The block diagram explaining the concept is shown in Figs. 3.22 and 3.23.

(a) After summing point

The changes to be done in the block diagram in shifting the take-off point after a summing point is explained in Figs. 3.22(a) and (b).

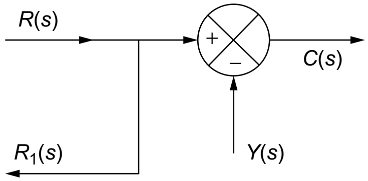


Fig. 3.22(a)

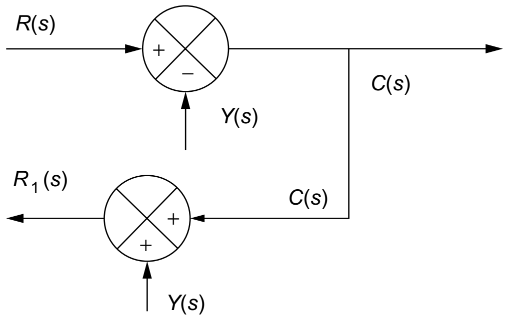


Fig. 3.22(b)

Proof:

The output $C(s) = R(s) - Y(s)$

The output of the take-off point

$R_1(s) = R(s)$ (3.16)

The output $C(s) = R(s) - Y(s)$

The output of the take-off point

$R_1(s) = C(s) + Y(s)$

$= R(s) - Y(s) + Y(s) = R(s)$ (3.17)

From Eqs. (3.16) and (3.17), the output of the take-off point is same even though the take-off point is shifted after a summing point.

(b) Before summing point

The changes to be done in the block diagram in shifting the take-off point before a summing point is explained in Figs. 3.23(a) and (b).

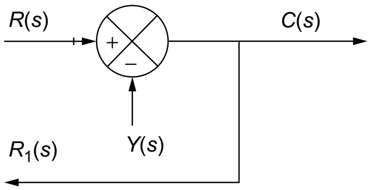


Fig. 3.23(a)

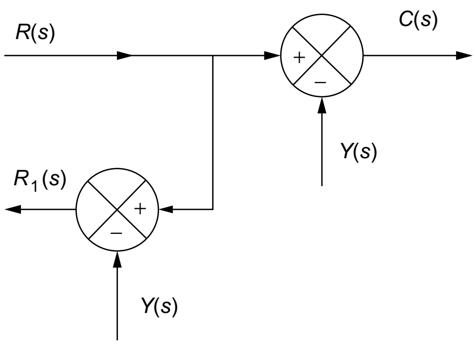


Fig. 3.23(b)

Proof: The output $C(s) = R(s) - Y(s)$ The output of the take-off point $R_1(s) = C(s)$ $= R(s) - Y(s)$ (3.18)	The output $C(s) = R(s) - Y(s)$ The output of the take-off point $R_1(s) = R(s) - Y(s)$ (3.19)

From Eqs. (3.18) and (3.19), the output of the take-off point is same even though the take-off point is shifted before a summing point.

(viii) Exchanging two or more summing points:
Two or more summing points present in the block diagram can be interchanged without changing the output of the block diagram. The block diagram explaining the concept of exchanging the summing point is shown in Figs. 3.24(a) and (b).

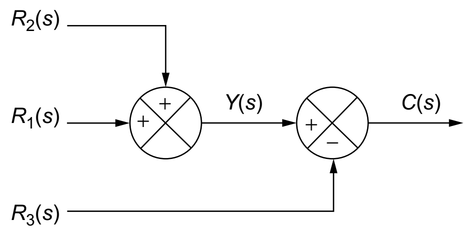


Fig. 3.24(a)

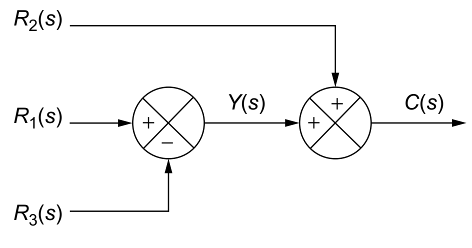


Fig. 3.24(b)

Proof: The intermediate output $Y(s) = R_1(s) + R_2(s)$ The output of the summing point $C(s) = Y(s) - R_3(s)$ $= R_1(s) + R_2(s) - R_3(s)$ (3.20)	The intermediate output $Y(s) = R_1(s) - R_3(s)$ The output of the summing point $C(s) = Y(s) + R_2(s)$ $= R_1(s) + R_2(s) - R_3(s)$ (3.21)

From Eqs. (3.20) and (3.21), it is clear that the output obtained after exchanging the summing points is same as the output of the summing point before exchanging.

3.4.4 Block Diagram Reduction for Complex Systems

The flow chart for reducing the complex block diagram to a simplified block diagram is shown in Fig. 3.25.

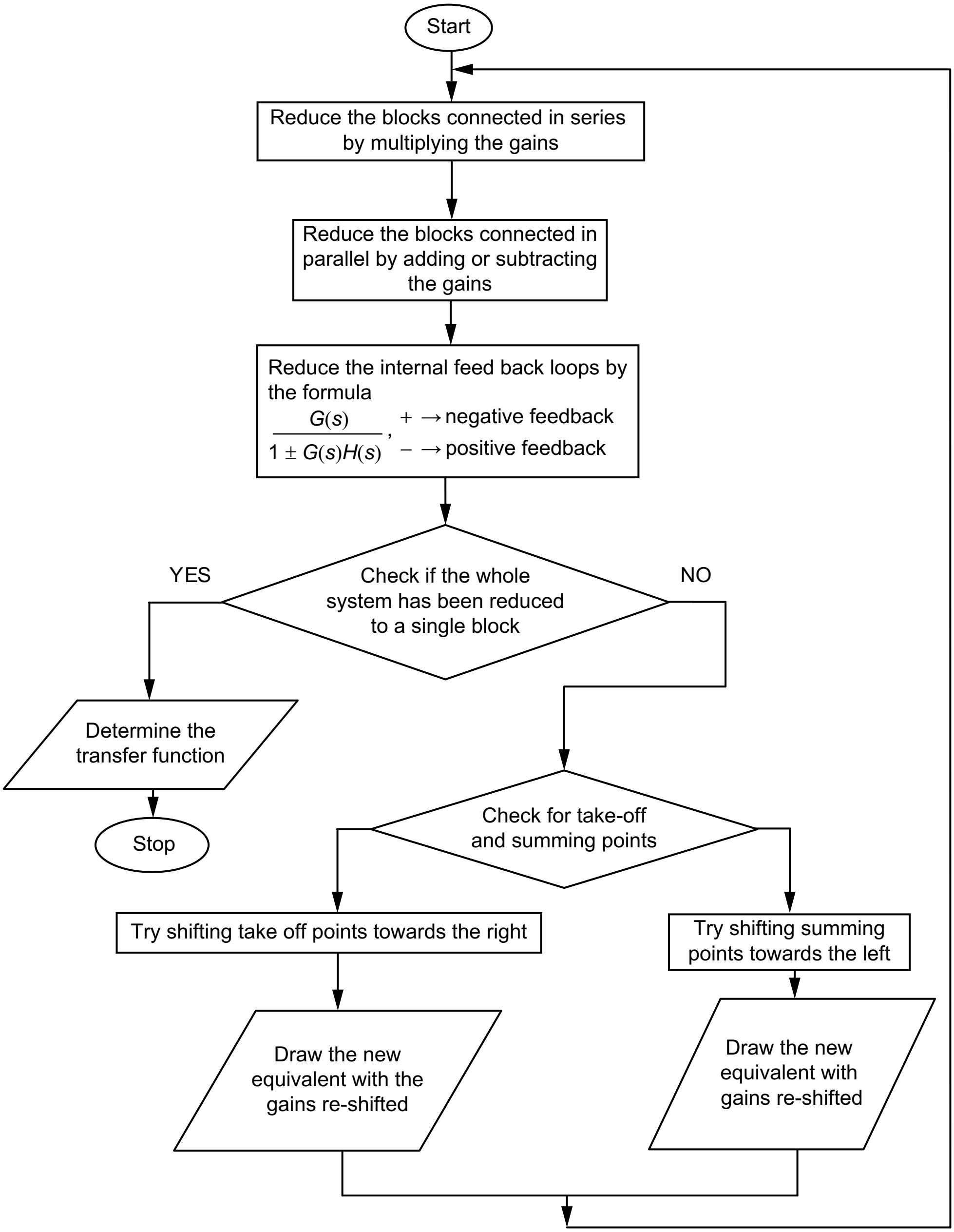


Fig. 3.25 | Block diagram reduction flow chart

Example 3.5: For the block diagram of the system shown in Fig. E3.5(a), determine the transfer function using block diagram reduction technique.

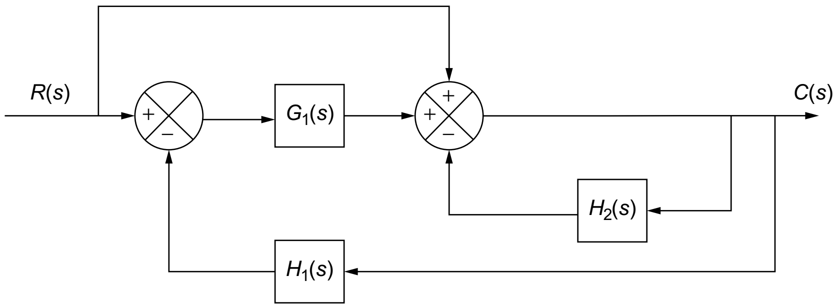


Fig. E3.5(a)

Solution:

Step 1: The summing point three input signals as represented by dotted lines in Fig. E3.5(b) can be separated by having the feedback signal separately and others separately. The resultant block diagram is shown in Fig. E3.5(c).

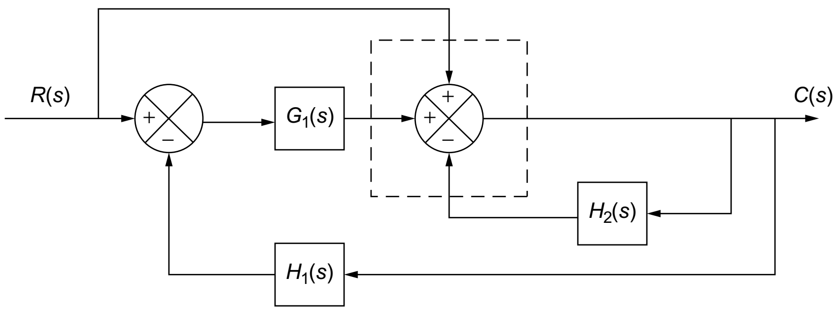


Fig. E3.5(b)

Step 2: The feedback path as represented by dotted lines in Fig. E3.5(c) can be reduced to a single block as shown by dotted lines in Fig. E3.5(d).

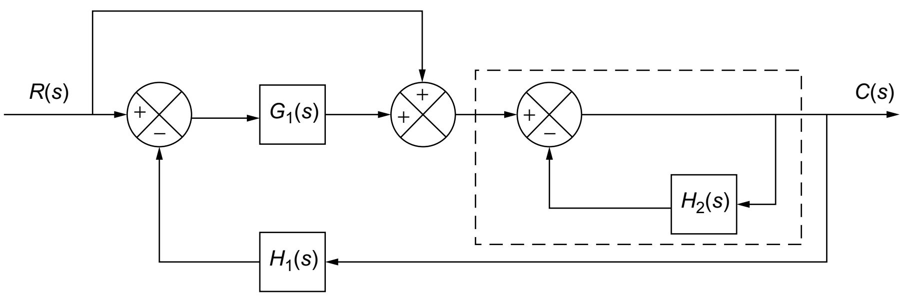


Fig. E3.5(c)

Step 3: Now, there exist two cases. One, the take-off point from the signal $R(s)$ can be moved after summing point and the other is the summing point that exists before the block $G_1(s)$ can be moved after the block $G_1(s)$. But interchanging the summing point and take-off point will make the reduction technique more complex. Therefore, the summing point as represented by dotted lines in Fig. E3.5(d) is moved after the block and the resultant block diagram is shown in Fig. E3.5(e).

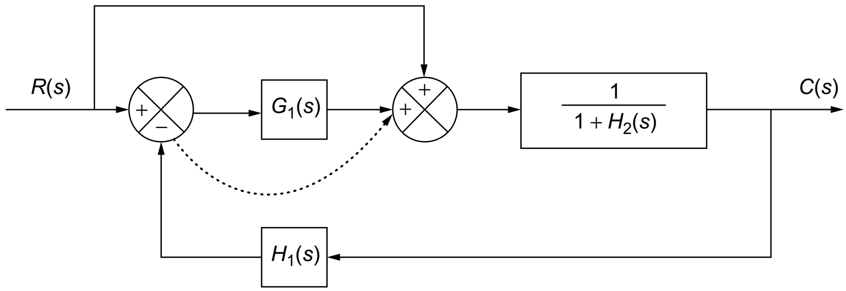


Fig. E3.5(d)

Step 4: The two summing points in series can be interchanged and also the blocks in series as represented by dotted lines in Fig. E3.5(e) can be reduced so that the resultant block diagram as represented in Fig. E3.5(f) is simple.

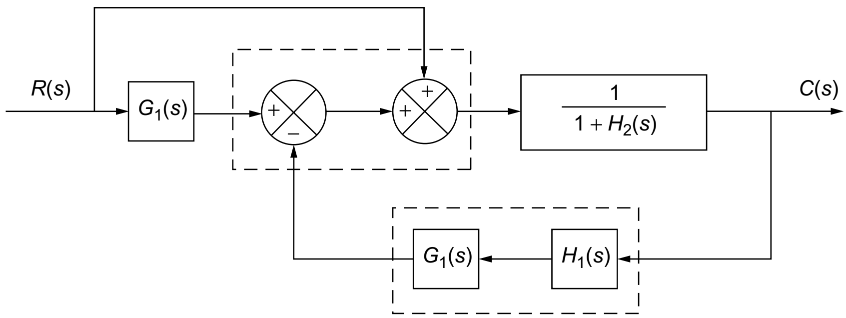


Fig. E3.5(e)

Step 5: The feedback path and the blocks in parallel as represented by dotted lines in Fig. E3.5(f) can be simplified and the resultant block diagram is shown in Fig. E3.5(g).

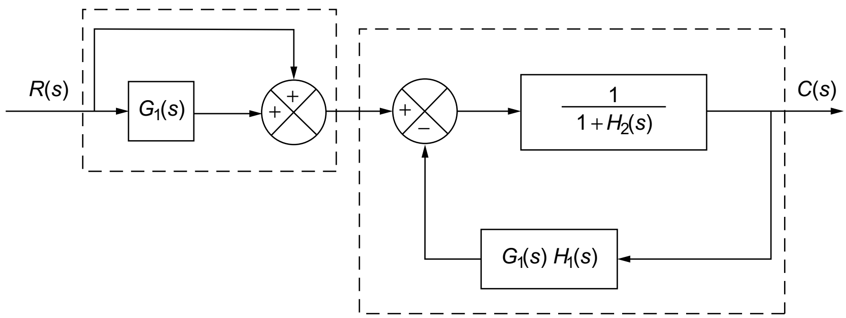


Fig. E3.5(f)

The resultant of the feedback loop as represented by dotted lines in Fig. E3.5(f) is

$$\frac{\frac{1}{1+H_2(s)}}{1+\frac{1}{1+H_2(s)}G_1(s)H_1(s)} = \frac{1}{1+H_2(s)+G_1(s)H_1(s)}$$

Step 6: The blocks in series as represented by dotted lines in Fig. E3.5(g) can be combined to get the transfer function for the block diagram shown in Fig. E3.5(a).

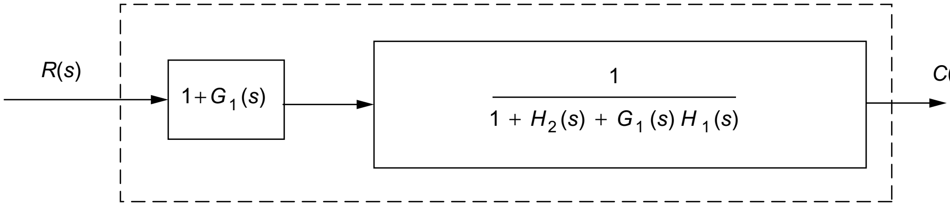


Fig. E3.5(g)

Hence, the overall transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{[G_1(s)+1]}{1+H_2(s)+G_1(s)H_1(s)}$$

Example 3.6: **For the block diagram of the system shown in Fig. E3.6(a), determine the closed-loop transfer function using the block diagram reduction technique.**

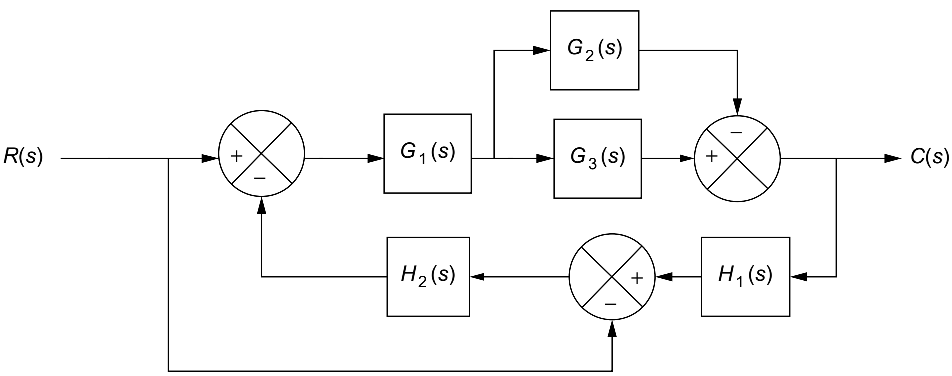


Fig. E3.6(a)

Solution:

Step 1: The blocks in parallel as represented by dotted lines in Fig. E3.6(b) can be reduced to a single block as indicated in Fig. E3.6(c).

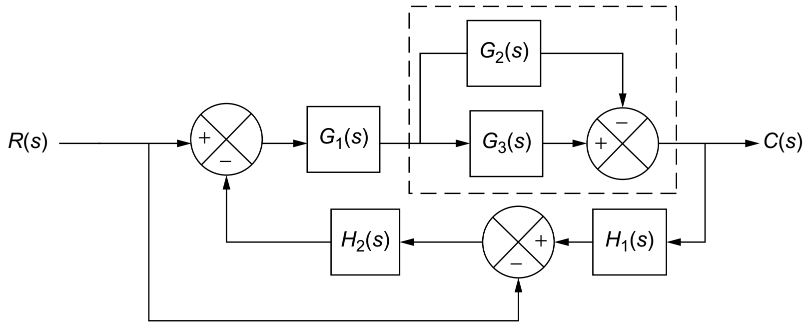


Fig. E3.6(b)

Step 2: The summing point as represented by dotted lines in Fig. E3.6(c) can be shifted after $H_2(s)$ block in the feedback path and the resultant block diagram is shown in Fig. E3.6(d).

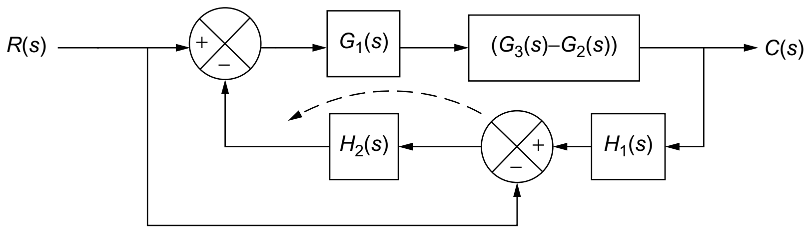


Fig. E3.6(c)

Step 3: The blocks in series as represented by dotted lines in Fig. E3.6(d) can be reduced to a single block as represented in Fig. E3.6(e).

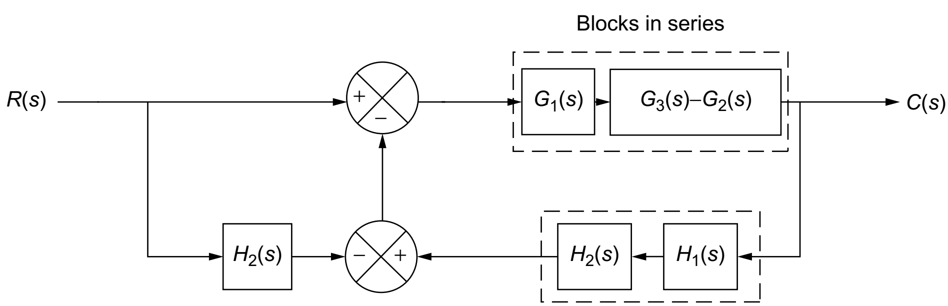


Fig. E3.6(d)

Step 4: The summing point in the feedback as represented by dotted lines in Fig. E3.6(e) can be removed and the resultant block diagram is shown in Fig. E3.6(f).

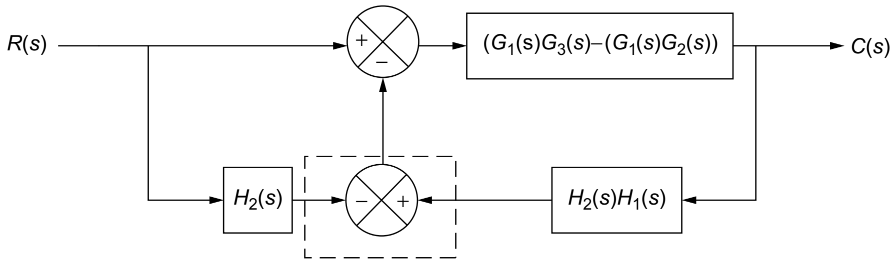


Fig. E3.6(e)

Step 5: In the block diagram as represented by dotted lines in Fig. E3.6(f), the blocks in parallel can be reduced to a single block. In addition, the feedback loop in the block diagram can be reduced. The resultant block diagram is shown in Fig. E3.6(g).

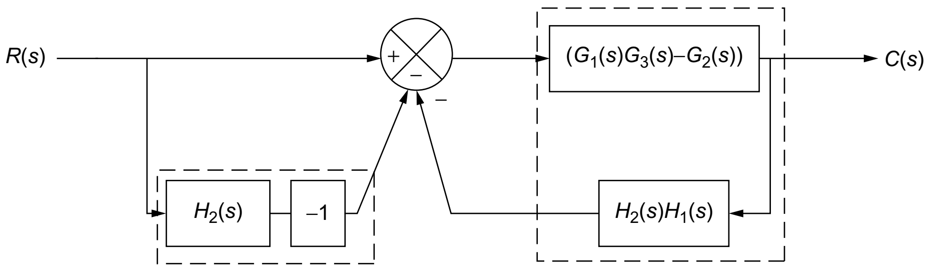


Fig. E3.6(f)

Step 6: The blocks in series as represented in Fig. E3.6(g) can be reduced to a single block to determine the transfer function of the system as shown in Fig. E3.6(a).

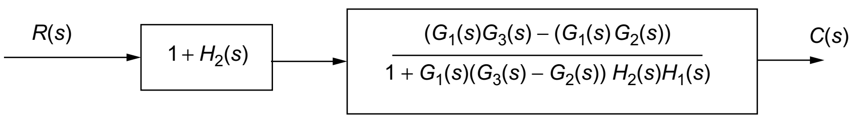


Fig. E3.6(g)

Hence, the transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G_1(s)\left(1+H_2(s)\right)\left(G_3(s)-G_2(s)\right)}{1+G_1(s)H_1(s)H_2(s)G_3(s)-G_1(s)H_1(s)H_2(s)G_2(s)}.$$

Example 3.7: For the block diagram of the system shown in Fig. E3.7(a). Determine the transfer function using the block diagram reduction technique when the input *R*(*s*) at any instant of time is applied to any one station.

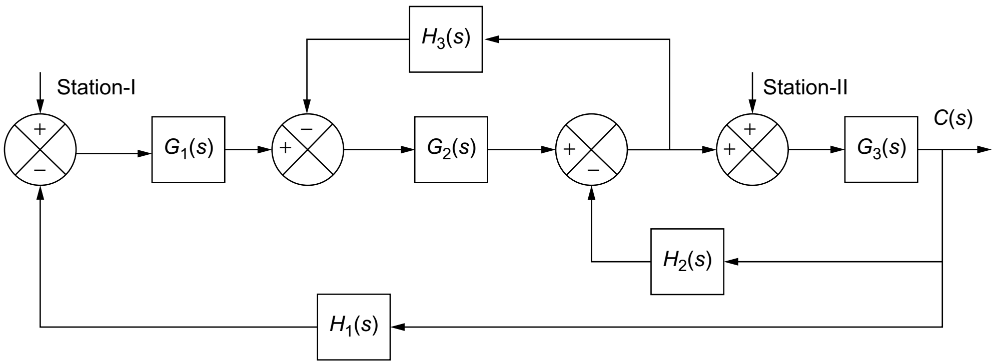


Fig. E3.7(a)

Solution: It is given that the input $R(s)$ at any instant is applied to any one station. Hence, we have two cases for which the transfer function is to be determined.

Case 1: Input $R(s)$ is applied to station I only

The input at station II is made zero and the summing point is removed. The output for the input at station I is $C_1(s)$. The block diagram representation with the input $R(s)$ at station-I is shown in Fig. E3.7(b). The transfer function for the block diagram is obtained by applying the block diagram reduction technique to the diagram shown in Fig. E3.7(b).

Step 1: The take-off point that exists before $G_3(s)$ block as represented by dotted lines in Fig. E3.7(b) is moved after $G_3(s)$ block and the take-off points that exist after $G_3(s)$ block are rearranged. The resultant block diagram is shown in Fig. E3.7(c).

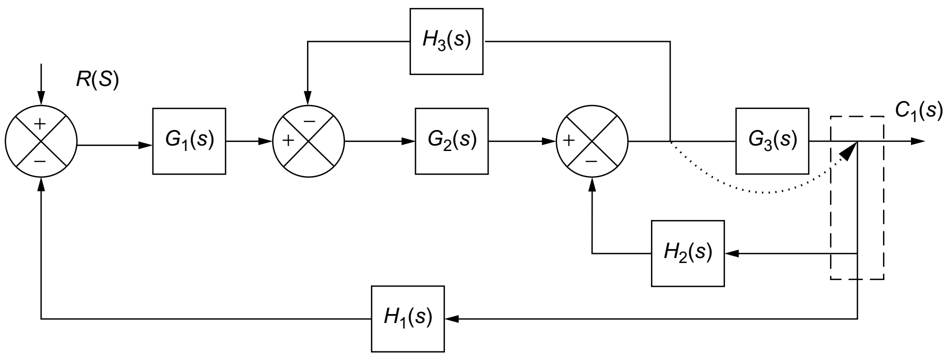


Fig. E3.7(b)

Step 2: The feedback loop with the feedback block $H_2(s)$ as represented by dotted lines in Fig. E3.7(c) can be reduced. After reduction of the feedback loop, the blocks in series as represented by dotted lines in Fig. E3.7(c) can be multiplied and the resultant block diagram is shown in Fig. E3.7(d).

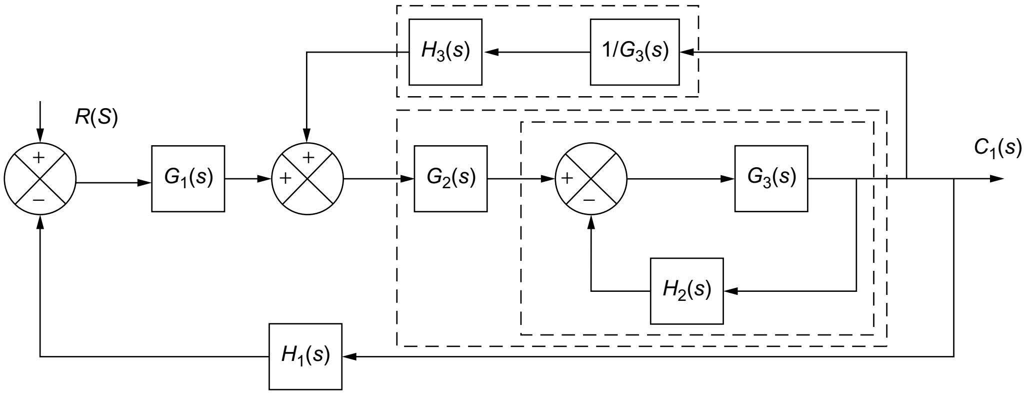


Fig. E3.7(c)

Step 3: The feedback loop as represented by dotted lines in Fig. E3.7(d) can be reduced. The block diagram after reduction of feedback loop is shown in Fig. E3.7(e).

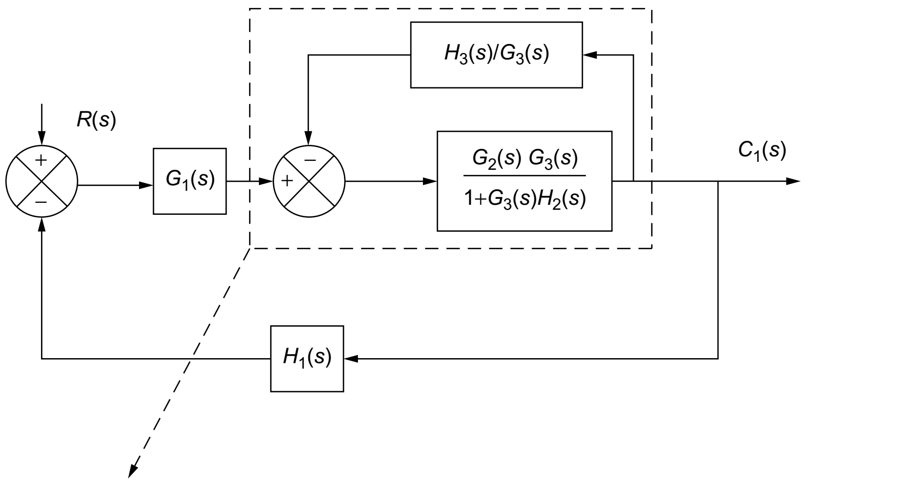


Fig. E3.7(d)

$$\frac{\dfrac{G_2(s)G_3(s)}{1+G_3(s)H_2(s)}}{1+\dfrac{G_2(s)G_3(s)}{1+G_3(s)H_3(s)}\times\dfrac{H_3(s)}{G_3(s)}}=\frac{\dfrac{G_2(s)G_3(s)}{1+G_3(s)H_2(s)}}{1+\dfrac{G_2(s)H_3(s)}{1+G_3(s)H_2(s)}}=\frac{G_2(s)G_3(s)}{1+G_3(s)H_2(s)+G_2(s)H_3(s)}$$

Step 4: The blocks in series as represented by dotted lines in Fig. E3.7(e) can be reduced to a single block as represented by dotted lines in Fig. E3.7(f).

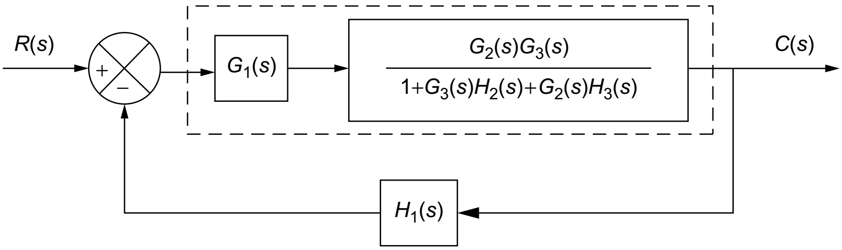


Fig. E3.7(e)

Step 5: The feedback loop with feedback block $H_1(s)$ as indicated by dotted lines in Fig. E3.7(f) can be reduced to determine the closed-loop transfer function of the system represented in Fig. E3.7(a) when the input is applied at station I only.

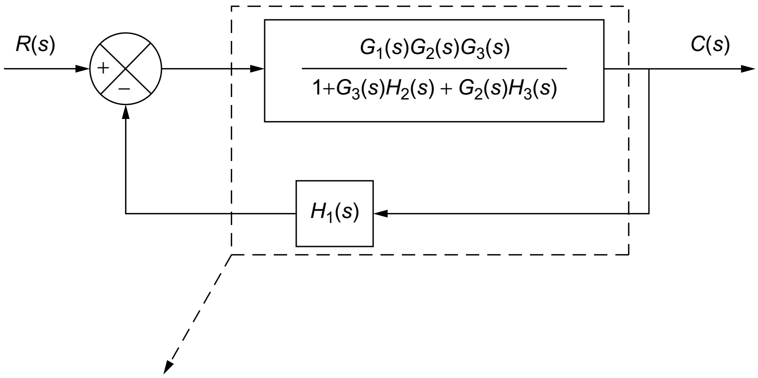


Fig. E3.7(f)

$$\frac{\dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)H_2(s)+G_2(s)H_3(s)}}{1+\dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)H_2(s)+G_2(s)H_3(s)}\times H_1(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)H_2(s)+G_2(s)H_3(s)+G_1(s)G_2(s)G_3(s)H_1(s)}$$

Therefore,

$$\frac{C_1(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)H_2(s)+G_2(s)H_3(s)+G_1(s)G_2(s)G_3(s)H_1(s)}$$

Case 2: Input $R(s)$ is applied to station II only

The input at station I is made zero and the summing point is removed. In addition, the negative sign in the summing point is attached to the block $H_1(s)$. The output for the input at station II is $C_2(s)$. The block diagram representation with the input $R(s)$ at station II is shown in Fig. E3.7(g). The transfer function for the block diagram is obtained by applying the block diagram reduction technique to the diagram shown in Fig. E3.7(g).

Step 1: In this step, the blocks in series as indicated by dotted lines in Fig. E3.7(g) can be reduced to a single block. In addition, the summing point that has one of the inputs as $H_2(s)$ is shifted before $G_2(s)$ and the take-off points that exist after $G_3(s)$ are rearranged. The resultant block diagram is shown in Fig. E3.7(h).

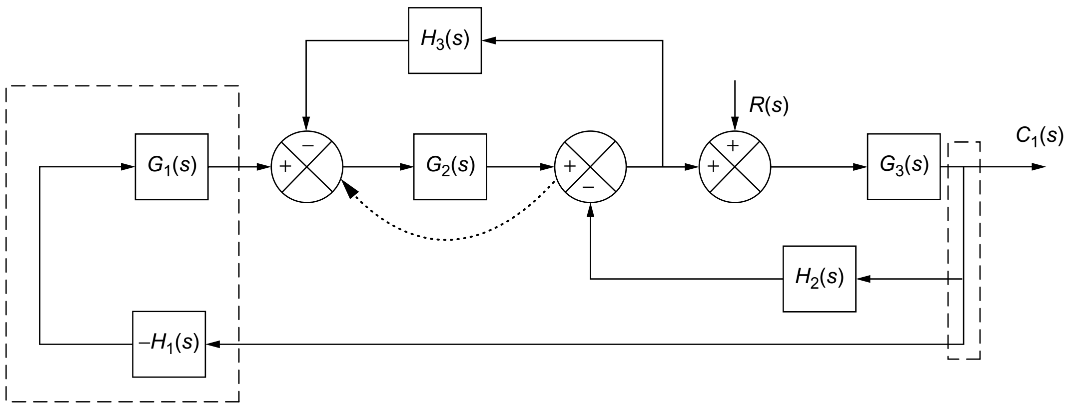


Fig. E3.7(g)

Step 2: The two summing points in series as indicated by dotted lines in Fig. E3.7(h) can be interchanged. In addition, the blocks in series can be reduced to a single block. The resultant block diagram is shown in Fig. E3.7(i).

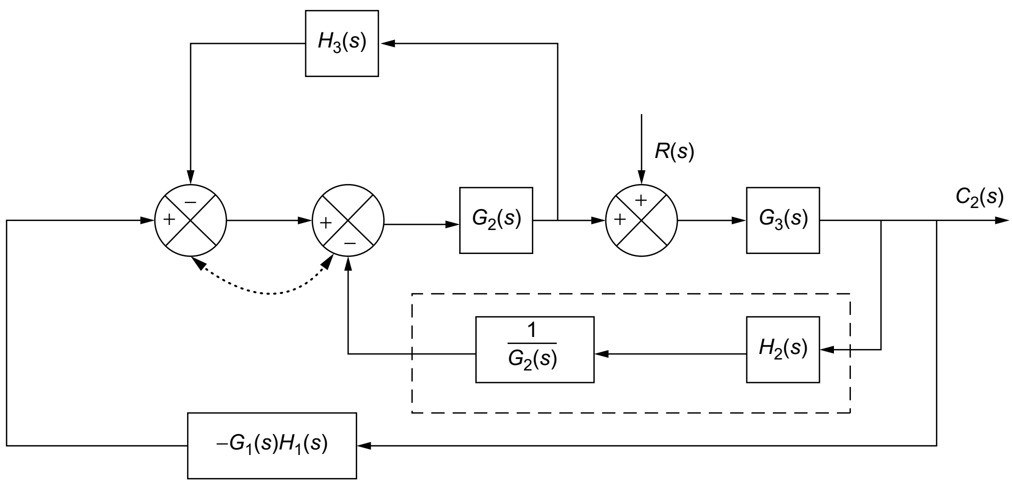


Fig. E3.7(h)

Step 3: The parallel blocks and the feedback loop that exists in the block diagram as indicated by dotted lines in Fig. E3.7(i) can be reduced to a single block as represented in Fig. E3.7(j).

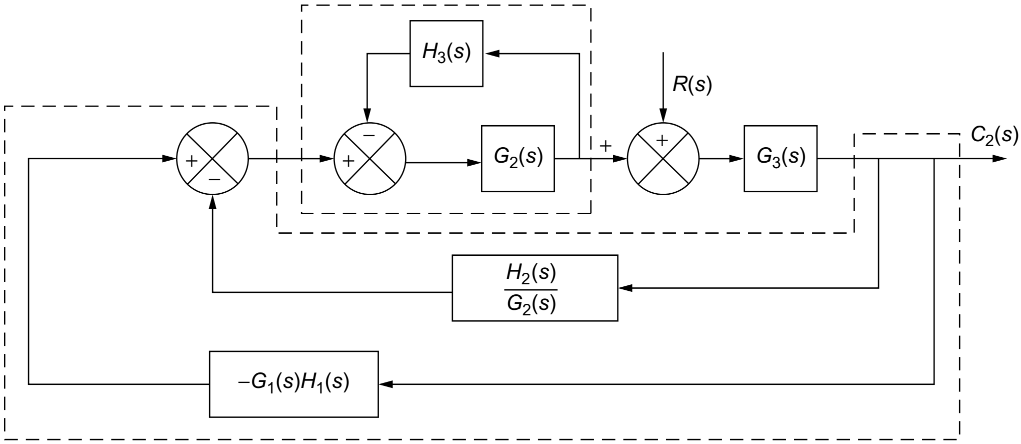


Fig. E3.7(i)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.7(j) can be reduced to a single block as shown in Fig. E3.7(k).

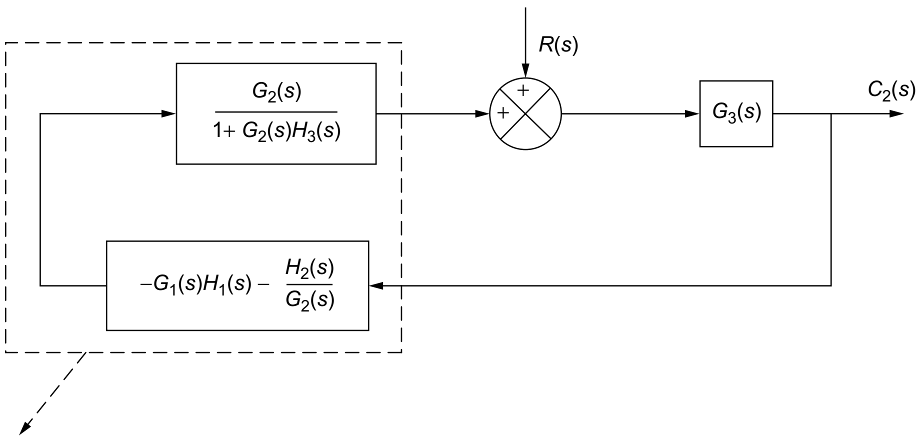


Fig. E3.7(j)

$$\left[\frac{G_2(s)}{1+G_2(s)H_3(s)}\right]\times\left[-G_1(s)H_1(s)-\frac{H_2(s)}{G_2(s)}\right]=\left[\frac{G_2(s)}{1+G_2(s)H_3(s)}\right]\times\left[\frac{-G_1(s)H_1(s)G_2(s)-H_2(s)}{G_2(s)}\right]$$

$$=\frac{-G_2(s)\big(G_1(s)G_2(s)H_1(s)+H_2(s)\big)}{\big(1+G_2(s)H_3(s)\big)G_2(s)}$$

Step 5: The feedback loop as indicated by dotted lines in Fig. E3.7(k) can be reduced in order to determine the transfer function of the system represented by block diagram as shown in Fig. E3.7(a) when the input $R(s)$ is applied at station II only.

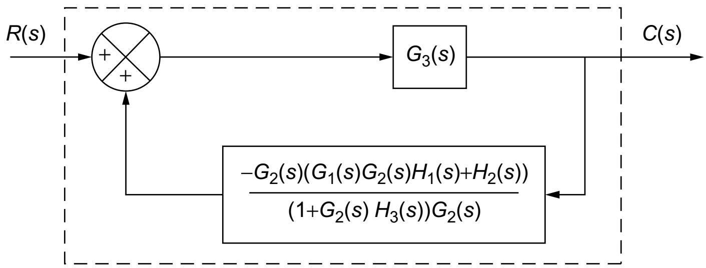


Fig. E3.7(k)

$$\frac{G_3(s)}{1-\left[\dfrac{-\left(G_1(s)G_2(s)H_1(s)+H_2(s)\right)}{1+G_2(s)H_3(s)}\right]G_3(s)} = \frac{G_3(s)}{\dfrac{1+G_2(s)H_3(s)+G_3(s)\left(G_1(s)G_2(s)H_1(s)+H_2(s)\right)}{1+G_2(s)H_3(s)}}$$

$$= \frac{G_3(s)\left(1+G_2(s)H_3(s)\right)}{1+G_2(s)H_3(s)+G_3(s)\left(G_1(s)G_2(s)H_1(s)+H_2(s)\right)}$$

$$\frac{C_2(s)}{R(s)} = \frac{G_3(s)\left(1+G_2(s)H_3(s)\right)}{1+G_2(s)H_3(s)+G_3(s)\left(G_1(s)G_2(s)H_1(s)+H_2(s)\right)}$$

Hence, the transfer function of the multi-input system shown in Fig. E3.7(a) is obtained by considering one input at a time.

Example 3.8: For the block diagram of the system shown in Fig. E3.8(a), determine the transfer function using the block diagram reduction technique.

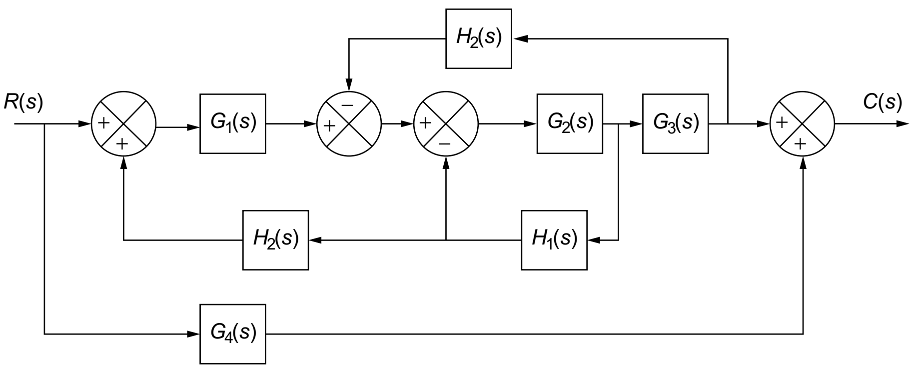


Fig. E3.8(a)

Solution:

Step 1: The take-off point that exists in the common feedback path as indicated by dotted lines in Fig. E3.8(b) is separated to form two individual feedback paths. The resultant block diagram is shown in Fig. E3.8(c).

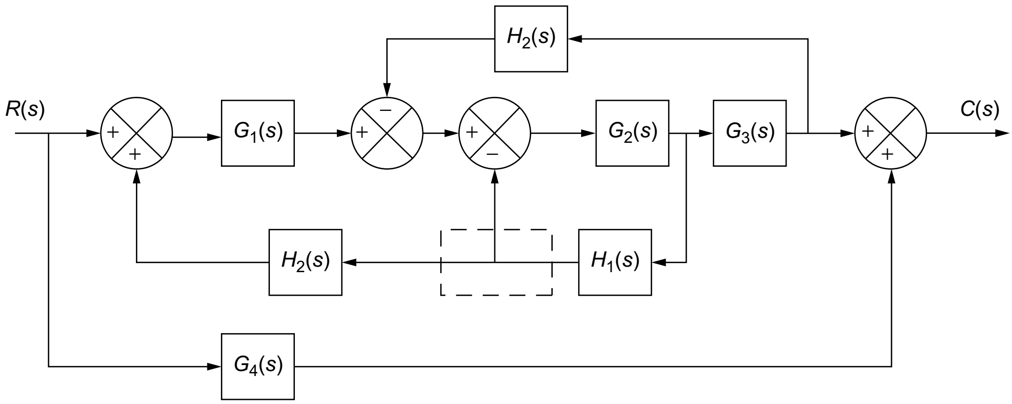


Fig. E3.8(b)

Step 2: The blocks in series and the feedback path as indicated by dotted lines in Fig. E3.8(c) can be reduced to form a single block and the resultant block diagram is shown in Fig. E3.8(d).

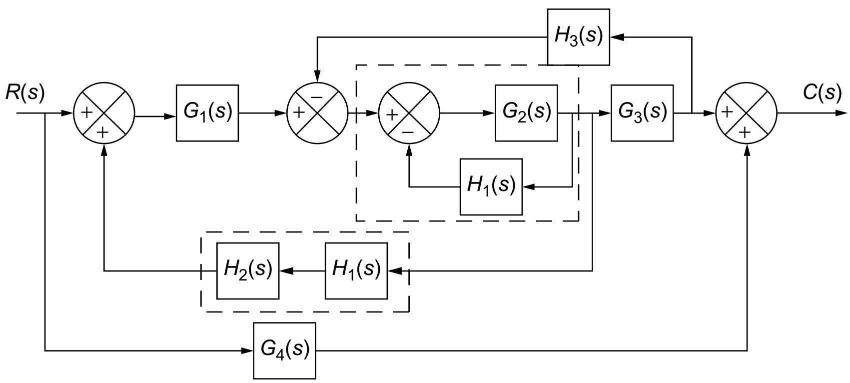


Fig. E3.8(c)

Step 3: The take-off point that exists between the blocks $\dfrac{G_2(s)}{1+G_2(s)H_1(s)}$ and $G_3(s)$ as represented by dotted lines in Fig. E3.8(d) can be shifted after the block $G_3(s)$. The resultant block diagram is shown in Fig. E3.8(e).

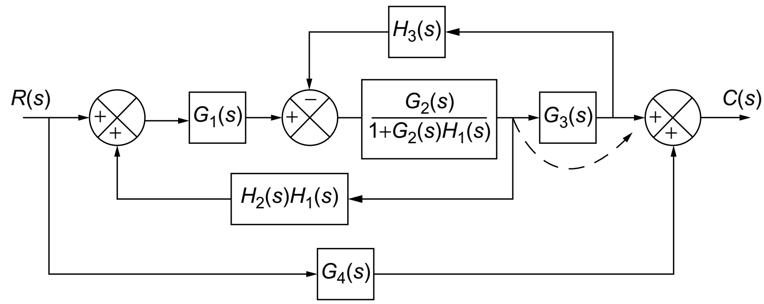


Fig. E3.8(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.8(e) can be combined to form a single block and the resultant block diagram is shown in Fig. E3.8(f).

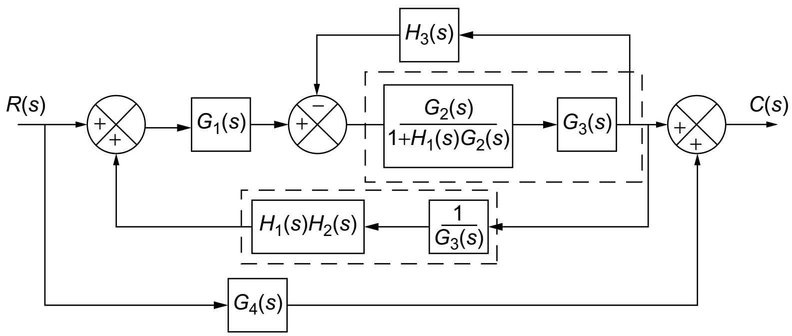


Fig. E3.8(e)

Step 5: The feedback path as represented by dotted lines in Fig. E3.8(f) can be reduced to form a single block and the resultant block diagram is shown in Fig. E3.8(g).

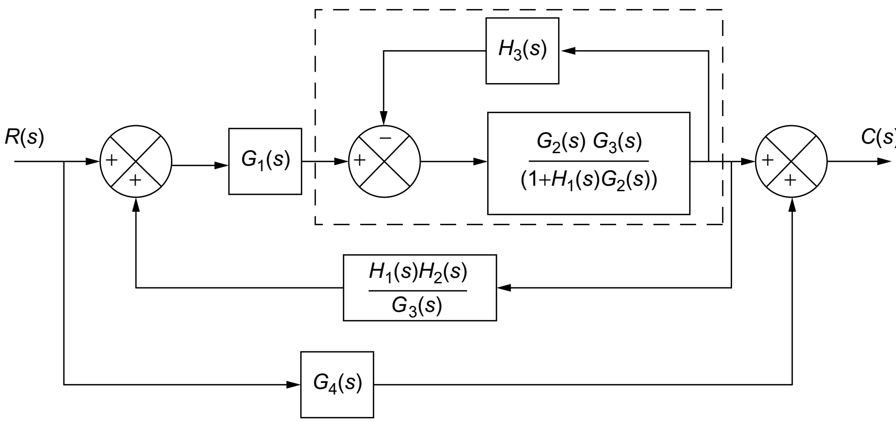


Fig. E3.8(f)

Step 6: The blocks in series as represented by dotted lines in Fig. E3.8(g) can be combined to form a single block and the resultant block diagram is shown in Fig. E3.8(h).

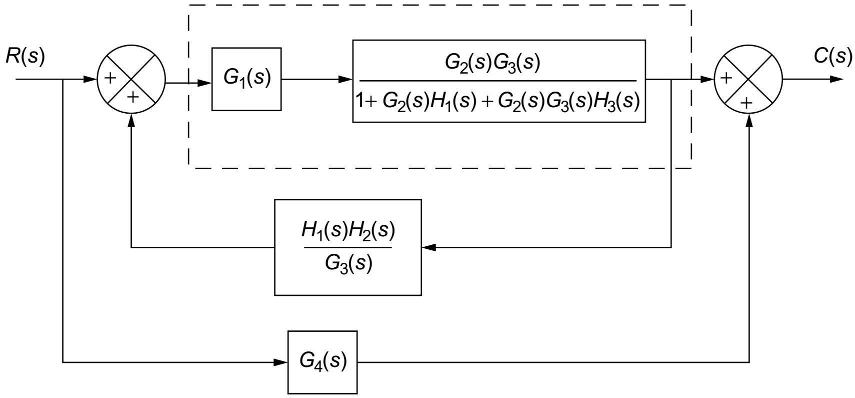


Fig. E3.8(g)

Step 7: The feedback path as indicated by dotted lines in Fig. E3.8(h) can be reduced to form a single block and the resultant block diagram is shown in Fig. E3.8(i).

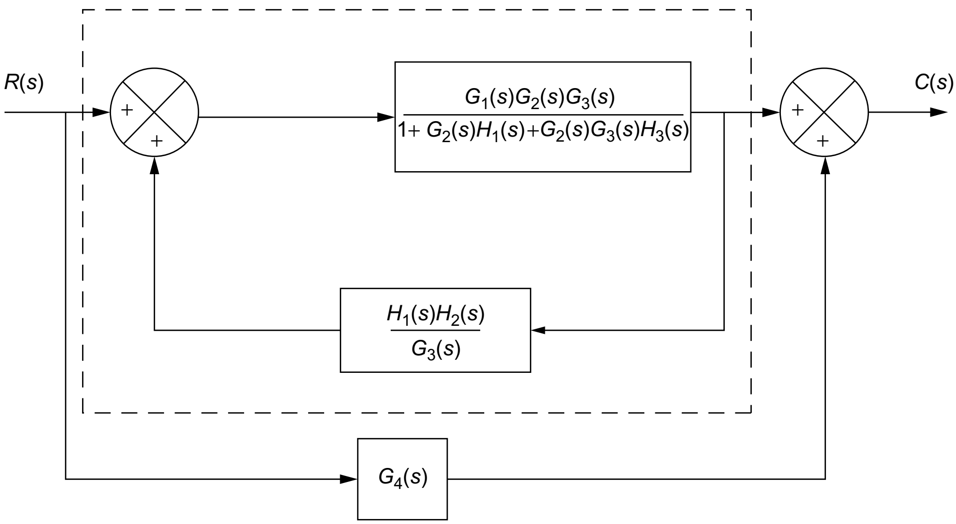


Fig. E3.8(h)

Step 8: The blocks in parallel as shown in Fig. E3.8(i) can be combined to determine the transfer function of the system.

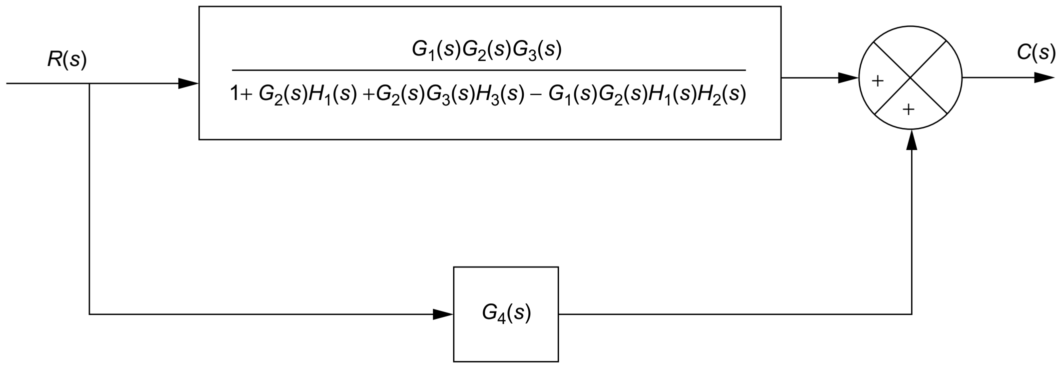


Fig. E3.8(i)

Hence, the transfer function of the system is given by

$$\frac{C(s)}{R(s)}=\frac{\begin{bmatrix}G_1(s)G_2(s)G_3(s)+G_4(s)+G_2(s)G_4(s)H_1(s)\\+G_2(s)G_3(s)G_4(s)H_3(s)-G_1(s)G_2(s)G_4(s)H_1(s)H_2(s)\end{bmatrix}}{1+G_2(s)H_1(s)+G_2(s)G_3(s)H_3(s)-G_1(s)G_2(s)H_1(s)H_2(s)}$$

Example 3.9: For the block diagram of the system as shown in Fig. E3.9(a), determine the transfer function using the block diagram reduction technique.

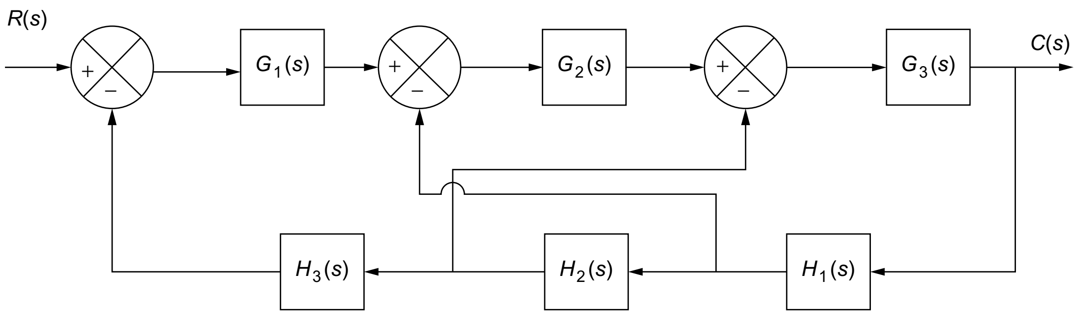


Fig. E3.9(a)

Solution:

Step 1: The take-off point that exists after the block $H_2(s)$ as indicated by dotted lines in Fig. E3.9(b) is shifted before the block $H_2(s)$ and the resultant block diagram is shown in Fig. E3.9(c).

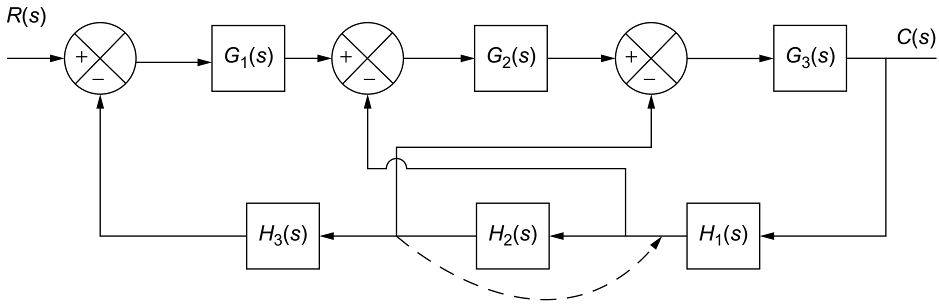


Fig. E3.9(b)

Step 2: The blocks in series as represented by dotted lines in Fig. E3.9(c) can be combined to form a single block. The resultant block diagram is shown in Fig. E3.9(d).

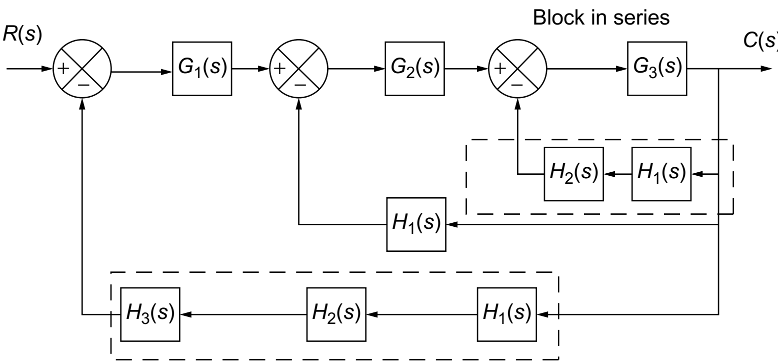


Fig. E3.9(c)

Step 3: The feedback path as indicated by dotted lines in Fig. E3.9(d) can be reduced and the resultant block diagram is shown in Fig. E3.9(e).

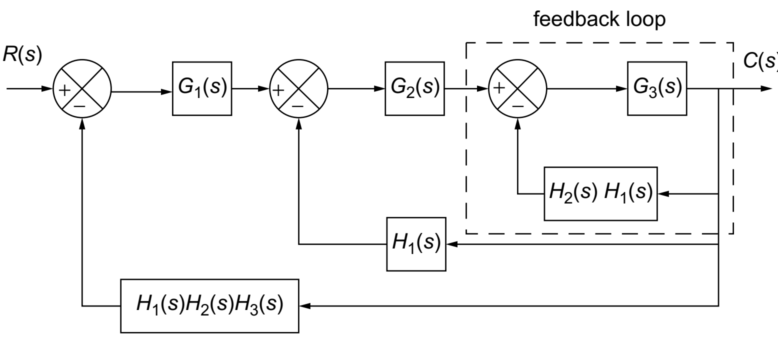


Fig. E3.9(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.9(e) can be combined to form a single block. The resultant block diagram is shown in Fig. E3.9(f).

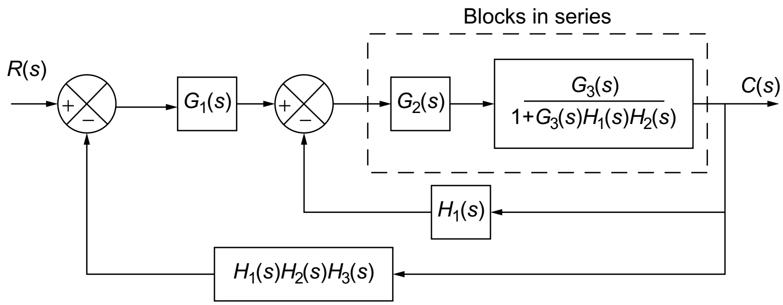


Fig. E3.9(e)

Step 5: The feedback path as represented by dotted lines in Fig. E3.9(f) can be reduced and the resultant block diagram is shown in Fig. E3.9(g).

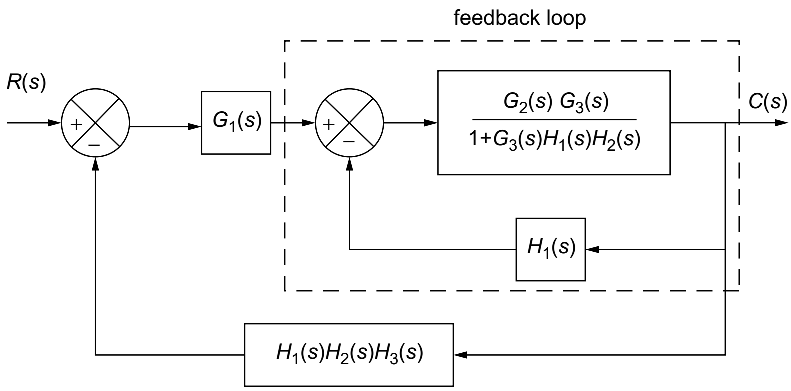


Fig. E3.9(f)

Step 6: The blocks in series as represented by dotted lines in Fig. E3.9(g) can be combined to form a single block. The resultant block diagram is shown in Fig. E3.9(h).

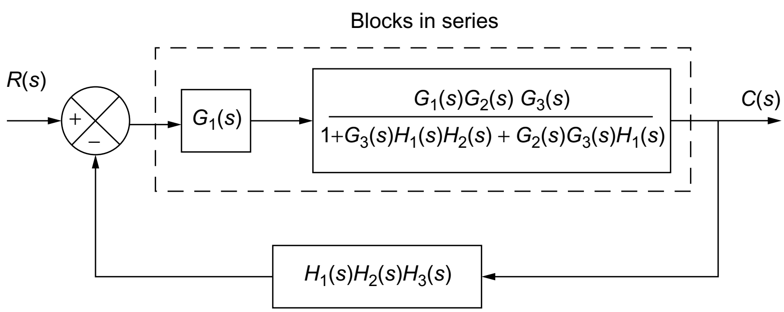


Fig. E3.9(g)

Step 7: The feedback path as indicated by dotted lines in Fig. E3.9(h) can be reduced and the resultant block diagram is shown in Fig. E3.9(i).

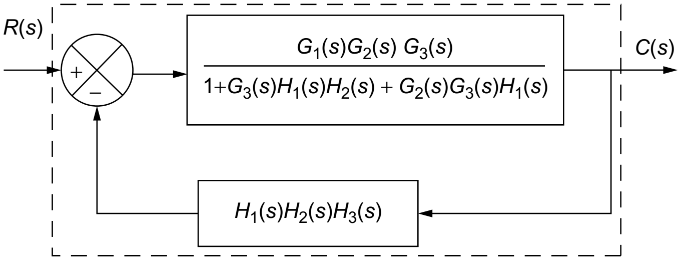


Fig. E3.9(h)

Step 8: The block diagram shown in Fig. E3.9(i) can be solved to determine the transfer function of the given system.

Fig. E3.9(i)

Hence, the transfer function of the given system is obtained as

$$\frac{C(s)}{R(s)}=\frac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)H_1(s)H_2(s)+G_2(s)G_3(s)H_1(s)+G_1(s)G_2(s)G_3(s)H_1(s)H_2(s)H_3(s)}$$

Example 3.10: For the block diagram of the system as shown in Fig. E3.10(a), determine the transfer function using the block diagram reduction technique.

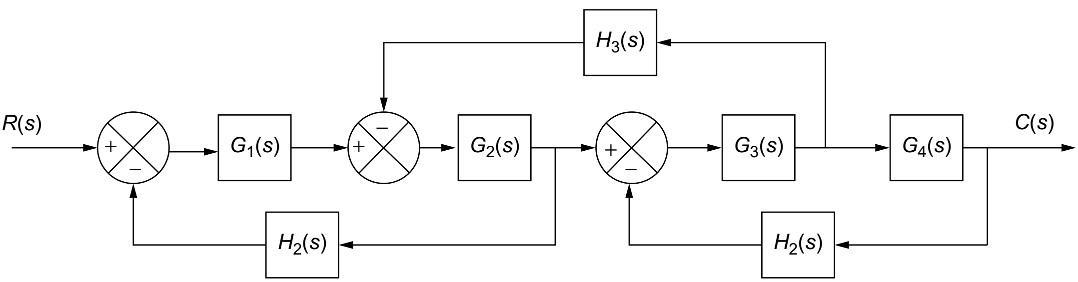


Fig. E3.10(a)

Solution:

Step 1: The summing point that exists after the block $G_1(s)$ and the take-off point that exists after the block $G_3(s)$ as indicated by dotted lines in Fig. E3.10(b) are shifted before the block $G_1(s)$ and after the block $G_4(s)$ respectively and the resultant block diagram is shown in Fig. E3.10(c).

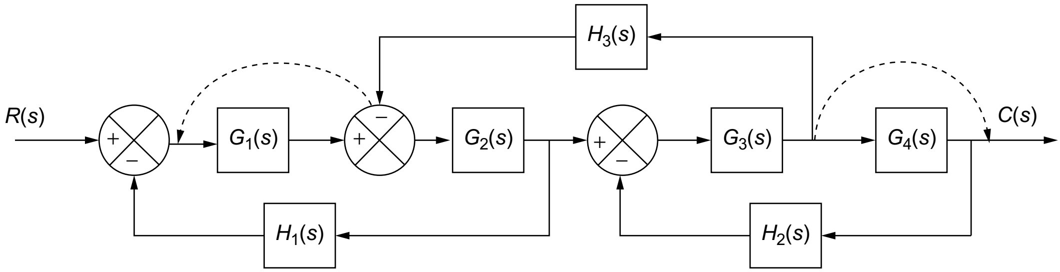


Fig. E3.10(b)

Step 2: The blocks in series as indicated by dotted lines in Fig. E3.10(c) are combined to form a single block. The resultant block diagram is shown in Fig. E3.10(d).

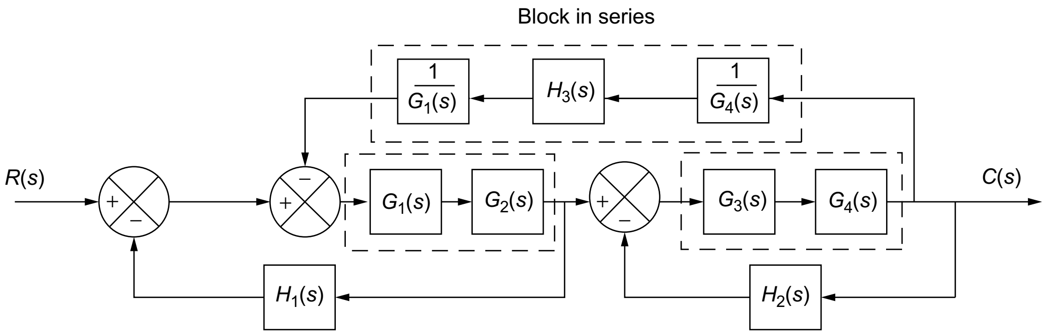


Fig. E3.10(c)

Step 3: The summing points that exist between the input signal and the block $G_1(s)G_2(s)$ as indicated by dotted lines in Fig. E3.10(d) are interchanged and the resultant block diagram is shown in Fig. E3.10(e).

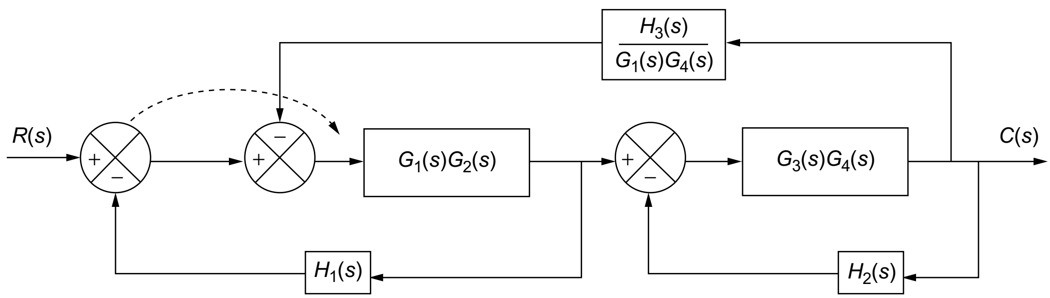


Fig. E3.10(d)

Step 4: The feedback path as represented by dotted lines in Fig. E3.10(e) can be reduced and the resultant block diagram is shown in Fig. E3.10(f).

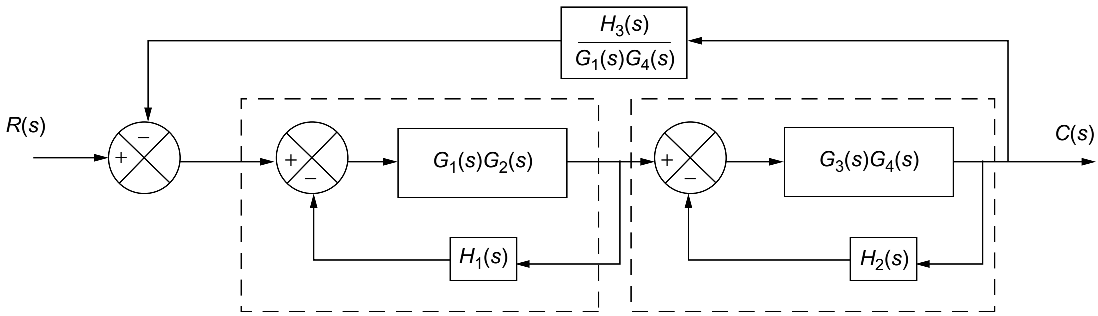


Fig. E3.10(e)

Step 5: The blocks in series as indicated by dotted lines in Fig. E3.10(f) are combined to form a single block. The resultant block diagram is shown in Fig. E3.10(g).

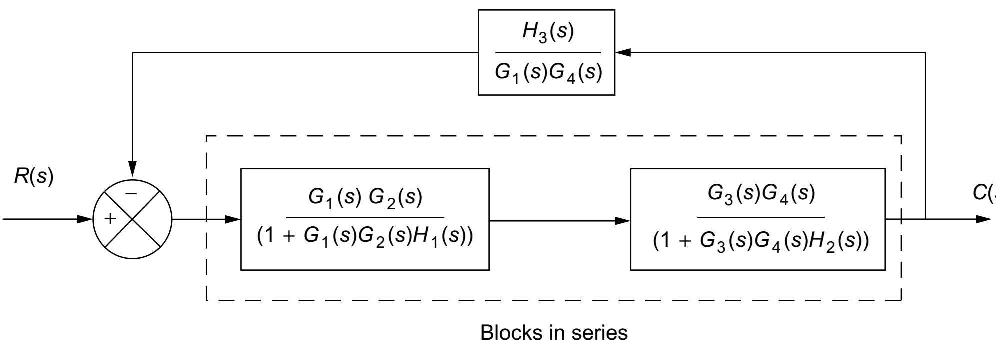


Fig. E3.10(f)

Step 6: The feedback path as indicated by dotted lines in Fig. E3.10(g) can be reduced in order to determine the transfer function of the system.

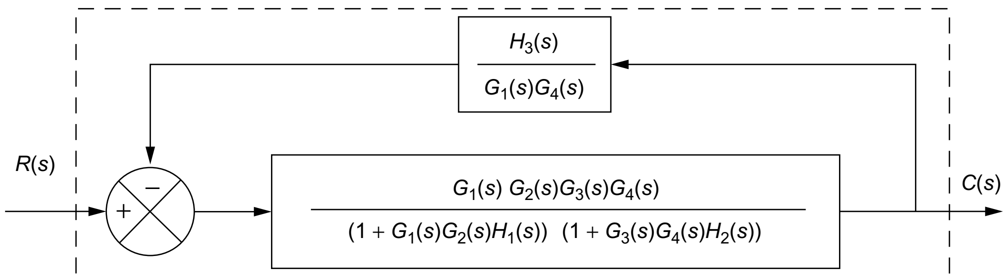


Fig. E3.10(g)

Hence, the transfer function of the given system is obtained as

$$\frac{C(s)}{R(s)}=$$

$$\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{\{[1+G_1(s)G_2(s)H_1(s)]\times[1+G_3(s)G_4(s)H_2(s)]\times[G_1(s)G_4(s)]\}+G_1(s)G_2(s)G_3(s)G_4(s)H_3(s)}$$

Example 3.11: For the block diagram of the system shown in Fig. E3.11(a), determine the transfer function using the block diagram reduction technique.

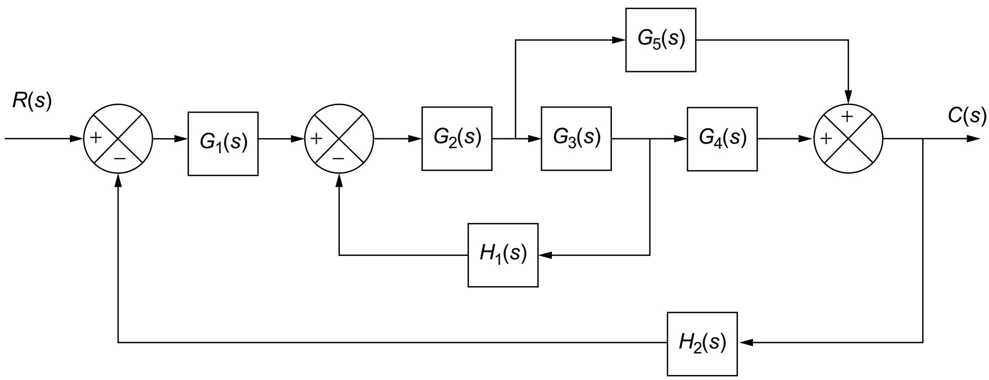


Fig. E3.11(a)

Solution:

Step 1: The take-off point that exists between the blocks $G_2(s)$ and $G_3(s)$ as indicated by dotted lines in Fig. E3.11(b) is shifted after the block $G_3(s)$. The resultant block diagram is shown in Fig. E3.11(c).

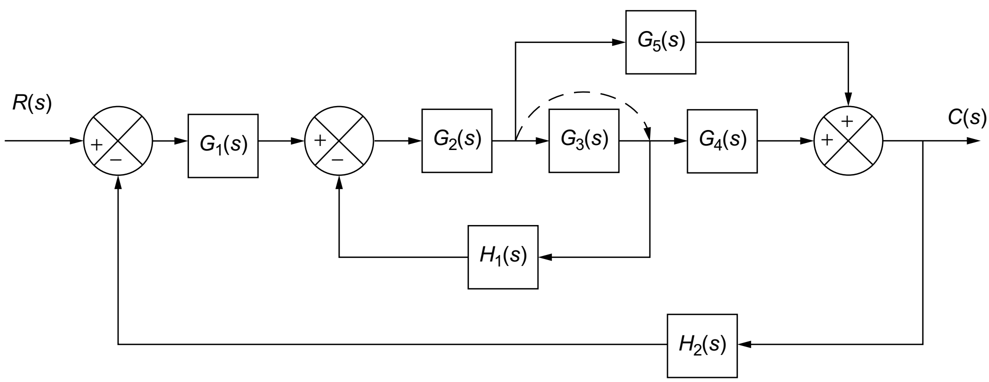


Fig. E3.11(b)

Step 2: The blocks in series as indicated by dotted lines in Fig. E3.11(c) can be combined to form a single block. The resultant block diagram is shown in Fig. E3.11(d).

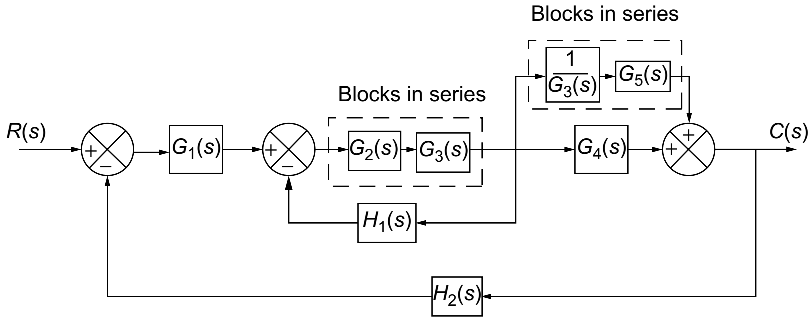


Fig. E3.11(c)

Step 3: The blocks in parallel and the feedback loop that exists in the block diagram as indicated by dotted lines in Fig. E3.11(d) can be reduced to form a single block. The resultant block diagram is shown in Fig. E3.11(e).

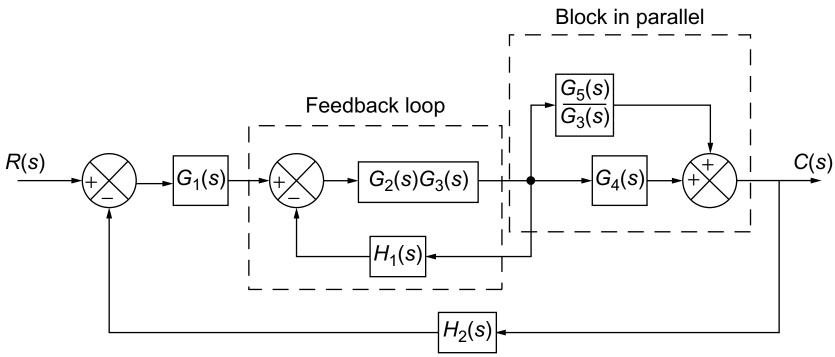


Fig. E3.11(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.11(e) can be reduced to form a single block. The resultant block diagram is shown in Fig. E3.11(f).

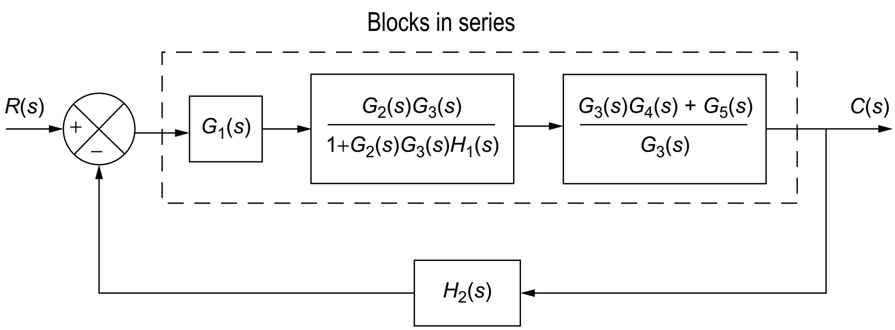


Fig. E3.11(e)

Step 5: The feedback loop that exists in the block diagram shown in Fig. E3.11(f) can be reduced in order to determine the transfer function of the system.

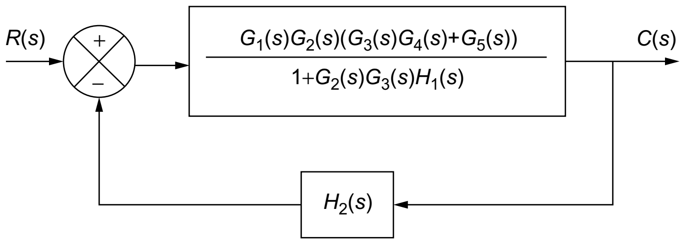


Fig. E3.11(f)

Hence, the transfer function of the system is obtained as

$$\frac{C(s)}{R(s)}=\frac{\dfrac{G_1(s)G_2(s)\left[\{G_3(s)G_4(s)\}+G_5(s)\right]}{1+G_2(s)G_3(s)H_1(s)}}{1+\dfrac{G_1(s)G_2(s)\left[\{G_3(s)G_4(s)\}+G_5(s)\right]}{1+G_2(s)G_3(s)H_1(s)}\times H_2(s)}$$

i.e., $$\frac{C(s)}{R(s)}=\frac{G_1(s)G_2(s)G_3(s)G_4(s)+G_1(s)G_2(s)G_5(s)}{1+G_2(s)G_3(s)H_1(s)+G_1(s)G_2(s)G_3(s)G_4(s)H_2(s)+G_1(s)G_2(s)G_5(s)H_2(s)}$$

Example 3.12: For the block diagram of the system shown in Fig. E3.12(a), determine the transfer function using the block diagram reduction technique.

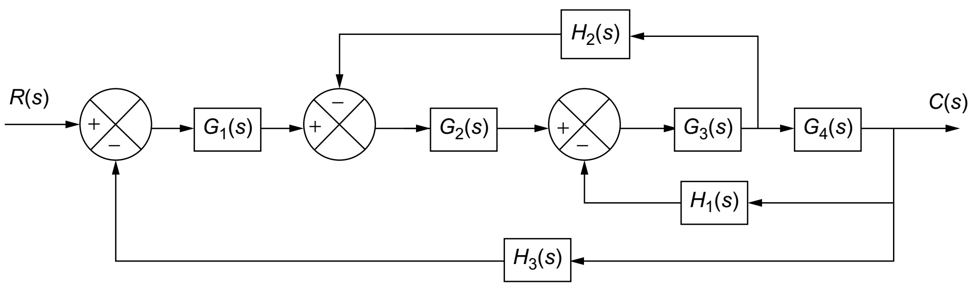


Fig. E3.12(a)

Solution:

Step 1: The take-off point that exists between the blocks $G_3(s)$ and $G_4(s)$ as indicated by dotted lines in Fig. E3.12(b) is shifted after the block $G_4(s)$. The resultant block diagram is shown in Fig. E3.12(c).

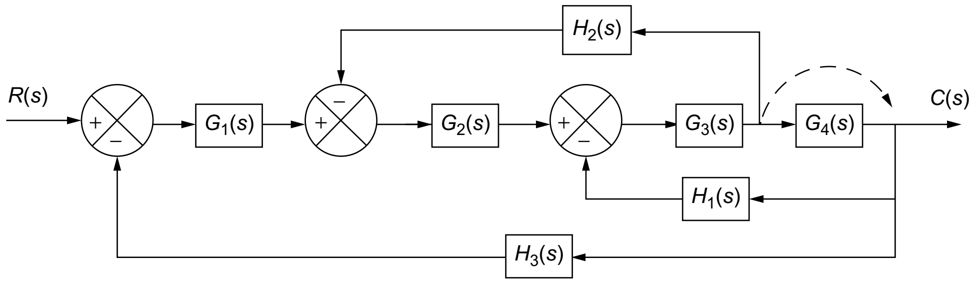


Fig. E3.12(b)

Step 2: The blocks in series as indicated by dotted lines in Fig. E3.12(c) can be combined to form a single block and the resultant block diagram is shown in Fig. E3.12(d).

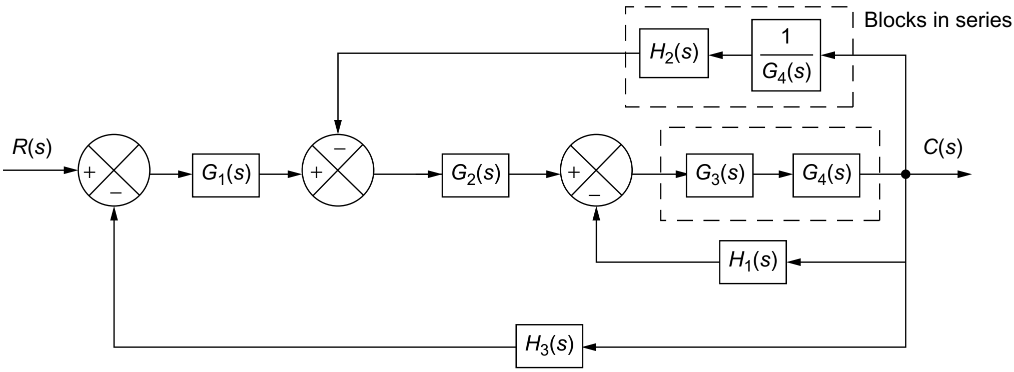


Fig. E3.12(c)

Step 3: The feedback loop as indicated by dotted lines in Fig. E3.12(d) can be reduced to a single block and the resultant block diagram is shown in Fig. E3.12(e).

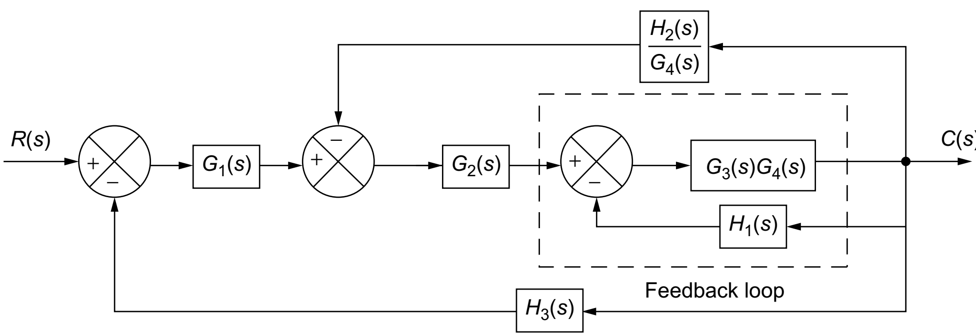


Fig. E3.12(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.12(e) can be reduced to a single block and the resultant block diagram is shown in Fig. E3.12(f).

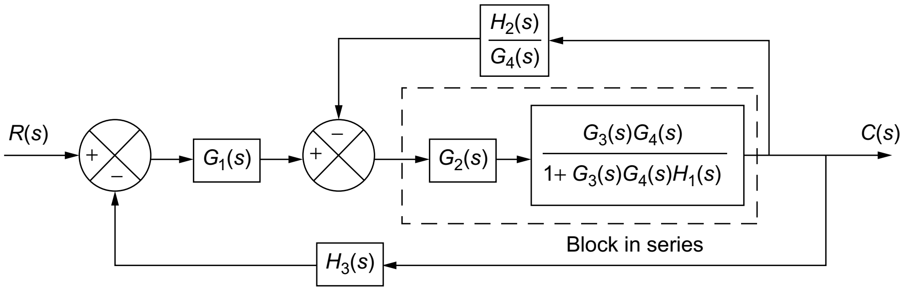


Fig. E3.12(e)

Step 5: The feedback path as indicated by dotted lines in Fig. E3.12(f) can be reduced to form a single block and resultant block diagram is shown in Fig. E3.12(g).

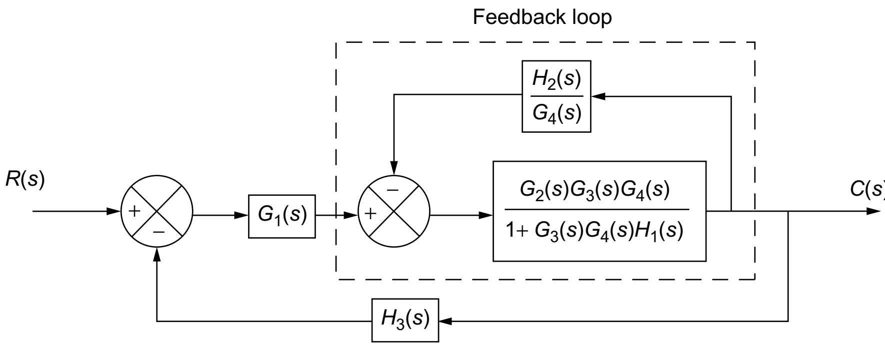


Fig. E3.12(f)

Step 6: The blocks in series as indicated by dotted lines in Fig. E3.12(g) can be reduced to a single block and the resultant block diagram is shown in Fig. E3.12(h).

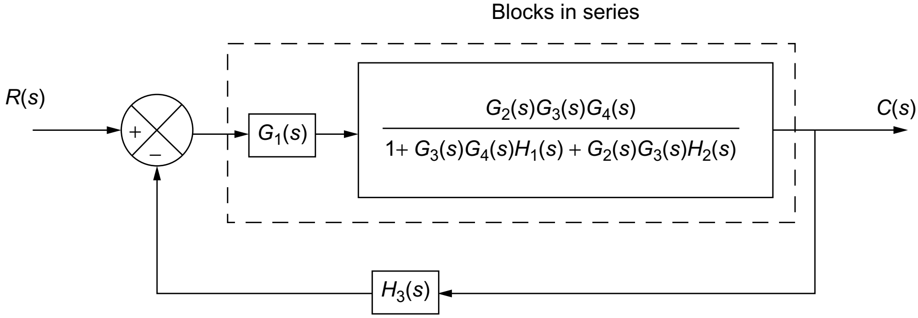


Fig. E3.12(g)

Step 7: The feedback path as indicated by dotted lines in Fig. E3.12(h) can be reduced to obtain transfer function of the system.

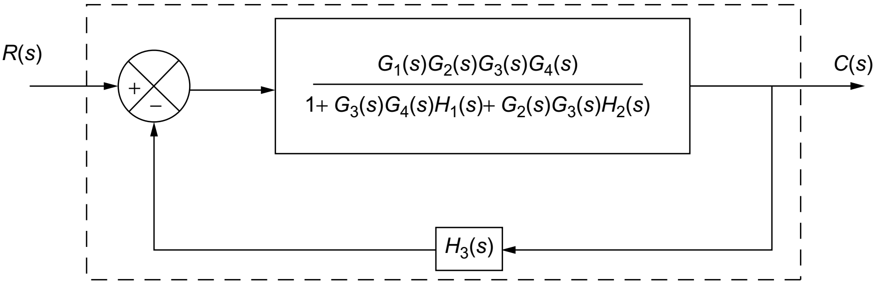


Fig. E3.12(h)

Hence, the transfer function of the system is obtained as

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1(s)G_2(s)G_3(s)G_4(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)}}{1+\left(\dfrac{G_1(s)G_2(s)G_3(s)G_4(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)}\right)\times H_3(s)}$$

i.e., $$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)G_4(s)}{1+G_3(s)G_4(s)H_1(s)+G_2(s)G_3(s)H_2(s)+G_1(s)G_2(s)G_3(s)G_4(s)H_3(s)}$$

Example 3.13: For the block diagram of the system shown in Fig. E3.13(a), determine the transfer function using the block diagram reduction technique.

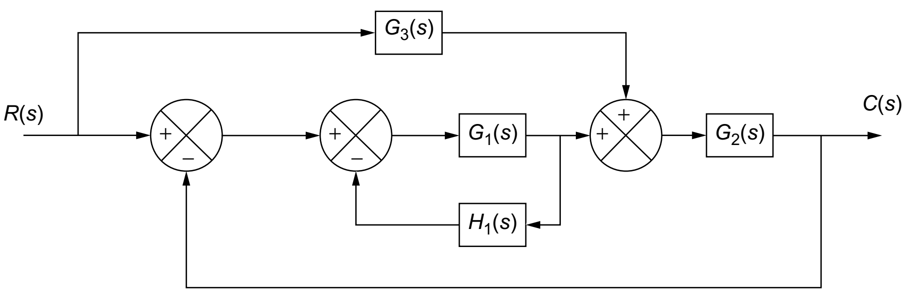


Fig. E3.13(a)

Solution:

Step 1: The feedback path as indicated by dotted lines in Fig. E3.13(b) can be reduced and the resultant block diagram is shown in Fig. E3.13(c).

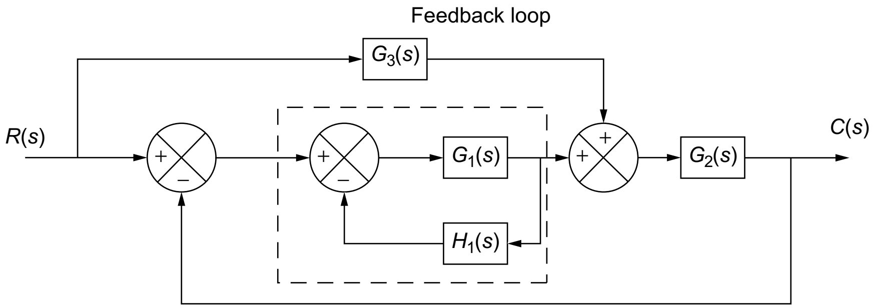


Fig. E3.13(b)

Step 2: The summing point that exists before the block $G_2(s)$ as indicated by dotted lines in Fig. E3.13(c) is shifted before the $\dfrac{G_1(s)}{1+\left[G_1(s)H_1(s)\right]}$ and the resultant block diagram is shown in Fig. E3.13(d).

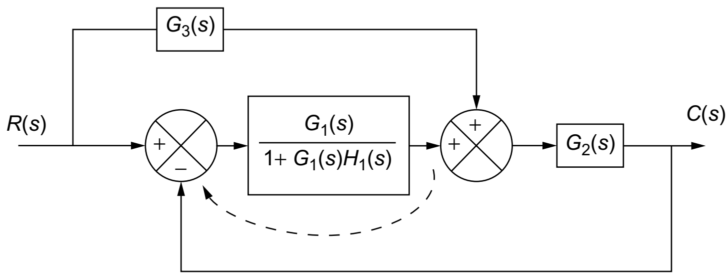


Fig. E3.13(c)

Step 3: The two summing points that exist in the block diagram as indicated by dotted lines in Fig. E3.13(d) are interchanged and the resultant block diagram is shown in Fig. E3.13(e).

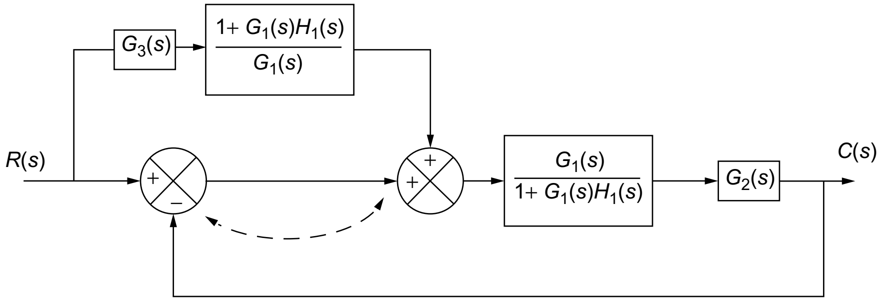


Fig. E3.13(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.13(e) can be reduced to form a single block and the resultant block diagram is shown in Fig. E3.13(f).

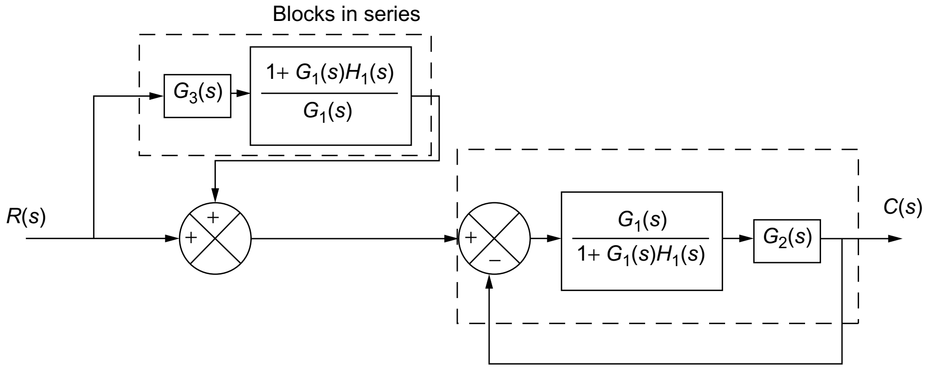


Fig. E3.13(e)

Step 5: The blocks in parallel and the feedback loop as indicated by dotted lines in Fig. E3.13(f) can be reduced and the resultant block diagram is shown in Fig. E3.13(g).

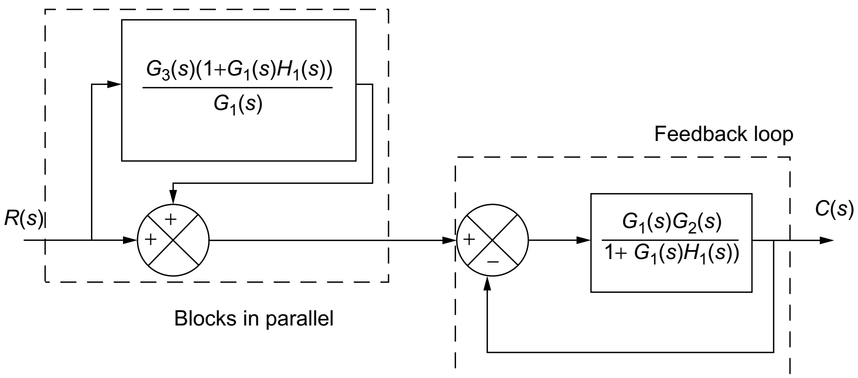


Fig. E3.13(f)

Step 6: The blocks in series as indicated by dotted lines in Fig. E3.13(g) can be reduced in order to determine the transfer function of the system.

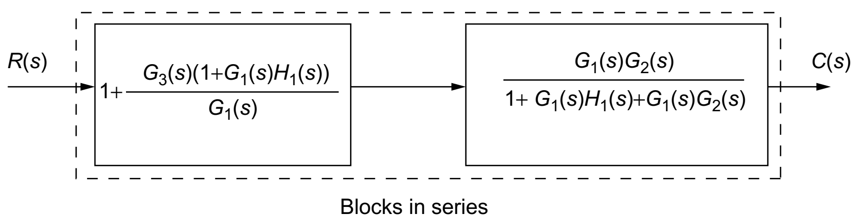


Fig. E3.13(g)

Hence, the transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{G_1(s)+\{G_3(s)\times(1+G_1(s)H_1(s))\}}{G_1(s)} \times \frac{G_1(s)G_2(s)}{(1+G_1(s)H_1(s)+G_1(s)G_2(s))}$$

$$\text{i.e., } \frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)+G_2(s)G_3(s)+G_1(s)G_2(s)G_3(s)H_1(s)}{1+G_1(s)H_1(s)+G_1(s)G_2(s)}$$

Example 3.14: For the block diagram of the system shown in Fig. E3.14(a), determine the transfer function using the block diagram reduction technique.

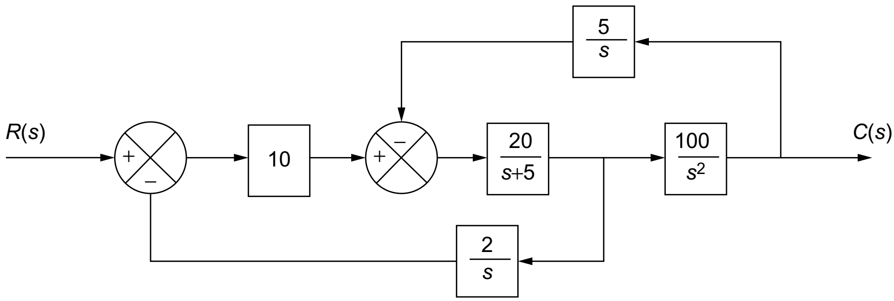


Fig. E3.14(a)

Solution:

Step 1: The take-off point that exists between the blocks $\frac{20}{(s+5)}$ and $\frac{100}{s^2}$ as indicated by dotted lines in Fig. E3.14(b) is shifted after the block $\frac{100}{s^2}$ and the resultant diagram is shown in Fig. E3.14(c).

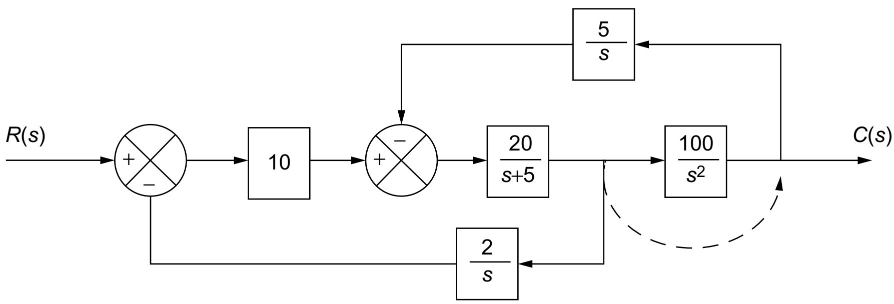


Fig. E3.14(b)

Step 2: The blocks in series as indicated by dotted lines in Fig. E3.14(c) can be combined to form a single block. The resultant diagram is shown in Fig. E3.14(d).

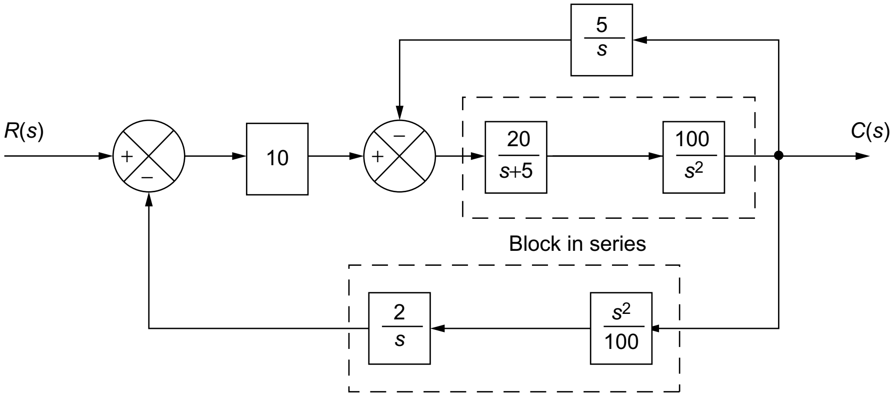


Fig. E3.14(c)

Step 3: The feedback path as indicated by dotted lines in Fig. E3.14(d) can be reduced and the resultant block diagram is shown in Fig. E3.14(e).

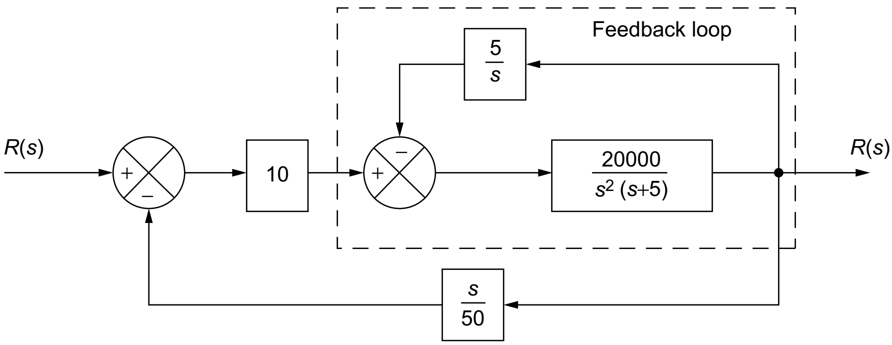


Fig. E3.14(d)

Step 4: The blocks in series as indicated by dotted lines in Fig. E3.14(e) can be combined to form a single block. The resultant block diagram is shown in Fig. E3.14(f).

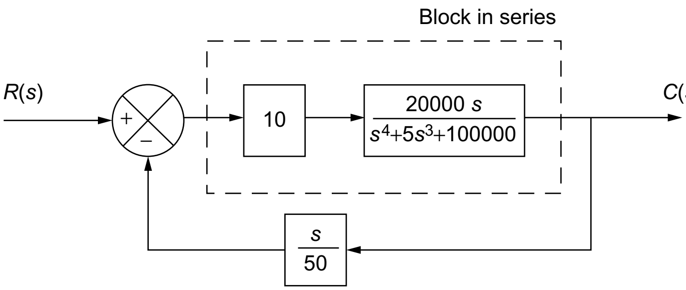


Fig. E3.14(e)

Step 5: The feedback path as indicated by dotted lines in Fig. E3.14(f) can be reduced in order to determine the transfer function of the system.

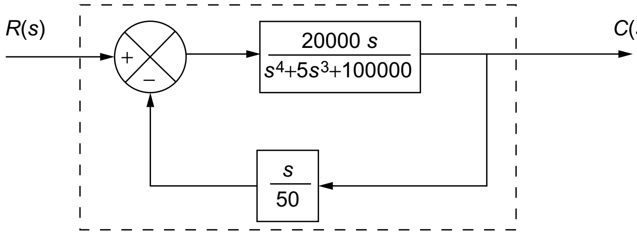


Fig. E3.14(f)

Hence, the transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{\dfrac{20000s}{s^4+5s^3+100000}}{1+\dfrac{20000s}{s^4+5s^3+10000}\times\dfrac{s}{50}}$$

$$\text{i.e., } \frac{C(s)}{R(s)} = \frac{20000s}{s^4+5s^3+400s^2+100000}$$

Review Questions

1. What do you mean by block diagram representation of a system?
2. Explain how an open-loop system can be represented by using a block diagram
3. Give the block diagrammatic representation of a generalized closed-loop system.
4. What are the advantages of the block diagram representation of a system?
5. What are the disadvantages of the block diagram representation of a system?
6. Explain with a neat flow chart, how to represent an electrical system with the help of a block diagram.
7. What do you mean by block diagram reduction technique?
8. What is the necessity for using the block diagram reduction technique?
9. What do you mean by block diagram algebra?
10. What are the elements present in the block diagram algebra?
11. Define: (i) block, (ii) input signal, (iii) forward path, (iv) feedback path, (v) feedback signal and (vi) error signal.
12. What do you mean by summing point?
13. Define output signal and take-off point.
14. What are the lists of rules used for reducing the complex block diagram to a simple one?
15. Explain with a neat diagram, the rules for reducing the complex block diagram.

16. Explain with a neat flow chart the procedure used for implementing the block diagram reduction technique.
17. Represent the electrical circuit shown in Fig. Q3.17 by a block diagram.

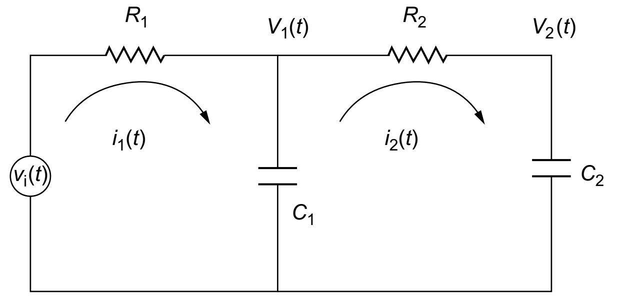


Fig. Q3.17

18. For the block diagram of the system shown in Fig. Q3.18, determine the transfer function using block diagram reduction technique.

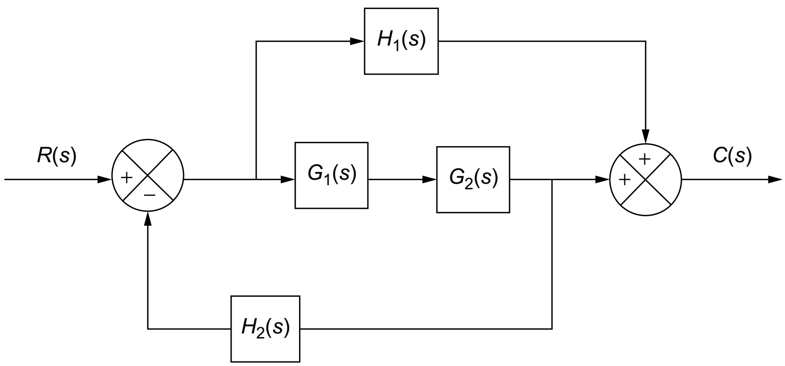


Fig. Q3.18

19. For the block diagram of the system shown in Fig. Q3.19, determine the transfer function using the block diagram reduction technique.

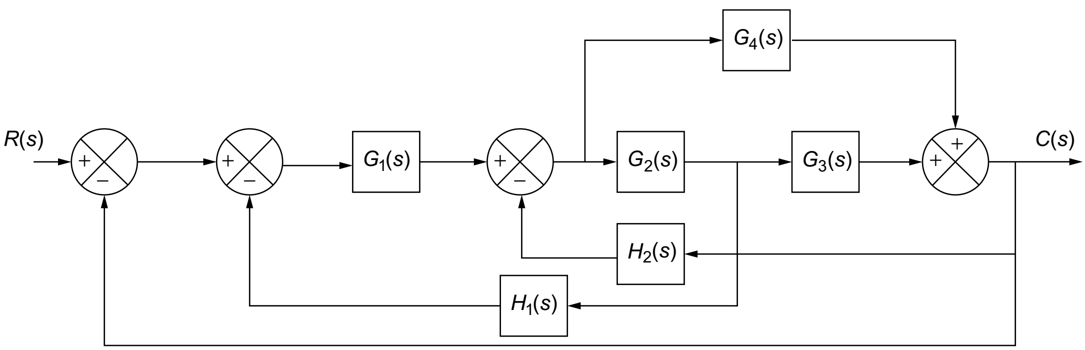


Fig. Q3.19

20. For the block diagram of the system shown in Fig. Q3.20, determine the transfer function using the block diagram reduction technique.

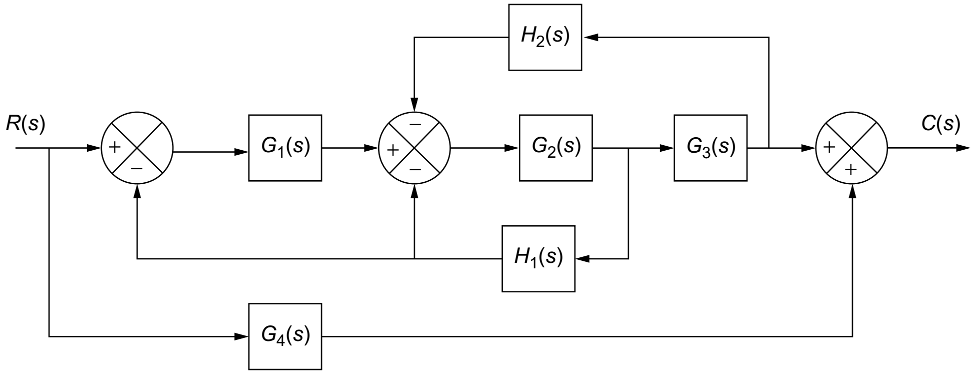


Fig. Q3.20

21. For the block diagram of the system shown in Fig. Q3.21, determine the transfer functions $\frac{C_1(s)}{R_1(s)}$ and $\frac{C_2(s)}{R_1(s)}$ using the block diagram reduction technique.

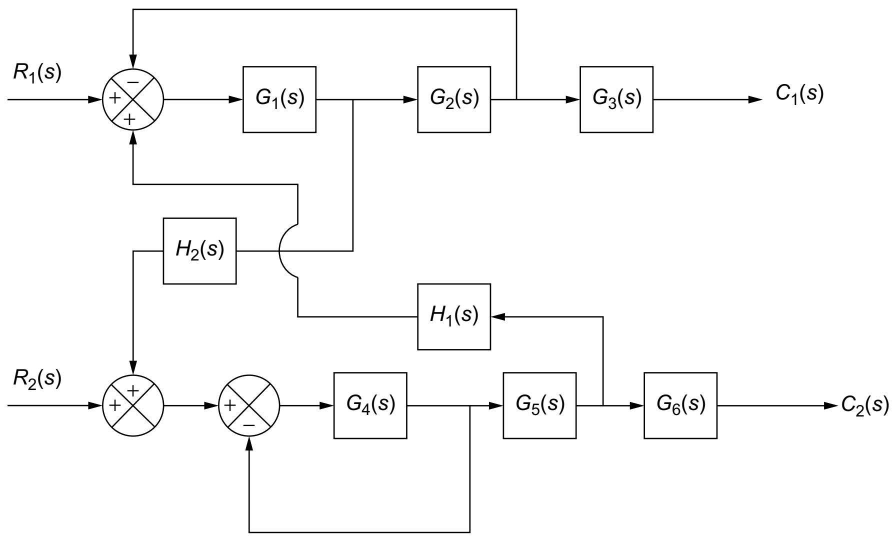


Fig. Q3.21

22. For the block diagram of the system shown in Fig. Q3.22, determine the transfer function using the block diagram reduction technique.

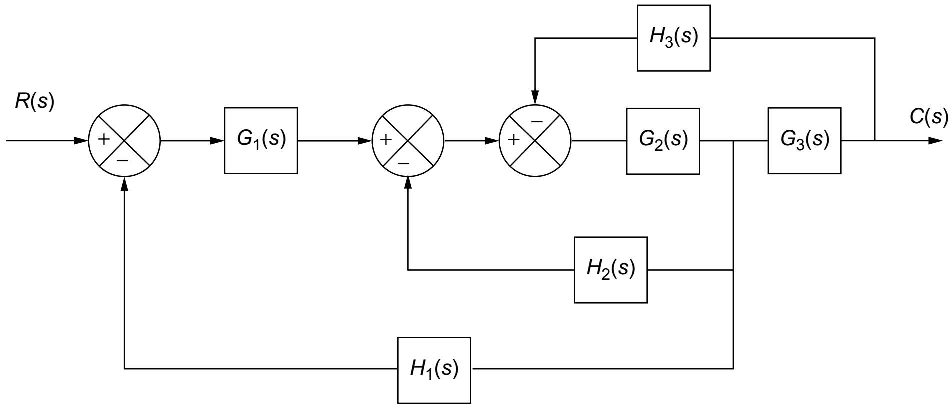


Fig. Q3.22

23. For the block diagram of the system shown in Fig. Q3.23, determine the transfer function using the block diagram reduction technique.

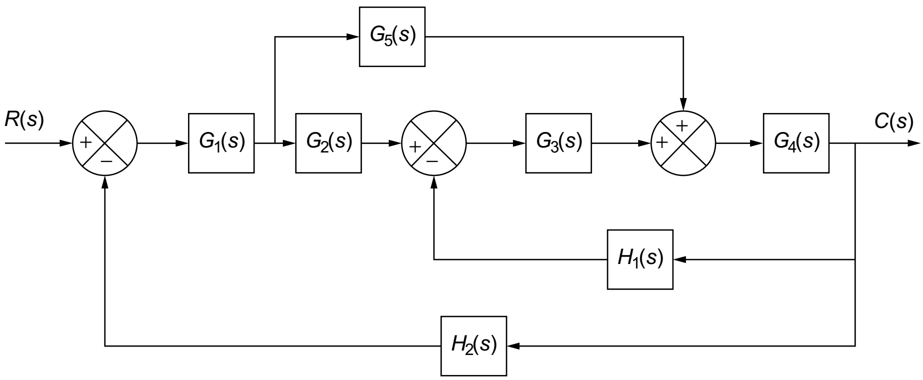


Fig. Q3.23

24. For the block diagram of the system shown in Fig. Q3.24, determine the transfer function using the block diagram reduction technique.

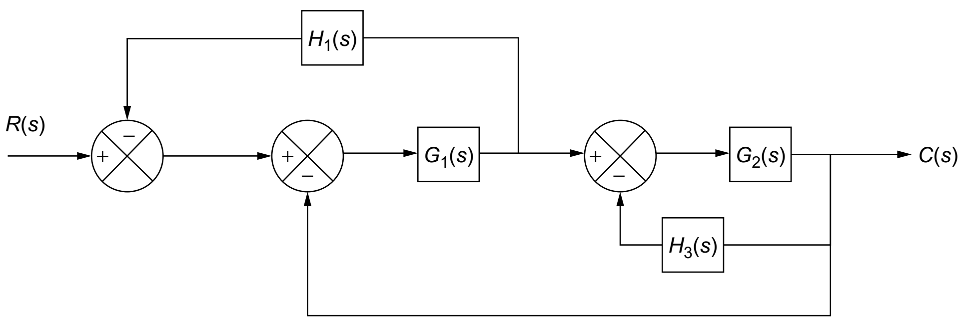


Fig. Q3.24

25. Represent the electrical circuit shown in Fig. Q3.25 by a block diagram.

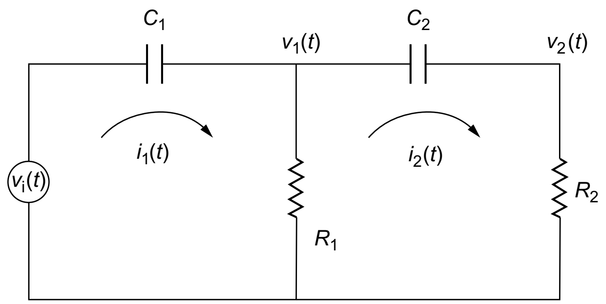


Fig. Q3.25

26. For the system shown in Fig. Q3.26, determine the transfer function using block diagram reduction technique.

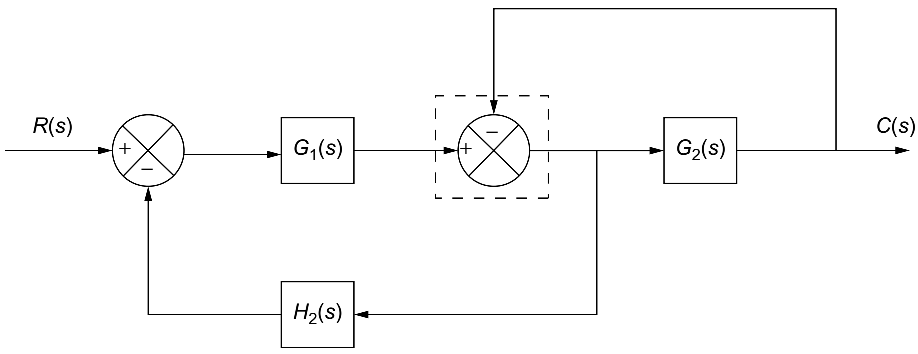


Fig. Q3.26

27. For the block diagram of the system shown in Fig. Q3.27, determine the transfer function using the block diagram reduction technique.

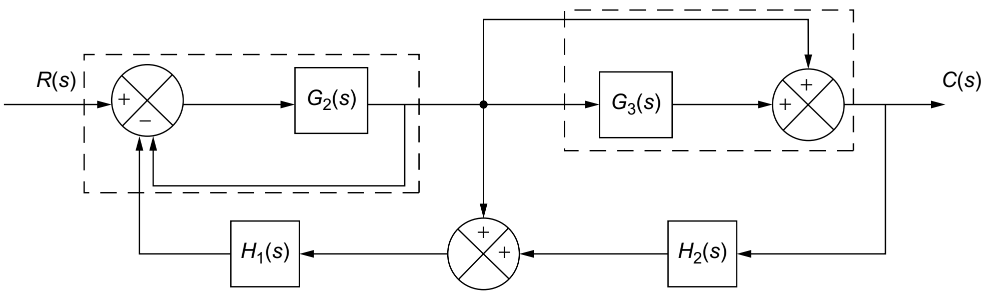


Fig. Q3.27

28. For the block diagram of the system shown in Fig. Q3.28, determine the output using the block diagram reduction technique when (i) each input acts separately on the system and (ii) both the inputs act on the system simultaneously.

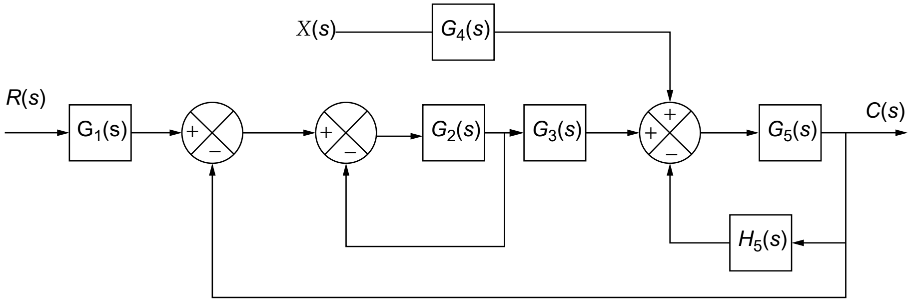


Fig. Q3.28

29. For the block diagram of the system shown in Fig. Q3.29, determine the output using the block diagram reduction technique when (i) only the input R(s) acts on the system, (ii) only the disturbance input D(s) acts on the system and (iii) both the inputs act on the system simultaneously.

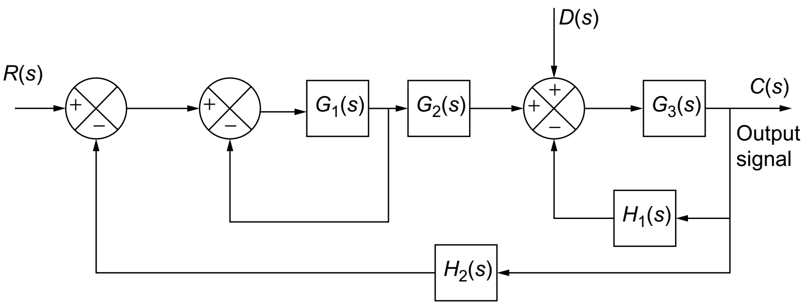


Fig. Q3.29

30. For the block diagram of the system shown in Fig. Q3.30, determine the transfer function using the block diagram reduction technique.

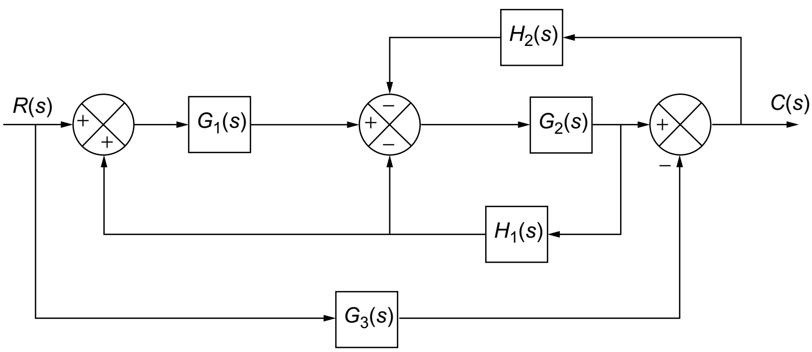


Fig. Q3.30

31. For the block diagram of the system shown in Fig. Q3.31, determine the transfer function using the block diagram reduction technique.

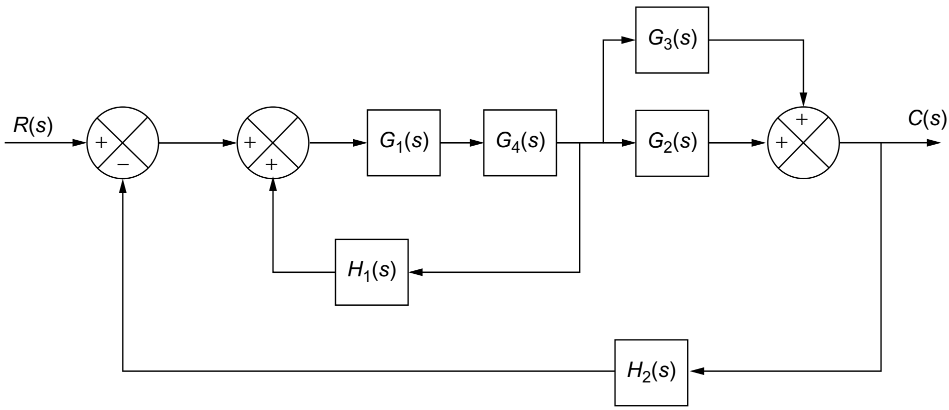


Fig. Q3.31

32. For the block diagram of the system shown in Fig. Q3.32, determine the transfer function using the block diagram reduction technique.

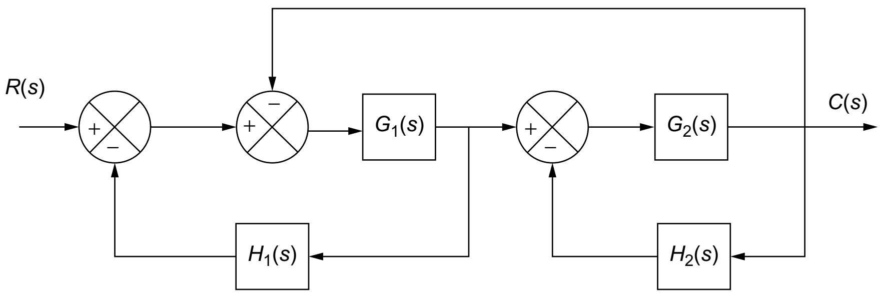


Fig. Q3.32

33. For the block diagram of the system shown in Fig. Q3.33, determine the transfer function using the block diagram reduction technique.

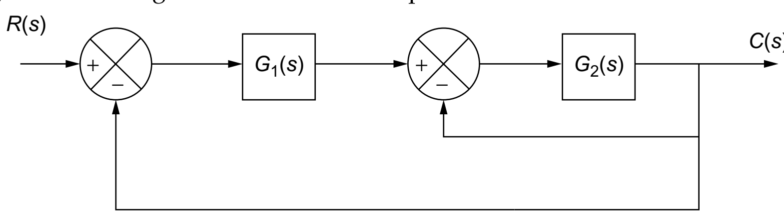


Fig. Q3.33

4

SIGNAL FLOW GRAPH

4.1 Introduction

The diagrammatic or pictorial representation of a set of simultaneous linear algebraic equations of a more complicated system is known as signal flow graph (SFG). It shows the flow of signals in the system. It is important to note that the flow of signals in SFG is only in one direction. To represent the set of algebraic equations using SFG, it is necessary that those algebraic equations are to be represented in the *s*-domain. The transfer function of the system which is represented by SFG can be obtained by using Mason's gain formula.

The dependent and independent variables in the set of algebraic equations are represented by the nodes in the SFG. The branches are used to connect different nodes present in SFG. The connection between the different nodes is based on the relationship given in the algebraic equation. The arrow and the multiplication factor indicated on the branch of SFG represent the signal direction.

The SFG and the block diagram representation of a system yield the same transfer function; but when a system is represented by SFG, the transfer function is obtained easily and quickly without using the SFG reduction techniques.

4.1.1 Signal Flow Graph Terminologies

The terminologies used in SFG are explained with the help of SFG of a system as shown in Fig. 4.1.

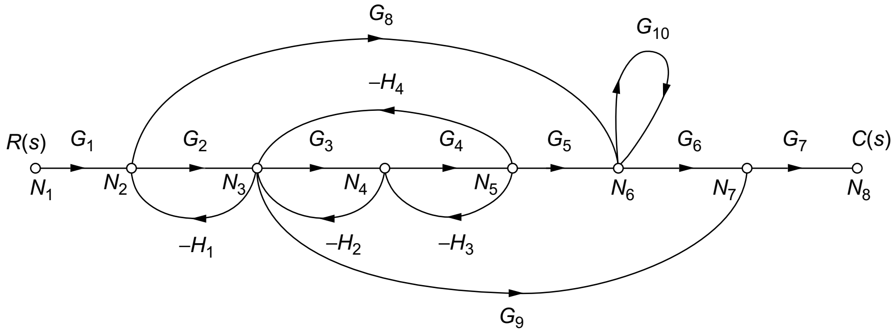


Fig. 4.1 | Signal flow graph of a system

Node

The variables present in the set of algebraic equations are represented by a point called node. The classification of nodes is shown in Fig. 4.2.

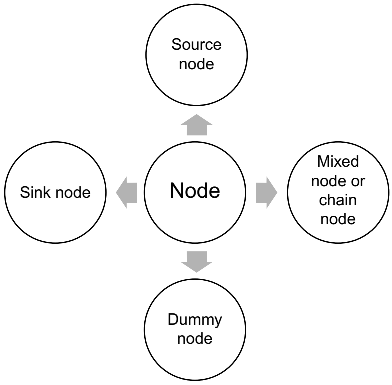


Fig. 4.2 | Classification of nodes

The different types of nodes shown in Fig. 4.2 are explained in detail as follows:

Dummy node – (Special case)

If there is no source or sink node in the SFG of a system, then the input and output nodes can be created by adding branches with gain 1, which are known as dummy nodes. A schematic diagram representing the dummy node is shown in Fig. 4.3.

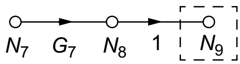


Fig. 4.3 | A schematic diagram representing dummy node

Branch

The line segment joining the two nodes with a specific direction is known as a branch. The specific direction is indicated by an arrow in the branch. A schematic diagram representing the branch is shown in Fig. 4.4.

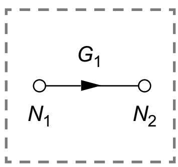


Fig. 4.4 | A schematic diagram representing a branch

Source Node or Input Node

The type of node that has only outgoing branches is known as a source node or input node. It is the only independent variable present in the system as it does not depend on any other variables. All the other nodes present in the given system are dependent variables. The dotted line shown in Fig. 4.5 highlights the source node or the input node.

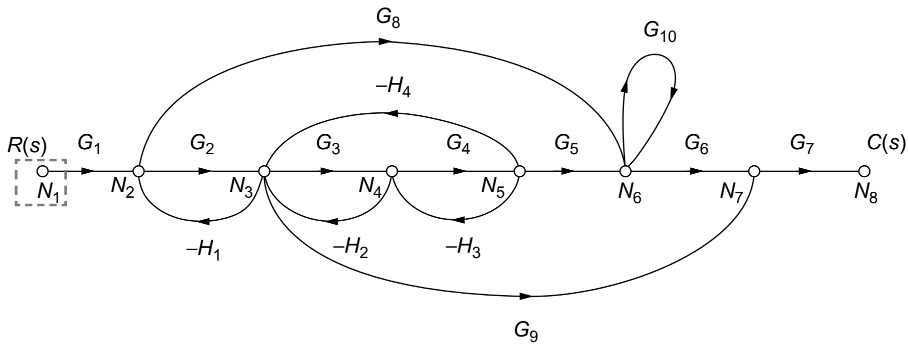


Fig. 4.5 | Source node or Input node

Sink Node or Output Node

Sink node or output node is the one that has only incoming branches. The dotted lines shown in Fig. 4.6 highlight the sink node or output node.

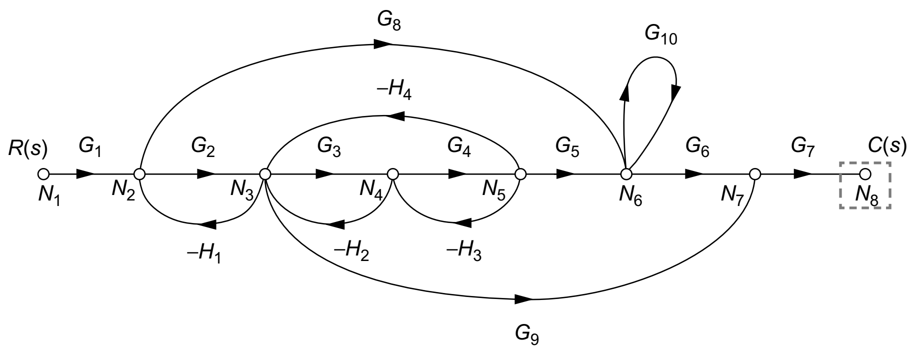


Fig. 4.6 | Sink node or output node

Mixed Node or Chain Node

Mixed or chain node is the one that has both incoming and outgoing branches. The dotted lines shown in Fig. 4.7 highlight the mixed node or chain node.

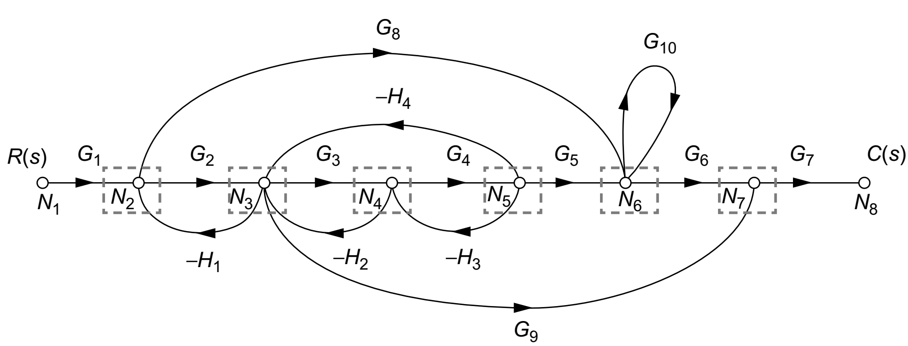


Fig. 4.7 | Mixed node or chain node

Transmittance

When the signal flows from one node to another, the signal acquires a multiplication factor or gain. This gain between the two nodes is known as transmittance and it is indicated above the arrow mark in the branch. The transmittance between a dummy node and any other node is 1. The dotted lines shown in Fig. 4.8 highlight the transmittance between nodes N_1 and N_2.

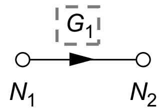


Fig. 4.8 | Transmittance

Path

When the branches between the nodes are connected continuously in a specific direction and indicated by arrows on the branch, it is known as a path. The classification of a path is shown in Fig. 4.9.

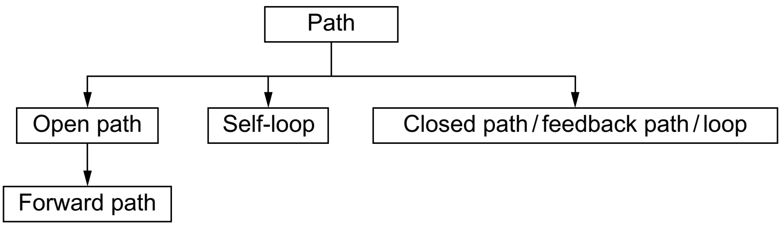


Fig. 4.9 | Classification of a path

Open Path

If the path starts from one node and ends at another node with none of the nodes traversed more than once is known as an open path.

Forward Path

In the open path, if the path starts from source or input node and ends at sink or output node with none of the nodes traversed more than once is known as forward path. The path represented by the dotted lines in Fig. 4.10 highlights one of the forward paths present in the SFG of the system shown in Fig. 4.1 in which the input node is N_1 and the output node is N_8.

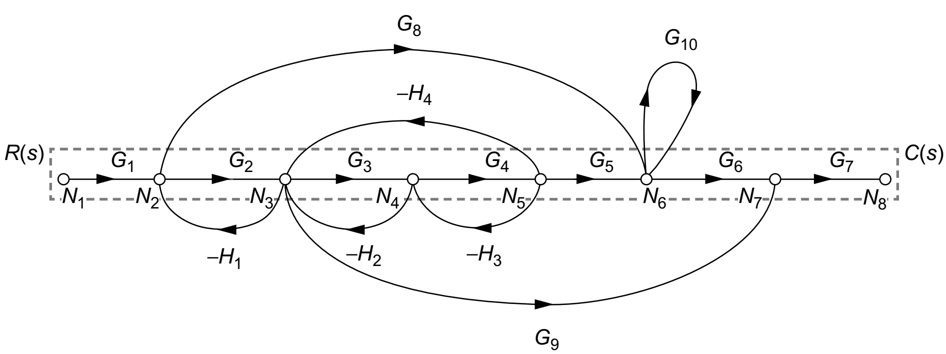


Fig. 4.10 | Forward path

The other types of forward paths are shown in Figs. 4.11 and 4.12.

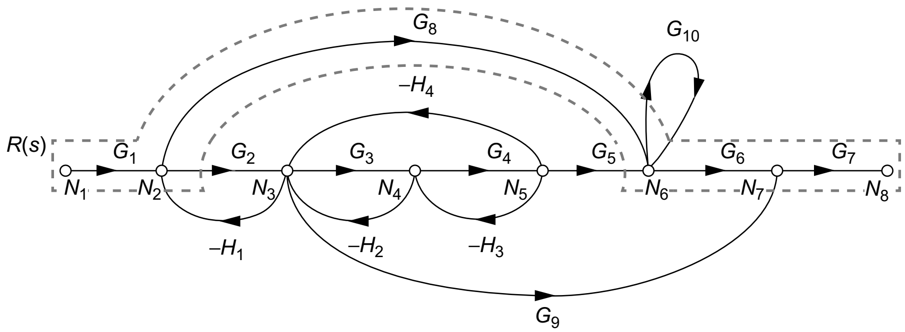


Fig. 4.11 | Forward path

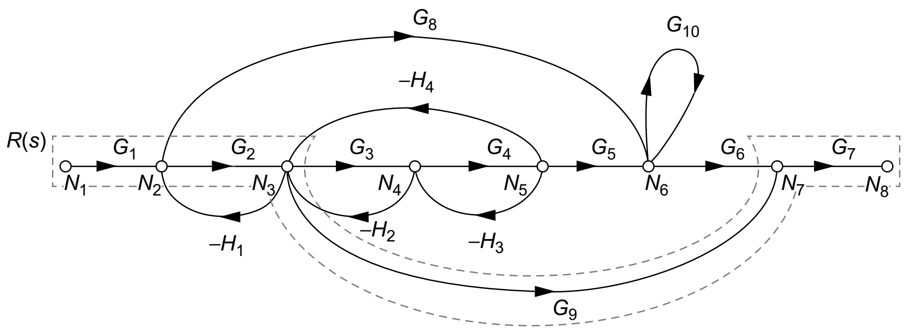


Fig. 4.12 | Forward path

Closed Path or Loop or Feedback Path

If the path starts from one node and ends at the same node while traversing the other nodes for only one time, then it is known as a closed path or loop or a feedback path. The paths that are encircled by dotted lines in the schematic diagrams of Figs. 4.13 through 4.15 highlight the closed path or loop or feedback path for the SFG of the system shown in Fig. 4.1.

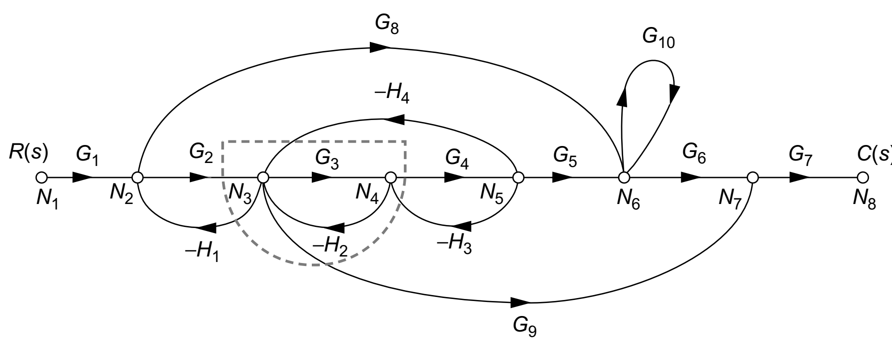


Fig. 4.13 | Closed path or loop or feedback path

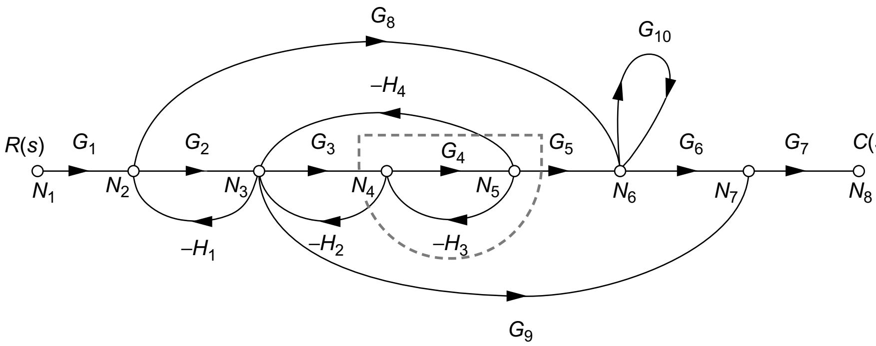


Fig. 4.14 | Closed path or loop or feedback path

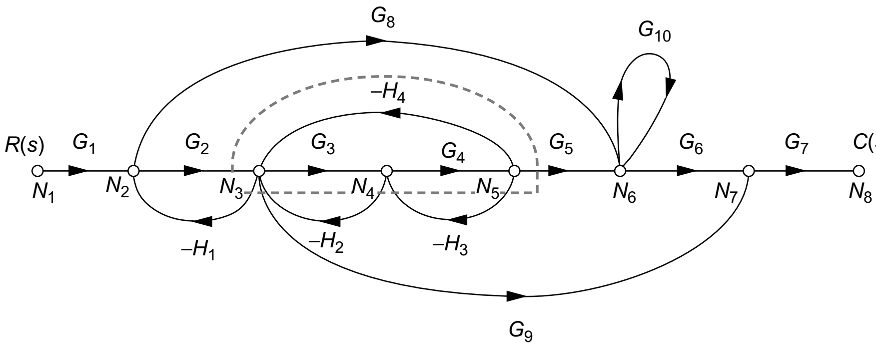


Fig. 4.15 | Closed path or loop or feedback path

Self-Loop

The path that starts from one node and ends at the same node without traversing any other nodes is known as self-loop. The self-loop is also a type of closed path, since the path starts and ends at the same node. The loop encircled by the dotted lines in Fig. 4.16 highlights the self-loop.

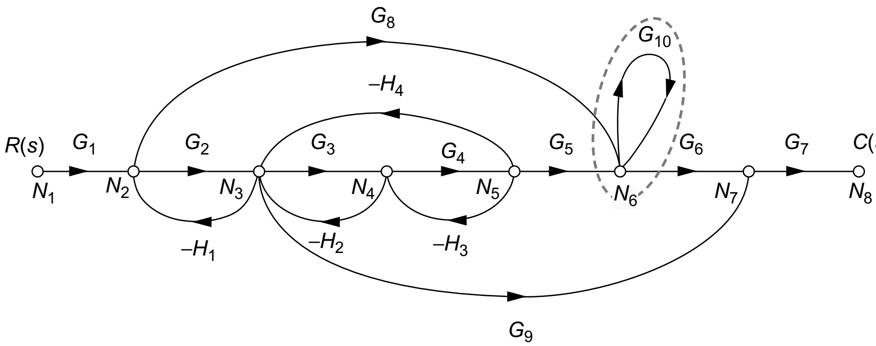


Fig. 4.16 | Self-loop

Non-Touching Loops

If two or more individual loops existing in the given system do not have any common nodes between them, then the loops are known as non-touching loops. The loops encircled by the dotted lines in Fig. 4.17 through Fig. 4.21 highlight the non-touching loops as there are no common nodes.

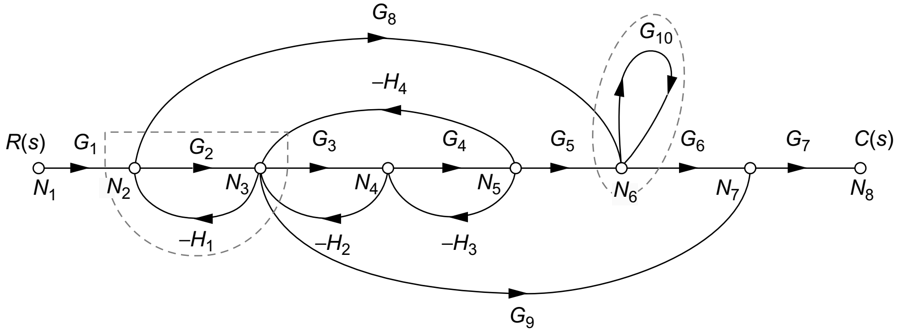


Fig. 4.17 | Non-touching loops

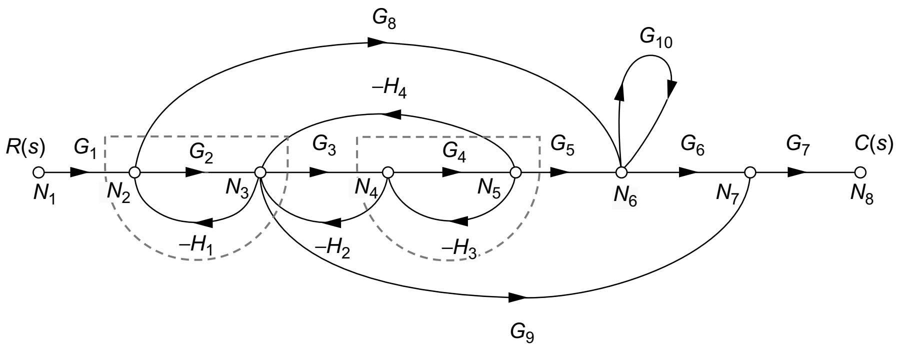


Fig. 4.18 | Non-touching loops

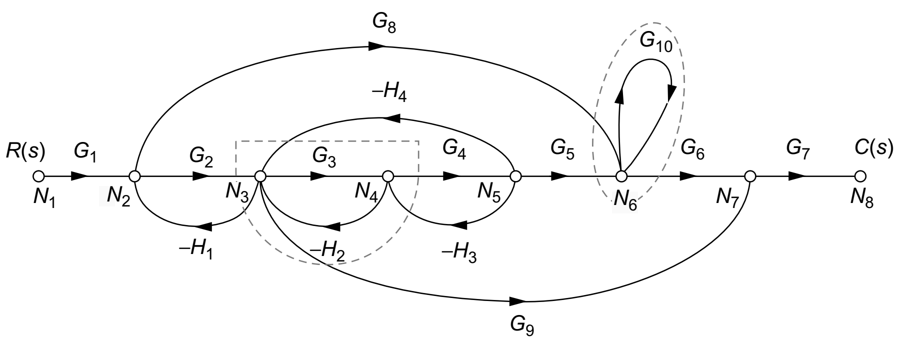


Fig. 4.19 | Non-touching loops

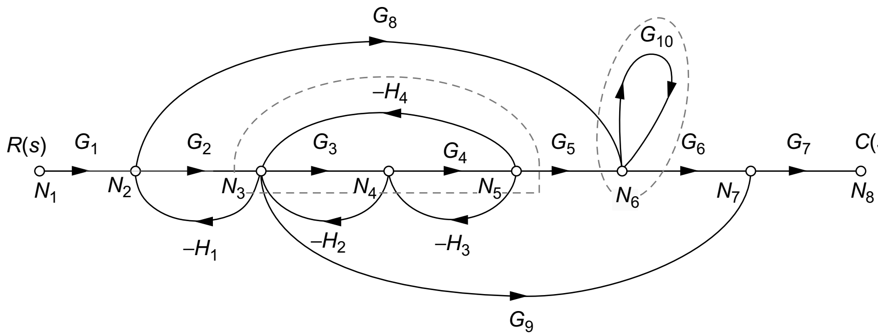


Fig. 4.20 | Non-touching loops

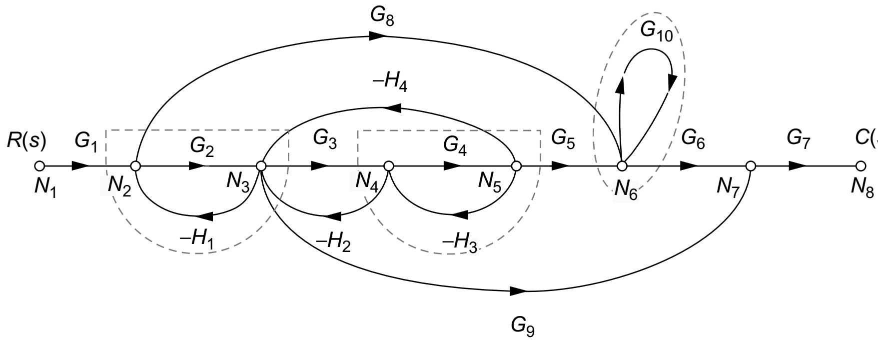


Fig. 4.21 | Non-touching loops

Gain

In a path, the transmittance existing in each of the branch can be multiplied to form a total transmittance of the path. This total transmittance is known as the gain of the path. The gains are classified as shown in Fig. 4.22 depending on the path.

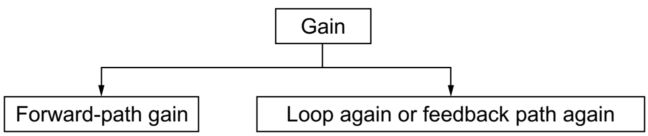
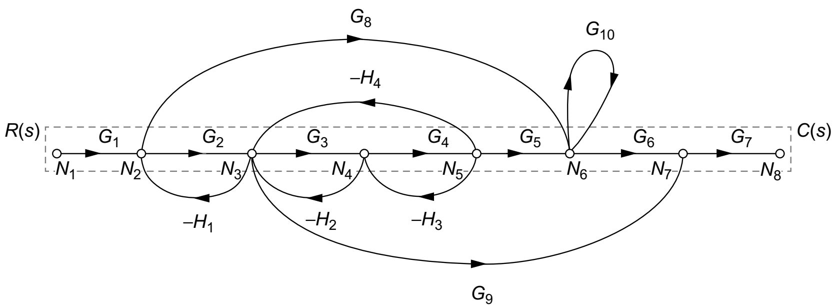
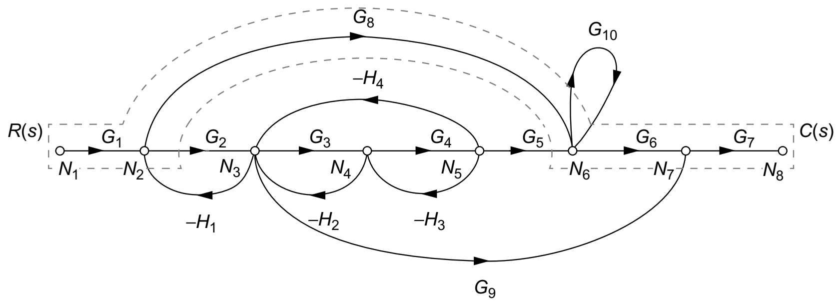
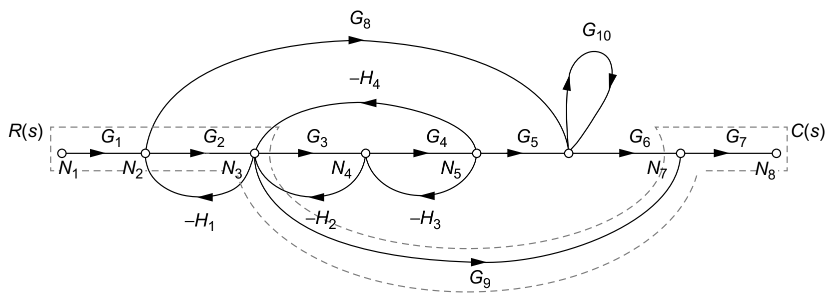


Fig. 4.22 | Classification of gain

Forward-Path Gain

If the transmittance of the branch existing in the forward path is multiplied, then the resultant gain is known as forward-path gain. The forward-path gains for different forward paths existing in the SFG of the system are given in Table 4.1.

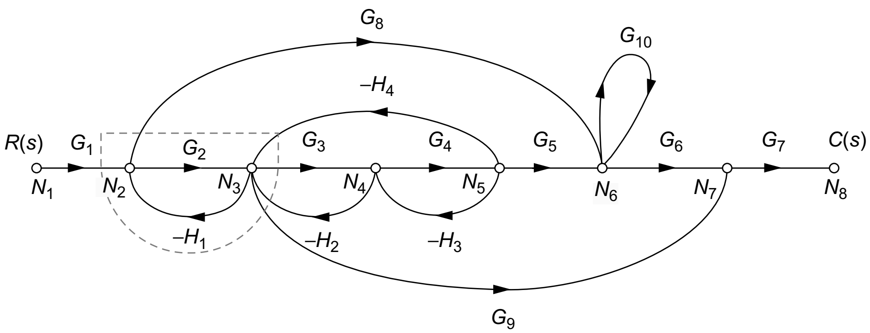
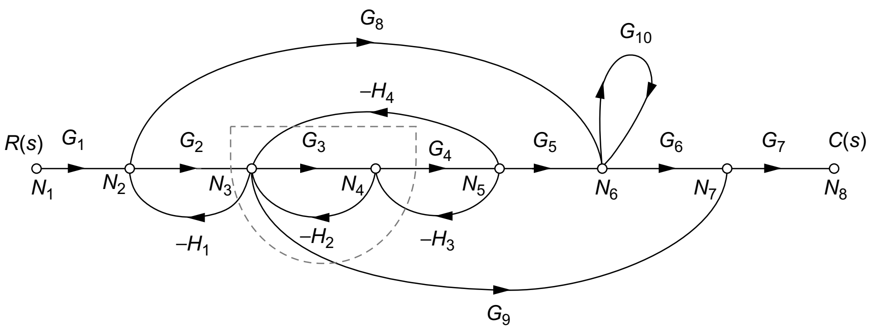
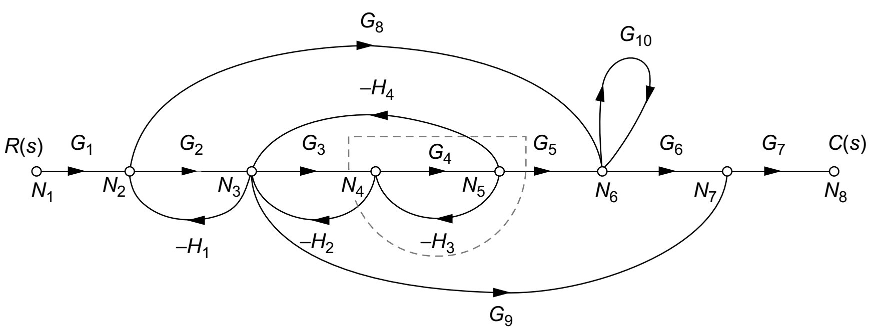
Table 4.1 | Forward-path gains

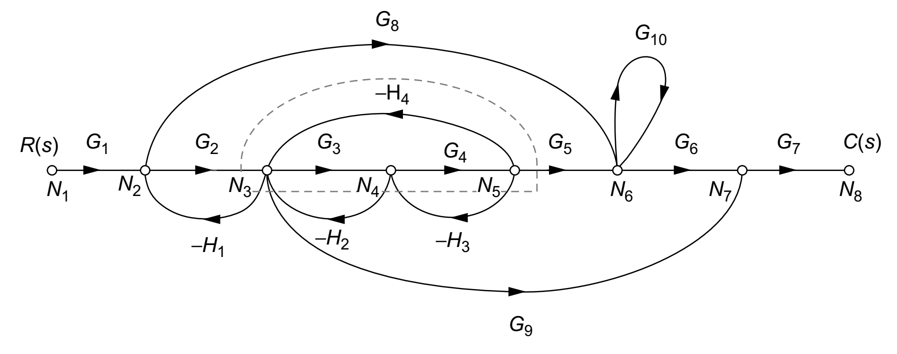
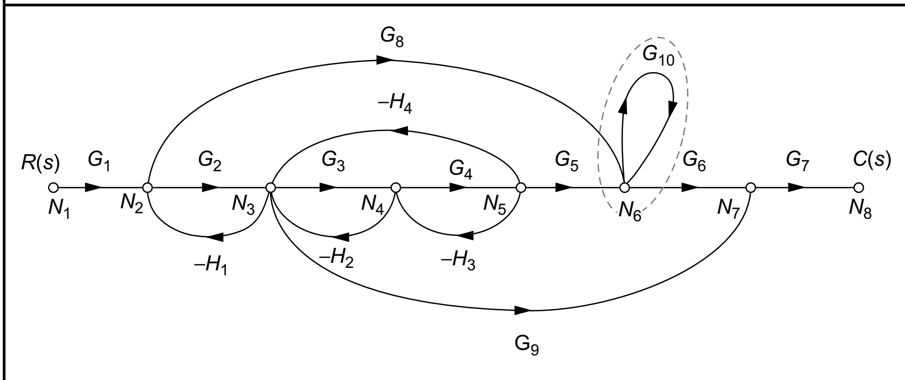
Forward path	Gain
	$G_1G_2G_3G_4\,G_5G_6G_7$
	$G_1G_6G_7G_8$
	$G_1G_2G_7G_9$

Loop Gain or Feedback Path Gain

If the transmittance of the branch existing in the loop is multiplied, the resultant gain is known as loop or feedback path gain. The loop or feedback path gains for different loops existing in the SFG of the system shown in Fig. 4.1 are given in Table 4.2.

Table 4.2 | Loop or feedback path gains

Feedback path	Loop or feedback path gain
	$-G_1H_1$
	$-G_3H_2$
	$-G_4H_3$

<table>
<tr>
<td>
</td>
<td>$-G_3G_4H_4$</td>
</tr>
<tr>
<td>
</td>
<td>$-G_{10}$</td>
</tr>
</table>

4.1.2 Properties of SFG

The important properties of SFG are:

(i) There must be a cause-and-effect relationship between the algebraic equations for which the SFG has to be constructed.
(ii) SFG is applicable to linear time invariant systems.
(iii) Flow of signal in the SFG of the system is indicated by the arrow present in the branch between the nodes.
(iv) SFG for a particular system is not unique since the differential equations of a particular system can be written in different ways.
(v) Variable or a signal given in the differential equation is indicated by a node.
(vi) Functional dependence of a node on another node is indicated by a branch.

4.1.3 SFG Algebra

The transfer function of the system represented using SFG can be obtained by applying certain algebraic rules to the SFG of the system. The following rules are used in Mason's gain formula for determining the transfer function of the given system.

Rule 1: Signal at a node: The signal at a node is obtained by multiplying the transmittance of the branch connecting the nodes and the signal at the previous node. The schematic diagram explaining the above concept is shown in Figs. 4.23(a) and 4.23(b).

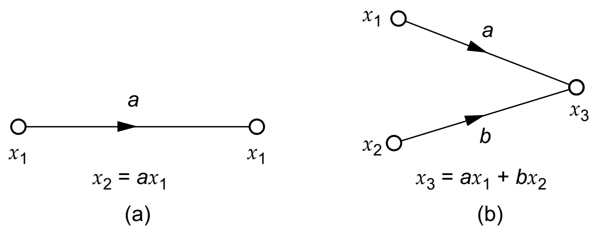


Fig. 4.23 | Signal at a node

Rule 2: When branches are in cascade connection: If the branches are pointing towards the same direction, then the branches are said to be in cascade connection as shown in Fig. 4.24(a). The transmittances indicated in each branch of the cascade connection can be multiplied to form a single branch as shown in Fig. 4.24(b).

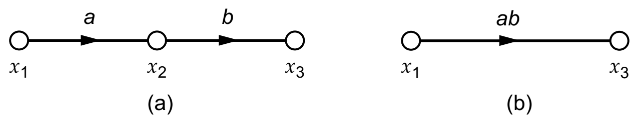


Fig. 4.24 | Branches in cascade connection

Rule 3: When branches are in parallel connection: If one or more branches pointing in the same direction originate and ends at the same node, then the branches in the SFG of the system are said to be in parallel connection as shown in Fig. 4.25(a). The transmittances indicated in each branch of the parallel connection can be added to form a single branch as shown in Fig. 4.25(b).

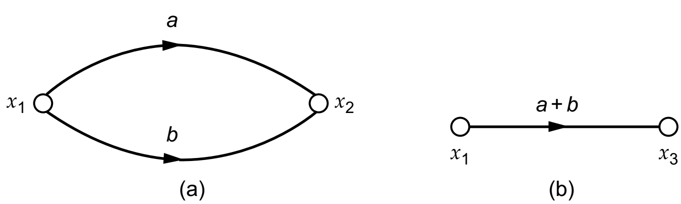


Fig. 4.25 | Branches in parallel connection

Rule 4: Elimination of loops: The loops existing in the SFG of the system as shown in Fig. 4.26(a) can be eliminated as shown in Fig. 4.26(b). But while determining the transfer function of the system, this rule is not used.

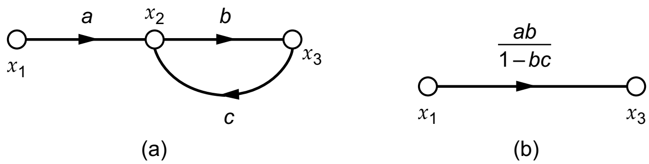


Fig. 4.26 | Elimination of loops

Rule 5: Elimination of mixed node: Mixed node with one or more incoming branches and an outgoing branch as shown in Fig. 4.27(a) can be eliminated by multiplying the transmittance of outgoing branch with transmittance of each of the incoming branches. The SFG of the system after eliminating such mixed node is shown in Fig. 4.27(b).

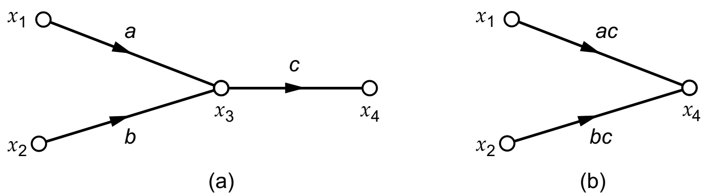


Fig. 4.27 | Elimination of mixed node

4.1.4 Mason's Gain Formula for SFG

The transfer function or overall gain of the system represented by SFG can be obtained using Mason's gain formula given by

$$P = \frac{1}{\Delta}\sum_{k=1}^{n} P_k \Delta_k$$

where P_k is the k^{th} forward-path gain, n is the total number of forward paths and $\Delta = 1 - S_1 + S_2 - S_3 + \cdots$.

Here S_1 is the sum of all individual loop gains, S_2 is the sum of the product of gains of all possible combination of two non-touching loops and S_3 is the sum of the product of gains of all possible combination of three non-touching loops. Therefore,

$\Delta_k = 1-$ (sum of individual loop gains that do not touch the k^{th} forward path)

4.1.5 Signal Flow Graph From Differential Equation

The flowchart to determine the signal flow graph from a set of differential equations is shown in Fig. 4.28.

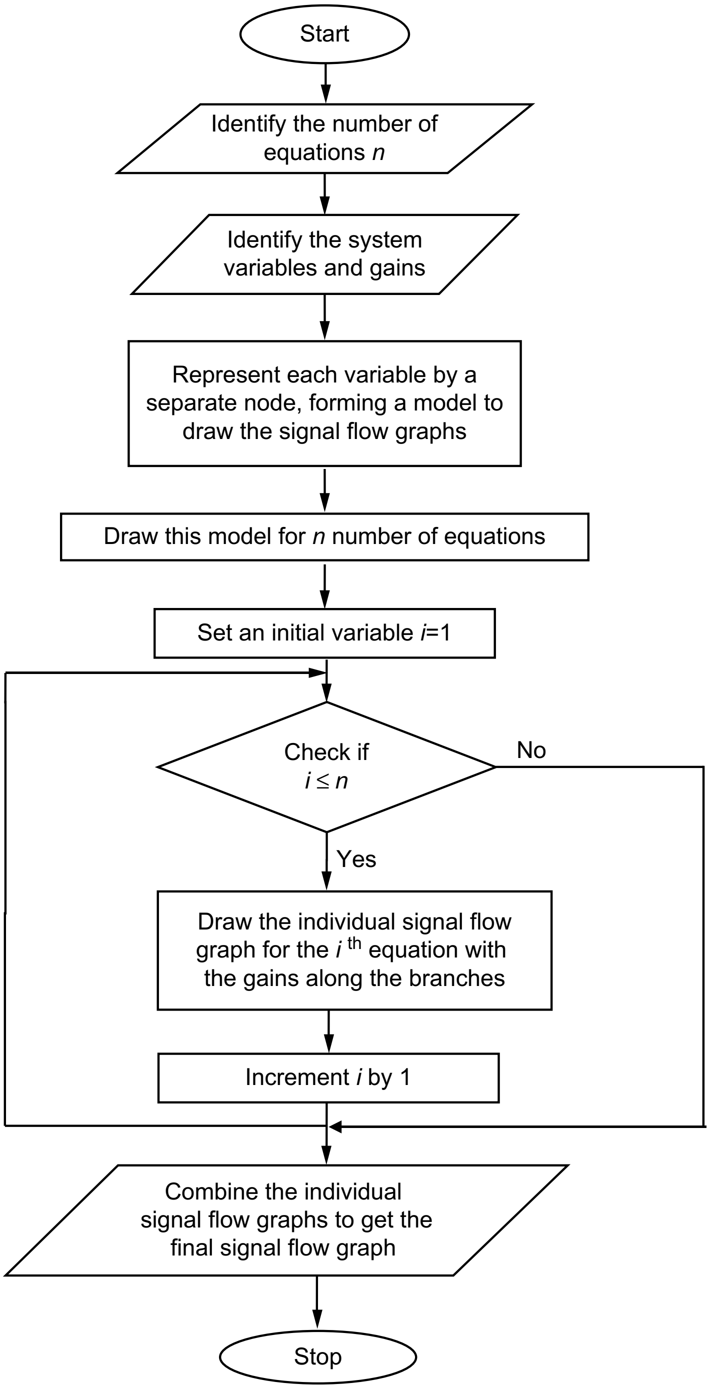


Fig. 4.28 | Flow chart to determine SFG from a set of differential equations.

Example 4.1: The equations describing the system are given below:

$$Y_1(s) = a_{11}Y_1(s) + Y_2(s) + b_1U_1(s)$$

$$Y_2(s) = a_{21}Y_1(s) + a_{22}Y_2(s) + b_2U_2(s)$$

Obtain (a) SFG for the given system and (b) transfer functions of the system using Mason's gain formula:

$$\left.\frac{Y_2(s)}{U_1(s)}\right| \text{ for } U_2(s) = 0 \text{ and } \left.\frac{Y_2(s)}{U_2(s)}\right| \text{ for } \quad U_1(s) = 0$$

Solution:

Given
$$Y_1(s) = a_{11}Y_1(s) + Y_2(s) + b_1U_1(s) \qquad (1)$$

$$Y_2(s) = a_{21}Y_1(s) + a_{22}Y_2(s) + b_2U_2(s) \qquad (2)$$

(a) To obtain SFG for the given system:

For the given equations, the inputs are $U_1(s)$ and $U_2(s)$. In addition, the outputs for the given equations are $Y_1(s)$ and $Y_2(s)$ respectively.

Step 1: The SFG for Eqn. (1) is shown in Fig. E4.1(a).

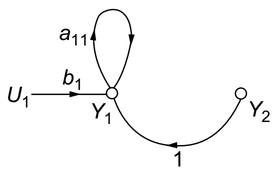


Fig. E4.1(a)

Step 2: The SFG for Eqn. (2) is shown in Fig. E4.1(b).

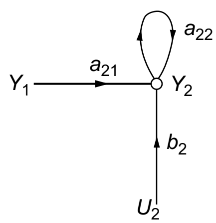


Fig. E4.1(b)

Step 3: The SFG for the system represented by the Eqs. (1) and (2) can be obtained by combining the individual signal flow graphs that are shown in Figs. E4.1(a) and (b). The SFG for the system is shown in Fig. E4.1(c).

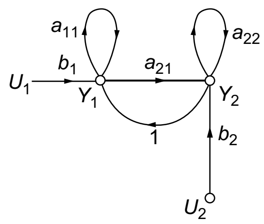


Fig. E4.1(c)

(b) To obtain the transfer function of the system:

(1) To determine $\frac{Y_2(s)}{U_1(s)}$

To determine the required transfer function, it is necessary to make the input $U_2(s)$ zero. Hence, the required transfer function will be

$$\left.\frac{Y_2(s)}{U_1(s)}\right|_{U_2(s)=0}$$

Step 1: The modified SFG from the original SFG with nodes marked is shown in Fig. E4.1(d).

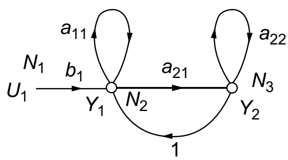
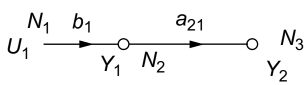


Fig. E4.1(d)

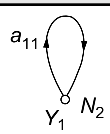
Step 2: The number of forward paths n in the SFG shown in Fig. E4.1(d) is one and the gain corresponding to the forward path is given in Table E4.1(a).

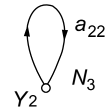
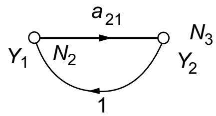
Table E4.1(a) | Gain for the forward path

Forward path	Gain
	$P_1 = b_1 a_{21}$

Step 3: The number of individual loops m present in the SFG shown in Fig. E4.1(d) is $m = 3$ and the gain corresponding to each loop is given in Table E4.1(b).

Table E4.1(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = a_{11}$

2.		$L_2 = a_{22}$
3.		$L_3 = a_{21}$

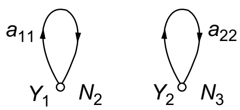
Step 4: Then the individual loop gains are added together to determine the total individual loop gain S_1. Therefore,

$$S_1 = \sum_{i=1}^{m} L_{\text{i}}, \text{ where } m = 3$$

$$= a_{11} + a_{22} + a_{21}$$

Step 5: Determine the number of possible combination of loops (j), where $j = 2$ *to* m, which has no node in common in the SFG. For the given combination of loop j, determine the product of gains of all possible j combination of non-touching loops $\left(L_{\text{ji}}\right)$, where $i = 1, 2, \cdots$. For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.1(c).

Table E4.1(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = a_{11}a_{22}$

Step 6: Determine the sum of gain of loops $\left(S_{\text{j}}\right)$, where $j = 2$ *to* m, for each combination of non-touching loops j. In the given problem, there exists only one different combination of two loops that have no node in common. Hence, S_{j}, where $j = 3$ to m is zero.

Therefore, $$S_2 = a_{11}a_{22}$$

Step 7: Determine Δ_{i} where $i = 1$ to n for the given SFG. Since there exists no part of the graph that does not touch the forward path in the given SFG, the value of Δ_1 is one.

Step 8: The transfer function of the system using Mason's gain formula is given by

$$\left.\frac{Y_2(s)}{U_1(s)}\right| (\text{for} \quad U_2(s) = 0) = \frac{\Delta_1 P_1}{1 - S_1 + S_2} = \frac{b_1 a_{21}}{1 - a_{21} - a_{11} - a_{22} + a_{11}a_{22}}$$

(2) To determine $\frac{Y_2(s)}{U_2(s)}$

To determine the required transfer function, it is necessary to make the input $U_1(s)$ zero. Hence, the required transfer function will be $\frac{Y_2(s)}{U_2(s)}$ *for* $U_1(s) = 0$.

Step 1: The modified SFG from the original SFG with nodes marked is shown in Fig. E4.1(e).

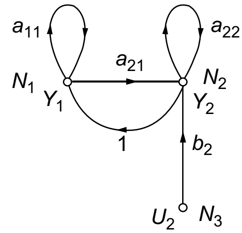
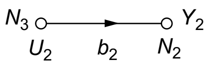


Fig. E4.1(e)

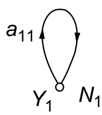
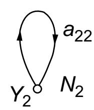
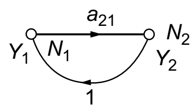
Step 2: The number of forward paths n in the SFG shown in Fig. E4.1(d) is 1 and the gain corresponding to the forward path is given in Table E.4.1(d).

Table E4.1(d) | Gain for the forward path

Forward path	Gain
	$P_1 = b_2$

Step 3: The number of individual loops m present in the SFG shown in Fig. E4.1(d) is $m = 3$ and the gain corresponding to each loop is given in Table E4.1(e).

Table E4.1(e) | Individual loop gains

Individual loops			
Gain	$L_1 = a_{11}$	$L_2 = a_{22}$	$L_3 = a_{21}$

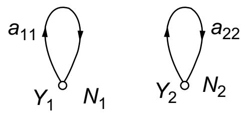
Step 4: Then, the individual loop gains are added together to determine the total individual loop gain S_1. Therefore,

$$S_1 = \sum_{i=1}^{m} L_i, \text{ where } m = 3$$

$$= a_{11} + a_{22} + a_{21}$$

Step 5: Determine the number of possible combination of loops (j), where $j = 2$ *to* m, which has no node in common in the SFG. For the given combination of loop j, determine the product of gains of all possible j combination of non-touching loops (L_{ji}), where $i = 1, 2, \cdots$. For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.1(f).

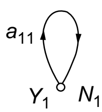
Table E4.1(f) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = a_{11}a_{22}$

Step 6: Determine the sum of gain of loops (S_j), where $j = 2$ to m, for each combination of non-touching loops j. In the given problem, there exists only one different combination of two loops that have no node in common. Hence, (S_j), where $j = 3$ to m is zero. Therefore, $S_2 = a_{11}a_{22}$.

Step 7: Determine (Δ_i), where $i = 1$ to n for the given SFG. The part of the graph that does not touch the forward path and its corresponding gain are given in Table E4.1(g).

Table E4.1(g) | Determination of Δ_2

Loop/part of graph	Gain	Δ_I
	a_{11}	$\Delta_1 = 1 - a_{11}$

Step 8: The transfer function of the system using Mason's gain formula is given by

$$\frac{Y_2(s)}{U_2(s)}\bigg|(for \quad U_1(s) = 0) = \frac{\Delta_1 P_1}{1 - S_1 + S_2} = \frac{b_2(1 - a_{11})}{1 - a_{21} - a_{11} - a_{22} + a_{11}a_{22}}$$

Example 4.2: The SFG for a particular system is shown in Fig. E4.2(a). Determine the block diagram for the system.

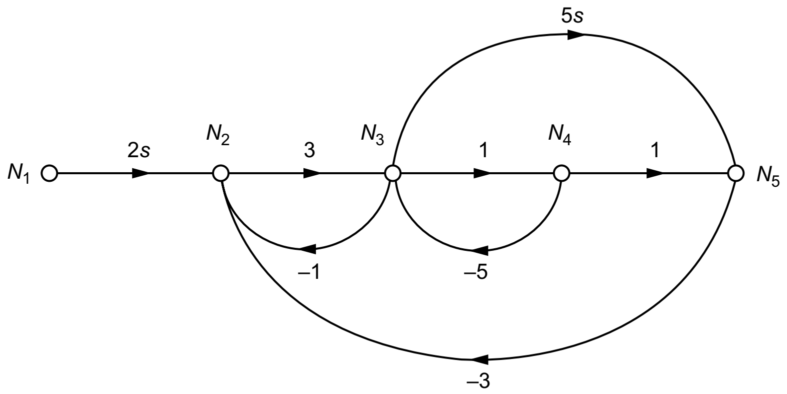


Fig. E4.2(a)

Solution:

Step 1: The SFG shown in Fig. E4.2(a) can be modified as shown in Fig. E4.2(b) by using different variables at different nodes.

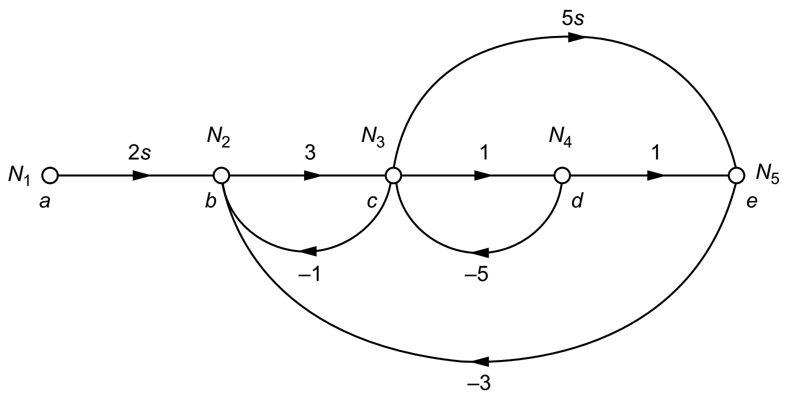


Fig. E4.2(b)

Step 2: To obtain the individual block diagram for the variables in the SFG, it is necessary to obtain its corresponding equations. For the SFG shown in Fig. E4.2(b), there are five variables of which a is the input variable and others are output variables.
The equations for each variable present in the SFG are given below:
For variable b, the equation is given by

$$b = 2sa - c - 3e \tag{1}$$

For variable c, the equation is given by

$$c = 3b - 5d \tag{2}$$

For variable d, the equation is given by

$$d = c \tag{3}$$

For variable e, the equation is given by

$$e = 5sc + d \tag{4}$$

Step 3: Once the equation of each variable present in the SFG is obtained, the individual block diagram for each of the equation is drawn. Hence, the individual block diagrams for the Eqs. (1) to (4) are shown in Figs. E4.2(c) through (f).

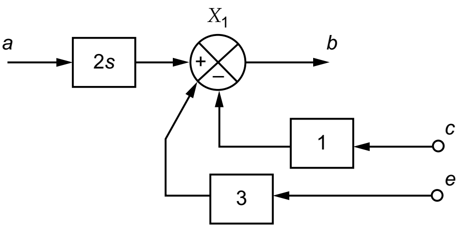


Fig. E4.2(c)

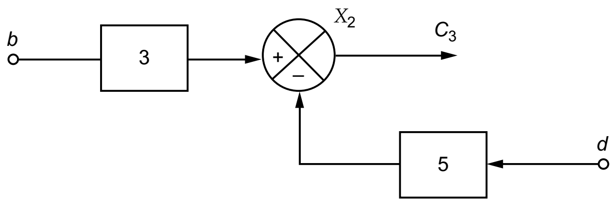


Fig. E4.2(d)

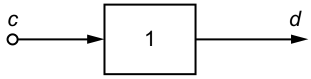


Fig. E4.2(e)

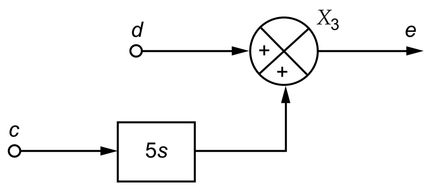


Fig. E4.2(f)

Step 4: The individual block diagrams shown in Figs. E4.2(c) through (f) are combined in an appropriate way and the block diagram equivalent to the given SFG is shown in Fig. E4.2(g).

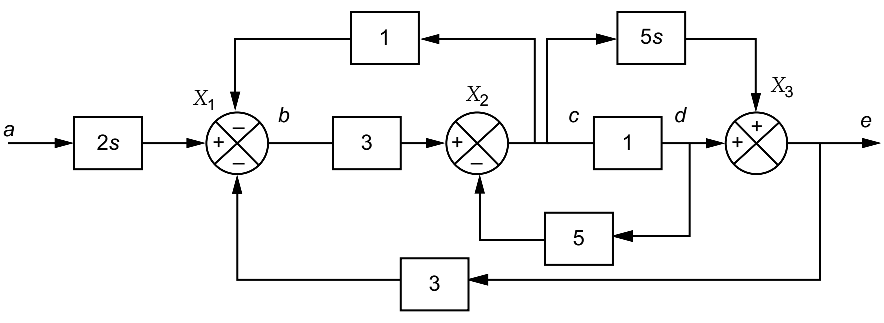


Fig. E4.2(g)

Example 4.3: **The signal flow diagram for a particular system is shown in Fig. E4.3. Determine the transfer function of the system using Mason's gain formula.**

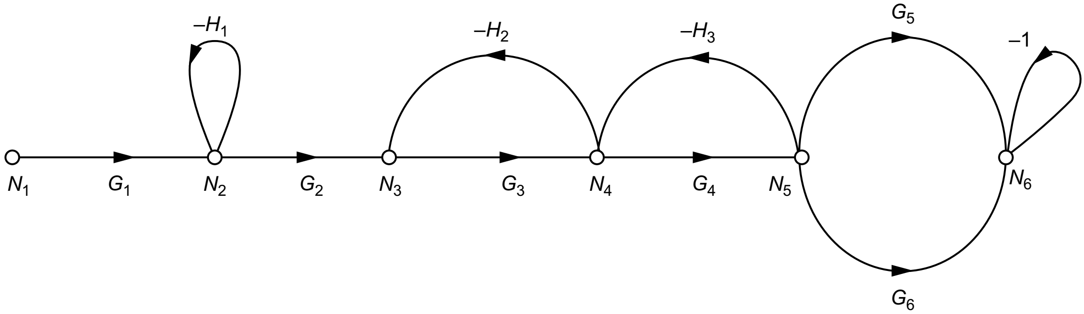
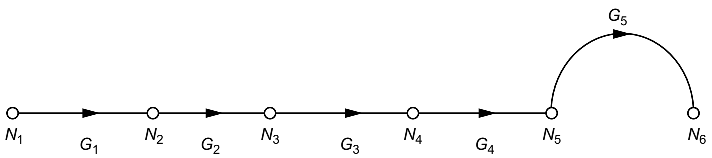
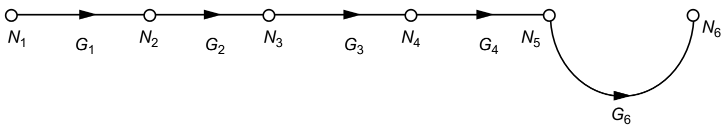


Fig. E4.3

Solution:

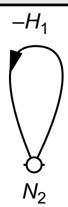
Step 1: For the given SFG, $n = 2$ and its corresponding gain is given in Table E4.3(a).

Table E4.3(a) | Gains for different forward paths

S. No	Forward path	Gain
1.		$P_1 = G_1G_2G_3G_4G_5$
2.		$P_2 = G_1G_2G_3G_4G_6$

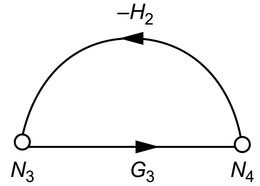
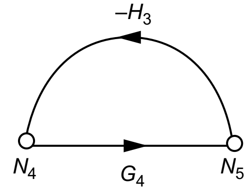
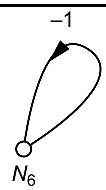
Step 2: For the given SFG, $m = 4$ and its corresponding gain is given in Table E4.3(b).

Table E4.3(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -H_1$

(*Continued*)

Table E4.3(b) | (Continued)

2.		$L_2 = -G_3H_2$
3.		$L_3 = -G_4H_3$
4.		$L_4 = -1$

Step 3: Then, the gains of individual loops are added together to get the total individual loop gain S_1. Therefore,

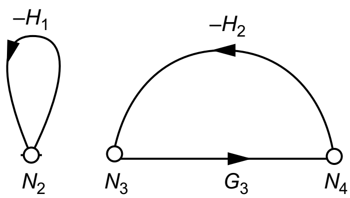
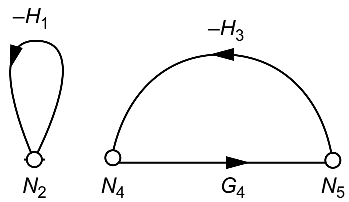
$$S_1 = \sum_{i=1}^{m} L_i, \text{ where } m = 4$$

$$= L_1 + L_2 + L_3 + L_4$$

$$= -H_1 - G_3H_2 - G_4H_3 - 1$$

Step 4: For the given SFG, the product of gains of all possible combination of two and three non-touching loops is given in Tables E4.3(c) and (d) respectively.

Table E4.3(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = H_1H_2G_3$
	$L_{22} = H_1H_3G_4$

(*Continued*)

Table E4.3(c) | (Continued)

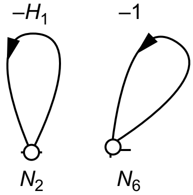
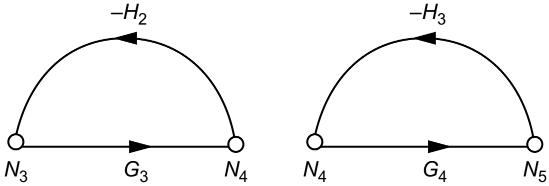
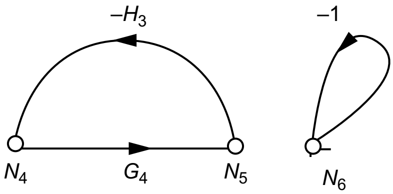
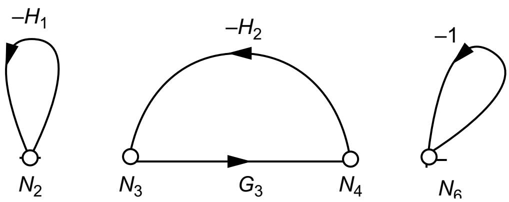
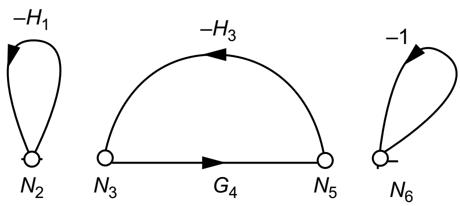
Combination of two non-touching loops	Product of gains
	$L_{23} = H_1$
	$L_{24} = H_2H_3G_3G_4$
	$L_{25} = H_3G_4$

Table E4.3(d) | Product of gains of combination of three non-touching loops

S. No	Combination of three non-touching loops	Product of gains
1.		$L_{31} = -H_1H_2G_3$
2.		$L_{32} = -H_1H_3G_4$

Step 5: In the given problem, there only exists a combination of two and three loops that have no node in common. Hence, S_j, where $j = 4$ to m is zero.

Therefore, $S_2 = L_{21} + L_{22} + L_{23} + L_{24} + L_{25}$

$$= H_1H_2G_3 + H_1H_3G_4 + H_1 + H_2H_3G_3G_4 + H_3G_4$$

and $$S_3 = L_{31} + L_{32}$$
$$= -H_1H_2G_3 - H_1H_3G_4$$

Step 6: Since there exists no part of the graph that does not touch the forward paths in the given SFG, the values of Δ_1 and Δ_2 are 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{\Delta_1 P_1 + \Delta_2 P_2}{1 - S_1 + S_2 - S_3}$$

$$= \frac{G_1G_2G_3G_4\left(G_5 + G_6\right)}{1 + \left(1 + H_1 + G_3H_2 + G_4H_3\right) + \left(H_1 + H_1H_2G_3 + H_1H_3G_4 + H_2H_3G_3G_4 + H_3G_4\right) + \left(H_1H_3G_4 + H_1H_2G_3\right)}$$

Example 4.4: **The SFG for a system is shown in Fig. E4.4. Obtain the transfer function of the given SFG using Mason's gain formula.**

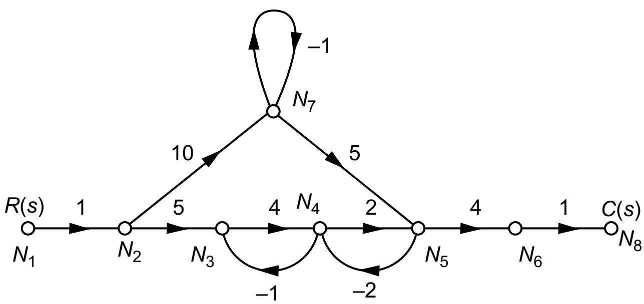
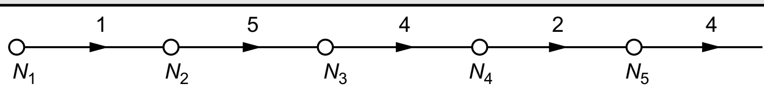
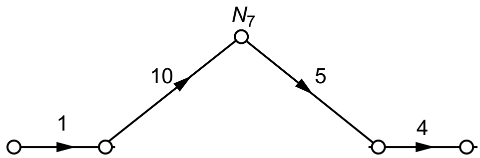


Fig. E4.4

Solution:

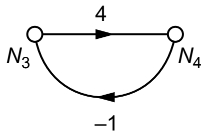
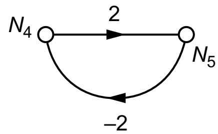
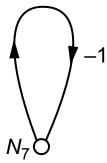
Step 1: The number of forward paths n in the SFG shown in Fig. E4.4 is $n = 2$ and the corresponding gain of the forward path is given in Table E4.4(a).

Table E4.4(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = 160$
2.		$P_2 = 200$

Step 2: The number of individual loops present in the SFG shown in Fig. E4.4 is $m = 4$ and the corresponding gain of each loop is given in Table E4.4(b).

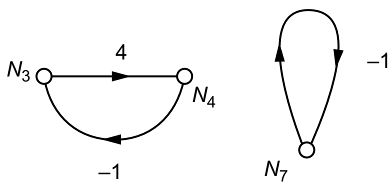
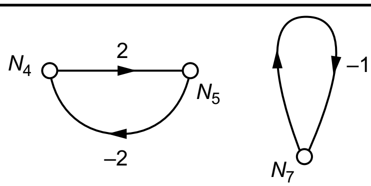
Table E4.4(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -4$
2.		$L_2 = -4$
3.		$L_3 = -1$

Step 3: For the given SFG, $S_1 = -9$.

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.4(c).

Table E4.4(c) | Product of gains of combination of two non-touching loops

S. No	Combination of two non-touching loops	Product of gains
1.		$L_{21} = 4$
2.		$L_{22} = 4$

Step 5: In the given problem, there exist two different combination of two loops that have no node in common. Hence, $S_2 = 8$ and S_3 is zero.

Step 6: The part of the graph that does not touch the first and second forward paths, the corresponding gain and respective Δ_i values are given in Tables E4.4(d) and (e) respectively.

Table E4.4(d) | Determination of Δ_1

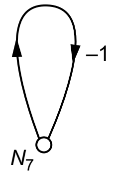
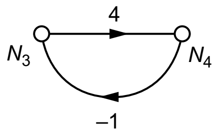
Loop/part of graph	**Gain**	Δ_I
	–1	$\Delta_2 = 2$

Table E4.4(e) | Determination of Δ_2

Loop/part of graph	**Gain**	Δ_I
	–4	$\Delta_2 = 5$

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{1320}{18}$$

Example 4.5: The SFG for a system is shown in Fig. E4.5. Obtain the transfer function of the given SFG using Mason's gain formula.

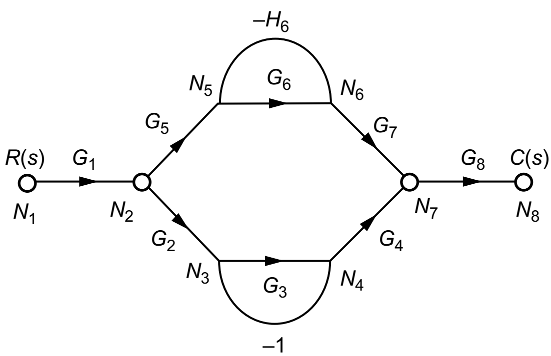
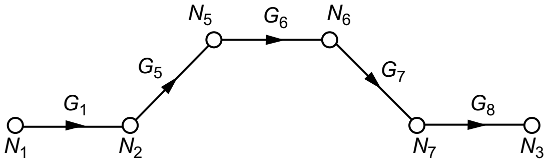
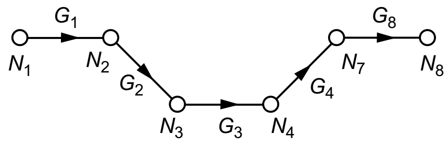


Fig. E4.5

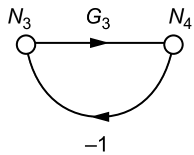
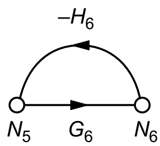
Step 1: The number of forward paths in the SFG shown in Fig. E4.5 is $n = 2$ and the gain corresponding to the forward path is given in Table E4.5(a).

Table E4.5(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = G_1G_5G_6G_7G_8$
2.		$P_1 = G_1G_2G_3G_4G_8$

Step 2: The number of individual loops present in the SFG shown in Fig. E4.5 is $m = 4$ and the gain corresponding to each loop is given in Table E4.5(b).

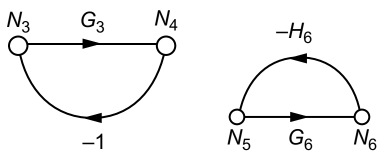
Table E4.5(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_3$
2.		$L_2 = -G_6H_6$

Step 3: For the given SFG, $S_1 = -G_3 - G_6H_6$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.5(c).

Table E4.5(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = G_3G_6H_6$

Step 5: In the given problem, there exists only one different combination of two loops that have no node in common. Hence, S_j, where $j = 3$ to m is zero.

Therefore, $$S_2 = G_3G_6H_6$$

Step 6: The part of the graph that does not touch the first and second forward path, its corresponding gain and respective Δ_i values are given in Tables E4.5(d) and (e).

Table E4.5(d) | Determination of Δ_1

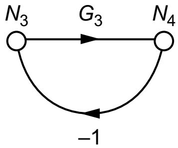
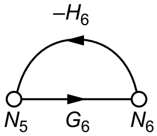
Loop/part of graph	Gain	Δ_1
	$-G_3$	$\Delta_1 = 1 + G_3$

Table E4.5(e) | Determination of Δ_2

Loop/part of graph	Gain	Δ_2
	$-G_6H_6$	$\Delta_2 = 1 + G_6H_6$

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_5G_6G_7G_8(1+G_3) + G_1G_2G_3G_4G_8(1+G_6H_6)}{1 + G_6H_6 + G_3 + G_3G_6H_6}.$$

Example 4.6: The signal flow diagram for a particular system is shown in Fig. E4.6. Determine the transfer function of the system using Mason's gain formula.

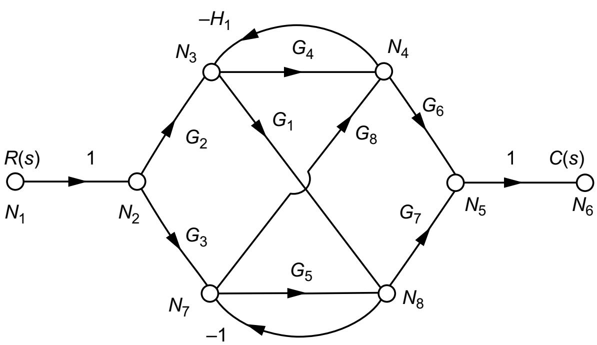
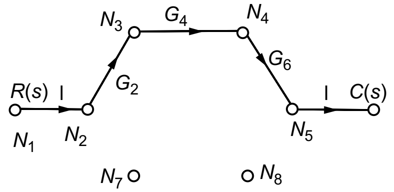
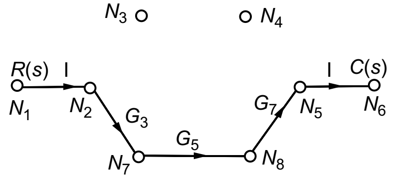
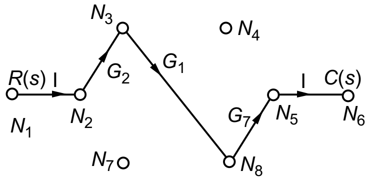
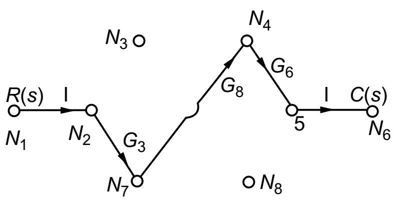
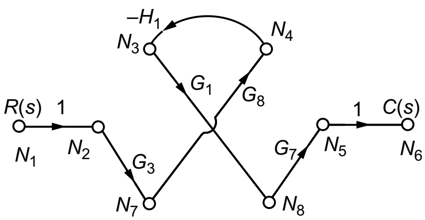
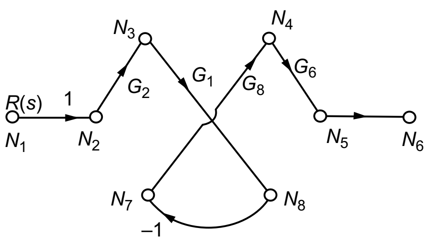


Fig. E4.6

Solution:

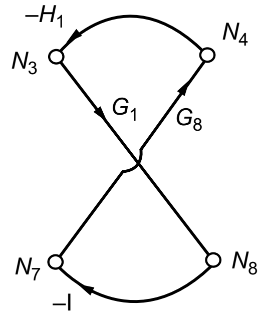
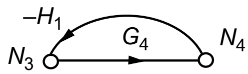
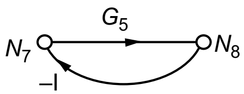
Step 1: For the given SFG, $n = 6$ and the corresponding gain is given in Table E4.6(a).

Table E4.6(a) | Gains for different forward paths

<table>
<tr><th>S. No</th><th>Forward path</th><th>Gain</th></tr>
<tr><td>1.</td><td></td><td>$P_1 = G_2G_4G_6$</td></tr>
<tr><td>2.</td><td></td><td>$P_2 = G_3G_5G_7$</td></tr>
<tr><td>3.</td><td></td><td>$P_3 = G_1G_2G_7$</td></tr>
<tr><td>4.</td><td></td><td>$P_4 = G_3G_8G_6$</td></tr>
<tr><td>5.</td><td></td><td>$P_5 = -G_1G_3G_7G_8H_1$</td></tr>
<tr><td>6.</td><td></td><td>$P_6 = -G_1G_2G_6G_8$</td></tr>
</table>

Step 2: For the given SFG, $m = 3$ and its corresponding gain is given in Table E4.6(b).

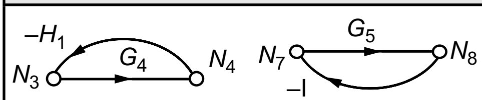
Table E4.6(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = G_1G_8H_1$
2.		$L_2 = -G_4H_1$
3.		$L_3 = -G_5$

Step 3: For the given SFG, $S_1 = G_1G_8H_1 - G_4H_1 - G_5$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.6(c).

Table E4.6(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = (-G_4H_1)(-G_5)$ $= G_4G_5H_1$

Step 5: In the given problem, there exists only one combination of two loops that have no node in common. Hence, S_j, where $j = 3$ to m is zero. Therefore,

$$S_2 = L_{21} = G_4G_5H_1$$

Step 6: The part of the graph that does not touch the first and second forward path, its corresponding gain and respective Δ_i values are given in Tables E4.6(d) and (e).

Table E4.6(d) | Determination of Δ_1

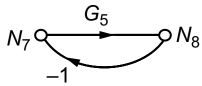
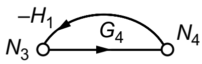
Loop/part of graph	Gain	Δ_1
	$-G_5$	$\Delta_1 = 1 + G_5$

Table E4.6(e) | Determination of Δ_2

Loop/part of graph	Gain	Δ_2
	$-G_4H_1$	$\Delta_2 = 1 + G_4H_1$

But there is no part of the graph that does not touch the other forward paths in the given SFG, the value of Δ_p, where $p = 3$ *to* n is one.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{\Delta_1 P_1 + \Delta_2 P_2 + \Delta_3 P_3 + \Delta_4 P_4 + \Delta_5 P_5 + \Delta_6 P_6}{1 - S_1 + S_2}$$

$$= \frac{G_2G_4G_6\left(1+G_5\right) + G_3G_5G_7\left(1+G_4H_1\right) + G_1G_2G_7(1) + G_3G_6G_8(1) - G_1G_3G_7G_8H_1(1) - G_1G_2G_6G_8(1)}{1 + G_4H_1 + G_5 - G_1G_8H_1 + G_4G_5H_1}$$

$$= \frac{G_2G_4G_6\left(1+G_5\right) + G_3G_5G_7\left(1+G_4H_1\right) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8}{1 + G_4H_1 + G_5 - G_1G_8H_1 + G_4G_5H_1}.$$

Example 4.7: **The SFG for a system is shown in Fig. E4.7. Obtain the transfer function of the given SFG using Mason's gain formula.**

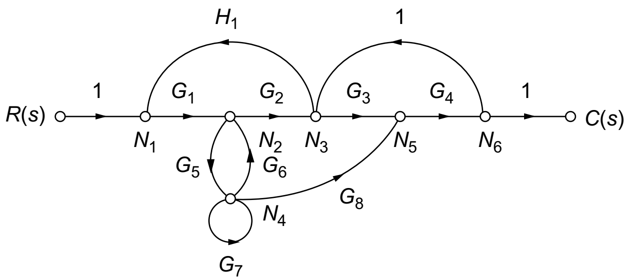
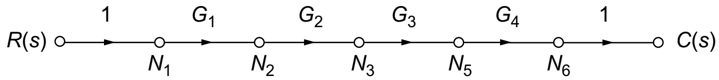
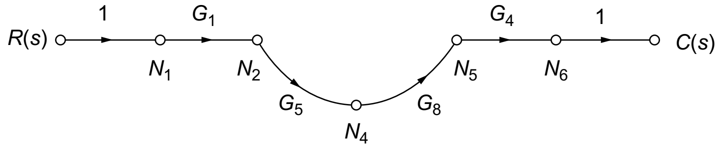


Fig. E4.7

Solution:

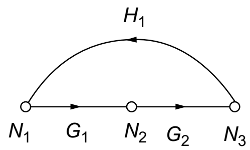
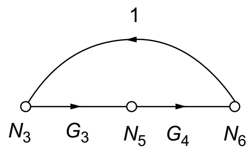
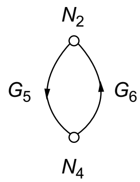
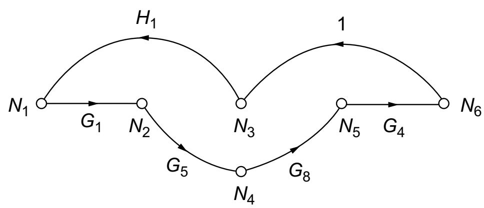
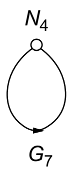
Step 1: The number of forward paths in the SFG shown in Fig. E4.7 is $n = 2$ and the gain corresponding to the forward path is given in Table E4.7(a).

Table E4.7(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = G_1G_2G_3G_4$
2.		$P_2 = G_1G_4G_5G_8$

Step 2: The number of individual loops present in the SFG shown in Fig. E4.7 is $m = 5$ and the gain corresponding to each loop is given in Table E4.7(b).

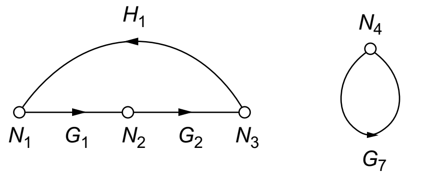
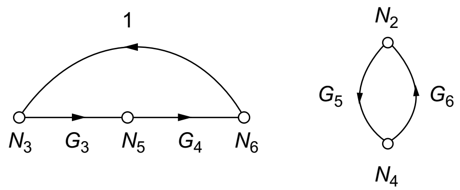
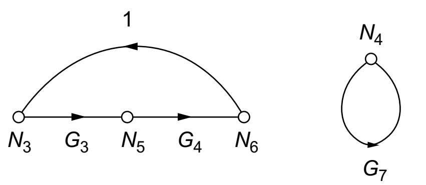
Table E4.7(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = G_1G_2H_1$
2.		$L_2 = G_3G_4$
3.		$L_3 = G_5G_6$
4.		$L_4 = G_1G_4G_5G_8H_1$
5.		$L_3 = G_7$

Step 3: For the given SFG, $S_1 = G_1G_2H_1 + G_3G_4 + G_1G_5G_4G_8H_1 + G_5G_6 + G_7$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.7(c).

Table E4.7(c) | Product of gains of combination of two non-touching loops

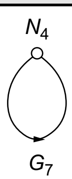
S. No	Combination of two non-touching loops	Product of gains
1.		$L_{21} = G_1G_2G_7H_1$
2.		$L_{22} = G_3G_4G_5G_6$
3.		$L_{22} = G_3G_4G_7$

Step 5: In the given problem, there exist different combination of two loops which have no node in common. Hence, S_j, where $j = 3$ to m is zero. Therefore,

$$S_2 = G_1G_2G_7H_1 + G_3G_4G_5G_6 + G_3G_4G_7$$

Step 6: Since there exist no part of the graph which does not touch the second forward path, the value of Δ_2 is 1. The part of the graph which does not touch the first forward path and its corresponding gain are given in Table E4.7(d).

Table E4.7(d) | Determination of Δ_1

Loop/part of graph	Gain	Δ_2
	G_7	$\Delta_2 = 1 - G_7$

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4-G_1G_2G_3G_4G_7+G_1G_4G_5G_8}{1-(G_1G_2H_1+G_3G_4+G_1G_4G_5G_8H_1+G_5G_6+G_7)+G_1G_2G_7H_1+G_3G_4G_5G_6+G_3G_4G_7}$$

Example 4.8: For the electrical circuit shown in Fig. E4.8(a), determine the transfer function of the system, $\frac{V_2(s)}{V_i(s)}$ using Mason's gain formula.

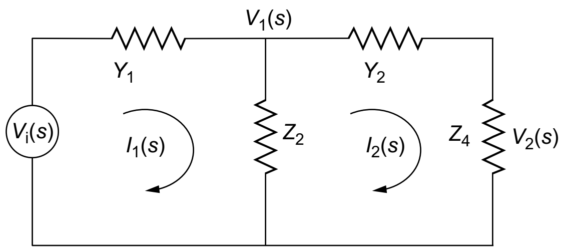


Fig. E4.8(a)

Solution: To determine the transfer function of an electrical circuit, the following procedure has to be followed:

Step A: Determine the block diagram of the given electrical circuit.
Step B: Determine the SFG of the system equivalent to the block diagram.
Step C: Determine the transfer function of the system using Mason's gain formula.

Step A: To determine the block diagram of the given electrical circuit: Admittances Y_1 and Y_2 are in series path. Impedances Z_2 and Z_4 are in shunt path. The node voltages in the given electrical circuit are $v_1(t)$ and $v_2(t)$. The loop currents in the given electrical circuit are $i_1(t)$ and $i_2(t)$.

The current flowing through each element present in the series path is found by using node voltages.

(a) The current $i_1(t)$ flowing through admittance Y_1 is given by

$$i_1(t)=[v_i(t)-v_1(t)]\times Y_1 \tag{1}$$

(b) The current $i_2(t)$ flowing through admittance Y_2 is given by

$$i_2(t)=[v_1(t)-v_2(t)]\times Y_2 \tag{2}$$

The voltages across each element present in the shunt path are found out by using loop currents.

(a) The voltage $v_1(t)$ across Z_2 is given by

$$v_1(t) = [i_1(t) - i_2(t)] \times Z_2 \tag{3}$$

(b) The voltage $v_2(t)$ across Z_4 is given by

$$v_2(t) = i_2(t) \times Z_4 \tag{4}$$

Individual block diagrams representing each of the above equations can be determined by examining the input and output of the respective equation. Then, the determined individual block diagrams can be combined in an appropriate way to represent the given electrical circuit.

To determine the individual block diagrams:

For Eqn. (1):

Taking Laplace transform of Eqn. (1), we obtain

$$I_1(s) = [V_i(s) - V_1(s)] \times Y_1 \tag{5}$$

where the inputs are $V_i(s)$, $V_1(s)$ and the output is $I_1(s)$. Hence, the block diagram for the above equation is shown in Fig. E4.8(b).

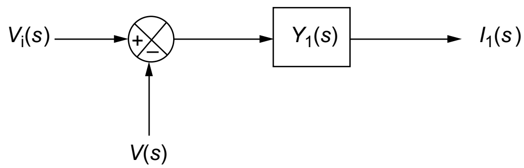


Fig. E4.8(b)

For Eqn. (2):

Taking Laplace transform of Eqn. (2), we obtain

$$I_2(s) = [V_1(s) - V_2(s)] \times Y_2 \tag{6}$$

where the inputs are $V_1(s)$, $V_2(s)$ and the output is $I_2(s)$. Hence, the block diagram for the above equation is shown in Fig. E4.8(c).

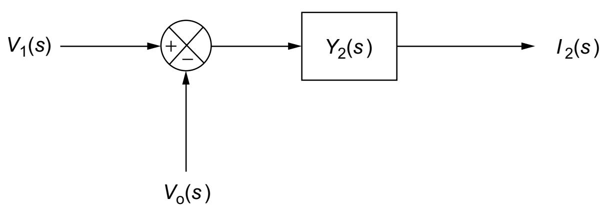


Fig. E4.8(c)

For Eqn. (3):
Taking Laplace transform of Eqn. (3), we obtain

$$V_1(s) = [I_1(s) - I_2(s)] \times Z_2 \tag{7}$$

where the inputs are $I_1(s)$ and $I_2(s)$ and the output is $V_1(s)$. Hence, and the block diagram for the above equation is shown in Fig. E4.8(d).

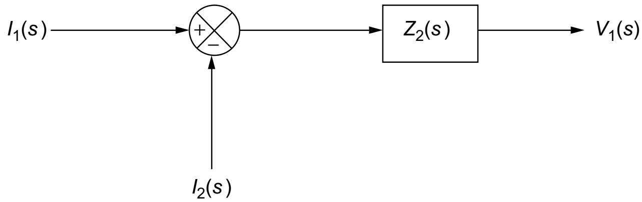


Fig. E4.8(d)

For Eqn. (4):
Taking Laplace transform of Eqn. (4), we obtain

$$V_2(s) = I_2(s) \times Z_4 \tag{8}$$

where the input is $I_2(s)$ and the output is $V_2(s)$. Hence, the block diagram for the above equation is shown in Fig. E4.8(e).

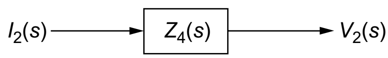


Fig. E4.8(e)

Hence, the overall block diagram for the given electrical circuit shown in Fig. E4.8(a) is obtained by interconnecting the block diagrams shown in Fig. E4.8(b) through (e). The overall block diagram of the given electrical circuit is shown in Fig. E4.8(f).

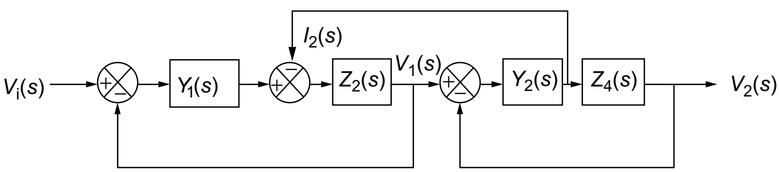


Fig. E4.8(f)

Step B: SFG for the given electrical circuit: The SFG equivalent of the block diagram shown in Fig. E4.8(f) is shown in Fig. E4.8(g).

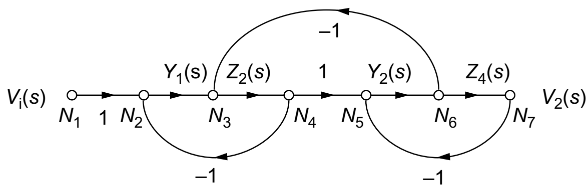
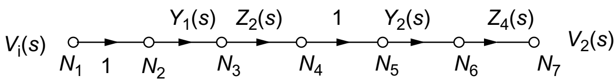


Fig. E4.8(g)

Step C: To determine the transfer function of the given electrical circuit

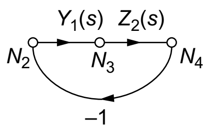
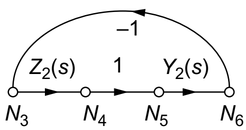
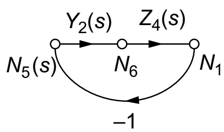
Step 1: For the given SFG, $n = 1$ and the gain corresponding to the forward path is given in Table E4.8(a).

Table E4.8(a) | Gain corresponding to the forward path

Forward path	Gain
	$P_1 = Z_2Z_4Y_1Y_2$

Step 2: For the given SFG, $m = 3$ and the corresponding loop gains are given in Table E4.8(b).

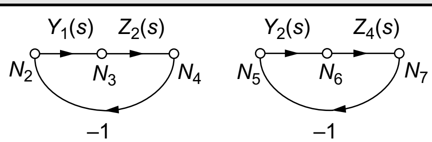
Table E4.8(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -Y_1Z_2$
2.		$L_2 = -Y_2Z_2$
3.		$L_3 = -Y_2Z_4$

Step 3: For the given SFG, $S_1 = -Y_1Z_2 - Y_2Z_2 - Y_2Z_4$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.8(c).

Table E4.8(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = Y_1Y_2Z_2Z_4$

Step 5: In the given problem, there exists one combination of two loops that have no node in common. Hence, S_j, where $j = 3$ to m is zero. Therefore,

$$S_2 = L_{21} = Y_1Y_2Z_2Z_4$$

Step 6: Since there exists no part of the graph that does not touch the first forward path in the given SFG, the value of Δ_1 is 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{Y_1Y_2Z_2Z_4}{1+Y_1Z_2+Y_2Z_2+Y_2Z_4+Y_1Y_2Z_2Z_4}$$

Example 4.9: The transfer function of the system is given by $T(s) = \dfrac{s^2+3s+3}{s^3+2s^2+3s+1}$. Determine the SFG for the given transfer function.

Solution: As the given transfer function is difficult to factorize, we can use the third method (observer canonical form) to obtain SFG of the system. The steps to obtain SFG of the system from transfer function are given below:

Step 1: The transfer function of the system (both numerator and denominator) should be divided by the highest power present in the respective terms. Therefore,

$$\frac{C(s)}{R(s)} = \frac{1}{s} \times \frac{\left(1+\dfrac{3}{s}+\dfrac{3}{s^2}\right)}{\left(1+\dfrac{2}{s}+\dfrac{3}{s^2}+\dfrac{1}{s^3}\right)} \tag{1}$$

Step 2: Cross-multiplying the Eqn. (1), we obtain

$$C(s)\times\left(1+\frac{2}{s}+\frac{3}{s^2}+\frac{1}{s^3}\right) = R(s)\times\frac{1}{s}\times\left(1+\frac{3}{s}+\frac{3}{s^2}\right) \tag{2}$$

Step 3: Combining the terms of like powers of integration gives

$$C(s) = \frac{1}{s}\times\left(1+\frac{3}{s}+\frac{3}{s^2}\right)R(s) - \left(\frac{2}{s}+\frac{3}{s^2}+\frac{1}{s^3}\right)C(s)$$

$$C(s) = \frac{1}{s}\times\big(R(s)-2C(s)\big)+\frac{1}{s^2}\times\big(3R(s)-3C(s)\big)+\frac{1}{s^3}\times\big(3R(s)-C(s)\big)$$

$$= \frac{1}{s}\times\left\{\big(R(s)-2C(s)\big)+\frac{1}{s}\times\big(3R(s)-3C(s)\big)+\frac{1}{s^2}\times\big(3R(s)-C(s)\big)\right\} \tag{3}$$

Step 4: From Eqn. (3), the SFG of the system can be drawn. It is very clear that the system requires three integration (highest power of $\frac{1}{s}$). Hence, as an initial step to draw the SFG of the system, a straight line with three integration is drawn as shown in Fig. E4.9(a).

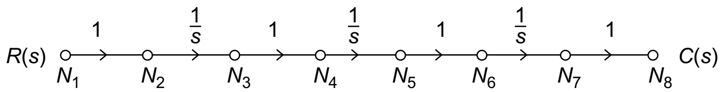


Fig. E4.9(a)

Step 5: By integrating the first term of Eqn. (3), i.e., $R(s)-2C(s)$ once, part of an output can be obtained. The integration of the first term is shown in Fig. E4.9(b).

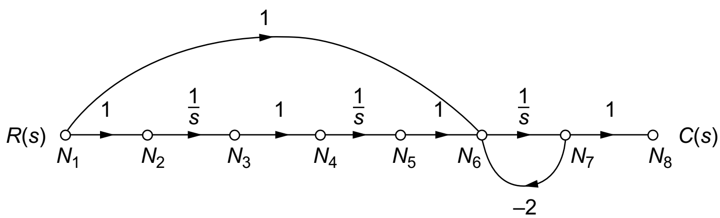


Fig. E4.9(b)

Step 6: By integrating the second term of Eqn. (3), i.e., $3R(s) - 3C(s)$, another part of the output can be obtained. The integration of the second term is shown in Fig. E4.9(c). As the second term has to be integrated twice, it must be present in the existing loop, i.e., in Fig. E4.9(b).

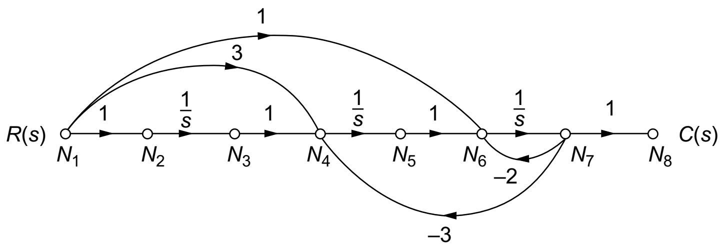


Fig. E4.9(c)

Step 7: By integrating the third term of Eqn. (3), i.e., $3R(s) - 3C(s)$, the whole output can be obtained. The integration of the third term is shown in Fig. E4.9(d). As the second term has to be integrated twice, it must be present in the existing loop, i.e., in Fig. E4.9(c).

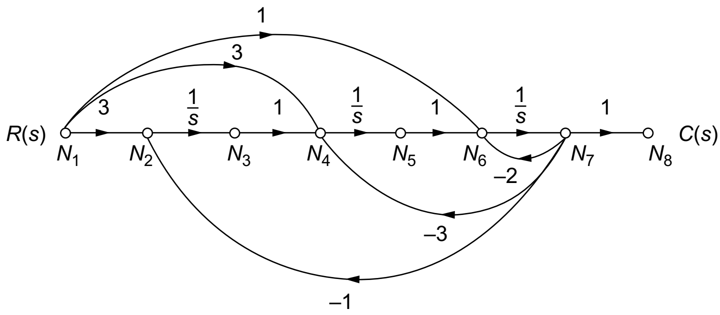


Fig. E4.9(d)

Hence, the diagram shown in Fig. E4.9(d) is the SFG representation of the given transfer function.

Example 4.10: The block diagram for a particular system is shown in Fig. E4.10(a). (i) Obtain the SFG equivalent to the block diagram, (ii) determine the transfer function of the system using Mason's gain formula and (iii) verify using block diagram reduction technique.

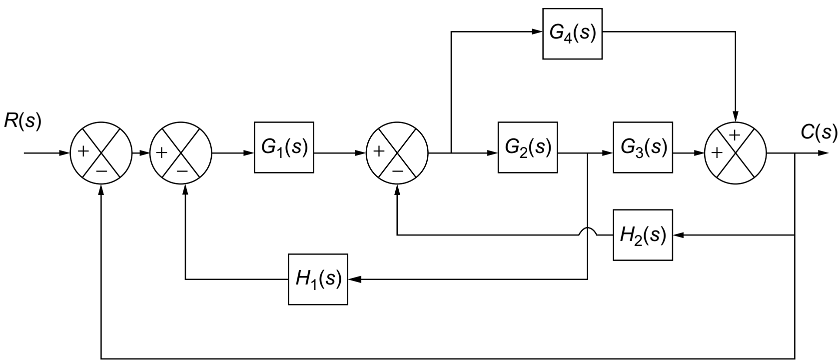


Fig. E4.10(a)

Solution:

Step 1: The SFG for the block diagram shown in Fig. E4.10(a) is shown in Fig. E4.10(b).

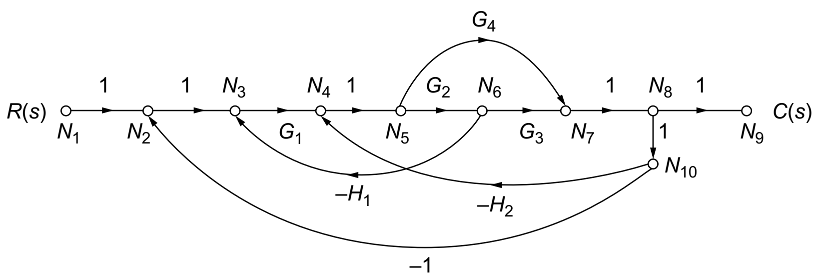
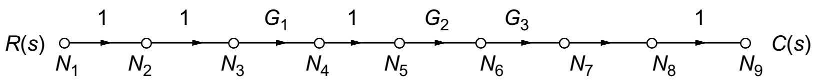
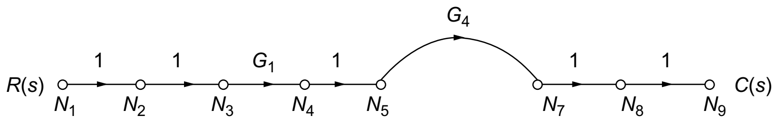


Fig. E4.10(b)

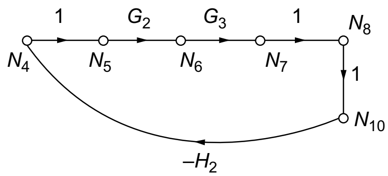
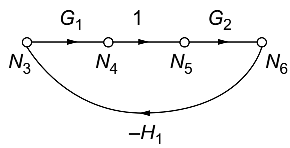
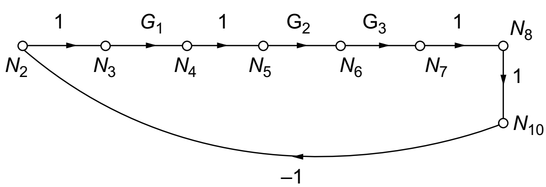
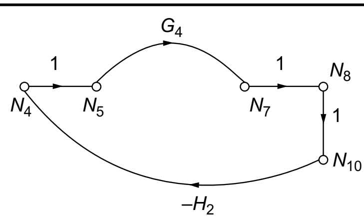
Step 2: For the given SFG, $n = 2$ and the gains corresponding to each forward path are given in Table E4.10(a).

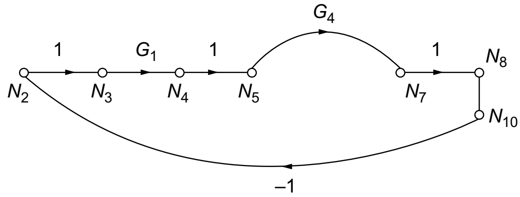
Table E4.10(a) | Gains for different forward paths

S. No	Forward path	Gain
1.		$P_1 = G_1G_2G_3$
2.		$P_2 = G_1G_4$

Step 3: For the given SFG, $m = 5$ and the corresponding loop gains are given in Table E4.10(b).

Table E4.10(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_2G_3H_2$
2.		$L_2 = -G_1G_2H_1$
3.		$L_3 = -G_1G_2G_3$
4.		$L_4 = -G_4H_2$

S. No	Individual loops	Gain
5.		$L_5 = -G_1G_4$

Step 4: For the given SFG, $S_1 = -G_2G_3H_2 - G_1G_2H_1 - G_1G_2G_3 - G_4H_2 - G_1G_4$

Step 5: For the given SFG, $S_j = 0,\ j = 2$ to m

Step 6: Determine Δ_i, where $i = 1$ to n for the given SFG. Since there exists no part of the graph that does not touch the first and second forward path in the given SFG, the values of Δ_1 and Δ_2 are 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4} \tag{1}$$

Verification using block diagram reduction technique

Step 1: The SFG of a system is shown in Fig. E4.10(a). The equivalent block diagram of the system is shown in Fig. E4.10(c).

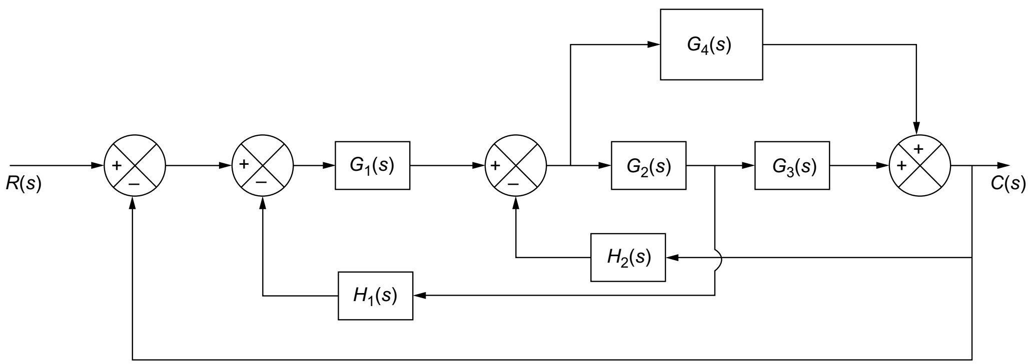


Fig. E4.10(c)

Step 2: The takeoff point that exists between the blocks $G_2(s)$ and $G_3(s)$ as shown in Fig. E4.10(d) is shifted before the block $G_2(s)$. The resultant block diagram is shown in Fig. E4.10(e).

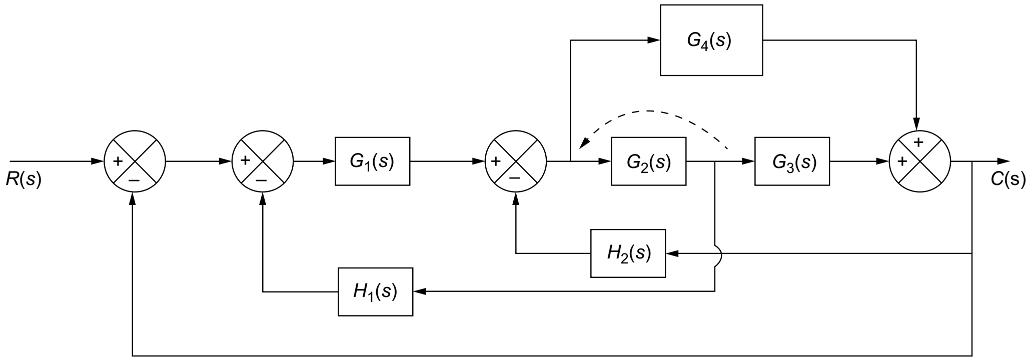


Fig. E4.10(d)

Step 3: The blocks in series as shown in Fig. E4.10(e) can be combined to form a single block and the resultant block diagram is shown in Fig. E4.10(f).

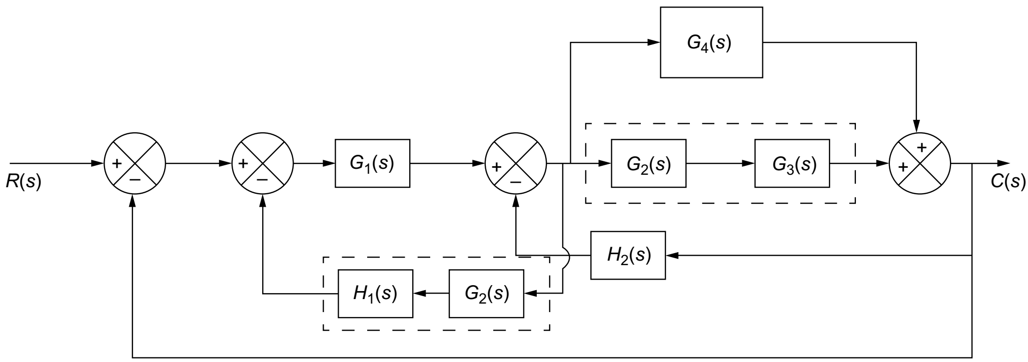


Fig. E4.10(e)

Step 4: The blocks in parallel as shown in Fig. E4.10(f) can be combined to form a single block. In addition, the summing point existing after the $G_1(s)$ block as shown in Fig. E4.10(f) is shifted before the $G_1(s)$ block. The resultant block diagram is shown in Fig. E4.10(g).

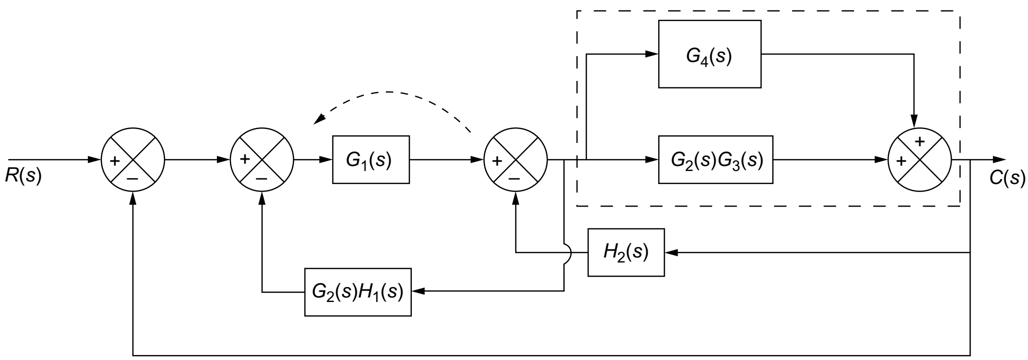


Fig. E4.10(f)

Step 5: The blocks in series as shown in Fig. E4.10(g) can be combined to form a single block. In addition, the two summing points that exist before the $G_1(s)$ block as shown in Fig. E4.10(g) can be interchanged. The resultant block diagram is shown in Fig. E4.10(h).

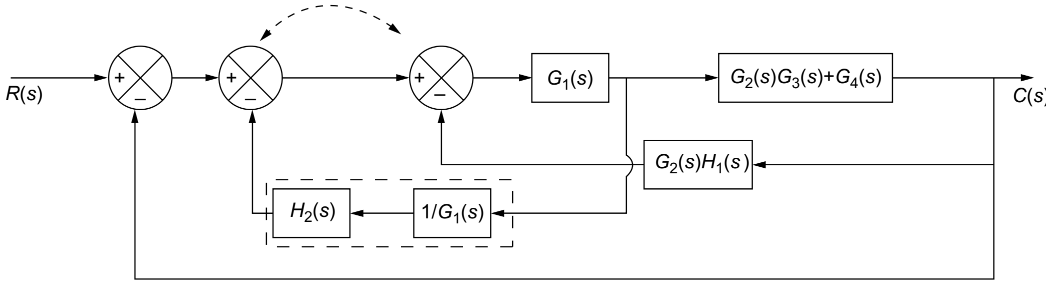


Fig. E4.10(g)

Step 6: The feedback path shown in Fig. E4.10(h) can be reduced to form a single block. The resultant block diagram is shown in Fig. E4.10(i).

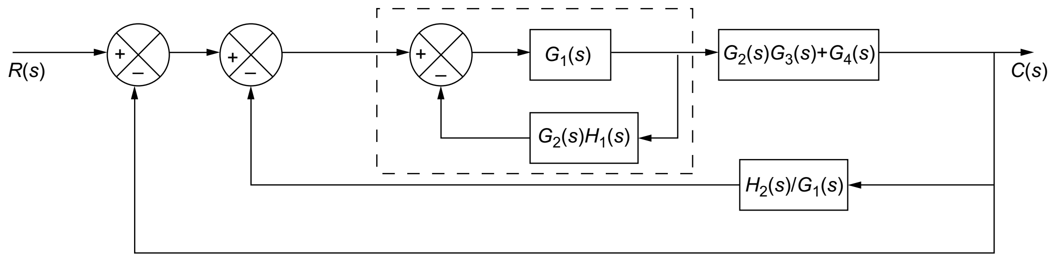


Fig. E4.10(h)

Step 7: The blocks in series as shown in Fig. E4.10(i) can be reduced to form a single block. In addition, the common takeoff point as shown in Fig. E4.10(i) can be separated. The resultant block diagram is shown in Fig. E4.10(j).

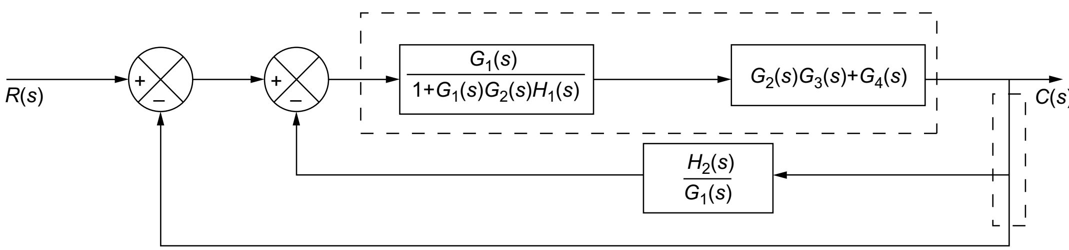


Fig. E4.10(i)

Step 8: The feedback loop existing in the block diagram as shown in Fig. E4.10(j) can be reduced to form a single block and the resultant block diagram is shown in Fig. E4.10(k).

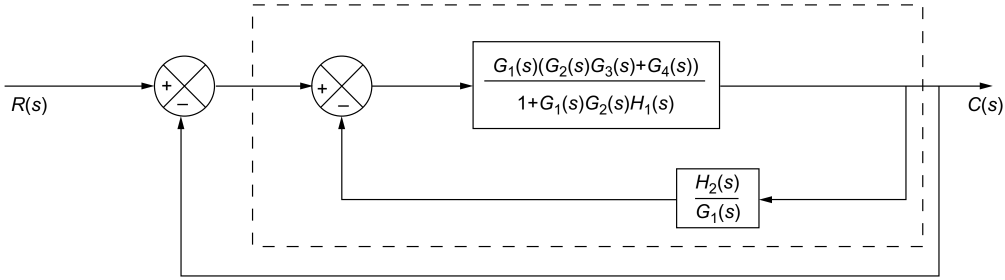


Fig. E4.10(j)

Step 9: The feedback loop existing in the block diagram as shown in Fig. E4.10(k) can be reduced to determine the transfer function of the system whose SFG is shown in Fig. E4.10(a).

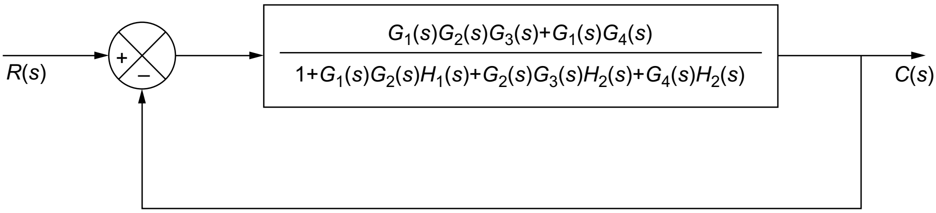


Fig. E4.10(k)

Hence, the transfer function of the system is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4} \tag{2}$$

From Eqs. (1) and (2), it is clear that the transfer function of the system shown in Fig. E4.10(a) using SFG and block diagram reduction techniques are the same.

Example 4.11: The block diagram of the system is shown in Fig. E4.11(a). Determine the transfer function by using block diagram reduction technique and verify it using SFG.

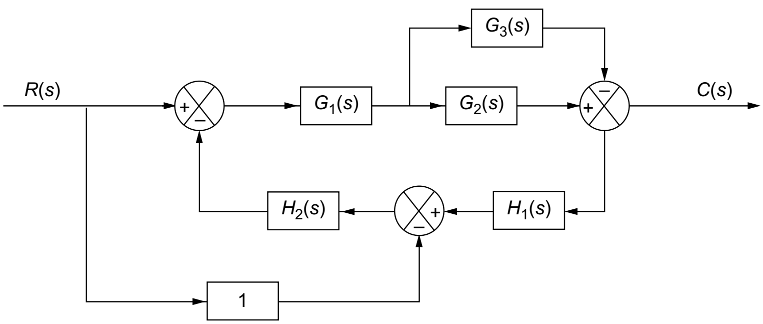


Fig. E4.11(a)

Solution:

(a) To determine the transfer function by using block diagram reduction technique

Step 1: The blocks in parallel as shown in Fig. E4.11(b) are combined to form a single block. The resultant block diagram is shown in Fig. E4.11(c).

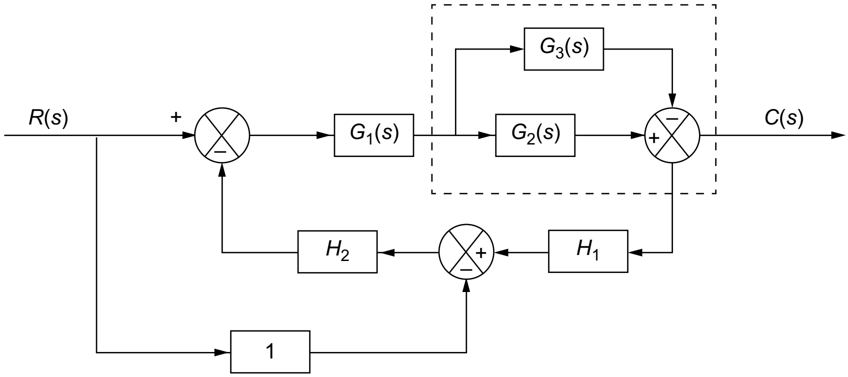


Fig. E4.11(b)

Step 2: The blocks in series as shown in Fig. E4.11(c) are combined to form a single block. The resultant block diagram is shown in Fig. E4.11(d).

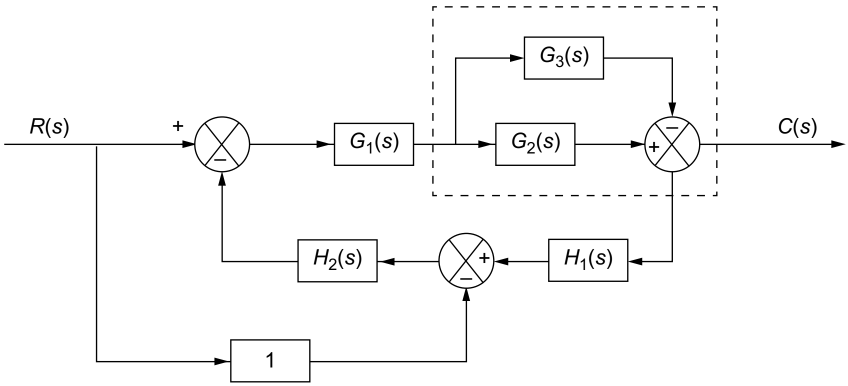


Fig. E4.11(c)

Step 3: The summing point present after the block $H_1(s)$ as shown in Fig. E4.11(d) can be moved after the block $H_2(s)$. The resultant block diagram is shown in Fig. E4.11(e).

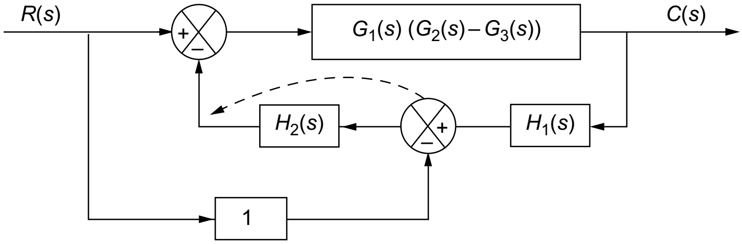


Fig. E4.11(d)

Step 4: The blocks in series as shown in Fig. E4.11(e) are combined to form a single block. Also, the two summing points that exist in the block diagram shown in Fig. E4.11(e) can be combined to obtain a single summing point and the resultant block diagram is shown in Fig. E4.11(f).

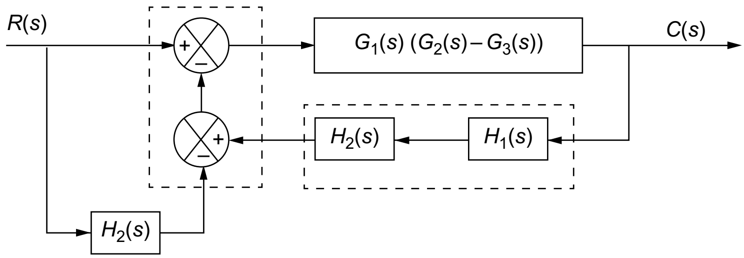


Fig. E4.11(e)

Step 5: The summing point with three inputs as shown in Fig. E4.11(f) is split into two summing points and the resultant block diagram is shown in Fig. E4.11(g).

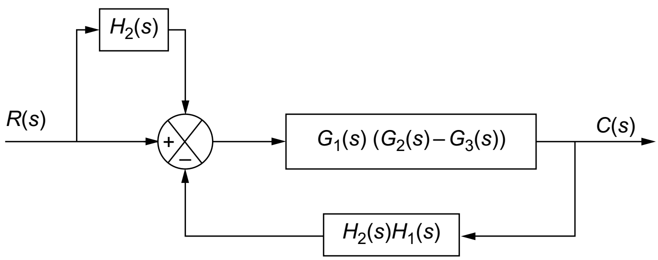


Fig. E4.11(f)

Step 6: The blocks in parallel and the feedback loop existing in the block diagram as shown in Fig. E4.11(g) can be reduced to form a single block and the resultant block diagram is shown in Fig. E4.11(h).

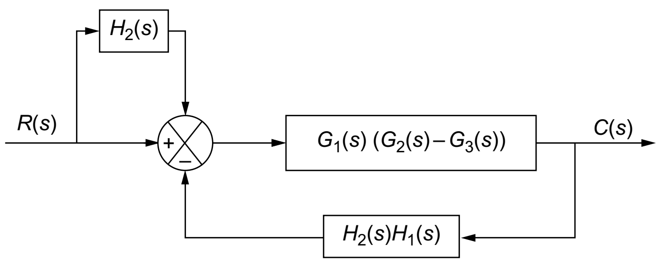


Fig. E4.11(g)

Step 7: The blocks in series as shown in Fig. E4.11(h) can be combined to determine the transfer function of the system.

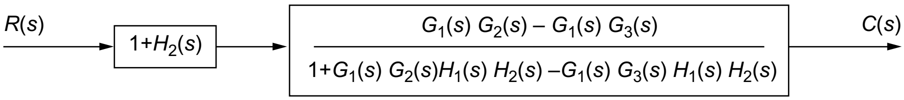


Fig. E4.11(h)

Hence, the transfer function of the given system is obtained as

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s) - G_1(s)G_3(s) + G_1(s)G_2(s)H_2(s) - G_1(s)G_3(s)H_2(s)}{1 + G_1(s)G_2(s)H_1(s)H_2(s) - G_1(s)G_3(s)H_1(s)H_2(s)} \quad (1)$$

(b) To determine transfer function by using SFG

Step 1: The SFG for the given block diagram shown in Fig. E4.11(a) is drawn and is shown in Fig. E4.11(i).

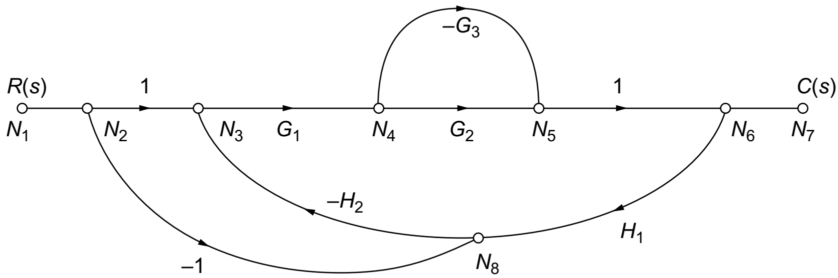
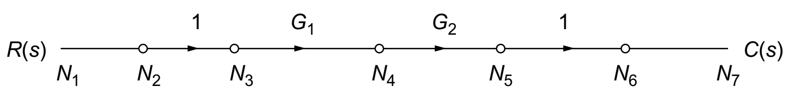
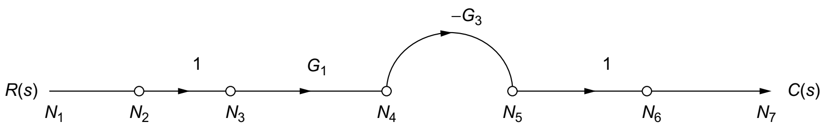
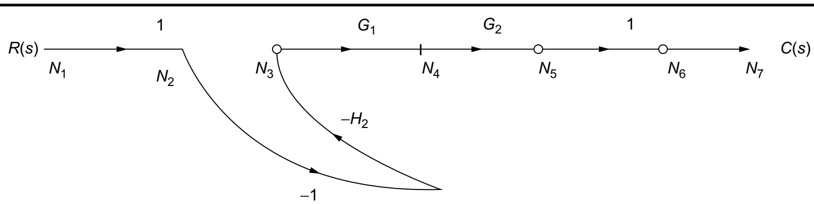


Fig. E4.11(i)

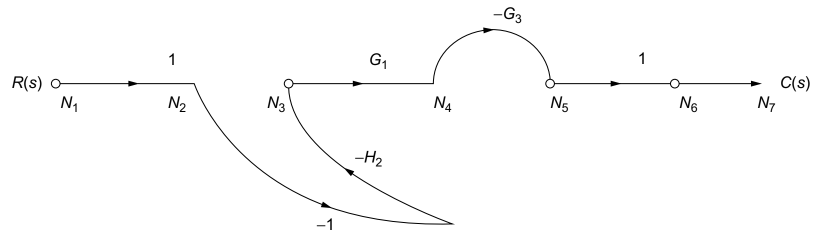
Step 2: The number of forward paths in the SFG, n shown in Fig. E4.11(i) is 4 and the gain corresponding to the forward path is given in Table E4.11(a)

Table E4.11(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = G_1G_2$
2.		$P_2 = -G_1G_3$
3.		$P_3 = G_1G_2H_2$

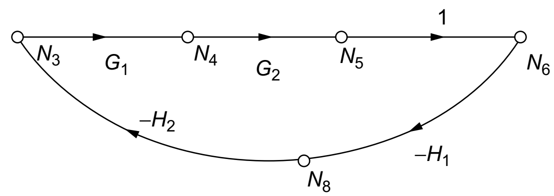
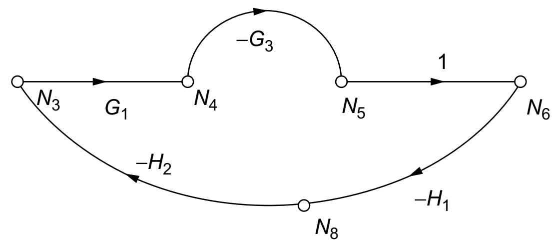
(*Continued*)

Table E4.11(a) | (Continued)

S. No	Forward path	Gain
4.		$P_4 = -G_1G_3H_2$

Step 3: The number of individual loops present in the SFG shown in Fig. E4.11(a) is $m = 2$ and the gain corresponding to each loop is given in Table E4.11(b).

Table E4.11(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_1G_2H_1H_2$
2.		$L_2 = G_1G_3H_1H_2$

Step 4: For the given SFG, $S_1 = -G_1G_2H_1H_2 + G_1G_3H_1H_2$

Step 5: Since there exists no combination of two non-touching loops without a common node in the given SFG, the product of gains of the non-touching loops is zero, i.e., higher orders of S_j are zero, where $j = 2, 3, \ldots, m$.

Step 6: Since there exists no part of the graph that does not touch the forward paths in the given SFG, the values of Δ_1 and Δ_2 are one.

Step 7: The transfer function of the system using Mason's gain formula is given

$$\frac{C(s)}{R(s)} = \frac{G_1G_2 - G_1G_3 + G_1G_2H_2 - G_1G_3H_2}{1 + G_1G_2H_1H_2 - G_1G_3H_1H_2} \qquad (2)$$

From Eqs. (1) and (2), it is clear that the transfer functions obtained for the system shown in Fig. E4.11(a) using block diagram reduction technique and SFG technique are same.

Example 4.12: The SFG for a system is shown in Fig. E4.12(a). Obtain the transfer function of the system represented by SFG using Mason's gain formula.

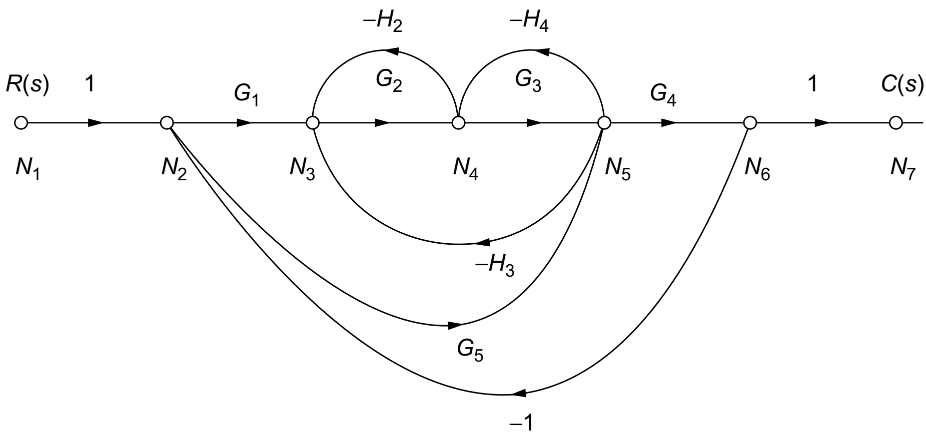
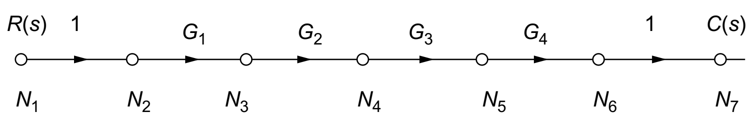
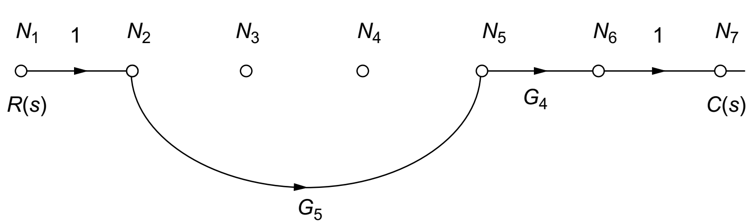


Fig. E4.12(a)

Solution:

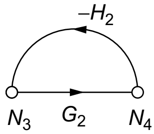
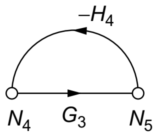
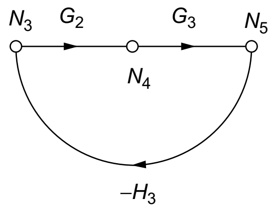
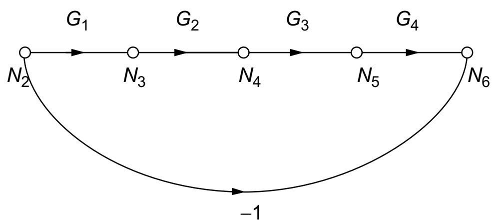
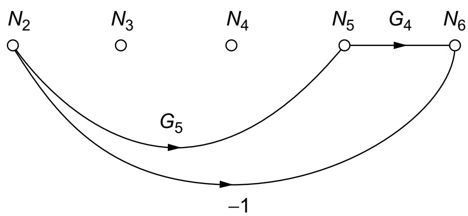
Step 1: For the given SFG, $n = 2$ and the gains corresponding to each forward path are given in Table E4.12(a).

Table E4.12(a) | Gains for different forward paths

S. No	Forward path	Gain
1.		$P_1 = G_1G_2G_3G_4$
2.		$P_2 = G_4G_5$

Step 2: For the given SFG, $m = 5$ and the corresponding loop gains are given in Table E4.12(b).

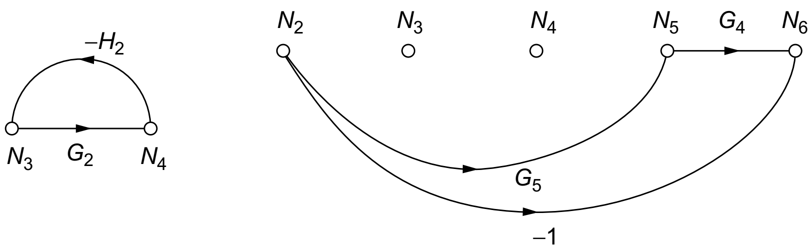
Table E4.12(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_2H_2$
2.		$L_2 = -G_3H_4$
3.		$L_3 = -G_2G_3H_3$
4.		$L_4 = -G_1G_2G_3G_4$
5.		$L_5 = -G_4G_5$

Step 3: For the given SFG, $S_1 = -G_2H_2 - G_3H_4 - G_2G_3H_3 - G_1G_2G_3G_4 - G_4G_5$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.12(c).

Table E4.12(c) | Product of gains of combination of two non-touching loops

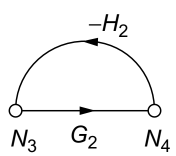
Combination of two non-touching loops	Product of gains
	$L_{21} = G_2G_4G_5H_2$

Step 5: In the given problem, there exist only two different combination of two loops that have no node in common. Hence, S_j, where $j = 3$ to m is zero. Therefore,

$$S_2 = L_{21} = G_2G_4G_5H_2$$

Step 6: Since there exists no part of the graph that does not touch the first forward path in the given SFG, the value of Δ_1 is 1.
But, the part of the graph that does not touch the second forward path and its corresponding gain are given in Table E4.12(d).

Table E4.12(d) | Determination of Δ_2

Loop/part of graph	Gain	Δ_2
	$-G_2 H_2$	$\Delta_2 = 1 + G_2H_2$

Step 7: The transfer function of the system using Mason's gain formula is

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_4G_5 + G_2G_4G_5H_2}{1 + G_2H_2 + G_3G_4 + G_2G_3H_3 + G_1G_2G_3G_4 + G_4G_5 + G_2G_4G_5H_2}$$

Example 4.13: The block diagram of the system is shown in Fig. E4.13(a). Determine (i) the SFG of the system equivalent to the block diagram and (ii) the transfer function using Mason's gain formula.

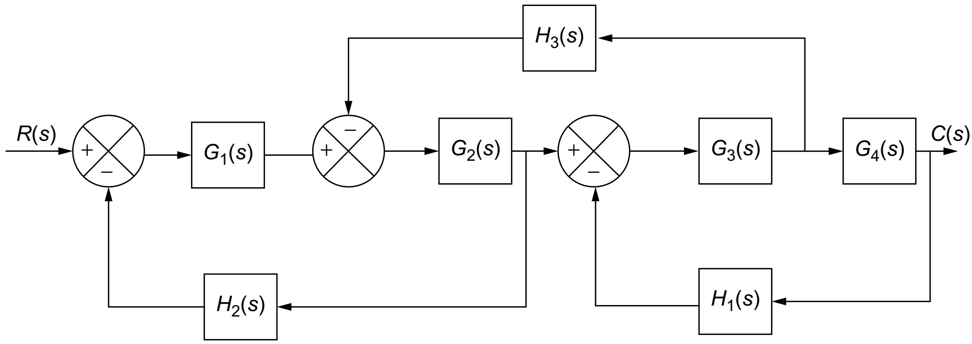


Fig. E4.13(a)

Solution:

Step 1: The SFG for the block diagram shown in Fig. E4.13(a) is shown in Fig. E4.13(b).

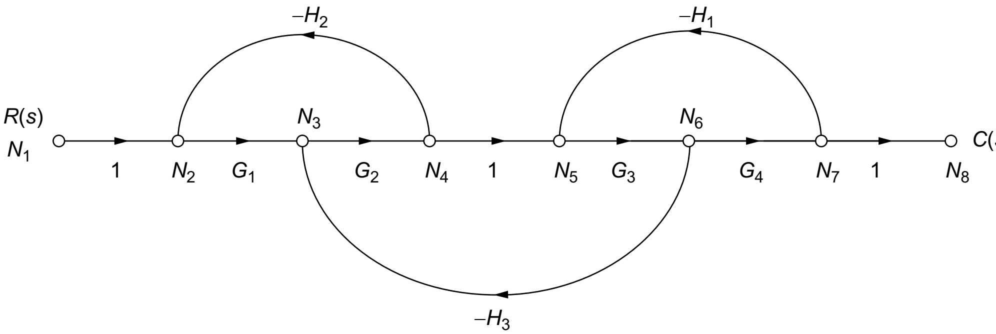
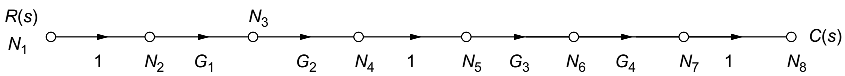


Fig. E4.13(b)

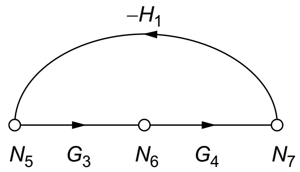
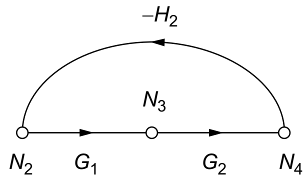
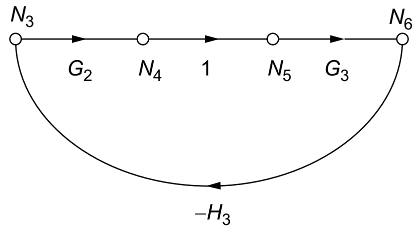
Step 2: The number of forward paths in the SFG, n shown in Fig. E4.13(b) is 1 and the gain corresponding to the forward path is given in Table E4.13(a).

Table E4.13(a) | Gain for the forward path

Forward path	Gain
	$P_1 = G_1G_2G_3G_4$

Step 3: The number of individual loops present in the SFG shown in Fig. E4.13(b) is $m = 3$ and the gain corresponding to each loop is given in Table E4.13(b).

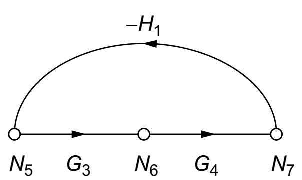
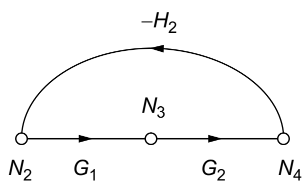
Table E4.13(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_3G_4H_1$
2.		$L_2 = -G_1G_2H_2$,
3.		$L_3 = -G_2G_3H_3$

Step 4: For the given SFG, $S_1 = -G_3G_4H_1 - G_1G_2H_2 - G_2G_3H_3$

Step 5: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.13(c).

Table E4.13(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
 	$L_{21} = G_1G_2G_3G_4H_1H_2$

Step 6: In the given problem, there exists one combination of two loops that have no node in common. Hence, S_j, where $j = 3$ to m is zero. Therefore,

$$S_2 = L_{21} = G_1G_2G_3G_4H_1H_2$$

Step 7: Since there exists no part of the graph that does not touch the forward paths in the given SFG, the values of Δ_1 and Δ_2 are 1.

Step 8: The transfer function of the system using Mason's gain formula is

$$\frac{C(s)}{R(s)} = \frac{Y_6}{Y_1} = \frac{G_1G_2G_3G_4}{1 + G_3G_4H_1 + G_1G_2H_2 + G_2G_3H_3 + G_1G_2G_3G_4H_1H_2}$$

Example 4.14: The SFG for a system is shown in Fig. E4.14(a). Obtain the transfer function of the system $\frac{Y_3}{Y_1}$ using Mason's gain formula.

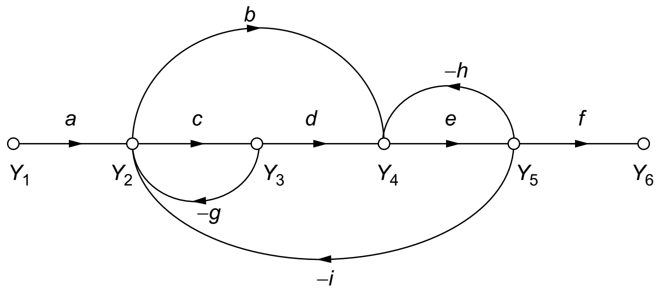


Fig. E4.14(a)

Solution: To determine the transfer function of the system, the following steps have to be followed:

(i) The transfer function of the system, $\frac{Y_6}{Y_1}$ is obtained by considering Y_1 as the input and Y_6 as the output.

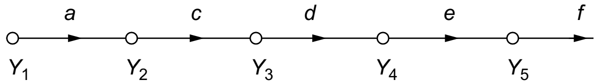
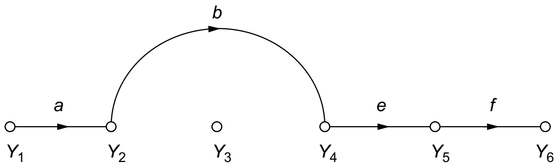
(ii) Then, the transfer function of the system, $\frac{Y_6}{Y_3}$ is obtained by considering Y_3 as the input and Y_6 as the output.

(iii) Then, the required transfer function of the system, $\frac{Y_3}{Y_1}$ is obtained as $\frac{Y_3}{Y_1} = \frac{\frac{Y_6}{Y_1}}{\frac{Y_6}{Y_3}}$

(a) To determine the transfer function of the system, $\frac{Y_6}{Y_1}$

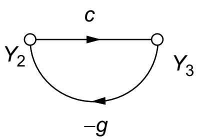
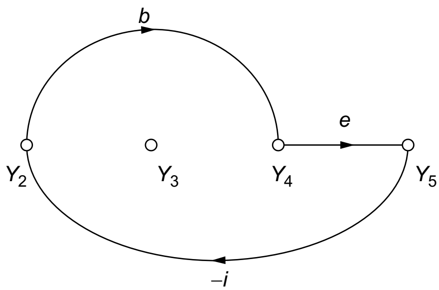
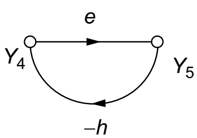
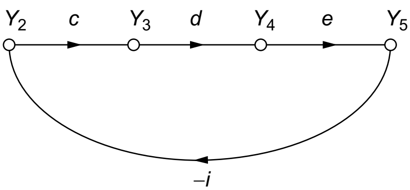
Step 1: The number of forward paths in the SFG, n shown in Fig. E4.14(a) is 2 and the gain corresponding to the forward path is given in Table E4.14(a).

Table E4.14(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = acdef$
2.		$P_2 = abef$

Step 2: The number of individual loops present in the SFG shown in Fig. E4.14(a) is $m = 4$ and the gain corresponding to each loop is given in Table E4. 14(b).

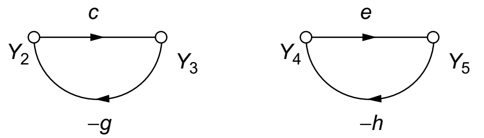
Table E4.14(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -cg$
2.		$L_2 = -bei$
3.		$L_3 = -eh$
4.		$L_4 = -cdei$

Step 3: For the given SFG, $S_1 = -cg - bei - eh - cdei$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.14(c).

Table E4.14(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
	$L_{21} = cegh$

Step 5: In the given problem, $S_2 = L_{21} = cegh$

Step 6: The next step is to determine Δ_i, where $i = 1$ to n for the given SFG. Since there exists no part of the graph that does not touch the forward paths in the given SFG, the values of Δ_1 and Δ_2 are one.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{Y_6}{Y_1} = \frac{acdef + abef}{1 + cg + bei + eh + cdei + cegh}$$

(b) To determine the transfer function of the system, $\frac{Y_6}{Y_3}$

Step 1: The modified SFG for determining the transfer function of the system, $\frac{Y_6}{Y_3}$ is shown in Fig. E4.14(b).

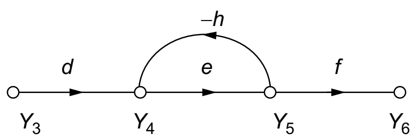
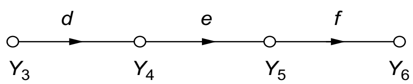


Fig. E4.14(b)

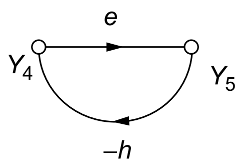
Step 2: The number of forward paths in the SFG, n shown in Fig. E4.14(b) is 1 and the gain corresponding to the forward path is given in Table E4.14(d).

Table E4.14(d) | Gain for the forward path

Forward path	Gain
	$P_1 = def$

Step 3: The number of individual loops present in the SFG shown in Fig. E4.14(a) is $m = 1$ and the gain corresponding to loop is given in Table E4.14(e).

Table E4.14(e) | Loop and its gain

Individual loops	Gain
	$L_1 = -eh$

Step 4: For the given SFG, $S_1 = -eh$

Step 5: Since there exists no combination of two non-touching loops without a common node in the given SFG, the product of gains of the non-touching loops is zero, i.e., higher orders of S_j are zero, where $j = 2, 3, \cdots, m$.

Step 6: The next step is to determine Δ_i, where, $i = 1$ to n for the given SFG. Since there exists no part of the graph that does not touch the forward paths in the given SFG, the value of Δ_1 is 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C(s)}{R(s)} = \frac{Y_6}{Y_3} = \frac{def}{1+eh}$$

(c) To determine the required transfer function of the system, $\frac{Y_3}{Y_1}$
Required transfer function,

$$\frac{Y_3}{Y_1} = \frac{\frac{Y_6}{Y_1}}{\frac{Y_6}{Y_3}} = \frac{acdef + abef + acde^2 fh + abe^2 fh}{def + cdefg + bde^2 fi + de^2 fh + cd^2e^2 fi + cde^2 fgh}$$

Example 4.15: For the electrical circuit shown in Fig. E4.15(a), determine the transfer function of the system, $\frac{V_3(s)}{V_1(s)}$ using Mason's gain formula.

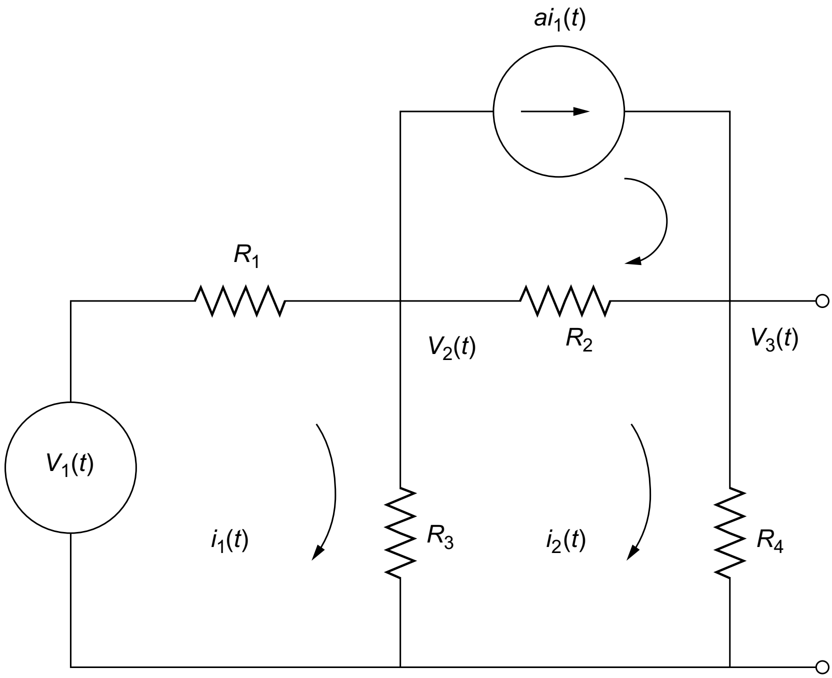


Fig. E4.15(a)

Solution: To determine the transfer function of an electrical circuit, the following procedure has to be followed:

Step A: Determine the individual SFG of the electrical equations.
Step B: Determine the SFG for the given electrical circuit.
Step C: Determine the transfer function of the given electrical circuit using Mason's gain formula.

Step A: To determine the individual SFG of the electrical equations.
The currents flowing through R_1 is $i_1(t)$ and R_2 is $i_2(t)$. Hence, using node voltages $v_1(t)$ $v_2(t)$ and $v_3(t)$, the currents $i_1(t)$ and $i_2(t)$ are obtained as

$$i_1(t) = \frac{v_1(t) - v_2(t)}{R_1} \tag{1}$$

$$i_{R2}(t) = \frac{v_2(t) - v_3(t)}{R_2} \tag{2}$$

Also, from Fig. E4.15(a), we obtain

$$i_{R2}(t) = i_2(t) - ai_i(t) \tag{3}$$

Equating Eqs. (2) and (3) and simplifying, we obtain

$$i_2(t) = \frac{v_2(t) - v_3(t)}{R_2} + ai_1(t) \tag{4}$$

The voltage $v_2(t)$ across R_3 is given by

$$v_2(t) = \left[i_1(t) - i_2(t)\right] R_3 \tag{5}$$

The voltage $v_3(t)$ across R_4 is given by

$$v_3(t) = i_2(t) R_4 \tag{6}$$

For Eqn. (1):
Taking Laplace transform of Eqn. (1), we obtain

$$I_1(s) = \frac{V_1(s) - V_2(s)}{R_1} \tag{7}$$

where the input is $V_1(s)$ and $V_2(s)$ and the output is $I_1(s)$. Hence, the SFG of the above equation is shown in Fig. E4.15(b).

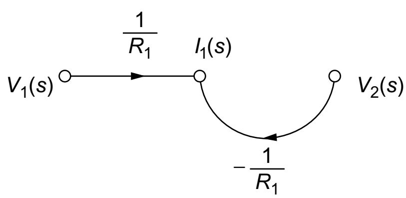


Fig. E4.15(b)

For Eqn. (4):
Taking Laplace transform of Eqn. (4), we obtain

$$I_2(s) = \frac{V_2(s) - V_3(s)}{R_2} + aI_1(s) \tag{8}$$

where the inputs are $V_3(s)$ and $V_2(s)$ and the output is $I_2(s)$. Hence, the SFG of the above equation is shown in Fig. E4.15(c).

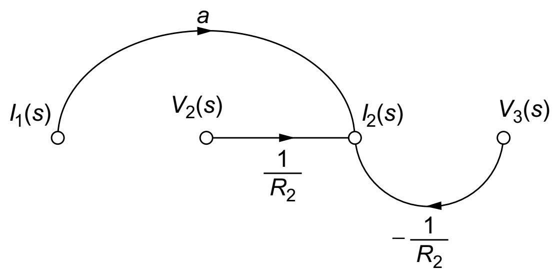


Fig. E4.15(c)

For Eqn. (5):
Taking Laplace transform of Eqn. (5), we obtain

$$V_2(s) = [I_1(s) - I_2(s)]R_3 \tag{9}$$

where the inputs are $I_2(s)$ and $I_1(s)$ and the output is $V_2(s)$. Hence, the SFG of the above equation is shown in Fig. E4.15(d).

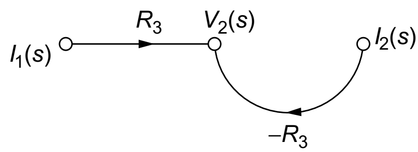


Fig. E4.15(d)

For Eqn. (6):
Taking Laplace transform of Eqn. (6), we obtain

$$V_3(s) = I_2(s)R_4 \tag{10}$$

where the input is $I_3(s)$ and the output is $V_3(s)$. Hence, the SFG of the above equation is shown in Fig. E4.15(e).

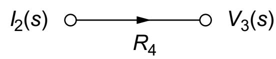


Fig. E4.15(e)

Step B: SFG for the given electrical circuit

Hence, the overall SFG for the given electrical circuit shown in Fig. E4.15(a) is obtained by interconnecting the individual SFGs shown in Figs. E4.15(b) through (e) and is shown in Fig. E4.15(f).

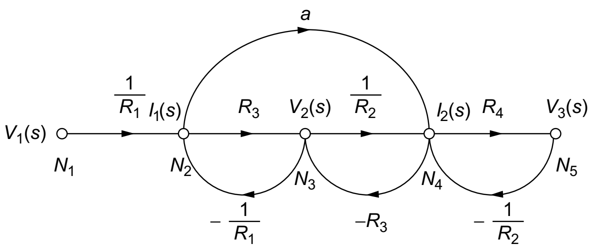
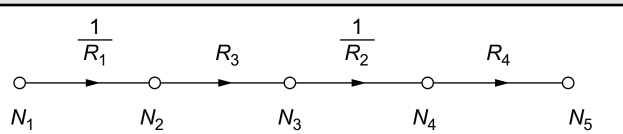


Fig. E4.15(f)

Step C: To determine the transfer function of the given electrical circuit using Mason's gain formula

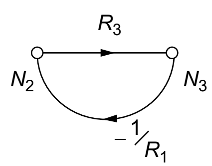
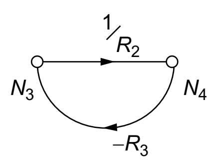
Step 1: For the given SFG, $n = 1$ and the gain corresponding to the forward path is given in Table E4.15(a).

Table E4.15(a) | Gain corresponding to the forward path

Forward path	Gain
	$P_1 = \dfrac{R_3 R_4}{R_1 R_2}$

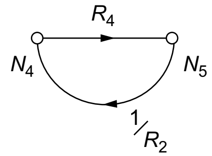
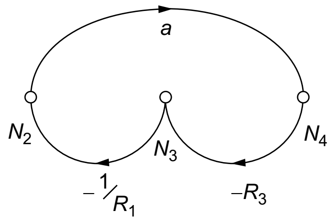
Step 2: For the given SFG, $m = 4$ and the corresponding loop gains are given in Table E4.15(b).

Table E4.15(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -\dfrac{R_3}{R_1}$
2.		$L_2 = -\dfrac{R_3}{R_2}$

(*Continued*)

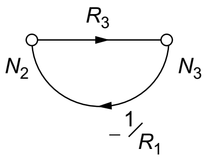
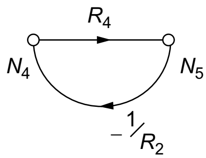
Table E4.15(b) | (Continued)

S. No	Individual loops	Gain
3.		$L_3 = -\dfrac{R_4}{R_2}$
4.		$L_4 = \dfrac{aR_3}{R_1}$

Step 3: For the given SFG, $S_1 = -\dfrac{R_3}{R_1} - \dfrac{R_3}{R_2} - \dfrac{R_4}{R_2} + \dfrac{aR_3}{R_1}$

Step 4: For the given SFG, the product of gains of all possible combination of two non-touching loops is given in Table E4.15(c).

Table E4.15(c) | Product of gains of combination of two non-touching loops

Combination of two non-touching loops	Product of gains
 	$L_{21} = \dfrac{R_3R_4}{R_1R_2}$

Step 5: In the given problem,

$$S_2 = L_{21} = \frac{R_3R_4}{R_1R_2}$$

Step 6: Since there exists no part of the graph that does not touch the first forward path in the given SFG, the value of Δ_1 is 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{V_3(s)}{V_1(s)} = \frac{R_3R_4}{R_1R_2 + R_3R_4 + R_3R_2 + R_3R_1 + R_4R_1 + aR_2R_3}$$

Example 4.16: The block diagram for a particular system is shown in Fig. E4.16(a). (i) Obtain the SFG equivalent to the block diagram and (ii) determine the output of the system using Mason's gain formula.

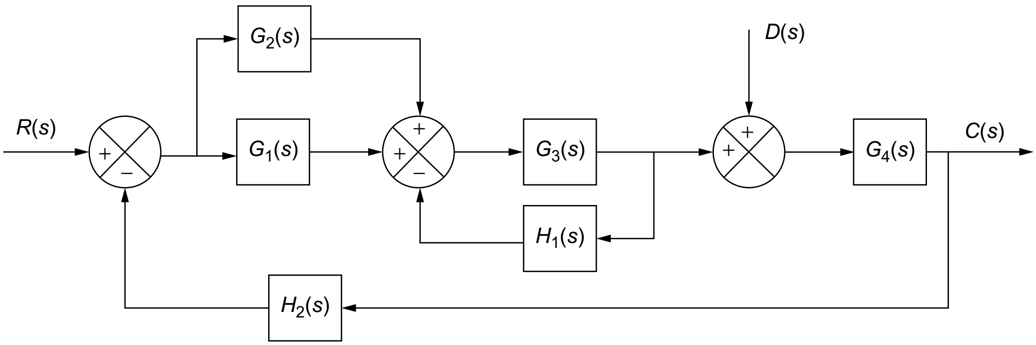


Fig. E4.16(a)

Solution: Since the given problem has two inputs $R(s)$ and $D(s)$, the total output of the system is determined by adding the individual outputs obtained by taking one input at a time. Hence, the total output of the system is given by

$$C(s) = C_1(s) + C_2(s) \tag{1}$$

where $C_1(s)$ is the output obtained by taking input $R(s)$ and $C_2(s)$ is the output obtained by taking input $D(s)$.

(a) Output when $D(s) = 0$:

Step 1: The modified SFG for the block diagram shown in Fig. E4.16(a) when the input $D(s) = 0$ is shown in Fig. E4.16(b).

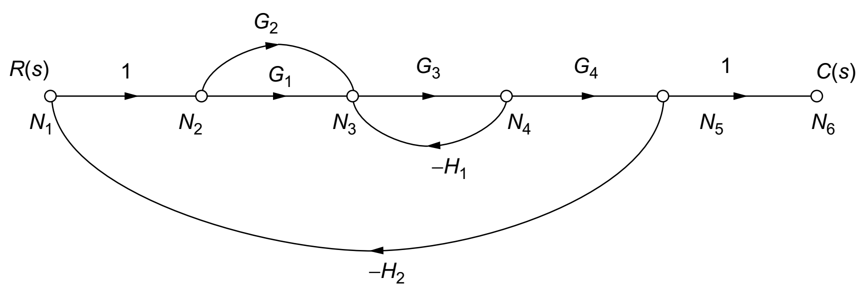
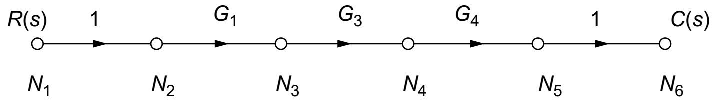
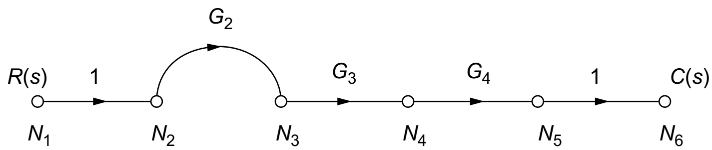


Fig. E4.16(b)

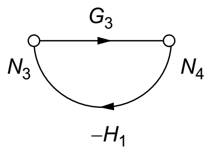
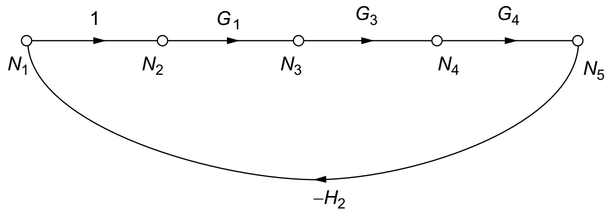
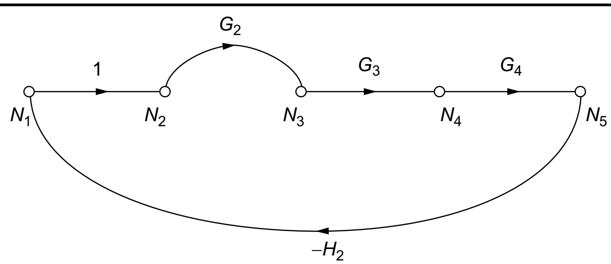
Step 2: The number of forward paths in the SFG, n shown in Fig. E4.16(a) is 2 and the gain corresponding to the forward path is given in Table E4.16(a).

Table E4.16(a) | Gain for the forward path

S. No	Forward path	Gain
1.		$P_1 = G_1G_3G_4$
2.		$P_2 = G_2G_3G_4$

Step 3: The number of individual loops present in the SFG shown in Fig. E4.16(a) is $m = 3$ and the gain corresponding to each loop is given in Table E4.16(b).

Table E4.16(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_3H_1$
2.		$L_2 = -G_1G_3G_4H_2$
3.		$L_3 = -G_2G_3G_4H_2$

Step 4: For the given SFG, $S_1 = -G_3H_1 - G_1G_3G_4H_2 - G_2G_3G_4H_2$

Step 5: Since there exist no combination of non-touching loops, higher orders of S_j are zero, where $j = 2, 3, \cdots, m$.

Step 6: Since there exists no part of the graph that does not touch the forward paths in the given SFG, the values of Δ_1 and Δ_2 are 1.

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C_1(s)}{R(s)} = \frac{G_1G_3G_4 + G_2G_3G_4}{1 + G_3H_1 + G_1G_3G_4H_2 + G_2G_3G_4H_2}$$

Hence, the output of the system when the input $R(s)$ is only applied to the system is given by

$$C_1(s) = \left[\frac{G_1G_3G_4 + G_2G_3G_4}{1 + G_3H_1 + G_1G_3G_4H_2 + G_2G_3G_4H_2}\right]R(s)$$

(b) When the input R(s) = 0:

Step 1: The modified SFG for the block diagram shown in Fig. E4.16 (a) when the input $R(s) = 0$ is shown in Fig. E4.16 (c).

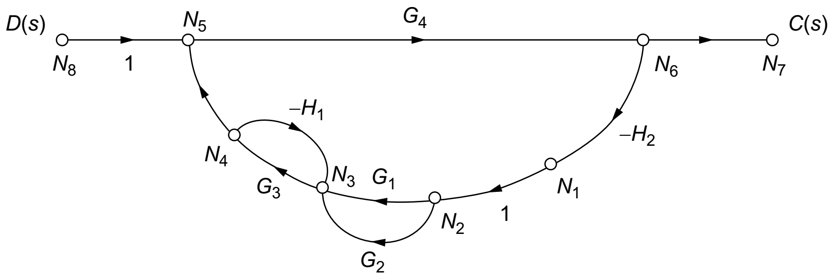
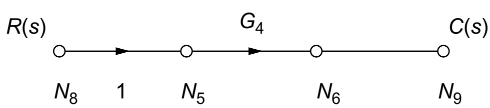


Fig. E4.16(c)

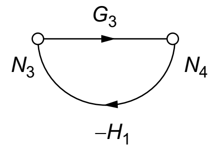
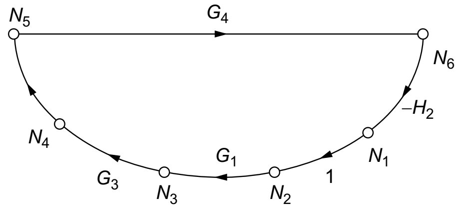
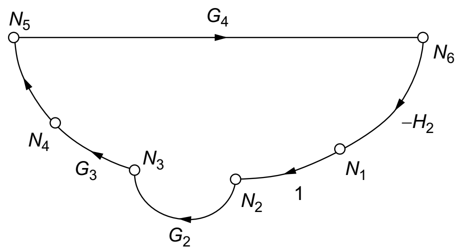
Step 2: The number of forward paths in the SFG, n, shown in Fig. E4.16(c) is 1 and the gain corresponding to the forward path is given in Table E4.16(c).

Table E4.16(c) | Gain for the forward path

Forward path	Gain
	$P_1 = G_4$

Step 3: The number of individual loops present in the SFG shown in Fig. E4.16(a) is $m = 3$ and the gain corresponding to each loop is given in Table E4.16(d).

Table E4.16(d) | Individual loop gains

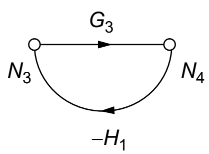
S. No	Individual loops	Gain
1.		$L_1 = -G_3H_1$
2.		$L_2 = -G_1G_3G_4H_2$
3.		$L_3 = -G_2G_3G_4H_2$

Step 4: For the given SFG, $S_1 = -G_3H_1 - G_1G_3G_4H_2 - G_2G_3G_4H_2$

Step 5: Since there exists no combination of non-touching loops, higher orders of S_j are zero, where $j = 2, 3, \cdots, m$.

Step 6: The part of the graph that does not touch the forward path one is given in Table E4.16(e). The corresponding value of Δ_1 is also shown in the table.

Table E4.16(e) | Determination of Δ_1

Part of the graph	Corresponding gain	Δ_1
	$-G_3H_1$	$\Delta_1 = 1 + G_3H_1$

Step 7: The transfer function of the system using Mason's gain formula is given by

$$\frac{C_2(s)}{D(s)} = \frac{G_4(1+G_3H_1)}{1+G_3H_1+G_1G_3G_4H_2+G_2G_3G_4H_2}$$

Hence, the output of the system when the input $R(s)$ is only applied to the system is given by

$$C_2(s) = \frac{D(s)\left(G_4(1+G_3H_1)\right)}{1+G_3H_1+G_1G_3G_4H_2+G_2G_3G_4H_2}$$

Hence, the total output of the system shown in Fig. E.4.16(a) is given by

$$C(s) = \left[\frac{\left(G_4(1+G_3H_1)\right)}{1+G_3H_1+G_1G_3G_4H_2+G_2G_3G_4H_2}\right]D(s) + \left[\frac{G_1G_3G_4+G_2G_3G_4}{1+G_3H_1+G_1G_3G_4H_2+G_2G_3G_4H_2}\right]R(s)$$

Example 4.17: For the electrical circuit shown in Fig. E4.17(a), determine the transfer function of the system, $\frac{V_2(s)}{V_i(s)}$ using Mason's gain formula.

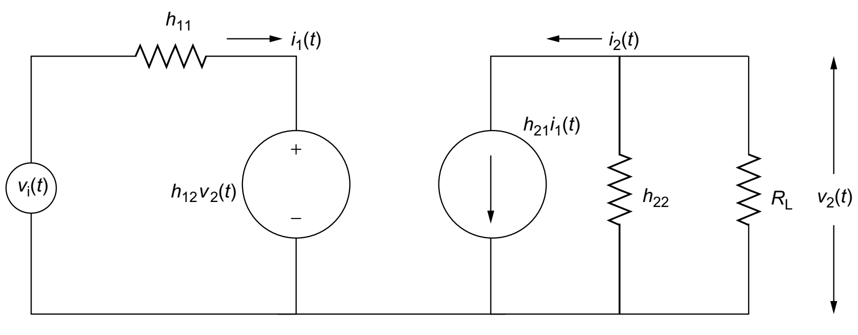


Fig. E4.17(a)

Solution: To determine the transfer function of an electrical circuit, the following procedure has to be followed:

Step A: Determine the SFG of the individual electrical equations.
Step B: Determine the SFG for the given electrical circuit.
Step C: Determine the transfer function of the given electrical circuit using Mason's gain formula.

Step A: To determine the SFG of the individual electrical equation
The loop equations for the given electrical circuit are
Loop 1:

$$-h_{11}i_1(t) - h_{12}v_2(t) + v_1(t) = 0$$

$$i_1(t) = \frac{v_1}{h_{11}} - \frac{h_{12}}{h_{11}} v_2(t) \tag{1}$$

Loop 2:

$$i_2(t) = h_{21} i_1(t) + h_{22} v_2(t) \tag{2}$$

Also, the output is given by

$$v_2(t) = -i_2(t) R_L \tag{3}$$

Taking Laplace transform on both sides of the above two equations, we obtain

$$I_1(s) = \frac{V_1(s)}{h_{11}} - \frac{h_{12}}{h_{11}} V_2(s) \tag{4}$$

$$I_2(s) = h_{21} I_1(s) + h_{22} V_2(s) \tag{5}$$

$$V_2(s) = -I_2(s) R_L \tag{6}$$

The SFGs for Eqs. (4) to (6) are shown in Figs. E4.47(b) through (d) respectively.

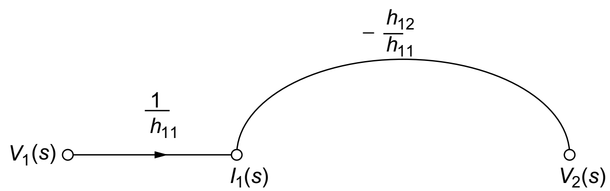


Fig. E4.17(b)

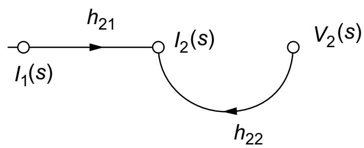


Fig. E4.17(c)

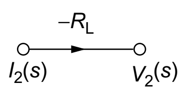


Fig. E4.17(d)

Step B: SFG for the given electrical circuit

The SFG of the given electrical circuit is obtained by combining the SFGs shown in Figs. E4.47(b) through (d) in an appropriate way and is shown in Fig. E4.17(e).

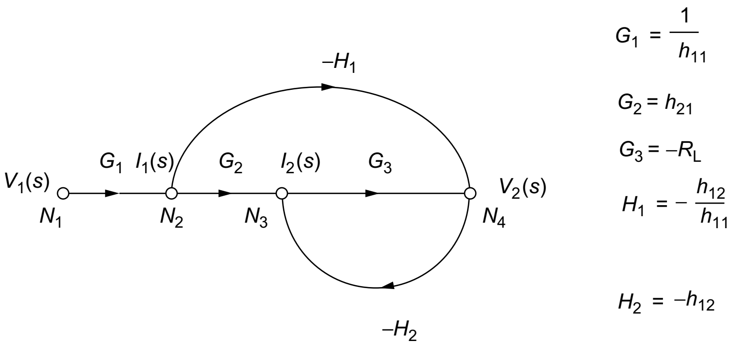


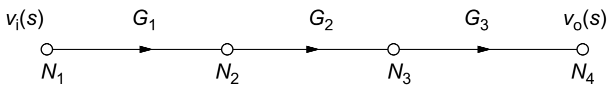
Fig. E4.17(e)

For the SFG shown in Fig. E4.17(e), $G_1 = \dfrac{1}{h_{11}}$, $G_2 = h_{21}$, $G_3 = -R_L$, $H_1 = -\dfrac{h_{12}}{h_{11}}$ and $H_2 = -h_{22}$.

Step C: To determine the transfer function of the given electrical circuit using Mason's gain formula

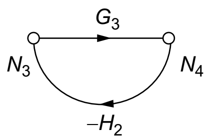
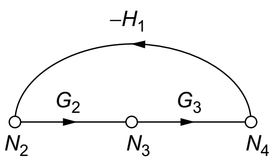
Step 1: For the given SFG, $n = 1$ and the gain corresponding to the forward path is given in Table E4.17(a).

Table E4.17(a) | Gain corresponding to the forward path

Forward path	Gain
	$P_1 = G_1G_2G_3$

Step 2: For the given SFG, $m = 2$ and the corresponding loop gains are given in Table E4.17(b).

Table E4.17(b) | Individual loop gains

S. No	Individual loops	Gain
1.		$L_1 = -G_3H_2$
2.		$L_2 = -G_2G_3H_1$

Step 3: For the given SFG, $S_1 = -G_3H_2 - G_2G_3H_1$

Step 4: Hence, $L_{ji} = 0$ and $S_2 = 0$.

Step 5: Since there exists no part of the graph that does not touch the first forward path in the given SFG, the value of Δ_1 is 1.

Step 6: The transfer function of the system using Mason's gain formula is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{G_1G_2G_3}{1 + G_3H_2 + G_2G_3H_1}$$

4.1.6 Comparison between SFG and Block Diagram

The comparison between SFG and block diagram based on certain characteristics is given in Table 4.3.

Table 4.3 | SFG vs block diagram

Characteristics	Block diagram	SFG
Time consumption	More since the diagrams have to be redrawn repeatedly	Less since there is no necessary to redraw the diagrams
Technique applied	Block diagram reduction technique	Mason's gain formula
Representation of elements	Blocks are used to represent the elements	Nodes are used to represent the elements
Representation of transfer function of each element	Represented inside the block of each element.	Represented along the branches above the arrow ahead
Feedback paths	Present, and hence the formula $\frac{G}{1 \pm GH}$ is used to reduce the paths	Present, but there is no need for any formulae to reduce the paths
Self-loops	Absence of self-loops	Presence of self-loops
Summing points and takeoff points	Present in block diagram	Absent in SFG

Review Questions

1. Define signal flow graph of a system.
2. What are the terminologies used in a signal flow graph of the system?
3. Define a node.
4. What are the different types of nodes present in a signal flow graph representation of a system?
5. Define different types of node present in the SFG of the system.

6. Define a path.
7. What are the different types of paths?
8. Define different types of paths present in SFG of the system.
9. Define self-loop and non-touching loops.
10. Define gain.
11. What are the different types of gains?
12. Define forward-path gain and loop gain/feedback path gain.
13. What are the properties of a signal flow graph?
14. What do you mean by a signal flow graph algebra?
15. What are the rules for deriving the transfer function of a system using SFG?
16. Explain the different rules used in SFG of the system.
17. Define the following terms with reference to signal flow graphs: path, input node, sink, path gain, and loop gain.
18. Write down the Mason's gain formula for signal flow graph. Give the meaning for each symbol.
19. Explain with a neat flow chart, the procedure for deriving the SFG from a differential equation.
20. Compare block diagram with SFG technique.
21. The block diagram for a particular system is shown in Fig. Q4.21. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

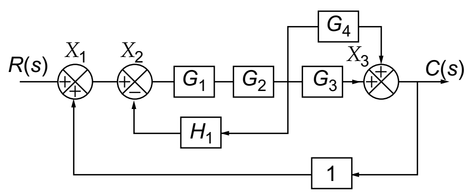


Fig. Q4.21

22. The block diagram representation of a system is shown in Fig. Q4.22. (i) Obtain the transfer function using block diagram reduction technique and (ii) verify the result obtained using SFG.

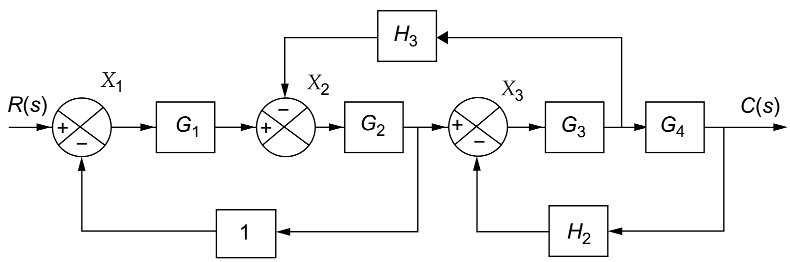


Fig. Q4.22

23. The block diagram for a particular system is shown in Fig. Q4.23. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

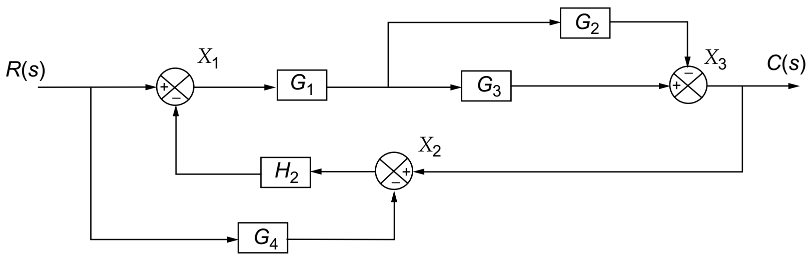


Fig. Q4.23

24. The block diagram for a particular system is shown in Fig. Q4.24. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

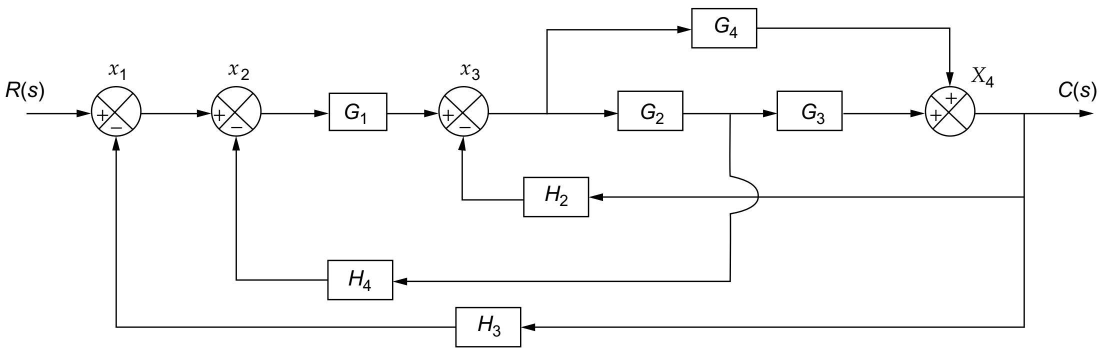


Fig. Q4.24

25. For the SFG of a system shown in Fig. Q4.25, obtain the transfer function of the system represented by SFG using Mason's gain formula.

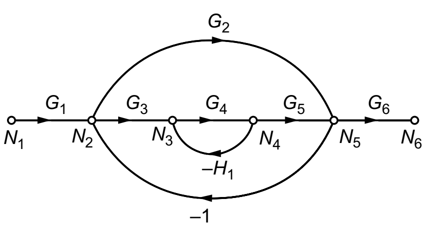


Fig. Q4.25

26. The SFG for a particular system is shown in Fig. Q4.26. Determine the transfer function of the system using Mason's gain formula.

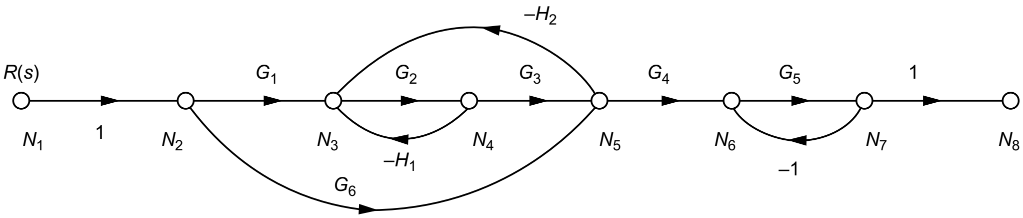


Fig. Q4.26

27. The SFG of a system is shown in Fig. Q4.27, obtain the transfer function of the given SFG using Mason's gain formula.

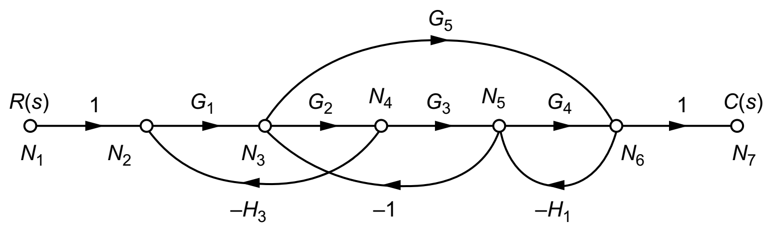


Fig. Q4.27

28. The signal flow diagram for a particular system is shown in Fig. Q4.28. Determine the transfer function of the system using Mason's gain formula.

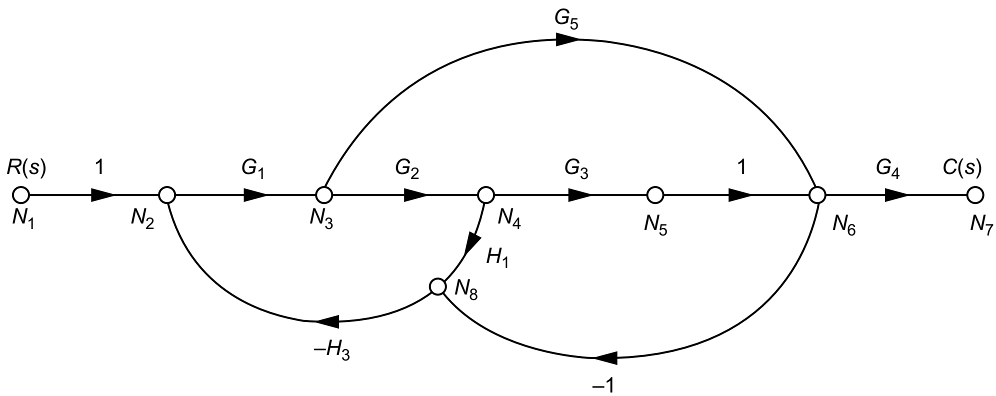


Fig. Q4.28

29. For the SFG of a system shown in Fig. Q4.29, obtain the transfer function of the system $\dfrac{N_7(s)}{N_1(s)}$ represented by SFG using Mason's gain formula.

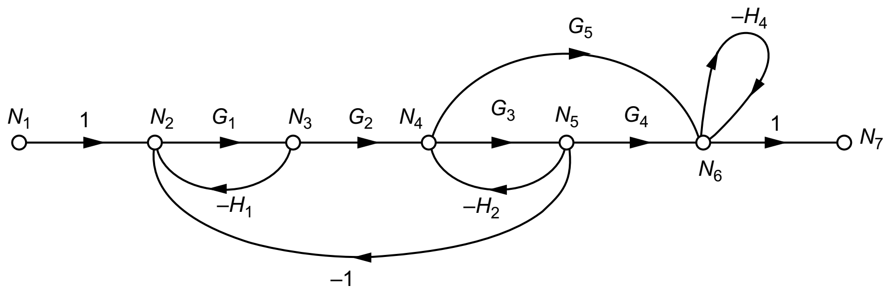


Fig. Q4.29

30. The SFG for a system is shown in Fig. Q4.30. Obtain the transfer function of the given SFG using Mason's gain formula.

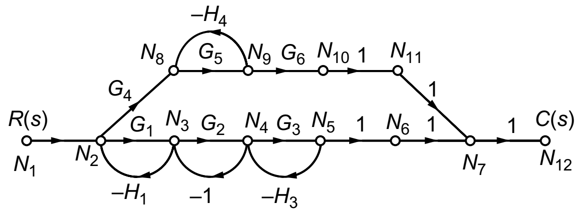


Fig. Q4.30

31. The SFG for a system is shown in Fig. Q4.31. Obtain the transfer function of the given SFG using Mason's gain formula.

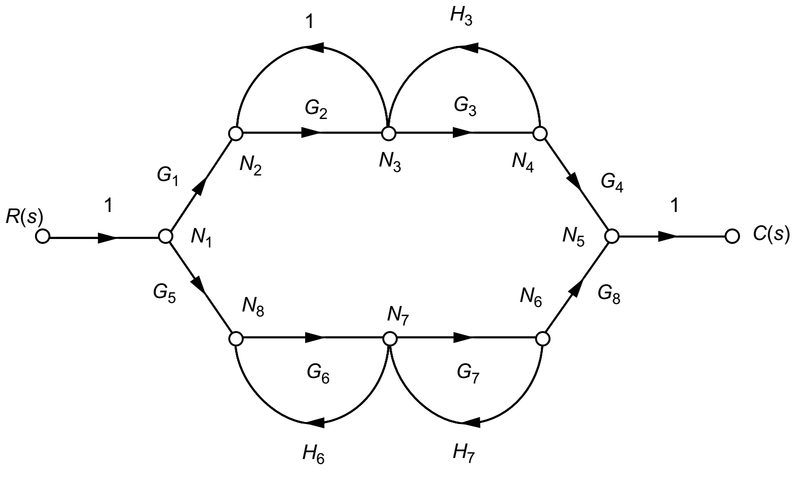


Fig. Q4.31

32. The SFG for a system is shown in Fig. Q4.32. Obtain the transfer function of the given SFG using Mason's gain formula.

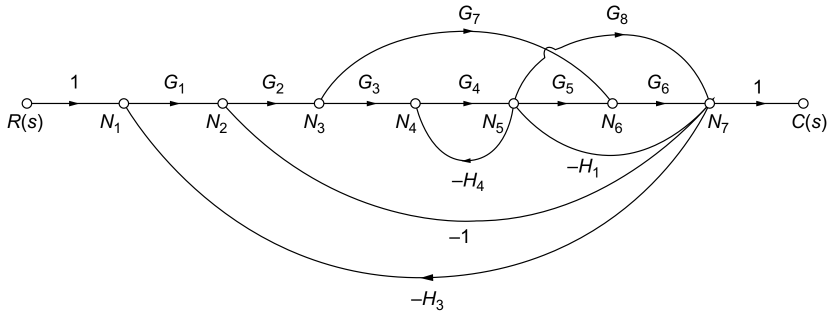


Fig. Q4.32

33. The block diagram for a particular system is shown in Fig. Q4.33. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

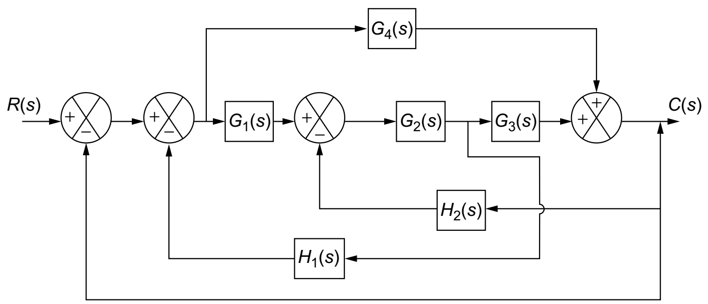


Fig. Q4.33

34. The block diagram for a particular system is shown in Fig. Q4.34. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

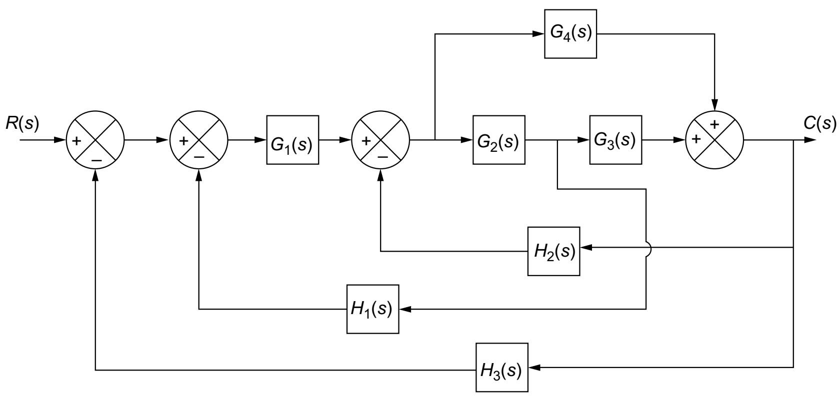


Fig. Q4.34

35. The block diagram of the system is shown in Fig. Q4.35. Determine the transfer function by using block diagram reduction technique and verify it using SFG technique.

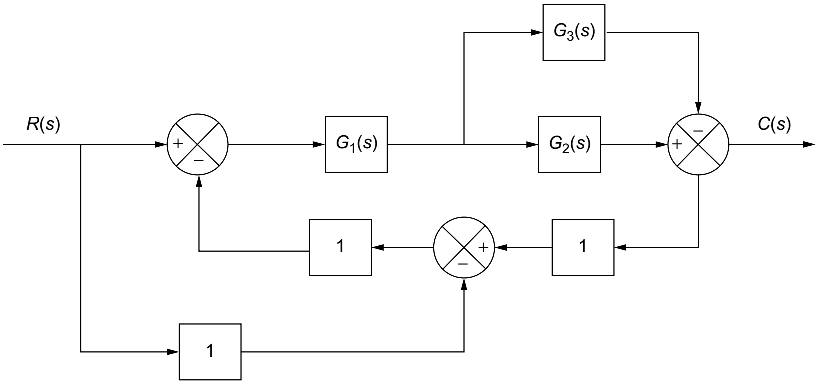


Fig. Q4.35

36. The SFG for a system is shown in Fig. Q4.36. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

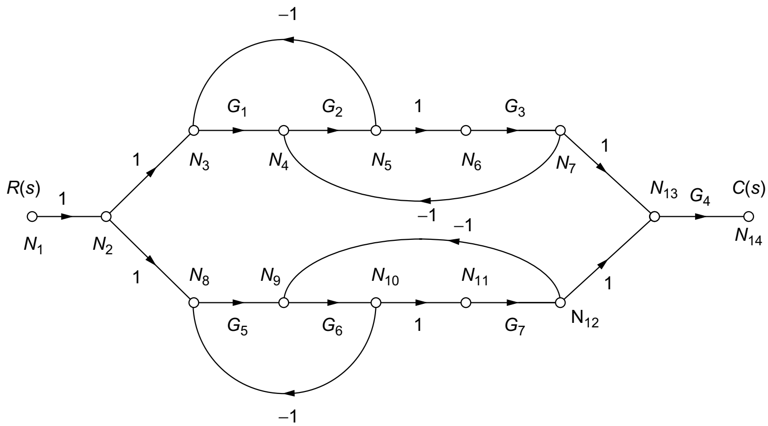


Fig. Q4.36

37. The SFG for a system is shown in Fig. Q4.37. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

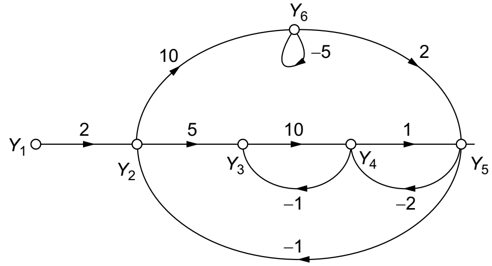


Fig. Q4.37

38. The SFG for a system is shown in Fig. Q4.38. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

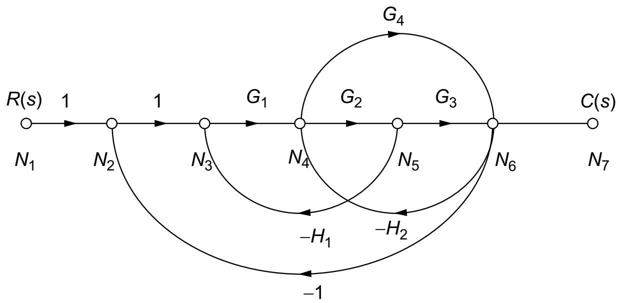


Fig. Q4.38

39. The SFG for a system is shown in Fig. Q4.39. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

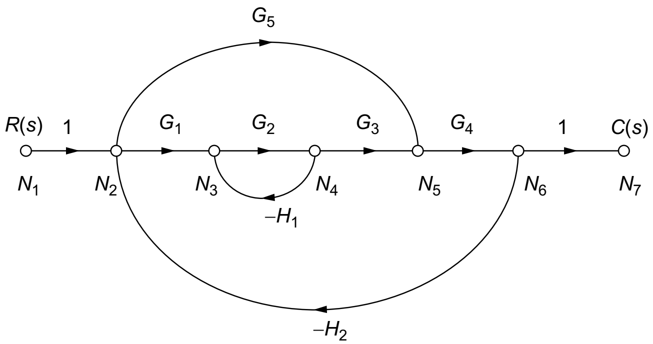


Fig. Q4.39

40. The SFG for a system is shown in Fig. Q4.40. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

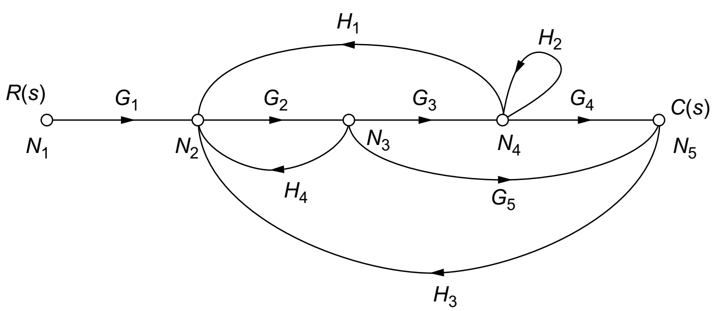


Fig. Q4.40

41. For the SFG system shown in Fig. Q4.41, obtain the transfer function of the given SFG using Mason's gain formula.

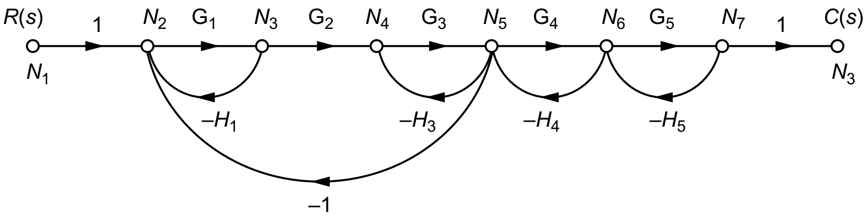


Fig. Q4.41

42. The SFG for a system is shown in Fig. Q4.42. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

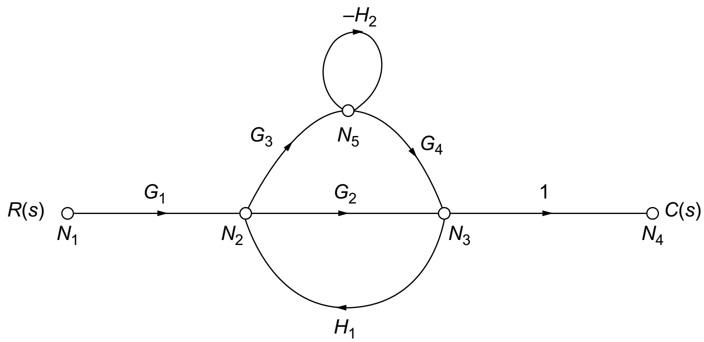


Fig. Q4.42

43. The block diagram for a particular system is shown in Fig. Q4.43. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

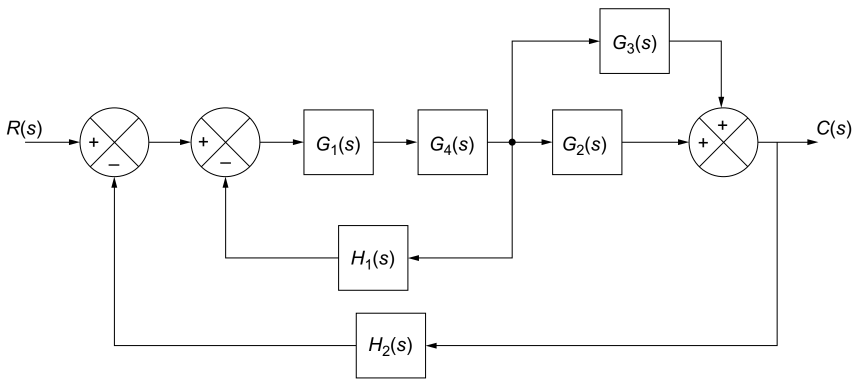


Fig. Q4.43

44. The SFG for a system is shown in Fig. Q4.44. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

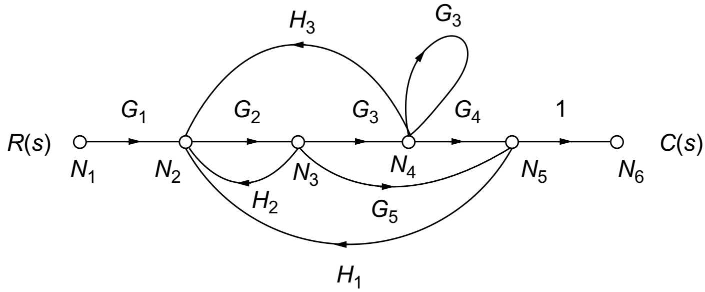


Fig. Q4.44

45. The SFG for a system is shown in Fig. Q4.45, obtain the transfer function of the system represented by SFG using Mason's gain formula.

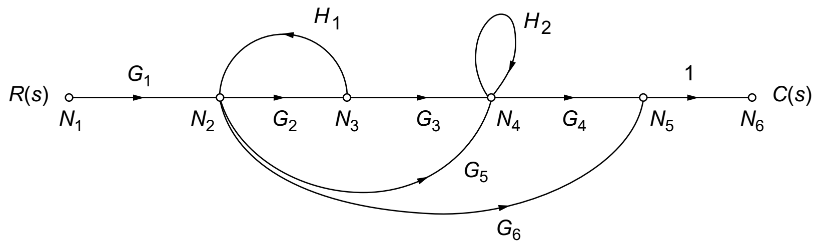


Fig. Q4.45

46. The SFG for a system is shown in Fig. Q4.46. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

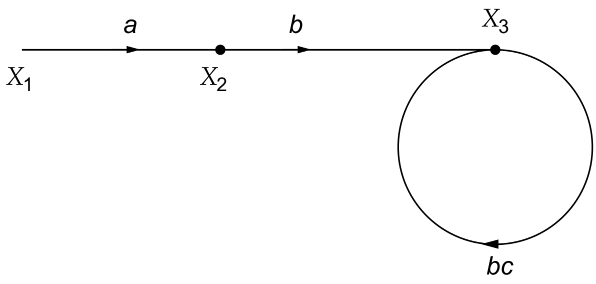


Fig. Q4.46

47. The SFG for a system is shown in Fig. Q4.47. (i) Obtain the transfer function of the system using Mason's gain formula; (ii) draw its equivalent block diagram and verify the transfer function obtained.

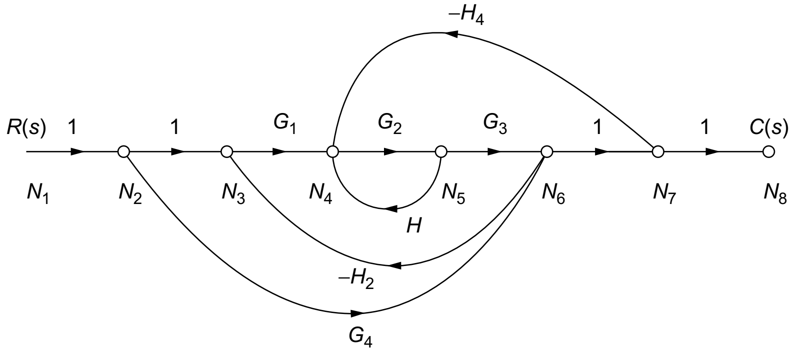


Fig. Q4.47

48. The block diagram for a particular system is shown in Fig. Q4.48. (i) Obtain the SFG equivalent to the block diagram and (ii) determine the transfer function of the system using Mason's gain formula.

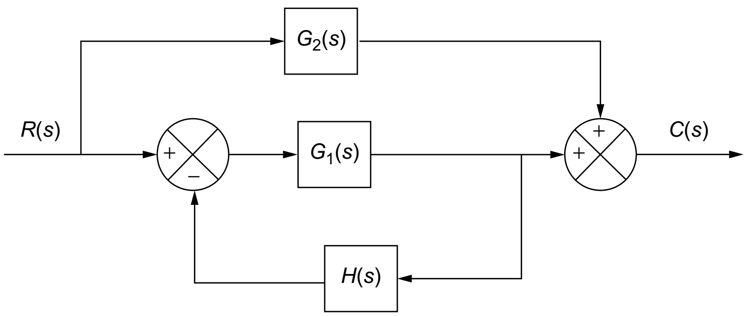


Fig. Q4.48

49. The SFG for a system is shown in Fig. Q4.49. Obtain the transfer function of the system represented by SFG using Mason's gain formula.

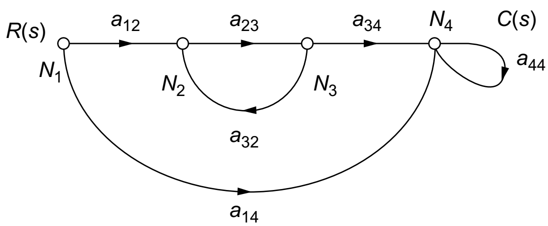


Fig. Q4.49

50. For the electrical circuit shown in Fig. Q4.50, determine the transfer function of the system, $\frac{V_2(s)}{V_i(s)}$ using Mason's gain formula.

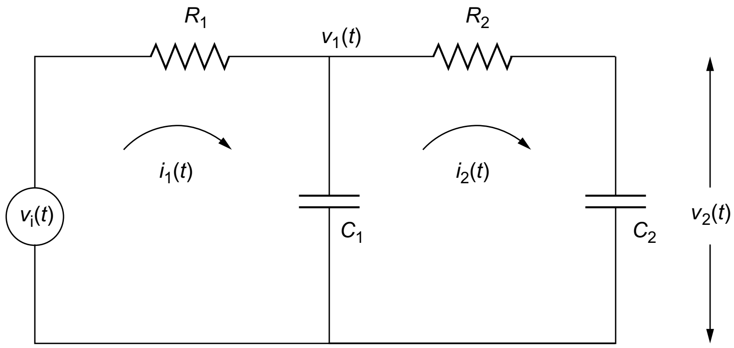


Fig. Q4.50

This Page is intentionally left blank.

5

TIME RESPONSE ANALYSIS

5.1 Introduction

The objective of this chapter is to analyse the performance of a system after the mathematical model of that system is obtained. The performance of a system is analysed based on the response of the system. The response of a system can be in the time domain or frequency domain that depends on the standard test signals applied to the system. In this chapter, the performance of a system is analysed based on the time response of the system. But the time response analysis of a system is based on the standard test signals that are random in nature. In addition to the randomness of the signals, these signals are not known in advance. It is necessary to have clear knowledge about standard test signals as the system performance varies in accordance with the standard signals that are applied to the system. Practically, if a signal is applied as input to any system, the system does not follow the input and a steady-state error exists between the input and output of the system. Controllers are used alongwith the system to reduce the steady-state error when a standard test signal is applied.

The mathematical modelling of any physical system was derived in Chapters 1 and 2. This chapter discusses various parts present in the time response of the modelled system, various standard test input signals, time-domain performance of the system, steady-state error developed in the system due to various test signals and controllers to reduce the developed steady-state error. In addition, this chapter analyses the transient and steady-state responses of first and second order systems subjected to various standard test signals. Moreover, the transient behaviour for underdamped, critically damped and overdamped cases of a second-order system and steady-state error of the first and second order control systems have also been discussed.

5.2 Time Response of the Control System

In control systems, the state and output responses are evaluated with respect to time as time is an independent variable. The evaluated state and output responses can be generally defined as the time response of a system. The time response of a control system, when a standard signal is applied to the system is denoted by $c(t)$.
The total time response of the system is given by

$$c(t) = c_t(t) + c_{ss}(t)$$

where $c_t(t)$ is the transient response and $c_{ss}(t)$ is the steady-state response. The time response of the system is used for evaluating the performance of the system. The performance indices of the system are usually obtained from transient and steady-state responses.

5.2.1 Transient Response

The part of the time response of the system that goes to zero as time t becomes very large is known as transient response. The transient response of the system is obtained before the system reaches steady-state value. Thus $c_t(t)$ has the property given by

$$\underset{t \to \infty}{Lt}\, c_t(t) = 0$$

The significant features of transient response of a system are:

(i) Transient response helps in understanding the nature of variation of the system.
(ii) It does not follow the standard signal applied due to the presence of inertia, friction and energy storage elements.
(iii) It exhibits damped oscillations.
(iv) The deviation between the applied standard signal and the output is high but not desirable and it has to be controlled before the steady state is reached.

5.2.2 Steady-State Response

The part of the time response of the system that is obtained as the independent variable time t approaches infinity is known as steady-state response of the system. It can also be defined as a part of the total response found after the transient response dies out.
The significant features of steady-state response of a system are:

(i) Steady-state response helps in giving an idea about the accuracy of the system.
(ii) As the independent variable time t, tends to infinity, the deviation between the input and output gives the steady-state error.
(iii) Steady-state response of the system follows the transient response of the system.
(iv) The pattern of the steady-state response of the system varies like sine wave or a ramp function.

Example 5.1: Determine the time response of the closed-loop system as the function of time, denoted by $c(t)$ for the system shown in Fig. E5.1.

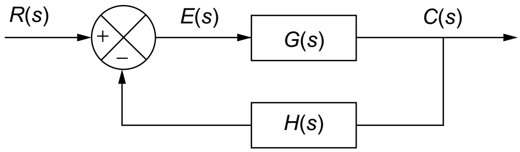


Fig. E5.1

Solution:
From Fig. E5.1, we have

$$E(s) = R(s) - C(s)H(S)$$

$$C(s) = E(s)G(s)$$

Using the above two equations, we obtain

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Therefore, the output response in s-domain is

$$C(s) = R(s) \times \frac{G(s)}{1+G(s)H(s)}$$

Taking inverse Laplace transform, we obtain output response in time domain as

$$c(t) = \mathcal{L}^{-1}\left[R(s)\frac{G(s)}{1+G(s)H(s)}\right]$$

Example 5.2: Determine the current $i(t)$ flowing through the series RC circuit shown in Fig. E5.2(b) when the periodic waveform shown in Fig. E5.2(a) is applied. Also, determine the transient and steady-state currents.

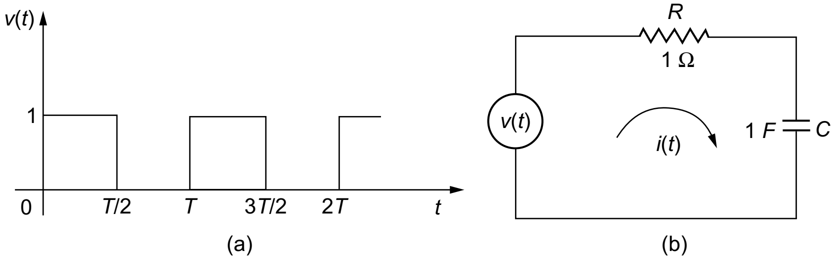


Fig. E5.2

Solution:

The function for the period of the given waveform is

$$v(t) = 1, \text{ for } 0 < t \le T/2$$

$$= 0, \text{ for } T/2 < t \le T$$

$$V(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}} \left[\int_0^{T/2} 1.e^{-st}dt + \int_{T/2}^{T} 0.e^{-st}dt \right]$$

$$= \frac{1}{1-e^{-sT}} \left[\frac{e^{-st}}{-s} \right]_0^{T/2}$$

$$= \frac{1}{1-e^{-sT}} \left[\frac{1-e^{-sT/2}}{s} \right]$$

Example 5.3: Determine the response for the system whose transfer function is given by $H(s) = \dfrac{1}{(s+1)(s^2+s+1)}$ when the excitation is $x(t) = (1+e^{-3t}-e^{-t})u(t)$.

Solution:

The given input function is $x(t) = 1 + e^{-3t} - e^{-t}$.

Taking Laplace transform, we obtain

$$X(s) = \frac{1}{s} + \frac{1}{s+3} - \frac{1}{s+1}$$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} = \frac{A_1}{s+1} + \frac{A_2 s + A_3}{s^2+s+1}$$

$$A_1(s^2+s+1) + (A_2 s + A_3)(s+1) = 1$$

Substituting $s = -1$, we obtain $A_1 = 1$.

Comparing the coefficients of s^2, we obtain $A_1 + A_2 = 0$

$$A_2 = -1$$

Comparing the coefficients of s, we obtain $A_1 + A_2 + A_3 = 0$

$$1 - 1 + A_3 = 0$$

$$A_3 = 0$$

Therefore, $H(s) = \dfrac{1}{(s+1)(s^2+s+1)} = \dfrac{1}{s+1} - \dfrac{s}{s^2+s+1}$

It is known that $Y(s) = H(s)X(s)$

$$Y(s) = \left(\frac{1}{s+1} - \frac{s}{s^2+s+1}\right)\left(\frac{1}{s} + \frac{1}{s+3} - \frac{1}{s+1}\right)$$

$$= \frac{1}{s(s+1)} + \frac{1}{(s+1)(s+3)} - \frac{1}{(s+1)^2} - \frac{1}{(s^2+s+1)} - \frac{s}{(s+3)(s^2+s+1)} + \frac{s}{(s+1)(s^2+s+1)}$$

Using partial fraction expansions, the above functions can be expanded as

(i) $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$

(ii) $\dfrac{1}{(s+1)(s+3)} = \dfrac{1}{2(s+1)} - \dfrac{1}{2(s+3)}$

(iii) $\dfrac{s}{(s+3)(s^2+s+1)} = \dfrac{-3/7}{s+3} + \dfrac{3/7s+1/7}{s^2+s+1}$

(iv) $\dfrac{s}{(s+1)(s^2+s+1)} = -\dfrac{1}{s+1} + \dfrac{s+1}{s^2+s+1}$

Therefore, $Y(s) = \dfrac{1}{s} - \dfrac{1}{s+1} + \dfrac{1}{2(s+1)} - \dfrac{1}{2(s+3)} - \dfrac{1}{(s+1)^2} - \dfrac{1}{s^2+s+1} + \dfrac{3/7}{s+3} - \dfrac{3/7s+1/7}{s^2+s+1}$

$$-\frac{1}{s+1} + \frac{s+1}{(s^2+s+1)}$$

where $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} - \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

Taking inverse Laplace transform of $Y(s)$, we obtain

$$y(t) = 1 - e^{-t} + \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} - te^{-t} - \frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t + \frac{3}{7}e^{-3t} - \frac{3}{7}\left(e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right)$$

$$-\frac{1}{7}\times\frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t - e^{-t} + e^{-t/2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t - \frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

$$y(t) = 1 + e^{-t}\left[-1+\frac{1}{2}-1-t\right] + \sin\frac{\sqrt{3}}{2}te^{-t/2}\times\left[-\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{7}-\frac{2}{7\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\right]$$

$$+\cos\frac{\sqrt{3}}{2}te^{-t/2}\left[1-\frac{3}{7}\right] + e^{-3t}\left[\frac{3}{7}-\frac{1}{2}\right]$$

$$y(t) = 1 - \left[\frac{3}{2}+t\right]e^{-t} - \frac{6}{7\sqrt{3}}\sin\frac{\sqrt{3}}{2}te^{-t/2} + \frac{4}{7}\cos\frac{\sqrt{3}}{2}te^{-t/2} + e^{-3t}\left[-\frac{1}{14}\right].$$

Example 5.4: The total response of the system when the system excited by an input is given by $c(t) = \frac{1}{2}\left(2 - e^{-3t}\right)$. Determine the steady-state response and transient response of the system.

Solution:

Given $c(t) = \frac{1}{2}\left(2 - e^{-3t}\right)$

We know that the total response of the system is composed of transient and steady-state responses, i.e., $c(t) = c_t(t) + c_{ss}(t)$.

From the definitions of transient and steady-state responses, we obtain

Transient response of the system $c_t(t) = -\frac{1}{2}e^{-3t}$

Steady-state response of the system $c_{ss}(t) = \frac{1}{2} \times 2 = 1$

5.3 Standard Test Signals

The time response of the control system depends on the type of standard test signals applied to the system. Control systems are analysed and designed depending on the performance indices of the systems. But performance indices of various control systems depend on the type of standard test signal applied to it. It is necessary to know the importance of test signals as a correlation exists between the output response of the system to a standard test input signal and the capability of the system to cope with applied standard test input signal.

As discussed earlier, the input signal to a control system is not known ahead of time, but it is random in nature and the instantaneous input cannot be expressed analytically. Only in some special cases, the input signal is known in advance and expressible analytically or by curves, such as in the case of the automatic control of cutting tools.

The standard test signals that can be applied to the control system are (i) impulse signal, (ii) step signal, (iii) ramp signal, (iv) parabolic signal and (v) sinusoidal signal.
However, to determine the time response characteristic of the control system, the first four of the above standard test signals are applied and to determine the frequency response characteristic, standard sinusoidal signal is applied.

5.3.1 Impulse Signal

The signal that is also called shock input, having its value as zero at all times except at $t = 0$ is called impulse signal and it is shown in Fig. 5.1. This type of signal is not practically available since the occurrence of this type of signal is for a very small interval of time.

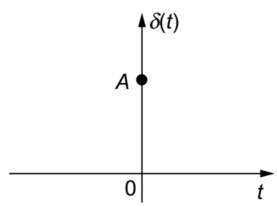


Fig. 5.1 | Impulse signal

Mathematically, a unit-impulse signal is represented by

$$r(t) = \delta(t)$$

$$\delta(t) = \begin{cases} A, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

and $$\int_{-\infty}^{\infty} \delta(t)dt = A.$$

From the above equations, it is clear that the area of the impulse function is unity. In addition, the area is confined to an infinitesimal interval on the *t*-axis and is concentrated only at $t = 0$.
Taking Laplace transform, we obtain $R(s) = A$.
When $A = 1$, the impulse signal becomes unit-impulse signal i.e., $R(s) = 1$.
The impulse response of a system with transfer function $G(s)$ is given by

$$G(s) = C(s)$$

Taking inverse Laplace transform, we obtain $c(t) = g(t)$,

where $g(t)$ is called "weighting function" of the system when unit-impulse signal is applied to the system.
The impulse signal is important due to the following reasons:

(i) The impulse signal is very useful for analysing the system in time domain.
(ii) It is used to generate the system response if provided with the fundamental information about the system characteristic.
(iii) Transfer function of the system can be obtained once the impulse response of the system is known.
(iv) Impulse responses of the system can be obtained by taking the derivative of the step response of the system.
(v) Depending on the area under the impulse response of the system, the stability of the system can be determined. For example, if the area under the impulse response curve is finite, then the system is stable with bounded input and bounded output.

5.3.2 Step Signal

The signal that resembles a sudden change by changing its value from one level (usually zero) to another level (may be *A*) in very small duration of time is known as step signal. The schematic representation of step signal is shown in Fig. 5.2.

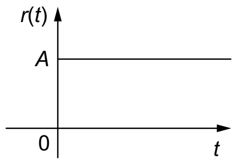


Fig. 5.2 | Step signal

The step signal is represented mathematically

$$r(t) = A\,u(t) \tag{5.1}$$

The step signal can be obtained by integrating the impulse signal $\delta(t)$ as given by

$$u(t) = \int_{-\infty}^{\infty} \delta(t)dt = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

when $A = 1$, the step signal becomes unit-step signal $u(t)$.
Taking Laplace transform of Eqn. (5.1), we obtain

$$R(s) = \frac{\mathrm{A}}{\mathrm{s}}$$

The step response of a system with transfer function $G(s)$ is given by

$$G(s) = \frac{C(s)}{R(s)}$$

Therefore,
$$C(s) = \frac{AG(s)}{s}$$

Importance of Step Signal:

(i) It represents an instantaneous change in the reference input signal of the system. For example, if the input is an angular position of a mechanical shaft, a step input represents the sudden rotation of the shaft.
(ii) It is useful in revealing the quickness of the system when an abrupt change is applied to the input.

5.3.3 Ramp Signal

The signal that depends linearly on the independent variable of the system, time t is known as ramp signal. The ramp signal that starts at zero resembles a constant velocity function. The schematic representation of ramp signal is shown in Fig. 5.3.

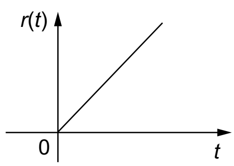


Fig 5.3 | Ramp signal

The ramp signal is represented mathematically as

$$r(t) = \begin{cases} At, & t > 0 \\ 0, & t < 0 \end{cases}$$

The unit-ramp signal $r(t)$ can be obtained by integrating an unit-impulse signal twice or integrating a unit-step signal once, i.e.,

$$r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\alpha} \delta(\tau) d\tau d\alpha$$

$$= \int_{-\infty}^{t} u(\alpha) d\alpha$$

$$= \int_{0^{+}}^{t} 1 d\alpha$$

When $A = 1$, the ramp signal becomes unit-ramp signal.

i.e., $$r(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$$

Taking Laplace transform, we obtain $R(s) = \frac{A}{s^2}$

The ramp response of a system with transfer function $G(s)$ is given by

$$G(s) = \frac{C(s)}{R(s)}$$

Therefore, $$C(s) = \frac{AG(s)}{s^2}$$

Importance of Ramp Signal:

(i) Ramp signal starts at time $t = 0$ and varies linearly with it.
(ii) It helps in testing the response of the system when a ramp signal is applied to it. For example, if the input variable represents the angular displacement of the shaft, the ramp input denotes the constant speed rotation of the shaft.

5.3.4 Parabolic Signal

The signal that depends on the square of the independent variable of the system, time t is known as parabolic signal. The parabolic signal that starts at zero resembles a constant acceleration function. The schematic representation of parabolic signal is shown in Fig. 5.4.

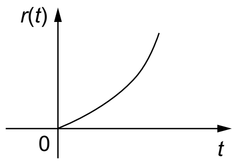


Fig.5.4 | Parabolic signal

The parabolic signal is mathematically represented as

$$r(t)=\begin{cases}\dfrac{At^2}{2}, & t>0\\ 0, & t<0\end{cases}$$

Taking Laplace transform, we obtain $R(s)=\dfrac{\mathrm{A}}{\mathrm{s}^3}$

When $A=1$, the parabolic signal is unit-parabolic signal. The parabolic signal can be obtained by integrating the ramp signal.
The parabolic response of a system with transfer function $G(s)$ is given by

$$G(s)=\frac{C(s)}{R(s)}$$

Therefore, $C(s)=\dfrac{AG(s)}{\mathrm{s}^3}$

The different types of standard test signals with the corresponding $r(t)$ in time-domain and $R(s)$ in frequency-domain are given in Table 5.1.

Table 5.1 | Standard test signals

Type of the signal	$r(t)$ in time domain	$R(s)$ in frequency-domain
Impulse	$\delta(t)$	1
Step	$Au(t)$	$\dfrac{A}{s}$
Ramp	$A\,t$	$\dfrac{A}{s^2}$
Parabolic	$\dfrac{At^2}{2}$	$\dfrac{A}{s^3}$

Example 5.5: Determine $i(t)$ for the step and impulse responses of the series RL circuit shown in Fig. E5.5.

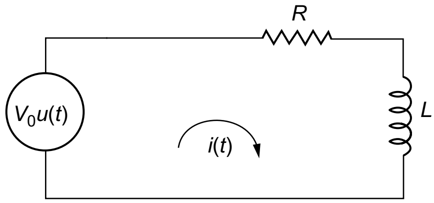


Fig. E5.5

Solution:

Step Response

For the step response, the input excitation is $V_0 u(t)$.

Applying Kirchhoff's voltage law to the circuit shown in Fig. E5.5, we obtain

$$L\frac{di(t)}{dt} + Ri(t) = V_0 u(t)$$

Taking Laplace transform, we obtain

$$L\left\{sI(s) - i(0^+)\right\} + RI(s) = \frac{V_0}{s}$$

Because of the presence of inductance $i(0^+) = 0$, i.e., the current through an inductor cannot change instantaneously due to the conservation of flux linkages.

Therefore, $$I(s)[sL + R] = \frac{V_0}{s}$$

Hence, $$I(s) = \frac{V_0}{s(sL+R)} = \frac{V_0}{L} \times \frac{1}{s\left(s + \frac{R}{L}\right)}$$

$$= \frac{V_0}{L} \times \frac{L}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right] = \frac{V_0}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right]$$

Taking inverse Laplace transform, we obtain

$$i(t) = \frac{V_0}{R}\left[1 - e^{-\frac{R}{L}t}\right]$$

Impulse Response:

For the impulse response, the input excitation is $\delta(t)$.

Applying Kirchhoff's voltage law to the circuit, we obtain

$$L\frac{di(t)}{dt}+Ri(t)=\delta(t)$$

Taking Laplace transform, we obtain

$$L\left\{sI(s)-i(0^+)\right\}+RI(s)=1$$

Since $i(0^+)=0$,
$$I(s)=\frac{1}{R+Ls}=\frac{1}{L}\cdot\frac{1}{s+R/L}$$

Taking inverse Laplace transform, we obtain

$$i(t)=\frac{1}{L}e^{-(R/L)t}u(t).$$

Example 5.6: Determine $i(t)$ for the step and impulse responses of the series RLC circuit shown in Fig. E5.6.

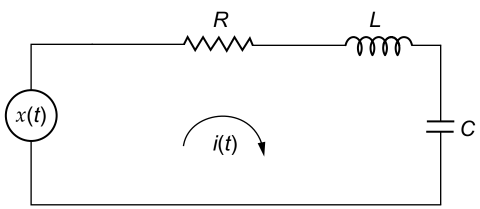


Fig. E5.6

Solution:

Step Response

For the step response, the input signal $x(t)=V_0u(t)$

In the series RLC circuit shown in Fig. E5.6, the integro-differential equation is

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{-\infty}^{t}i(t)dt=V_0u(t)$$

or
$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{-\infty}^{0}i(t)dt+\frac{1}{C}\int_{0}^{t}i(t)dt=V_0u(t)$$

Taking Laplace transform, we obtain

$$L\left[sI(s)-i(0^+)\right]+RI(s)+\frac{1}{C}\mathcal{L}\left[q(0^+)\right]+\frac{1}{C}\frac{I(s)}{s}=\frac{V_0}{s}$$

or $$L\left[sI(s)-i(0^+)\right]+RI(s)+\frac{1}{C}\frac{q(0^+)}{s}+\frac{1}{C}\frac{I(s)}{s}=\frac{V_0}{s}$$

Due to the presence of inductor $L, i(0^+)=0$. Also, $q(0^+)$ is the charge on the capacitor C at $t=0^+$. If the capacitor is initially uncharged, then $q(0^+)=0$. Substituting these two initial conditions, we obtain

$$LsI(s)+RI(s)+\frac{I(s)}{sC}=\frac{V_0}{s}$$

$$I(s)\left[Ls+R+\frac{1}{sC}\right]=\frac{V_0}{s}$$

Therefore, $$I(s)=\frac{V_0}{Ls^2+Rs+\frac{1}{C}}=\frac{V_0}{L(s-p_1)(s-p_2)}$$

where $p_1, p_2=\frac{-R}{2L}\pm\frac{1}{2L}\sqrt{R^2-4\frac{L}{C}}$.

Taking inverse Laplace transform, we obtain

$$i(t)=\frac{\frac{V_0}{L}}{(p_1-p_2)}\left[e^{p_1t}-e^{p_2t}\right]$$

Impulse Response

For the impulse response, the input signal $x(t)=\delta(t)$.
Applying Kirchhoff's law to the circuit, we obtain

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{-\infty}^{t}i(t)dt=\delta(t)$$

or $$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{-\infty}^{0}i(t)dt+\frac{1}{C}\int_{0}^{t}i(t)dt=\delta(t)$$

Taking Laplace transform, we obtain

$$L\left[sI(s)-i(0^+)\right]+RI(s)+\frac{1}{C}L\left[q(0^+)\right]+\frac{1}{C}\frac{I(s)}{s}=1$$

or
$$L\left[sI(s)-i(0^+)\right]+RI(s)+\frac{1}{C}\frac{q(0^+)}{s}+\frac{1}{C}\frac{I(s)}{s}=1$$

Since $q(0^+)=0$ and $i(0^+)=0$, we obtain

$$LsI(s)+RI(s)+\frac{I(s)}{sC}=1$$

$$I(s)\left[Ls+R+\frac{1}{sC}\right]=1$$

Taking inverse Laplace transform, we obtain

$$i(t)=\frac{1}{L}e^{-\alpha t}\cos h\omega t$$

where $\alpha=\dfrac{R}{2L}$ and $\omega=\sqrt{\dfrac{R^2}{4L^2}-\dfrac{1}{LC}}$.

Example 5.7: For the mechanical translational system shown in Fig. E5.7, obtain time response of the system subjected to (i) unit-step input and (ii) unit-impulse input.

Solution:
For the system given in Fig. E5.7, the input is $f(t)$ and the output is $x_1(t)$ and $x_2(t)$ respectively.
Using D´Alemberts principle, we obtain

$$K\left[x_1(t)-x_2(t)\right]=f(t) \tag{1}$$

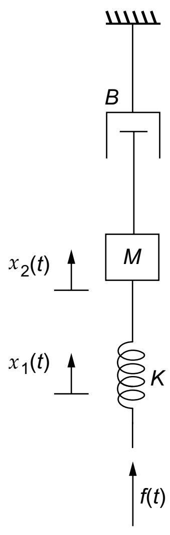


Fig.E5.7

For mass M, $$f_K(t)+f_M(t)+f_B(t)=0$$

$$K\left[x_2(t)-x_1(t)\right]+M\frac{d^2x_2(t)}{dt^2}+B\frac{dx_2(t)}{dt}=0 \quad (2)$$

Taking Laplace transform of Eqs. (1) and (2), we obtain

$$KX_1(s)-KX_2(s)=F(s) \quad (3)$$

$$-KX_1(s)+(s^2M+sB+K)X_2(s)=0 \quad (4)$$

Representing Eqs. (3) and (4) in matrix form, we obtain

$$\begin{bmatrix} K & -K \\ -K & s^2M+sB+K \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}=\begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

Using Cramer's rule, we obtain $$X_2(s)=\frac{\Delta_2}{\Delta}$$

$$\Delta_2=\begin{bmatrix} K & F(s) \\ -K & 0 \end{bmatrix}=KF(s)$$

$$\Delta=\begin{vmatrix} K & -K \\ -K & s^2M+sB+K \end{vmatrix}=K\left(s^2M+sB\right)$$

Therefore, $$X_2(s)=\frac{\Delta_2}{\Delta}=\frac{KF(s)}{K\left(Ms^2+Bs\right)}$$

i.e., $$\frac{X_2(s)}{F(s)}=\frac{1}{(Ms^2+Bs)}$$

Also, $$X_1(s)=\frac{\Delta_1}{\Delta}$$

$$\Delta_1=\begin{vmatrix} F(s) & -K \\ 0 & s^2M+sB+K \end{vmatrix}=F(s)\left\{s^2M+sB+K\right\}$$

Therefore, $$X_1(s)=\frac{\Delta_1}{\Delta}=\frac{F(s)\left\{s^2M+sB+K\right\}}{K(s^2M+sB)}$$

Hence, $$\frac{X_1(s)}{F(s)}=\frac{\left\{s^2M+sB+K\right\}}{K(s^2M+sB)}$$

Therefore, the transfer functions of the given mechanical system are

$$\frac{X_1(s)}{F(s)} = \frac{\{s^2M + sB + K\}}{K(s^2M + sB)} \text{ and } \frac{X_2(s)}{F(s)} = \frac{1}{(Ms^2 + Bs)}.$$

(i) For unit-step input:

When a unit-step signal is applied to the system, $F(s) = \frac{1}{s}$.

To determine $x_1(t)$	To determine $x_2(t)$
Hence, $X_1(s) = \frac{1}{s} \times \frac{\{s^2M + sB + K\}}{K(s^2M + sB)}$	$X_2(s) = \frac{1}{s} \times \frac{1}{s^2M + sB}$
i.e., $X_1(s) = \frac{1}{K}\left[\frac{1}{s} + \frac{\frac{K}{M}}{s\left(s + \frac{B}{M}\right)}\right]$	i.e., $X_2(s) = \frac{\frac{1}{M}}{s^2\left(s + \frac{B}{M}\right)}$
Using partial fractions, we obtain	Using partial fractions, we obtain
$X_1(s) = \frac{1}{K}\left[\frac{1}{s} + \frac{M}{Bs} - \frac{M}{B\left(s + \frac{B}{M}\right)}\right]$	$X_2(s) = \frac{\frac{-M}{B^2}}{s} + \frac{\frac{1}{B}}{s^2} + \frac{\frac{M}{B^2}}{\left(s + \frac{B}{M}\right)}$
Taking inverse Laplace transform, we obtain	Taking inverse Laplace transform, we obtain
$x_1(t) = \frac{1}{K}\left[1 + \frac{M}{B} - \frac{M}{B}e^{-\left(\frac{B}{M}\right)t}\right]$	$x_2(t) = \left(\frac{1}{B}\right)t + \frac{M}{B^2}\left(e^{-\left(\frac{B}{M}\right)t} - 1\right)$

(ii) For unit-impulse input:

When a unit-impulse signal is applied to the system $F(s) = 1$

To determine $x_1(t)$	To determine $x_2(t)$
Hence, $X_1(s) = 1 \times \frac{\{s^2M + sB + K\}}{K(s^2M + sB)}$	$X_2(s) = 1 \times \frac{1}{s^2M + sB}$

(*Continued*)

(Continued)

To determine $x_1(t)$	To determine $x_2(t)$
i.e., $X_1(s)=\frac{1}{K}\left[1+\frac{\frac{K}{M}}{s\left(s+\frac{B}{M}\right)}\right]$ Using partial fractions, we obtain $X_1(s)=\frac{1}{K}\left[1+\frac{\frac{K}{B}}{s}-\frac{\frac{K}{B}}{\left(s+\frac{B}{M}\right)}\right]$ Taking inverse Laplace transform, we obtain $x_1(t)=\frac{1}{K}\left[\delta(t)+\frac{K}{B}-\frac{K}{B}e^{-\left(\frac{B}{M}\right)t}\right]$	i.e., $X_2(s)=\frac{\frac{1}{M}}{s\left(s+\frac{B}{M}\right)}$ Using partial fractions, we obtain $X_2(s)=\frac{\frac{1}{B}}{s}-\frac{\frac{1}{B}}{\left(s+\frac{B}{M}\right)}$ Taking inverse Laplace transform, we obtain $x_2(t)=\frac{1}{B}\left(1-e^{-\left(\frac{B}{M}\right)t}\right)$

Example 5.8: **Determine the response of the system whose transfer function is given by $\frac{C(s)}{R(s)}=\frac{20}{s+10}$ when the system is subjected to unit-impulse, step and ramp inputs with zero initial conditions.**

Solution:

(i) For a unit-impulse input:

The transfer function of the system is given by $\frac{C(s)}{R(s)}=\frac{20}{(s+10)}$. For a unit-impulse input, $R(s)=1$.

Hence,
$$C(s)=\frac{20}{(s+10)}$$

Taking inverse Laplace transform, we obtain impulse response as

$$c(t)=20\times e^{-10t}$$

(ii) For a unit-step input:

The transfer function of the system is given by $\frac{C(s)}{R(s)}=\frac{20}{(s+10)}$. For a unit-step input, $R(s)=\frac{1}{s}$.

Hence,
$$C(s)=\frac{20}{s(s+10)}$$

Using partial fractions, we obtain

$$C(s) = \frac{20}{s(s+10)} = \frac{A_1}{s} + \frac{A_2}{s+10}$$

Here, $A_1 = 2$ and $A_2 = -2$

Hence, $C(s) = \frac{2}{s} - \frac{2}{s+10}$

Taking inverse Laplace transform, we obtain step response as

$$c(t) = 2 - 2e^{-10t}$$

(iii) For a unit-ramp input:

The transfer function of the system is given by $\frac{C(s)}{R(s)} = \frac{20}{(s+10)}$.

For a unit-ramp input, $R(s) = \frac{1}{s^2}$

Hence, $C(s) = \frac{20}{s^2(s+10)}$

Using partial fractions, we obtain

$$C(s) = \frac{20}{s(s+10)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s+10}$$

Here, $A_1 = -0.2$, $A_2 = 2$ and $A_3 = 0.2$

Hence, $C(s) = -\frac{0.2}{s} + \frac{2}{s^2} + \frac{0.2}{s+10}$

Taking inverse Laplace transform, we obtain ramp response as

$$c(t) = -0.2 + 2t + 0.2e^{-10t}$$

The unit-impulse, unit-step and unit-ramp responses of the given system are plotted and shown in Fig. E5.8.

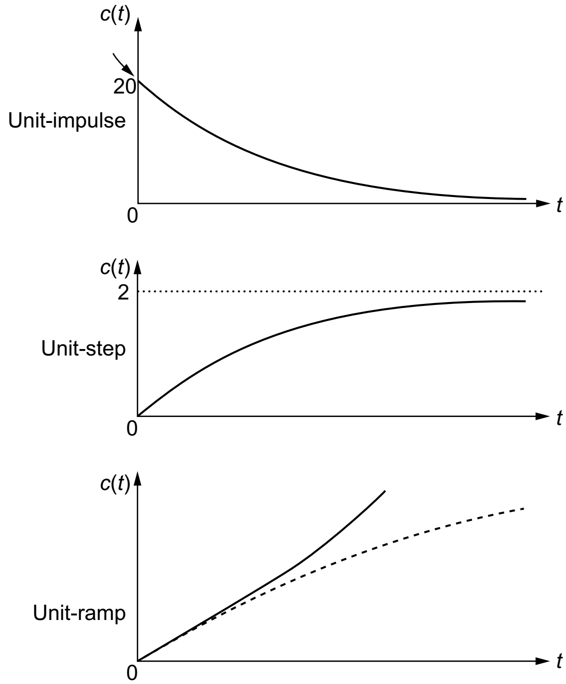


Fig. E5. 8

Example 5.9: **Determine the unit-step response of a unity feedback system whose transfer function is given by** $\dfrac{C(s)}{R(s)} = \dfrac{10s+10}{s^2+10s+10}$**.**

Solution:

The closed-loop transfer function of the system is given by $\dfrac{C(s)}{R(s)} = \dfrac{10s+10}{s^2+10s+10}$

For a unit-step input $\qquad R(s) = \dfrac{1}{s}$

Therefore, $\qquad C(s) = \dfrac{1}{s} \times \dfrac{10s+10}{s^2+10s+10} = \dfrac{1}{s} \times \dfrac{10(s+1)}{(s+1.127)(s+8.873)}$

Using partial fractions, we obtain

$$C(s) = \frac{A_1}{s} + \frac{A_2}{(s+1.127)} + \frac{A_3}{(s+8.873)}$$

Here, $A_1 = 1,\ A_2 = 0.1454$ and $A_3 = -1.1454$

Therefore, $C(s) = \dfrac{1}{s} + \dfrac{0.1454}{(s+1.127)} - \dfrac{1.1454}{(s+8.873)}$

Taking inverse Laplace transform, we get a unit-step response as

$$c(t) = 1 + 0.1454e^{-1.127t} - 1.145e^{-8.873t}$$

Example 5.10: The open-loop transfer function of the system with unity feedback is given by $G(s) = \dfrac{9}{(s^2+4s)}$. Determine the response of the system subjected to unit-impulse input.

Solution:

The closed-loop transfer function of the system is given by $\dfrac{C(s)}{R(s)} = \dfrac{9}{(s^2+4s+9)}$

For a unit-impulse $R(s) = 1$. Hence, $C(s) = \dfrac{9}{(s+2)^2+5} = \dfrac{9}{\sqrt{5}} \dfrac{\sqrt{5}}{(s+2)^2 + (\sqrt{5})^2}$

Taking inverse Laplace transform of the above equation, we obtain response of the system as

$$c(t) = \frac{9}{\sqrt{5}} e^{-2t} \sin\sqrt{5}t$$

5.4 Poles, Zeros and System Response

Time response analysis of any system for standard test input signals can be obtained using inverse Laplace transform or differential equations. But these techniques are time-consuming and laborious. These issues can be overcome by employing pole-zero method. The system response can be determined in a short time by using pole-zero method.

5.4.1 Poles and Zeros of a Transfer Function

The transfer function of the system is expressed as

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

where a_n, a_{n-1}, ···, a_0 and b_m, b_{m-1}, ···, b_0 are constants.

The above transfer function can also be expressed as

$$G(s) = k\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

where k is a real number. The constants $z_1,\ z_2,\ z_3 \cdots$ are called the zeros of $G(s)$, as they are the values of s at which $G(s)$ becomes zero. Conversely, $p_1,\ p_2,\ p_3 \cdots$ are called the poles of $G(s)$, as they are the values of s at which $G(s)$ becomes infinity.

The poles and zeros may be plotted in a complex plane diagram, referred to as the s-plane. In the complex s-plane where $s = \sigma + j\omega$, a pole is denoted by a small cross and a zero by a small circle. This diagram gives a good indication of the degree of stability of a system.
For example, the pole-zero plot for the given transfer function

$$G(s)H(s) = \frac{(s+3)(s+1)}{s(s^2+2s+2)(s+2)(s-3)}$$ is plotted in Fig. 5.5.

For the given system, poles are $0,\ -1 \pm j1,\ -2$ and 3 and zeros are -3 and -1.

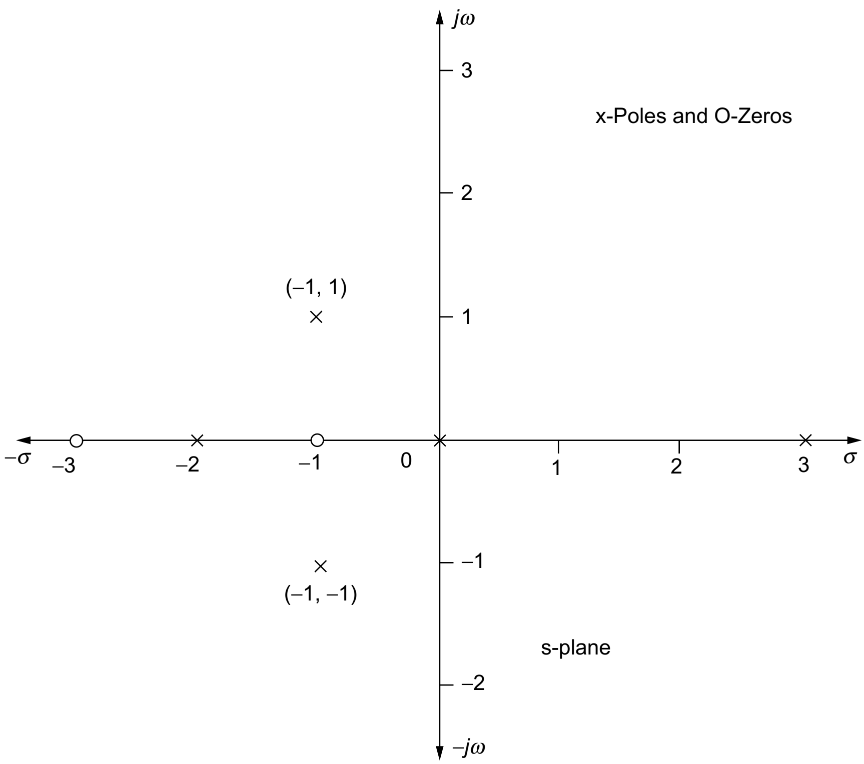


Fig. 5.5 | Poles and zeros of a system

5.4.2 Stability of the System

The stability of the system can be analysed based on the location of roots of the denominator polynomial or poles of the transfer function of the system. In addition, it does not depend on the input excitation applied to the system. The roots of the denominator polynomial or the poles of the transfer function of the system will determine whether the system is stable, unstable or marginally stable based on the location of poles in the s-plane, provided the degree of the denominator polynomial is greater than or equal to the degree of the numerator polynomial.

By finding the location of the poles, i.e., the roots of the denominator polynomial of the transfer function, the stability of the system can be determined.

(i) For a stable system, all the poles of the transfer function must lie in the left half of the s-plane.
(ii) A system is said to be unstable, if any of the poles of transfer function are located in the right half of the s-plane.
(iii) A system is said to be marginally stable, if transfer function has any poles on the $j\omega$-axis in the s-plane provided the other poles of transfer function lie in the left half of the s-plane.

Although computer programmes are available for finding the roots of the denominator polynomial of order higher than three, it is difficult to find the range of a parameter for stability. In such cases, especially in control system design an analytical procedure called the *Routh–Hurwitz stability criterion* is used to analyse the stability of the system.

5.5 TYPE and ORDER of the System

The feed-forward transfer function of the system is $G(s)$ and the system has a feedback transfer function $H(s)$.

5.5.1 TYPE of the System

Generally, the loop transfer function is expressed as

$$G(s)H(s)=\frac{K_{\mathrm{n}}\left(1+A_1s+A_2s^2+\cdots+A_{\mathrm{m}}s^m\right)}{s^n\left(1+B_1s+B_2s^2+\cdots+B_{\mathrm{n}}s^n\right)}$$

where $A_1, A_2, \cdots, A_{\mathrm{m}}$ and $B_1, B_2, \cdots, B_{\mathrm{n}}$ are the coefficients that are constant and K_{n} is the overall gain of the transfer function.

The TYPE of the system is decided based on the value n present in the general form of transfer function. The TYPE of the system and its corresponding general form of transfer function for some systems are given in Table 5.2.

Table 5.2 | TYPE of the system and the corresponding general transfer function

Value of *n*	TYPE of the system	Corresponding general transfer function
0	TYPE ZERO	$G(s)H(s)=\frac{K_0(1+A_1s)(1+A_2s)\cdots}{s^0(1+B_1s)(1+B_2s)\cdots}$

(*Continued*)

Table 5.2 | (Continued)

Value of *n*	TYPE of the system	Corresponding general transfer function
1	TYPE ONE	$G(s)H(s)=\dfrac{K_1(1+A_1s)(1+A_2s)\cdots}{s^1(1+B_1s)(1+B_2s)\cdots}$
2	TYPE TWO	$G(s)H(s)=\dfrac{K_2(1+A_1s)(1+A_2s)\cdots}{s^2(1+B_1s)(1+B_2s)\cdots}$

5.5.2 ORDER of the System

For the system described earlier, the generalized closed-loop transfer function is given by

$$\frac{C(s)}{R(s)}=\frac{K\left(A_m s^m + A_{m-1}s^{m-1}+\cdots+A_0\right)}{\left(B_n s^n + B_{n-1}s^{n-1}+\cdots+B_0\right)}$$

where K is the gain factor. The ORDER of the system corresponds to the maximum power of s in the denominator polynomial.

Example 5.11: Find the type of the loop transfer function for the following systems:

$$\text{(i) } G(s)H(s)=\frac{5(1+sT_1)}{s(1+sT_2)} \qquad \text{(ii) } G(s)H(s)=\frac{9(s+5)}{(s+6)}$$

$$\text{(iii) } G(s)H(s)=\frac{10(1+5s)}{s^3(1+6s)} \qquad \text{(iv) } G(s)H(s)=\frac{50(s+5s^2)}{s^2(1+7s)}$$

Solution:

(i) Since the number of poles at origin is 1, the transfer function is TYPE ONE system.
(ii) Since there is no pole at origin, the transfer function is TYPE ZERO system.
(iii) Since the number of poles at origin is 3, the transfer function is TYPE THREE system.
(iv) Since the number of poles at origin is 2, the transfer function is TYPE TWO system.

Example 5.12: Find the order of the closed-loop transfer functions for the systems given by (i) $\dfrac{C(s)}{R(s)}=\dfrac{10\left[1+2s+s^2\right]}{\left[1+3s+s^2+s^3\right]}$ **and (ii)** $\dfrac{C(s)}{R(s)}=\dfrac{3\left[1+2s\right]}{\left[1+10s\right]}$.

Solution:

(i) Since the maximum power of s in denominator polynomial is 3, the given transfer function is third-order system.
(ii) Since the maximum power of s in denominator polynomial is 1, the given transfer function is first-order system.

5.6 First-Order System

The transfer function of the first-order system relating the input and output is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(A_m s^m + A_{m-1} s^{m-1} + \cdots + A_0\right)}{\left(B_1 s + B_0\right)}$$

If no zeros exist in the system, the transfer function of the first-order system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{K(A_0)}{\left(B_1 s + B_0\right)}$$

In the first-order system, the feed-forward transfer function of the system is $\frac{1}{sT}$ and it has unity feedback system. Hence, the transfer function of the system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{(sT+1)}$$

where T is the time constant.
The block diagram of the first-order system is shown in Fig. 5.6.

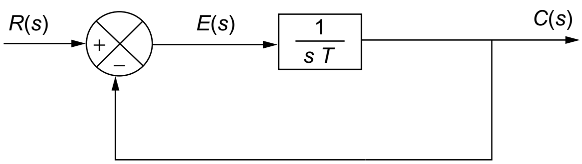


Fig. 5.6 | First-order system

5.6.1 Performance Parameters of First-Order System

(i) Rise Time t_r : The time taken by the response of a system to reach 90% of the final value from 10% is known as the rise time t_r. It can be obtained by equating the response of the system in time domain $c(t)$ to 0.1 and 0.9 respectively. Hence, t_r is given by

$$t_r = t\big|_{c(t)=0.9} - t\big|_{c(t)=0.1}$$

(ii) Delay Time t_d : The time taken by the response of a system to reach 50 % of the final value is known as the delay time t_d. It can be obtained by equating the response of the system in time domain $c(t)$ to 0.5.

$$t_d = t\big|_{c(t)=0.5}$$

(iii) Settling Time t_s : The time taken by the response of a system to reach 98% of the final value is known as the settling time. It can be obtained by equating the response of the system in time domain $c(t)$ to 0.98. Hence, t_s is given by

$$t_s = t\big|_{c(t)=0.98}$$

(iv) Steady-State Error e_{ss} : It is the measurement of a difference existing between the standard input signal $r(t)$ applied and the output response obtained from the system $c(t)$ when such standard signals are applied.

$$e_{ss} = \underset{t\to\infty}{Lt}\left[r(t) - c(t)\right]$$

5.6.2 Time Response of a First-Order System

The time response of a first-order system when subjected to standard test signals such as unit-impulse, unit-step and unit-ramp inputs is discussed.

Time response of a first-order system subjected to unit-impulse input:
The transfer function of a first-order system is given by

$$\frac{C(s)}{R(s)} = \frac{1}{sT+1}$$

Therefore,
$$C(s) = \frac{R(s)}{1+sT}$$

For unit-impulse input, $R(s) = 1$. Therefore, $C_i(s) = \dfrac{1}{(1+sT)}$

Taking inverse Laplace transform, we obtain $c_i(t) = \dfrac{1}{T}e^{-t/T}$ for $t \geq 0$

Time response of first-order system subjected to unit-step input:

We know that, for the first-order system $C(s) = \dfrac{R(s)}{1+sT}$

For unit-step input, $r(t) = 1$ and $R(s) = \dfrac{1}{s}$.

Substituting $R(s) = \dfrac{1}{s}$, we obtain $C_s(s) = \dfrac{1}{s(1+sT)}$

Using partial fractions, we obtain $C_s(s) = \dfrac{1}{s} - \dfrac{T}{1+sT} = \dfrac{1}{s} - \dfrac{1}{s+\dfrac{1}{T}}$

Taking inverse Laplace transform, we obtain $c_s(t) = 1 - e^{-t/T}$ for $t \geq 0$

It is clear that the output value starts at zero and reaches unity as the independent variable time t approaches infinity as shown in Fig. 5.7.

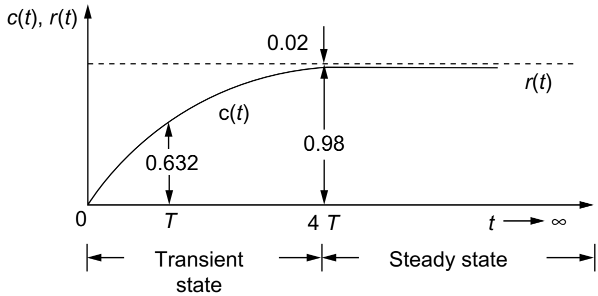


Fig. 5.7 | Time response of first-order system subjected to step input

The steady-state error e_{ss} is given by

$$e_{ss} = \underset{t\to\infty}{Lt}\left[r(t) - c_s(t)\right] = e^{-\infty} = 0$$

Also, the steady-state error of the system is determined using final value theorem as

$$\begin{aligned} e_{ss} &= \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s) \\ &= \underset{s\to 0}{Lt}\, s\left(R(s) - C_r(s)\right) \\ &= \underset{s\to 0}{Lt}\, s\left(\frac{1}{s} - \frac{1}{s(Ts+1)}\right) = 0 \end{aligned}$$

Since $e_{ss} = 0$, the first-order system is capable of responding to a unit-step input without any steady-state error.

Substituting $t = T$ in the time response of the system, we obtain

$$c_s(t) = 1 - e^{-1} = 0.632$$

Here, the response of the system reaches 63.2% of the final value for the time constant T as shown in Fig. 5.7. The response of the system $c_s(t)$ and the time constant of the system T are inversely proportional to each other.

Time response of first-order system subjected to unit-ramp input:

We know that, for the first-order system, $C(s) = \dfrac{R(s)}{1+sT}$

For unit-ramp input or velocity input, $r(t) = t$, $R(s) = \dfrac{1}{s^2}$

Therefore, $$C_r(s) = \frac{1}{s^2(1+sT)}$$

Using partial fractions, we obtain

$$C_r(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Taking inverse Laplace transform, we obtain

$$c_r(t) = t - T + Te^{-t/T} = t - T\left(1 - e^{-t/T}\right)$$

The error signal is

$$e(t) = r(t) - c_r(t) = t - \left(t - T\left(1 - e^{-t/T}\right)\right) = T\left(1 - e^{-t/T}\right)$$

The steady-state error e_{ss} is given by

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = T$$

Also, the steady-state error of the system is determined using final value theorem as

$$\begin{aligned} e_{ss} &= \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s) \\ &= \underset{s\to 0}{Lt}\, s\left(R(s) - C_r(s)\right) \\ &= \underset{s\to 0}{Lt}\, s\left(\frac{1}{s^2} - \frac{1}{s^2(Ts+1)}\right) = T \end{aligned}$$

Thus, the first-order system under consideration will track a unit-ramp input with a steady-state error T, which is the time constant of the system.

The input signal, output signal and steady-state error for the first-order system subjected to ramp input are shown in Fig. 5.8.

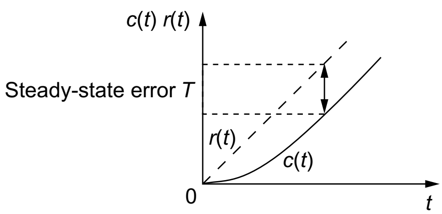


Fig. 5.8 | Time response of first-order system subjected to ramp input

The time response of first-order system when subjected to various standard input signal is given in Table 5.3.

Table 5.3 | Time response of first-order system subjected to various test signals

Type of test signal	Response of the first-order system $c(t)$	Relationship in terms of	
		Differentiation	Integration
Unit-impulse	$c_i(t)=\frac{1}{T}e^{-t/T}$	$c_i(t)=\frac{d(c_s(t))}{dt}$ or $c_i(t)=\frac{d^2(c_r(t))}{dt^2}$	$c_i(t)$
Unit-step	$c_s(t)=1-e^{-t/T}$	$c_s(t)=\frac{d(c_r(t))}{dt}$	$c_s(t)=\int_0^t c_i(t)\,dt$
Unit-ramp	$c_r(t)=t-T+Te^{-t/T}$	$c_r(t)$	$c_r(t)=\int_0^t c_s(t)dt$ $=\int_0^t\int_0^t c_i(t)\,dt$

Time-domain specifications for a first-order system subjected to unit-impulse input:

The time response of the first-order system subjected to unit-impulse input is

$$c(t)=\frac{1}{T}e^{-t/T}$$

The time-domain specifications are given below:

(i) Rise Time t_r:

From the definition, $t_r = t\big|_{c(t)=0.9} - t\big|_{c(t)=0.1}$

Hence, $t\big|_{c(t)=0.9}$ and $t\big|_{c(t)=0.1}$ are obtained as follows:

When the response of a system reaches 90% of the final value, i.e., $c(t)=0.9$, the time $t_{0.9}$ is obtained as

$$0.9=\frac{1}{T}e^{-t_{0.9}/T}$$

$$t_{0.9}=\frac{T}{\ln 0.9T}$$

When the response of a system reaches 10% of the final value, i.e., $c(t)=0.1$, the time $t_{0.1}$ obtained as

$$0.1=\frac{1}{T}e^{-t_{0.1}/T}$$

$$t_{0.1}=\frac{T}{\ln 0.1\,\mathrm{T}}$$

Hence, rise time, $t_{\mathrm{r}} = T\left(\dfrac{1}{\ln 0.9T} - \dfrac{1}{\ln 0.1\,\mathrm{T}}\right)$ sec.

(ii) Delay Time t_{d} :
From the definition, $t_{\mathrm{d}} = t\big|_{c(t)=0.5}$

When the response of a system reaches 50% of the final value, i.e., $c(t) = 0.5$, the delay time $t_{0.5}$, is obtained as

$$0.5 = \frac{1}{T} e^{-t_{0.5}/T}$$

$$t_{0.5} = \frac{T}{\ln 0.5T}$$

Hence, delay time, $t_{\mathrm{d}} = T\dfrac{1}{\ln 0.5T}$ sec.

(iii) Settling Time t_{s} :
From the definition, for 2% tolerance, the settling time is given by

$$t_{\mathrm{s}} = t\big|_{c(t)=0.98}$$

When the response of a system reaches 98% of the final value, i.e., $c(t) = 0.98$, the time $t_{0.98}$ is obtained as

$$0.98 = \frac{1}{T} e^{-t_{0.98}/T}$$

$$t_{0.98} = \frac{T}{\ln 0.98T}$$

Hence, settling time, $t_{\mathrm{s}} = T\dfrac{1}{\ln 0.98\,\mathrm{T}}$ sec for 2% tolerance

Similarly, for 5% tolerance, the settling time, $t_{\mathrm{s}} = T\dfrac{1}{\ln 0.95T}$ sec.

Time-domain specifications for a first-order system subjected to unit-step input:
The time response of a first-order system subjected to unit-impulse input is

$$c(t) = 1 - e^{-t/T}$$

The time-domain specifications are given below:

(i) Rise Time t_{r} :

From the definition, $t_{\mathrm{r}} = t\big|_{c(t)=0.9} - t\big|_{c(t)=0.1}$

Hence, $t\big|_{c(t)=0.9}$ and $t\big|_{c(t)=0.1}$ are obtained as follows.

When the response of a system reaches 90% of the final value, i.e., $c(t) = 0.9$, the time $t_{0.9}$, obtained as

$$0.9 = 1 - e^{-t_{0.9}/T}$$

$$t_{0.9} = -0.434T$$

When the response of a system reaches 10% of the final value, i.e., $c(t) = 0.1$, the time $t_{0.1}$ obtained as

$$0.1 = 1 - e^{-t_{0.1}/T}$$

$$t_{0.1} = -9.491T$$

Hence, rise time, $t_{\mathrm{r}} = t_{0.9} - t_{0.1} = -0.434T + 9.491T = 9.057T$ sec.

(ii) Delay Time t_{d} :

From the definition, $t_{\mathrm{d}} = t\big|_{c(t)=0.5}$

When the response of a system reaches 50% of the final value, i.e., $c(t) = 0.5$, the delay time $t_{0.5}$ obtained as

$$0.5 = 1 - e^{-t_{0.5}/\mathrm{T}}$$

$$t_{0.5} = -1.442T$$

Hence, delay time, $t_{\mathrm{d}} = -1.442\ \mathrm{T}$ sec.

(iii) Settling Time t_{s} :

From the definition, for 2% tolerance, the settling time is given by

$$t_{\mathrm{s}} = t\big|_{c(t)=0.98}$$

When the response of a system reaches 98% of the final value, i.e., $c(t) = 0.98$, the settling time $t_{0.98}$ obtained as

$$0.98 = 1 - e^{-t_{0.98}/T}$$

$$t_{0.98} = -0.2556\ \mathrm{T}$$

Hence, settling time, $t_{\mathrm{s}} = -0.2556\ \mathrm{T}$ sec for 2% tolerance.

Similarly, for 5% tolerance, the settling time, $t_{\mathrm{s}} = -0.3338\ \mathrm{T}$ sec.

5.7 Second-Order System

The transfer function of the second-order system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(A_m s^m + A_{m-1} s^{m-1} + \cdots + A_0\right)}{\left(B_2 s^2 + B_1 s + B_0\right)}$$

If no zeros exist in the system, the transfer function of the second-order system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(A_0\right)}{\left(B_2 s^2 + B_1 s + B_0\right)}$$

For determining the time response and analysing the performance of the second-order system, let us assume that the feed-forward transfer function of the system is $\frac{\omega_n^2}{s(s+2\xi\omega_n)}$ and it has a unity feedback. Hence, the transfer function of the second-order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$$

where ω_n is an undamped natural frequency and ξ is the damping ratio which is given by

$$\xi = \frac{\text{Actual damping}}{\text{Initial damping}}$$

The block diagram of the second-order system is shown in Fig. 5.9(a).

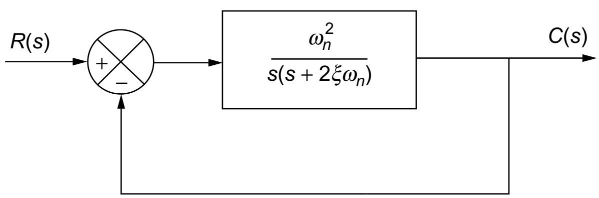


Fig. 5.9(a) | Second-order system

5.7.1 Classification of Second-Order System

The second-order system can be classified into four types based on the roots of the characteristic equation of the system.

The characteristic equation of the second-order system is given by $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$.

Its corresponding roots are given by

$$s_1, s_2 = -\xi\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{\xi^2 - 1}$$

Here, the roots of the characteristic equation of the second-order system is dependent on the damping ratio ξ.
The classification of the second-order system based on the roots of characteristic equation which depends on the damping ratio ξ is given in Table 5.4.

Table 5.4 | Classification of second-order system

Nature of the system	Damping ratio ξ	Roots of the characteristic equation s_1, s_2	Nature of roots
Undamped or oscillating system	0	$\pm j\omega_n$	Purely imaginary
Underdamped	<1	$-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ or $-\sigma \pm j\omega_d$	Complex
Critically damped	1	$-\omega_n$	Real and equal
Overdamped	>1	$-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$	Real and unequal

5.7.2 Performance Parameters of Second-Order System

Delay Time t_{d}: The time taken by the response of a system to reach 50% of the final value is known as the delay time (t_{d}). It can be obtained by equating the response of the system $c(t)$ to 0.5, given by

$$t_{\mathrm{d}} = t\big|_{c(t)=0.5}$$

Rise Time t_{r}: The time taken by the response of a system to reach 90% of the final value from 10% is known as the rise time. In addition, it can be defined as the difference between the time at which the response of the system has reached 10% and 90% of final value. The time at which response of the system reaches 10% and 90% can obtained by equating the response of the system $c(t)$ to 0.1 and 0.9 respectively. Hence, t_{r} is given by

$$t_{\mathrm{r}} = t\big|_{c(t)=0.9} - t\big|_{c(t)=0.1}$$

Peak Time t_{p}: The time taken by the response of a system to reach the first peak overshoot is known as the peak time. It can be obtained by differentiating the time response of the system and equating it to zero. i.e., $t_{\mathrm{p}} = t\big|_{\frac{dc(t)}{dt}=0}$

Maximum Peak Overshoot M_p: The maximum peak overshoot is obtained at the peak time of the response of a system, which is given by

$$M_p = c(t_p) - c(t_\infty)$$

where $c(t_p)$ is the value obtained from the response of the system at peak time and $c(t_\infty)$ is the steady-state value of the signal applied to the system.

The maximum peak overshoot is always represented as the percentage peak overshoot as

$$\% \; M_p = \frac{c(t_p) - c(t_\infty)}{c(t_\infty)} \times 100\%$$

Special Case: If a standard unit signal is applied to a system, the steady-state value of the signal applied to the system $c(t_\infty)$ will be 1 and hence the maximum peak overshoot will be

$$M_p = c(t_p) - 1$$

The relationship between the damping ratio of the second-order system ξ and peak overshoot M_p is shown in Fig. 5.9(b), irrespective of the input applied to the system. It is inferred that for an undamped system ($\xi = 0$), the peak overshoot is maximum; and for the critically damped system ($\xi = 1$), the peak overshoot is minimum. In addition, the peak overshoot does not exist for the second-order overdamped system ($\xi > 1$). Hence, peak overshoot varies only for the second order underdamped system ($0 < \xi < 1$).

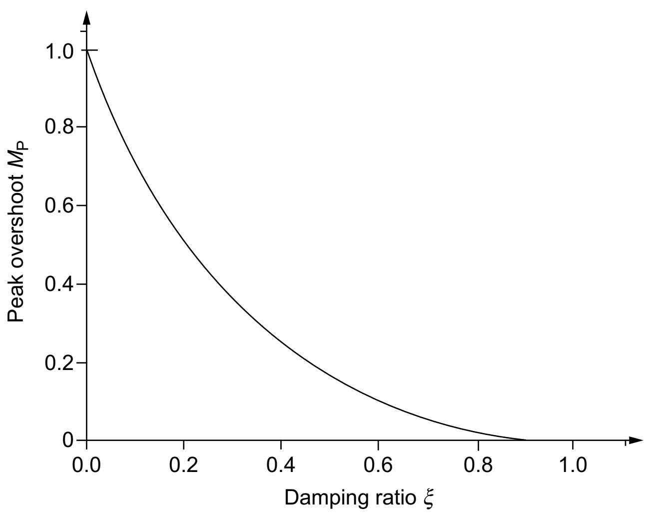


Fig. 5.9(b) | Relation between peak overshoot and damping ratio

Settling Time t_s: The time taken by the response of a system to reach 98% of the final value is known as the settling time. It can be obtained by equating the response of the system $c(t)$ to 0.98.

$$t_s = t\big|_{c(t)=0.98}$$

Steady-State Error e_{ss} : It is the measurement of the difference existing between the standard input signal *r(t)* applied and the output response obtained from the system *c(t)* when such standard signals are applied.

$$e_{ss} = \underset{t\to\infty}{Lt}\left[r(t) - c(t)\right]$$

5.7.3 Time Response of the Second-Order System

The time response of a second-order system (undamped, underdamped, critically damped and overdamped) when the system is subjected to standard test signals such as unit-step, unit-ramp and unit-impulse inputs is discussed in this section.

Time response of the second-order system subjected to unit-impulse input:
The closed-loop transfer function of the second-order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{5.1}$$

and its characteristic equation is given by

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of the equation are given by

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

For unit-impulse input $r(t) = \delta(t)$ and $R(s) = 1$.

Case 1: Undamped System

From Table 5.4, it is clear that for an undamped system, $\xi = 0$.

Substituting $\xi = 0$ in Eqn. (5.1), we obtain $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + \omega_n^2}$

The roots of its characteristic equation are given by

$$s_1, s_2 = \pm j\omega_n$$

which are purely imaginary.

Substituting $R(s) = 1$, we obtain $C(s) = \dfrac{\omega_n^2}{\left(s^2 + \omega_n^2\right)}$

Taking inverse Laplace transform, we obtain the time response of the second-order un-damped system as

$$c(t) = \omega_n^2 t e^{-\omega_n t}$$

Case 2: Underdamped System
From Table 5.4, it is clear that for underdamped system, $\xi < 1$
For $\xi < 1$, the roots of the characteristic equation are

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -\sigma \pm j\omega_d$$

which are complex conjugate.

Here, $\sigma = \xi\omega_n$ and $\omega_d = \omega_n\sqrt{1-\xi^2}$

Substituting $R(s) = 1$ in Eqn. (5.1), we obtain $C(s) = \dfrac{\omega_n^2}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$

Adding and subtracting $\xi^2\omega_n^2$ to the denominator of the above equation, we obtain

$$C(s) = \frac{\omega_n^2}{(s+\xi\omega_n)^2 + \omega_n^2\left(1-\xi^2\right)}$$

Multiplying and dividing by ω_d, we obtain $C(s) = \dfrac{\omega_n^2}{\omega_d} \cdot \dfrac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}$

Taking inverse Laplace transform, we obtain the response of an underdamped system as

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\omega_d t$$

Case 3: Critically Damped
From Table 5.4, it is clear that for critically damped system, $\xi = 1$.
For $\xi = 1$, the roots of the characteristic equation are given by

$$s_1, s_2 = -\omega_n$$

which are real and equal.
Substituting $\xi = 1$ in Eqn. (5.1), we obtain

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

Substituting $R(s) = 1$ in the above equation, we obtain $C(s) = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

Taking inverse Laplace transform, we obtain $c(t) = \omega_n^2 .te^{-\omega_n t}$

The response of critically damped second-order system has no oscillations.

Case 4: Overdamped System

From Table 5.4, it is clear that for an overdamped system, $\xi > 1$.

For $\xi > 1$, the roots of the characteristic equation are given by

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

which are real and unequal.

Substituting $R(s) = 1$ in Eqn. (5.1), we obtain $C(s) = \dfrac{\omega_n^2}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$

Assuming $s_a = -s_1$ and $s_b = -s_2$, we obtain $C(s) = \dfrac{\omega_n^2}{(s + s_a)(s + s_b)}$

Using partial fractions, we obtain

$$C(s) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{(s + s_a)} - \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{(s + s_b)}$$

Taking inverse Laplace transform, we obtain the response of an overdamped system as

$$c(t) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}}\left(e^{-s_a t} - e^{-s_b t}\right)$$

The input and output signals for the second-order system subjected to unit-impulse input are shown in Fig. 5.10.

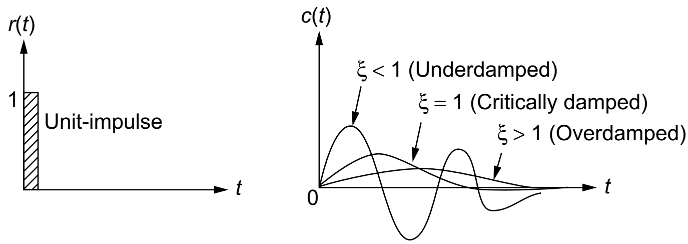


Fig. 5.10 | Responses of the second-order system subjected to impulse input for different cases

Time response of a second-order system subjected to a unit-step input:

The closed-loop transfer function of the second-order system is $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

and its characteristic equation is given by

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of the equation are given by

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

For unit-step input $r(t) = 1$ and $R(s) = \frac{1}{s}$.

Case 1: Undamped System

From Table 5.4, for an undamped system, $\xi = 0$

Substituting $\xi = 0$, we obtain $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + \omega_n^2}$

and the roots of its characteristic equation will be purely imaginary and it is given by

$$s_1, s_2 = \pm j\omega_n$$

Substituting $R(s) = \dfrac{1}{s}$ in Eqn. (5.1), we obtain $C(s) = \dfrac{\omega_n^2}{s\left(s^2 + \omega_n^2\right)}$

Using partial fractions, we obtain

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Taking inverse Laplace transform, we obtain time response of the second-order undamped system as

$$c(t) = 1 - \cos\omega_n t$$

Therefore, the response of an undamped second-order system when subjected to unit-step input is purely oscillatory which is shown in Fig. 5.11.

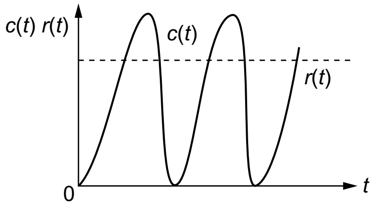


Fig. 5.11 | Response of undamped second-order system subjected to unit-step input

Case 2: Underdamped System

From the Table 5.4, it is clear that for underdamped system, $\xi < 1$.

Substituting $\xi < 1$, the roots of the characteristic equation are given by

$s_1, s_2 = -\xi\omega_{\mathrm{n}} \pm j\omega_{\mathrm{n}}\sqrt{1-\xi^2} = -\sigma \pm j\omega_{\mathrm{d}}$,

which are complex conjugates where $\sigma = \xi\omega_{\mathrm{n}}$ and $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\xi^2}$.

Substituting $R(s) = \frac{1}{s}$ in Eqn. (5.1), we obtain $C(s) = \frac{1}{s}\frac{\omega_{\mathrm{n}}^2}{\left(s^2 + 2\xi\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2\right)}$

Using partial fractions, we obtain

$$C(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_{\mathrm{n}})}{s^2 + 2\xi\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2}$$

Adding and subtracting $\xi^2\omega_{\mathrm{n}}^2$ to the denominator of second term in the above equation, we obtain

$$C(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_{\mathrm{n}})}{s^2 + 2\xi\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2 + \xi^2\omega_{\mathrm{n}}^2 - \xi^2\omega_{\mathrm{n}}^2}$$

Rearranging, we obtain

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_{\mathrm{n}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{n}}^2(1-\xi^2)}$$

Hence,

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_{\mathrm{n}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{d}}^2}$$

where $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\xi^2}$.

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_{\mathrm{n}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{d}}^2} - \frac{\xi\omega_{\mathrm{n}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{d}}^2}$$

Multiplying and dividing by ω_{d} in the third term of the above equation, we obtain

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_{\mathrm{n}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{d}}^2} - \frac{\xi\omega_{\mathrm{n}}}{\omega_{\mathrm{d}}}\frac{\omega_{\mathrm{d}}}{(s + \xi\omega_{\mathrm{n}})^2 + \omega_{\mathrm{d}}^2}$$

Taking inverse Laplace transform, we obtain

$$c(t) = 1 - e^{-\xi\omega_{\mathrm{n}}t}\cos\omega_{\mathrm{d}}t - \frac{\xi\omega_{\mathrm{n}}}{\omega_{\mathrm{d}}}e^{-\xi\omega_{\mathrm{n}}t}\sin\omega_{\mathrm{d}}t$$

$$= 1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

Simplifying, we obtain

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\xi\sin\omega_d t + \sqrt{1-\xi^2}\cos\omega_d t\right) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

where $\phi = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$.

The response of an underdamped second-order system oscillates before settling to a final value is shown in Fig. 5.12. The oscillation depends on the damping ratio.

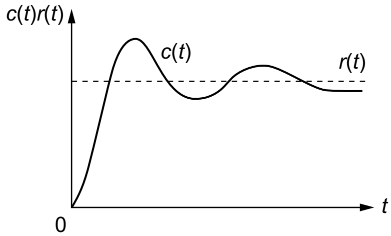


Fig. 5.12 | Response of underdamped second-order system subjected to unit-step input

Case 3: Critically Damped System

From Table 5.4, it is clear that for critically damped system, $\xi = 1$.
For $\xi = 1$, the roots of the characteristic equation are given by

$$s_1, s_2 = -\omega_n$$

which are real and equal.

Substituting $\xi = 1$ in Eqn. (5.1), we obtain $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

Since $R(s) = \dfrac{1}{s}$, $C(s) = \dfrac{\omega_n^2}{s(s+\omega_n)^2}$

Using partial fraction expansion, we obtain

$$C(s) = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

Taking inverse Laplace transform, we obtain

$$c(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)$$

The response of a critically damped second-order system has no oscillations as shown in Fig. 5.13.

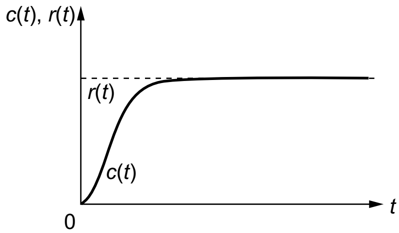


Fig. 5.13 | Response of a critically damped second-order system subjected to unit-step input

Case 4: Overdamped System

From Table 5.4, it is clear that for underdamped system, $\xi > 1$.
For $\xi > 1$, the roots of the characteristic equation are given by

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

which are real and unequal.

Substituting $R(s) = \frac{1}{s}$ in Eqn. (5.1), we obtain

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$$

$$= \frac{A_1}{s} + \frac{A_2}{\left(s + \omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)\right)} + \frac{A_3}{\left(s + \omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)\right)}$$

Using partial fractions, we obtain

$$A_1 = 1,\ A_2 = \frac{1}{2\left(\xi^2 - 1 + \xi\sqrt{\xi^2 - 1}\right)} \text{ and } A_3 = \frac{1}{2\left(\xi^2 - 1 - \xi\sqrt{\xi^2 - 1}\right)}$$

Therefore,

$$C(s) = \frac{1}{s} + \frac{1}{2\left(\xi^2 - 1 + \xi\sqrt{\xi^2 - 1}\right)} \times \frac{1}{s + \omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)} + \frac{1}{2\left(\xi^2 - 1 - \xi\sqrt{\xi^2 - 1}\right)} \times \frac{1}{s + \omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)}$$

Taking inverse Laplace transform, we get the response of the overdamped system as

$$c(t) = 1 + \frac{e^{-\omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)t}}{2\left(\xi^2 - 1 + \xi\sqrt{\xi^2 - 1}\right)} + \frac{e^{-\omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)t}}{2\left(\xi^2 - 1 - \xi\sqrt{\xi^2 - 1}\right)}$$

Therefore, the response of an overdamped second-order system when subjected to unit-step input has no oscillations but it takes longer time to reach the final steady value as shown in Fig. 5.14.

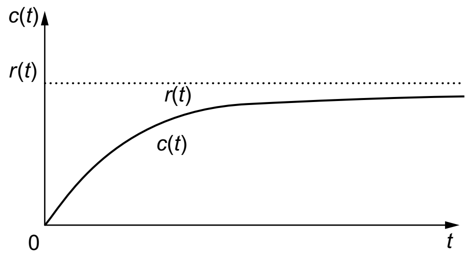


Fig. 5.14 | Response of an overdamped second-order system when subjected to unit-step input

The responses of the second-order system when subjected to a unit-step input are shown in Fig. 5.15.

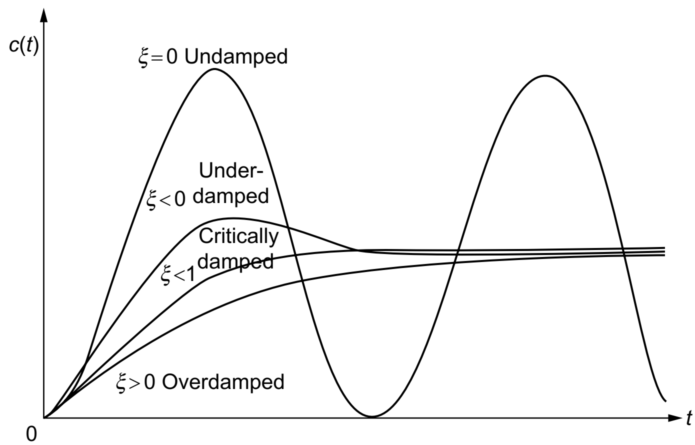


Fig. 5.15 | Responses of different second-order systems when the system is excited by unit-step input

Time response of a second-order system subjected to a unit-ramp input

The closed-loop transfer function of the second-order system is $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ and its characteristic equation is given by $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$.

The roots of the equation are given by

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

For a unit-impulse input, $r(t) = t$ and $R(s) = \frac{1}{s^2}$.

Case 1: Undamped System
From Table 5.4, it is clear that for an undamped system, $\xi = 0$.

Substituting $\xi = 0$ in Eqn. (5.1), we obtain $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + \omega_n^2}$

and the roots of its characteristic equation will be purely imaginary as given by

$$s_1, s_2 = \pm j\omega_n$$

Substituting $R(s) = \dfrac{1}{s^2}$ in Eqn.(5.1), we obtain $C(s) = \dfrac{1}{s^2} \times \dfrac{\omega_n^2}{\left(s^2 + \omega_n^2\right)}$

Using partial fractions, we obtain $C(s) = \dfrac{1}{s^2} - \dfrac{\omega_n^2}{\left(s^2 + \omega_n^2\right)}$

Taking inverse Laplace transform, we obtain the time response of the second-order undamped system as

$$c(t) = t\left(1 - \omega_n^2 e^{-\omega_n t}\right)$$

Case 2: Underdamped System
From Table 5.4, it is clear that for an underdamped system, $\xi < 1$.

For $\xi < 1$, the roots of the characteristic equation are given by $s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ which are complex conjugate.

Substituting $R(s) = \dfrac{1}{s^2}$ in Eqn. (5.1), we obtain $C(s) = \dfrac{1}{s^2}\left(\dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right)$

$$= \frac{1}{s^2} \times \frac{\omega_n^2}{\left(s + \xi\omega_n + j\omega_n\sqrt{1-\xi^2}\right)\left(s + \xi\omega_n - j\omega_n\sqrt{1-\xi^2}\right)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{\left(s + \xi\omega_n + j\omega_n\sqrt{1-\xi^2}\right)} + \frac{A_4}{\left(s + \xi\omega_n - j\omega_n\sqrt{1-\xi^2}\right)}$$

Using partial fractions, we obtain

$$A_1 = -\frac{2\xi}{\omega_n},\ A_2 = 1,\ A_3 = \frac{e^{-j\left(\frac{\pi}{2}+2\phi\right)}}{2\omega_n\sqrt{1-\xi^2}} \text{ and } A_4 = \frac{e^{j\left(\frac{\pi}{2}+2\phi\right)}}{2\omega_n\sqrt{1-\xi^2}}$$

Taking inverse Laplace transform, we obtain the time response of an underdamped system as

$$c(t) = t - \frac{2\xi}{\omega_n} + \frac{e^{-\xi\omega_n}}{\omega_d}\sin\left(\omega_d t + 2\phi\right)$$

Case 3: Critically Damped System
From Table 5.4, it is clear that for critically damped system, $\xi = 1$.

For $\xi = 1$, the roots of the characteristic equation are given by $s_1, s_2 = -\omega_n$ which are real and equal.

Substituting $\xi = 1$ in Eqn. (5.1), we obtain $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \dfrac{\omega_n^2}{(s+\omega_n)^2}$

Substituting $R(s) = \dfrac{1}{s^2}$ in above equation, we obtain $C(s) = \dfrac{1}{s^2} \times \dfrac{\omega_n^2}{(s+\omega_n)^2}$

Using partial fractions, we obtain

$$C(s) = \frac{-2}{s\omega_n} + \frac{1}{s^2} + \frac{2}{\omega_n(s+\omega_n)} + \frac{1}{(s+\omega_n)^2}$$

Taking inverse Laplace transform, we obtain

$$c(t) = \frac{2}{\omega_n}\left[e^{-\omega_n t} - 1\right] + t\left[1 + e^{-\omega_n t}\right]$$

Case 4: Overdamped System
From Table 5.4, it is clear that for underdamped system, $\xi > 1$.
For $\xi > 1$, the roots of the characteristic equation are given by $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$ which are real and unequal.

Substituting $R(s) = \dfrac{1}{s^2}$ in Eqn. (5.1), we obtain $C(s) = \dfrac{1}{s^2} \times \dfrac{\omega_n^2}{\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}$

Using partial fractions, we obtain $C(s) = \dfrac{A_1}{s} + \dfrac{A_2}{s^2} + \dfrac{A_3}{s - s_1} + \dfrac{A_4}{s - s_2}$

Taking inverse Laplace transform, we obtain the time response of the overdamped system as

$$c(t) = A_1 + A_2 t + A_3 e^{-\left(\omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)\right)} + A_4 e^{-\omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)}$$

where $A_1 = -\omega_n^2\left(\dfrac{1}{s_1^2(s_1 - s_2)} + \dfrac{1}{s_2^2(s_2 - s_1)}\right)$ $A_2 = \dfrac{\omega_n^2}{s_1 s_2}$, $A_3 = \dfrac{\omega_n^2}{s_1^2(s_1 - s_2)}$ and $A_4 = \dfrac{\omega_n^2}{s_2^2(s_2 - s_1)}$.

5.7.4 Time-Domain Specifications for an Underdamped Second-Order System

The time-domain specifications of an underdamped second-order system when subjected to different standard signals are discussed below.
Underdamped second-order system subjected to a unit-impulse input:
The time response of an underdamped second-order system is given by

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\omega_d t$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$.

Peak Time t_p:

The peak time, $t_p = t\Big|_{\frac{dc(t)}{dt}=0}$

$$\frac{dc(t_p)}{dt} = \frac{\omega_n}{\sqrt{1-\xi^2}}\left[e^{-\xi\omega_n t_p}\cos\omega_d t_p \times \omega_d + \sin\omega_d t_p\left(-\xi\omega_n\right)e^{-\xi\omega_n t_p}\right] = 0$$

$$\frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n} = \tan\omega_d t_p$$

$$t_p = \frac{1}{\omega_d}\left[\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right] = \frac{\phi}{\omega_d}$$

where $\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{\xi}\right)$.

Maximum Peak Overshoot M_p:

The maximum peak overshoot M_p when a standard unit signal is applied, is given by

$$M_p = c(t_p) - 1 = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_p}\sin\omega_d t_p - 1$$

Substituting the peak time $t_p = \dfrac{\phi}{\omega_d}$ in the above equation, we obtain

$$M_p = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n \frac{\phi}{\omega_d}}\sin\phi - 1$$

Using the right angled triangle shown in Fig. 5.16, we obtain

$$M_p = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\frac{\xi\phi}{\sqrt{1-\xi^2}}}\sin\phi - 1$$

Also, the percentage peak overshoot is given by

$$\%M_p = 100\times\left(\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\frac{\xi\phi}{\sqrt{1-\xi^2}}}\sin\phi - 1\right)$$

Settling Time t_s:
For 2 % of tolerance, the settling time is given by

$$t_s = t\big|_{c(t)=0.98}$$

For determining the settling time, the oscillatory component present in the response of the system is eliminated. Only the exponential component is considered because as time t tends to infinity, the response of the system tries to follow the input.
Hence, the modified time response of the system is given by

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}$$

Settling time for 2 % of tolerance is determined as

$$c(t_s) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_s} = 0.98$$

Solving the above equation, we obtain

$$t_s = \frac{1}{\xi\omega_n} \times \frac{1}{\ln \dfrac{0.98\sqrt{1-\xi^2}}{\omega_n}} \text{ sec for } 2\% \text{ tolerance}$$

Similarly, we can solve for 5% of tolerance and its settling time is

$$t_s = \frac{1}{\xi\omega_n} \times \frac{1}{\ln \dfrac{0.95\sqrt{1-\xi^2}}{\omega_n}} \text{ sec}$$

Underdamped second-order system subjected to a unit-step input:
The time response of an underdamped second-order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left(\xi \sin \omega_d t + \sqrt{1-\xi^2} \cos \omega_d t\right)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)$$

where $\phi = \tan^{-1} \dfrac{\sqrt{1-\xi^2}}{\xi}$.

Rise Time t_r:
From the definition, $t_r = t\big|_{c(t)=0.9} - t\big|_{c(t)=0.1}$

But for an underdamped second-order system, $t_r = t\big|_{c(t)=1}$

Using the above equation, the response of the second-order system,

$$c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\,t_r + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) = 1$$

$$-\frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\,t_r + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) = 0$$

$$\sin\left(\omega_n\sqrt{1-\xi^2}\,t_r + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) = \sin n\pi$$

$$\omega_n\sqrt{1-\xi^2}\,t_r + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = n\pi$$

For $n = 1$,

$$t_r = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_d} \text{ sec.}$$

Peak Time t_p:

From the definition, $t_p = t\big|_{\frac{dc(t)}{dt}=0}$

$$\frac{dc(t_p)}{dt} = \xi\omega_n \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$-\frac{\omega_n\sqrt{1-\xi^2}\,e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \cos\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Equating the above equation to zero, the peak time t_p can be determined:

$$\xi\omega_n \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) - \frac{\omega_n\sqrt{1-\xi^2}\,e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \cos\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) = 0$$

Simplifying, we obtain

$$\xi \sin\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) - \sqrt{1-\xi^2}\cos\left(\omega_n\sqrt{1-\xi^2}\,t_p + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) = 0$$

Using the term $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$, a right-angled triangle can be formed as shown in Fig. 5.16. The terms $\sqrt{1-\xi^2}$ and ξ can be replaced by $\sin\phi$ and $\cos\phi$ respectively.

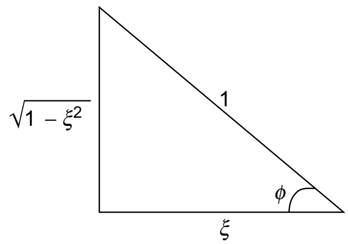


Fig. 5.16 | Right angled triangle for damping ratio

Rewriting the above equation, we obtain

$$\sin\left(\omega_n\sqrt{1-\xi^2}\,t_p + \phi\right)\cos\phi - \cos\left(\omega_n\sqrt{1-\xi^2}\,t_p + \phi\right)\sin\phi = 0$$

Using the formula $\sin(A-B) = \sin A\cos B - \sin B\cos A$, we obtain

$$\sin\left(\omega_n\sqrt{1-\xi^2}\,t_p\right) = 0$$

or $\qquad \omega_n\sqrt{1-\xi^2}\,t_p = 0, \pi, 2\pi, \cdots.$

We know that the peak time is the time taken by the response to reach the first peak overshoot. Therefore,

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d}\ \text{sec.}$$

The peak time t_p given by the above equation corresponds to one-half cycle of the frequency of damped oscillations.

Maximum Peak Overshoot M_p:
From the definition, the maximum peak overshoot M_p when a standard unit-step signal is applied is given by

$$M_\mathrm{p} = c\left(t_\mathrm{p}\right) - 1$$

$$= -\frac{e^{-\xi\omega_\mathrm{n}t_\mathrm{p}}}{\sqrt{1-\xi^2}}\sin\left(\omega_\mathrm{n}\sqrt{1-\xi^2}\,t_\mathrm{p} + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Substituting the peak time $t_\mathrm{p} = \dfrac{\pi}{\omega_\mathrm{d}}$ in the above equation, we obtain

$$M_\mathrm{p} = -\frac{e^{-\xi\omega_\mathrm{n}\frac{\pi}{\omega_\mathrm{d}}}}{\sqrt{1-\xi^2}}\sin\left(\pi + \varphi\right)$$

Using the formula $\sin\left(\pi + \phi\right) = -\sin\phi$ and using the right-angled triangle i.e., shown in Fig. 5.16, we obtain

$$M_\mathrm{p} = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

Also, the percentage peak overshoot is given by

$$\%M_\mathrm{p} = 100 \times e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\ \%$$

Settling Time t_s:
For 2 % of tolerance, the settling time is given by

$$t_\mathrm{s} = t\big|_{c(t)=0.98}$$

For determining the settling time, the oscillatory component present in the response of the system is eliminated and only exponential component is considered because as time t tends to infinity, the response of the system tries to follow the input.

Hence, the modified time response of the system is given by $c\left(t\right) = 1 - e^{-\xi\omega_\mathrm{n}t}$

Settling time for 2 % of tolerance is determined as

$$c\left(t_\mathrm{s}\right) = 1 - e^{-\xi\omega_\mathrm{n}t_\mathrm{s}} = 0.98$$

Solving the above equation, we obtain $\quad t_s = \dfrac{3.912}{\xi\omega_n} = \dfrac{4}{\xi\omega_n}$ sec for 2% of tolerance

Similarly, the settling time for 5 % of tolerance is given by

$$t_s = \frac{3}{\xi\omega_n}\ \text{sec.}$$

Steady-State Error:
The steady-state error is given by

$$e_{ss} = \underset{t\to\infty}{Lt}\left[r(t)-c(t)\right]$$

$$= \underset{t\to\infty}{Lt}\left[1-c(t)\right] = 0$$

Hence, the second-order underdamped system has zero steady-state error.

Example 5.13: If the characteristic equation of a closed-loop system is $s^2 + 2s + 2 = 0$, then check whether the system is either overdamped, critically damped, underdamped or undamped.

Solution:

The characteristic equation is $s^2 + 2s + 2 = 0$

Comparing the above equation with the standard form $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, we obtain

$$2\xi\omega_n = 2 \text{ and } \omega_n^2 = 2$$

$$\omega_n = \sqrt{2} \text{ and } \xi = \frac{1}{\sqrt{2}}$$

Since $\xi < 1,$ the given system is underdamped.

Example 5.14: The block diagram of the closed-loop servo system is shown in Fig. E5.14. The system has a servo amplifier to amplify the weak error signal $E(s)$, a servomotor to actuate the load and a reduction gear to reduce the load velocity and to increase the load torque. Determine the transfer function of the system and hence determine the steady-state error of the system subjected to unit-step and unit-ramp input.

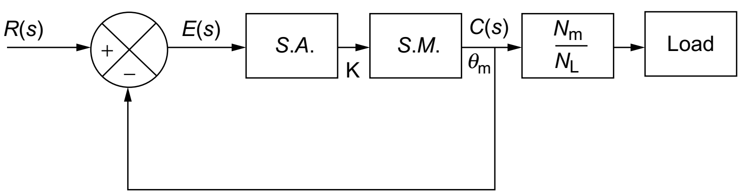


Fig. E5.14

Solution:

Let us assume that

N_{m} is the number of teeth of the gear in the motor side.

N_{L} is the number of teeth in the load side.

$a = \dfrac{N_{\mathrm{m}}}{N_{\mathrm{L}}}$ is the gear ratio.

$T_{\mathrm{L}}(s)$ is the load torque in Newton-metre.

K is the forward loop gain in Newton-metre/rad error.

J_{m} is the moment of inertia in kg-m^2.

B_{m} is the friction coefficient in Newton metre/rad/sec.

J_{L} is the moment of inertia of the load.

B_{L} is friction coefficient of the load.

$T(s) = aT_{\mathrm{L}}(s)$ is the load torque referred to the motor shaft.

B is the friction coefficient at the motor shaft and

$J = J_{\mathrm{m}} + a^2 J_{\mathrm{L}}$ is the moment of inertia at the motor shaft.

If $e(t)$ is the error signal, the actuating signal to drive the motor and the load is $Ke(t)$. If the system is a second-order system, then the differential equation of the system is

$$J\frac{d^2c(t)}{dt^2} + B\frac{dc(t)}{dt} + T(t)$$

which is referred to the motor shaft.

Equating the above equation to the actuating signal $Ke(t)$, we obtain

$$Ke(t) = J\frac{d^2c(t)}{dt^2} + B\frac{dc(t)}{dt} + T(t)$$

Taking Laplace transform on both the sides, we obtain

$$KE(s) = \left(Js^2 + Bs\right)C(s) + \frac{T}{s} \tag{1}$$

But $E(s) = R(s) - C(s)$

Substituting the error signal in Eqn. (1), we obtain

$$K \times \left[R(s) - C(s)\right] = \left(Js^2 + Bs\right) \times C(s) \text{ (neglecting load torque)}$$

$$K \times R(s) = \left(Js^2 + Bs + K\right) \times C(s)$$

Therefore, the transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{(Js^2 + Bs + K)} = \frac{\frac{K}{J}}{\left(s^2 + \frac{B}{J}s + \frac{K}{J}\right)},$$

Comparing the above equation with standard closed-loop transfer function of the second-order system $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, we obtain

$$\frac{K}{J} = \omega_n^2 \text{ and } \frac{B}{J} = 2\xi\omega_n$$

Therefore, $$\omega_n = \sqrt{\frac{K}{J}} \text{ and } \xi = \frac{B}{2\sqrt{JK}}$$

Also, using the above relations, we obtain $\omega_n = \frac{B}{2\xi J}$.

Determination of steady-state error:

(i) For unit-step input:

Substituting $C(s) = R(s) - E(s)$ in Eqn. (1), we obtain

$$KE(s) = (Js^2 + Bs)(R(s) - E(s)) + \frac{T}{s}$$

Solving the above equation, we obtain

$$E(s) = \frac{[Js^2 + B(s)]R(s) + \frac{T}{s}}{(Js^2 + Bs + K)}$$

For unit-step input, we have $$R(s) = \frac{1}{s}$$

Hence, $$E(s) = \frac{(Js^2 + Bs)\frac{1}{s} + \frac{T}{s}}{(Js^2 + Bs + K)}$$

Therefore, the steady-state error of the system is

$$e_{ss} = \underset{s\to 0}{Lt}\, sE(s) = \underset{s\to 0}{Lt}\, s\left[\frac{(Js^2 + Bs)\frac{1}{s} + \frac{T}{s}}{(Js^2 + Bs + K)}\right] = \frac{T}{K}$$

But when T is neglected, $e_{ss} = 0$

(ii) For a ramp input with slope ω_{i} (input shaft speed),

$$e_{\mathrm{ss}} = \frac{2\xi\omega_{\mathrm{i}}}{\omega_{\mathrm{n}}} + \frac{T}{K}.$$

Example 5.15 **The moment of inertia and retarding friction of a servo mechanism are 10×10^{-6} slug-ft² and 400×10^{-6} lb.ft/rad sec respectively. Also, the output torque of the system is 0.004 lb.ft/rad error. Determine ω_{n} and ξ for the system.**

Solution:

The formulae for ω_{n} and ξ for the servo system are

$$\omega_{\mathrm{n}} = \sqrt{\frac{K}{J}} \text{ and } \xi = \frac{\mathrm{B}}{2\sqrt{\mathrm{JK}}}$$

Substituting the known values in the above equations, we obtain

$$\omega_{\mathrm{n}} = \sqrt{\frac{K}{J}} = \sqrt{\frac{4\times10^{-3}}{10\times10^{-6}}} = 20 \text{ rad / sec}$$

$$\xi = \frac{B}{2J\omega_n} = \frac{400\times10^{-6}}{2\times10\times10^{-6}\times20} = 1.$$

Example 5.16: **The values of different parameters of the closed-loop servo system shown in Fig. E5.16 are B = 66×10^{-6} N.m/rad/sec, ξ = 0.25, J = 1.35 × 10^{-6} kg-m², T = 2.72×10^{-4} Nm/volt, gear ratio = 10:1 synchro constant K_{S} = 57.4 V/rad/error and constant error detector = 1 V/degree. The system has servo amplifier that amplifies the weak error signal $E(s)$ and a servomotor to actuate the load. Determine (i) ω_{n} (ii) the gain of the amplifier and (iii) e_{ss} when the input shaft rotates at a speed of 10 rpm.**

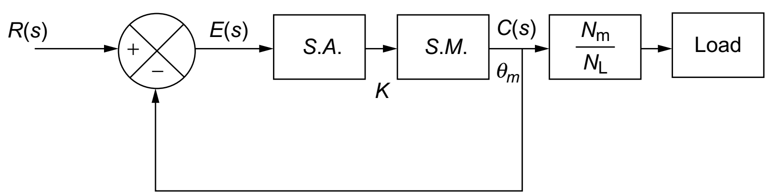


Fig. E5.16

Solution:

Input velocity, $\omega_i = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 10}{60} = 1.05\ \text{rad/sec}$

Motor constant, $T = 2.72 \times 10^{-4}$ N.m/volt

The formulae for ω_n and ξ for the servo system are $\omega_n = \sqrt{\dfrac{K}{J}}$ and $\xi = \dfrac{B}{2\sqrt{JK}}$

Substituting the known values in the above equations, we obtain

$$\omega_n = \sqrt{\frac{K}{J}} = 100\ \text{rad/sec}$$

and $$K = \frac{B^2}{4\xi^2 J} = \frac{66^2 \times 10^{-12}}{4(0.25)^2 \times 1.35 \times 10^{-6}} = 1.3 \times 10^{-2}\,\text{N.m/rad at the motor shaft.}$$

Gain of the amplifier

$$= \frac{\text{K}}{\text{gain of the error detector} \times \text{Gain of the motor} \times \text{Gain of the feedback loop}}$$

$$= \frac{1.3 \times 10^{-2}}{57.4 \times 2.72 \times 10^{-4} \times 1} = 0.83$$

Steady-state error at motor shaft $e_{ss} = \dfrac{2\xi\omega_i}{\omega_n} = \dfrac{2 \times 0.25 \times 1.05}{100} = 0.005$ rad.

Example 5.17: The closed-loop servo system that controls the position of the system as shown in Fig. E5.17 has servomotor and servo amplifier. Determine (i) amplifier gain and (ii) steady-state error, given that the moment of inertia of the motor $J_m = 1.4 \times 10^{-6}$ Kg – m², $J_L = 0.0027$ Kg – m², gear ratio is 100:1, synchro constant, $K_s = 57.4$ V/rad/error, motor constant $K_m = 3.3 \times 10^{-4}$ N.m/V, $B = 68 \times 10^{-6}$ N.m, $\xi = 0.2$ and the input shaft rotates at a speed of 20 rpm.

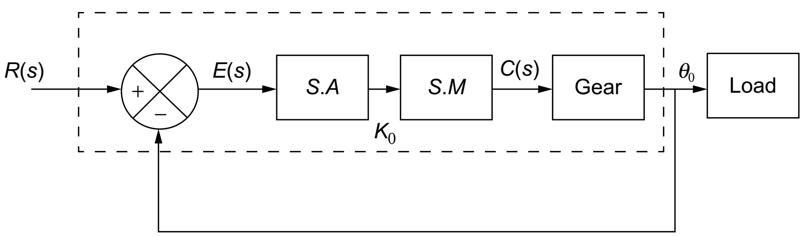


Fig. E5.17

Solution:

Referring to the load side, the moment of inertia at the load shaft is

$$J_o = \left(100^2 \times 1.4 \times 10^{-6}\right) + 0.0027 = 0.0167$$

The total friction at the load shaft

$$B_o = B = 68 \times 10^{-6} \times 10^4 = 0.68 \text{ N.m/rad/sec}$$

Therefore, $$\omega_n = \frac{B_o}{2\xi J_o} = \frac{0.68}{2 \times 0.2 \times 0.0167} = 102 \text{ rad/sec}$$

$$\omega_i = \text{Input speed} = \frac{2\pi \times 20}{60} = 2.1 \text{ rad/sec}$$

Steady-state error, $e_{ss} = \dfrac{2\xi\omega_i}{\omega_n} = \dfrac{2 \times 0.2 \times 2.1}{105} \times 57.4 = 0.45$

If K_o is the loop gain measured at the load side (feedback is given from the gear), then

$$K_o = K_s \times K_A \times K_m \times \text{gear ratio} \times 1$$

But $K_o = \omega_n^2 J_o = (102)^2 \times 1.67 \times 10^{-2} = 174 \text{ N.m/rad error}$

Therefore, $$K_A = \frac{174}{57.4 \times 3.3 \times 10^{-4} \times 100} = 92.$$

Example 5.18: The closed-loop transfer function of the mechanical system is given by $\dfrac{Q(s)}{I(s)} = \dfrac{1}{Js^2 + Bs + K}$. The following time-domain specifications are obtained when a step input of magnitude 10 Nm is applied to the system (i) maximum overshoot = 6%, (ii) time of peak overshoot = 1 sec and (iii) the steady-state value of the output is 0.5 rad. Determine *J*, *B* and *K*.

Solution:

The closed-loop transfer function can be rearranged as $\dfrac{Q(s)}{I(s)} = \dfrac{\frac{1}{J}}{\left(s^2 + \frac{B}{J}s + \frac{K}{J}\right)}$

Comparing the above equation with standard closed-loop transfer function of the second-order system $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, we obtain

$$\omega_n^2 = \frac{K}{J} \text{ i.e., } \omega_n = \sqrt{\frac{K}{J}} \tag{1}$$

and $$2\xi\omega_n = \frac{B}{J} \text{ i.e., } \xi = \frac{B}{2\sqrt{KJ}} \tag{2}$$

Given the maximum overshoot = 6 % = 0.06

Therefore, $0.06 = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

Taking natural logarithm on both sides, we obtain

$$\ln(0.06) = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

Solving the above equation, we obtain $\xi = 0.667$ (3)

Also, given the time for peak overshoot $t_p = 1$ sec.

Therefore, $$1 = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Solving the above equation, we obtain

$$\omega_n = \frac{\pi}{\sqrt{1-(0.667)^2}} = 4.2165 \text{ rad/sec} \tag{4}$$

For a step input of 10 Nm applied to the system, $I(s) = \frac{10}{s}$

Therefore, $$Q(s) = \frac{10}{s(Js^2 + Bs + K)}$$

The K can be obtained by determining the steady-state error of the system. It is obtained by using final value theorem.

Steady-state error $= \underset{s\to 0}{Lt}\, sQ(s)$

$$0.5 = \underset{s\to 0}{Lt} \frac{s.10}{s(Js^2 + Bs + K)} = \frac{10}{K} \tag{5}$$

Therefore, $K = 20$

Using Eqn. (1) and Eqn. (4), we obtain

$$4.2165 = \sqrt{\frac{K}{J}} \Rightarrow J = 1.1249$$

Similarly, substituting the known values in Eqn. (2), we obtain

$$B = 6.3274$$

Therefore, the overall transfer function of the given system is

$$\frac{Q(s)}{I(s)} = \frac{1}{1.1249\, s^2 + 6.3274\, s + 20}$$

Example 5.19: For the electrical system shown in Fig. E5.19, obtain the expressions for ω_n and ξ.

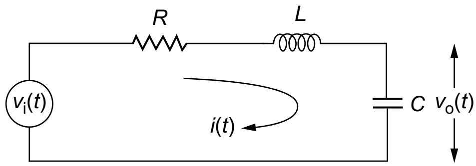


Fig.E5.19

Solution:

The output voltage across the capacitor is $v_o(t) = \frac{1}{C}\int i(t)\,dt$

Applying Kirchhoff's voltage law, we obtain

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace transform of the above equations, we obtain

$$V_o(s) = \frac{I(s)}{Cs} \tag{1}$$

and
$$V_i(s) = \left(R + Ls + \frac{1}{Cs}\right)I(s)$$

Simplifying, we obtain

$$I(s) = \frac{V_i(s)}{\left(R + Ls + \frac{1}{Cs}\right)} \tag{2}$$

Substituting Eqn. (2) in Eqn. (1), we obtain

$$V_o(s) = \frac{V_i(s)}{\left(R + Ls + \frac{1}{Cs}\right) \times \frac{1}{Cs}}$$

Hence, the transfer function for the system is obtained by

$$\frac{V_o(s)}{V_i(s)} = \frac{C^2s^2}{LCs^2 + RCs + 1}$$

Comparing the above equation with standard closed-loop transfer function of the second-order system $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, we obtain

$$\omega_n^2 = \frac{1}{LC} \text{ and } 2\xi\omega_n = \frac{R}{L}$$

Therefore,
$$\omega_n = \frac{1}{\sqrt{LC}} \text{ and } \xi = \frac{R}{2\sqrt{L/C}}.$$

Example 5.20: For the electrical system shown in Fig. E5.20, obtain the expressions for ω_n and ξ. Also, determine (i) peak overshoot and (ii) settling time for 2% tolerance.

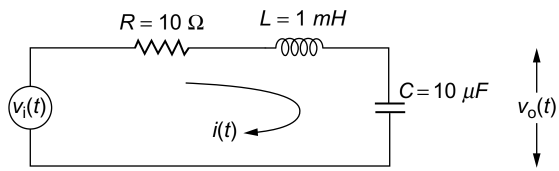


Fig.E5.20

Solution:

We know that, for the given system, the damped natural frequency and damping ratio are

$$\omega_n = \frac{1}{\sqrt{LC}} \text{ and } \xi = \frac{R}{2\sqrt{L/C}}$$

Substituting the known values, we obtain

$$\zeta = \frac{10}{2} \times \sqrt{\frac{10 \times 10^{-6}}{1 \times 10^{-3}}} = 0.5 \text{ and } \omega_n = \frac{1}{\sqrt{1 \times 10^{-3} \times 10 \times 10^{-6}}} = 10^4 \text{ rad/sec}$$

The time-domain specifications are

$$M_P = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{-\pi \times 0.5/\sqrt{1-0.5^2}} = 0.163 = 16.3\%$$

and
$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{10^4 \times 0.5} = 0.8 \text{ m sec}.$$

Example 5.21: A system with a unity feedback system has an open-loop transfer function $G(s)=\dfrac{144}{s(s+2)}$. Determine time-domain specifications of the system subjected to unit-step input.

Solution:
The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{144}{s^2+2s+144}$$

Comparing the above equation with standard closed-loop transfer function of the second-order system, we obtain

$$\omega_n^2=144, \omega_n=12 \text{ rad/sec}, \ 2\xi\omega_n=2, \quad \xi=0.08333, \ \omega_d=\omega_n\sqrt{1-\xi^2}=11.958 \text{ rad/s}.$$

Hence, the time-domain specifications are obtained as

Peak time, $$t_p=\frac{\pi}{\omega_d}=0.2627\,\text{sec}$$

$$\phi=\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)=1.487 \text{ rad}$$

Rise time, $$t_r=\frac{\pi-\phi}{\omega_d}=0.1383 \text{ sec}$$

Percentage peak overshoot, $M_p=e^{-\pi\xi/\sqrt{1-\xi^2}}\times 100=76.897\,\%$

Settling time for 2 % of tolerance, $t_s=\dfrac{4}{\xi\omega_n}=4\,\text{sec}$

and for 5 % of tolerance, $t_s=\dfrac{3}{\xi\omega_n}=3\,\text{sec}$.

Example 5.22: The system with unity feedback control system has an open-loop transfer function $G(s)=\dfrac{10}{s(s+2)}$. Determine the time-domain specifications when the system is subjected to a step input of 12 units.

Solution:

The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{10}{s^2+2s+10}$$

Comparing the above equation with standard closed-loop transfer function of the second-order system, we obtain

$$\omega_n^2 = 10, \quad \omega_n = 3.162 \text{ rad/sec} \;\; 2\xi\omega_n = 2, \;\; \xi = 0.31$$

and
$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.162\sqrt{1-0.31^2} = 3 \text{ rad/sec}$$

Hence, the time-domain specifications of the system subjected to unit-step input are obtained as

$$\phi = \cos^{-1} 0.31 = 72^\circ = \frac{72}{57.3} = 1.256 \text{ rad}$$

Rise time, $t_r = \dfrac{\pi - \phi}{\omega_d} = 0.628\,\text{sec}.$

Percentage peak overshoot, $\%M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100 = 36\%.$

Peak time, $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1.05\,\text{sec}.$

Time delay, $t_d = \dfrac{(1+0.7\xi)}{\omega_n} = 0.3849 \text{ sec}.$

Settling time, $t_s = \dfrac{4}{\xi\omega_n} = 4\,\text{sec}.$

Example 5.23: A system described by the differential equation is given by $\dfrac{d^2y(t)}{dt^2} + 8\dfrac{dy(t)}{dt} + 25y(t) = 50x(t)$. Determine (i) response of the system and (ii) the maximum output of the system when a step input of magnitude 2.5 units is applied to the system.

Solution:

Given $\quad \dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 25y(t) = 50x(t)$

Taking Laplace transform on both the sides of the given differential equation, we obtain

$$\left(s^2+8s+25\right)Y(s)=50X(s)$$

Therefore, the transfer function of the system can be obtained as

$$\frac{Y(s)}{X(s)}=\frac{50}{s^2+8s+25}=2\times\frac{25}{s^2+8s+25}$$

Comparing the above equation with standard closed-loop transfer function of the second-order system, we obtain

$$\omega_n^2=25,\text{ i.e., }\omega_n=5\text{ and }2\xi\omega_n=8\text{ i.e., }\xi=0.8$$

$$\omega_{\mathrm{d}}=\omega_{\mathrm{n}}\sqrt{1-\xi^2}=5\sqrt{1-(0.8)^2}=3\,\mathrm{rad/sec}$$

Therefore,

$$\theta=\tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]=36.869^0$$

Since for the given second-order system, the damping ratio is $\xi=\frac{8}{2\times5}=0.8$, the system is an underdamped system. Hence, the response for the second-order underdamped system when a unit-step signal applied is given by

$$y(t)=1-\frac{e^{-\xi\omega_{\mathrm{n}}t}}{\sqrt{1-\xi^2}}\sin\left(\omega_{\mathrm{d}}t+\theta\right)$$

But in this system, as the transfer function is multiplied by two, the response must also be multiplied by two.
Therefore, by substituting the known values in the above equation, we obtain

$$y(t)=2\left[1-\frac{e^{-0.8\times5t}}{\sqrt{1-(0.8)^2}}\sin\left(3t+36.869^0\right)\right]=2\left[1-1.667e^{-0.8\times5t}\sin\left(3t+36.869^0\right)\right]$$

The above equation is the response of the system subjected to a unit-step input. Hence, when the system is subjected to a step input of magnitude 2.5 units, we obtain the response of the system as

$$y(t)=2.5\left\{2\left[1-1.667e^{-4t}\sin\left(3t+36.86^0\right)\right]\right\}$$

$$=5\left[1-1.667e^{-4t}\sin\left(3t+36.86^0\right)\right]$$

The response of the system will be at its maximum value at $t=t_{\mathrm{p}}=$ peak time.

Therefore, $$t_{\mathrm{p}}=\frac{\pi}{\omega_{\mathrm{d}}}=\frac{\pi}{3}\,\mathrm{sec}.$$

Therefore, the maximum output is obtained as

$$y_{\max}(t) = y(t)\Big|_{\text{at } t=t_p} = 5 - 5e^{-4\pi/3}\left\{\cos 3\times\frac{\pi}{3} + 1.333\sin 3\times\frac{\pi}{3}\right\} = 5.07582 \text{ units.}$$

Example 5.24: A system with a unity feedback has an open-loop transfer function $G(s) = \dfrac{K}{(1+Ts)s}$, T is the time constant. Determine the factor by which the gain K should be multiplied so that the overshoot of a unit-step response is to be reduced from 75 % to 25 %.

Solution:
The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = \frac{K}{T}, \omega_n = \sqrt{\frac{K}{T}} \text{ and } \xi = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{KT}}$$

For 75% maximum overshoot M_p	For 25% maximum overshoot M_p
Using the formula, we obtain $0.75 = e^{-\pi\xi\sqrt{1-\xi^2}}$ Solving for ξ, we obtain $\xi = 0.0911$	$0.25 = e^{-\pi\xi_1\sqrt{1-\xi_1^2}}$ Solving for ξ_1, we obtain $\xi_1 = 0.4037$

Hence, the value must be changed from 0.0911 to 0.4037 to reduce the percentage peak overshoot from 75 % to 25 %.

Hence, by taking the ratio for different values of ξ, we obtain

$$\frac{\xi}{\xi_1} = \frac{\frac{1}{\left(2\sqrt{KT}\right)}}{\frac{1}{\left(2\sqrt{K_1T}\right)}}$$

$$0.2256554 = \sqrt{\frac{K_1}{K}}$$

$$K_1 = 0.05092\,\mathrm{K}$$

where K is the gain for ξ and K_1 is the gain for ξ_1.

Hence, the gain K must be multiplied by 0.05092 to change percentage peak overshoot from 75% to 25%.

Example 5.25: A system with a unity feedback has an open-loop transfer function $G(s) = \dfrac{K}{s(sT+2)}$ and T is the time constant. (i) Determine the factor by which the gain K should be multiplied so that the damping ratio is to be increased from 0.15 to 0.5 and (ii) determine the factor by which the time constant T should be multiplied so that the damping ratio is to be decreased from 0.8 to 0.4.

Solution:

The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{T}}{s^2 + \frac{2}{T}s + \frac{K}{T}}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = \frac{K}{T}, \text{i.e., } \omega_n = \sqrt{\frac{K}{T}} \text{ and } \xi = \frac{1}{\omega_n T} = \frac{1}{\sqrt{KT}}$$

Case (i):

Consider the time constant T to be constant. Let $K = K_1$ for $\xi = \xi_1 = 0.15$ and $K = K_2$ for $\xi = \xi_2 = 0.5$.

Using the formulae for ξ and taking the ratio, we obtain

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{K_2}{K_1}}$$

Solving the above equation, we obtain

$$K_2 = 0.09\,\mathrm{K}_1$$

Hence, the gain K has to be multiplied by the factor 0.09 to reduce the damping ratio from 0.5 to 0.15.

Case (ii):
Consider the gain K to be constant. Let $T = T_1$ for $\xi = \xi_1 = 0.8$ and $T = T_2$ for $\xi = \xi_2 = 0.4$.
Using the formula for ξ and taking the ratio, we obtain

$$\frac{\xi_1}{\xi_2} = \sqrt{\frac{T_2}{T_1}}$$

Solving the above equation, we obtain $T_2 = 4\,\mathrm{T}_1$.

Hence, the time constant T has to be multiplied by the factor 4 to reduce the damping ratio from 0.8 to 0.4.

Example 5.26: When a unit-step signal is applied, the time response of the second-order system is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$. Determine (i) the closed-loop transfer function of the system, (ii) undamped natural frequency ω_n and (iii) damping ratio ξ of the system.

Solution:
The Laplace transform of a unit-step signal applied to the system is given by

$$R(s) = \frac{1}{s}$$

Taking Laplace transform of the time response of the system $c(t)$, we obtain

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

Hence, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}}{\left(\frac{1}{s}\right)} = \frac{600}{s^2 + 70s + 600}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = 600 \qquad \text{i.e., } \omega_n = \sqrt{600} = 24.494 \text{ rad/sec}$$

and $$2\xi\omega_n = 70 \qquad \text{i.e., } \xi = \frac{70}{2 \times \sqrt{600}} = 1.4288$$

Since the damping ratio of the system $\xi > 1$, the system is overdamped.

Example 5.27: **A unity feedback system shown in Fig. E5.27 has an open-loop transfer function $G(s)=\dfrac{25}{s(s+2)}$. Determine (i) response $c(t)$ of the system subjected to a unit-step input, (ii) the damping ratio and (iii) undamped natural frequency of the system.**

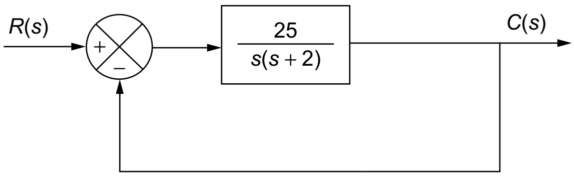


Fig. E5.27

Solution:
For the given system, the closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{25}{s^2+2s+25}$$

Comparing the above equation with the standard form, we obtain

$$\omega_{\mathrm{n}}^2=25 \text{ i.e., } \omega_{\mathrm{n}}=5 \text{ rad/sec} \text{ and } 2\xi\omega_{\mathrm{n}}=2 \text{ i.e., } \xi=0.2$$

Also, $\omega_{\mathrm{d}}=\omega_{\mathrm{n}}\sqrt{1-\xi^2}=4.8989 \text{ rad/sec}$

$$\theta=\tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]=1.3694 \text{ rad}$$

The response of the second-order underdamped system is given by

$$c(t)=1-\frac{e^{-\xi\omega_{\mathrm{n}}t}}{\sqrt{1-\xi^2}}\sin(\omega_{\mathrm{d}}t+\theta)$$

Substituting the known values in the above equation, we obtain

$$c(t)=1-1.0206e^{-t}\sin(4.8989t+1.3694)$$

Example 5.28: The mechanical system shown in Fig. E5.28 has the parameter values as $M = 100$ kg, $f = 1000$ N/m/sec and $K = 10000$ N/m. A step force of 100 N is applied to the system. Determine the (i) damping factor, (ii) undamped natural frequency and (iii) damped natural frequency and the step response of the system as a function of time.

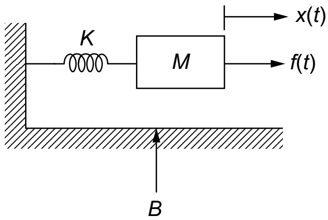


Fig. E5.28

Solution:
Using Newton's second law to the above system, we obtain

$$f(t) = M\frac{d^2x(t)}{dt^2} + Kx(t) + B\frac{dx(t)}{dt}$$

Taking Laplace transform and using the given values, we obtain

$$\frac{X(s)}{F(s)} = \frac{1}{100s^2 + 1000s + 10000}$$

$$= \frac{0.01}{s^2 + 10s + 100}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = 100 \text{ i.e., } \omega_n = 10$$

and $$2\xi\omega_n = 10 \text{ i.e., } \xi = \frac{10}{2\times 10} = 0.5$$

Damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2} = 8.6602\,\text{rad/sec}$

For a second-order underdamped system, the step response of the system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)$$

The given transfer function is modified as given in Eqn.(5.1) as

$$\frac{X(s)}{F(s)} = \frac{0.01}{100} \times \left[\frac{100}{s^2 + 10s + 100}\right]$$

Then, the response of the system is given by

$$x(t) = 100 \times \frac{0.01}{100} \times \left\{1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right\}$$

where $\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 1.047 \text{ rad.}$

Hence, $x(t) = 0.01\{1 - 1.1547e^{-5t} \sin(8.6602t + 1.047)\}$.

Example 5.29: For the system shown in Fig. E5.29, determine the time response $c(t)$ of the system.

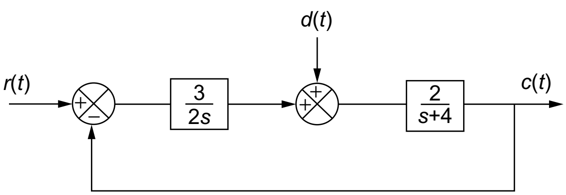
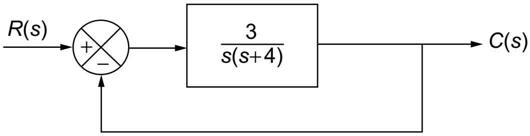
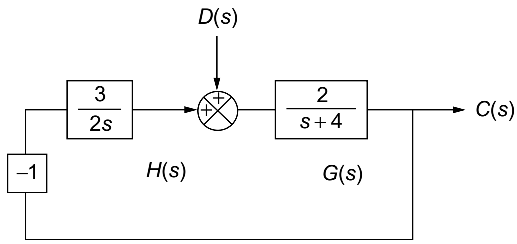


Fig. E5.29

Solution:

<table>
<tr><th>When d(t) = 0</th><th>When r(t) = 0</th></tr>
<tr><td>Equivalent block diagram

</td><td>Equivalent block diagram

</td></tr>
</table>

(*Continued*)

(Continued)

When $d(t) = 0$	When $r(t) = 0$
To determine time response $c(t)$: The closed-loop transfer function of the system is given by $$\frac{C(s)}{R(s)} = \frac{3}{(s+1)(s+3)}$$ For a step input, $r(t) = 1$, $R(s) = 1/s$ Therefore, $$C(s) = \frac{1}{s} \times \frac{3}{(s+1)(s+3)}$$ Using partial fractions, we obtain $$C(s) = \frac{1}{s} - \frac{1.5}{s+1} + \frac{0.5}{s+3}$$ Taking inverse Laplace transform, we obtain $c(t) = 1 - 1.5e^{-t} + 0.5e^{-3t}$.	To determine time response $c(t)$: The closed-loop transfer function of the system is given by $$\frac{C(s)}{D(s)} = \frac{2s}{s^2 + 4s + 3}$$ For a step input, $d(t) = 1$, $D(s) = 1/s$ Therefore, $$C(s) = \frac{1}{s} \times \frac{2s}{s^2 + 4s + 3}$$ Using partial fractions, we obtain $$C(s) = \frac{1}{s+1} - \frac{1}{s+3}$$ Taking inverse Laplace transform, we obtain $c(t) = e^{-t} - e^{-3t}$.

Example 5.30: For the mechanical system shown in Fig. E5.30, determine the expression for (i) damping factor, (ii) undamped natural frequency and (iii) damped natural frequency.

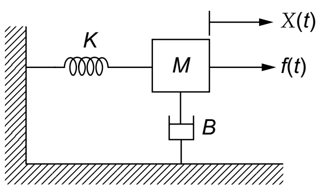


Fig. E5.30

Solution:

Using Newton's second law to the mass M, we obtain

$$f(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t)$$

Taking Laplace transform, we obtain

$$\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}=\frac{\frac{1}{M}}{s^2+\frac{B}{M}s+\frac{K}{M}}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2=\frac{K}{M}\text{ and }2\xi\omega_n=\frac{B}{M}$$

Hence, undamped natural frequency, $\omega_n=\sqrt{\frac{K}{M}}$.

Damping factor, $\xi=\frac{B}{2\sqrt{KM}}$.

Damped natural frequency, $\omega_d=\omega_n\sqrt{1-\xi^2}$.

Example 5.31: A unity feedback system has an open-loop transfer function $G(s)=\frac{K}{s(s+10)}$. Determine the gain K so that the system has a damping ratio of 0.5. Also, determine (i) settling time t_s, (ii) peak overshoot M_p and (iii) peak time t_p subjected to a unit-step input.

Solution:
For the given system, the closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{K}{s^2+10s+K}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2=K\text{ i.e., }\omega_n=\sqrt{K}\text{ and }2\xi\omega_n=10\text{ i.e., }\xi=\frac{5}{\sqrt{K}}$$

For the given damping ratio $\xi=0.5$, $K=100$

Hence, undamped natural frequency, $\omega_n=10\text{ rad/sec}$.

Damped natural frequency, $\omega_d=\omega_n\sqrt{1-\xi^2}=8.66\text{ rad/sec}$.

Settling time, $t_s=\frac{4}{\xi\omega_n}=0.8\text{ sec}$.

Percentage peak overshoot, $\% M_{\mathrm{p}} = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100 = 16.303\%$.

Peak time, $t_{\mathrm{p}} = \dfrac{\pi}{\omega_{\mathrm{d}}} = 0.3627\,\mathrm{sec}$.

Example 5.32: The differential equation for a system is given by $\dfrac{d^2y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 16y(t) = 9x(t)$. Determine (i) the time-domain specifications such as settling time, peak time, delay time, rise time and % peak overshoot and (ii) output response of the system subjected to a unit-step input.

Solution:

Given $\dfrac{d^2y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 16y(t) = 9x(t)$

Taking Laplace transform, we obtain

$$s^2\,Y(s) + 5s\,Y(s) + 16Y(s) = 9X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{9}{s^2 + 5s + 16} = \frac{9}{16}\left[\frac{16}{s^2 + 5s + 16}\right]$$

Comparing the above equation with the standard form $\dfrac{Y(s)}{X(s)} = \dfrac{\omega_{\mathrm{n}}^2}{s^2 + 2\xi\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2}$, we obtain

$$\omega^2_{\mathrm{n}} = 16 \text{ i.e., } \omega_{\mathrm{n}} = 4\,\mathrm{rad/sec} \text{ and } 2\xi\omega_{\mathrm{n}} = 5 \text{ i.e., } \xi = \frac{5}{2\times 4} = 0.625$$

Hence, undamped natural frequency, $\omega_{\mathrm{n}} = 4$ rad/sec.

Damping ratio, $\xi = 0.625$

Damped natural frequency, $\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\xi^2} = 3.1225$ rad/sec, $\theta = \tan^{-1}\left[\dfrac{\sqrt{1-\xi^2}}{\xi}\right] = 0.8950$ rad

The time-domain specifications are

Delay time, $t_{\mathrm{d}} = \dfrac{1 + 0.7\xi}{\omega_{\mathrm{n}}} = 0.3593$ sec

Rise time, $t_{\mathrm{r}} = \dfrac{\pi - \theta}{\omega_{\mathrm{d}}} = 0.7192$ sec

Peak time, $t_p = \dfrac{\pi}{\omega_d} = 1.0061$ sec

Settling time, $t_s = \dfrac{4}{\xi\omega_n} = 1.6$ sec

% peak overshoot, $\% M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \text{ x } 100 = 8.083\ \%$

Thus, the output response of the second-order system subjected to unit-step signal is given by

$$c(t) = A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

where $A = \dfrac{9}{16} = 0.5625.$

Substituting the known values, we obtain

$$c(t) = 0.5625\left[1 - 1.281\, e^{-2.5t}\, \sin(3.1225\, t + 0.8956\,)\right].$$

Example 5.33: Determine the values of K and a of the closed-loop system shown in Fig. E5.33(a), so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1$ kg.m^2.

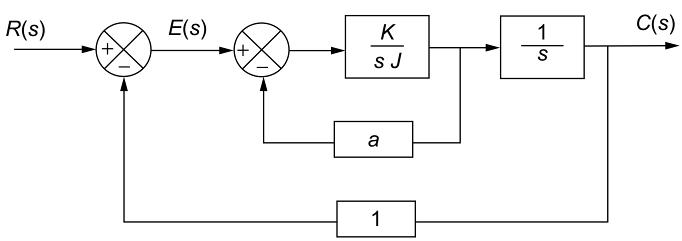


Fig. E5.33(a)

Solution: From Fig. E5.33(a),

$$\frac{C(s)}{E(s)} = \frac{K/Js}{\left[1 + \dfrac{Ka}{Js}\right]}\frac{1}{s} = \frac{K}{s(Js + Ka)}$$

After successive reduction, the block diagram is shown as

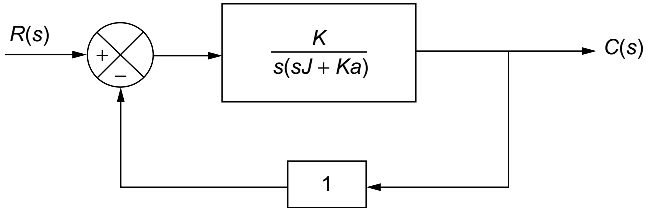


Fig. E5.33(b)

From Fig. E5.33(b), the following closed-loop transfer function is obtained:

$$\frac{C(s)}{R(s)} = \frac{K}{\left(Js^2 + Kas + K\right)} = \frac{K/J}{\left(s^2 + \frac{Ka}{J}s + \frac{K}{J}\right)}$$

The standard second-order transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}$$

Comparing the above equations, we obtain

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$2\xi\omega_n = \frac{Ka}{J} \text{ and } \xi = \frac{a}{2}\sqrt{\frac{K}{J}}$$

Substituting $J = 1 \text{ kgm}^2$, we obtain

$$\omega_n = \sqrt{K} \text{ and } \xi = \frac{a}{2}\sqrt{K}$$

For 25% overshoot, the equation is written as

$$e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} = 0.25$$

Taking natural logarithm on both the sides, we obtain

$$\frac{-\xi\pi}{\sqrt{1-\xi^2}} = \ln\ 0.25 = -1.386$$

Solving for ξ, we obtain $\xi = 0.4$

Also, $$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 2\,\text{sec}.$$

Solving the above equation for ω_n, we obtain

$$\omega_n = 1.717 \text{ rad / sec}$$

Therefore, $$K = \omega_n{}^2 = 1.717^2 = 2.95$$

and $$a = \frac{2\xi}{\omega_n} = \frac{2\times 0.4}{1.717} = 0.466.$$

Example 5.34: The system shown in Fig. E5.34 has a damping ratio of 0.7 and an undamped natural frequency of 4 rad/sec. Determine the gain *K* and constant *a*.

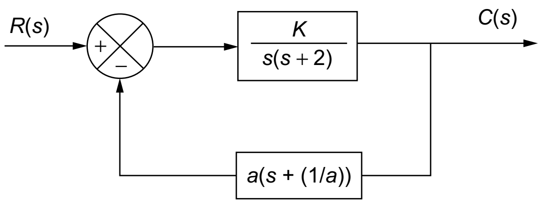


Fig. E5.34

Solution: The closed-loop transfer function for the given system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2+s(2+aK)+K}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = K \text{ i.e., } \omega_n = \sqrt{K}$$

and $$2\xi\omega_n = 2+aK \text{ i.e., } \xi = \frac{2+aK}{2\sqrt{K}}$$

Here, undamped natural frequency, $\omega_n = \sqrt{K}$

Given $\omega_n = 4$ rad/sec. Hence, $K = 16$.

Given $\xi = 0.7$.

The damping ratio, $\xi = \dfrac{2 + aK}{2\sqrt{K}}$

i.e.,
$$0.7 = \frac{2 + 16a}{2\sqrt{16}}$$

Therefore,
$$a = 0.225.$$

Example 5.35: The closed-loop transfer function of a second-order system is given by $\dfrac{C(s)}{R(s)} = \dfrac{25}{s^2 + 6s + 25}$. Determine the time-domain specifications such as rise time, peak time, peak overshoot and settling time when the system is subjected to unit-step input. Also, determine the output response of the given second-order system.

Solution: The closed-loop transfer function of the second-order system is given by

$$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 6s + 25}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^2 = 25 \text{ and } 2\xi\omega_n = 6$$

Hence, undamped natural frequency, $\omega_n = 5$

Damping ratio, $\xi = 0.6$

Damped natural frequency, $\omega_d = \omega_n\sqrt{1 - \xi^2} = 4$ rad/sec

$$\theta = \tan^{-1}\left[\frac{\sqrt{1 - \xi^2}}{\xi}\right] = 0.9272 \text{ rad}$$

The time-domain specifications are obtained as

Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d} = 0.5535$ sec

Peak time, $t_p = \dfrac{\pi}{\omega_d} = 0.7853$ sec

% maximum peak overshoot, $\%M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 9.478$ %

Settling time, $t_s = \dfrac{4}{\xi\omega_n} = 1.333$ sec

When the second-order system is subjected to a unit-step input, the output response is obtained as

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d\, t + \theta)$$

Substituting the known values in the above equation, we obtain

$$c(t) = 1 - 1.25\, e^{-3t} \sin(4\,t + 0.9272).$$

Example 5.36: The open-loop transfer function of the system with a unity feedback is given by $G(s) = 10 / s(s+2)$. Determine the time-domain specifications such as rise time, percentage peak overshoot, peak time and settling time when the system is subjected to a step unit of 12 units.

Solution: Given the open-loop transfer function of the given system is

$$G(s) = \frac{10}{s(s+2)}$$

The system has a unity feedback, $H(s) = 1$
Hence, the closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{10}{s^2 + 2s + 10}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n^{\,2} = 10 \text{ and } 2\xi\omega_n = 2$$

Hence, undamped natural frequency $\omega_n = 3.1622$ rad/sec.

Damping ratio, $\xi = \dfrac{2}{2 \text{ x } \sqrt{10}} = 0.3162.$

Damped natural frequency, $\omega_d = \omega_n\sqrt{1-\xi^2} = 3$ rad/sec.

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 1.249 \text{ rad.}$$

It is important to note that the formula for determining the time-domain specifications except for % peak overshoot of the given second-order system is same irrespective of the magnitude of the input signal applied to the system.
Hence, the time-domain specifications for the given system are obtained as

Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d} = 0.6308 \text{ sec}.$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.0472 \text{ sec}.$

% percentage peak overshoot, $\% M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100\% = 35.093\ \%.$

Example 5.37: In the block diagram representation of a servo system shown in Fig. E5.37, determine the values of K and K_1 so that the maximum overshoot of the system is 25% and the peak time is 2 sec when the system is subjected to a unit-step input.

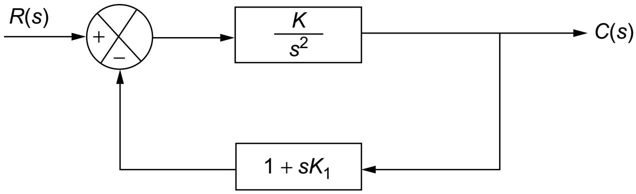


Fig. E5.37

Solution: The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2 + K + KK_1 s}$$

The characteristic equation of the system is

$$s^2 + KK_1 s + K = 0$$

Comparing the above equation with the standard form $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$, we obtain

$$2\xi\omega_n = KK_1 \text{ and } \omega_n^2 = K$$

Therefore, $$\omega_n = \sqrt{K},\ \xi = \frac{\sqrt{K}K_1}{2}$$

and $$\omega_d = \frac{\sqrt{K}}{\sqrt{2}}\sqrt{2 - KK_1^2}$$

Using peak time and maximum peak overshoot formula, we obtain

$$19.739 = 2K - K^2K_1^2 \text{ and } KK_1^2 = 0.652$$

Solving the above equation, we obtain

$$K_1 = 0.252 \text{ and } K = 10.267.$$

Example 5.38: The data of a closed-loop servo system are inertia J = 20 kg - m^2, viscous damping coefficient B = 100 Nw - m /rad/sec and a proportional controller that has a forward loop gain of K = 100 Nw – m/rad error. If the system is initially at rest, determine (i) percentage overshoot and (ii) settling time for 5% oscillation subjected to unit-step input.

Solution: For the given closed-loop servo system, we obtain

$$\xi = \frac{B}{2\sqrt{KJ}} \text{ and } \omega_n = \sqrt{\frac{K}{J}}$$

Substituting the known values, we obtain

$$\xi = \frac{100}{2\times\sqrt{1000\times 20}} = 0.357 \text{ and } \omega_n = \sqrt{\frac{1000}{20}} = 7.07 \text{ rad/ sec}$$

Therefore, % overshoot = $e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100 = e^{-0.357\pi\sqrt{(1-0.357^2)}} \times 100 = 30.09\%$

To find settling time for 5% oscillation
It is known that

$$e^{-m} \times 100 = 5 \text{ and } t_s = mT$$

Therefore, $\quad m = 2.996 \approx 3$

and the time constant, $T = \dfrac{1}{\xi\omega_n} = \dfrac{1}{0.357\times 7.07} = 0.4\,\text{sec}.$

Hence, $t_s = mT = 3\times 0.4\,\text{sec} = 1.2\,\text{sec}.$

Example 5.39: The open-loop transfer function of a second-order system is $G(s) = \dfrac{4}{s(s+2)}$. Determine (i) maximum peak overshoot M_p and (ii) the time to reach the maximum overshoot t_p when a step displacement of 20° is given to the system. Also, determine (iii) rise time t_r and (iv) settling time t_s for an error of 6% by determining the time constant of the system.

Solution: The closed-loop transfer function of a unity feedback system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{4}{s^2+2s+4}$$

Comparing the above equation with the standard form, we obtain

$$\omega_n = \sqrt{4} = 2\,\text{rad / sec} \text{ and } 2\xi\omega_n = 2 \Rightarrow \xi = 0.5$$

The maximum overshoot is given by $M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 0.159$

For an input of 20°, the overshoot = $0.159 \times 20 = 3.18$
The time at which the maximum overshoot occurs is

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{2\sqrt{1-0.25}} = 1.81 \text{ sec}$$

Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d}$ sec

where $\theta = \cos^{-1}\xi = \cos^{-1} 0.5 = \dfrac{\pi}{3}$.

Hence, $t_r = \dfrac{\pi - \frac{\pi}{3}}{\omega_n\sqrt{1-\xi^2}} = 1.21\,\text{sec}$.

For 6% error $e^{-m} = 0.06 \Rightarrow m = 2.81$

Time constant, $T = \dfrac{1}{\xi\omega_n} = 1$ sec

Settling time, $t_s = mT = 2.81 \times 1 = 2.81$ sec

5.8 Steady-State Error

The objective of the control system is to make the system response follow the specific standard reference signal applied to the system in a steady state without any deviation. The steady state occurs due to the presence of non-linearities in the system. The accuracy of the control system is measured by steady-state error. The steady-state error is defined as the difference between the output response of the control system and the standard reference signal applied to it. The occurrence of steady-state error in the control system is almost inevitable. But the steady-state error can be maintained at a minimum value by satisfying the specifications of the transient response of the system.

5.8.1 Characteristic of Steady-State Error

For the system shown in Fig. 5.17(a), $R(s)$ is the step input, $C(s)$ is the output and $E(s) = R(s) - C(s)$ is the error. As $c(t)$ equals $r(t)$ in the steady-state condition, there will be no error existing in the system (i.e., $e(t)$ = 0) which is practically impossible with the pure gain K. Thus, in the steady-state condition, the relation between the steady-state value of the error and the steady-state value of the output is given by $e_{ss} = \dfrac{1}{K}c_{ss}$. Thus, for achieving a smaller steady-state value of the error e_{ss}, a large value of gain K is used. Hence, a steady-state error will be present in the system if a step input is applied to the system that has a gain in the forward path.

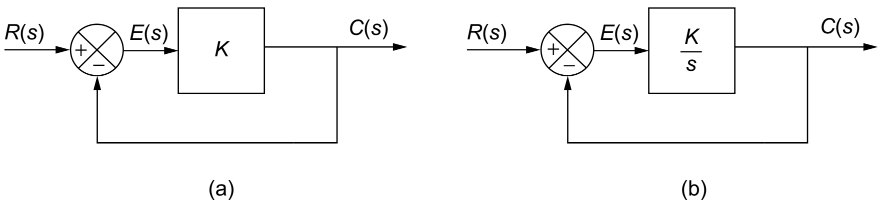


Fig 5.17 | A simple closed-loop system

When a step input is applied, the steady error of the system becomes zero if the gain of the system is replaced by an integrator as shown in Fig. 5.17(b). Thus, the steady-state error varies with the gain value associated with the system.

5.8.2 Determination of Steady-State Error

The steady-state error of a system when different types of standard test input signals are applied to it is discussed in this section. The steady-state error can be determined for the system with unity feedback and also for the system with non-unity feedback in terms of closed-loop transfer function of the system.

5.8.3 Steady-State Error in Terms of *G*(*s*)

In this section, we will be determining the steady-state error of the system with unity feedback path. Let $G(s)$ be the transfer function of the forward path and $H(s)=1$. Although, the steady-state error is calculated in terms of closed-loop transfer function of the system for a unity feedback system, it is better to use the open-loop transfer function $G(s)$.

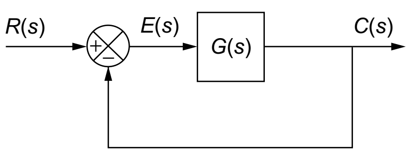


Fig 5.18 | A simple closed-loop system with unity feedback

Consider a system with unity feedback as shown in Fig. 5.18. The error $E(s)$ is the difference between the input $R(s)$ and the output $C(s)$. Thus, by solving for $E(s)$, it is possible to get an expression for the error.

Therefore, $E(s)=R(s)-C(s)$

$$C(s)=E(s)G(s)$$

Solving the above equations, we obtain $E(s)=\dfrac{R(s)}{1+G(s)}$

Applying the final value theorem to above the equation, we get the steady-state error of the system as

$$e(\infty)=\underset{s\to 0}{Lt}\frac{sR(s)}{1+G(s)}$$

Thus, the steady-state error $e(\infty)$ can be determined for different standard test input signals $R(s)$ and systems $G(s)$. Let us consider the system with forward path transfer function as $G(s)=\dfrac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$ and we can determine the relationship between the steady-state error and $G(s)$ by applying some standard signals such as unit-step input, unit-ramp input and unit-parabolic input.

Step Input

For a unit-step signal, $R(s)=\dfrac{1}{s}$,

$$e(\infty)=e_{ss}(\infty)=\underset{s\to 0}{Lt}\frac{s\left(\frac{1}{s}\right)}{1+G(s)}=\frac{1}{1+\underset{s\to 0}{Lt}G(s)}$$

Case 1: if at least one pole exists at the origin, i.e., $n\geq 1$

$$\underset{s\to 0}{Lt}G(s)=\infty$$

The steady-state error, $e_{ss}(\infty)=0$

Case 2: if there does not exist a pole at origin, i.e., $n=0$

$$\underset{s\to 0}{Lt}G(s)=\frac{z_1z_2\cdots}{p_1p_2\cdots}$$

which is finite and yields a finite steady-state error.

Ramp Input

For a unit-ramp signal, $R(s)=\dfrac{1}{s^2}$,

$$e(\infty)=e_{\text{ramp}}(\infty)=\underset{s\to 0}{Lt}\frac{s\frac{1}{s^2}}{1+G(s)}=\frac{1}{\underset{s\to 0}{Lt}sG(s)}$$

Case 1: If there are two or more poles at the origin, i.e., $n \geq 2$.

$$\underset{s \to 0}{Lt}\, sG(s) = \infty$$

The steady-state error $e_{ss}(\infty) = 0$
Case 2: If there is only one pole at the origin, i.e., $n = 1$

$$\underset{s \to 0}{Lt}\, sG(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

which is finite and leads to constant steady-state error
Case 3: If there is no pole at the origin,

$$\underset{s \to 0}{Lt}\, sG(s) = 0$$

which leads to an infinite steady-state error.

Parabolic Input:
For a unit-parabolic signal, $R(s) = \frac{1}{s^3}$,

$$e(\infty) = e_{\text{parabola}}(\infty) = \underset{s \to 0}{Lt} \frac{s \frac{1}{s^3}}{1 + G(s)} = \frac{1}{\underset{s \to 0}{Lt}\, s^2 G(s)}$$

Case 1: If there are three or more poles at the origin, i.e., $n \geq 3$

$$\underset{s \to 0}{Lt}\, s^2 G(s) = \infty$$

The steady-state error will be zero i.e., $e_{\text{steady}}(\infty) = 0$.
Case 2: If there are two poles at the origin, i.e., $n = 2$

$$\underset{s \to 0}{Lt}\, s^2 G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

The steady-state error will be constant finite value.
Case 3: If there is less than one pole at the origin, i.e., $n \leq 1$

$$\underset{s \to 0}{Lt}\, s^2 G(s) = 0$$

The steady-state error will be infinite, $e_{ss}(\infty) = \infty$.

5.8.4 Steady-State Error in Terms of *T*(*s*)

Let $G(s)$ be the transfer function of the forward path and $H(s)$ be the transfer function of the feedback path. In this case, the steady-state error is calculated in terms of closed-loop transfer function of the system $T(s)$.

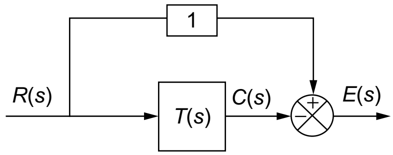


Fig 5.19 | A simple closed-loop system

Referring to Fig. 5.19, the error $E(s)$ is the difference between the input $R(s)$ and the output $C(s)$. Thus, by solving for $E(s)$, it is possible to get an expression for the error. Then, by applying final value theorem, the steady-state error is obtained as

$$E(s) = R(s) - C(s)$$

where $C(s) = R(s)T(s)$.

Therefore, $E(s) = R(s)[1 - T(s)]$

Applying the final value theorem, we obtain steady-state error of the system as

$$e(\infty) = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s) = \underset{s\to 0}{Lt}\, sR(s)[1 - T(s)]$$

5.8.5 Static Error Constants and System Type

In general, the response of a system consists of transient response and steady-state response. In the previous sections, we have discussed about the transient response of the system and its performance specifications such as damping ratio, natural frequency, settling time and percent overshoot subjected to different standard signals. Similarly, the performance specifications of steady-state error called static error constants can be determined as given below.

Static Error Constants

For a step input, $r(t) = u(t)$, its corresponding steady-state error is

$$e(\infty) = e_{\text{step}}(\infty) = \frac{1}{1 + \underset{s\to 0}{Lt}\, G(s)}$$

For a ramp input, $r(t) = tu(t)$, its corresponding steady-state error is

$$e(\infty) = e_{\text{step}}(\infty) = \frac{1}{\underset{s\to 0}{Lt}\, sG(s)}$$

For a parabolic input, $r(t) = \frac{1}{2}t^2u(t)$, its corresponding steady-state error is

$$e(\infty) = e_{\text{parabolola}}(s) = \frac{1}{\underset{s\to 0}{Lt}\, s^2G(s)}$$

$\underset{s\to 0}{Lt}\, G(s)$, $\underset{s\to 0}{Lt}\, sG(s)$ and $\underset{s\to 0}{Lt}\, s^2G(s)$ are collectively known as static error constants and they are separately termed as position error constant, velocity error constant and acceleration error constant respectively.

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)$

Since the static error constants are present at the denominator of the steady-state error equation, the steady-state error will decrease as the static error constant increases. It is clear that the above three static error constants are dependent on the transfer function of the system $G(s)$ and the number of poles present at the origin. But it is known that the number of poles present at the origin determines the TYPE of the system. Hence, the relationship among the TYPE of the system, static error constant and steady-state error of the system are given in Table 5.5.

The significant features of static error constants:

(i) Steady-state error characteristics are specified using static error constants.
(ii) The TYPE of the system, the type of input applied to the system can be determined by knowing the static error constants.
(iii) These coefficients are used for error analysis only if the input is step, ramp and parabolic inputs.
(iv) The values of static error constants given in Table 5.5 are for the systems only with unity feedback.
(v) If the input applied to a system is a linear combination of the inputs (step, ramp and parabolic), then the steady-state error can be obtained by superimposing the errors obtained due to each input.

Example 5.40: The open-loop transfer function of a control system is $G(s) = \dfrac{20}{(s+1)(s^2+10s+6)}$. Determine the steady-state error of the system when the transfer function of the feedback path is $H(s) = \dfrac{5}{s+3}$ and the system inputs are (i) $r(t) = 5$, (ii) $r(t) = 4t$ and (iii) $r(t) = \dfrac{3t^2}{2}$.

Solution: The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{100}{(s+1)(s^2+10s+6)(s+3)}$$

$$= \frac{(s+1)(s+3)(s^2+10s+6)+100}{(s+1)(s+3)(s^2+10s+6)}$$

Table 5.5 | Relationship among the input, system type, static error constant and steady-state error

Input	Steady-state error	TYPE 0		TYPE 1		TYPE 2		TYPE 3	
		Static error constant	Steady-state error	Static error constant	Steady-state error	Static error constant	Steady-state error	Static error constant	Steady-state error
Step $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0	$K_p = \infty$	0
Ramp $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = Constant$	$\frac{1}{K_v}$	$K_v = \infty$	0	$K_v = \infty$	0
Parabolic $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$	$K_a = \infty$	0

Therefore, the steady-state error of the system is

$$e_{ss} = \underset{s\to 0}{Lt}\, s\frac{R(s)}{1+G(s)H(s)} = \underset{s\to 0}{Lt}\, sR(s)\frac{(s+1)(s+3)(s^2+10s+6)}{(s+1)(s+3)(s^2+10s+6)+100}$$

Steady-state error for different inputs:

(i) For step input $r(t)=5$, $R(s)=\dfrac{5}{s}$

Therefore, $e_{ss} = \underset{s\to 0}{Lt}\, 5\dfrac{(s+1)(s+3)(s^2+10s+6)}{(s+1)(s+3)(s^2+10s+6)+100} = \dfrac{5\times 1\times 3\times 6}{1\times 3\times 6+100} = \dfrac{90}{118}$

(ii) For ramp input $r(t)=4t$, $R(s)=\dfrac{4}{s^2}$

$$e_{ss} = \underset{s\to 0}{Lt}\, s\frac{R(s)}{1+G(s)H(s)} = \underset{s\to 0}{Lt}\, \frac{4}{s}\frac{(s+1)(s+3)(s^2+10s+6)}{(s+1)(s+3)(s^2+10s+6)+100} = \infty$$

(iii) For acceleration input $r(t)=\dfrac{3t^2}{2}$, $R(s)=\dfrac{3}{s^3}$

$$e_{ss} = \underset{s\to 0}{Lt}\, s\frac{R(s)}{1+G(s)H(s)} = \underset{s\to 0}{Lt}\, \frac{3}{s^2}\frac{(s+1)(s+3)(s^2+10s+6)}{(s+1)(s+3)(s^2+10s+6)+100} = \infty.$$

Example 5.41: The open-loop transfer function of a unity feedback system is $G(s)=\dfrac{K}{s(1+0.025s)}$ and damping ratio $\xi = 0.4$. Determine K and the steady-state error for the ramp input.

Solution: The characteristic equation of the system,

$$1+G(s)H(s) = 1+\frac{K}{s(1+0.025s)} = s(1+0.025s)+K = 0$$

$$s^2+40s+40K=0$$

Comparing the above equation with the standard form $s^2+2\xi\omega_n s+\omega_n^2=0$, we obtain

$$2\xi\omega_n = 40$$

Substituting known values, we obtain $\omega_n = \dfrac{40}{2\times 0.4} = \dfrac{400}{8} = 50$

Also, $\quad 40K = \omega_n^2 = 50^2$

Hence, $K = \dfrac{50\times 50}{40} = \dfrac{250}{4} = 62.5$

Steady-state error for the ramp input:

Velocity error constant $K_V = \underset{s\to 0}{Lt}\, sG(s)H(s) = \underset{s\to 0}{Lt}\, \dfrac{K}{(1+0.025s)} = K = 62.5$

Therefore, $$e_{ss} = \frac{R}{K_V} = \frac{1}{62.5}.$$

Example 5.42: **For a unity feedback system shown in Fig. E5.42(a), B = 70 × 10⁻⁶ Nm/rad/sec, J = 1.4 × 10, motor constant $K_m = T = 0.03$ Nm / 100 V, gain of the error detector or synchro constant K_s = 57.4 V / rad / error, ξ = 0.3 and error = 1 V. Input speed = 20 rpm. Determine the steady-state error of the system.**

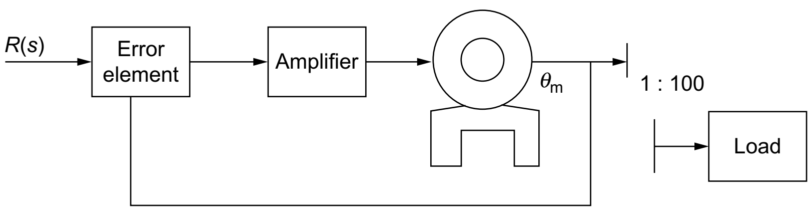


Fig. E5.42(a)

Solution: The equivalent block diagram of the system is shown in Fig. E5.42(b).

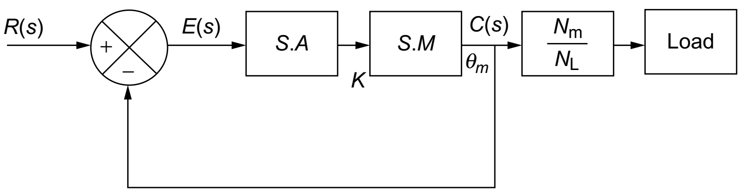


Fig. E5.42(b)

Input velocity, $\omega_i = \dfrac{2\pi N}{60} = \dfrac{2\pi\times 20}{60} = 2.1\ \text{rad/sec}$

Motor constant $K_m = T = 3 \times 10^{-4}$ Nm/volt

We know that, for the given system $2\xi\omega_n = \dfrac{F}{J}$ and $\omega_n = \sqrt{\dfrac{K}{J}}$

Therefore,
$$\xi = \frac{F}{2\sqrt{KJ}}$$

$$K = \frac{F^2}{4\xi^2 J} = \frac{70^2 \times 10^{-12}}{4(0.3)^2 \times 1.4 \times 10^{-6}} = \frac{35}{36} \times 10^{-2} \cong 1 \text{ Nm/rad at the motor shaft}$$

$$\text{Gain of the amplifier} = \frac{\text{K}}{\text{Gain of the error detector} \times \text{gain of the motor} \times \text{Gain of the feedback loop}}$$

$$= \frac{1}{57.4 \times 3 \times 10^{-4} \times 1} = 58.07$$

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{1}{1.4 \times 10^{-6}}} = 100\sqrt{\frac{100}{1.4}} = 850$$

Steady-state error at motor shaft, $e_{ss} = \dfrac{2\xi\omega_i}{\omega_n} = \dfrac{2 \times 0.3 \times 2.1\sqrt{1.4}}{100\sqrt{100}} = \dfrac{1.26}{850} = 1.482 \times 10^{-3}$.

Example 5.43: A control system is designed to keep the antenna of a tracking radar pointed at a flying target. The system must be able to follow a straight line course with speed up to 1,000 km/hr with a maximum permissible error of 0.01°. If the shortest distance from antenna to target is 300 m, then determine the ramp error constant K_v in order to satisfy the requirements.

Solution:

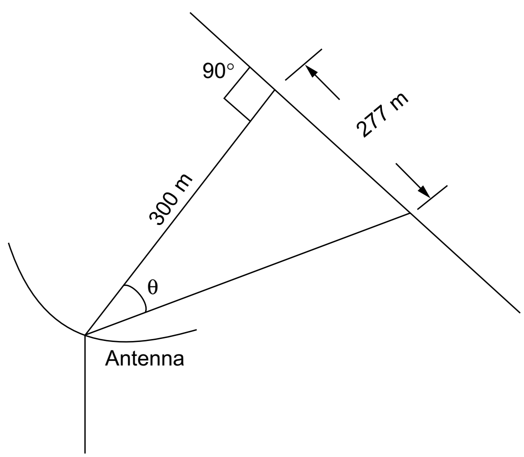


Fig. E5.43

Speed = 1000 km/hr $= \dfrac{10^6}{3600} = 277$ m/sec

Therefore, the distance travelled by the target for 1 sec = 277 m

From Fig. E5.43, we obtain $\tan\theta = \dfrac{277}{300} = 0.923$

Therefore, $\theta = \tan^{-1} 0.923 = 42°42' = 42.7°$

But we know that for a ramp input, the steady-state error is obtained as

$$e_{ss} = \frac{R}{K_V}$$

Therefore, for the given system where the input R is an angle θ, we obtain

$$e_{ss} = \frac{\theta}{K_v}$$

Therefore, the ramp error constant, $K_v = \dfrac{\theta}{0.01°} = \dfrac{42.7°}{0.01°} = 4270.$

Example 5.44: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{10}{s(0.1s+1)}$ and the input signal to be applied to the system is given by $r(t) = a_0 + a_1 t + \dfrac{a_2}{2} t^2$ Determine the steady-state error of the system.

Solution:

Given the open-loop transfer function $G(s) = \dfrac{10}{s(0.1s+1)}$.

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s) = \dfrac{10}{0(0+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s) = \dfrac{10}{(0+1)} = 10\,\text{sec}^{-1}$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2 G(s) = \dfrac{0\times 10}{(0+1)} = 0$

Hence, the steady-state error for

(i) the input a_0 is given by $e_{ss1} = \dfrac{a_0}{1+K_p} = 0$

(ii) the input $a_1 t$ is given by $e_{ss2} = \dfrac{a_1}{K_v} = 0.1a_1$

(iii) the input $\dfrac{1}{2}a_2 t^2$ is given by $e_{ss3} = \dfrac{a_2}{K_a} = \infty$

Hence, the total steady-state error is given by

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = 0 + 0.1a_1 + \infty = \infty .$$

Example 5.45: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{40(s+2)}{s(s+1)(s+4)}$. Determine (i) all the error constants and (ii) error for ramp input with magnitude 4.

Solution: (i) Given the open-loop transfer function $G(s) = \dfrac{40(s+2)}{s(s+1)(s+4)}$.

(a) Position error constant, $K_p = \underset{s \to 0}{Lt}\, G(s) = \infty$

(b) Velocity error constant, $K_v = \underset{s \to 0}{Lt}\, sG(s) = 20$

(c) Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\, s^2 G(s) = 0$

(ii) The steady-state error for ramp input is given by

$$e_{ss} = \frac{A}{K_v}$$

Give the magnitude of ramp input, A = 4

Therefore, $e_{ss} = \dfrac{4}{20} = 0.2.$

Example 5.46: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{10(s+2)(s+3)}{s(s+1)(s+5)(s+4)}$. Determine all the error constants and hence the steady-state error when the system is subjected to an input signal $r(t) = 3 + t + t^2$.

Solution: Given the open-loop transfer function $G(s)=\dfrac{10(s+2)(s+3)}{s(s+1)(s+5)(s+4)}$.

(i) Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)=\infty$

(ii) Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)=3$

(iii) Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)=0$

Hence, the steady-state error for

(a) the input 3 is given by $e_{ss1} = \dfrac{3}{1+K_p}=0$

(b) the input t is given by $e_{ss2} = \dfrac{1}{K_v}=0.333$

(c) the input t^2 is given by $e_{ss3} = \dfrac{2}{K_a}=\infty$

Hence, the total steady-state error is given by $e_{ss} = e_{ss1}+e_{ss2}+e_{ss3} = 0+0.333+\infty=\infty$.

Example 5.47: The block diagram shown in Fig. E5.47 represents a heat treating oven. The set point (desired temperature) is 1,000°C. What is the steady-state temperature?

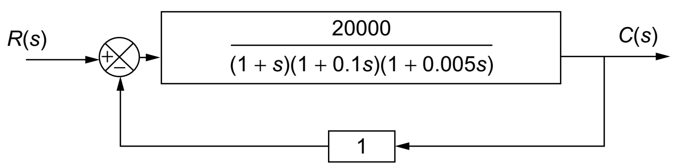


Fig. E5.47

Solution: Given $G(s)=\dfrac{20000}{(1+s)(1+0.1s)(1+0.005s)}$ $H(s)=1$ and input is step of 1,000

The input $R(s)=\dfrac{1{,}000}{s}$

For step input,

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, \frac{20{,}000}{(1+s)(1+0.1s)(1+0.005s)} = 20{,}000$$

Therefore, steady-state error, $$e_{ss} = \frac{A}{1+K_p}$$

where A = magnitude of step input.

i.e., $$e_{ss} = \frac{1{,}000}{1+20{,}000} = 0.04999$$

Therefore, the steady-state temperature can be determined as

$$\underset{t\to\infty}{Lt}\, c(t) = C_{ss} = \text{ desired output} - \mathrm{e}_{ss}$$

Hence, $$C_{ss} = 1{,}000 - 0.04999 = 999.95°C.$$

Example 5.48: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{10}{(s+2)(s+3)}$. Determine the type of signal for which the system gives a constant steady-state error and calculate its value.

Solution: Since there exist no poles at the origin, the system is TYPE 0 system.
From Table 5.5, the unit-step input gives a constant steady-state error for TYPE 0 system.
Therefore,

$$\text{Position error constant, } K_p = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt} \frac{10}{(s+2)(s+3)} = \frac{5}{3} = 1.667$$

and the steady-state error, $e_{ss} = \dfrac{1}{1+K_p} = 0.375.$

Example 5.49: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{K}{s^2(s+1)(s+4)}$ and the input signal to be applied to the system is given by $r(t) = 1+8t+9t^2$. Determine K so that the steady-state error of the system is 0.8.

Solution:

Given the open-loop transfer function $G(s) = \dfrac{K}{s^2(s+1)(s+4)}$ and $r(t) = 2+8t+9t^2$.

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s) = \dfrac{K}{0(0+1)(0+4)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s) = \dfrac{K}{0(0+1)(0+4)} = \infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s) = \dfrac{K}{(0+1)(0+4)} = \dfrac{K}{4}$

Hence, the steady-state error for

(i) the input 1 is given by $e_{ss1} = \dfrac{1}{1+K_p} = 0$

(ii) the input $8\,t$ is given by $e_{ss2} = \dfrac{8}{K_v} = 0$

(iii) the input $9t^2$ is given by $e_{ss3} = \dfrac{18}{K_a} = \dfrac{72}{K}$

Hence, the total steady-state error is given by $e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$

$$= 0 + 0 + \frac{72}{K}$$

$$e_{ss} = \frac{72}{K}$$

Given the steady-state error of the system $e_{ss} = 0.8$. Therefore,

$$\frac{72}{K} = 0.8$$

Hence, $K = 90.$

Example 5.50: The open-loop transfer function of a system with unity feedback is given by $G(s) = \dfrac{K_1(2s+1)}{s(5s+1)(1+s)^2}$ and the input signal to be applied to the system is given by $r(t) = (1+6t)$. Determine the K_1 so that the steady-state error of the system is 0.1.

Solution:

Given the open-loop transfer function $G(s) = \dfrac{K_1(2s+1)}{s(5s+1)(1+s)^2}$ and $r(t) = 1+6t.$

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s) = \dfrac{K_1(0+1)}{0(0+1)(0+1)^2} = \infty$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\, sG(s) = \dfrac{K_1(0+1)}{(0+1)(0+1)^2} = K_1$

Therefore, the steady-state error for

(i) the input 1 is given by $e_{ss1} = \dfrac{1}{1+K_p} = 0$

(ii) the input $6\,t$ is given by $e_{ss2} = \dfrac{6}{K_v} = \dfrac{6}{K_1}$

Hence, the total steady-state error is given by $e_{ss} = e_{ss1} + e_{ss2}$

$$= 0 + \frac{6}{K_1}$$

$$e_{ss} = \frac{6}{K_1}$$

Given the steady-state error of the system is 0.1. Therefore,

$$0.1 = \frac{6}{K_1}$$

Therefore, $K_1 = 60$.

Example 5.51: A control system with PD controller is shown in the Fig. E5.51. If the velocity error constant $K_v = 1{,}000$ and the damping ratio $\xi = 0.5$, find the values of K_p and K_D.

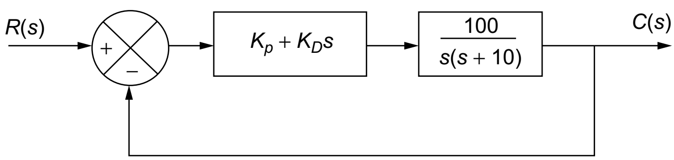


Fig. E5.51

Solution:

We have $K_v = \underset{s \to 0}{Lt}\, sG(s)H(s)$

or $$1{,}000 = \underset{s \to 0}{Lt}\, s\frac{(K_p + K_D s)100}{s(s+100)}$$

$$K_p = 100$$

The characteristic equation is $1+G(s)H(s)=0$

or $$1+\frac{(100+K_D s)100}{s(s+10)}=0$$

or $$s^2+(10+100\ K_D)s+10^4=0$$

Comparing with $s^2+2\xi\omega_n+\omega_n^2=0$, we obtain

$$2\xi\omega_n=10+100\ K_D$$

or $K_D=0.9.$

Example 5.52: The open-loop transfer function of a system with unity feedback is given by $G(s)=\frac{K(s+2)}{s^2(s^2+7s+12)}$ and a unit-parabolic signal is applied to the system. Determine the static error constants and steady-state error.

Solution:

Given the open-loop transfer function $G(s)=\frac{K(s+2)}{s^2(s^2+7s+12)}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)=\frac{K(0+2)}{0(0+0+12)}=\infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)=\frac{K(0+2)}{0(0+0+12)}=\infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)=\frac{K(0+2)}{(0+0+12)}=\frac{K}{12}$

The steady-state error of the system subjected to parabolic input is given by

$$e_{ss}=\frac{1}{K_a}=\frac{12}{K}$$

5.8.6 Generalized or Dynamic Error Coefficients

The error coefficients discussed in the previous sections are known as static error coefficients since each of the input applied to the system to get the steady-state error is not time-varying quantity. Also, it is impossible to design such control systems to maintain zero steady-state error when the input applied is a time-varying quantity. The steady-state error of such systems is determined as follows:

The error $E(s)$ of a unity feedback control system as discussed in the static error coefficients is given by

$$E(s)=\frac{R(s)}{1+G(s)}$$

The error transfer function, $W_e(s)=\frac{E(s)}{R(s)}=\frac{1}{1+G(s)}$ can be expressed as

$$W_e(s)=\frac{1}{1+G(s)}=\frac{1}{1+\frac{K\sum_{i=1}^{m}(T_i s+1)}{s^N\sum_{i=1}^{n}(T_i s+1)}}=\frac{s^N(1+a_1 s+\cdots+a_n s^n)}{s^N(1+a_1 s+\cdots+a_n s^n)+Ks^N(1+b_1 s+\cdots+b_m s^m)}$$

$$=\frac{(\alpha_0+\alpha_1 s+\alpha_2 s^2+\cdots)}{(\beta_0+\beta_1 s+\beta_2 s^2+\cdots)}$$

$$=C_0+C_1 s+\frac{C_2}{2!}s^2+\frac{C_3}{3!}s^3+\cdots$$

where, in general, the coefficients C_0, C_1, C_2, C_3, $\cdots$ are defined as *dynamic error coefficients* or *generalized error coefficients*.

The generalized error coefficients are obtained using the formula

$$C_m=\left(\frac{d^m}{ds^m}W_e(s)\right)_{at\, s=0} \tag{5.2}$$

To determine the steady-state error using the generalized or dynamic error coefficients, the following steps are followed:

$$E(s)=C_0R(s)+C_1sR(s)+\frac{C_2}{2!}s^2R(s)+\frac{C_3}{3!}s^3R(s)+\cdots$$

Taking inverse Laplace transform, we obtain

$$e(t)=C_0r(t)+C_1r'(t)+\frac{C_2}{2!}r''(t)+\frac{C_3}{3!}r'''(t)+\cdots$$

Therefore, the steady-state error is given by

$$e_{ss} = \underset{t \to 0}{Lt}\, e(t) = \underset{t \to 0}{Lt}\, C_0 r(t) + C_1 r'(t) + \frac{C_2}{2!} r''(t) + \cdots$$

The relationship between the static and dynamic (generalized) error coefficients are given below which are not difficult to prove.

$$C_0 = \frac{1}{1+K_p},\ C_1 = \frac{1}{K_v} \text{ and } C_2 = \frac{1}{K_a}.$$

Example 5.53: The open-loop transfer function of a unity feedback control system is $G(s) = \dfrac{9}{(s+1)}$. Determine the steady-state error of the system when the system is excited by (i) $r(t) = 2$, (ii) $r(t) = t$, (iii) $r(t) = \dfrac{3t^2}{2}$ and (iv) $r(t) = 1 + 2t + \dfrac{3t^2}{2}$ using the generalized error series.

Solution: The transfer function of the system relating the error and the input of the system is given by

$$W_e(s) = \frac{1}{1+G(s)} = \frac{(s+1)}{(s+10)}$$

Also, the error series of the system is given by $e(t) = C_0 r(t) + C_1 r'(t) + \dfrac{C_2}{2!} r''(t) + \dfrac{C_3}{3!} r'''(t) + \cdots$

Here, the coefficients C_0, C_1, C_2, C_3, $\cdots$ are obtained using the formula

$$C_m = \left(\frac{d^m}{ds^m} W_e(s) \right)_{\text{at } s=0}$$

(i) For input $r(t) = 2$

From the given input, we obtain $r'(t) = 0,\ r''(t) = 0, \cdots$

Therefore, steady-state error $e_{ss}(t) = C_o r(t)$

Hence, $C_0 = \left[W_e(s) \right]_{s=0} = \dfrac{1}{10}$

And, $e_{ss}(t) = C_o r(t) = \dfrac{1}{5}$

(ii) For the input $r(t) = t$

From the given input, we obtain

$r'(t) = 1$ and other differentials as zero.

Therefore, $$e_{ss}(t) = C_o t + C_1$$

Hence, $$C_1 = \left[\frac{dW_e(s)}{ds}\right]_{s=0} = \left[\frac{d}{ds}\left(\frac{s+1}{s+10}\right)\right]_{s=0} = \left[\frac{9}{(s+10)^2}\right]_{s=0} = \frac{9}{100}$$

and, $$e_{ss}(t) = C_o(t) + C_1 = \frac{1}{10}t + \frac{9}{100} = \frac{10t+9}{100}$$

(iii) For the input $r(t) = \frac{3t^2}{2}$

From the given input, we obtain

$r'(t) = 3t$, $r''(t) = 3$ and other higher differentials are zero.

Therefore, $$e_{ss}(t) = \frac{3t^2}{2}C_o + 3tC_1 + 3C_2$$

where $$C_2 = \left[\frac{d^2}{ds^2}W_e(s)\right] = \left[\frac{d}{ds}\left(\frac{9}{(s+10)^2}\right)\right]_{s=0} = -\frac{9}{500}$$

Hence, $$e_{ss}(t) = \frac{3}{20}t^2 + \frac{27}{100}t - \frac{27}{500} = \frac{3}{20}\left[\left(t^2 + 1.8t\right) - 0.36\right]$$

(iv) For the input $r(t) = 1 + 2t + \frac{3t^2}{2}$

From the given input, we obtain

$$r'(t) = 2 + 3t,\ r''(t) = 3 \text{ and other higher differentials are zero}$$

Therefore, $$\begin{aligned} e_{ss}(t) &= C_o r(t) + C_1 r'(t) + C_2 r''(t) \\ &= C_o\left(1 + 2t + \frac{3t^2}{2}\right) + \frac{9}{100}(2+3t) + 3C_2 \end{aligned}$$

Substituting the known values in the given equation, we obtain

$$\begin{aligned} e_{ss}(t) &= \frac{1}{10}\left(1 + 2t + \frac{3t^2}{2}\right) + \frac{9}{100}(2+3t) - \frac{9\times 3}{500} \\ &= \frac{1}{10}\left[1 + 2t + 1.5t^2 + 0.18 + 0.27t - 0.054\right] \\ &= \frac{1}{10}\left[1.234 + 2.27t + 1.5t^2\right] \end{aligned}$$

Example 5.54: The block diagram of a fire-control system with unity feedback is shown in Fig. E5.54. Determine the steady-state error of the system when the system input is $r(t) = \left(5 + 4t + \frac{3t^2}{2}\right)$ using generalized error series method.

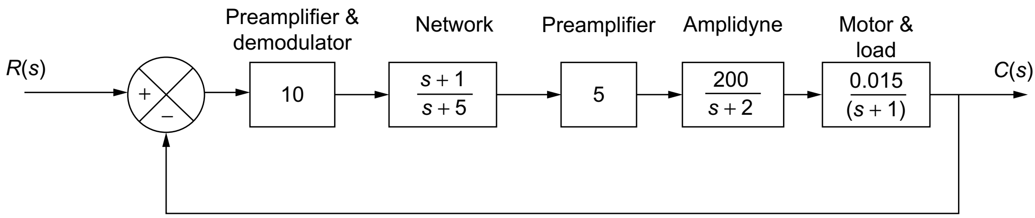


Fig. E5.54

Solution: From the block diagram shown in Fig. E5.54, we obtain

$$G(s) = 10 \times \left[\frac{(s+1)}{(s+5)}\right] \times 5 \times \left[\frac{200}{(s+2)}\right] \times \left[\frac{0.015}{(s+1)}\right] = \frac{150}{(s^2 + 7s + 10)}$$

For the negative feedback system with $H(s) = 1$, the function $W_e(s)$ is

$$W_e(s) = \frac{1}{1 + G(s)} = \frac{s^2 + 7s + 10}{s^2 + 7s + 160}$$

For the given input $r(t) = 5 + 4t + \frac{3t^2}{2}$, we have

$$r'(t) = (4 + 3t) \text{ and } r''(t) = 3 \text{ and other higher order differentials are zero}$$

Therefore, the steady-state error is given by

$$e_{ss}(t) = C_o\left(5 + 4t + \frac{3t^2}{2}\right) + C_1(4 + 3t) + 3C_2$$

Therefore, the generalized error coefficients are

$$C_o = [W_e(s)]_{s=0} = \frac{1}{16}$$

$$C_1 = \left[\frac{dW_e(s)}{ds}\right]_{s=0} = 0.0436$$

$$C_2 = \left[\frac{d^2 W_e(s)}{ds^2}\right]_{s=0} = 0.0121$$

Hence, the steady-state error of the system is

$$e_{ss}(t) = 0.0436(4 + 3\,t) = 0.1744 + 0.01308\,t.$$

Example 5.55: The open-loop transfer function of the system $G(s)$ for a type-2 unity feedback system is given by $G(s) = \dfrac{10(1+s)}{s^2(6+5s)}$. Determine the steady-state error to an input $r(t) = 1 + 3t + 4t^2$ using the generalized error coefficients.

Solution: The error transfer function for the system is given by

$$W_e(s) = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10(s+1)}{s^2(5s+6)}} = \frac{6s^2 + 5s^3}{10 + 10s + 6s^2 + 5s^3}$$

Dividing the numerator by denominator directly, we obtain

$$W_e(s) = \frac{3}{5}s^2 - \frac{1}{10}s^3 - \frac{13}{50}s^4 + \frac{1}{50}s^5 + \cdots$$

But the error transfer function is given by

$$W_e(s) = \frac{E(s)}{R(s)}$$

Using the above two equations, we obtain

$$E(s) = \left(\frac{3}{5}s^2 - \frac{1}{10}s^3 - \frac{13}{50}s^4 + \frac{1}{50}s^5 + \cdots\right)R(s)$$

Taking inverse Laplace transform, we obtain

$$e(t) = \frac{3}{5}r''(t) - \frac{1}{10}r'''(t) + \cdots$$

From the given input $r(t) = 1 + 3t + 4t^2$, we obtain $r''(t) = 8$ and $r'''(t) = 0$

Hence, the steady-state error of the system is

$$e_{ss} = \frac{3}{5}\times 8 - \frac{1}{10}\times 0 + \cdots = 4.8.$$

Example 5.56: The open-loop transfer function for a unity feedback control system is given by $G(s)=\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$. Determine (i) generalized error coefficients $C_0, C_1, \cdots, C_n$ and (ii) error series $e(t)$.

Solution: The open-loop transfer function of a unity feedback system is given by

$$G(s)H(s)=\frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

Therefore, the error transfer function of the system is given by

$$W_e(s)=\frac{1}{1+G(s)H(s)}=\frac{1}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}=\frac{s^2+2\xi\omega_n s}{s^2+2\xi\omega_n s+\omega_n^2}$$

The generalized error coefficients for the given system are determined using Eqn. (5.2) as

$$C_m=\left(\frac{d^m}{ds^m}W_e(s)\right)_{\text{at } s=0}$$

Therefore,

$$C_0=\left[W_e(s)\right]_{s=0}$$

$$C_1=\left[\frac{dW_e(s)}{ds}\right]_{s=0}$$

$$C_2=\left[\frac{d^2(W_e(s))}{ds^2}\right]_{s=0}$$

Thus,

$$\frac{dW_e(s)}{ds}=\frac{(s^2+2\xi\omega_n s+\omega_n^2)(2s+2\xi\omega_n)-(s^2+2\xi\omega_n s)(2s+2\xi\omega_n)}{\left[(s^2+2\xi\omega_n s+\omega_n^2)\right]^2}=\frac{2\omega_n^2 s+2\xi\omega_n^3}{\left[(s^2+2\xi\omega_n s+\omega_n^2)\right]^2}$$

$$\frac{d^2(W_e(s))}{ds^2}=\frac{d}{ds}\left\{\frac{2\omega_n^2 s+2\xi\omega_n^3}{\left[(s^2+2\xi\omega_n s+\omega_n^2)\right]^2}\right\}$$

$$= \frac{\left[\left[\left(s^2+2\xi\omega_{\text{n}}s+\omega_{\text{n}}^2\right)\right]^2\times\left(2\omega_{\text{n}}^2\right)\right]-\left(2\omega_{\text{n}}^2 s+2\xi\omega_{\text{n}}^3\right)\times 2\times\left(s^2+2\xi\omega_{\text{n}}s+\omega_{\text{n}}^2\right)\times\left(2s+2\xi\omega_{\text{n}}\right)}{\left[\left(s^2+2\xi\omega_{\text{n}}s+\omega_{\text{n}}^2\right)\right]^4}$$

Therefore, the generalized error coefficients are

$$C_0 = \left[W_{\text{e}}(s)\right]_{s=0} = 0$$

$$C_1 = \left[\frac{dW_{\text{e}}(s)}{ds}\right]_{s=0} = \frac{2\xi\omega_{\text{n}}^3}{\omega_{\text{n}}^4} = \frac{2\xi}{\omega_{\text{n}}}$$

$$C_2 = \left[\frac{d^2\left(W_{\text{e}}(s)\right)}{ds^2}\right]_{s=0} = \frac{2\omega_{\text{n}}^6 - 8\xi^2\omega_{\text{n}}^6}{\omega_{\text{n}}^8} = \frac{2-8\xi^2}{\omega_{\text{n}}^2}$$

Thus, the error series of the system using the generalized error coefficients is

$$e(t) = C_0 r(t) + C_1 r'(t) + \frac{C_2}{2!}r''(t) + \cdots = \frac{2\xi}{\omega_n}r'(t) + \left(\frac{2-8\xi^2}{2\omega_n^2}\right)r''(t) + \cdots$$

Example 5.57: The open-loop transfer function of a system is $G(s) = \dfrac{50}{s^2\left(s^2+4s+200\right)}$, and it has a feedback with a transfer function of $H(s) = 0.6$ s. Determine (i) position, velocity, acceleration coefficient and (ii) the steady-state error for the system when the system is subjected to an input $r(t) = 1 + t + \dfrac{t^2}{2}$ using generalized error series.

Solution: Given $G(s) = \dfrac{50}{s^2(s^2+4s+200)}$ and $H(s) = 0.6s$.

Therefore,
$$G(s)H(s) = \frac{50\times 0.6\, s}{s^2\left(s^2+4s+200\right)}$$

$$= \frac{30}{s\left(s^2+4s+200\right)}$$

Hence, the static error constants are $K_{\text{p}} = \underset{s\to 0}{Lt}\, G(s)H(s) = \dfrac{30}{(0)(0+0+200)} = \infty$

$$K_{\mathrm{v}} = \underset{s\to 0}{Lt}\, sG(s)H(s) = \frac{30s}{s\left(s^2+4s+200\right)} = \frac{30}{200} = 0.15$$

$$K_{\mathrm{a}} = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \frac{30s^2}{s\left(s^2+4s+200\right)} = \frac{0}{200} = 0$$

To determine e_{ss}:

For the input $r(t) = 1 + t + \frac{t^2}{2}$, we obtain $R(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}$

Error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$= \frac{\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}}{1+\frac{30s}{s^2\left(s^2+4s+200\right)}}$$

$$= \frac{1}{s}\left[\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right] + \frac{1}{s^2}\left[\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right] + \frac{1}{s^3}\left[\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right]$$

Therefore, steady-state error

$$e_{ss} = \underset{s\to 0}{Lt}\, s\left[\frac{1}{s}\left(\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right) + \frac{1}{s^2}\left(\frac{s\left(s^2+4s+200\right)}{s\left(s^2+rs+200\right)+30}\right) + \frac{1}{s^3}\left(\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right)\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right] + \frac{1}{s}\left(\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right) + \frac{1}{s^2}\left(\frac{s\left(s^2+4s+200\right)}{s\left(s^2+4s+200\right)+30}\right)$$

$$= 0 + \frac{200}{30} + \infty = \infty$$

5.9 Effect of Adding Poles and Zeros in the Second-Order System

Consider a second-order system with the open-loop transfer function as $G(s) = \dfrac{\omega_{\mathrm{n}}^2}{s\left(s+2\xi\omega_{\mathrm{n}}\right)}$ and unity feedback. The closed-loop transfer function of the system is obtained as $\dfrac{C(s)}{R(s)} = \dfrac{\omega_{\mathrm{n}}^2}{s^2+2\xi\omega_{\mathrm{n}}s+\omega_{\mathrm{n}}^2}$. The time-domain specifications of the second-order system are rise time, peak time, peak overshoot, settling time, etc. These time-domain specifications can be modified (either be increased or decreased) by adding either a pole or a zero.

5.9.1 Effect of Adding Poles

If a pole T_p is added to the above-mentioned system, the closed-loop transfer function of the system will be modified as $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\xi\omega_n s + \omega_n^2)(1 + sT_p)}$. Assuming a fixed value for ξ and ω_n and varying T_p, the response of the system can be determined. Fig. 5.20 shows the response of the second-order system subjected to a unit-step input by varying the pole T_p.

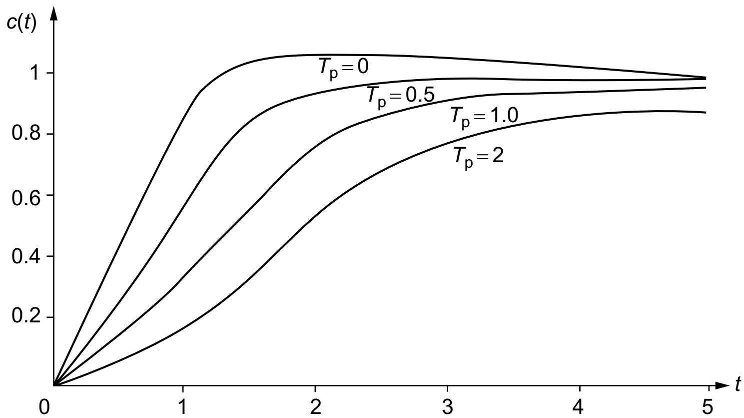


Fig. 5.20 | Step response of the second-order system for different pole location

The following changes occur by adding pole to the existing system:

(i) Peak overshoot decreases as T_p decreases.
(ii) Number of oscillations existing in the system becomes less as T_p decreases.
(iii) Increase in rise time.
(iv) System becomes more sluggish as T_p decreases.

Also, it is important to know the fact that by adding pole to the system in the forward path, the system will have the opposite effect.

5.9.2 Effect of Adding Zeros

If a zero T_z is added to the above-mentioned system, the closed-loop transfer function of the system will be modified as $\frac{C(s)}{R(s)} = \frac{\omega_n^2(1 + sT_z)}{(s^2 + 2\xi\omega_n s + \omega_n^2)}$. Assuming a fixed value for ξ and ω_n and varying T_z, the response of the system can be determined. Fig. 5.21 shows the response of the second-order system subjected to a unit-step input by varying the zero T_z.

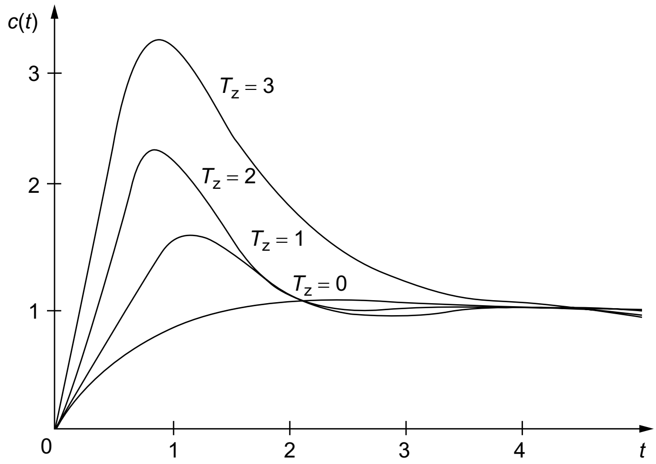


Fig. 5.21 | Step response of the second-order system for different zero location

The following changes occur by adding pole to the existing system:

(i) Increases rise time as T_z increases.
(ii) Increases peak overshoot and number of oscillations as T_z increases.

5.10 Response with P, PI and PID Controllers

The transient response specifications and steady-state response specifications of the system are to be maintained at a specific value for maintaining the system stability. Among the different transient response specifications discussed earlier in this chapter, the value of maximum overshoot is taken as the criteria to maintain stability and for steady-state response, steady-state error is taken as the criteria. The controllers are used to maintain those specifications at a specific value instead of zero or infinity. Here, a general second-order system is used and different types of controllers are used to maintain the steady-state error at a specific value. The output signal of the controller or the input signal of the system is known as actuating signal.

5.10.1 Proportional Derivative Control

The derivative controller alone is not used in the control system due to some reasons. One of the reasons is that if there is a constant steady-state error in the system, the derivative controller does not take any corrective measure since the derivative of a constant value is zero, which makes the error to prevail in the system even though a derivative controller is used. Hence, the derivative controller is used alongwith some of the other controllers.

The block diagram of the system with proportional derivative controller with unity feedback system is shown in Fig. 5.22. The system is a general second-order system. The actuating signal of the controller consists of the sum of the control action given by the proportional derivative controller. Hence, the actuating signal is given by

$$e_a(t) = e(t) + T_D \frac{de(t)}{dt}$$

Taking Laplace transform, we obtain

$$E_a(s) = E(s) + sT_D E(s)$$

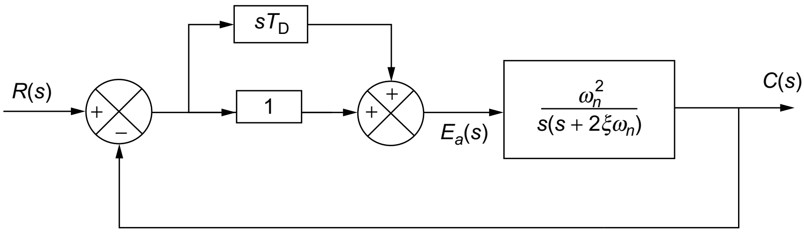


Fig. 5.22 | Proportional derivative controller of second-order system

The closed-loop transfer function of the system using proportional derivative control is given by

$$\frac{C(s)}{R(s)} = \frac{(1+sT_D)\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 T_D)s + \omega_n^2}$$

Comparing the above equation with the standard form $\frac{\omega_n^2}{s^2 + 2\xi'\omega_n s + \omega_n^2}$, we obtain

$$2\xi'\omega_n = 2\xi\omega_n + \omega_n^2 T_D$$

$$\xi' = \xi + \frac{\omega_n T_D}{2}$$

where ξ' is the damping ratio of the system with the controller.

From the above equation, it is clear that effective damping ratio has been increased by using the controller, thereby reducing the maximum overshoot.

The forward path transfer function of the system shown in Fig. 5.22 is given by

$$G(s) = (1+sT_D).\frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

and the feedback path transfer function is $H(s) = 1$.
Hence, the error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\left[\frac{(1+sT_D).\omega_n^2}{s(s+2\xi\omega_n)}\right]}$$

$$\frac{E(s)}{R(s)} = \frac{s(s+2\xi\omega_n)}{s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

For a unit-ramp input, $r(t) = \frac{t^2}{2}$.

Therefore, $R(s) = \frac{1}{s^3}$

Therefore, $E(s) = \frac{1}{s^2} \cdot \frac{s(s+2\xi\omega_n)}{\left[s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2\right]}$

The steady-state error for unit-ramp input is given by

$$e_{ss} = \underset{s\to 0}{Lt}\ sE(s)$$

$$= \underset{s\to 0}{Lt}\left[\frac{1}{s^2} \cdot \frac{s(s+2\xi\omega_n)}{\left[s^2+(2\xi\omega_n+\omega_n^2T_d)s+\omega_n^2\right]}\right]$$

$$e_{ss} = \frac{2\xi}{\omega_n}$$

It is clear that by introducing the proportional derivative controller in the system, the damping ratio ξ has been increased without changing the TYPE of the system and undamped natural frequency ω_n of the system. Also, due to the increase in the damping ratio of the system, the maximum peak overshoot and settling time that have direct impact on ξ have been reduced. Also, the steady-state error of the system remains unchanged even though a proportional derivative controller has been used.

5.10.2 Proportional Integral Control

The block diagram of a unity feedback system with proportional integral controller is shown in Fig. 5.23. The system is a general second-order system. The actuating signal of the controller consists of sum of the control action given by the proportional and integral controllers. Hence, the actuating signal is given by

$$e_a(t) = e(t) + K_i\int e(t)\,dt$$

where K_i is a constant.

Taking Laplace transform, we obtain

$$E_a(s) = E(s) + K_i\frac{E(s)}{s}$$

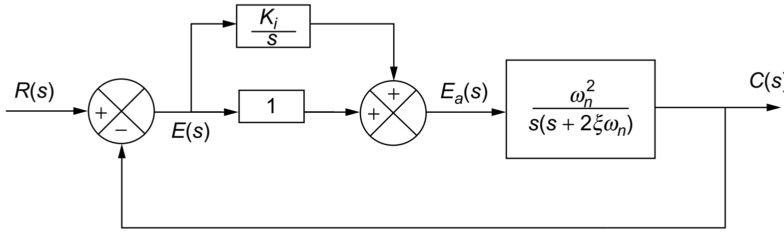


Fig. 5.23 | Proportional integral controller of second-order system

The closed-loop transfer function of the control system is given by

$$\frac{C(s)}{R(s)} = \frac{(s+K_i)\omega_n^2}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$$

The characteristic equation of the system with integral control is given by

$$s^3 + 2\xi\omega_n^2 + \omega_n^2 s + K_i\omega_n^2 = 0$$

The order of the system has been increased due to the inclusion of integral controller in the existing second-order system.

The forward path transfer function of the system shown in Fig. 5.23 is given by

$$G(s) = \frac{(s+K_i)\omega_n^2}{s^2(s+2\xi\omega_n)}$$

and the feedback path transfer function, $H(s) = 1$

Hence, the error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)}$$

$$\frac{E(s)}{R(s)} = \frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$$

For a unit-ramp input $r(t) = t, \quad R(s) = \frac{1}{s^2}$

Hence, $E(s) = \frac{1}{s^2}\frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$

The steady-state error is given by

$$e_{ss} = \underset{s\to 0}{Lt}\ sE(s) = 0$$

For unit-parabolic input, $r(t) = \frac{t^2}{2}$

$$E(s) = \frac{1}{s^3}\left[\frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}\right]$$

The steady-state error e_{ss} is given by $e_{ss} = \underset{s \to 0}{Lt}\ sE(s) = \dfrac{2\xi}{K_i\omega_n}$

It is clear that by introducing proportional integral controller in the system, the type and order of the system can been increased. Also, the steady-state error of the system by introducing proportional integral controller in the system has been decreased for certain types of inputs such as parabolic input.

5.10.3 Proportional Plus Integral Plus Derivative Control (PID Control)

The block diagram of a unity feedback system with proportional integral derivative controller is shown in Fig. 5.24. The system is a general second-order system. The actuating signal of the controller consists of sum of the control action given by the proportional, integral and derivative controllers. Hence, the actuating signal is given by

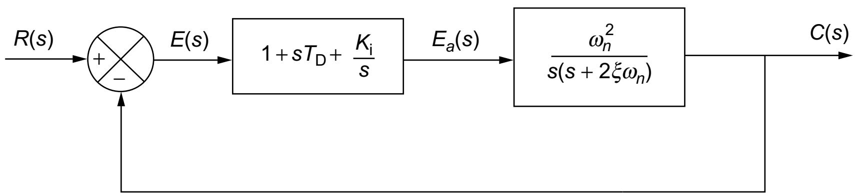


Fig. 5.24 | Proportional integral derivative controller of second-order system

The actuating signal for the PID control is given by $e_a(t) = e(t) + T_D\dfrac{de(t)}{dt} + K_i\int e(t)\,dt$

Taking Laplace transform, we obtain $E_a(s) = E(s)\left[1 + sT_D + \dfrac{K_i}{s}\right]$

From the above sections, we come to know that proportional derivative controller improves the transient response specifications (peak overshoot, settling time) and proportional integral controller improves the steady-state response specifications (steady-state error), but the proportional integral derivative controller that combines the actions of above-mentioned controllers helps in improving both the transient and steady-state response specifications.

Example 5.58: **The block diagram of a system with unity feedback is shown in Fig. E5.58. A derivative control is used in the given control system to make the damping ratio 0.8. Determine (i) T_D and (ii) compare the rise time, peak time and maximum overshoot of the system with and without the derivative control when the system is subjected to unit-step input.**

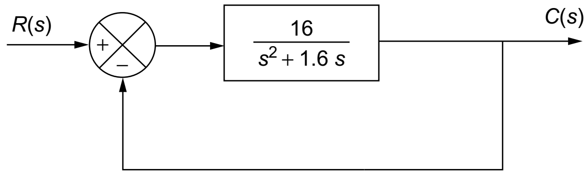
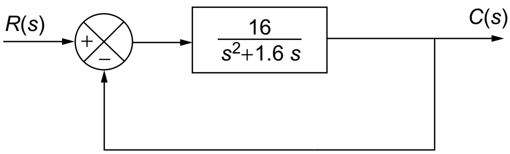
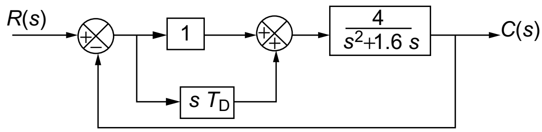


Fig. E5.58

Solution:

Without derivative control	**With derivative control**
	
The closed-loop transfer of the system is given by $$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{16}{s^2+1.6s+16}$$	The closed-loop transfer function of the system is given by $$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{16(1+sT_D)}{s^2+s(1.6+16T_D)+16}$$
Comparing the above equation with the standard form $\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, we obtain	Comparing the above equation with the standard form $\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, we obtain
$\omega_n^2=16$, $\omega_n=4$ and $2\xi\omega_n=16$, $\xi=0.2$	$\omega_n^2=16$, $\omega_n=4$ and $2\xi\omega_n=(1.6+16T_D)$.
Hence, the time-domain specifications for the system are given by	Here the damping ratio of the system with derivative control is 0.8. Therefore, $T_D=0.3$
Damped natural frequency	Damped natural frequency
$\omega_d=\omega_n\sqrt{1-\xi^2}=3.9192\,\text{rad / sec},$	$\omega_d=\omega_n\sqrt{1-\xi^2}=2.4\text{ rad/sec}$
$\theta=\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)=1.3694\text{ rad}$	$\theta=\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)=0.6435\,\text{rad}$
Delay time $t_d=\frac{1+0.7\xi}{\omega_n}=0.285\text{ sec}$	Delay time $t_d=\frac{1+0.7\xi}{\omega_n}=0.39\,\text{sec}$

(*Continued*)

(Continued)

Without derivative control	With derivative control
Peak time, $t_p = \dfrac{\pi}{\omega_d} = 0.8016$ sec	Peak time, $t_p = \dfrac{\pi}{\omega_d} = 1.3089\,\text{sec}$
Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d} = 0.4521$ sec	Rise time, $t_r = \dfrac{\pi - \theta}{\omega_d} = 1.041\,\text{sec}$
Maximum peak overshoot, $\%M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 52.662\%$	Maximum peak overshoot, $\%M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 1.516\%$

Example 5.59: The block diagram of the system shown in Fig. E5.59 has PD controller to control the time-domain specifications of the system. Determine T_d such that the system will be critically damped and also, calculate its settling time.

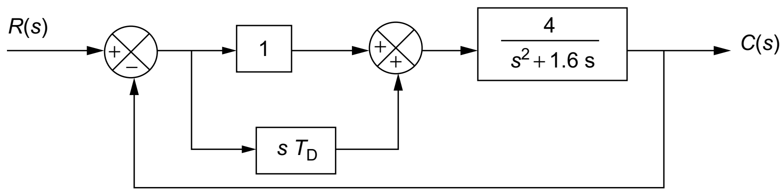


Fig. E5.59

Solution: The closed-loop transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{(1+sT_D)4}{s^2+1.6s+4T_Ds+4}$$

Comparing the above equation with the standard form $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, we obtain

$$\omega_n^2 = 4\,,\ \omega_n = 2 \text{ and } 2\xi\omega_n = (1.6+4T_D).$$

Since the damping ratio of the system with derivative control is given as 1 (For critically damped system) substituting 1 in the above equation we obtain $T_D = 0.6$

Hence, settling time is given by $t_s = \dfrac{4}{\xi\omega_n} = 2\,\text{sec}$

5.11 Performance Indices

The qualitative measure of the performance of a system is very important in designing a control system. Performance index is a qualitative measure of the performance of the system where certain parameters of the system can be varied for getting the optimum design and hence emphasizing on system specifications. A particular performance index referred as cost function should be chosen and has to be measured for obtaining the optimal design. There are many cost functions of which some are discussed below:

(i) Integral of the absolute magnitude of the error (IAE)

$$IAE = \int_0^\infty |e(t)|\ dt$$

where error $e(t) = r(t) - c(t)$.

The integral value is minimum for a damping factor of 0.7 for a second-order system.

(ii) Integral of the square of the error (ISE)

$$ISE = \int_0^\infty e^2(t)\ dt$$

The integral value is minimum for a damping factor of 0.5 for a second-order system.

(iii) Integral of time multiplied by the absolute value of error (ITAE), which is given by

$$ITAE = \int_0^\infty t|e(t)|\ dt$$

The integral value is minimum for a damping factor of 0.7 for a second-order system.

(iv) Integral of time multiplied by the squared error (ITSE)

$$\text{ITSE} = \int_0^\infty te^2(t)\ dt$$

(v) Integral of squared time multiplied by the absolute value of error square (ISTAE)

$$\text{ISTAE} = \int_0^\infty t^2|e(t)|\ dt$$

(vi) Integral of squared time multiplied by square error (ISTSE)

$$\text{ISTSE} = \int_0^\infty t^2e^2(t)\ dt$$

The last three performance indices are not used due to the difficulty existing in determining the values. Figure. 5.25 shows the performance indices of the system obtained using the response $c(t)$.

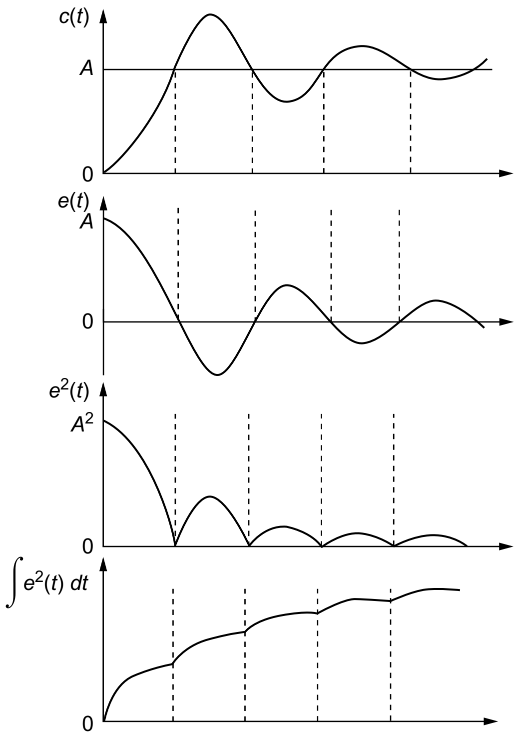


Fig. 5.25 | Performance indices obtained using the response of the system

Review Questions

1. What do you mean by time response of a system and what are its components?
2. Define transient response system and steady-state response of a system. In addition, mention the significance of these responses of the system.
3. Define the standard tests signals used for time-domain studies:
   (a) Impulse signal and unit-impulse signal, (b) step signal and unit-step signal, (c) ramp signal and unit-ramp signal and (d) parabolic signal.
4. What is the importance of impulse signal, step signal, ramp signal and parabolic signal?
5. What is the relation existing between the standard test signals?
6. Define the type and the order of a system.
7. What do you mean by first- and second-order systems? Give examples.
8. List out the performance indices used for evaluating the performance of first-order system.
9. Derive the response of a first-order system subjected to the following:
   (a) Impulse input and unit-impulse input, (b) step input and unit-step input and (c) ramp input and unit-ramp input.
10. Derive the performance indices of a first-order system when the system is excited by the following inputs: (a) Unit-impulse, (b) step input and (c) ramp input.

11. By deriving the unit-step response of a first-order system, obtain the response of the system when it is excited by (a) unit-impulse input and (b) unit-ramp input.
12. Sketch the responses of a first-order system subjected to a (a) unit-impulse input, (b) unit-step input and (c) unit-ramp input.
13. How is second-order system classified based on the damping ratio ξ ?
14. What are the effects of damping ratio on the response of a second-order system?
15. List out the performance indices used for evaluating the performance of a second-order system.
16. Define the following performance indices and give its expression:
    (a) Delay time, (b) rise time, (c) peak time, (d) maximum peak overshoot and (e) settling time.
17. Discuss the relationship between the maximum peak overshoot and damping ratio.
18. Derive the response of a second-order underdamped system when the system is subjected to the following inputs: (a) Unit-impulse (b) unit-step and (c) Unit-ramp
    Also, derive it for overdamped, undamped and critically damped cases of second-order system.
19. Derive the expression for a unit-impulse response and unit-ramp response of a standard second-order system using the unit-step response of the system.
20. Derive the expression for time-domain specifications for evaluating the performance of second-order underdamped system when subjected to the following inputs: (a) Unit-impulse, (b) unit-step and (c) unit-ramp. Also, derive it for overdamped, undamped and critically damped cases of second-order system.
21. What do you mean by steady-state error?
22. Discuss the concept of steady-state error of a system in terms of forward-path transfer function $G(s)$ of the system and in terms of closed-loop transfer function $T(s)$ of the system.
23. Discuss the concept of steady-state error when the system is excited by following inputs: (a) Unit-step, (b) unit-ramp and (c) unit-parabolic.
24. What are the types of static error constants?
25. What are the units of K_p, K_v and K_a ?
26. Give the relationship between the static error constants, standard test signals, steady-state error and TYPE of the system.
27. What is the significance of static error constants?
28. What do you mean by generalized or dynamic error constants and why are they required?
29. Discuss the concept of generalized error constants using a derivation.
30. How are the generalized or dynamic error constants and static error constants related?
31. Discuss the effect of adding a pole and a zero to an existing system.
32. How does the controller play a significant role in the time-domain analysis of the system?
33. How does the steady-state value of the system changes when the following controllers are added to the system: (a) PD controller, (b) PI controller and (c) PID controller
34. What do you mean by the performance indices of the system?
35. What are the different cost functions used in evaluating the performance of a system? Also, sketch the different cost functions.

36. The specification given on a certain second-order feedback control system is that overshoot of the step response should not exceed 25%. What are the corresponding limiting values of the damping ratio ξ and peak resonance M_r ?
37. Derive the error constants for step, ramp and parabolic inputs to a system. Also, deduce the effects of these constants on Type 0, 1, 2 systems.
38. The unit-step response of a system is found to produce an overshoot of 10%, rise time of 1 sec and zero steady-state error. Identify the system transfer function.
39. The open-loop transfer function of a closed-loop system with unity feedback is given. Evaluate the static error constants and estimate the steady-state errors for step, ramp and acceleration inputs.

(a) $G(s)=\dfrac{10}{s(0.1s+1)}$	(f) $G(s)H(s)=\dfrac{(s+2)}{s(1+0.5s)(1+0.2s)}$
(b) $G(s)=\dfrac{K}{s+1}$	(g) $G(s)=\dfrac{K(1+0.5s)(1+2s)}{s^2(s^2+4s+5)}$
(c) $G(s)=\dfrac{\frac{1}{3}}{s\left(\frac{1}{3}s+1\right)}$	(h) $\dfrac{10}{s(s+1)(s+2)}$
(d) $G(s)=\dfrac{20s}{s^2(s^2+3s+2)(s+4)}$	(i) $G(s)=\dfrac{20s}{s^2(s^2+3s+2)(s+4)}$
(e) $G(s)=\dfrac{K(s+2)}{s^2(s^2+7s+12)}$	(j) $G(s)=\dfrac{10}{s(0.1s+1)}$

40. The system shown in Fig. Q5.40 is a unity feedback control system with a minor feedback loop (output derivative feedback).

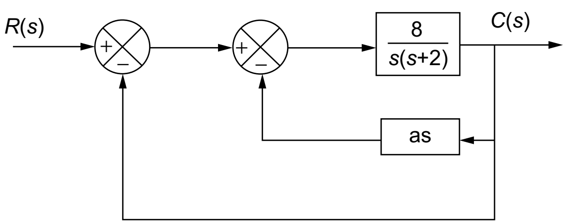


Fig. Q5.40

(a) In the absence of derivative feedback $(a=0)$, determine the damping ratio and natural frequency. Also, determine the steady-state error resulting from a unit-ramp input.

(b) Determine the derivative feedback control a, which will increase the damping ratio of the system to 0.7. What is the steady-state error to unit-ramp input with this setting of the derivative feedback constant.

41. The value of data of the components of a servomechanism are $J = 18 \times 10^{-6}$ slug-ft^2, $F = 102 \times 10^{-6}\, lb.ft/rad/sec$, $\xi = 0.35$, $e_{ss} = 0.5\text{deg}$ and input speed is 10 rpm. Determine (a) forward loop gain K and (b) change in the friction coefficient value to maintain the damping value as 0.358.
42. A unity feedback second-order servo system has the transient response curve as shown in Fig. Q5.42 given below for unit-step input. The motor torque is proportional to the error. For constant velocity input of 1 rad/sec, a position lag error 2° is observed. The steady-state error for step input is 0.1 rad for each lb.ft of applied torque. Find (a) ω_n, (b) ξ, (c) J and (d) F.

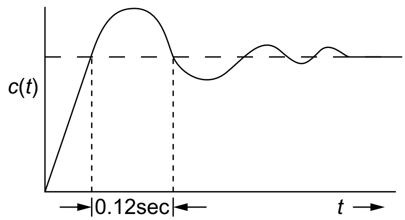


Fig. Q5.42

43. Estimate the static error constants and steady-state error for the systems with unity feedback for which the input to the system and the forward-path transfer function of the system are given in Table Q5.43.

Table Q5.43

Sl. No	System transfer function	System input
1	$G(s) = \dfrac{100}{s^2(1+2s)(1+5s)}$	$r(t) = 5 + 3t + 2t^2$
2	$G(s) = \dfrac{20}{s(1+0.01s)}$	$r(t) = 1 + t + t^2$
3	$G(s) = \dfrac{100}{s^2(1+2s)(1+5s)}$	$r(t) = 5 + 3t + 2t^2$
4	$G(s) = \dfrac{s}{s+1}$	$r(t) = 1 + 3t + \dfrac{5t^2}{2}$
5	$G(s) = \dfrac{20}{s(0.01s+1)}$	$r(t) = 1 + t + t^2$
6	$G(s) = \dfrac{10}{s(0.1s+1)}$	$r(t) = a_0 + a_1 t + a_2 t^2$
7	$G(s) = \dfrac{s}{s+1}$	$r(t) = 1 + 3t + \dfrac{5t^2}{2}$

(*Continued*)

Table Q5.43 | (Continued)

Sl. No	System transfer function	System input
8	$G(s)=\dfrac{20}{s(0.01s+1)}$	$r(t)=1+t+t^2$
9	$G(s)=\dfrac{40}{s(0.2s+1)}$	$r(t)=4t^2$
10	$G(s)=\dfrac{40}{s(0.2s+1)}$	$r(t)=(3+4t)t$

44. The open-loop transfer functions for different systems with unity feedback are given in Table Q5.44. The inputs to be applied to the system are also given. Determine the generalized error coefficients and the steady-state error.

Table Q5.44

Sl. No	Transfer function of the system	Input to be applied
1	$G(s)=\dfrac{10}{s+1}$	$r(t)=t.$
2	$G(s)=\dfrac{50}{s(1+0.1s)}$	$r(t)=1+2t+t^2$
3	$G(s)=\dfrac{10}{s(0.1s+1)}$	$r(t)=A_0+A_1t+\dfrac{A_2}{2}t^2$
4	$G(s)=\dfrac{K}{s+1}$	$r(t)=1.$
5	$G(s)=\dfrac{150}{s(0.25s+1)}$	$r(t)=(1+t^2)u(t)$

45. For a second-order system whose open-loop transfer function $G(s)=\dfrac{4}{s(s+2)}$, determine the maximum overshoot and the rise time to reach the maximum overshoot when a step displacement of $18°$ is given to the system. Find the rise time and the setting time for an error of 7% and the time constant.

46. Calculate the steady-state error of a TYPE 0 system having unit-step input and position error constant of $\dfrac{1}{19}$.

47. Consider a unity feedback system with a closed-loop transfer function $\dfrac{C(s)}{R(s)}=\dfrac{(Ks+b)}{s^2+as+b}$. Determine the open-loop transfer function $G(s)$. Show that the steady-state error with unit-ramp input is $(a-K)/b$.

48. Determine a unit-impulse response of a second-order system whose transfer function: (a) $G(s)=\dfrac{9}{s^2+4s+9}$ and (b) $F(s)=\dfrac{A}{s-a}$.
49. The open-loop transfer function of a unity feedback system is given by $G(s)=\dfrac{2}{s(s+3)}$. Determine the response of the system subjected to unit-step input.
50. Derive the time response of a system with $\dfrac{C(s)}{R(s)}=\dfrac{K}{Js^2+Bs+K}$, subjected to a step input, assuming the system is underdamped.
51. A unity feedback system has a loop transfer function $\dfrac{K}{s(s+1)(s+2)}$. The steady-state error for a ramp input of $r(t)=10t$ is required to be less than 1.0. Find K.
52. The open-loop transfer function of a unity feedback control system is $\dfrac{K}{s(1+T_s)}$. By what factor should K be multiplied so that the overshoot of unit-step response is reduced from 80% to 20%?
53. Determine the TYPE and ORDER of the following systems:

    (a) $G(s)H(s)=\dfrac{K(1+0.5s)}{s(1+s)(1+2s)}$, (b) $G(s)H(s)=\dfrac{(s+4)}{(s-2)(s+0.25)}$, (c) $G(s)=\dfrac{35(s+4)}{s(s+2)(s+5)}$,

    (d) $F(s)=\dfrac{500(s+10)}{s(s^2+10s+100)}$ and (e) $G(s)H(s)=\dfrac{(s+4)}{(s+2)(s+0.25)}$.
54. The transfer function of a system with unity feedback is given below. Determine the time response of the system and also, determine the rise time, settling time and peak overshoot of the system subjected to unit-step input.

    (a) $G(s)=\dfrac{(0.4s+1)}{s(s+0.6)}$, (b) $G(s)=\dfrac{2}{s(s+1)}$, (c) $\dfrac{C(s)}{R(s)}=\dfrac{25}{s^2+6s+25}$, (d) $G(s)=\dfrac{44.5}{(s+2)(s+5)}$,

    (e) $G(s)=\dfrac{(0.4s+1)}{s(s+0.6)}$, (f) $\dfrac{C(s)}{R(s)}=\dfrac{25}{s^2+6s+25}$, (g) $G(s)=\dfrac{10}{(s+2)(s^2+5s+6)}$,

    (h) $\dfrac{C(s)}{R(s)}=\dfrac{1000}{(s^2+34.5s+1000)}$ and (i) $G(s)=\dfrac{2}{s(s+3)}$.
55. The characteristic equation of a second-order system is given by $s^2+6s+36=0$. Estimate the peak overshoot for a unit-step input.
56. Determine the current $i(t)$ flowing through the series RL circuit shown in Fig. Q5.56.

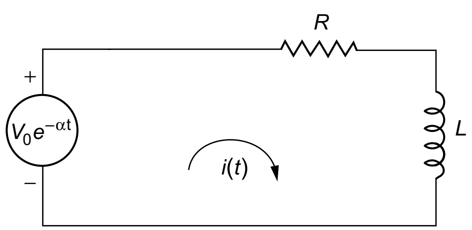


Fig. Q5.56

57. Determine the current $i(t)$ flowing through the series RL circuit shown in Fig. Q5.57 when the circuit is excited by the following inputs: (i) $\sin\omega t$, (ii) $\cos\omega t$, (iii) $V_m \cos(\omega t + \theta)$ and (iv) $V_m \sin(\omega t + \theta)$.

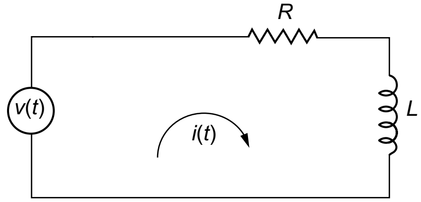


Fig. Q5.57

58. Determine the current $i(t)$ flowing through the series RL circuit shown in Fig. Q5.58(b) when a triangular wave shown in Fig. Q5.58(a) is applied as an input.

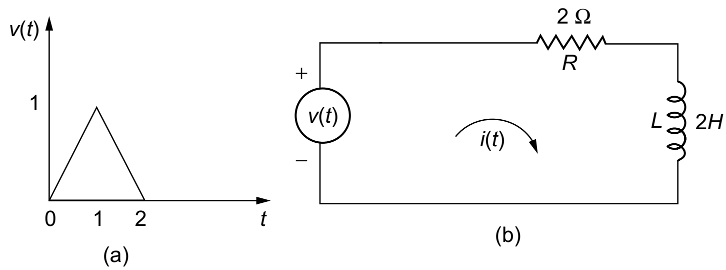


Fig. Q5.58

59. Determine the current $i(t)$ flowing through the series RC circuit shown in Fig. Q5.59 when the circuit is excited by the following inputs: (i) $\sin\omega t$, (ii) $\cos\omega t$, (iii) $V_m\cos(\omega t + \theta)$ and (iv) $V_m\sin(\omega t + \theta)$.

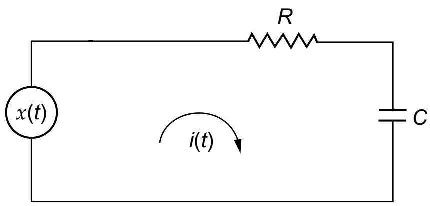


Fig. Q5.59

60. Determine the step and impulse responses of the series RC circuit shown in Fig. Q5.59.

61. Determine the step and impulse responses of the parallel RLC circuit shown in Fig. Q5.61.

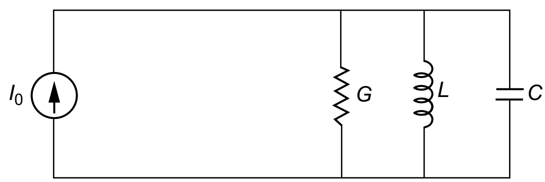


Fig. Q5.61

6

STABILITY AND ROUTH–HURWITZ CRITERION

6.1 Introduction

A system is said to be stable if it does not exhibit large changes in its output for a small change in its input, initial conditions or its system parameters. In a stable system, the output is predictable and finite for a given input. The definition of stability depends on the type of system. Generally, the stability of a system is classified as *stable*, *unstable* and *marginally stable*.

A linear time-invariant system is said to be stable if the output remains bounded when the system is excited by a bounded input. This is called the *Bounded-Input Bounded-Output (BIBO) stability criterion*. Further,

(i) When there is no input, the system should produce a zero output irrespective of initial conditions. This is called the *asymptotic stability criterion*.
(ii) If a bounded input is applied, the system remains stable for all values of system parameters. This is called the *absolute stability.*
(iii) The stability of a system that exists for a particular range of parameters is called *conditional stability*.
(iv) The relative stability indicates how close the system is to instability.
(v) Limitedly stable system produces output that has constant amplitude of oscillations.

The system can also be classified as Single-Input Single-Output (SISO), Multi-Input Multi-Output (MIMO), linear, nonlinear, time-invariant and time variant systems. In this chapter, the stability of linear SISO time-invariant systems has been discussed.

6.2 Concept of Stability

Let us consider the following two practical examples to have a clear idea on the stability of a system before going into the stability of linear SISO time-invariant systems.

Practical Example 1: Consider a ball that rests on different types of surfaces under the influence of the gravitational force as shown in Fig. 6.1.

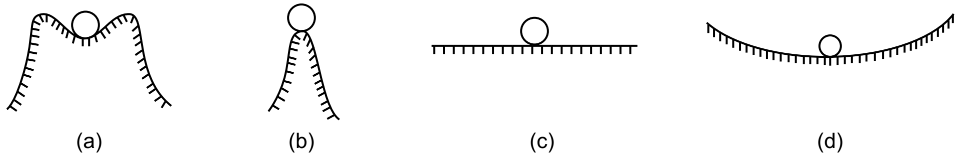


Fig. 6.1 | Different equilibrium conditions: (a) Stable (asymptotic), (b) Unstable, (c) Marginally stable (non-asymptotic) and (d) Oscillatory

Stable System: A system is said to be stable if it maintains its equilibrium position (original position) when a small disturbance is applied to it. As shown in Fig. 6.1(a), even if the ball is disturbed, the equilibrium position will not change. Hence, the system is called *asymptotically stable*.

Unstable System: A system is said to be unstable if it attains a new equilibrium position that does not resemble the original equilibrium position when a small disturbance is applied to it. As shown in Fig. 6.1(b), even if the ball is slightly disturbed, it will upset the equilibrium condition and the ball will roll down from the hill and it will never return back to the equilibrium state. Hence, the system is called *unstable*.

Marginally Stable System: A system is said to be marginally stable if it attains a new equilibrium position that resembles the original equilibrium position when a small disturbance is applied to it. As shown in Fig. 6.1(c), when an impulsive force is applied, the ball will move for a fixed distance and then stop due to friction. Although the position is changed, the ball will come to a new equilibrium position. Hence, the system is called *marginally or non-asymptotically stable*.

Oscillatory System: As shown in Fig. 6.1(d), when we apply a disturbing force, the ball will move in both the directions (oscillatory motion). Eventually, these oscillations will cease due to the presence of friction and the ball will return back to its stable equilibrium state.

Practical Example 2: A conical structure shown in Fig. 6.2 is used to explain the concept of stability.

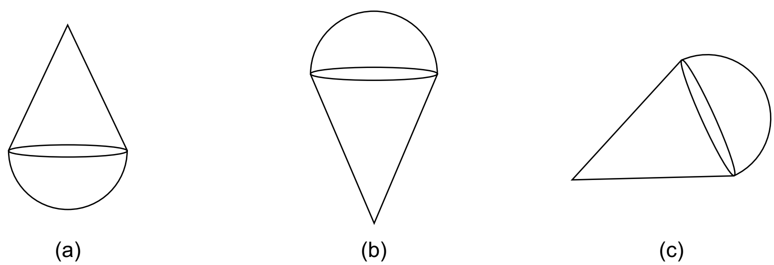


Fig. 6.2 | Classification of system stability: Conical structure placed (a) on its base (stable), (b) on its tip (unstable) and (c) on its horizontal side (marginally stable)

Stable System: Consider a conical structure placed on its base as shown in Fig. 6.2(a). When a small disturbance is applied to the cone, a slight displacement can be observed before the cone returns back to its original position. Since the cone comes back its original position, it is said to be in the stable state.

Unstable System: Consider a conical structure placed on its tip as shown in Fig. 6.2(b). When a small disturbance is applied to the cone, it falls and rolls. Since the cone attains a new equilibrium point that does not resemble its original equilibrium point, it is said to be in the unstable state.

Marginally Stable System: Consider a conical structure placed on a horizontal surface as shown in Fig. 6.2(c). When a small disturbance is applied to the cone, a slight displacement can be observed before the cone attains a new position (equilibrium point). Since the new equilibrium point resembles the original equilibrium point, it is said to be in marginally stable state.

6.3 Stability of Linear Time-Invariant System

The most important specification of a system is its stability. Stability of the system depends on the total response of the system. The total response of the system is the summation of forced and natural responses of the system and is given by

$$c(t) = c(t)_{\text{forced}} + c(t)_{\text{natural}}$$

The stability of the system can be defined based on either the natural response of the system or the total response of the system.

6.3.1 Stability Based on Natural Response of the System, $c(t)_{\text{natural}}$

The stability of the linear time invariant (LTI) system based on the natural response of the system is shown in Table 6.1. It is clear that for a stable LTI system, only the forced response remains as it is when the natural response approaches zero as time tends to infinity.

6.3.2 Stability Based on the Total Response of the System, $c(t)$

The above definition for the stability of the system is based on the natural response of the system. But when the total response of the system is given, it is very difficult to get the natural and forced responses separately. The stability of the system based on the total response is dependent on the input applied to the system and is shown in Table 6.2.

The output or the total response of the system is said to be bounded, if the output or the total response of the system has a finite area under the curve. In addition, the output or the total response of the system is said to be unbounded, if the output or the total response of the system has an infinite area. The bounded and unbounded outputs are graphically shown in Table 6.2.

When the input to the system is unbounded, the output or the total response of the system is also unbounded. But from the total response of the system, it is very difficult to suggest whether the natural response of the system is unbounded or the forced response of the system is unbounded. Hence, for an unbounded input, we cannot suggest whether the system is stable, unstable or marginally stable.

Table 6.1 | Classification of stability of LTI system

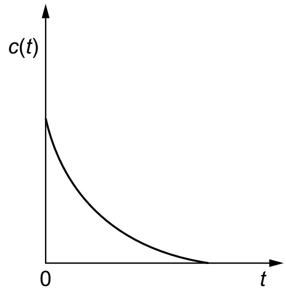
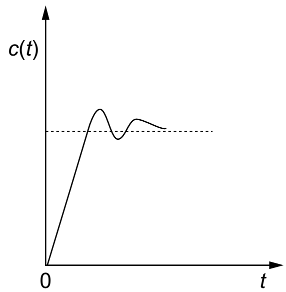
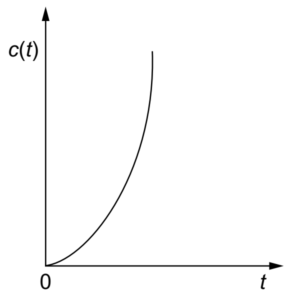
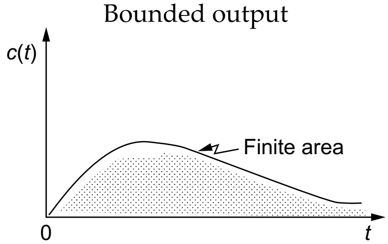
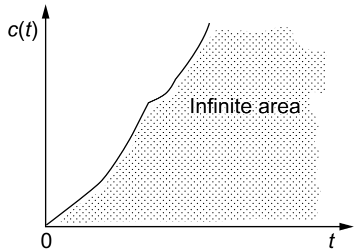
Natural Response of the System, $c(t)_{\text{natural}}$	**Nature of the System**	**Response of the System**
$c(t)_{natural} \to 0$ as $t \to \infty$	Stable	
$c(t)_{natural} \to$ grows without bound as $t \to \infty$	Unstable	
$c(t)_{natural} \to$ either decays or grows as $t \to \infty$	Marginally stable	

Table 6.2 | Stability classification based on input and total response

Input Applied to the System *u(t)*	**Output or Total Response of the System *c(t)***	**Stability of the System**
Bounded input	Bounded output 	Stable

(*Continued*)

Table 6.2 | (Continued)

Bounded input	Unbounded output  	Unstable

Thus, the above examples define the stability of the system based on the bounded input and bounded output. Hence, this is known as BIBO stability.

6.4 Mathematical Condition for the Stability of the System

Let $R(s)$, $C(s)$ and $G(s)$ be the input, output and the transfer function of the LTI SISO system respectively. The transfer function of the system is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

For the impulse input, $R(s) = 1$. Therefore, $C(s) = G(s)$. Taking inverse Laplace transform, we have $c(t) = g(t)$. For any input $r(t)$, the output $c(t)$ can be represented in the form of convolution as

$$c(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau \tag{6.1}$$

where $g(\tau)$ is the impulse response of the system output. If $r(t)$ and $c\ (t)$ remain bounded, i.e., finite for all t, then

$$\int_0^\infty g(\tau) d\tau = \text{finite}$$

The above equation may mathematically be derived as follows:

Taking absolute value on both sides of Eqn. (6.1), we obtain

$$|c(t)| = \left| \int_0^\infty g(\tau) r(t-\tau) d\tau \right|$$

According to an axiom, the absolute value of an integral cannot be greater than the integral of the absolute value of the integrand. Hence,

$$|c(t)| \le \int_0^\infty |g(\tau) r(t-\tau) d\tau|$$

$$\leq \int_0^\infty |g(\tau)||r(t-\tau)|d\tau$$

If $r(t)$ is bounded, then

$$|r(t) \leq M < \infty| \text{ for } t \geq t_0$$

$$|c(t)| \leq \int_0^\infty M|g(\tau)|d\tau$$

$$\leq \int_0^\infty |g(\tau)|d\tau$$

If $c(t)$ remains bounded, then

$$|c(t)| \leq N < \infty \text{ for } t \geq t_0$$

Combining the above two equations, we obtain

$$\int_0^\infty |g(\tau)|d\tau \leq P < \infty$$

Hence,

$$\int_0^\infty |g(\tau)|d\tau = \text{finite}$$

6.5 Transfer Function of the System, *G*(*s*)

From Eqn. (6.1), it is clear that the mathematical condition for a stable system depends on the nature of $g(t)$. But the nature of $g(t)$ depends on the poles of the transfer function of the system $G(s)$ or the roots of the characteristic equation of the system. The roots of the characteristic equation may be purely real, purely imaginary or complex conjugate with single root or multiple roots. In the section that follows, we will be discussing about the response of the system when an impulse signal is applied to the system.

6.5.1 Effects of Location of Poles on Stability

Single Pole at the Origin: When there is a single pole at the origin, the transfer function is given by

$$\frac{C(s)}{R(\mathrm{s})} = \frac{1}{s}$$

For impulse response, $R(\mathrm{s}) = 1$

Then, $C(s) = \frac{1}{s}$

Therefore, $c(t) = 1$

The location of the pole in the s-plane and its corresponding impulse response of the system having a single pole at the origin are shown in Fig. 6.3(a) and (b).

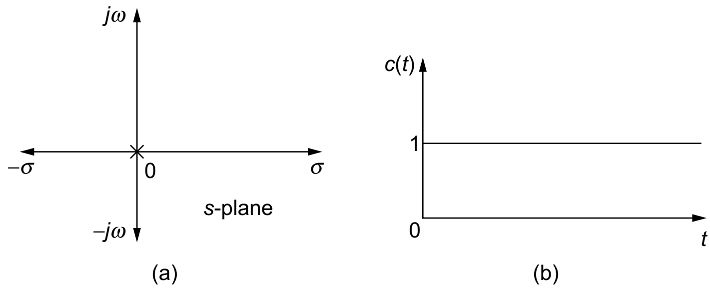


Fig. 6.3 | (a) Location of pole in the s-plane and (b) Impulse response of the system having a single pole at the origin

A Pair of Poles at the Origin: When a pair of poles is located at the origin, the transfer function is

$$\frac{C(s)}{R(s)} = \frac{1}{s^2}$$

When $R(s) = 1$, the corresponding impulse response is the ramp function, i.e.,

$$c(t) = t$$

The location of poles in the s-plane and its corresponding impulse response of the system having a pair of poles at the origin are shown in Fig. 6.4(a) and (b).

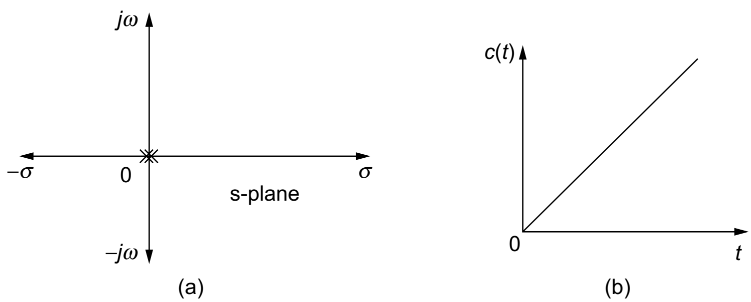


Fig. 6.4 | (a) Location of poles in the s-plane and (b) Impulse response of the system having a pair of poles at the origin

Single Real Pole at the Right Half of the s-Plane: Let us consider the simple transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{s-1}$$

When $R(s) = 1$, the impulse response is $c(t) = e^t$.

The location of pole in the s-plane and its corresponding impulse response for the system having a single pole at the right half of the s-plane are shown in Fig. 6.5(a) and (b). The response clearly indicates that the output is not bounded and thus the system is unstable.

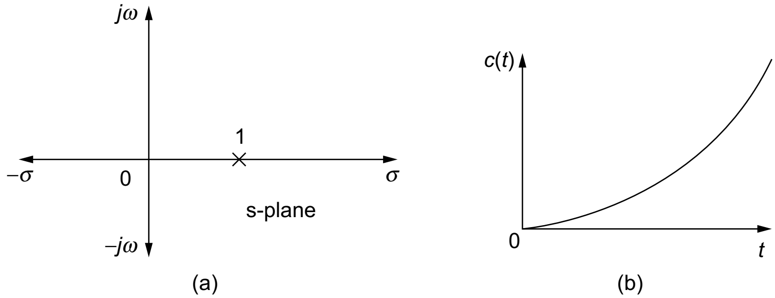


Fig. 6.5 | (a) Location of a single pole in the right half of the s-plane and (b) Impulse response of the system having a single pole in the right half of the s-plane

Single Real Pole at the Left Half of the s-Plane: Let us consider the simple transfer function as

$$\frac{C(s)}{R(s)} = \frac{1}{s+1}$$

When $R(s) = 1$, the impulse response is $c(t) = e^{-t}$.

The location of pole in the s-plane and its corresponding impulse response for having a single pole at the left half of the s-plane are shown in Fig. 6.6(a) and (b). The response clearly indicates that the output decays to zero and the system is stable.

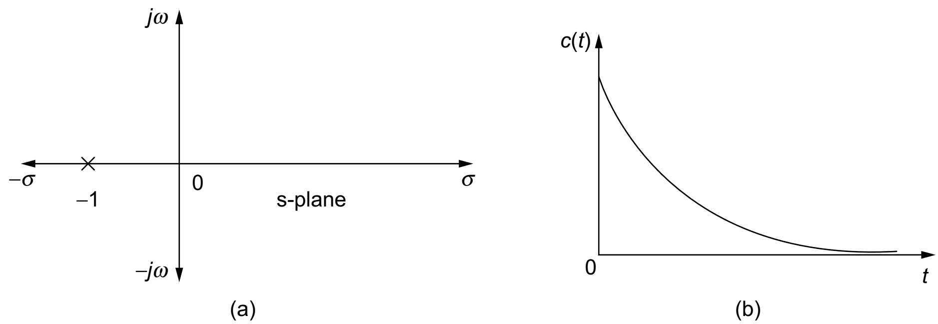


Fig. 6.6 | (a) Location of a single pole in the left half of the s-plane and (b) Impulse response of the system having a single pole in the left half of the s-plane

A Pair of Poles on the Imaginary Axis: A transfer function of the system having a pair of poles on the imaginary axis is given by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2+1}$$

When $R(s) = 1$, the impulse response is $c(t) = \sin t$.

The location of poles in the s-plane and its corresponding impulse response are shown in Fig. 6.7(a) and (b). The response, which oscillates between +1 and −1, indicates that the system is marginally stable.

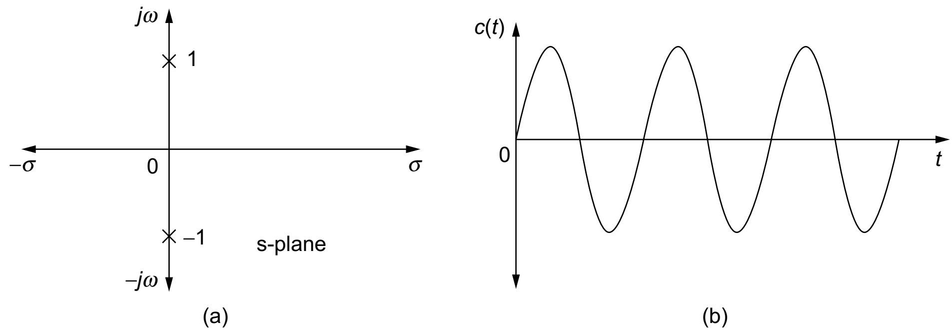


Fig. 6.7 | (a) Location of poles in the s-plane and (b) Impulse response of the system having a pair of poles on the imaginary axis

Two Pairs of Repeated Poles on the Imaginary Axis: Let the transfer function be

$$\frac{C(s)}{R(s)} = \frac{1}{\left(s^2+1\right)^2}$$

When $R(s) = 1$, the impulse response is $c(t) = t\sin t$.

The location of poles in the s-plane and its corresponding impulse response when there are two pairs of repeated poles on the imaginary axis are shown in Fig. 6.8(a) and (b). Here, the system is unstable if it possesses two pairs of repeated poles on the imaginary axis.

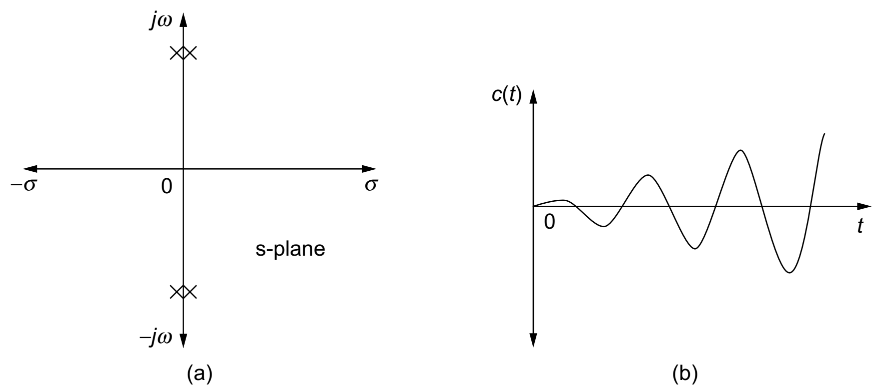


Fig. 6.8 | (a) Location of poles in the s-plane and (b) Impulse response of the system when there are two pairs of repeated poles on the imaginary axis

A Pair of Complex Poles in the Right Half of the *s*-Plane: Let the transfer function be

$$\frac{C(s)}{R(s)} = \frac{1}{(s-1)^2+1}$$

When $R(s)=1$, the impulse response is $c(t)=e^{t}\sin t$.

The location of poles in the s-plane and its corresponding impulse response when there are repeated poles in the right half of the s-plane are shown in Fig. 6.9(a) and (b). The output clearly shows that the system having this kind of transfer function is unstable.

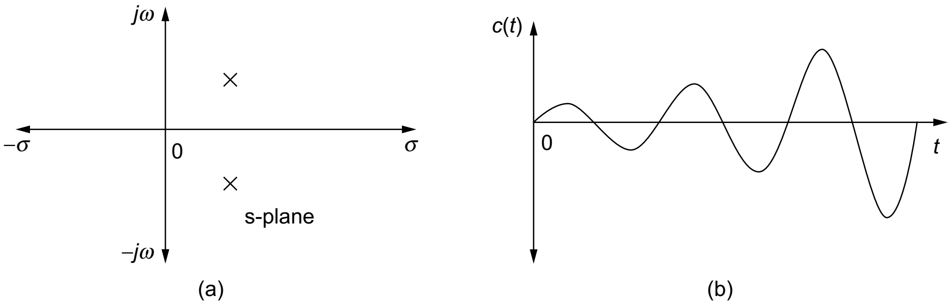


Fig. 6.9 | (a) Location of poles in the s-plane and (b) Impulse response of the system when there are repeated poles in the right half of the s-plane

A Pair of Complex Poles in the Left Half of the s-Plane: Let the transfer function be

$$\frac{C(s)}{R(s)}=\frac{1}{(s+1)^2+1}$$

When $R(s)=1$, the impulse response is $c(t)=e^{-t}\sin t$.

The location of poles in the s-plane and its corresponding impulse response when there are repeated poles in the left half of the s-plane are shown in Fig. 6.10(a) and (b). The output clearly shows that the system having this kind of transfer function is stable.

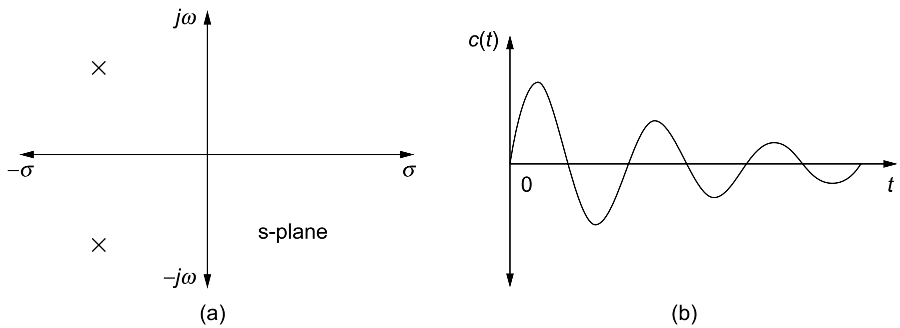
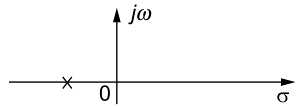
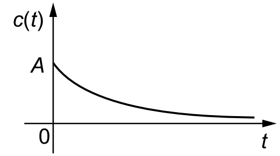
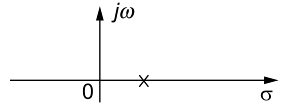
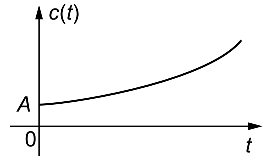
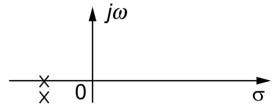
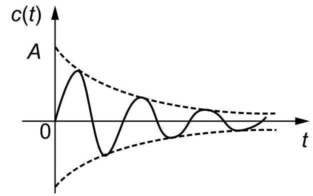
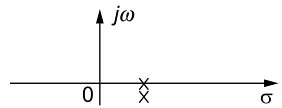
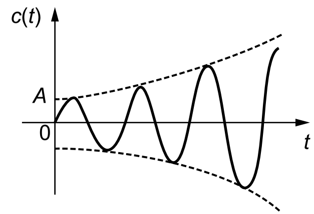


Fig. 6.10 | (a) Location of poles in the s-plane and (b) Impulse response of the system when there are repeated poles in the left half of the s-plane

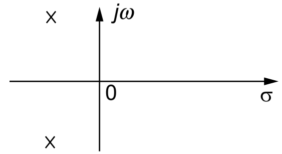
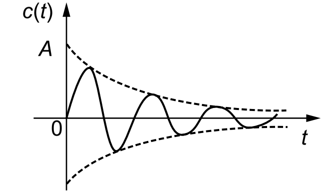
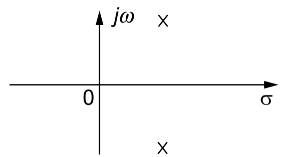
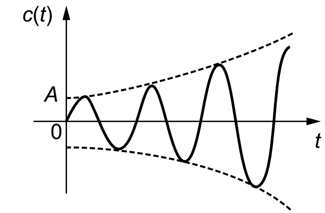
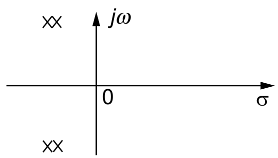
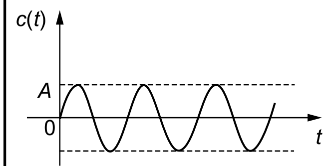
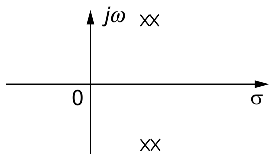
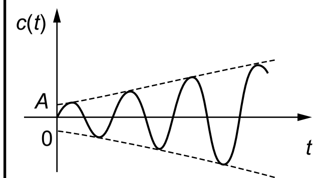
The mathematical expression for the response of a system and the graphical representation of the output response of the system corresponding to different types of roots are shown in Table 6.3.

Table 6.3 | Nature of poles and the corresponding output responses of the system

Nature of Poles in the *s*-plane		Mathematical Expression	Location of Poles in the *s*-plane	Output Response of the System $c(t)$
Single real root at $s = \pm\sigma$	At $s = -\sigma$	$c(t) = Ae^{-\sigma t}$		
	At $s = \sigma$	$c(t) = Ae^{\sigma t}$		
Multiple real roots at $s = \pm\sigma$	At $s = -\sigma$	$c(t) = \begin{pmatrix} A_1 + A_2 t \\ + \cdots + A_k t^{k-1} \end{pmatrix} e^{-\sigma t}$ where k is the number of roots		
	At $s = \sigma$	$c(t) = \begin{pmatrix} A_1 + A_2 t \\ + \cdots + A_k t^{k-1} \end{pmatrix} e^{\sigma t}$ where k is the number of roots		

(*Continued*)

Table 6.3 | (Continued)

Nature of Poles in the *s-plane*		**Mathematical Expression**	**Location of Poles in the *s*-plane**	**Output Response of the System $c(t)$**
Complex roots at $s = \pm\sigma \pm j\omega$	At $s = -\sigma \pm j\omega$	$c(t) = Ae^{-\sigma t}\sin(\omega t + \beta)$		
	At $s = \sigma \pm j\omega$	$c(t) = Ae^{\sigma t}\sin(\omega t + \beta)$		
Multiple complex roots at $s = \pm\sigma \pm j\omega$	At $s = -\sigma \pm j\omega$	$c(t) = \begin{pmatrix} A_1 \sin(\omega t + \beta_1) + \\ A_2 t \sin(\omega t + \beta_2) + \cdots \\ A_k t^{k-1} \sin(\omega t + \beta_k) \end{pmatrix} e^{-\sigma t}$ where k is the number of roots		
	At $s = \sigma \pm j\omega$	$c(t) = \begin{bmatrix} A_1 \sin(\omega t + \beta_1) + \\ A_2 t \sin(\omega t + \beta_2) + \cdots + \\ A_k t^{k-1} \sin(\omega t + \beta_k) \end{bmatrix} e^{\sigma t}$ where k is the number of roots		

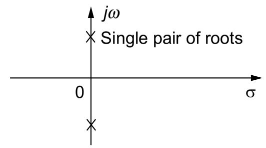
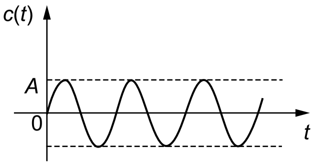
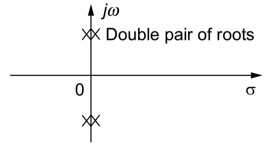
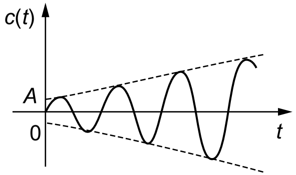
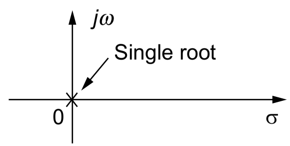
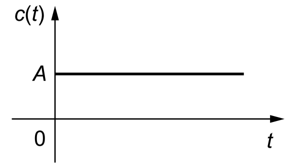
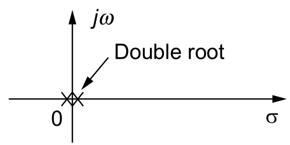
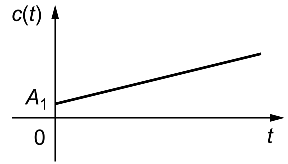
Single imaginary root	At $s = \pm j\omega$	$c(t) = A\sin(\omega t + \beta)$		
Multiple imaginary roots	At $s = \pm j\omega$	$c(t) = \begin{bmatrix} A_1\sin(\omega t + \beta_1) + \\ A_2 t\sin(\omega t + \beta_2) + \cdots + \\ A_k t^{k-1}\sin(\omega t + \beta_k) \end{bmatrix}$ where k is the number of roots		
Single root at the origin	At $s = 0$	$c(t) = A$		
Multiple roots at the origin	At $s = 0$	$c(t) = (A_1 + A_2 + \cdots + A_k t^{k-1})$ where k is the number of roots		

Table 6.4 lists the conditions for stability of the system based on the location of poles in the s-plane.

Table 6.4 | Location of poles in the s-plane and their stability

Sl. No.	Location of Poles in the s-plane	Nature of Stability
1	Left half of the *s*-plane	Stable
2	Right half of the *s*-plane	Unstable
3	Conjugate pair of poles on the imaginary axis	Continuous oscillation and hence marginally stable
4	Repeated poles on the imaginary axis	Unstable
5	Repeated poles at the origin	Unstable

It is evident that the roots of the characteristic equation, i.e., the poles of the transfer function *G(s)* should lie in the left half of the *s*-plane for the system to be stable, or on the imaginary axis for the system to be marginally stable.

The conditions for stable and unstable regions in the *s*-plane are graphically shown in Fig. 6.11.

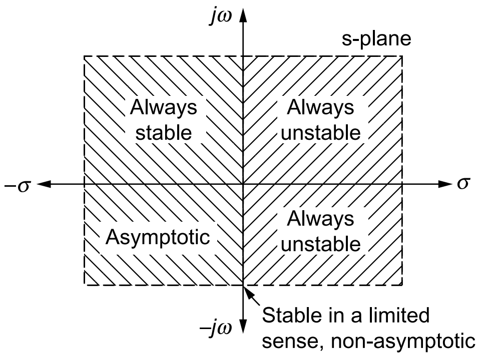


Fig. 6.11 | Stable and unstable regions in the *s*-plane

6.6 Zero-Input Stability or Asymptotic Stability

Condition of asymptotic stability is defined by the following two responses of LTI system:

(i) **Zero-State Response**: Response of a system due to input alone with all the initial conditions of the system assigned to zero is known as zero-state response.

(ii) **Zero-Input Response**: Response of a system due to initial conditions alone when the input of the system is assigned to zero is known as zero-input response. When the system is subjected to both input and initial conditions, the total response of the system can be obtained by using the principle of superposition as follows:

Total response = zero-state response + zero-input response.

Zero-Input Stability

Condition for stability of a system when the input is zero and when it is driven only by its initial conditions is known as *zero-input stability* or *asymptotic stability*. The zero-input stability also depends on the poles of the transfer function *G(s)*. The output of the system due to initial condition when the input of n^{th} order system is assumed to be zero is given by

$$c(t) = \sum_{k=0}^{n-1} g_k(t)\, y^{(k)}(t_0) \tag{6.2}$$

where
$$y^{(k)}(t_0) = \left.\frac{d^k y(t)}{dt^k}\right|_{t=0}$$

and $g_k(t)$ is the zero-input response due to $y^{(k)}(t_0)$.

A system is said to be zero-input stable if the zero-input response of the system $y(t)$, is subjected to finite initial conditions and $y^{(k)}(t_o)$ reaches zero as the time approaches infinity.

The condition for the system to be zero-input stable is given by

(i) $|y(t)| \le M < \infty$, for $t \ge t_o$, where M is a positive number.

(ii) $\underset{t \to \infty}{Lt} |y(t)| = 0$

From the second condition, it is clear that the zero-input stability is also known as asymptotic stability.

Taking absolute value on both sides of Eqn. (6.2), we obtain

$$|c(t)| = \left|\sum_{k=0}^{n-1} g_k(t)\, y^{(k)}(t_0)\right| \le \sum_{k=0}^{n-1} |g_k(t)|\,|y^{(k)}(t_0)|$$

As all the initial conditions are finite, the required condition for the system to be stable is given by

$$\sum_{k=0}^{n-1} |g_k(t)| < \infty, \ \ for \ \ t \ge 0$$

6.6.1 Importance of Asymptotic Stability

(i) The asymptotic stability depends on the poles of the transfer function of the system.
(ii) The dependence of BIBO stability condition based on the location of poles of the system is applicable to asymptotic stability.
(iii) If a system is BIBO stable, then it must be asymptotically stable.

Example 6.1: The transfer function of a closed-loop system is $\dfrac{2s^2+6s+5}{(s+1)^2(s+2)}$. Find the characteristic equation of the system.

Solution: The denominator of transfer function when equated to zero is called the characteristic equation of the system. Hence, the characteristic equation of the system is $(s+1)^2(s+2)=0$.

Example 6.2: Comment on the stability of the systems whose transfer functions are given by

(i) $G(s)=\dfrac{5(s+2)}{(s+1)(s+4)(s+5)}$

(ii) $G(s)=\dfrac{8(s-2)}{(s+1)(s^2+4)}$

(iii) $G(s)=\dfrac{15}{(s^2+1)^2(s+4)}$

(iv) $G(s)=\dfrac{20}{(s-3)(s+4)}$

(v) $G(s)=\dfrac{10}{s(s+4)}$

Solution:

(i) Poles are at $s=-1, s=-4$ and $s=-5$. Hence, the system is asymptotically stable or simply stable.
(ii) Poles are at $s=-1$ and $s=\pm j2$. Hence, the system is marginally stable.
(iii) Poles are at $s=-4$ and $s=\pm j1$ (double). Hence, the system is unstable.
(iv) Poles are at $s=3$ and $s=-4$. Hence, the system is unstable.
(v) Poles are at $s=0$ and $s=-4$. Hence, the system is stable.

6.7 Relative Stability

Based on the location of poles of the transfer function, we can determine whether the system is stable, unstable or marginally stable. In practical systems, it is not sufficient to know that the system is stable, but a stable system must meet the requirements on relative stability which is a quantitative measure of how fast the transients die out in the system.

Relative stability is measured by relative settling times of each root or a pair of roots. A system is said to be relatively more stable if the settling time of the system is less when compared to other systems. The settling time of the system varies inversely to the real part of the roots. Hence, the roots located nearer to the imaginary axis will have a larger settling time and hence the system is relatively less stable.

For a system, the relative stability improves as the closed-loop poles of the transfer function moves away from the imaginary axis in the left half of the s-plane as shown in Fig. 6.12.

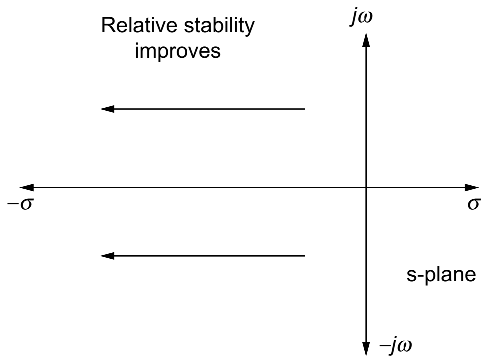


Fig. 6.12 | Relation between the stability and closed-loop poles

Example 6.3: Consider the system, $G(s)H(s) = \dfrac{A}{(s+p)}$ where p is the real pole. Assume two real poles with values p_1 and p_2 where $|p_2| > |p_1|$. Give your comments on their relative stability.

Solution: Given two poles with values p_1 and p_2 where $p_1 > p_2$, the values of poles p_1 and p_2 are plotted in the s-plane as shown in Fig. E6.3(a). For each value of the pole, when substituted in the system transfer function, the response of the system can be obtained as shown in Fig. E6.3(b). It is clear that the system with the pole value equal to p_2 settles much quicker than the system with the pole value equal to p_1. Hence, from the definition of relative stability, the system with lesser time is relatively more stable than the other one. From Fig. E6.3(a) and Fig. E6.3(b), it is inferred that the system with a less pole value is relatively more stable.

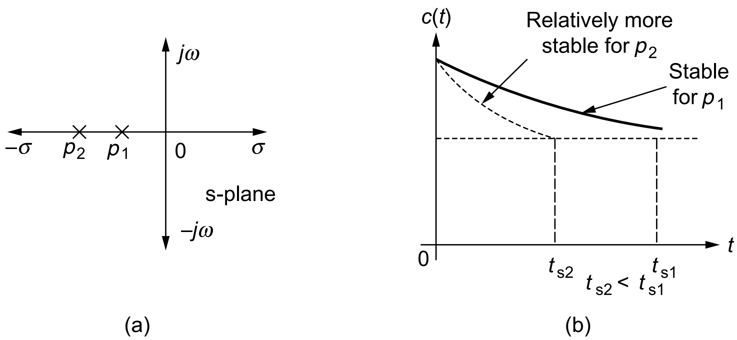


Fig. E6.3 | System with real poles: (a) location of poles in the s-plane and (b) their responses with corresponding settling times

Example 6.4: Consider the system $G(s)H(s) = \frac{A}{(s+p)}$, where p is the complex conjugate pole. Assume two conjugate poles with real parts having values p_1 and p_2 where $|p_2| > |p_1|$. Give your comments on their relative stability.

Solution: Given two complex conjugate poles with real parts having values of p_1 and p_2 where $p_1 > p_2$, the poles are plotted in the s-plane as shown in Fig. E6.4(a). For each value of pole, when substituted in the system transfer function, the response of the system can be obtained as shown in Fig. E6.4(b). It is clear that the system with pole value equal to p_2 settles much quicker than the system with pole value equal to p_1 Hence, from the definition of relative stability, the system with lesser time is relatively more stable than the other one. From Fig. E6.4(a) and (b), it is inferred that the system whose pole value has a lesser value or the complex conjugate pole which is at a higher distance from the imaginary axis is relatively more stable.

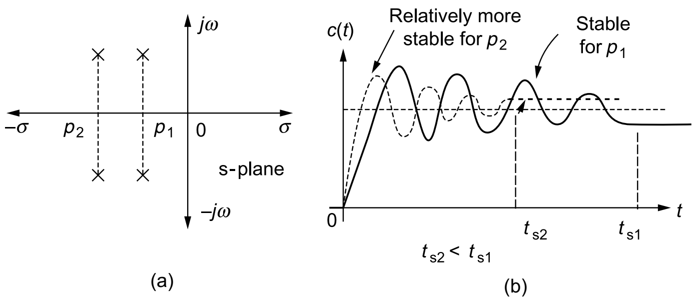


Fig. E6.4 | System with complex conjugate poles: (a) location of poles in the s-plane and (b) their responses with corresponding settling times.

Example 6.5: The transfer function of a plant is $G(s) = \frac{5}{(s+5)(s^2+s+1)}$. Find the second-order approximation of $G(s)$ using dominant pole concept.

Solution: Given

$$G(s) = \frac{5}{(s+5)(s^2+s+1)}$$

$$= \frac{5}{5\left(1+\frac{s}{5}\right)(s^2+s+1)} = \frac{1}{(s^2+s+1)\left(1+\frac{s}{5}\right)}$$

In the given transfer function, denominator is $(s+5)\left[(s+0.5)^2 + \frac{3}{4}\right]$. It can be observed that the pole at $s = -0.5 \pm j\frac{\sqrt{3}}{2}$ is more dominant than the pole at $s = -5$. Thus the second-order approximation is obtained.

6.8 Methods for Determining the Stability of the System

When the characteristic equation of the system has known parameters, it is very easy to determine the stability of the system by finding the roots of characteristic equation. But when there are some unknown parameters in the equation, the stability of the system can be determined by using the methods given below.

(i) **Routh–Hurwitz Criterion:** Routh–Hurwitz criterion was developed independently by E. J. Routh (1892) in the United States of America and A. Hurwitz (1895) in Germany, which is useful to determine the stability of the system without solving the characteristic equation. This is an algebraic method that provides stability information of the system that has the characteristic equation with constant coefficients. It indicates the number of roots of the characteristic equation, which lie on the imaginary axis and in the left half and right half of the s-plane without solving it.

(ii) **Root Locus Technique:** The technique which was first introduced by W. R. Evans in 1946 for determining the trajectories of the roots of the characteristic equation is known as root locus technique. Root locus technique is also used for analyzing the stability of the system with variation of gain of the system.

(iii) **Bode Plot:** This is a plot of magnitude of the transfer function in decibel versus ω and the phase angle versus ω, as ω is varied from zero to infinity. By observing the behaviour of the plots, the stability of the system is determined.

(iv) **Polar Plot**: It provides the magnitude and phase relationship between the input and output response of the system, i.e., a plot of magnitude versus phase angle in the polar co-ordinates. The magnitude and phase angle of the output response is obtained from the steady-state response of a system on a complex plane when it is subjected to sinusoidal input.

(v) **Nyquist Criterion:** This is a semi-graphical method that gives the difference between the number of poles and zeros of the closed-loop transfer function of the system, which are in the right half of the s-plane.

The Routh–Hurwitz criterion for examining the stability of the system is discussed in the following section.

6.9 Routh–Hurwitz Criterion

Routh–Hurwitz criterion gives the information on the absolute stability of a system without any necessity to solve for the closed-loop system poles. This method helps in determining the number of closed-loop system poles in the left half of the s-plane, the right half of the s-plane and on the $j\omega$ axis, but not their co-ordinates. Stability of the system can be determined by the location of poles.

The transfer function of the linear closed loop system is represented as

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{B(s)}{F(s)}$$

where a_n and b_m are the real coefficients and $n, m = 0, 1, 2, \cdots$.

Let $F(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$ be the characteristic equation of the given linear system. When this characteristic equation is solved for the roots, it gives the closed-loop poles of the given linear system, which decides the absolute stability of the system.

In order that all the roots of the characteristic equation be *pseudo-negative*, two conditions are necessary although they are not sufficient. These conditions are as follows:

(i) All the coefficients from a_n to a_0 of the characteristic equation must have the same sign.
(ii) All the coefficients from a_n to a_0 of the characteristic equation must be present.

These conditions are based on the laws of algebra which relate the coefficients of the characteristic equation as follows:

$$\frac{a_{n-1}}{a_n} = -\sum \text{roots of the equation}$$

$$\frac{a_{n-2}}{a_n} = \sum \text{products of the roots taken two at a time}$$

$$\frac{a_{n-3}}{a_n} = -\sum \text{products of the roots taken three at a time, etc.}$$

$$\frac{a_0}{a_n} = (-1)^n \text{ products of all roots}$$

Thus, if the above ratios are positive and non-zero, then all the roots of the characteristic equation lie in the left half of the s-plane.

Example 6.6: For the characteristic equation $F(s) = s^4 + 4s^3 + 3s + 6$, check whether it has roots in the left half of the *s*-plane.

Solution: Given the characteristic equation $F(s) = s^4 + 4s^3 + 3s + 6$. It does not have the term s^2. Hence, it does not have all the roots in the left half of the s-plane. Therefore, there are some pseudo-negative roots.

Example 6.7: For the characteristic equation $F(s) = 2s^4 - 5s^3 + 2s^2 + 7s + 3$, check whether it has roots in the left half of the *s*-plane.

Solution: In the given characteristic equation $F(s) = 2s^4 - 5s^3 + 2s^2 + 7s + 3$, the coefficient of s^3 is negative, whereas all others are positive. Hence, it does not have all the roots in the left half of the s-plane.

6.9.1 Minimum-Phase System

If all the poles and zeros of the system lie in the left half of the s-plane, then the system is called *minimum-phase*. For example, if the transfer function of a system is $G(s)=\frac{(1+s\tau_1)(1+s\tau_3)}{(1+s\tau_2)(1+s\tau_4)}$ and $\tau_1,\tau_2,\tau_3,\tau_4>0$, then the system is a minimum-phase system because the two poles at $s=-\frac{1}{\tau_2}$ and $s=-\frac{1}{\tau_4}$ and two zeros at $s=-\frac{1}{\tau_1}$ and $s=-\frac{1}{\tau_3}$ are in the left half of the s-plane.

6.9.2 Non-Minimum-Phase System

If a system has at least one pole or zero in the right half of the s-plane, then the system is called the *non-minimum-phase system*. For example, if the transfer function of a system is $G(s)=\frac{1-s\tau_1}{1+s\tau_2}$, then the system is a non-minimum-phase system because there is a zero at $s=\frac{1}{\tau_1}$ in the right half of the s-plane.

Example 6.8: The closed-loop transfer function of a control system is given by $G(s)=\frac{s-5}{(s+2)(s+3)}$. Check whether the system is a minimum phase system or a non-minimum phase system.

Solution: In a minimum phase system, all the poles as well as zeros should be in the left half of the s-plane. In the given system, there is a zero at $s = 5$ which is in the right half of the s-plane. Hence, the system is a non-minimum phase system.

6.10 Hurwitz Criterion

Hurwitz criterion states that the necessary and sufficient conditions for a system to be stable with all roots of the characteristics equation lying in the left half of the s-plane, the determinant values of all the sub-matrices of the Hurwitz matrix D_k, $k = 1, 2, \ldots, n$ must be positive.

6.10.1 Hurwitz Matrix Formation

The transfer function of the linear closed-loop system is represented as follows:

$$\frac{C(s)}{R(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_0}=\frac{B(s)}{F(s)}$$

where a_n and b_m are the real coefficients and $n, m = 0, 1, 2, \cdots$.

Here, $F(s)=a_n s^n+a_{n-1}s^{n-1}+\cdots+a_0=0$ is the characteristic equation of the given linear system.

The Hurwitz matrix, H formed by the coefficients of the characteristic equation is given by the following matrix:

$$H = D_n = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}$$

where n is the order of matrix.

The sub-matrices formed from the Hurwitz matrix H are as follows:

$$D_1 = a_{n\text{-}1}$$

$$D_2 = \begin{bmatrix} a_{n\text{-}1} & a_{n\text{-}3} \\ a_n & a_{n\text{-}2} \end{bmatrix}$$

$$D_3 = \begin{bmatrix} a_{n\text{-}1} & a_{n\text{-}3} & a_{n\text{-}5} \\ a_n & a_{n\text{-}2} & a_{n\text{-}4} \\ 0 & a_{n\text{-}1} & a_{n\text{-}3} \end{bmatrix}$$

The condition for stability is $D_1 > 0, D_2 > 0, D_3 > 0, \cdots, D_n > 0$. Thus, if the determinant values of the matrices D_k, where $k = 1, 2, \cdots n$, are all positive, then the roots of the characteristic equation or the poles of the system lie in the left half of the s-plane and hence the system is stable.

6.10.2 Disadvantages of Hurwitz Method

The following disadvantages exist in the Hurwitz method of examining the stability of the system:

(i) Time consuming and complication exist in determining the determinant value of higher order matrices for higher order systems (i.e., higher value of n).
(ii) It is difficult to predict the number of roots of the characteristic equation or poles of the system existing in the right-half of the s-plane.
(iii) It examines whether the system is stable or unstable and does not examine the marginal stability of the system.

To overcome the above disadvantages, a new method called Routh's method is suggested by the scientist Routh and is also called Routh–Hurwitz method.

Example 6.9: Examine the stability of the system by Hurwitz method whose characteristic equation is given by $F(s) = s^3 + 5s^2 + 2s + 4 = 0$.

Solution: Comparing the given characteristic equation with the standard equation $F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$, the values of the coefficients are:

$$a_3 = 1, a_2 = 5, a_1 = 2, a_0 = 4 \text{ and the order of the equation is } n = 3.$$

Hurwitz matrix is obtained as

$$D_n = \begin{bmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{bmatrix}$$

The sub-matrices formed from the Hurwitz matrix are:

$$D_1 = a_2$$

$$D_2 = \begin{bmatrix} a_2 & a_0 \\ a_3 & a_1 \end{bmatrix}$$

and

$$D_3 = D_n$$

Substituting the given values, we obtain

$$|D_1| = 1,\ |D_2| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 6 \text{ and } |D_3| = \begin{vmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 24$$

The determinant values of the above matrices are $|D_1| = 1, |D_2| = 6$ and $|D_3| = 24$.

Since the determinant values of all the matrices are positive, all the poles lie in the left half of the s-plane. Hence, the system is stable.

6.11 Routh's Stability Criterion

This method of examining the stability criterion of the system is known as Routh's array method or Routh–Hurwitz method. In this method, the coefficient of the characteristic equation of the system are tabulated or framed in a particular way. The array formed by tabulating the coefficients of the characteristic equation is called Routh's array.

The general characteristic equation of the system is represented by:

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

Routh's array is formed as:

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \\
s^{n-3} & c_1 & c_2 & c_3 & & \\
. & . & . & . & & \\
. & . & . & . & & \\
. & . & . & . & & \\
s^0 & a_0 & & & &
\end{array}$$

For the above array, the coefficients of first two rows are obtained directly from the characteristic equation of the system.

The coefficients of the third row is obtained from the coefficients of the first two rows using the following formulae:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}},\ b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \quad \text{and} \quad b_3 = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

Similarly, the coefficients of the m^{th} row are obtained from the coefficients of $(m-1)^{th}$ and $(m-2)^{th}$ rows. The coefficients of the fourth row are obtained as

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1},\ c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} \quad \text{and} \quad c_3 = \frac{b_1 a_{n-7} - b_4 a_{n-1}}{b_1}$$

The above process is continued until the coefficient for s^0 is obtained, which will be equal to a_0. Now, the stability of the system can be examined by using this Routh's array.

6.11.1 Necessary Condition for the Stability of the System

If all the terms existing in the first column of Routh's array have the same sign (i.e., either positive or negative), it suggests that there exists no pole in the right half of the s-plane and hence the system is stable.

If there is any sign change (i.e., either from positive to negative or from negative to positive) in the first column of the array or tabulation, then the following inferences are made:

(i) Number of sign changes is equal to the number of roots lying in the right half of the s-plane.
(ii) The system is unstable.

Example 6.10: For the characteristic equation $s^3 + 5s^2 + 7s + 3 = 0$, find the number of roots in the left half of the s-plane.

Solution: By applying Routh's criterion for $s^3 + 5s^2 + 7s + 3 = 0$, we have

$$\begin{array}{c|cc} s^3 & 1 & 7 \\ s^2 & 5 & 3 \\ s^1 & \dfrac{5\times 7 - 1\times 3}{5} = \dfrac{32}{5} & 0 \\ s^0 & 3 & \end{array}$$

As there is no sign change in the first column, there is no pole in the right-half of the s-plane. Hence, all the three poles are in the left half of the s-plane.

Example 6.11: The characteristic equation of a system is given by $s^4+8s^3+18s^2+16s+5=0$. Determine the stability of the system and comment on the location of the roots of the characteristic equation (poles of the system) using Routh's array.

Solution: The characteristic equation of the system is $s^4+8s^3+18s^2+16s+5=0$

Comparing the given characteristic equation with the standard equation $F(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots+a_0=0$, Routh's array is formed as:

$$\begin{array}{c|ccc} s^4 & 1 & 18 & 5 \\ s^3 & 8 & 16 & 0 \\ s^2 & 16 & 5 & 0 \\ s^1 & 13.5 & 0 & \\ s^0 & 5 & & \end{array}$$

Since there is no sign change in the first column of Routh's array, no root exists in the right half of the s-plane. Here, all the four poles of the system are located in the left half of the s-plane and therefore the system is stable.

Example 6.12: Determine the stability of the following system using Routh's criterion:

(i) $G(s)\,H(s)=\dfrac{1}{(s+2)\,(s+4)}$ **(ii) $G(s)\,H(s)=\dfrac{9}{s^2\,(s+2)}$**

Solution:

(i) The characteristic equation of the system is

$$1+G(s)H(s)=0$$

$$1+\frac{1}{(s+2)(s+4)}=0$$

i.e.,
$$s^2+6s+9=0$$

Therefore, comparing this characteristic equation with the standard equation $F(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots+a_0=0$, Routh's array is formed as:

$$\begin{array}{c|cc} s^2 & 1 & 9 \\ s^1 & 6 & 0 \\ s^0 & 9 & \end{array}$$

As there is no sign change in the first column of Routh's array, there is no root in the right half of the s-plane. Hence, all the poles of the system are located in the left half of the s-plane and therefore the system is stable.

(ii) The characteristic equation of the system is

$$1+G(s)H(s)=0$$

$$1+\frac{9}{s^2(s+2)}=0$$

i.e., $$s^3+2s^2+0s+9=0$$

For this characteristic equation, Routh's array is formed as:

$$\begin{array}{c|cc} s^3 & 1 & 0 \\ s^2 & 2 & 9 \\ s^1 & -4.5 & 0 \\ s^0 & 9 & \end{array}$$

Since there are two sign changes i.e., from 2 to −4.5 and from −4.5 to 9 in the first column of Routh's array, two poles of the system are located in the right half of the s-plane. Hence, the system is unstable.

Example 6.13: The characteristic equations of a system are given by

(i) $s^4+s^3+20s^2+9s+100=0$ and

(ii) $5s^6+8s^5+12s^4+20s^3+100s^2+150s+100=0$.

Determine the stability of the system and comment on the location of the roots of the characteristic equation (poles of the system) using Routh's array.

Solution:

(i) For the given characteristic equation $s^4+s^3+20s^2+9s+100=0$, Routh's array is formed as:

s^4	1	20	100
s^3	1	9	0
s^2	11	100	0
s^1	$-\frac{1}{11}$	0	
s^0	100		

Since there are two sign changes in the first column of Routh's array i.e., from 11 to $-\frac{1}{11}$ and from $-\frac{1}{11}$ to 100, two poles of the system are located in the right half of the s-plane. Hence, the system is unstable.

(ii) For the given characteristic equation $5s^6+8s^5+12s^4+20s^3+100s^2+150s+100=0$, Routh's array is formed as:

s^6	5	12	100	100
s^5	8	20	150	0
s^4	−0.5	6.25	100	
s^3	120	1750		
s^2	13.54	100		
s^1	863.8	0		
s^0	100			

Since there are two sign changes in the first column of Routh's array i.e., from 8 to −0.5 and from −0.5 to 120, two poles of the system are located in the right half of the s-plane. Hence, the system is unstable.

Example 6.14: For the characteristic equation $s^3+7s^2+25s+39=0$, find the number of roots lying in the right half of the vertical line through 1.

Solution: To find the number of roots lying in the right half of the vertical line through 1, replace s with $(x-1)$ in the given characteristic equation. Therefore, the new characteristic equation becomes

$$(x-1)^3+7(x-1)^2+25(x-1)+39=0$$

i.e., $x^3+4x^2+14x+20=0$
For the above characteristic equation, Routh's array is

x^3	1	14	
x^2	(4) 1	(20) 5	← divide by 4
x^1	9		
x^0	5		

Since the order of the equation is three, there exist three roots for the above characteristic equation. As there is no sign change in the first column of Routh's array, all the three roots lie in the left half of the vertical line through 1. Hence, there is no root in the right half of the vertical line through 1.

6.11.2 Special Cases of Routh's Criterion

In this section, let us see the various cases in which difficulties exist in examining the stability of the system.

Special Case 1: If the first element of any row of Routh's array becomes zero (other elements of the row are non zero), then it becomes difficult in examining the stability of the system since the terms in the next row becomes infinite. Consider a system whose characteristic equation is given by $s^5+2s^4+3s^3+6s^2+2s+1=0$. Examine the stability of the system using Routh's array.

Solution: For the given characteristic equation, Routh's array is formed as:

s^5	1	3	2
s^4	2	6	1
s^3	0	1.5	0
s^2	∞	$\cdots$	$\cdots$

Since the first value in the third row of Routh's array is zero, the first value in the fourth row of Routh's array becomes infinite and makes the examination of the stability of the system difficult. There are three methods to solve and investigate the stability of the system.

Method 1: Replace the zero (first element in the third row) with a small positive real number ε and continue the evaluation of the array. After completing the array, examine the possibility of occurrence of a sign change using $\underset{\varepsilon \to 0}{Lt}$. Considering the same example, replace the first element of the third row by ε and continue evaluating the array.

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 2 \\ s^4 & 2 & 6 & 1 \\ s^3 & \varepsilon & 1.5 & 0 \\ s^2 & \dfrac{6\varepsilon-3}{\varepsilon} & 1 & 0 \\ s^1 & \dfrac{\left(\dfrac{1.5(6\varepsilon-3)}{\varepsilon}-\varepsilon\right)}{\left(\dfrac{6\varepsilon-3}{\varepsilon}\right)} & 0 & \\ s^0 & 1 & & \end{array}$$

Now, for examining the occurrence of a sign change, consider all the elements in the array that has ε in it and apply the limits.

$$\underset{\varepsilon\to 0}{Lt}\left(\frac{6\varepsilon-3}{\varepsilon}\right)=6-\underset{\varepsilon\to 0}{Lt}\frac{3}{\varepsilon}=-\infty$$

$$\underset{\varepsilon\to 0}{Lt}\frac{1.5(6\varepsilon-3)-\varepsilon^2}{6\varepsilon-3}=+1.5$$

Then, replace these values in Routh's array to have a clear idea about the sign changes occurring in the system. The modified Routh's array will be as follows:

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 2 \\ s^4 & 2 & 6 & 1 \\ s^3 & \varepsilon & 1.5 & 0 \\ s^2 & -\infty & 1 & 0 \\ s^1 & 1.5 & 0 & \\ s^0 & 1 & & \end{array}$$

Since there are two sign changes in the first column of Routh's array i.e., ε to $-\infty$ and $-\infty$ to $+1.5$, two roots lie in the right half of the s-plane. Hence the system is unstable.

Method 2: In this method, the variable s in the characteristic equation of the system is replaced by the variable $\frac{1}{x}$. After rearranging the characteristic equation using the variable, x, Routh's array is formed and then the stability of the system is examined.

Consider the same example with the characteristic equation $F(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

Substituting $s = \frac{1}{x}$ in the given characteristic equation and rearranging, we obtain the new characteristic equation as follows:

$$x^5 + 2x^4 + 6x^3 + 3x^2 + 2x + 1 = 0$$

Then Routh's array is formed as:

x^5	1	6	2
x^4	2	3	1
x^3	4.5	1.5	0
x^2	2.33	1	0
x^1	−0.429	0	
x^0	1		

Since there are two sign changes i.e., from 2.33 to −0.429 and from −0.429 to 1 in the first column of Routh's array, two roots lie in the left half of the s-plane. Hence, the system is unstable.

Method 3: When Routh's array has a zero in the first column, we multiply $F(s)$ by a factor $(s+a)$, where a is positive, say 1. Since $(s+a)F(s)$ has all zeros of $F'(s)$ and zero at $-a$, the change of sign in Routh's array of $(s+1)F(s)$ will indicate the existence of pseudo-positive roots in $F(s)$. The following example illustrates this point.

Consider the same example with the characteristic equation $F(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

Multiply $F(s)$ by a factor $(s+1)$ to obtain the new characteristic equation as

$$F'(s) = s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 3s + 1 = 0$$

Routh array for the modified characteristics equation is

s^6	1	5	8	1
s^5	(3)	(9)	(3)	← Divide by 3

$$\begin{array}{c|ccc}
 & 1 & 3 & 1 \\
s^4 & 2 & 7 & 1 \\
s^3 & \left(-\frac{1}{2}\right) & \left(\frac{1}{2}\right) & \leftarrow \quad \text{Divide by } \frac{1}{2} \\
 & -1 & 1 & \\
s^2 & 9 & 1 & \\
s^1 & \frac{10}{9} & & \\
s^0 & 1 & &
\end{array}$$

Since there are two sign changes i.e., from 2 to −1 and from −1 to 9 in the first column of Routh's array, two roots lie in the right half of the *s*-plane. Hence, the system is unstable.

Example 6.15: The open-loop transfer function of a unity feedback system is $G(s) = \dfrac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$. Find the number of roots that lie in the right half of the *s*-plane.

Solution: Given $G(s) = \dfrac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$ and $H(s) = 1$.

The characteristic equation is

$$1 + G(s)H(s) = 0$$

or $$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 13 = 0$$

Routh's array for the characteristic equation is

$$\begin{array}{c|ccc}
s^5 & 1 & 3 & 5 \\
s^4 & 2 & 6 & 13 \\
s^3 & 0 & -1.5 & \\
s^2 & \infty & \cdots &
\end{array}$$

Since the first value in the third row of Routh's array is zero, the first value in the fourth row of Routh's array becomes infinite. Hence, it is difficult to examine the stability of the system. The new characteristic equation can be obtained by replacing s by $\frac{1}{x}$ as follows:

$$13x^5 + 5x^4 + 6x^3 + 3x^2 + 2x + 1 = 0$$

Then, Routh's array is formed as:

x^5	13	6	2
x^4	5	3	1
x^3	−1.8	−0.6	
x^2	1.3	1	
x^1	0.75		
x^0	1		

Since there are two sign changes i.e., from 5 to −1.8 and from −1.8 to $\frac{4}{3}$ in the first column of Routh's array, two poles lie in the right half of the s-plane.

Example 6.16: Examine whether the system with the following characteristic equation $s^4 + s^3 + s^2 + s + 2 = 0$ is stable. Also, find the nature of the roots of the equation.

Solution: Routh's array for the given characteristic equation is tabulated as follows:

s^4	1	1	2
s^3	1	1	0
s^2	0	2	

When a zero comes in the first column of Routh's array, the method of testing the stability is as follows:

Substituting $s = \frac{1}{x}$ in the characteristic equation, we obtain

$$\frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 2 = 0$$

$$1 + x + x^2 + x^3 + 2x^4 = 0$$

$$2x^4 + x^3 + x^2 + x + 1 = 0$$

Routh's array for the above equation is tabulated as follows:

$$\begin{array}{c|ccc} x^4 & 2 & 1 & 1 \\ x^3 & 1 & 1 & 0 \\ x^2 & -1 & 1 & 0 \\ x^1 & 2 & 0 & \\ x^0 & 1 & & \end{array}$$

There are two sign changes i.e., from 1 to −1 and from −1 to 2 exist in the first column of Routh's array. Therefore, there are two roots in the right half of the *s*-plane. Hence, the system is unstable.

To find the Number of Roots in the Left Half of the *s*-Plane

To find the number of roots existing in the left half of the s-plane, replace s with -s in the given characteristic equation

$$F(-s) = s^4 - s^3 + s^2 - s + 2 = 0$$

Routh's array for the above equation is tabulated as

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 2 \\ s^3 & -1 & -1 & 0 \\ s^2 & 0 & & \end{array}$$

When a zero comes in the first column of Routh's array, the method of testing the stability is as follows:

Substituting $s = \dfrac{1}{x}$ in the characteristic equation, we obtain

$$2x^4 - x^3 + x^2 - x + 1 = 0$$

Routh's array for the above equation is tabulated as

$$\begin{array}{c|ccc} x^4 & 2 & 1 & 1 \\ x^3 & -1 & -1 & 0 \\ x^2 & -1 & 1 & 0 \\ x^1 & -2 & 0 & \\ x^0 & 1 & & \end{array}$$

There are two sign changes i.e., from 2 to −1 and −2 to 1 in the first column of Routh's array. Therefore, there are two roots in the left half of the s-plane.

It is already known that there are two roots in the right half of the s-plane. There are totally four roots for the above characteristic equation. Therefore, there is no root on the imaginary axis. The result is that the system is unstable.

Example 6.17: The characteristic equation of the system is given by $s^5+s^4+2s^3+2s^2+3s+15=0$. Investigate the stability of the system using Routh's array.

Solution: For the given characteristic equation $s^5+s^4+2s^3+2s^2+3s+15=0$, Routh's array is formed as:

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & 0 & -12 & 0 \\ s^2 & \infty & \cdots & \cdots \end{array}$$

Since the first value in the third row of Routh's array is zero, the first value in the fourth row of Routh's array becomes infinite. Hence, the above characteristic equation can be investigated for the stability of the system using the following two methods tabulated below.

Method 1	**Method 2**
Replace the zero (first element in the row) by a small positive real number ε and continue the evaluation of the array. $$\begin{array}{c\|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & \varepsilon & -12 & 0 \\ s^2 & \dfrac{(2\varepsilon+12)}{\varepsilon} & 15 & 0 \\ s^1 & \dfrac{(-24\varepsilon-144-15\varepsilon^2)}{2\varepsilon+12} & 0 & \\ s^0 & 15 & & \end{array}$$	Substitute $s=\dfrac{1}{x}$ in the given characteristic equation and rearranging for the new characteristic equation, we obtain $$15x^5+3x^4+2x^3+2x^2+x+1=0$$ Routh's array for the above characteristic equation is formed as follows: $$\begin{array}{c\|ccc} x^5 & 15 & 2 & 1 \\ x^4 & 3 & 2 & 1 \\ x^3 & -8 & -4 & 0 \\ x^2 & 0.5 & 1 & 0 \\ x^1 & 12 & 0 & \\ x^0 & 1 & & \end{array}$$

<table>
<tr>
<td>

Examining the sign changes:

$$\underset{\varepsilon \to 0}{Lt} = \frac{2\varepsilon + 12}{\varepsilon} = +\infty$$

$$\underset{\varepsilon \to 0}{Lt} \frac{-24\varepsilon - 144 - 15\varepsilon^2}{2\varepsilon + 12} = -12$$

Rewriting Routh's array using the values obtained from the limits

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & \varepsilon & -12 & 0 \\ s^2 & +\infty & 15 & 0 \\ s^1 & -12 & 0 & \\ s^0 & 15 & & \end{array}$$

There are two sign changes in the first column (from $+\infty$ to -12 and -12 to 15)

</td>
<td>

There are two sign changes in the first column (from 3 to −8 and −8 to 0.5).

</td>
</tr>
</table>

Since there are two sign changes in the first column of the modified Routh's array obtained from both the methods, two poles of the system are located in the right half of the s-plane. Hence, the system is unstable.

Example 6.18: Investigate the stability of a system using Routh's criterion for the characteristic equation of a system given by $s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15 = 0$

Solution: Given the characteristic equation of the system is

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 10s + 15 = 0$$

Then, Routh's array is formed as:

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 10 \\ s^4 & 2 & 6 & 15 \\ s^3 & 0 & 2.5 & 0 \\ s^2 & +\infty & \cdots & \cdots \end{array}$$

Since the first value in the third row of Routh's array is zero, the first value in the fourth row of Routh's array becomes infinite.

The above characteristic equation can be investigated for the stability of the system using the following two methods tabulated below.

<table>
<tr><th>Method 1</th><th>Method 2</th></tr>
<tr>
<td>
Replace zero (first element in the third row) by a small positive real number ε and continue the evaluation of the array.

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 10 \\ s^4 & 2 & 6 & 15 \\ s^3 & \varepsilon & 2.5 & 0 \\ s^2 & \dfrac{(6\varepsilon-5)}{\varepsilon} & 15 & 0 \\ s^1 & \dfrac{(15\varepsilon-12.5-15\varepsilon^2)}{(6\varepsilon-5)} & 0 & \\ s^0 & 15 & & \end{array}$$

Examining for the sign changes:

$$\underset{\varepsilon\to 0}{Lt} = \frac{6\varepsilon-5}{\varepsilon} = -\infty$$

$$\underset{\varepsilon\to 0}{Lt} = \frac{15\varepsilon-12.5-15\varepsilon^2}{6\varepsilon-5} = 2.5$$

Rewriting Routh's array using the values obtained from the limits

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 10 \\ s^4 & 2 & 6 & 15 \\ s^3 & \varepsilon & 2.5 & 0 \\ s^2 & -\infty & 15 & 0 \\ s^1 & 2.5 & 0 & \\ s^0 & 15 & & \end{array}$$

There are two sign changes from ε to $-\infty$ and $-\infty$ to 2.5.
</td>
<td>
Substituting $s = \frac{1}{x}$ in the given characteristic equation and rearranging, we get the new characteristic equation as follows:

$$15x^5 + 10x^4 + 6x^3 + 3x^2 + 2x + 1 = 0$$

Routh's array for the new characteristic equation is formed as:

$$\begin{array}{c|ccc} x^5 & 15 & 3 & 2 \\ x^4 & 10 & 3 & 1 \\ x^3 & 1.5 & 0.5 & 0 \\ x^2 & -0.333 & 1 & 0 \\ x^1 & 5 & 0 & \\ x^0 & 1 & & \end{array}$$

There are two sign changes from +1.5 to −0.333 and −0.333 to 5.
</td>
</tr>
</table>

Since there are two sign changes in the first column of the modified Routh's arrays obtained from both the methods, two poles of the system are located in the right half of the s-plane. Hence, the system is unstable.

Special Case 2: If all the elements of any row of Routh's array become zero or if all the elements of $(n\text{-}1)^{th}$ row is an integral multiple of n^{th} row, then it becomes difficult to examine the stability of the system as determining the terms in the next row becomes difficult.

Consider a system whose characteristic equation is given by $s^3+10s^2+50s+500=0$.

Examine the stability of the system using Routh's array.

Solution: For the characteristic equation $s^3+10s^2+50s+500=0$, Routh's array is formed as:

$$\begin{array}{c|ccc} s^3 & 1 & 50 & 0 \\ s^2 & 10 & 500 & 0 \\ s^1 & 0 & 0 & 0 \end{array}$$

Since all the elements in the third row of Routh's array is zero, it becomes difficult to determine the values of the element present in the next row and makes the examination of the stability of the system difficult.

Solution for Such Cases Consider a Routh's array in which all the elements of third row are zero as given below:

$$\begin{array}{c|ccc} s^5 & a & b & c \\ s^4 & d & e & f \\ s^3 & 0 & 0 & 0 \end{array}$$

Elimination Procedure to overcome the Difficulty: The steps involved in eliminating the above-mentioned difficulty are listed below:

(i) An auxiliary equation, $A(s)$ is formed with the coefficients present in the row i.e., just above the row of zeros. The auxiliary equation is formed with the alternate powers of s starting from the power indicated against it. For the given example, the auxiliary equation is given by $A(s)=ds^4+es^2+f$.

(ii) The first-order derivative of the auxiliary equation with respect to s is determined, which is given by $\dfrac{dA(\mathrm{s})}{ds}=4ds^3+2es$.

(iii) The coefficients of $\dfrac{dA(\mathrm{s})}{ds}$ is used to replace the row of zeros. Hence, now Routh's array is given as follows:

$$\begin{array}{c|ccc} s^5 & a & b & c \\ s^4 & d & e & f \\ s^3 & 4d & 2e & 0 \end{array}$$

(iv) Now Routh's array is completed with the help of these new coefficients and the stability of the system is examined.

Importance of an Auxiliary Equation $A(s)$: Auxiliary equation is a part of the original characteristic equation i.e., the roots of the auxiliary equation belong to the original characteristic equation also. In addition, the roots of the auxiliary equation are the most dominant roots of the original characteristic equation. The roots of the auxiliary equation are called the dominant roots since it decides the stability of the system.

Hence, the stability of the system can be predicted easily by determining the roots of the auxiliary equation $A(s)$ (dominant roots) rather than determining all the roots of the characteristic equation. The remaining roots of the characteristic equation do not play any significant role in the stability analysis.

For example, if the order of the characteristic equation $F(s)$ is n and the order of auxiliary equation $A(s)$ is m, then the number of roots that play a significant role in the stability analysis of the system is given by m. Hence,

Number of dominant roots = m.
Number of non-dominant roots = $(n-m)$.

On completion of Routh's array with the help of auxiliary equation, two suggestions over the stability are as follows:

(i) If there is any sign change in the first column of the modified Routh's array, then we can conclude that the system is unstable since some of the roots lie in the right side of the s-plane.
(ii) If there is no sign change in the first column of the modified Routh's array, we cannot suggest that the system is stable. Only after finding the roots of the auxiliary equation (dominant roots), we can find the stability of the system.

The different location of roots of auxiliary equation in the s-plane and the corresponding nature of stability of the system are shown in Table 6.4.

Example 6.19: The characteristic equation of a system is given by $s^3 + 10s^2 + 50s + 500 = 0$. Investigate the stability of the system using Routh's array.

Solution: For the characteristic equation of the system $s^3+10s^2+50s+500=0$, Routh's array is formed as follows:

$$\begin{array}{c|ccc} s^3 & 1 & 50 & 0 \\ s^2 & 10 & 500 & 0 \\ s^1 & 0 & 0 & 0 \end{array}$$

As all the elements in the third row of Routh's array are zero, it becomes difficult to determine the values of the elements present in the next row. This makes the examination of the stability of the system difficult. To eliminate this difficulty, the following steps are carried out:

Step 1: Auxiliary equation of the system is given by $A(s) = 10s^2 + 500$.

Step 2: First-order derivative of the auxiliary equation is given by $\frac{dA(s)}{ds} = 20s$.

Step 3: The coefficient of the first-order derivative of the auxiliary equation is used to replace the row of zeros. The modified Routh's array is given by

$$\begin{array}{c|ccc} s^3 & 1 & 50 & 0 \\ s^2 & 10 & 500 & 0 \\ s^1 & 20 & 0 & 0 \end{array}$$

Step 4: With the help of new coefficients, the complete Routh's array is formed as:

$$\begin{array}{c|ccc} s^3 & 1 & 50 & 0 \\ s^2 & 10 & 500 & 0 \\ s^1 & 20 & 0 & \\ s^0 & 500 & & \end{array}$$

Although there is no sign change in the first column of Routh's array, the system cannot be suggested as a stable system. For suggesting the stability of the system, the roots of the auxiliary equation $A(s)$ are to be determined.

Upon solving $A(s) = 0$, i.e., $10s^2 + 500 = 0$, we obtain $s = \pm j\sqrt{50}$, which are the dominant roots of the given characteristic equation. The given system is marginally stable because the dominant roots are on the imaginary axis of the s-plane.

Example 6.20: The characteristic polynomial of a system is $2s^5 + s^4 + 2s^2 + 2s + 1 = 0$. Check whether the system is stable, marginally stable, unstable or oscillatory.

Solution: For the characteristic polynomial of a system $2s^5 + s^4 + 2s^2 + 2s + 1 = 0$, Routh's array is formed as:

$$\begin{array}{c|ccc} s^5 & 2 & 0 & 2 \\ s^4 & 1 & 2 & 1 \\ s^3 & 0 & 0 & 0 \end{array}$$

Since all the elements in the third row are zero, the auxiliary equation $A(s)$ is formed as $A(s) = s^4 + 2s^2 + 1$. i.e., $A(s) = (s^2 + 1)^2 = 0$.

The dominant roots of the characteristics equation (or) roots of auxiliary equation are $j, j, -j, -j$.

Since repeated roots exist on the imaginary axis, the given system is unstable.

Example 6.21: The characteristic equation of the system is given by $s^5+6s^4+15s^3+30s^2+44s+24=0$. Investigate the stability of the system using Routh's array.

Solution: For the characteristic equation of the system $s^5+6s^4+15s^3+30s^2+44s+24=0$, Routh's array is formed as:

$$\begin{array}{c|ccc} s^5 & 1 & 15 & 44 \\ s^4 & 6 & 30 & 24 \\ s^3 & 10 & 40 & 0 \\ s^2 & 6 & 24 & 0 \\ s^1 & 0 & 0 & \end{array}$$

Here, all the elements in the fifth row of Routh's array are zero. Hence, the following steps are carried out to determine the values of the elements present in the next row.

Step 1: Auxiliary equation for the system is given by $A(s)=6s^2+24$.

Step 2: First-order derivative of the auxiliary equation is given by $\frac{dA(s)}{ds}=12s$.

Step 3: The coefficient of the first-order derivative of the auxiliary equation is used to replace the row of zeros. The modified Routh's array is given by

$$\begin{array}{c|ccc} s^5 & 1 & 15 & 44 \\ s^4 & 6 & 30 & 24 \\ s^3 & 10 & 40 & 0 \\ s^2 & 6 & 24 & 0 \\ s^1 & 12 & 0 & \end{array}$$

Step 4: The complete Routh's array is formed with the help of new coefficients and is given by

$$\begin{array}{c|ccc} s^5 & 1 & 15 & 44 \\ s^4 & 6 & 30 & 24 \\ s^3 & 10 & 40 & 0 \\ s^2 & 6 & 24 & 0 \\ s^1 & 12 & 0 & \\ s^0 & 24 & & \end{array}$$

Although there is no sign change in the first column of Routh's array, the system cannot be suggested as a stable system. For suggesting the stability of the system, the roots of the auxiliary equation $A(s)$ are to be determined.

Upon solving, $A(s)=0$, i.e., $6s^2+24=0$, we obtain $s=\pm j2$, which are the dominant roots of the given characteristic equation obtained using auxiliary equation.

The given system is marginally stable because the dominant roots are on the imaginary axis of the s-plane.

Example 6.22: **For the characteristic equation $s^6+s^5-2s^4-3s^3-7s^2-4s-4=0$, find the number of roots which lie in the right half and the left half of the s-plane.**

Solution:

(i) To find the number of roots in the right of the s-plane

For the characteristic equation of the system, $s^6+s^5-2s^4-3s^3-7s^2-4s-4=0$, Routh's array is formed as:

$$\begin{array}{c|cccc} s^6 & 1 & -2 & -7 & -4 \\ s^5 & 1 & -3 & -4 & \\ s^4 & 1 & -3 & -4 & \\ s^3 & 0 & 0 & & \end{array}$$

In the above Routh's array, all the elements in the fourth row are zero. Therefore, the auxiliary equation for the system is given by

$$A(s)=s^4-3s^2-4$$

i.e., $$\frac{dA(s)}{ds}=4s^3-6s$$

The coefficient of the first-order derivative of the auxiliary equation is used to replace the row of zeros. The modified Routh's array is formed as:

$$\begin{array}{c|cc} s^3 & 4 & -6 \\ s^2 & -1.5 & -4 \\ s^1 & -16.67 & \\ s^0 & -1 & \end{array}$$

As there is only one sign change i.e., from 4 to −1.5 in the first column, only one root falls in the right half of the s-plane.

(ii) To find the number of roots in the left half of the s-plane

In order to find the number of roots in the left half of the s-plane, $F(-s)$ is obtained and corresponding Routh's array is formed. The number of sign changes indicates the number of roots in the left half of the s-plane. Therefore,

$$F(-s) = s^6 - s^5 - 2s^4 + 3s^3 - 7s^2 + 4s - 4$$

Routh's array is formed as:

$$\begin{array}{c|cccc} s^6 & 1 & -2 & -7 & -4 \\ s^5 & -1 & 3 & 4 & \\ s^4 & 1 & -3 & -4 & \\ s^3 & 0 & 0 & & \end{array}$$

In the above Routh's array, all the elements in the third row are zero. Therefore, the auxiliary equation of the system is $A(s) = s^4 - 3s^2 - 4$.

i.e.,
$$\frac{dA(s)}{ds} = 4s^3 - 6s$$

The coefficient of the first order derivative of the auxiliary equation is used to replace the row of zeros. The modified Routh's array is formed as:

$$\begin{array}{c|cc} s^3 & 4 & -6 \\ s^2 & -1.5 & -4 \\ s^1 & -1 & \\ s^0 & -1 & \end{array}$$

There are three sign changes (from 1 to −1, −1 to 1 and 4 to −1.5) in the first column of Routh's array. Hence, there are three roots in the left half of the s-plane. We have already seen that there is one root in the right half of the s-plane. Since the order of the equation is six, there are totally six roots for the above characteristic equation. Therefore, the remaining two roots lie on the imaginary axis.

6.11.3 Applications of Routh's Criterion

(i) **Relative Stability Analysis:** Routh's criterion helps in determining the relative stability of the system about a particular line $s = -a$. Axis of the s-plane is shifted by $-a$ and then Routh's array is formed by substituting $s = s - a, (a \rightarrow \text{constant})$ in the characteristic equation. The number of sign changes in the first column of the new Routh's array is equal to the number of roots located on the right side of the vertical line $s = -a$.

The shifting of the s-plane by $-a$ is clearly shown in Fig. 6.13.

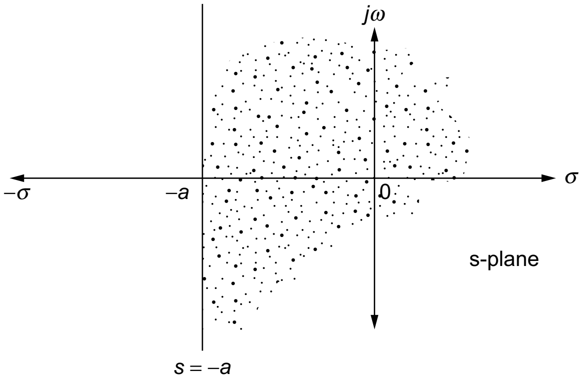


Fig. 6.13 | Relative stability

(ii) **To Determine the Range of Values *of K*:** Consider a practical system whose block diagram is shown in Fig. 6.14. The range of gain, K could be determined using Routh's array.

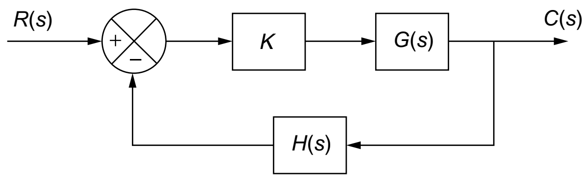


Fig. 6.14 | Block diagram of a control system

The closed-loop transfer function of the system shown in Fig. 6.14 is given by

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Hence, the characteristic equation of the system is given by

$$F(s) = 1 + KG(s)H(s) = 0$$

The stability of the system or the location of roots of the characteristic equation (poles of the system) depends on the proper selection of value of gain, K. To determine the range of K, following steps are used:

(a) Routh's array is completed in terms of gain value K.

(b) Range of values of K is determined such that the sign of all values present in the first column of Routh's array remains same.

Since the stability of the system depends on the value of the gain K, the system is conditionally stable.

6.11.4 Advantages of Routh's Criterion

The advantages of Routh's criterion of examining the stability for the given system are as follows:

(i) It does not require solving the characteristic equation to examine the stability of the system.
(ii) It is less time-consuming method as there is no requirement of finding the determinant values.
(iii) Relative stability of the system is examined.
(iv) The value of system gain for which the system has sustained oscillations could be determined and hence the frequency of oscillations can also be determined.
(v) Helps in determining the range of system gain for which the system is stable.

6.11.5 Limitations of Routh's Criterion

The following are the limitations of Routh's criterion:

(i) Stability of the system can be examined if and only if the characteristic equation has real coefficients.
(ii) There is difficulty in providing the exact location of the closed-loop poles.
(iii) It does not suggest any method for stabilizing the unstable system.
(iv) It is applicable only to LTI systems.

Example 6.23: The characteristic equation of the system is given by $F(s) = s(s^2 + s + 1)(s + 4) + K = 0$. Investigate the range of values of gain, K, using Routh's array for the system to be stable.

Solution: The characteristic equation for the system is

$$s^4 + 5s^3 + 5s^2 + 4s + K = 0$$

Then, Routh's array is formed as:

$$\begin{array}{c|ccc} s^4 & 1 & 5 & K \\ s^3 & 5 & 4 & 0 \\ s^2 & 4.2 & K & 0 \\ s^1 & \dfrac{(16.8-5K)}{4.2} & 0 & \\ s^0 & K & & \end{array}$$

The condition for the system to be stable is that the sign of all the values in the first column should be the same.

Hence, $K > 0,\ 16.8 - 5K > 0$

Therefore, the range of values of gain, K for the system to be stable is $0 < K < 3.36$

Example 6.24: **A closed-loop system described by the transfer function is given by $\dfrac{C(s)}{R(s)} = \dfrac{1}{s^3 + \alpha s^2 + Ks + 3}$ is stable. Determine the constraints on α and K.**

Solution: The characteristic equation is $s^3 + \alpha s^2 + Ks + 3 = 0$

Routh's array is formed as follows:

$$\begin{array}{c|cc} s^3 & 1 & K \\ s^2 & \alpha & 3 \\ s^1 & \dfrac{\alpha K - 3}{\alpha} & 0 \\ s^0 & 3 & \end{array}$$

For the system to be stable, $\alpha > 0$ and $\dfrac{\alpha K - 3}{\alpha} > 0$

Thus, $\alpha > 0$ and $\alpha K > 3$

Example 6.25: **The characteristic equation of the system is given by $s^4 + s^3 + 3Ks^2 + (K + 2)s + 4 = 0$. Examine the range of values of gain K using Routh's array for the system to be stable.**

Solution: The characteristic equation given for the system is

$$s^4 + s^3 + 3Ks^2 + (K+2)s + 4 = 0$$

Then, Routh's array is formed as:

$$\begin{array}{c|ccc} s^4 & 1 & 3K & 4 \\ s^3 & 1 & K+2 & 0 \\ s^2 & 2K-2 & 4 & 0 \\ s^1 & \dfrac{(2K^2 + 2K - 8)}{(2K-2)} & 0 & \\ s^0 & 4 & & \end{array}$$

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array.

Hence, $$2K-2>0, \left(\left(2K^2+2K-8\right)/\left(2K-2\right)\right)>0$$

From the first condition, we obtain

$$K>1$$

From the second condition, we obtain

$$K^2 + K - 4 > 0 \text{ i.e., } K > 1.5615 \text{ and } K > -2.5615$$

Therefore, the range of values of gain K for the system to be stable is

$$K > 1.5615$$

Example 6.26: The feedback control system is shown in Fig. E6.26. Find the range of values of K for the system to be stable.

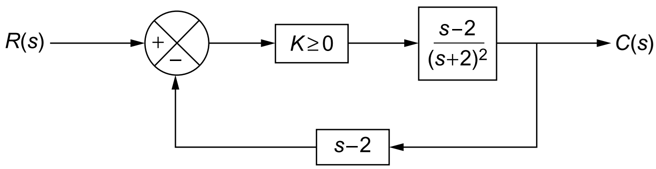


Fig. E6.26

Solution: From the block diagram, the transfer function is

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

where $$G(s) = \frac{K(s-2)}{(s+2)^2}$$

and $$H(s) = (s-2)$$

The characteristic equation is given by

$$1+G(s)H(s) = 0$$

$$1+\frac{K(s-2)}{(s+2)^2}(s-2) = 0$$

or $$(s+2)^2 + K(s-2)^2 = 0$$

or $$(1+K)s^2 + 4(1-K)s + 4K + 4 = 0$$

Routh's array is formed as:

$$\begin{array}{c|cc} s^2 & 1+K & 4+4K \\ s^1 & 4-4K & 0 \\ s^0 & 4+4K & \end{array}$$

For the system to be stable, $1+K>0$, $4+4K>0$ and $4-4K>0$. This gives $-1<K<1$.
For $0 \le K < 1$, the system will remain stable.

Example 6.27: For the function F(s) = $s^4 + Ks^3 + (K+4)s^2 + (K+3)s + 4 = 0$, find the real value of K so that the system is just oscillatory.

Solution:

$$\begin{array}{c|ccc} s^4 & 1 & (K+4) & 4 \\ s^3 & K & (K+3) & \\ s^2 & K^2+3K-3 & 4 & \\ s^1 & \dfrac{K^3+2K^2+6K-9}{K^2+3K-3} & & \\ s^0 & 4 & & \end{array}$$

From the above Routh's array, it is clear that $K>0$.

Also, $K^2+3K-3>0$

and $K^3+2K^2+6K-9>0$

$(K-1)(K^2+3K+9)>0$

The condition $K^2+3K-3>0$ and $K>0$ imposes that (K^2+3K+9) is positive. Hence, $(K-1)>0$

(K^2+3K-3) imposes that $K>-3.79$ and $K>0.79$

$(K-1)>0$ imposes that $K>1$

Hence, $K>1$ for absolute stability.
At $K=1$, the system oscillates.

Example 6.28: **Determine the positive values of K and a so that the system shown in the Fig. E6.28 oscillates at a frequency of 2 rad/sec,**

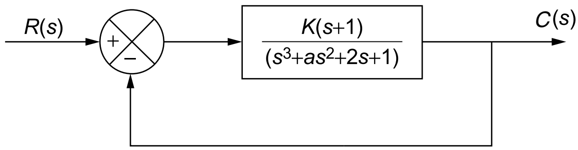


Fig. E6.28

Solution: The characteristic equation is given by

$$1+G(s)H(s)=0$$

$$1+\frac{K(s+1)}{s^3+as^2+2s+1}=0$$

$$s^3+as^2+(2+K)s+K+1=0$$

Routh's array is formed as:

$$\begin{array}{c|cc}
s^3 & 1 & 2+K \\
s^2 & a & 1+K \\
s^1 & \dfrac{(2+K)a-(1+K)}{a} & \\
s^0 & 1+K &
\end{array}$$

For the system to have oscillations,

$$\frac{a(2+K)-(K+1)}{a}=0$$

or $$a=\frac{K+1}{K+2} \tag{1}$$

Then, we have $$as^2+K+1=0 \tag{2}$$

Given $\omega = 2$ rad/sec, substituting $s=j\omega$,

$$s^2=-\omega^2=-4,$$

Thus, $$-4a+K+1=0$$

Upon solving Equations (1) and (2), we obtain $K=2$ and $a=0.75$.

Example 6.29: Determine the range of values of $K(K > 0)$ such that the characteristic equation $s^3 + 3(K + 1)s^2 + (7K + 5)s + (4K + 7) = 0$ has roots more negative than $s = -1$.

Solution: Given the characteristic equation $s^3 + 3(K+1)s^2 + (7K+5)s + (4K+7) = 0$. To find the roots more negative than $s = -1$, replace s with $(x-1)$. Then, the new characteristic equation becomes

$$(x-1)^3 + 3(K+1)(x-1)^2 + (7K+5)(x-1) + (4K+7) = 0 .$$

Simplifying, we obtain

$$x^3 + 3Kx^2 + (K+2)x + 4 = 0$$

For the new characteristic equation, Routh's array is formed as:

$$\begin{array}{c|cc} x^3 & 1 & K+2 \\ x^2 & 3K & 4 \\ x^1 & \dfrac{3K(K+2)-4}{3K} & \\ x^0 & 4 & \end{array}$$

Here, $K > 0$ and $\dfrac{3K(K+2)-4}{3K} > 0$ i.e., $K > 0.53$ and $K > -2.52$

Therefore, the range of values of K is $\infty > K > 0.53$.

Example 6.30: The open-loop transfer function of the system is given by $G(s) = \dfrac{K}{s\left[s(s+10)+T\right]}$, which has a unity feedback system.

(i) Determine the relation between K and T so that the system is stable.
(ii) Determine the relation between K and T if all the roots of characteristic equation obtained for the same system as given above have to lie on the left side of the line $s = -1$ in the s-plane.

Solution:

(i) The characteristic equation of the system with unity feedback is given by

$$1+G(s)H(s)=0$$

$$1+\frac{K}{s\left(s(s+10)+T\right)}=0$$

$$s^3+10s^2+Ts+K=0$$

Therefore, Routh's array is formed as:

s^3	1	T
s^2	10	K
s^1	$\frac{(10T-K)}{10}$	0
s^0	K	

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array. Therefore,

$$K>0, (10T-K)/10>0$$

The relation between the gain values of K and T for the system to be stable is given by $K<10T$ and its pictorial representation is shown in Fig. E6.30(a).

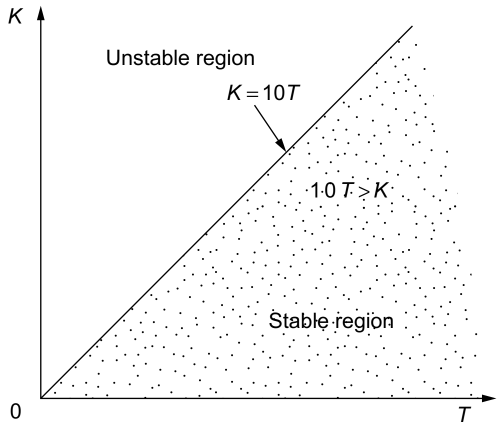


Fig. E6.30(a)

(ii) The stability condition can be determined by estimating the roots that lie on the right side of the line, $s=-1$. The following steps are carried out:

(a) Replace s by $(x-1)$ in the characteristic equation and determine the new characteristic equation:

$$(x-1)^3+10(x-1)^2+T(x-1)+K=0$$

Therefore, $$x^3+7x^2+(T-17)x+(K-T+9)=0$$

(b) Routh's array is formed as:

$$\begin{array}{c|cc} x^3 & 1 & T\text{-}17 \\ x^2 & 7 & K-T+9 \\ x^1 & \dfrac{(8T-K-128)}{7} & 0 \\ x^0 & K-T+9 & \end{array}$$

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array.

Hence, $K-T+9>0$ and $8T-K-128>0$

From the first condition, we obtain

$$K>T-9$$

From the second condition, we obtain

$$K<8T-128$$

The relation between the gain values K and T for the system to be stable is given by the above two relations and its pictorial representation is shown in Fig. E6.30(b).

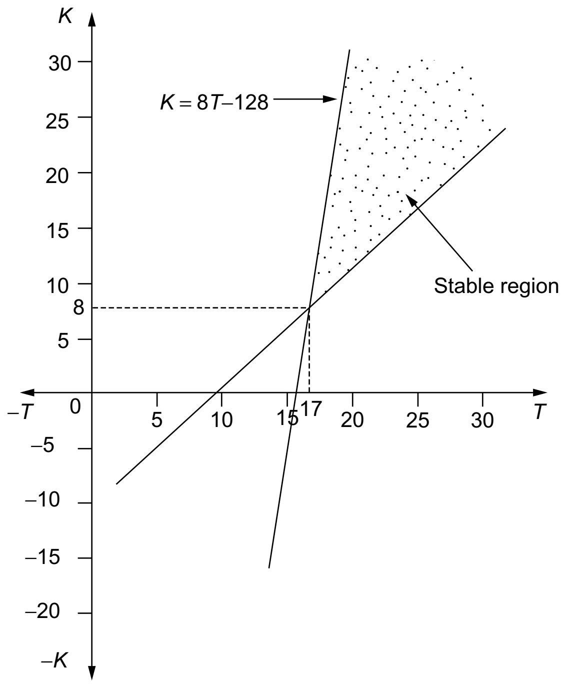


Fig. E6.30(b)

Example 6.31 The open-loop transfer function of a unity feedback control system is given by $G(s)=\dfrac{e^{-st}}{s(s+2)}$. Examine the stability of the system using Routh's array.

Solution: The characteristic equation of the unity feedback control system is given by

$$1+G(s)=0$$

Therefore, $$1+\frac{e^{-sT}}{s(s+2)}=s^2+2s+e^{-sT}=0$$

Here, $$e^{-st}=1-sT+\frac{s^2T^2}{2!}+\cdots$$

Considering the first three terms of the above expression, the characteristic equation is

$$s^2\left(1+\frac{T^2}{2}\right)+s(2-T)+1=0$$

Then, Routh's array is formed as:

$$\begin{array}{c|cc} s^2 & 1+\dfrac{T^2}{2} & 1 \\ s^1 & 2-\mathrm{T} & 0 \\ s^0 & 1 & \end{array}$$

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array.

Hence, $$T<2 \text{ and } T<\sqrt{2}$$

Thus, for all values of $T<\sqrt{2}$, the system remains stable.

Example 6.32: The open-loop transfer function of a unity feedback control system is given by $G(s)=\dfrac{K}{(s+2)(s^3+10s^2+49s+100)}$.

(i) Examine the range of values of gain K, using Routh's table for the system to be stable.

(ii) Also, determine the value of K which causes sustained oscillation in the system and then determine the frequency of sustained oscillations.

Solution:

(i) The characteristic equation of the unity feedback control system is given by

$$1+G(s)H(s)=0$$

$$1+\frac{K}{(s+2)(s^3+10s^2+49s+100)}=0$$

$$s^4+12s^3+69s^2+198s+(200+K)=0$$

Then, Routh's array is formed as:

$$\begin{array}{c|ccc} s^4 & 1 & 69 & 200+K \\ s^3 & 12 & 198 & 0 \\ s^2 & 52.5 & 200+K & 0 \\ s^1 & \dfrac{(7995-12K)}{52.5} & 0 & \\ s^0 & 200+K & & \end{array}$$

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array. Hence,

(a) $200+K>0$, therefore, $K>-200$

(b) $\dfrac{(7995-12K)}{52.5}>0$, therefore, $K<666.25$

Thus, the range of values of gain K for which the system remains stable is

$$-200<K<666.25$$

(ii) To have a sustained oscillations in the system, all the values in any row in Routh's array must be zero. Therefore, in the fourth row, if we substitute a proper value of K, we can make all the elements in that row to be zero. Hence, for $K = 666.25$, Routh's array is

$$\begin{array}{c|ccc} s^4 & 1 & 69 & 866.25 \\ s^3 & 12 & 198 & 0 \\ s^2 & 52.5 & 866.25 & 0 \\ s^1 & 0 & 0 & \end{array}$$

Hence, the system has sustained oscillations for the value of $K = 666.25$. To determine the frequency of the sustained oscillations, the auxiliary equation $A(s)$ is solved for the roots.

The auxiliary equation for the above Routh's array is given by

$$A(s) = 52.5s^2 + 866.5$$

Upon solving, we get

$$s = \pm j4.062$$

Hence, the frequency of the sustained oscillations is 4.062 rad/sec.

Example 6.33: The block diagram representation of a unity feedback control system is shown in Fig. E6.33. Determine the range of values of gain *K* using Routh's array for the system to be stable.

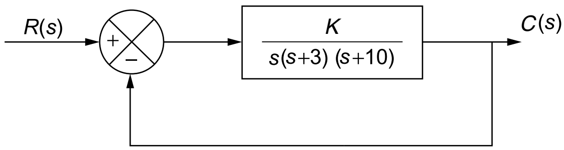


Fig. E6.33

Solution: The characteristic equation of the unity feedback control system is given:

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+3)(s+10)} = 0$$

$$s^3 + 13s^2 + 30s + K = 0$$

Then, Routh's array is formed as:

$$\begin{array}{c|cc} s^3 & 1 & 30 \\ s^2 & 13 & K \\ s^1 & \dfrac{(390-K)}{13} & 0 \\ s^0 & K & \end{array}$$

The condition for the system to be stable is that there should not be any sign change in the first column of Routh's array. Hence,

(i) $K > 0$

(ii) $\dfrac{(390-K)}{52.5} > 0$

Therefore,

$$K < 390$$

The range of values of gain for which the system remains stable is $0 < K < 390$.

Review Questions

1. Define stability?
2. What is BIBO stability?
3. Define asymptotic stability of a system.
4. Distinguish between absolute stability and relative stability.
5. Explain the concept of stability of feedback control system.
6. Write down the general form of the characteristic equation of a closed-loop system.
7. Write down the characteristic equation for $\dfrac{C(s)}{R(s)} = \dfrac{9}{s^2 + 20s + 9}$
8. What is Hurwitz criterion?
9. What are the conditions required for a system to be stable? How does Routh–Hurwitz criterion help in deciding a stable system?
10. What are the difficulties faced while applying Routh–Hurwitz criterion?
11. What are the two special cases in applying Routh–Hurwitz criteria?
12. Is Routh–Hurwitz criterion applicable if
    (i) the characteristic equation is non-algebraic?
    (ii) a few of the coefficients are not real?
13. Using Routh's criterion, determine the stability of the system represented by the characteristic equation:
    $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$. Comment on the locations of the roots of the characteristic equation.
14. Define dominant poles and zeros.
15. Explain Routh's Hurwitz stability criterion.
16. The characteristic equation of a system is given by $s^3 + 25s^2 + 100s + 4000 = 0$. Using Routh–Hurwitz criterion, find out whether the system is stable or not. Also, find the number of roots lying in the left half of the s-plane.
17. Check whether the characteristic equation $s^4 + 2s^3 + 6s^2 + 7s + 5 = 0$ represents a stable system or not.
18. For the characteristic equation $s^4 + 4.8s^3 + 4.2s^2 - 5.2s - 4.8 = 0$, find the number of roots lying to the left half of the vertical line through −1.
19. Investigate the stability using Routh–Hurwitz stability criterion, for the closed-loop system whose characteristic equation is $s^4 + s^3 - 3s^2 - s + 2 = 0$.
20. Check the stability of the following system whose characteristic equations is $s^4 - 1.7s^3 + 1.04s^2 - 0.268s + 0.024 = 0$.
21. By means of Routh–Hurwitz stability criterion, determine the stability of the system with the characteristic equation $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$. Also determine the number of roots of the equation that are in the right half of the s-plane.
22. Apply Routh's criteria to determine the number of roots in the positive half of the s-plane for the polynomial $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$.

23. The characteristic equation of a system is given by $s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10 = 0$. Determine the number of roots in the left and right half of the s-plane.
24. Examine the stability of the system whose characteristic equation is given by $s^6 + 5s^5 + 7s^4 + 22s^3 + 16s^2 + 8s + 16 = 0$
25. Investigate the characteristic equation for the distribution of its roots: $2s^7 + s^6 + 2s^5 + 6s^4 + 7s^3 + 10s^2 + 6s + 4 = 0$.
26. Consider the unity feedback system with the open-loop transfer function: $G(s) = \dfrac{10}{s(s-1)(2s+3)}$. Is this system stable?
27. Study the stability of a feedback control system whose loop transfer function is $G(s)H(s) = \dfrac{75(1+0.2s)}{s(s^2+16s+100)}$.
28. The characteristic equation of a control system is $2s^3 + 6s^2 + 6s + (K+1) = 0$. Determine the range of K for which the system is stable.
29. The characteristic equation of a system is $s^4 + 6s^3 + 12s^2 + 12s + K = 0$. Find the range of values of K for which the system will be stable.
30. Determine the range of K such that the feedback system with characteristic equation $s(s^2+s+1)(s+4) + K = 0$ is stable.
31. For the system whose characteristic equation is $s^4 + 20Ks^3 + 5s^2 + 10s + 15 = 0$, find the range of values of K for the system to be stable.
32. The open-loop transfer function of a unity feedback control system is given by $G(s) = \dfrac{K(1+\alpha s)}{s(s+1)(s+2)}$ where α and K are both positive. Represent the region of stability in the parameter plane (K, α).
33. A system oscillates with a frequency ω if it has poles at $s = \pm j\omega$ and no poles in the right half of the s-plane. Determine the values of K and a so that the system whose open-loop transfer function $G(s) = \dfrac{K(s+1)}{s^3 + as^2 + 2s + 1}$ oscillates at frequency 2 rad/sec.
34. The closed-loop transfer function of a system is given by $\dfrac{C(s)}{R(s)} = \dfrac{K}{s(s+4)(s^2+s+1)+K}$. Determine the range of K for which the system is stable.
35. Obtain the value of K for the system whose characteristic equation given by $s^3 + 3Ks^2 + (K+2)s + 4 = 0$ is to be stable.
36. Determine the range of K such that the feedback system having the characteristic equation $s(s^2+s+1)(s+4) + K = 0$ is stable.
37. Find the range of K for stability of unity feedback control system having open-loop transfer function as $G(s) = \dfrac{K}{s(s+1)(s+2)}$.

38. The characteristic equation for a feedback control system is given by $s^4 + 20s^3 + 15s^2 + 2s + K = 0$.
    (i) Find the range of K for stability.
    (ii) What is the frequency in rad/sec at which the system will oscillate?
    (iii) How many roots of the characteristic equation lie in the right half of the s-plane when $K = 5$?
39 Investigate the stability condition for the system with the characteristic equation $F(s) = (s - 1)^2 (s + 2) = 0$.
40 The open-loop transfer function of a unity feedback system is $G(s) = \dfrac{K}{s(s^2 + s + 2)(s + 3)}$.

Determine the range of values of K for which the system is stable.

This Page is intentionally left blank.

7

ROOT LOCUS TECHNIQUE

7.1 Introduction

The closed-loop poles or the roots of the characteristic equation of the system determines the absolute and relative stability of linear control systems. Investigating the trajectories of the roots of the characteristic equation is an important study in linear control systems. Root loci introduced by W. R. Evans in 1946 are the trajectories of the roots of the characteristic equation or the graphical representation of the closed-loop poles when certain system parameter varies. In addition, it is a powerful method of analysis and design for stability and transient response.

The location of closed-loop poles in the control system depends on the variable loop gain in the system. Hence, it is necessary to have a clear knowledge about how the closed-loop poles move in the s-plane when the loop gain is varied. The closed-loop poles can be moved to the desired locations by making simple adjustment in the gain. Hence, it is important to select an appropriate gain value such that the transient response characteristics are satisfactory.

In this chapter, the technique for the construction of root locus for the characteristic equation of the control system with the help of simple rules is discussed.

7.2 Advantages of Root Locus Technique

The advantages of root locus technique are:

(i) It gives a clear idea about the effect of variable loop gain adjustment with relatively small effort.
(ii) The stability of the system can be understood clearly.
(iii) It gives an idea about the transient response of the system, i.e., whether the system is over-damped, under-damped or critically damped.
(iv) Root locus technique can be applied to the discrete control system using z-transform.
(v) Root contours (Root loci for multivariable parameters) are obtained by varying one parameter at a time.

(vii) It helps in designing the system accurately by selecting gain K for a particular damping ratio.
(viii) Gain and phase margins can be determined using the root locus technique.

7.3 Categories of Root Locus

The general root locus problem is formulated by referring to

$$F(s) = D(s) + KN(s) = 0$$

where $D(s)$ is the n^{th} order polynomial of s given by $D(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$,

$N(s)$ is the m^{th} order polynomial of s given by $N(s) = s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m$,

n and m are positive integers and K is a variable loop gain that can vary from $-\infty$ to $+\infty$.

Depending on the variable loop gain in the closed-loop control system, the root locus is categorized as given in Table 7.1.

Table 7.1 | Classification of root loci based on K

Variable loop gain K	**Category**
Positive $0 \le K < \infty$	Root loci (RL) with positive K
Negative $-\infty < K \le 0$	Complementary root loci (CRL)
More than one gain value	Root contours (RC)
$-\infty < K < \infty$	Complete root loci

7.3.1 Variation of Loop Gain with the Root Locus

The block diagram of a closed-loop control system is shown in Fig. 7.1(a) and its resultant block diagram is shown in Fig. 7.1(b). The transfer function of the closed-loop control system is given by

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + 20s + K}$$

where $K = K_1 K_2$.

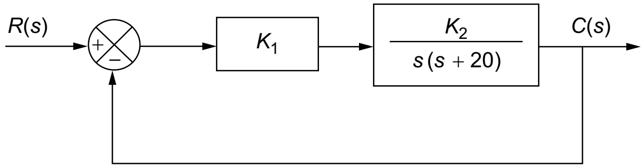


Fig. 7.1(a) | A simple closed-loop system

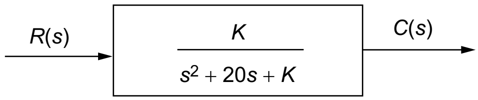


Fig. 7.1(b) | Reduced form of a closed-loop system

It is clear that the closed-loop poles of the control system depend on the gain K. But the root locus depends on the values of closed-loop poles. Let us see how the closed-loop poles and hence the root locus varies with closed-loop gain.

The values of two poles (p_1 and p_2) for different values of gain K are given in Table 7.2. The corresponding plot for the poles is shown in Fig. 7.2.

Table 7.2 | Values of poles for different gain values K

Gain K	**Pole 1**	**Pole 2**
20	−18.944	−1.055
40	−17.745	−2.254
60	−16.324	−3.675
80	−14.4721	−5.5278
100	−10	−10
120	−10 + j4.472	−10 − j4.472
140	−10 + j6.324	−10 − j6.324
160	−10 + j7.7459	−10 − j7.7459

From Table 7.2, it is clear that one of the poles moves from left to right and the other one moves from right to left as closed-loop gain increases. But when $K = 100$, both the values of poles are equal. If K is increased further, poles moves into the complex plane where one pole moves in the direction of positive imaginary axis and the other one moves in the direction of negative imaginary axis.

The root loci for the above closed-loop control system with the closed-loop poles are shown in Fig. 7.2. The gain $K = 100$, is the breakaway point.

The changes happening in the transient response of the system as the gain value varies can be inferred from the root locus. From Table 7.2, the following information is inferred and tabulated in Table 7.3.

Table 7.3 | Nature of the system based on gain K

Gain K	**Nature of poles**	**Nature of the system**
< 100	Real	Over-damped
=100	Multiple real poles	Critically damped
> 100	Conjugate complex poles	Under-damped

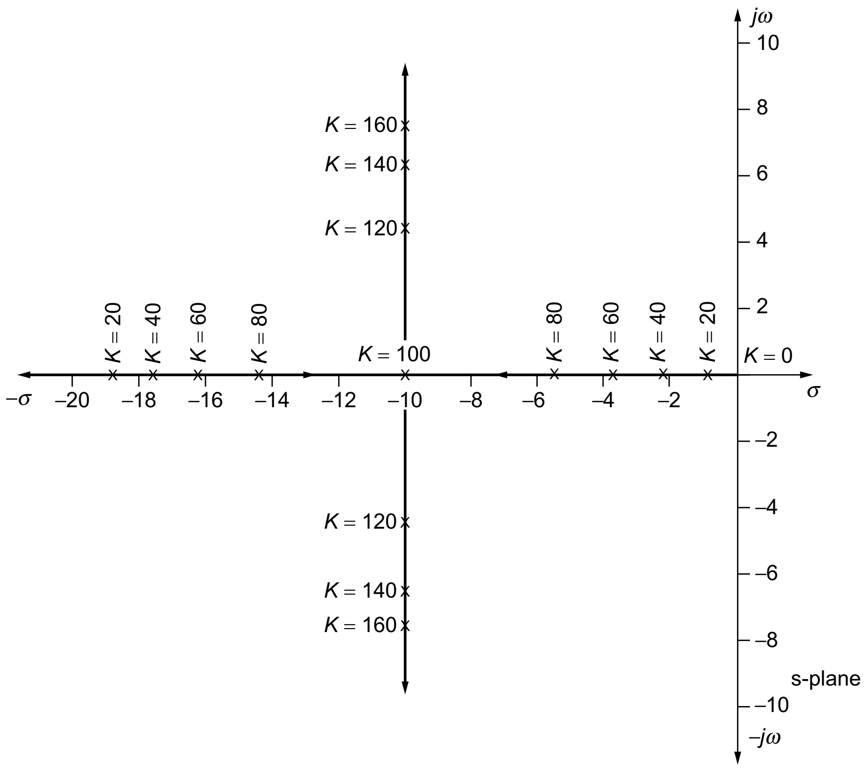


Fig. 7.2 | Root loci of the closed-loop system

7.4 Basic Properties of Root Loci

The closed-loop transfer function of a simple control system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{7.1}$$

and its characteristic equation is given by

$$1+G(s)H(s) = 0 \tag{7.2}$$

The formulation of the root loci is based on the algebraic equation expressed as

$$F(s) = D(s) + KN(s) = 0 \tag{7.3}$$

If $G(s)H(s)$ contains a variable parameter K, then the function $G(s)H(s)$ can be rewritten as

$$G(s)H(s) = \frac{KN(s)}{D(s)} \tag{7.4}$$

Substituting Eqn. (7.4) in Eqn. (7.2), we obtain

$$1+\frac{KN(s)}{D(s)}=0$$

Hence, the characteristic equation of the system becomes

$$\frac{D(s)+KN(s)}{D(s)}=0$$

Here, the numerator is similar to the general algebraic equation used for formulating the root loci. Thus, the root loci for a control system can be identified by writing the loop transfer function $G(s)H(s)$ in the form of $F(s)=D(s)+KN(s)=0$.

7.4.1 Conditions Required for Constructing the Root Loci

The characteristic equation of the closed-loop control system is expressed as the general algebraic equation used for constructing the root locus.

Here $G(s)H(s)=KG_1(s)H_1(s)$ where $G_1(s)H_1(s)$ does not contain any variable parameter K and s is a complex variable.

Hence, the characteristic equation of the system becomes

$$1+KG_1(s)H_1(s)=0$$

Therefore, $$G_1(s)H_1(s)=-\frac{1}{K}$$

From the above equation, two Evans conditions for constructing the root loci for the given system are obtained as follows:

Condition on Magnitude:

$$|G_1(s)H_1(s)|=\frac{1}{|K|} \quad \text{for } -\infty<K<\infty$$

Conditions on Angles:

(i) $\angle G_1(s)H_1(s)=(2k+1)\pi, \quad \text{for } K\geq 0$

$= \text{Odd multiple of } \pi \text{ radians or } 180^\circ$

(ii) $\angle G_1(s)H_1(s)=2k\pi, \quad \text{for } K\leq 0$

$= \text{Even multiple of } \pi \text{ radians or } 180^\circ$

where $k=0, \pm 1, \pm 2 \cdots (\text{any integer})$.

7.4.2 Usage of the Conditions

The following points help in the construction of root locus diagram in s - plane based on the above conditions.

(i) The trajectories of the root loci are determined by using the conditions on angles.
(ii) The variable parameter K on the root loci is determined by using the condition on magnitude. The value of K is determined once the root loci are drawn.

7.4.3 Analytical Expression of the Conditions

Based on the knowledge of the poles and zeros of the loop transfer function, $G(s)H(s)$, the root locus for the system is graphically constructed. Hence, the conditions required for constructing the root loci are to be determined in the form of poles and zeros.

Let us express the transfer functions $G(s)H(s)$ as

$$G(s)H(s) = KG_1(s)H_1(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where z_j is the zeros of $G(s)H(s)$ for $i=1,2,\cdots,m$ and p_j is the poles of $G(s)H(s)$ for $j=1,2,\cdots,n$

The poles and zeros of $G(s)H(s)$ can be real or purely imaginary or complex conjugate pairs.

Using the previous conditions for constructing the root loci, the conditions in terms of poles and zeros of $G(s)H(s)$ are determined.

(i) $$\left|G_1(s)H_1(s)\right| = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+p_j|} = \frac{1}{|K|}, \text{ for } \quad -\infty < K < \infty$$

The conditions on angles for different root loci are

(ii) For $0 \le K < \infty$ (RL):

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = (2k+1)\times 180°$$

(iii) For $-\infty < K \le 0$ (CRL)

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = 2k\times 180°$$

where $k=0, \pm 1, \pm 2, \cdots$

Consider a point s_1 on the root loci drawn for a system whose characteristic equation is $1+G(s)H(s)=0$.

For a positive K, any point on the RL should satisfy the following condition:

The difference in radians or degrees between the sum of angles drawn from the zeros and sum of angles drawn from poles of $G(s)H(s)$ to any point s_1 in the RL should be an odd multiple of π or 180°.

For a negative K, any point on the CRL should satisfy the following condition:

The difference in radians or degrees between the sum of angles drawn from the zeros and sum of angles drawn from poles of $G(s)H(s)$ to any point s_1 in the CRL should be an even multiple of π or 180°.

7.4.4 Determination of Variable Parameter *K*

The variable parameter K in the characteristic equation $1+G(s)H(s)=0$ can be determined for each point s_1 in the root loci by

$$|K| = \frac{\prod_{j=1}^{n} |s+p_j|}{\prod_{i=1}^{m} |s+z_i|}$$

The value of K at any point s_1 in the root loci is obtained by substituting the value s_1 in the above equation. The numerator and denominator of the above equation is graphically interpreted as the product of the length of the vectors drawn from the poles and zeros of $G(s)H(s)$ respectively.

If the point s_1 lies on the RL, then the variable parameter K has a positive value. In addition, if the point s_1 lies on the CRL, then the variable parameter K has a negative value.

Example of Constructing a Root Loci

Consider the function $G(s)H(s) = K\dfrac{(s+z_1)(s+z_2)}{s(s+p_2)(s+p_3)}$.

Here the number of zeros m and number of poles n are 2 and 3 respectively.

Consider an arbitrary point s_1 in the s-plane to verify the conditions. The vector lines are drawn from that point to each pole and zero present in the function as shown in Fig. 7.3.

The angles from each pole or zero to that arbitrary point s_1 are determined with respect to positive real axis and are denoted as $\theta_{p1}, \theta_{p2}, \theta_{p3}$ for poles and θ_{z1}, θ_{z2} for zeros as represented in Fig. 7.3.

If the arbitrary point s_1 is located in the RL, the angle condition for constructing the root loci is given by

$$\angle G(s)H(s) = \angle(s_1+z_1) + \angle(s_1+z_2) - \angle s_1 - \angle(s_1+p_2) - \angle(s_1+p_3)$$

$$= \theta_{z1} + \theta_{z2} - \theta_{p1} - \theta_{p2} - \theta_{p3} = (2k+1)\times 180^\circ$$

where $k = 0, \pm 1, \pm 2, \cdots$.

Similarly, if the arbitrary point s_1 is located in the CRL, the angle condition for constructing the root loci is given by

$$\angle G(s)H(s) = \angle(s_1+z_1) + \angle(s_1+z_2) - \angle s_1 - \angle(s_1+p_2) - \angle(s_1+p_3)$$

$$= \theta_{z1} + \theta_{z2} - \theta_{p1} - \theta_{p2} - \theta_{p3} = 2k \times 180^\circ$$

where $k = 0, \pm 1, \pm 2, \cdots$

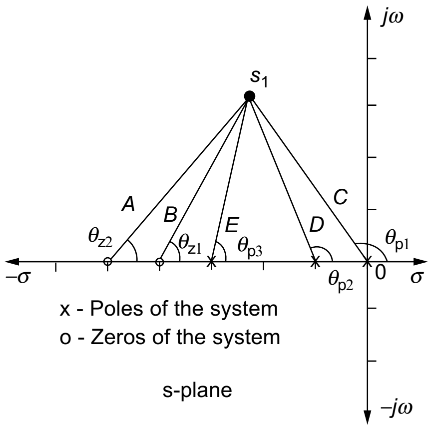


Fig. 7.3 | Vector lines from an arbitrary point

Using the arbitrary point s_1, the magnitude of the variable parameter K is obtained by

$$|K| = \frac{\prod_{j=1}^{n} |s + p_j|}{\prod_{i=1}^{m} |s + z_i|}$$

From Fig. 7.3, the lengths of vectors drawn from each pole and zero are noted down. Now, the variable parameter K is given by

$$|K| = \frac{|s_1 + p_1||s_1 + p_2||s_1 + p_3|}{|s_1 + z_1||s_1 + z_2|} = \frac{C \times D \times E}{A \times B}$$

The sign of the variable parameter K depends on whether the root loci are RL or CRL.

In general, for the function $G(s)H(s)$ with multiplying factor as the variable parameter K with the known poles and zeros of the function, the construction of root loci follows the two steps as given below:

(i) The arbitrary point s_1 in the s-plane is identified, which satisfies the respective angle condition for different root loci (RL or CRL).

(ii) The variable parameter K on the root loci is obtained using the formula.

Example 7.1: **The loop transfer function of the system with a negative feedback system is given by $G(s)H(s) = \frac{K(s+1)}{s(s+2)}$. Consider an arbitrary point $s_1 = -3 + j4$ in s-plane. Determine whether the point s_1 lies in the RL or CRL.**

Solution: The phase angle of the loop transfer function at s_1 is

$$\begin{aligned}\angle G_1(s)H_1(s)\Big|_{s=-3+j4} &= \angle(-3+j4+1)-\angle(-3+j4+2)-\angle(-3+j4)\\ &= \angle(-2+j4)-\angle(-1+j4)-\angle(-3+j4)\\ &= \tan^{-1}\left(-\frac{4}{2}\right)-\tan^{-1}(-4)-\tan^{-1}\left(-\frac{4}{3}\right)\\ &= -63.43^\circ-\left(-75.96^\circ\right)-\left(-53.10^\circ\right)=65.63^\circ\end{aligned}$$

Since this angle is neither an odd multiple nor an even multiple of 180°, the arbitrary point s_1 does not lie either on RL or CRL.

7.4.5 Minimum and Non-Minimum Phase Systems

If all the closed-loop zeros of the system lie in the left half of the *s*-plane, then the system is called non-minimum phase system; and if any one of the closed-loop zeros of the system lies in the right half of the *s*-plane, then the system is called minimum phase system.

7.5 Manual Construction of Root Loci

The trial-and-error method of determining the variable parameter K and an arbitrary point s_1 in the *s*-plane that satisfies the angle condition for different types of root loci (RL or CRL) is a tedious process. To overcome these difficulties, Evans invented a special tool called Spirule (which assists in addition and subtraction of angles of vectors). But this method also has its difficulties to determine the unknown parameter K.

7.5.1 Properties / Guidelines for Constructing the Root Loci

It is necessary to understand the properties, guidelines or rules for constructing the root loci for a system manually. The properties are based on the relation between the poles and zeros of $G(s)H(s)$ and the roots of the characteristic equation $1+G(s)H(s)=0$.

Property 1: The points on the root loci at $K=0$ and $K=\pm\infty$

(i) The points on the root loci corresponding to the variable parameter $K=0$ are the poles of $G(s)H(s)$.

(ii) The points on the root loci corresponding to the variable parameter $K=\pm\infty$ are the zeros of $G(s)H(s)$.

Consider a system whose characteristic equation is given by

$$(s+a)(s+b)(s+c)+K(s+d)(s+e)=0$$

When $K=0$, the roots of the given equation are $-a,-b$ and $-c$. Similarly, when $K=\pm\infty$, the roots of the equation are $-d,-e$ and $-\infty$. Since the order of the equation is 3, the third root lies at ∞.

The above equation can be written as

$$1+\frac{K(s+d)(s+e)}{(s+a)(s+b)(s+c)}=0$$

Therefore, $$G(s)H(s)=\frac{K(s+d)(s+e)}{(s+a)(s+b)(s+c)}$$

Thus, the poles and zeros of the given system are $-a,-b,-c$ and $-d,-e$ respectively.

Hence, it is clear that the roots of the characteristic equation at $K=0$ are the poles of the transfer function $G(s)H(s)$ and the roots of the characteristic equation at $K=\pm\infty$ including one at ∞ are the zeros of the of the transfer function $G(s)H(s)$.

Property 2: Branches on the root loci

The locus of any one of the roots of the characteristic equation when the variable parameter K varies from $-\infty$ to $+\infty$ is called the branch of the root loci.

The number of branches present in the root loci is equal to the number of poles or zeros present in the loop-transfer function depending on the values. Therefore, the condition by which the number of branches can be determined is given below:

Let n be the number of poles present in the loop transfer function and
m be the number of zeros present in the loop transfer function.

The number of branches of root loci is determined as:

If $n>m$, number of branches of root loci = n
If $m>n$, number of branches of root loci = m
If $m=n$, number of branches of root loci = m or n.

For the same characteristic equation of the system discussed in the previous property, $(s+a)(s+b)(s+c)+K(s+d)(s+e)=0$, the number of branches of the root loci is equal to 3, since the order of the characteristic equation is 3.

Property 3: Root locus symmetry

The root locus of any system is symmetrical on the real axis of the s-plane. The root locus of the system may be symmetrical about the poles and zeros of the system or to any point on the real axis of the s-plane.

The root locus of any system may be symmetrical to one or more points on the real axis of the plane. It is also possible that the root locus may be symmetrical to a complex conjugate point.

Property 4: Intersection of asymptotes with the real axis of *s*-plane

All the asymptotes present in the root loci of any system will intersect at a particular point on the real axis of the s-plane. The particular point on the real axis where the asymptotes intersect is called centroid.

The formula for determining the centroid is given by

$$\sigma_1=\frac{\sum \text{finite poles of } G(s)H(s)-\sum \text{finite zeros of } G(s)H(s)}{n-m}$$

The centroid obtained using the above formula is a real number even when the poles or zeros are either real or complex conjugate pairs.

Property 5: Asymptotes of root loci and its angles and behaviour of root loci at $|s| = \infty$

We know that the characteristic equation of any system $1+G(s)H(s)=0$ can be replaced by $1+K\dfrac{N(s)}{D(s)}=0$, i.e., $1+K\dfrac{s^m+b_1s^{m-1}+\cdots+b_{m-1}s^1+s^0}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s^1+s^0}=0$. If the order of the polynomials $D(s)$ and $N(s)$ are not equal (i.e., $m\neq n$), then some of the branches of the root loci approach infinity in the s-plane.

Asymptotes are used to describe the properties of root loci that approach infinity in the s-plane when $|s|\to\infty$. The number of asymptotes N_a for a system when $m\neq n$ will be $|n-m|$, which describes the root loci behaviour at $|s|=\infty$. The angle of asymptotes is determined by

$$\theta_i = \frac{2i+1}{|n-m|}\times 180^\circ, \qquad \text{for } n\neq m \tag{7.5}$$

where $i = 0, 1, 2, \cdots, (|n-m|-1)$, n is the number of finite poles of $G(s)H(s)$ and m is the number of finite poles of $G(s)H(s)$

The asymptotes are drawn from the centroid σ by using the angle of asymptotes.

The asymptotes, the angle of asymptotes and the centroid (i.e., intersection of asymptotes on the real axis) for a transfer function $G(s)H(s)=K\dfrac{(s+1)}{s(s+3)(s^2+2s+2)}$ are shown in Fig. 7.4.

For the given example, the number of asymptotes N_a, is 3(since $|n-m|=3$).

In Fig. 7.4, A_1, A_2 and A_3 are the asymptotes whose angles are 60°, 180° and 300° which are obtained using Eqn. (7.5).

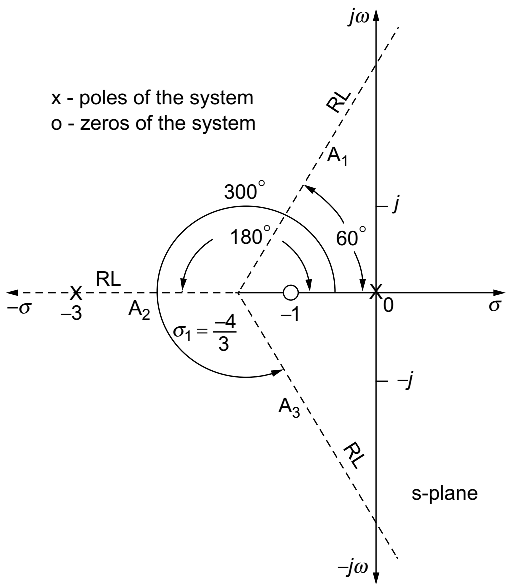
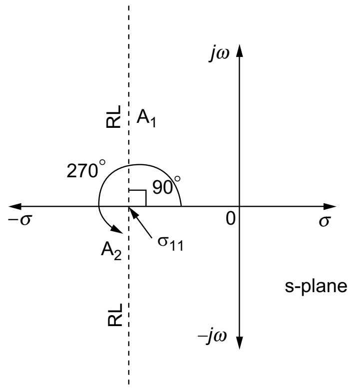
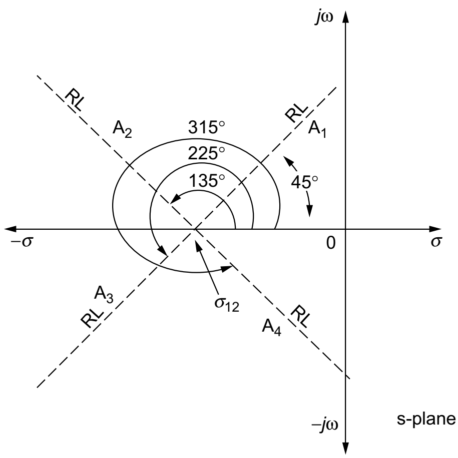


Fig. 7.4 | Asymptotes, angle of asymptotes and centroid

Figs 7.5(a) through (c) show the asymptotes, the angle of asymptotes and the centroid for different types of system.

Root loci	**Number of asymptotes, centroid and its corresponding transfer function**
 **Fig. 7.5(a)**	Transfer function, $G(s)H(s)=\dfrac{K}{s(s+p_1)}$ Number of asymptotes $N_a = 2-0 = 2$ (A_1 and A_2) Centroid $\sigma_{11} = -\dfrac{p_1}{2}$ Angle of asymptotes: $\theta_i = \dfrac{(2i+1)\times 180^\circ}{2}$ for $i = 0$ to 1 Hence, $\theta_0 = 90^\circ$ and $\theta_1 = 270^\circ$
 **Fig. 7.5(b)**	Transfer function, $G(s)H(s)=\dfrac{K}{s(s+p_1)(s+p_2)(s+p_3)}$ or $=\dfrac{K(s+z_1)}{s^2(s+p_1)(s+p_2)(s+p_3)}$ Number of asymptotes $N_a = 4$ (A_1, A_2, A_3, and A_4) Centroid, $\sigma_{12} = -\dfrac{(p_1+p_2+p_3)}{4}$ or $=\dfrac{z_1-(p_1+p_2+p_3)}{4}$ Angle of asymptotes: $\theta_i = \dfrac{(2i+1)\times 180^\circ}{4}$, for $i = 0$ to 3 Hence, $\theta_0 = 45^\circ, \theta_1 = 135^\circ, \theta_2 = 225^\circ$ and $\theta_3 = 315^\circ$

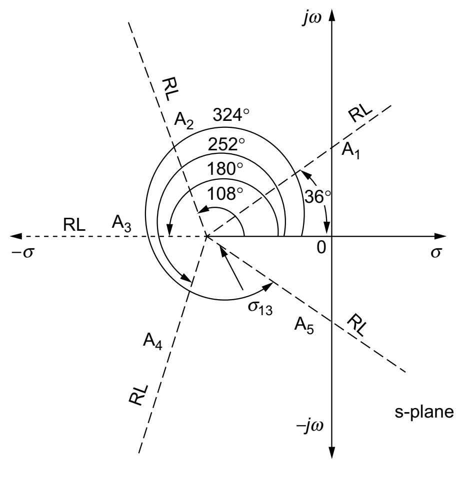


Fig.7.5(c)

Transfer function,

$$G(s)H(s)=\frac{K}{s^2(s+p_1)(s+p_2)(s+p_3)}$$

Number of asymptotes N_a = 5 – 0 = 5 (A_1, A_2, A_3, A_4 and A_5)

Centroid, $\sigma_{13}=-\dfrac{(p_1+p_2+p_3)}{5}$

Angle of asymptotes:

$$\theta_i=\frac{(2i+1)\times180^\circ}{5},\ \text{for } i=0 \text{ to } 4$$

Hence,

$\theta_0=36^\circ$, $\theta_1=108^\circ$, $\theta_2=180^\circ$, $\theta_3=252^\circ$ and $\theta_4=324^\circ$

Property 6: Root loci on the real axis

In the root loci, each branch on the real axis starts from a pole and ends at a zero or infinity. Each branch on the real axis is determined based on a condition given below by assuming that we are searching for a branch of root loci on the real axis from right to left.

For a branch to exist on the real axis after a particular point, the summation of number of poles and zeros of $G(s)H(s)$ to the right of that particular point should be an odd number.

The point at which the above condition is checked may be pole or zero which lies on the real axis. This condition for determining the branches of root loci on the real axis does not get affected by the complex poles and zeros of $G(s)H(s)$.

For the system whose transfer function is given by $G(s)H(s)=K\dfrac{(s+1)}{s(s+3)(s^2+2s+2)}$, the branches of root loci are indicated in Fig. 7.6.

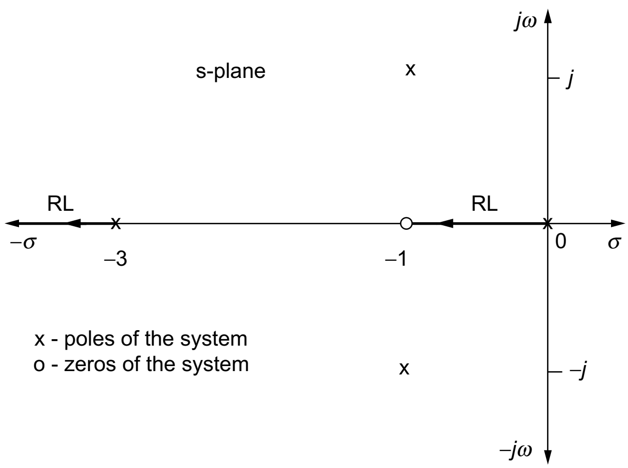


Fig. 7.6 | Indication of branches of root loci on the real axis

Here, if we look to the right of the point –1 (zero of the system), only one pole exists (i.e., odd number). Hence, a branch of root loci exists between –1 and 0. Now, if we look from infinity, the total number of poles and zeros existing to the right of the point is 5 (odd number). Hence, a branch of root loci exists between $-\infty$ and -3.

Figures 7.7(a) and (b) indicate the branch of root loci existing on the real axis for two general systems.

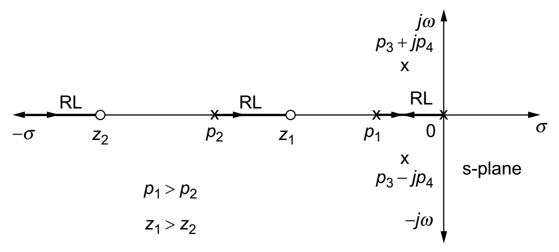


Fig. 7.7(a)

Transfer function of the system

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)}{s(s+p_1)(s+p_2)(s+p_3 \pm jp_4)}$$

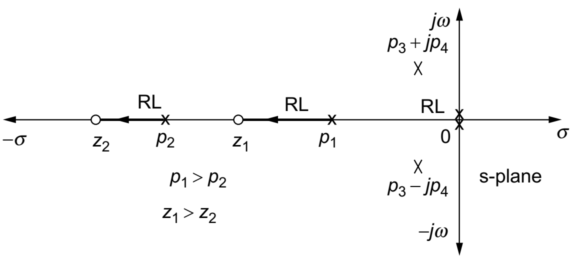


Fig. 7.7(b)

Transfer function of the system

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)(s+p_3 \pm jp_4)}$$

From Figs. 7.7(a) and (b), it can be inferred that the addition of one pole at the origin changes the branch of root loci existing on the real axis.

Property 7: Angle of departure and angle of arrival in root loci

In a root locus, the angle of departure or arrival is determined for a complex pole or complex zero respectively. This angle indicates the angle of the tangent to the root locus.

The angle of departure θ_d and angle of arrival θ_a is determined using

$$\theta_d = 180^\circ - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$\theta_a = 180^\circ + \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex zero} - \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex zero}$$

For the system whose poles are denoted as shown in Fig. 7.8(a), the angle of departure of the complex pole is determined using the formula.

The angle of departure for the pole $-a + j\omega b$ is $\theta_{d1} = -\left(\phi_{p1} + \phi_{p2}\right)$ and the angle of departure for the pole $-a - j\omega b$ is $\theta_{d2} = -\theta_{d1}$.

For the system whose zeros are denoted as shown in Fig. 7.8(b), the angle of arrival of the complex zero is determined using the formula.

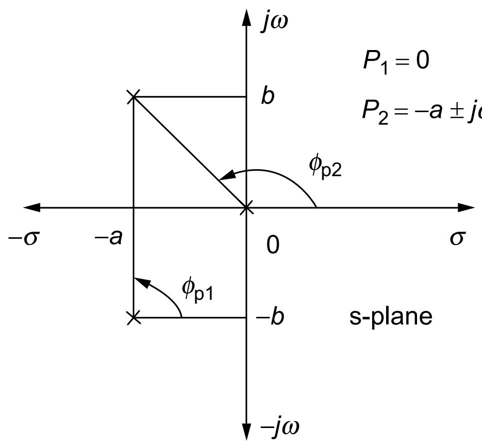


Fig. 7.8(a) | Angle of departure

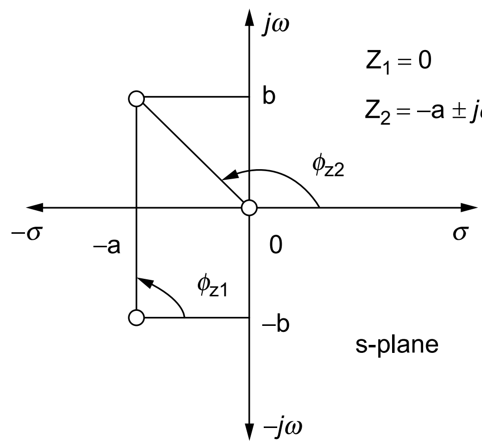


Fig. 7.8(b) | Angle of arrival

The angle of arrival for the zero $-a + j\omega b$ is $\theta_{a1} = -\left(\phi_{z1} + \phi_{z2}\right)$ and the angle of arrival for the zero $-a - j\omega b$ is $\theta_{a2} = -\theta_{a1}$.

Property 8: Intersection of root loci with the imaginary axis

The intersection point of the root loci on the imaginary axis and the corresponding variable parameter K can be determined using two ways.

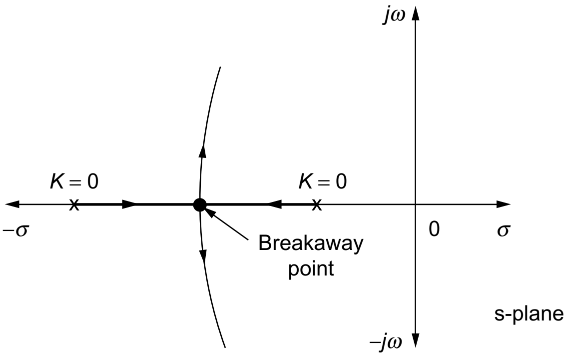
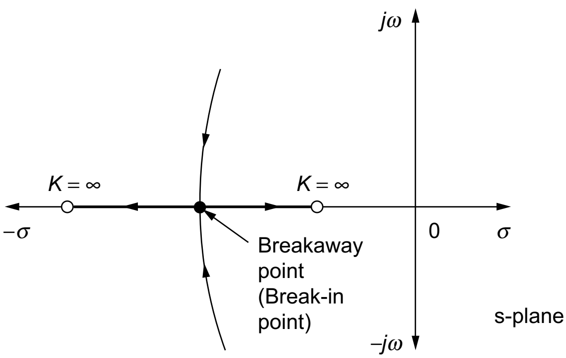
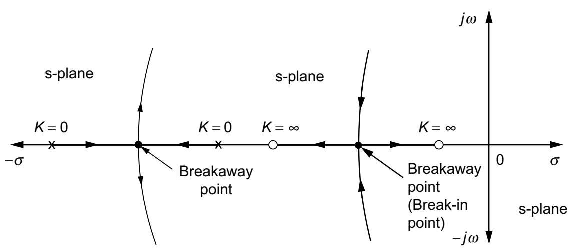
(i) Using Routh–Hurwitz criterion.

(ii) Substituting $s = j\omega$ in the characteristic equation and equating the real and complex parts to zero, the intersection point on the imaginary axis ω can be obtained and its corresponding gain value K.

Property 9: Breakaway or break-in points on the root loci

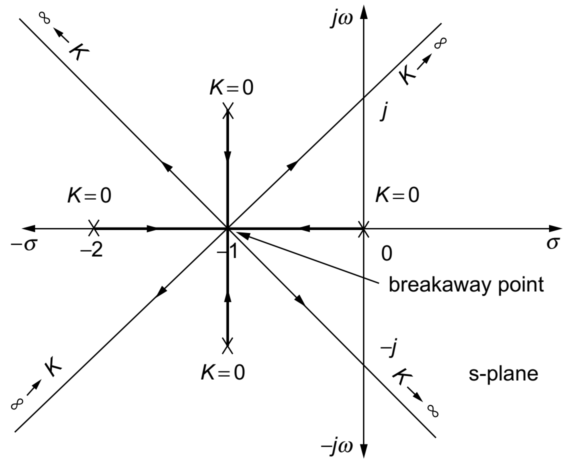
The point on the s-plane where two or more branches of the root loci arrive and then depart in the opposite direction is called breakaway point or saddle point. The point on the s-plane where two or more branches of the root loci arrive and then depart in opposite direction is called break-in point. The illustrations of different types of breakaway points are given in Table 7.4.

Table 7.4 | Types of breakaway points

Indication	Graphical indication	Inference
When a branch exists between two adjacent poles		A breakaway point exists between those two adjacent poles. (Root locus starts from those two poles, meets at breakaway point and departs in the opposite direction)
When a branch exists between two adjacent zeros		A breakaway point (break-in point) exists between those two adjacent zeros. (Root locus starts from two poles, meets at break-in point and departs in the opposite direction along the real axis)
When a branch exists between two adjacent poles and branch between two adjacent zeros		From the diagram, it is clear that for a system both breakaway and break-in points exist.

(*Continued*)

Table 7.4 | (Continued)

Breakaway point can exist when the branch of root locus from complex pole reaches the branch of root loci between the two real poles		From the diagram, it is clear that for a system, a common breakaway point may exist for more than one branch.

The root locus diagram for a particular system can have more than one breakaway point/break-in point and it is not necessary to have the breakaway point/break-in point only on the real axis. Due to the complex symmetry of root loci, the breakaway point/break-in point may be a complex conjugate pair.

The breakaway/break-in points on the root loci for a particular system whose characteristic equation is represented by $1+KG_1(s)H_1(s)=0$ must satisfy the following condition:

$$\frac{dG_1(s)H_1(s)}{ds}=0 \tag{7.6}$$

It is noted that all the breakaway/break-in points on the branch of the root loci must satisfy the above condition, but not all the solutions obtained using the equation are breakaway points. In other words, we can say that the above condition is necessary but not sufficient. Hence, in addition to the above condition, the breakaway point must also satisfy the condition, $1+KG_1(s)H_1(s)=0$, for some real variable parameter K.

We know that $K=-\dfrac{1}{G_1(s)H_1(s)}$

Taking the derivative on both sides of the equation with respect to the variable s, we obtain

$$\frac{dK}{ds}=\frac{dG_1(s)H_1(s)/ds}{\left(G_1(s)H_1(s)\right)^2}$$

Thus, the condition for breakaway point can also be written as

$$\frac{dK}{ds}=0 \tag{7.7}$$

All the solutions obtained by solving Eqn. (7.6) or Eqn. (7.7) cannot be valid breakaway/break-in points. Let $s_1, s_2, \cdots$ be the solutions obtained by solving the equation. Each solution s_i is said to be a valid or invalid breakaway point/break-in points based on the condition given in Table 7.5.

Table 7.5 | Validation of breakaway point

S. no.	Value of s_i	Gain K at s_i	Angle condition	Breakaway/ break-in points
1	Real value	Positive ($K > 0$)	Not required	Valid
2	Real value	Negative ($K < 0$)	Not required	Invalid
3	Complex number	Positive ($K > 0$)	$\angle G_1(s)H_1(s)\big\|_{at\ s_i} = k\pi$ where $k = 1,3,5\cdots$.	Valid
4	Complex number	Positive ($K > 0$)	$\angle G_1(s)H_1(s)\big\|_{at\ s_i} = k\pi$ where $k = 2,4,6\cdots$.	Invalid
5	Complex number	Negative ($K < 0$)	Not required	Invalid
6	Complex number	Complex number	$\angle G_1(s)H_1(s)\big\|_{at\ s_i} = k\pi$ where $k = 1,3,5\cdots$.	Valid
7	Complex number	Complex number	$\angle G_1(s)H_1(s)\big\|_{at\ s_i} = k\pi$ where $k = 2,4,6\cdots$.	Invalid

Inference about the breakaway point

(i) If the solutions obtained using Eqn. (7.6) or Eqn. (7.7) are all real values and the value of K at those real values are positive, then those values are called breakaway points on the root loci.

(ii) If the solutions obtained using Eqn. (7.6) or Eqn. (7.7) are complex conjugate values and if they also satisfy the condition $\angle G_1(s)H_1(s) = k\pi$ where $k = 1, 3, 5\cdots$ (i.e., angle of $G_1(s)H_1(s)$ at the breakaway point should be an odd multiple of 180°), then it is said to be a breakaway point of the root loci.

Gain at the breakaway point

$$|K| = \frac{1}{\left|G_1(s)H_1(s)\right|_{s=\text{breakaway point}}}$$

Property 10: Root sensitivity

The root sensitivity is defined as the sensitivity of the roots of the characteristic equation when the variable parameter K varies. The root sensitivity is given by

$$S_K = \frac{ds/s}{dK/K} = \frac{K}{s}\frac{ds}{dK}$$

Hence, at the breakaway point, the root sensitivity of the characteristic equation becomes infinite (since $\frac{dK}{ds} = 0$ at breakaway point). The value of K should not be selected to operate the system at breakaway point.

Property 11: Determination of variable parameter K on the root loci

If the root locus for the system is constructed manually, then the variable parameter K at any point s_1 on the root loci can be determined by

$$|K| = \frac{\prod_{j=1}^{n} |s + p_j|}{\prod_{i=1}^{m} |s + z_i|}$$

But graphically, the variable parameter K is determined by

$$|K| = \frac{\prod \text{Length of vectors drawn from the poles of } G_1(s)H_1(s) \text{ to } s_1}{\prod \text{Length of vectors drawn from the zeros of } G_1(s)H_1(s) \text{ to } s_1}$$

The properties to plot the root loci are tabulated in Table 7.6.

Table 7.6 | Properties to construct root loci

S. no.	Property	Inference
1	At starting point $K = 0$	Poles of $G(s)H(s)$ including at ∞
2	At terminating point $K = \infty$	Zeros of $G(s)H(s)$ including at ∞
3	Branches of root loci	Number of branches = Order of the system
4	Symmetry of root loci	Symmetrical to the real axis in the s-plane.
5	Asymptotes of root loci	Number of asymptotes $N_a = \|n - m\|$ where n is the number of poles and m is the number of zeros
6	Angle of asymptotes	$\theta_i = \frac{2i+1}{\|n-m\|} \times 180^\circ, \quad \textit{for } n \neq m$ where $i = 0, 1, 2, \cdots, (\|n-m\| - 1)$

(*Continued*)

Table 7.6 (Continued)

S. no.	Property	Inference
7	Intersection of asymptotes on real axis (centroid point) σ_1	$\dfrac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$
8	Branches of root loci on the real axis	Condition for a branch to exist: Total number of poles and zeros to the right side of the point should be an odd number.
9	Angle of departure or angle of arrival for a complex pole or complex zero	$\theta_d = 180^\circ - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$ $\theta_a = 180^\circ + \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex zero} - \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex zero}$
10	Intersection of root loci with imaginary axis	Determined using Routh–Hurwitz criterion or by substituting $s = j\omega$ in the characteristic equation and then comparing the real and imaginary parts.
11	Breakaway point or break-in point	From the characteristic equation, determine K and then solve $\dfrac{dK}{ds} = 0$ to determine the breakaway/break-in points.
12	Variable parameter K in the root loci	$\lvert K \rvert = \dfrac{\prod_{j=1}^{n} \lvert s + p_j \rvert}{\prod_{i=1}^{m} \lvert s + z_i \rvert}$ $= \dfrac{\prod \text{Length of vectors drawn from the poles of } G_1(s)H_1(s) \text{ to } s_1}{\prod \text{Length of vectors drawn from the zeros of } G_1(s)H_1(s) \text{ to } z_1}$

7.5.2 Flow Chart for Constructing the Root Locus for a System

The flow chart for constructing the root locus for a system whose loop transfer function is known is shown in Fig. 7.9.

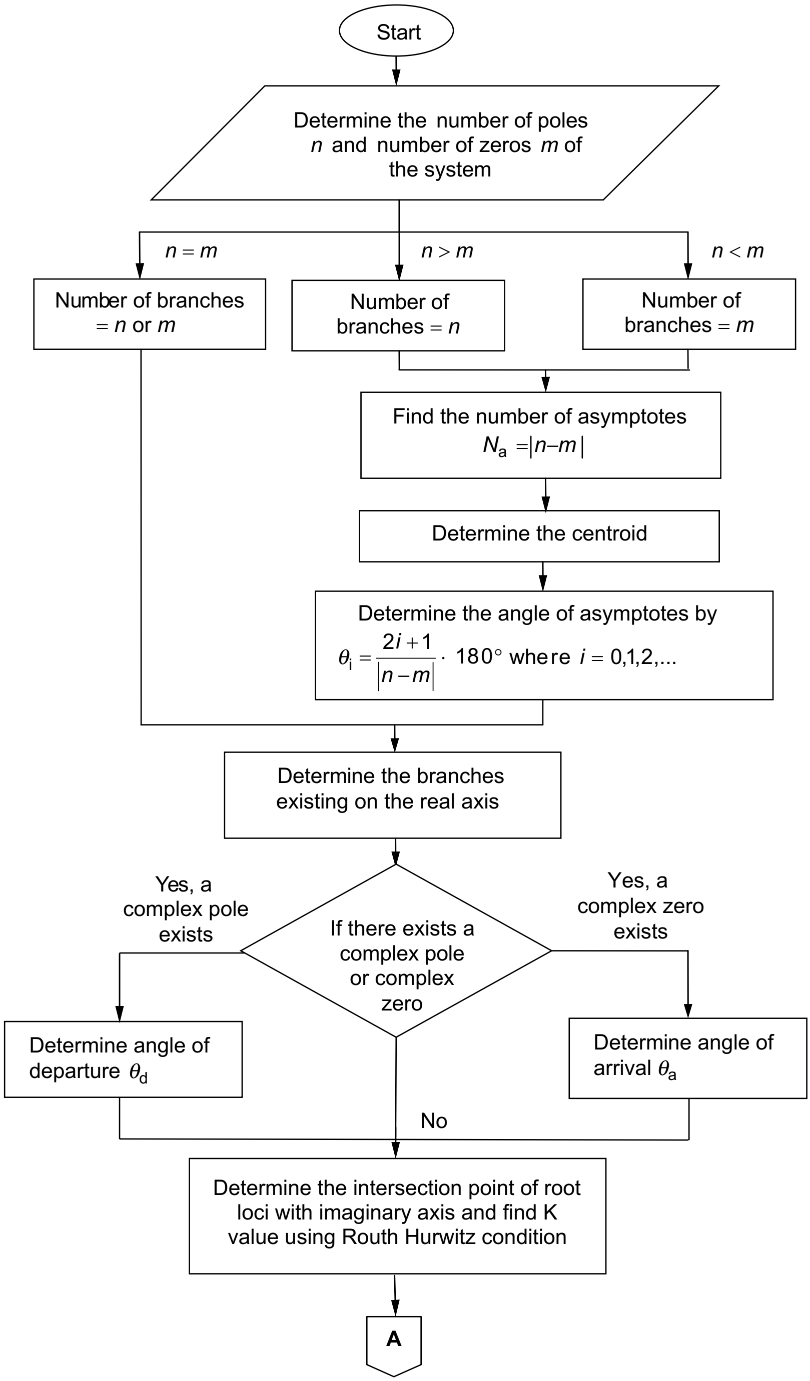


Fig. 7.9 | Flow chart for construction of root locus

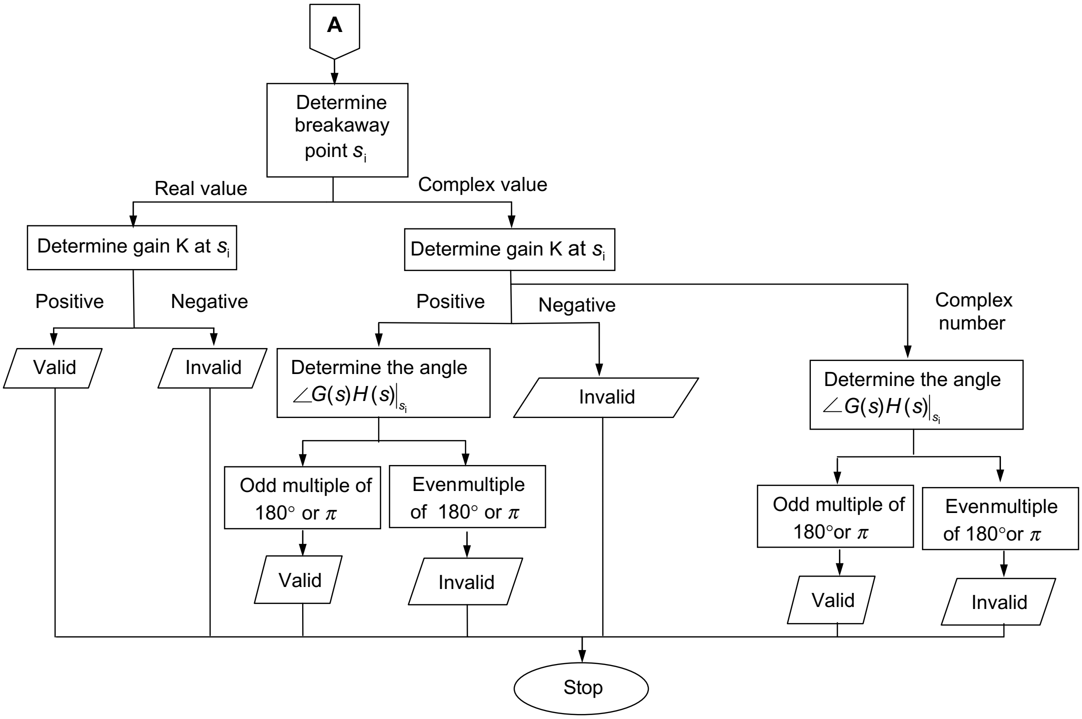
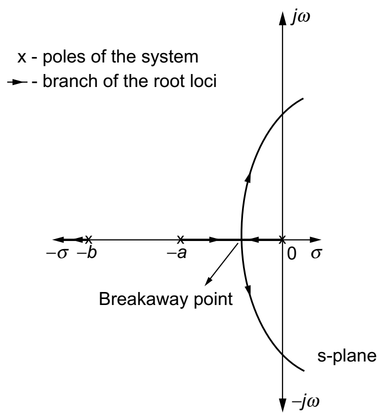


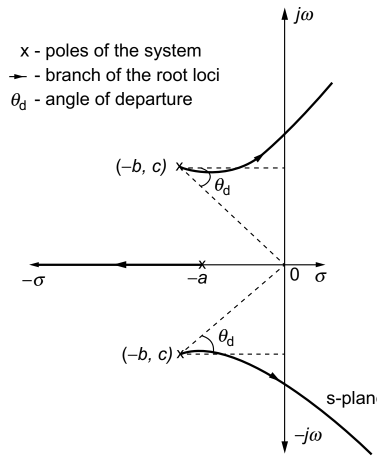
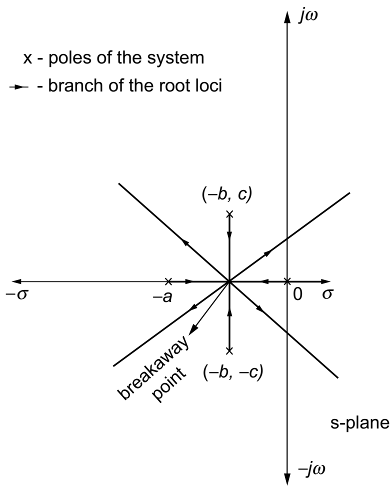
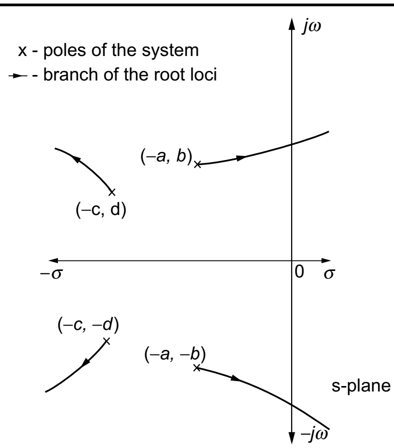
Fig. 7.9 | (Continued)

7.6 Root Loci for different Pole-Zero Configurations

The root loci for different pole-zero configuration are given in Table 7.7. The shape of the root loci for the system depends on the relative position of poles and zeros of the system.

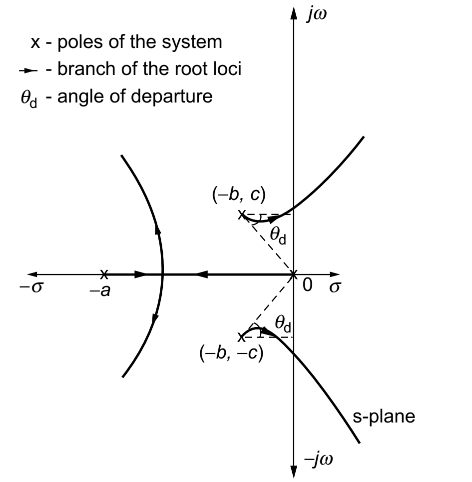
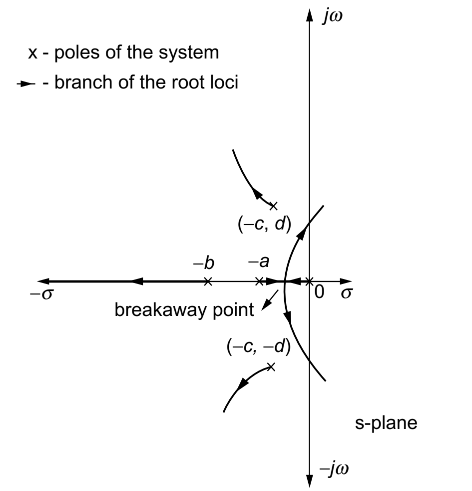
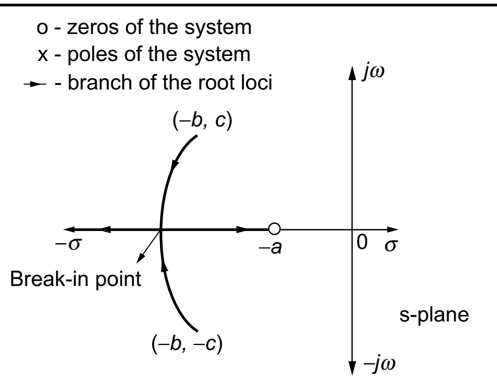
Table 7.7 | Root loci for different pole-zero configuration of the system

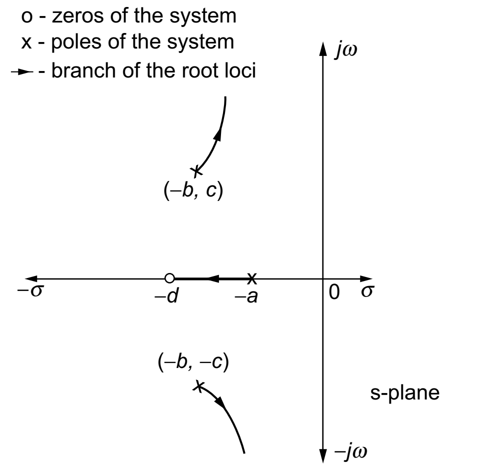
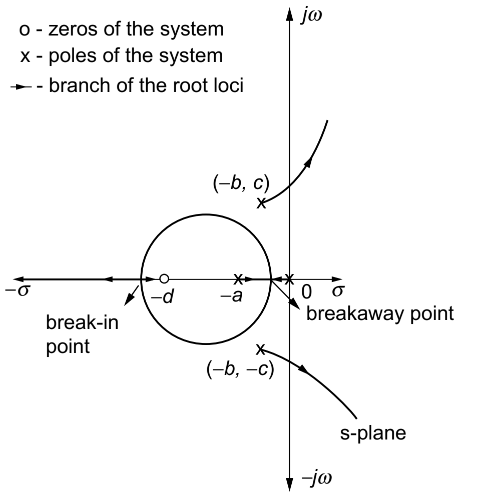
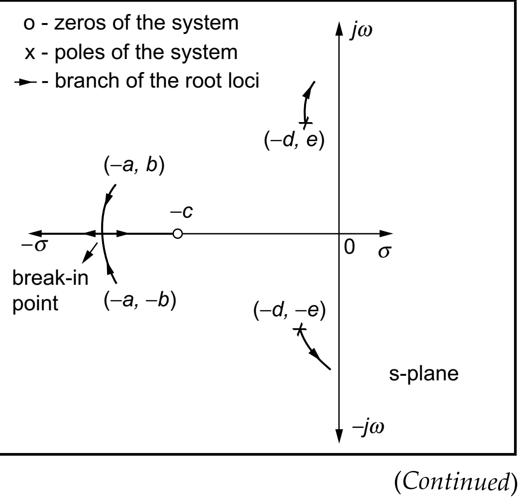
Loop transfer function of the system $G(s)H(s)$	Root loci for the system
$\dfrac{1}{s(s+a)(s+b)}, b > a$	

$\dfrac{1}{(s+a)(s+b+jc)(s+b-jc)}, b>a$	
$\dfrac{1}{s(s+a)(s+b+jc)(s+b-jc)}, a>b$	
$\dfrac{1}{(s+a+jb)(s+a+jb)(s+c+jd)(s+c-jd)}$, $\mathrm{c}>a, \mathrm{b}>\mathrm{d}$	

(*Continued*)

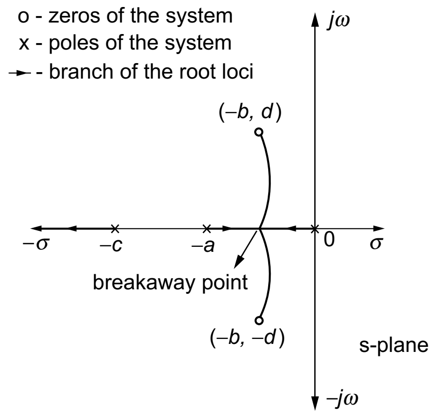
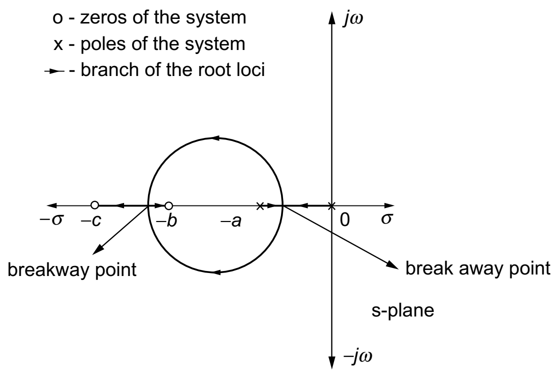
Table 7.7 (Continued)

Loop transfer function of the system $G(s)H(s)$	Root loci for the system
$\dfrac{1}{s(s+a)(s+b+jc)(s+b-jc)}, a>b$	
$\dfrac{1}{s(s+a)(s+b)(s+c+jd)(s+c-jd)}, b>a>c$	
$\dfrac{(s+a)}{(s+b+jc)(s+b-jc)}, b>a$	

$\dfrac{(s+d)}{(s+a)(s+b+jc)(s+b-jc)}, d>b>a$	
$\dfrac{(s+d)}{s(s+a)(s+b+jc)(s+b-jc)}, d>a>b$	
$\dfrac{(s+c)}{(s+a+jb)(s+a-jb)(s+d+je)(s+d-je)},$ $a>c>d\ ,\ b<e$	

(*Continued*)

Table 7.7 (Continued)

Loop transfer function of the system $G(s)H(s)$	**Root loci for the system**
$\dfrac{(s+b+jd)(s+b-jd)}{s(s+a)(s+c)}, c>b>a$	
$\dfrac{(s+b)(s+c)}{s(s+a)}, c>b>a$	

Example 7.2: Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s+4)(s^2+4s+20)}$.

Solution:

(i) For the given system, poles are at $0, -4, -2+j4, -2-j4$, i.e., $n=4$ and zeros do not exist, i.e., $m=0$.

(ii) Since $n>m$, the number of branches of the root loci for the given system is $n=4$

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 4-0=4$

(b) Angles of asymptotes,

$$\theta_i = \frac{2i+1}{|n-m|}\times 180^\circ, \quad \text{for} \quad n\neq m,\ i=0,1,2 \text{ and } 3$$

Therefore, $\theta_0 = 45^\circ$, $\theta_1 = 135^\circ$, $\theta_2 = 225^\circ$ and $\theta_3 = 315^\circ$.

(c) Centroid, $\sigma_1 = \dfrac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zero of } G(s)H(s)}{n-m}$

$$= \frac{-4-2+4j-2-4j-0}{4} = -2$$

(iv) Since complex poles exist, angle of departure exists for the system.
To determine the angle of departure θ_d:
(a) For the pole $-2+j4$,

$$\theta_{d1} = 180° - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$= 180° - (\phi_{p1} + \phi_{p2} + \phi_{p3}) = 180° - (90 + \alpha_1 + 180° - \alpha_2)$$

$$= 180° - \left(90° + \tan^{-1}\left(\frac{4}{2}\right) + 180° - \tan\left(\frac{4}{2}\right)\right) = -90°$$

(b) For the pole $-2-j4$, $\theta_{d2} = -\theta_{d1} = 90°$

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = 0$, the total number of poles and zeros existing on the right of 0 is zero (nor an even number or an odd number). Therefore, a branch of root loci does not exist between 0 and ∞.

Similarly, if we look from the pole $p = -4$, the total number of poles and zeros existing on the right of -4 is three (odd number). Therefore, a branch of root loci exists between 0 and -4.

Hence, only one branch of root loci exists on the real axis for the given system.

(vi) The breakaway and break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K}{s(s+4)(s^2+4s+20)} = 0$$

Therefore,
$$K = -\left(s^4 + 8s^3 + 36s^2 + 80s\right) \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$-\left(4s^3 + 24s^2 + 72s + 80\right) = 0$$

$$s^3 + 6s^2 + 18s + 20 = 0$$

Upon solving, we obtain

$$s = -2,\ -2 - j2.45,\ -2 + j2.45$$

Substituting $s = -2$ in Eqn. (3), we get $K = 64$. Since K is positive, the point $s = -2$ is a breakaway point.

Since two of the three obtained solution are complex numbers, it is necessary to check the angle condition also. Therefore,

$$\angle G(s)H(s)\Big|_{s=-2-j2.45} = \frac{0°}{\angle -2-2.45j \times \angle 2-2.45j \times \angle -6.45j \times \angle 1.55j}$$

$$= \frac{0°}{3.16\angle -129.22° \times 3.16\angle -50.77° \times 6.45\angle -90° \times 1.55\angle 90°} = 180°$$

Also, $$\angle G(s)H(s)\Big|_{s=-2+j2.45} = \frac{0°}{\angle -2+2.45j \times \angle 2+2.45j \times \angle -1.55j \times \angle 6.45j}$$

$$= \frac{0°}{3.16\angle 129.22° \times 3.16\angle 50.77° \times 1.55\angle -90° \times 6.45\angle 90°} = -180°$$

Since the angle for the given system at the complex numbers are odd multiple of 180°, the obtained complex numbers are also breakaway points.

Hence, $s = -2$ and $s = -2 \pm j2.45$ are the breaking away points for the given system.

(vii) The point at which the branch of root locus intersects the imaginary axis is determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0 \tag{4}$$

Routh array for the above equation,

$$\begin{array}{c|ccc} s^4 & 1 & 36 & K \\ s^3 & 8 & 80 & \\ s^2 & 26 & K & \\ s^1 & \dfrac{2080-8K}{26} & & \\ s^0 & K & & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows of Routh array must be either zero or greater than zero. i.e.,

$$\frac{2080-8K}{26} = 0. \text{ Therefore, } K = 260$$

Substituting K in the characteristic equation, we obtain

$$s^4 + 8s^3 + 36s^2 + 80s + 260 = 0$$

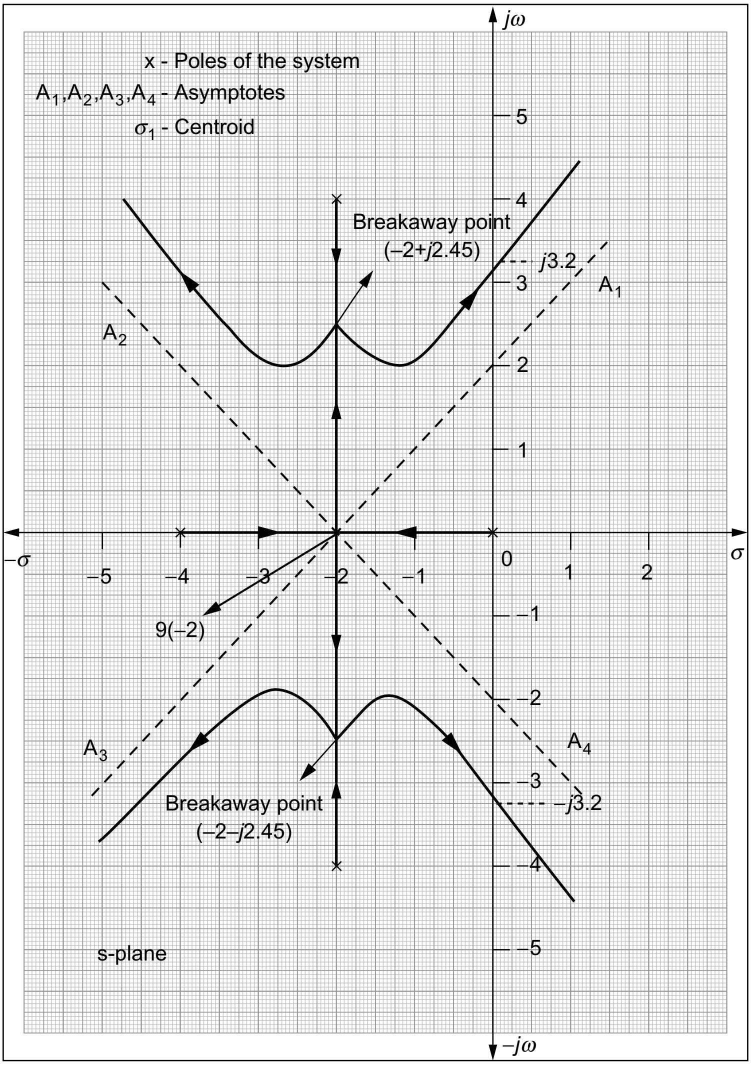


Fig. E7.2

Substituting $s = j\omega$ in the above equation, we get

$$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + 260 = 0$$

$$\omega^4 - j8\omega^3 - 36\omega^2 + j80\omega + 260 = 0$$

Equating the imaginary part of the above equation to zero, we get

$$-j8\omega^3 + j80\omega = 0$$

$$\omega^2 = 10$$

or
$$\omega = \sqrt{10} = \pm 3.2$$

Therefore, the point at which the root locus crosses the imaginary axis is $\pm j3.2$.

The entire root locus for the system is shown in Fig. E7.2.

Example 7.3: A pole-zero plot of a loop transfer function with gain K in the forward path of a control system is shown in Fig. E7.3(a). Sketch the root locus for the system.

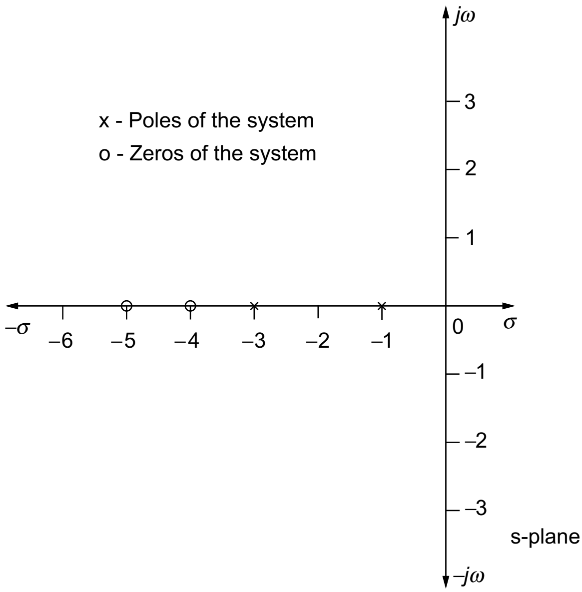


Fig. E7.3(a)

Solution:

(i) In the pole-zero plot shown in Fig. E7.3(a), the poles are at $-3, -1$ i.e., $n = 2$ and zeros are at -4 , -5 i.e., $m = 2$. Therefore, the loop transfer function, $G(s)H(s) = \dfrac{K(s+4)(s+5)}{(s+3)(s+1)}$.

(ii) Since $n = m$ the number of branches of the root loci for the given system is n or $m = 2$.

(iii) For the given system, as the number of poles and number of zeros are equal, the number of asymptotes for the given system is zero. Hence, it is not necessary of calculating the angle of asymptotes and centroid for the given system.

(iv) As all the poles and zeros are real values, there is no necessity in calculating the angle of departure and angle of arrival.

(v) Though the number of branches of the root loci for the given system is 2, it is necessary to determine the number of branches of the root loci existing on the real axis.

If we look from the point $p = -1$, the total number of poles and zeros existing on the right of -1 is zero (neither odd nor even). Therefore, a branch of root locus does not exists between -1 and $+\infty$.

Similarly, if we look from the point $p = -3$, the total number of poles and zeros existing on the right of -3 is one (odd number). Therefore, a branch of root loci exists between -3 and -1.

Similarly, if we look from the point $z = -4$, the total number of poles and zeros existing on the right of -4 is two (even number). Therefore, a branch of root loci does not exist between -4 and -3.

Similarly, if we look from the point $z = -5$, the total number of poles and zeros existing on the right of -5 is three (odd number). Therefore, a branch of root loci exists between -5 and -4.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of the point is four (even number). Therefore, a branch of root loci does not exist between $-\infty$ and -5.

Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway/break-in points for the given system are determined by,

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K(s+4)(s+5)}{(s+3)(s+1)} = 0$$

Therefore,
$$K = \frac{-s^2 - 4s - 3}{s^2 + 9s + 20} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we get

$$\frac{dK}{ds} = \frac{\left(s^2 + 9s + 20\right)\left(-2s - 4\right) - \left(-s^2 - 4s - 3\right)\left(2s + 9\right)}{\left(s^2 + 9s + 20\right)^2} = 0$$

$$\left(s^2 + 9s + 20\right)\left(-2s - 4\right) - \left(-s^2 - 4s - 3\right)\left(2s + 9\right) = 0$$

i.e.,
$$-5s^2 - 34s - 53 = 0$$

Upon solving, we get

$$s = -2.42 \text{ and } s = -4.38$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K = 0.202 \text{ for } s = -2.42 \text{ and } K = 143.07 \text{ for } s = -4.38$$

The breakaway point exists only for the positive values of K. Hence, $s = -2.42$ and $s = -4.38$ are the two breakaway points that exist for the given system.

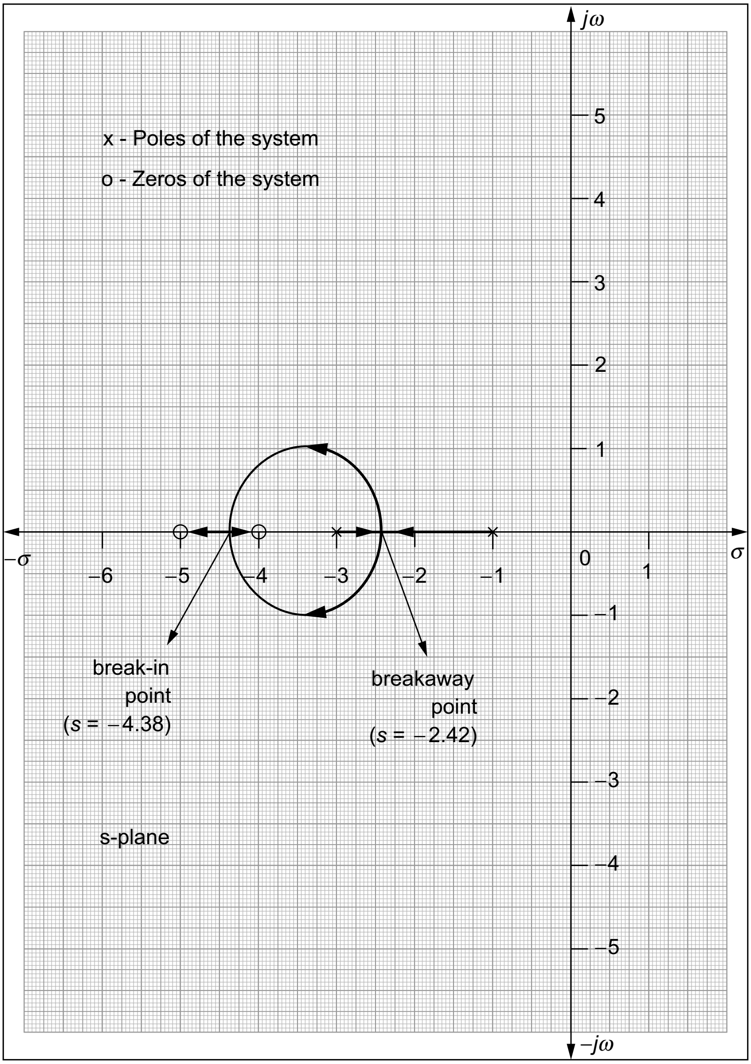


Fig. E7.3(b)

(vii) The point at which the branch of root loci intersects the imaginary axis is determined as follows:
From Eqn. (2), the characteristic equation for the given system is

$$(s+3)(s+1)+K(s+4)(s+5)=0$$

i.e., $$s^2(1+K)+s(4+9K)+(3+20K)=0$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & 1+K & 3+20K \\ s^1 & 4+9K & 0 \\ s^0 & 3+20K & \end{array}$$

For s^0, $3+20K>0$, $K>-\dfrac{3}{20}$

For s^1, $4+9K>0$, $K>-\dfrac{4}{9}$

For s^2, $1+K>0$, $K>-1$

As there exists no positive marginal value of K, the root locus does not get intersected with the imaginary axis. Also, the complete root locus lies in the left half of s-plane.

The final root locus for the system is shown in Fig. E7.3(b).

Example 7.4: A unity feedback control system shown in Fig. E7.4(a) has a controller with a transfer function $G_1(s)=\dfrac{K}{s}$, which controls a process with the transfer function $G_2(s)=\dfrac{1}{(s^2+2s+2)}$. Find the loop transfer function of the system and construct the root locus of the system.

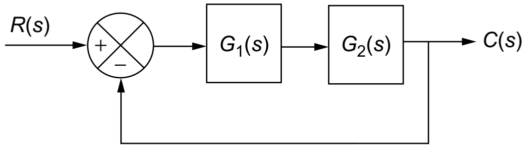


Fig. E7.4(a)

Solution: For the given system $H(s)=1$ and $G(s)=G_1(s)G_2(s)=\dfrac{K}{s(s^2+2s+2)}$. Therefore, the loop transfer function $G(s)H(s)=\dfrac{K}{s(s^2+2s+2)}$.

(i) For the loop transfer function of the system, the poles are at 0, $-1+j$ and $-1-j$ i.e., $n=3$ and zero does not exist i.e., $m=0$.

(ii) Since $n>m$ the number of branches of the root loci for the given system is $n=3$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system is $N_a = n-m = 3$.

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|}\times 180^\circ, \text{ for } n \neq m,\ i=0 \text{ and } 1$$

Therefore, $\theta_0 = 60°$, $\theta_1 = 180°$ and $\theta_2 = 300°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{-2}{3} = -0.67$$

(iv) Since complex poles exist, the angle of departure exists for the system.

To determine the angle of departure θ_d:

(a) For the pole $-1+j$,

$$\theta_{d1} = 180° - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$= 180° - \left(\phi_{p1} + \phi_{p2}\right) = 180° - \left(135° + 90°\right) = -45°$$

(b) For the pole $-1-j$, $\theta_{d2} = -\theta_{d1} = 45°$

(v) *To determine the number of branches existing on the real axis:*

If we look from the point $p=0$, the total number of poles and zeros existing on the right of 0 is zero (neither odd nor an even number). Therefore, a branch of root loci does not exist between 0 and ∞.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of the point is three. Therefore, a branch of root loci exists between $-\infty$ and 0.

Hence, only one branch of root loci exists on the real axis for the given system.

(vi) The breakaway or break-in points are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K}{s(s^2+2s+2)} = 0$$

Therefore,
$$K = -s^3 - 2s^2 - 2s$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -3s^2 - 4s - 2 = 0$$

$$3s^2 + 4s + 2 = 0$$

Upon solving, we obtain

$$s = -0.67 + j0.4714 \text{ and } s = -0.67 - j0.4714$$

Since the above points are complex conjugate values, the angle condition should be checked.

i.e., $$\angle G(s)H(s)\Big|_{s=-0.67+j0.4714} = \frac{1}{\angle(-0.67+j0.4714)\angle(0.33+j1.47)\angle(0.33-j0.53)}$$

$$= \frac{0°}{\angle 144.87° \ \angle 77.34° \ \angle -58.09°} = -164.11°$$

Also, $$\angle G(s)H(s)\Big|_{s=-0.67-j0.4714} = 164.11°$$

Since the angle of $G(s)H(s)$ is not an odd multiple of 180°, the points obtained using Eqn. (1) is not a breakaway point.

Hence, no breakaway point exists for the given system.

(vii) The point at which the branch of root loci intersects the imaginary axis is determined as follows:

From Eqn. (2), the characteristic equation for the given system is

$$s^3 + 2s^2 + 2s + K = 0.$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 2 & K \\ s^1 & \dfrac{4-K}{2} & 0 \\ s^0 & K & \end{array}$$

Depending on K, the system can be stable or marginally stable or unstable.

From Routh array, if the value of

(a) K is between 0 and 3, the system is stable.
(b) K is greater than 4, the system is unstable.

Now, if $K = 4$, the third row of Routh array becomes 0. Hence, the auxiliary equation is $2s^2 + K = 0$ i.e., $2s^2 + 4 = 0$.

Solving the above equation, we obtain $s = \pm j1.414$, which is the intersection point on the imaginary axis.

The final root locus for the system is shown in Fig. E7.4(b).

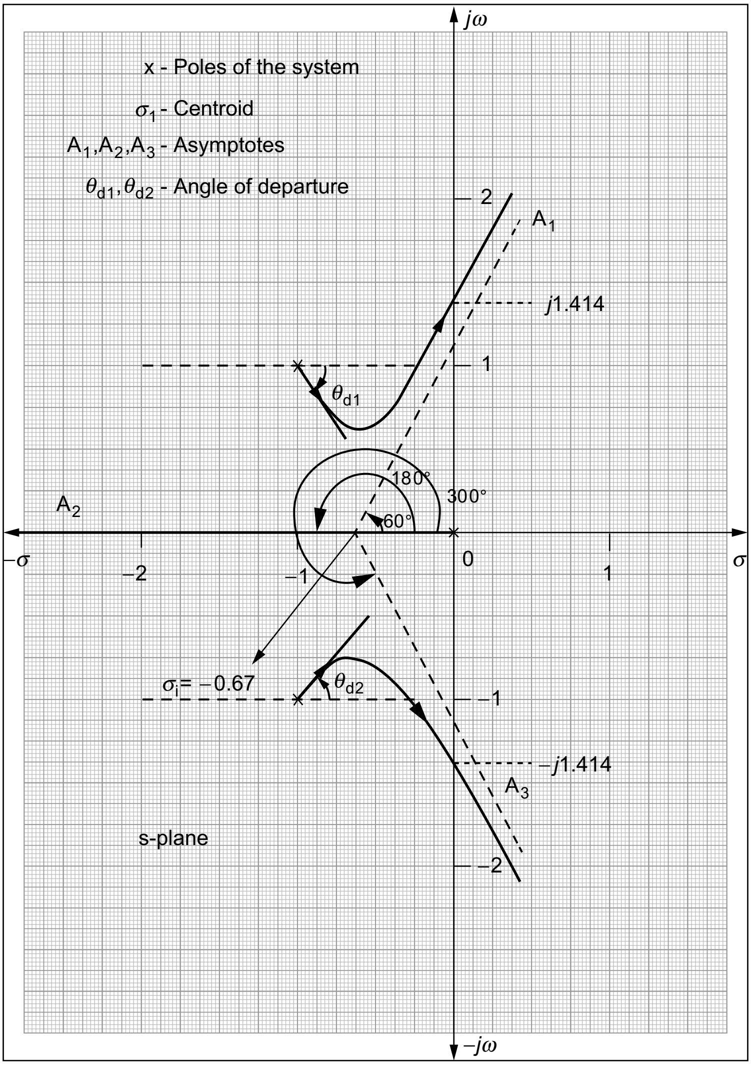


Fig. E7.4(b)

Example 7.5: Develop the root locus for the system whose loop transfer function is given by $G(s)H(s) = \dfrac{K(s+2)}{s(s+1)(s+4)}$

Solution:

(i) For the given system, the poles are at 0, −1 and −4 i.e., $n = 3$ and zero is at −2 i.e., $m = 1$.

(ii) Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(iii) The details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = n - m = 2$.

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } \; n \neq m, \; i = 0 \text{ and } 1$$

Therefore, $\theta_0 = 90°$ and $\theta_1 = 270°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{-3}{2} = -1.5$$

(iv) As all the poles and zeros are real values, there is no necessity to calculate the angle of departure and angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -1$, the total number of poles and zeros existing on the right of -1 is one (odd number). Therefore, a branch of root loci exists between -1 and 0.

Similarly, if we look from the pole $p = -4$, the total number of poles and zeros existing on the right of -4 is three (odd number). Therefore, a branch of root loci exists between -4 and -2.

Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway point for the given system is determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K(s+2)}{s(s+1)(s+4)} = 0$$

Therefore,
$$K = \frac{-s^3 - 5s^2 - 4s}{(s+2)}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -3s^3 - 6s^2 - 10s^2 - 20s - 4s - 8 + s^3 + 5s^2 + 4s = 0$$

$$2s^3 + 11s^2 + 20s + 8 = 0$$

The above equation can be solved using Lin's method (trial-and-error method). The third-order polynomial of the above equation will have at least one real root. The trial and error method for determining the roots is shown below:

Trial 1:

First trial divisor = $20s + 8 = s + 0.4.$

$$
\begin{array}{r|l}
 & 2s^2 + 10.2s + 15.92 \\ \hline
s + 0.4 & 2s^3 + 11s^2 + 20s + 8 \\
 & \underline{2s^3 + 0.8\, s^2} \\
 & 10.2\, s^2 + 20\, s \\
 & \underline{10.2\, s^2 + 4.08\, s} \\
 & 15.92\, s + 8 \\
 & \underline{15.92\, s + 6.368} \\
 & \underline{1.632}
\end{array}
$$

Trial 2:

Second trial divisor = $15.92s + 8 = s + 0.5$

$$
\begin{array}{r|l}
 & 2s^2 + 10\, s + 15 \\ \hline
s + 0.5 & 2s^3 + 11s^2 + 20s + 8 \\
 & \underline{2s^3 + 1\, s^2} \\
 & 10\, s^2 + 20\, s \\
 & \underline{10\, s^2 + 5\, s} \\
 & 15\, s + 8 \\
 & \underline{15.s + 7.5} \\
 & \underline{0.5}
\end{array}
$$

Trial 3:

Third trial divisor $= 15s + 8 = s + 0.53.$

$$
\begin{array}{r|l}
 & 2s^2 + 9.94s + 15. \\ \hline
s + 0.53 & 2s^3 + 11s^2 + 20s + 8 \\
 & \underline{2s^3 + 1.06\, s^2} \\
 & 9.94\, s^2 + 20\, s \\
 & \underline{9.94\, s^2 + 5.2682\, s} \\
 & 14.7318\, s + 8 \\
 & \underline{14.7318.s + 7.8} \\
 & \underline{0.2}
\end{array}
$$

Trial 4:

Fourth trial divisor $= 14.73s + 8 = s + 0.543.$

$$
\begin{array}{r|l}
 & 2s^2 + 9.914s + 14.16 \\ \hline
s + 0.543 & 2s^3 + 11s^2 + 20s + 8 \\
 & \underline{2s^3 + 1.086\, s^2} \\
 & 9.914s^2 + 20s \\
 & \underline{9.914s^2 + 5.383s} \\
 & 14.61\, s + 8 \\
 & \underline{14.61\, s + 7.93} \\
 & \underline{0.07}
\end{array}
$$

Trial 5

Fifth trial divisor = $14.61s + 8 = s + 0.547$

$$
\begin{array}{r|l}
 & 2s^2 + 9.906s + 14.581 \\ \hline
s + 0.547 & 2s^3 + 11s^2 + 20s + 8 \\
 & \underline{2s^3 + 1.064\, s^2} \\
 & 9.906\, s^2 + 20s \\
 & \underline{9.906\, s^2 + 5.428\, s} \\
 & 14.581\, s + 8 \\
 & \underline{14.581\, s + 7.97} \\
 & \underline{0.03}
\end{array}
$$

Since the remainder obtained in the fifth trial is 0.03, the point $-\dfrac{8}{14.581} = -0.549$ is one of the roots of the equation.

Hence, rewriting the above cubic equation, we obtain

$$(s+0.549)(s^2+4.95s+7.27)=0$$

Solving the above equation, we obtain

$$s=-0.549, -2.475 \pm j1.07$$

Substituting $s=-0.549$ in Eqn. (3), we get $K = 0.5888$. Since K is positive, the point $s=-0.549$ is a breakaway point.

As the other two points are complex conjugate points, it is necessary to determine the angles of the system at those points.

i.e., $$\angle G(s)H(s)\Big|_{s=-2.475+j1.07} = \frac{(-0.475+j1.07)}{(-2.475+j1.07)(-1.475+j1.07)(1.525+j1.07)}$$
$$=-221.78^\circ$$

Also, $$\angle G(s)H(s)\Big|_{s=-2.475-j1.07} = 221.78^\circ$$

Since the angles obtained are not odd multiple of 180°, the complex points obtained using Eqn. (1) are not a breakaway or break-in points.

Hence, only one breakaway point exists for the given system.

(vii) The point at which the branch of root loci intersects the imaginary axis is obtained as follows:

Using Eqn. (2), the characteristic equation for the given system is

$$s(s+1)(s+4)+K(s+2)=0$$

i.e., $$s^3+5s^2+s(K+4)+2K=0$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 4+K \\ s^2 & 5 & 2K \\ s^1 & \dfrac{20+5K-2K}{5} & 0 \\ s^0 & 2K & \end{array}$$

To determine the point at which the root locus crosses the imaginary axis, the first element in the third row must be zero, i.e., $20+3K=0$. Therefore, $K=-\dfrac{20}{3}=-6.67$. As K is a negative real value, the root locus do not cross the imaginary axis.

The final root locus for the system is shown in Fig. E7.5.

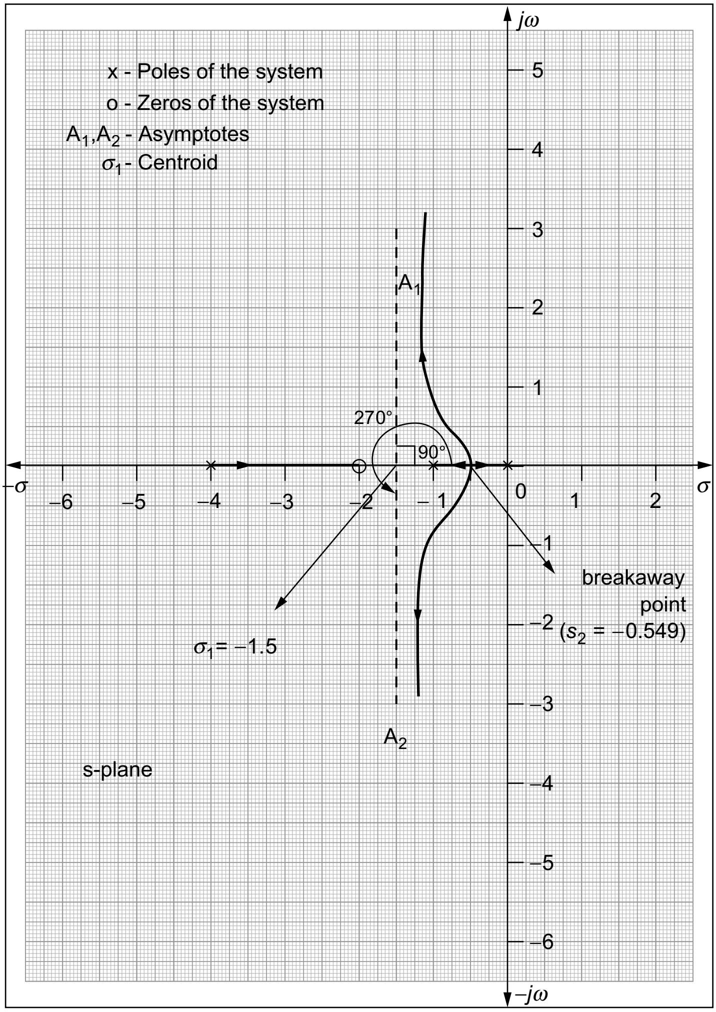


Fig. E7.5

Example 7.6: Sketch the root locus for the system whose loop transfer function $G(s)H(s)=\dfrac{K}{s(s+2)(s+4)}$. Also, (i) determine gain K when $M_p = 16.3\%$, (ii) determine the settling time t_s and peak time t_p using the gain value obtained in (i).

Solution:

(a) For the given system $G(s)H(s)=\dfrac{K}{s(s+2)(s+4)}$, the poles are at 0, −2 and −4 i.e., $n=3$ and zeros do not exist i.e., $m=0$.

(b) Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(c) The details of the asymptotes are given below:

(1) Number of asymptotes for the given system $N_a = n - m = 3$.

(2) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m, \; i = 0, 1 \text{ and } 2$$

Therefore, $\theta_0 = 60°$, $\theta_1 = 180°$ and $\theta_2 = 300°$

(3) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{-6}{3} = -2$$

(d) As all the poles and zeros are real values, there is no necessity in calculating the angle of departure and angle of arrival.

(e) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -2$, the total number of poles and zeros existing on the right of -2 is one (odd number). Therefore, a branch of root loci exists between -2 and 0.

Similarly, if we look from the point at $-\infty$, the total number of poles and zero existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between $-\infty$ and -4.

Hence, two branches of root loci exist on the real axis for the given system.

(f) The breakaway/break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

Therefore,

$$K = -s^3 - 6s^2 - 8s \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -3s^2 - 12s - 8 = 0$$

$$3s^2 + 12s + 8 = 0$$

Solving for s, we get

$$s = -0.8453 \text{ and } s = -3.1547$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K = 3.0792 \text{ for } s = -0.8453 \text{ and } K = -3.0792 \text{ for } s = -3.1547.$$

The breakaway point exists only for the positive values of K. Hence, $s = -0.8453$ is the only one breakaway point that exists for the given system.

(g) The point at which the branch of root loci intersects the imaginary axis is determined as follows:

From Eqn. (2), the characteristic equation for the given system is

$$s(s+2)(s+4)+K=0$$

i.e.,
$$s^3+6s^2+8s+K=0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48-K}{6} & 0 \\ s^0 & K & \end{array}$$

To determine the point at which the root locus crosses the imaginary axis, the first element in the third row must be zero, i.e., $48-K=0$. Therefore, $K = 48$.
Substituting K in Eqn. (3), we get

$$s^3 + 6s^2 + 8s + 48 = 0$$

Upon solving, we get

$$s = -6,\ j2.828,\ -j2.828.$$

Hence, the root locus crosses the imaginary axis at $s = \pm j2.828$.
The complete root locus for the system is shown in Fig. E7.6(a).

(i) It is given that the maximum peak overshoot for the system is 16.3%. We know that the maximum peak overshoot for a second-order system is

$$M_p\% = 100 \times e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

Substituting the known values, we obtain

$$16.3 = 100 \times e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

$$0.163 = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

Solving the above equation, we obtain

$$\xi = 0.5$$

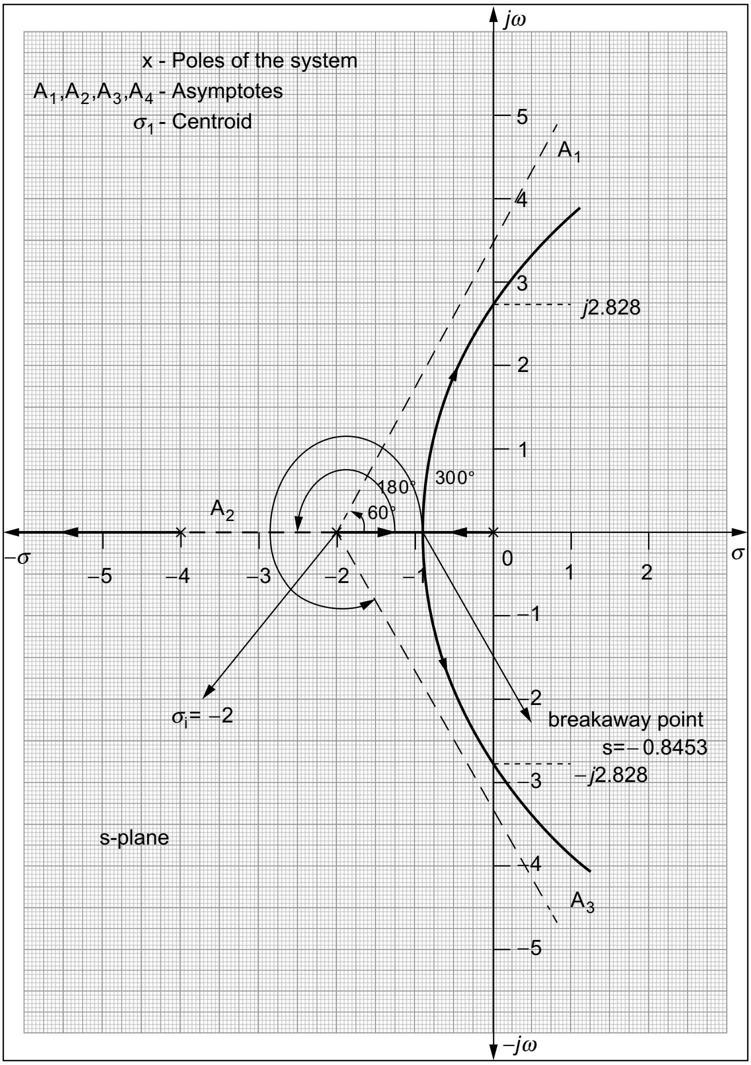


Fig. E7.6(a)

For this damping ratio $\xi = 0.5$, a straight line can be drawn from the origin with the angle, θ, obtained by $\theta = \cos^{-1}(\xi)$. The point at which the straight line crosses on the root locus is $P = -0.675 + j1.15$ as shown in Fig. E7.6(b).

We know that the magnitude condition at any point of the system

$$|G(s)H(s)|_{at\ any\ point} = 1$$

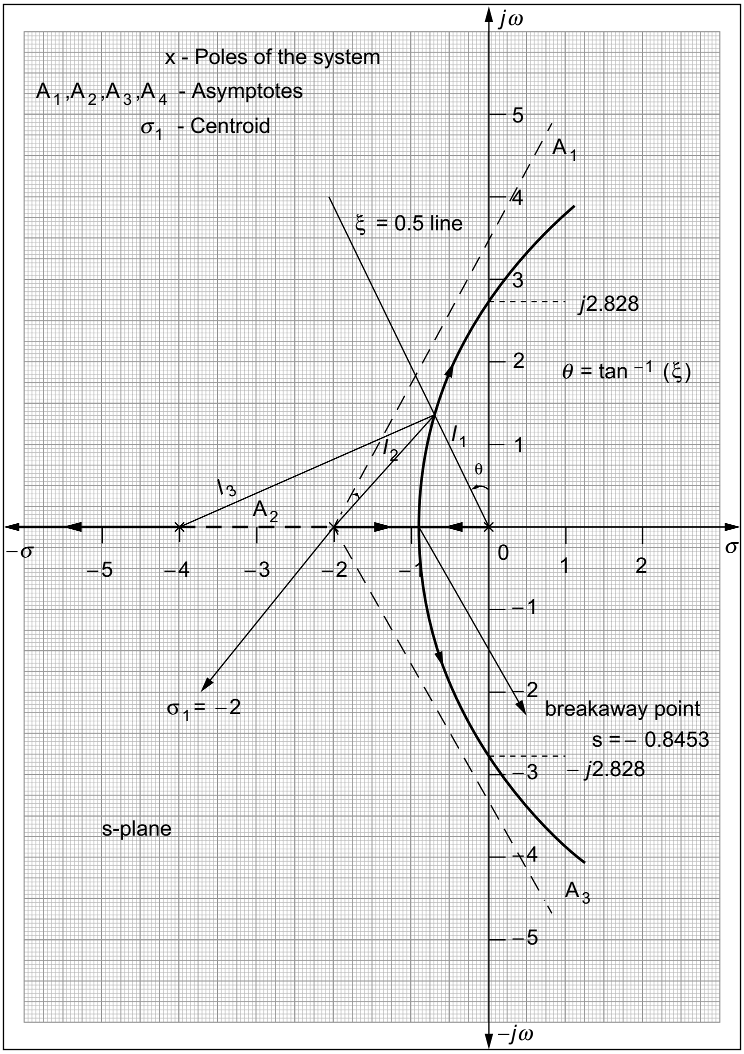


Fig. E7.6(b)

i.e., $$\left|\frac{K}{s(s+2)(s+4)}\right|_{p=-0.675+j1.25} = 1$$

Therefore, $$\frac{|K|}{|-0.675+j1.25|\,|-0.675+j1.25+2|\,|-0.675+j1.25+4|} = 1$$

Therefore, gain K corresponding to the given peak overshoot, $M_p = 16.3\%$ is $K = 9.174$.

(ii) The point at which the line for $M_p = 16.3\%$ cuts the root locus is $p = -0.675 + j1.25$. The settling time and peak time of the system is obtained as

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{|\text{Real part of } p|} = \frac{4}{|0.675|} = 5.92 \text{ sec.}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{|\text{Imaginary part of } p|} = \frac{\pi}{|1.25|} = 2.513 \text{ sec}$$

Example 7.7: **Prove that the breakaway points for a system in the root locus are obtained by solving for the solutions of $\frac{dK}{ds} = 0$ where K is the open-loop gain (variable parameter) of the system whose open-loop transfer function is $G(s)$.**

Solution: The characteristic equation of the system whose open-loop transfer function of the system given by $G(s)$ is obtained as $1+G(s)H(s)=0$ which is the combination of s terms and open-loop gain K terms. It can be arranged as

$$F(s) = P(s) + KQ(s) = 0 \tag{1}$$

where $P(s)$ is the polynomial containing K and s terms and

$KQ(s)$ is the polynomial containing K and s terms

Now at breakaway points, multiple roots occur. Mathematically, this is possible if $\frac{dF(s)}{ds} = 0.$

Differentiating Eqn. (1) with respect to s, we get

$$\frac{dF(s)}{ds} = \frac{dP(s)}{ds} + K\frac{dQ(s)}{ds} = 0$$

Therefore,

$$K = \frac{-\left(\frac{dP(s)}{ds}\right)}{\left(\frac{dQ(s)}{ds}\right)} \tag{2}$$

Substituting Eqn. (2) in Eqn. (1) at breakaway point, we can write

$$P(s) - \frac{\left(\frac{dP(s)}{ds}\right)}{\left(\frac{dQ(s)}{ds}\right)}Q(s) = 0 \tag{3}$$

Solving Eqn. (3) for *s*, breakaway points can be obtained.
Now, equating Eqn. (1) to zero and solving for K, we get

$$K = -\frac{P(s)}{Q(s)}$$

Therefore, $$\frac{dK}{ds} = -\left\{\frac{Q(s)\left(+\frac{dP(s)}{ds}\right) - \left[+P(s)\right]\left(\frac{dQ(s)}{ds}\right)}{Q(s)^2}\right\} \tag{4}$$

i.e., $$\frac{dK}{ds} = \frac{dQ(s)}{ds}P(s) - \frac{dP(s)}{ds}Q(s) = 0 \tag{5}$$

This is same as Eqn. (3) which yields breakaway points. This proves that the roots of $\frac{dK}{ds} = 0$ are the actual breakaway points.

Example 7.8: (i) Prove that the root locus of the system $G(s)H(s) = \frac{K(s+b)}{s(s+a)}$ is a circle. (ii) Assuming *a* = 4 and *b* = 6 for the above loop transfer function, develop the root locus. (iii) Determine the range of variable gain, *K* for which the system is under-damped. (iv) Determine the range of variable gain, *K* for which the system is critically damped and (*v*) Determine the minimum value of damping ratio.

Solution: (i) Let $s = \sigma + j\omega$ be a complex point on the root locus for the given system.

Therefore, $$G(s)H(s)\Big|_{s=\sigma+j\omega} = \frac{K(\sigma + j\omega + b)}{(\sigma + j\omega)(\sigma + j\omega + a)}$$

To prove that the root locus of the system is a circle, the angle of the system at any point must be 180°.

i.e., $$\angle G(s)H(s)\Big|_{\text{at any point}} = 180° \tag{1}$$

Therefore, $$\angle G(s)H(s)\Big|_{s=\sigma+j\omega} = \frac{\tan^{-1}\left[\frac{\omega}{\sigma+b}\right]}{\tan^{-1}\left[\frac{\omega}{\sigma}\right]\tan^{-1}\left[\frac{\omega}{\sigma+a}\right]}$$

$$= \tan^{-1}\left[\frac{\omega}{\sigma+b}\right] - \left\{\tan^{-1}\left[\frac{\omega}{\sigma}\right] + \tan^{-1}\left[\frac{\omega}{\sigma+a}\right]\right\}$$

$$= \tan^{-1}\left[\frac{\omega}{\sigma+b}\right] - \tan^{-1}\left\{\frac{\frac{\omega}{\sigma}+\frac{\omega}{\sigma+a}}{1-\frac{\omega}{\sigma}-\frac{\omega}{\sigma+a}}\right\}$$

$$\left(\text{since} \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left\{\frac{A+B}{1-AB}\right\}\right)$$

$$= \tan^{-1}\left[\frac{\omega}{\sigma+b}\right] - \tan^{-1}\left\{\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\frac{\omega}{\sigma+b}-\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}{1+\frac{\omega}{\sigma+b}\times\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}\right\}$$

From Eqn. (1),

$$180^\circ = \tan^{-1}\left\{\frac{\frac{\omega}{\sigma+b}-\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}{1+\frac{\omega}{\sigma+b}\times\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}\right\}$$

Therefore,

$$\tan(180^\circ) = \left(\frac{\frac{\omega}{\sigma+b}-\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}{1+\frac{\omega}{\sigma+b}\times\frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2}}\right)$$

Since $\tan(180^\circ) = 0$, we obtain

$$\frac{\omega}{\sigma+b} - \frac{\omega(2\sigma+a)}{\sigma(\sigma+a)-\omega^2} = 0$$

Therefore, $\sigma^2 + a\sigma - \omega^2 - 2\sigma^2 - 2b\sigma - a\sigma - ab = 0$

Adding and subtracting b^2 on both sides, we obtain

$$\sigma^2 + 2b\sigma + b^2 + \omega^2 - b^2 + ab = 0$$

Therefore, $$(\sigma + b)^2 + (\omega - 0)^2 = \left(\sqrt{b(b-a)}\right)^2$$

The above equation is the equation of a circle with centre $(-b, 0)$ which is the location of open loop zero and radius $\sqrt{b(b-a)}$. This proves that root locus of the given system is a circle.

(ii) Assuming a = 4 and b = 6, the loop transfer function of the system becomes $G(s)H(s) = \dfrac{K(s+6)}{s(s+4)}$.

(a) For the given system, the poles are at 0 and −4, i.e., $n = 2$ and zero is at −6, i.e., $m = 1$.
(b) Since $n > m$, the number of branches of the root loci for the given system $n = 2$.
(c) For the given system, the details of the asymptotes are given below.
   (1) Number of asymptotes for the given system $N_a = 1$.
   (2) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m \text{ and } i = 0$$

   Therefore, $\theta_0 = 180°$
   (3) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{2}{1} = 2$$

(d) Since no complex pole or complex zero exists for the given system, there is no necessity of calculating the angle of departure or angle of arrival.
(e) *To determine the number of branches existing on the real axis:*
   If we look from the pole $p = -4$, the total number of poles and zeros existing on the right of the point is one (odd number). Therefore, a branch of root loci exists between −4 and 0.
   Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between −6 and $-\infty$
   Hence, two branches of root loci exist on the real axis for the given system.
(f) The breakaway and break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e., $$1+\frac{K(s+6)}{s(s+4)}=0$$

Therefore, $$K=\frac{-s^2-4s}{s+6} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds}=\frac{(s+6)(-2s-4)-(-s^2-4s)(1)}{(s+6)^2}=0$$

$$s^2+12s+24=0$$

Upon solving, we obtain

$$s=-2.5358 \text{ and } s=-9.4641$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K=1.071 \text{ for } s=-2.5358 \text{ and } K=14.928 \text{ for } s=-9.4641.$$

The breakaway point exists only for the positive values of K. Hence, $s = -2.5358$ and $s = -9.4641$ are the two breakaway points that exist for the given system.

(g) The point at which the branch of root loci intersects the imaginary axis is determined as follows:
From Eqn. (2), the characteristic equation for the given system is

$$s^2+(4+K)s+6K=0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & 1 & 6K \\ s^1 & 4+K & 0 \\ s^0 & 6K & \end{array}$$

To determine the point at which the root locus crosses the imaginary axis, the first element in the second row must be zero, i.e., $K = -4$.

As K is a negative real value, the root loci does not cross the imaginary axis.

Hence, the final root loci for the system are shown in Fig. E7.8.

(iii) For the system to be under-damped, the range of gain K is

$$1.0717 < K < 14.928$$

(iv) The system is critically damped when the gain $K = 1.0717$ or 14.928.

(v) Determination of minimum value of damping ratio ξ

For the root locus of the given system a tangent is drawn from the origin to the root locus and the angle from the real axis to the tangent is measured as $\theta = 35°$ as shown in Fig. E7.8.

Therefore, the damping ratio, $\xi = \cos\theta = \cos 35° = 0.819$.

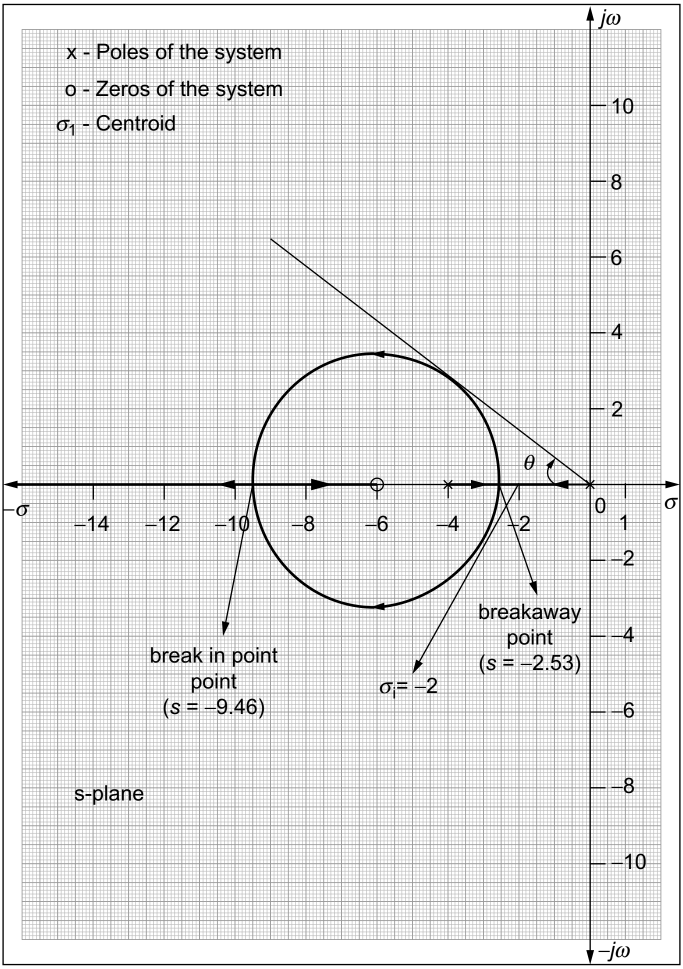


Fig. E7.8

Example 7.9: **Plot the root locus for the system whose loop transfer function is $G(s)H(s)=\dfrac{K}{s(s+1)(s+3)}$. Also, determine the variable parameter K (i) for marginal stability of the system and (ii) at the point $s = -4$.**

Solution:

(a) For the given system, the poles are at 0, −1 and −3 i.e., $n = 3$ and zero does not exist i.e., $m = 0$.

(b) Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(c) For the given system, the details of the asymptotes are given below:
  (1) Number of asymptotes for the given system $N_a = (n-m) = 3$.
  (2) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m, i = 0, 1 \text{ and } 2$$

  Therefore, $\theta_0 = 60°$, $\theta_1 = 180°$, and $\theta_2 = 300°$
  (3) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{-4}{3} = -1.333$$

(d) Since no complex pole or complex zero exists for the given system, there is no necessity of calculating the angle of departure or angle of arrival.

(e) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -1$, the total number of poles and zeros existing on the right of -1 is one (odd number). Therefore, a branch of root loci exists between -1 and 0.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of the point is three. Therefore, a branch of root loci exists between $-\infty$ and -3.

Hence, two branches of root loci exist on the real axis for the given system.

(f) The breakaway/break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \qquad (1)$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \qquad (2)$$

i.e., $$1 + \frac{K}{s(s+1)(s+3)} = 0$$

Therefore, $$K = -s^3 - 4s^2 - 3s \qquad (3)$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -3s^2 - 8s - 3 = 0$$

$$3s^2 + 8s + 3 = 0$$

Upon solving, we obtain

$$s = -2.215 \text{ and } s = -0.451$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K = 0.63311 \text{ for } s = -0.451 \text{ and } K = -2.11 \text{ for } s = -2.215$$

The breakaway point exists only for the positive values of K. Hence, $s = -0.451$ is the only one breakaway point that exists for the given system.

(g) The point at which the branch of root locus crosses the imaginary axis is determined as follows:
From Eqn. (2), the characteristic equation for the given system is

$$s(s+1)(s+3)+K=0$$

Therefore,
$$s^3+4s^2+3s+K=0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 4 & K \\ s^1 & \dfrac{12-K}{4} & 0 \\ s^0 & K & \end{array}$$

To determine the point at which the root locus crosses the imaginary axis, the first element in the third row must be zero, i.e., $12-K=0$. Therefore, $K=12$.

Substituting K in Eqn. (3), we get

$$s^3 + 4s^2 + 3s + 12 = 0$$

Upon solving, we get

$$s = -4,\ j1.732\ ,\ -j1.732.$$

Hence, the root locus crosses the imaginary axis at $s = \pm j1.732$.
The root locus plot for the system is shown in Fig. E7.9.

(1) The marginal value of variable gain $K = 12$.
(2) The variable gain K at any point s is obtained by

$$|K| = \left|\frac{1}{G(s)H(s)}\right|_{\text{at any point}}$$

Hence, at $s = -4$, we obtain

$$|K| = \left| \frac{1}{\frac{1}{s(s+1)(s+3)}} \right|_{s=-4}$$

$$K = 12.$$

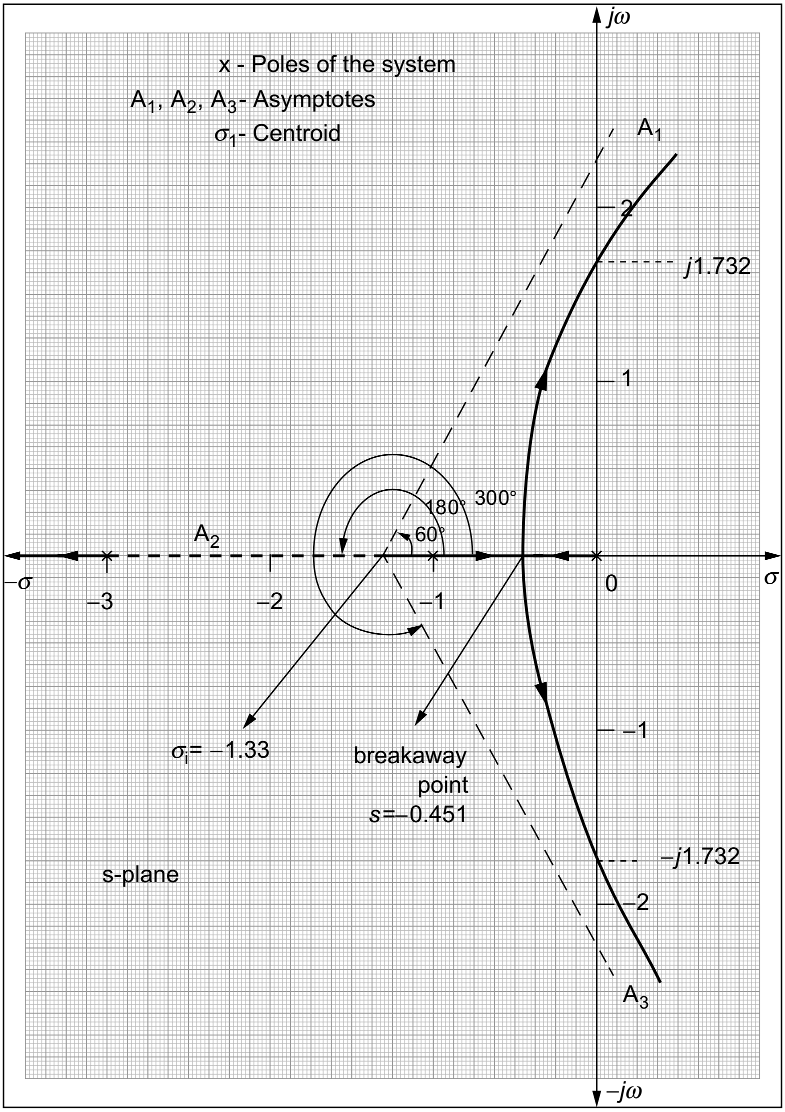


Fig. E7.9

Example 7.10: Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K(s+2)}{s^2+2s+3}$. Also, determine ω at $K=1.33$.

Solution:

(i) For the given system, poles are at $\left(-1-j\sqrt{2}\right)$ and $\left(-1+j\sqrt{2}\right)$ i.e., $n=2$ and zero is at -2 i.e., $m=1$.

(ii) Since $n>m$, the number of branches of the root loci for the given system is $n=2$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a=n-m=1$

(b) Angles of asymptotes

$$\theta_i=\frac{2i+1}{|n-m|}\times 180°,\quad \text{for } n\neq m \text{ and } i=0$$

Therefore, $\theta_0=180°$.

(c) Centroid

$$\sigma_1=\frac{\sum \text{finite poles of } G(s)H(s)-\sum \text{finite zeros of } G(s)H(s)}{n-m}$$

$$=\frac{-1-j\sqrt{2}-1+j\sqrt{2}-(-2)}{1}=0$$

(iv) Since complex poles exist, the angle of departure exists for the system.

To determine the angle of departure θ_d:

(a) For the pole $-1+j\sqrt{2}$,

$$\theta_{d1}=180°-\sum_{j=1}^{n}\text{Angle from pole } j \text{ to the complex pole }-\sum_{i=1}^{m}\text{Angle from zero } i \text{ to the complex pole}$$

$$=180°-\left(\phi_{p1}+\phi_{p2}\right)=180°-\left(\tan^{-1}\left(\sqrt{2}\right)+90°\right)=144.73°$$

(b) For the pole $-1-j\sqrt{2}$, $\theta_{d2}=-\theta_{d1}=-144.73°$.

(v) *To determine the number of branches existing on the real axis:*

If we look from point at $-\infty$, the total number of poles and zeros existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between $-\infty$ and -2.

Hence, one branch of root loci exists on the real axis for the given system.

(vi) The breakaway/break-in points for the given system are determined by

$$\frac{dK}{ds}=0 \tag{1}$$

For the given system, the characteristic equation is

$$1+G(s)H(s)=0 \tag{2}$$

i.e., $$1+\frac{K(s+2)}{(s^2+2s+3)}=0$$

Therefore, $$K=-\frac{\left(s^2+2s+3\right)}{s+2} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds}=-\left[\frac{(s+2)(2s+2)-\left(s^2+2s+3\right)}{s+2}\right]=0$$

$$s^2+4s+1=0$$

Upon solving, we obtain

$$\text{s} = -0.26 \text{ and } -3.73$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K=10.62 \text{ for } s=-3.73 \text{ and } K=-1.464 \text{ for } s=-0.26$$

The breakaway point exists only for the positive values of K. Hence, $s = -3.73$ is the only one breakaway point that exists for the given system.

(vii) The point at which the branch of root locus crosses the imaginary axis is determined as follows:
From Eqn. (2), the characteristic equation of the system is

$$s^2+(2+K)s+(3+2K)=0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & 1 & 3+2K \\ s^1 & 2+K & 0 \\ s^0 & 3+2K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in the second row and third row must be zero, i.e., $2+K=0$ and $3+2K=0$. Solving for K, we obtain $K=-2$ and $K=-1.5$. Since $K=-1.5$ lies within $K=-2$, we take$K=-2$.

As K is negative real value, the root loci do not cross the imaginary axis.

The final root locus for the system is shown in Fig. E7.10.

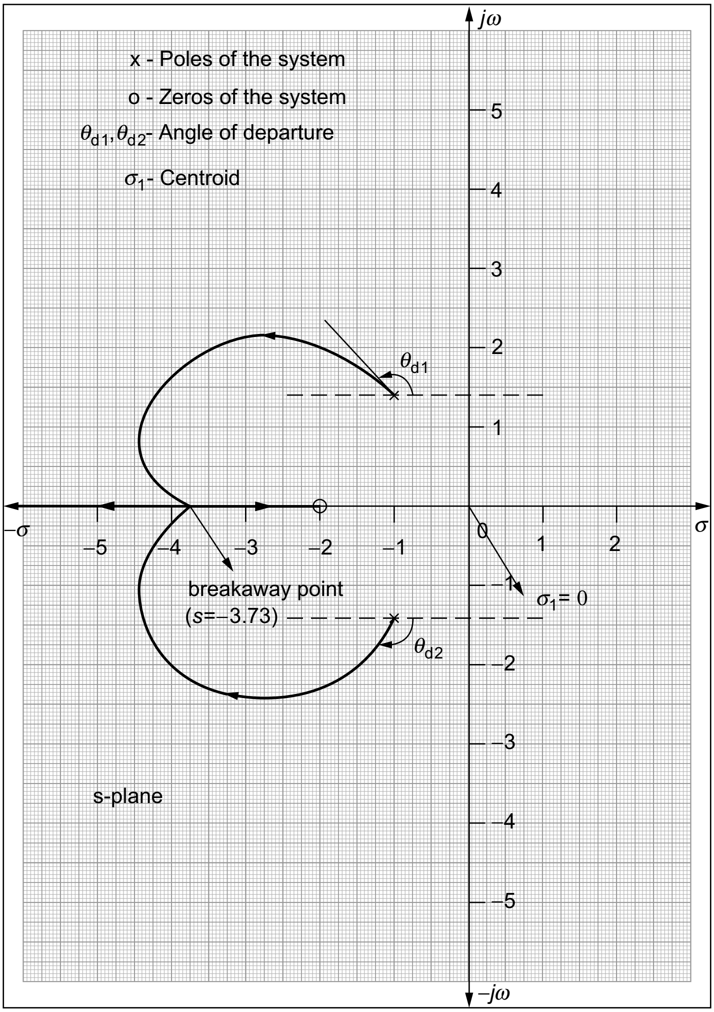


Fig. E7.10

To determinine damping ratio ω

The characteristic equation of a second-order system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \tag{5}$$

where ω_n is the undamped natural frequency of the system and ξ is the damping ratio. The characteristic equation of the given system after substituting $K = 1.33$, is

$$s^2 + (2+K)s + (3+2K) = 0 \tag{6}$$

Comparing Eqs. (5) and (6), we obtain

$$\omega_n^{\ 2} = 3 + 2K$$

Given $K = 1.33$, Hence $\omega_n^2 = 5.66$ i.e., $\omega_n = 2.379$

Then, $$2\xi\omega_n = 2 + K = 3.33$$

Therefore, the damping ratio, $\xi = \dfrac{3.33}{2\omega_n} = \dfrac{3.33}{2 \times 2.379} = 0.699$

Example 7.11: Plot the root locus of the system whose loop transfer function is given by $G(s)H(s) = \dfrac{K}{s(s+1)(s+3)(s+5)}$.

Solution:

(i) For the given system, poles are at 0, –1, –3 and –5 i.e., $n = 4$ and zeros do not exist i.e., $m = 0$.
Since $n > m$, the number of branches of the root loci for the given system is $n = 4$.

(ii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 4 - 0 = 4$.

(b) Angles of asymptotes,

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m,\ i = 0, 1, 2 \text{ and } 3$$

Therefore, $\theta_0 = 45°$, $\theta_1 = 135°$, $\theta_2 = 225°$ and $\theta_3 = 315°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

$$= \frac{-0-1-3-5-0}{4} = -2.25$$

(iii) Since no complex pole or complex zero exists for the system, there is no necessity of determining angle of departure/angle of arrival.

(iv) *To determine the number of branches existing on the real axis:*

Since the total number of poles and zeros existing on the right of –1 and –5 are 1 and 3 (odd numbers), a branch of root loci exists between –1 and 0, and –5 and –3.

Hence, two branches of root loci exist for a given system on the real axis.

(v) The breakaway/break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e., $$1+\frac{K}{s(s+1)(s+3)(s+5)}=0$$

Therefore, $$K=-\left[s(s+1)(s+3)(s+5)\right]=-\left[s^4+9s^3+23s^2+15s\right] \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dk}{ds}=-(4s^3+27s^2+46s+15)=0$$

Upon solving, we obtain

$$s=-0.425,\ -4.253,\ -2.07$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$K = 2.878$ for $s = -0.425$, $K = 12.949$ for $s = -4.253$ and $K = -6.035$ for $s = -2.07$

The breakaway points exist only for the positive values of K. Hence, $s = -0.425$ and $s = -4.253$ are the two breakaway points that exist for the given system.

(vi) The point at which the branch of root locus crosses the imaginary axis is determined as follows:
To find the crossing point on the imaginary axis, substitute $s = j\omega$ in

$$s^4+9s+23s^2+15s+K=0$$

$$(j\omega)^4+9(j\omega)^2+23(j\omega)^2+15(j\omega)+K=0$$

$$\omega^4-j9\omega^3-23\omega^2+j15\omega+K=0$$

Equating imaginary part to zero, we obtain

$$-j9\omega^3+j15\omega=0$$

$$-j9\omega^3=-j15\omega$$

$$\omega^2=\frac{15}{9}$$

or $$\omega=1.29$$

Hence, the root locus intersects at $\pm j1.29$.
Equating real parts to zero, we obtain

$$\omega^4-23\omega^2+K=0$$

$$K=35.55.$$

Hence, the gain of the system at which the root locus intersects the imaginary axis is 35.55.

The complete root locus for the system is shown in Fig. E7.11.

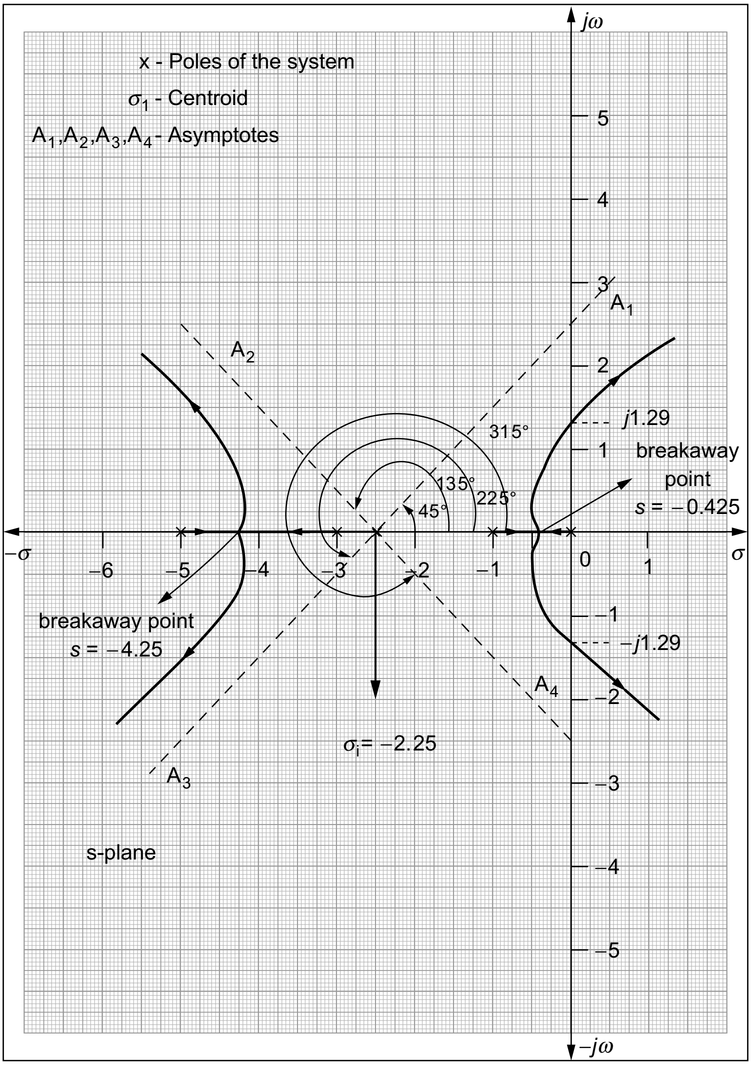


Fig. E7.11

Example 7.12: Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s+1)(s+2)}$.

Solution:

(i) For the given system, poles are at 0, −1 and −2, i.e., $n = 3$ and zeros do not exist i.e., $m = 0$.
Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(ii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 3 - 0 = 3$

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m, i = 0, 1 \text{ and } 2$$

Therefore, $\theta_0 = 60°$, $\theta_1 = 180°$ and $\theta_2 = 300°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

$$= \frac{-0-1-2-0}{3} = -1$$

(iii) Since no complex pole or complex zero exists for the system, there is no necessity in determining the angle of departure/angle of arrival.

(iv) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -1$, the total number of poles and zeros existing on the right of −1 is one (odd number). Therefore, a branch of root loci exists between −1 and 0.

Similarly, if we look from the pole $p = -\infty$, the total number of poles and zeros existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between $-\infty$ and −2.

Hence, two branches of root loci exist on the real axis for the given system.

(v) The breakaway/break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

Therefore,
$$K = -\left[(s+1)(s+2)s\right] = -\left[s^3 + 3s^2 + 2s\right] \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -\left(3s^2 + 6s + 2\right) = 0$$

Upon solving, we obtain

$$s = -1.5, -0.42$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K = 0.384 \text{ for } s = -0.42 \text{ and } K = -0.375 \text{ for } s = -1.5$$

The breakaway point exists only for the positive values of K. Hence, $s = -0.42$ is the only one breakaway point that exists for the given system.

(vi) The point at which the branch of root locus crosses the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^3 + 3s^2 + 2s + K = 0 \tag{4}$$

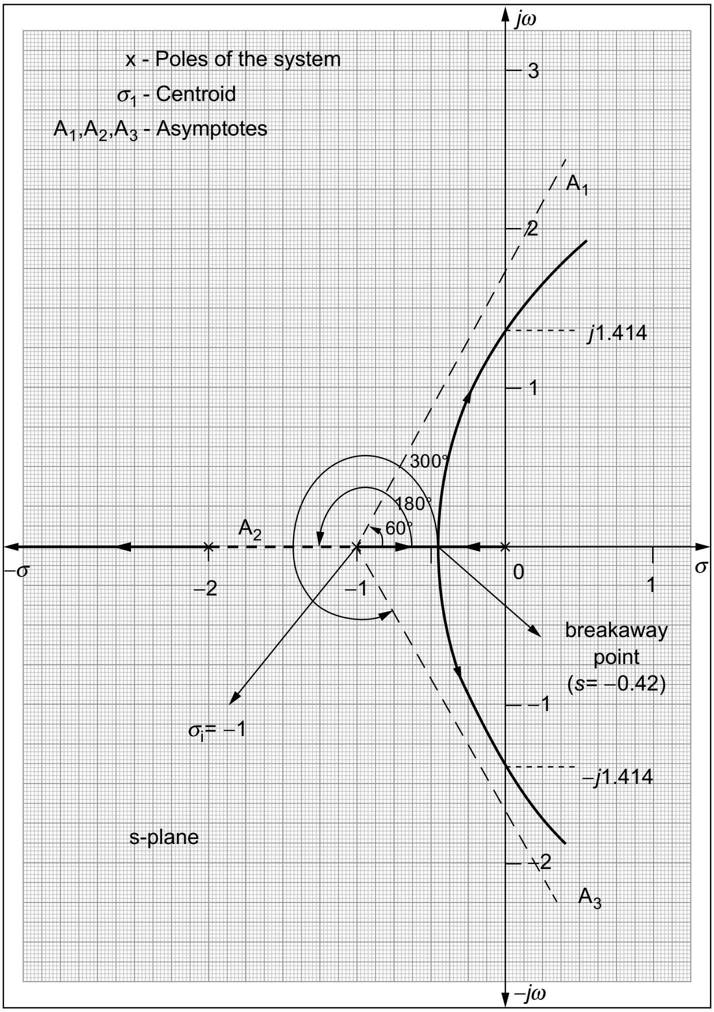


Fig. E7.12

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in the third row must be zero, i.e., $6-K=0$. Therefore, $K=6$

Substituting K in Eqn. (4) and solving for s, we obtain

$$s = -3,\ -j1.414 \text{ and } j1.414$$

Hence, the root locus crosses the imaginary axis at $\pm j1.414$.

The final root locus for the system is shown in Fig. E7.12.

Example 7.13: The characteristic polynomial of a feedback control system is given by $s^2(1+K)+s(1+5K)+6K=0$. Plot the root locus for this system.

Solution: The characteristic polynomial for the given system is

$$s^2(1+K)+s(1+5K)+6K=0$$

i.e.,
$$s^2+Ks^2+s+5Ks+6K=0$$

$$s(s+1)+K(s^2+5s+6)=0$$

$$s(s+1)+K(s+2)(s+3)=0$$

i. e.,
$$1+\frac{K(s+2)(s+3)}{s(s+1)}=0$$

We know that the characteristic equation is $1+G(s)H(s)=0$.

Therefore, $G(s)H(s)=\dfrac{K(s+2)(s+3)}{s(s+1)}$

(i) For $G(s)H(s)=\dfrac{K(s+2)(s+3)}{s(s+1)}$, poles are at 0 and −1, i.e., $n=2$ and zeros are at −2 and −3, i.e., $m=2$.

(ii) Since $n=m$, the number of branches of the root loci for the given system is n or $m=2$.

(iii) Since the number of poles and number of zeros present in the system are equal, no asymptote exists for the system and hence the angle of asymptotes and centroid need not be determined for the system.

(iv) Since no complex pole or complex zero exists for the system, there is no necessity of determining angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*
Since the total number of poles and zeros existing on the right of –1 and –3 are 1 and 3 (odd numbers), a branch of root loci exists between –1 and 0 and –3 and –2.
Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway and break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{Ks(s+1)}{(s+2)(s+3)} = 0$$

Therefore,
$$K = -\left[\frac{s^2 + s}{s^2 + 5s + 6}\right] \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -\left[\frac{(s^2 + 5s + 6)(2s+1) - (s^2 + s)(2s+5)}{(s^2 + 5s + 6)^2}\right] = 0$$

$$2s^3 + s^2 + 10s^2 + 5s + 12s + 6 - (2s^3 + 5s^2 + 2s^2 + 5s) = 0$$

$$4s^2 + 12s + 6 = 0$$

Upon solving, we get

$$s = -0.63, -2.366$$

For $s = -0.63$ and $s = -2.366$, the values of K are 0.0717 and 13.93 respectively. Since the values of K are positive, the points $s = -0.63$ and $s = -2.366$ are breakaway point and break-in point.

(vii) The point at which the branch of root loci crosses the imaginary axis is determined as follows:
From Eqn. (2), the characteristic equation of the system is

$$s^2(K+1) + s(5K+1) + 6K = 0 \tag{4}$$

Routh array for the above equation,

$$\begin{array}{c|cc} s^2 & K+1 & 6K \\ s^1 & 5K+1 & \\ s^0 & 6K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $5K+1=0$. Therefore, $K = -0.2$.
Since K is negative, the root locus does not cross the imaginary axis.
The complete root locus for the system is shown in Fig. E7.13.

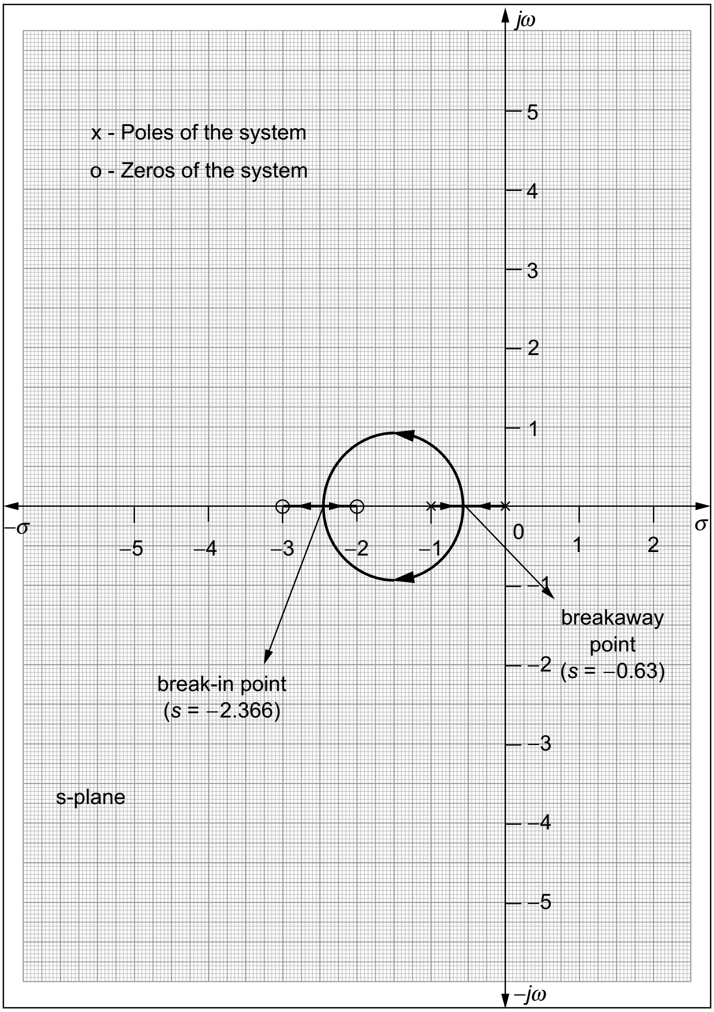


Fig. E7.13

Example 7.14: Sketch the root locus of the system whose transfer function is given by $\frac{C(s)}{R(s)} = \frac{K}{s(s+4)(s^2+s+1)+K}$.

Solution: We know that, the closed-loop transfer function of the system with the feedback transfer function $H(s)$ and open loop transfer function $G(s)$ is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Converting the given transfer function to the above form, we obtain

$$\frac{C(s)}{R(s)} = \frac{\left(\dfrac{K}{s(s+4)(s^2+s+1)}\right)}{1+\left(\dfrac{K}{s(s+4)(s^2+s+1)}\right)}$$

Hence, the function $G(s)H(s) = \left(\dfrac{K}{s(s+4)(s^2+s+1)}\right)$ for which the root locus has been drawn in the subsequent steps:

(i) For the given system $G(s)H(s) = \left(\dfrac{K}{s(s+4)(s^2+s+1)}\right)$, poles are at $0, -4, -0.5 - j0.86$, $-0.5 + j0.86$ i.e., $n = 4$ and zeros do not exist i.e., $m = 0$.

(ii) Since $n > m$, the number of branches of the root loci for the given system is $n = 4$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 4 - 0 = 4$

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{for } n \neq m,\ i = 0, 1,\ 2 \text{ and } 3$$

Therefore, $\theta_0 = 45°, \theta_1 = 135°, \theta_2 = 225°, \theta_3 = 315°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{-4 - 0.5 + j0.86 - 0.5 - j0.86 - 0}{4} = -1.25$$

(iv) Since complex poles exist, the angle of departure exists for the system.

To determine the angle of departure θ_d:

(a) For the pole $-0.5 + j0.866$

$$\theta_{d1} = 180° - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$= 180° - (\phi_{p1} + \phi_{p2} + \phi_{p3}) = 180° - (90 + \alpha_1 + 180° - \alpha_2)$$

$$= 180° - \left(90° + \tan^{-1}\left(\frac{3.5}{0.86}\right) + 180° - \tan\left(\frac{0.5}{0.86}\right)\right) = -44°$$

(b) For the pole $-0.5 - j0.866$, $\theta_{d2} = -\theta_{d1} = 44°$

(v) *To determine the number of branches existing on the real axis:*

Since the total number of poles and zeros existing on the right of –4 is 3 (odd number), a branch of root loci exists between –4 and 0.

Hence, one branch of root loci exists on the real axis for the given system.

(vi) The breakaway point/break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 + \frac{K}{s(s+4)(s^2+s+1)} = 0$$

Therefore,
$$K = -\left[s^4 + 5s^3 + 5s^2 + 4s\right] \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$-\left(4s^3 + 15s^2 + 10s + 4\right) = 0$$

Upon solving, we obtain

$$s = -3.03, -0.357 \pm j0.449$$

For $s = -3.03$, K is 21.01. Since K is positive, the point at $s = -3.03$ is a breakaway point.

Since the other two points obtained by solving Eqn. (1) are complex numbers, it is necessary to check the angle condition also.

Hence,

$$\angle G(s)H(s)\Big|_{s=-0.357+j0.449} = \frac{1}{\angle(-0.357 + j0.449) \times \angle(3.643 + j0.449) \times \angle(0.143 + j1.315) \times \angle(0.143 - j0.417)}$$

$$= -147°$$

Similarly,
$$\angle G(s)H(s)\Big|_{s=-0.357-j0.449} = 147°$$

Since the angles obtained for the given system at the complex numbers obtained on solving Eqn. (1) are not odd multiple of 180°, the complex numbers are not breakaway points.

Hence, only one breakaway point exists for the given system.

(vii) The point at which the branch of root loci intersects the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^4 + 5s^3 + 5s^2 + 4s + K = 0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|ccc} s^4 & 1 & 5 & K \\ s^3 & 5 & 4 & \\ s^2 & 4.2 & K & \\ s^1 & \dfrac{16.8-5K}{4.2} & & \\ s^0 & K & & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $\dfrac{16.8-5K}{4.2} = 0$. Therefore, $K = 3.36$.

Substituting K in the characteristic equation, we obtain

$$s^4 + 5s^3 + 5s^2 + 4s + 3.36 = 0$$

Substituting s = jω in the above equation, we obtain

$$(j\omega)^4 + 5(j\omega)^3 + 5(j\omega)^2 + 4(j\omega) + 3.36 = 0$$

$$\omega^4 - j5\omega^3 - 5\omega^2 + j4\omega + 3.36 = 0$$

Equating the imaginary part of the above equation to zero, we obtain

$$-5\omega^3 + 4\omega = 0$$

$$\omega^2 = 0.8$$

or $$\omega = \pm 0.894$$

Therefore, the point at which the root locus crosses the imaginary axis is $\pm j0.894$.

The complete root locus for the system is shown in Fig. E7.14.

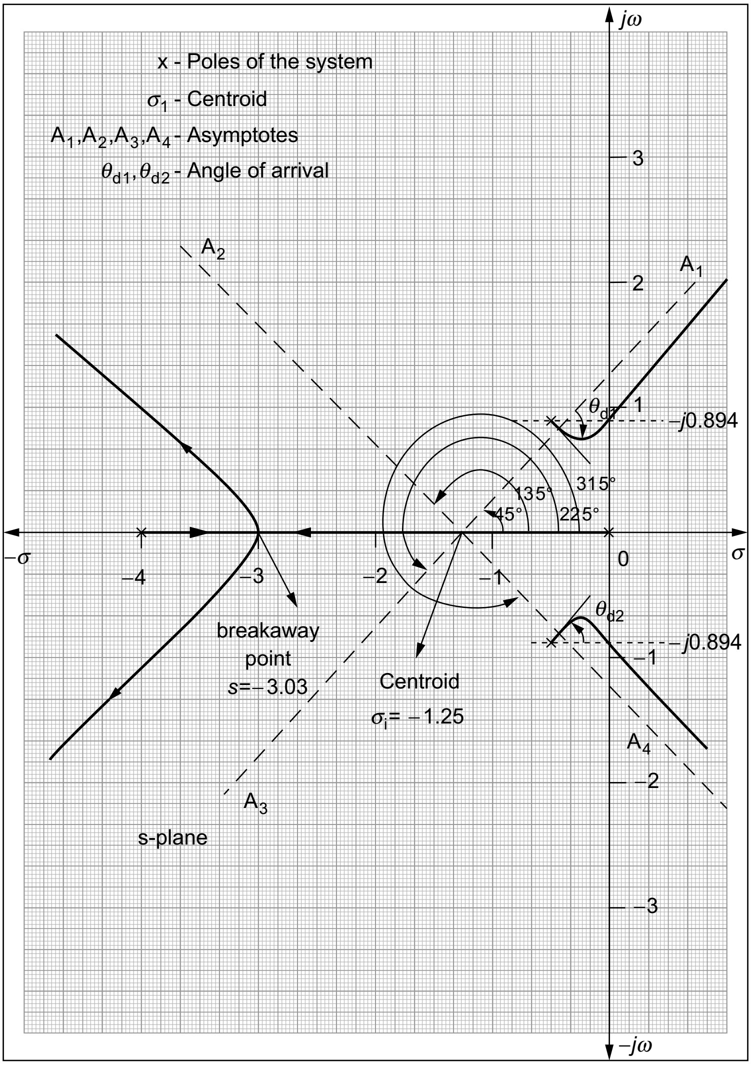


Fig. E7.14

Example 7.15: **Sketch the root locus of the system whose loop transfer function is given by** $G(s)H(s) = \dfrac{K(s^2+1)}{s(0.5s+1)}$**. Also, determine gain** K

(i) when the damping ratio is $\xi = 0.707$ **and (ii) for which repetitive roots occur.**

Solution The given function can be rewritten as

$$G(s)H(s)=\frac{2K(s^2+1)}{s(s+2)}$$

Replacing $\frac{2K}{2}$ by K' in the above equation, we obtain

$$G(s)H(s)=\frac{K'(s^2+1)}{s(s+2)}$$

(a) For the given system, poles are at 0 and −2 i.e., $n=2$ and zeros are at $+i$ and $-i$ i.e., $m=2$.
(b) Since $n=m$, the number of branches of the root loci for the given system is n *or* $m=2$.
(c) Since the number of poles and zeros present in the system are equal, the given system does not have any asymptote. Also, for the given system, it is not possible to determine the angle of asymptotes and centroid
(d) Since complex zeros exist, the angle of arrival exists for the system.

To determine the angle of arrival θ_a:

(1) For the zero j,

$$\theta_{a1}=180°-\sum_{i=1}^{n}\text{Angle from zero } i \text{ to the complex zero}+\sum_{j=1}^{m}\text{Angle from pole } j \text{ to the complex zero}$$

$$=180°-(\phi_{z1})+(\phi_{p1}+\phi_{p2})=180°-(90°)+(90°+\alpha_1)$$

$$=180°-(90°)+\left(90°+\tan^{-1}\left(\frac{1}{2}\right)\right)=206.56°$$

(2) For the zero $-j$, $\theta_{a2}=-\theta_{a1}=-206.56°$

(e) *To determine the number of branches existing on the real axis:*

Since the total number of poles and zeros existing on the right of −2 is 3 (odd number), a branch of root loci exists between −2 and 0.

Hence, one branch of root loci exists on the real axis for the given system.

(f) The breakaway/break-in points for the given system can be determined by

$$\frac{dK}{ds}=0 \tag{1}$$

For the given system, the characteristic equation is

$$1+G(s)H(s)=0 \tag{2}$$

i.e., $$1+\frac{K(s^2+1)}{s(s+2)}=0$$

Therefore, $$K^{'}=-\frac{\left(s^2+2s\right)}{\left(s^2+1\right)} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we get

$$-\left[\frac{\left(s^2+1\right)(2s+2)-\left(s^2+2s\right)(2s)}{\left(s^2+1\right)}\right]=0$$

$$-\left(2s^3+2s^2+2s+2-2s^3-4s^2\right)=0$$

$$s^2-s-2=0$$

Upon solving, we obtain

$$s=2,-1$$

Substituting the values of s in Eqn. (3), we get the corresponding values of K as

$$K=-1.6 \text{ for } s=2 \text{ and } K=0.5 \text{ for } s=-1$$

The breakaway point exists only for the positive values of K. Hence, $s=-1$ is the only one breakaway point that exists for the given system.

(g) The point at which the root loci intersects the imaginary axis can be obtained as follows:

From Eqn. (2), the characteristic equation of the system is

$$\left(1+K^{'}\right)s^2+2s+K^{'}=0$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & \left(1+K^{'}\right) & K' \\ s^1 & 2 & \\ s^0 & K' & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $1+K^{'}=0$. Therefore, $K^{'}=-1$.

Thus, $K=\dfrac{K'}{2}=-0.5$

Since K is negative, the root locus does not cross the imaginary axis at any point.

The complete root locus for the system is shown in Fig. E7.15.

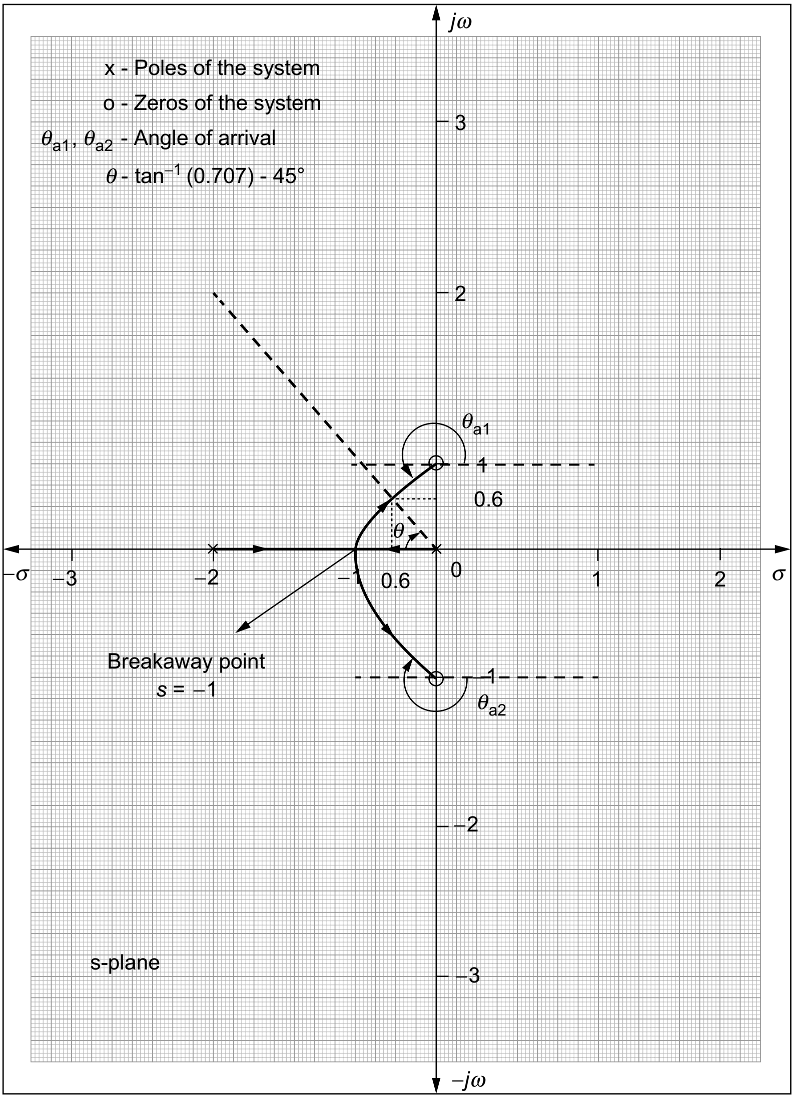


Fig. E7.15

(i) Determinine K:

Given the damping ratio $\xi = 0.707$, angle $\alpha = \cos^{-1}\xi = \cos^{-1}(0.707) = 45^{\circ}$.

Draw a line from origin at an angle of 45° so that the line bisects the root locus.

Determine the point at which the root locus has been bisected. For the given problem, the root locus gets bisected at $0.6 + j0.6$.

Hence, the magnitude of the given transfer function at that particular point is 1.

$$\left|G(s)H(s)\right| \text{ at } 0.6 + j0.6 = 1$$

$$\left|\frac{K'\left((0.6+j0.6)^2+1\right)}{(0.6+j0.6)\left((0.6+j0.6)+2\right)}\right|=1$$

$$\left|\frac{K'(1+j0.72)}{1.2+j1.92}\right|=1$$

$$K'=|1.7+j0.69|$$

$$=1.834$$

Therefore, the value of K for the given damping ratio is $K=0.917$

(ii) Finding gain at which the repetitive roots occur:

For a given system, the repetitive roots occur only at breakaway or break-in points.

Therefore, $$\left|G(s)H(s)\right|_{s=-1}=1$$

i.e., $$\left|\frac{K'\left((-1)^2+1\right)}{(-1)(-1+2)}\right|=1$$

$$\left|\frac{K'\left((-1)^2+1\right)}{(-1)(-1+2)}\right|=1$$

$$K'=\frac{1}{|-2|}=0.5$$

Therefore, the value of K at which repetitive roots occur is $K=0.25$.

Example 7.16: Construct the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K(s+2)}{s^2(s+10)}$.

Solution:

(i) For the given system, poles are at 0, 0 and −10 i.e., $n=3$ and zero is at −2 i.e., $m=1$.

(ii) Since $n>m$, the number of branches of the root loci for the given system is $n=3$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a=3-1=2$

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m,\ i = 0 \text{ and } 1$$

Therefore, $\theta_0 = 90°$ and $\theta_1 = 270°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{0+0-10-(-2)}{2} = -4$$

(iv) Since a complex pole or complex zero does not exist for the given system, there is no angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -10$, the total number of poles and zeros existing on the right of −10 is three (odd number). Therefore, a branch of root loci exists between −10 and −2.

Hence, only one branch of root loci exists on the real axis for the given system.

(vi) The breakaway/break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,

$$1 + \frac{K(s+2)}{s^2(s+10)} = 0$$

Therefore

$$K = -\frac{\left(s^3 + 10s^2\right)}{(s+2)} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\left(\left(3s^2 + 20s\right)(s+2)\right) - \left(s^3 + 10s^2\right) = 0$$

$$\left(3s^3 + 6s^2 + 20s^2 + 40s\right) - s^3 - 10s^2 = 0$$

$$2s^3 + 16s^2 + 40s = 0$$

$$2s\left(s^2 + 8s + 20\right) = 0$$

Upon solving, we obtain

$$s = 0, \ -4 \pm j2$$

Substituting $s = 0$ in Eqn. (3), we obtain $K = 0$. Hence, the point is not a valid breakaway point.

For the other two points, i.e., $s = -4 \pm j2$, it is necessary to check the angle condition.

$$\angle G(s)H(s)\Big|_{s=-4+j2} = \frac{\angle(-2+j2)}{\angle(-4+j2)\times\angle(-4+j2)\times\angle(6+j2)} = -190°$$

Similarly, $\angle G(s)H(s)\Big|_{s=-4-j2} = 190°$

Since the angle of the system at the complex numbers is not an odd multiple of 180°, these complex numbers are not valid breakaway points.

Hence, no breakaway point exists for the given system.

(vii) The point at which the branch of root loci intersects the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^3 + 10s^2 + Ks + 2K = 0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & K \\ s^2 & 10 & 2K \\ s^1 & \dfrac{8K}{10} & \\ s^0 & 2K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero i.e., $\dfrac{8K}{10} = 0$. Therefore, $K = 0$.

Substituting the value $K = 0$ in Eqn. (4), we obtain

$$s^2(s+10) = 0$$

Upon solving, $s = 0, 0, -10$.

Since all the values of s are real, the root locus does not cross the imaginary axis.

The complete root locus for the given system is shown in Fig. E7.16.

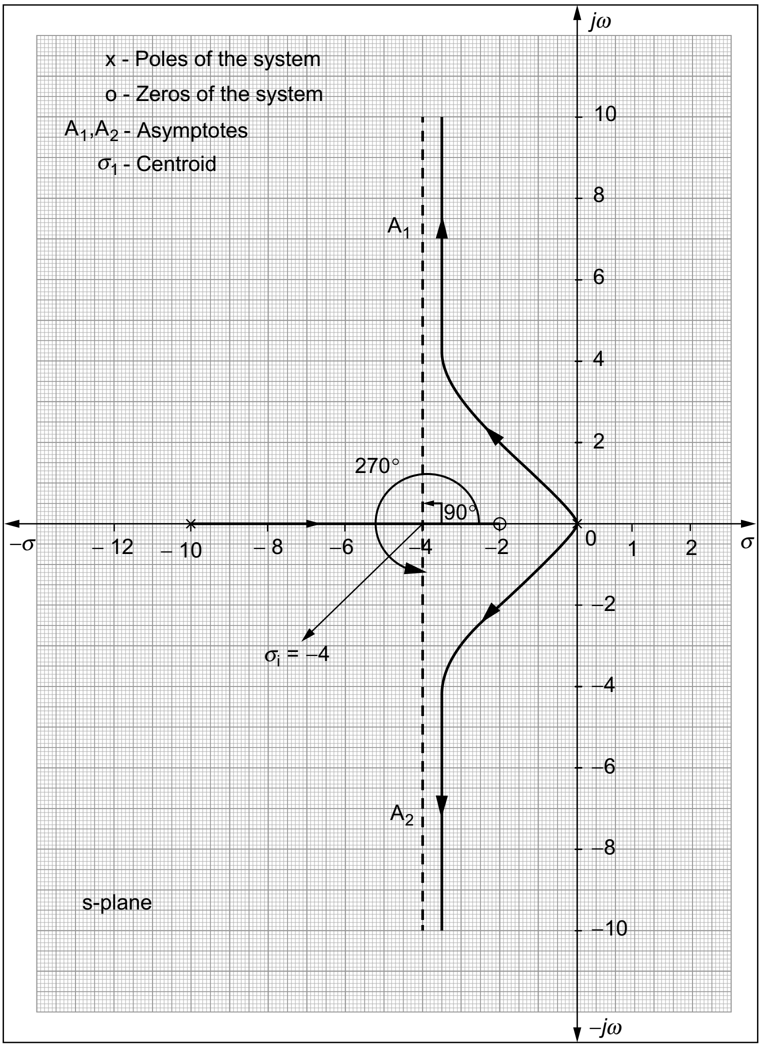


Fig. E7.16

Example 7.17: The loop transfer function of a system is given by $G(s)H(s) = \dfrac{K(s+1)}{s(s-1)}$. Construct the root locus of the system.

Solution:

(i) For the given system, poles are at 0 and 1, i.e., $n = 2$ and zero is at -1 i.e., $m = 1$.

(ii) Since $n > m$, the number of branches of the root loci for the given system is $n = 2$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 2 - 1 = 1.$

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m \text{ and } i = 0$$

Therefore, $\theta_o = 180°$

(c) Centroid

$$\sigma_1 = \frac{\Sigma \text{ finite poles of } G(s)H(s) - \Sigma \text{ finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{0+1-(-1)}{1} = 2$$

(iv) Since a complex pole or complex zero does not exist for the given system, there is no angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = 0$, the total number of poles and zeros existing on the right of 0 is one (odd number). Therefore, a branch of root loci exists between 0 and 1.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of that point is three (odd number). Therefore, a branch of root loci exists between -1 and $-\infty$.

Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway and break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,

$$1 + \frac{K(s+1)}{s(s-1)} = 0$$

Therefore,

$$K = -\left[\frac{s(s-1)}{s+1}\right] \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$-\left[\frac{(s+1)(2s-1) - (s^2 - s)(1)}{(s+1)^2}\right] = 0$$

$$s^2 + 2s - 1 = 0$$

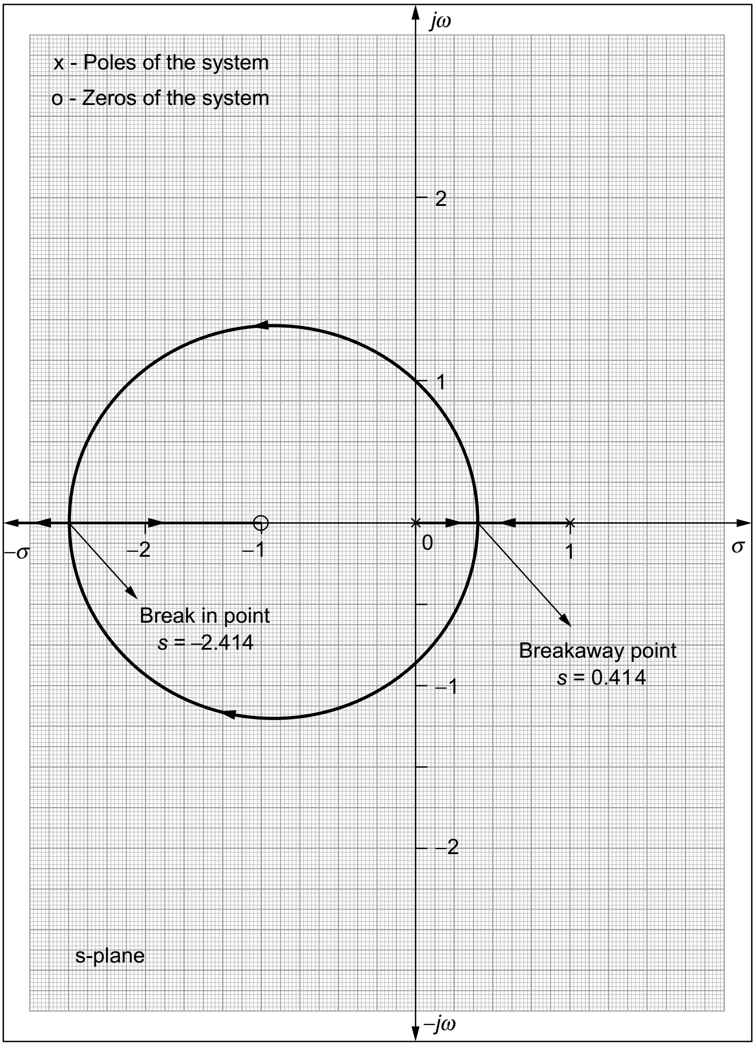


Fig. E7.17

Therefore, $s = 0.414, \ -2.414$.

For $s = 0.414$ and $s = -2.414$, the values of K are 0.1715 and 5.828 respectively. Since the values of K are positive, $s = 0.414$ and $s = -2.414$ are the breakaway and break-in points respectively.

(vii) The point at which the branch of root loci intersects the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^2 + s\left(K - 1\right) + K = 0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & 1 & K \\ s^1 & K-1 & \\ s^0 & K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $K-1=0$. Therefore, $K=1$.

Substituting K in Eqn. (4), we obtain

$$s^2+1=0$$

Upon solving, we get

$$s=\pm j$$

Therefore, the root locus crosses the imaginary axis at $\pm j$.

The complete root locus for the system is shown in Fig. E7.17.

Example 7.18: Develop the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{65000K_1}{s(s+25)(s^2+100s+2600)}$ using the properties of root loci.

Solution:

(i) For the given system, poles are at $0, -25, -50\pm j10$ i.e., $n=4$ and zero does not exist i.e., $m=0$.

(ii) Since $n>m$, the number of branches of root loci for the given system is $n=4$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a=4-0=4$.

(b) Angles of asymptotes

$$\theta_i=\frac{2i+1}{|n-m|}\times 180°, \text{ for } n\neq m,\ i=0,1,2 \text{ and } 3.$$

Therefore, $\theta_0=45°, \theta_1=135°, \theta_2=225°$ and $\theta_3=315°$.

(c) Centroid

$$\sigma_1=\frac{\sum \text{finite poles of } G(s)H(s)-\sum \text{finite zero of } G(s)H(s)}{n-m}$$

$$=\frac{0-25-50-10j-50+10j-0}{4}=-31.25$$

(iv) Since complex poles exist, the angle of departure exists for the system.

To determine the angle of departure θ_d:

(a) For the pole $-50+j10$

$$\theta_{d1} = 180° - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole } + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$= 180° - \left(\phi_{p1} + \phi_{p2} + \phi_{p3}\right)$$

$$= 180° - \left(90 + 180 - \tan^{-1}\left(\frac{10}{25}\right) + 180° - \tan^{-1}\left(\frac{10}{50}\right)\right) = -236.88°$$

(b) For the pole $-50 - j10$, $\theta_{d2} = -\theta_{d1} = 236.88°$

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -25$, the total number of poles and zeros existing on the right of −25 is one (odd number). Therefore, a branch of root loci exists between −25 and 0.

Hence, only one branch of root loci exists on the real axis for the given system.

(vi) The breakaway/break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \qquad (1)$$

The characteristic equation of the given system is

$$1 + G(s)H(s) = 0$$

i.e.,
$$1 + \frac{K}{s(s+25)(s^2 + 100s + 2600)} = 0$$

Therefore,
$$K = -\left(s^4 + 125s^3 + 5100s^2 + 65000s\right)$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -\left(4s^3 + 375s^2 + 10200s + 65000\right) = 0$$

Upon solving, we obtain

$$s = -9.15, -45.95, -38.64.$$

Substituting $s = -9.15$, $s = -45.95$ and $s = -38.64$, the values of K are 2.56×10^5, -1.12×10^5 and -1.2×10^5 respectively. Since K is positive only at $s = -9.15$, the point $s = -9.15$ is a valid breakaway point.

(vii) The point at which the branch of root locus crosses the imaginary axis is obtained by substituting $s = j\omega$ in the characteristic equation and then comparing the real and imaginary parts of the equation.

$$\omega^4 - j125\omega^3 - 5100\omega^2 + j65000\omega + K = 0$$

Comparing the imaginary parts on both sides, we obtain

$$\omega^2 = \frac{65000}{125}.$$

or, $$\omega = \pm 22.8.$$

Hence, the root locus crosses the imaginary axis at $\pm j22.8$.
The complete root locus plot for the given system is shown in Fig. E7.18.

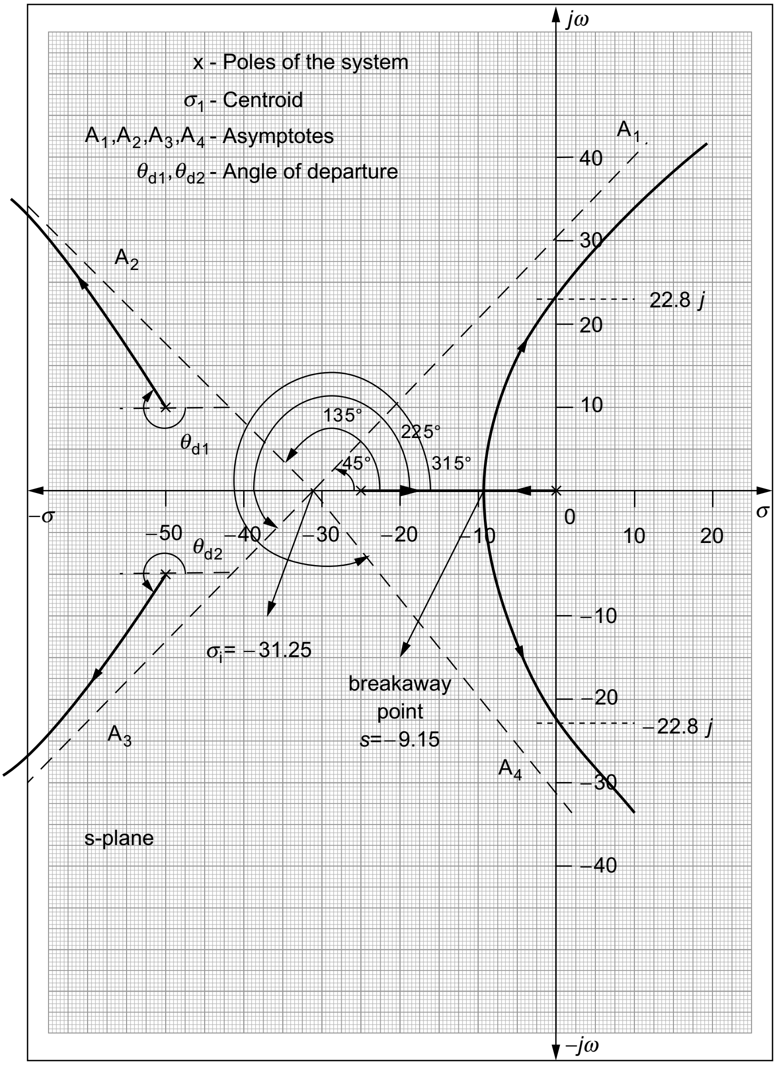


Fig. E7.18

Example 7.19: Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K(s+a)}{s(s^2+2s+2)}$ by varying the parameter, $a=-4,-2,$ -1 and 0. Also, comment on the root locus of the different systems.

Solution: The values obtained for constructing the root loci for different values of the parameter, a, are given in Table E7.19.

Table E7.19 | Root loci for different systems

Properties of root loci	$a=-4$	$a=-2$	$a=-1$	$a=0$
$G(s)H(s)$	$\dfrac{K(s-4)}{s(s^2+2s+2)}$	$\dfrac{K(s-2)}{s(s^2+2s+2)}$	$\dfrac{K(s-1)}{s(s^2+2s+2)}$	$\dfrac{K}{(s^2+2s+2)}$
Poles of the system	$0, -1\pm j$	$0, -1\pm j$	$0, -1\pm j$	$-1\pm j$
Zeros of the system	4	2	1	No zeros
Branches of root loci	3	3	3	2
Number of asymptotes N_a	2	2	2	2
Angle of asymptotes θ_i	90° and 270°	90° and 270°	90° and 270°	90° and 270°
Centroid σ_i	−3	−2	−1.5	−1
Branch of root loci on the real axis	1	1	1	NIL
Angle of departure, θ_i	123.7° and −123.7°	116.74° and −116.74°	108.43° and −108.43°	90°
Breakaway points	Does not exist	Does not exist	Does not exist	Does not exist
Intersection point on the imaginary axis	Does not cross the imaginary axis	Does not cross the imaginary axis	Does not cross the imaginary axis	Does not cross the imaginary axis

The root loci of the systems with $a = -4$, $a = -2$, $a = -1$ and $a = 0$ are shown in Figs. E7.19(a), (b), (c) and (d) respectively.

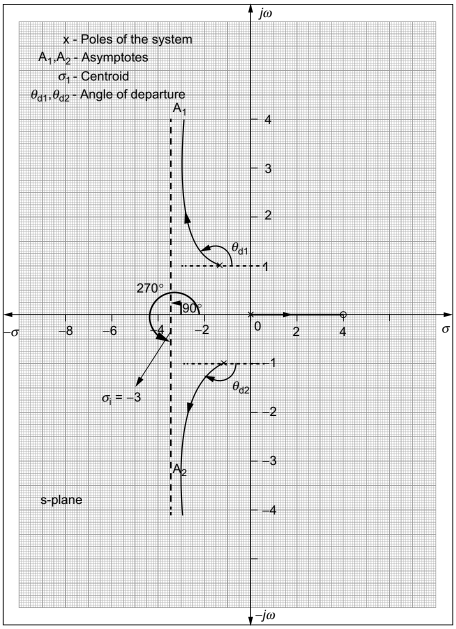


Fig. E7.19(a)

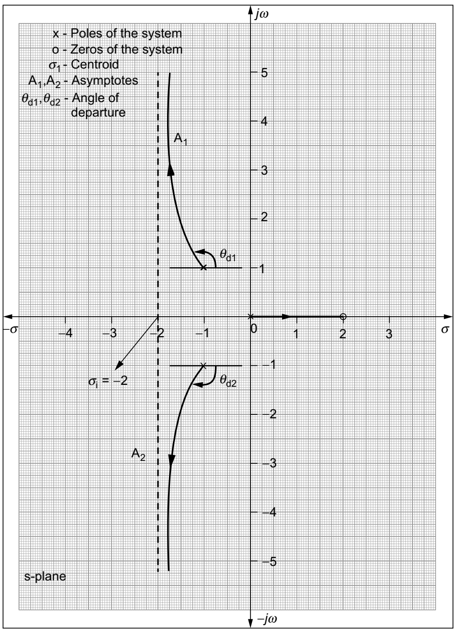


Fig. E7.19(b)

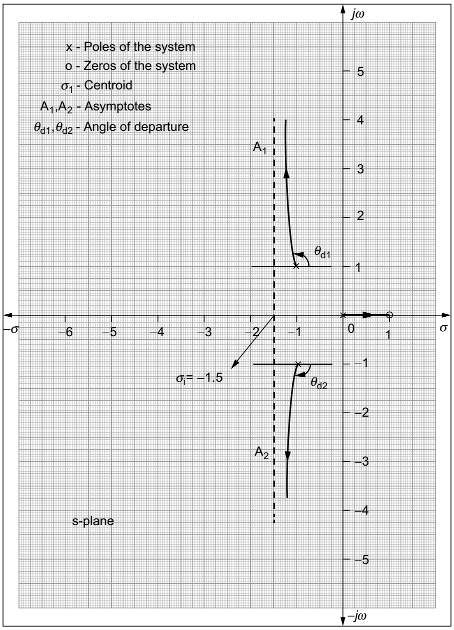


Fig. E7.19(c)

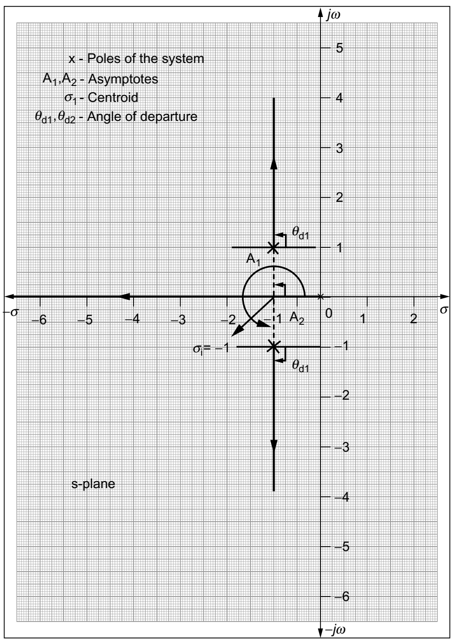


Fig. E7.19(d)

7.7 Effect of Adding Poles and Zeros in the System

Consider a system whose loop transfer function is given by, $G(s)H(s) = \dfrac{K}{s(s+1)}$. The root loci for the given system are shown in Fig. 7.10. For designing a controller to the system given above, it is necessary to have a clear study over the effects of root loci when extra poles or zeros or both get added to the loop transfer function. The above system is stable for all the values of variable gain $K > 0$.

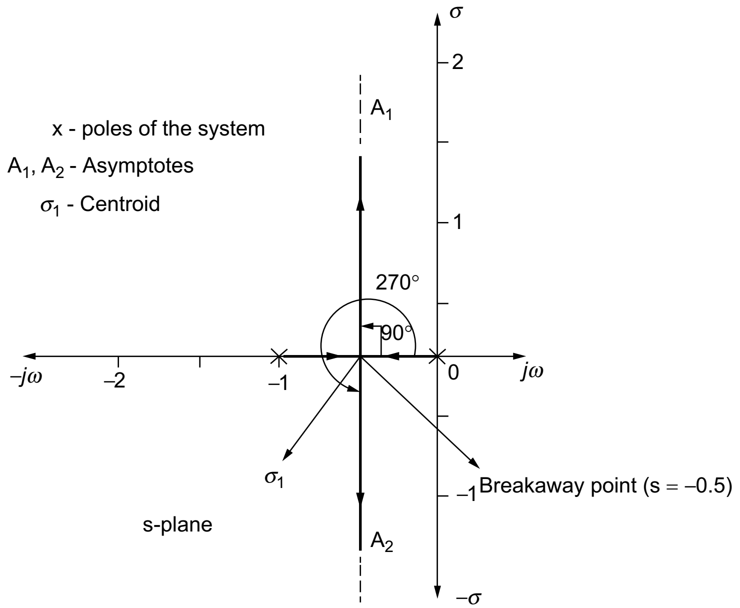


Fig. 7.10 | Root locus for $G(s)H(s) = \dfrac{K}{s(s+1)}$

7.7.1 Addition of Poles to the Loop Transfer Function, $G(s)H(s)$

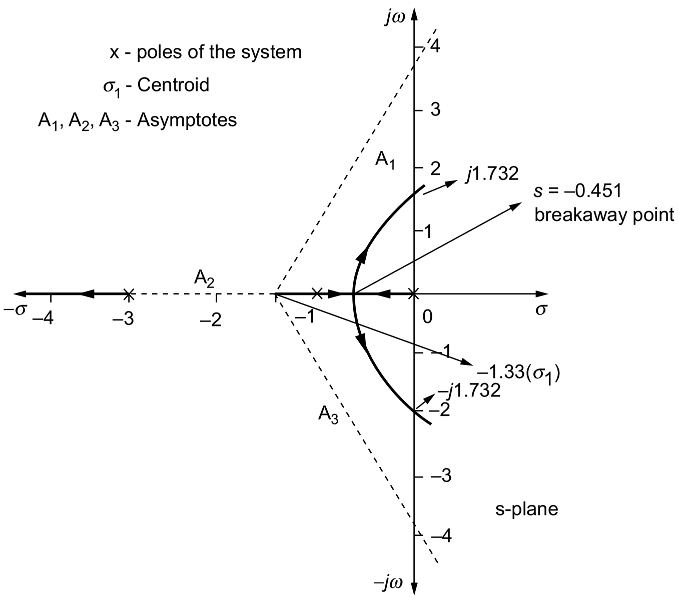
The effects of addition of extra poles in the root loci of the existing system are studied for three cases and they are given in Table 7.8.

7.7.2 Effect of Addition of Poles

If a pole is added to the system, the following effects take place in the root loci of the original system.

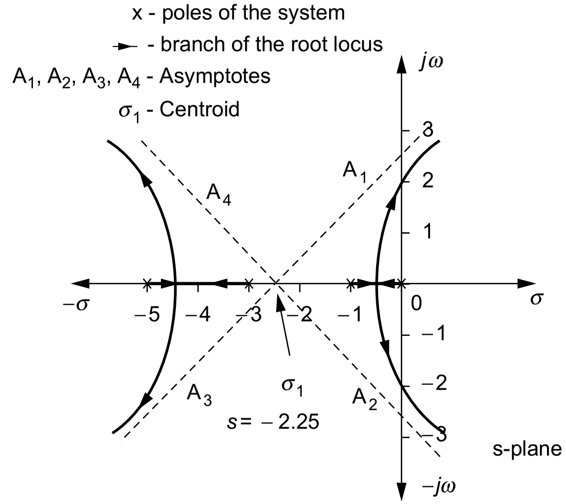
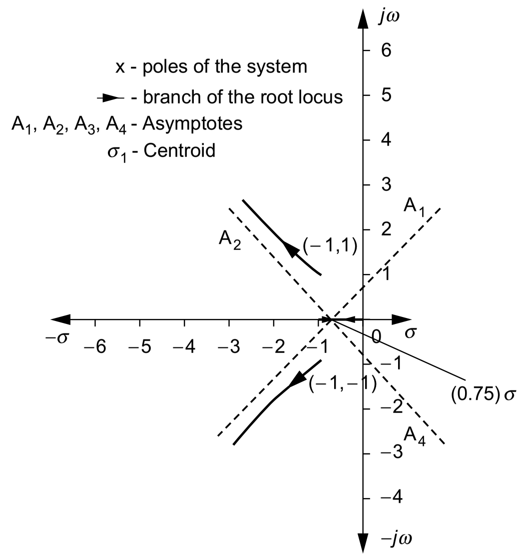
(i) Root locus gets shifted towards the right-half s-plane.
(ii) Stability of the system gets gradually decreased.
(iii) Nature of the system response becomes more oscillatory.
(iv) Marginal value of K decreases.

Table 7.8 | Effect of addition of poles to the system

Extra pole in *s*-plane	Loop transfer function of the system $G(s)H(s)$	Root loci for the given system	Effect of addition of poles to the system
At $s = -3$	$\dfrac{K}{s(s+1)(s+3)}$		✓ Root loci bends towards the RHS of *s*-plane ✓ Angle of asymptotes $(\pm 90°$ to $(\pm 60°$ and $180°))$ ✓ Number of asymptotes $N_a = 2$ to 3 ✓ Centroid $\sigma_1 = (-0.5)$ to (-1.33) ✓ Marginal value of $K = 12$, above which the system becomes unstable.

(*Continued*)

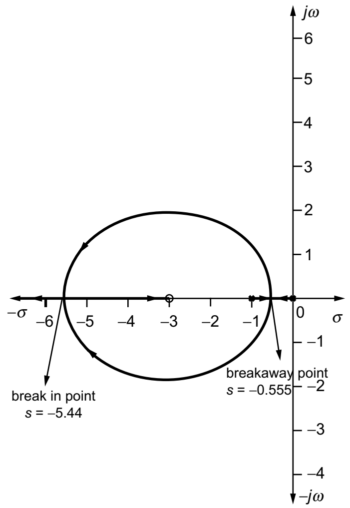
Table 7.8 | (Continued)

At $s=-5$	$\dfrac{K}{s(s+1)(s+3)(s+5)}$		✓ Root loci bends further more towards the RHS of *s*-plane ✓ Angle of asymptotes $(\pm 45°, 135° \text{ and } 225°)$ ✓ Centroid $\sigma_1 = (-0.5)$ to (-0.75) ✓ Stability of the system decreases further more.
Complex conjugate poles at $s = 1 \pm j$	$\dfrac{K}{s(s+1)(s+1+j)(s+1-j)}$		✓ Root loci bends further more towards the RHS of *s*-plane ✓ Angle of asymptotes $(\pm 45°, 135° \text{and } 225°)$ ✓ Centroid $\sigma_1 = (-0.5)$ to (-0.75) ✓ Marginal value of $K = 2.222$ above which the system becomes unstable

7.7.3 Addition of Zero to the Loop Transfer Function

The effects of addition of an extra zero in the root loci of the existing system are studied and given in Table 7.9.

Table 7.9 | Effect of addition of zero to the system

Extra zero in *s*-plane	System	Root loci	Effect
At $s = -3$	$G(s)H(s) = \dfrac{K(s+3)}{s(s+1)}$		✓ Root locus bends towards the LHS of *s*-plane ✓ Relative stability of the system gets improved.

7.7.4 Effect of Addition of Zeros

If a zero is added to the left half of system, the following effects take place in the root loci of the original system.

(i) Root locus gets shifted towards the left-half of *s*-plane.
(ii) Stability of the system gradually increases.
(iii) Nature of the system response becomes less oscillatory.
(iv) Marginal value K increases.

Example 7.20: Sketch the root locus of the system whose loop transfer function is $G(s)H(s) = \dfrac{K}{s^2(s+2)}$. Also, plot the root locus when a zero at –1 is added to the existing system.

Solution:

(i) For the given system, poles are at 0, 0 and –2, i.e., $n = 3$ and zeros do not exist, i.e., $m = 0$.
(ii) Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.
(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes for the given system $N_a = 3 - 0 = 3$

(b) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m, \ i = 0, 1 \text{ and } 2$$

Therefore, $\theta_0 = 60°, \theta_1 = 180°$ and $\theta_2 = 300°$.

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

$$= \frac{-0-0-2-0}{3} = -0.666$$

(iv) Since no complex pole or complex zero exists for the system, there is no necessity of determining angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the point $-\infty$, the total number of poles and zeros existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between $-\infty$ and -2.

Hence, only one branch of root loci exists on the real axis for the given system.

(vi) The breakaway/break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

The characteristic equation of the system is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,

$$1 + \frac{K}{s^2(s+2)} = 0$$

Therefore,

$$K = -\left(s^3 + 2s^2\right) \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds} = -\left(3s^2 + 4s\right) = 0$$

Upon solving, we obtain

$$s = 0, -1.33$$

For $s = 0$ and $s = -1.33$, the values of K are 0 and -1.18 respectively. Hence, the point $s = 0$ is a breakaway point and $s = -1.33$ is not a breakaway point.

(vii) The point at which the root locus crosses the imaginary axis is determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^3 + 2s^2 + K = 0 \tag{4}$$

Routh array for the above equation,

$$\begin{array}{c|cc} s^3 & 1 & 0 \\ s^2 & 2 & K \\ s^1 & -\dfrac{K}{2} & \\ s^0 & K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $-\dfrac{K}{2} = 0$. Therefore, $K = 0$.

Substituting K in the characteristic equation and solving for s, we obtain

$$s = 0,\ 0,\ -2.$$

Hence, the root locus of the system does not cross the imaginary axis.

The final root locus for the system is shown in Fig. E7.20(a).

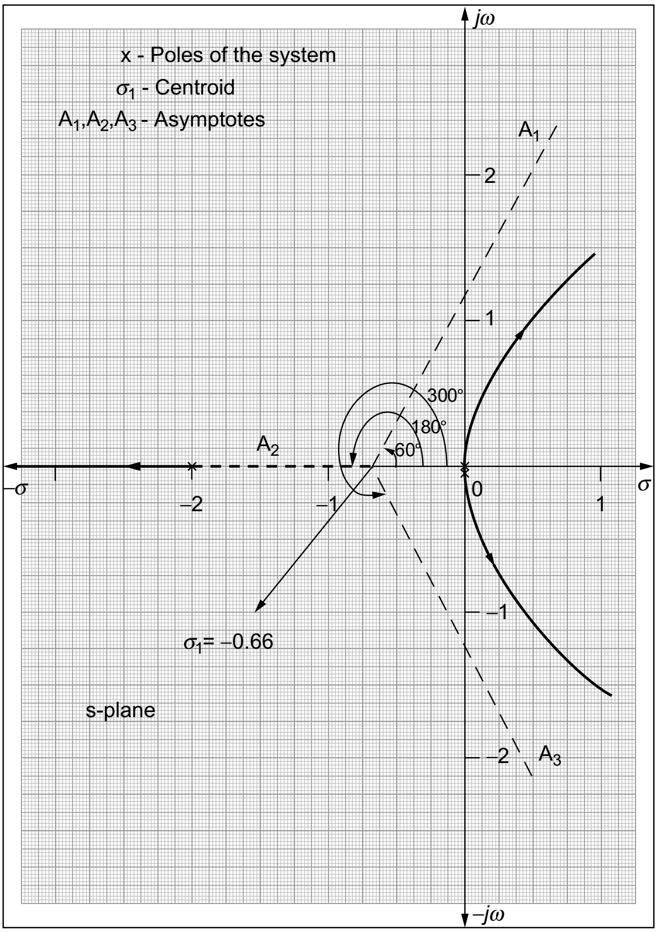


Fig. E7.20(a)

(ii) When a zero at s = −1 is added to the existing system, the transfer function of the system becomes

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+2)}$$

(a) For the given system, poles are at 0, 0 and −2 i.e., $n = 3$ and zero is at −1 i.e., $m = 1$.

(b) **S**ince $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(c) For the given system, the details of the asymptotes are given below:

(1) Number of asymptotes for the given system $N_a = 3 - 1 = 2$.

(2) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|} \times 180°, \text{ for } n \neq m, i = 0 \text{ and } 1$$

Therefore, $\theta_0 = 90°$ and $\theta_1 = 270°$.

(3) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

$$= \frac{-0-0-2-(-1)}{2} = -0.5$$

(d) Since no complex pole or complex zero exists for the system, there is no necessity of determining angle of departure/angle of arrival.

(e) *To determine the number of branches of root loci on the real axis:*

If we look from the pole $p = -2$, the total number of poles and zeros existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between −2 and −1.

Hence, one branch of root loci exists on the real axis for the given system.

(f) The breakaway/break-in points for the given system can be obtained by

$$\frac{dK}{ds} = 0 \quad (1)$$

The characteristic equation of the given system is

$$1 + G(s)H(s) = 0 \quad (2)$$

i.e., $$1 + \frac{K(s+1)}{s^2(s+2)} = 0$$

Therefore, $$K = \frac{-(s^3 + 2s^2)}{(s+1)}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds}=\frac{(s+1)(-3s^2-4s)-(-s^3-2s^2)}{(s+1)^2}=\frac{-2s^3-5s^2-4s}{(s+1)^2}=0$$

Upon solving, we obtain

$$s=0,-1.25\pm j0.66.$$

For $s=0$, K is 0. Since K is neither positive nor negative, the point $s=0$ is not a valid breakaway point.

Since the other two points are complex numbers, it is necessary to check the angle condition also.

$$\angle(G(s)H(s))\Big|_{s=-1.25+0.66j}=\frac{\angle(-0.25+0.66j)}{\angle(-.125+0.66j)\times\angle(-.125+0.66j)\times\angle(0.75+0.66j)}=-235°$$

Similarly, $\angle(G(s)H(s))\Big|_{s=-1.25-0.66j}=235°$

Since the angle of the transfer function $G(s)H(s)$ at the complex numbers is not an odd multiple of 180°, the complex numbers are not valid breakaway points.

Hence, no breakaway points exist for the given system.

(g) The point at which the branch of root loci intersects the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^3+2s^2+Ks+K=0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & K \\ s^2 & 2 & K \\ s^1 & \frac{K}{2} & \\ s^0 & K & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in the third row must be zero, i.e., $\frac{K}{2}=0$. Therefore, $K=0$.

Substituting $K=0$ in the characteristic equation, we obtain

$$s^3+2s^2=0$$

Upon solving, we obtain $s=0,-2$. As there is no imaginary part in the solution, the branch of root loci does not cross the imaginary axis.

The final root locus for the system is shown in Fig. E7.20(b).
It is inferred that, when a zero is added, root locus shifts left and stability increases.

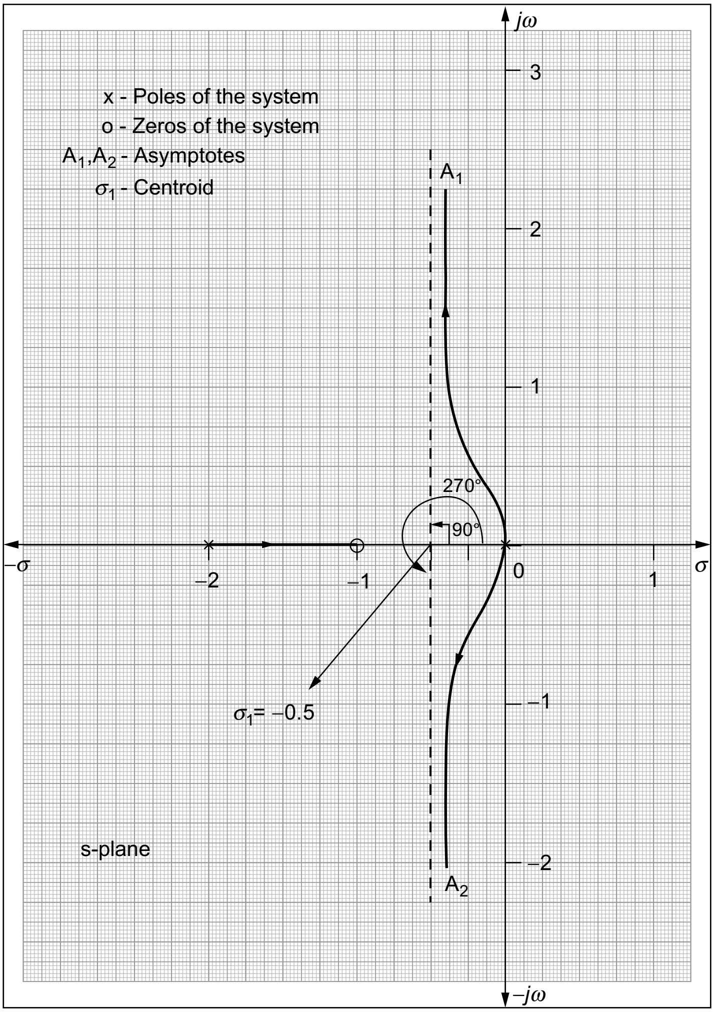


Fig. E7.20(b)

7.8 Time Response from Root Locus

The pole-zero configuration of a loop transfer function, $G(s)H(s)$, plays a major role in determining the time response of the system. The closed-loop transfer function of the system with forward loop transfer function $G(s)$ and negative feedback transfer function $H(s)$ is

$$\frac{C(s)}{R(s)} = T(s) = \frac{G(s)}{1+G(s)H(s)}$$

where $C(s)$ is the Laplace transform of the output signal and $R(s)$ is the Laplace transform of the input signal.

For a unity feedback system, $H(s)=1$ and if there exists n closed-loop poles and m closed-loop zeros, then the transfer function $T(s)$ can be expressed as

$$T(s)=K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$=K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)} \tag{7.8}$$

where K is the closed-loop gain which can be expressed in terms of z_i and p_j.

The time response of the closed-loop transfer function, $c(t)$ approaches 1 as $t\to\infty$ if an unit step input is applied to the system. Applying final value theorem, we get

$$\underset{t\to\infty}{Lt}\, c(t)=\underset{s\to 0}{Lt}\, s.R(s)T(s)$$

Here, $R(s)=\dfrac{1}{s}$ and $\underset{t\to\infty}{Lt}\, c(t)=1$

Therefore,
$$1=\underset{s\to 0}{Lt}\, T(s)=\underset{s\to 0}{Lt}\frac{G(s)}{1+G(s)}$$

$$1=\underset{s\to 0}{Lt}\, K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}$$

$$1=K\frac{\prod_{i=1}^{m}(-z_i)}{\prod_{j=1}^{n}(-p_j)}$$

or
$$K=\frac{\prod_{j=1}^{n}(-p_j)}{\prod_{i=1}^{m}(-z_i)} \tag{7.9}$$

Substituting Eqn. (7.9) in Eqn. (7.8), we obtain

$$T(s)=\frac{\prod_{j=1}^{n}(-p_j)}{\prod_{i=1}^{m}(-z_i)}\times\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}$$

If the closed-loop poles existing in the system are distinct, then

$$C(s) = \frac{1}{s}T(s)$$

$$= \frac{1}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n}$$

Taking inverse Laplace transform, we obtain

$$c(t) = 1 + A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} \tag{7.10}$$

where $A_k = -1 \times \dfrac{\prod_{\substack{j=1 \\ j\neq k}}^{n}(-p_j)}{\prod_{i=1}^{m}(-z_i)} \times \dfrac{\prod_{i=1}^{m}(s-z_i)}{\prod_{\substack{j=1 \\ j\neq k}}^{n}(s-p_j)}$.

Using the above two equations, the response of the system can be obtained if the closed-loop poles-zeros are known.

Example 7.21: The pole-zero plot of a loop transfer function of a system is shown in Fig. E7.21. Determine the response of the system when it is subjected to unit step input.

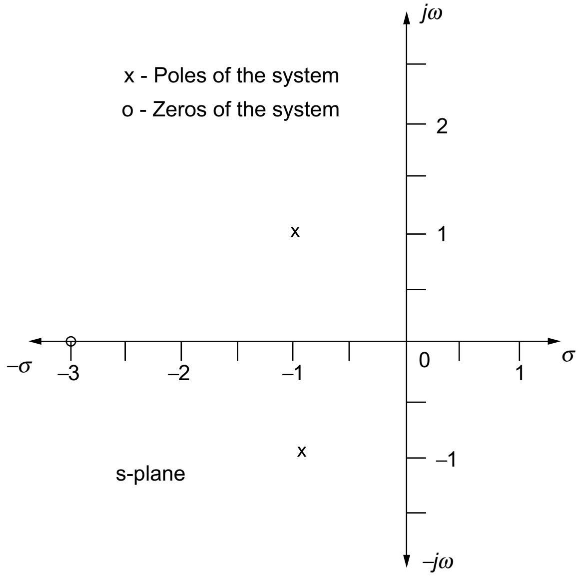


Fig. E7.21

Solution:

From Fig. E7.21, the poles are at $-1 \pm j1$ and zero is at -3

Using Eqn. (7.10), we obtain the response of the system for the unit step input as

$$c(t) = 1 + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

where $$A_k\Big|_{s=p_k} = -1 \times \frac{\prod\limits_{\substack{j=1 \\ j \neq k}}^{n} (-p_j)}{\prod\limits_{i=1}^{m} (-z_i)} \times \frac{\prod\limits_{i=1}^{m} (s - z_i)}{\prod\limits_{\substack{j=1 \\ j \neq k}}^{n} (s - p_j)}$$

To determine, A_k:

$$A_1\Big|_{s=-1+j1} = -1 \times \frac{-(-1-j1)}{-(-3)} \times \frac{(-1+j1-(-3))}{(-1+j1-(-1-j1))}$$

$$= -1 \times \frac{(1+j1)}{3} \times \frac{(2+j1)}{j2} = -0.5 + j0.166$$

$$A_2\Big|_{s=-1-j1} = -1 \times \frac{-(-1+j1)}{-(-3)} \times \frac{(-1-j1-(-3))}{(-1-j1-(-1+j1))}$$

$$= -1 \times \frac{(1-j1)}{3} \times \frac{(2-j1)}{-j2} = -0.5 - j0.166$$

Therefore,

$$c(t) = 1 + (-0.5 + j0.166)e^{(-1+j1)t} + (-0.5 - j0.166)e^{(-1-j1)t}$$

$$= 1 - 0.5 \times (e^{-t} \times e^{jt}) + j0.166 \times (e^{-t} \times e^{jt}) - 0.5 \times (e^{-t} \times e^{-jt}) - j0.166 \times (e^{-t} \times e^{-jt})$$

$$= 1 - \left(0.5 \times e^{-t} \times (e^{jt} + e^{-jt})\right) + j0.166 \times e^{-t} \times (e^{jt} - e^{-jt})$$

$$= 1 - e^{-t}\cos t + j0.332 e^{-t}\sin t \qquad (\text{since } e^{jt} + e^{-jt} = 2\cos t \text{ and } e^{jt} - e^{-jt} = 2\sin t)$$

7.9 Gain Margin and Phase Margin of the System

The gain margin and phase margin for a system can be determined using root locus method.

7.9.1 Gain Margin of the System

The maximum allowable loop gain value in decibels (dB) before the closed-loop system becomes unstable is called gain margin. The formula for determining the gain margin in dB is

$$\text{Gain margin in } dB,\ g_m\Big|_{dB} = 20\log_{10}(g_m)$$

where $$g_m = \frac{K_{\text{marginal}}}{K_{\text{design}}}$$

K_{marginal} = gain K at which the root locus crosses the imaginary axis and

K_{design} = required gain

It is to be noted that if the root locus of the system does not cross the imaginary axis, then K_{marginal} is ∞ and hence the gain margin of the system is also ∞.

7.9.2 Phase Margin of the System

The maximum allowable phase angle in degrees of $G(s)H(s)$ before the closed-loop system becomes unstable is called phase margin. The phase margin of the system can be calculated using the following steps:

Step 1: Substitute $s = j\omega$ in the given loop transfer function and determine $G(j\omega)H(j\omega)$.

Step 2: Substitute K_{design} in the given loop transfer function and determine the frequency ω, for which $|G(j\omega)H(j\omega)| = 1$.

Step 3: Determine the angle of $G(j\omega)H(j\omega)$ at the frequency obtained in the previous step.

Step 4: Determine the phase margin of the system as

Phase margin, $p_m = 180° + \angle G(j\omega)H(j\omega)$.

Example 7.22: Determine gain margin and phase margin of the system whose loop transfer function is given by $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$ and the required/desired gain value is 3.

Solution: The final root locus of the system, whose loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$ is shown in Fig. E7.12. Also, from Example 7.12, it is known that the root locus crosses the imaginary axis at $s = \pm j1.414$. Substituting $s = \pm j1.414$ in the characteristic equation of the system, we obtain the marginal gain K_{marginal} as 6. Hence, the gain margin of the system in dB is

$$\text{Gain margin in } dB,\ g_m\big|_{dB} = 20\log_{10}(g_m)$$

where $$g_m = \frac{K_{\text{marginal}}}{K_{\text{design}}}$$

In this case, $K_{\text{marginal}} = 6$ and $K_{\text{desired}} = 3$. Therefore,

$$g_m = \frac{K_{\text{marginal}}}{K_{\text{design}}} = \frac{6}{3} = 2$$

$$\text{Gain margin in } dB,\ g_m\big|_{dB} = 20\log(2) = 6.02 \text{ dB}$$

To determine phase margin of the system:

Step 1: Substituting $s = j\omega$ in the loop transfer function, we obtain

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega)(j\omega+1)(j\omega+2)}$$

Step 2: Substituting $K_{\text{desired}} = 3$ in the above equation and using the condition $|G(j\omega)H(j\omega)| = 1$, we get the value of frequency ω.

$$\frac{3}{\omega \times \sqrt{\omega^2+1} \times \sqrt{\omega^2+4}} = 1$$

Squaring on both sides of the equation, we obtain

$$9 = \omega^2 \times (\omega^2+1) \times (\omega^2+4)$$
$$= \omega^6 + 5\omega^4 + 4\omega^2$$

Therefore, $\omega^6 + 5\omega^4 + 4\omega^2 - 9 = 0$

Substituting $x = \omega^2$ in the above equation, we obtain

$$x^3 + 5x^2 + 4x - 9 = 0$$

Upon solving, we obtain

$$x = 0.939, -2.969 \pm j0.872$$

Neglecting the complex numbers and negative numbers, we obtain the frequency as

$$\omega = \sqrt{x} = \sqrt{0.939} = 0.969 \text{ rad/sec.}$$

Step 3: $\angle G(j\omega)H(j\omega)\big|_{\omega=0.969} = 0° - 90° - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)\Big|_{\omega=0.969}$

$$= 0° - 90° - 44.09° - 25.85° = -159.94°$$

Step 4: Phase margin of the system, $\text{pm} = 180° - 159.94° = 20.06°$.

7.10 Root Locus for *K* < 0 Inverse Root Locus or Complementary Root Loci

From Table 7.1, it is clear that the complementary root loci (CRL) or inverse root loci for a system is the simple root locus with gain value $K<0$. In the previous sections, we have discussed the properties of root loci and steps for constructing the root loci manually for a negative feedback system with the gain value $K > 0$. Hence, the characteristic equation taken for constructing the root loci is $1 + G(s)H(s) = 0$.

But the characteristic equation for constructing the inverse or complementary root loci is $1-G(s)H(s)=0$ (i.e., negative feedback system with the gain value $K > 0$). Also, if the root loci constructed for the positive feedback system whose characteristic equation is $1-G(s)H(s)=0$, which is called complementary or inverse root loci.

The rules for constructing the CRL or inverse root loci are same as the rules for constructing the ordinary root locus with small modifications.

The following are the properties that differ from the ordinary properties of the root loci.

Property 1: Number of branches of root loci on the real axis

If the total number of poles and zeros lying to the right side of any point on the real axis is an even number, then a branch exists on the real axis.

Property 2: Angle of each asymptote present in the system is determined by

$$\theta_i = \frac{2i}{|n-m|}\times 180°, \quad \text{for} \quad n \neq m$$

where $i = 0, 1, 2, \cdots, (|n-m|-1), m$ is the number of zeros and n is the number of poles .

Property 3: Angle of departure/angle of arrival for the complex pole and complex zero is determined using the following formula:

$$\theta_d = 360° - \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex pole} + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$\theta_a = 360° + \sum_{j=1}^{n} \text{Angle from pole } j \text{ to the complex zero} - \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex zero}$$

7.10.1 Steps in Constructing the Inverse Root Loci Manually

(i) Plot the poles and zeros of the system $G(s)H(s)$ or determine and plot the points at $K=0$ and $K=\pm\infty$. The points at $K=0$ and $K=\pm\infty$ are marked using $\times$ and o respectively (i.e., poles and zeros of the system). Let n be the number of finite poles of $G(s)H(s)$ and m be the number of finite zeros of $G(s)H(s)$.

(ii) The total number of branches of the root loci is determined by
Number of branches = Number of poles or number of zeros of the system, whichever is greater

(iii) The total number of asymptotes in the root loci of the system is determined by
Number of asymptotes $N_a = |n-m|$

(iv) The intersection of the asymptotes with the real axis (centroid) is determined by using

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m}$$

(v) Angles of asymptotes for different root loci are to be determined.

$$\theta_i = \frac{2i}{|n-m|}\times 180°, \text{ for } n \neq m,\ i = 0,1,2\cdots(|n-m|-1)$$

(vi) Determine of branches of root loci existing on the real axis

(vii) Determine of angle of departure or angle of arrival if complex pole or complex zero exists for the system

$$\theta_d = 360° - \sum_{i=1}^{n} \text{Angle from pole } i \text{ to the complex pole } + \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex pole}$$

$$\theta_a = 360° + \sum_{i=1}^{n} \text{Angle from pole } i \text{ to the complex zero } - \sum_{i=1}^{m} \text{Angle from zero } i \text{ to the complex zero}$$

(viii) Determine the breakaway points in the root locus of the system and check for validity of breakaway points. Also, determine gain K at the breakaway point by using

$$\frac{dK}{ds} = 0$$

(ix) Determine the intersection point of the root loci with imaginary axis and its corresponding K is obtained using Routh–Hurwitz criterion.

Example 7.23: Sketch the root locus of the positive unity feedback system $G(s)H(s) = \dfrac{K}{s(s+4)}$.

Solution:

(i) For the given system, poles are at 0 and −4 i.e., $n = 2$ and zero does not exist i.e., $m = 0$.

(ii) Since $n > m$, the number of branches of root loci for the given system is $n = 2$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes $N_a = n - m = 2$

(b) Angles of asymptotes

$$\theta_i = \frac{2i}{|n-m|} \times 180°, \text{ for } n \neq m, \; i = 0 \text{ and } 1$$

Therefore, $\theta_0 = 0°$ and $\theta_1 = 180°$.

(c) Centroid

$$\sigma_1 = \frac{\Sigma \text{ finite poles of } G(s)H(s) - \Sigma \text{ finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{-4-0}{2} = -2$$

(iv) Since complex pole/complex zero does not exist for the given system, there is no angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = 0$, the total number of poles and zeros existing on the right of 0 is zero (nor an even number or a odd number). Therefore, a branch of root loci exists between 0 and ∞.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of $-\infty$ is 2 (even number). Therefore, a branch of root loci exists between $-\infty$ and -4.

Hence, two branch of root loci exists on the real axis for the given system.

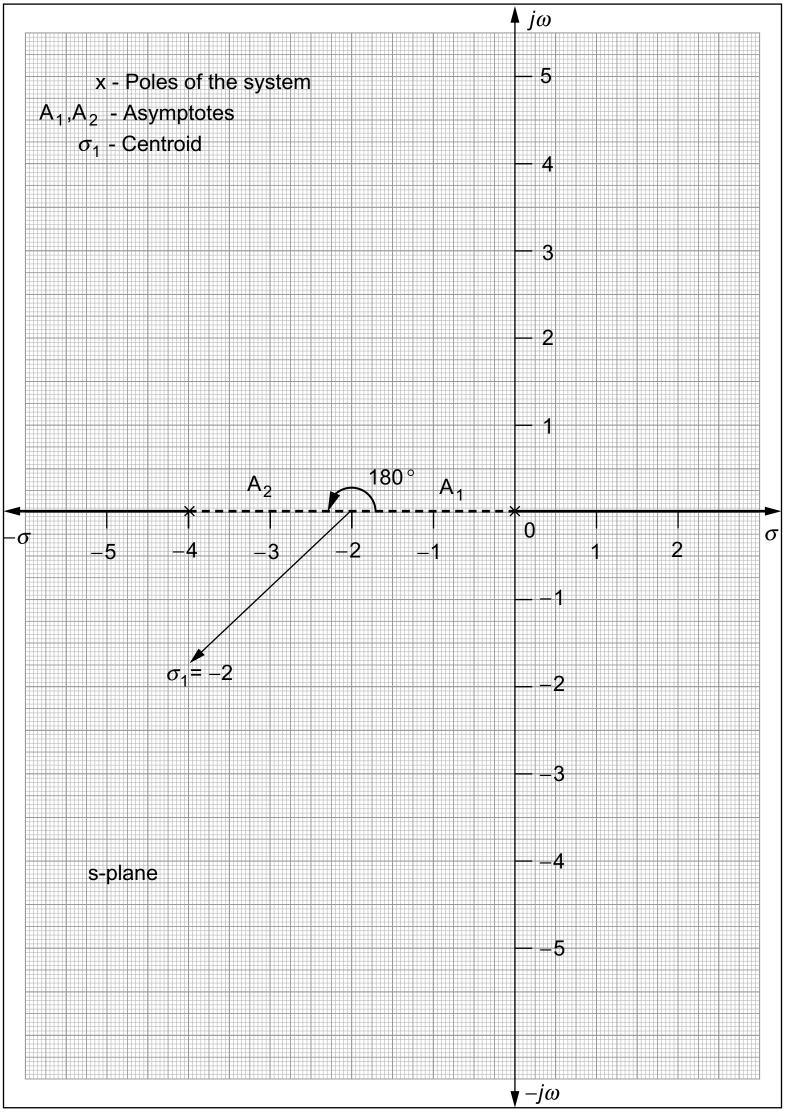


Fig. E7.23

(vi) The breakaway or break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \quad (1)$$

For the given system, the characteristic equation is

$$1 - G(s)H(s) = 0 \quad (2)$$

i.e.,
$$1 - \frac{K}{s(s+4)} = 0$$

Therefore,
$$K = \left(s^2 + 4s\right) \quad (3)$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$2s + 4 = 0$$

Therefore, $s = -2$.

We know that the breakaway/break-in points for a system should lie in the branches of root loci on the real axis. But the point $s = -2$ does not lie in the branch of root loci on the real axis.

Since $s = -2$ is not a breakaway point for the given system, the breakaway or break-in point does not exist.

(vii) As the branches of root locus lie only on the real axis, there is no point on the imaginary axis at which the root locus crosses.

The entire root locus for the system $G(s)H(s) = \frac{K}{s(s+4)}$ is shown in Fig. E7.23.

Example 7.24: Develop the root locus for a positive unity feedback system whose loop transfer function is given by $G(s)H(s) = \frac{K}{s(s+2)(s+5)}$.

Solution:

(i) For the given system, poles are at 0, −2 and −5, i.e., $n = 3$ and zero does not exist i.e., $m = 0$.

(ii) Since $n > m$, the number of branches of root loci for the given system is $n = 3$.

(iii) For the given system, the details of the asymptotes are given below:

(a) Number of asymptotes $N_a = 3$

(b) Angles of asymptotes

$$\theta_i = \frac{2i}{|n-m|} \times 180°, \text{ for } n \neq m, i = 0, 1 \text{ and } 2$$

Therefore, $\theta_0 = 0°$, $\theta_1 = 120°$ and $\theta_2 = 240°$

(c) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{-2-5}{3} = -2.33.$$

(iv) Since complex pole/complex zero does not exist for the system, there is no angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = 0$, the total number of poles and zeros existing on the right of 0 is zero (nor an even number or an odd number). Therefore, a branch of root loci exists between 0 and ∞.

Similarly, if we look from the pole $p = -5$, the total number of poles and zeros existing on the right of −5 is 2 (even number). Therefore, a branch of root loci exists between −5 and −2.

Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway or break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 - G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 - \frac{K}{s(s+2)(s+5)} = 0$$

Therefore,
$$K = (s^3 + 7s^2 + 10s) \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$3s^2 + 14s + 10 = 0$$

Therefore, $s = -0.88$ and $s = -3.78$.

We know that, the breakaway/break-in points for a system should lie in the branches of root loci on the real axis. But the point $s = -0.88$ does not lie in the branch of root loci on the real axis.

Hence, $s = -0.88$ is not a breakaway point and for the given system, the valid breakaway point is $s = -3.78$.

(vii) As the branches of root loci follows the asymptotes for the given system, it is clear that the no branch of root loci crosses the imaginary axis.

The complete plot of the system is shown in Fig. E7.24.

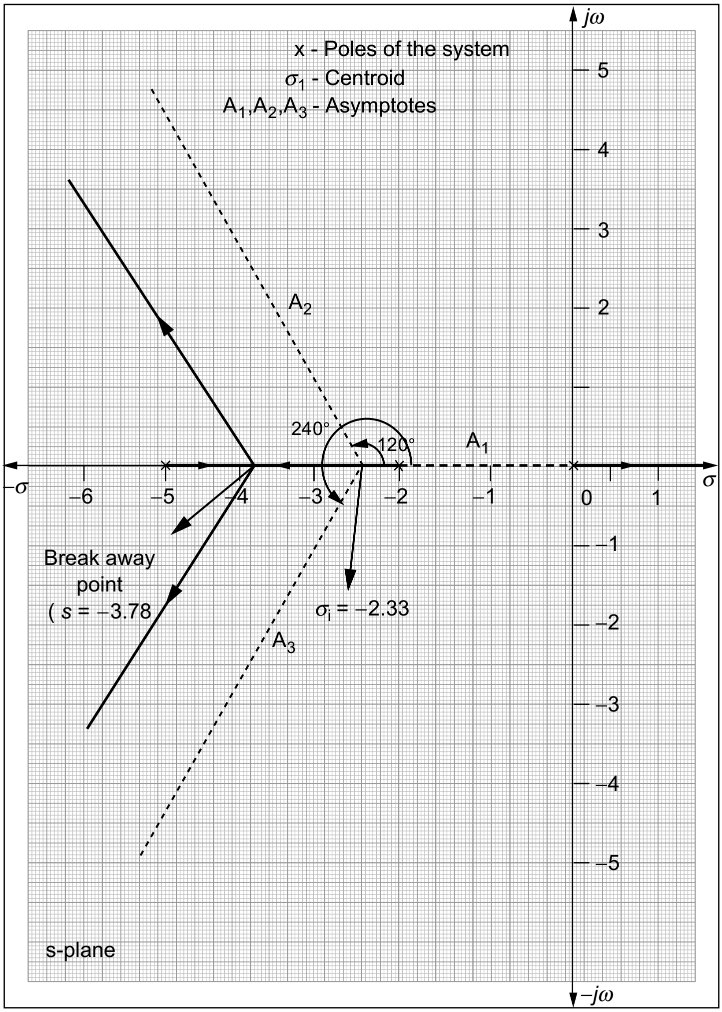


Fig. E7.24

7.11 Pole-Zero Cancellation Rules

The root locus of a system is constructed/developed based on the loop transfer function of the system $G(s)H(s)$. If the open-loop transfer function of the system $G(s)$ and feedback transfer function of the system $H(s)$ are given separately, there is always a possibility of cancellation of pole-zero between $G(s)$ and $H(s)$. But simply cancelling the common poles and zeros between $G(s)$ and $H(s)$ is not a advisable one. Therefore, two rules in cancelling the common poles and zeros existing between $G(s)$ and $H(s)$ are given below with proof.

Rule 1: The poles present in $G(s)$ should not be cancelled by zeros present in $H(s)$.

Rule 2: The zeros present in $G(s)$ can be cancelled by poles present in $H(s)$.

Proof for rule 1:

Consider a system with $G(s)=\dfrac{K}{s(s+p_1)(s+p_2)}$ and $H(s)=\dfrac{(s+z_1)}{(s+p)}$. If $p_1=z_1=a$ and pole-zero cancellation is allowed, then $G(s)H(s)=\dfrac{K}{s(s+p_2)(s+p)}$.

Now, the characteristic equation of the system with the loop transfer function $G(s)H(s)$ is

$$1+G(s)H(s)=0$$

$$s(s+p_2)(s+p)+K=0 \tag{7.11}$$

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{\dfrac{K}{s(s+a)(s+p_2)}}{1+\dfrac{K(s+a)}{s(s+a)(s+p_2)(s+p)}}$$

$$=\frac{K(s+p)}{s(s+a)(s+p_2)(s+p)+K(s+a)}.$$

The characteristic equation for the above function is

$$s(s+a)(s+p_2)(s+p)+K(s+a)=0 \tag{7.12}$$

Comparing Eqn. (7.11) and Eqn. (7.12), it is clear that in Eqn. (7.11), a pole at $s=-a$ is missing which is due to the pole-zero cancellation. Therefore, pole-zero cancellation of the system is not allowed.

Hence, the loop transfer function of the system is $G(s)H(s)=\dfrac{K}{s(s+a)(s+p_2)(s+p)}$ instead of $G(s)H(s)=\dfrac{K}{s(s+p_2)(s+p)}$.

Proof for rule 2:

Consider a system with $G(s)=\dfrac{K(s+z)}{s(s+p_1)(s+p_2)}$ and $H(s)=\dfrac{(s+z_1)}{(s+p)}$. If $p=z=a$ and zero-pole cancellation is allowed, then $G(s)H(s)=\dfrac{K(s+z_1)}{s(s+p_2)(s+p_1)}$.

Now, the characteristic equation of the system with the loop transfer function $G(s)H(s)$ is

$$1+G(s)H(s)=0$$

$$s(s+p_2)(s+p_1)+K(s+z_1)=0 \tag{7.13}$$

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{\dfrac{K(s+a)}{s(s+p_1)(s+p_2)}}{1+\dfrac{K(s+a)(s+z_1)}{s(s+p_1)(s+p_2)(s+a)}}$$

$$=\frac{K(s+a)}{s(s+p_1)(s+p_2)+K(s+z_1)}.$$

The characteristic equation for the above function is

$$s(s+p_1)(s+p_2)+K(s+z_1)=0 \tag{7.14}$$

Since Eqn. (7.13) and Eqn. (7.14) are same, zero-pole cancellation of the $G(s)$ and $H(s)$ is allowed.

Example 7.25: Consider a system with open-loop transfer function $G(s)=\dfrac{K(s+3)}{(s+1)}$ and a feedback transfer function $H(s)=\dfrac{(s+1)}{(s+2)}$. Plot the root locus for the system.

Solution: Although there exists a common point where pole and zero exist, due to rule 1 discussed in Section 7.11, the loop transfer function of the system is considered as $G(s)H(s)=\dfrac{K(s+3)}{(s+2)(s+1)}$

(i) For the given system, poles are at −1 and −2 i.e., $n=2$ and zero is at −3 i.e., $m=1$.
(ii) Since $n>m$, the number of branches of the root loci for the given system is $n=2$.
(iii) For the given system, the details of the asymptotes are given below:
  (a) Number of asymptotes for the given system $N_a=2-1=1$.
  (b) Angles of asymptotes

$$\theta_i=\frac{2i+1}{|n-m|}\times 180°, \text{ for } n\neq m \text{ and } i=0$$

Therefore, $\theta_\circ=180°$.

(c) Centroid

$$\sigma_1 = \frac{\Sigma \text{ finite poles of } G(s)H(s) - \Sigma \text{ finite zero of } G(s)H(s)}{n-m}$$

$$= \frac{-1-2-(-3)}{1} = 0.$$

(iv) Since a complex pole or complex zero does not exist for the system, there is no angle of departure/angle of arrival.

(v) *To determine the number of branches existing on the real axis:*

If we look from the pole $p = -2$, the total number of poles and zeros existing on the right of -2 is one (odd number). Therefore, a branch of root loci exists between -2 and -1.

Similarly, if we look from the point at $-\infty$, the total number of poles and zeros existing on the right of that point is three (odd number). Therefore, a branch of root loci exists between -3 and $-\infty$.

Hence, two branches of root loci exist on the real axis for the given system.

(vi) The breakaway and break-in points for the given system can be determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2}$$

i.e.,
$$1 - \frac{K(s+3)}{(s+1)(s+2)} = 0$$

Therefore,
$$K = -\frac{(s^2+3s+2)}{s+3} \tag{3}$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$-\left[\frac{(s+3)(2s+3)-(s^2+3s+2)}{(s+3)^2}\right] = 0$$

$$s^2 + 6s + 7 = 0$$

Therefore, $s = -1.58, \; -4.414$

For $s = -1.58$ and $s = -4.414$, the values of K are 0.1715 and 7.2 respectively. Since the values of K are positive, $s = -1.58$ and $s = -4.414$ are the breakaway and break-in points respectively.

(vii) The point at which the branch of root loci intersects the imaginary axis can be determined as follows:

From Eqn. (2), the characteristic equation of the system is

$$s^2 + (3+K)s + (3K+2) = 0 \tag{4}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^2 & 1 & (3K+2) \\ s^1 & 3+K & \\ s^0 & 3K+2 & \end{array}$$

To determine the point at which root locus crosses the imaginary axis, the first element in all the rows must be zero, i.e., $K = -3$ or $K = -0.66$.
Since K is negative, the branch of root loci does not cross the imaginary axis at any point.
The complete root locus for the given system is shown in Fig. E7.25.

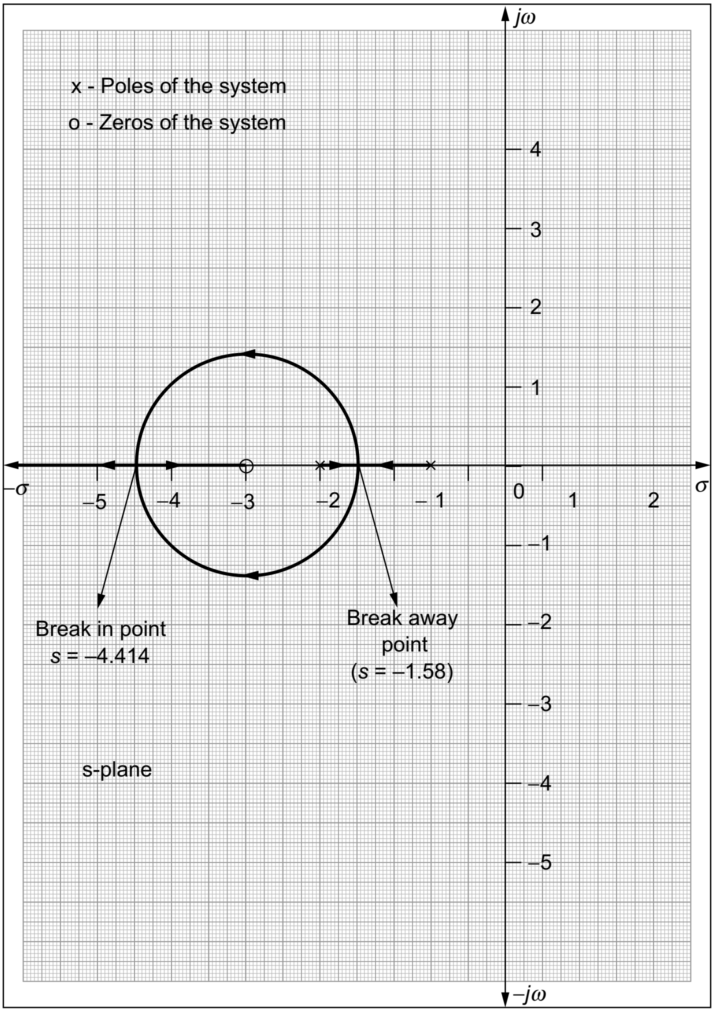


Fig. E7.25

7.12 Root Contours (Multi-Variable System)

The root locus problems discussed so far has a single variable parameter K, which can be varied for designing the system accurately. But in a general control system problem, there exist number of varying parameters whose effects are to be investigated.

In this section, let us discuss how a root locus problem is carried out for a system with two variable parameters present in it. Root locus is called root contour when one or more parameter is varied. Root contour will posses all the properties of root locus and hence the method of construction of root loci is also applicable in constructing root contour.

The general root-contour problem is formulated by referring to the equation:

$$F(s) = D(s) + K_1 N_1(s) + K_2 N_2(s) = 0 \tag{7.15}$$

where

$D(s)$ is the n^{th} order polynomial of s given by $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$

$N_1(s)$ is the $m_1{}^{\text{th}}$ order polynomial of s given by $s^{m_1} + b_1 s^{m_1-1} + \cdots + b_{m_1-1}s + b_{m_1}$

$N_2(s)$ is the $m_2{}^{\text{th}}$ order polynomial of s given by $s^{m_2} + b_1 s^{m_2-1} + \cdots + b_{m_2-1}s + b_{m_2}$

where n, m_1 and m_2 are positive integers and K_1 and K_2 are variable loop gains which vary from $-\infty$ to $+\infty$.

The following steps are used to draw the root contour for the multi-variable system:

Step 1: Set variable parameter K_2 to zero which makes Eqn. (7.15) as

$$D(s) + K_1 N_1(s) = 0$$

Step 2: The equation now has a single variable parameter K_1 and the root locus of the single parameter system is determined using the equation $1 + \dfrac{K_1 N_1(s)}{D(s)} = 0$ as the equation is similar to $1 + K_1 G_1(s) H_1(s) = 0$.

Step 3: Now, dividing the original Eqn. (7.15) by $D(s) + K_1 N_1(s)$, we obtain

$$1 + \frac{K_2 N_2(s)}{D(s) + K_1 N_1(s)} = 0$$

with which the root loci for the system with the single parameter K_2 can be constructed as the equation is similar to $1 + K_2 G_2(s) H_2(s) = 0$.

Review Questions

1. Define root loci.
2. What are the advantages of root locus technique?
3. How the root loci of a system are categorized?
4. Discuss the necessary conditions for formulating the root loci for a system.

5. What are the properties to be known in constructing the root locus manually?
6. Define the following terms:
   (i) asymptotes, (ii) centroid and (iii) breakaway point.
7. Explain the steps involved in calculating the breakaway points.
8. Explain the procedure in constructing the root locus for a particular system.
9. Explain the procedure for obtaining K for a specified damping ratio ξ.
10. The loop transfer function of a unity feedback system is $G(s)H(s)=\dfrac{K(s+3)}{s(s+2)}$. Determine the breakaway and break-in points for the system.
11. Sketch the root locus of the system $G(s)H(s)=\dfrac{K}{s(s+3)(s^2+3s+4.5)}$.
12. Construct the root locus of the system $G(s)H(s)=\dfrac{K(s+1)}{s(s-1)(s^2+5s+20)}$.
13. Plot the root locus of the system whose characteristic equation is given by $s^3+9s^2+Ks+K=0$.
14. The characteristic equation of a feedback control system is given by $s^4+3s^3+12s^2+(K-16)s+K=0$. Construct the root locus for $0\le K\le\infty$ and discuss about the stability of the system.
15. Sketch the root locus of the system $G(s)H(s)=\dfrac{K}{(s+3)(s+5)(s^2+2s+5)}$.
16. The block diagram of a control system is shown in Fig. Q7.16. Construct the root locus of the system with $K=10$, $G_1(s)=\dfrac{1}{s^2(s+5)}$ and $K_1\ge 0$.

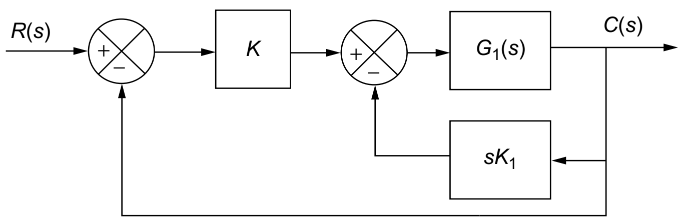


Fig. Q7.16

17. The block diagram of a control system is shown in Fig. Q7.17. The transfer function of a system $G_1(s)=\dfrac{1}{s^2(s+2)}$. (i) When the switch, S is open, construct the root locus for the system and discuss about the stability of the system when K varies and (ii) when the switch S is closed, plot the root locus for the system with $K=1$ and discuss about the stability of the system when K_1 varies.

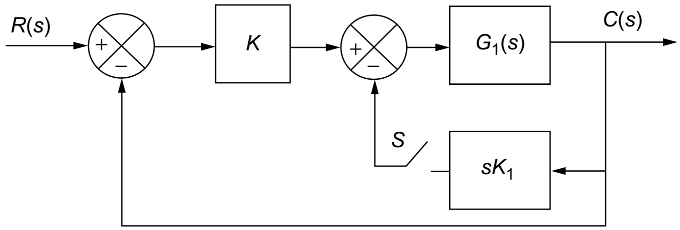


Fig. Q7.17

18. The characteristic equation of a feedback control system is given by $s^3 + K_2 s^2 + K_1(s+1) = 0$, where K_1 and K_2 are the variable parameters that vary from 0 to ∞. Construct the root contours when (i) $K_2 = 0$ and (ii) K_1 is constant.

19. The loop transfer function of a unity feedback system is given by $G(s) = \dfrac{K(s+2)}{s(s+3)(s+4)(s^2+2s+2)}$.
    (i) Plot the root locus diagram as a function of K.
    (ii) Determine gain K when the damping ratio of the system $\xi = 0.707$.

20. Construct the root locus of the system $G(s)H(s) = \dfrac{K}{s(s+2)(s+8)}$. Also, determine gain K when the damping ratio of the system $\xi = 0.707$.

21. Sketch the root locus of the system $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$. Also, determine gain K when the damping ratio of the system, $\xi = 0.5$.

22. The loop transfer function of a unity feedback system is $G(s)H(s) = \dfrac{K(s+2)}{s^2+2s+3}$. Construct the root locus for the system and determine gain K (i) for which repetitive roots occur, (ii) for which the closed-loop system becomes under-damped and (iii) for which the system will have $\xi = 0.7$.

23. Explain the steps how the gain margin and phase margin of a system can be determined.

24. Plot the root locus of the system whose loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+1)(s+3)}$. Determine K for (i) critical damping, $\xi = 1$ and for (ii) zero damping, $\xi = 0$. Also, determine the gain margin of the system for $K_{desired} = 5$.

25. Determine the gain margin and phase margin with $K_{desired} = 12$ for the system whose loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$.

26. Plot the root locus of the system whose loop transfer function is $G(s)H(s) = \dfrac{K}{s(s^2+6s+25)}$. Determine the gain margin and phase margin of the system for (i) $K_{desired} = 15$ and (ii) $K_{desired} = 1500$.

27. The loop transfer function of the system is given by $G(s)H(s) = \dfrac{K(s+0.1)}{s(s-0.2)(s^2+s+0.6)}$. Determine the stability of the system.

28. The characteristic equation of a system is $1+\dfrac{K}{s(s^2+2s+4)}$. Determine the loop transfer function of the system and construct the root locus for the same.

29. A system with a unity negative feedback system has the loop transfer function $G(s)H(s) = \dfrac{K}{(s+1)(s+2)(s+3)}, K > 0$. Construct the root locus for the system and (i) determine gain K for M_p = 20 per cent (ii) determine settling time, t_s and peak time, t_p for gain K obtained in (i).

30. The loop transfer function of a system is $G(s)H(s) = \dfrac{K}{s(0.2s+1)(0.05s+1)}$. Construct the root locus for the system and determine gain K when the phase margin of the system is $40°$.

31. The loop transfer function of a system is $G(s)H(s) = \dfrac{K(s+20)}{s(s^2+4s+20)}$. Plot the root locus for the system and determine gain K for an oscillatory response of the system.

32. Show that the root locus of the system $G(s)H(s) = \dfrac{K(s+3)}{s(s+2)}$ is a circle with radius $\sqrt{3}$ and centre (–3, 0).

33. The loop transfer function of the system is $G(s)H(s) = \dfrac{K(s^2+1.5s+1.5625)}{(s-0.75)(s+0.25)(s+1.25)(s+2.25)}$. Construct the root locus for the system and determine gain K for a stable closed-loop system.

34. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{K(s+a)}{s(s^2+2s+2)}$. Sketch the root locus for the system with $a = -3, -2, -1, -0.5$ and 0.

35. The unity feedback system has a loop transfer function $G(s)H(s) = \dfrac{K}{s^2(s+2)}$. (i) construct the root locus for the system, (ii) show how the addition of zero at $s = -a$ $(0 \le a < 2)$ stabilizes the system and (iii) plot the root locus with $a = 1$ and determine gain K, damping ratio ξ, un-damped natural frequency ω_n which gives greatest damping ratio in oscillatory mode.

36. The loop transfer function of a system is $G(s)H(s)=\dfrac{K}{s(s^2+2s+2)(s^2+6s+10)}$. Construct the root locus of the system.
37. A unity feedback system has a loop transfer $G(s)H(s)=\dfrac{K(s^2+6s+10)}{(s^2+2s+10)}$. Plot the root locus of the system.
38. Discuss how the root locus of a system will be affected by adding poles to the existing system.
39. Discuss how the root locus of a system will be affected by adding zeros to the system.
40. What are the effects of adding poles and zeros to the existing system?
41. How the time response of a system can be determined using root locus of a system?
42. The block diagram of a system is shown in Fig. Q7.42. Construct the root locus for the system with $G(s)=\dfrac{(s^2+2s+4)}{s(s+4)(s+5)(s^2+1.4s+1)}$.

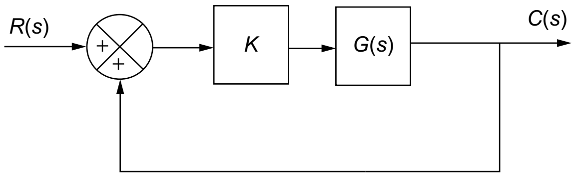


Fig. Q7.42

43. What are the rules to be considered in determining the loop transfer function of a system when $G(s)$ and $H(s)$ are given?
44. The transfer function of a system $G(s)=\dfrac{K}{s(s+2)(s+3)}$ and feedback transfer function, $H(s)=(s+2)$. Determine the loop transfer function of the system and construct the root locus for the same.
45. Explain the concept of root contour.
46. For the following systems, determine the number of asymptotes, angle of asymptotes and centroid.

    (i) $s^2+4s+(K+4)=0$

    (ii) $s^3+6s^2+(K+5)s+5K=0$

    (iii) $(s^3+6s^2+12s+8)+K(s+1)=0$

    (iv) $s^5+4s^4-5s^3+K(s^2+3s+2)=0$

47. Determine the angle of departure or angle of arrival or both the angles for the following systems.

    (i) $G(s)H(s)=\dfrac{Ks}{(s+1)(s^2+1)}$

(ii) $G(s)H(s)=\dfrac{Ks}{(s-1)(s^2+1)}$

(iii) $G(s)H(s)=\dfrac{Ks}{s(s+2)(s^2+2s+2)}$

(iv) $G(s)H(s)=\dfrac{Ks}{s^2(s^2+2s+2)}$

(v) $G(s)H(s)=\dfrac{K(s^2+2s+2)}{s^2(s+2)(s+3)}$

48. Determine the branches of root loci existing on the real axis and hence determine the breakaway or break-in points for the systems given below:

(i) $G(s)H(s)=\dfrac{Ks(s+2)}{(s+4)(s^2+2s+2)}$

(ii) $G(s)H(s)=\dfrac{Ks}{(s+4)(s^2+2s+2)}$

(iii) $G(s)H(s)=\dfrac{Ks(s+3)}{s^2(s+4)(s^2+2s+2)}$

(iv) $G(s)H(s)=\dfrac{Ks}{(s+4)(s+5)}$

(v) $G(s)H(s)=\dfrac{K(s+2)}{(s+7)}$

49. Plot the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K(s+0.5)}{s^2(s+4.5)}$.

50. Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{(s+5)(s+10)}$.

51. Sketch the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s+3)(s^2+2s+2)}$

52. Construct the root locus of the system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s+10)(s+50)}$.

This Page is intentionally left blank.

8

FREQUENCY RESPONSE ANALYSIS

8.1 Introduction

After obtaining the mathematical model of the system, its performance is analysed based on the time response or frequency response. Time response of the system is defined as the response of the system when standard test signals such as impulse, step and so on are applied to it. Frequency response of the system is defined as the response of the system when standard sinusoidal signals are applied to it with constant amplitude over a range of frequencies. The frequency response indicates the steady-state response of a system to a sinusoidal input. The characteristics and performance of the industrial control system are analysed by using the frequency response techniques. Frequency response analysis is used in numerous systems and components such as audio and video amplifiers, speakers, sound cards and servomotors. The different techniques available for frequency response analysis lend themselves to a simplest procedure for experimental testing and analysis. In addition, the stability and relative stability of the system for the sinusoidal input can be analysed by using different frequency plots.

This chapter introduces the frequency response of the system and explains the procedure for construction of Bode plot to determine frequency domain specifications such as gain crossover frequency, phase crossover frequency, gain margin and phase margin.

8.1.1 Advantages of Frequency Response Analysis

The time response of the system can be obtained only if the transfer function of the system is known earlier, which is always not possible. But it is possible to obtain frequency response of the system even though the transfer function of the system is not known. The reasons for determining the frequency response of the system are:

(i) Analytically, it is more difficult to determine the time response of the system for higher order systems.

(ii) As there exists numerous ways of designing a control system to meet the time domain performance specifications, it becomes difficult for the designer to choose a suitable design for a particular system.
(iii) The transfer function of a higher order system can be identified by computing the frequency response of the system over a wide range of frequencies ω.
(iv) The time-domain specifications of a system can be met by using the frequency domain specifications as a correlation exists between the frequency response and time response of a system.
(v) The stability of a non-linear system can be analysed by the frequency response analysis.
(vi) The transfer function of a higher order system can be obtained using frequency response analysis which makes use of physical data when it is difficult to obtain using differential equations.
(vii) The frequency response analysis can be applied to the system that has no rational transfer function (i.e., a system with transportation lag).
(viii) The frequency response analysis can be applied to the system even when the input is not deterministic.
(ix) The frequency response analysis is very convenient in measuring the system sensitivity to noise and parameter variations.
(x) In frequency response analysis, stability and relative stability of a system can be analysed without evaluating the roots of the characteristic equation of the system.
(xi) The frequency response analysis is simple and accurate.

8.1.2 Disadvantages of Frequency Response Analysis

The disadvantages of frequency response analysis are:

(i) Frequency response analysis is not recommended for the system with very large time constants.
(ii) It is not useful for non-interruptible systems.
(iii) It can generally be applied only to linear systems. When this approach is applied to a non-linear system, the result obtained is not exact.
(iv) It is considered as outdated when compared with the methods developed for digital computer and modelling.

The comparison between the time response analysis and frequency response analysis is listed in Table 8.1.

Table 8.1 | Comparison between the time response analysis and frequency response analysis

S. No	Time response analysis	Frequency response analysis
1	Determining the response of higher order systems is more difficult.	Determining the response of higher order systems is much easier.
2	No unified method exists for designing a system which meets the time-domain specifications.	Graphical method exits for designing the system in order to meet its specifications.

3	Less convenient in measuring the sensitivity of the system for noise and parameter variations.	More convenient in measuring the sensitivity of the system for noise and parameter variations.
4	Mathematical model is necessary for analysing the performance of the system.	Real data obtained for the physical systems are enough for analysing the performance of the system.
5	Cannot be applied to a non-linear system.	Can be applied to a non-linear system.

8.2 Importance of Sinusoidal Waves for Frequency Response Analysis

In frequency response analysis, the signal used as an input to the system is a sinusoidal wave because of the following reasons:

(i) Fourier theory (infinite sum of sine wave and cosine wave).
(ii) Any input signal applied to the circuit can be represented as a combination of sine and cosine waves of different frequencies and amplitudes.

Consider a ramp signal as shown in Fig. 8.1(a) and constant sine and cosine waves of varying amplitude as shown in Fig. 8.1(b).

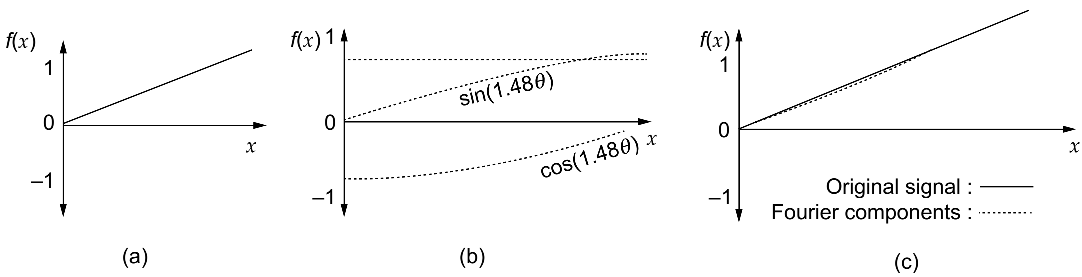


Fig. 8.1 | (a) Ramp signal, (b) Fourier components and (c) original and sum of fourier components

If the sine wave and cosine wave as shown in Fig. 8.1(b) are added, the resultant will be a ramp signal indicated using a dotted line along with the original ramp signal shown as a continuous line depicted in Fig. 8.1(c). Since any signal can be represented by varying the frequency and amplitude of the sine and cosine waves, the sinusoidal wave is chosen for frequency domain analysis.

8.3 Basics of Frequency Response Analysis

The steady-state output of the stable linear system is also a sine wave of same frequency when a sinusoidal input is applied to the system. The output and input of the system generally differs in both magnitude and phase. Generally, the frequency response of the system can be obtained by replacing the variable s in the transfer function with $j\omega$ as given by

$$G(j\omega) = G(s)\Big|_{s=j\omega} = M(\omega)\ e^{j\omega\phi} = M\angle\phi \tag{8.1}$$

where M is the magnitude or gain i.e., the sinusoidal amplitude ratio of output to input. In addition, ϕ is the angle by which the output leads the input. The parameters M and ϕ are functions of the angular frequency ω. The Laplace variable s is a complex number that is represented as $s = \sigma + j\omega$. If a linear time-invariant system is subjected to a pure sinusoidal input, $A\ \sin\omega t$, the output response of the system contains both the transient part and the steady-state part. But at $t \to \infty$, the transient part dies out because the roots have negative parts for a stable system and only the steady-state part of the output response exists. Therefore, as the frequency response of the system deals with steady-state analysis of the system, it is enough to substitute $s = j\omega$ instead of $s = \sigma + j\omega$.

Example 8.1 **Consider a low-pass RC network with $R = 2\Omega$ and $C = 0.5F$ as shown in Fig. E8.1 and is driven by the input voltage $10\cos 4t$. Determine the output voltage v_0(t) across the capacitor, C.**

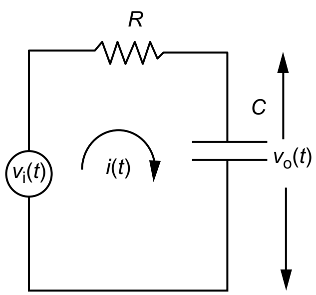


Fig. E8.1

Solution: The transfer function of the low-pass RC network is

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

Substituting the values of R and C, we obtain

$$G(s) = \frac{1}{s+1}$$

The Laplace transform of the input, $V_i(s) = \mathcal{L}(10\cos 4t) = \dfrac{10s}{s^2+16}$

Hence, the output of the system, $V_o(s) = G(s)V_i(s) = \dfrac{10s}{(s^2+16)(s+1)}$

Using partial fractions, we obtain $V_o(s) = \dfrac{\frac{-10}{17}}{(s+1)} + \dfrac{\frac{10}{17}s}{s^2+16} + \dfrac{\frac{160}{17}}{s^2+16}$

Taking inverse Laplace transform, we obtain $v_o(t) = \dfrac{-10}{17}e^{-t} + \left(\dfrac{10}{17}\cos 4t + \dfrac{40}{17}\sin 4t\right)$

where $\dfrac{-10}{17}e^{-t}$ is the transient part of the output and $\left(\dfrac{10}{17}\cos 4t + \dfrac{40}{17}\sin 4t\right)$ is the steady-state part of the output.

8.4 Frequency Response Analysis of Open-Loop and Closed-Loop Systems

The frequency response analysis of both open-loop and closed-loop systems is discussed below.

8.4.1 Open-Loop System

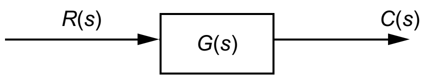


Fig. 8.2 | A simple open-loop system

Consider an open-loop linear time-invariant system as shown in Fig. 8.2. The input signal i.e., sinusoidal with amplitude R and frequency ω is given by

$$r(t) = R\sin\omega t$$

The steady-state output of the system $c(t)$ will also be a sinusoidal function with the same frequency ω_o but possibly with a different amplitude and phase, i.e.,

$$c(t) = \text{C} \sin(\omega t + \phi)$$

where C is the amplitude of the output sinusoidal wave and ϕ is the phase shift in degrees or radians.

Let the transfer function of a linear single-input single-output (SISO) system be $G(s)$. Then, the relation between the Laplace transforms of the input and the output is

$$C(s) = G(s)R(s)$$

For sinusoidal steady-state analysis i.e., frequency response analysis, s is replaced with $j\omega$. Then, the above equation becomes

$$C(j\omega) = G(j\omega)R(j\omega) \tag{8.2}$$

Here, $C(j\omega)$ can be written in terms of magnitude and phase angle as

$$C(j\omega) = |C(j\omega)|\angle C(j\omega) \tag{8.3}$$

Similarly, Eqn. (8.2) can be re-written as

$$C(j\omega) = |G(j\omega)|\angle G(j\omega) \times |R(j\omega)|\angle R(j\omega) \tag{8.4}$$

Hence, comparing Eqs. (8.3) and (8.4), we obtain

$$|C(j\omega)| = |G(j\omega)| \times |R(j\omega)| \tag{8.5}$$

and the phase angle is

$$\phi = \angle C(j\omega) = \angle G(j\omega) + \angle R(j\omega) \tag{8.6}$$

From the transfer function $G(s)$ of a linear time-invariant system with sinusoidal input, the magnitude and phase angle of the output can be obtained using Eqs. (8.5) and (8.6) respectively.

8.4.2 Closed-Loop System

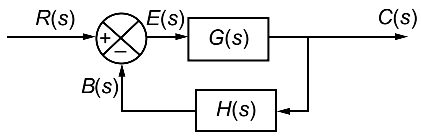


Fig. 8.3 | A simple closed-loop system

The closed-loop system is shown in Fig. 8.3.

The transfer function $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

For the frequency response analysis, substituting $s = j\omega$, we get

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

The transfer function $T(j\omega)$ can be expressed in terms of its magnitude and phase as $T(j\omega) = |T(j\omega)|\angle T(j\omega)$

Therefore, the magnitude of $T(j\omega)$ is

$$M = |T(j\omega)| = \left|\frac{G(j\omega)}{1+G(j\omega)H(j\omega)}\right| = \frac{|G(j\omega)|}{1+|G(j\omega)||H(j\omega)|} \tag{8.7}$$

The phase angle of $T(j\omega)$ is

$$\phi = \angle T(j\omega) = \angle G(j\omega) - \angle\left(1 + G(j\omega)H(j\omega)\right) = \angle G(j\omega) - \angle G(j\omega)H(j\omega) \tag{8.8}$$

Thus, by knowing the transfer function $G(s)$ of a linear time-invariant system and the feedback transfer function $H(s)$, the magnitude and phase angle of the output for the closed-loop system can be obtained using Eqs. (8.7) and (8.8) respectively, provided the input to the system is sinusoidal.

8.4.3 Closed-Loop System with Poles and Zeros

Consider a negative feedback closed-loop system as shown in Fig. 8.3 with the closed-loop transfer function as

$$T(s) = \frac{C(s)}{R(s)} \tag{8.9}$$

where $C(s)$ is the Laplace transform of the output and $R(s)$ is the Laplace transform of the input.

For a closed-loop stable system,

$$T(s) = \frac{Z(s)}{P(s)} \tag{8.10}$$

where, $Z(s) = K\prod_{i=1}^{m}(s + z_i)$ and $P(s) = \prod_{j=1}^{n}(s + p_j)$

where m and n are positive and distinct integers with $m \le n$.
Consider the input $r(t)$ applied to the system as

$$r(t) = \begin{cases} A\cos\omega t, \ t \ge 0 \\ 0, \text{otherwise} \end{cases}$$

Taking Laplace transform, we obtain $R(s) = \dfrac{As}{s^2 + \omega^2}$

Hence, the Laplace transform of the output using Eqn. (8.9) is $C(s) = T(s)\,R(s)$

Substituting Eqn. (8.10) in the above equation, we obtain

$$C(s) = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} \times \frac{As}{s^2 + \omega^2}$$

Using partial fractions for the above equation, we obtain

$$C(s) = \frac{C_1}{(s + p_1)} + \frac{C_2}{(s + p_2)} + \cdots + \frac{C_n}{(s + p_n)} + \frac{K_1}{(s - j\omega)} + \frac{K_1^*}{(s + j\omega)}$$

$$= F(s) + \frac{K_1}{(s - j\omega)} + \frac{K_1^*}{(s + j\omega)} \tag{8.11}$$

Here, $F(s) = \sum_{i=1}^{n} \frac{C_i}{(s + p_i)}$, where $C_i = (s + p_i)C(s)\Big|_{s=-p_i}$

In addition, $K_1 = (s - j\omega)C(s)\Big|_{s=j\omega}$

$$= (s - j\omega)\frac{As}{s^2 + \omega^2}T(s)\Big|_{s=j\omega} = (s - j\omega)\frac{As}{(s + j\omega)(s - j\omega)}T(s)\Bigg|_{s=j\omega}$$

$$= \frac{A \times j\omega}{2j\omega}T(j\omega) = \frac{A}{2}T(j\omega)$$

As the constant K_1 is a complex number, it is convenient to represent it in polar form as

$$K_1 = \frac{A}{2}\left|T(j\omega)\right| e^{j\phi}, \text{ where } \phi = \angle T(j\omega)$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega} + F(s)\right\}$$

$$= A\left|T(j\omega)\right|\left[\mathcal{L}^{-1}\left\{\frac{e^{j\phi}}{2(s - j\omega)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-j\phi}}{2(s + j\omega)}\right\}\right] + \mathcal{L}^{-1}\left\{\sum_{i=1}^{n}\frac{C_i}{(s + p_i)}\right\}$$

$$= A\left|T(j\omega)\right|\left[\frac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2}\right] + \sum_{i=1}^{n} C_i e^{-p_i t}$$

$$= A\left|T(j\omega)\right|\cos(\omega t + \phi) + \sum_{i=1}^{n} C_i e^{-p_i t}$$

Since we have assumed all the values of $p_i > 0$, the steady-state response is

$$\underset{t\to\infty}{Lt}\ c(t) \cong c_{ss} = A\left|T(j\omega)\right|\cos(\omega t + \phi)$$

Thus, if a sinusoidal input is applied to a system that has negative real value of poles, the steady-state response is a scaled, phase-shifted version of the input. The scaling factor is $\left|T(j\omega)\right|$ and the phase shift ϕ is the phase of $T(j\omega)$.

8.5 Frequency Response Representation

The frequency response analysis of a system is used to determine the system gain and phase angle of the system at different frequencies. Hence, the system gain and phase angle can be represented either in a tabular form or graphical form.

Tabular form: It is useful in representing the system gain and phase angle of a system at different frequencies only if the data set is relatively small. It is also useful in experimental measurement.

Graphical form: It provides a convenient way to view the frequency response data. There are many ways of representing the frequency response in the graphical form.

8.5.1 Determination of Frequency Response

The frequency response analysis of a system can be determined using (i) experimental determination and (ii) mathematical determination.

(i) Experimental determination of frequency response

This method is used only when the system transfer function is not known. This method is used for a plant testing and verification of the plant model. Determining the frequency response of a the small system experimentally in a laboratory environment is not very difficult. A typical set-up for the experimental determination of the frequency response is shown in Fig. 8.4.

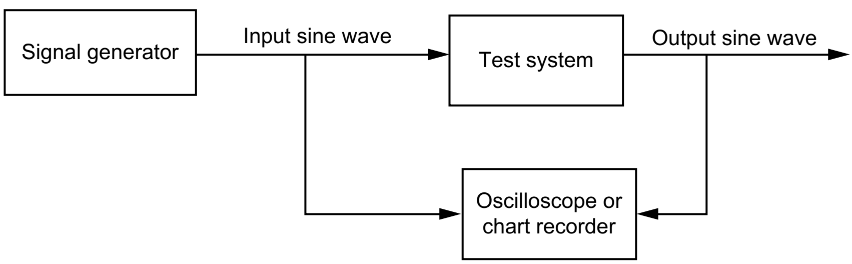


Fig. 8.4 | Experimental set-up for frequency response

The requirement for determining frequency response varies for different systems. The general requirements for determining the frequency response of systems are power supply, signal generator and a chart recorder or dual trace oscilloscope. The experimental set-up varies depending on the test systems.

The steps to be followed for determining the frequency response are:

(a) Test frequency range and input signal amplitude are established.
(b) Sinusoidal signal with low frequency is applied and sufficient time is allowed for settling of transients present in the system.
(c) After a stable output is reached, using the input and output waveforms, the system gain and phase angle are determined.
(d) The same procedure can be applied for the input signal with different frequencies and the frequency response analysis of the system can be determined.

Thus, the frequency response of the system whose transfer function is not known can be determined.

(ii) Mathematical evaluation of frequency response

This method is applicable to the system whose transfer function is known. In this method, by substituting $s = j\omega$, the transfer function is considered to be a function of frequency and it is treated as a complex variable. The system gain and phase angle at a particular frequency is same as the magnitude and phase angle of the complex number.

The mathematical procedure for determining the frequency response of simple and complex transfer functions are given below:

(a) Simple transfer function

1. Choose the frequency for which the frequency response has to be determined.
2. Substitute $s = j\omega$ in the given transfer function.
3. Convert the resultant complex number into a polar form.
4. Substitute the chosen frequency ω in the above equation and determine the magnitude and angle of the complex number.
5. Then, the system gain and phase angle at that particular frequency are determined.
6. Repeat the above steps for different frequencies and the corresponding system gain and the phase angle can be tabulated.

(b) Complex transfer function

A complex transfer function can be separated in two simple transfer functions and each simple transfer function can be converted into the polar form. The complex number in polar form allows easier manipulation of magnitude and angle. The procedure is as follows:

1. Convert the complex transfer function into a product of simple transfer functions.
2. For each simple transfer function, follow the steps that have been discussed earlier to determine the individual magnitude and angle of the simple transfer functions.
3. Then, the overall gain and phase angle of the complex transfer function at a particular frequency can be obtained by multiplying the individual gain and adding the phase angles, respectively.
4. The above steps can be repeated for different frequencies and the system gain and phase angle can be determined.

Example 8.2 **The transfer function of the system is given by** $G(s) = \dfrac{5(s+1)}{(s+2)(s+3)}$. **Determine the frequency response of the system over a frequency range of 0.1–10 rad/sec.**

Solution:

Gain of the complex system = product of gains of simple systems

Phase angle of the complex system = addition of phase angles of simple systems

The given complex system can be divided into four simple functions as

$$G(s) = 5 \times (s+1) \times \frac{1}{(s+2)} \times \frac{1}{(s+3)}$$

Simple system 1:

$$G_1(s) = 5$$

Substituting $s = j\omega$, we obtain $G_1(j\omega) = 5$

Therefore, gain of the system $M_1 = 5$ and phase angle of the system $\phi_1 = 0°$.

Simple system 2

$$G_2(s) = (s+1)$$

Substituting $s = j\omega$, we obtain $G_2(j\omega) = (j\omega + 1)$

Therefore, gain of the system $M_2 = \sqrt{1+\omega^2}$ and phase angle of the system $\phi_2 = \tan^{-1}(\omega)$.

Simple system 3

$$G_3(s) = \frac{1}{(s+2)}$$

Substituting $s = j\omega$, we obtain $G_3(j\omega) = \frac{1}{(j\omega+2)} = \frac{(2-j\omega)}{(2+j\omega)(2-j\omega)} = \frac{(2-j\omega)}{(4+\omega^2)} = \frac{2}{4+\omega^2} - j\frac{\omega}{4+\omega^2}$

Therefore, gain of the system, $M_3 = \sqrt{\left(\frac{2}{(4+\omega^2)}\right)^2 + \left(\frac{\omega}{(4+\omega^2)}\right)^2}$

and phase angle of the system $\phi_3 = -\tan^{-1}\left(\frac{\omega}{2}\right)$.

Simple system 4

$$G_4(s) = \frac{1}{(s+3)}$$

Therefore, $G_4(j\omega) = \frac{1}{(j\omega+3)} = \frac{(3-j\omega)}{(3+j\omega)(3-j\omega)} = \frac{(3-j\omega)}{(9+\omega^2)} = \frac{3}{9+\omega^2} - j\frac{\omega}{9+\omega^2}$

Therefore, gain of the system $M_4 = \sqrt{\left(\frac{3}{(9+\omega^2)}\right)^2 + \left(\frac{\omega}{(9+\omega^2)}\right)^2}$

and phase angle of the system $\phi_4 = -\tan^{-1}\left(\frac{\omega}{3}\right)$.

The gain and the phase angle of various simple systems at different frequencies between 0.1 and 10 rad/sec are calculated and tabulated as given in Tables E8.3(a) and (b) respectively.

Table E8.3(a) | Gain of the simple systems at different frequencies

Frequency (ω in rad/sec)	Gain of $G_1(s) = 5$ M_1	Gain of $G_2(s) = s + 1$ M_2	Gain of $G_3(s) = \frac{1}{s+2}$ M_3	Gain of $G_4(s) = \frac{1}{s+3}$ M_4
0.1	5	1.0050	0.4994	0.3331
0.5	5	1.1180	0.4851	0.3288
1	5	1.4142	0.4472	0.3162
2	5	2.2361	0.3536	0.2774
3	5	3.1623	0.2774	0.2357
5	5	5.0990	0.1857	0.1715
8	5	8.0622	0.1213	0.1170
10	5	10.0498	0.0981	0.0958

Table E8.3(b) | Phase angle of the system at different frequencies

Frequency (ω in rad/sec)	Phase angle of $G_1(s) = 5$ ϕ_1 (in deg.)	Phase angle of $G_2(s) = s + 1$ ϕ_2 (in deg.)	Phase angle of $G_3(s) = \frac{1}{s+2}$ ϕ_3 (in deg.)	Phase angle of $G_4(s) = \frac{1}{s+3}$ ϕ_4 (in deg.)
0.1	0	5.7105	–2.8624	–1.9092
0.5	0	26.5650	–14.0362	–9.4623
1	0	45.0000	–26.5651	–18.4349
2	0	63.4349	–45.0000	–33.6901
3	0	71.5650	–56.3099	–45.0000
5	0	78.6900	–68.1986	–59.0362
8	0	82.8749	–75.9638	–69.4440
10	0	84.2894	–78.6901	–73.3008

Therefore, the overall gain and phase angle of the complex system is determined using the following expressions and are tabulated in Table E8.3(c).

Gain of the system, $M = M_1 \times M_2 \times M_3 \times M_4$

Phase angle of the system, $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$.

Table E8.3(c) | Gain and phase angle of the complex system at different frequencies

Frequency (rad/sec)	Gain *(M)*	Gain in dB = 20 log *M*	Phase angle (ϕ in deg.)
0.1	0.8359	–1.5569	0.9389
0.5	0.8916	–0.9965	3.0665
1	0.9998	–0.0017	0.01
2	1.0960	0.7962	–15.2552
3	1.0338	0.2887	–29.7449
5	0.8119	–1.8099	–48.5448
8	0.5721	–4.8507	–62.5329
10	0.4722	–6.5174	–67.7015

8.6 Frequency Domain Specifications

Frequency domain specifications of a system are necessary to determine the quality of the system and to design the linear control systems using frequency domain analysis.

Consider a simple closed-loop system as shown in Fig. 8.3. Its transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The magnitude or gain of the above transfer function as a function of frequency is given by

$$|M(j\omega)| = \left|\frac{G(j\omega)}{1+G(j\omega)H(j\omega)}\right| = \frac{|G(j\omega)|}{|1+G(j\omega)H(j\omega)|} = \frac{|G(j\omega)|}{1+|G(j\omega)||H(j\omega)|}$$

The phase angle of the transfer function as a function of frequency is given by

$$\angle M(j\omega) = \phi(j\omega) = \angle G(j\omega) - \angle\left[1+G(j\omega)H(j\omega)\right] = \angle G(j\omega) - \angle\left[G(j\omega)H(j\omega)\right]$$

The typical gain–phase characteristics of a feedback control system are shown in Figs. 8.5(a) and (b).

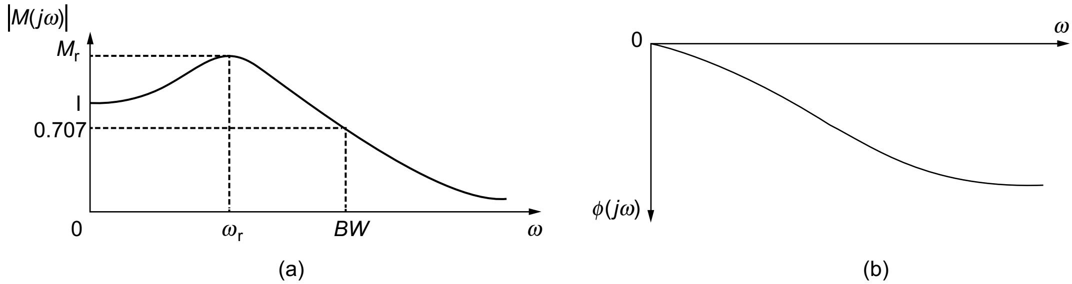


Fig. 8.5 | Gain–phase characteristics

The different frequency domain specifications that are required for designing a control system and determining its performance are:

(i) Resonant peak M_r

The maximum value of gain $M(j\omega)$ as the frequency of the system ω is varied over a range is known as resonant peak M_r. The relative stability of the system can be determined based on this value. There exists a direct relationship between the maximum overshoot in the time-domain analysis of the system and the resonant peak in the frequency domain analysis (i.e., a large value of M_r indicates that the maximum overshoot of the system is also large). The resonant peak lies between 1.1 and 1.5.

(ii) Resonant frequency ω_r

The frequency of the system at which the resonant peak occurs is known as resonant frequency ω_r. The frequency of oscillations in the time domain is related to the resonant frequency ω_r. When resonant frequency is high, the time response or transient response of the system is fast.

(iii) Cut-off frequency ω_c

The frequency at which the gain of the system is 3 dB or 0.707 times the gain of the system at zero frequency is known as cut-off frequency ω_c.

(iv) Bandwidth

The range of frequencies that lie between zero and ω_c is known as bandwidth. It is also defined as the range of frequencies over which the magnitude response of the system is flat.

The value of bandwidth indicates the ability of the system to reproduce the input signal and it is a measure of the noise rejection characteristics. In time-domain analysis of the system for a given damping factor, bandwidth in the frequency domain indicates the rise time in the time domain. If the bandwidth of the system in the frequency domain is large, then the rise time of the system in time domain is small and the system will be faster.

(v) Cut-off rate

The slope of the magnitude curve obtained near the cut-off frequency is called cut-off rate. The cut-off rate indicates the ability of the system to distinguish the signal from noise.

(vi) Gain crossover frequency ω_{gc}

The frequency at which the gain of the system is unity is called the gain crossover frequency ω_{gc}.

If the gain of the system is expressed in dB, then the gain crossover frequency is defined as the frequency at which the gain of the system is 0 dB (since 20 log 1 = 0 dB).

(vii) Phase crossover frequency ω_{pc}
The frequency at which the phase angle of the system is −180° is called phase crossover frequency.

(viii) Gain margin g_m
In root locus technique, there exists a relationship between the gain and stability of the system. Similarly, in frequency response method, there exists a value of K, beyond which the system becomes unstable. Hence, the gain margin g_m is defined as the factor by which the gain of the system can be increased before the system becomes unstable. Mathematically, it is defined as the reciprocal of the gain of the system at the phase crossover frequency ω_{pc}.

The stability of the system is directly proportional to the gain margin of the system. In addition, the high gain margin results in an unacceptable response and the result becomes sluggish in nature.

(ix) Phase margin p_m
In frequency response analysis, gain margin alone is not sufficient to comment on the relative stability of the system. The other factor that affects the relative stability of the system is phase margin p_m. It is defined as the additional phase lag that makes the system marginally stable. In addition, it can be defined as the addition of phase angle of the system at gain crossover frequency and 180°.

As a thumb rule, for a good overall stability of the system, the gain margin of the system should be around 12 dB and the phase margin should be between 45° and 60°.

8.7 Frequency and Time Domain Interrelations

The interrelation between the time-domain analysis and frequency domain analysis of the system is explicit for the first order. In this section, the interrelation existing between the time domain and frequency domain for a second-order system is discussed. In addition, the frequency domain specifications for the second-order system are derived using the frequency response analysis.

Consider a second-order system as shown in Fig. 8.6 with the feed-forward transfer function as $\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$ and with a unity feedback.

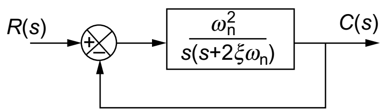


Fig. 8.6 | A simple second order system

Hence, the transfer function of the second-order system is given by

$$M(s)=\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \tag{8.12}$$

where ξ is the damping ratio of the system and ω_n is the undamped natural frequency of the system.

In addition, $\xi = \dfrac{\text{Actual damping}}{\text{Initial damping}}$

The transfer function of the system in frequency domain is obtained by substituting $s = j\omega$ in Eqn. (8.12) given by

$$M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{\left[-\omega^2 + j2\xi\ \omega_n\omega + \omega_n^2\right]} \tag{8.13}$$

Substituting $u = \dfrac{\omega}{\omega_n}$ in Eqn. (8.13), we obtain $M(j\omega) = \dfrac{C(j\omega)}{R(j\omega)} = \dfrac{1}{1 + 2\xi uj - u^2}$ (8.14)

where u is the normalized driving signal frequency.

Therefore,
$$M(j\omega) = |M(j\omega)|\angle M(j\omega) \tag{8.15}$$

Using Eqs. (8.14) and (8.15), we obtain $|M(j\omega)| = M = \dfrac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}}$ (8.16)

and
$$\angle M(j\omega) = \phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right) \tag{8.17}$$

The steady-state output of the system when the system is excited by the sinusoidal input with unit magnitude and variable frequency ω (i.e., $r(t) = \sin(\omega t)$) is given by

$$c(t) = \frac{1}{\sqrt{(1-u^2)^2 + (2\xi u)^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)\right)$$

Using Table 8.1, the magnitude and phase angle plots with respect to the normalized frequency u are shown in Figs. 8.7(a) and 8.7(b) respectively.

Table 8.2 | Magnitude and phase values

u	M	ϕ
0	1	0
1	$1/(2\xi)$	$-\pi/2$
∞	0	$-\pi$

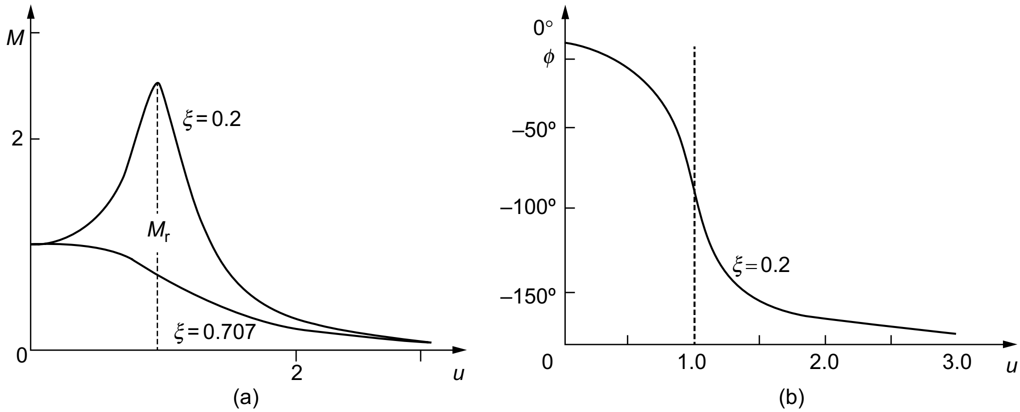


Fig. 8.7 | (a) Magnitude and (b) phase angle plots of second-order system

8.7.1 Frequency Domain Specifications

(i) Resonant frequency ω_r

At resonant frequency ω_r, the first-order derivative of the magnitude of frequency domain analysis is zero.

i.e.,
$$\left.\frac{dM(\omega)}{d\omega}\right|_{\omega=\omega_r} = 0$$

Since the magnitude in Eqn. (8.16) depends on the normalized frequency u, the above equation can be written as $\left.\frac{dM(u)}{du}\right|_{u=u_r} = 0$

where u_r is the normalized resonant frequency.

Therefore,
$$-\frac{1}{2} \times \frac{2(1-u_r^2)(-2u_r) + 8\xi^2 u_r}{\left[(1-u_r^2)^2 + 4\xi^2 u_r^2\right]^{\frac{3}{2}}} = 0$$

Simplifying, we obtain
$$u_r^2 = 1 - 2\xi^2$$
$$u_r = \sqrt{1-2\xi^2}$$

We know that, $u_r = \frac{\omega_r}{\omega_n}$.

Therefore, the resonant frequency is given by $\omega_r = \omega_n\sqrt{1-2\xi^2}$ (8.18)

(ii) Resonant peak M_r

The magnitude at resonant frequency ω_r, is known as the resonant peak M_r.

i.e., $$M_r = M\big|_{\omega = \omega_r}$$

Since the magnitude depends on the normalized frequency, the above equation can be written as $$M_r = M\big|_{u = u_r}$$

Substituting the u_r in Eqs. (8.16) and (8.17), we obtain

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \tag{8.19}$$

$$\phi_r = -\tan^{-1}\frac{\sqrt{1-2\xi^2}}{\xi} \tag{8.20}$$

But we know that in time domain analysis of a second-order system,

Peak overshoot, $M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$ (8.21)

Damped natural frequency, $\omega_d = \omega_n\sqrt{1-\xi^2}$ (8.22)

From Eqs. (8.19) to (8.22), it is clear that there exists a relationship between the time-domain analysis and frequency domain analysis of a system.

Interesting facts about the interrelation between time-domain analysis and frequency domain analysis of a second-order system are given below.

(a) For relative stability of the system, the values of peak overshoot and resonant peak of the system are:

Peak overshoot $M_p = 10 \text{ to } 15\%$

Resonant peak $M_r = 1 \text{ to } 1.4 = 0 \text{ to } 3 \text{ dB}$

If M_r is greater than 1.5 and the system is subjected to noise signals, then the system may face serious problems.

(b) Variation of M_r and M_p with respect to the variation in ξ

(1) As ξ increases, the value of peak overshoot, M_p gets decreased and it becomes zero or gets vanished at $\xi = 1$.

The resonant peak M_r will also get vanished as the damping ratio ξ increases. But the value of ξ at which the resonant peak vanishes is derived as

$$\sqrt{1-2\xi^2} = 0$$

Therefore, $$\xi = 0.707$$

Hence, at $\xi = 0.707$ and $\xi = 1$, the resonant peak and peak overshoot vanished.

This concept is shown in Figs. 8.8(a) and (b).

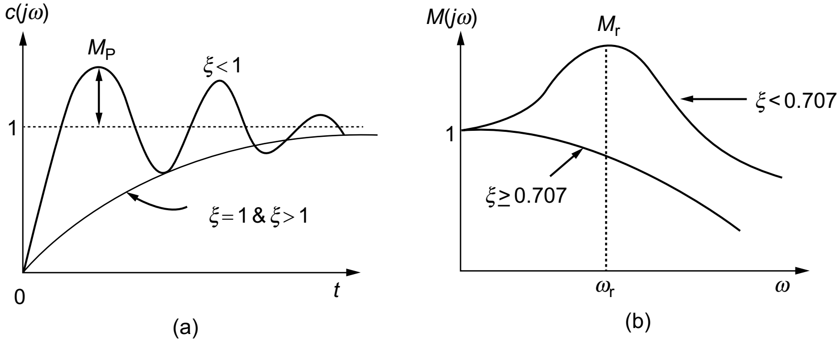


Fig. 8.8 | M_p and M_r for different values of ξ

(2) When $\xi = 0$:
The M_p gets the maximum value i.e., $M_p = 1$. In addition, M_r approaches infinity as ξ decreases to zero. This concept is shown in Figs. 8.9(a) and (b) respectively.

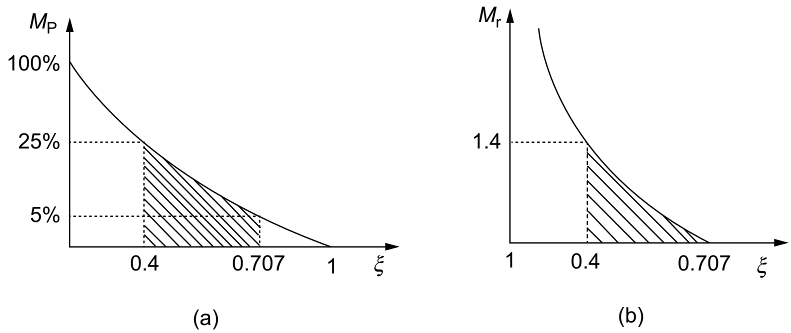


Fig. 8.9 | M_p and M_r for $\xi = 0$

(c) Choosing the value of damping ratio ξ:
The damping ratio ξ should be chosen between 0.4 and 0.707 (i.e., $0.4 < \xi < 0.707$) to have a tolerable M_p and M_r. If the chosen ξ is less than 0.4, both M_p and M_r have larger values that are not desirable for the system.

(d) Variations of ω_r and ω_d with respect to ξ:
When the value of ξ is chosen between 0.4 and 0.707, the value of ω_r and ω_d are comparable to each other.
When $\xi = 0$, both the values of ω_r and ω_d approach ω_n.
When ξ is small, then the values of ω_r and ω_d are:

(1) ω_d will be large and hence the rise time is small.
(2) ω_r will be large and the response of the system will be faster.

Here, the value of ω_r indicates the speed of the response. The above concept is shown in Figs. 8.10(a) and (b).

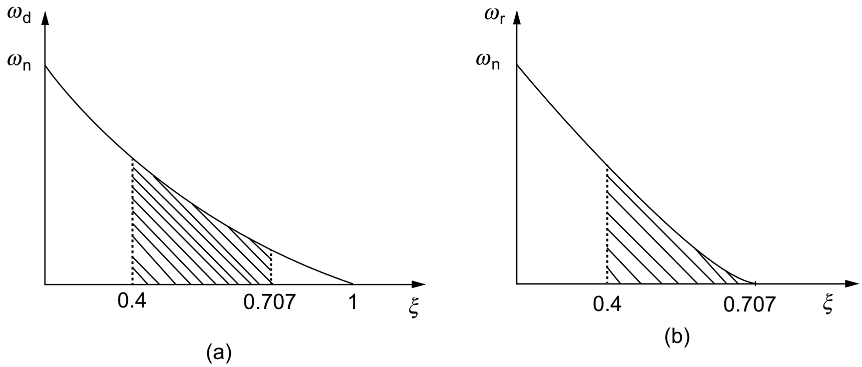


Fig. 8.10 | ω_d and ω_r for different values of ξ

(iii) Bandwidth

The range of frequencies between zero and cut-off frequency ω_c, is known as bandwidth and the cut-off frequency is defined as the frequency at which the magnitude of the system is 3 dB down the magnitude of the system at zero frequency. Hence, bandwidth is nothing but the cut-off frequency ω_c.

Assuming the magnitude of the system at zero frequency as 1, the cut-off frequency is derived as

$$\left|M(j\omega)\right|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} \times \left|M(j\omega)\right|_{\omega=0} = \frac{1}{\sqrt{2}}$$

But from Eqn. (8.16), it is clear that the magnitude depends on the normalized frequency. Hence, the above equation can be written as

$$\left|M(j\omega)\right|_{u=u_c} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{\left(1-u_c^2\right)^2 + \left(2\xi u_c\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left(1-u_c^2\right)^2 + \left(2\xi u_c\right)^2 = 2$$

$$1 + u_c^4 - 2u_c^2 + 4\xi^2 u_c^2 = 2$$

$$u_c^4 + 2u_c^2\left(2\xi^2 - 1\right) - 1 = 0$$

Comparing the above equation with the quadratic equation $ax^2 + bx + c = 0$, we obtain

$$a = 1,\ b = 2\left(2\xi^2 - 1\right),\ c = -1 \text{ and } x = u_c^2$$

Hence, solving this quadratic equation, we obtain

$$u_c^2 = \frac{-2\left(2\xi^2 - 1\right) \pm \sqrt{4\left(2\xi^2 - 1\right)^2 + 4}}{2}$$

Considering only the positive values, we obtain $u_c^2 = 1 - 2\xi^2 + \sqrt{\left(2\xi^2 - 1\right)^2 + 1}$

Therefore, $$u_c = \sqrt{1 - 2\xi^2 + \sqrt{\left(2\xi^2 - 1\right)^2 + 1}} \tag{8.23}$$

We know that, $u_c = \dfrac{\omega_c}{\omega_n}$

Therefore, the cut-off frequency is given by $$\omega_c = \omega_n\sqrt{1 - 2\xi^2 + \sqrt{\left(2\xi^2 - 1\right)^2 + 1}} \tag{8.24}$$

The expression for bandwidth is also same as that of the cut-off frequency. From Eqn. (8.23), the normalized bandwidth u_c is equal to 1. The graphical idea about the bandwidth with respect to the damping ratio ξ is shown in Fig. 8.11.

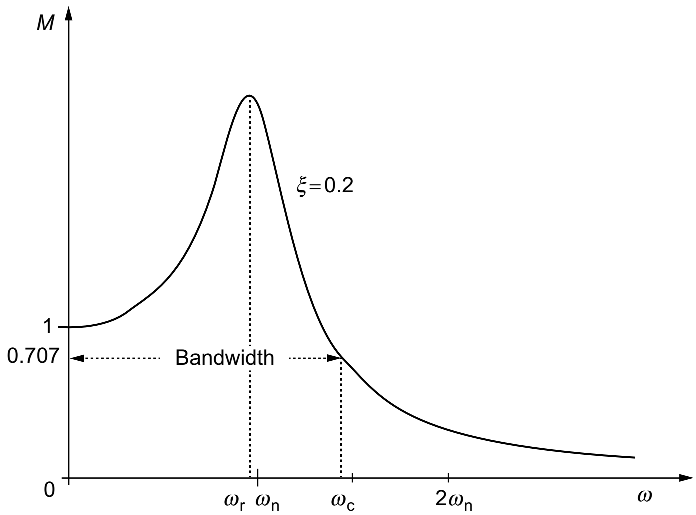


Fig. 8.11 | Magnitude versus frequency

The graphical idea between the normalized bandwidth $\dfrac{\omega_c}{\omega_n}$ and the damping ratio ξ is shown in Fig. 8.12.

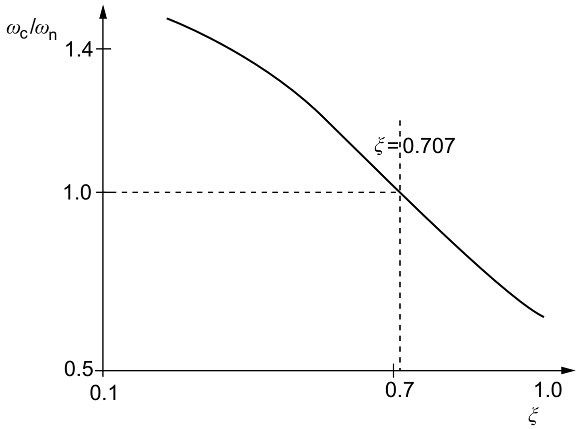


Fig. 8.12 | ξ vs normalized bandwidth

Example 8.3 **The open-loop or the feed-forward transfer function of a unity feedback system is given by $G(s) = \dfrac{81}{s(s+8)}$. Determine the resonant frequency and resonant peak for the given system.**

Solution: Given $G(s) = \dfrac{81}{s(s+8)}$ and $H(s) = 1$.

Hence, the closed-loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{81}{s^2 + 8s + 81}$

The characteristic equation of the given system is $s^2 + 8s + 81 = 0$

Comparing the above equation with the standard second-order characteristic $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$, we obtain

$$\omega_n^2 = 81\text{, i.e., } \omega_n = 9 \text{ rad/sec and } 2\xi\omega_n = 8\text{, i.e., } 2\xi \times 9 = 8$$

Hence, $\xi = 0.44$

Resonant peak, $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}} = \dfrac{1}{2 \times 0.44\sqrt{1-(0.44)^2}} = 1.265$

Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\xi^2} = 9\sqrt{1-2\times(0.44)^2} = 7.045$ rad/sec

Example 8.4 The closed-loop poles of a system are at $s = -2 \pm j3$. Determine (i) bandwidth, (ii) normalized peak driving signal frequency and (iii) resonant peak for such a system.

Solution: The closed-loop transfer function of any system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

and the characteristic equation of the system is given by

$$1+G(s)H(s)=0$$

The closed-loop poles of a system are obtained by equating the characteristic equation to zero. Hence, the characteristic equation of the given system using the given closed-loop poles is obtained as $\quad (s+2-j3)(s+2+j3)=0$

i.e., $$s^2+4s+13=0$$

Comparing the above equation with the standard second-order characteristic equation, we obtain $\omega_n^2 = 13$. Therefore, $\omega_n = 3.605$ rad/sec

$$2\xi\omega_n = 4$$

Therefore, $$\xi = \frac{2}{\omega_n} = 0.5547$$

Using the damping ratio and natural frequency, the frequency domain specifications can be determined as given below:

(i) Bandwidth (BW) $= \omega_n\sqrt{1-2\xi^2 \pm \sqrt{(1-2\xi^2)^2+1}}$

$$= 3.605 \times \sqrt{1-\left(2\times 0.5547^2\right) \pm \sqrt{\left(1-\left(2\times 0.5547^2\right)\right)^2+1}} = 4.350 \text{ rad/sec}$$

(ii) Normalized peak driving signal frequency (u_p) is

$$u_p = \sqrt{1-2\xi^2} = \sqrt{1-2(0.5547)^2} = 0.3846$$

(iii) Resonant peak, $M_p = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

Therefore, $$M_p = \frac{1}{2\times 0.5547\times\sqrt{1-(0.5547)^2}} = 1.083$$

Example 8.5 **Consider a second-order system with a natural frequency of 4 rad/sec and damped natural frequency of 1.6 rad/sec. Determine (i) the percentage of peak overshoot when the system is subjected to a unit step input and (ii) the resonant peak value when the system is subjected to sinusoidal input.**

Solution: Given $\omega_n = 4$ rad/sec and $\omega_d = 1.6$ rad/sec

We know that $\omega_d = \omega_n\sqrt{1-\xi^2}$

Therefore, $$\left(1-\xi^2\right) = \frac{\omega_d^2}{\omega_n^2} = \frac{(1.6)^2}{4^2} = 0.16$$

Upon solving, we obtain $\xi = 0.916$

(a) Peak overshoot when subjected to step input $= e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100 = e^{\frac{-0.916\pi}{\sqrt{1-0.839}}} \times 100 = 0.076\%$

(b) Resonance peak, $M_p = \dfrac{1}{2\xi\sqrt{1-\xi^2}} = \dfrac{1}{2 \times 0.916 \times 0.401} = 1.36$

Example 8.6 **Consider a second-order system with resonant peak 2 and resonant frequency of 6 rad/sec. Determine the transfer function of the given second-order system and hence determine (i) rise time t_r, (ii) peak time t_p, (iii) settling time t_s and (iv) % peak overshoot M_p of the given system when the system is subjected to step input. In addition, determine the time of oscillation and number of oscillations before the response of the system gets settled.**

Solution: Given $M_r = 2$ and $\omega_r = \omega_n\sqrt{1-2\xi^2} = 6$

We know that $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

i.e., $$2 = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

i.e., $$4 = \frac{1}{4\xi^2\left(1-\xi^2\right)}$$

Upon solving, we obtain $\xi^2 = \dfrac{16 \pm \sqrt{256-64}}{32} = \dfrac{16}{32} \pm \dfrac{\sqrt{192}}{32} = 0.5 \pm 0.433$

i.e., $\xi^2 = 0.067$ or 0.933

Therefore, $\xi = 0.2588$ or 0.966

The system with the damping ratio greater than 0.707 does not exhibit any peak in the frequency response of the system. Hence, the damping ratio of the given system is $\xi = 0.2588$.

We know that, $\omega_r = \omega_n\sqrt{1-2\xi^2}$

Substituting the known values, we obtain $6 = \omega_n\sqrt{1-2(0.2588)^2}$

Upon solving, we obtain $\omega_n = 6.447$ rad/sec

The standard second-order transfer function of the system is $\dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$
Substituting the known values, we obtain

$$\frac{C(s)}{R(s)} = \frac{6.447^2}{s^2+(2\times 0.2588\times 6.447)s+41.563} = \frac{41.563}{s^2+3.336s+41.563}$$

Here, $$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\text{ rad}$$

Therefore, $$\theta = \tan^{-1}\left[\frac{\sqrt{1-(0.2588)^2}}{0.2588}\right] = \tan^{-1}\left[\frac{0.9659}{0.2588}\right] = 1.3\text{ rad}$$

Also, $\omega_d = \omega_n\sqrt{1-\xi^2} = 6.447\times\sqrt{1-(0.2588^2)} = 6.227$ rad/sec

Rise time, $t_r = \dfrac{\pi-\theta}{\omega_d} = \dfrac{\pi-1.3}{6.227} = 0.2957$ sec

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{6.227} = 0.5045$ sec

Settling time for a system can be determined for 2% error tolerance and 5% error tolerance.

Hence, $$t_s = \frac{4}{\xi\omega_n} = 2.3973\text{ sec for 2\% tolerance}$$

$$t_s = \frac{3}{\xi\omega_n} = 1.798\text{ sec for 5\% tolerance}$$

Period of oscillation, $$T_{osc} = \frac{2\pi}{\omega_d} = 1.009 \text{ sec}$$

Number of oscillations before the response of the system gets settled is given by

$$N = \frac{t_s}{T_{osc}} = \frac{2.3973}{1.009} = 2.37 \simeq 3$$

Peak overshoot of the system is $M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = e^{\frac{-0.2588\pi}{\sqrt{1-(0.2588)^2}}} = 0.4309$

Example 8.7 The time response of a second-order system when the system is subjected to a unit step input is given below:

Time (sec)	0	0.05	0.1	0.15	0.2	0.25	0.3	0.36	0.4	0.45	0.5
c(t)	0	0.25	0.8	1.08	1.12	1.02	0.98	0.98	1.0	1.0	1.0

Determine the frequency response parameters of the system (i) peak resonance M_r, (ii) resonant frequency ω_r and (iii) cut-off frequency of the system ω_c.

Solution:

From the given Table, the maximum value of the time response and the peak time of the system are 1.12 and 0.2 s, respectively.

Hence, there exists a peak overshoot of 0.12 at 0.2 sec.

Therefore, $0.12 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$

Solving the above equation, we obtain $\xi = 0.5594$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

i.e., $$0.2 = \frac{\pi}{\omega_n\sqrt{1-(0.5594^2)}}$$

Therefore, $\omega_n = 18.95$ rad/sec

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.5594\times\sqrt{1-(0.5594^2)}} = 1.0779$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 18.95\times\sqrt{1-2(0.5594^2)} = 11.5912 \text{ rad/sec}$$

$$\omega_c = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = 18.95\sqrt{1-2(0.5594)^2+\sqrt{2-4(0.5594)^2+4(0.5594)^4}}$$
$$= 22.75 \text{ rad/sec}$$

Example 8.8 **Consider a unity feedback system as shown in Fig. E8.9. Determine the K and a that satisfies the frequency domain specifications as $M_r = 1.04$ and $\omega_r = 11.55$ rad/sec. In addition, for the determined values of K and a, determine settling time and bandwidth of the system.**

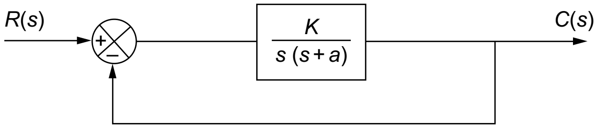


Fig. E8.9

Solution: Given $G(s) = \dfrac{K}{s(s+a)}$ and $H(s) = 1$

Hence, the closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K}{s(s+1)}}{1+\dfrac{K}{s(s+a)}} = \frac{K}{s^2+as+K}$$

Comparing the above equation with the standard second-order transfer function, we obtain

$$\omega_n^2 = K \text{ i.e., } \omega_n = \sqrt{K} \tag{1}$$

$$2\xi\omega_n = a \text{ i.e., } \xi = \frac{a}{2\omega_n} = \frac{a}{2\sqrt{K}} \tag{2}$$

Using the values of resonant peak and the resonant frequency, the values of K and a can be determined.

(i) $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

i.e., $1.04 = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

Upon solving, we obtain $\xi = 0.6021$

(ii) $\omega_r = \omega_n\sqrt{1-2\xi^2}$

i.e., $11.55 = \omega_n\sqrt{1-2(0.6021)^2}$

Upon solving, we obtain $\omega_n = 22.027$ rad/sec

Substituting the values of ξ and ω_n in Eqs. (1) and (2), we obtain

$$K = \omega_n^2 = (22.027)^2 = 485.1887 \text{ and } a = 2\xi\omega_n = 2 \times 0.6021 \times 22.027 = 26.525$$

(iii) Settling time, $t_s = \dfrac{4}{\xi\omega_n} = \dfrac{4}{0.6021 \times 22.027} = 0.3016$ sec

(iv) Bandwidth, $BW = \omega_n\sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}} = 25.2309$ rad/sec

Example 8.9 **The damping ratio and natural frequency of oscillation of a second-order system is 0.5 and 8 rad/sec respectively. Determine the resonant peak and resonant frequency.**

Solution: Given $\xi = 0.5$ and $\omega_n = 8$ rad/sec

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2 \times 0.5\sqrt{1 - 0.5^2}} = 1.154$$

$$\omega_r = \omega_n\sqrt{1 - 2\xi^2} = 8 \times \sqrt{1 - 2 \times 0.5^2} = 5.6568 \text{ rad/sec}$$

Example 8.10 **The specification given on a certain second-order feedback control system is that the overshoot of the step response should not exceed 25 per cent. What are the corresponding limiting values of the damping ratio ξ and peak resonance M_r?**

Solution:

Given $M_p = 25\%$

$$\%M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100 \qquad \text{i.e., } 0.25 = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

Therefore, $$\ln(0.25) = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

i.e., $$(-1.3863)^2 = \frac{\pi^2\xi^2}{(1-\xi^2)}$$

or $$1.9218\left(1-\xi^2\right)=\pi^2\xi^2$$

i.e., $$\xi^2 = 0.162983$$

Therefore, damping ratio, $\xi = 0.4037$

Hence, resonant peak, $M_\mathrm{r} = \dfrac{1}{2\xi\sqrt{1-\xi^2}} = \dfrac{1}{2\times 0.4037\times\sqrt{1-(0.4037)^2}} = 1.3537$

Example 8.11 **Determine the frequency specifications of a second-order system when closed transfer function is given by $\dfrac{C(s)}{R(s)} = \dfrac{64}{s^2+10s+64}$**

Solution: Comparing denominator of the transfer function with $s^2+2\xi\omega_\mathrm{n}s+\omega_\mathrm{n}^2$, we obtain

$$\omega_\mathrm{n}^2 = 64 \text{ i.e., } \omega_n = 8 \text{ and } 2\xi\omega_n = 10 \text{ i.e., } \xi = 0.625$$

$$M_\mathrm{r} = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.625\times\sqrt{1-0.625^2}} = 1.0248$$

and $$\omega_\mathrm{r} = \omega_\mathrm{n}\sqrt{1-2\xi^2} = 8\times\sqrt{1-2(0.625)^2} = 3.741 \text{ rad/sec}$$

$$BW = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = 8\sqrt{1-2(0.625)^2+\sqrt{2-4(0.625)^2+4(0.625)^4}} = 8.917 \text{ rad/sec}$$

8.8 Effect of Addition of a Pole to the Open-Loop Transfer Function of the System

Consider a system with the open-loop transfer function $G(s)=\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$. When a pole at $s=-\dfrac{1}{T_\mathrm{p}}$ is added to such a system, the open-loop transfer function of the system becomes $G(s)=\dfrac{\omega_n^2}{s(s+2\xi\omega_n)(1+T_\mathrm{p}s)}$. When a sinusoidal input is applied to such a system, the system becomes less stable compared to the stability of the previous system. In addition, the specifications of the frequency domain and time domain vary depending on the time constant T_p.

For larger values of T_p, we obtain

(i) Larger rise time that in turn decreases the bandwidth of the system.

(ii) Larger value of resonant peak that corresponds to a larger value of maximum overshoot in the time response of the system.

8.9 Effect of Addition of a Zero to the Open-Loop Transfer Function of the System

Consider a system with the open-loop transfer function $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$. When a zero at $s = -\dfrac{1}{T_z}$ is added, the open-loop transfer function of the system becomes $G(s) = \dfrac{\omega_n^2(1+T_z s)}{s(s+2\xi\omega_n)}$.

When a sinusoidal input is applied to the system, the following changes occur in the frequency domain specifications.

(i) Bandwidth of the system gets increased.

(ii) Increase in time constant for higher values of T_z.

(iii) Settling time of the system gets increased.

8.10 Graphical Representation of Frequency Response

Determining the frequency response of a system, i.e., the magnitude and phase angle of a system for different frequencies from 0 to ∞ by using tabulation method becomes more complicated when more number of poles and zeros exist in the system. An alternative method that eliminates the difficulty of the tabulation method is the graphical representation of frequency response.

There are different graphical methods by which the frequency response can be represented. They are

(i) Bode plot (asymptotic plots)

(ii) Polar plot

(iii) Nyquist plot

(iv) Constant M and N circles

(v) Nichols chart

The Bode plot of representing frequency response of a system is discussed in this chapter and the other plots will be discussed in the subsequent chapters.

8.11 Introduction to Bode Plot

Bode plot introduced by H.W. Bode was first used in the study of feedback amplifiers. It is one of the popular graphical methods used for determining the stability of the system when the system is subjected to sinusoidal input. The stability of the closed-loop system is determined based on the frequency response of the loop transfer function of the system, i.e., $G(s)H(s)$. The gain or magnitude and phase angle of the system can be easily represented as a function of frequency using Bode plot. It is also a very useful graphical tool in analysing and designing of linear control systems.

The Bode plot consists of two plots:

(i) Magnitude plot: To plot the logarithmic magnitude or gain of the loop transfer function in dB versus frequency ω i.e., $M = 20 \log G(j\omega)H(j\omega)|$ in dB versus ω.

(ii) Phase plot: To plot the phase angle of the loop transfer function versus frequency ω i.e., $\phi = \angle G(j\omega)H(j\omega)$ versus ω.

In Bode plot, both the magnitude and phase plots are plotted against the frequency in the logarithmic scale. In addition, the magnitude of the system is plotted in dBs (decibels). Hence, the Bode plot is also called logarithmic plot. Since the magnitude and phase plots of a system are sketched based on the asymptotic properties instead of detailed plotting, the Bode plot is also called asymptotic plots.

8.11.1 Reasons for Using Logarithmic Scale

The reasons for plotting the magnitude and phase angle plots of the Bode plot in a logarithmic scale are:

(i) In higher order systems, magnitudes of the individual subsystem have to be multiplied to get the magnitude of the higher order system that is a tedious process. But if we use the logarithmic scale, the multiplication part can be replaced by the addition that makes the process of determining the magnitude of the higher order system easier.

(ii) In addition, the variation of frequency in a large scale is easier if we use a logarithmic scale rather than the ordinary scale.

It is noted that in Bode plot, the magnitude of the loop transfer function is taken in terms of decibels (complex logarithm) rather than the simple logarithm. The magnitude used to plot the magnitude plot in Bode plot is determined using

$$M = 20 \log\left|G(j\omega)H(j\omega)\right| \text{ dB}$$

Table 8.3 shows the importance of logarithmic scale rather than the ordinary scale. Any real value existing between $\frac{1}{100}$ and 100 can be plotted between −40 to 40 dB.

Table 8.3 | The dB values for the original magnitude

Magnitude, $M = \lvert G(j\omega)H(j\omega) \rvert$	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	2	3.16	5	10	100
M in dB	−40	−20	−6	−3	0	3	6	10	14	20	40

Thus a wide range of magnitude can be plotted easily using logarithmic magnitude scale.

8.11.2 Advantages of Bode Plot

The advantages of using Bode plot in plotting the frequency response of a system are:

(i) Expansion of frequency range of a system is simple.

(ii) Experimental determination of transfer function of a system is simpler if the frequency response of the system is shown using Bode plot.

(iii) Usage of asymptotic straight lines for approximating the frequency response of the system.

(iv) Bode plot can be plotted for complicated systems.

(v) Relative stability of the closed-loop system can be analysed by plotting the frequency response of the loop transfer function of the system using Bode plot.

(vi) Frequency domain specifications such as gain crossover frequency, phase crossover frequency, gain margin and phase margin can easily be determined.

(vii) The variation of frequency domain specifications can easily be viewed when a controller is added to the existing system.

(viii) The system gain K can be designed based on the required gain and phase margins.

(ix) Polar plot and Nyquist plot can be constructed based on the data obtained from Bode plot.

8.11.3 Disadvantages of Bode Plot

The disadvantages of using Bode plot in plotting the frequency response of a system are:

(i) Using Bode plot, it is possible only to determine the absolute and relative stability of the minimum phase system.

(ii) Corrections are to be made in the obtained plot to meet the desired frequency plot.

8.12 Determination of Frequency Domain Specifications from Bode Plot

The different frequency domain specifications that can easily be determined using Bode plot are gain margin, phase margin, gain crossover frequency and phase crossover frequency. The plot shown in Fig. 8.13 indicates the determination of the above said frequency domain specifications.

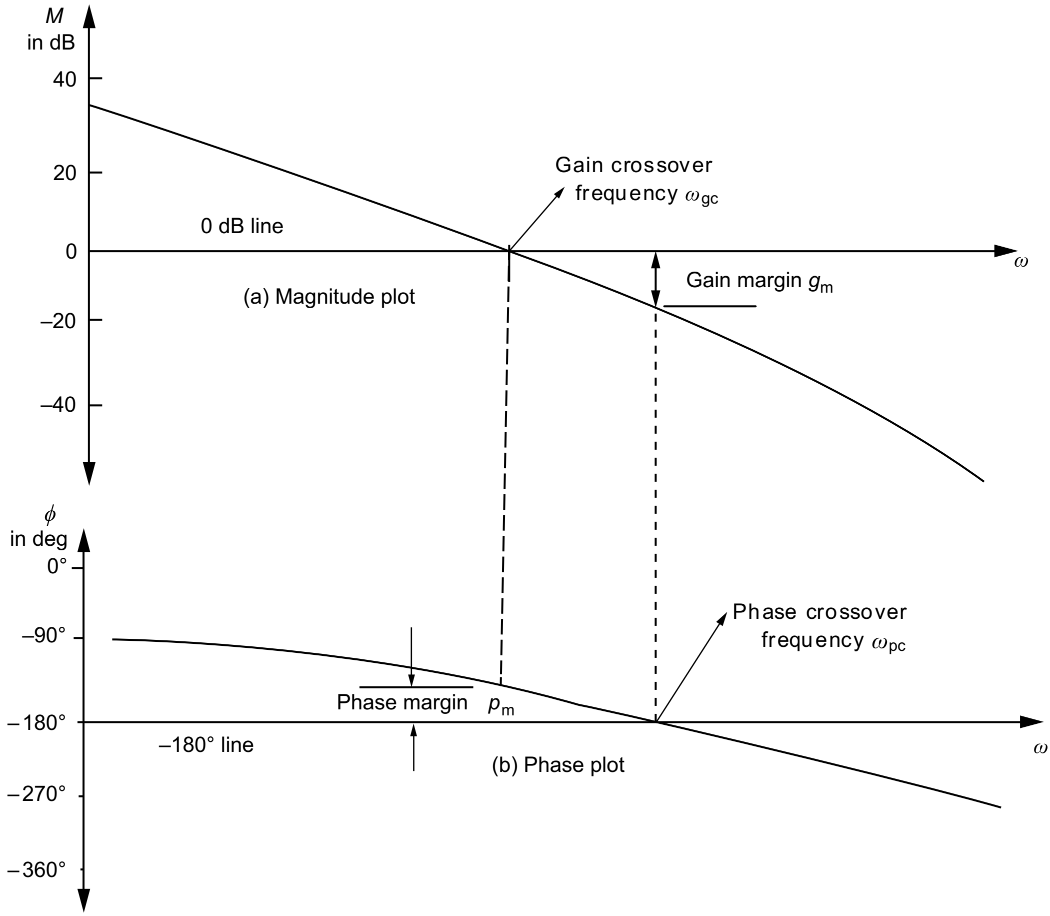


Fig. 8.13 | Frequency domain specifications

From the above plot, the formula for determining the frequency domain specifications can be obtained as follows:

Gain crossover frequency, ω_{gc} = frequency at which $M = 0$.

Phase crossover frequency, ω_{pc} = frequency at which $\phi = -180°$.

Gain margin, $g_m = 0\text{ dB} - M\big|_{\omega = \omega_{pc}}$.

Phase margin, $p_m = \phi\big|_{\omega = \omega_{gc}} - (-180°) = 180° +$ phase angle at ω_{gc} .

8.13 Stability of the System

The stability of the system is easier to determine using Bode plot once the frequency domain specifications are obtained. The stability of the system can be analysed on the basis of crossover frequencies (ω_{pc} and ω_{gc}) or gain and phase margins.

8.13.1 Based on Crossover Frequencies

The system can either be a stable system, marginally stable system or unstable system. The stability of the system based on the relation between crossover frequencies is given in Table 8.4.

Table 8.4 | Stability of the system based on crossover frequencies

S. No	Relation between ω_{gc} and ω_{pc}	Stability of the system
1.	$\omega_{gc} < \omega_{pc}$	Stable system
2.	$\omega_{gc} > \omega_{pc}$	Unstable system
3.	$\omega_{gc} = \omega_{pc}$	Marginally stable system

8.13.2 Based on Gain Margin and Phase Margin

The stability of the system based on the gain margin and phase margin is given in Table 8.5.

Table 8.5 | Stability of the system based on g_m and p_m

S. No.	Gain margin g_m	Phase margin p_m	Relation between ω_{gc} and ω_{pc}	Stability of the system
1	Positive	Positive	$\omega_{gc} < \omega_{pc}$	Stable system
2	Positive	Negative	$\omega_{gc} < \omega_{pc}$	Unstable system
3	Negative	Positive	$\omega_{gc} > \omega_{pc}$	Unstable system
4	0 dB	0°	$\omega_{gc} = \omega_{pc}$	Marginally stable system

8.14 Construction of Bode Plot

The construction of Bode plot can be illustrated by considering the generalized form of loop transfer function is given by:

$$G(s)H(s) = \frac{e^{-sT} \times s^Z K_s \prod_{k=1}^{Q}\left(s^2 + 2\xi_k \omega_{nk} s + \omega_{nk}^2\right)\prod_{i=1}^{M}\left(sT_i + 1\right)}{s^N \prod_{m=1}^{V}\left(s^2 + 2\xi_m \omega_{nm} s + \omega_{nm}^2\right)\prod_{l=1}^{U}\left(sT_l + 1\right)}$$

where $K_s, T_i, T_l, \xi_k, \omega_{nk}$ and ω_{nm} are real constants,
Z is the number of zeros at the origin,
N is the number of poles at the origin or the TYPE of the system,
M is the number of simple poles existing in the system,
U is the number of simple zeros existing in the system,

V is the number of complex poles existing in the system
Q is the number of complex zeros existing in the system and
T is the time delay in seconds.
Substituting $s = j\omega$ and simplifying, we obtain

$$G(j\omega)H(j\omega) = \frac{e^{-(j\omega)T} \times (j\omega)^Z K_s \prod_{k=1}^{Q}\left((j\omega)^2 + 2\xi_k \omega_{nk}(j\omega) + \omega_{nk}^2\right)\prod_{i=1}^{M}(j\omega T_i + 1)}{(j\omega)^N \prod_{m=1}^{V}\left((j\omega)^2 + 2\xi_m \omega_{nm}(j\omega) + \omega_{nm}^2\right)\prod_{l=1}^{U}(j\omega T_l + 1)}$$

Rearranging the above equation, we obtain

$$G(j\omega)H(j\omega) = \frac{e^{-(j\omega)T} \times K(j\omega)^Z \prod_{k=1}^{Q}\left(1 + j2\xi_k u_{nk} - u_{nk}^2\right)\prod_{i=1}^{M}(j\omega T_i + 1)}{(j\omega)^N \prod_{m=1}^{V}\left(1 + j2\xi_m u_{nm} s - u_{nm}^2\right)\prod_{l=1}^{U}(j\omega T_l + 1)} \tag{8.25}$$

where $K = \dfrac{K_s \prod_{k=1}^{Q}\omega_{nk}^2}{\prod_{i=1}^{v}\omega_{nm}^2}$, $u_{nk} = \dfrac{\omega}{\omega_{nk}}$ and $u_{nm} = \dfrac{\omega}{\omega_{nm}}$

The magnitude of Eqn. (8.25) in dB is given by

$$G(j\omega)H(j\omega)\Big|_{dB} = 20\log|K| + \begin{Bmatrix} 20\log(j\omega)^Z \\ \text{or} \\ 20Z\log|jw| \end{Bmatrix} + \sum_{k=1}^{Q} 20\log\left|\left(1 + j2\xi_k u_{nk} - u_{nk}^2\right)\right| + \sum_{i=1}^{M} 20\log\left|(1 + j\omega T_i)\right|$$
$$- \begin{Bmatrix} 20\log(j\omega)^N \\ \text{or} \\ 20N\log|j\omega| \end{Bmatrix} - \sum_{m=1}^{V} 20\log\left|\left(1 + j2\xi_m u_{nm} - u_{nm}^2\right)\right| - \sum_{l=1}^{U} 20\log\left|(1 + j\omega T_l)\right|$$

The phase angle of Eqn. (8.25) is given by

$$\angle G(j\omega)H(j\omega) = -57.32\omega T + \left(Z \times \left(\frac{\pi}{2}\right)\right) + \sum_{k=1}^{Q}\tan^{-1}\left(\frac{2\xi_k u_{nk}}{1 - u_{nk}^2}\right) + \sum_{i=1}^{M}\tan^{-1}(\omega T_i) - \left(N \times \left(\frac{\pi}{2}\right)\right)$$
$$- \sum_{m=1}^{V}\tan^{-1}\left(\frac{2\xi_m u_{nm}}{1 - u_{nm}^2}\right) - \sum_{l=1}^{U}\tan^{-1}(\omega T_l)$$

where magnitude of the term e^{-sT} is 1 and phase angle of the term e^{-sT} is $-57.32 \times \omega T$.

It is noted that phase angle of the term $20\log|K| = 0^\circ$.

From Eqn. (8.25), it is clear that the open-loop transfer function $G(j\omega)H(j\omega)$ may contain the combination of any of the following five factors:

(i) Constant K.

(ii) Zeros at the origin $(j\omega)^{Z}$ and poles at the origin $(j\omega)^{-N}$.

(iii) Simple zero $(1+j\omega T)$ and simple pole $(1+j\omega T)^{-1}$.

(iv) Complex zero $(1+j2\xi u_n - u_n^2)$ and complex pole $(1+j2\xi u_n - u_n^2)^{-1}$.

(v) Transportation lag $e^{-(j\omega)T}$.

Hence, it is necessary to have a complete study (magnitude and phase plots) of these factors that can be utilized in constructing the plot (magnitude and phase plots) of a composite loop transfer function $G(j\omega)H(j\omega)$. The composite plot for the loop transfer function $G(j\omega)H(j\omega)$ is constructed by adding the plots of individual factors present in the function. Thus Bode plot is an approximate asymptotic plot of individual factors.

Factor 1: Constant *K*

The magnitude and phase angle of the constant K which are to be plotted in a semi log graph sheet are:

Magnitude in dB = $20\log K$.

Phase angle = 0°.

The magnitude and phase plots corresponding to the values obtained using the above equation are shown in Figs. 8.14(a) and (b) respectively.

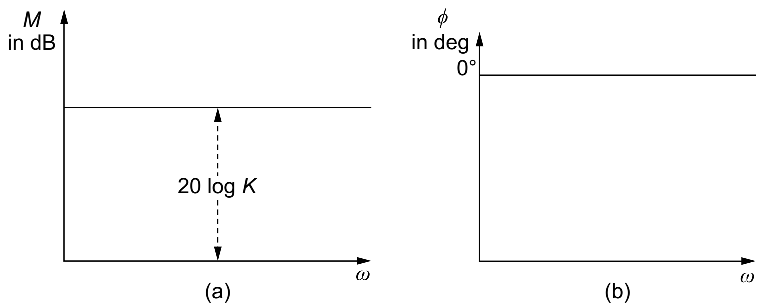


Fig. 8.14 | Bode plot for factor K

Note:

(i) The constant K can have a value that is either greater than 1 or less than 1. The magnitude corresponding to the gain that is greater than 1 is positive and gain that is less than 1 is negative.

(ii) If the gain K is varied, the corresponding changes occur only in the magnitude plot (i.e., either the plot raises or lowers). If the gain value is increased by a factor 10, then the corresponding magnitude plot gets raised by 20 dB. Similarly, if the gain value is decreased by a factor 10, then the corresponding magnitude plot gets decreased by 20 dB. The proof of this concept is given below.

$$20\log(K\times 10) = 20\log K + 20\log 10 = 20\log K + 20$$

Similarly, $\quad 20\log(K\times 10^{n}) = 20\log K + 20\times n$

(iii) In addition, if the magnitude of a constant K is expressed in dB, the magnitude of reciprocal of the particular constant K in dB will have the same value but the sign differs.

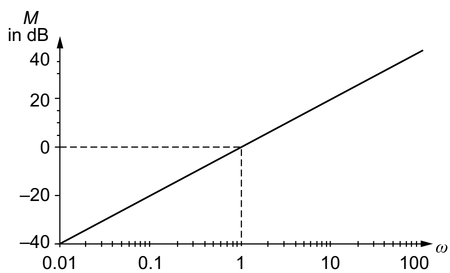
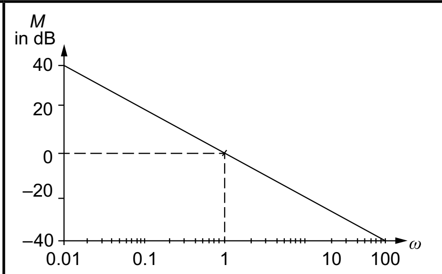
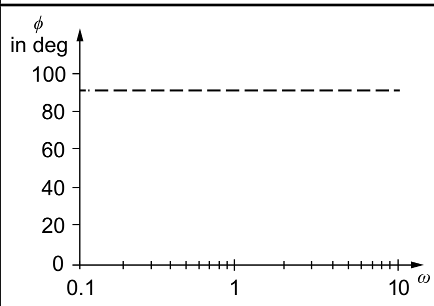
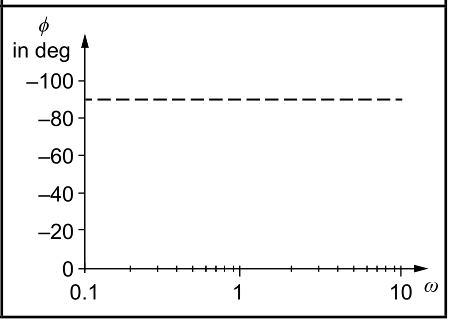
i.e., $20\log K = -20\log\left(\dfrac{1}{K}\right)$

Factor 2: Zeros at the origin $(j\omega)^{Z}$ or poles at the origin $(j\omega)^{-N}$

In Bode plots, the frequency ratios are expressed in terms of octaves or decades. When the frequency band is from ω_1 to $10\omega_1$, it is called decade.

The details for plotting the magnitude and phase plots when only one pole or zero exists at the origin are given in Table 8.6.

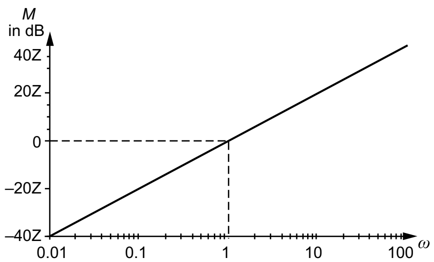
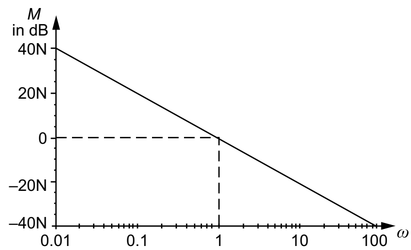
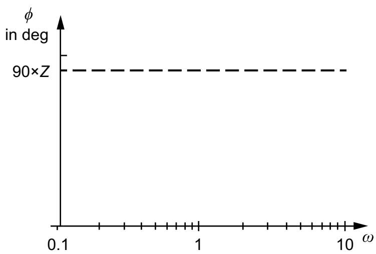
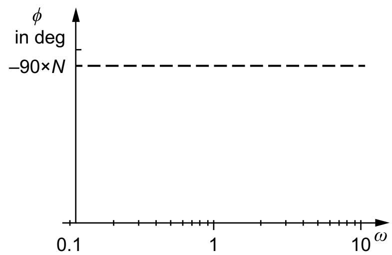
Table 8.6 | Magnitude and phase plots for $N = 1$ and $Z = 1$

Index	Zero at the origin	Pole at the origin
Magnitude in dB	$20\log\lvert j\omega\rvert = 20\log(\omega)$	$20\log\left\lvert\dfrac{1}{j\omega}\right\rvert = -20\log(\omega)$
Phase angle ϕ	90°	–90°
Slope of the straight line	20 dB/decade	–20 dB/decade
Magnitude plot		
Phase plot		

The magnitude in dB will be equal to zero at the frequency value of 1 rad/sec.

If multiple poles or zeros exist at the origin, the corresponding changes in the magnitude and phase plots are given in Table 8.7. Let the number of zeros at the origin existing in the system be Z and the number of poles at the origin be N.

Table 8.7 | Magnitude and phase plots for $N > 1$ and $Z > 1$

<table>
<tr><th>Index</th><th>Zero at the origin</th><th>Pole at the origin</th></tr>
<tr><td>Magnitude in dB</td><td>$20\log(j\omega)^Z = 20 \times Z \times \log(\omega)$</td><td>$20\log\left(\frac{1}{j\omega}\right)^N = -20 \times N \times \log(\omega)$</td></tr>
<tr><td>Phase angle ϕ</td><td>$Z \times 90°$</td><td>$N \times (-90°)$</td></tr>
<tr><td>Slope of the straight line</td><td>$(20 \times Z)$ dB/decade</td><td>$(-20 \times N)$ dB/decade</td></tr>
<tr><td>Magnitude plot</td><td>
</td><td>
</td></tr>
<tr><td>Phase plot</td><td>
</td><td>
</td></tr>
</table>

Factor 3: Simple pole $(1 + j\omega T)^{-1}$ or simple zero $(1 + j\omega T)$

The details for plotting the magnitude and phase plots when only one simple pole or simple zero exists are given in Table 8.8.

Table 8.8 | Magnitude and phase angle for simple pole and simple zero

Index	Simple zero	Simple pole
Magnitude in dB	$20\log(1 + j\omega T) = 20\log\left(\sqrt{1+(\omega T)^2}\right)$	$20\log\left(\frac{1}{1 + j\omega T}\right) = -20\log\left(\sqrt{1+(\omega T)^2}\right)$
Phase angle ϕ	$\tan^{-1}(\omega T)$	$-\tan^{-1}(\omega T)$

It is known that the magnitude of a number is same as the magnitude of the reciprocal of a number with the opposite sign. Hence, in this case the step-by-step procedure for plotting the magnitude and phase plots for a simple pole are discussed.

For simple pole $(1+j\omega T)^{-1}$

Magnitude plot

(i) The magnitude of the simple pole in dB is $20\log\left(\frac{1}{1+j\omega T}\right) = -20\log\left(\sqrt{1+(\omega T)^2}\right)$.

(ii) Consider two cases:

Case 1: For low frequencies $\omega << \frac{1}{T}$

Magnitude = $-20\log 1 = 0\,dB$

Slope = 0 dB / decade

Case 2: For high frequencies $\omega \gg \frac{1}{T}$

Magnitude = $-20\log \omega T\ dB$

The slope of the line is determined as follows:

At $\omega = \frac{1}{T}$, magnitude = 0 dB

At $\omega = \frac{10}{T}$, magnitude = –20 dB

Hence, the slope of the line is –20dB/decade.

(iii) Hence, the magnitude plot of the simple pole can be approximated using the two straight-line asymptotes

(a) Straight line at 0 dB $\left(0 < \omega < \frac{1}{T}\right)$.

(b) Straight line with the slope of –20dB/decade $\left(\frac{1}{T} < \omega < \infty\right)$

The approximate magnitude plot of the simple pole is shown in Fig. 8.15(a).

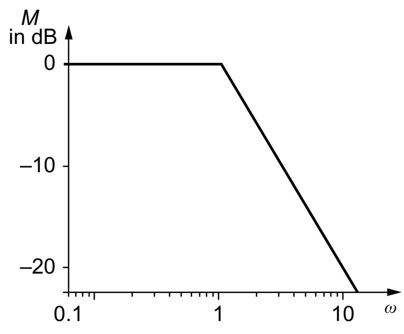


Fig. 8.15 (a) | Approximate magnitude plot for $(1+j\omega T)^{-1}$

Corner frequency

The frequency at which two asymptotes meet is called corner frequency or break frequency. In this case, the corner frequency is at $\omega = \frac{1}{T}$. The corner frequency divides the frequency response curve of the system as low-frequency region and high-frequency region.

Error in the magnitude plot

The magnitude plot obtained using the above three steps is an approximated magnitude curve. The actual magnitude curve can be obtained by substituting different values of ω in magnitude equation of simple pole as given in Table 8.9.

Table 8.9 | Actual magnitude curve

Frequency, ω	0	$\frac{1}{T}$	$\frac{10}{T}$	$\frac{100}{T}$
Magnitude in dB, $-20\log\left(\sqrt{1+(\omega T)^2}\right)$	0	−3.01	−20.04	−40.00

Hence, an error exists in the magnitude plot. The error in the magnitude plot is maximum at the corner frequency and the value of the error is obtained by

$$\text{Error} = \left\{-20\log\left(\sqrt{1+(\omega T)^2}\right) - \left(-20\log(1)\right)\right\}_{\omega=\frac{1}{T}} = -3.03 \text{ dB}$$

The values of error at different frequencies are given in Table 8.10.

Table 8.10 | Error versus Frequency ω

Frequency, ω	$\frac{0.5}{T}$	$\frac{1}{T}$	$\frac{2}{T}$
Error	−1 dB	−3 dB	−1 dB

The magnitude plot with the approximate curve and actual curve is shown in Fig. 8.15(b).

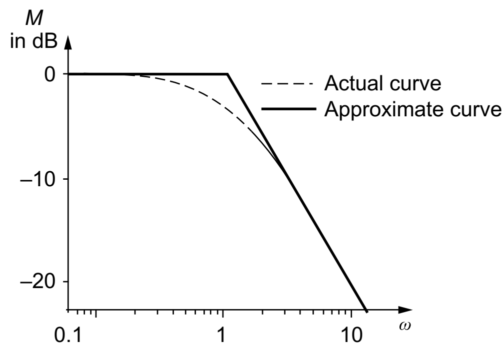


Fig. 8.15 (b) | Magnitude plot for $(1 + j\omega T)^{-1}$

Phase plot

(i) The phase angle for a simple pole is $-\tan^{-1}(\omega T)$.

(ii) The phase plot for a simple pole is obtained by substituting different values of ω.

At $\omega = 0$, phase angle = 0°

At $\omega = \dfrac{1}{T}$, phase angle = –45°

At $\omega = \infty$, phase angle = –90°

Hence, the phase plot of a simple pole is shown in Fig. 8.15(c).

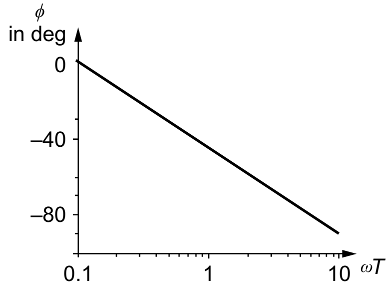


Fig. 8.15 (c) | Approximate phase plot for $(1 + j\omega T)^{-1}$

The phase plot of a simple pole is skew symmetric about the inflection point at phase angle = –45°.

Some errors exist in the phase plot when the actual value is approximated and the approximate phase angle value and actual phase angle at different frequencies are given in Table 8.11.

Table 8.11 | Exact and approximate phase values for simple zero

Frequency, ω	$\frac{0.1}{T}$	$\frac{1}{T}$	$\frac{2}{T}$	$\frac{10}{T}$
Actual phase angle (in deg.)	–5.7°	–45°	–64°	–84.3°
Approximate phase angle (in deg.)	–0°	–45°	–60°	–90°

The phase plot with the approximate curve and actual curve are shown in Fig. 8.15(d).

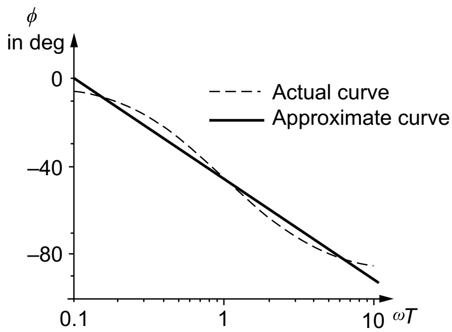


Fig. 8.15 (d) | Phase plot for a simple pole

For simple zero $\left(1+j\omega T\right)$

The reciprocal of a simple pole is a simple zero. Hence, the magnitude and phase plots of a simple zero are just the mirror image of the plots of simple pole. The magnitude and phase plots of a simple zero with the actual curve and approximated curve are shown in Figs. 8.15(e) and (f) respectively.

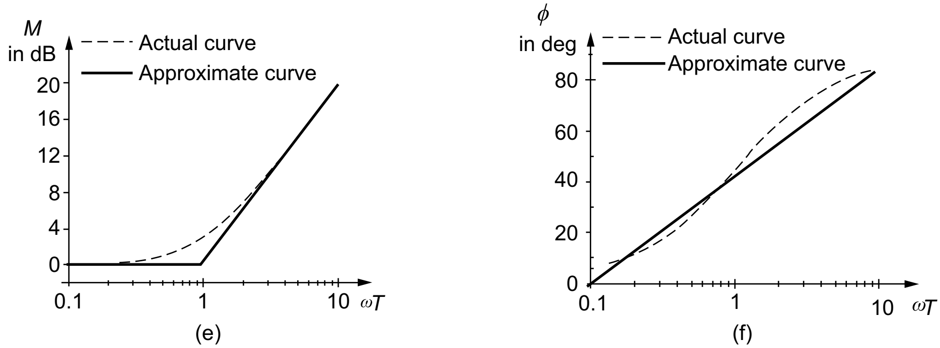


Fig. 8.15 | Bode plot for simple zero $(1+j\omega T)^{-1}$

If the number of simple poles and simple zeros of same value existing on the system is p, then the magnitude and phase plots are obtained as

(i) Corner frequency is the same i.e., $\omega=\dfrac{1}{T}$.
(ii) Low-frequency asymptote is a horizontal straight line at $0\,\text{dB}$.
(iii) High-frequency asymptote is a straight line with the slope of $-20\times p$ dB/decade or $20\times p$ dB/decade.
(iv) The phase angle in the phase plot of the system is p times the phase angle of the simple pole or simple zero at each frequency.
(v) Error in each plot is p times the error of simple pole or simple zero.

Factor 4: Complex zero $\left(1+j2\xi u_n-u_n^2\right)$ and complex pole $\left(1+j2\xi u_n-u_n^2\right)^{-1}$

The details for plotting the magnitude and phase plots of complex pole and complex zero is given in Table 8.12.

Table 8.12 | Magnitude and phase angle for complex zero and complex pole

Index	Complex zero	Complex pole
Magnitude (in dB)	$20\log\left(1+j2\xi u_n-u_n^2\right)$ $=20\log\left(\sqrt{\left(1-u_n^2\right)^2-\left(2\xi u_n\right)^2}\right)$	$20\log\left(\dfrac{1}{1+j2\xi u_n-u_n^2}\right)$ $=20\log\left(\sqrt{\left(1-u_n^2\right)^2-\left(2\xi u_n\right)^2}\right)$
Phase angle ϕ	$\tan^{-1}\left(\dfrac{2\xi u_n}{1-u_n^2}\right)$	$-\tan^{-1}\left(\dfrac{2\xi u_n}{1-u_n^2}\right)$

In Table 8.12, ξ is the damping ratio and $u_n = \frac{\omega}{\omega_n}$.

It is known that the magnitude of a number is same as the magnitude of the reciprocal of that number with opposite sign. The magnitude and phase plots of the quadratic factor depend on the corner frequency and the damping factor. Hence, in this case, the step-by-step procedure for plotting the magnitude and phase plots for simple poles has been discussed.

8.14.1 Effect of Damping Ratio ξ

When the damping ratio is greater than 1, i.e., $\xi > 1$, the quadratic factor can be written as the product of two first-order factors with real poles. When the damping ratio is within a range, i.e., $0 < \xi < 1$, the quadratic factor can be written as the product of two complex conjugate factors. The asymptotic approximation of the plots is not accurate for this factor with lower values of ξ.

Complex pole $\left(1 + j2\xi u_n - u_n^2\right)^{-1}$

The step-by-step procedure for plotting the magnitude and phase plots for a complex pole is discussed below.

Magnitude plot:

(i) The magnitude of the simple pole in dB is

$$20\log\left(\frac{1}{\left(1+j2\xi u_n - u_n^2\right)}\right) = -20\log\left(\sqrt{\left(1-u_n^2\right)^2 + \left(2\xi u_n\right)^2}\right) \tag{8.26}$$

(ii) Consider two cases:

Case 1: For lower frequencies $\omega << \omega_n$,

Magnitude = $-20\log 1 = 0$ dB

Slope = 0 dB/decade.

Case 2: For higher frequencies $\omega >> \omega_n$,

$$\text{Magnitude} = -20\log\left(\frac{\omega}{\omega_n}\right)^2 dB = -40\log\left(\frac{\omega}{\omega_n}\right) dB$$

The slope of the line is determined as

At $\omega = \omega_n$, magnitude = 0 dB

At $\omega = 10\omega_n$, magnitude = −40 dB

Hence, the slope of the line is −40 dB/decade.

(iii) Hence, the magnitude plot of the simple pole can be approximated using the two straight-line asymptotes

(a) Straight line at 0 dB $\left(0 < \omega < \omega_n\right)$.

(b) Straight line with the slope of −40dB/decade $\left(\omega_n < \omega < \infty\right)$.

The approximate magnitude plot of the complex pole is shown in Fig. 8.16(a).

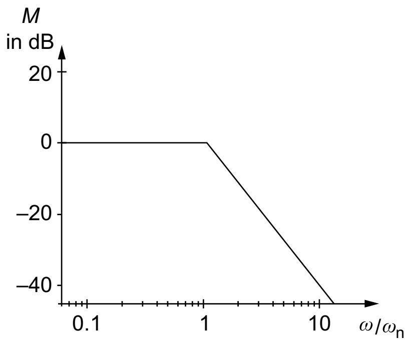


Fig. 8.16 (a) | Approximate magnitude plot for $\left(1+j2\xi u_n - u_n^2\right)^{-1}$

At the corner frequency i.e., $\omega = \omega_n$, the resonant peak occurs and its magnitude depends on ξ. Also, error exists in the approximation of two asymptotes and the magnitude of the error depends inversely on damping ratio ξ.

The actual magnitude curve can be obtained by substituting different values of ω in Eqn. (8.26) as given in Table 8.13.

Table 8.13 | Actual magnitude curve

Frequency ω	0	ω_n	$10\omega_n$
Magnitude in dB, $-20\log\left(\sqrt{\left(1-u_n^2\right)^2+\left(2\xi u_n\right)^2}\right), u_n = \dfrac{\omega}{\omega_n}$	0	$-20\log(2\xi)$	$-20\log\left(\sqrt{(99)^2+(20\xi)^2}\right)$

Hence, error always exists in the magnitude plot of the system. The error at $\omega = \omega_n$ is calculated as given below:

$$\text{Error} = \text{actual value} - \text{approximate value}$$
$$= -20\log\left(\sqrt{4\xi^2}\right) - 0 = -20\log(2\xi)\ \text{dB}$$

The error for different values of ξ at $\omega = \omega_n$ is given in Table 8.14. Similarly, error will vary depending on ξ and ω.

Table 8.14 | Error versus damping ratio at $\omega = \omega_n$

Damping ratio ξ	0.1	0.3	0.5	0.7	0.9	1.0
Error in dB	13.98	4.44	0	–2.92	–5.10	–6.02

The magnitude plot with the actual and asymptotic value is shown in Fig. 8.16(b).

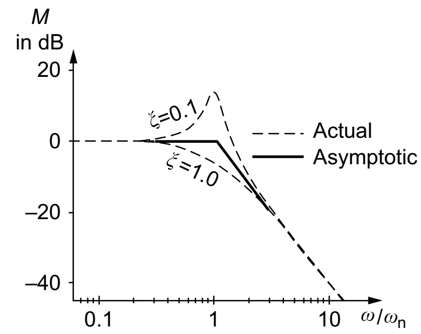


Fig. 8.16 (b) | Magnitude plot for $\left(1+j2\xi u_n-u_n^2\right)^{-1}$

Phase plot

(i) The phase angle for the simple pole is given by

(a) $\phi=-\tan^{-1}\left(\dfrac{2\xi u_n}{1-u_n^2}\right)$ for $\omega<\omega_n$

(b) $\phi=-\tan^{-1}\left(\dfrac{2\xi u_n}{1-u_n^2}\right)-180^\circ$ for $\omega>\omega_n$

(ii) The phase plot for the simple pole is obtained by substituting different values of ω.
At $\omega=0$, phase angle = 0°
At $\omega=\omega_n$, phase angle = –90°
At $\omega=\infty$, phase angle = –180°

Hence, the phase plot of a complex pole is shown in Fig. 8.16(c).

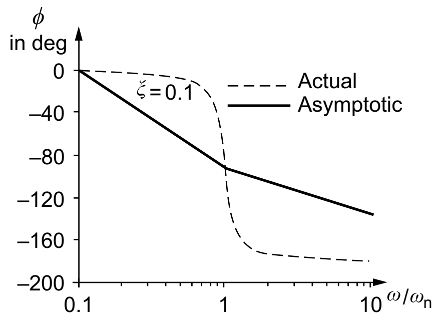


Fig. 8.16 (c) | Phase plot for $\left(1+j2\xi u_n-u_n^2\right)^{-1}$

The phase plot of a simple pole is skew symmetric about the inflection point at phase angle = –90°.

Complex zero $\left(1+j2\xi u_n-u_n^2\right)$

Plots for a quadratic zero are mirror images of those for a pole. The magnitude and phase plots for a complex zero are shown in Figs. 8.16(d) and (e) respectively.

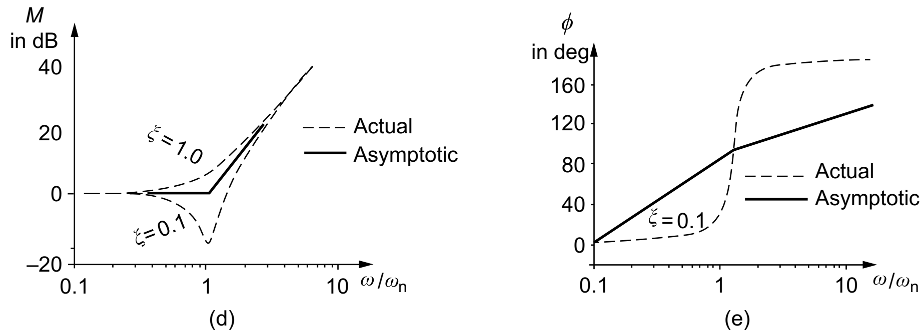


Fig. 8.16 | Bode plot for complex zero $\left(1+j2\xi u_n - u_n^2\right)$

It is noted that in the quadratic zero when the damping ratio is less than 0.7 (i.e., $\xi < 0.7$), a dip has to be drawn at frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$ with the amplitude of $20\log\left(2\xi\sqrt{1-\xi^2}\right)$.

Similarly, for quadratic pole when the damping ratio is less than 0.7 (i.e., $\xi < 0.7$), a peak has to be drawn at frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$ with the amplitude of $-20\log\left(2\xi\sqrt{1-\xi^2}\right)$.

Factor 5: Transportation lag, $e^{-(j\omega)T}$

Consider a system with loop transfer function as

$$G(s)H(s) = e^{-sT}$$

In frequency domain, the loop transfer function is given by

$$G(j\omega)H(j\omega) = e^{-(j\omega)T}$$

where T is the time delay in seconds.

Therefore, $$G(j\omega)H(j\omega) = \cos(\omega T) - j\sin(\omega T)$$

Hence, the magnitude and phase angle of the factor is given by

$$\left|G(j\omega)H(j\omega)\right| = \sqrt{\cos^2(\omega T) + \sin^2(\omega T)} = 1$$

and $$\phi = \angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{-\sin(\omega T)}{\cos(\omega T)}\right) = -\omega T.$$

Since the phase angle obtained is in the unit of radians, the phase angle can be calculated in degrees as

$$\phi = \angle G(j\omega)H(j\omega) = -\omega T \times \frac{180^\circ}{\pi} = -57.32^\circ \times \omega T.$$

Table 8.15 shows the phase angle for different values of ω.

Table 8.15 | Phase angle versus frequency ω

Frequency ω in rad/sec	0	$\frac{1}{T}$	∞
Phase angle (ϕ in deg.)	0°	–57.32°	$-\infty$

It is clear that for the transportation lag factor, the magnitude plot is zero for all the values of frequency ω and the phase plot is varying linearly with the frequency ω.

Table 8.16 shows the different factors with its corner frequency, slope of the magnitude curve, values of the magnitude in dB and phase angle in degrees.

Table 8.16 | Possible factors present in a given system

Factor	**Corner frequency (rad/sec)**	**Slope of magnitude (dB/decade)**	**Value of the magnitude (in dB)**	**Phase angle (in deg.)**
K	–	0	0	0
$(j\omega)^N$	–	$20\times N$	$N\times 20\log(\omega)$	$90°\times N$
$(j\omega)^{-N}$	–	$20\times N$	$-N\times 20\log(\omega^N)$	$-90°\times N$
$1+j\omega T$	$\frac{1}{T}$	20	$20\log\left(\sqrt{1+(\omega T)^2}\right)$	$\tan^{-1}\omega T$
$(1+j\omega T)^{-1}$	$\frac{1}{T}$	–20	$-20\log\left(\sqrt{1+(\omega T)^2}\right)$	$-\tan^{-1}\omega T$
$(1+j2\xi u_n-u_n^2)$	ω_n	40	$20\log\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}\right)$	$\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right], \omega<\omega_n$, $\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]+180°, \omega>\omega_n$

(*Continued*)

Table 8.16 | (Continued)

Factor	Corner frequency (rad/sec)	Slope of magnitude (dB/decade)	Value of the magnitude (in dB)	Phase angle (in deg.)
$\left(1+j2\xi u_n - u_n^2\right)^{-1}$	ω_n	–40	$-20\log\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}\right)$	$-\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right], \omega < \omega_n$ $-\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]-180°, \omega < \omega_n$
$e^{-(j\omega)T}$	–	–	1	$-57.32\times\omega T$

8.15 Constructing the Bode Plot for a Given System

Consider the loop transfer function as

$$G(s)H(s) = \frac{K(1+sT_1)}{s^2(1+sT_2)\left(1+\frac{s^2}{\omega_n^2}+2\xi\frac{s}{\omega_n}\right)}$$

Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{K(1+j\omega T_1)}{(j\omega)^2(1+j\omega T_2)\left(1-\frac{\omega^2}{\omega_n^2}+2\xi\frac{j\omega}{\omega_n}\right)}$$

Now, the corner frequencies of the given system are

$$\omega_{c1} = \frac{1}{T_1},\ \omega_{c2} = \frac{1}{T_2} \text{ and } \omega_{c3} = \omega_n$$

The phase plot is independent of the corner frequency. Since the magnitude of each term present in the transfer function is calculated in the increasing order of corner frequency, the magnitude plot depends on the corner frequency.

Hence, the relation between the corner frequencies is considered as $\omega_{c1} < \omega_{c2} < \omega_{c3}$.

8.15.1 Construction of Magnitude Plot

For each term present in the loop transfer function, the slope of the magnitude curve varies. To combine the slope of the different terms present in the given system, the following steps are followed:

(i) Slope of each term is calculated.
(ii) Change in slope from one term to another in the order of increasing corner frequencies has to be determined using Table 8.17. Usually, the constant term along with the pole or zero at the origin is taken as a term.

Table 8.17 | Determination of change in slope at different corner frequencies

Term	Corner frequency ω	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\dfrac{K}{(j\omega)^2}$	–	–40 (since -20×2)	–
$1+j\omega T_1$	$\omega_{c1}=\dfrac{1}{T_1}$	+20	–40 + 20 = –20
$1+j\omega T_2$	$\omega_{c2}=\dfrac{1}{T_2}$	–20	–20 – 20 = –40
$\left(1-\dfrac{\omega^2}{\omega_n^2}+2\xi\dfrac{j\omega}{\omega_n}\right)$	$\omega_{c3}=\omega_n$	–40	–40 – 40 = –80

(iii) Two different frequencies other than the corner frequencies are chosen arbitrarily as lower frequency value ω_l and higher frequency value ω_h such that the relation between different frequencies is

$$\omega_l<\omega_{c1}<\omega_{c2}<\omega_{c3}<\omega_h$$

(iv) Determine the gain at different frequencies in increasing order. In this case, there are five different frequencies and have only four terms. Hence, the magnitude of the first term in Table 8.17 is calculated at the first two frequencies. In this case, the magnitude of $\dfrac{K}{(j\omega)^2}$ is calculated at ω_l and ω_{c1}. The calculation of magnitude at other frequencies ω_{c2}, ω_{c3} and ω_h is done using the formula

$$\text{Gain at } \omega_i=\left\{\left[\text{Change in slope from } \omega_{i-1} \text{ to } \omega_i\right]\times\log\left(\frac{\omega_i}{\omega_{i-1}}\right)\right\}+\text{gain at } \omega_{i-1}$$

where $i=3,4,\ldots$.

The gain at different frequencies are tabulated as given in Table 8.18.

Table 8.18 | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{K}{(j\omega)^2}$	$\omega_1 = \omega_l$	–	$A_1 = -20\log(\omega_l^2)$
$\frac{K}{(j\omega)^2}$	$\omega_2 = \omega_{c1}$	–	$A_2 = -20\log(\omega_2^2)$
$1 + j\omega T_1$	$\omega_3 = \omega_{c2}$	–20	$A_3 = \left[-20 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2$
$1 + j\omega T_2$	$\omega_4 = \omega_{c3}$	–40	$A_4 = \left[-40 \times \log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3$
$\left(1 - \frac{\omega^2}{\omega_n^2} + 2\xi\frac{j\omega}{\omega_n}\right)$	$\omega_5 = \omega_h$	–80	$A_5 = \left[-80 \times \log\left(\frac{\omega_5}{\omega_4}\right)\right] + A_4$

(v) Any term in the estimation of magnitude has to be included only after its corresponding corner frequency is crossed.

(vi) Using Table 8.18, the magnitude plot for the given system can be drawn.

8.15.2 Construction of Phase Plot

The construction of phase plot is much simpler when compared to construction of magnitude plot.

The phase angle of the given system as a function of frequency ω has to be determined as

$$\phi = -180^\circ + \tan^{-1}(\omega T)_1 - \tan^{-1}(\omega T_2) - \tan^{-1}\left(\frac{2\xi\frac{\omega}{\omega_1}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) \text{ for } \omega \le \omega_n$$

$$\text{and } \phi = -180^\circ + \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}\left(\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) - 180^\circ \text{ for } \omega > \omega_n$$

The values of ϕ at different frequencies are calculated including the frequencies mentioned in Table 8.18 and tabulated.

8.16 Flow Chart for Plotting Bode Plot

The flow chart for plotting the Bode plot (both magnitude and phase angle plot) for a given system is shown in Fig. 8.17.

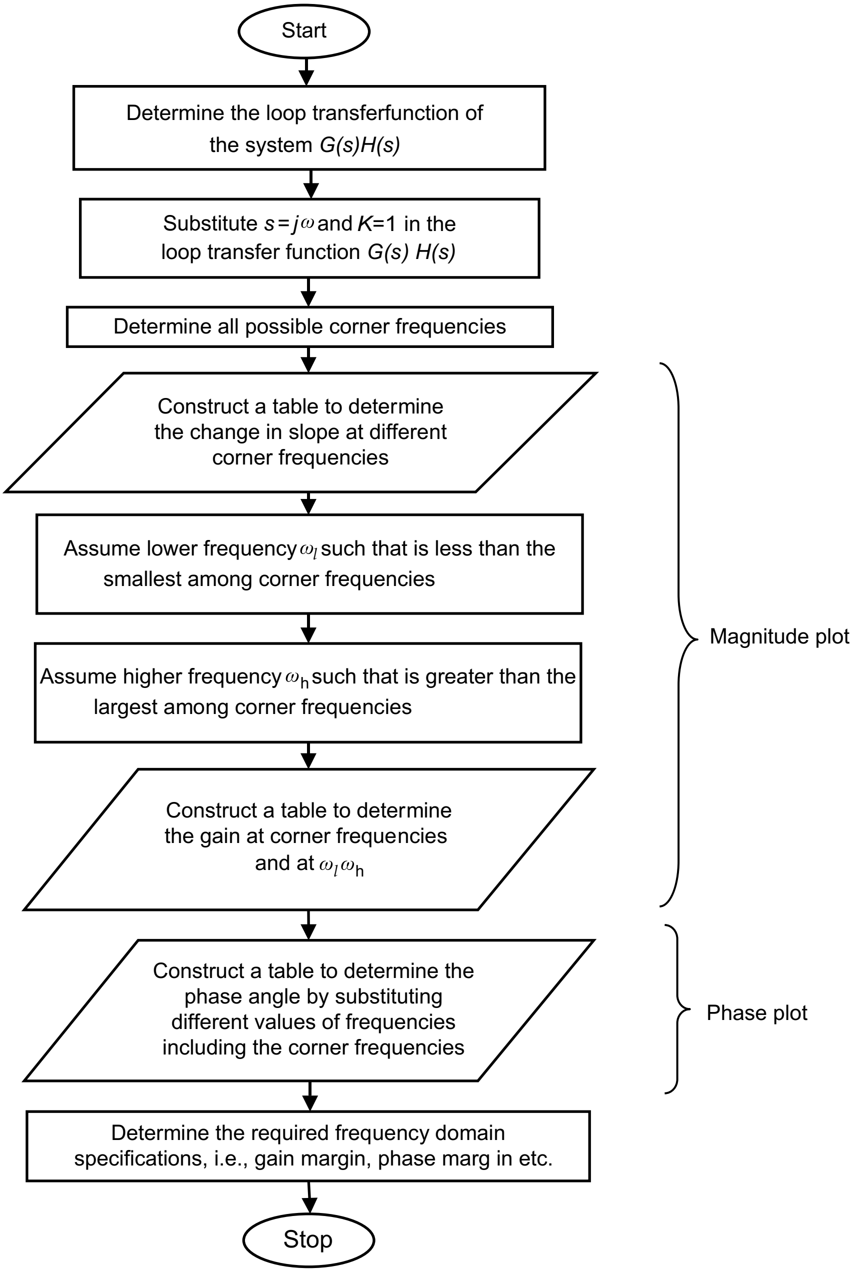


Fig. 8.17 | Flow chart for plotting the Bode plot for a system

8.17 Procedure for Determining the Gain *K* from the Desired Frequency Domain Specifications

The flow chart for determining the gain *K* for the desired ω_{gc} is given in Fig. 8.18(a).

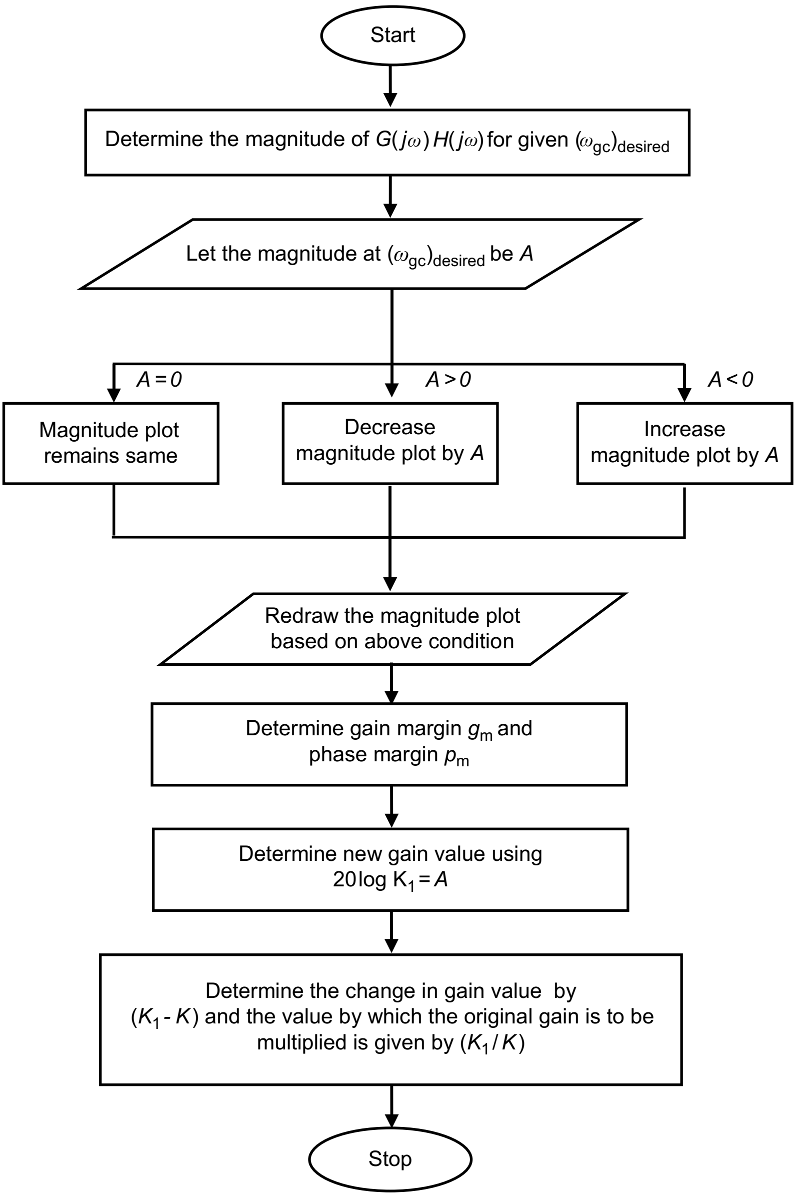


Fig. 8.18 (a) | Determination of *K* for the desired ω_{gc}

The flow chart for determining the gain K for the desired ω_{gc} is given in Fig. 8.18(b).

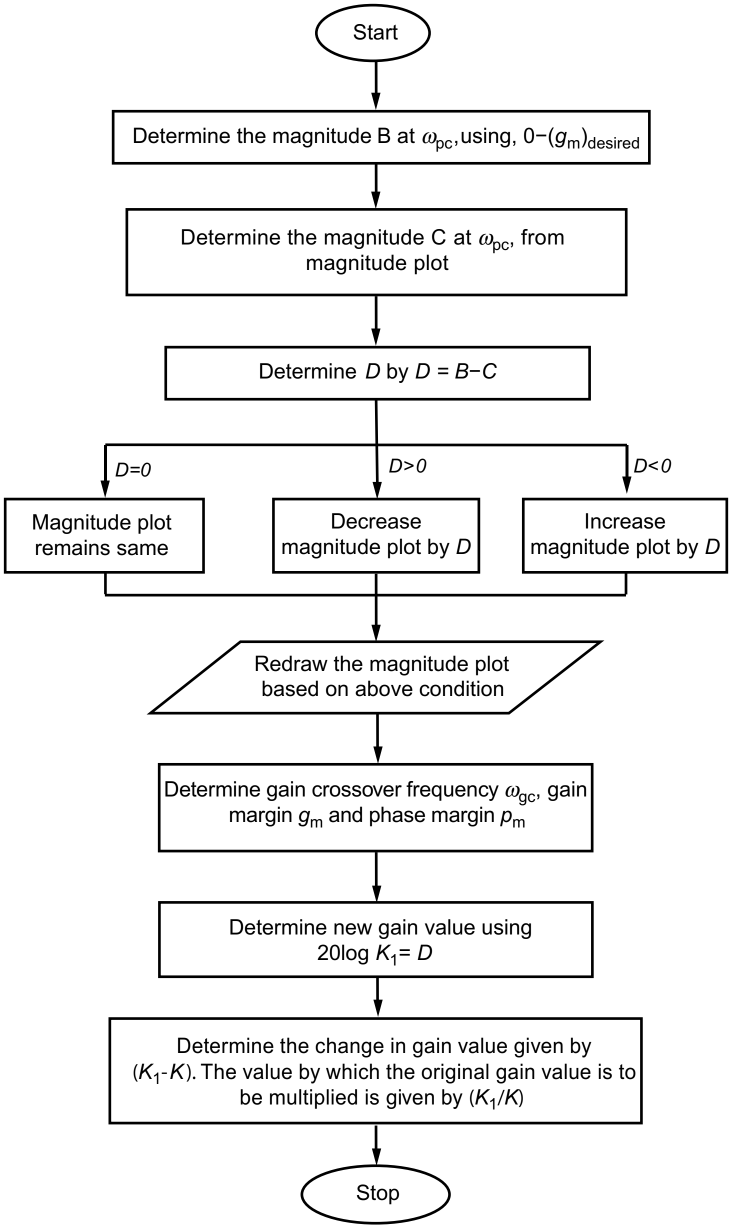


Fig. 8.18 (b) | Determination of K for the desired g_m

The flow chart for determining the gain K_1 for the desired p_m is shown in Fig. 8.18(c).

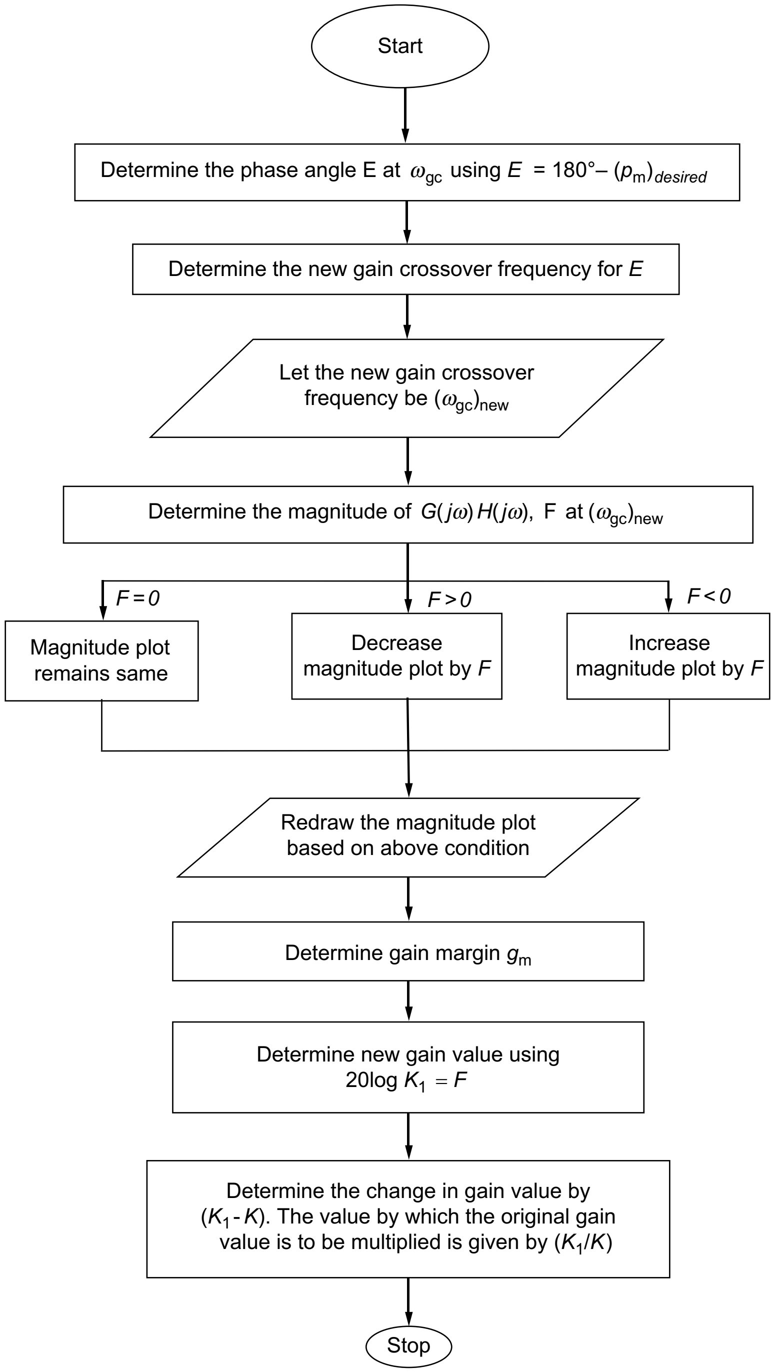


Fig. 8.18 (c) | Determination of K for the desired p_m

8.18 Maximum Value of Gain

The gain K has an effect only on the magnitude plot of a loop transfer function of a system. The magnitude plot of a loop transfer function can be shifted vertically upwards or vertically downwards depending on the gain K. If there is an increase in gain value from the original value, then the magnitude plot gets shifted vertically upward and if there is a decrease in gain from the original value, then the magnitude plot gets shifted vertically downward. The gain value cannot be increased or decreased infinitely. There exists some limits in doing so. The gain can be increased or decreased till the magnitude reaches 0 dB at phase crossover frequency. The new gain can be determined as

$$20\log K = A$$

where A is the value by which the magnitude plot has been raised or lowered to reach 0 dB at ω_{gc}.

8.19 Procedure for Determining Transfer Function from Bode Plot

The step-by-step procedure for determining the transfer function of a system from its magnitude plot are:

Step 1: Determine the number of corner frequencies existing in the system (the point at which the slope of the system changes). If there exist n different slopes for the given system, then there exist $(n-1)$ different corner frequencies.

Step 2: Determine the initial slope and final slope of the system.

Step 3: Construct a table for the given system in the format given below to determine the different factors present in the system.

Initial slope (dB/decade)	Final slope (dB/decade)	Change in slope = Final slope–initial slope (dB/decade)	Factor present in the transfer function	Corner frequency ω, at which the change in slope occurs

Step 4: Determine the gain K, corner frequencies, unknown frequencies and unknown magnitude values using the equation of a straight line, $y = mx + c$, where y is the magnitude in dB, m is the slope of the line in dB/dec, x is the logarithmic value of frequency ω, i.e., $\log\omega$ in rad/sec and c is the constant.

8.20 Bode Plot for Minimum and Non-Minimum Phase Systems

Consider the loop transfer function of a minimum phase transfer function as

$$\left(G(s)H(s)\right)_{\text{min}} = \frac{1+sT}{1+sT_1} \tag{8.27}$$

and the loop transfer function of a non-minimum phase system as

$$\left(G(s)H(s)\right)_{\text{non-min}} = \frac{1-sT}{1+sT_1} \tag{8.28}$$

where $T_1 > T > 0$.
The pole-zero plots of the minimum and non-minimum systems are shown in Fig. 8.19.

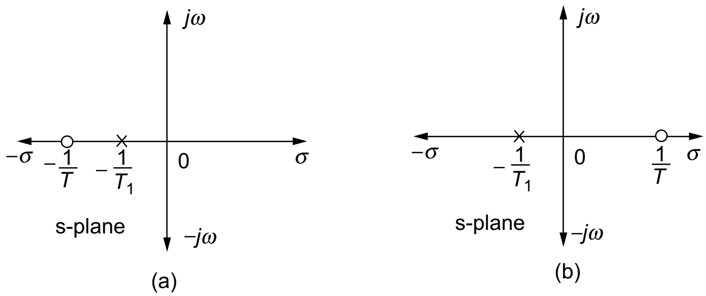


Fig. 8.19 | Pole-zero plots of (a) minimum and (b) non-minimum phase systems

The magnitude and phase angle of the transfer function can be obtained by substituting $s = j\omega$ in Eqs. (8.27) and (8.28) as

$$\left|G(j\omega)H(j\omega)\right|_{\min} = \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\omega T_1)^2}} \text{ and } \left|G(s)H(s)\right|_{\text{non-min}} = \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\omega T_1)^2}} \tag{8.29}$$

$$\begin{aligned} \left(\angle G(j\omega)H(j\omega)\right)_{\min} &= \tan^{-1}(\omega T) - \tan^{-1}(\omega T_1) \\ \text{and } \left(\angle G(j\omega)H(j\omega)\right)_{\text{non-min}} &= -\tan^{-1}(\omega T) - \tan^{-1}(\omega T_1) \end{aligned} \tag{8.30}$$

From Eqs. (8.29) and (8.30), it is clear that the magnitudes of minimum phase system and non-minimum phase system are same, whereas the phase angles of minimum phase system and non-minimum phase system are different. The magnitude and phase plots of minimum and non-minimum phase systems are shown in Fig. 8.20.

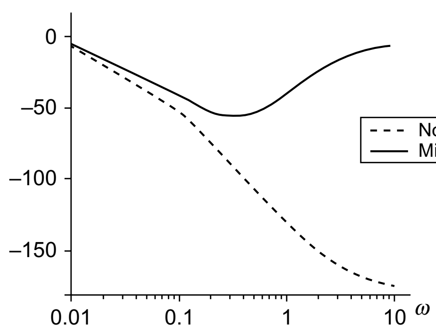


Fig. 8.20 | Phase plot of minimum and non-minimum phase systems

Example 8.12 **The loop transfer function of a system is given by $G(s)H(s)=\dfrac{80}{s(s+2)(s+20)}$. Sketch the Bode plot and determine the following frequency domain specifications (i) gain margin g_m, (ii) phase margin p_m, (iii) gain crossover frequency ω_{gc} and (iv) phase crossover frequency ω_{pc}. Also, comment on the stability of the system.**

Solution:

(i) The loop transfer function of a system is given by

$$G(s)H(s)=\frac{80}{s(s+2)(s+20)}=\frac{2}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{20}\right)}$$

(ii) Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega)=\frac{2}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{20}\right)}$$

(iii) Two corner frequencies exist in the given loop transfer function are:

$$\omega_{c1}=\frac{1}{\frac{1}{2}}=2 \text{ rad/sec and } \omega_{c2}=\frac{1}{\frac{1}{20}}=20 \text{ rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies are given in Table E8.12(a).

Table E8.12(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\frac{2}{(j\omega)}$	–	−20	–
$\frac{1}{\left(1+\frac{j\omega}{2}\right)}$	$\omega_{c1}=2$	−20	$-20-20=-40$
$\frac{1}{\left(1+\frac{j\omega}{20}\right)}$	$\omega_{c2}=20$	−20	$-40-20=-60$

(v) Assume the lower frequency as $\omega_l = 1$ rad/sec and higher frequency as $\omega_h = 30$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.12(b).

Table E8.12(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain A_i
$\frac{2}{(j\omega)}$	$\omega_1 = \omega_l = 1$	–	$A_1 = 20\log\left(\frac{2}{\omega_1}\right) = 6.02$ dB
$\frac{2}{(j\omega)}$	$\omega_2 = \omega_{c1} = 2$	–	$A_2 = 20\log\left(\frac{2}{\omega_2}\right) = 0$ dB
$\frac{1}{\left(1+\frac{j\omega}{2}\right)}$	$\omega_3 = \omega_{c2} = 20$	–40	$A_3 = \left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -40$ dB
$\frac{1}{\left(1+\frac{j\omega}{20}\right)}$	$\omega_4 = \omega_h = 30$	–60	$A_4 = \left[-60\times\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -50.56$ dB

(vii) The magnitude plot of the given system is plotted using Table E8.12(b) and is shown in Fig. E8.12(a).

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -90^{\circ} - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{20}\right)$$

(ix) The phase angle at different frequencies are obtained by using the above equation and the values are tabulated as given in Table E8.12(c).

Table E8.12(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	$-\tan^{-1}\left(\frac{\omega}{2}\right)$	$-\tan^{-1}\left(\frac{\omega}{20}\right)$	Phase angle ϕ (in deg.)
0. 2	–5.7°	–0.57°	–96.27°
2	–4.5°	–5.7°	–140.7°
8	–75.96°	–21.8°	–187.76°
10	–78.69°	–26.56°	–195.29°
20	–84.28°	–45°	–219.28°
40	–87.13°	–63.43°	–240.58°
∞	–90°	–90°	–270°

(x) The phase plot of the given system is plotted using Table E8.12(c) and is shown in Fig. E8.12(b).

Hence, the magnitude and phase plots for the loop transfer function of the system are shown in Fig. E8.12.

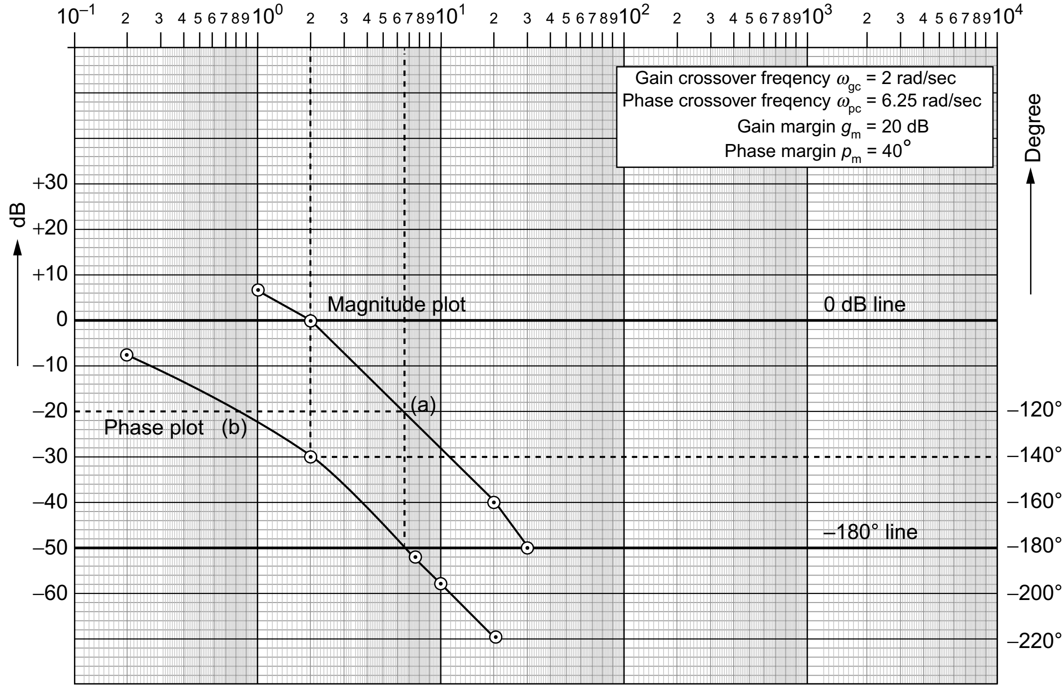


Fig. E8.12

(xi) The frequency domain specifications for the given system can be obtained by drawing two horizontal lines across the two plots of the given system (one across the magnitude plot at 0 dB and another across the phase plot at −180°).

From the definition of gain crossover frequency, phase crossover frequency, gain margin, phase margin and from the graph shown in Fig. E8.12, the frequency domain specifications values are obtained as

Gain crossover frequency, $\omega_{gc} = 2$ rad/sec

Phase crossover frequency, $\omega_{pc} = 6.25$ rad/sec

Gain margin, $g_m = 0-(-20) = 20$ dB

Phase margin, $p_m = 180° + (-140°) = 40°$

(xii) The stability of the system can be determined based on either the frequency or the gain.

Based on frequency:

Since $\omega_{gc} < \omega_{pc}$, the system is stable.

Based on frequency domain specifications:

Since the gain margin and phase margin are greater than zero and $\omega_{gc} < \omega_{pc}$, the system is stable.

Example 8.13 **The loop transfer of a given system is given by $G(s)H(s) = \dfrac{K}{s(1+0.1s)(1+s)}$. Sketch the Bode plot for given system. In addition, (i) determine the gain K and the phase margin so that gain margin is +30 dB and (ii) determine the gain K and the gain margin so that phase margin is $30°$.**

Solution:

(i) Given $G(s)H(s) = \dfrac{K}{s(1+0.1s)(1+s)}$.

(ii) Substituting $s = j\omega$ and $K = 1$, we obtain

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+0.1j\omega)(1+j\omega)}$$

(iii) The two corner frequencies in the given system are

$$\omega_{c1} = \frac{1}{0.1} = 10 \text{ rad/sec and } \omega_{c2} = \frac{1}{1} = 1 \text{ rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.13(a).

Table E8.13(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\dfrac{1}{(j\omega)}$	–	–20	–
$\dfrac{1}{(1+j\omega)}$	$\omega_{c1} = 1$	–20	$-20 - 20 = -40$
$\dfrac{1}{(1+0.1j\omega)}$	$\omega_{c2} = 10$	–20	$-40 - 20 = -60$

(v) Assume the lower frequency as $\omega_l = 0.1$ rad/sec and higher frequency as $\omega_h = 20$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.13(b).

Table E8.13(b) | Gain at different frequencies

Term	**Frequency**	**Change in slope (in dB)**	**Gain, A_i**
$\frac{1}{(j\omega)}$	$\omega_1 = \omega_l = 0.1$	-	$A_1 = 20\log\left(\frac{1}{\omega_1}\right) = 20\text{ dB}$
$\frac{1}{(j\omega)}$	$\omega_2 = \omega_{c1} = 1$	-	$A_2 = 20\log\left(\frac{1}{\omega_2}\right) = 0\text{ dB}$
$\frac{1}{(1+j\omega)}$	$\omega_3 = \omega_{c2} = 10$	–40	$A_3 = \left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -40\text{ dB}$
$\frac{1}{(1+0.1j\omega)}$	$\omega_4 = \omega_h = 20$	–60	$A_4 = \left[-60\times\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -58.06\text{ dB}$

(vii) The magnitude plot of the given system is plotted using Table E8.13(b) and is shown in Fig. E8.13(a).

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -90^\circ - \tan^{-1}(0.1\omega) - \tan^{-1}(\omega)$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.13(c).

Table E8.13(c) | Phase angle of the system for different frequencies

Frequency ω, (rad/sec)	**$-\tan^{-1}(\omega)$**	**$-\tan^{-1}(0.1\omega)$**	**Phase angle ϕ (in deg.)**
0.1	–5.7°	–0.57°	–96.2°
1	–45°	–5.7°	–140.7°
3	–71.56°	–16.6°	–178.25°
5	–78.6°	–26.5°	–195.16°
10	–84.2°	–45°	–219.2°
∞	–90°	–90°	–270°

(x) The phase plot of the given system is plotted using Table E8.13(c) and is shown in Fig. E8.13(a).

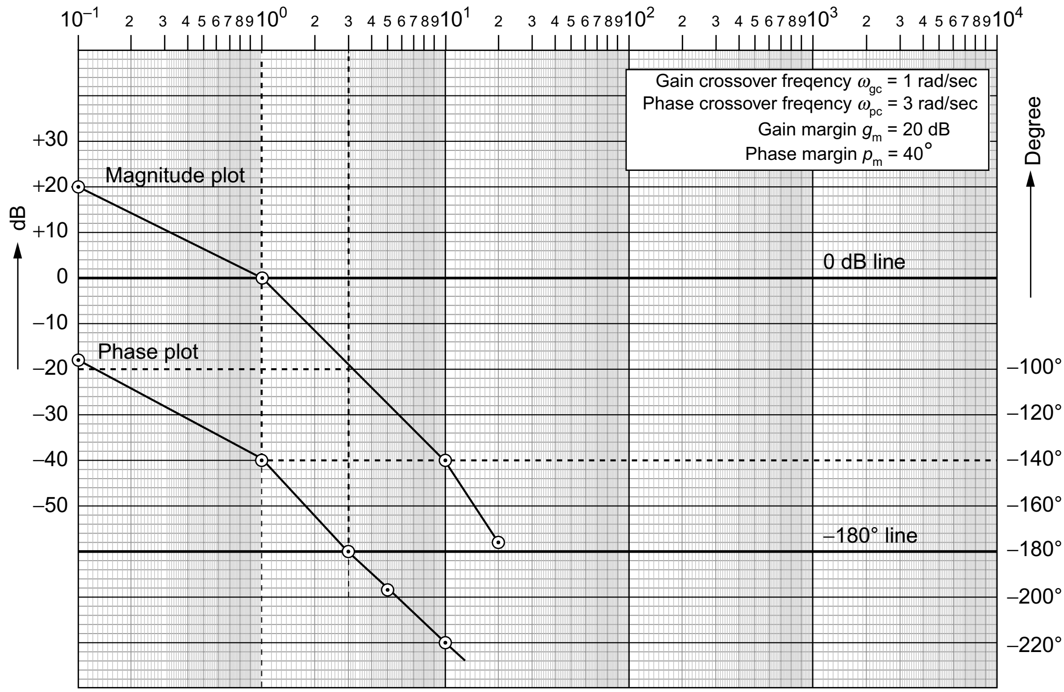


Fig. E8.13 (a)

(xi) The frequency domain specifications for the given system can be obtained by drawing two horizontal lines across the two plots of the given system (one across the magnitude plot at 0 dB and the other across the phase plot at −180°).

From the definitions of gain crossover frequency, phase crossover frequency, gain margin, phase margin and the graph shown in Fig. E8.13(a), the frequency domain specifications values are obtained as

Gain crossover frequency, $\omega_{gc} = 1$ rad/sec

Phase crossover frequency, $\omega_{pc} = 3.15$ rad/sec

Gain margin, $g_m = 20$ dB

Phase margin, $p_m = 40°$

(xii) The next step is to determine the gain K corresponding to a particular gain margin and phase margin.

(i) To determine the gain value *K* and gain margin corresponding to desired phase margin: The required phase margin for the system is $(p_m)_r = 30°$.

But it is known that, phase margin $(p_m)_r = 180° + \text{New phase angle at} (\omega_{gc})_{new}$.

Therefore, New Phase angle at $(\omega_{gc})_{new} = 180° - (p_m)_r = 180° - 30° = 150°$.

Hence, from Fig. E8.13(a), the new gain crossover frequency for a phase angle of 150° is $(\omega_{gc})_{new} = 1.3$ rad/sec.

It is proven that at the gain crossover frequency, the magnitude must be 0 dB. But from Fig. E8.13(a), the magnitude at $\left(\omega_{gc}\right)_{new} = -5\,\text{dB}$. Hence, the magnitude plot shown in Fig. E8.13(a) is raised by 5dB.

The modified Bode plot is shown in Fig. E8.13(b).

From Fig. E8.13(b), the new frequency domain specifications are

New phase margin, $\left(p_{m}\right)_{new} = 30°$

New gain margin, $\left(g_{m}\right)_{new} = 16$ dB

New gain crossover frequency, $\left(\omega_{gc}\right)_{new} = 1.3$ rad/sec

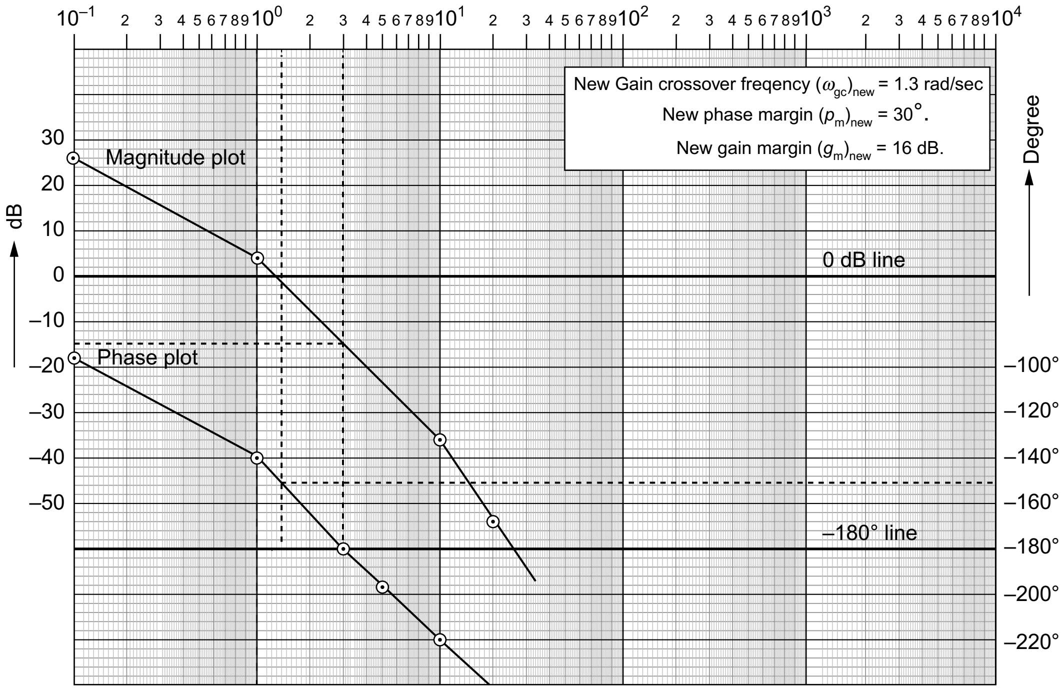
The value of K for the newly obtained Bode plot is determined as

$20\log K = B$ where B is the value in dB by which the original magnitude plot has been raised or lowered. In this case, the value of B is 5 dB.

Hence, $20\log K = 5$

$$\log K = \frac{5}{20} = 0.25$$

Therefore, the value of K is $10^{0.25} = 1.77$.



(b)

Fig. E8.13 (b)

(ii) To determine the gain K and phase margin corresponding to the gain margin

The required gain margin $(g_m)_{new} = 30$ dB.

We know that, the gain margin is obtained using the formula,

$$g_m = 0 - (\text{gain at } \omega_{pc})$$

Therefore, required gain at ω_{pc} is -30 dB.

But for the given system from the Bode plot shown in Fig. E8.13(a), the gain at ω_{gc} is -20 dB. Hence, it is clear that every point in the magnitude plot shown in Fig. E8.13(a), has to be reduced by 10 dB. The modified Bode plot is shown in Fig. E8.13(c).

From the Bode plot shown in Fig. E8.13(c), the obtained frequency domain specifications are:

New phase margin, $(p_m)_{new} = 64^\circ$

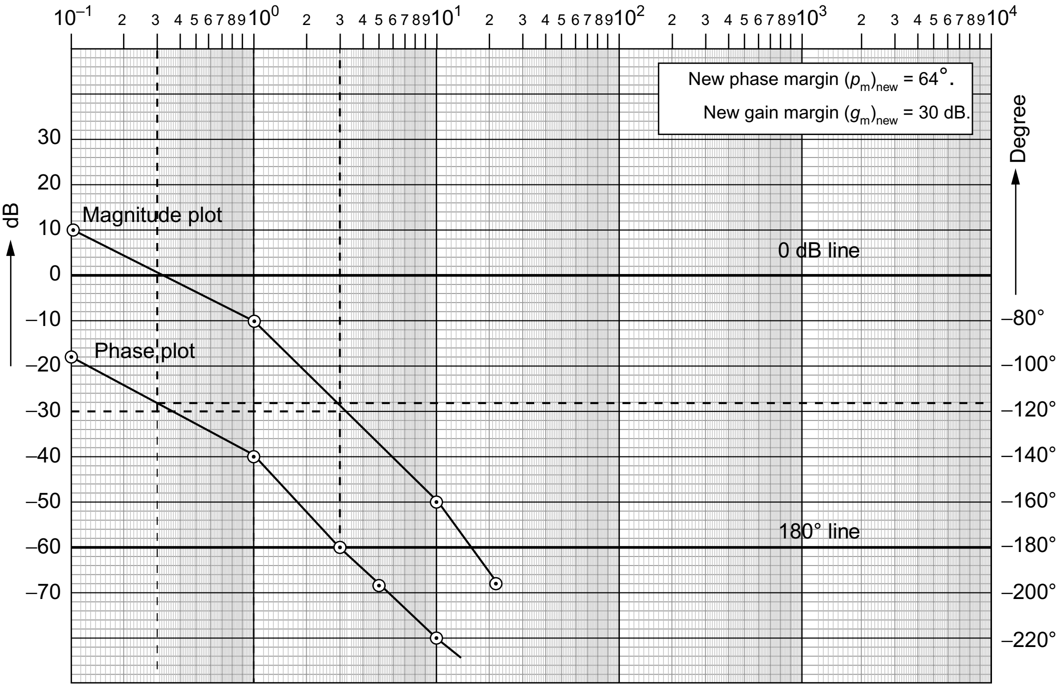
New gain margin, $(g_m)_{new} = 30$ dB

The value of K for the newly obtained Bode plot is determined as

$20 \log K = C$, where C is the value in dB by which the original magnitude plot has been raised or lowered. In this case, the value of C is -10.

Hence,
$$20 \log K = -10$$
$$\log K = \frac{-10}{20} = -0.5$$

Therefore, the value of K is $10^{-0.5} = 0.316$.



(c)

Fig. E8.13 (c)

Example 8.14 **The loop transfer function of a system is given by $G(s)H(s)=\dfrac{Ks^2}{(1+0.2s)(1+0.02s)}$. Sketch the Bode plot for the given system. Also, determine the gain value K for the gain crossover frequency 5 rad/sec.**

Solution:

(i) The loop transfer function of the system is given by $G(s)H(s)=\dfrac{Ks^2}{(1+0.2s)(1+0.02s)}$.

(ii) Substituting $s = j\omega$ and $K = 1$ in the given system, we obtain

$$G(j\omega)H(j\omega)=\frac{(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

(iii) The two corner frequencies existing in the system are

$$\omega_{c1}=\frac{1}{0.2}=5\text{ rad/sec and }\omega_{c2}=\frac{1}{0.02}=50\text{ rad/sec.}$$

(iv) The change in slope at different corner frequencies is given in Table E8.14(a).

Table E8.14(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$(j\omega)^2$	–	+40	–
$\dfrac{1}{(1+0.2j\omega)}$	$\omega_{c1}=5$	–20	+ 40 – 20 = + 20
$\dfrac{1}{(1+0.02j\omega)}$	$\omega_{c2}=50$	–20	+ 20 – 20 = 0

(v) Assume $\omega_l = 1$ rad/sec and $\omega_h = 100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.14(b).

Table E8.14(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain A_i
$(j\omega)^2$	$\omega_1=\omega_l=1$	–	$A_1=20\log\left(\omega_1^2\right)=0$ dB

(*Continued*)

Table 8.14(b) | (Continued)

Term	Frequency	Change in slope in dB	Gain A_i
$(j\omega)^2$	$\omega_2 = \omega_{c1} = 5$	–	$A_2 = 20\log(\omega_2^2) = 27.95\text{ dB}$
$\dfrac{1}{(1+0.2j\omega)}$	$\omega_3 = \omega_{c2} = 50$	+20	$A_3 = \left[+20\times\log\left(\dfrac{\omega_3}{\omega_2}\right)\right] + A_2 = 47.95\text{ dB}$
$\dfrac{1}{(1+0.02j\omega)}$	$\omega_4 = \omega_h = 100$	0	$A_4 = \left[0\times\log\left(\dfrac{\omega_4}{\omega_3}\right)\right] + A_3 = 47.95\text{ dB}$

(vii) The magnitude plot of the given system is plotted using Table E8.14(b) and is shown in Fig. E8.14(a).

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = 180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.02\omega)$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.14(c).

Table E8.14(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	$-\tan^{-1}(0.2\omega)$	$-\tan^{-1}(0.02\omega)$	Phase angle ϕ (in deg.)
0.1	–1.14°	–0.114°	178.74°
1	–11.30°	–1.145°	167.55°
5	–45°	–5.71°	129.29°
10	–63.43°	–11.30°	105.27°
50	–84.28°	–45°	50.72°
100	–87.13°	–63.43°	29.44°
∞	–90°	–90°	0°

(x) The phase plot of the given system is plotted using Table E8.14(c) and is shown in Fig. E8.14(a).

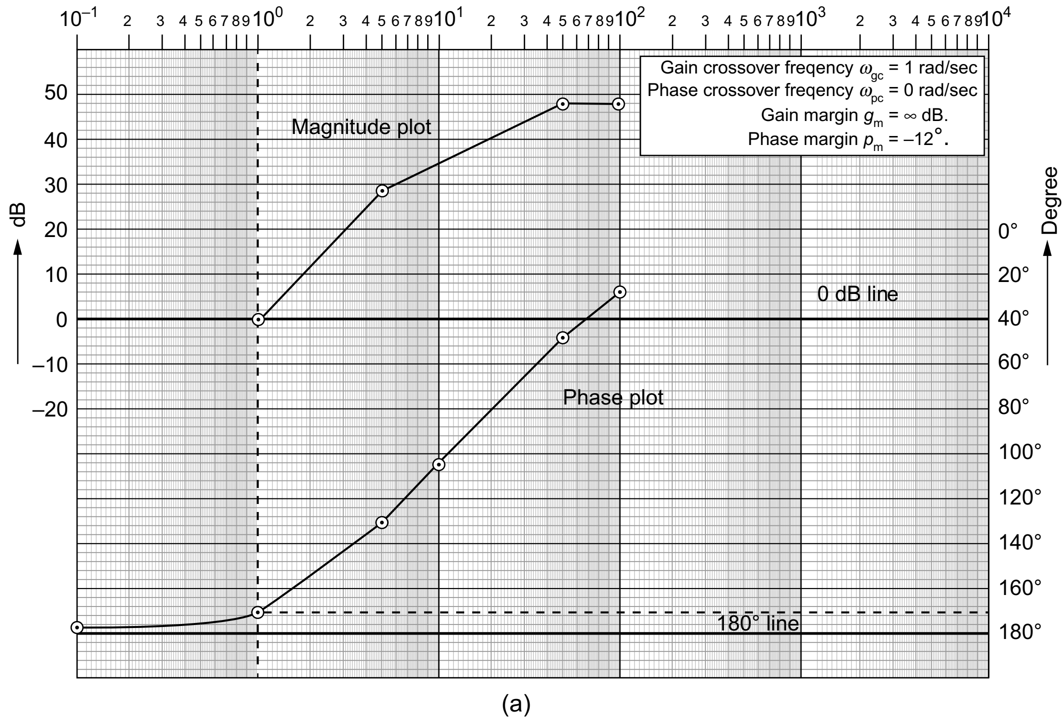


Fig. E8.14 (a)

(xi) The frequency domain specifications for the given system are:

Gain crossover frequency, (ω_{gc})=1 rad/sec.

Phase crossover frequency, (ω_{pc}) =0 rad/sec.

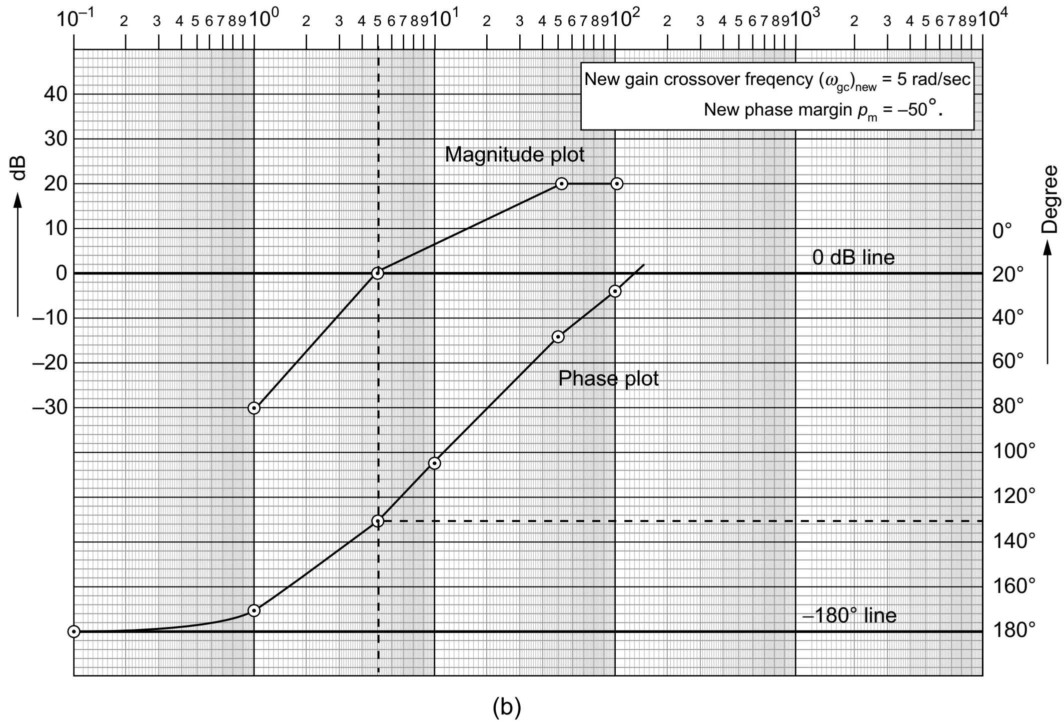
Gain margin, (g_m) = ∞ dB.

Phase margin, (p_m) = – 50°.

(xii) The next step is to determine the gain K corresponding to a particular gain cross-over frequency.

It is given that the required gain crossover frequency of the system is 5 rad/sec, i.e., $\left(\omega_{gc}\right)_{new} = 5$ rad/sec. Hence, it is necessary that the magnitude of the system at $\left(\omega_{gc}\right)_{new}$ must be 0 dB. But from the Bode plot of the system shown in Fig. E8.14(a), the magnitude of the system is 28 dB.

Therefore, each and every point present in the magnitude plot must be added by –28 dB. Hence, the modified Bode plot with $\omega_{gc} = 5$ rad/sec is shown in Fig. E8.14(b).



(b)

Fig. E8.14 (b)

The value of K for the new Bode plot is determined as follows:

$20\log K = D$ where D is the value by which the magnitude plot has been raised or lowered. In this case, $D = -28$ dB.

Hence, $20\log K = -28$

$$\log K = \frac{-28}{20} = -1.4$$

Therefore, the $K = 10^{-1.4} = 0.0398$.

Example 8.15 **The loop transfer function of the system is given by $G(s)H(s) = \frac{0.75(1+0.2s)}{s(1+0.5s)(1+0.1s)}$. Sketch the Bode plot for the system. Also, determine the frequency domain specifications and hence determine the stability of the system.**

Solution:

(i) The loop transfer function of the system is $G(s)H(s) = \frac{0.75(1+0.2s)}{s(1+0.5s)(1+0.1s)}$.

(ii) Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

(iii) The three corner frequencies existing in the system are

$$\omega_{c1} = \frac{1}{0.5} = 2 \text{ rad/sec}, \ \omega_{c2} = \frac{1}{0.2} = 5 \text{ rad/sec and } \omega_{c3} = \frac{1}{0.1} = 10 \text{ rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.15(a).

Table E8.15(a) | Determination of change in slope at different corner frequencies

Term	**Corner frequency ω (rad/sec)**	**Slope of the term (in dB/decade)**	**Change in slope (in dB/decade)**
$\frac{0.75}{j\omega}$	–	–20	–
$\frac{1}{(1+0.5j\omega)}$	$\omega_{c1} = 2$	–20	$-20 - 20 = -40$
$(1+0.2j\omega)$	$\omega_{c2} = 5$	+20	$-40 + 20 = -20$
$\frac{1}{(1+0.1j\omega)}$	$\omega_{c3} = 10$	–20	$-20 - 20 = -40$

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.15(b).

Table E8.15(b) | Gain at different frequencies

Term	**Frequency**	**Change in slope (in dB)**	**Gain, A_i**
$\frac{0.75}{j\omega}$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\frac{0.75}{\omega_1}\right) = 17.50 \text{ dB}$
$\frac{0.75}{j\omega}$	$\omega_2 = \omega_{c1} = 2$	–	$A_2 = 20\log\left(\frac{0.75}{\omega_2}\right) = -8.51 \text{ dB}$

(*Continued*)

Table 8.15(b) | (Continued)

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{1}{(1+0.5j\omega)}$	$\omega_3 = \omega_{c2} = 5$	–40	$A_3 = \left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -24.42\ \text{dB}$
$(1+0.2j\omega)$	$\omega_4 = \omega_{c3} = 10$	–20	$A_4 = \left[-20\times\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -30.44\ \text{dB}$
$\frac{1}{(1+0.1j\omega)}$	$\omega_5 = \omega_h = 100$	–40	$A_5 = \left[-40\times\log\left(\frac{\omega_5}{\omega_4}\right)\right] + A_4 = -70.44\ \text{dB}$

(vii) The magnitude plot of the given system is plotted using Table E8.15(b) and is shown in Fig. E8.15.

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -90^\circ + \tan^{-1}(0.2\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}(0.1\omega)$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.15(c).

Table E8.15(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	$\tan^{-1}(0.2\omega)$	$-\tan^{-1}(0.5\omega)$	$-\tan^{-1}(0.1\omega)$	Phase angle ϕ (in deg.)
0.1	1.14°	–2.86°	–0.57°	–92.29
2	21.80°	–45°	–11.30°	–124.5
5	45°	68.19	–26.56°	–139.75
10	63.43°	78.69	–45°	–150.26
50	84.28°	87.70	–78.69°	–172.11
100	87.13°	88.85	–84.28°	–176
∞	90°	–90°	–90°	–180°

(x) The phase plot of the given system is plotted using Table E8.15(c) and is shown in Fig. E8.15.

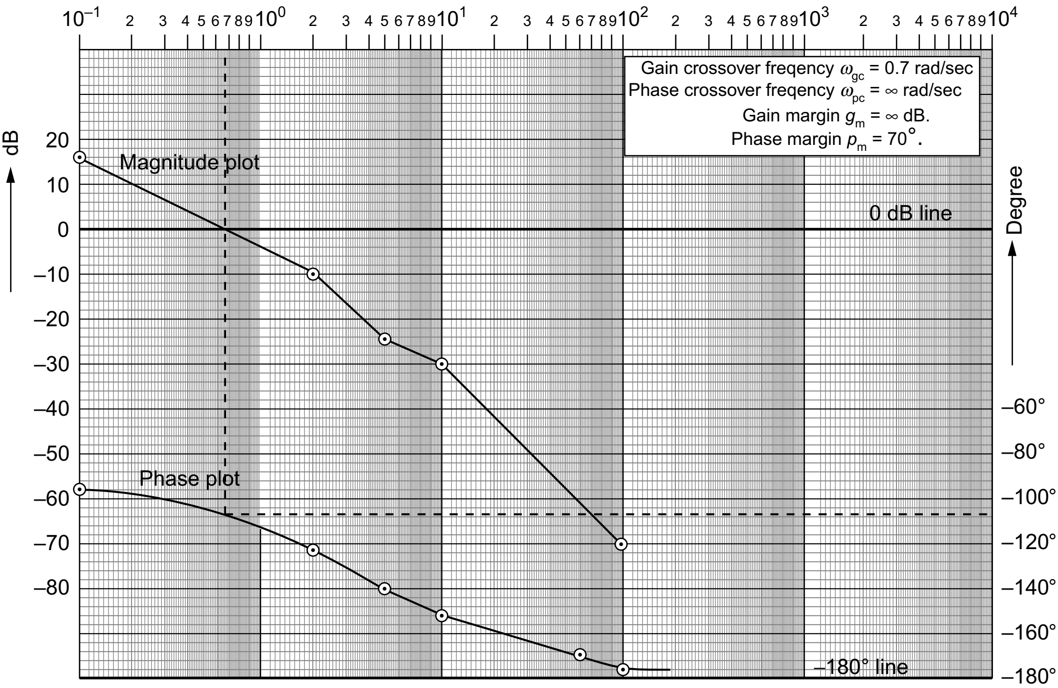


Fig. E8.15

(xi) The frequency domain specifications for the given system are:

Gain crossover frequency, $\omega_{gc} = 1.7$ rad/sec

Phase margin, $p_m = 180° - 110° = 70°$

From Table E8.15(c), it is clear that the phase angle of the system becomes –180° only when the frequency of the system is ∞ rad/sec. Hence, phase crossover frequency of the system is ∞ rad/sec. Therefore, the gain margin of the system is also ∞ dB.

(xii) The stability of the system can be determined based on either the frequency or the gain.

Based on frequency: Since $\omega_{gc} < \omega_{pc}$, the system is stable.

Based on frequency domain specifications: Since the gain margin and phase margin are greater than zero and $\omega_{gc} < \omega_{pc}$, the system is stable.

Example 8.16 **The loop transfer function of a system is given by** $G(s)H(s) = \dfrac{40(1+s)}{(5s+1)(s^2+2s+4)}$. **Sketch the Bode plot for the given system and determine the frequency domain specifications.**

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{40(1+s)}{(5s+1)(s^2+2s+4)}$.

(ii) Substituting $s = j\omega$ and rearranging, we obtain

$$G(j\omega)H(j\omega) = \frac{10(1+j\omega)}{(1+5j\omega)\left(0.25(j\omega)^2+0.5j\omega+1\right)}$$

(iii) The three corner frequencies existing in the system are

$\omega_{c1} = \dfrac{1}{5} = 0.2$ rad/sec, $\omega_{c2} = \dfrac{1}{1} = 1$ rad/sec and the third corner frequency is obtained as follows.

Comparing the standard second-order characteristic equation with the given quadratic factor, we obtain

$$\omega_n = 2 \text{ rad/sec and } 2\xi\omega_n = 2$$

Hence, the third corner frequency existing in the system is $\omega_{c3} = \omega_n = 2$ rad/sec.

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies are given in Table E8.16(a).

Table E8.16(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
10	–	0	–
$\dfrac{1}{(1+5j\omega)}$	$\omega_{c1} = 0.2$	–20	$-20+0=-20$
$(1+j\omega)$	$\omega_{c2} = 1$	+20	$-20+20=0$
$\left(0.25(j\omega)^2+0.5j\omega+1\right)$	$\omega_{c3} = \omega_n = 2$	–40	$0-40=-40$

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.16(b).

Table E8.16(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain, A_i
10	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log(10) = 20$ dB
10	$\omega_2 = \omega_{c1} = 0.2$	–	$A_2 = 20\log(10) = 20$ dB

(*Continued*)

Table E8.16(b) | (Continued)

$\frac{1}{(1+5j\omega)}$	$\omega_3 = \omega_{c2} = 1$	–20	$A_3 = \left[-20 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = 6.02 \text{ dB}$
$(1+j\omega)$	$\omega_4 = \omega_{c3} = 2$	0	$A_4 = \left[0 \times \log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = 6.02 \text{ dB}$
$\frac{1}{\left(0.25(j\omega)^2 + 0.5j\omega + 1\right)}$	$\omega_5 = \omega_h = 100$	–40	$A_5 = \left[-40 \times \log\left(\frac{\omega_5}{\omega_4}\right)\right] + A_4 = -61.93 \text{ dB}$

(vii) The magnitude plot of the given system is plotted using Table E8.16(b) and is shown in Fig. E8.16.

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = \begin{cases} \tan^{-1}(\omega) - \tan^{-1}(5\omega) - \tan^{-1}\left(\dfrac{0.5\omega}{1-0.25\omega^2}\right) & \text{for } \omega \le \omega_n \\ \tan^{-1}(\omega) - \tan^{-1}(5\omega) - \tan^{-1}\left(\dfrac{0.5\omega}{1-0.25\omega^2}\right) - 180^\circ & \text{for } \omega > \omega_n \end{cases}$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.16(c).

Table E8.16(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	**$\tan^{-1}(\omega)$**	**$-\tan^{-1}(5\omega)$**	**$-\tan^{-1}\left(\frac{0.5\omega}{1-0.25\omega^2}\right)$**	**Phase angle ϕ (in deg.)**
0.1	5.71°	–26.56°	–2.86°	–23.71°
0.2	11.30°	–45°	–5.76°	–39.46°
2	63.43°	–84.28°	–90°	–110.85°
5	78.69°	–87.70°	–154.53°	–163.54°
10	84.28°	–88.85°	–168.23°	–172.8°
50	88.85°	–89.77°	–177.70°	–178.62°
100	89.42°	–89.88°	–178.85°	–179.31°
∞	90°	–90°	–180°	–180°

(x) The phase plot of the given system is plotted using Table E8.16(c) and is shown in Fig. E8.16.

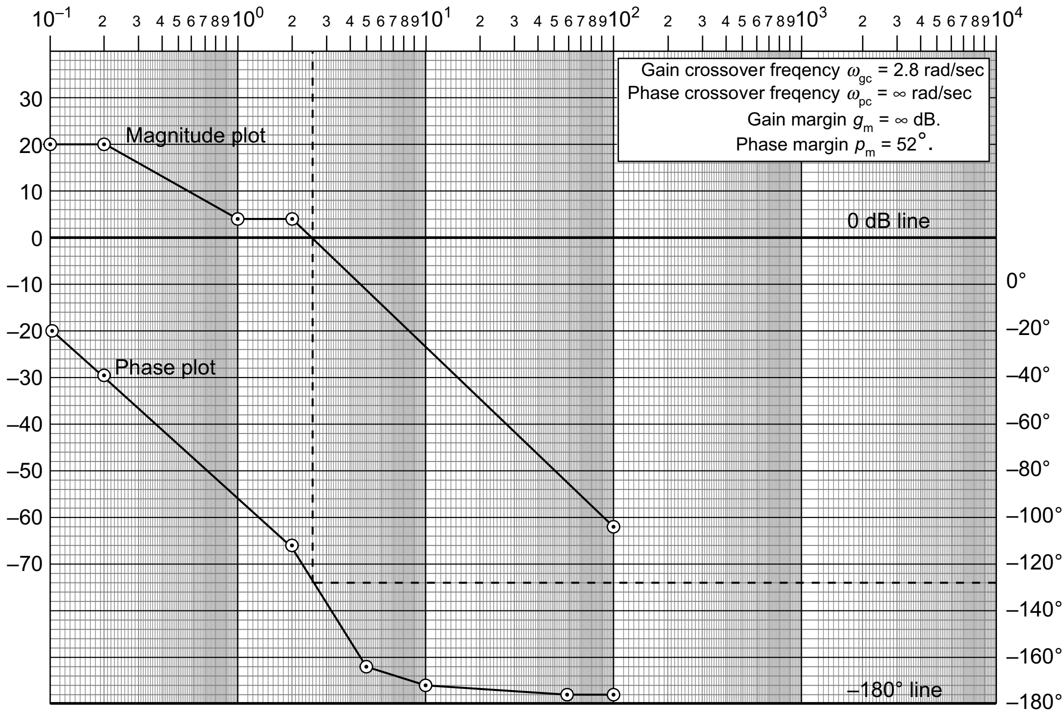


Fig. E8.16

(xi) The frequency domain specifications are:

Gain crossover frequency, $\omega_{gc} = 2.8$ rad/sec

Phase crossover frequency, $\omega_{pc} = \infty$ rad/sec

Gain margin, $g_m = \infty$ dB

Phase margin, $p_m = 52°$

Example 8.17 **The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10(s+10)}{s^2(s+0.1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.**

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{10(s+10)}{s^2(s+0.1)}$

(ii) Substituting $s = j\omega$ and rearranging, we obtain

$$G(j\omega)H(j\omega)=\frac{100(0.1j\omega+1)}{(j\omega)^2(10j\omega+1)}$$

(iii) The two corner frequencies existing in the system are

$$\omega_{c1}=\frac{1}{10}=0.1\ \text{rad/sec and}\ \omega_{c2}=\frac{1}{0.1}=10\ \text{rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.17(a).

Table E8.17(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (dB/decade)	Change in slope (dB/decade)
$\frac{100}{(j\omega)^2}$	–	–40	–
$\frac{1}{(1+10j\omega)}$	$\omega_{c1}=0.1$	–20	–40 – 20 = –60
$(1+0.1j\omega)$	$\omega_{c2}=10$	+20	–60 + 20 = –40

(v) Assume $\omega_l=0.01$ rad/sec and $\omega_h=100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.17(b).

Table E8.17(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{100}{(j\omega)^2}$	$\omega_1=\omega_l=0.01$	–	$A_1=20\log\left(\frac{100}{\omega_1^2}\right)=120$ dB
$\frac{100}{(j\omega)^2}$	$\omega_2=\omega_{c1}=0.1$	–	$A_2=20\log\left(\frac{100}{\omega_2^2}\right)=80$ dB

(Continued)

Table 8.17(b) | (Continued)

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{1}{(1+10j\omega)}$	$\omega_3 = \omega_{c2} = 10$	–60	$A_3 = \left[-60 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -40$ dB
$(1+0.1j\omega)$	$\omega_4 = \omega_h = 100$	–40	$A_4 = \left[-40 \times \log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -80$ dB

(vii) The magnitude plot of the given system is plotted using Table E8.17(b) and is shown in Fig. E8.17.

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -180^\circ - \tan^{-1}(10\omega) + \tan^{-1}(0.1\omega).$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.17(c).

Table E8.17(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	$-\tan^{-1}(10\omega)$	$-\tan^{-1}(0.1\omega)$	Phase angle ϕ (in deg.)
0	0°	0°	–180°
0.1	–45°	0.572°	–224.42°
0.2	–63.42°	1.14°	–242.28°
2	–87.13°	11.30°	–255.83°
5	–88.85°	26.56°	–242.29°
10	–89.42°	45°	–224.42°
50	–89.88°	78.69°	–191.19°
100	–89.94°	84.28°	–185.66°
∞	–90°	90°	–180°

(x) The phase plot of the given system is plotted using Table E8.17(c) and is shown in Fig. E8.17.

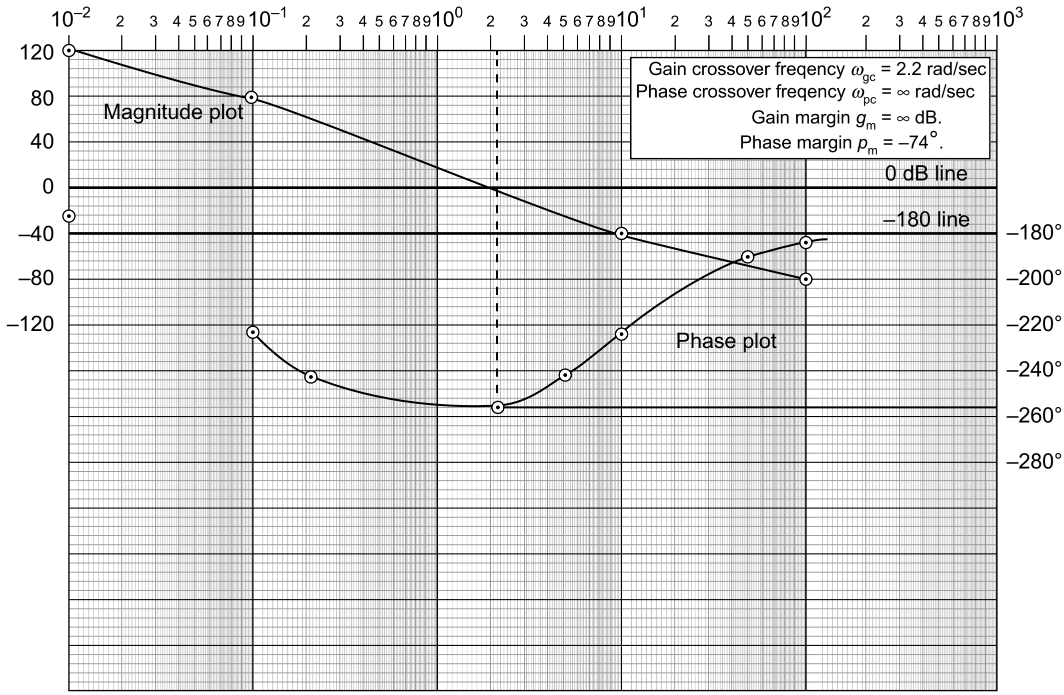


Fig. E8.17

(xi) The frequency domain specifications of the system are:
Gain crossover frequency, $\omega_{gc} = 2.2$ rad/sec
Phase crossover frequency, $\omega_{pc} = \infty$ rad/sec
Gain margin, $g_m = \infty$ dB
Phase margin, $p_m = -74°$

Example 8.18 The loop transfer function of a given system is $G(s)H(s) = Ks^n$. Sketch the Bode plot for the given system.

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = Ks^n$.

(ii) Substituting $s = j\omega$, we obtain $G(j\omega)H(j\omega) = K(j\omega)^n$

(iii) It is known that for a simple zero, the slope is 20 dB/dec and there exists no corner frequency for the given system. Hence, for the given system the slope is $20 \times n$.

(iv) As there exists only one factor in the given system, the change in slope cannot be determined.

(v) The magnitude for the given system is $20\log(j\omega)^n$

(vi) The phase angle for the given system is $\angle G(j\omega) = \phi = 90° \times n$
Hence, the magnitude and phase plots for the given system depends on n.

Example 8.19 The frequency response of a unity feedback system is given in Table E8.19.

Table E8.19 | Frequency response of a system

ω	2	3	4	5	6	8	10
$G(j\omega)H(j\omega)$	7.5	4.8	3.15	2.25	1.70	1.00	0.64
$\angle G(j\omega)H(j\omega)$	–118°	–130°	–140°	–150°	–157°	–170°	–180°

Determine (a) gain margin and phase margin of the system and (b) change in gain so that the gain margin of the system is 20 dB.

Solution:

(a) From Table E8.19, the frequency domain specifications are:
Gain crossover frequency, $\omega_{gc} = \infty$ rad/sec
Phase crossover frequency, $\omega_{pc} = 10$ rad/sec
Gain margin, $g_m = -0.64$ dB
Phase margin, $p_m = \infty$

(b) For the gain $K = 1$, the gain margin $g_m = -0.64$ dB. But the required gain margin is 20 dB. Therefore, the magnitude at different frequency should increase by $20 - (-0.64) = 20.64$ dB.

Hence, $20 \log K = 20.64$

i.e., $K = 10.76$.

Therefore, change in gain required to have the gain margin of the system as 20 dB is $10.76 - 1 = 9.76$.

Example 8.20 The loop transfer function of a system is given by $G(j\omega)H(j\omega) = \dfrac{10(3+j\omega)}{j\omega(j\omega+2)\left((j\omega)^2 + j\omega + 2\right)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.

Solution:

(i) The loop transfer function of the given system is

$$G(j\omega)H(j\omega) = \frac{10(3+j\omega)}{j\omega(j\omega+2)\left((j\omega)^2 + j\omega + 2\right)}$$

(ii) Rearranging the above equation, we obtain

$$G(j\omega)H(j\omega)=\frac{7.5(1+j0.33\omega)}{j\omega(j0.5\omega+1)\left((j\omega)^2 0.5+j0.5\omega+1\right)}$$

(iii) The three corner frequencies existing in the system are $\omega_{c2}=\frac{1}{0.33}=3$ rad/sec, $\omega_{c3}=\frac{1}{0.5}=2$ rad/sec and the third corner frequency is obtained as follows.

Comparing the standard second-order characteristic equation with the given quadratic factor, we obtain

$$(\omega_n)^2=2 \text{ rad/sec and } 2\xi\omega_n=1$$

Hence, the third corner frequency existing in the system is $\omega_{c3}=\omega_n=1.414$ rad/sec.

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.20(a).

Table E8.20(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\frac{7.5}{j\omega}$	–	–20	–
$\frac{1}{\left(0.5(j\omega)^2+j0.5\omega+1\right)}$	$\omega_{c1}=\omega_n=1.414$	–40	$-40-20=-60$
$\frac{1}{(1+j0.5\omega)}$	$\omega_{c2}=2$	–20	$-40-20=-80$
$(1+j0.33\omega)$	$\omega_{c3}=3$	+20	$-80+20=-60$

(v) Assume $\omega_l=0.1$ rad/sec and $\omega_h=50$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.20(b).

Table E8.20(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{7.5}{j\omega}$	$\omega_1=\omega_l=0.1$	–	$A_1=20\log\left(\frac{7.5}{\omega_1}\right)=37.5$ dB

(*Continued*)

Table 8.20(b) | (Continued)

Term	**Frequency**	**Change in slope (in dB)**	**Gain, A_i**
$\frac{7.5}{j\omega}$	$\omega_2 = \omega_{c1} = 1.414$	–	$A_2 = 20\log\left(\frac{7.5}{\omega_2}\right) = 14.49\ \text{dB}$
$\frac{1}{\left(0.5(j\omega)^2 + j0.5\omega + 1\right)}$	$\omega_3 = \omega_{c2} = 2$	–60	$A_3 = \left[-60\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = 5.45\ \text{dB}$
$\frac{1}{(1 + j0.5\omega)}$	$\omega_4 = \omega_{c3} = 3$	–80	$A_4 = \left[-80\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -8.64\ \text{dB}$
$(1 + j0.33\omega)$	$\omega_5 = \omega_h = 50$	–60	$A_5 = \left[-60\log\left(\frac{\omega_5}{\omega_4}\right)\right] + A_4 = -81.95\ \text{dB}$

(vii) The magnitude plot of the given system is plotted using Table E8.20(b) and is shown in Fig. E8.20.

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = \begin{cases} -90^\circ + \tan^{-1}(0.33\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\dfrac{0.5}{1 - 0.5\omega^2}\right) & \text{for } \omega \le \omega_n \\ -90^\circ + \tan^{-1}(0.33\omega) - \tan^{-1}(0.5\omega) - \tan^{-1}\left(\dfrac{0.5}{1 - 0.5\omega^2}\right) - 180^\circ & \text{for } \omega > \omega_n \end{cases}$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.20(c).

Table E8.20(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	**$-\tan^{-1}(0.33\omega)$**	**$-\tan^{-1}(0.5\omega)$**	**$-\tan^{-1}\left(\frac{0.5\omega}{1-0.5\omega^2}\right)$**	**Phase angle ϕ (in deg.)**
0.1	1.89°	–2.86°	–2.87°	–23.71°
1.414	25.01°	–35.26°	–89.97°	–39.46°
2	33.42°	–45°	–135°	–236.8°
3	44.7°	–56.31°	–156.81°	–258.42°
50	86.5°	–87.7°	–178.85°	–270.05°
∞	90°	–90°	–180°	–270°

(x) The phase plot of the given system is plotted using Table E8.20(c) and is shown in Fig. E8.20.

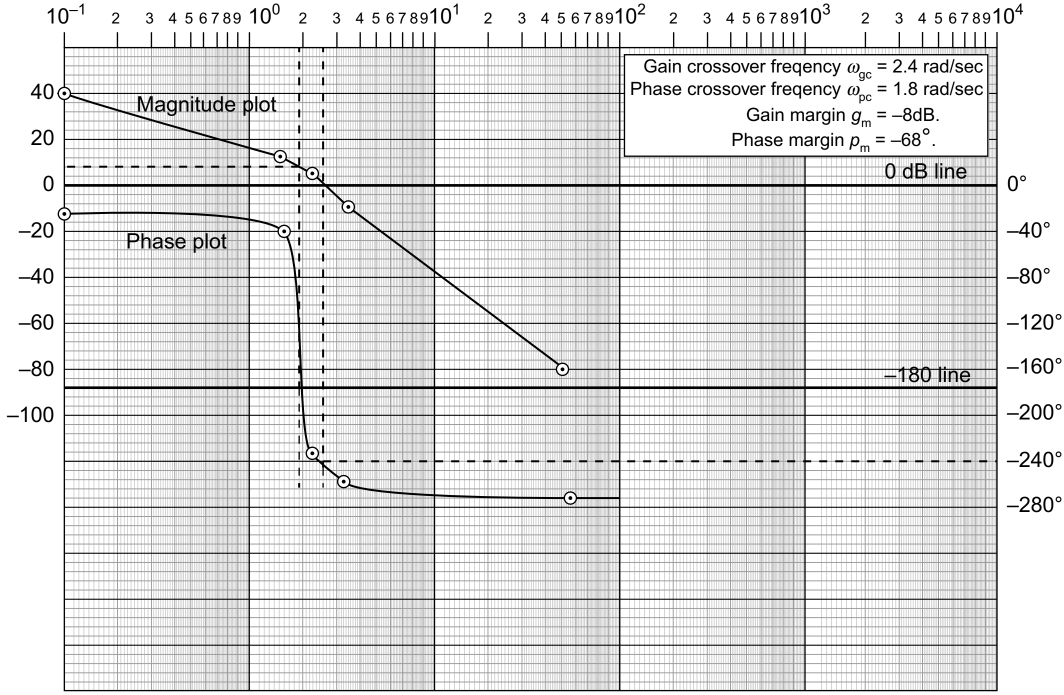


Fig. E8.20

(xi) The frequency domain specifications for the system are:

Gain crossover frequency, $\omega_{gc} = 2.4$ rad/sec

Phase crossover frequency, $\omega_{pc} = 1.8$ rad/sec

Gain margin, $g_m = -8$ dB

Phase margin, $p_m = -68°$

Example 8.21 The loop transfer function of a system is given by $G(s)H(s) = \dfrac{s^2}{(1+0.2s)(1+0.2s)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{s^2}{(1+0.2s)(1+0.2s)}$

(ii) Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{(j\omega)^2}{(1+0.2j\omega)(1+0.2j\omega)}$$

(iii) The corner frequency that exists in the system is $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.21(a).

Table E8.21(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$(j\omega)^2$	–	+ 40	–
$\frac{1}{(1+0.2j\omega)^2}$	$\omega_{c1} = 5$	–40 (–20 *2 = – 40)	40 – 40 = 0

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.21(b).

Table E8.21(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
$(j\omega)^2$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\omega_1^2\right) = -40$ dB
$(j\omega)^2$	$\omega_2 = \omega_{c1} = 5$	–	$A_2 = 20\log\left(\omega_2^2\right) = 27.95$ dB
$\frac{1}{(1+0.2j\omega)}$	$\omega_3 = \omega_h = 10$	0	$A_3 = \left[0 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = 27.95$ dB

(vii) The magnitude plot of the given system is plotted using Table E8.21(b) and is shown in Fig. E8.21.

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = \left(2 \times 90°\right) - \left(2 \times \tan^{-1}(0.2\omega)\right) = 180° - \left(2 \times \tan^{-1}(0.2\omega)\right)$$

(ix) The phase angle at different frequencies are tabulated as given in Table E8.21(c).

Table E8.21(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	**$-\tan^{-1}(0.2\omega)$**	**Phase angle ϕ in deg.**
0.1	$-1.14°$	$177.72°$
0.2	$-2.29°$	$175.42°$
2	$-21.8°$	$136.4°$
5	$-45°$	$90°$
10	$-63.43°$	$53.14°$
50	$-84.28°$	$11.44°$
100	$-87.13°$	$5.74°$
∞	$-90°$	$0°$

(x) The phase plot of the given system is plotted using Table E8.21(c) and is shown in Fig. E8.21.

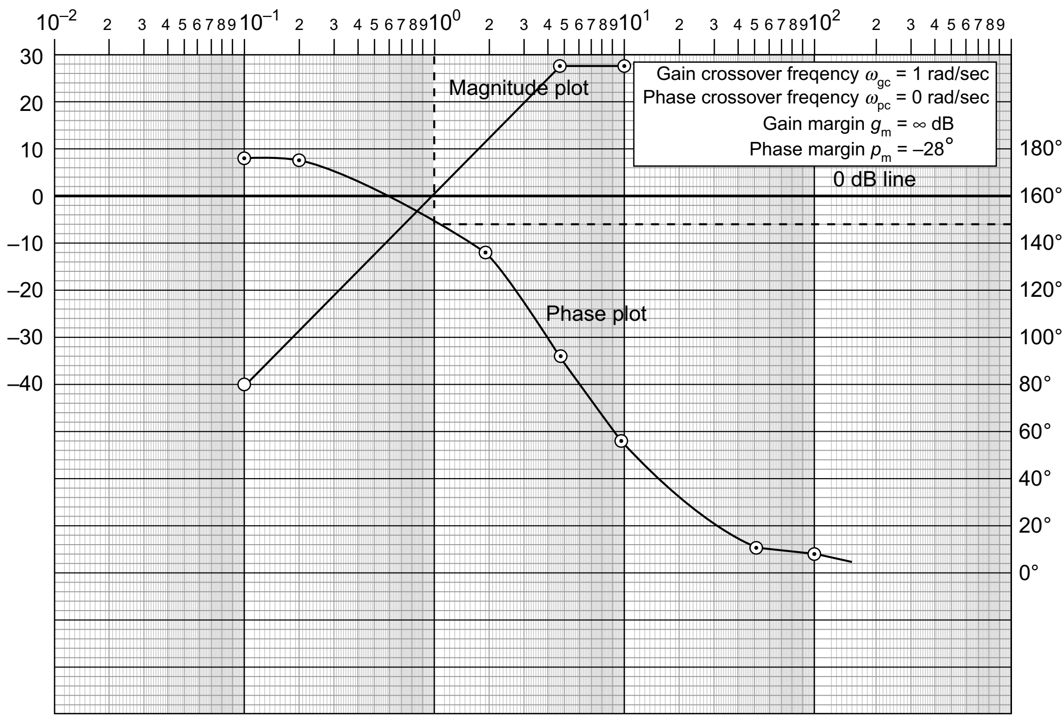


Fig. E8.21

(xi) The frequency domain specifications of the system are:
Gain crossover frequency, $\omega_{gc} = 1$ rad/sec
Phase crossover frequency, $\omega_{pc} = 0$ rad/sec
Gain margin, $g_m = \infty$ dB
Phase margin, $p_m = -28°$

Example 8.22 **The loop transfer function of a system is given by $G(s)H(s) = \dfrac{2000(s+0.3)}{(s+4)(s^2+30s+20)}$. Sketch the Bode plot for the given system. Determine the frequency domain specifications.**

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{2000(s+0.3)}{(4+s)(s^2+30s+20)}$

(ii) As the roots of the quadratic factor are real, we can split the quadratic factor as

$$(s^2+30s+20) = (s+0.682)(s+29.32)$$

(iii) Substitute $s = j\omega$ and rearranging, we obtain

$$G(j\omega)H(j\omega) = \frac{\frac{2000}{0.3}(1+3.33j\omega)}{4(1+0.25j\omega)\times 0.682(1+1.466j\omega)\times 29.32(1+0.034j\omega)}$$

$$= \frac{83.33(1+3.33j\omega)}{(1+0.25j\omega)(1+1.466j\omega)(1+0.034j\omega)}$$

(iv) The four corner frequencies existing in the system are

$\omega_{c1} = \dfrac{1}{3.33} = 0.3$ rad/sec, $\omega_{c2} = \dfrac{1}{1.466} = 0.682$ rad/sec, $\omega_{c3} = \dfrac{1}{0.25} = 4$ rad/sec and

$\omega_{c4} = \dfrac{1}{0.034} = 29.32$ rad/sec.

To sketch the magnitude plot

(v) The change in slope at different corner frequencies is given in Table E8.22(a).

(vi) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 500$ rad/sec.

Table E8.22(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
83.33	–	0	–
$(1+3.33j\omega)$	$\omega_{c1} = 0.3$	+ 20	0 + 20 = + 20
$\dfrac{1}{(1+1.466j\omega)}$	$\omega_{c2} = 0.682$	–20	+ 20 – 20 = 0
$\dfrac{1}{(1+0.25j\omega)}$	$\omega_{c3} = 4$	–20	0 – 20 = –20
$\dfrac{1}{(1+0.034j\omega)}$	$\omega_{c4} = 29.32$	–20	–20 – 20 = –40

(vii) The values of gain at different frequencies are determined and given in Table E8.22(b).

Table E8.22(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
83.33	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log(83.33) = 38.41\text{dB}$
83.33	$\omega_2 = \omega_{c1} = 0.3$	–	$A_2 = 20\log(83.33) = 38.41\text{dB}$
$(1+3.33j\omega)$	$\omega_3 = \omega_{c2} = 0.682$	+20	$A_3 = \left[20\times\log\left(\dfrac{\omega_3}{\omega_2}\right)\right] + A_2 = 45.54\text{dB}$
$\dfrac{1}{(1+1.466j\omega)}$	$\omega_4 = \omega_{c3} = 4$	0	$A_4 = \left[0\times\log\left(\dfrac{\omega_4}{\omega_3}\right)\right] + A_3 = 45.54\text{dB}$
$\dfrac{1}{(1+0.25j\omega)}$	$\omega_5 = \omega_{c4} = 29.32$	–20	$A_5 = \left[-20\times\log\left(\dfrac{\omega_5}{\omega_4}\right)\right] + A_4 = 28.23\text{dB}$
$\dfrac{1}{(1+0.34j\omega)}$	$\omega_6 = \omega_h = 500$	–40	$A_6 = \left[-40\times\log\left(\dfrac{\omega_6}{\omega_5}\right)\right] + A_5 = 21.04\text{dB}$

(viii) The magnitude plot of the given system is plotted using Table E8.22(b) and is shown in Fig. E8.22.

To sketch the phase plot

(ix) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = \tan^{-1}(3.33\omega) - \tan^{-1}(1.466\omega) - \tan^{-1}(0.25\omega) - \tan^{-1}(0.034\omega)$$

(x) The phase angles at different frequencies are tabulated as given in Table E8.22(c).

Table E8.22(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	$\tan^{-1}(3.33\omega)$	$-\tan^{-1}(1.466\omega)$	$-\tan^{-1}(0.25\omega)$	$-\tan^{-1}(0.034\omega)$	Phase angle ϕ (in deg.)
0.1	18.41°	–8.34°	–1.43°	–0.194°	8.446°
2	81.46°	–71.16°	–26.56°	–3.89°	–20.15°
5	86.56°	–82.23°	–51.34°	–9.648°	–56.65°
10	88.27°	–86.09°	–68.19°	–18.78°	–84.79°
50	89.65°	–89.21°	–85.42°	–59.53°	–144.51
100	89.82°	–89.60°	–87.70°	–73.61°	–161.09°
∞	90°	–90°	–90°	–90°	–180°

(xi) The phase plot of the given system is plotted using Table E8.22(c) and is shown in Fig. E8.22.

(xii) The frequency domain specifications of the given system are:

Gain crossover frequency, $\omega_{gc} = 140$ rad/sec

Phase crossover frequency, $\omega_{pc} = \infty$ rad/sec

Gain margin, $g_m = \infty$ dB

Phase margin, $p_m = 14°$

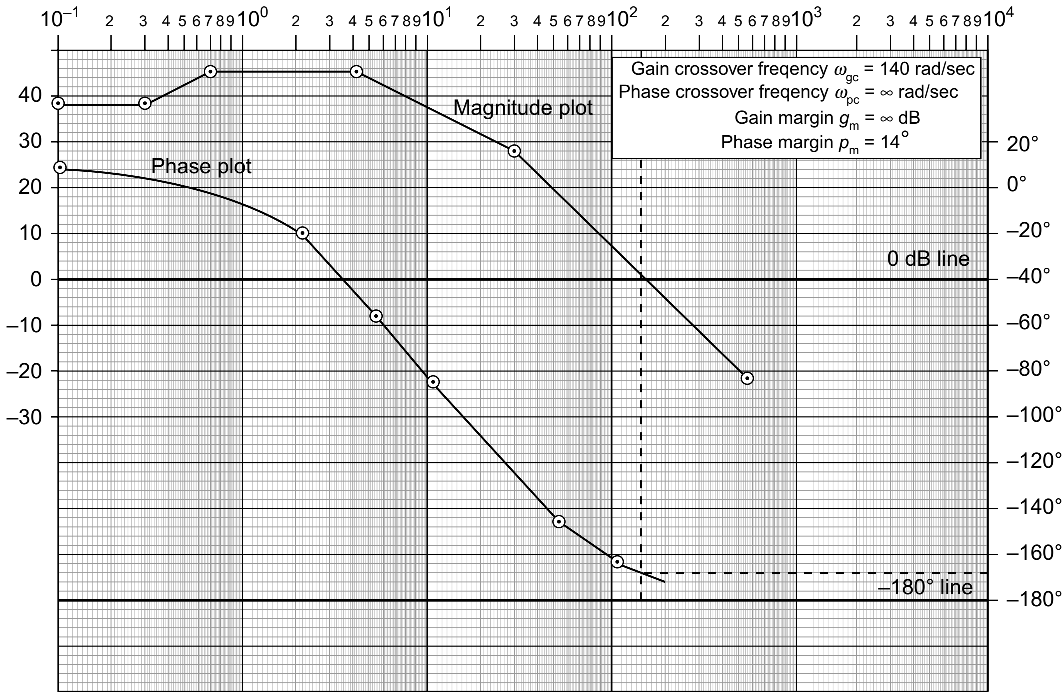


Fig. E8.22

Example 8.23 **The loop transfer function of a system is given by $G(s)H(s) = \dfrac{Ke^{-0.2s}}{s(2+s)(8+s)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.**

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{Ke^{-0.2s}}{s(2+s)(8+s)}$

(ii) Substituting $s = j\omega$ and $K = 1$ and rearranging, we obtain

$$G(j\omega)H(j\omega) = \frac{\frac{K}{16}e^{-0.2j\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)} = \frac{K'e^{-0.2j\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$$

where $K' = \dfrac{K}{16}$.

(iii) The two corner frequencies existing in the system are

$$\omega_{c1} = \frac{1}{0.5} = 2 \text{ rad/sec and } \omega_{c2} = \frac{1}{0.125} = 8 \text{ rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.23(a).

Table E8.23(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\frac{1}{(j\omega)}$	–	−20	–
$\frac{1}{(1+0.5j\omega)}$	$\omega_{c1} = 2$	−20	$-20 - 20 = -40$
$\frac{1}{(1+j0.125\omega)}$	$\omega_{c2} = 8$	−20	$-40 - 20 = -60$

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.23(b).

Table E8.23(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain, A_i
$\frac{1}{(j\omega)}$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\frac{1}{\omega_1}\right) = 20\text{ dB}$
$\frac{1}{(j\omega)}$	$\omega_2 = \omega_{c1} = 2$	–	$A_2 = 20\log\left(\frac{1}{\omega_2}\right) = -6.02\text{ dB}$
$\frac{1}{(1+0.5j\omega)}$	$\omega_3 = \omega_{c2} = 8$	−40	$A_3 = \left[-40 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -30.01\text{ dB}$
$\frac{1}{(1+j0.125\omega)}$	$\omega_4 = \omega_h = 50$	−60	$A_4 = \left[-60 \times \log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -77.76\text{ dB}$

(vii) It is known that the delay term ($e^{-0.2s}$) present in the loop transfer function will contribute only to phase angle plot. Hence, the magnitude plot of any system with or without delay terms is same.

(viii) The magnitude plot of the given system is plotted using Table E8.23(b) and is shown in Fig. E8.23.

To sketch the phase plot

(ix) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = \left(-57.32 \times 0.2\omega\right) - 90° - \tan^{-1}\left(0.5\omega\right) - \tan^{-1}\left(0.125\omega\right)$$
$$= \left(-11.464\omega\right) - 90° - \tan^{-1}\left(0.5\omega\right) - \tan^{-1}\left(0.125\omega\right)$$

(x) The phase angles at different frequencies are tabulated as given in Table E8.23(c).

Table E8.23(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	**-11.464ω**	**$-\tan^{-1}(0.5\omega)$**	**$-\tan^{-1}(0.125\omega)$**	**Phase angle, ϕ (in deg.)**
0.1	–1.14°	–2.86°	–0.586°	–94.7°
1	–11.45°	–26.56°	–1.953°	–135.13°
2	–22.91°	–45°	–18.83°	–171.94°
4	–45.83°	–63.43°	–73.66°	–225.82°

(xi) The phase plot of the given system is plotted using Table E8.23(c) and is shown in Fig. E8.23.

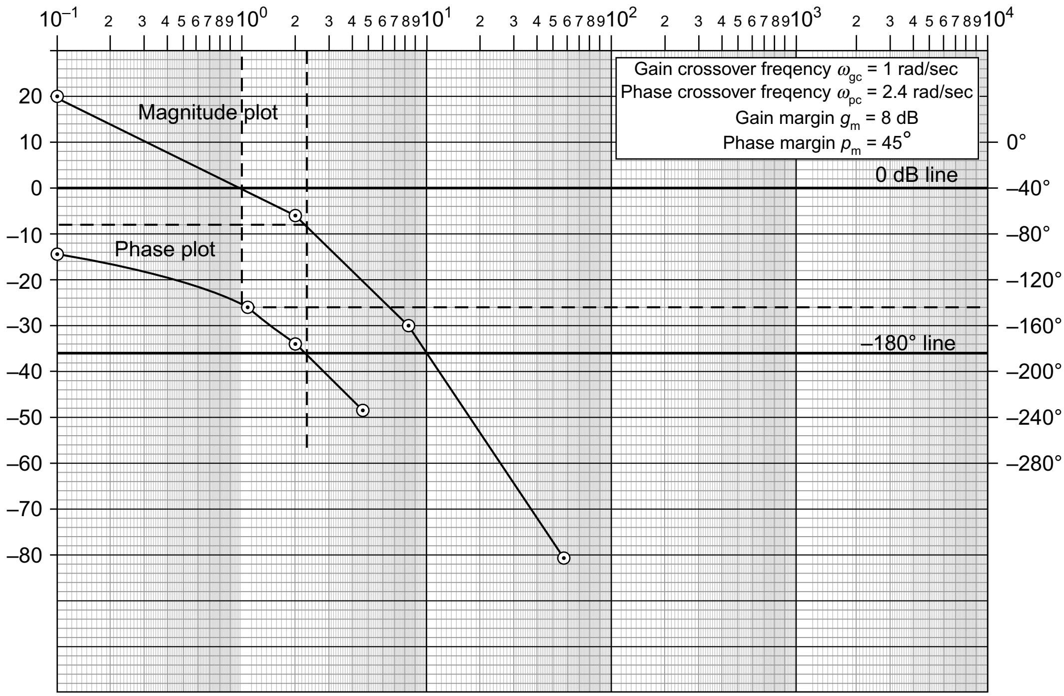


Fig. E8.23

(xii) The frequency domain specifications of the system are:

Gain crossover frequency, $\omega_{gc} = 1$ rad/sec

Phase crossover frequency, $\omega_{pc} = 2.4$ rad/sec

Gain margin, $g_m = 8$ dB

Phase margin, $p_m = 45°$

Example 8.24 **The block diagram of a system with $G_1(s) = \dfrac{2.5}{(s+1)(5s+12)}$ and $H(s) = \dfrac{2}{3s+4}$ is shown in Fig. E8.24(a). Determine the loop transfer function and sketch the Bode plot for the same. Also, determine the maximum gain K before the system becomes unstable.**

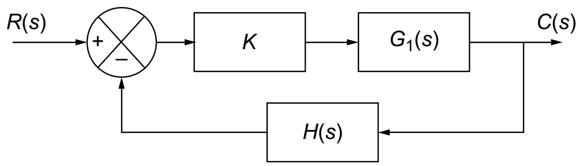


Fig. E8.24(a)

Solution:

(i) The loop transfer function of the system shown in Fig. E8.24(a) is $G(s)H(s) = \dfrac{5K}{(s+1)(5s+12)(3s+4)}$.

(ii) Substituting $s = j\omega$ and $K = 1$ and rearranging, we obtain

$$G(j\omega)H(j\omega) = \frac{0.1041}{(1+j\omega)(1+j0.416\omega)(1+j0.75\omega)}$$

(iii) The three corner frequencies existing in the system are

$$\omega_{c1} = \frac{1}{1} = 1\,\text{rad/sec},\ \omega_{c2} = \frac{4}{3} = 1.33\,\text{rad/sec and}\ \omega_{c3} = \frac{12}{5} = 2.4\,\text{rad/sec.}$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.24(a).

Table E8.24(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
0.1041	–	0	–
$\frac{1}{(1+j\omega)}$	$\omega_{c1} = 1$	–20	$0 - 40 = -20$
$\frac{1}{(1+j0.75\omega)}$	$\omega_{c2} = 1.33$	–20	$-20 - 20 = -40$
$\frac{1}{(1+j0.416\omega)}$	$\omega_{c3} = 2.4$	–20	$-40 - 20 = -60$

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 100$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.24(b).

Table E8.24(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
0.1041	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log(0.1041) = -19.65$ dB
0.1041	$\omega_2 = \omega_{c1} = 1$	–	$A_2 = 20\log(0.1041) = -19.65$ dB
$\frac{1}{(1+j0.75\omega)}$	$\omega_3 = \omega_{c2} = 1.33$	–20	$A_3 = \left[-20 \times \log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -22.12$ dB
$\frac{1}{(1+j0.416\omega)}$	$\omega_4 = \omega_{c3} = 2.4$	–40	$A_4 = \left[-40 \times \log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -32.37$ dB
$\frac{1}{(1+j\omega)}$	$\omega_5 = \omega_h = 100$	–60	$A_5 = \left[-60 \times \log\left(\frac{\omega_5}{\omega_4}\right)\right] + A_4 = -129.55$ dB

(vii) The magnitude plot of the given system is plotted using Table E8.24(b) and is shown in Fig. E8.24(b).

To sketch the phase plot

(viii) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -\tan^{-1}(\omega) - \tan^{-1}(0.416\omega) - \tan^{-1}(0.75\omega)$$

(ix) The phase angles at different frequencies are tabulated as given in Table E8.24(c).

Table E8.24(c) | Phase angle of the system for different frequencies

Frequency ω (rad/sec)	**$-\tan^{-1}(\omega)$**	**$-\tan^{-1}(0.416\omega)$**	**$-\tan^{-1}(0.75\omega)$**	**Phase angle ϕ (in deg.)**
0.1	–5.71°	–2.38°	–4.28°	–12.37°
0.2	–11.30°	–4.75°	–8.53°	–24.58°
2	–63.43°	–39.76°	–56.30°	–159.49°
5	–78.69°	–64.32°	–75.06°	–218.07°
10	–84.28°	–76.48°	–82.4°	–243.16°
50	–88.85°	–87.24°	–88.47°	–264.56°
100	–89.42°	–88.62°	–89.23°	–267.27°
∞	–90°	–90°	–90°	–270°

(x) The phase plot of the given system is plotted using Table E8.24(c) and is shown in Fig. E8.24(b).

(xi) The frequency domain specifications of the system are:

Gain crossover frequency, $\omega_{gc} = \infty$ rad/sec

Phase crossover frequency, $\omega_{pc} = 2.9$ rad/sec

Gain margin, $g_m = 37$ dB

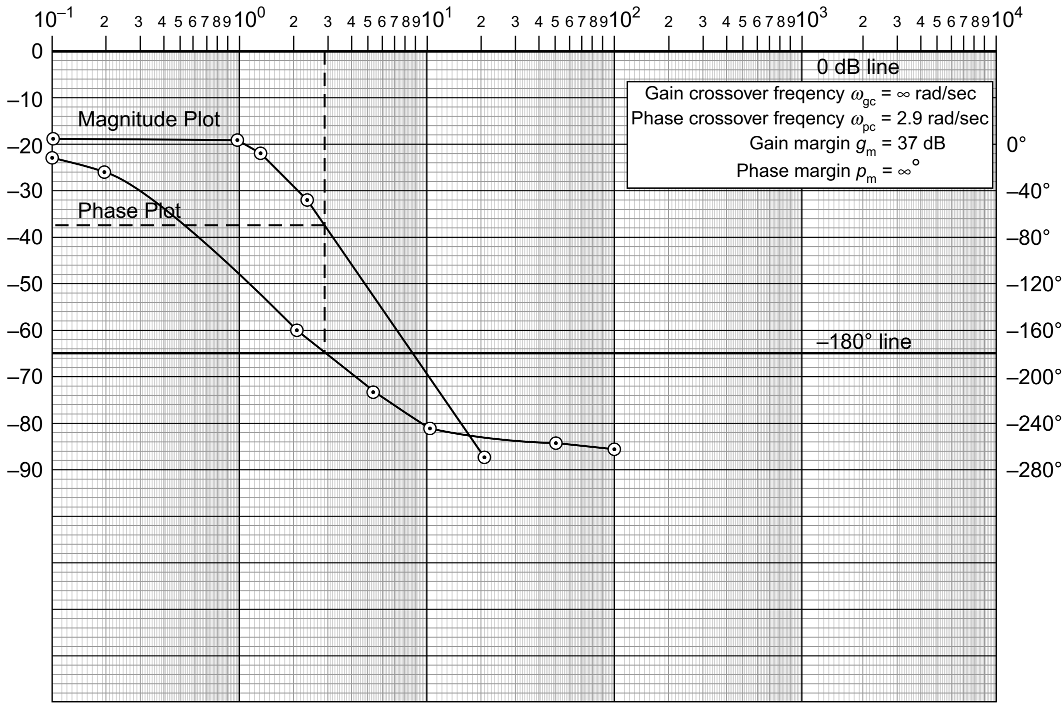
Phase margin, $p_m = \infty°$

To determine the maximum gain K

(xii) The magnitude in dB at ω_{pc} is 37 dB. Therefore, the magnitude plot can be raised by dB until the magnitude becomes 0 dB at ω_{pc}. Therefore, the maximum gain K is determined as

$$20 \log K = 37$$

$$K = 70.79$$



(b)

Fig. E8.24(b)

Example 8.25 **The loop transfer function of a system is given by** $G(s)H(s) = \dfrac{10e^{-sT}}{s(0.1s+1)(0.01s+1)}$**. Determine the maximum value of time delay** T**, for the closed-loop system to be stable.**

Solution:

(i) The loop transfer function of the given system is $G(s)H(s) = \dfrac{10e^{-sT}}{s(0.1s+1)(0.01s+1)}$

(ii) Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{10e^{-j\omega T}}{s(j0.1\omega+1)(j0.01\omega+1)}$$

(iii) The two corner frequencies existing in the system are

$$\omega_{c1} = \frac{1}{0.1} = 10 \text{ rad/sec and } \omega_{c2} = \frac{1}{10.01} = 100 \text{ rad/sec}.$$

To sketch the magnitude plot

(iv) The change in slope at different corner frequencies is given in Table E8.25(a).

Table E8.25(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency, ω (rad/sec)	Slope of the term (in dB/decade)	Change in slope (in dB/decade)
$\frac{10}{j\omega}$	–	–20	–
$\frac{1}{(1+0.1j\omega)}$	$\omega_{c1} = 10$	–20	$-20-20=-40$
$\frac{1}{(1+0.01j\omega)}$	$\omega_{c2} = 100$	–20	$-40-20=-60$

(v) Assume $\omega_l = 0.1$ rad/sec and $\omega_h = 1{,}000$ rad/sec.

(vi) The values of gain at different frequencies are determined and given in Table E8.25(b).

Table E8.25(b) | Gain at different frequencies

Term	Frequency	Change in slope (in dB)	Gain, A_i
$\frac{10}{j\omega}$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\frac{10}{\omega_1}\right) = 40\ \text{dB}$
$\frac{10}{j\omega}$	$\omega_2 = \omega_{c1} = 10$	–	$A_2 = 20\log\left(\frac{10}{\omega_2}\right) = 0\ \text{dB}$
$\frac{1}{(1+0.1j\omega)}$	$\omega_3 = \omega_{c2} = 100$	–40	$A_3 = \left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = -40\ \text{dB}$
$\frac{1}{(1+0.01j\omega)}$	$\omega_4 = \omega_h = 1{,}000$	–60	$A_4 = \left[-60\times\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -100\ \text{dB}$

(vii) It is known that the delay term (e^{-sT}) present in the loop transfer function will contribute only to the phase angle plot. Hence, the magnitude plot of any system with or without delay term is same.

(viii) It is clear from Table E8.25(b) that the gain crossover frequency ω_{gc} is 10 rad/sec.

(ix) The phase angle of the system is

$$\phi = (-57.32\times\omega T) - 90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.01\omega) \tag{1}$$

(x) To determine the maximum value of time delay, T :
The phase margin for the system to be stable is $0°$. Therefore,

$$180° + \angle G(j\omega)H(j\omega)\Big|_{\omega = \omega_{gc}} = 0°$$

Substituting Eqn. (1) in the above equation, we obtain

$$180°\left(-57.32 \times \omega_{gc} T\right) - 90° - \tan^{-1}\left(0.1\omega_{gc}\right) - \tan^{-1}\left(0.01\omega_{gc}\right) = 0°$$

Substituting $\omega_{gc} = 10$ rad/sec, we obtain

$$180° - \left(57.32 \times 10 \times T\right) - 90° - \tan^{-1}(1) - \tan^{-1}(0.1) = 0°$$

$$573.2T = 39.289 \qquad \text{i.e., } T = 0.068 \text{ sec.}$$

Therefore, maximum time delay, T , for the closed-loop system to be stable is 0.068 sec.

Example 8.26 **The magnitude plot of a system is shown in Fig. E8.26(a). Determine the transfer function of the system.**

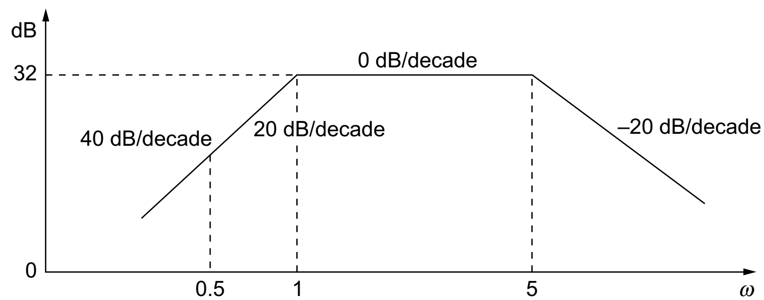


Fig. E8.26 (a)

Solution:
From Fig. E8.26(a), we can obtain the following information:

(i) Three corner frequencies exist in the system
(ii) Initial slope of the plot = 40 dB/decade
(iii) Final slope of the plot = –20 dB/decade

The change in slope from initial to final slope occurs in three steps and the factor that could be present in the transfer function for that change in slope to occur is given in Table E8.26.

Table E8.26 | Factor for the change in slope

Initial slope (dB/decade)	Final slope (dB/decade)	Change in slope = final slope–initial slope (dB/decade)	Factor present in the transfer function	Corner frequency at which the change in slope occurs
–	40	40	s^2	–
40	20	–20	$\frac{1}{(1+sT_1)}$	$\omega_{c1} = 0.5$
20	0	–20	$\frac{1}{(1+sT_2)}$	$\omega_{c2} = 1$
0	–20	–20	$\frac{1}{(1+sT_3)}$	$\omega_{c3} = 5$

Therefore, from the above table, we can determine the loop transfer function of the system as a function of gain value K as

$$G(s)H(s) = \frac{Ks^2}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

The values of time constants T_1, T_2 and T_3 can be determined from their respective corner frequencies as

$$T_1 = \frac{1}{\omega_{c1}} = \frac{1}{0.5} = 2,\ T_2 = \frac{1}{\omega_{c2}} = \frac{1}{1} = 1 \text{ and } T_3 = \frac{1}{\omega_{c3}} = \frac{1}{5} = 0.2\,.$$

Hence, the loop transfer function of the system as a function of K is

$$G(s)H(s) = \frac{Ks^2}{(1+2s)(1+s)(1+0.2s)}$$

To determine the gain K

The gain K can be determined using the general equation of straight line, i.e., $y = mx + c$ where y is the magnitude of $G(j\omega)H(j\omega)$ in dB, m is the slope of the line, x is the logarithmic frequency in rad/sec and c is the constant.

The given magnitude plot of a system is redrawn as shown in Fig. E8.26(b).

For the line segment bc, we have

Slope of the line, $m = 20$.

Therefore, $y = 20 \log \omega + c$

At $\omega = 1$ rad/sec, $y = 32$ dB.

$$\text{Therefore, } 32 = (20 \times \log 1) + c \quad \text{i.e., } c = 32.$$

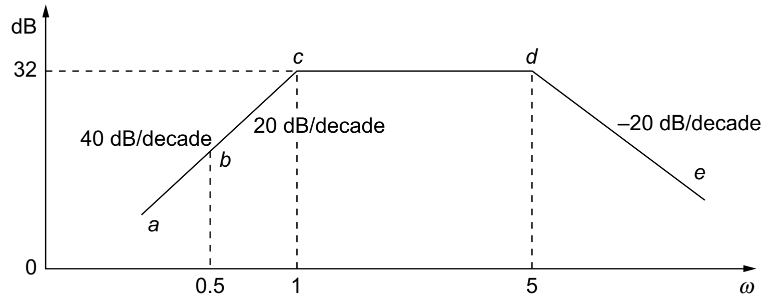


Fig. E8.26(b)

The magnitude at 0.5 rad/sec is $y = \left(20 \times \log\left(0.5\right)\right) + 32$

Solving the above equation, we obtain $y = 25.98$ dB.

At $\log\omega = 0.5$ rad/sec, only the gain and poles at the origin exist in the system, i.e., Ks^2.

Therefore, $20\log_{10}\left(K\left(0.5\right)^2\right) = 25.98$

$$\log_{10}\left(K\left(0.5\right)^2\right) = 1.2991$$

$$K\left(0.5\right)^2 = 10^{1.2991} = 19.906$$

Therefore, $K = 79.624$.

Hence, the transfer function for the given magnitude plot of a system is

$$G\left(s\right)H\left(s\right) = \frac{79.624s^2}{\left(1+2s\right)\left(1+s\right)\left(1+0.2s\right)}$$

Example 8.27 The magnitude plot of a particular system is shown in Fig. E8.27(a). Determine the transfer function of the system.

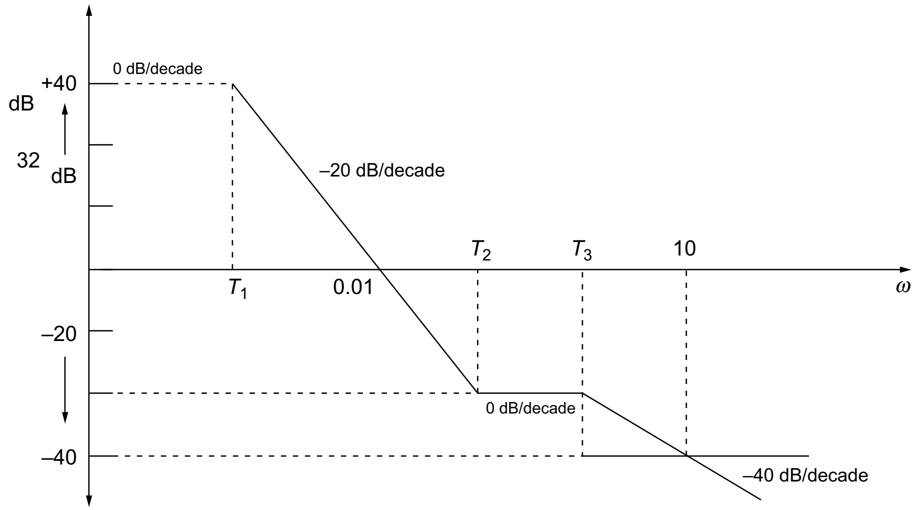


Fig. E8.27 (a)

Solution:

From Fig. E8.27(a), we can obtain the following information:

(i) Three corner frequencies exist in the system

(ii) Initial slope of the plot = 0 dB/decade

(iii) Final slope of the plot = –40 dB/decade

The change in slope from initial to final slope occurs in three steps and the factor that could be present in the transfer function for that change in slope to occur is given in Table E8.27.

Table E8.27 | Factor for the change in slope

Initial slope (dB/decade)	Final slope (dB/decade)	Change in slope = Final slope–Initial slope (dB/decade)	Factor present in the transfer function	Corner frequency at which the change in slope occurs
–	0	0	K	–
0	–20	–20	$\frac{1}{(1+sT_1)}$	$\omega_{c1} = \frac{1}{T_1}$
–20	0	20	$(1+sT_2)$	$\omega_{c2} = \frac{1}{T_2}$
0	–40	–40	$\frac{1}{(1+sT_3)^2}$	$\omega_{c3} = \frac{1}{T_3}$

Therefore, from Table E8.27, we can determine the loop transfer function of the system as a function of gain value K as

$$G(s)H(s) = \frac{K\left(1+\dfrac{s}{T_2}\right)}{\left(1+\dfrac{s}{T_1}\right)\left(1+sT_3\right)^2}$$

To determine the gain K and corner frequencies:

(i) As the initial slope of the magnitude plot is 0 dB/decade and the magnitude is 40 dB, we obtain the gain K as

$$20\log K = 40$$

Therefore, $K = 100$.

(ii) Corner frequencies

The gain K can be determined using the general equation of straight line, i.e., $y = mx + c$ where y is the magnitude of $G(j\omega)H(j\omega)$ in dB, m is the slope of the line, x is the logarithmic frequency in rad/sec and c is the constant.

The magnitude plot given in Fig. E8.27(a) can be redrawn as shown in Fig. E8.27(b).

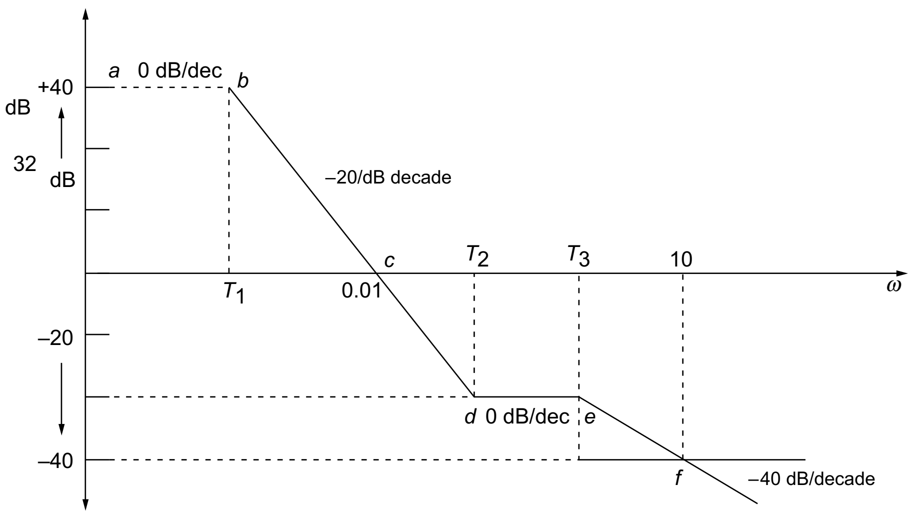


Fig. E8.27 (b)

For the line segment bc , we have $m = -20$
Therefore, $y = -20\log\omega + c$

At $\omega = 0.01$, $y = 0$ dB.

Therefore, $0 = -20\log(0.01) + c$

Solving the above equation, we obtain $c = -40$
Also, at $\omega = T_1$, $y = 40$ dB

Therefore, $40 = -20\log(T_1) - 40$

Solving the above equation, we obtain $T_1 = 0.0001$ s.
Similarly, for line segment cd , we have $m = -20$
Therefore, $y = -20\log\omega + c$

At $\omega = 0.01$, $y = 0$ dB.

Therefore, $0 = -20\log(0.01) + c$

Solving the above equation, we obtain $c = -40$
Also, at $\omega = T_2$, $y = -30$ dB

Therefore, $-30 = -20\log(T_2) - 40$

Solving the above equation, we obtain $T_2 = 0.316$ s.
Similarly, for line segment ef , we have

$$m = -40$$

Therefore, $y = -40\log\omega + c$

At $\omega = 10$, $y = -40$ dB.

Therefore, $-40 = -20\log(10) + c$

Solving the above equation, we obtain $c = -20$
Also, at $\omega = T_3$, $y = -30$ dB

Therefore, $-30 = -40\log(T_3) - 20$

Solving the above equation, we obtain $T_3 = 1.778$ s.
Therefore, the required transfer function will be

$$G(s) = \frac{100(1+10s)}{(1+10000s)(1+0.316s)^2}$$

Review Questions

1. Define frequency response of a system and discuss about the advantages of frequency response method for the analysis of linear systems.
2. Discuss the correlation between frequency response and transient response for a second-order system. Derive any expression used.
3. Explain the procedure to get the closed-loop frequency response from the loop transfer function.
4. Explain the procedure for experimentally determining the frequency response of an electric circuit containing several interconnected components. Draw a complete schematic diagram of the experimental set-up.
5. What are the different methods available for frequency response plot?
6. Discuss the advantages of frequency response analysis over the root locus technique.
7. Write down the frequency domain specifications.
8. Define the frequency domain specification for bandwidth.
9. What are the advantages and disadvantages of having large bandwidth?
10. Express the resonant peak M_{p} and the resonant frequency ω_{r} in terms of natural frequency of oscillation ω_n and damping ratio ξ for a unity feedback second-order system.
11. If a control system is required to have a given value for the resonant peak, describe how you would get the required value of the static loop gain K, given the loop transfer function.
12. Explain the effect of feedback on the gain and bandwidth of the system.
13. What do you understand by frequency response plot?
14. What are the unique features of Bode plots?
15. What is corner frequency in Bode plot?
16. Define minimum phase transfer function.
17. Explain the gain margin and phase margin.
18. Define gain and phase crossover frequencies.
19. Explain the significance of phase margin and gain margin in determining the relative stability of closed-loop systems.
20. What are the advantages and disadvantages of Bode plot?
21. What is the major difference between Bode magnitude plots for first-order system and second-order system?
22. Describe the change in open-loop frequency response magnitude plot if the time delay is added to the plant.
23. If the phase response of a pure time delay is plotted on a linear phase versus linear frequency plot, what would be the shape of the curve?
24. What will be initial slope of the Bode magnitude plot for a system having no poles and no zeros at origin?
25. Tabulate the phase angle versus frequency for the loop transfer function of a system given by $G(s)H(s)=\dfrac{s^2(s+8)}{(s+4)^2(s+0.2)}$.
26. A second-order system has a natural frequency of oscillation $\omega_n = 2.5$ rad/sec and damped frequency of oscillation $\omega_{\mathrm{d}} = 2.0$ rad/sec. Determine (i) when it is subjected to a step input, calculate its percentage overshoot and (ii) when it is subjected to a sinusoidal input, calculate the maximum ratio to its output (resonant peak).

27. An under damped second-order system exhibited 17% overshoot for a step input, and the frequency of oscillations was 0.5 Hz. Calculate the resonant peak M_{p} and the resonant frequency ω_{p} of the frequency response of this system.
28. A closed-loop transfer function of an automatic control system is given by

$$T(s) = \frac{1}{(1+0.1s)(1+0.15s+0.1s^2)}$$

Calculate (i) the peak response M_{p} and resonant frequency ω_{r} of the system and (ii) damping ratio and natural frequency of oscillation ω_{n} of the equivalent second-order system.
29. The overshoot of the unit step response of a second-order feedback system is 30 per cent. Determine (a) damping ratio ξ, (b) resonant peak M_{p}, (c) resonant frequency ω_{r}, (d) damping frequency ω_{d} and (e) time at which peak overshoot occurs. Assume natural frequency of oscillation $\omega_n = 10$ rad/sec.
30. Determine the bandwidth for the closed-loop transfer function given by

$$T(s) = \frac{5}{s^2+4s+5}$$

31. The transfer function of a system is given by $G(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$. Show that $\left|G(j\omega_n)\right| = \dfrac{1}{2\xi}$.
32. The closed-loop transfer function of a unity feedback control system is $T(s) = \dfrac{K}{s^2+As+K}$. The peak overshoot for a unit step input is 25 per cent. The damped frequency of oscillation is 10 rad/sec. Determine A, K, M_{p}, bandwidth and the time at which the first overshoot occurs. In addition, find the steady state error of the system if $r(t) = 10t$.
33. For $T(s) = \dfrac{K}{s^2+As+K}$, if $M_{\mathrm{p}} = 1.04$ and $\omega_{\mathrm{r}} = 11.55$ rad/sec, determine K and A and hence calculate the settling time and bandwidth.
34. A control system is represented by Fig. Q8.34.

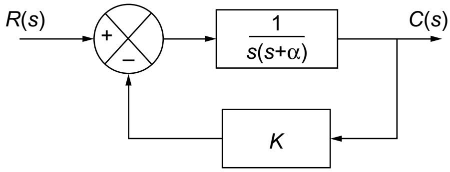


Fig. Q8.34

Determine (i) the values of K and α so that $M_{\mathrm{r}} = 1$ and $\omega_{\mathrm{r}} = 12$ rad/sec and (ii) corresponding bandwidth.

35. The transfer function of a system is given by $G(s)=\dfrac{100Ke^{-Ts}}{s(s^2+10s+100)}$. (a) When $K=1$, determine the maximum time delay T in seconds for the closed-loop system to be stable and (b) when $T=1$ sec determine the maximum gain K for the system to be stable.
36. For a unity feedback system shown in Fig. Q8.36, determine (a) M_r, (b) ω_r and (c) bandwidth of the system for $K = 2, 21.39$ and 100.

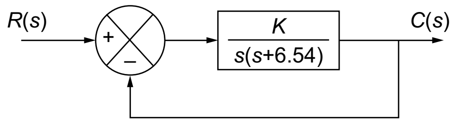


Fig. Q8.36

37. Consider a unity feedback system whose loop transfer function is $G(s)=\dfrac{K}{s(s^2+s+0.5)}$.

    Determine gain K such that the resonant peak magnitude in the frequency response is 2 dB or $M_r = 2$ dB.
38. A unit step input is applied to a feedback control system whose open-loop transfer function is given by $G(s)=\dfrac{K}{s(sT+1)}$. Determine K and T given that maximum overshoot is 26 per cent and resonant frequency is $\omega_r = 8$ rad/sec. Calculate resonance peak M_p, gain crossover frequency and phase margin.
39. For each closed-loop system with the following performance characteristics, determine the closed-loop bandwidth for (a) $\xi=0.2, T_s=3\,\text{sec}$, (b) $\xi=0.2, T_p=3\,\text{sec}$, (c) $T_s=4\,\text{sec}, T_p=2\,\text{sec}$ and (d) $\xi=0.3, T_r=4\,\text{sec}$.
40. Determine the resonant frequency ω_r, resonant peak M_p and bandwidth for the system whose transfer function is given by $\dfrac{C(s)}{R(s)}=\dfrac{5}{s^2+2s+5}$.
41. Determine the phase crossover frequency, gain crossover frequency, gain margin and phase margin for the given system with $H(s)=1$ and $G(s)=\dfrac{1}{s(s+1)(s+0.5)}$ without plotting the Bode plot. Also comment on the stability of the stability.
42. Determine analytically the gain margin and phase margin for the system with the following transfer function $G(s)H(s)=\dfrac{5}{s\left(1+\dfrac{s}{2}\right)\left(1+\dfrac{s}{6}\right)}$. Also determine the gain crossover frequency and phase cross over frequency.
43. The transfer function of an electric circuit is $\dfrac{20}{10s^2+4s+1}$. Determine (a) gain margin and phase margin and (b) gain and phase angle at a frequency of 1 kHz.
44. An amplifier is subjected to a sinusoidal voltage gain as $v = 1.25\sin 100t$. Determine the amplitude of output signal and the phase difference between output and input. The amplifier transfer function is $\dfrac{10}{0.01s+1}$.

45. A sinusoidal waveform $10 \sin 120t$ is fed to a constant block with a transfer function of 0.675. Determine (a) signal gain, (b) output amplitude and (c) phase difference between input and output.
46. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10}{s(1+0.4s)(1+0.1s)}$. Sketch the Bode plot for the given system. Also, determine the frequency domain specifications.
47. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{5}{s(0.6s+1)(1+0.1s)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
48. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10}{s(0.05s+1)(1+0.1s)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
49. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10(1+s)}{s(0.5s+1)(5s+1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
50. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{2}{s(s+1)(s+0.2)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
51. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10(1+0.5s)}{s(0.1s+1)(0.2s+1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
52. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{100}{s(s+1)(0.2s+1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
53. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{2}{s(0.5s+1)(0.2s+1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
54. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10}{s(0.1s+1)(0.01s+1)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.
55. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{200(s+2)}{s(s^2+10s+100)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.

56. Determine the gain and phase margins of the system using Table Q8.56.

Table Q8.56

Frequency ω (rad/sec)	**Gain**	**Magnitude (in dB)**	**Phase angle (in degrees)**
0. 1	5	14	0
1	1.5	3.5	–90
10	1	0	–135
20	0.25	–12	–180
100	0.1	–20	–235

57. The loop transfer function of a system is $G(s)H(s)=\dfrac{300e^{-0.05s}}{s(s+6)(s+10)}$. Determine the gain margin and phase margin using Bode plot.
58. A constant block has a magnitude of 20 dB at 100 rad/sec. Determine (a) block transfer function, (b) magnitude at a frequency of 1,000 rad/sec and (c) phase angle at 1,000 rad/sec.
59. Determine the stability of the system using Bode plot whose block diagram is given in Fig. Q8.59.

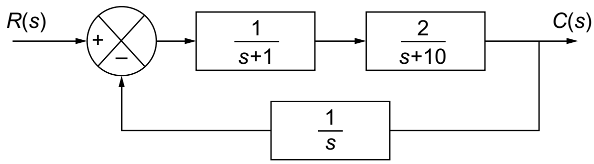


Fig. Q8.59

61. A unity feedback control system is characterized by an open-loop transfer function

$$G(s)H(s)=\frac{K}{s(1+T_1s)(1+T_2s)}T_1,T_2>0, K>0 .$$

Show that the gain margin of the system is $\dfrac{T_1+T_2}{K\sqrt{T_1T_2}}$.

62. A negative feedback control system is characterized by an open-loop transfer function $G(s)H(s)=\dfrac{K}{s(s+1)(0.1s+1)}$. Determine (a) K for a gain margin g_m of 10 dB, (b) K for a g_m of 30 dB, (c) K to give phase margin p_m of 24° and (d) the marginal value of K for stability.
63. At the phase crossover frequency, the gain of the system is $\dfrac{KT_1T_2}{(T_1+T_2)}$. What is the corresponding gain margin in dB?
64. The block diagram of a floppy disc is shown in Fig. Q8.64.

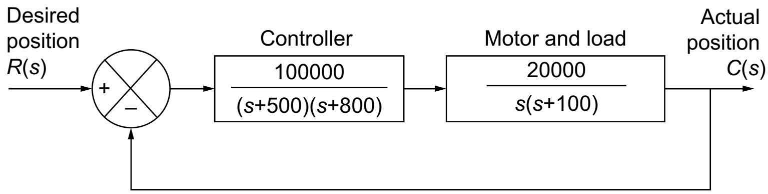


Fig. Q8.64

Use frequency response technique to determine (a) g_m, p_m, zero dB frequency 180° frequency, (b) closed-loop bandwidth, percentage overshoot, settling time and peak time and also comment on the stability of the system.

65. Draw the Bode plot for the transfer function $G(s) = \dfrac{K}{s(1+sT_1)(1+sT_2)}$.

66. The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s(T_1 s+1)(T_2 s+1)}.$$

Derive an expression for gain K in terms of T_1 *and* T_2 and specified gain margin g_m.

67. For the given transfer function $G(s) = \dfrac{10(1+s)e^{-0.01s}}{s(1+0.5s)}$, sketch the Bode magnitude and phase angle plot. Obtain gain margin and phase margin.

68. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{K}{s(s+4)(s+10)}$. Sketch the Bode plot for the given system. Also, determine the gain K when the gain margin is 10 dB.

69. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{K}{s(s+4)(s+10)}$. Sketch the Bode plot for the given system. Also, determine the gain K when the phase margin is 30°.

70. The loop transfer of a given system is given by $G(s)H(s) = \dfrac{K}{s(1+0.2s)(1+0.02s)}$. Sketch the Bode plot for the given system. Also, determine the gain margin and phase margin of the system.

71. The loop transfer function of a system is given by $G(s)H(s) = \dfrac{10}{s(0.5s+1)(1+0.1s)}$. Sketch the Bode plot for the given system and determine the frequency domain specifications.

72. A servo control system of type 0, has a transfer function $G(s)H(s) = \dfrac{100(1+0.1s)}{(1+s)}$. Sketch the Bode plot and comment on the stability of the system.

73. Explain the Bode plot for the following functions of a unity feedback system.

(i) $G(j\omega) = K$ (ii) $G(j\omega) = (j\omega)^{\pm n}$ and (iii) $G(j\omega) = (1 + j\omega T)^{\pm n}$.

74. Explain the Bode plot for the following functions of a unity feedback system:

(i) $G(j\omega) = e^{-j\omega T}$ and (ii) $G(j\omega) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$.

75. The closed-loop control system incorporating an amplifier with a gain of K is shown in Fig. Q8.75.

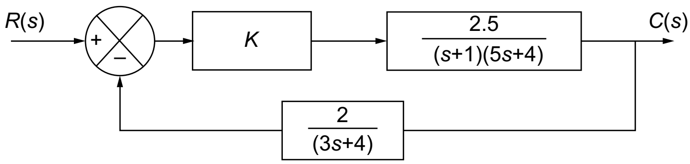


Fig. Q8.75

Using Bode plot, determine (a) is the system stable for a gain $K = 1$ and (b) what is the maximum value of K before the system becomes unstable.

76. Draw the Bode plot for the system shown in Fig. Q8.76. Determine (i) g_m and p_m (ii) whether the closed-loop system is stable (iii) obtain the closed-loop frequency response and plot on a Bode diagram. Find the resonance frequency and bandwidth. What are the values of δ, ω_n and settling time?

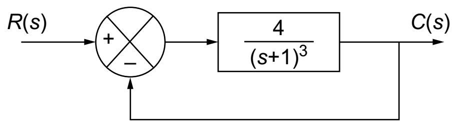


Fig. Q8.76

77. Sketch the Bode plot for the loop transfer function $G(s)H(s) = \dfrac{2e^{-sT}}{s(s+1)(1+0.5s)}$. Determine the maximum value of T for the system to be stable.

78. Draw the Bode plot for the transfer function $G(s)H(s) = \dfrac{(1+sT_a)(1+sT_b)}{(1+sT_1)(1+sT_2)}$, where $T_1 > T_a > T_b > T_2$.

79. For the following transfer function, identify the corner frequencies, the initial slope and the final slope of the magnitude plot $G(s)H(s) = \dfrac{s(s+8)^2}{(s^2+2s+2)(s^2+2s+4)}$

80. Consider a system as shown in Fig. Q8.80.

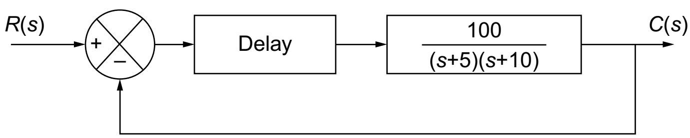


Fig. Q8.80

Determine (a) phase margin for the time delay of 0, 0.1, 0.2, 0.5 and 1 s (b) gain margin for the time delay mentioned in (a), (c), for what time delay mentioned in (a), the system is stable and (d) for each time delay that makes the system unstable, how much reduction in gain is required for the system to be stable?

81. The open-loop transfer function of a unity feedback system is $G(s)=\dfrac{Ke^{-0.1s}}{s(1+0.1s)(1+s)}$.

    Using Bode plot, determine (a) K so that the g_m is 20 dB, (b) K so that the p_m is 60° and (c) K so that the resonant peak M_p of the system is 1 dB. What are the corresponding values of ω_r and ω_n and (d) K so that the bandwidth ω_b of the system is 1.5 rad/sec.

82. Describe the asymptotic phase response of a system with a single pole at −2.

83. For a system with three poles at −4, what is the maximum difference between the asymptotic approximation and the actual magnitude response?

84. The loop transfer function of a system is given by $G(s)H(s)=\dfrac{K(T_a s+1)(T_b s+1)}{s^2(Ts+1)}$. Sketch the Bode plot for the following two cases: (i) $T_b > T_a > T > 0$ and (ii) $T > T_a > T_b > 0$.

85. Consider a system as shown in Fig. Q8.85. Determine the loop transfer function and sketch the Bode diagram for the same. Also, determine the value of gain K such that the phase margin is 50°. What is the gain margin of this system with this gain K?

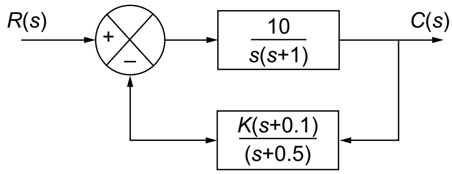


Fig. Q8.85

86. The block diagram of a control system is shown in Fig. Q8.86. Determine the range of the gain K for stability.

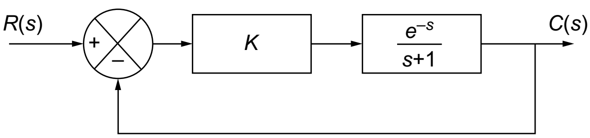


Fig. Q8.86

87. Determine the gain margin, phase margin, damping factor and critical frequency of oscillation for the system with unity feedback and has a transfer function $G(s)=\dfrac{4(s+1)}{s^2}$.

88. Sketch the Bode plot for the loop transfer function of the system

$$G(s)H(s)=\frac{100K(s+5)(s+40)}{s^3(s+100)(s+200)}.$$

Determine the range of K for closed-loop stability.

89. Determine the gain and phase angle of the system at a frequency of 1 rad/sec. The transfer function of the system is $G(s) = (s+1)$.
90. The transfer function of the system is given by $G(s) = \frac{1}{(4s+1)}$. Determine the frequency response of the system over a frequency range of 0.1–10 rad/sec.
91. Determine the gain and phase angle of the given system at a frequency of 0.1 Hz. The transfer function of the system is $G(s) = (10s+1)(2s+5)$.

This Page is intentionally left blank.

9

POLAR AND NYQUIST PLOTS

9.1 Introduction to Polar Plot

The presence of magnitude plot and phase plot for showing the variation of gain and phase angle of a system with respect to the change in frequency is the major disadvantage of the Bode plot. The plot that combines both the plots to a single plot without losing any information is called polar plot. The polar plot can be plotted either on a polar graph or on an ordinary graph. Hence, polar plot for a particular system can be defined in two ways based on the graph used for plotting. The two different definitions for polar plot are given below:

The polar plot of a loop transfer function $G(j\omega)H(j\omega)$ is defined as a plot of $\left|G(j\omega)H(j\omega)\right|$ versus $\angle G(j\omega)H(j\omega)$ on the polar coordinates as the frequency ω varies from zero to infinity when it is plotted on a polar graph. Also, it is defined as a plot of real part of $G(j\omega)H(j\omega)$ versus imaginary part of $G(j\omega)H(j\omega)$ as the frequency ω varies from zero to infinity when it is plotted on an ordinary graph. The real and imaginary parts of the transfer function $G(j\omega)H(j\omega)$ can be denoted as $G_R(j\omega)H_R(j\omega)$ and $G_I(j\omega)H_I(j\omega)$ respectively.

The concentric circles present in the polar graph represent the magnitude of the transfer function and the radial lines crossing the circles represent the phase angle of the transfer function. The phase angle of the transfer function $\angle G(j\omega)H(j\omega)$ can either be positive or negative. Hence, it can be noted that on a polar graph, positive phase angle is measured in counterclockwise direction and negative phase angle is measured in clockwise direction. A simple example of polar plot of a transfer function on an ordinary graph and on a polar graph is shown in Figs. 9.1(a) and (b) respectively.

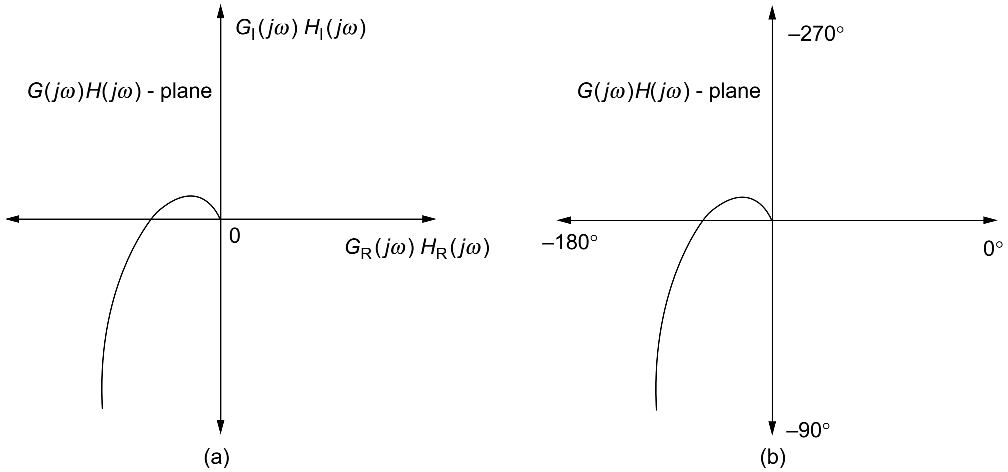


Fig. 9.1 | Polar plot on a graph

9.2 Starting and Ending of Polar Plot

The polar plot of any system starts from one quadrant on an ordinary graph and ends at the other quadrant. The starting and ending point of the polar plot depends on the TYPE and ORDER of the loop transfer function.

The starting point of the polar plot depends on the TYPE of the loop transfer function of a given system as shown in Fig. 9.2(a). If the TYPE of the loop transfer function of a given system n is greater than 3, then the starting point of such system follows the system with TYPE $(n-4)$.

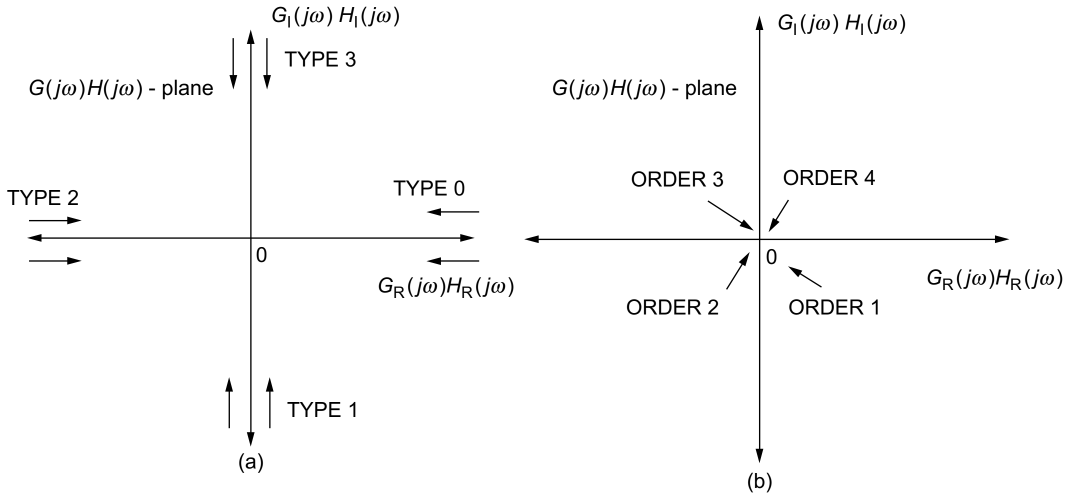


Fig. 9.2 | Starting and ending points of polar plot

The ending point of the polar plot depends on the ORDER of the loop transfer function as shown in Fig. 9.2(b). Similar to the starting point of the polar plot, if the ORDER of the loop transfer function of the given system $m > 4$, then the ending point of such system follows the system with ORDER $(m-5)$.

It can be noted that this method of determining the starting and ending points of the polar plot is applicable to the system whose loop transfer function has only poles. In addition, it is applicable only if the plot is plotted on an ordinary graph sheet.

9.3 Construction of Polar Plot

The construction of polar plot can be illustrated by considering the generalized form of loop transfer function as

$$G(s)H(s) = \frac{e^{-sT} \times s^Z K_s \prod_{k=1}^{Q}\left(s^2 + 2\xi_k \omega_{nk} s + \omega_{nk}^2\right)\prod_{i=1}^{M}\left(sT_i + 1\right)}{s^N \prod_{m=1}^{V}\left(s^2 + 2\xi_m \omega_{nm} s + \omega_{nm}^2\right)\prod_{l=1}^{U}\left(sT_l + 1\right)}$$

where K_s, T_i, T_l, ξ_k, ξ_m, ω_{nk} and ω_{nm} are real constants,

Z is the number of zeros at the origin,
N is the number of poles at the origin or the TYPE of the system,
M is the number of simple poles existing in the system,
U is the number of simple zeros existing in the system,
V is the number of complex poles existing in the system,
Q is the number of complex zeros existing in the system and
T is the time delay in seconds.

Substituting $s = j\omega$ in the above equation and simplifying, we obtain

$$G(j\omega)H(j\omega) = \frac{e^{-(j\omega)T} \times (j\omega)^Z K_s \prod_{k=1}^{Q}\left((j\omega)^2 + 2\xi_k \omega_{nk}(j\omega) + \omega_{nk}^2\right)\prod_{i=1}^{M}\left(j\omega T_i + 1\right)}{(j\omega)^N \prod_{m=1}^{V}\left((j\omega)^2 + 2\xi_m \omega_{nm}(j\omega) + \omega_{nm}^2\right)\prod_{l=1}^{U}\left(j\omega T_l + 1\right)} \tag{9.1}$$

Its magnitude is

$$\left|G(j\omega)H(j\omega)\right| = \frac{(\omega)^Z K_s \prod_{k=1}^{Q}\left(\sqrt{(2\xi_k \omega_{nk}\omega)^2 + \left(\omega_{nk}^2 - \omega^2\right)^2}\right)\prod_{i=1}^{M}\left(\sqrt{(\omega T_i)^2 + 1}\right)}{(\omega)^N \prod_{m=1}^{V}\left(\sqrt{(2\xi_m \omega_{nm}\omega)^2 + \left(\omega_{nm}^2 - \omega^2\right)}\right)\prod_{l=1}^{U}\left(\sqrt{(\omega T_l)^2 + 1}\right)} \tag{9.2}$$

and the phase angle is

$$\angle G(j\omega)H(j\omega) = -57.32\omega T + \left(Z \times \left(\frac{\pi}{2}\right)\right) + \sum_{k=1}^{Q} \tan^{-1}\left(\frac{2\xi_{\kappa} u_{nk}}{1-u_{nk}^2}\right) + \sum_{i=1}^{M} \tan^{-1}(\omega T_i) - \left(N \times \left(\frac{\pi}{2}\right)\right)$$
$$- \sum_{m=1}^{V} \tan^{-1}\left(\frac{2\xi_m u_{nm}}{1-u_{nm}^2}\right) - \sum_{l=1}^{U} \tan^{-1}(\omega T_l) \tag{9.3}$$

where the magnitude of the term e^{-sT} is 1 and phase angle of the term e^{-sT} is $-57.32 \times \omega T$.

From Eqn. (9.1), it is clear that the open-loop transfer function $G(j\omega)H(j\omega)$ may contain the combination of any of the following five factors:

(i) Constant K
(ii) Zeros at the origin $(j\omega)^{Z}$ and poles at the origin $(j\omega)^{-N}$
(iii) Simple zero $(1+j\omega T)$ and simple pole $(1+j\omega T)^{-1}$
(iv) Complex zero $(1+j2\xi u_n - u_n^2)$ and complex pole $(1+j2\xi u_n - u_n^2)^{-1}$
(v) Transportation lag $e^{-(j\omega)T}$

Hence, it is necessary to have a complete study of magnitude and phase angle of these factors that can be utilized in constructing the polar plot of a composite loop transfer function $G(j\omega)H(j\omega)$. The polar plot for the loop transfer function $G(j\omega)H(j\omega)$ is constructed by determining its magnitude and phase angle. The magnitude and phase angle of the loop transfer function can be obtained by the multiplication of magnitudes and addition of the phase angles of individual factors present in the function.

Also, if the polar plot of $G(j\omega)H(j\omega)$ is to be plotted on an ordinary graph, the magnitude and phase angle of $G(j\omega)H(j\omega)$ obtained at different frequencies are converted to the real and imaginary parts of $G(j\omega)H(j\omega)$. Hence, it becomes necessary to determine the magnitude and phase angle of each factor which can be possibly present in a system.

Factor 1: Constant K

The constant K is independent of frequency ω and hence the magnitude and phase angle of the factor are:

Magnitude : K
Phase angle : 0°

In addition, the real and imaginary parts of constant K are K and 0 respectively.

Since the magnitude, real part and imaginary part of constant K are independent of frequency, the polar plot of constant K is a point on both polar graph and ordinary graph. The polar plot for constant K on polar graph and on an ordinary graph is shown in Figs. 9.3(a) and (b) respectively.

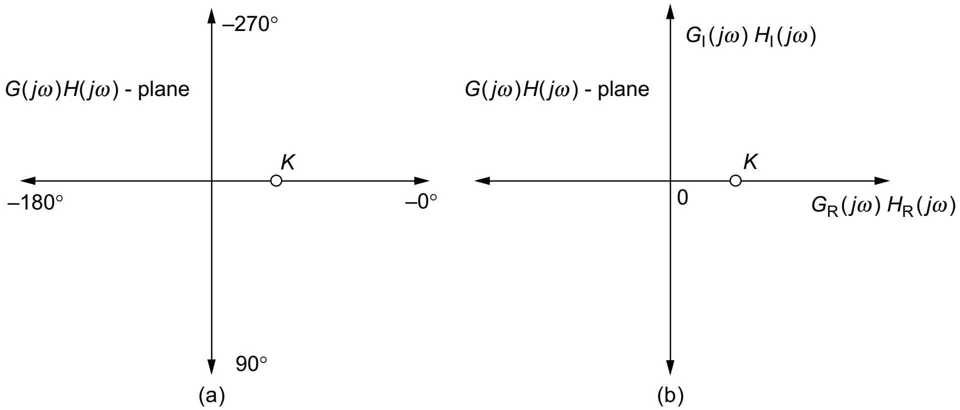


Fig. 9.3 | Polar plot for constant K

Factor 2: Zeros at the origin $(j\omega)^Z$ and poles at the origin $(j\omega)^{-N}$

The zeros at the origin $(j\omega)^Z$ and poles at the origin $(j\omega)^{-N}$ are dependent on frequency ω. Hence, the magnitude and phase angle when one pole or zero exists at the origin are given in Table 9.1(a).

Since the magnitude and phase angle of the factors are dependent on frequency ω, the polar plot for the factor on polar graph can be drawn with the help of Table 9.1(a).

Table 9.1(a) | Magnitude and Phase angle for $j\omega$ and $(j\omega)^{-1}$ for different values of frequency ω

Frequency ω	Zero at the origin $j\omega$		Pole at the origin $(j\omega)^{-1}$	
	Magnitude, ω	Phase angle, 90°	Magnitude ω^{-1}	Phase angle, 90°
0.01	0.01	90°	100	–90°
0.1	0.1	90°	10	–90°
1	1	90°	1	–90°
∞	∞	90°	0	–90°

The polar plot for $j\omega$ and $(j\omega)^{-1}$ on polar graph is shown in Figs. 9.4(a) and (b) respectively.

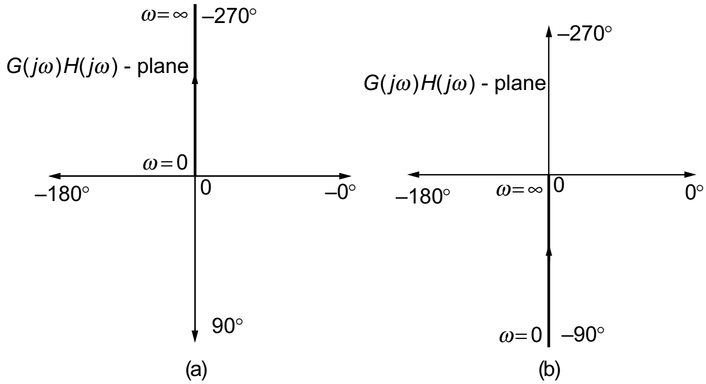


Fig. 9.4 | Polar plot for $j\omega$ and $(j\omega)^{-1}$ on polar graph

When more than one pole or zero exist at the origin, the changes to be done in plotting the polar plot on polar graph is given in Table 9.1(b).

Table 9.1(b) | Magnitude and phase angle of $(j\omega)^{Z}$ and $(j\omega)^{-N}$

Factors	Magnitude	Phase angle
Zeros at the origin $(j\omega)^{Z}$	$(\omega)^{Z}$	$(90^\circ \times Z)$
Poles at the origin $(j\omega)^{-N}$	$(\omega)^{-N}$ or $\left(\frac{1}{\omega}\right)^{N}$	$(-90^\circ \times N)$

When more than one pole or zero exist at the origin, the changes to be done in plotting the polar plot on an ordinary graph are not as easy as they are on polar graph. The reason is that depending on Z and N, the real and imaginary parts vary. For example, if $Z=1$, the real part is zero, whereas if $Z=2$, the imaginary part is zero.

Therefore, the polar plot for different values of Z and N rotates by 90° either in the clockwise direction or in the anticlockwise direction on an ordinary graph. The polar plots of $(j\omega)^{Z}$ for different values of Z on an ordinary graph are shown in Figs. 9.4(c) through (f).

Similarly, the polar plot of $(j\omega)^{-N}$ for different values of N can be drawn on an ordinary graph. In addition, the polar plots of $(j\omega)^{Z}$ and $(j\omega)^{-N}$ for different values of Z and N can be drawn on polar graph.

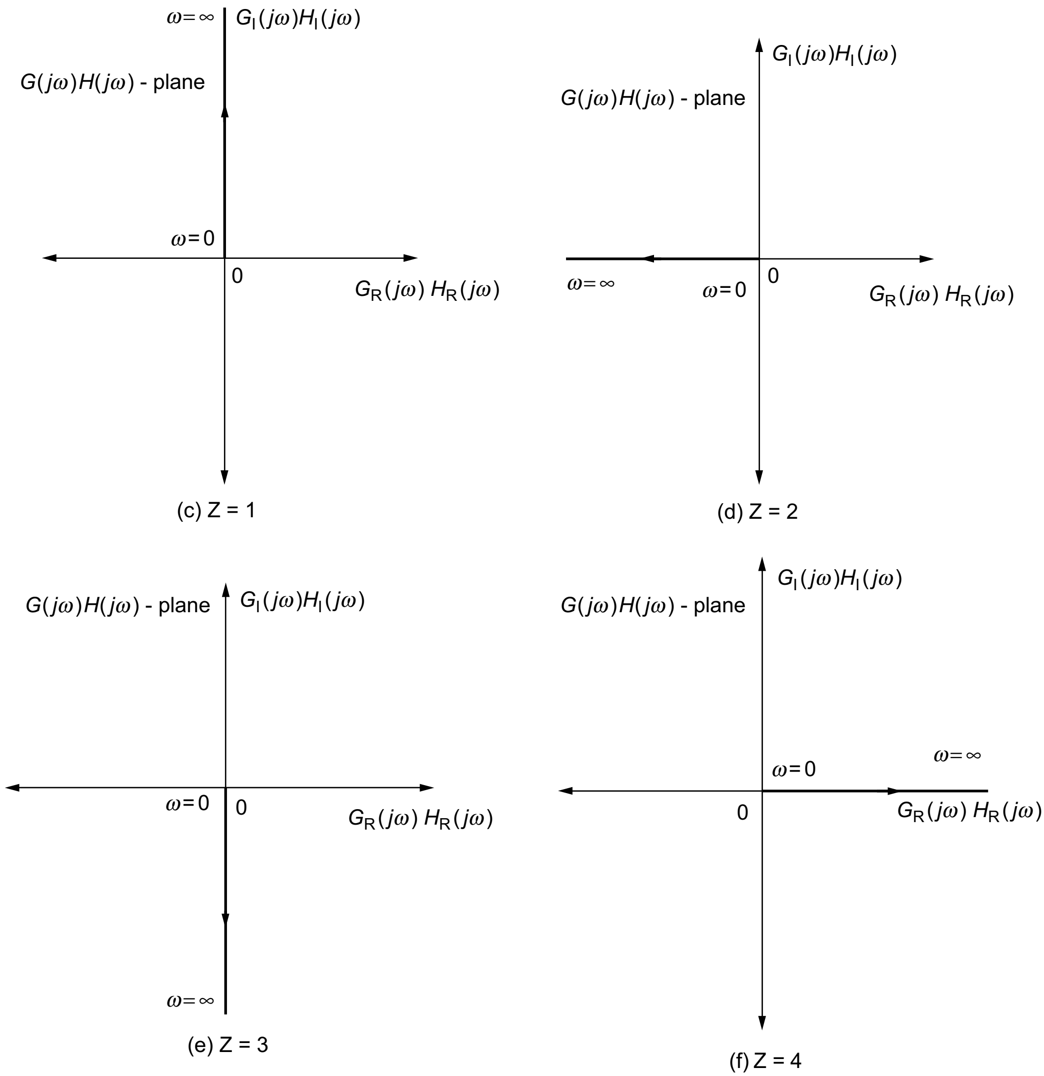


Fig. 9.4 | Polar plot for different values of Z

It can be noted that if the polar plot of $\left(j\omega\right)^{Z}$ is drawn on an ordinary graph and it is rotated by $180°$ in the clockwise or anticlockwise direction, the polar plot of $\left(j\omega\right)^{N}$ can be obtained, provided $Z = N$.

Factor 3: Simple zero $(1 + j\omega T)$ and simple pole $(1 + j\omega T)^{-1}$

The simple zero $\left(1+j\omega T\right)$ and simple pole $\left(1+j\omega T\right)^{-1}$ are dependent on frequency ω. Hence, the magnitude and phase angle of simple zero and simple pole are given in Table 9.2.

Since the magnitude and phase angle are dependent on frequency ω, the polar plot for a simple zero and simple pole on polar graph can be drawn using Table 9.2.

Table 9.2 | Magnitude and Phase angle for $(1+j\omega T)$ and $(1+j\omega T)^{-1}$ for different values of frequency ω

Frequency ω	Simple zero $(1+j\omega T)$		Simple pole $(1+j\omega T)^{-1}$	
	Magnitude, $\sqrt{1+(\omega T)^2}$	Phase angle, $\tan^{-1}(\omega T)$	Magnitude, $\frac{1}{\sqrt{1+(\omega T)^2}}$	Phase angle, $-\tan^{-1}(\omega T)$
0	1	0°	1	–0°
$\frac{1}{T}$	$\sqrt{2}$	45°	$\frac{1}{\sqrt{2}}$	–45°
∞	∞	90°	0	–90°

The polar plots for $(1+j\omega T)$ and $(1+j\omega T)^{-1}$ on the polar graph are shown in Figs. 9.5(a) and (b) respectively.

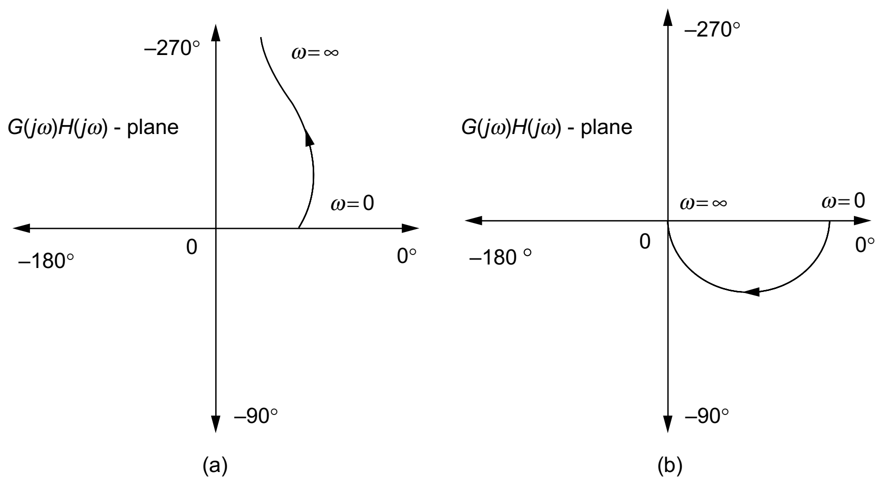


Fig. 9.5 | Polar plot for $(1+j\omega T)$ and $(1+j\omega T)^{-1}$ on polar graph

Factor 4: Complex zero $(1 + j2\xi u_n - u_n^2)$ and complex pole $(1 + j2\xi u_n - u_n^2)^{-1}$

The complex zero $\left(1+j2\xi u_n-u_n^2\right)$ and complex pole $\left(1+j2\xi u_n-u_n^2\right)^{-1}$ are dependent on frequency ω. Hence, the magnitude and phase angle of the factors are given in Table 9.3(a).

Table 9.3(a) | Magnitude and phase angle of $\left(1+j2\xi u_n-u_n^2\right)$ and $\left(1+j2\xi u_n-u_n^2\right)^{-1}$

Factors	Magnitude	Phase angle
Complex zero $\left(1+j2\xi u_n-u_n^2\right)$	$\sqrt{\left(1-u_n\right)^2+\left(2\xi u_n\right)^2}$	$\tan^{-1}\left(\frac{2\xi u_n}{1-u_n}\right)$
Complex pole $\left(1+j2\xi u_n-u_n^2\right)^{-1}$	$\frac{1}{\sqrt{\left(1-u_n\right)^2+\left(2\xi u_n\right)^2}}$	$-\tan^{-1}\left(\frac{2\xi u_n}{1-u_n}\right)$

In Table 9.3(a), ξ is the damping ratio and $u_n=\frac{\omega}{\omega_n}$.

Since the magnitude and phase angle of the factors are dependent on frequency ω, the polar plot for the factor on the polar graph can be drawn using Table 9.3(b).

Table 9.3(b) | Magnitude and Phase angle for $\left(1+j2\xi u_n-u_n^2\right)$ and $\left(1+j2\xi u_n-u_n^2\right)^{-1}$ for different values of frequency ω

Frequency ω	Complex zero $(1 + j2\xi u_n - u_n^2)$		Complex pole $(1 + j2\xi u_n - u_n^2)^{-1}$	
	Magnitude	Phase angle	Magnitude	Phase angle
0	1	0°	1	0°
ω_n	2ξ	90°	$\frac{1}{2\xi}$	–90°
∞	∞	180°	0	–180°

The polar plots for $\left(1+j2\xi u_n-u_n^2\right)$ and $\left(1+j2\xi u_n-u_n^2\right)^{-1}$ on the polar graphs for different values of ξ are shown in Figs. 9.6(a) and (b) respectively.

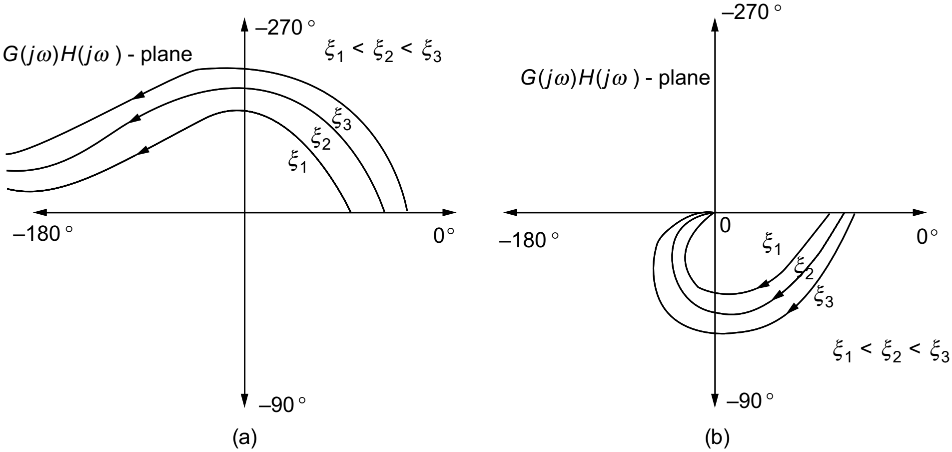


Fig. 9.6 | Polar plots for $\left(1+j2\xi u_n-u_n^2\right)$ and $\left(1+j2\xi u_n-u_n^2\right)^{-1}$ on the polar graph

Factor 5: Transportation lag $e^{-j\omega T}$

The factor $e^{-j\omega T}$ can be rewritten as $e^{-j\omega T}=\cos(\omega T)-j\sin(\omega T)$. The magnitude and phase angle of the factor are

$$\text{Magnitude of } e^{-j\omega T}=\cos^2(\omega T)+\sin^2(\omega T)=1$$

and $$\text{Phase angle}=\tan^{-1}\left(-\frac{\sin(\omega T)}{\cos(\omega T)}\right)=\tan^{-1}\left(-\tan(\omega T)\right)=\tan^{-1}\left(\tan(-\omega T)\right)=-\omega T$$

But the phase angle obtained is in the unit of radians. Therefore, the phase angle in degrees is obtained as

$$\angle G(j\omega)H(j\omega)=-\omega T\times\frac{180°}{\pi}=-57.32°\times\omega T$$

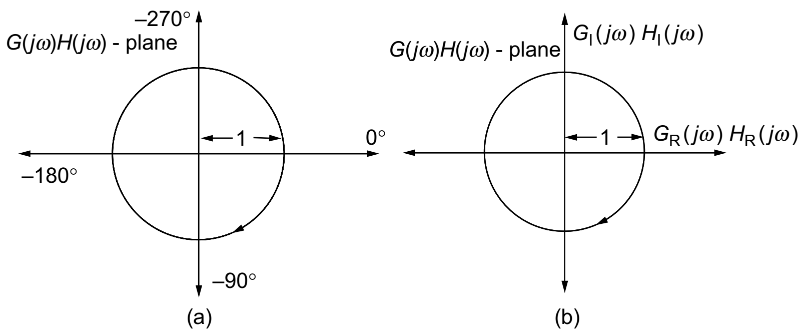


Fig. 9.7 | Polar plot of transportation lag

Since the magnitude of the factor is independent of frequency ω, the polar plot for the factor on polar graph and on an ordinary graph is a unit circle, which are shown in Figs. 9.7(a) and (b) respectively.

9.4 Determination of Frequency Domain Specification from Polar Plot

The different frequency domain specifications such as gain margin, phase margin, phase crossover frequency and gain crossover frequency can be determined by using polar plot. As the polar plots can be plotted both on polar graph and ordinary graph, the determination of frequency domain specification also differs.

9.4.1 Gain Crossover Frequency ω_{gc}

It is the frequency at which the magnitude of the loop transfer function $|G(j\omega)H(j\omega)|$ is unity. This frequency can be obtained by substituting random frequencies in Eqn. (9.2). But by this process, we can obtain only an approximate value of frequency and also this process is time-consuming. Hence, there exist two alternate methods for determining ω_{gc} which are discussed below:

Method 1: It is known that $|G(j\omega)H(j\omega)|=1$ at ω_{gc}. Also, Eqn. (9.2) is a function of frequency. Therefore, by solving the equation, we obtain ω_{gc}. This method consumes more time.
Method 2: From the polar plot drawn on the polar graph, it is easy to determine the $\angle G(j\omega)H(j\omega)$ when $|G(j\omega)H(j\omega)|=1$. Substituting the obtained angle in Eqn. (9.3) and using $\tan(A+B)$, it is easy to determine ω_{gc}.

Hence, method 2 is used in this chapter for determining the gain crossover frequency ω_{gc} when a polar plot for a loop transfer function is drawn.

9.4.2 Phase Crossover Frequency ω_{pc}

It is the frequency at which the phase angle of the loop transfer function $\angle G(j\omega)H(j\omega)$ is $180°$ or $-180°$. This frequency ω_{pc} can be obtained by substituting random frequencies in Eqn. (9.3). But by this process, we can obtain only an approximate value of frequency and also this process is a time-consuming one. Hence, an alternate method exists for determining ω_{pc} as discussed below:

Alternate method
It is known that $\angle G(j\omega)H(j\omega)=180°$ at ω_{pc}. Hence, using $\tan(A+B)$, the value of ω_{pc} can be determined.

9.4.3 Gain Margin g_m

The gain margin g_m of the system obtained by taking the inverse of the magnitude of $|G(j\omega)H(j\omega)|$ at phase crossover frequency ω_{pc}.

$$g_m = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

9.4.4 Phase Margin p_m

The phase margin p_m of the system is obtained by

$$p_m = 180^\circ + \angle G(j\omega)H(j\omega)\Big|_{\omega=\omega_{gc}}$$

9.5 Procedure for Constructing Polar Plot

The flow chart for constructing the polar plot of a system on a polar graph is shown in Fig. 9.8(a).

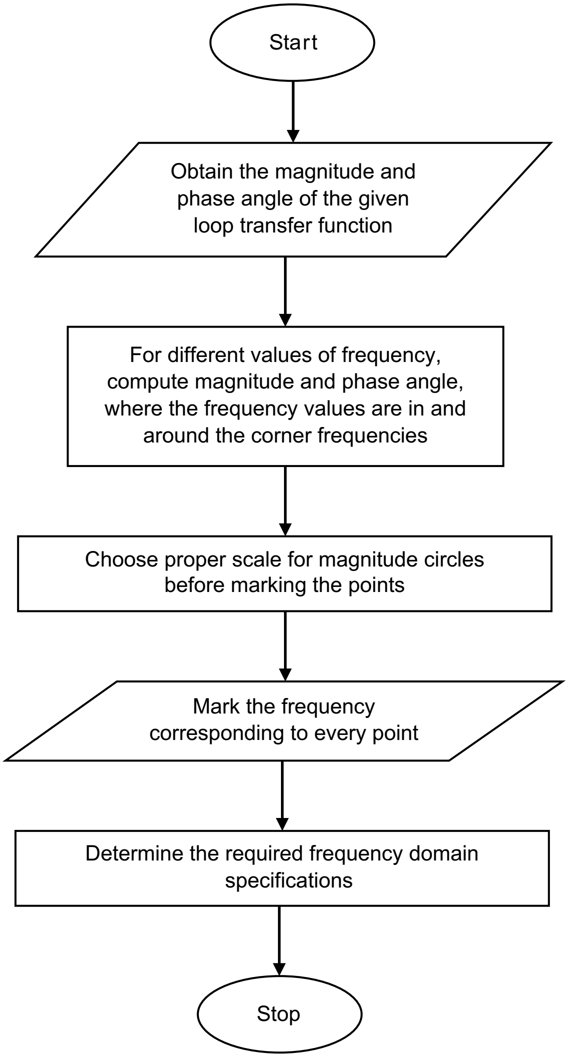


Fig. 9.8(a) | Flow chart for constructing the polar plot on polar graph

The flowchart for constructing the polar plot of a system on an ordinary graph is shown in Fig. 9.8(b).

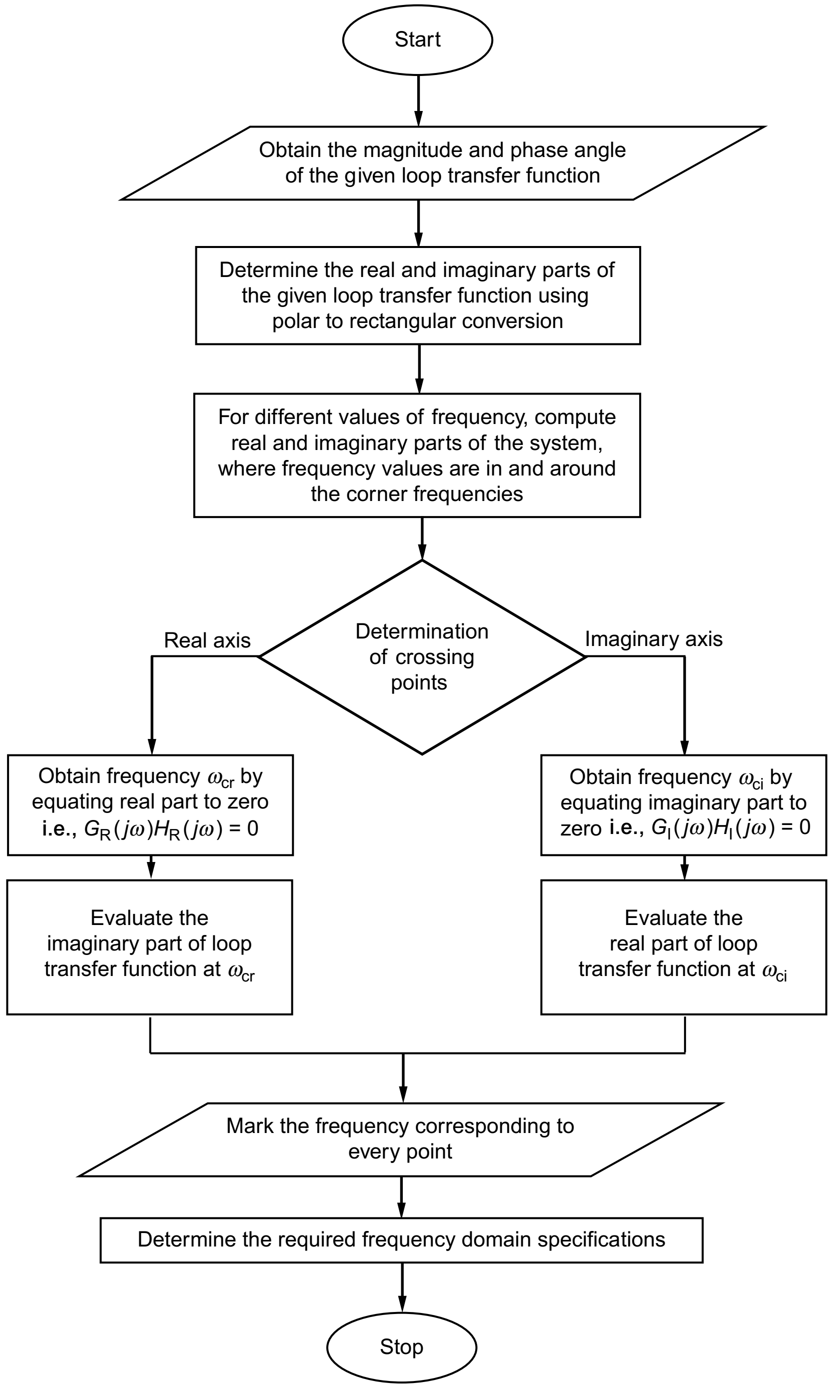


Fig. 9.8(b) | Flow chart for constructing the polar plot on an ordinary graph

Intersection of polar plot for a system

If the polar plot intersects $0°$ line on a polar graph, it implies that the polar plot intersects the positive real axis line on an ordinary graph. Therefore, there exists a relation between the intersection points on an ordinary graph and on a polar graph which is listed in Table 9.4.

Table 9.4 | Relation between intersection points

Intersection point on polar graph	Intersection point on an ordinary graph
0°	Positive real axis
–90°	Negative imaginary axis
–180°	Negative real axis
–270°	Positive imaginary axis

In addition, it is to be noted that if n number of intersection points exists in plotting the polar plot on a polar graph, same number of intersection points exists in plotting the polar plot on an ordinary graph.

9.6 Typical Sketches of Polar Plot on an Ordinary Graph and Polar Graph

The typical sketches of polar plot on an ordinary and polar graph based on the TYPE and ORDER of a system are given below.

TYPE 0 ORDER 2 System

Let the loop transfer function of TYPE 0 ORDER 2 system be

$$G(s)H(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

The magnitude and phase angle of the system are given by

$$|G(j\omega)H(j\omega)| = \left|\frac{1}{(1+j\omega T_1)(1+j\omega T_2)}\right|$$

and

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

The magnitude and phase angle at $\omega = 0$ and $\omega = \infty$ are given in Table 9.5.

Table 9.5 | Magnitude and phase angle of the system

Frequency ω	Magnitude $\lvert G(j\omega)\ H(j\omega)\rvert$	Phase angle $\angle G(j\omega)\ H(j\omega)$
0	1	0°
∞	0	–180°

To determine the intersection point on real and imaginary axis

The real and imaginary part of loop transfer function can be determined as

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} \times \frac{(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1-j\omega T_2)}$$

$$= \frac{1-\omega^2 T_1 T_2 - j\omega(T_1+T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$$

Therefore, real part of loop transfer function $G_R(j\omega)H_R(j\omega) = \dfrac{1-\omega^2 T_1 T_2}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$

and imaginary part of loop transfer function $G_I(j\omega)H_I(j\omega) = \dfrac{-\omega(T_1+T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$

Intersection point on imaginary axis

Step 1: Equating the real part of loop transfer function to zero and determining the frequency ω, i.e., $G_R(j\omega)H_R(j\omega) = 0$.

Therefore, $\dfrac{1-\omega^2 T_1 T_2}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} = 0$

Upon solving, we obtain

$$\omega = \frac{1}{\sqrt{T_1 T_2}}$$

Step 2: Substituting $\omega = \dfrac{1}{\sqrt{T_1 T_2}}$ in the imaginary part of loop transfer function, we obtain the intersection of the polar plot on imaginary axis.

Intersection point on imaginary axis = $G_I(j\omega)H_I(j\omega)\Big|_{\omega=\frac{1}{\sqrt{T_1 T_2}}}$

Substituting $\omega = \dfrac{1}{\sqrt{T_1T_2}}$ in $G_I(j\omega)H_I(j\omega) = \dfrac{-\omega(T_1+T_2)}{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$, we obtain

$$G_I(j\omega)H_I(j\omega) = -\frac{\sqrt{T_1T_2}}{T_1+T_2}$$

Step 3: Therefore, magnitude and phase angle of loop transfer function at $\omega = \dfrac{1}{\sqrt{T_1T_2}}$ is

Magnitude $G(j\omega)H(j\omega) = \dfrac{\sqrt{T_1T_2}}{T_1+T_2}$

and Phase angle $\angle G(j\omega)H(j\omega) = -90°$

Using the magnitude and phase angle at different frequencies, we can determine the real and imaginary parts of the loop transfer function as given in Table 9.6.

Table 9.6 | Real and imaginary parts of the system

Frequency ω	**Real part $G_R(j\omega)$ $H_R(j\omega)$**	**Imaginary part $G_I(j\omega)$ $H_I(j\omega)$**
0	1	0
Intersection point on imaginary axis, $\dfrac{1}{\sqrt{T_1T_2}}$	0	$-\dfrac{\sqrt{T_1T_2}}{T_1+T_2}$
∞	0	0

Using Table 9.6, the polar plot of TYPE 0 ORDER 2 system on an ordinary graph is shown in Fig. 9.9(a).

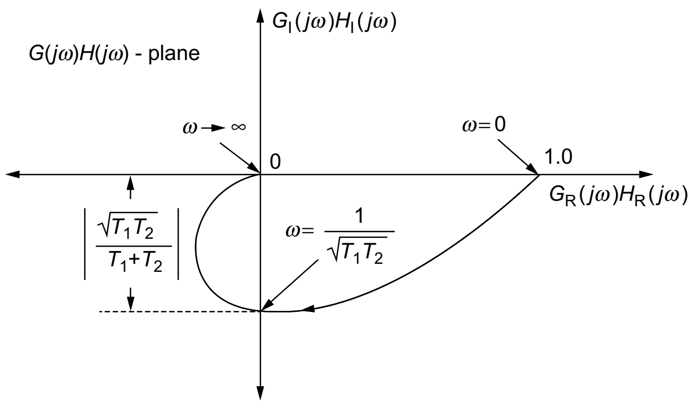


Fig. 9.9(a) | Polar plot of the system

The magnitude and phase angle at different frequencies are given in Table 9.7.

Table 9.7 | Magnitude and phase angle

Frequency ω	**Magnitude $\lvert G(j\omega)\ H(j\omega)\rvert$**	**Phase angle $\angle G(j\omega)\ H(j\omega)$**
0	1	0°
Intersection point on imaginary axis, $\frac{1}{\sqrt{T_1T_2}}$	$\frac{\sqrt{T_1T_2}}{T_1+T_2}$	–90°
∞	0	–180°

Using Table 9.7, the polar plot of TYPE 0 ORDER 2 system on polar graph is shown in Fig. 9.9(b).

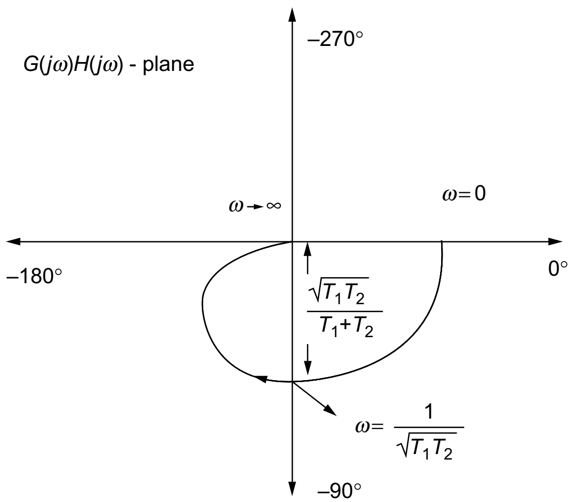


Fig. 9.9(b) | Polar plot of the system

Similarly, the polar plot on a polar graph and intersection points on the different axes for different systems can be determined.

9.7 Stability Analysis using Polar Plot

The stability of a system can be examined from the polar plot once the frequency domain specifications are obtained. The stability of the system can be analyzed by using the crossover frequencies (ω_{pc} and ω_{gc}) or the gain and phase margins.

9.7.1 Based on Crossover Frequencies

A system can either be a stable system or marginally stable system or unstable system. The stability of the system based on the relation between crossover frequencies is given in Table 9.8.

Table 9.8 | Stability of the system based on crossover frequencies

S. No	Relation between ω_{pc} and ω_{gc}	Stability of the system
1.	$\omega_{gc} < \omega_{pc}$	Stable system
2.	$\omega_{gc} > \omega_{pc}$	Unstable system
3.	$\omega_{gc} = \omega_{pc}$	Marginally stable system

9.7.2 Based on Gain Margin and Phase Margin

The stability of a system based on the gain margin and phase margin is given in Table 9.9.

Table 9.9 | Stability of the system based on g_m and p_m

S. No.	Gain margin g_m	Phase margin p_m	Relation between ω_{pc} and ω_{gc}	Stability of the system
1.	Positive	Positive	$\omega_{gc} < \omega_{pc}$	Stable system
2.	Positive	Negative	$\omega_{gc} < \omega_{pc}$	Unstable system
3.	Negative	Positive	$\omega_{gc} > \omega_{pc}$	Unstable system
4.	0	0°	$\omega_{gc} = \omega_{pc}$	Marginally stable system

But the problem that exists in examining the stability of the system from polar plot is the determination of ω_{gc} and ω_{pc} which is a tedious process. Hence, an alternate way of examining the stability of the system using polar plot exists, which does not require the determination of ω_{gc} and ω_{pc} is discussed in the following section.

9.7.3 Based on the Location of Phase Crossover Point

Consider the polar plot of a loop transfer function drawn on an ordinary graph as shown in Fig. 9.10.

Let A and B be the points on the polar plot when the plot crosses the real axis and unit circle as shown in Fig. 9.10. Using Table 9.5, the points A and B are called phase crossover and gain crossover points respectively. Let the frequency ω at the point A is the phase crossover frequency ω_{pc} and magnitude of the loop transfer function at the point A be $\left|G(j\omega_{pc})H(j\omega_{pc})\right|$.

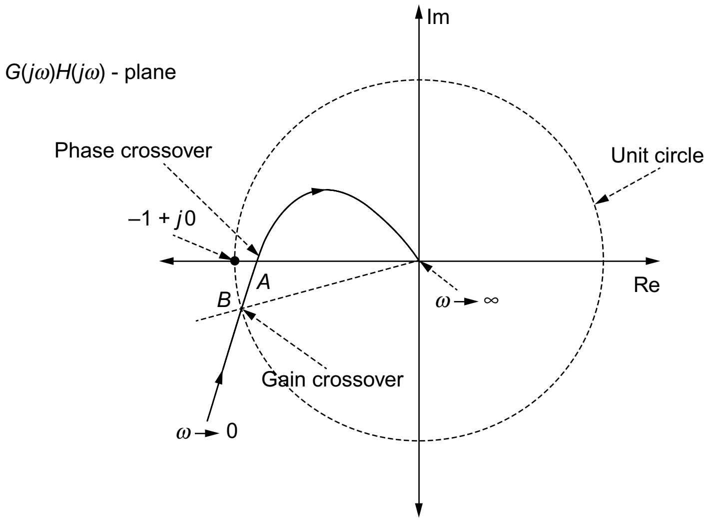


Fig. 9.10 | Polar plot of a loop transfer function in an ordinary graph

Now, based on the location of point A, the stability of the system can be examined. The stability of the system based on the location of point A is listed in Table 9.10.

Table 9.10 | Stability of the system based on phase crossover point, A

Location of point A	**Stability of the system**	**Gain margin of the system in dB**
Polar plot does not cross the negative real axis	Stable	$g_m = \infty$
Between the origin and the point $-1+j0$	Stable	$g_m > 0$
Beyond the point $-1+j0$	Unstable	$g_m < 0$
At $-1+j0$	Marginally stable	$-20\log\left\|G(j\omega_{pc})H(j\omega_{pc})\right\|$

9.8 Determining the Gain *K* from the Desired Frequency Domain Specifications

The frequency domain specifications for which the gain K can be determined are gain margin and phase margin. The procedure for determining the gain K for the desired g_m and p_m using polar plot is not so tedious as it was using Bode plot. The following sections describe how the gain K can be determined for desired g_m and p_m using polar plot.

9.8.1 When the Desired Gain Margin of the System is Specified

The step-by-step procedure to determine the gain K for a specified gain margin is explained below:

Step 1: Assume $K = 1$ for the given loop transfer function and construct the polar plot of the system either on the polar graph or on the ordinary graph.

Step 2: Determine the phase crossover point, A from the polar plot. If the polar plot is plotted on the polar graph, the point A refers to the point at which the polar plot crosses $-180°$ line and if the polar plot is plotted on the ordinary graph, the point A refers to the point at which the polar plot crosses negative real axis.

Step 3: Determine the magnitude of the loop transfer function at A from the polar plot. Let it be $\left|G(j\omega)H(j\omega)\right|_A$.

Step 4: Let the desired gain margin of the system be $\left(g_{\mathrm{m}}\right)_{\mathrm{desired}}$ dB. Let B be the phase crossover point on the polar plot corresponding to $\left(g_{\mathrm{m}}\right)_{\mathrm{desired}}$. The magnitude of the loop transfer function at B can be determined by using $-20\log\left|G(j\omega)H(j\omega)\right|_B = \left(g_{\mathrm{m}}\right)_{\mathrm{desired}}$.

Step 5: Now, the gain K for the desired gain margin can be determined as

$$K = \frac{\left|G(j\omega)H(j\omega)\right|_A}{\left|G(j\omega)H(j\omega)\right|_B}$$

Step 6: With the gain K, the new polar plot can be drawn. The polar plot for different values of K is shown in Fig. 9.11(a).

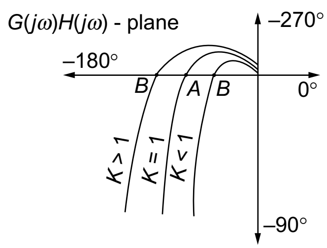


Fig. 9.11(a) | Polar plot for different values of K

9.8.2 When the Desired Phase Margin of the System is Specified

The step-by-step procedure to determine the gain K for a specified phase margin is explained below:

Step 1: Assume $K = 1$ for the given loop transfer function and construct the polar plot of the system either on the polar graph or on the ordinary graph.

Step 2: Determine the gain crossover point, C from the polar plot. If the polar plot is plotted on the polar graph or on the ordinary graph, the point C refers to the point at which the polar plot crosses unity circle.

Step 3: Determine the magnitude of the loop transfer function at C from the polar plot. Let it be $\left|G(j\omega)H(j\omega)\right|_C$.

Step 4: Let the desired phase margin of the system be $(p_{\text{m}})_{\text{desired}}$ dB. Let D be the gain cross-over point on the polar plot corresponding to $(p_{\text{m}})_{\text{desired}}$.

Step 5: Determine the phase angle of the system corresponding to $(p_{\text{m}})_{\text{desired}}$. Let it be E deg. The value of E is determined by using, $(p_{\text{m}})_{\text{desired}} = 180° + E°$.

Step 6: Draw a radial line using E deg from origin that cuts the polar plot of the loop transfer function $G(j\omega)H(j\omega)$ with $K = 1$ at F. The magnitude of the loop transfer function at F can be determined from the polar plot and let it be $\left|G(j\omega)H(j\omega)\right|_F$.

Step 7: Now, the gain K for the desired gain margin can be determined as

$$K = \frac{\left|G(j\omega)H(j\omega)\right|_C}{\left|G(j\omega)H(j\omega)\right|_F} = \frac{1}{\left|G(j\omega)H(j\omega)\right|_F}$$

Step 8: With the gain K, the new polar plot can be drawn. The polar plot for different values of K is shown in Fig. 9.11(b).

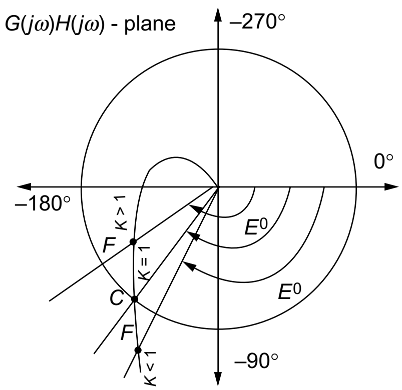
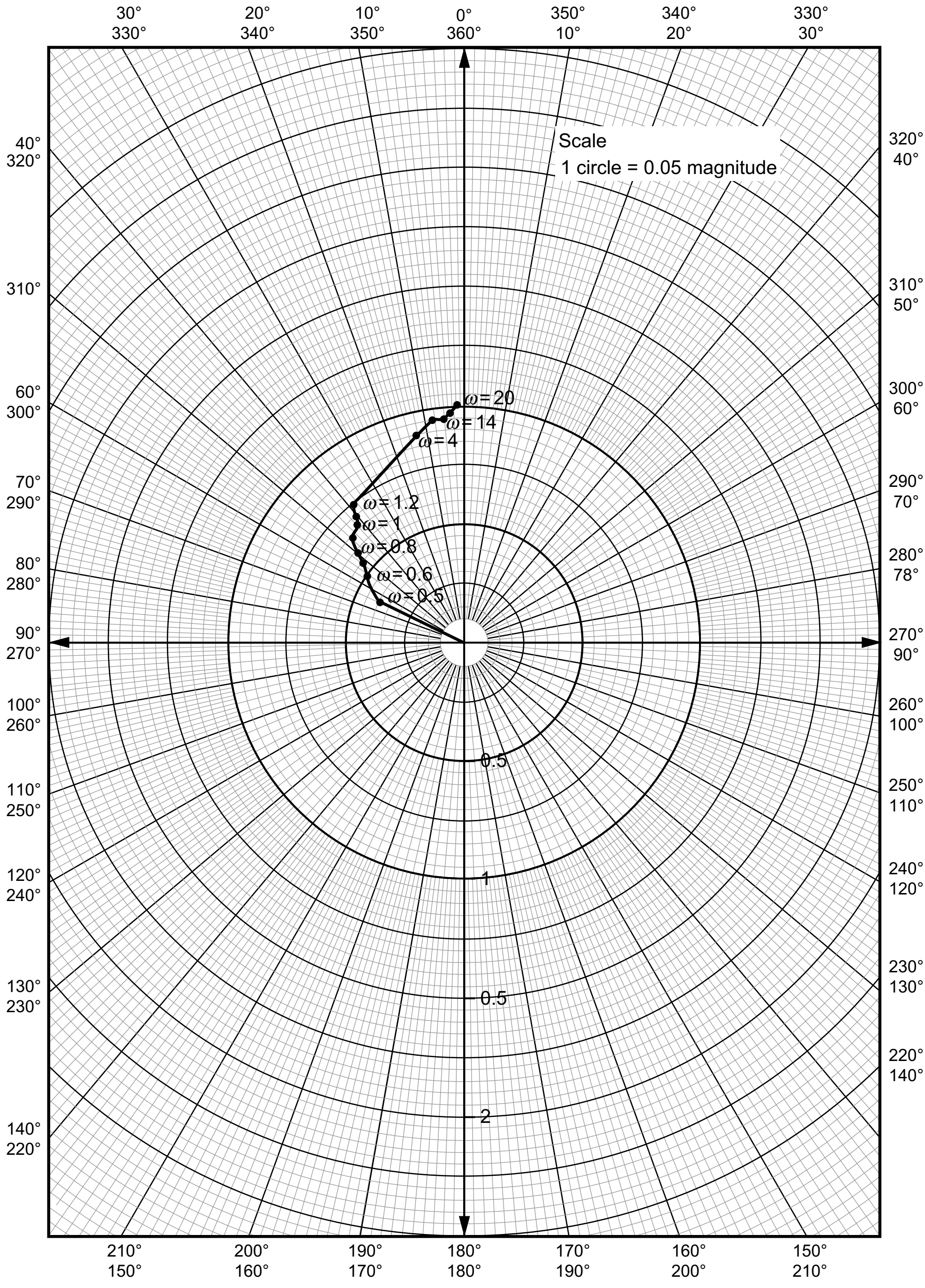


Fig. 9.11(b) | Polar plot for different values of K

Example 9.1: The loop transfer function of a system is $G(s)H(s) = \dfrac{s}{s+1}$. Sketch the polar plot for the system.

Solution:

(i) The loop transfer function of the system $G(s)H(s) = \dfrac{s}{s+1}$.



Scale
1 circle = 0.05 magnitude
ω = 20
ω = 14
ω = 4
ω = 1.2
ω = 1
ω = 0.8
ω = 0.6
ω = 0.5
0.5
1
0.5
2
0°
360°
10°
350°
20°
340°
30°
330°
350°
10°
340°
20°
330°
30°
40°
320°
310°
60°
300°
70°
290°
80°
280°
90°
270°
100°
260°
110°
250°
120°
240°
130°
230°
140°
220°
320°
40°
310°
50°
300°
60°
290°
70°
280°
78°
270°
90°
260°
100°
250°
110°
240°
120°
230°
130°
220°
140°
210°
150°
200°
160°
190°
170°
180°
180°
170°
190°
160°
200°
150°
210°

Fig. E9.1

(ii) Substituting $s = j\omega$ in the loop transfer function, we obtain

$$G(j\omega)H(j\omega) = \frac{j\omega}{j\omega + 1}$$

(iii) The only one corner frequency existing in the system is $\omega_{c1} = 1$ rad/sec.

(iv) The magnitude and phase angle of the system are $M = \frac{\omega}{\sqrt{1+\omega^2}}$

and $\phi = 90° - \tan^{-1}\omega$

(v) For different values of frequency ω, the magnitude and phase angle of the system are calculated and tabulated in Table E9.1.

Table E9.1 | Magnitude and phase angle of the system

Frequency ω in rad/sec	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**	**1.1**	**1.2**	**4**	**6**	**10**	**12**	**14**	**20**
Magnitude M	0.4	0.5	0.56	0.6	0.67	0.7	0.73	0.8	0.97	0.98	0.99	0.99	0.99	0.99
Phase angle ϕ in deg.	63.4	59	55	51	48	45	42	40	14	9.46	5.7	4.76	4	2.86

The polar plot of the given system using Table E9.1 is shown in Fig. E9.1.

Example 9.2: The loop transfer function of a system is $G(s)H(s) = \frac{K(s+1)}{s(1+0.2s)(1+0.5s)}$. Sketch the polar plot for the system.

Solution:

1. The loop transfer function of the system $G(s)H(s) = \frac{K(s+1)}{s(1+0.2s)(1+0.5s)}$.

2. Substituting $s = j\omega$ and $K = 1$ in the loop transfer function, we obtain

$$G(j\omega)H(j\omega) = \frac{(j\omega+1)}{j\omega(1+j0.2\omega)(1+j0.5\omega)}$$

3. The three corner frequencies existing in the system are

$$\omega_{c1} = 1 \text{ rad/sec, } \omega_{c2} = \frac{1}{0.5} = 2 \text{ rad/sec and } \omega_{c3} = \frac{1}{0.2} = 5 \text{ rad/sec}$$

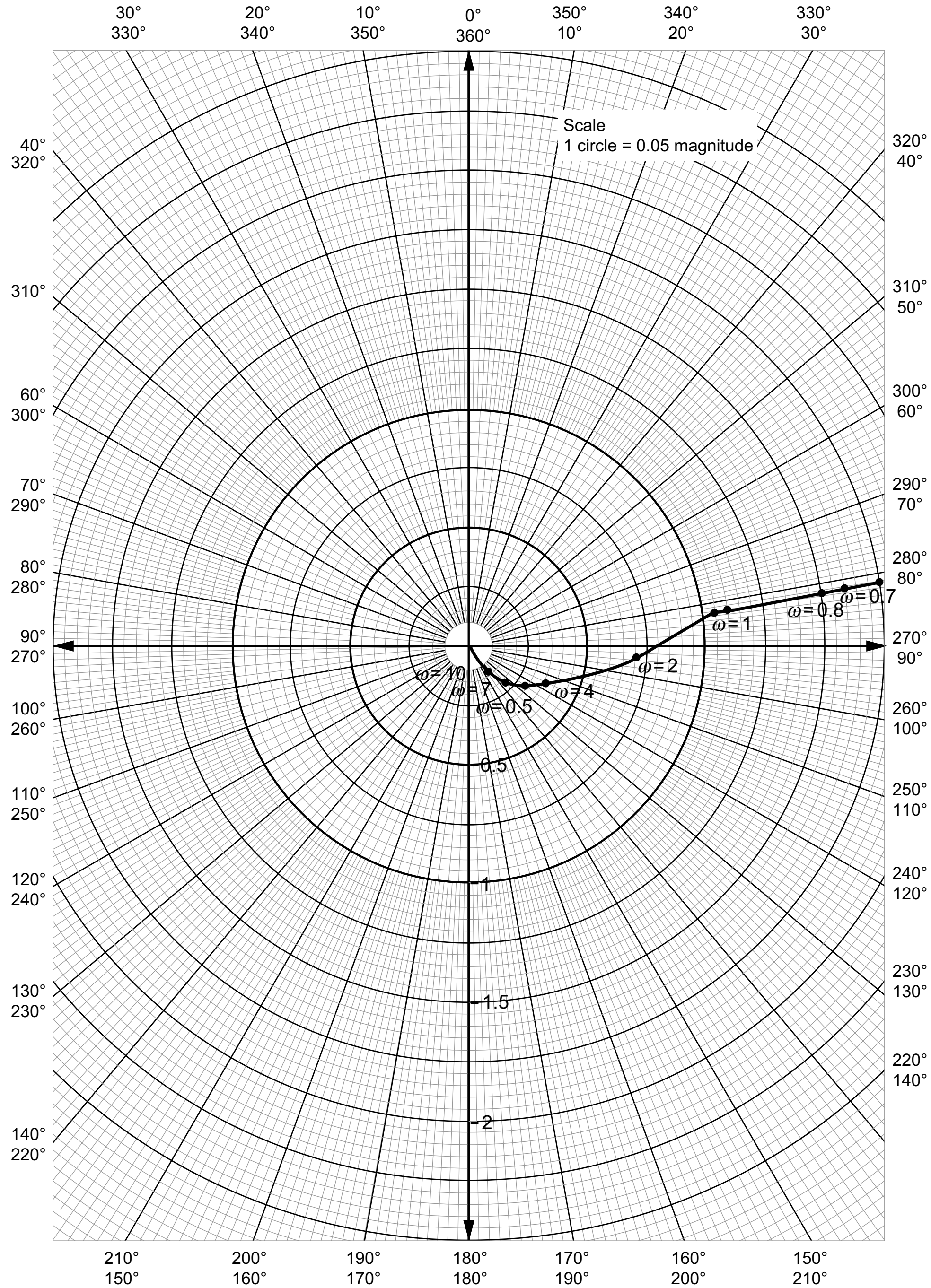


Fig. E9.2 | Polar plot for $G(s)H(s)=\dfrac{K(s+1)}{s(1+0.2s)(1+0.5s)}$

4. The magnitude and phase angle of the system are

$$M = \frac{\sqrt{1+\omega^2}}{\omega\sqrt{1+0.04\omega^2}.\sqrt{1+0.25\omega^2}}$$

and $$\phi = \tan^{-1}(\omega) - 90° - \tan^{-1}(0.2\omega) - \tan^{-1}(0.5\omega)$$

5. For different values of frequency ω, the magnitude and phase angle of the system are calculated and tabulated in Table E9.2.

Table E9.2 | Magnitude and phase angle of the system

Frequency ω in rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1	2	4	5	7	10
Magnitude *M*	2.6	2.2	1.8	1.6	1.5	1.3	1.24	0.7	0.35	0.3	0.2	0.1
Phase angle ϕ in deg.	−84	−83	−82	−82.3	−82	−82	−82.8	−93	−11	−124	−136	−148

The polar plot of the given system using Table E9.2 is shown in Fig. E9.2.

Example 9.3: The loop transfer function of a unity feedback system is given by $G(s)H(s) = \dfrac{(s+2)}{s^2(s+1)(2s+1)}$. Sketch the polar plot for the system.

Solution:

1. The loop transfer function of a system is $G(s)H(s) = \dfrac{(s+2)}{s^2(s+1)(2s+1)}$.
2. Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{(2+j\omega)}{j\omega^2(1+j\omega)(1+j2\omega)}$$

3. The three corner frequencies existing in the system are $\omega_{c1} = \frac{1}{2} = 0.5$ rad/sec, $\omega_{c2} = 1$ rad/sec and $\omega_{c3} = 2$ rad/sec.
4. The magnitude and phase angle of the system are

$$M = \frac{2\sqrt{1+0.25\omega^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} = \frac{2\sqrt{1+0.25\omega^2}}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

and $$\phi = \tan^{-1} 0.5\omega - 180° - \tan^{-1}\omega - \tan^{-1} 2\omega$$

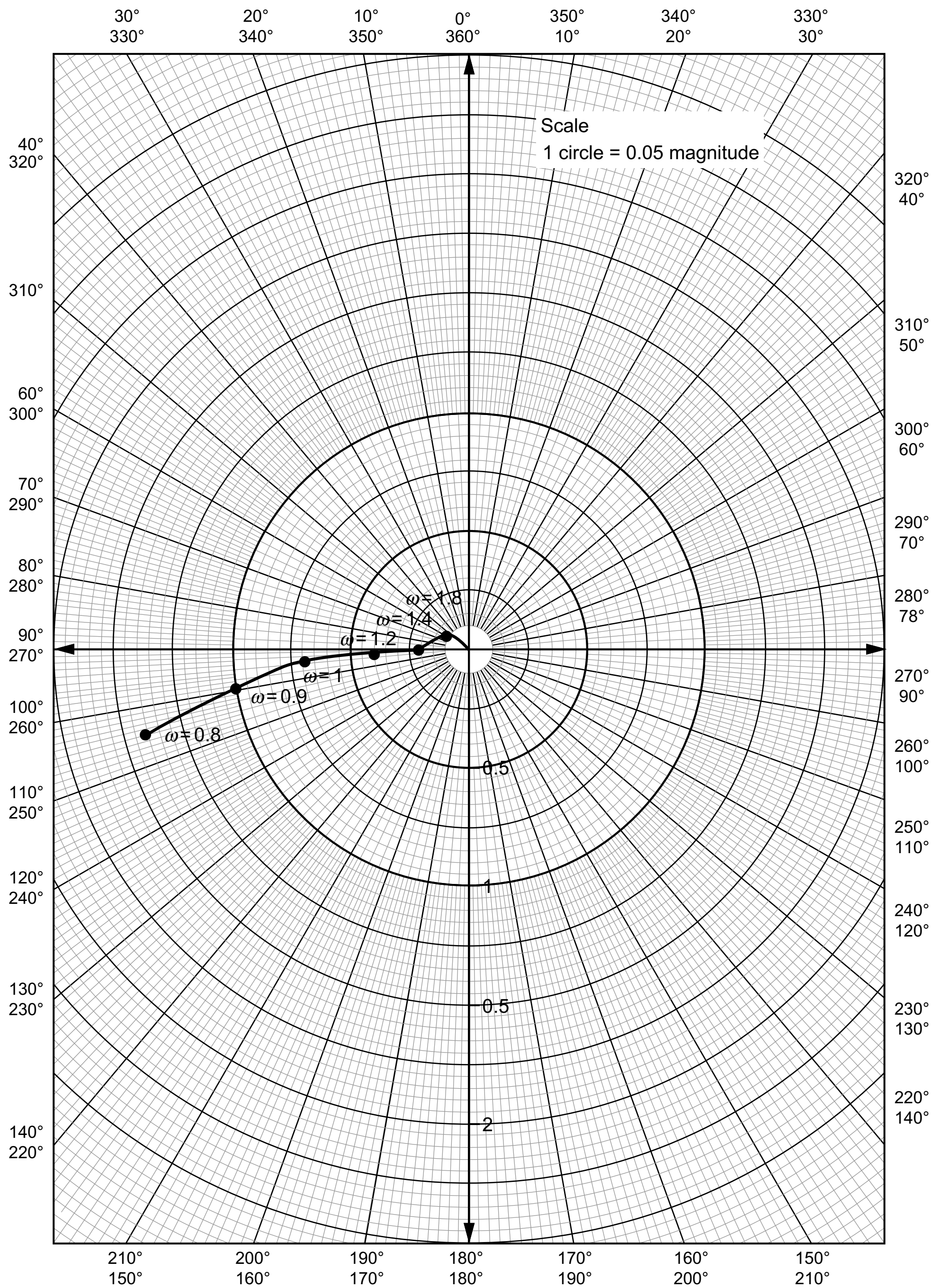


Fig. E9.3 | Polar plot for $G(s)H(s) = \dfrac{(s+2)}{s^2(s+1)(2s+1)}$

5. For different values of frequency ω, the magnitude and phase angle of the system are calculated and tabulated in Table E9.3.

Table E9.3 | Magnitude and phase angle of the system

Frequency ω in rad/sec	0.7	0.8	0.9	1	1.2	1.4	1.8	2
Magnitude *M*	2.1	1.4	1	0.7	0.4	0.24	0.1	0.07
Phase angle ϕ in deg.	−250	−255	−259	−267	−267	−269	−273	−274

The polar plot of the given system using the data in Table E9.3 is shown in Fig. E9.3.

Example 9.4: The loop transfer function of a unity feedback system is given by $G(s)H(s)=\dfrac{1}{s(1+s)^2}$. Sketch the polar plot for the system and determine the gain and phase margin of the system.

Solution:

1. The loop transfer function of a system is $G(s)H(s)=\dfrac{1}{s(1+s)^2}$.

2. Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega)=\frac{1}{j\omega(1+j\omega)^2}$$

3. The corner frequency of the given system is $\omega_{c1}=1$ rad/sec.

4. The magnitude and phase angle of the system are

$$M=\frac{1}{\omega(1+\omega^2)}=\frac{1}{(\omega+\omega^3)}$$

and $$\phi=-90°-2\tan^{-1}\omega$$

5. For different values of frequency ω, the magnitude and phase angle of the system are calculated and tabulated in Table E9.4.

Table E9.4 | Magnitude and phase angle of the system

Frequency ω in rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
Magnitude *M*	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4	0.3	0.26
Phase angle ϕ in deg.	−134	−143	−151	−159	−167	−174	−180	−185	−190	−195

The polar plot of the given system using Table E9.4 is shown in Fig. E9.4.

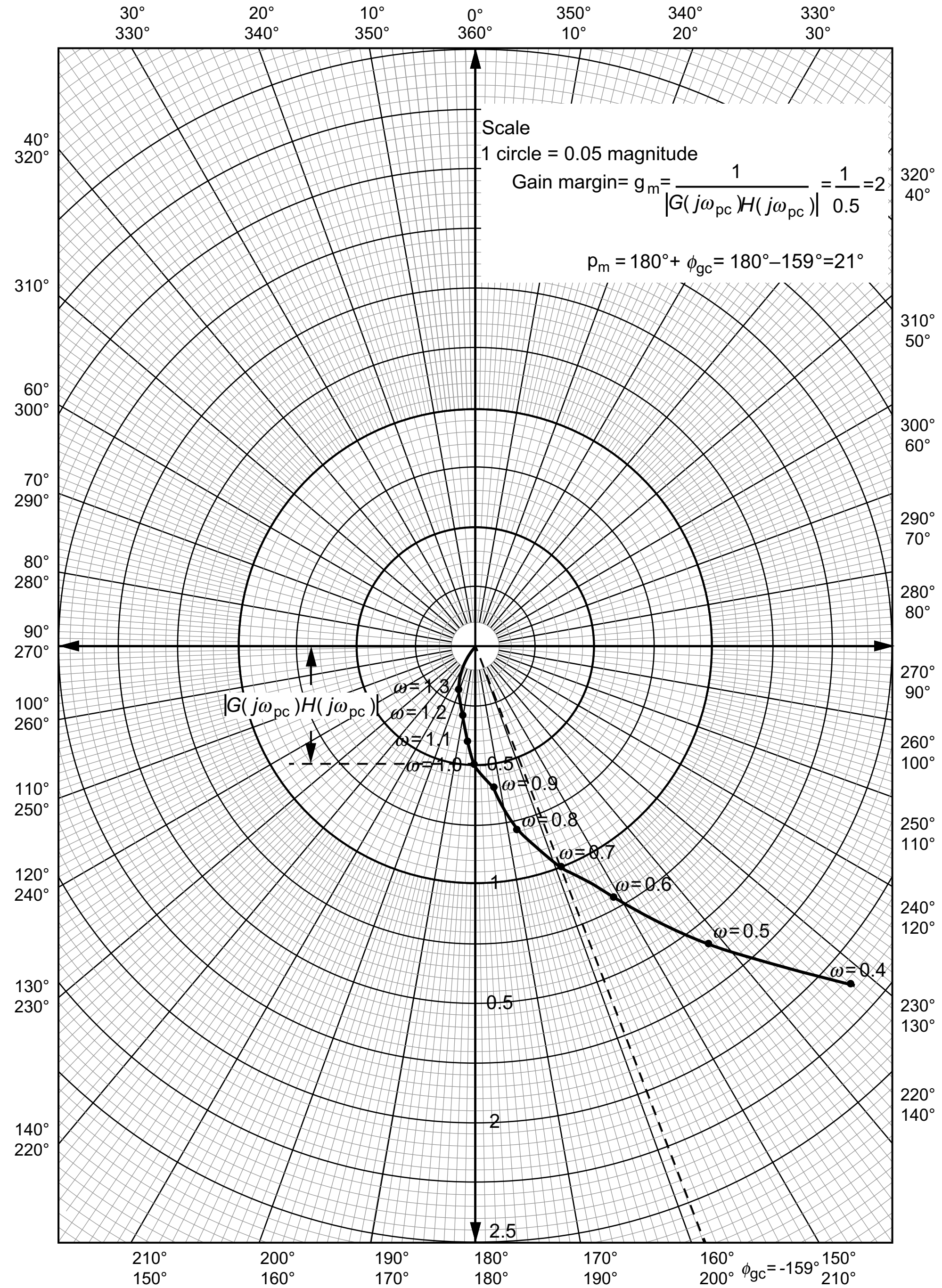

Scale
1 circle = 0.05 magnitude
Gain margin= $g_m = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|} = \frac{1}{0.5} = 2$
$p_m = 180° + \phi_{gc} = 180° – 159° = 21°$
$|G(j\omega_{pc})H(j\omega_{pc})|$
$\omega = 1.3$
$\omega = 1.2$
$\omega = 1.1$
$\omega = 1.0$
$\omega = 0.9$
$\omega = 0.8$
$\omega = 0.7$
$\omega = 0.6$
$\omega = 0.5$
$\omega = 0.4$
0.5
1
0.5
2
2.5
$\phi_{gc} = -159°$

Fig. E9.4

Determination of Gain Margin and Phase Margin

The gain margin and phase margin of the system are calculated at phase crossover frequency and gain crossover frequency respectively. The phase crossover frequency is the frequency at which the phase angle of the system is $180°$ or $-180°$. Also, the gain crossover frequency is the frequency at which the magnitude of the system is 1.

Using the above definition and Table E9.4, we obtain

Gain crossover frequency $\omega_{gx} = 0.7$ rad/sec and Phase crossover frequency $\omega_{pc} = 1$ rad/sec.

Hence, gain margin and phase margin of the system are given by

$$\text{Gain margin, } g_m = \frac{1}{\left|G(j\omega)H(j\omega)\right|_{\omega_{pc}}} = \frac{1}{0.5} = 2$$

and Phase margin, $p_m = 180° + \left(\angle G(j\omega)H(j\omega)\right)_{\omega_{gc}} = 180° - 159° = 21°$

Example 9.5: The loop transfer function of a unity feedback system is given by $G(s)H(s) = \dfrac{8}{(s+1)(s+2)}$. Sketch the polar plot for the system.

Solution:

1. The loop transfer function of a system is $G(s)H(s) = \dfrac{8}{(s+1)(s+2)}$
2. Substituting $s = j\omega$, we obtain

$$G(j\omega)H(j\omega) = \frac{8}{(j\omega+1)(j\omega+2)}$$

3. The two corner frequencies existing in the given system are

$$\omega_{c1} = 1 \text{ rad/sec and } \omega_{c2} = 2 \text{ rad/sec respectively.}$$

4. The magnitude and phase angle of the system are

$$M = \frac{4}{\sqrt{1+\omega^2}\sqrt{1+0.25\omega^2}} = \frac{4}{\sqrt{1+1.25\omega^2+0.25\omega^4}}$$

and $$\phi = -\tan^{-1}\omega - \tan^{-1}0.5\omega$$

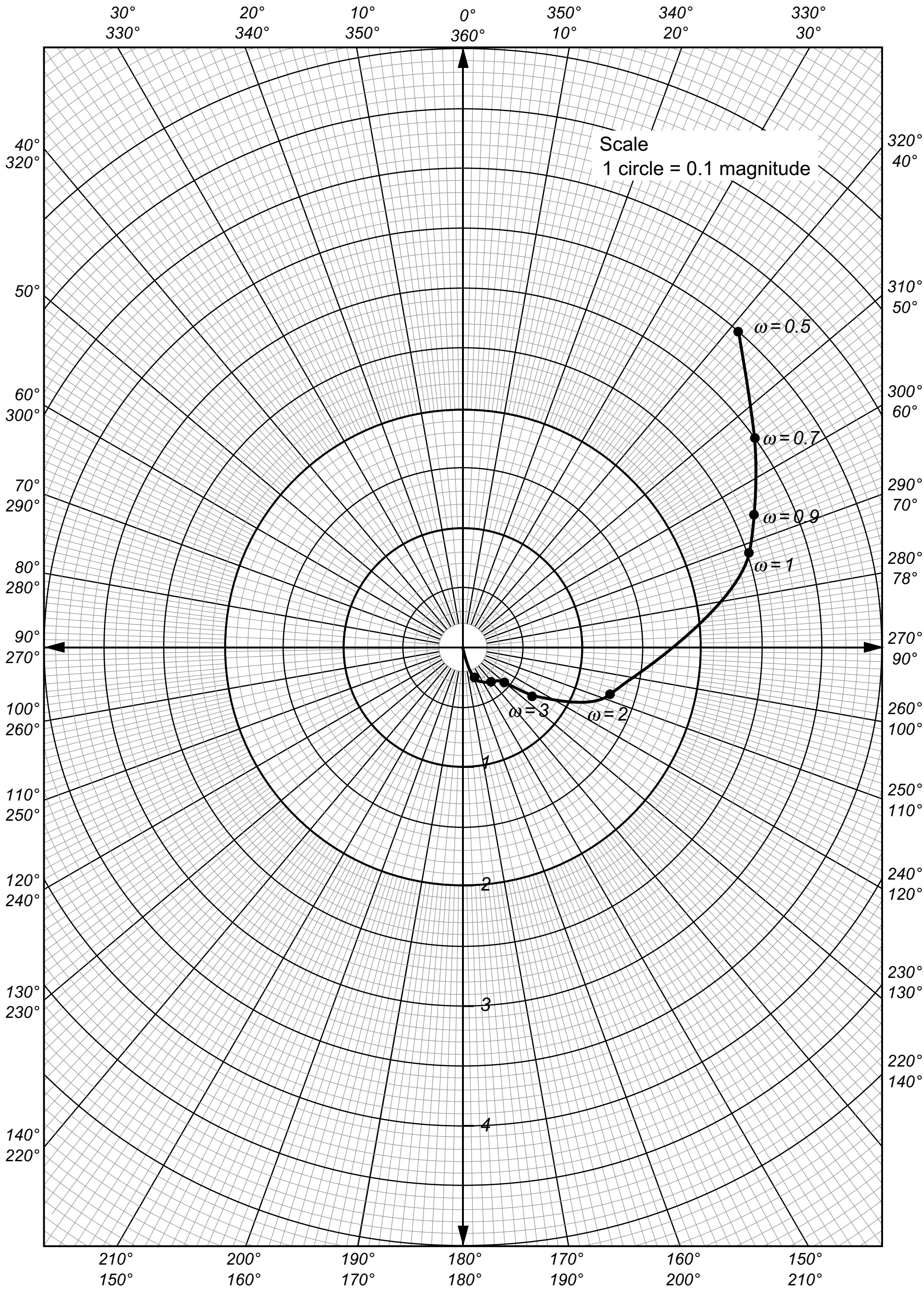

Scale
1 circle = 0.1 magnitude
ω = 0.5
ω = 0.7
ω = 0.9
ω = 1
ω = 2
ω = 3
1
2
3
4
0° 360°
10° 350°
20° 340°
30° 330°
350° 10°
340° 20°
330° 30°
40° 320°
50°
60° 300°
70° 290°
80° 280°
90° 270°
100° 260°
110° 250°
120° 240°
130° 230°
140° 220°
320° 40°
310° 50°
300° 60°
290° 70°
280° 78°
270° 90°
260° 100°
250° 110°
240° 120°
230° 130°
220° 140°
210° 150°
200° 160°
190° 170°
180° 180°
170° 190°
160° 200°
150° 210°

Fig. E9.5

5. For different values of frequency ω, the magnitude and phase angle of the system are calculated and tabulated in Table E9.5.

Table E9.5 | Magnitude and phase angle of the system

Frequency ω in rad/sec	0.5	0.7	0.9	1	2	3	4	5	7	8
Magnitude *M*	3.5	3	2.7	2.5	1.26	0.7	0.4	0.3	0.2	0.1
Phase angle ϕ in deg.	−41	−54	−66	−72	−108	−128	−139	−147	−150	−159

The polar plot of the given system using the data in Table E9.5 is shown in Fig. E9.5.

9.9 Introduction to Nyquist Stability Criterion

The Routh–Hurwitz criterion and root locus methods are used to determine the stability of the linear single-input single-output (SISO) system by finding the location of the roots of the characteristic equation in s-plane. The Nyquist stability criterion derived by H. Nyquist is a semi-graphical method that helps in determining the absolute stability of the closed-loop system graphically from frequency response of loop transfer function (Nyquist plot) without determining the closed-loop poles. The Nyquist plot of a loop transfer function $G(s)H(s)$ is a graphical representation of the frequency response analysis when the frequency ω is varied from $-\infty$ to ∞. Since most of the linear control systems are analyzed by using their frequency responses, Nyquist plot will be convenient in determining the stability of the system.

9.10 Advantages of Nyquist Plot

The advantages of Nyquist plot are:

(i) Nyquist plot helps in determining the relative stability of the system in addition to the absolute stability of the system.

(ii) It determines the stability of the closed-loop system from open-loop transfer function without calculating the roots of characteristic equation.

(iii) It gives the degree of instability of an unstable system and indicates the ways in which the stability of the system can be improved.

(iv) It gives information related to frequency domain characteristics such as M_r, ω_r, BW etc..

(v) It can easily be applied to systems with pure time delay that cannot be analyzed using root locus method or Routh–Hurwitz criterion.

9.11 Basic Requirements for Nyquist Stability Criterion

The Nyquist stability criterion will be useful in analyzing the stability of both the open-loop and closed-loop systems. The concepts such as encirclement, enclosement, number of encirclements around a point, mapping from one plane to another plane and principle of argument are important to study the Nyquist stability criterion of a system.

9.12 Encircled and Enclosed

The concept of encircled and enclosed will be helpful to interpret the Nyquist plot for analyzing the stability of the system.

9.12.1 Encircled

A point or region in a complex function plane is said to be encircled by a closed path if the particular point or region is found inside the closed path irrespective of the direction of closed path. All other points existing outside the closed path is not encircled. The concept of encirclement with two different points and different directions of closed path is shown in Figs. 9.12(a) and (b).

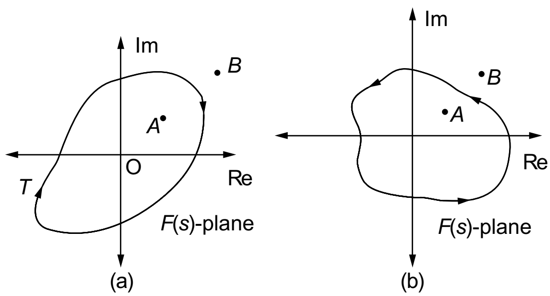


Fig. 9.12 | Encirclement concept

Irrespective of the direction of closed path, the point A shown in Figs. 9.12(a) and (b) is encircled and it is found inside the path. Since the point B shown in Figs. 9.12(a) and (b) lies outside the closed path, the point B is not encircled.

9.12.2 Enclosed

A point or region in a complex function plane is said to be enclosed by a closed path if the point or region is encircled by the closed path or if the point or region lies to the left of the closed path when the path is traversed in the counter clockwise (CCW) direction.

In addition, a point or region in a complex function plane is said to be enclosed by a closed path if the point or region is not encircled by the closed path or it lies to the left of the closed path when the path is traversed in the clockwise (CW) direction. The concept of enclosement with two different points and different directions of closed path is shown in Figs. 9.13(a) and (b) respectively.

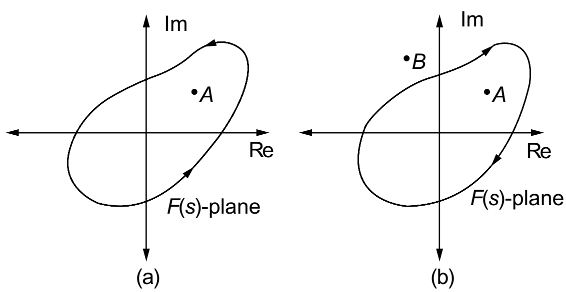


Fig. 9.13 | Enclosement concept

The point A in Fig.9.13(a) and point B in Fig. 9.13(b) is enclosed by the closed path.

9.13 Number of Encirclements or Enclosures

Let N be the number of encirclements or enclosures for a point, when the point is encircled or enclosed by the closed path. The sign of N depends on the direction of closed path. If the closed path is in the clockwise direction, it is $+N$; and if the closed path is in the counter clockwise direction, it is $-N$. The magnitude of N is determined as follows:

(i) Consider an arbitrary point s_1 on the closed path.

(ii) Let the point s_1 follow the closed path in the direction of closed path until it reaches the starting point.

Now, N is the net number of revolutions traversed by the arbitrary point s_1 or the number of encirclements/enclosures for a point. For example, consider two points A and B as shown in Figs. 9.14(a) and (b) in $F(s)$-plane.

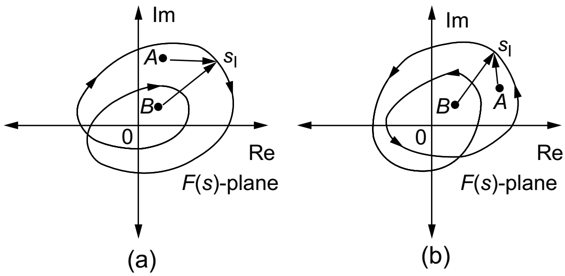


Fig. 9.14 | Number of encirclement concept

From Fig. 9.14(a), it is clear that the point A and B are encircled by the closed path and from Fig. 9.14(b), it is clear that the point A and B are enclosed by the closed path. Therefore, the net number of revolutions traversed by s_1 around the point A in Figs. 9.14(a) and (b) is -1 and $+1$ respectively. Similarly, the net number of revolutions traversed by s_1 around the point B in Figs. 9.14(a) and (b) is -2 and $+2$ respectively.

9.14 Mapping of s-Plane into Characteristic Equation Plane

The mapping of s-plane into characteristic equation plane will be helpful in mapping s-plane into $G(s)H(s)$-plane which is the basic Nyquist stability criterion for analyzing the stability of the open-loop and closed-loop systems.

Consider a simple closed-loop system as shown in Fig. 9.15.

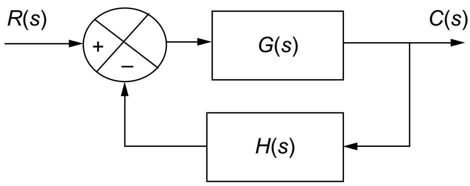


Fig. 9.15 | A simple closed-loop system

The closed-loop transfer function of the system is
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{9.5}$$

and the characteristic equation of the system is
$$F(s) = 1+G(s)H(s) \tag{9.6}$$

where $G(s)$ is the forward path transfer function, $H(s)$ is the feedback path transfer function and $G(s)H(s)$ is the loop transfer function of the system.
The general form of the loop transfer function is given by
$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)(1+T_c s)\ldots(1+T_m s)}{(1+T_1 s)(1+T_2 s)(1+T_3 s)\ldots(1+T_n s)} e^{-T_d s} = \frac{N(s)}{D(s)} \tag{9.7}$$

and T_d is the time delay in seconds.
Substituting Eqn. (9.7) in Eqn. (9.6), we obtain
$$F(s) = 1+G(s)H(s) = 1+\frac{N(s)}{D(s)} = \frac{N'(s)}{D(s)} \tag{9.8}$$

Since $s = \sigma + j\omega$ is a complex quantity, the function $F(s)$ is also a complex quantity that can be defined as $u + jv$ and represented on the complex $F(s)$-plane with u and v as the co-ordinates. Equations (9.7) and (9.8) indicate that for every point in s-plane at which $F(s)$ exists, we can determine a point in the $F(s)$-plane. Therefore, if a contour exists in the s-plane and does not go through any pole or zero of $G(s)H(s)$, a corresponding contour will exist in the $F(s)$-plane. These contours will be helpful in determining the stability of the system using Nyquist stability criterion.

Consider a characteristic equation of a system as $F(s) = \frac{1}{(s-3)}$ (only one pole) and a contour in s-plane as shown in Fig. 9.16(a). From Fig. 9.16(a), it is clear that, as the contour in s-plane encloses +3 (a singular point), a contour which encloses the origin will exist in $F(s)$-plane.

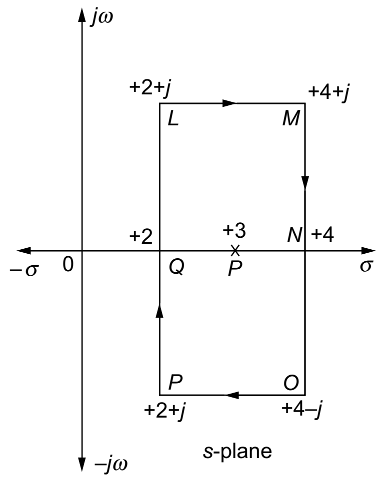


Fig. 9.16(a) | A contour in s-plane

Table 9.11 is used for transforming the contour in s-plane to a contour in $F(s)$-plane.

Table 9.11 | Transforming the contour from s-plane into $F(s)$-plane

Points in s-plane	**Points in $F(s)$-plane, $F(s) = \dfrac{1}{(s-3)}$**
$L = 2+j$	$L' = \dfrac{1}{(2+j-3)} = \dfrac{1}{(-1+j)} = \dfrac{1}{2}(1+j)$
$M = 4+j$	$M' = \dfrac{1}{(4+j-3)} = \dfrac{1}{(1+j)} = \dfrac{1}{2}(1-j)$
$N = 4$	$N' = \dfrac{1}{(4-3)} = 1$
$O = 4-j$	$O' = \dfrac{1}{(4-j-3)} = \dfrac{1}{(1-j)} = \dfrac{1}{2}(1+j)$
$P = 2-j$	$P' = \dfrac{1}{(2-j-3)} = \dfrac{1}{(-1-j)} = \dfrac{1}{2}(-1+j)$
$Q = 2$	$Q' = \dfrac{1}{(2-3)} = -1$

The contour in $F(s)$-plane that is obtained using s-plane is shown in Fig. 9.16(b).

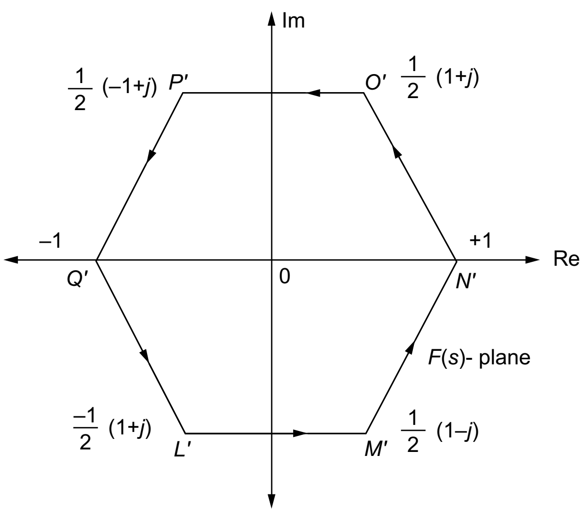


Fig. 9.16(b) | Contour in $F(s)$-plane

The conclusion obtained from Figs. 9.16(a) and (b) is that **if the contour in s-plane encloses a single pole and moves in the clockwise direction, then the contour in $F(s)$-plane will encircle the origin once in the counter clockwise direction.**

Similarly, if the contour in s-plane encloses a m poles and moves in the clockwise direction, then the contour in $F(s)$-plane will encircle the origin m-times in the counter clockwise direction. Table 9.11 shows the mapping of different s-plane contour to the $F(s)$-plane contour.

9.15 Principle of Argument

The principle of arguments will be useful in examining the stability of the system based on the Nyquist stability criterion. Its value is based on the mapping of contour from s-plane to the $F(s)$-plane. The principle of argument can be stated as follows:

Let Z be the number of zeros and P be the number of poles of the characteristic equation encircled by the s-plane contour, then the number of encirclements N, made by the $F(s)$-plane contour around the origin can be obtained as the difference between Z and P. The principle of argument in equation form can be obtained as

$$N = Z - P \tag{9.9}$$

where

N is the number of encirclements made by the $F(s)$-plane contour around the origin,

Z is the number of zeros of characteristic equation encircled by s-plane contour and

P is the number of poles of characteristic equation encircled by s-plane contour.

The principle of argument can be graphically proved when $F(s)$ of the system is represented in terms of polar co-ordinates (in terms of magnitude and phase angle) as explained below.

Consider $F(s) = \dfrac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$, where z_i is a zero of $F(s)$ and p_i is a pole of $F(s)$ and a contour with an arbitrary point s_1 as shown in Fig. 9.17.

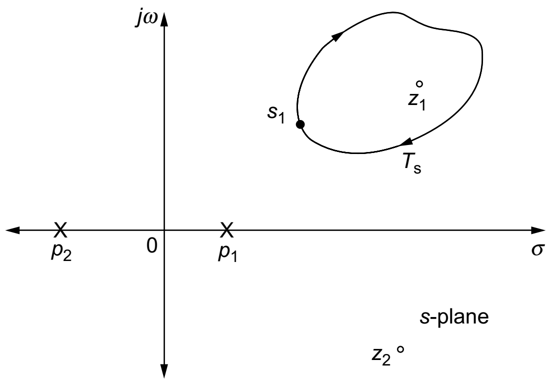


Fig. 9.17 | Contour in s-plane with poles and zeros

Then, $F(s)$ in polar co-ordinates can be obtained as

$$F(s) = |F(s)| \angle F(s) = \frac{|s+z_1|\,|s+z_2|}{|s+p_1|\,|s+p_2|}\left(\angle s+z_1 + \angle s+z_2 - \angle s+p_1 - \angle s+p_2\right)$$

$$= |F(s)|\left(\phi_{z1} + \phi_{z2} - \phi_{p1} - \phi_{p2}\right) \quad (9.10)$$

The function $F(s)$ at $s = s_1$ is given by

$$F(s_1) = \frac{(s_1+z_1)(s_1+z_2)}{(s_1+p_1)(s_1+p_2)} \quad (9.11)$$

Each factor present in Eqn. (9.11) can be represented graphically by the vector drawn from poles or zeros to the arbitrary point s_1 as shown in Fig. 9.18(a) which in turn represents $F(s_1)$ as shown in Fig. 9.18(b).

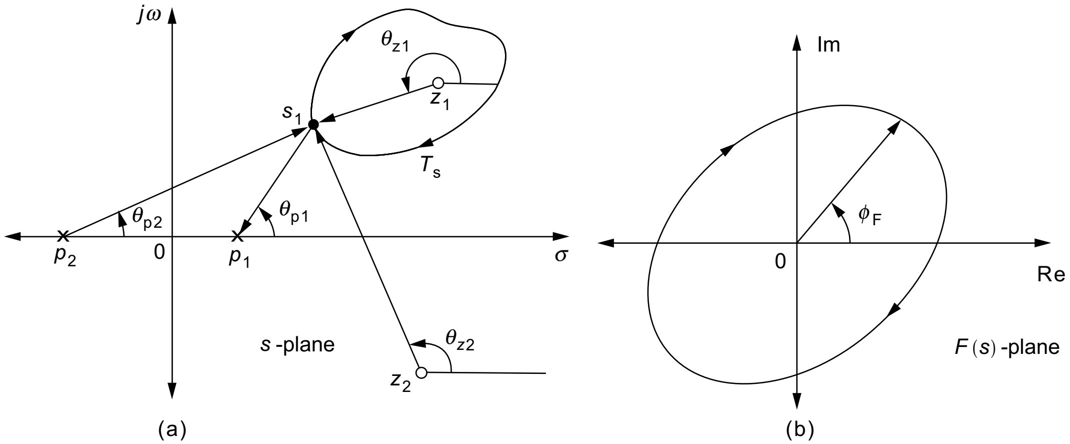


Fig. 9.18 | Relation between s -plane and $F(s)$ -plane

If the arbitrary point s_1 moves in the contour in the specified direction, then the angles generated by the vectors drawn from poles and zeros at s_1 can be determined till the arbitrary point reaches the initial position. The resultant angle generated by the poles and zeros that are not encircled by the contour and zero which is encircled by the contour will be 2π radians if we calculate it manually. Therefore, if Z number of zeros are encircled by the contour, then the angle generated by the poles and zeros when the point s_1 completes one full rotation in the contour will be $2\pi \times Z$ radians.

If the contour in s -plane encloses P number of poles and Z number of zeros, then the net angle generated by the poles and zeros when the point s_1 completes one full rotation in the contour will be

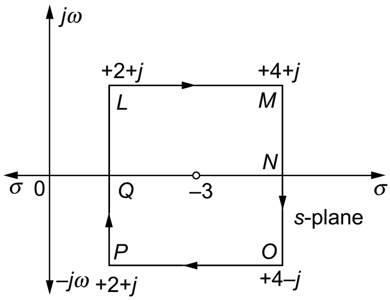
$$\phi_F = \phi_Z - \phi_P$$

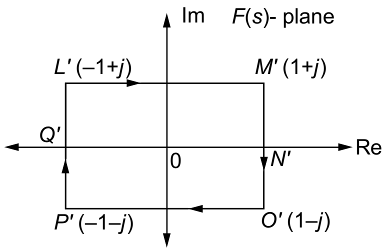
or
$$2\pi \times N = (2\pi \times Z) - (2\pi \times P) \tag{9.12}$$

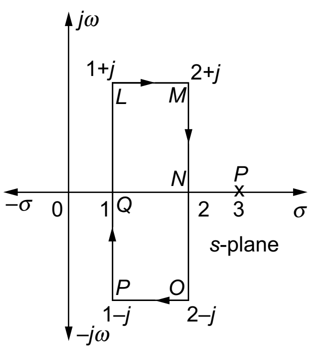
where ϕ_F is the angle made by the contour in $F(s)$-plane which is shown in Fig. 9.13(b).

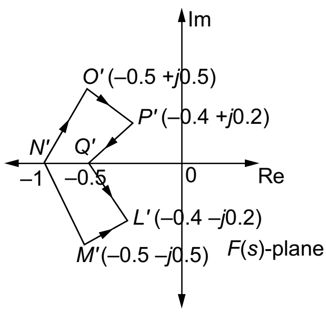
N is the net number of encirclements of the origin made by the contour in $F(s)$-plane. Hence, the net number of encirclements from Eqn. (9.12) will be $N = Z - P$, which is the same as Eqn. (9.9). Table 9.12 shows the number of encirclements made by the $F(s)$-plane contour around the origin for different s-plane contours.

Table 9.12 | Mapping of s-plane contour to the $F(s)$-plane contour alongwith the principle of arguments

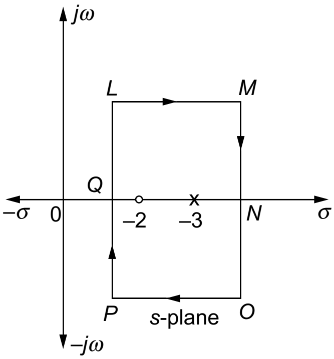
<table>
<tr><th>s-plane contour</th><th>F(s)-plane contour</th><th>Principle of argument</th></tr>
<tr><td>

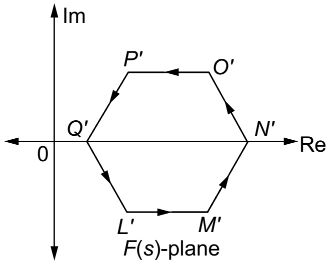
Clockwise contour encloses a zero at $s = -3$.</td><td>

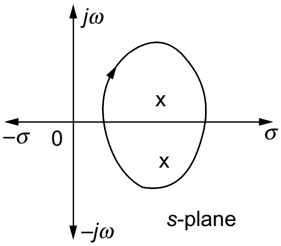
Clockwise contour encloses the origin</td><td>$N = Z - P = 1$</td></tr>
<tr><td>

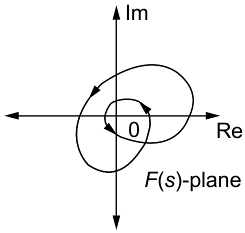
Clockwise contour does not enclose the pole at $s = -3$.</td><td>

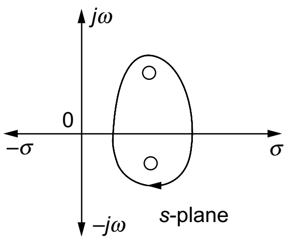
Clockwise contour does not enclose the origin.</td><td>$N = Z - P = 0$</td></tr>
</table>

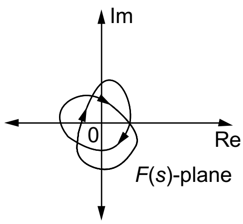
<table>
<tr>
<td>

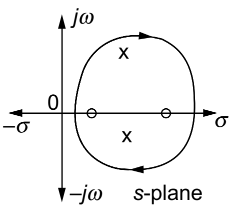
Clockwise contour encloses a pole and a zero.</td>
<td>

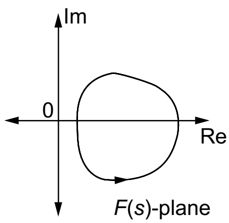
Counterclockwise contour does not enclose the origin.</td>
<td>$N = Z - P = 0$</td>
</tr>
<tr>
<td>

Clockwise contour encloses two poles</td>
<td>

Counterclockwise contour encloses the origin (twice)</td>
<td>$N = Z - P = -2$</td>
</tr>
<tr>
<td>

Clockwise contour encloses two zeros</td>
<td>

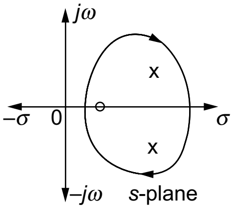
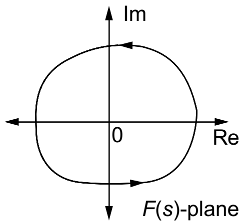
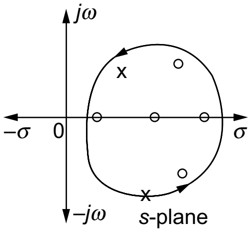
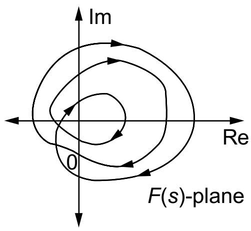
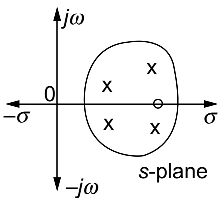
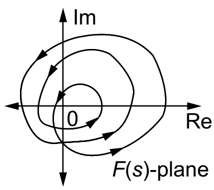
Clockwise contour encloses the origin (twice)</td>
<td>$N = Z - P = 2$</td>
</tr>
<tr>
<td>

Clockwise contour encloses two zeros and two poles.</td>
<td>

Counterclockwise contour does not enclose the origin.</td>
<td>$N = Z - P = 0$</td>
</tr>
</table>

(Continued)

Table 9.12 | (Continued)

<table>
<tr><td>Clockwise contour encloses two poles and a zero.</td><td>Counterclockwise contour encloses the origin.</td><td>$N = Z - P = 1$</td></tr>
<tr><td>Clockwise contour encloses zeros and poles with a pole on the contour.</td><td>Clockwise contour encloses the origin (thrice).</td><td>$N = Z - P = 3$</td></tr>
<tr><td>Counterclockwise contour encloses more poles than zeros.</td><td>Counterclockwise contour encloses the origin (thrice).</td><td>$N = Z - P = 3$</td></tr>
</table>

The conclusion obtained using Eqn. (9.9) and Table 9.12 is listed in Table 9.13.

Table 9.13 | Possible outcomes of principle of arguments

Relation between Z and P	**Value of N**	**Direction of contour**	
		s-plane contour	**F(s)-plane contour**
$Z > P$	Positive	CW	CW
		CCW	CW
$Z < P$	Negative	CW	CCW
		CCW	CW
$Z = P$	Zero	CW	CW
		CCW	CW

9.16 Nyquist Stability Criterion

The closed-loop system is said to be a stable system if all the roots of the characteristic equation $1+G(s)H(s)=0$ lie in the left half of the s-plane. The system is stable if all the closed-loop poles of the system or the roots of the characteristic equation lie in the left half of the s-plane even though the poles and zeros of the loop transfer function $G(s)H(s)$ lie in the right half of the s-plane. The relationship between the zeros of $F(s)$, poles of $F(s)$, characteristic equation $F(s)=0$ and loop transfer function can be obtained using Eqs. (9.7) and (9.8) as follows:

(i) Zeros of $F(s)$ = closed-loop poles of the system
= roots of characteristic equation

(ii) Poles of $F(s)$ = poles of loop transfer function $G(s)H(s)$

The two types of stability are defined as

(i) Open-loop stability: If all the poles of the loop transfer function $G(s)H(s)$ or poles of $F(s)$ lie in the left half of the s-plane and the value is less than zero, then the system is said to an open-loop stable system.

(ii) Closed-loop stability: If all the closed-loop poles of the system or the roots of characteristic equation or zeros of $F(s)$ lie in the left half of the s-plane and their values are less than zero, then the system is said to be a closed-loop stable system.

If the loop transfer function of a system is given by Eqn. (9.7), just by inspection or by using Routh's criterion, it is possible to determine the number of poles which does not lie in the left half of the s-plane. But determining the zeros of $F(s)$ or the roots of the characteristic equation of the system given by Eqn. (9.8) is difficult, if the order of the polynomial is greater than 3. Hence, to overcome these difficulties, the Nyquist stability criterion relates the frequency response of the loop transfer function $G(j\omega)H(j\omega)$ to the number of poles and zeros of $F(s)$ which will be helpful in determining the absolute stability of the system.

9.17 Nyquist Path

The semi-circular contour path of infinite radius in the right half of the s-plane with the entire $j\omega$ axis from $\omega=-\infty$ to $\omega=+\infty$ in the clockwise direction is called as Nyquist path. This Nyquist path will be helpful in analyzing the stability of the linear control system and it encloses the entire right half of s-plane and also encloses the zeros and poles of $F(s)$ that has positive real parts. The Nyquist path for a particular system is shown in Fig. 9.19(a). It is necessary that the Nyquist path should not pass through any poles and zeros of $F(s)$.

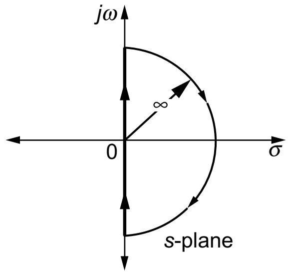


Fig. 9.19(a) | Nyquist path

If the loop transfer function $G(s)H(s)$ has a pole or poles at the origin in the s-plane, mapping of this point will become indeterminate. Hence, a contour is taken around the origin to avoid such situations, which is shown in Fig. 9.19(b) and the zoomed view of the contour is shown in Fig. 9.19(c).

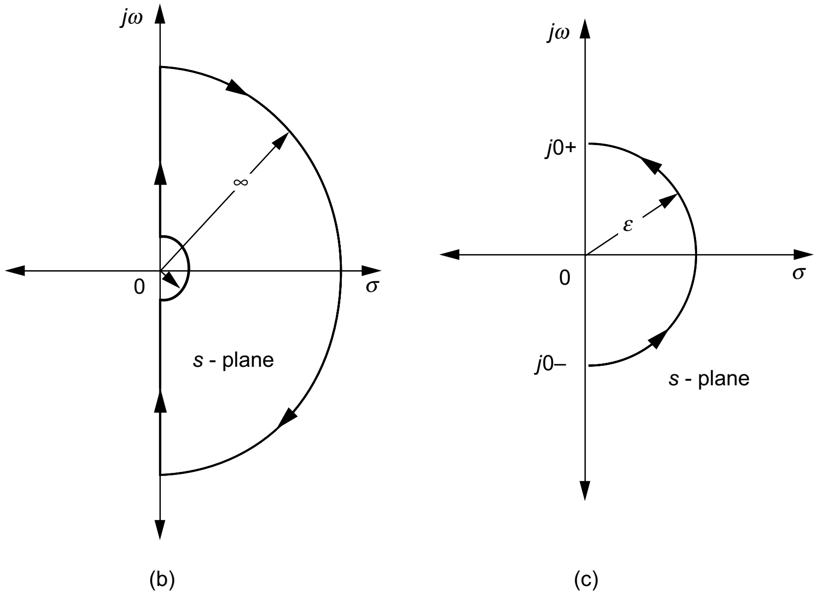


Fig. 9.19 | Nyquist contour for a system with a pole and/or zero at the origin

If the mapping theorem is applied to the system, then the following conclusion is made:
If the Nyquist path for a system is chosen as shown in Fig. 9.19(a), then the number of zeros of $F(s)$ which lies in the right half of s-plane is equal to the summation of the number of poles of $F(s)$ which lies in the right half of s-plane and number of clockwise encirclements of the origin made by the contour in $F(s)$-plane.

9.18 Relation Between G(s) H(s)-Plane and F(s)-Plane

We know that $F(s)=1+G(s)H(s)$, which is the vector sum of the unit vector and the vector $G(s)H(s)$. Hence, $1+G(s)H(s)$ is identical to the vector drawn from the point $-1+j0$ to the point in the vector $G(s)H(s)$ as shown in Fig. 9.20.
Therefore, encirclement of the origin by the contour in $F(s)$-plane is similar to the encirclement of the point $-1+j0$ by the contour in $G(s)H(s)$-plane. Hence, the stability of the closed-loop system can be examined by determining the number of encirclements of $-1+j0$ in $G(s)H(s)$-plane.

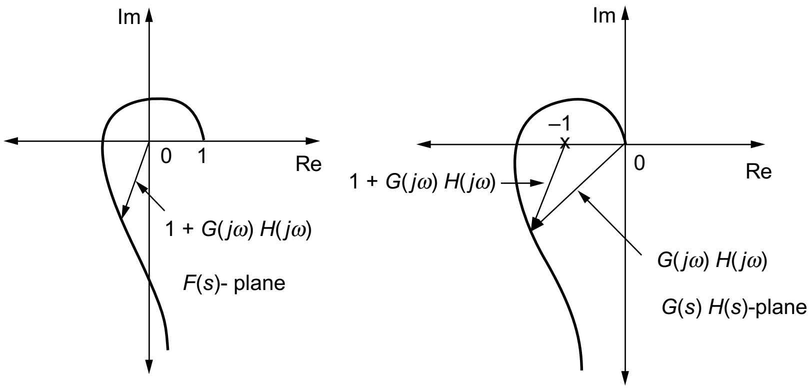


Fig. 9.20 | Relationship between $F(s)$-plane and $G(s)H(s)$-plane

9.19 Nyquist Stability Criterion Based on the Encirclements of $-1+j0$

The Nyquist stability criterion based on the number of encirclements of $-1+j0$ is analyzed for two cases as below:

Case 1: *When the loop transfer function has no poles or zeros on the imaginary axis*

If the loop transfer function of the system $G(s)H(s)$ has m number of poles in the right half of the s-plane, then for the system to be stable, the number of encirclements in the counterclockwise direction made by the locus in $G(s)H(s)$ plane around the point $-1+j0$ must be equal to m.

The above criterion can be expressed as

$$Z = N + P$$

where Z is the number of zeros of $F(s)$ present in the right half of s-plane, N is the number of clockwise encirclements made by the locus of $G(s)H(s)$ around the point $-1+j0$ and P is the number of poles of $G(s)H(s)$ present in the right half of s-plane.

Case 2: *When the loop transfer function has poles and/or zeros on the imaginary axis*

If the loop transfer function has poles and/or zeros on the imaginary axis, the Nyquist path is modified by drawing a small semicircle with very small radius ε as shown in Fig. 9.19(c).

If the loop transfer function of the system $G(s)H(s)$ has m number of poles in the right half of the s-plane, then for the system to be stable, the number of encirclements in the counterclockwise direction made by the modified locus in $G(s)H(s)$-plane around the point $-1+j0$ must be equal to m.

9.20 Stability Analysis of the System

Let Z be the number of zeros and P be the number of poles of the characteristic equation encircled by the s-plane contour (Nyquist path), then the number of encirclements, N made by the $G(s)H(s)$-plane contour around the point $-1+j0$ in the clockwise direction can be obtained as the difference between Z and P.

i.e., $N = Z - P$

If N is negative, it implies that the encirclement is in the counterclockwise direction. The different conditions for the system to be stable are discussed in Table 9.14.

Table 9.14 | Conditions for the system to be stable

Number of encirclements around $-1+j0$	Number of poles of $G(s)H(s)$ in the right half of s-plane	Number of zeros of $G(s)H(s)$ in the right half of s-plane	Stability of the system
$N=0$	$P\neq 0$	$Z>0$	System is unstable
$N=0$	$P=0$	$Z=0$	System is stable
$N=-m$ (counter clockwise)	$P<m$	$Z<0$	System is unstable
$N=-m$ (counter clockwise)	$P=m$	$Z=0$	System is stable
$N=-m$ (counter clockwise)	$P>m$	$Z>0$	System is unstable
$N=m$ (clockwise)	$P<$ m or P = m or P $>m$	$Z>0$ or $Z=2m$ or $Z>2m$	System is unstable

In general, the condition for the closed-loop stability is that Z must be equal to zero and the condition for the open-loop stability is that P must be equal to zero.

9.21 Procedure for Determining the Number of Encirclements

The procedure for determining the number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ is given below:

Step 1: Construct the contour in $G(s)H(s)$-plane based on the Nyquist path in the s-plane.

Step 2: Draw a dotted line from the point $-1+j0$ which is directed in the third quadrant of the $G(s)H(s)$-plane.

Step 3: Chose any arbitrary point P in the contour in the $G(s)H(s)$-plane.

Step 4: Starting from the point P traverse the contour in the $G(s)H(s)$-plane.

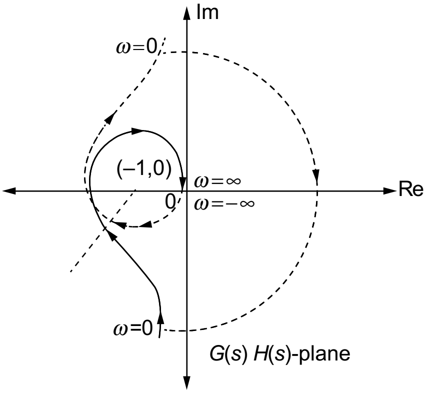
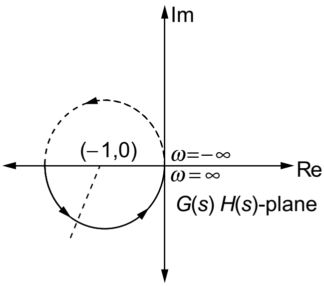
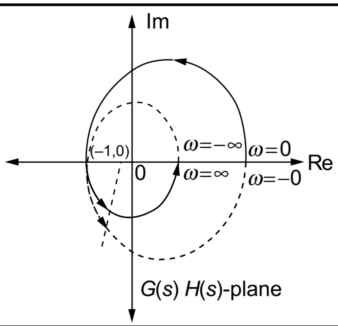
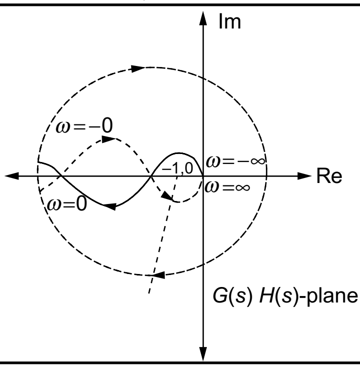
Step 5: If the dotted line is cut by the contour in the clockwise direction, the number of encirclements N is increased by one and if the dotted line is cut by the contour in the counterclockwise direction, the number of encirclements N is decreased by one.

Step 6: Repeat the above step until the arbitrary point reaches the initial position.

Step 7: The value of N present at the end of one complete rotation gives the number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$.

The number of encirclements made by the different contours in $G(s)H(s)$-plane is listed in Table 9.15.

Table 9.15 | Number of encirclements for different contour in $G(s)H(s)$-plane

Contour in $G(s)H(s)$-plane	**Number of encirclements N**
	$N = 2$
	$N = 1$
	$N = -2$
	$N = 0$

9.21.1 Flow chart for Determining the Number of Encirclements Made by the Contour in $G(s)H(s)$-Plane

The flow chart for determining the number of encirclements made by the contour in $G(s)H(s)$-plane is shown in Fig. 9.21.

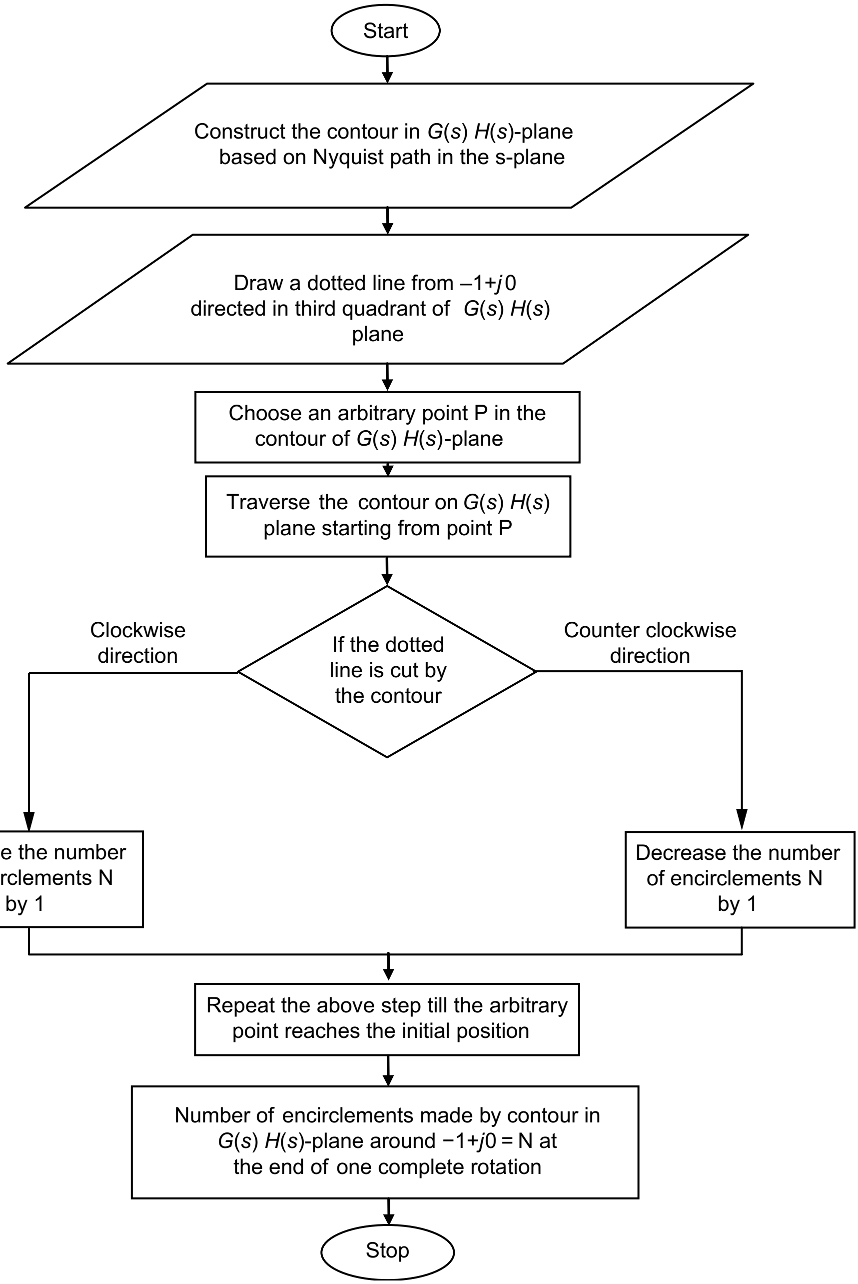
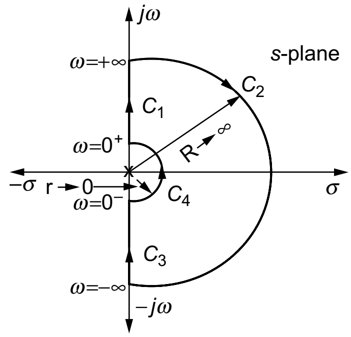
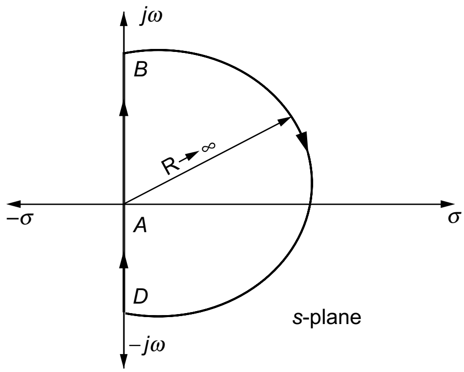


Fig. 9.21 | Flow chart for number of encirclements

9.22 General Procedures for Determining the Stability of the System Based on Nyquist Stability Criterion

Step 1: Construct the Nyquist path that covers the whole right-hand side of the s-plane based on the loop transfer function. The different Nyquist paths for different loop transfer functions are shown below:

<table>
<tr><td>When a pole or zero exist at the origin</td><td>
</td></tr>
<tr><td>When a pole or zero does not exist at the origin</td><td>
</td></tr>
</table>

Step 2: Divide the Nyquist path into different sections with the value of ω, s and radius of the semi-circular path.

Step 3: For each section present in the Nyquist path, determine the contour in the $G(s)H(s)$-plane.

Step 4: The intersection point of the contour on the real axis is determined by substituting ω at which the imaginary part of the transfer function is zero.

Step 5: Complete the contour in the $G(s)H(s)$-plane for the given system by joining the individual contours determined in the previous step.

Step 6: Determine the number of encirclements N made by the contour in $G(s)H(s)$-plane around the point $-1+j0$.

Step 7: For the given loop transfer function, determine the number of poles P existing in the right half of the s-plane.

Step 8: Determine the number of zeros, Z existing in the right half of the s-plane by using $Z=N+P$.

Step 9: Using Table 9.15, examine the closed-loop stability of the system.

9.22.1 Flow chart for Determining the Stability of the System Based on Nyquist Stability Criterion

The flow chart for determining the stability of the system based on Nyquist stability criterion is shown in Fig. 9.22.

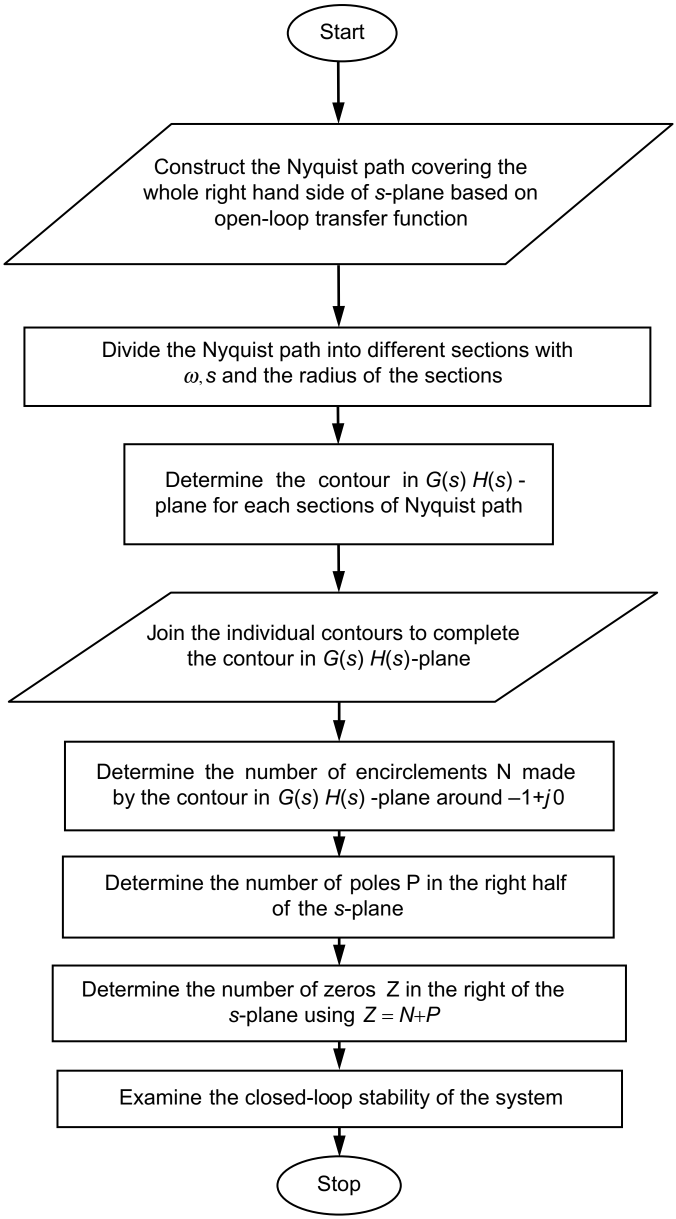


Fig. 9.22 | Flow chart for examining the stability of the system

Example 9.6: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{K}{s(s+2)(s+10)}$. Sketch the Nyquist plot and hence calculate the range of values of K for stability.

Solution:

(i) In general, the Nyquist path for a system is considered to have a complete right-hand side of the s-plane covering the entire imaginary axis including the origin. In addition, the Nyquist path should not pass through any poles and/or zeros on the imaginary axis. As the pole at the origin for the given system exists, a small semicircle with infinitesimal radius is drawn around the origin of the Nyquist path. The modified Nyquist path for the system is shown in Fig. E9.6(a).

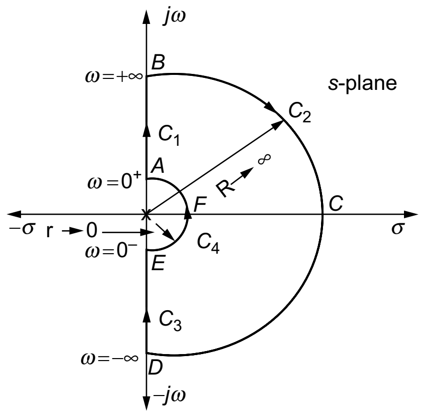


Fig. E9.6(a)

(ii) The different sections present in Fig. E9.6(a) alongwith its parameters are listed in Table E9.6(a).

Table E9.6(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s=j\omega$ and $\omega=0^+$ to ∞
Semicircle BCD	$s=\underset{R\to\infty}{Lt}\, Re^{j\theta}$ and $\dfrac{\pi}{2}\le\theta\le\dfrac{-\pi}{2}$
DE	$s=-j\omega$ and $\omega=-\infty$ to 0^-
Semicircle EFA	$s=\underset{r\to 0}{Lt}\, re^{j\theta}$ and $\dfrac{-\pi}{2}\le\theta\le\dfrac{\pi}{2}$

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.6(a) and the individual contours are listed in Table E9.6(b).

Table E9.6(b) | Contours for different sections

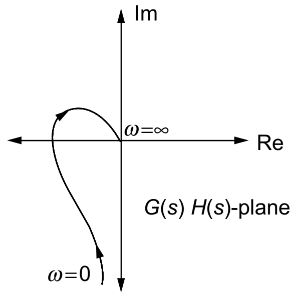
Points used for constructing the contour | **Contour**

Section AB: $s = j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{K}{\omega\sqrt{(\omega^2+2)}\times\sqrt{(\omega^2+100)}}$$

$$\angle G(j\omega)H(j\omega) = -90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

ω	$\left\|G(j\omega)H(j\omega)\right\|$	$\angle G(j\omega)H(j\omega)$
$\omega = 0^+$	∞	$-90°$
$\omega = \infty$	0	$-270°$



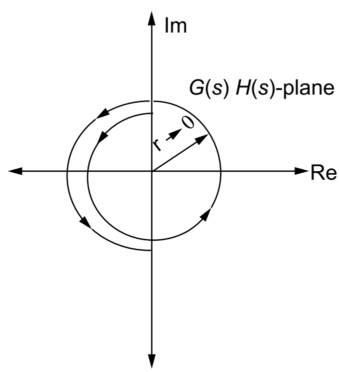
Section BCD: $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{K}{s^3}$

Substituting $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt}\ \frac{K}{R\left(e^{j\theta}\right)^3} = 0e^{-j3\theta}$$

θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\frac{3\pi}{2}}$	$0e^{j\frac{3\pi}{2}}$



(*Continued*)

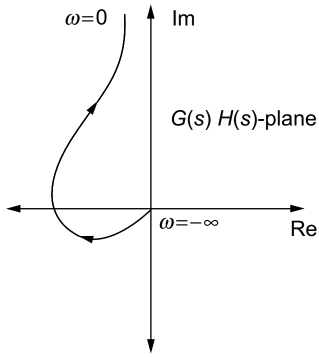
Table E9.6(b) | (Continued)

Section DE: $s=-j\omega$

$$\left|G(j\omega)H(j\omega)\right|=\frac{K}{\omega\sqrt{\left(\omega^2+2\right)}\times\sqrt{\left(\omega^2+100\right)}}$$

$$\angle G(j\omega)H(j\omega)=90^\circ-\tan^{-1}\left(-\frac{\omega}{2}\right)-\tan^{-1}\left(-\frac{\omega}{10}\right)$$

ω	$\left\|G(j\omega)H(j\omega)\right\|$	$\angle G(j\omega)H(j\omega)$
$\omega=-\infty$	0	–270°
$\omega=0^-$	∞	–90°



Section EFA: $s=\underset{r\to 0}{Lt}\, re^{j\theta}$ and $\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$

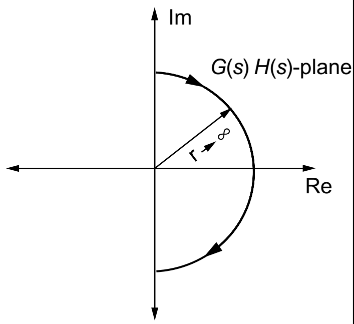
Replacing $s+B=B$ in the given loop transfer function,

we obtain $G(s)H(s)=\dfrac{K}{s(2)(10)}$

Substituting $s=\underset{r\to 0}{Lt}\, re^{j\theta}$, we obtain

$$G(s)H(s)=\underset{r\to 0}{Lt}\frac{K}{20\times re^{j\theta}}=\infty e^{-j\theta}$$

θ	$\frac{-\pi}{2}$	$\frac{\pi}{2}$
$G(s)H(s)$	$\infty e^{j\frac{\pi}{2}}$	$\infty e^{-j\frac{\pi}{2}}$



(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s)=\dfrac{K}{s(s+2)(s+10)}$

Therefore, $G(j\omega)H(j\omega)=\dfrac{K}{j\omega(j\omega+2)(j\omega+10)}$

$$=\frac{K}{j\omega(j\omega+2)(j\omega+10)}\times\frac{-j\omega(-j\omega+2)(-j\omega+10)}{-j\omega(-j\omega+2)(-j\omega+10)}$$

$$=\frac{-Kj\omega\left[20-12j\omega-\omega^2\right]}{\omega^2\left(3+\omega^2\right)\left(100+\omega^2\right)}$$

Hence, the real part of the transfer function = $\dfrac{-12\omega^2}{\omega^2\left(3+\omega^2\right)\left(100+\omega^2\right)}$

and the imaginary part of the transfer function = $\dfrac{-K\left(20\omega-\omega^3\right)}{\omega^2\left(3+\omega^2\right)\left(100+\omega^2\right)}$

Equating the imaginary part of the above transfer function to zero, we obtain

$$\omega\left(20-\omega^2\right)=0$$

$$\omega^2=20 \text{ or } \omega=\sqrt{20}$$

Substituting $\omega=\sqrt{20}$ in the real part of the transfer function, we obtain the intersection point Q as $\dfrac{-K}{240}$.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.6(b).

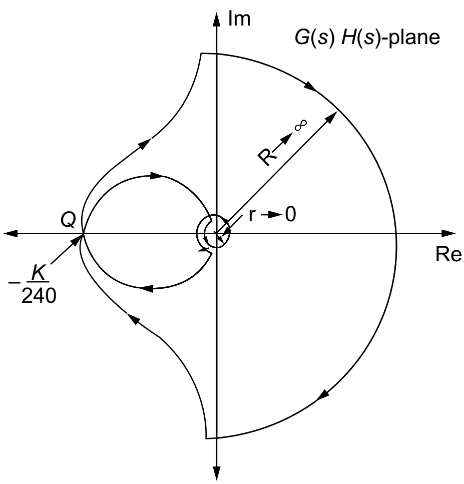


Fig. E9.6(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system cannot be determined as the intersection point depends on the gain K.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane cannot be determined for the given system.

(ix) Hence, the stability of the system depends on the gain K.

(x) To determine the range of gain K for the system to be stable:
For the system to be stable, Q should be less than -1

$$\text{i.e., } -\frac{K}{240} < -1 \text{ or } K < 240$$

Therefore, for the system to be stable, the range of gain K is $0 < K < 240$.

Example 9.7: **The loop transfer function of a certain control system is given by $G(s)H(s) = \dfrac{1}{(1+s)(1+2s)(1+3s)}$. Sketch the Nyquist plot and comment on the stability of the system.**

Solution:

(i) As there exists no pole on the imaginary axis for the given system, the Nyquist path for the original system is shown in Fig. E9.7(a).

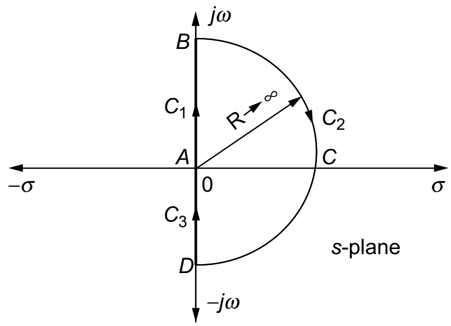


Fig. E9.7(a)

(ii) The different sections present in Fig. E9.7(a) alongwith parameters are listed in Table E9.7(a).

Table E9.7(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\ \mathrm{Re}^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DA	$s = -j\omega$ and $\omega = -\infty$ to 0

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.7(a) and the individual contours are listed in Table E9.7(b).

Table E9.7(b) | Contours for different sections

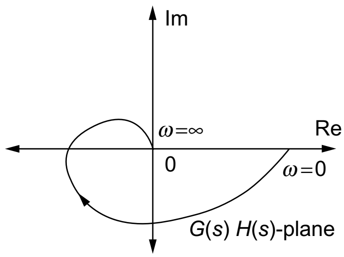
Points used for constructing the contour	Contour

Section AB: $s = j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{1}{\sqrt{(\omega^2+1)} \times \sqrt{(4\omega^2+1)} \times \sqrt{(9\omega^2+1)}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}(2\omega) - \tan^{-1}(3\omega)$$

ω	$\left|G(j\omega)H(j\omega)\right|$	$\angle G(j\omega)H(j\omega)$
$\omega = 0$	1	0°
$\omega = \infty$	0	–270°



Section BCD: $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

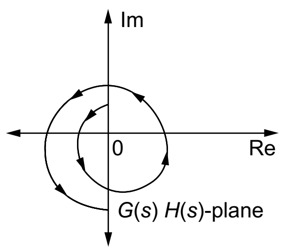
Replacing $s + A = s$ in the given loop transfer function,

we obtain $G(s)H(s) = \frac{1}{6s^3}$

Substituting $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt} \frac{1}{6R\left(e^{j\theta}\right)^3} = 0e^{-j3\theta}$$

θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\frac{3\pi}{2}}$	$0e^{+j\frac{3\pi}{2}}$

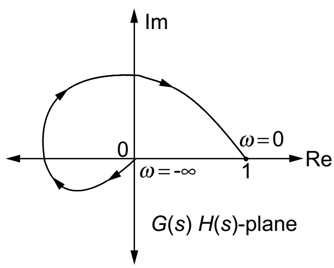


Section DA: $s = -j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{1}{\sqrt{(\omega^2+1)} \times \sqrt{(4\omega^2+1)} \times \sqrt{(9\omega^2+1)}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}(-\omega) - \tan^{-1}(-2\omega) - \tan^{-1}(-3\omega)$$

ω	$\left|G(j\omega)H(j\omega)\right|$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	270°
$\omega = 0$	1	0°



(iv) To determine the intersection point of the complete contour in the real axis:
The loop transfer function of the given system is

$$G(s)H(s)=\frac{1}{(s+1)(2s+1)(3s+1)}$$

Therefore, $G(j\omega)H(j\omega)=\dfrac{1}{(j\omega+1)(j2\omega+1)(j3\omega+1)}$

$$=\frac{1}{(j\omega+1)(j2\omega+1)(j3\omega+1)}\times\frac{(-j\omega+1)(-j2\omega+1)(-j3\omega+1)}{(-j\omega+1)(-j2\omega+1)(-j3\omega+1)}$$

$$=\frac{(1-j\omega)\left[1-5j\omega-6\omega^2\right]}{(1+\omega^2)(1+4\omega^2)(1+9\omega^2)}$$

$$=\frac{1-11(\omega^2)}{(1+\omega^2)(1+4\omega^2)(1+9\omega^2)}-\frac{6j\omega(1-\omega^2)}{(1+\omega^2)(1+4\omega^2)(1+9\omega^2)}$$

Hence, the real part of the transfer function = $\dfrac{1-11(\omega^2)}{(1+\omega^2)(1+4\omega^2)(1+9\omega^2)}$

and the imaginary part of the transfer function = $-\dfrac{6j\omega(1-\omega^2)}{(1+\omega^2)(1+4\omega^2)(1+9\omega^2)}$

Equating the imaginary part of the transfer function to zero, we obtain

$$1-\omega^2=0$$

$$\omega^2=1 \text{ or } \omega=1.$$

Substituting $\omega=1$ in the real part of the transfer function, we obtain the intersection point Q as

$$Q=\frac{1-11(1)}{(1+1)(1+4(1))(1+9)}=-\frac{10}{2\times5\times10}=-0.1$$

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.7(b).

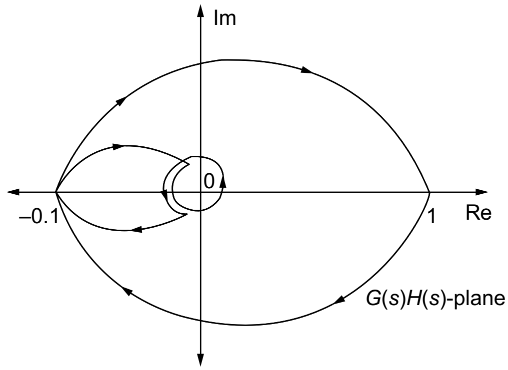


Fig. E9.7(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is 0, i.e., $N=0$.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane is determined by using $Z=N+P=0$.

(ix) Using Table 9.15 and the value of Z obtained in the previous step, we may conclude that the system is stable.

Example 9.8: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{10}{s^2(1+0.2s)(1+0.5s)}$. Sketch the Nyquist plot and comment on the stability of the system.

Solution:

(i) As two poles exist at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.8(a).

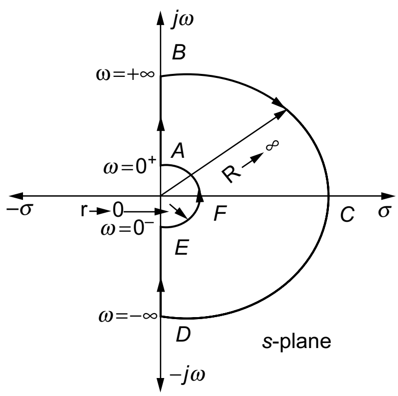


Fig. E9.8(a)

(ii) The different sections present in Fig. E9.8(a) alongwith its parameters are listed in Table E9.8(a).

Table E9.8(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0^+$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DE	$s = -j\omega$ and $\omega = -\infty$ to 0^-
Semicircle EFA	$s = \underset{r\to 0}{Lt}\ re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.8(a) and the individual contours are listed in Table E9.8(b).

Table E9.8(b) | Contours for different sections

<table>
<tr><th>Points used for constructing the contour</th><th>Contour</th></tr>
<tr><td>

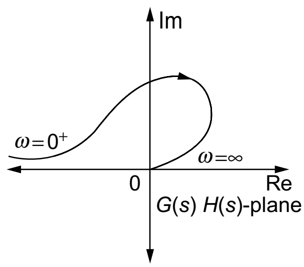
Section AB: $s = j\omega$

$$|G(j\omega)H(j\omega)| = \frac{10}{\omega^2 \times \sqrt{(0.04\omega^2 + 1)} \times \sqrt{(0.25\omega^2 + 1)}}$$

$$\angle G(j\omega)H(j\omega) = -180^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.5\omega)$$

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = 0^+$	∞	-180°
$\omega = \infty$	0	-360°

</td><td>

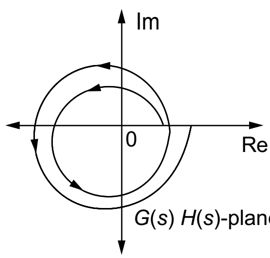


</td></tr>
<tr><td>

Section BCD: $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{10}{0.1s^4}$

</td><td>



</td></tr>
</table>

(*Continued*)

Table E9.8(b) | (Continued)

Substituting $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt} \frac{10}{0.1R^4\left(e^{j\theta}\right)^4} = 0e^{-j4\theta}$$

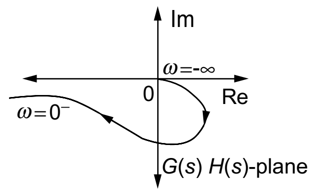
θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j2\pi}$	$0e^{+j2\pi}$

Section DE: $s = -j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{10}{\omega^2 \times \sqrt{\left(0.04\omega^2+1\right)} \times \sqrt{\left(0.25\omega^2+1\right)}}$$

$$\angle G(j\omega)H(j\omega) = -180° - \tan^{-1}(-0.2\omega) - \tan^{-1}(-0.5\omega)$$

ω	$\left|G(j\omega)H(j\omega)\right|$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	0°
$\omega = 0^-$	∞	–180°



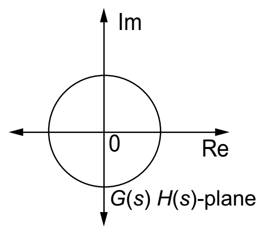
Section EFA: $s = \underset{r\to 0}{Lt}\, re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{10}{s^2}$

Substituting $s = \underset{r\to 0}{Lt}\, re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{r\to 0}{Lt} \frac{10}{r^2\left(e^{j\theta}\right)^2} = \infty e^{-j2\theta}$$

θ	$\frac{-\pi}{2}$	$\frac{\pi}{2}$
$G(s)H(s)$	$\infty e^{j\pi}$	$\infty e^{-j\pi}$



(iv) To determine the intersection point of the complete contour in the real axis:

The loop transfer function of the given system is $G(s)H(s) = \dfrac{10}{s^2(1+0.2s)(1+0.5s)}$

Therefore, $G(j\omega)H(j\omega) = \dfrac{10}{(j\omega)^2(1+j0.2\omega)(1+j0.5\omega)}$

$$= \frac{10}{(j\omega)^2(1+j0.2\omega)(1+j0.5\omega)} \times \frac{(1-j0.2\omega)(1-j0.5\omega)}{(1-j0.2\omega)(1-j0.5\omega)}$$

$$= \frac{10\left[1-0.7j\omega-0.1\omega^2\right]}{(-\omega^2)(1+0.4\omega^2)(1+0.25\omega^2)}$$

$$= \frac{10-1(\omega^2)}{(-\omega^2)(1+0.4\omega^2)(1+0.25\omega^2)} - \frac{j7\omega}{(-\omega^2)(1+0.4\omega^2)(1+0.25\omega^2)}$$

Hence, the real part of the transfer function = $\dfrac{10-(\omega^2)}{(-\omega^2)(1+0.4\omega^2)(1+0.25\omega^2)}$

and the imaginary part of the transfer function = $-\dfrac{j7\omega}{(-\omega^2)(1+0.4\omega^2)(1+0.25\omega^2)}$

Equating the imaginary part of the transfer function to zero, we obtain $\omega = 0$.

Substituting $\omega = 0$ in the real part of the transfer function, we obtain the intersection point $Q = \infty$. Hence, there is no valid intersection point of contour on the imaginary axis.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane as shown in Fig. E9.8(b).

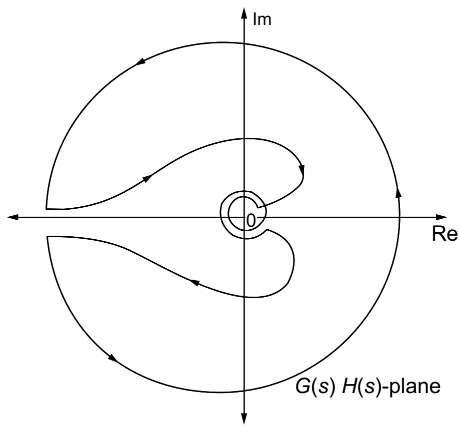


Fig. E9.8(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is 2, i.e., $N=-2$.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is 0, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane is determined by using $Z=N+P=-2$.

(ix) Using the Table 9.15 and the value of Z obtained in the previous step, we may conclude that the system is unstable.

Example 9.9: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{50}{(s+1)(s+2)}$. Sketch the Nyquist plot and examine the stability of the system.

Solution:

(i) As no pole exists at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.9(a).

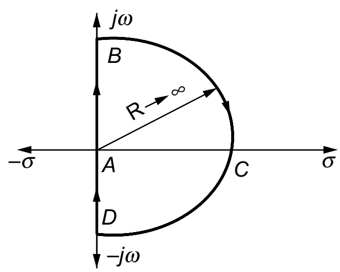


Fig. E9.9(a)

(ii) The different sections present in Fig. E9.9(a) alongwith its parameters are listed in Table E9.9(a).

Table E9.9(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s=j\omega$ and $\omega=0^{+}$ to ∞
Semicircle BCD	$s=\underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\dfrac{\pi}{2}\le\theta\le\dfrac{-\pi}{2}$
DA	$s=-j\omega$ and $\omega=-\infty$ to 0^{-}

(iii) Construct a contour in $G(s)H(s)$ -plane for each section given in Table E9.9(a) and the individual contours are listed in Table E9.9(b).

Table E9.9(b) | Contours for different sections

<table>
<tr><th>Points used for constructing the contour</th><th>Contour</th></tr>
<tr><td>

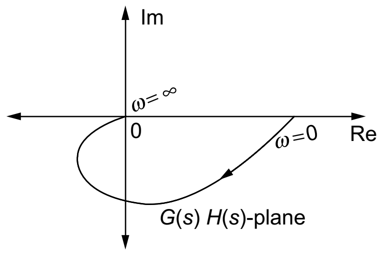
Section AB: $s = j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{50}{\sqrt{(\omega^2+1)} \times \sqrt{(\omega^2+4)}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

<table>
<tr><td>ω</td><td>$\left|G(j\omega)H(j\omega)\right|$</td><td>$\angle G(j\omega)H(j\omega)$</td></tr>
<tr><td>$\omega = 0$</td><td>25</td><td>0°</td></tr>
<tr><td>$\omega = \infty$</td><td>0</td><td>–180°</td></tr>
</table>

</td><td>



</td></tr>
<tr><td>

Section BCD: $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

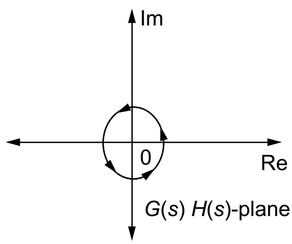
Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{50}{s^2}$

Substituting $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt} \frac{50}{R^2\left(e^{j\theta}\right)^2} = 0e^{-j2\theta}$$

<table>
<tr><td>θ</td><td>$\frac{\pi}{2}$</td><td>$\frac{-\pi}{2}$</td></tr>
<tr><td>$G(s)H(s)$</td><td>$0e^{-j\pi}$</td><td>$0e^{+j\pi}$</td></tr>
</table>

</td><td>



</td></tr>
</table>

(*Continued*)

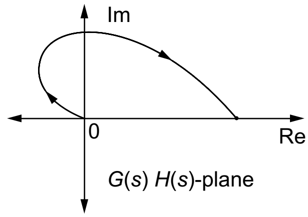
Table E9.9(b) | (Continued)

Section DA: $s=-j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{50}{\sqrt{(\omega^2+1)}\times\sqrt{(\omega^2+4)}}$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1}(-\omega)-\tan^{-1}\left(-\frac{\omega}{2}\right)$$

ω	$\left\|G(j\omega)H(j\omega)\right\|$	$\angle G(j\omega)H(j\omega)$
$\omega=-\infty$	0	180°
$\omega=0^-$	∞	0°



(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s)=\dfrac{50}{(s+1)(s+2)}$

Therefore,

$$G(j\omega)H(j\omega)=\frac{50}{(1+j\omega)(2+j\omega)}\times\frac{(1-j\omega)(2-j\omega)}{(1-j\omega)(2-j\omega)}$$

$$=\frac{50\left[2-3j\omega-\omega^2\right]}{(1+\omega^2)(4+\omega^2)}=\frac{50(2-\omega^2)-150j\omega}{(1+\omega^2)(4+\omega)}$$

Hence, the real part of the transfer function = $\dfrac{50(2-\omega^2)}{(1+\omega^2)(4+\omega)}$

and the imaginary part of the transfer function = $\dfrac{-150\omega}{(1+\omega^2)(4+\omega)}$

Equating the imaginary part of the above transfer function to zero, we obtain

$$\omega=0$$

Substituting $\omega=0$ in the real part of the transfer function, we obtain the intersection point Q as 25. Hence, other than the point $Q=25$, we have no intersection of the contour in the $F(s)$-plane.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.9(b).

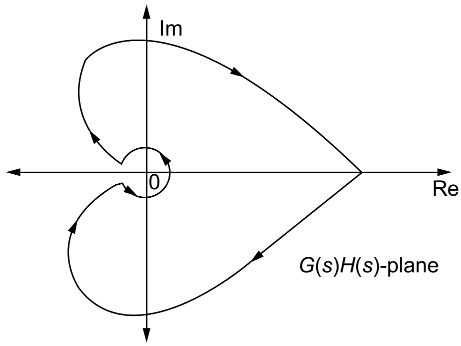


Fig. E9.9(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is zero, i.e., $N=0$

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane is $Z=N+P=0$.

(ix) As the value of zero which lies in the right half of the s-plane is zero, the system is stable.

Example 9.10: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{1}{s(s+2)}$. Sketch the Nyquist plot and examine the stability of the system.

Solution:

(i) As no pole exists at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.10(a).

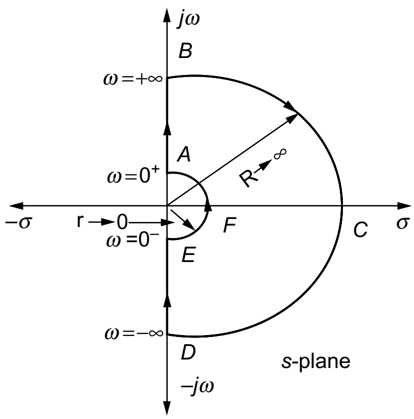


Fig. E9.10(a)

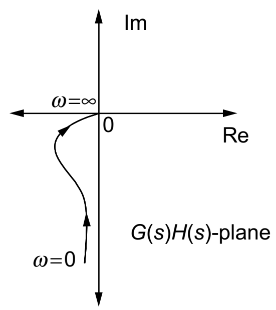
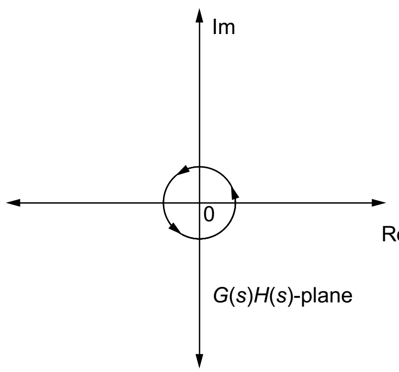
(ii) The different sections present in Fig. E9.10(a) alongwith its parameters are listed in Table E9.10(a).

Table E9.10(a) Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0^{+}$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DA	$s = -j\omega$ and $\omega = -\infty$ to 0^{-}

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.10(a) and the individual contours are listed in Table E9.10(b).

Table E9.10(b) Contours for different sections

<table>
<tr><th>Points used for constructing the contour</th><th>Contour</th></tr>
<tr><td>Section AB: $s = j\omega$
$$\left|G(j\omega)H(j\omega)\right| = \frac{1}{\omega \times \sqrt{(\omega^2+4)}}$$
$$\angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right)$$
<table><tr><th>ω</th><th>$\left|G(j\omega)H(j\omega)\right|$</th><th>$\angle G(j\omega)H(j\omega)$</th></tr><tr><td>$\omega = 0$</td><td>$\infty$</td><td>$-90^\circ$</td></tr><tr><td>$\omega = \infty$</td><td>0</td><td>$-180^\circ$</td></tr></table></td><td></td></tr>
<tr><td>Section BCD: $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{1}{s^2}$
Substituting $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$, we obtain
$$G(s)H(s) = \underset{R\to\infty}{Lt}\ \frac{1}{R^2\left(e^{j\theta}\right)^2} = 0e^{-j2\theta}$$</td><td></td></tr>
</table>

(*Continued*)

Table E9.10(b) | (Continued)

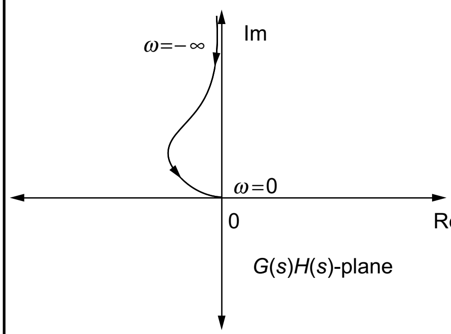
θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\pi}$	$0e^{+j\pi}$

Section DA: $s = -j\omega$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega \times \sqrt{(\omega^2+4)}}$$

$$\angle G(j\omega)H(j\omega) = 90° - \tan^{-1}\left(-\frac{\omega}{2}\right)$$

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	180°
$\omega = 0^-$	∞	90°



(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s) = \dfrac{1}{s(s+2)}$

Therefore,
$$G(j\omega)H(j\omega) = \frac{1}{j\omega(2+j\omega)} \times \frac{-j\omega(2-j\omega)}{-j\omega(2-j\omega)}$$

$$= \frac{-\omega^2 - 2j\omega}{\omega^2(4+\omega^2)}$$

Hence, the real part of the transfer function = $\dfrac{-\omega^2}{\omega^2(4+\omega^2)}$

and the imaginary part of the transfer function = $\dfrac{-2\omega}{\omega^2(4+\omega^2)}$

Equating the imaginary part of the above transfer function to zero, we obtain $\omega = 0$. Substituting $\omega = 0$ in the real part of the transfer function, we obtain the intersection point Q as ∞. Therefore, the contour crosses the real axis only at ∞.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.10(b).

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is zero, i.e., $N=0$.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane is $Z=N+P=0$.

(ix) As the value of zero which lies in the right half of the s-plane is zero, the system is stable.

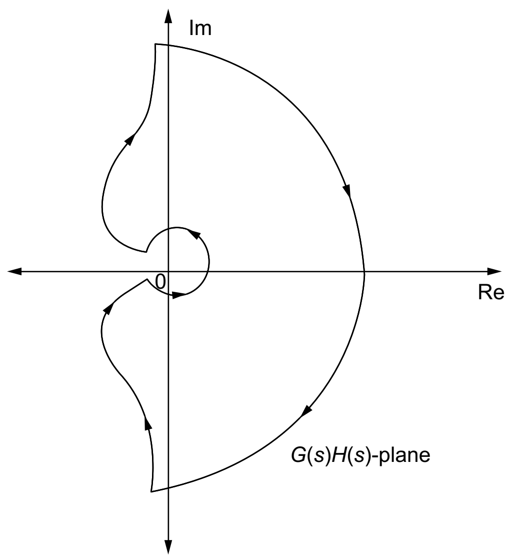


Fig. E9.10(b)

Example 9.11: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{4s+1}{s^2(s+1)(2s+1)}$. Sketch the Nyquist plot and examine the stability of the system.

Solution:

(i) As two poles exist at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.11(a).

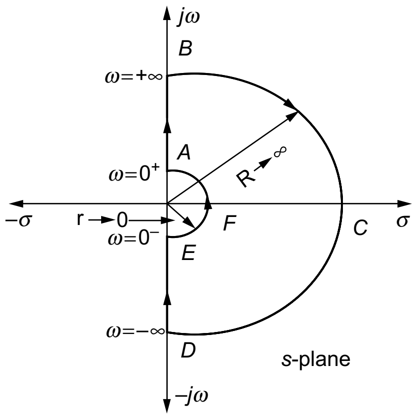


Fig. E9.11(a)

(ii) The different sections present in Fig. E9.11(a) alongwith its parameters are listed in Table E9.11(a).

Table E9.11(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0^+$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\ \mathrm{Re}^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DE	$s = -j\omega$ and $\omega = -\infty$ to 0^-
Semicircle EFA	$s = \underset{r\to 0}{Lt}\ re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.11(a) and the individual contours are listed in Table E9.11(b).

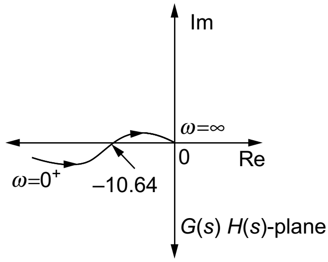
Table E9.11(b) | Contours for different sections

Points used for constructing the contour	Contour
Section AB: $s = j\omega$ $\lvert G(j\omega)H(j\omega)\rvert = \dfrac{\sqrt{(16\omega^2+1)}}{\omega^2 \times \sqrt{(\omega^2+1)} \times \sqrt{(4\omega^2+1)}}$ $\angle G(j\omega)H(j\omega) = \tan^{-1}(4\omega) - 180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$	

(*Continued*)

Table E9.11(b) | (Continued)

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = 0$	∞	$-180°$
$\omega = \infty$	0	$-270°$



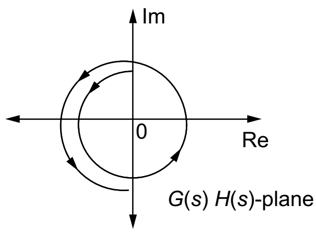
Section BCD: $s = \underset{R\to\infty}{Lt}\ \text{Re}^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{2}{s^3}$

Substituting $s = \underset{R\to\infty}{Lt}\ \text{Re}^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt} \frac{1}{R^3\left(e^{j\theta}\right)^3} = 0e^{-j3\theta}$$

θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\frac{3\pi}{2}}$	$0e^{+j\frac{3\pi}{2}}$

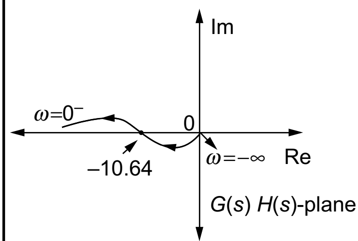


Section DE: $s = -j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{\sqrt{\left(16\omega^2+1\right)}}{\omega^2 \times \sqrt{\left(\omega^2+1\right)} \times \sqrt{\left(4\omega^2+1\right)}}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}(4\omega) + 180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	270°
$\omega = 0^-$	∞	180°



Section EFA: $s = \underset{r\to 0}{Lt}\ re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Replacing $s + A = A$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{1}{s^2}$

(*Continued*)

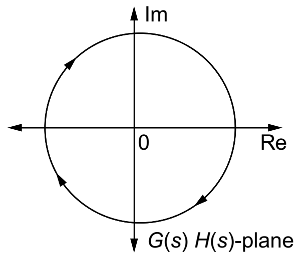
Table E9.11(b) | (Continued)

<table>
<tr><td>

Substituting $s = \underset{r\to 0}{Lt}\, re^{j\theta}$, we obtain

θ	$\frac{-\pi}{2}$	$\frac{\pi}{2}$
$G(s)H(s)$	$\infty e^{j\pi}$	$0e^{-j\pi}$

</td><td>



</td></tr>
</table>

(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s) = \dfrac{4s+1}{s^2(s+1)(2s+1)}$

We know that when the $G(j\omega)H(j\omega)$ contour crosses the real axis, the phase angle of the system will be $-180°$.
Hence,

$$\tan^{-1}(4\omega) - 180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega) = -180°$$

$$\tan^{-1}(4\omega) = \tan^{-1}(\omega) + \tan^{-1}(2\omega)$$

Taking tan on both sides of the equation, we obtain

$$\tan\left(\tan^{-1}(4\omega)\right) = \tan\left(\tan^{-1}(\omega) + \tan^{-1}(2\omega)\right)$$

We know that, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Using the above equation, we obtain $4\omega = \dfrac{\omega + 2\omega}{1-(\omega \times 2\omega)}$

Upon solving, we obtain

$$\omega = 0.354 \text{ rad/sec}$$

Substituting $\omega = 0.354$ in the magnitude of the transfer function, we obtain

$$\left|G(j\omega)H(j\omega)\right| = 10.64.$$

Therefore, the contour intersects the real axis at -10.64

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.11(b).

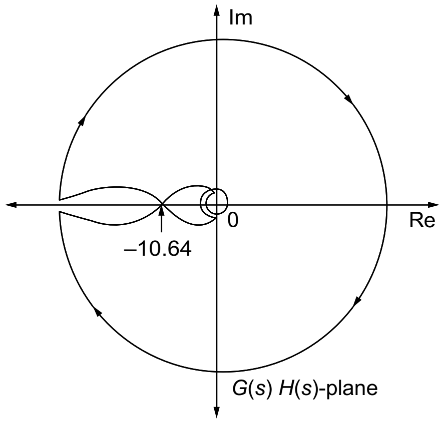


Fig. E9.11(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is zero, i.e., $N=2$.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane is $Z=N+P=2$.

(ix) As the value of zero which lies in the right half of the s-plane is zero, the system is unstable.

Example 9.12: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{K(s+1)^2}{s^3}$. Sketch the Nyquist plot and examine the stabilityof the system.

Solution:

(i) As three poles exist at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.12(a).

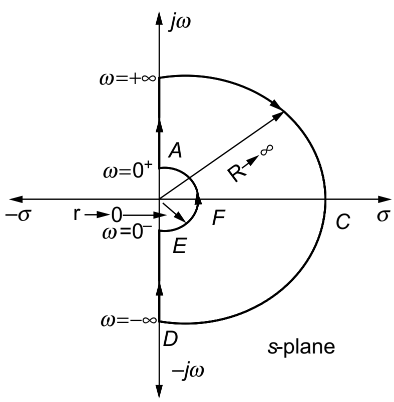


Fig. E9.12(a)

(ii) The different sections present in Fig. E9.12(a) alongwith its parameters are listed in Table E9.12(a).

Table E9.12(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0^+$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DE	$s = -j\omega$ and $\omega = -\infty$ to 0^-
Semicircle EFA	$s = \underset{r\to 0}{Lt}\, re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.12(a) and the individual contours are listed in Table E9.12(b).

Table E9.12(b) | Contours for different sections

<table>
<tr><th>Points used for constructing the contour</th><th>Contour</th></tr>
<tr><td>

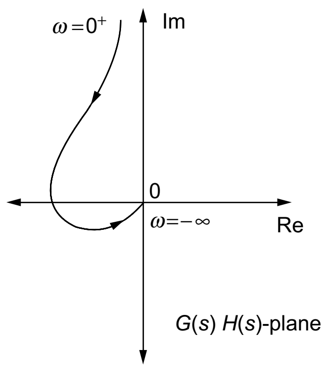
Section AB: $s = j\omega$

$$|G(j\omega)H(j\omega)| = \frac{K(\omega^2+1)}{\omega^3}$$

$$\angle G(j\omega)H(j\omega) = 2\tan^{-1}(\omega) - 270°$$

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = 0$	∞	–270°
$\omega = \infty$	0	–90°

</td><td>



</td></tr>
<tr><td>

Section BCD: $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function,

we obtain $G(s)H(s) = \frac{2}{s}$

</td><td></td></tr>
</table>

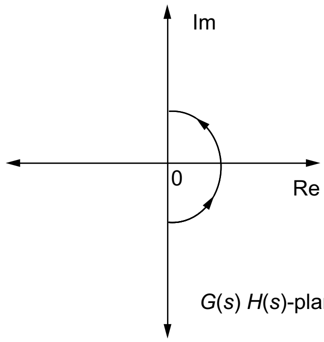
(*Continued*)

Table E9.12(b) | (Continued)

Substituting $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt}\frac{1}{R\left(e^{j\theta}\right)} = 0e^{-j\theta}$$

θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\frac{\pi}{2}}$	$0e^{+j\frac{\pi}{2}}$

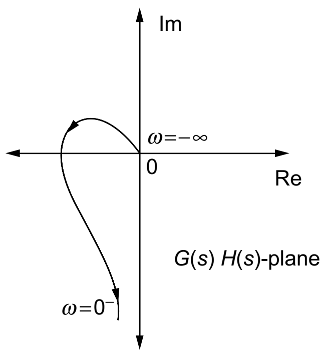


Section DE: $s = -j\omega$

$$\left|G(j\omega)H(j\omega)\right| = \frac{K\left(\omega^2+1\right)}{\omega^3}$$

$$\angle G(j\omega)H(j\omega) = 2\tan^{-1}(-\omega) + 270^\circ$$

ω	$\left\|G(j\omega)H(j\omega)\right\|$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	90°
$\omega = 0^-$	∞	270°



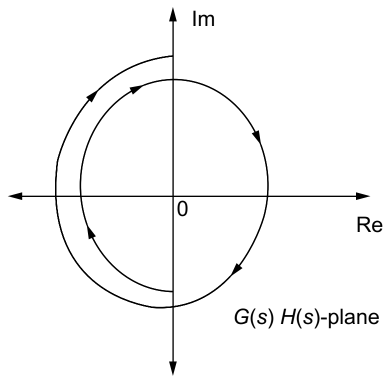
Section EFA: $s = \underset{r\to 0}{Lt}\, re^{j\theta}$ and $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Replacing $s + A = A$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{1}{s^3}$

Substituting $s = \underset{R\to\infty}{Lt}\, Re^{j\theta}$, we obtain

$$G(s)H(s) = \underset{r\to 0}{Lt}\frac{1}{r^3\left(e^{j\theta}\right)^3} = \infty e^{-j3\theta}$$

θ	$\frac{-\pi}{2}$	$\frac{\pi}{2}$
$G(s)H(s)$	$\infty e^{\frac{j3\pi}{2}}$	$0e^{\frac{-j3\pi}{2}}$



(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s) = \frac{K(s+1)^2}{s^3}$

Therefore,
$$G(j\omega)H(j\omega) = -\frac{K(1-\omega^2+j2\omega)}{j\omega}$$

$$= -\frac{K(1-\omega^2+j2\omega)}{j\omega} \times \frac{-j\omega}{-j\omega}$$

$$= -\frac{2K}{\omega^2} + j\frac{K(1-\omega^2)}{\omega^3}$$

Hence, the real part of the transfer function = $-\frac{2K}{\omega^2}$

and the imaginary part of the transfer function = $\frac{K(1-\omega^2)}{\omega^2}$.

Equating the imaginary part of the transfer function, we obtain

$$\omega = 1$$

Substituting $\omega = 1$ in the real part of the transfer function, we obtain the intersection point Q as $-2K$.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.12(b).

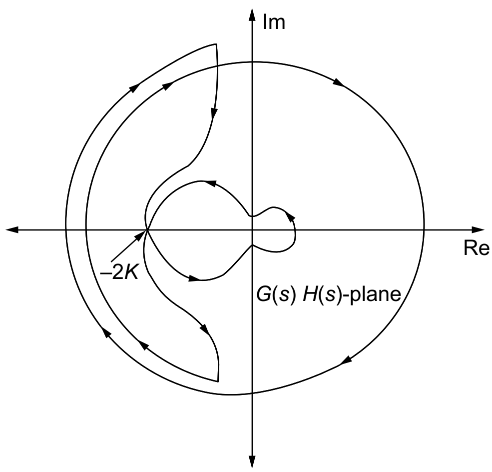


Fig. E9.12(b)

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system cannot be determined as the intersection point depends on the gain K.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P=0$.

(viii) The number of zeros which lies in the right half of the s-plane cannot be determined for the given system.

(ix) As the value of zero which lies in the right half of the s-plane is not determined, the stability of the system cannot be analyzed. The stability of the system depends on the K.

(x) To determine the range of gain K for the system to be stable:
For the system to be stable, Q should be less than −1.
i.e., $-2K < -1$ or $K < 0.5$

Therefore, for the system to be stable, the range of gain K is $0 < K < 0.5$.

Example 9.13: The loop transfer function of a certain control system is given by $G(s)H(s)=\dfrac{(s+2)}{(s+1)(s-1)}$. Sketch the Nyquist plot and examine the stability of the system.

Solution:

(i) As no poles exist at the origin for the given system, the Nyquist path for the system is shown in Fig. E9.13(a).

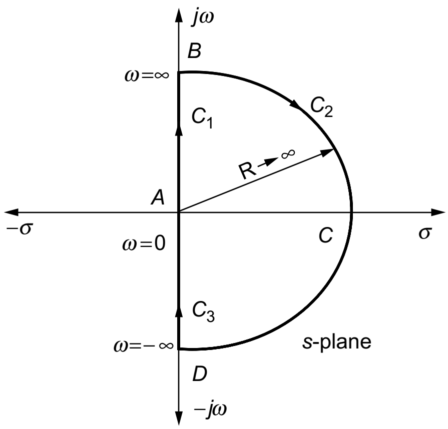


Fig. E9.13(a)

(ii) The different sections present in Fig. E9.13(a) alongwith parameters are listed in Table E9.13(a).

Table E9.13(a) | Sections present in the Nyquist path

Section	Parameter
AB	$s = j\omega$ and $\omega = 0^+$ to ∞
Semicircle BCD	$s = \underset{R\to\infty}{Lt}\ \mathrm{R}e^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$
DA	$s = -j\omega$ and $\omega = -\infty$ to 0^-

(iii) Construct a contour in $G(s)H(s)$-plane for each section given in Table E9.13(a) and the individual contour is listed in Table E9.13(b).

Table E9.13(b) | Contour for different sections

<table>
<tr><th>Points used for constructing the contour</th><th>Contour</th></tr>
<tr><td>

Section AB: $s = j\omega$

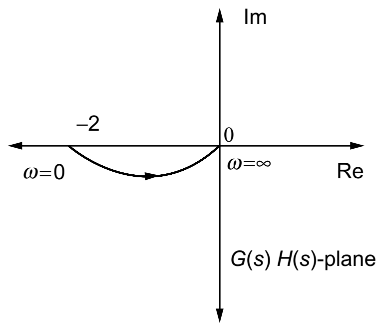
$$\left|G(j\omega)H(j\omega)\right| = \frac{\sqrt{(\omega^2+4)}}{(\omega^2+1)}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}(\omega) - \left(180^\circ - \tan^{-1}(\omega)\right)$$

$$= \tan^{-1}\left(\frac{\omega}{2}\right) - 180^\circ$$

ω	$\left\|G(j\omega)H(j\omega)\right\|$	$\angle G(j\omega)H(j\omega)$
$\omega = 0$	2	–180°
$\omega = \infty$	0	–90°

</td><td>



</td></tr>
<tr><td>

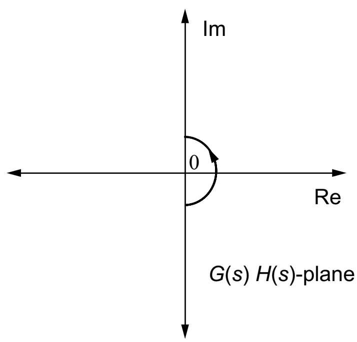
Section BCD: $s = \underset{R\to\infty}{Lt}\ \mathrm{R}e^{j\theta}$ and $\frac{\pi}{2} \le \theta \le \frac{-\pi}{2}$

Replacing $s + A = s$ in the given loop transfer function, we obtain $G(s)H(s) = \frac{1}{s}$

Substituting $s = \underset{R\to\infty}{Lt}\ \mathrm{R}e^{j\theta}$, we obtain

$$G(s)H(s) = \underset{R\to\infty}{Lt}\ \frac{1}{R\left(e^{j\theta}\right)} = 0e^{-j\theta}$$

</td><td>



</td></tr>
</table>

(*Continued*)

Table E9.13(b) | (Continued)

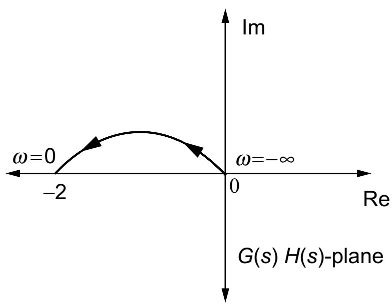
θ	$\frac{\pi}{2}$	$\frac{-\pi}{2}$
$G(s)H(s)$	$0e^{-j\frac{\pi}{2}}$	$0e^{+j\frac{\pi}{2}}$

Section DA: $s = -j\omega$

$$|G(j\omega)H(j\omega)| = \frac{\sqrt{(\omega^2+4)}}{(\omega^2+1)}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(-\frac{\omega}{2}\right) + 180°$$

ω	$\lvert G(j\omega)H(j\omega)\rvert$	$\angle G(j\omega)H(j\omega)$
$\omega = -\infty$	0	90°
$\omega = 0$	2	180°



(iv) To determine the intersection point of the contour in the real axis:

The loop transfer function of the given system is $G(s)H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$

Therefore, $G(j\omega)H(j\omega) = \dfrac{(j\omega+2)}{(-\omega^2-1)}$

Hence, the real part of the transfer function = $\dfrac{-2}{(1+\omega^2)}$

and the imaginary part of the transfer function = $\dfrac{-\omega}{(1+\omega^2)}$.

Equating the imaginary part of the transfer function, we obtain $\omega = 0$.
As the real part of the transfer function is independent of ω, there is no intersection point on the real axis other than –2.

(v) The individual contours obtained in step (iii) are combined together to construct the complete contour in $G(s)H(s)$-plane alongwith the intersection point as shown in Fig. E9.13(b).

(vi) The number of encirclements made by the contour in $G(s)H(s)$-plane around the point $-1+j0$ for the given system is $N = -1$.

(vii) For the given system, the number of poles which lies in the right half of the s-plane is zero, i.e., $P = 1$.

(viii) The number of zeros which lies in the right half of the s-plane for the given system is $Z = N + P = 0$.

(ix) As the value of zero which lies in the right half of the s-plane is zero the given system is stable.

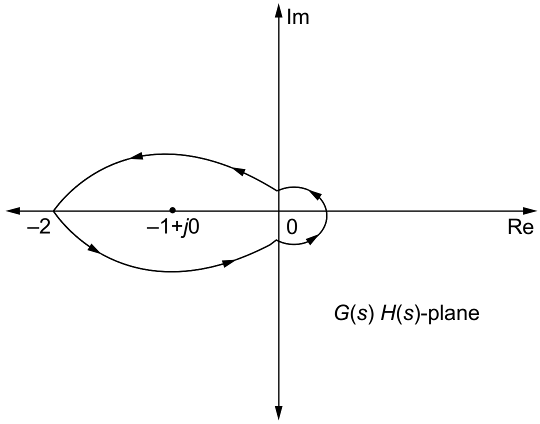


Fig. E9.13(b)

Review Questions

1. A feedback control system is characterized by the open-loop transfer function

$$G(s)H(s) = \frac{100}{s(s+4)(s+20)}$$

   (a) Draw the polar plot of the open-loop system. Calculate the magnitude and phase at 10 equi-spaced frequency points, including the point where the plot intersects the imaginary axis and use them to construct the plot.

   (b) From the polar plot, how will you determine the gain margin, phase margin, gain crossover frequency and phase crossover frequency? Comment on the system stability.

2. Draw the polar plot of each of the following complex functions:

   (a) $G(j\omega) = \omega^2 \angle 45°$

   (b) $G(j\omega) = \omega^2 \left(\cos 45° + j \sin 45°\right)$

   (c) $G(j\omega) = 0.707\omega^2 + j0.707\omega^2$

   (d) $G(j\omega) = 0.707\omega^2 (1 + j) + 1$

3. Sketch the polar plot for $G(s)H(s) = \dfrac{1}{s^4(s+p)}$, $p > 0$.
4. Determine: (a) the phase crossover frequency, (b) the gain crossover frequency, (c) the gain margin and (d) phase margin for the system of $G(s)H(s) = \dfrac{1}{s(s+p_1)(s+p_2)}$ with $p_1 = 1$ and $p_2 = \dfrac{1}{2}$.
5. Sketch the polar plot for

(a) $G(s)H(s) = \dfrac{K(s+z_1)}{s^2(s+p_1)(s+p_2)(s+p_3)}$, $z_1, p_i > 0$.

(b) $G(s)H(s) = \dfrac{s+z_1}{s(s+p_1)}, z_1, p_1 > 0.$

(c) $G(s)H(s) = \dfrac{s+z_1}{s(s+p_1)(s+p_2)}, z_1, p_i > 0.$

(d) $G(s)H(s) = \dfrac{K}{s^2(s+p_1)(s+p_2)}, p_i > 0.$

(e) $G(s)H(s) = \dfrac{K}{s^3(s+p_1)(s+p_2)}, p_i > 0.$

(f) $G(s)H(s) = \dfrac{s+z_1}{s^4(s+p_1)}, z_1, p_1 > 0.$

(g) $G(s)H(s) = \dfrac{e^{-Ts}(s+z_1)}{s^2(s^2+a)(s^2+b)}, z_1, a, b > 0.$

(h) $G(s)H(s) = \dfrac{s}{(s+p_1)(s-p_2)}, p_i > 0.$

(i) $G(s)H(s) = \dfrac{s+z_1}{s^2(s+p_1)}, z_1, p_1 > 0.$

(j) $G(s)H(s) = \dfrac{s+z_1}{s^2(s+p_1)}, z_1, p_1 > 0.$

(k) $G(s)H(s) = \dfrac{s+z_1}{s^2(s+p_1)(s+p_2)}, z_1, p_i > 0.$

(l) $G(s)H(s) = \dfrac{s+z_1}{s^3(s+p_1)(s+p_2)}, z_1, p_i > 0.$

(m) $G(s)H(s) = \dfrac{(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)(s+p_3)}$, $z_i, p_i > 0$

(n) $G(s)H(s) = \dfrac{e^{-Ts}(s+z_1)}{s^2(s+p_1)}, z_1, p_1 > 0.$

(o) $G(s)H(s) = \dfrac{(s-z_1)}{s^2(s^2+p_1)}, z_1, p_1 > 0.$

6. Sketch the polar plot for the following loop transfer function (a) $G(s) = \dfrac{20}{s(s+1)}$ and (b) $G(s) = \dfrac{20}{s(s+1)(s+2)}$.
7. Draw the polar plot for the unity feedback system given by $G(s) = \dfrac{K}{s(s+2)(s+5)}$ and determine the range of for the stability of the system.

8. Sketch the polar plot for the unity feedback system given by $G(s)=\dfrac{1}{(s+1)^2}$ and find the phase margin.

9. Draw the polar plot for the transfer function of a system $G(s)=\dfrac{10(s+2)}{s(s+1)(s+3)}$.

10. Consider the transfer function $G(s)=\dfrac{(1+4s)(1-6s)}{(1-4s)(1+6s)}$. Draw the polar plot for $0 \le \omega \le \infty$.

11. Consider the non-minimum phase transfer function $G(s)=\dfrac{10(s-2)}{s^2(4-s)(s+5)}$. Sketch the polar plot.

12. Draw the polar plot for the transfer function $G(s)=\dfrac{10(s+1)}{(s+10)}$.

13. Consider the following transfer function $G(s)=\dfrac{K}{1+sT}$. Show that the polar plot of $G(j\omega)$in the $G(s)$plane is a circle $\left(\dfrac{K}{2},0\right)$and diameter Kas ω varies from $-\infty$ to ∞.

14. Determine graphically the gain margin and phase margin and gain and phase crossover frequencies for the system whose open-loop transfer function is given as $G(s)H(s)=\dfrac{8}{s(s+1)(s+4)}$.

15. The open-loop transfer function of a unity feedback control system is given by $G(s)=\dfrac{1}{s(s+1)(2s+1)}$. Sketch the polar plot and determine the gain margin and phase margin.

16. Determine the phase margin ad gain margin for the system whose transfer function is $G(s)=\dfrac{2}{s(1+s)(4+s)}$. Use polar plot.

17. The open-loop transfer function of a unity feedback control system is given by $G(s)=\dfrac{K}{s(1+sT_1)(1+sT_2)}$. Derive an expression for the gain K in terms of T_1 and $T_2$and specify the gain margin G_m.

18. Draw the polar plot for the following transfer functions (a)$G(s)=e^{-s}$ and b) $G(s)=\dfrac{e^{-s}}{s+2}$.

19. For the following transfer function draw the actual polar plot by calculating the magnitude and phase for $G(s)=\dfrac{10(s+2)}{(s+1)(s-4)}$

20. Determine analytically phase margin, gain margin, phase crossover frequency and gain-crossover frequency for the transfer function given below. Draw the polar plot.

$$G(s) = \frac{12}{(s+4)(s^2+2s+2)}$$

21. Sketch the polar plot for the following transfer function:

$$G(s) = \frac{300(s^2+2s+4)}{s(s+10)(s+20)}$$

22. For an RLC series network, the output is taken across the capacitor. The transfer function is $G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ where the undamped natural frequency $\omega_n = \frac{1}{\sqrt{LC}}$ and damping coefficient $\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$. Draw the polar plot.
23. Obtain the polar plot for the unity feedback control system having open-loop transfer function as $G(s)H(s) = \frac{K}{s(s+2)(s+50)}$, assume $K = 1{,}300$.
24. State and explain the Nyquist criterion for stability.
25. State the Nyquist stability criterion for the stability of a closed-loop system and compare it with Routh's criterion.
26. Given the open-loop transfer function $G(s)H(s) = \frac{K}{s(T_1 s+1)(T_2 s+1)}$, determine the condition of stability from the polar plot. Verify this condition using Routh–Hurwitz criterion.
27. Write a note on Nyquist path.
28. Define gain margin and phase margin with reference to Nyquist plot.
29. Explain clearly with the necessary mathematical preliminaries the Nyquist criterion for stability.
30. Explain the conditionally stable system. With the help of a Nyquist plot, show the behaviour of such systems.
31. What are the effects of additional poles and zeros on the shape of the Nyquist loci? Explain using appropriate diagrams.
32. Define principle of arguments.
33. Using Nyquist criterion, investigate the stability of the system $G(s)H(s) = \frac{K}{(1+sT_1)(1+sT_2)}$
34. If a position control system has the forward loop transfer function $G(s) = \frac{s}{s(1+s)(1+0.1s)}$, determine the gain margin and phase margin the system using Nyquist plot.

35. Given $G(s)=\dfrac{K(s+2)}{(s^2+3s+6.25)(s+1)}$, determine the stability range of K using Nyquist's stability criterion.
36. The loop transfer function of a unity feedback system is given by $G(s)H(s)=\dfrac{1}{s^2(1+s)(1+2s)}$. Sketch the polar plot for the system and determine the gain and phase margin of the system.
37. The loop transfer function of a unity feedback system is given by $G(s)H(s)=\dfrac{10}{s(1+0.1s)(1+s)}$. Sketch the polar plot for the system and determine the gain and phase margin of the system.
38. The loop transfer function of a unity feedback system is given by $G(s)H(s)=\dfrac{1}{s(s+1)(2s+1)}$. Sketch the polar plot for the system.
39. The open-loop transfer function of a feedback system is $GH(s)=\dfrac{50}{s(1+0.4s)(1+0.8s)}$, determine its stability using Nyquist criterion.
40. Sketch the Nyquist plot for $G(s)H(s)=\dfrac{200}{s(s+5)(s+10)}$ and determine whether the closed-loop system is stable or not.
41. The loop transfer function is $G(s)H(s)=\dfrac{K}{s(1+s)(1+2s)(1+3s)}$. Determine, using Nyquist stability criterion, the critical value of K for stability of closed-loop system.
42. A position control system has the forward loop transfer function $G(s)=\dfrac{2}{s(s+1)(s+5)}$. Draw the Nyquist plot and determine the stability of the closed-loop system.
43. Sketch the Nyquist plot for the system whose open-loop transfer function is given by $G(s)H(s)=\dfrac{10}{s^2(s+2)(s+10)}$ and using Nyquist stability criterion, check whether system is stable.
44. The open-loop transfer function of a unity feedback control system is given by $G(s)=\dfrac{10}{s(1+0.1s)(1+0.8s)}$. Draw the Nyquist plot and calculate the gain margin.
45. The loop transfer function of a feedback system is given by $G(s)H(s)=\dfrac{10}{s(s+2)(s+5)}$. Sketch the Nyquist plot and determine the stability and hence the gain and phase margins.

46. Sketch the Nyquist plot for the following open-loop transfer function and comment on the stability $GH(s) = \frac{1}{s(s+1)(s+2)}$. Determine the gain and phase margins.
47. A control system has the loop transfer function $\frac{K(s+1)}{s^2(s+2)(s+3)}$. Sketch the Nyquist diagram and determine its stability.
48. The loop transfer function of a feedback system is $\frac{K(s+1)}{s^2(s+5)(s+10)}$. Using Nyquist method, determine the stability of the system.

10

CONSTANT *M*- AND *N*-CIRCLES AND NICHOLS CHART

10.1 Introduction

The frequency-response techniques discussed in the previous chapters are the open-loop frequency plots that have obtained the information about the closed-loop stability using the open-loop frequency response of the system. In these methods, the frequency response of the loop transfer function $G(j\omega)H(j\omega)$ is plotted and information regarding the closed-loop stability is derived from the plot. In this chapter, constant M-circles (magnitude), constant N-circles (phase) and Nichols chart are discussed, which directly give the information regarding the closed-loop stability of the system.

10.2 Closed-Loop Response from Open-Loop Response

The closed-loop transfer function of a system with open-loop transfer function $G(s)$ and feedback transfer function $H(s)$ is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Substituting $s = j\omega$ in the above equation, we obtain

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

Letting $\frac{C(j\omega)}{R(j\omega)} = M(j\omega)$, we obtain

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)} = M\angle\phi \tag{10.1}$$

where M is the magnitude of the closed-loop transfer function and ϕ is the phase angle of the closed-loop transfer function.

Since the magnitude and phase angle of the closed-loop system are the function of frequency ω, the closed-loop frequency plot of a system is the plot of magnitude and phase angle of the closed-loop system with respect to frequency ω. Analytically or graphically, the magnitude and phase angle of the closed-loop system for different values of ω can be calculated. In analytical method, tedious calculations are involved in determining the magnitude and phase angle of the closed-loop system for different values of ω. The graphical methods that eliminate the disadvantages of the analytical method are constant *M*- and *N*-circles and Nichols chart.

10.3 Constant *M*-Circles

Let the open-loop transfer function of the system be $G(s)$. In frequency-response analysis, s is replaced with $j\omega$. Therefore,

$$G(j\omega) = u + jv \tag{10.2}$$

where u is the real part of $G(j\omega)$ and v is the imaginary part of $G(j\omega)$.

The closed-loop transfer function of the system with unity feedback is obtained as

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Substituting Eqn. (10.2) and using Eqn. (10.1) in the above equation, we obtain

$$\frac{C(j\omega)}{R(j\omega)} = M(j\omega) = \frac{u+jv}{1+(u+jv)}$$

The magnitude of the above equation is

$$|M(j\omega)| = M = \frac{\sqrt{u^2+v^2}}{\sqrt{(1+u)^2+v^2}}$$

Squaring both sides of the above equation and cross multiplying, we obtain

$$M^2\left((1+u)^2+v^2\right) = u^2+v^2 \tag{10.3}$$

Simplifying the above equation, we get

$$M^2\left(1+u^2+2u+v^2\right) = u^2+v^2$$

i.e.,
$$\left(1-M^2\right)u^2+\left(1-M^2\right)v^2-2uM^2 = M^2$$

Dividing the above equation by $\left(1-M^2\right)$, we obtain

$$u^2+v^2-2u\frac{M^2}{\left(1-M^2\right)}=\frac{M^2}{\left(1-M^2\right)}$$

Adding the term $\left(\frac{M^2}{\left(1-M^2\right)}\right)^2$ on both sides of the above equation, we obtain

$$u^2+v^2-2u\frac{M^2}{\left(1-M^2\right)}+\left(\frac{M^2}{\left(1-M^2\right)}\right)^2=\frac{M^2}{\left(1-M^2\right)}+\left(\frac{M^2}{\left(1-M^2\right)}\right)^2$$

Upon simplifying, we obtain

$$\left(u-\frac{M^2}{\left(1-M^2\right)}\right)^2+v^2=\frac{M^2\left(1-M^2\right)+M^4}{\left(1-M^2\right)^2}$$

$$=\frac{M^2}{\left(1-M^2\right)^2}$$

i.e.,

$$\left(u-\frac{M^2}{\left(1-M^2\right)}\right)^2+v^2=\left(\frac{M}{\left(1-M^2\right)}\right)^2,\ M\neq 1 \tag{10.4}$$

Equation (10.4) resembles the equation of a circle with centre $\left(\left(\frac{M^2}{1-M^2}\right),0\right)$ and radius $\frac{M}{1-M^2}$.

The constant M-loci or constant M-circles are the family of circles described by Eqn. (10.4) in $G(j\omega)$-plane when M takes different values. The constant M-loci or constant M-circles for different values of M are shown in Fig. 10.1.

The following three cases can be inferred from Fig. 10.1.

Case 1: $M>1$

When the magnitude M increases, the radius of M-circles decreases and the centre of the M-circles move towards the point $\left(-1+j0\right)$. Therefore, when $M=\infty$, the radius of M-circle is zero and centre is at $\left(-1+j0\right)$. This implies that the constant M-circle at $M=\infty$ represents the point $\left(-1+j0\right)$. These circles lie to the left of the $M=1$ circle.

Case 2: $M=1$

When the magnitude M is equal to 1, the radius of M-circle is infinity and the centre of the M-circle is $-\infty$. This implies that the constant M-circles at $M=1$ represents a straight line parallel to imaginary axis and the intersection of the parallel line with the real axis can be determined by substituting $M=1$ in Eqn. (10.3). Thus, the intersection point of $M=1$ circle in the real axis is at $u=\frac{-1}{2}$.

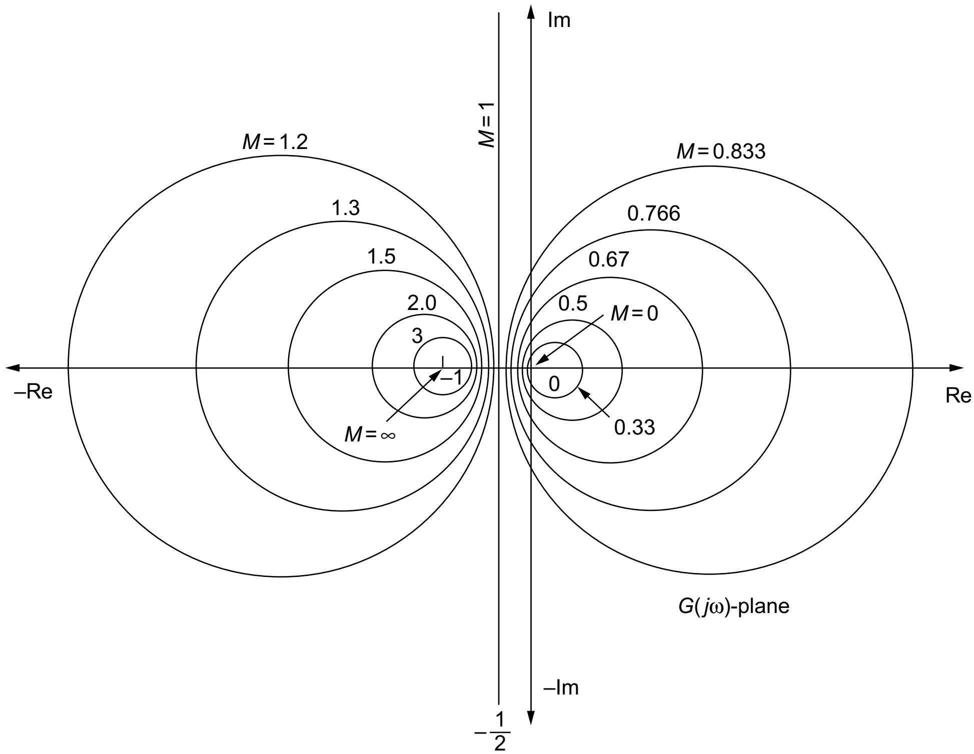


Fig. 10.1 | Constant *M*-circles or constant *M*-loci

Case 3: $M < 1$

When the magnitude M decreases, the radius of M-circles decreases and the centre of the M-circles moves towards the origin. Therefore, when $M = 0$, the radius of M-circles is zero and centre is at origin. This implies that the constant M-circles at $M = \infty$ represents the origin.

The above three cases can be explained more clearly with the help of Table 10.1 which gives the centre and radius of M-circles for different values of M.

Table 10.1 | Centre and radius of *M*-circles

Value of *M*	Centre of *M*-circle	Radius of *M*-circle
0.1	(0.0101, 0)	0.101
0.2	(0.041, 0)	0.208
0.5	(0.33, 0)	0.667
1	$-\infty$	∞
2	(–1.33, 0)	0.667
5	(–1.041, 0)	0.208

10.3.1 Applications of Constant *M*-Circles

(i) The constant *M*-circles can be used to determine resonant peak M_r.
(ii) The resonant frequency ω_r can be determined by using the constant *M*-circles.
(iii) The gain *K* of open-loop transfer function with unity feedback system for a desired M_r can be determined.
(iv) The magnitude plot of the system can be determined easily using constant *M*-circles.

10.3.2 Resonant Peak M_r and Resonant Frequency ω_r from Constant *M*-Circles

The polar plot of a loop transfer function $G(j\omega)H(j\omega)$ and the constant *M*-circles for the same system are shown in Fig. 10.2.

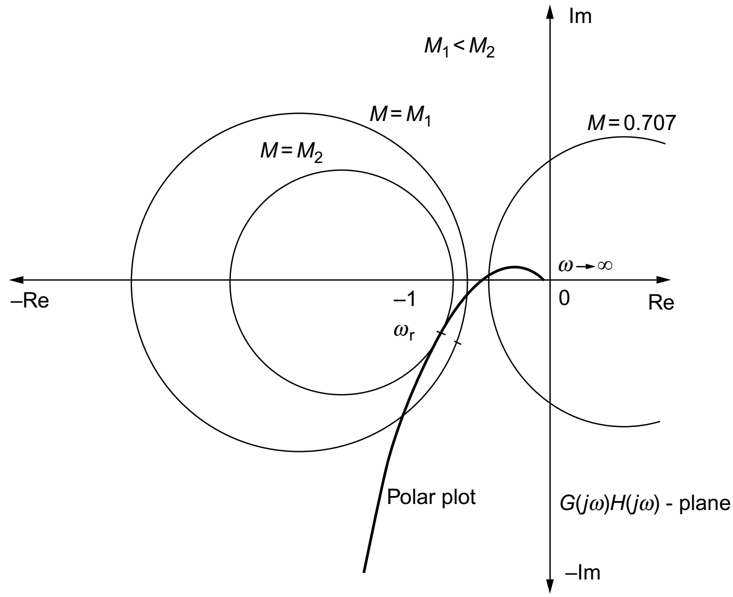


Fig. 10.2 | Polar plot and constant *M*-circles

The resonant peak M_r and resonant frequency ω_r of the system are determined by using the following steps:

Step 1: The polar plot and constant *M*-circles for different values of *M* are drawn as shown in Fig. 10.2.

Step 2: The polar plot will intersect the constant *M* circles at more than one point.

Step 3: The frequency corresponding to the different intersection points is determined.

Step 4: The circles at which the polar plot is a tangent to the circle are determined and the corresponding *M* values are found.

Step 5: The value of *M* with smaller value is the resonant peak M_r of the system.

Step 6: The frequency corresponding to the particular intersection point is the resonant frequency of the system M_r.

For the polar plot and the constant *M*-circles shown in Fig. 10.2, the resonant peak M_r is at M_2 and the resonant frequency is ω_r.

The flow chart for determining M_r and ω_r from constant *M*-circles is shown in Fig. 10.3.

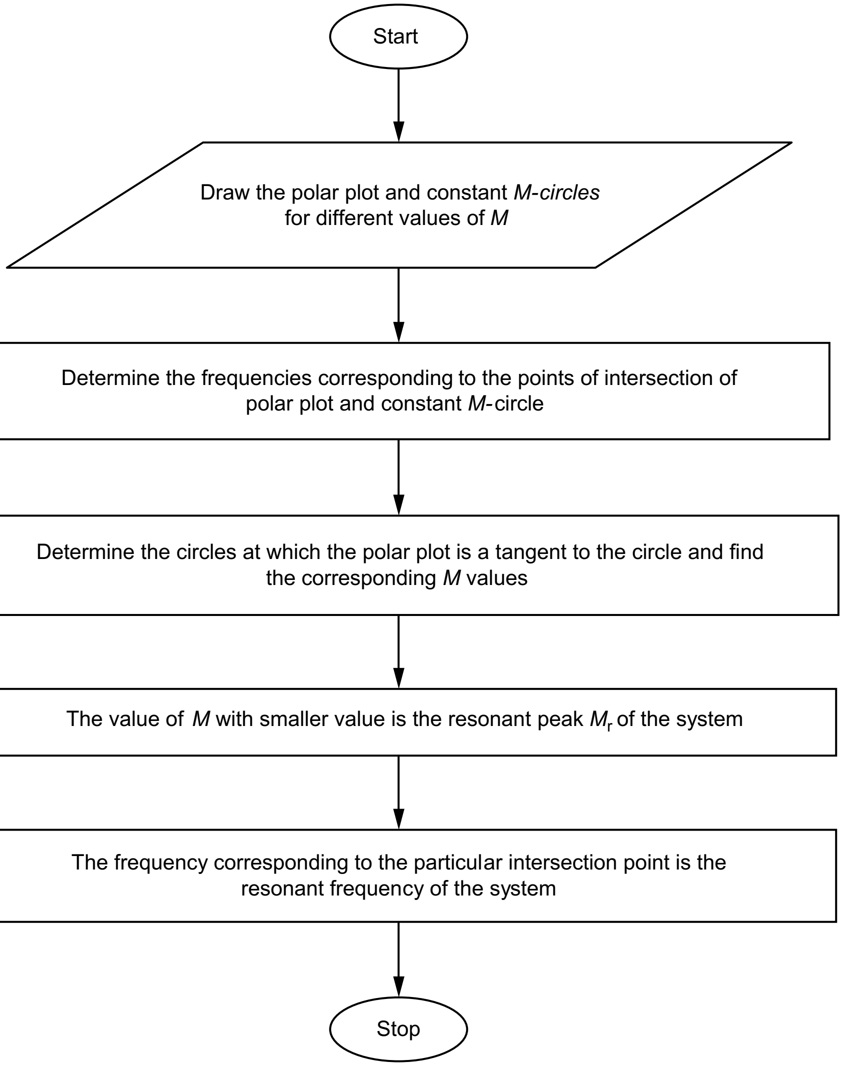


Fig. 10.3 | Flow chart for determining M_r and ω_r

10.3.3 Variation of Gain *K* with M_r and ω_r

The variation of gain *K* with the polar plot is shown in Fig. 10.4. It is evident that as the gain *K* increases, the values of resonant peak and resonant frequency will also increase.

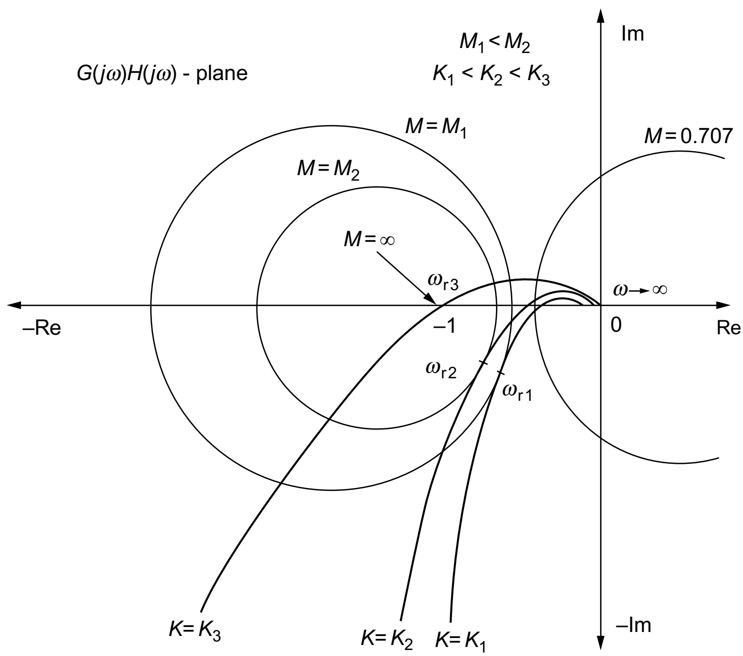


Fig. 10.4 | Polar plot and constant M-circles for different values of gain K

10.3.4 Bandwidth of the System

When the variation of magnitude of the system with respect to frequency ω alone is plotted as shown in Fig. 10.5, the bandwidth of the system can be calculated and this is depicted in Fig. 10.5.

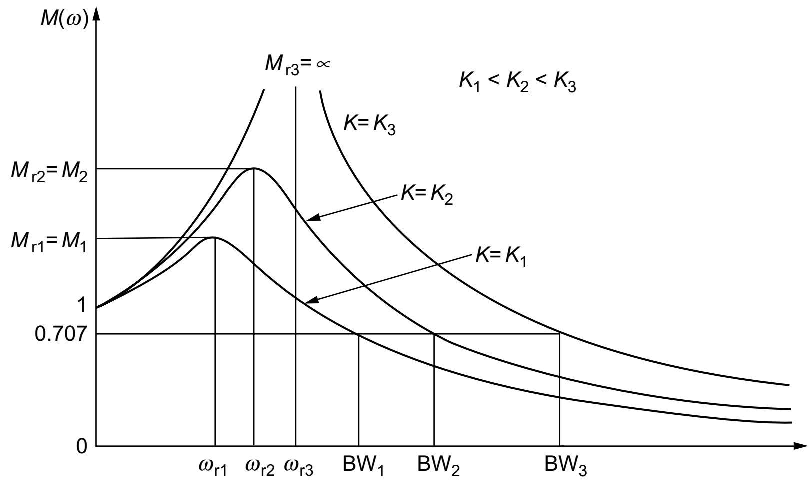


Fig. 10.5 | Magnitude versus ω curve

Also, the resonant peak and resonant frequency are also shown in Fig. 10.5.

10.3.5 Stability of the System

The variation of resonant peak and resonant frequency with respect to variation in gain K is shown in Fig. 10.4. It is clear that, at gain $K = K_3$, the resonant peak is ∞. At this point, the system becomes marginally stable. Also, the system will be unstable when its gain is greater than K_3.

10.3.6 Determination of Gain K Corresponding to the Desired Resonant Peak $(M_r)_{\text{desired}}$

The step-by-step procedure to determine the value of gain K corresponding to the desired resonant peak is given below:

Step 1: The polar plot of the system is drawn by assuming gain $K = K_1$.

Step 2: The angle ϕ is determined by using the formula, $\phi = \sin^{-1}\left(\dfrac{1}{(M_r)_{\text{desired}}}\right)$.

Step 3: A radial line OA is drawn from the origin as shown in Fig. 10.6(a).

Step 4: The constant M-circle with $M = (M_r)_{\text{desired}}$ with centre on the negative real axis is drawn, which is tangent to both the radial line OA and polar plot as shown in Fig. 10.6(a).

Step 5: Let the constant M-circle touch the radial line OA at B.

Step 6: A perpendicular line is drawn from the point B so that the line intersects the negative real axis at C.

Step 7: Thus, the desired gain K is obtained using the formula, $K = \dfrac{K_1}{\text{OC}}$.

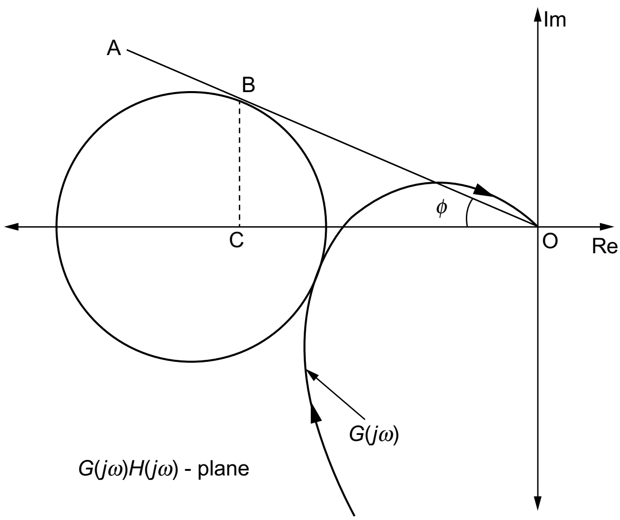


Fig. 10.6(a) | Gain K corresponding to $(M_r)_{\text{desired}}$

The flow chart for determining gain K for desired resonant peak is given in Fig. 10.6(b).

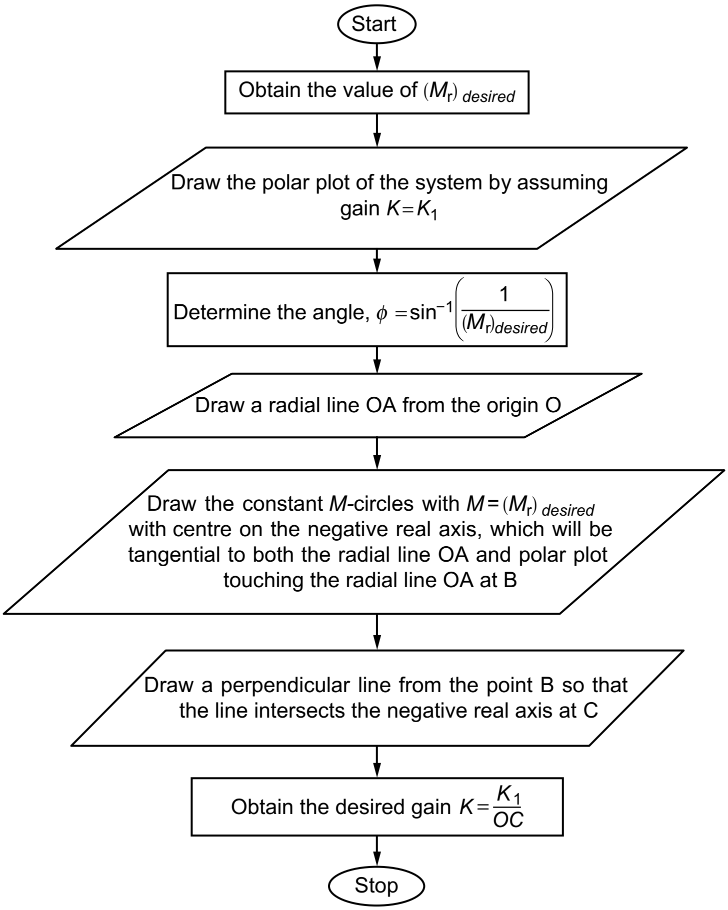


Fig. 10.6(b) | Flow chart for determining gain K for $(M_r)_{\text{desired}}$

10.3.7 Magnitude Plot of the System from Constant *M*-Circles

The step-by-step procedure for determining the magnitude plot of the system using constant M-circles is given below:

Step 1: The polar plot of a system for the given gain is plotted in $G(j\omega)H(j\omega)$-plane.

Step 2: The constant M-circles for different values of M are plotted in the same $G(j\omega)H(j\omega)$-plane.

Step 3: The intersection points of the polar plot and constant M-circles are noted down.

Step 4: The frequency corresponding to an intersection point is noted down from polar plot and the magnitude corresponding to the intersection point is the value of M.

Step 5: Once the value of M and its corresponding frequency are noted down from the intersection points, the magnitude plot of the system is obtained by taking the value of M along Y-axis and corresponding frequency along X-axis.

The constant M-circles with polar plot of a system and its corresponding magnitude plot are shown in Figs. 10.7 and 10.8(a) respectively.

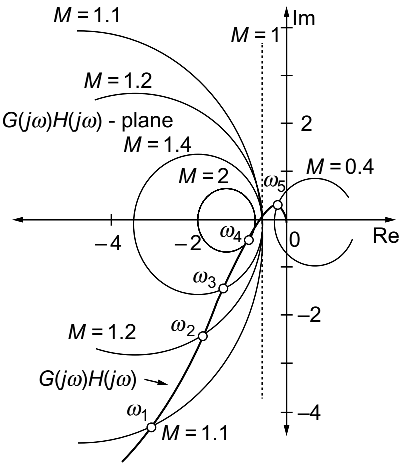


Fig. 10.7 | Polar plot and constant *M*-circles

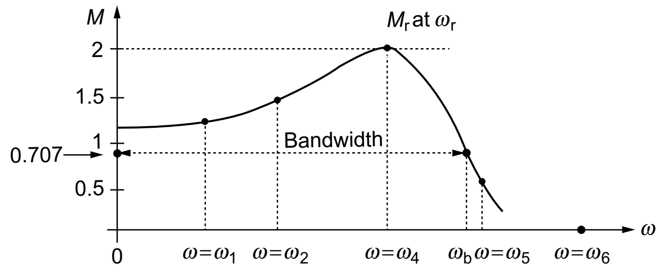


Fig. 10.8(a) | Magnitude plot of the system

The flow chart for the determination of magnitude plot from constant M-circles is shown in Fig. 10.8(b).

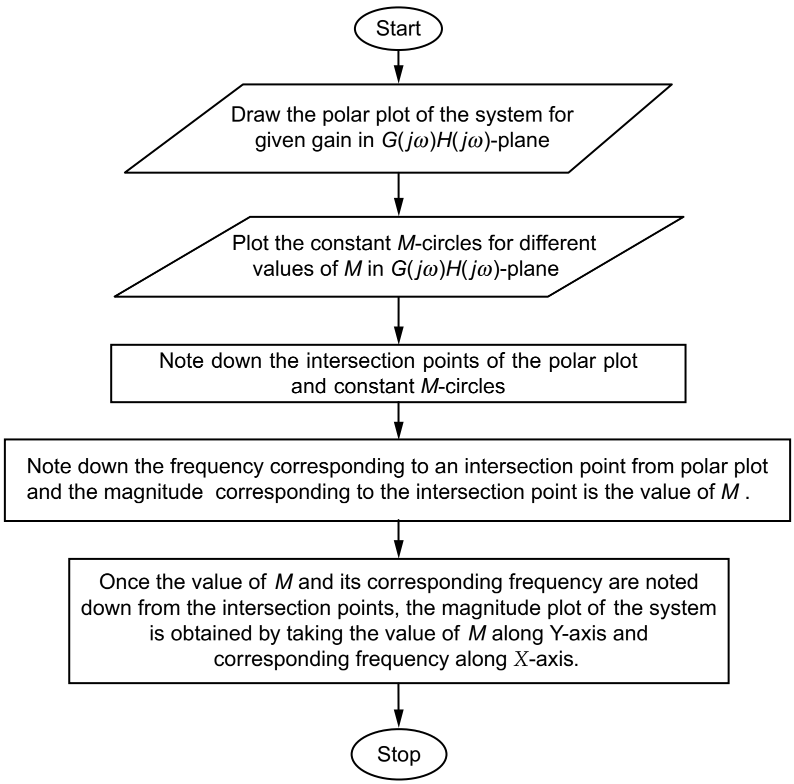


Fig. 10.8(b) | Flow chart for obtaining the magnitude plot from constant *M*-circles

10.4 Constant *N*-Circles

Let the open-loop transfer function of the system be $G(s)$. In the frequency response analysis of the system s is replaced with $j\omega$. Therefore,

$$G(j\omega) = u + jv \tag{10.5}$$

where u is the real part of $G(j\omega)$and v is the imaginary part of $G(j\omega)$.

The closed-loop transfer function of the system with unity feedback is obtained as

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Substituting Eqn. (10.5) and using Eqn. (10.1) in the above equation, we obtain

$$\frac{C(j\omega)}{R(j\omega)} = M(j\omega) = \frac{u+jv}{1+(u+jv)}$$

The phase angle of the system is

$$\phi = \tan^{-1}\left(\frac{v}{u}\right) - \tan^{-1}\left(\frac{v}{1+u}\right)$$

Taking tan on both sides of the above equation, we obtain

$$\tan(\phi) = \tan\left(\tan^{-1}\left(\frac{v}{u}\right) - \tan^{-1}\left(\frac{v}{1+u}\right)\right)$$

Substituting $N = \tan(\phi)$ in the above equation, we obtain

$$N = \tan\left(\tan^{-1}\left(\frac{v}{u}\right) - \tan^{-1}\left(\frac{v}{1+u}\right)\right)$$

Using the formula $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \ \tan B}$in the above equation, we obtain

$$N = \frac{\frac{v}{u} - \frac{v}{1+u}}{1+\left(\frac{v}{u}\right)\left(\frac{v}{1+u}\right)}$$

Simplifying the above equation, we obtain

$$N = \frac{v(1+u) - vu}{u(1+u)+v^2}$$

$$= \frac{v}{u^2+v^2+u}$$

Therefore, $$u^2 + v^2 + u - \frac{v}{N} = 0$$

Adding $\left[\frac{1}{4} + \frac{1}{(2N)^2}\right]$ on both sides of the above equation, we obtain

$$u^2 + v^2 + u - \frac{v}{N} + \frac{1}{4} + \frac{1}{(2N)^2} = \frac{1}{4} + \frac{1}{(2N)^2}$$

Rearranging the above equation, we obtain

$$\left(u + \frac{1}{2}\right)^2 + \left(v - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \tag{10.6}$$

The above equation resembles the equation of a circle with centre $\left(\frac{-1}{2}, \frac{1}{2N}\right)$ and radius $\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}$.

It is to be noted that irrespective of the value of *N*, Eqn. (10.6) gets satisfied for $(u = 0, v = 0)$ and $(u = -1, v = 0)$. Therefore, each constant *N*-circle will pass through the origin and $(-1 + j0)$ in the $G(j\omega)\ H(j\omega)$-plane.

The constant *N*-loci or constant *N*-circles are the family of circles described by Eqn. (10.6) in $G(j\omega)\ H(j\omega)$-plane when *N* takes different values. The constant *N*-loci or constant *N*-circles for different values of *N* are shown in Fig. 10.9.

The centre and radius of constant *N*-circles for different values of *N* are shown in Table 10.2.

Table 10.2 | Centre and radius of *N*-circles

Value of α	Value of *N*	Centre of *N*-circle	Radius of *N*-circle
20°	0.364	(0.5, 1.374)	1.462
40°	0.839	(0.5, 0.596)	0.778
60°	1.732	(0.5, 0.289)	0.577
–20°	–0.364	(0.5, –1.374)	1.462
–40°	–0.839	(0.5, –0.596)	0.778
–60°	–1.732	(0.5, –0.289)	0.577

The constant *N*-circles shown in Fig. 10.9 for a given value of ϕ are only arc and not entire circles. Thus constant *N*-circles for $\phi = 60°$ and $\phi = 60° - 180° = -120°$ are parts of same circle.

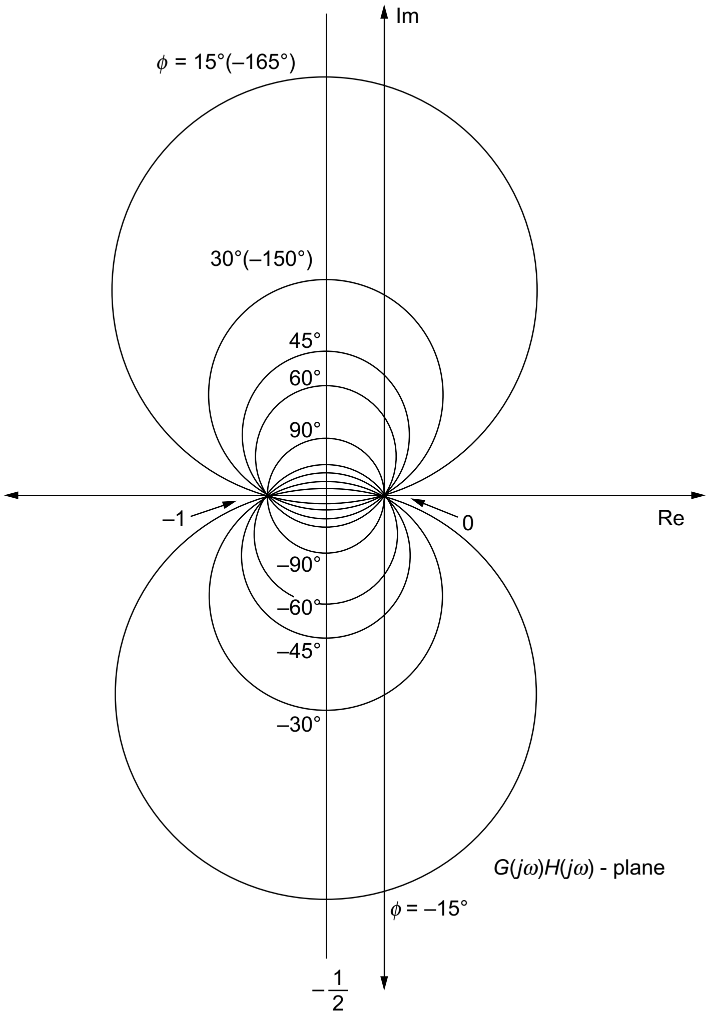


Fig. 10.9 | Constant N-circles or constant N loci

10.4.1 Phase Plot of the System from Constant *N*-Circles

The step-by-step procedure for determining the phase plot of the system using constant N-circles is given below:

Step 1: The polar plot of a system for the given gain is plotted in $G(j\omega)\ H(j\omega)$-plane.

Step 2: The constant N-circles for different values of ϕ are plotted in the same $G(j\omega)$ $H(j\omega)$-plane.

Step 3: The intersection points of the polar plot and constant N-circles are noted down.

Step 4: The frequency corresponding to an intersection point is noted down from polar plot and the magnitude corresponding to the intersection point is the value of ϕ.

Step 5: Once the value of ϕ and its corresponding frequency are noted down from the intersection points, the magnitude plot of the system is obtained by taking the value of ϕ along Y-axis and corresponding frequency along X-axis.

The phase plot for the system using constant N-circles is shown in Fig. 10.10(a).

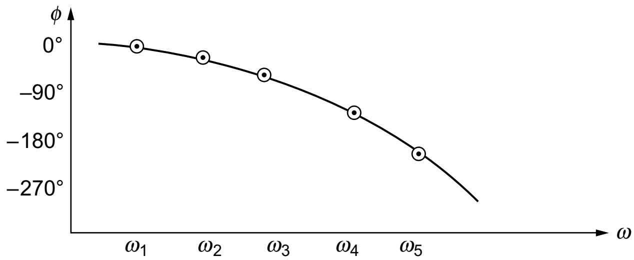


Fig. 10.10(a) | Phase angle plot of the system

The flow chart for determining the phase angle plot from constant N-circles is shown in Fig. 10.10(b).

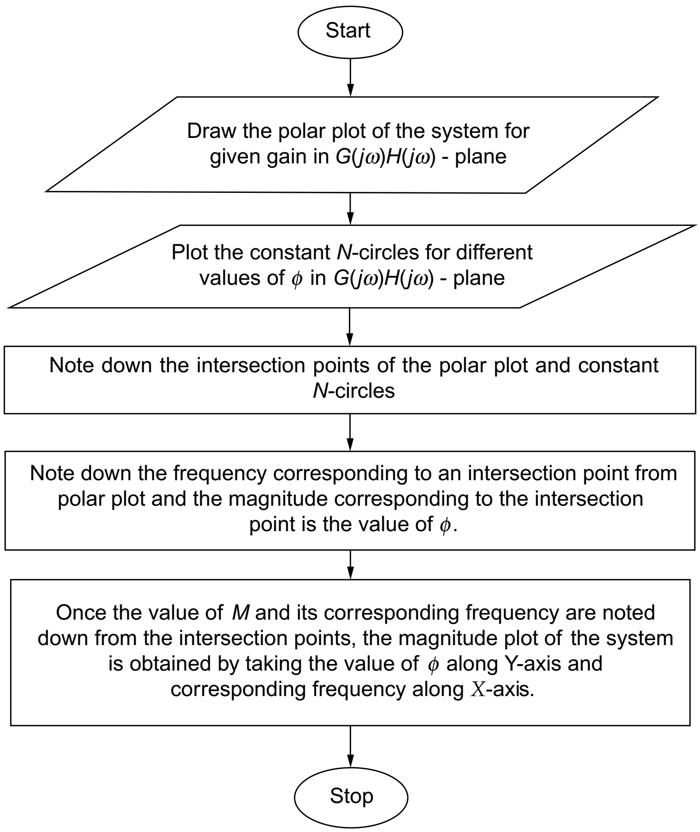


Fig. 10.10(b) | Flow chart for plotting phase angle plot of the system

10.5 Nichols Chart

The chart that results after the transformation of constant *M*- and *N*-circles to log-magnitude and phase angle coordinates is called Nichols chart. This transformation is first introduced by N.B. Nichols. In addition, the Nichols chart can be defined as the chart that consists of constant *M*- and *N*-circles superimposed on a graph sheet. The graph sheet consists of magnitude of the system in decibels which is obtained from constant *M*-circles along the *Y*-axis and phase angle of the system in degrees which is obtained from constant *N*-circles along the *X*-axis. The typical Nichols chart with constant *M*- and *N*-circles is shown in Fig. 10.11.

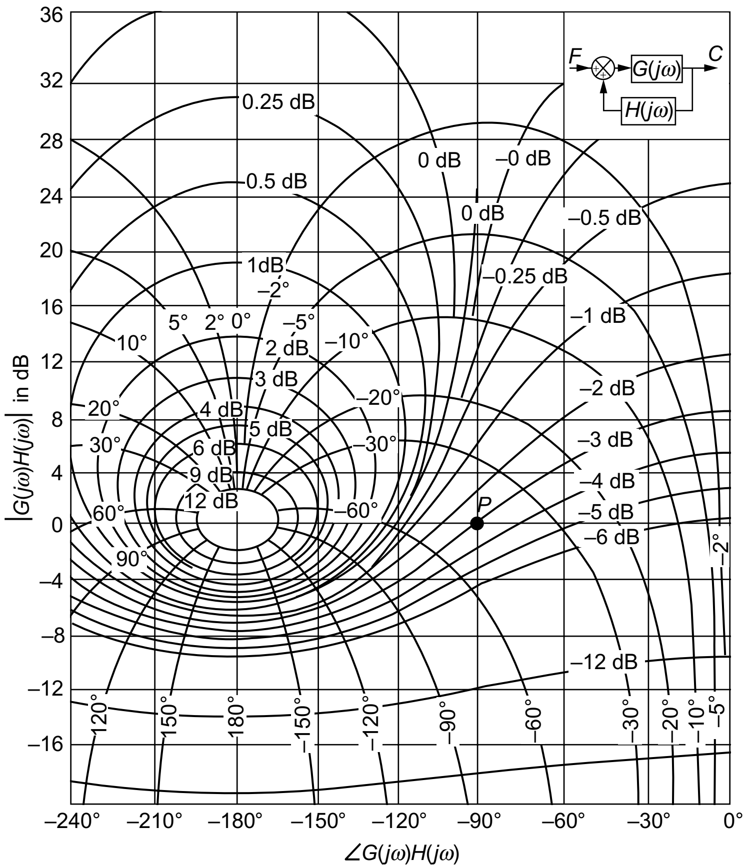


Fig. 10.11 | Nichols chart

10.5.1 Reason for the Usage of Nichols Chart

The reason for using Nichols chart for determining the stability of the system is due to the disadvantages that exist in working with the polar coordinates. The disadvantage is that when the loop gain of the system is changed, the polar plot will tend to change its shape.

But in Bode plot or in magnitude versus phase plot, when the loop gain of the system is altered, the curve gets shifted up or down vertically. Therefore, it is convenient to work with Nichols chart compared to polar plot.

10.5.2 Advantages of Nichols Chart

The advantages of Nichols chart are:

(i) The magnitude and phase plot of the system can be obtained.
(ii) The values of resonant peak, resonant frequency and bandwidth can be obtained from the Nichols chart for the given $G(j\omega)$.
(iii) The various time-domain specifications can be obtained once the value of resonant peak and resonant frequency are determined.
(iv) The gain of the system for the desired resonant peak can be determined.

10.5.3 Transformation of Constant *M*- and *N*-Circles into Nichols Chart

The transformation of constant M-circle into Nichols chart is explained using the step-by-step procedure as follows:

Step 1: The constant M-circles for $M > 1$ are plotted in Fig. 10.12(a).

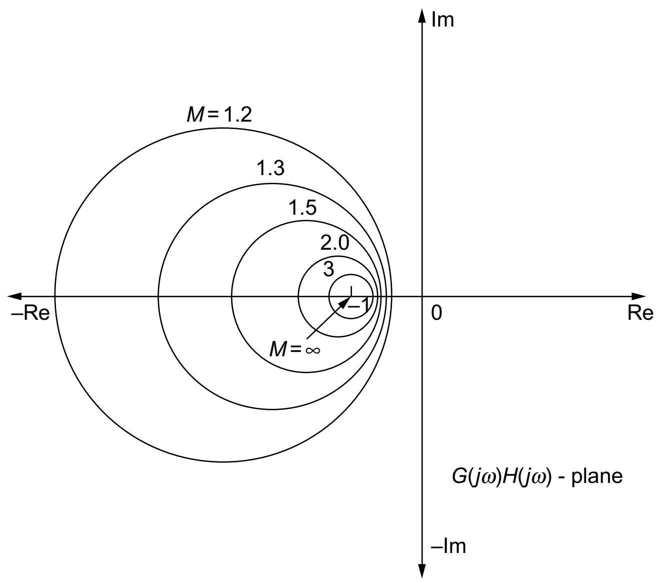


Fig. 10.12(a) | Constant *M*-circles of a system

Step 2: A line is drawn from the origin to any point on the constant M-circles as shown in Fig. 10.12(b).

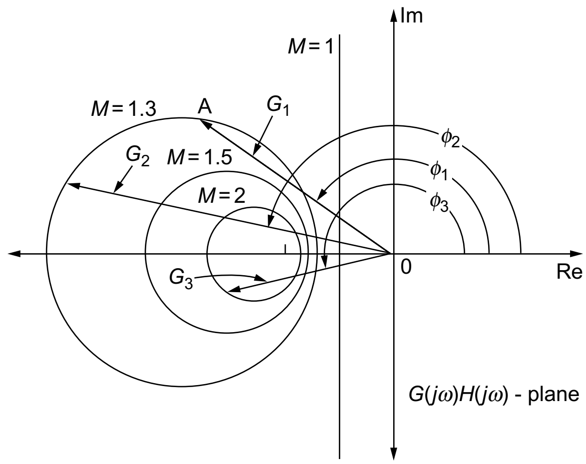


Fig. 10.12(b) | Line from origin to constant *M*-circles

Step 3: Consider a point A in the constant *M*-circles.

Step 4: The magnitude in dB and phase angle in degrees for that particular point are

Magnitude in dB = $20 \log G_1$

Phase angle in degrees = ϕ_1

where G_1 is the length of the line OA and ϕ_1 is the angle from the positive real axis to the line OA taken in the counterclockwise direction.

Step 5: Similarly, any point in the constant *M*-circles can be plotted in the Nichols chart.

Step 6: The constant *M*-circles shown in Fig. 10.12(a) are transformed to Nichols chart as shown in Fig. 10.12(c).

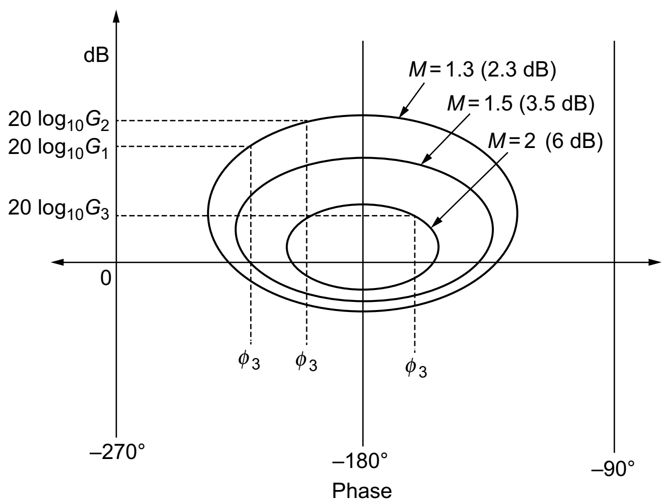


Fig. 10.12(c) | Constant *M* circle in Nichols chart

It is noted that the critical point $(-1+j0)$ is plotted in Nichols chart as the point $(0\text{dB}, 180°)$. Similarly, the constant N-circles can be transformed into Nichols chart.

10.5.4 Determination of Frequency Domain Specifications from Nichols Chart

The different frequency domain specifications that can be determined by using Nichols chart are gain margin g_m, phase margin p_m, gain crossover frequency ω_{gc}, phase crossover frequency ω_{pc} and bandwidth ω_b.

The step-by-step procedure to determine the frequency domain specifications using Nichols chart is given below:

Step 1: Draw the $G(j\omega)H(j\omega)$ plot in the Nichols chart.

Step 2: Transform the constant M-circles into the Nichols chart.

Step 3: Determine the value of M_{p1} at which the locus $G(j\omega)H(j\omega)$ is a tangent to that particular M-circle.

Step 4: Determine the frequency of that particular point ω_{p1} from $G(j\omega)H(j\omega)$ plot.

Step 5: Then, the resonant peak M_r and resonant frequency ω_r are determined as $M_r = M_{p1}$ and $\omega_r = \omega_{p1}$.

Step 6: The gain crossover frequency and phase crossover frequency are the frequencies at which the $G(j\omega)$ plot crosses the 0dB line and $-180°$ line in Nichols chart.

Step 7: Determine the gain margin and phase margin as shown in Fig. 10.13.

Step 8: The bandwidth of the system is determined as the frequency at which $G(j\omega)H(j\omega)$ plot crosses the –3 dB or 0.707 circle.

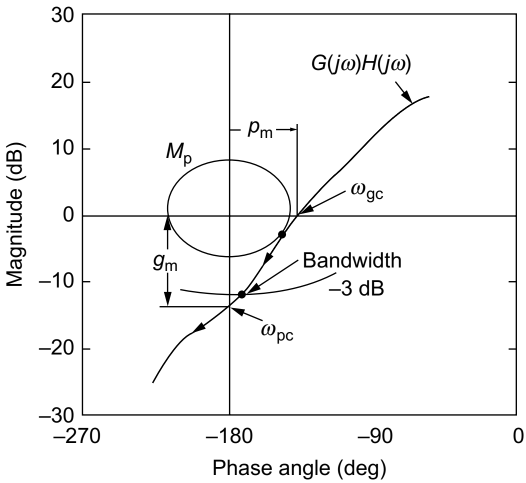


Fig. 10.13 | Determination of frequency domain specifications using Nichols chart

10.5.5 Determination of Gain *K* for a Desired Frequency Domain Specifications

The gain K of the system can be determined for the desired value of frequency domain specifications. The step-by-step procedure for determining the gain K for the desired value of frequency domain specifications is discussed in this section.

Case 1: To determine the gain K for a desired gain margin $(g_m)_{desired}$

The step-by-step procedure to determine the gain K for $(g_m)_{desired}$ is given below.

Step 1: Draw the $G(j\omega)H(j\omega)$ plot for gain $K = 1$.

Step 2: Determine the magnitude A in dB of $G(j\omega)H(j\omega)$ for a phase angle of –180°.

Step 3: Determine the magnitude of $G(j\omega)H(j\omega)$ in dB for $(g_m)_{desired}$ which is given by $B = 0 - (g_m)_{desired}$.

Step 4: Determine $C = B - A$.

Step 5: The $G(j\omega)H(j\omega)$ plot will be shifted vertically upward or vertically downward based on the value of C.

If $C > 0$, the plot is shifted vertically upward by C dB.
If $C < 0$, the plot is shifted vertically downward by C dB.
If $C = 0$, the plot is neither shifted upward nor downward vertically.

Step 6: The gain K for $(g_m)_{desired}$ is determined using the formula

$$20\log K = C$$

The flow chart for determining the gain K for $(g_m)_{desired}$ is shown in Fig. 10.14(a).

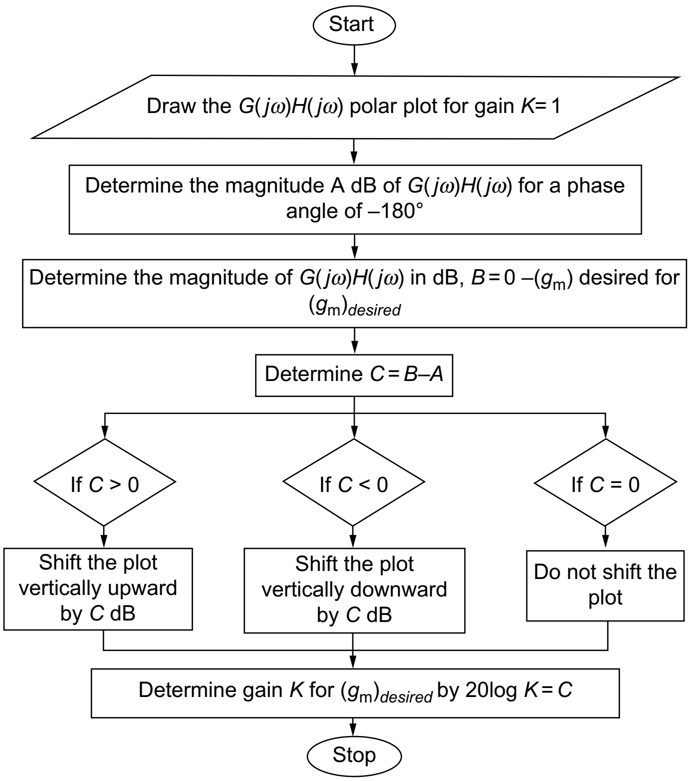


Fig. 10.14(a) | Flow chart for determining K for $(g_m)_{desired}$

Case 2: To determine the gain K for a desired phase margin $(p_{\mathrm{m}})_{\mathrm{desired}}$

The step-by-step procedure to determine the gain K for $(p_{\mathrm{m}})_{\mathrm{desired}}$ is given below.

Step 1: Draw the $G(j\omega)H(j\omega)$ plot for gain $K=1$.

Step 2: Determine the angle of $G(j\omega)H(j\omega)$ for $(p_{\mathrm{m}})_{\mathrm{desired}}$ as $D=(p_{\mathrm{m}})_{\mathrm{desired}}-180°$.

Step 3: Determine the magnitude E of $G(j\omega)H(j\omega)$ in dB for D degrees.

Step 4: The $G(j\omega)H(j\omega)$ plot will be shifted vertically upward or vertically downward based on the value of E.
If $E>0$, the plot is shifted vertically downward by EdB.
If $E<0$, the plot is shifted vertically upward by EdB.
If $E=0$, the plot is neither shifted upward nor downward vertically.

Step 6: The gain K for $(p_{\mathrm{m}})_{\mathrm{desired}}$ is determined by using the formula

$$20\log K=-E$$

The flow chart for determining the gain K for $(p_{\mathrm{m}})_{\mathrm{desired}}$ is shown in Fig. 10.14(b).

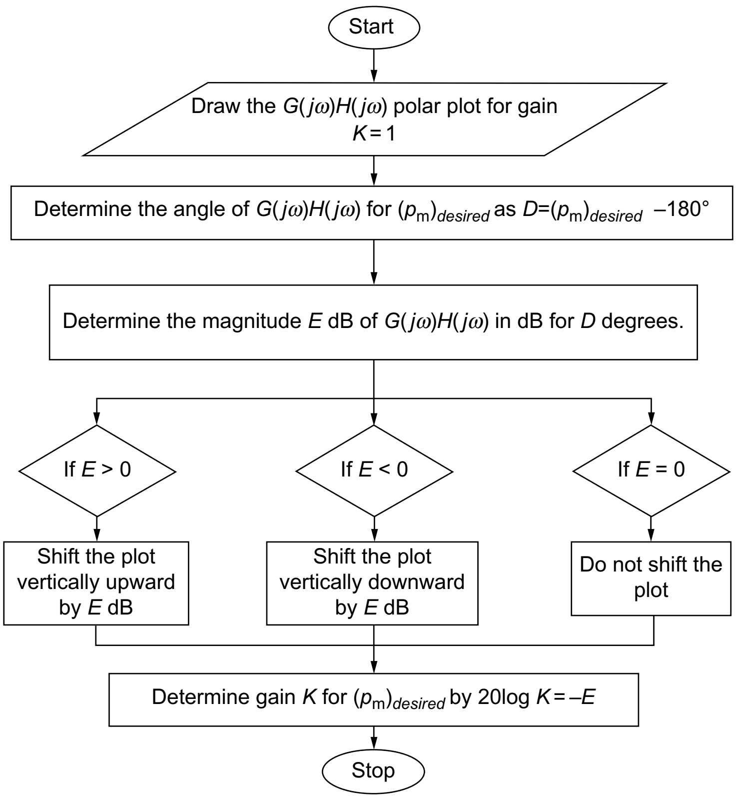


Fig. 10.14(b) | Flow chart for determining K for $(p_m)_{\mathrm{desired}}$

Case 3: To determine the gain K for a desired resonant peak $(M_r)_{desired}$

The step-by-step procedure to determine the gain for $(M_r)_{desired}$ is given below.

Step 1: Draw the $G(j\omega)H(j\omega)$ plot for gain $K = 1$.

Step 2: Determine the resonant peak for the system M_r.

Step 3: Trace the plot of $G(j\omega)H(j\omega)$ with gain $K = 1$ and move the plot either upward or downward based on $(M_r)_{desired}$ so that the traced plot is a tangent to the $(M_r)_{desired}$ contour.

Step 4: Determine the value F in dB by which the plot has moved.

Step 5: The gain K for $(M_r)_{desired}$ is determined by using the formula,

$$20\log K = F$$

The flow chart for determining the gain K for $(M_r)_{desired}$ is shown in Fig. 10.14(c).

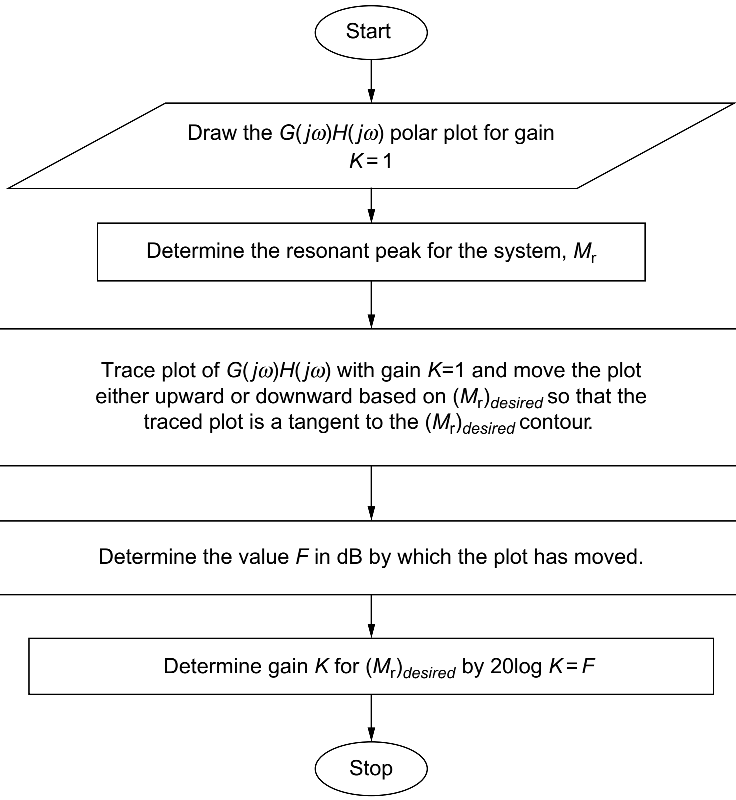


Fig. 10.14(c) | Flow chart for determining K for $(M_r)_{desired}$

Case 4: To determine the gain K for a desired bandwidth $(\omega_b)_{desired}$

The step-by-step procedure to determine the gain K for $(\omega_b)_{desired}$ is given below.

Step 1: Draw the $G(j\omega)H(j\omega)$ plot for gain $K = 1$.

Step 2: Determine the magnitude of $G(j\omega)H(j\omega)$ in dB for $(\omega_b)_{desired}$. Let it be GdB.

Step 3: If the value of G is equal to –3dB, then the plot is neither shifted vertically upward or downward. But, if the value of G is not equal to –3dB, then the following steps are to be followed.

Step 4: Determine the point H in the M = –3dB contour when it crosses GdB.

Step 5: Trace the plot of $G(j\omega)H(j\omega)$ with gain $K = 1$ and move the plot either upward or downward so that it passes through the point H.

Step 6: Determine the value I in dB by which the plot has moved.

Step 7: The gain K for $(\omega_b)_{desired}$ is determined by using the formula

$$20\log K = I$$

The flow chart for determining the gain K for $(\omega_b)_{desired}$ is shown in Fig. 10.14(d).

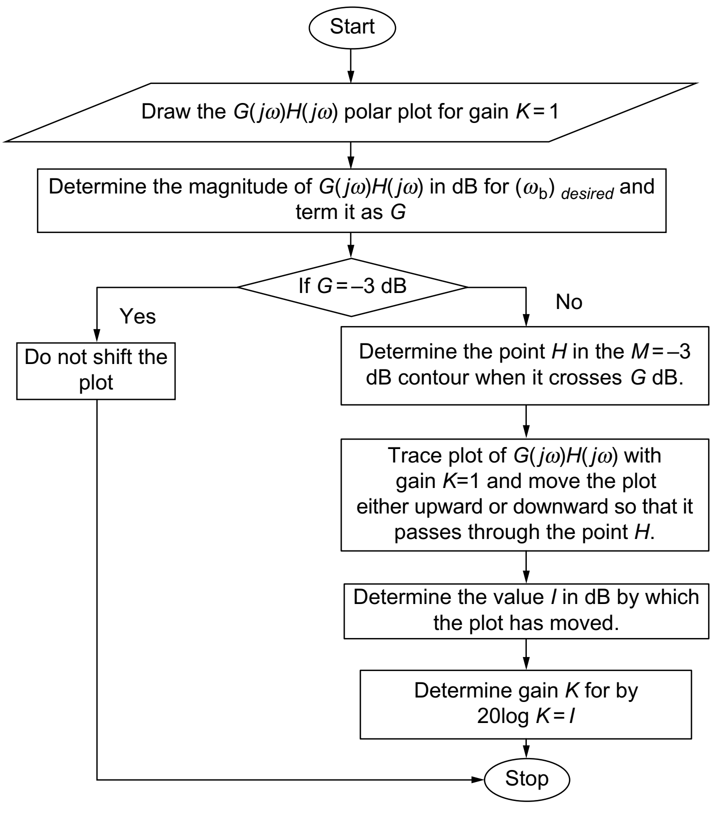


Fig. 10.14(d) | Flow chart for determining K for $(\omega_b)_{desired}$

Example 10.1: **The loop transfer function of a system with a unity feedback is given by $G(s)H(s)=\dfrac{2}{s(s+1)\left(1+\dfrac{s}{3}\right)}$. Determine the gain margin, phase margin, gain crossover frequency, phase crossover frequency and bandwidth of the system using Nichols chart.**

Solution:

The $G(j\omega)H(j\omega)$ plot is plotted in the Nichols chart.

(a) Substituting $s=j\omega$ in the given transfer function, we obtain

$$G(j\omega)H(j\omega)=\frac{2}{j\omega(j\omega+1)\left(1+\dfrac{j\omega}{3}\right)}$$

(b) The magnitude and phase angle of the system are

$$\left|G(j\omega)H(j\omega)\right|=\frac{2}{\omega\left(\sqrt{\omega^2+1}\right)\left(\sqrt{1+\dfrac{\omega^2}{9}}\right)}=\frac{6}{\omega\left(\sqrt{\omega^2+1}\right)\left(\sqrt{9+\omega^2}\right)}$$

and $\angle G(j\omega)H(j\omega)=-90°-\tan^{-1}(\omega)-\tan^{-1}\left(\dfrac{\omega}{3}\right)$

(c) Substituting the different values of frequency ω in the above equations, we obtain the magnitude and phase angle of the system. The magnitude in dB and phase angle of the system in degrees for different frequency values are shown in Table E10.1.

Table E10.1 | Magnitude and phase angle of the system

Frequency ω rad/s	0.1	0.2	0.3	0.5	1	1.2	1.5	1.732	2
Magnitude of the system	19.88	9.78	6.35	3.53	1.34	0.99	0.66	0.5	0.37
Magnitude of the system in dB	25.97	19.81	16.06	10.95	2.55	–0.08	–3.58	–6.02	–8.58
Phase angle of the system in degrees	–98°	–105°	–112°	–126°	–153°	–162°	–173°	–180°	–187°

(d) Using the data obtained in Table E10.1, the plot for $G(j\omega)H(j\omega)$ is plotted in Nichols chart as shown in Fig. E10.1.

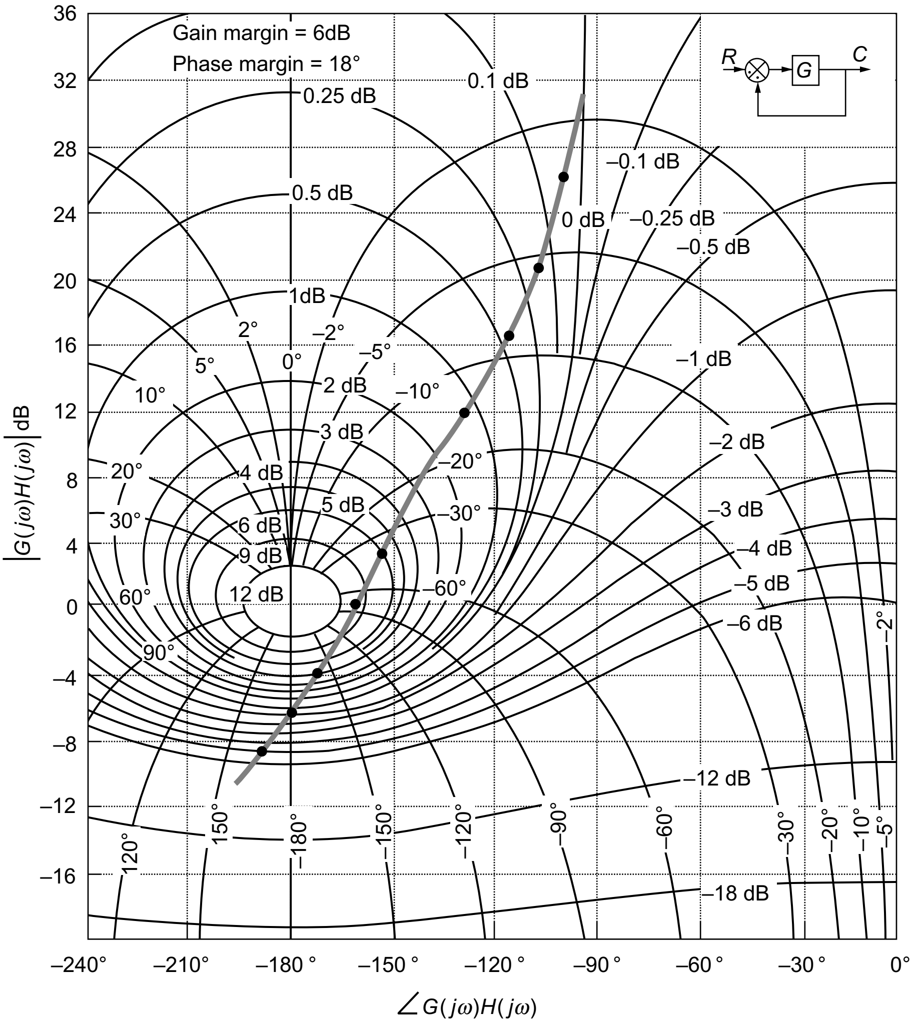


Fig. E10.1 | $G(jw)H(jw)$ plot in Nichols chart

(e) Using the definition, the different frequency domain specifications are:

Gain margin, $g_m = 6$ dB

Phase margin, $p_m = 18°$

Gain crossover frequency, $\omega_{gc} = 1.2$ rad/s

Phase crossover frequency, $\omega_{pc} = 1.732$ rad/s

Example 10.2: The loop transfer function of an unity feedback system is $G(s)H(s) = \frac{Ke^{-0.2s}}{s(1+0.25s)(1+0.1s)}$. Using Nichols chart, determine the gain K so that (a) the desired gain margin of the system is 16 dB, (b) the desired phase margin of the system is 40° and (c) the desired resonant peak of the system is 5dB.

Solution:

1. The $G(j\omega)H(j\omega)$ plot is plotted in the Nichols chart.

(a) Substituting $s = j\omega$ and $K = 1$ in the given loop transfer function, we obtain

$$G(j\omega)H(j\omega) = \frac{e^{-j0.2\omega}}{j\omega(1+j0.25\omega)(1+j0.1\omega)}$$

(b) The magnitude and phase angle of the system are

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega\left(\sqrt{1+0.0625\omega^2}\right)\left(\sqrt{1+0.01\omega^2}\right)}$$

and $\angle G(j\omega)H(j\omega) = -11.464\omega - 90° - \tan^{-1}(0.25\omega) - \tan^{-1}(0.1\omega)$

(c) Substituting different values of ω in the above two equations, we obtain the magnitude and phase angle of the system as given in Table E10.2.

Table E10.2 | Magnitude and phase angle of the system

Frequency ω rad/s	0.01	0.02	0.1	0.5	1	1.2	1.5	2	4
Magnitude of the system	99.99	49.99	9.99	1.98	0.96	0.79	0.61	0.43	0.16
Magnitude of the system in dB	40	34	20	6	0	–2	–4	–7	–16
Phase angle of the system in degrees	–90°	–91°	–93°	–106°	–121°	–127°	–136°	–151°	–203°

(d) Using the data obtained in Table E10.2, the plot for $G(j\omega)H(j\omega)$ is plotted in Nichols chart as shown in Fig. E10.2(a).

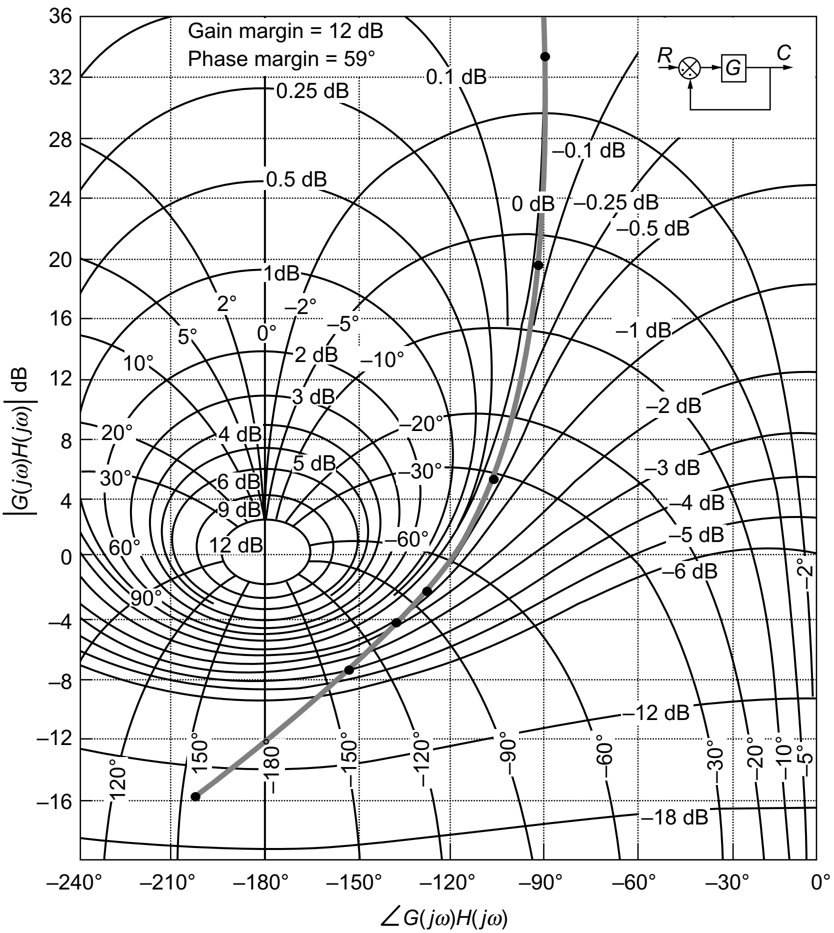


Fig. E10.2(a)

2. Using the plot shown in Fig. E10.2(a), the frequency domain specifications for the system with gain $K = 1$, are

   Gain margin, $g_m = 12$ dB

   Phase margin, $p_m =\ 59°$ and

   Gain crossover frequency, $\omega_{gc} = 1$ rad/s

3. The gain values for the desired frequency domain specifications are determined as follows:

Case 1: When the desired gain margin of the system is 16 dB.

1. The magnitude of $G(j\omega)H(j\omega)$ at –180°, $A = -12$ dB.
2. For the desired gain margin of 20 dB, $B = -16$ dB.

3. Therefore, $C = -16 - (-12) = -4$ dB.
4. As the value of C is less than zero, the $G(j\omega)H(j\omega)$ plot is to be shifted vertically downward by 4 dB. The modified plot in Nichols chart is shown in Fig. E10.2(b).

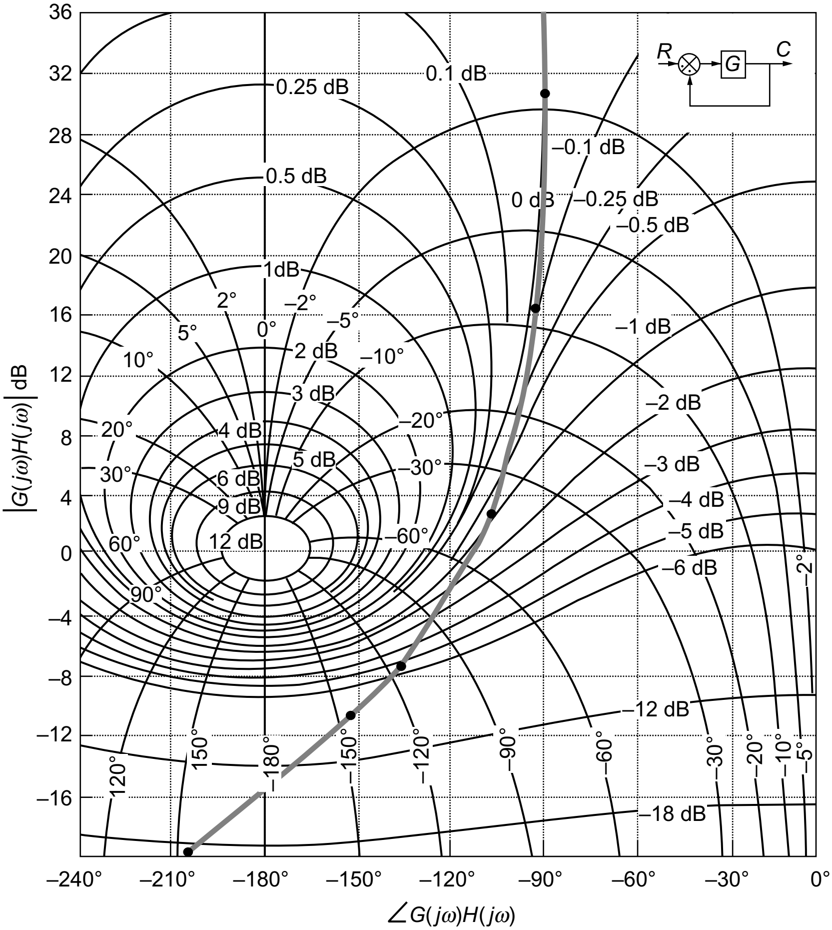


Fig. E10.2(b)

5. Therefore, the gain for $(g_m)_{\text{desired}} = 16$ dB is

$$20 \log K = 4$$

i.e., $$K = 10^{\frac{4}{20}} = 10^{0.2} = 1.584.$$

Case 2: When the phase margin of the system is 40°.

1. The angle of $G(j\omega)H(j\omega)$ for $(p_m)_{\text{desired}}$, $D = 40° - 180° = -140°$.
2. The magnitude of $G(j\omega)H(j\omega)$ in dB for Ddegrees, $E = -5$dB.

3. As the value of E is less than zero, the plot is to be shifted vertically upward by 5dB.
4. The modified plot of $G(j\omega)H(j\omega)$ is shown in Fig. E10.2(c).

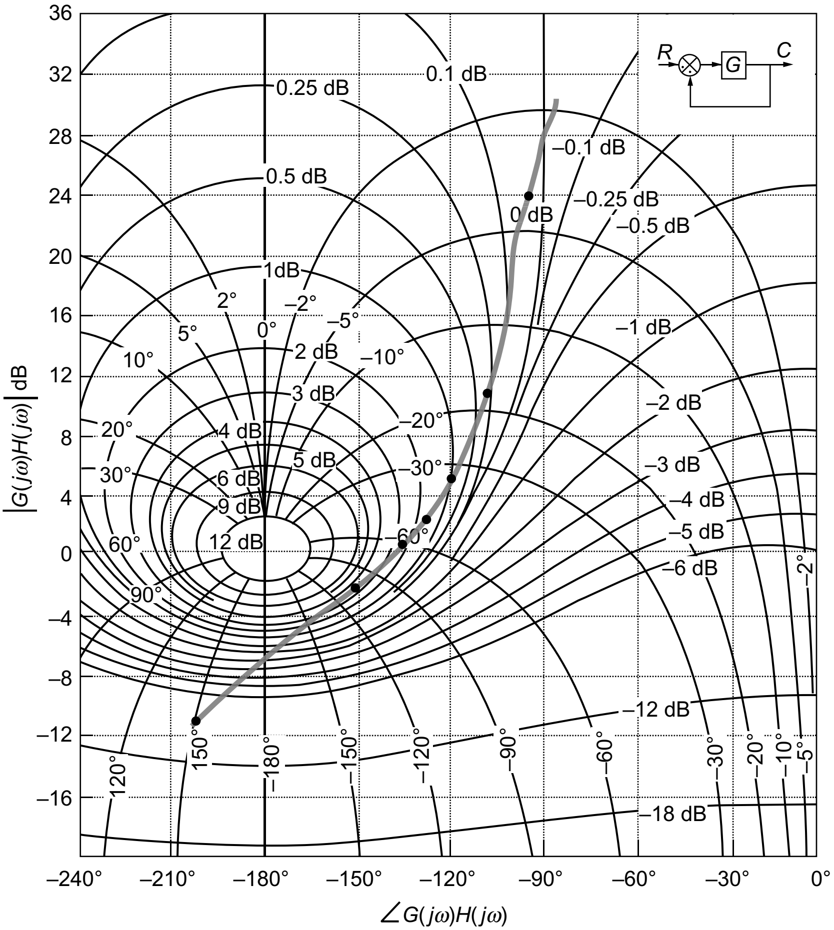


Fig. E10.2(c)

5. Therefore, the gain of the system for desired phase margin is

$$20\log K = 5$$

i.e.,
$$K = 10^{\frac{5}{20}} = 10^{0.25} = 1.778.$$

Case 3: When the desired resonant peak of the system, $(M_r)_{\text{desired}}$ = 6 dB.

1. The resonant peak of the system is 0.25 dB.
2. The plot of $G(j\omega)H(j\omega)$ shown in Fig. E10.2(a) is traced in a tracing paper.
3. The plot in the tracing paper is moved in the Nichols chart so that the resonant peak of the system is 6 dB and the plot is plotted in the Nichols chart as shown in Fig. E10.2(d).

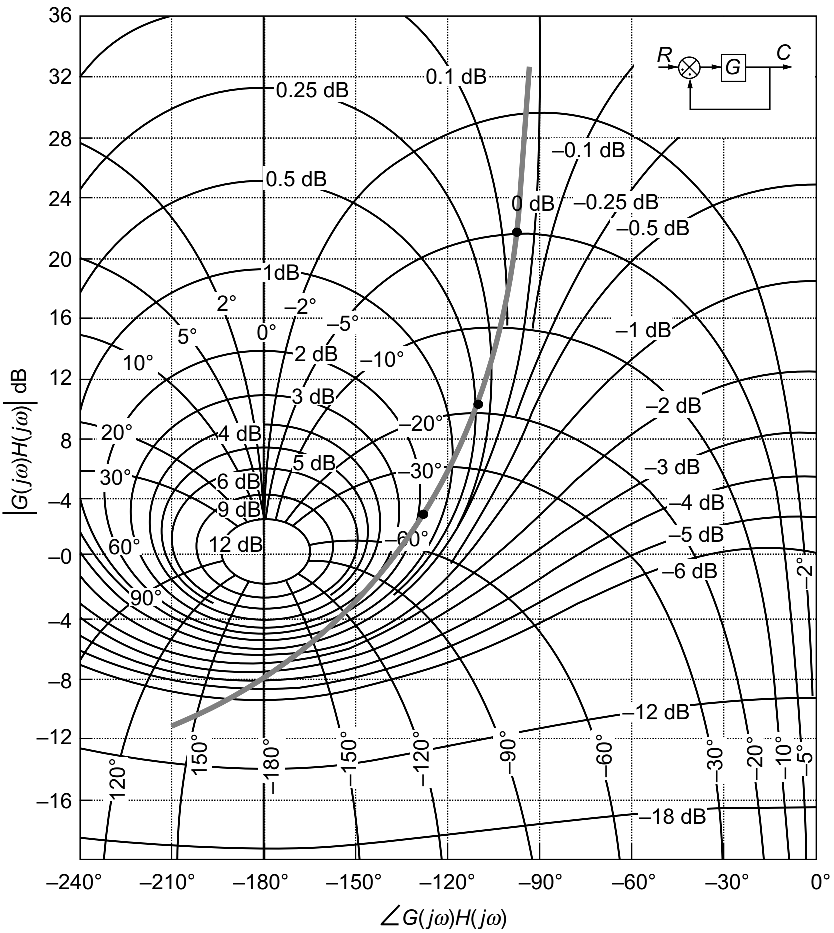


Fig. E10.2(d)

4. If we check the plots in Figs. E10.2(a) and E10.2(d), it is clear that the plot has been increased by 5 dB. Therefore, let $F = 5$ dB.

5. Therefore, the gain of the system for $\left(M_{\mathrm{r}}\right)_{\text{desired}}$ is

$$20 \log K = 5$$

i.e.,
$$K = 10^{\frac{5}{20}} = 10^{0.25} = 1.778.$$

Thus, the frequency domain specifications for the system are determined and the gain values for the various desired frequency domain specifications are also determined.

Review Questions

1. Define constant M-circles.
2. Define constant N-circles.
3. How can the closed-loop frequency response of the system be obtained from the open-loop frequency response using constant M- and constant N-circles?
4. What do you mean by Nichols Chart?
5. How can the closed-loop frequency response of the system be obtained from the open-loop frequency response using Nichols chart?
6. What are the advantages of Nichols chart?
7. Explain with a neat diagram, how the frequency domain specifications are obtained using Nichols chart.
8. How are the constant M-circles marked in the Nichols chart?
9. The loop transfer function of an unity feedback system is $G(s)H(s)=\dfrac{10K}{s(s+10)}$. Using Nichols chart, determine the gain K so that resonant peak is 1.4. Also, determine the corresponding frequency.
10. Determine the frequency domain specification of an unity feedback system whose loop transfer function is given by $G(s)H(s)=\dfrac{2(s+1)}{s(0.1s+1)^2}$ using Nichols chart.
11. The loop transfer function of an unity feedback system is given by $G(s)H(s)=\dfrac{54}{\left(\dfrac{s}{10}+1\right)(s^2+8s+25)}$. Determine the frequency domain specifications of the system using Nichols chart.
12. Determine the frequency domain specifications of an unity feedback system whose loop transfer function is given by $G(s)H(s)=\dfrac{2000K}{(s+20)(s+10)(3s+10)}$ using Nichols chart with $K=1$. Also, determine the gain K so that the gain margin of the system is 10 dB.
13. Determine the frequency domain specification of an unity feedback system whose loop transfer function is given by $G(s)H(s)=\dfrac{K}{s(s^2+s+4)}$ using Nichols chart with $K=1$. Also, determine the gain K so that the phase margin of the system is 50°.
14. In the Nichols chart shown in Fig. Q10.14, the plot for $\dfrac{G(j\omega)}{K}$ is plotted for different frequencies. Determine the frequency domain specifications of the system.

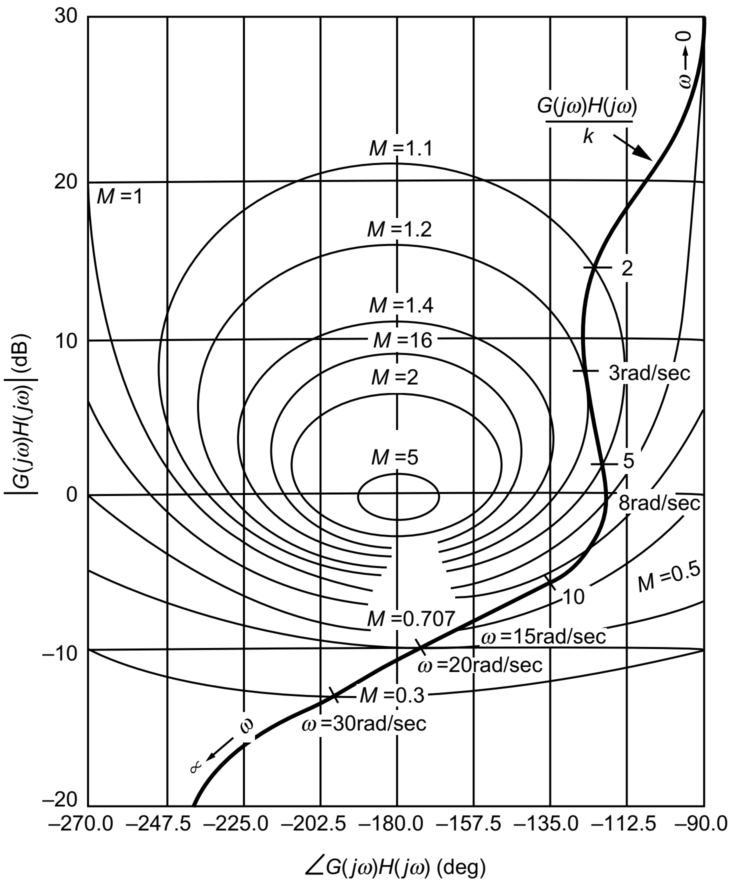

30
20
10
0
–10
–20
–270.0
–247.5
–225.0
–202.5
–180.0
–157.5
–135.0
–112.5
–90.0
|G(jω)H(jω)| (dB)
∠G(jω)H(jω) (deg)
ω → 0
G(jω)H(jω) / k
M = 1.1
M = 1
M = 1.2
M = 1.4
M = 16
M = 2
M = 5
2
3rad/sec
5
8rad/sec
10
M = 0.5
M = 0.707
ω = 15rad/sec
ω = 20rad/sec
M = 0.3
ω = 30rad/sec
ω
∞

Fig. Q10.14

11

COMPENSATORS

11.1 Introduction

The ultimate goal of examining the time domain analysis, frequency domain analysis and stability of a system in the previous chapters is to design a control system. The specific tasks for a system or requirements of a system can be achieved by a proper design of control systems. The performance specifications of a system generally relate to accuracy, stability and response time and these also vary based on the domain by which the system is to be designed. When the performance specifications of a system are given in time domain, the design can be carried out using any time-domain techniques and also if the performance specifications of a system are given in frequency domain, then the design can be carried out using any frequency domain techniques.

The general step followed in designing a control system is adjusting the gain of the system to meet its requirements. In practical situations, adjustment of gain may lead a system to unstable region or it may not be sufficient to meet the desired specifications. In such situations, additional devices or components can be added to the system to meet the specifications or to make the system stable. The additional devices or components that are added to a system are called compensators. The procedure for designing the compensators of a single-input single-output linear time invariant control system is discussed in this chapter. The steps to be followed in designing a control system are:

(i) Determine how the system functions and what the system has to do.
(ii) Determine the controller or compensator that can be more suitable for the given system.
(iii) Determine the parameters or constants present in the controller or compensator transfer function that has been chosen in the previous step.

11.2 Compensators

The compensator circuits basically introduce a pole and/or zero to the existing system to meet the desired specifications. The compensators may be designed using electrical, mechanical, pneumatic or any other components. The basic idea of determining the transfer function of the compensator is to suitably place the dominant closed-loop poles of a system. Compensated systems can be classified into six categories based on the location of compensators in the system as follows:

(i) Series or cascade compensation
(ii) Feedback or parallel compensation
(iii) Load or series-parallel compensation
(iv) State feedback compensation
(v) Forward compensation with series compensation
(vi) Feed-forward compensation

11.2.1 Series or Cascade Compensation

In this type of compensated system, the compensator is included in the feed-forward path of the system as shown in Fig. 11.1(a).

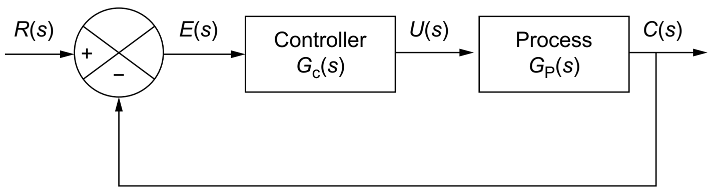


Fig. 11.1(a) | Series or cascade compensation

The addition of compensator in the feed-forward path adjusts the gain of a system which reduces the response time and peak overshoot of the system. In addition, the stability of the system gets reduced.

11.2.2 Feedback or Parallel Compensation

In this type of compensated system, the compensator is included in the feedback path of a system as shown in Fig. 11.1(b).

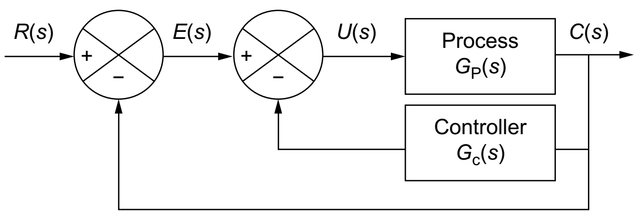


Fig. 11.1(b) | Parallel or feedback compensation

The addition of compensator in the feedback path increases the response time of the system that makes it accurate and more stable.

11.2.3 Load or Series-Parallel Compensation

The combination of both series and parallel compensation shown in Fig. 11.1(c) is known as load or series - parallel compensation.

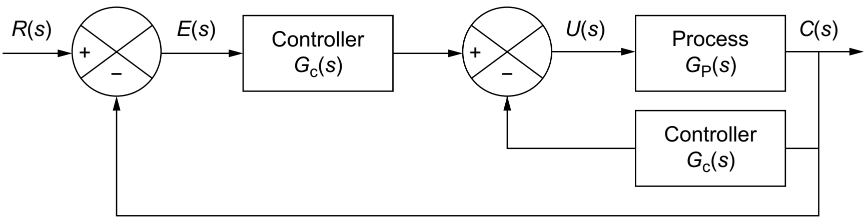


Fig. 11.1(c) | Series-parallel or load compensation

11.2.4 State Feedback Compensation

In this type of compensation, the control signal in the form of state variable is fed back as a control signal through the constant real gain as shown in Fig. 11.1(d).

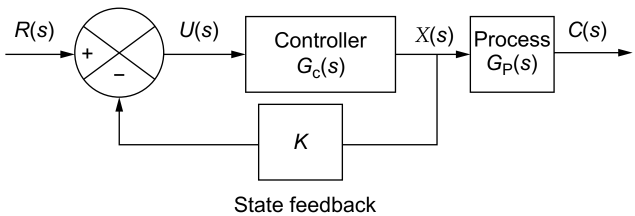


Fig. 11.1(d) | State feedback compensation

The implementation of the state feedback compensation is costly and impractical for higher order systems.

11.2.5 Forward Compensation with Series Compensation

When a simple closed-loop system is in series with the feed-forward controller, $G_{cf}(s)$, the resultant compensation is the forward compensation with series compensation that is shown in Fig. 11.1(e).

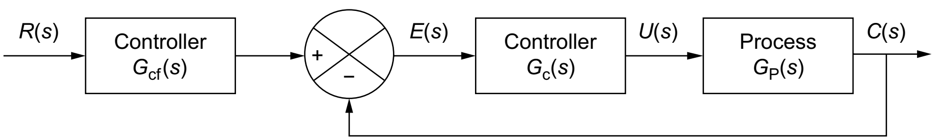


Fig. 11.1(e) | Forward compensation with series compensation

11.2.6 Feed-forward Compensation

When the feed-forward controller $G_{cf}(s)$ is placed in parallel with the forward path of a simple closed-loop system, the resultant compensation is the feed-forward compensation that is shown in Fig. 11.1(f).

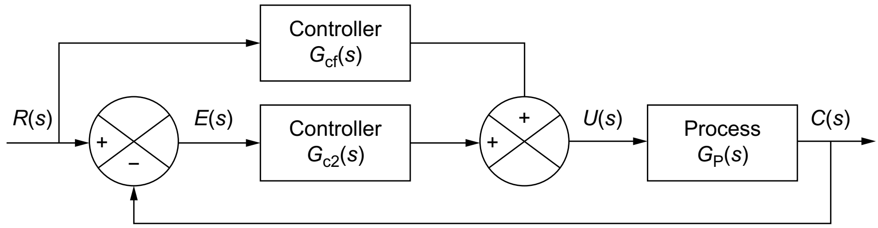


Fig. 11.1(f) | Feed-forward compensation

The compensated systems shown in Figs. 11.1(a), 11.1(b) and 11.1(d) have one degree of freedom which intimates that the system has single controller. The disadvantage of one degree of freedom controller is that the performance criteria realized using these compensation techniques are limited.

In simple words, the compensators introduce additional poles/zeros to an existing system so that the desired specification is achieved.

11.2.7 Effects of Addition of Poles

The following are the effects of addition of poles to an existing system:

(i) The root locus of a compensated system will be shifted towards the right-hand side of the *s*-plane.
(ii) Stability of a system gets lowered.
(iii) Settling time of a system increases.
(iv) Accuracy of a system is improved by the reduction of steady-state error.

11.2.8 Effects of Addition of Zeros

The following are the effects of addition of zeros to an existing system:

(i) The root locus of a compensated system will be shifted towards the left-hand side of the *s*-plane.
(ii) Stability of a system gets increased.
(iii) Settling time of a system decreases.
(iv) Accuracy of a system is lowered as steady-state error of the system increases.

In this chapter, three types of compensators used in the electrical systems are discussed. They are:

(i) Lag compensators
(ii) Lead compensators
(iii) Lag–lead compensators

11.2.9 Choice of Compensators

The choice of compensators from the different categories discussed in the previous sections is based on the following factors:

(i) Nature of signal to the system
(ii) Available components
(iii) Experience of the designer
(iv) Cost
(v) Power levels at different points and so on

11.3 Lag Compensator

The lag compensator is one that has a simple pole and a simple zero in the left half of the *s*-plane with the pole nearer to the origin. The term *lag* in the lag compensator means that the output voltage lags the input voltage and the phase angle of the denominator of the transfer function is greater than that of numerator. The general transfer function of the lag compensator is given by

$$G_{la}(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}, \quad \beta > 1 \tag{11.1}$$

where β and T are the constants and $K_c = \frac{1}{\beta}$.

The pole-zero configuration of the lag compensator is shown in Fig. 11.2.

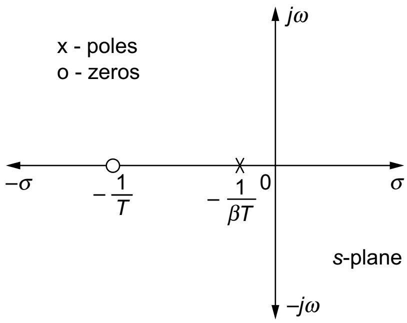


Fig. 11.2 | Pole-zero configuration of $G_{la}(s)$

The corner frequencies present in the lag compensator, whose transfer function is given by Eqn. (11.1) are at $\omega = \frac{1}{T}$ and $\omega = \frac{1}{\beta T}$. The Bode plot and polar plot of the lag compensator are shown in Figs. 11.3(a) and 11.3(b) respectively.

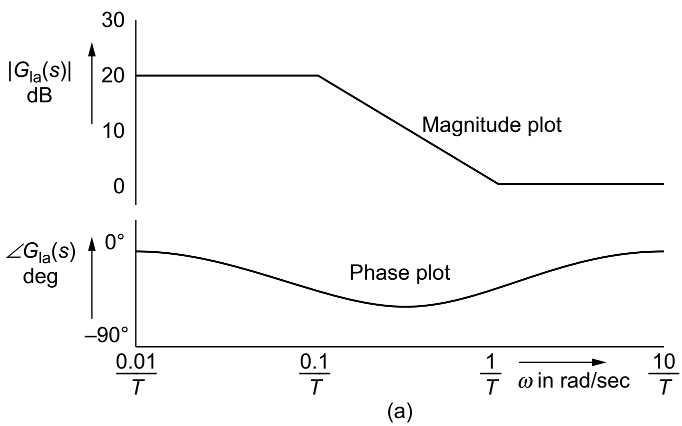
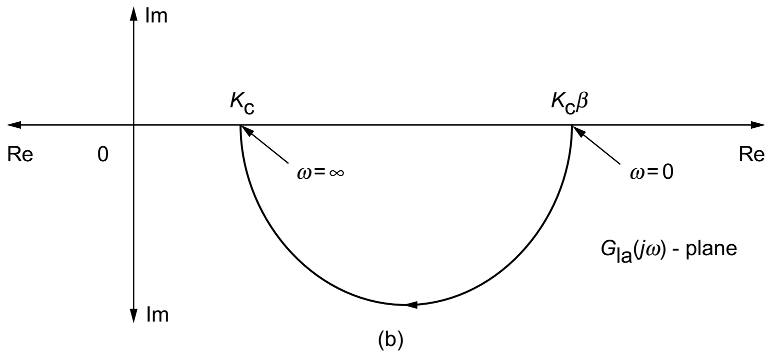


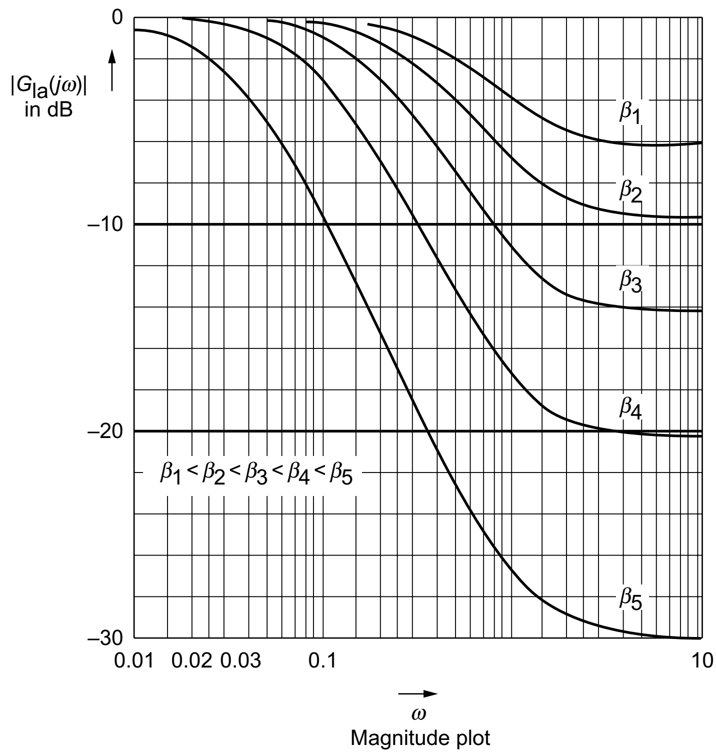
Fig. 11.3 | Plots of lag compensator



(b)

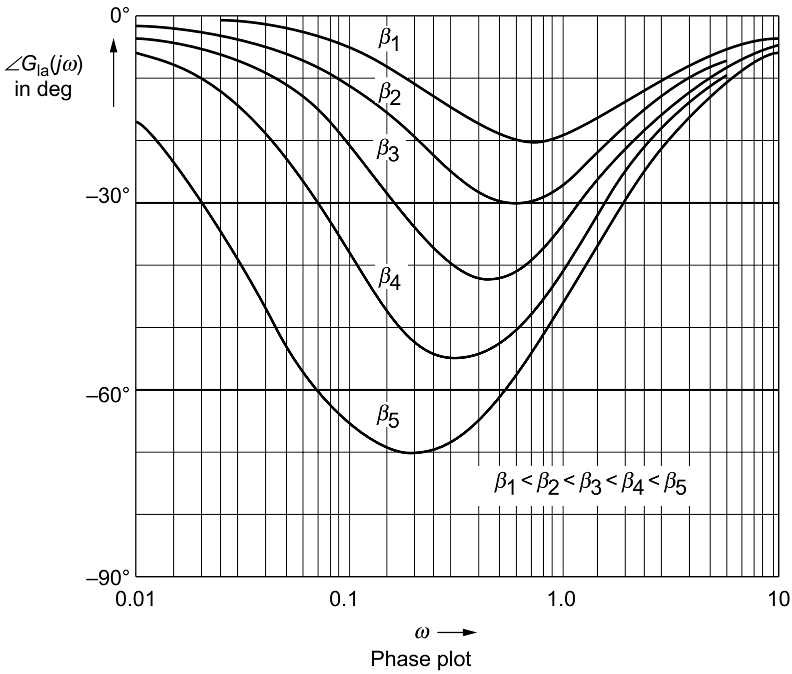
Fig. 11.3 | (Continued)

The Bode plot shown in Fig. 11.3(a) is plotted with $K_c = 0.1$ and $\beta = 10$. It is inferred that (i) magnitude of lag compensator is high at low frequencies and (ii) magnitude of lag compensator is zero at high frequencies. Hence, from the above conclusions, it is clear that the lag compensator behaves like a low-pass filter. The value of β is chosen between 3 and 10. The magnitude plot and phase plot of the compensator with different values of β are shown in Figs. 11.4(a) and 11.4(b) respectively.



(a)

Fig. 11.4 | Bode plot for different values of β (a) magnitude plot and (b) phase plot



(b)

Fig. 11.4 | (Continued)

11.3.1 Determination of Maximum Phase Angle ϕ_m

The modified transfer function of the lag compensator is

$$G_{la}(s) = \left(\frac{1+sT}{1+s\beta T}\right) \tag{11.2}$$

The magnitude and phase angle of the lag compensator are

$$M = \frac{\sqrt{1+\omega^2 T^2}}{\sqrt{1+\omega^2 \beta^2 T^2}} \tag{11.3}$$

and

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\omega \beta T) \tag{11.4}$$

The maximum phase angle ϕ_m occurs at

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_m} = 0$$

Differentiating Eqn. (11.4) with respect to ω, we obtain

$$\frac{T}{1+\omega_{\mathrm{m}}^2 T^2}-\frac{\beta T}{1+\beta^2\omega_{\mathrm{m}}^2 T^2}=0$$

$$\frac{1}{1+\omega_{\mathrm{m}}^2 T^2}=\frac{\beta}{1+\beta^2\omega_{\mathrm{m}}^2 T^2}$$

$$\omega_{\mathrm{m}}^2\left(\beta T^2-\beta^2 T^2\right)=1-\beta$$

Solving the above equation, we obtain

$$\omega_{\mathrm{m}}=\frac{1}{T\sqrt{\beta}}=\left(\frac{1}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{\beta T}}\right) \tag{11.5}$$

Substituting the above equation in Eqn. (11.4), we obtain

$$\tan\phi_{\mathrm{m}}=\frac{1-\beta}{2\beta} \tag{11.6}$$

Thus, Eqn. (11.5) gives the frequency at which the phase angle of the system is maximum and Eqn. (11.6) gives the maximum phase angle of the lag compensator.

11.3.2 Electrical Representation of the Lag Compensator

A simple lag compensator using resistor and capacitor is shown in Fig. 11.5.

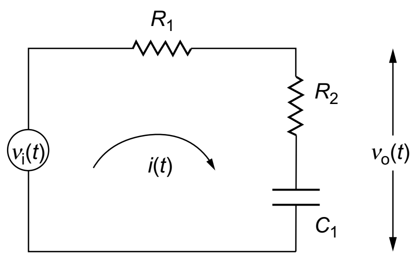


Fig. 11.5 | A simple lag compensator

Applying Kirchoff's voltage law to the above circuit, we obtain

$$v_i(t)=(R_1+R_2)i(t)+\frac{1}{C_1}\int i(t) \tag{11.7}$$

and

$$v_0(t)=R_2 i(t)+\frac{1}{C_1}\int i(t) \tag{11.8}$$

Taking Laplace transform on both sides of Eqs. (11.7) and (11.8), we obtain

$$V_i(s) = \left(R_1 + R_2 + \frac{1}{sC_1}\right) I(s) \tag{11.9}$$

and
$$V_0(s) = \left(R_2 + \frac{1}{sC_1}\right) I(s) \tag{11.10}$$

Substituting $I(s)$ from Eqs. (11.10) to (11.9), we obtain

$$V_i(s) = \left(R_1 + R_2 + \frac{1}{sC_1}\right) \frac{V_0(s)}{\left(R_2 + \frac{1}{sC_1}\right)}$$

Therefore, the transfer function of the above circuit is

$$\frac{V_0(s)}{V_i(s)} = \frac{\left(R_2 + \frac{1}{sC_1}\right)}{\left(R_1 + R_2 + \frac{1}{sC_1}\right)}$$

Rearranging the above equation, we obtain

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{\left(\frac{R_1 + R_2}{R_2}\right)} \left(\frac{s + \frac{1}{R_2 C_1}}{s + \frac{1}{\left(\frac{R_1 + R_2}{R_2}\right) R_2 C_1}} \right) \tag{11.11}$$

Comparing Eqn. (11.1) and Eqn. (11.11), we obtain

$$T = R_2 C_1,\ \beta = \frac{R_1 + R_2}{R_2} \text{ and } K_c = \frac{R_2}{R_1 + R_2} = \frac{1}{\beta}.$$

11.3.3 Effects of Lag Compensator

The following are the effects of adding lag compensator to a given system are:

(i) The lag compensator attenuates the high-frequency noise signals in the control loop.
(ii) It increases the steady-state error constants of a system.

(iii) Gain crossover frequency of a compensated system gets lowered.
(iv) Bandwidth of the compensated system decreases.
(v) Maximum peak overshoot, rise time and settling time of the system increases.
(vi) The system becomes more sensitive to the parameter variations.
(vii) As it acts like a proportional integral controller, it makes the system less stable.
(viii) Transient response of the compensated system becomes slower.

11.3.4 Design of Lag Compensator

The objective of designing the lag compensator is to determine the values of β and T for an uncompensated system $G(s)$ based on the desired system requirements. The design of lag compensator is based on the frequency domain specifications or time-domain specifications. The Bode plot is used for designing the lag compensator based on the frequency domain specifications, whereas the root locus technique is used for designing the lag compensator based on the time-domain specifications. Once the values of β and T are determined, the transfer function of the lag compensator $G_{\mathrm{la}}(s)$ can be obtained. The transfer function of the compensated system is $G_{\mathrm{c}}(s) = G_{\mathrm{la}}(s)G(s)$. If the Bode plot or root locus technique is plotted for the compensated system, it will satisfy the desired system requirements.

11.3.5 Design of Lag Compensator Using Bode Plot

Consider the open-loop transfer function of the uncompensated system $G(s)$. The objective is to design a lag compensator $G_{\mathrm{la}}(s)$ for $G(s)$ so that the compensated system $G_{\mathrm{c}}(s)$ will satisfy the desired system requirements. The steps for determining the transfer function of the compensated system $G_{\mathrm{la}}(s)$ using Bode plot are explained below:

Step 1: If the open-loop transfer function of the system $G(s)$ has a variable K, then $G_1(s) = G(s)$ or $G_1(s) = KG(s)$.

Step 2: Depending on the input and TYPE of the system, the variable K present in the transfer function of the uncompensated system $G_1(s)$ is determined based on either the steady-state error or the static error constant of the system.
The static error constants of the system are:

Position error constant, $K_{\mathrm{p}} = \underset{s\to 0}{Lt}\, G_1(s)$

Velocity error constant, $K_{\mathrm{v}} = \underset{s\to 0}{Lt}\, sG_1(s)$

Acceleration error constant, $K_{\mathrm{a}} = \underset{s\to 0}{Lt}\, s^2G_1(s)$

The relation between the steady-state error and static error constant based on the TYPE of the system and input applied to the system can be referred to Table 5.5 of Chapter 5.

Step 3: Construct the Bode plot for $G_1(s)$ with gain K obtained in the previous step and determine the frequency domain specifications of the system (i.e., phase margin, gain margin, phase crossover frequency and gain crossover frequency).

Step 4: Let the desired phase margin of the system be $(p_m)_d$. With a tolerance ε, determine $((p_m)_d)_{new}$ as

$$((p_m)_d)_{new} = (p_m)_d + \varepsilon$$

where $\varepsilon = 5$ to 10.

Step 5: Determine the new gain crossover frequency of the system for the phase margin $((p_m)_d)_{new}$. Let it be $(\omega_{gc})_{new}$.

Step 6: Determine the magnitude A of the system in dB from the magnitude plot corresponding to $(\omega_{gc})_{new}$.

Step 7: As the magnitude of system at the gain crossover frequency must be zero, the Bode plot must be either increased or decreased by A dB.

Step 8: The value of β in the transfer function of the lag compensator will be determined as

$$-20\log\beta = \pm A$$

Here $+A$ is when the magnitude plot is to be increased and $-A$ is when the magnitude plot is to be decreased.

Step 9: The value of T in the transfer function of the lag compensator will be determined by using the equation, $\frac{1}{T} = \frac{(\omega_{gc})_{new}}{10}$.

Step 10: Thus, the transfer function of the lag compensator will be determined as

$$G_{la}(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}$$

Step 11: The transfer function of the compensated system will be $G_c(s) = G_1(s)G_{la}(s)$. If the Bode plot for the compensated system is drawn, it will satisfy the desired system requirements.

The flow chart for determining the parameters present in the transfer function of the lag compensator using Bode plot is shown in Fig. 11.6.

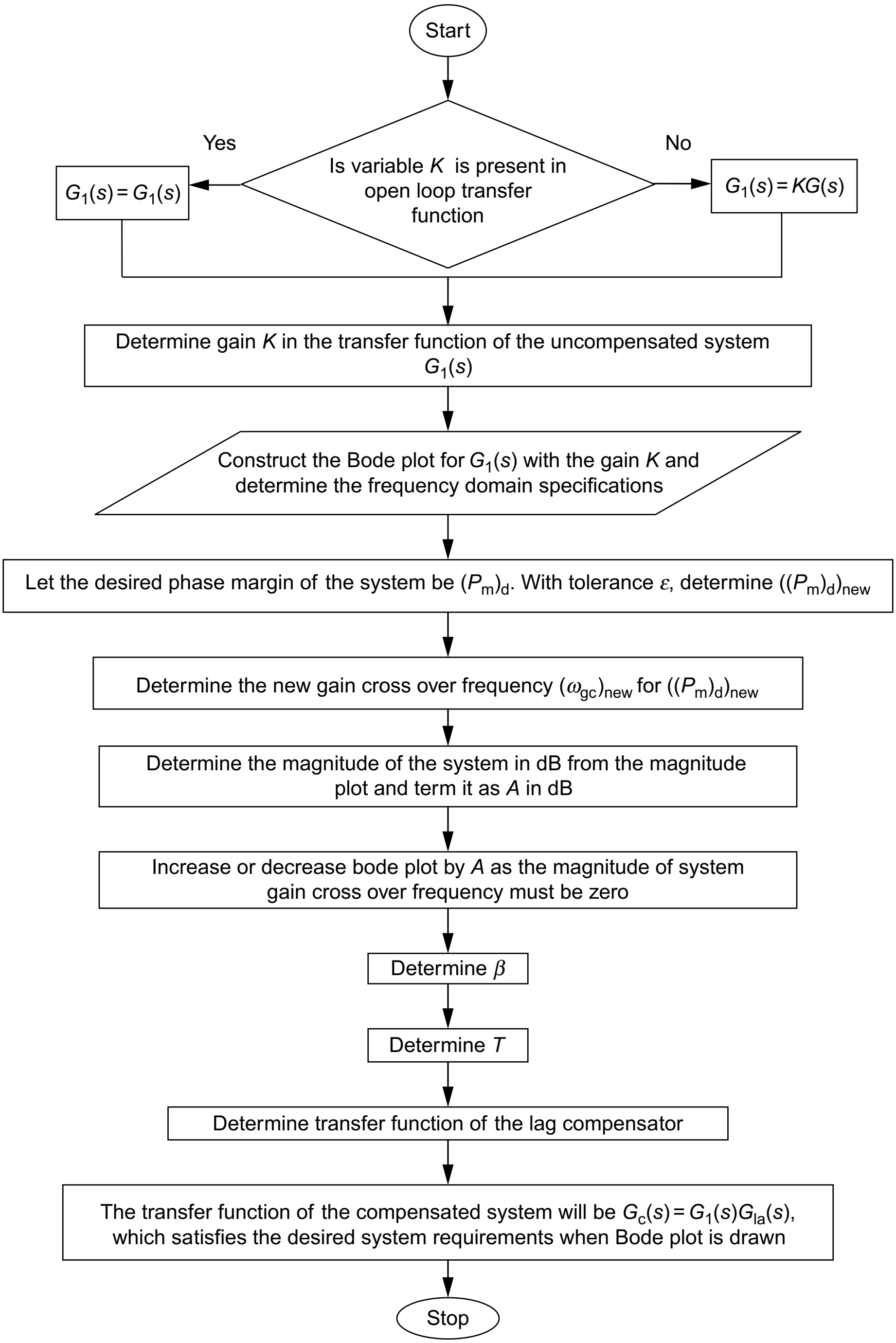


Fig. 11.6 | Flow chart for designing the lag compensator using Bode plot

Example 11.1 **Consider a unity feedback uncompensated system with the open-loop transfer function as $G(s) = \frac{5}{s(s+2)}$. Design a lag compensator for the system such that the compensated system has static velocity error constant K_v = 20 sec^{-1}, phase margin p_m = 55° and gain margin g_m = 12 dB.**

Solution

1. Let $G_1(s) = KG(s) = \frac{5K}{s(s+2)}$ and desired phase margin $(p_m)_d = 55^\circ$.
2. The value of K is determined by using $K_v = \underset{s \to 0}{Lt}\, sG_1(s)$ as

$$20 = \underset{s \to 0}{Lt}\, s\frac{5K}{s(s+2)}$$

   Solving the above equation, we obtain $K = 8$.

3. The Bode plot for $G_1(s) = \frac{40}{s(s+2)}$ $H(s) = 1$ is drawn.

   (a) Given $G_1(s)H(s) = \frac{40}{s(s+2)}$

   (b) Substituting $s = j\omega$ and $K = 1$ in the above equation, we obtain

$$G_1(j\omega)H(j\omega) = \frac{20}{j\omega(1+j0.5\omega)}$$

   (c) The corner frequency existing in the given system is

$$\omega_{c1} = \frac{1}{0.5} = 2 \text{ rad/sec}$$

To sketch the magnitude plot:

   (d) The changes in slope at different corner frequencies are given in Table E11.1(a).

Table E11.1(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω rad/sec	Slope of the term in dB/decade	Change in slope in dB/decade
$\frac{1}{j\omega}$	–	−20	–
$\frac{1}{(1+j0.5\omega)}$	$\omega_{c1} = 2$	−20	−20 − 20 = −40

(e) Assume the lower frequency as $\omega_l = 0.1$ rad/sec and higher frequency as $\omega_h = 20$ rad/sec.

(f) The values of gain at different frequencies are determined and given in Table E11.1(b).

Table E11.1(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain A_i
$\dfrac{20}{(j\omega)}$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\dfrac{20}{\omega_1}\right) = 46.02$ dB
$\dfrac{20}{(j\omega)}$	$\omega_2 = \omega_{c1} = 2$	–	$A_2 = 20\log\left(\dfrac{20}{\omega_2}\right) = 20$ dB
$\dfrac{1}{(1+j0.5\omega)}$	$\omega_3 = \omega_h = 20$	–40	$A_3 = \left[-40\times\log\left(\dfrac{\omega_3}{\omega_2}\right)\right] + A_2 = -20$ dB

(g) The magnitude plot of the given system is plotted using Table E11.1(b) and is shown in Fig. E11.1.

To sketch the phase plot:

(h) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -90 - \tan^{-1}(0.5\omega)$$

(i) The phase angle at different frequencies is obtained using the above equation and the values are tabulated as shown in Table E11.1(c).

Table E11.1(c) | Phase angle of a system for different frequencies

Frequency ω rad/sec	$-\tan^{-1}(0.5\omega)$	Phase angle ϕ (in degree)
0.2	–5.71°	–95.71
2	–45°	–135
4	–63.53°	–153.43
10	–78.69°	–168.69
∞	–90°	–180

(j) The phase plot of a given system is plotted with the help of Table E11.1(c) and is shown in Fig. E11.1(a).

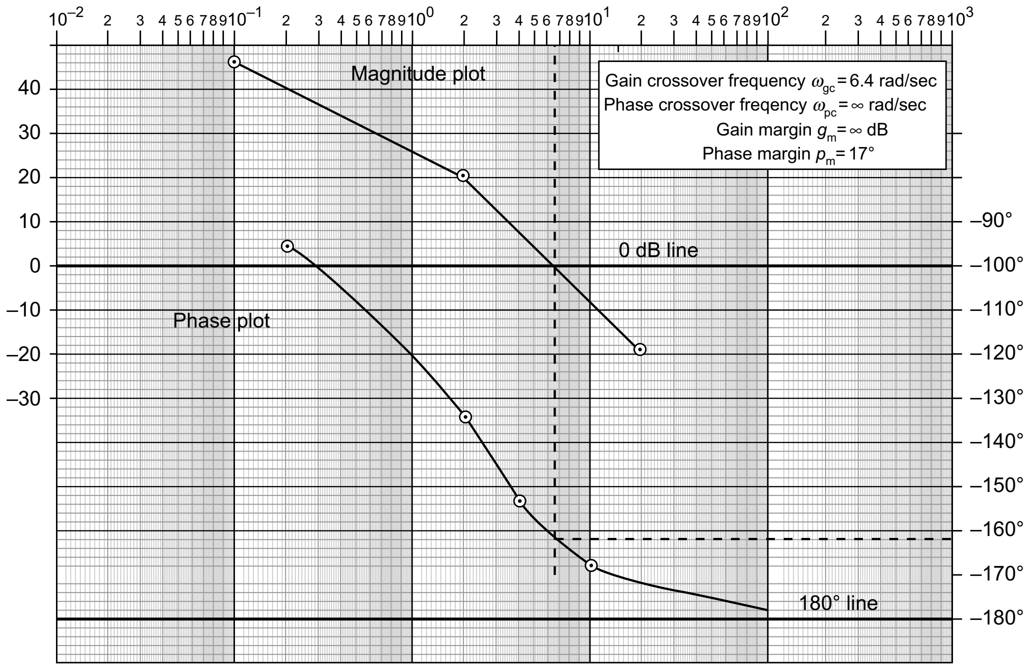


Fig. E11.1(a)

(k) The frequency domain specifications of the given system are:

Gain crossover frequency, ω_{gc} = 6.4 rad/sec.

Phase crossover frequency, ω_{pc} = ∞ rad/sec.

Gain margin, g_m = ∞ dB

Phase margin, p_m = 17°

It can be noted that the uncompensated system is stable, but the phase margin of the system is less than the desired phase margin which is 55°.

4. Let $\left((p_m)_d\right)_{new} = (p_m)_d + \varepsilon = 55^\circ + 5^\circ = 60^\circ$.

5. The new gain crossover frequency for $\left((p_m)_d\right)_{new}$ is $(\omega_{gc})_{new}$ = 0.92 rad/sec.

6. The magnitude of the system corresponding to $(\omega_{gc})_{new}$ is A = 27 dB.

7. The magnitude plot of the uncompensated system is decreased by 27 dB so that the gain at $(\omega_{gc})_{new}$ is zero dB.

8. The value of β in the lag compensator will be determined as

$$-20\log\beta = -27$$

i. e., $\beta = 22.38$

9. The value of T in the transfer function of the lag compensator is determined as

$$\frac{1}{T} = \frac{(\omega_{gc})_{new}}{10}$$

i. e., $T = 10.86$

10. Thus, the transfer function of the lag compensator is

$$G_{la}(s) = \frac{\left(s+\dfrac{1}{T}\right)}{\left(s+\dfrac{1}{\beta T}\right)} = \frac{(s+0.092)}{(s+0.0041)}$$

11. Thus, the transfer function of the compensated system is $G_c(s) = \dfrac{40(s+0.092)}{s(s+2)(s+0.0041)}$

11.3.6 Design of Lag Compensator Using Root Locus Technique

When a system is desired to meet the static error constant alongwith other time-domain specifications such as peak overshoot, rise time, settling time, damping ratio of the system and undamped natural frequency of oscillation, then lag compensator will be designed using root locus technique. The step-by-step procedure for designing the lag compensator using root locus technique is discussed below:

Step 1: The root locus of an uncompensated system with the loop transfer function $G(s)H(s)$ is constructed.

Step 2: Determine θ using $\theta = \cos^{-1}(\xi)$.

Step 3: Draw a line from origin with an angle θ from the negative real axis and determine the point at which it cuts the root locus of the uncompensated system. Let that point be the dominant pole of the closed-loop system P.

Step 4: If the uncompensated system has a gain K, the gain K is determined by using the formula $|G(s)H(s)|_{s=P} = 1$, or else, we can proceed to the next step.

Step 5: The static error constant of the uncompensated system is determined by

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG(s)$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)$

Let the static error constant determined for the system be K_{static}.

Step 6: Determine the factor by which the static error constant is to be increased is determined by using

$$\text{Factor} = \frac{K_{desired}}{K_{static}}$$

Step 7: Select the zero and pole of the lag compensator which lie very close to the origin such that the pole lies right to zero of the compensator. Let z and p be the zero and pole of the compensator. The pole and zero are chosen such that $z = 10p$.

Step 8: The transfer function of the lag compensator is obtained as $G_{la}(s) = K_c \dfrac{s+z}{s+p}$.

Step 9: The transfer function of the compensated system is obtained as $G_c(s) = G(s)G_{la}(s)$.

Step 10: The root locus of the compensated system is drawn and with the help of the damping ratio ξ, the dominant pole of the compensated system is determined. The new dominant pole obtained is P_{new}.

Step 11: The constant K_c is determined by using $\left|G_c(s)\right|_{s=P_{new}} = 1$.

Step 12: If the static error constant of the compensated system is determined, it will satisfy the desired specifications.

Step 13: If the static error constant does not satisfy the desired specifications, then alternative values of z and p are chosen and step 7–12 will be continued; otherwise, we can stop the procedure.

Thus, the transfer function of the lag compensator using root locus technique is determined.

The flow chart for designing the lag compensator using root locus technique is shown in Fig. 11.7.

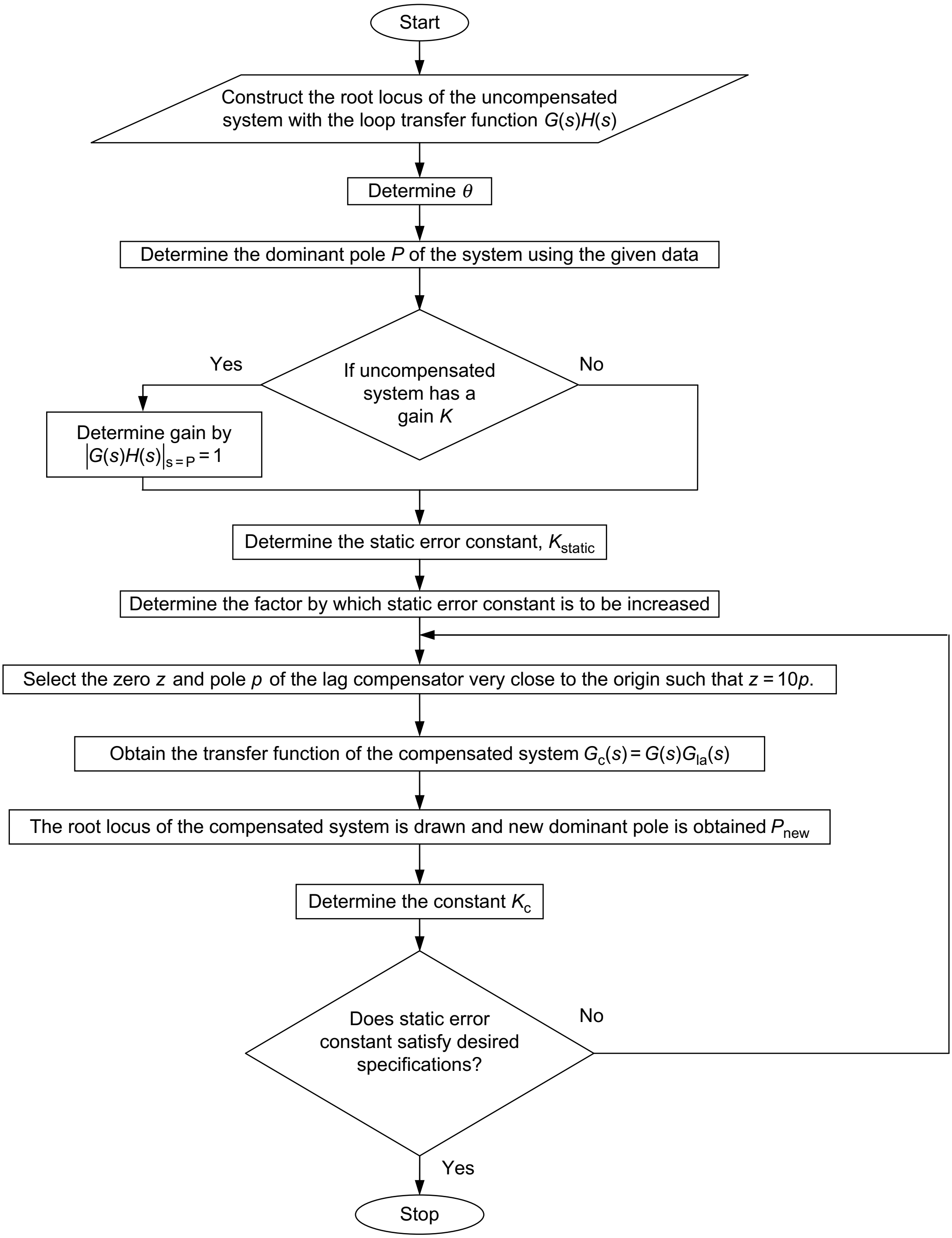


Fig. 11.7 | Flow chart for designing the lag compensator

Example 11.2: Consider a unity feedback uncompensated system with the open-loop transfer function as $G(s)=\dfrac{K}{s(s+1)(s+4)}$. Design a lag compensator for the system such that the compensated system has static velocity error constant $K_v = 12\text{sec}^{-1}$, damping ratio $\xi = 0.5$ and settling time $t_s = 10\text{sec}$.

Solution:

1. The root locus for an uncompensated system is drawn.

(a) For the given system $G(s)=\dfrac{K}{s(s+1)(s+4)}$, the poles are at 0, –1 and –4 i.e., $n = 3$ and zeros does not exist i.e., $m = 0$.

(b) Since $n > m$, the number of branches of the root loci for the given system is $n = 3$.

(c) The details of the asymptotes are:

(i) Number of asymptotes for the given system $N_a = (n-m) = 3$.

(ii) Angles of asymptotes

$$\theta_i = \frac{2i+1}{|n-m|}\times 180^\circ, \text{ for } n \neq m,\ i = 0 \text{ and } 1$$

Therefore, $\theta_0 = 60^\circ$, $\theta_1 = 180^\circ$ and $\theta_2 = 300^\circ$

(iii) Centroid

$$\sigma_1 = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{n-m} = \frac{-5}{3} = -1.667$$

(d) As all the poles and zeros are real values, there is no necessity to calculate the angle of departure and angle of arrival.

(e) *To determine the number of branches existing on the real axis.*

If we look from the pole $p = -1$, the total number of poles and zeros existing on the right of –1 is one (odd number). Therefore, a branch of root loci exists between –1 and 0.

Similarly, if we look from the point at $-\infty$, the total number of poles and zero existing on the right of the point is three (odd number). Therefore, a branch of root loci exists between $-\infty$ and –4.

Hence, two branches of root loci exist on the real axis for the given system.

(f) The breakaway/break-in points for the given system are determined by

$$\frac{dK}{ds} = 0 \tag{1}$$

For the given system, the characteristic equation is

$$1+G(s)H(s)=0 \tag{2}$$

i.e.,
$$1+\frac{K}{s(s+1)(s+4)}=0$$

Therefore,
$$K=-s^3-5s^2-4s$$

Differentiating the above equation with respect to s and using Eqn. (1), we obtain

$$\frac{dK}{ds}=-3s^2-10s-4=0$$

$$3s^2+10s+4=0$$

Solving for s, we obtain

$$s=-0.4648 \text{ and } s=-2.8685$$

(i) For $s=-0.4648$, K is 0.8794. Since K is positive, the point $s=-0.4648$ is a breakaway point. (ii) For $s=-2.8685$, K is –6.064. Since K is negative, the point $s=-2.8685$ is not a breakaway point.

Hence, only one breakaway point exists for the given system.

(g) The point at which the branch of root loci intersects the imaginary axis is determined as follows:

Using Eqn. (2), the characteristic equation for the given system is

$$s(s+1)(s+4)+K=0$$

i.e.,
$$s^3+5s^2+4s+K=0 \tag{3}$$

Routh array for the above equation is

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 5 & K \\ s^1 & \dfrac{20-K}{5} & 0 \\ s^0 & K & \end{array}$$

To determine the point at which the root loci crosses the imaginary axis, the first element in the third row must be zero i.e., $20-K=0$. Therefore, $K=20$.

As K is a positive real value, the root locus crosses the imaginary axis and the point at which it crosses imaginary axis is obtained by substituting K in Eqn. (3) and solving for s.

The solutions for the cubic equation $s^3+5s^2+4s+20=0$ are $s_1=-5$, $s_2=j2$ and $s_3=-j2$.

Hence, the point in the imaginary axis where the root loci crosses is $\pm j2$.

The complete root locus for the system is shown in Fig. E11.2.

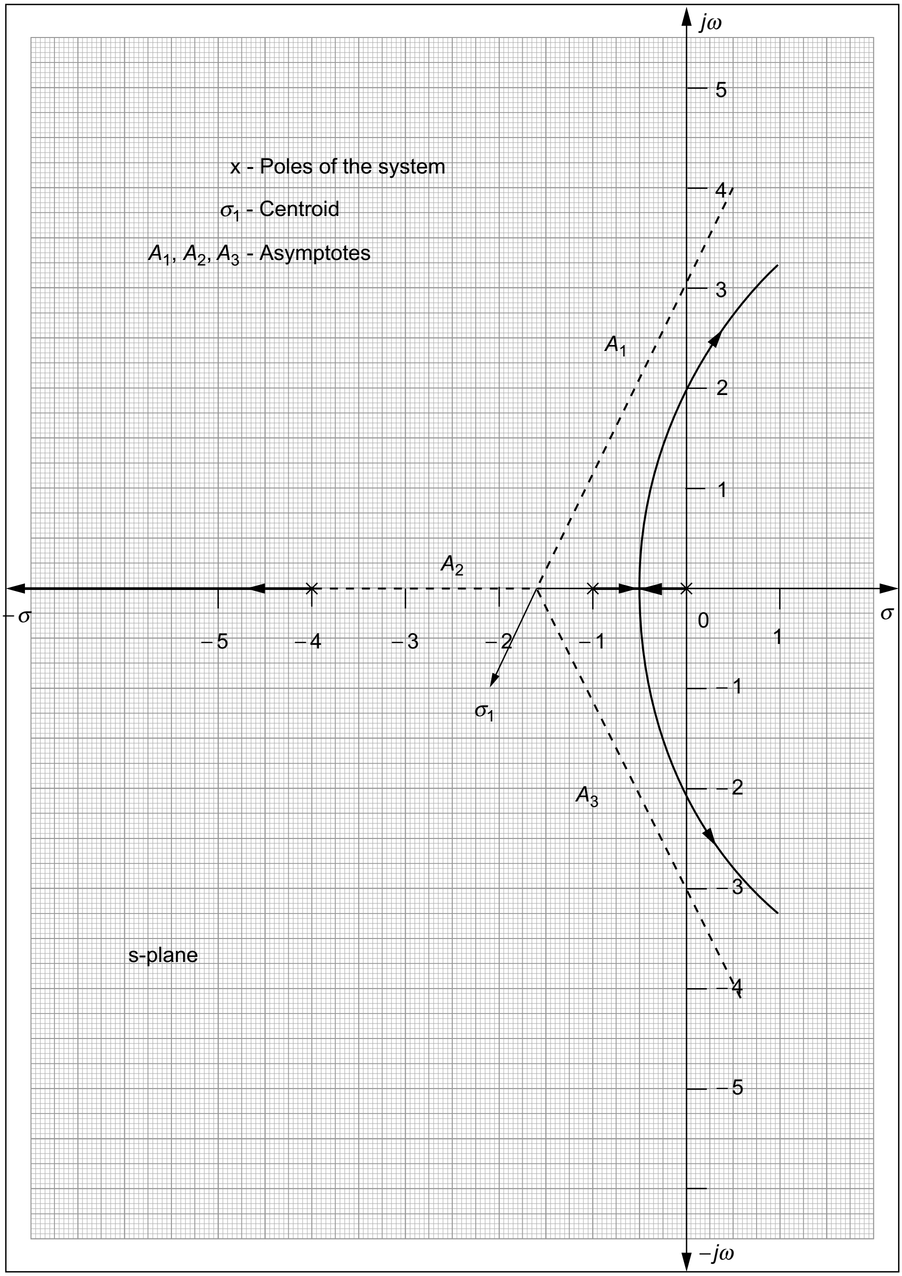


Fig. E11.2

2. Using $\xi = 0.5$ and settling time $t_s = \dfrac{4}{\xi\omega_n} = 10$, the dominant closed-loop poles are

$$-\xi\omega_n \pm \omega_n\sqrt{1-\xi^2} = -0.4 \pm j0.6993 \text{ (since } \omega_n = 0.8).$$

3. The gain K at dominant closed-loop poles is obtained using the magnitude condition,

$$\left|G(s)H(s)\right|_{s=-0.4+j0.6993} = 1$$

$$\frac{|K|}{|-0.4+j0.6993||0.6+j0.6993||3.6+j0.6993|} = 1$$

$$\frac{K}{0.8\times0.9167\times3.667} = 1$$

i. e., $K = 2.6892.$

Therefore, the transfer function of the compensated system is $G(s) = \dfrac{2.6892}{s(s+1)(s+4)}$.

4. The static error constant for the uncompensated system is

$$K_{\text{static}} = \underset{s\to0}{Lt}\, sG(s) = \frac{2.6892}{1\times4} = 0.6723.$$

5. The factor by which the static error constant to be increased is determined by using

$$\text{Factor} = \frac{K_{\text{desired}}}{K_{\text{static}}} = \frac{5}{0.6723} = 7.437$$

6. Let the zero of the compensator be at 0.1 and the pole of the compensator be at 0.01.
7. Therefore, the transfer function of the lag compensator is

$$G_{\text{la}}(s) = K_{\text{c}}\frac{(s+0.1)}{(s+0.01)}.$$

8. Thus, the transfer function of the compensated system is

$$G_c(s) = K_c\frac{(s+0.1)}{(s+0.01)}\times\frac{2.6892}{s(s+1)(s+4)} = \frac{K(s+0.1)}{s(s+1)(s+4)(s+0.01)}$$

where $K = 2.6892K_{\text{c}}$

11.4 Lead Compensator

The lead compensator is one that has a simple pole and a simple zero in the left half of the s-plane with the zero nearer to the origin. The term *lead* in the lead compensator refers that the output voltage leads the input voltage and the phase angle of the numerator of the transfer function is greater than that of denominator. The general transfer function of the lead compensator is given by

$$G_{le}(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}, 0 < \alpha < 1 \tag{11.12}$$

where α, T and K_c are constants.

The pole-zero configuration of the lead compensator is shown in Fig. 11.8.

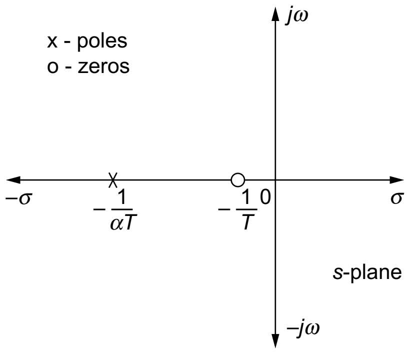
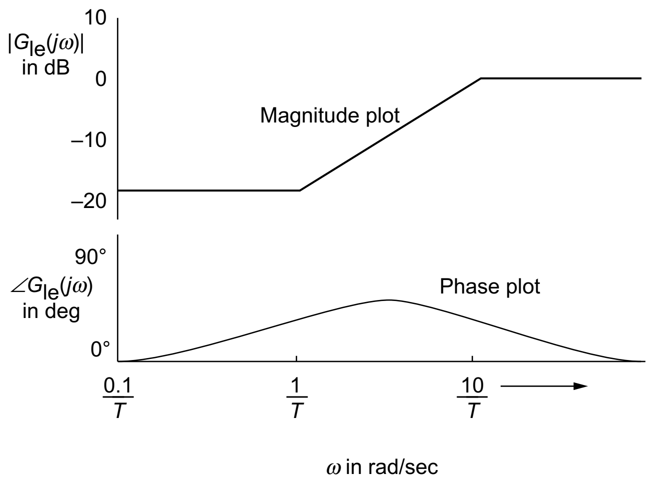


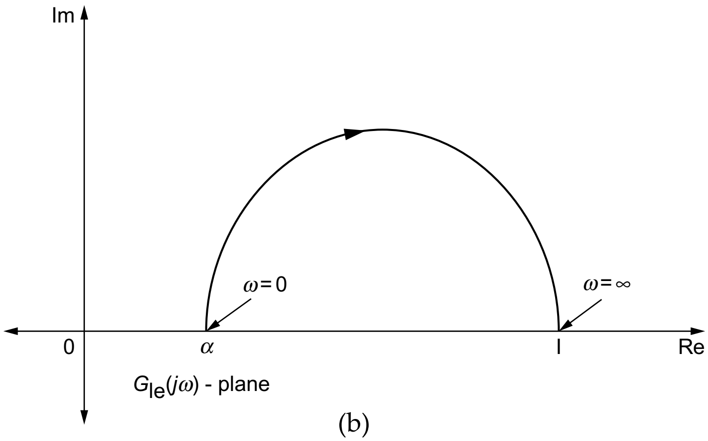
Fig. 11.8 | Pole-zero configuration of $G_{le}(s)$

The corner frequencies present in the lead compensator whose transfer function is given by Eqn. (11.12) are at $\omega = \frac{1}{T}$ and $\omega = \frac{1}{\alpha T}$. The Bode plot and polar plot of the lead compensator are shown in Figs. 11.9(a) and 11.9(b) respectively.



(a)

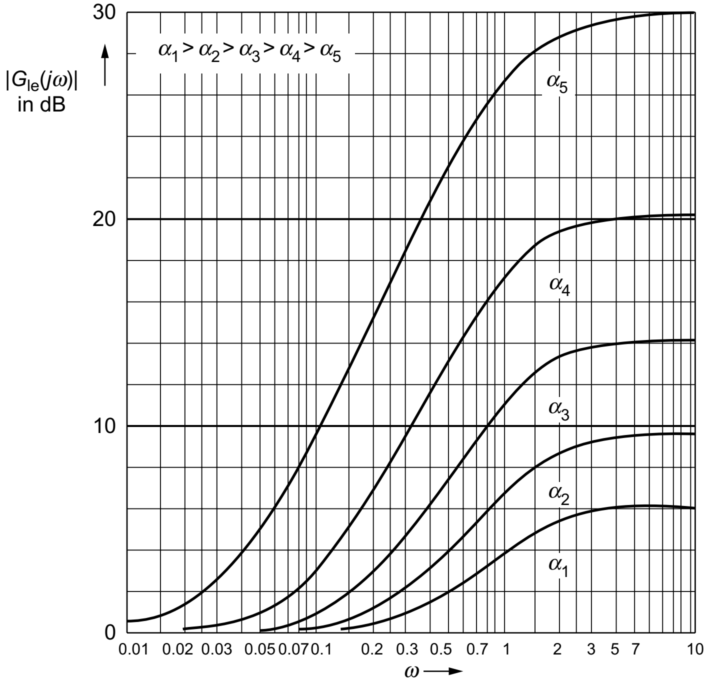
Fig. 11.9 | Plots of lead compensator



(b)

Fig. 11.9 | (Continued)

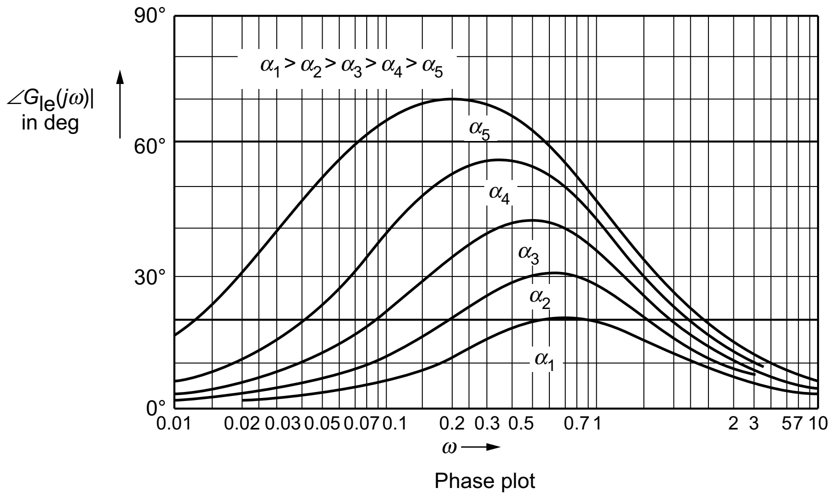
The Bode plot shown in Fig. 11.9(a) is plotted with $K_c = 1$ and $\alpha = 0.1$. It is inferred that (i) magnitude of lead compensator is low at low frequencies and (ii) magnitude of lead compensator is zero at high frequencies. Hence, the lead compensator behaves like a high-pass filter. The magnitude and phase plots for different values of α are shown in Figs. 11.10(a) and 11.10(b) respectively.



Magnitude plot

(a)

Fig. 11.10 | Bode plot for different values of α (a) magnitude plot and (b) phase plot



(b)

Fig. 11.10 | (Continued)

11.4.1 Determination of Maximum Phase Angle ϕ_m

The modified transfer function of the lead compensator is

$$G_{le}(s) = \alpha\left(\frac{1+sT}{1+s\alpha T}\right) \tag{11.13}$$

The magnitude and phase angle of the lead compensator are

$$M = \frac{\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2\alpha^2T^2}} \tag{11.14}$$

and

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\omega\alpha T) \tag{11.15}$$

The maximum phase angle ϕ_m occurs at

$$\left.\frac{d\phi}{d\omega}\right|_{\omega=\omega_m} = 0$$

Differentiating Eqn. (11.15) with respect to ω, we obtain

$$\frac{T}{1+\omega_m^2T^2} - \frac{\alpha T}{1+\alpha^2\omega_m^2T^2} = 0$$

$$\frac{1}{1+\omega_m^2T^2} = \frac{\alpha}{1+\alpha^2\omega_m^2T^2}$$

$$\omega_m^2\left(\alpha T^2 - \alpha^2T^2\right) = 1-\alpha$$

Solving the above equation, we obtain

$$\omega_{\mathrm{m}} = \frac{1}{T\sqrt{\alpha}} = \left(\frac{1}{\sqrt{T}}\right)\left(\frac{1}{\sqrt{\alpha T}}\right) \tag{11.16}$$

Substituting the above equation in Eqn. (11.15), we obtain

$$\tan\phi_{\mathrm{m}} = \frac{1-\alpha}{2\alpha} \tag{11.17}$$

Thus, Eqn. (11.16) gives the frequency at which the phase angle of the system is maximum and Eqn. (11.17) gives the maximum phase angle of the lag compensator.

11.4.2 Electrical Representation of the Lead Compensator

A simple lead compensator using resistor and capacitor is shown in Fig. 11.11.

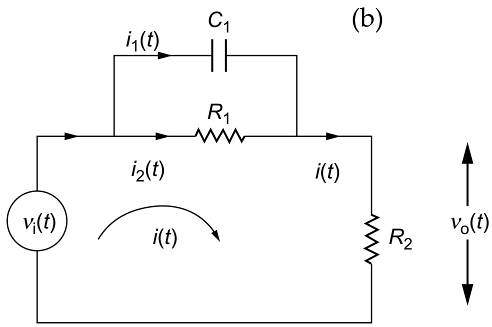


Fig. 11.11 | A simple lead compensator

Applying Kirchoff's current law to the above circuit, we obtain

$$i_1(t) + i_2(t) = i(t)$$

$$C_1 \frac{d\left(v_i(t) - v_0(t)\right)}{dt} + \frac{1}{R_1}\left(v_i(t) - v_0(t)\right) = \frac{1}{R_2} v_o(t) \tag{11.18}$$

Taking Laplace transform on both sides, we obtain

$$sC_1\left(V_i(s) - V_o(s)\right) + \frac{1}{R_1}\left(V_i(s) - V_0(s)\right) = \frac{1}{R_2} V_0(s) \tag{11.19}$$

$$sC_1 V_i(s) - sC_1 V_o(s) + \frac{1}{R_1} V_i(s) - \frac{1}{R_1} V_0(s) = \frac{1}{R_2} V_0(s)$$

$$\left(sC_1 + \frac{1}{R_1}\right) V_i(s) = \left(\frac{1}{R_2} + sC_1 + \frac{1}{R_1}\right) V_0(s)$$

Therefore, the transfer function of the above circuit is

$$\frac{V_0(s)}{V_i(s)} = \frac{\left(s + \frac{1}{R_1 C_1}\right)}{\left(s + \frac{1}{\left(\frac{R_2}{R_1 + R_2}\right) R_1 C_1}\right)} \tag{11.20}$$

Comparing Eqs. (11.12) and (11.20), we obtain

$$T = R_1 C_1 \text{ and } \alpha = \frac{R_2}{R_1 + R_2}$$

11.4.3 Effects of Lead Compensator

The effects of adding lead compensator to the given system are:

(i) Damping of the closed-loop system increases since a dominant zero is added to the system.
(ii) Peak overshoot of the system, rise time and settling time of the system decrease and as a result of which the transient response of the system gets improved.
(iii) Gain margin and phase margin of the system get increased.
(iv) Improves the relative stability of the system.
(v) Increases the bandwidth of the system that corresponds to the faster time response.

11.4.4 Limitations of Lead Compensator

The limitations of adding lead compensator to the given system are:

(i) Single-phase lead compensator can provide a maximum phase lead of 90°. If a phase lead of more than 90° is required, multistage compensator must be used.
(ii) There is always a possibility of reaching the conditionally stable condition even though the desired system requirements are achieved.

11.4.5 Design of Lead Compensator

The objective of designing the lead compensator is to determine the values of α and T for an uncompensated system $G_1(s)$ based on the desired system requirements. The design of lead compensator can be either based on the frequency domain specifications or time- domain specifications. The Bode plot is used for designing the lead compensator based on the frequency domain specifications and root locus technique is used for designing the lead compensator based on the time-domain specifications. Once α and T are determined, the transfer function of the lead compensator $G_{\text{le}}(s)$ can be obtained. The transfer function of the compensated system $G_c(s) = G_{\text{le}}(s) G_1(s)$. If the Bode plot or root locus technique is plotted for the compensated system, it will satisfy the desired system requirements.

11.4.6 Design of Lead Compensator Using Bode Plot

Let the open-loop transfer function of the uncompensated system be $G(s)$. The objective is to design a lead compensator $G_{le}(s)$ for $G(s)$ so that the compensated system $G_c(s)$ will satisfy the desired system requirements. The steps for determining the transfer function of the compensated system $G_{le}(s)$ using Bode plot are explained below:

Step 1: If the open-loop transfer function of the system $G(s)$ has a variable K, then $G_1(s) = G(s)$; otherwise $G_1(s) = KG(s)$.

Step 2: Depending on the input and TYPE of the system, the variable K present in the open-loop transfer function of the uncompensated system is determined based on either the steady-state error or the static error constant of the system.

The static error constants of the system are

$$\text{Position error constant, } K_p = \underset{s\to 0}{Lt}\, G_1(s)$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{Lt}\, sG_1(s)$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{Lt}\, s^2G_1(s)$$

The relation between the steady-state error and static error constant based on the TYPE of the system and input applied to the system can be referred to Table 5.5 of Chapter 5.

Step 3: Construct the Bode plot for the uncompensated system with gain K obtained in the previous step and determine the frequency domain specifications of the system (i.e., phase margin, gain margin, phase crossover frequency and gain crossover frequency). Let the phase margin of the uncompensated system be p_m.

Step 4: Let the desired phase margin of the system be $(p_m)_d$. With a tolerance ε, determine $\left((p_m)_d\right)_{new}$ as

$$\left((p_m)_d\right)_{new} = (p_m)_d - p_m + \varepsilon$$

where $\varepsilon = 5$ to 10.

Step 5: Determine α using $\sin\left((p_m)_d\right)_{new} = \dfrac{1-\alpha}{1+\alpha}$.

Step 6: Determine $-10\log\left(\dfrac{1}{\alpha}\right)$ in dB. Let it be A.

Step 7: Determine the frequency from the magnitude plot of the uncompensated system for the magnitude of A dB. Let this frequency be the new gain crossover frequency $(\omega_{gc})_{new}$.

Step 8: Determine T using $(\omega_{gc})_{new} = \dfrac{1}{T\sqrt{\alpha}}$.

Step 9: Determine K_c using $K = K_c\alpha$.

Step 10: Thus, the transfer function of the lead compensator will be determined as

$$G_{le}(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c\alpha\left(\frac{1+sT}{1+s\alpha T}\right)$$

Step 11: The transfer function of the compensated system will be $G_c(s) = G(s)G_{le}(s)$ $= K_c\alpha\left(\frac{1+sT}{1+s\alpha T}\right)G_{le}(s)$. If the Bode plot for the compensated system is drawn, it will satisfy the desired system requirements.

Flow chart for designing the lead compensator using Bode plot

The flow chart for determining the parameters present in the transfer function of the lead compensator using Bode plot is shown in Fig. 11.12.

Example 11.3: Consider a unity feedback uncompensated system with the open-loop transfer function as $G(s) = \frac{K}{s(s+1)}$. Design a lead compensator for the system such that the compensated system has static velocity error constant $K_v = 12\text{sec}^{-1}$ and phase margin $p_m = 40°$.

Solution:

1. Let $G_1(s) = G(s) = \frac{K}{s(s+1)}$.
2. The value of K is determined by using $K_v = \underset{s\to 0}{Lt}\, sG_1(s)$ as

$$12 = \underset{s\to 0}{Lt}\, s\frac{K}{s(s+1)}$$

   Solving the above equation, we obtain $K = 12$.

3. The Bode plot for $G_1(s) = \frac{12}{s(s+1)}$ $H(s) = 1$ is drawn.

   (a) Given $G_1(s)H(s) = \frac{12}{s(s+1)}$

   (b) Substituting $s = j\omega$ in the above equation, we obtain

$$G_1(j\omega)H(j\omega) = \frac{12}{j\omega(1+j\omega)}$$

   (c) The corner frequency existing in the given system is

$$\omega_{c1} = 1 \text{ rad/sec}$$

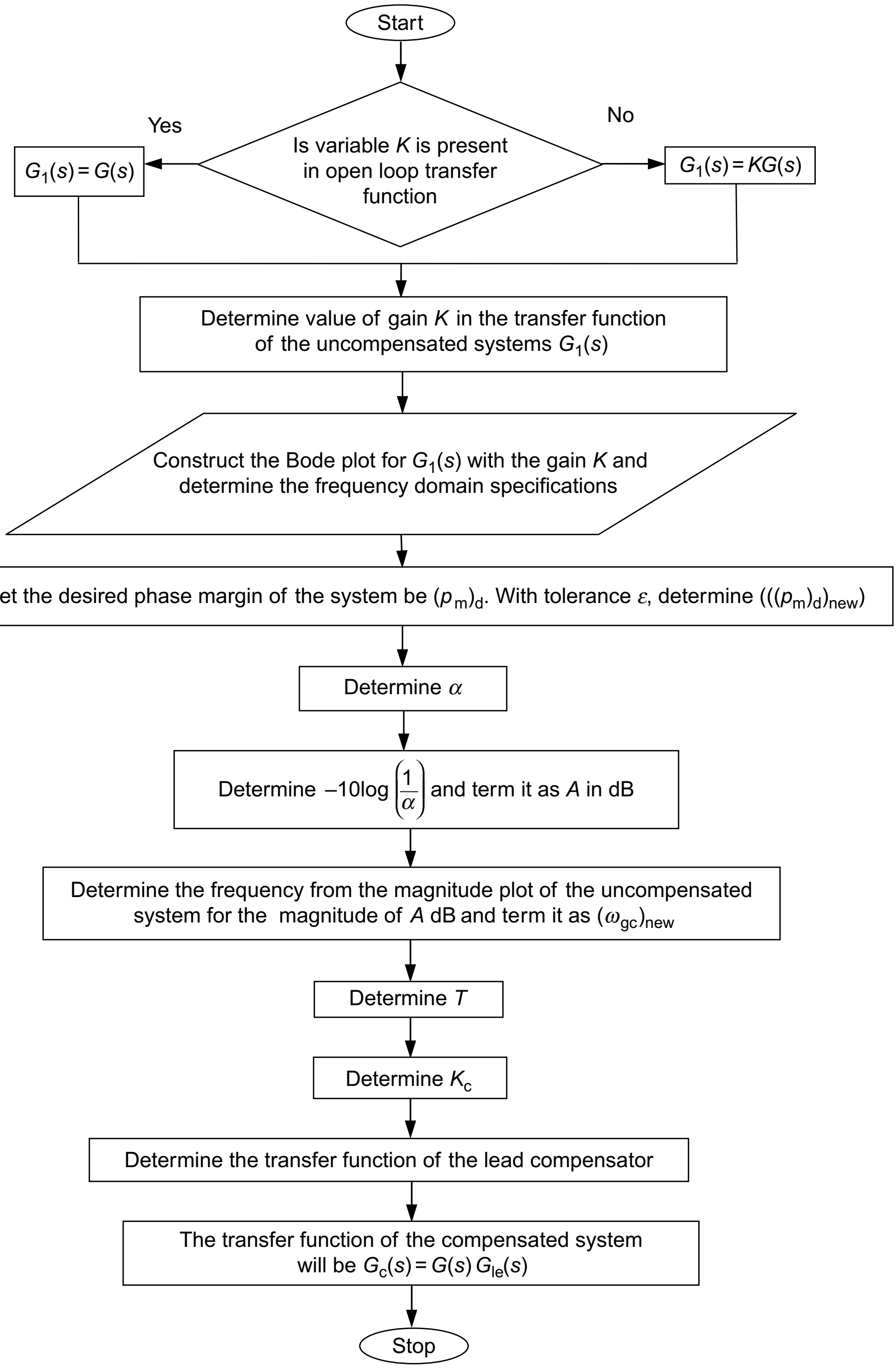


Fig. 11.12 | Flow chart for designing the lead compensator using Bode plot

To sketch the magnitude plot:

(d) The changes in slope at different corner frequencies are given in Table E11.3(a).

Table E11.3(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω rad/sec	Slope of the term in dB/decade	Change in slope in dB/decade
$\frac{1}{j\omega}$	–	–20	–
$\frac{1}{(1+j\omega)}$	$\omega_{c1}=1$	–20	$-20-20=-40$

(e) Assume the lower frequency as ω_l = 0.1 rad/sec and higher frequency as ω_h = 20 rad/sec.

(f) The values of gain at different frequencies are determined and given in Table E11.3(b).

Table E11.3(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain A_i
$\frac{12}{(j\omega)}$	$\omega_1=\omega_l=0.1$	–	$A_1=20\log\left(\frac{12}{\omega_1}\right)=41.58\ \text{dB}$
$\frac{12}{(j\omega)}$	$\omega_2=\omega_{c1}=1$	–	$A_2=20\log\left(\frac{12}{\omega_2}\right)=21.58\ \text{dB}$
$\frac{1}{(1+j\omega)}$	$\omega_3=\omega_h=20$	–40	$A_3=\left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right]+A_2=-30.46\ \text{dB}$

(g) The magnitude plot of the given system is plotted using Table E11.3(b) and is shown in Fig. E11.3.

To sketch the phase plot:

(h) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi=-90-\tan^{-1}(\omega)$$

(i) The phase angle at different frequencies is obtained using the above equation and the values are tabulated as shown in Table E11.3(c).

Table E11.3(c) | Phase angle of the system for different frequencies

Frequency ω rad/sec	$-\tan^{-1}(\omega)$	**Phase angle ϕ (in degree)**
0.2	–11.30°	–101.3
2	–63.43°	–153.43
5	–75.69°	–165.69°
10	–84.28°	–174.28
∞	–90°	–180

(j) The phase plot of the given system is plotted with the help of Table E11.3(c) and is shown in Fig. E11.3.

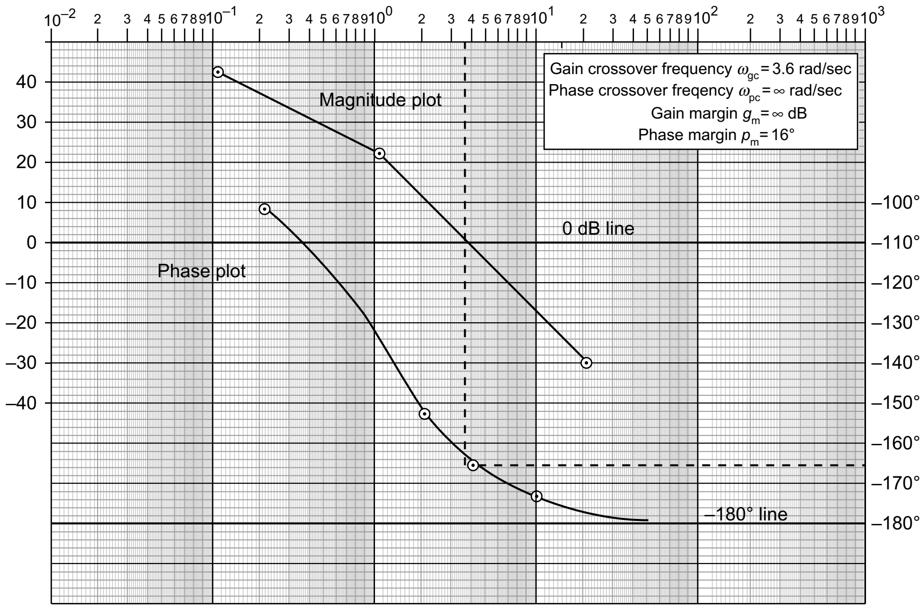


Fig. E11.3

(k) The frequency domain specifications of the given system are

Gain crossover frequency, $\omega_{gc} = 3.6$ rad/sec

Phase crossover frequency, $\omega_{pc} = \infty$ rad/sec

Gain margin, $g_m = \infty$ dB and

Phase margin, $p_m = 16^\circ$

It can be noted that the uncompensated system is stable, but the phase margin of the system is less than the desired phase margin.

4. With tolerance value $\varepsilon = 5^{\circ}$, $\left((p_{\mathrm{m}})_{\mathrm{d}}\right)_{\mathrm{new}}$ is calculated as

$$\text{i. e., } \left((p_{\mathrm{m}})_{\mathrm{d}}\right)_{\mathrm{new}} = 40^{\circ} - 16^{\circ} + 5^{\circ} = 29^{\circ}.$$

5. The value of α in the lead compensator is determined by using, $\sin\left(29^{\circ}\right) = \dfrac{1-\alpha}{1+\alpha}$.

$$\text{i.e., } 0.484\left(1+\alpha\right) = 1-\alpha$$

$$0.484 + 0.484\,\alpha = 1-\alpha$$

Solving the above equation, we obtain

$$\alpha = 0.3477$$

6. Let $B = -10\log\left(\dfrac{1}{\alpha}\right) = -4.58$ dB

7. The new gain crossover frequency $\left(\omega_{\mathrm{gc}}\right)_{\mathrm{new}}$ corresponding to B from the magnitude plot of the uncompensated system is 5 rad/sec.

8. Using $\left(\omega_{\mathrm{gc}}\right)_{\mathrm{new}} = \dfrac{1}{T\sqrt{\alpha}}$, determine T as $5 = \dfrac{1}{T\sqrt{0.3477}}$.

$$\text{i. e., } T = 0.339.$$

9. The value of K_{c} is determined as $12 = K_{\mathrm{c}}\left(0.3477\right)$ i.e., $K_{\mathrm{c}} = 34.51$.

10. The transfer function of the lead compensator is

$$G_{\mathrm{le}}\left(s\right) = K_{\mathrm{c}}\frac{\left(s+\dfrac{1}{T}\right)}{\left(s+\dfrac{1}{\alpha T}\right)} = 12\frac{\left(1+0.339s\right)}{\left(1+0.1178s\right)}$$

11. Thus, the transfer function of the compensated system is

$$G_{\mathrm{c}}\left(s\right) = 12\frac{\left(1+0.339s\right)}{s\left(s+1\right)\left(1+0.1178s\right)}$$

11.4.7 Design of Lead Compensator Using Root Locus Technique

The desired time-domain specifications that can be specified for designing the lead compensator are peak overshoot, settling time, rise time, damping ratio and undamped natural frequency of the system. The step-by-step procedure for designing the lead compensator for an uncompensated system whose open-loop transfer function is given by $G(s)$ using root locus technique is given below:

Step 1: Determine the damping ratio ξ and undamped natural frequency ω_n of the system based on the given time-domain specifications.

Step 2: Determine the dominant closed-loop poles of the system using $-\xi\omega_n \pm j\omega_d$ or $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$. Let P_1 and P_2 be the dominant closed-loop poles of the system that is marked in s-plane as shown in Fig. 11.13(a).

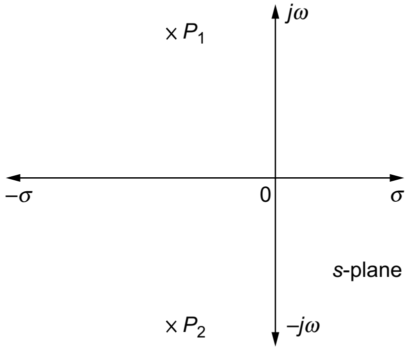


Fig. 11.13(a)

Step 3: Determine the angle of the loop transfer function $G(s)H(s)$ at any one of the dominant pole i.e., either P_1 or P_2. Let the angle obtained be M degrees i.e., $\angle G(s)H(s)\big|_{s=P_1}$

Step 4: The angle obtained in the previous step, i.e., M should be an odd multiple of 180°. If it is not so, some values of angle can be added or subtracted from M to make it an odd multiple of 180°. The angle by which M has to be added or subtracted is obtained using $\phi = 180° - M$. The value of ϕ is always less than 60° and if ϕ is greater than 60°, then multiple lead compensators can be used.

Step 5: Draw a line parallel to the X-axis from point P_1 to A and also join the point P_1 with the origin as shown in Fig. 11.13(b).

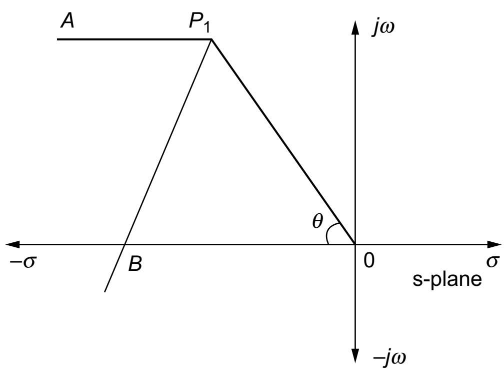


Fig. 11.13(b)

Step 6: Determine $\angle AP_1O$ and using $\dfrac{\angle AP_1O}{2}$ draw a line from the point P_1 that bisects the negative X-axis at point B as shown in Fig. 11.13(b).

Step 7: Draw two lines P_1C and P_1D from point P_1 such that $\angle BP_1C = \angle BP_1D = \dfrac{\phi}{2}$, which is shown in Fig. 11.13(c).

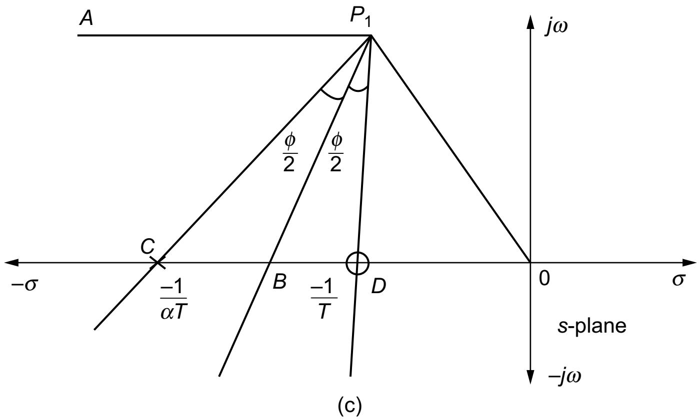


Fig. 11.13(c)

Step 8: The points C and D in the negative real axis corresponds to the pole and zero of the lead compensator respectively.

Step 9: The constants in the lead compensator (α and T) are determined by using $\dfrac{1}{T} = |D|$ and $\dfrac{1}{\alpha T} = |C|$.

Step 10: Using the constants determined in the previous step, the transfer function of the lead compensator is obtained as

$$G_{le}(s) = K_c \left(\frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\alpha T}} \right).$$

Step 11: The transfer function of the compensated system is obtained using $G_c(s) = G_{le}(s)G(s)$.

Step 12: The constant K_c of the system is determined by using $\left|G_c(s)\right|_{s=P_1} = 1$.

Step 13: If the root locus of the compensated system is drawn, we can see that the root locus of the compensated system will pass through the dominant closed-loop poles of the system.

The flow chart for designing the lead compensator using the root locus technique is shown in Fig. 11.14.

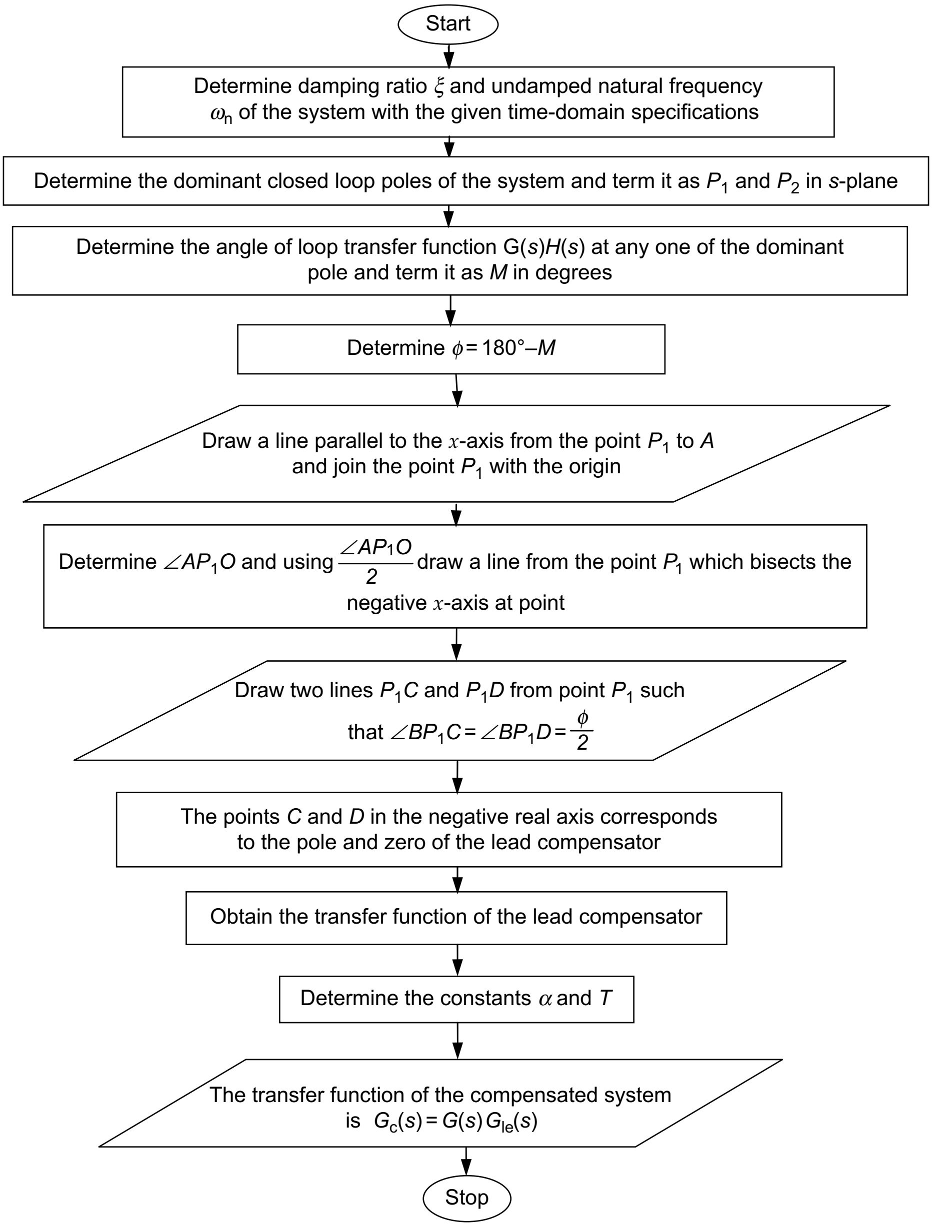


Fig. 11.14 | Flow chart for designing the lead compensator

Example 11.4 **Consider a unity feedback uncompensated system with the open-loop transfer function as $G(s)=\dfrac{K}{s(s+1)(s+4)}$. Design a lead compensator for the system such that the compensated system has damping ratio $\xi = 0.5$ and un-damped natural frequency $\omega_n = 2$ rad/sec.**

Solution

1. Refer to the solution of Example 11.2 and follow the steps to obtain the root locus of the system.

   The complete root locus for the system is shown in Fig. E11.4.

2. Using $\xi = 0.5$ and undamped natural frequency ω_n = 2, the dominant closed-loop poles are $-\xi\omega_n \pm \omega_n\sqrt{1-\xi^2} = -1 \pm j1.73$.

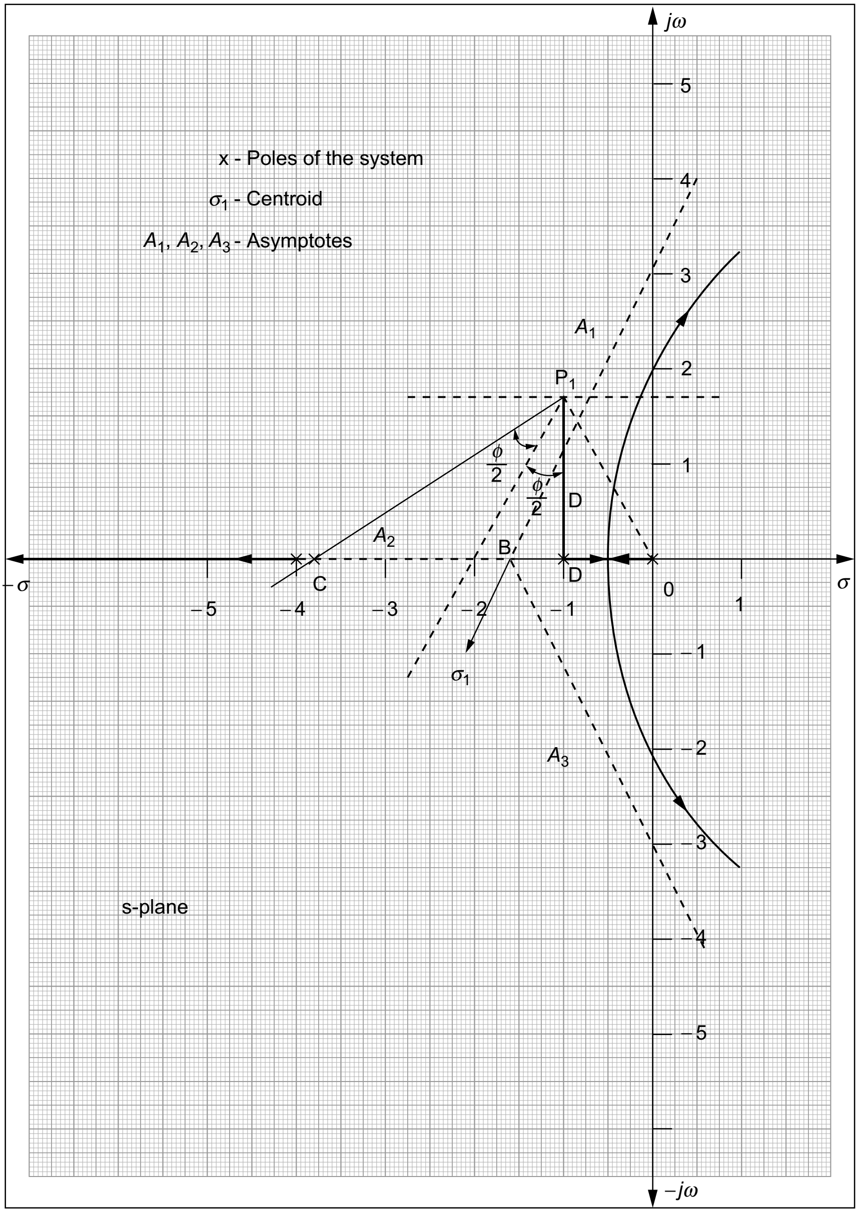


Fig. E11.4

3. The angle of the system at dominant closed-loop pole is

$$\angle G(s)H(s)\Big|_{s=-1+j1.73} = \frac{\angle K}{(\angle -1+j1.73)(\angle j1.73)(\angle 3+j1.73)} = -240^\circ$$

4. Since the angle obtained in the previous step is not an odd multiple of 180°, the angle to be contributed by lead compensator is given by $\phi = -180^\circ - (-240^\circ) = 60^\circ$.
5. A line parallel to the X-axis from point P_1 to A and also the point P_1 is joined with the origin.
6. The angle $\angle AP_1O$ is determined as $120^\circ$and using $\frac{\angle AP_1O}{2} = 60^\circ$, a line from the point P_1 is drawn which bisects the negative X-axis at point B.
7. Two lines P_1C and P_1D are drawn from point P_1 such that $\angle BP_1C = \angle BP_1D = \frac{\phi}{2} = 30^\circ$
8. The points C and D are the poles and zeros of the lead compensator respectively. Since the zero is at –1 and there exists pole-zero cancellation, the zero of lead compensator is taken slightly left of point –1. Let it be –1.1. Therefore, $C = -3.8$ and $D = -1.1$
9. The constants in the lead compensator are determined by using $\frac{1}{T} = |D|$ and $\frac{1}{\alpha T} = |C|$.

   Therefore, $T = 0.909$ and $\alpha = 0.289$.
10. Therefore, the transfer function of lead compensator is $G_{\mathrm{le}}(s) = K_c\left(\frac{s+1.1}{s+3.8}\right)$
11. Thus, the transfer function of the compensated system is $G_c(s) = \frac{K_c(s+1.1)}{s(s+1)(s+4)(s+3.8)}$.
12. Using the magnitude condition, $|G_c(s)|_{s=P_1} = 1$, the constant K_c is determined as

$$\frac{K_c(0.1+j1.73)}{(-1+j1.73)(j1.73)(3+j1.73)(2.8+j1.73)} = 1$$

$$K_c = 22.73.$$

Thus, the transfer function of the compensated system is

$$G_c(s) = \frac{22.73(s+1.1)}{s(s+1)(s+4)(s+3.8)}$$

11.5 Lag–Lead Compensator

In the previous sections, the unique advantages, disadvantages and limitations of the lead compensator and lag compensator have been discussed. Lead compensator will improve the rise time and damping and also affects natural frequency of the system and lag compensator will improve the damping of the system, but it also increases the rise time and settling time. Therefore, lag–lead compensator is a combination of lag and lead compensators which is used for a system to gain the individual advantages of each compensator. Also, the use of both the compensators is necessary for some systems as the desired result cannot be achieved when the compensators are used alone. In general, in lag–lead compensator, the lead compensator is used for achieving a shorter rise time and higher bandwidth and lag compensator is used for achieving good damping for a system.

The general transfer function of the lag–lead compensator is given by

$$G_{la-le}(s) = K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{1}{\alpha T_1}} \right) \left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right) = K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \right) \left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right) \tag{11.21}$$

where $\alpha, \beta, \gamma, T_1, T_2, K_c$ are constants and $\alpha < 1, \beta > 1, \gamma > 1, \alpha = \frac{1}{\gamma}$.

The term in Eqn. (11.21) which produces the effect of lead compensator is

$$\left(\frac{s+\frac{1}{T_1}}{s+\frac{1}{\alpha T_1}} \right) = \alpha \left(\frac{T_1 s+1}{\alpha T_1 s+1} \right)$$

The term in Eqn. (11.21) that produces the effect of lag compensator is

$$\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right) = \beta \left(\frac{T_2 s+1}{\beta T_2 s+1} \right)$$

The pole-zero configuration of the lag–lead compensator is shown in Fig. 11.15.

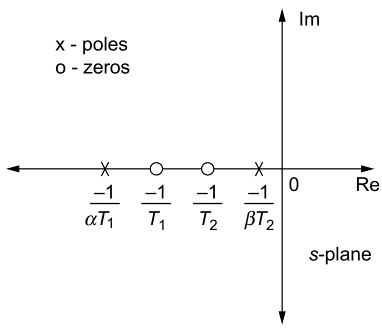
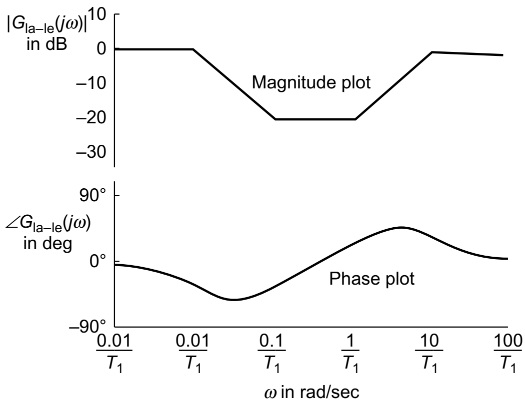
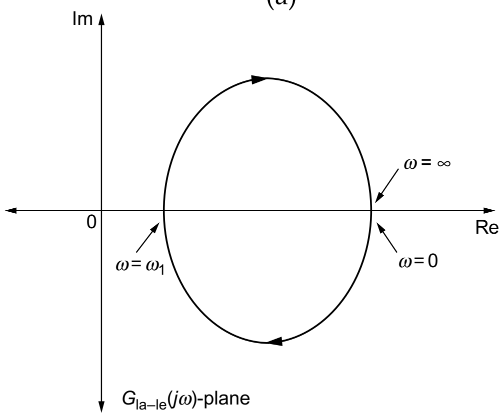


Fig. 11.15 | Pole-zero configuration of $G_{la-le}(s)$

The Bode plot and polar plot of the lag–lead compensator is shown in Figs. 11.16(a) and 11.16(b) respectively.



(a)



(b)

Fig. 11.16 | Plots of lag–lead compensator

The Bode plot shown in Fig. 11.16(a) is plotted with $\beta = 10$ and $\alpha = 0.1$. The conclusion inferred from the Bode plot of the lag compensator shown in Fig. 11.16(a) is that magnitude of lag–lead compensator is zero at low and high frequencies.

11.5.1 Electrical Representation of the Lag–Lead Compensator

A simple lag-lead compensator using resistor and capacitor is shown in Fig. 11.17.

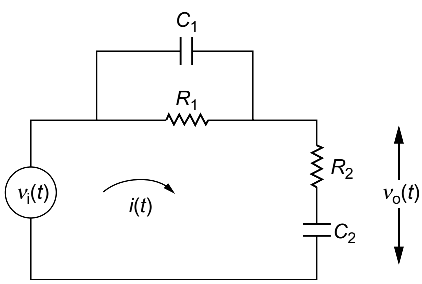


Fig. 11.17 | A simple lag–lead compensator

Applying Kirchoff's current law to the above circuit as shown in Fig. 11.17, we obtain

$$C_1 \frac{d\left(v_i(t) - v_0(t)\right)}{dt} + \frac{1}{R_1}\left(v_i(t) - v_0(t)\right) = i(t) \tag{11.21}$$

The output voltage of the circuit is given by

$$v_0(t) = R_2 i(t) + \frac{1}{C_2}\int i(t)\,dt \tag{11.22}$$

Taking Laplace transform on both sides of the above equations, we obtain

$$sC_1\left(V_i(s) - V_o(s)\right) + \frac{1}{R_1}\left(V_i(s) - V_0(s)\right) = I(s) \tag{11.23}$$

$$V_0(s) = \left(R_2 + \frac{1}{sC_2}\right) I(s)$$

$$I(s) = \frac{V_0(s)}{\left(R_2 + \frac{1}{sC_2}\right)} \tag{11.24}$$

Substituting Eqn. (11.24) in Eqn. (11.23), we obtain

$$sC_1\left(V_i(s) - V_o(s)\right) + \frac{1}{R_1}\left(V_i(s) - V_0(s)\right) = \frac{V_0(s)}{\left(R_2 + \frac{1}{sC_2}\right)}$$

$$\left(sC_1 + \frac{1}{R_1}\right)V_i(s) - \left(sC_1 + \frac{1}{R_1}\right)V_0(s) = \frac{V_0(s)}{\left(R_2 + \frac{1}{sC_2}\right)}$$

Simplifying the above equation, we obtain

$$\left\{\left(sC_1 + \frac{1}{R_1}\right)V_i(s) - \left(sC_1 + \frac{1}{R_1}\right)V_0(s)\right\}\left(R_2 + \frac{1}{sC_2}\right) = V_0(s)$$

$$\left(sC_1 + \frac{1}{R_1}\right)\left(R_2 + \frac{1}{sC_2}\right)V_i(s) = V_0(s)\left\{1 + \left(sC_1 + \frac{1}{R_1}\right)\left(R_2 + \frac{1}{sC_2}\right)\right\}$$

$$\frac{(sR_1C_1 + 1)(sR_2C_2 + 1)}{sR_1C_2}V_i(s) = V_0(s)\left\{1 + \frac{(sR_1C_1 + 1)(sR_2C_2 + 1)}{sR_1C_2}\right\}$$

Therefore, the transfer function of the above circuit is

$$\frac{V_0(s)}{V_i(s)} = \frac{(sR_1C_1 + 1)(sR_2C_2 + 1)}{sR_1C_2 + (sR_1C_1 + 1)(sR_2C_2 + 1)}$$

$$= \frac{(sR_1C_1 + 1)(sR_2C_2 + 1)}{s^2 R_1R_2C_1C_2 + s\left(R_1C_1 + R_2C_2 + R_1C_2\right) + 1}$$

Therefore,

$$\frac{V_0(s)}{V_i(s)} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}} \tag{11.25}$$

Rearranging Eqn. (11.21), we obtain

$$G_{la-le}(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{s^2+s\left(\frac{\gamma}{T_1}+\frac{1}{\beta T_2}\right)+\frac{\gamma}{\beta}T_1T_2} \tag{11.26}$$

Comparing Eqn. (11.26) with Eqn. (11.25), we get

$$T_1 = R_1C_1, \qquad T_2 = R_2C_2,$$

$$\frac{\gamma}{T_1}+\frac{1}{\beta T_2} = \frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1},$$

$$\frac{\gamma}{\beta}T_1T_2 = R_1R_2C_1C_2 \text{ and } \beta = \gamma.$$

From the above representation, it can be noted that the values of β and γ should be same for the lag–lead compensator.

11.5.2 Effects of Lag–Lead Compensator

The following are the effects of adding lag–lead compensator to the given system:

(i) Used when both the fast response and good static accuracy for a system is required.
(ii) Increases low-frequency gain that improves steady state.
(iii) Increases the bandwidth of the system that results in faster response of the system.

11.5.3 Design of Lag–Lead Compensator

The objective of designing the lag–lead compensator is to determine the values of constants (β, α, T_1 and T_2) present in the transfer function of a compensator for a uncompensated system $G_1(s)$ based on the desired system requirements. The design of lag–lead compensator can be either based on the frequency domain specifications or based on time-domain specifications. The Bode plot is used for designing the lag–lead compensator based on the frequency domain specifications and root locus technique is used for designing the lag-lead compensator based on the time-domain specifications. Once the constant is determined, the transfer function of the lag–lead compensator $G_{\text{la-le}}(s)$ can be obtained. The transfer function of the compensated system is $G_c(s) = G_{\text{la-le}}(s)G_1(s)$; and if the Bode plot or root locus technique is plotted for the compensated system, it will satisfy the desired system requirements.

11.5.4 Design of Lag–Lead Compensator Using Bode Plot

Consider the open-loop transfer function of the uncompensated system be $G(s)$. The objective is to design a lag–lead compensator $G_{\text{la-le}}(s)$ for $G(s)$ so that the compensated system $G_c(s)$

will satisfy the desired system requirements. The step by step procedure for determining the transfer function of the compensated system $G_{la-le}(s)$ using Bode plot is explained below:

Step 1: If the open-loop transfer function of the system $G(s)$ has a variable K, then $G_1(s) = G(s)$; otherwise $G_1(s) = KG(s)$.

Step 2: Depending on the input and TYPE of the system, the variable K present in the open-loop transfer function of the uncompensated system is determined based on either the steady-state error or the static error constant of the system.

The static error constants of the system are

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G_1(s)$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, sG_1(s)$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G_1(s)$

The relation between the steady-state error and static error constants based on the TYPE of the system and input applied to the system can be referred to Table 5.5 of Chapter 5.

Step 3: Construct the Bode plot for the uncompensated system with gain K obtained in the previous step and determine the frequency domain specifications of the system (i.e., phase margin, gain margin, phase crossover frequency and gain crossover frequency). Let the phase margin, gain margin, phase crossover frequency and gain crossover frequency of the uncompensated system be p_m, g_m, ω_{pc} and ω_{gc}.

Step 4: Let the desired phase margin of the system be $(p_m)_d$.

Step 5: Determine β using $\sin(p_m)_d = \dfrac{1-\alpha}{1+\alpha}$ and $\alpha\beta = 1$.

Step 6: Determine the frequency of the system at which the phase plot of the uncompensated system is $-180°$. Let the frequency be the new gain crossover frequency $(\omega_{gc})_{new}$.

Step 7: Determine the constant, $T_2 = \dfrac{10}{(\omega_{gc})_{new}}$.

Step 8: Determine the transfer function of the lag compensator as $\dfrac{(1+T_2s)}{(1+\beta T_2 s)}$.

Step 9: Determine the magnitude of uncompensated system at $(\omega_{gc})_{new}$. Let it be A dB.

Step 10: Draw a line from the point $((\omega_{gc})_{new}, A)$ so that it bisects the 0 dB line and –20 dB line and determine the frequencies corresponding to the intersection points. Let B rad/sec and C rad/sec be the frequencies at which the line intersects 0 dB line and –20 dB line respectively.

Step 11: Determine the constants T_1 and α using $T_1 = \dfrac{1}{C}$ and $\alpha T_1 = \dfrac{1}{B}$.

Step 12: Thus, the transfer function of the lag–lead compensator will be determined as

$$G_{la\text{-}le}(s) = K_c\left(\frac{s+\dfrac{1}{T_1}}{s+\dfrac{1}{\alpha T_1}}\right)\left(\frac{s+\dfrac{1}{T_2}}{s+\dfrac{1}{\beta T_2}}\right)$$

Step 10: The transfer function of the compensated system will be $G_c(s) = G(s)G_{la\text{-}le}(s)$. If the Bode plot for the compensated system is drawn, it will satisfy the desired system requirements.

Flow chart for designing the lag–lead compensator using Bode plot

The flow chart for determining the parameters present in the transfer function of the lag–lead compensator using Bode plot is shown in Fig. 11.18.

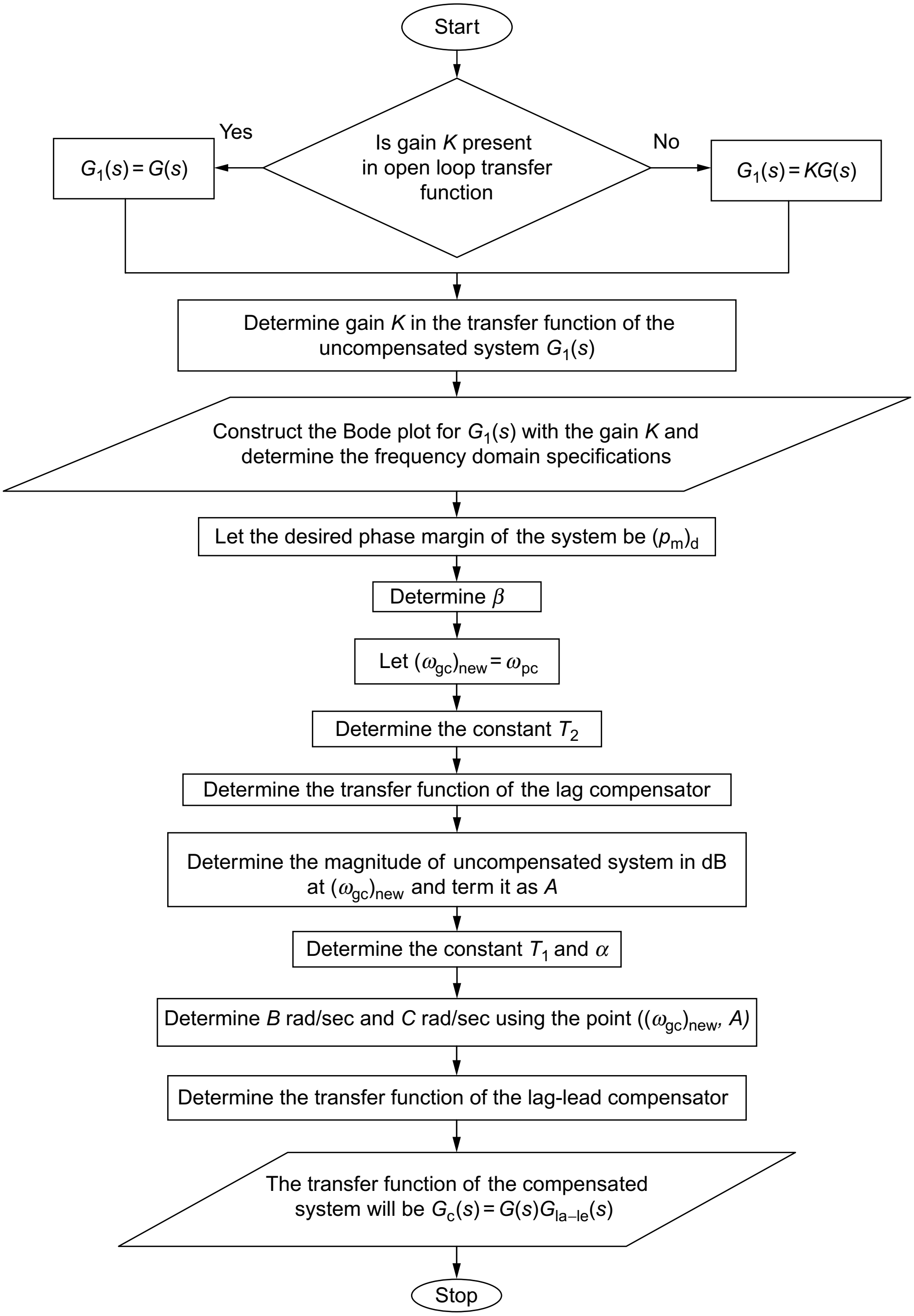


Fig. 11.18 | Flow chart for designing the lag–lead compensator using Bode plot

Example 11.5: **Consider a unity feedback uncompensated system with the open-loop transfer function as $G(s)=\dfrac{K}{s(s+1)(s+2)}$. Design a lag–lead compensator for the system such that the compensated system has static velocity error constant $K_v = 10\text{sec}^{-1}$, gain margin $g_m \geq 10$ dB and phase margin $p_m = 50°$.**

Solution:

1. Let $G_1(s)=G(s)=\dfrac{K}{s(s+1)(s+2)}$.
2. The gain K is determined by using $K_v = \underset{s\to 0}{Lt}\, sG_1(s)$ as
$$10 = \underset{s\to 0}{Lt}\, s\frac{K}{s(s+1)(s+2)}$$
Solving the above equation, we obtain $K = 20$.
3. The Bode plot for $G_1(s)=\dfrac{20}{s(s+1)(s+2)}$ $H(s) = 1$ is drawn.
   (a) Given $G_1(s)H(s)=\dfrac{20}{s(s+1)(s+2)}$
   (b) Substituting $s = j\omega$ in the above equation, we obtain
$$G_1(j\omega)H(j\omega)=\frac{10}{j\omega(1+j\omega)(1+j0.5\omega)}$$
   (c) The corner frequencies existing in the given system are
$$\omega_{c1} = 1 \text{ rad/sec and } \omega_{c2} = 2 \text{ rad/sec.}$$

To sketch the magnitude plot:

   (d) The changes in slope at different corner frequencies are given in Table E11.5(a).

Table E11.5(a) | Determination of change in slope at different corner frequencies

Term	Corner frequency ω rad/sec	Slope of the term in dB/decade	Change in slope in dB/decade
$\dfrac{1}{j\omega}$	–	–20	–
$\dfrac{1}{(1+j\omega)}$	$\omega_{c1} = 1$	–20	–20 – 20 = –40
$\dfrac{1}{(1+j0.5\omega)}$	$\omega_{c2} = 2$	–20	–40 – 20 = –60

(e) Assume the lower frequency as $\omega_l = 0.1$ rad/sec and higher frequency as $\omega_h = 20$ rad/sec.
(f) The gain at different frequencies are determined and given in Table E11.5(b).
(g) The magnitude plot of the given system is plotted using Table E11.1(b) and is shown in Fig. E11.5.

Table E11.5(b) | Gain at different frequencies

Term	Frequency	Change in slope in dB	Gain A_i
$\frac{10}{(j\omega)}$	$\omega_1 = \omega_l = 0.1$	–	$A_1 = 20\log\left(\frac{10}{\omega_1}\right) = 40$ dB
$\frac{10}{(j\omega)}$	$\omega_2 = \omega_{c1} = 1$	–	$A_2 = 20\log\left(\frac{10}{\omega_2}\right) = 20$ dB
$\frac{1}{(1+j\omega)}$	$\omega_3 = \omega_{c2} = 2$	–40	$A_3 = \left[-40\times\log\left(\frac{\omega_3}{\omega_2}\right)\right] + A_2 = 7.95$ dB
$\frac{1}{(1+j0.5\omega)}$	$\omega_4 = \omega_h = 20$	–60	$A_4 = \left[-60\times\log\left(\frac{\omega_4}{\omega_3}\right)\right] + A_3 = -52.05$ dB

To sketch the phase plot:

(h) The phase angle of the given loop transfer function as a function of frequency is obtained as

$$\phi = -90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.5\omega)$$

(i) The phase angle at different frequencies is obtained using the above equation and the values are tabulated as shown in Table E11.5(c).

Table E11.5(c) | Phase angle of the system for different frequencies

Frequency ω rad/sec	$-\tan^{-1}(\omega)$	$-\tan^{-1}(0.5\omega)$	Phase angle ϕ (in degree)
0.1	–5.71°	–2.86°	–98.5
1	–45°	–26.56°	–161.5
2	–63.4°	–45°	–198.4
∞	–90°	–90°	–270

(j) The phase plot of the given system is plotted with the help of Table E11.5(c) and is shown in Fig. E11.5.

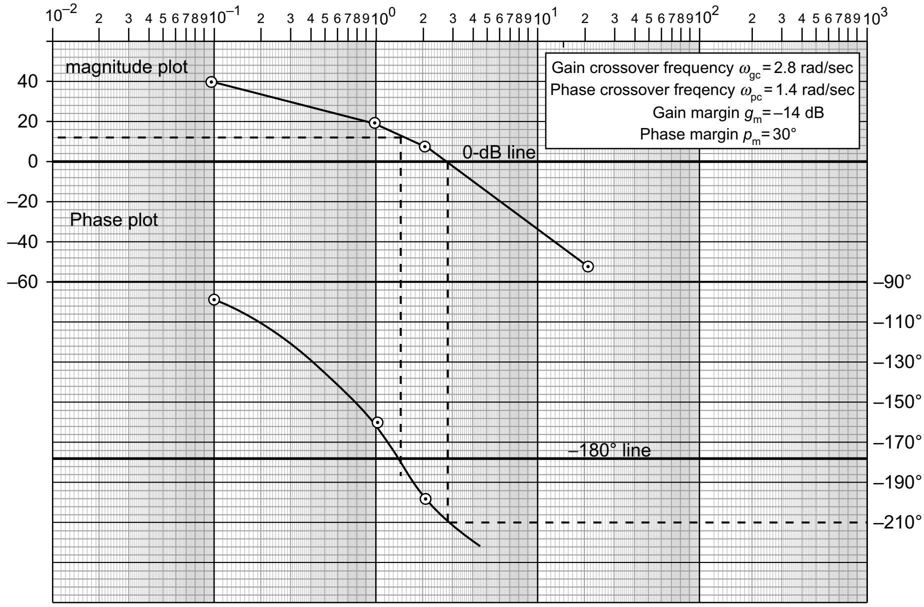


Fig. E11.5

(k) The frequency domain specifications of the given system are:
Gain crossover frequency, $\omega_{gc} = 2.8$ rad/sec.
Phase crossover frequency, $\omega_{pc} = 1.4$ rad/sec.
Gain margin, $g_m = -14$ dB
Phase margin, $p_m = -30^\circ$.

It is to be noted that the uncompensated system is stable, but the phase margin of the system is less than the desired phase margin which is 40°.

4. The desired phase margin for the compensated system is $(p_m)_d = 50^\circ$.
5. Using $\sin(p_m)_d = \dfrac{1-\alpha}{1+\alpha}$ and $\alpha\beta = 1$, β is determined as 7.54.
6. Let $(\omega_{gc})_{new} = \omega_{pc}$=1.4 rad/sec
7. The time constant of lag compensator is determined as

$$T_2 = \frac{10}{(\omega_{gc})_{new}} = \frac{10}{1.4} = 7.142 \text{ rad / sec}$$

8. Therefore, the transfer function of lag compensator is $G_{la}(s) = 7.54\dfrac{(1+7.142s)}{(1+53.85s)}$.
9. The magnitude of the system at $(\omega_{gc})_{new}$, $A = 14$dB.

10. If the slope of +20 dB/decade is drawn from (1.4 rad/sec, –14 dB), the slope cuts the 0 dB line and –20 dB line at $\omega = 7.5$ rad/sec and at $\omega = 0.75$ rad/sec respectively.
11. Thus, the transfer function of the lead portion of the lag–lead compensator is

$$G_{\text{le}}(s) = \frac{s+0.75}{s+7.5} = \frac{1}{10}\left(\frac{1+1.333s}{1+0.1333s}\right)$$

12. Therefore, the transfer function of the lag–lead compensator is

$$G_c(s) = G_{\text{la}}(s)G_{\text{le}}(s) = 0.754\frac{(1+1.333s)(1+7.142s)}{(1+0.1333s)(1+53.85s)}$$

11.5.5 Design of Lag–Lead Compensator Using Root Locus Technique

The transfer function of the lag–lead compensator is given by $G_{\text{la-le}}(s) = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\gamma}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}$, $\gamma > 1, \beta > 1$ and $\alpha = \frac{1}{\gamma}$. The design of lag–lead compensator using root locus technique is based on the relation between β and γ. Therefore, two different cases exist in designing the lag–lead compensator for a system.

Case 1: When $\beta \neq \gamma$

The step-by-step procedure for designing the lag–lead compensator using root locus technique when $\beta \neq \gamma$ is discussed below.

Step 1: Determine the damping ratio ξ and undamped natural frequency ω_n of the system based on desired time-domain specifications of the system.

Step 2: Determine the dominant pole of the system using $-\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$. Let the dominant poles of the system be P_1 and P_2.

Step 3: Follow the procedure shown in Fig. 11.14 in determining the constants of lead compensator, i.e., T_1 and γ.

Step 4: Determine the constant K_c using the condition $\left| K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right) G(s) \right|_{s=P_1} = 1.$

Step 5: Using the static error constant and constants determined in the previous steps, the value of β is determined.

Step 6: Determine T_2 using $\left| \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right|_{s=P_1} = 1$

Step 7: Now, determine the transfer function of the lag–lead compensator $G_{\text{la-le}}(s)$.

Step 8: The transfer function of the compensated system is $G_c(s) = G_{la-le}(s)G(s)$.

Flow chart for designing the lag–lead compensator using root locus technique when $\gamma \neq \beta$
The step-by-step procedure for designing the lag–lead compensator using root locus technique when $\gamma \neq \beta$ is shown in Fig. 11.19.

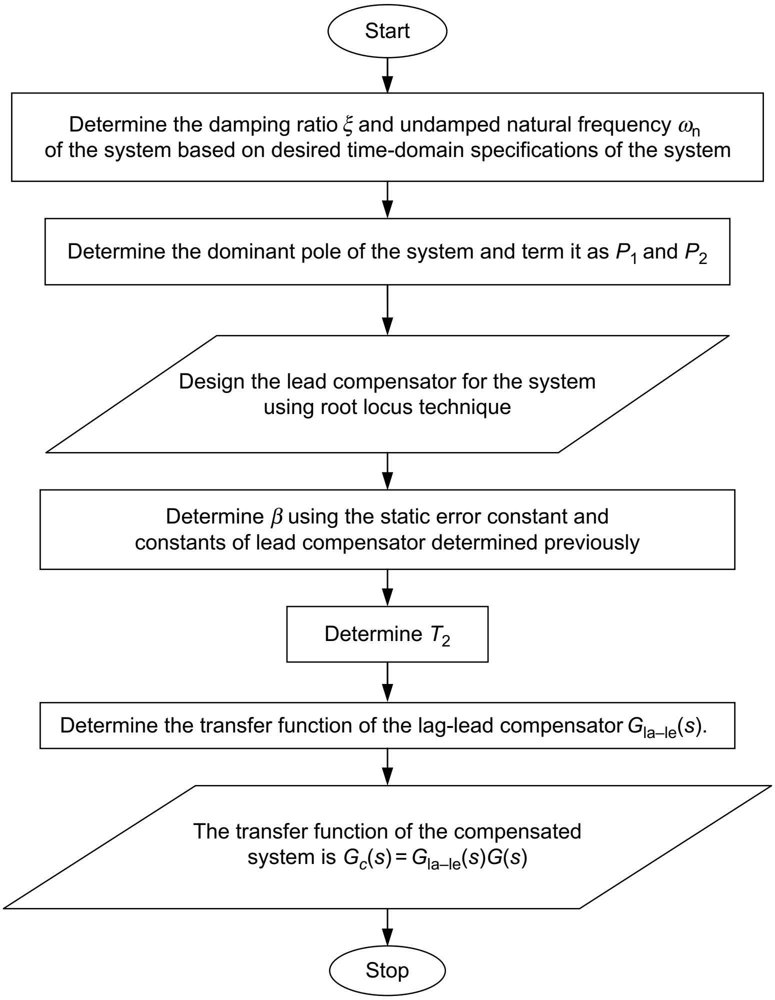


Fig. 11.19 | Flow chart for designing lag-lead compensator when $\gamma \neq \beta$

Case 2: When $\beta = \gamma$

The step-by-step procedure for designing the lag–lead compensator using root locus technique when $\beta = \gamma$ is discussed below:

Step 1: Determine the damping ratio ξ and undamped natural frequency ω_n of the system based on desired-time domain specifications of the system.

Step 2: Determine the dominant pole of the system using $-\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}$. Let the dominant poles of the system be P_1 and P_2.

Step 3: Using the formula for static error constant, determine the value of constant K_c.

Step 4: Determine the angle of the loop transfer function $G(s)H(s)$ at any one of the dominant pole, i.e., either P_1 or P_2. Let the angle obtained be M degrees.

Step 5: The angle obtained in the previous step, i.e., M should be an odd multiple of 180°. If it is not so some value of angle can be added or subtracted from M to make it an odd multiple of 180°. The angle by which M has to be added or subtracted is obtained using the formula, $\phi = 180° - M$. The value of ϕ is always less than 60° and if ϕ is greater than 60°, then multiple lead compensators can be used.

Step 6: Determine the length of points OA and OB such that $\left|\dfrac{\left(s+\dfrac{1}{T_1}\right)}{\left(s+\dfrac{\beta}{T_1}\right)}\right| = \dfrac{1}{\left|G(s)\right|_{s=P_1}} = \dfrac{P_1A}{P_1B}$ and $\angle AP_1B = M$. The points in the s-plane are shown in Fig. 11.20.

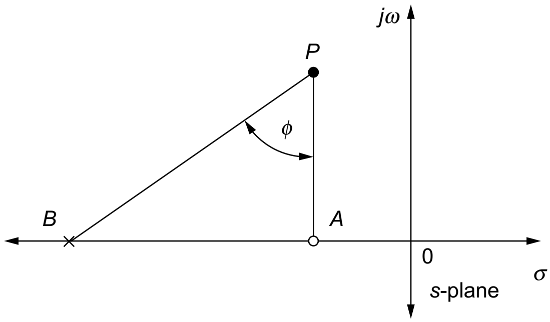


Fig. 11.20

Step 7: Determine the constants β and T_1 using the formula $T_1 = \dfrac{1}{|OA|}$ and $\dfrac{\beta}{T_1} = |OB|$.

Step 8: Determine T_2 using $\left|\dfrac{s+\dfrac{1}{T_2}}{s+\dfrac{1}{\beta T_2}}\right|_{s=P_1} = 1$.

Step 9: The transfer function of the lag–lead compensator is determined as

$$G_{\text{la-le}}(s) = K_c \frac{\left(s+\dfrac{1}{T_1}\right)\left(s+\dfrac{1}{T_2}\right)}{\left(s+\dfrac{\beta}{T_1}\right)\left(s+\dfrac{1}{\beta T_2}\right)}.$$

Flow chart for designing the lag–lead compensator using root locus technique when $\gamma = \beta$
The step-by-step procedure for designing the lag–lead compensator using root locus technique when $\gamma = \beta$ is shown in Fig. 11.21.

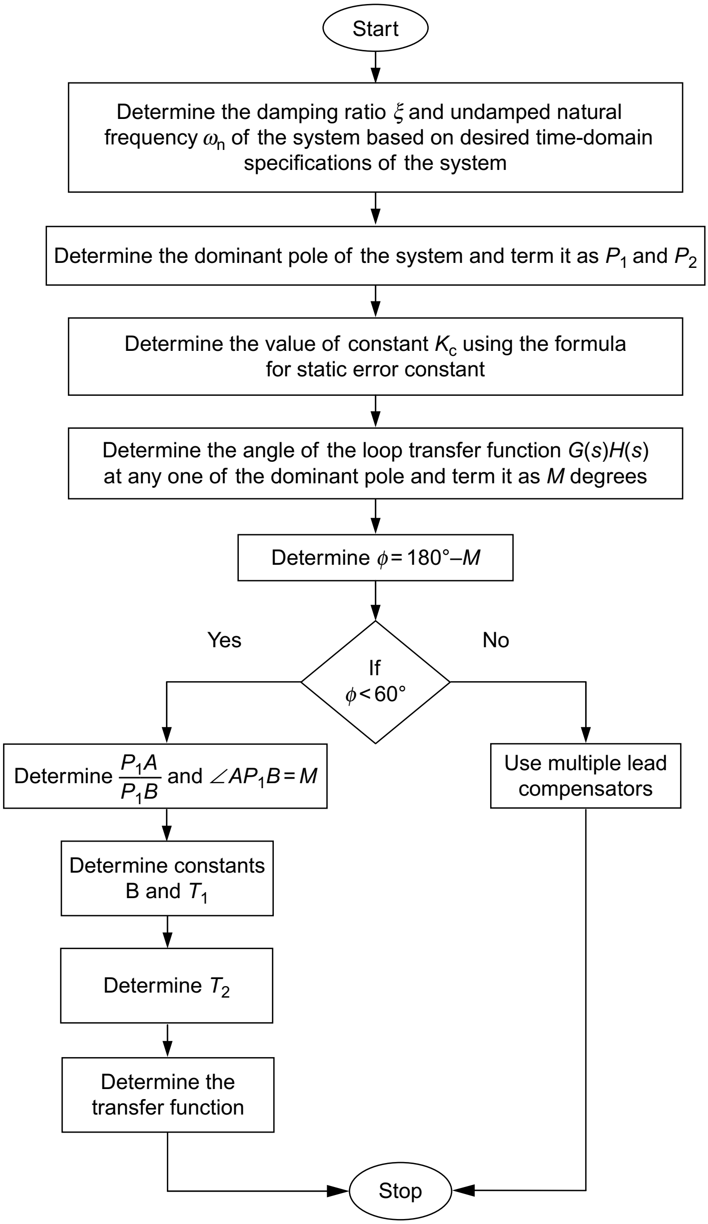


Fig. 11.21 | Flow chart for designing lag-lead compensator when $\gamma = \beta$

Review Questions

1. What is the principle of compensation? What are the types of compensation?
2. Draw the electric network that could be used for phase lag compensation. Derive its transfer function and hence obtain Bode plot.
3. Derive the transfer function of phase lag network and sketch the frequency response curves to the same.
4. Under what circumstances a lag compensator is preferred? Why?
5. What are the steps involved in designing a phase-lag compensator?
6. What are the advantages of the lag compensator?
7. Give the diagram to lead compensator.
8. How are lag and lead compensators realized using electric circuits?
9. Under what circumstances is a lead compensator preferred? Why?
10. Show how a phase lead compensation network can be designed.
11. Derive the transfer function of phase lead network and obtain the necessary expressions for system compensation with lead network.
12. Enumerate the design steps involved in the phase lead compensation.
13. Compare lead, lag and lag-lead compensation.
14. Write a short note on series compensation.
15. Explain the features of feedback compensation.
16. Write short notes on the feedback compensation.
17. Show the realization of a lag–lead compensator using electrical circuit.
18. Write a note on series compensation using lag-lead networks.
19. What is meant by feedback compensation?
20. Consider a TYPE 1 system with open loop transfer function $G(s) = \dfrac{K}{s(s+1)(s+4)}$ Design a lag compensator to meet the following specifications:
    Damping ratio = 0.707
    Settling time = 10 sec
    Velocity error constant $\geq 5\ \text{sec}^{-1}$
21. For a unity feedback system with open-loop gain $G(s) = \dfrac{K}{s(s+1)(s+4)}$ design a compensator to meet the following:
    Damping ratio = 0.5
    Settling time = 10 sec
    Velocity error co-efficient $\geq 5\ \text{sec}^{-1}$
22. Determine the transfer function of a lead compensator that will provide a phase lead of 50° and gain of 8 db at $\omega = 5$ rad/sec.
23. A unity feedback network system has $G(s) = \dfrac{K}{s(s+2)}$. Design a compensator network for the system so that the static velocity error constant $K_v = 20\ \text{sec}^{-1}$ and the phase margin is at least 45°.
24. Design a lead compensator for a unity-feedback open-loop transfer function $G(s) = \dfrac{K}{s(s+1)}$ to meet the following specifications:

Damping ratio $\xi = 0.7$
Settling time $t_s = 1.4\,\text{sec}$
and Velocity error constant $K = 2\,\text{sec}^{-1}$

25. Design a lag compensator for the system $G(s) = \dfrac{K}{s(s+1)(0.2s+1)}$ with unity feedback to achieve the following specifications:
Velocity error constant $K_v = 8\,\text{sec}^{-1}$
and phase margin $\approx 40^\circ$
26. A unity feedback system with the open-loop transfer function $G(s) = \dfrac{K}{s(1+0.3s)(1+0.5s)}$ is required to have (i) velocity error constant $K_v \geq 10\,\text{sec}^{-1}$ and (ii) phase margin $\geq 28^\circ$. Design a suitable phase lag compensator to meet the above specifications.
27. Explain the steps involved in the time-domain design of phase lead compensation.
28. The open-loop transfer function of a system is given as $G(s) = \dfrac{K}{s^2(s+1.5)}$. Design a lead compensation for the system to meet the following specifications:
(i) Setting time 4 sec and (ii) peak over shoot for step input $\leq$ 20 %.
29. A unity feedback system with forward path transfer function $G(s) = \dfrac{K}{s(1+0.3s)(1+0.5s)}$ is to have (i) the velocity error constant $K_v = 10\,\text{sec}^{-1}$ and (ii) phase margin $\geq 30^\circ$. Design a suitable phase-lag compensator to meet the above specifications.
30. A unity feedback system with $G(s) = \dfrac{4}{s(2s+1)}$ is to be compensated to have a phase margin of 40° without sacrificing the velocity error constant. Design a suitable lag network and compute the component values for a suitable impedance level.
31. Design a compensator for a unity feedback system with open-loop gain given by $G(s) = \dfrac{10}{s(s+2)(s+8)}$ so that the static velocity error constant is 80 sec^{-1} and the dominant closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$.
32. For the system whose open-loop transfer function is $G(s)H(s) = \dfrac{K}{s(s+1)(s+4)}$, design a compensator so that the damping ratio is 0.5 and undamped natural frequency is 2.
33. Design a compensator for the system with $G(s) = \dfrac{250}{s(1+0.1s)}$ to have a phase margin of 30°.

This Page is intentionally left blank.

12

PHYSIOLOGICAL CONTROL SYSTEMS

12.1 Introduction

Physiological and Engineering control system look similar to each other. A physiological control system explores some control problem related to biological environment and provides solution to the control researchers. The involvement of control theory makes the biomedical application more efficient and natural. The control mechanism in the biomedical application that comprises components that are necessary for maintenance of homeostasis at all levels of living systems is discussed in this chapter. The positive and negative feedback control mechanisms for maintaining homeostasis are also discussed.

12.2 Physiological Control Systems

The collection of trillions of cells composes the human body functions in a pattern that is essential for continuation of life process. In physiological control system the human body is considered as a system and the internal environment of the human body refers to each and everything present within the body where the cells live and work together. A specific optimum condition should be maintained within the body for proper functioning of cells which in turn makes the tissues, organs and the total system work properly. It is also necessary that the composition and fluid temperature around the cells must be constant for the cells to survive and function properly.

Homeostasis is the property of a system that helps in maintaining the consistency of the internal environment of the system. In addition, homeostasis can be defined as the property in which the variables of the system are regulated for maintaining a stable and relatively constant environment in the internal environment of the system despite the changes in the external environment. At the optimal temperature, when the internal environment of a system has optimal concentration of gases, nutrients, ions and water, the system is in homeostasis condition. The relationship between the human body and homeostasis is shown in Fig. 12.1. The human body will be operated in such a way that the homeostasis is maintained and it is essential for the survival.

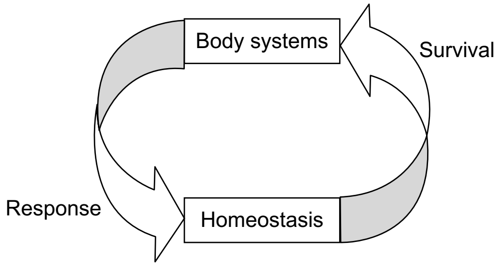


Fig. 12.1 | Relationship between human body and homeostasis

12.3 Properties of Physiological Control Systems

The properties of physiological control systems are:

(i) Physiological control system is corrected in such a way that the stability of the system is maintained.
(ii) Delay exists in correction of error sensed by a system.
(iii) The correction made by a system is proportional to the error.
(iv) Residual error exists even after the correction.
(v) The gain of a system is used for assessing the effectiveness of that system.

The characteristic curve of a physiological control system that depicts the properties of the system is shown in Fig. 12.2.

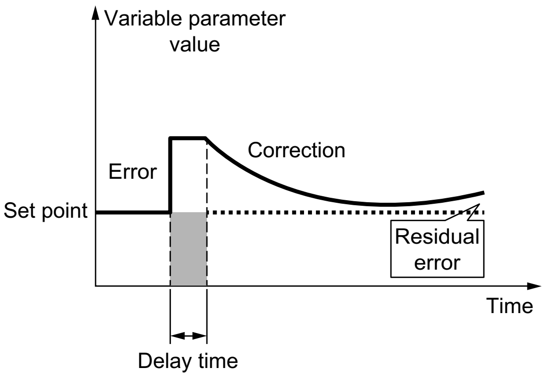


Fig. 12.2 | Characteristic curve of a control system

The special properties of a physiological control system are:

(i) A system has a variable gain that is used to control the oscillatory changes.
(ii) A system has a variable set point as the optimal value of a parameter varies for different conditions.

(iii) In certain conditions, a system anticipates the error and takes necessary action.
(iv) In a system, two or more control mechanisms are linked together for controlling a variable.

12.3.1 Target of the Homeostasis

The different systems in the human body—digestive system, circulatory system, respiratory system, excretory system, nervous system, endocrine system, immune system, musculo-skeletal system and integumentary system—help in their own way for maintaining the homeostasis in the internal environment. Therefore, the following points are the collective target of the human body or homeostasis:

(i) Maintaining adequate concentration of energy sources
(ii) Having adequate and continuous supply of oxidants
(iii) Limiting the concentration of waste materials
(iv) Sustaining optimum value of pH for proper functioning
(v) Maintaining concentration of water, salt and electrolytes in the system
(vi) Regulation of volume, pressure and temperature of the body within the sustainable range

12.3.2 Imbalance in the Homeostasis

All the organs present within a body act together through a combination of hormonal and nervous mechanisms for maintaining the homeostasis in the internal environment of the body. In day-to-day activities, the functions such as regulation of respiratory gases, protection against disease agents (pathogens), fluid and salt balance maintenance, regulation of energy and nutrient supply and maintenance of constant body temperature are regulated by the human body. In addition, when the human body gets injured, it must have the capability to repair itself. The following reasons cause an imbalance in the homeostasis:

(i) Due to external disturbance, the property of homeostasis gets disturbed and may cause diseases in the body.
(ii) Due to reduction in the efficiency of the system as the organism ages.
(iii) Resultant of unstable internal environment in the system due to reduction of efficiency which increases the risk of illness.
(iv) Physical changes associated with aging.

Some examples of homeostatic imbalances are high core temperature, increase in concentration of salt in the blood or decrease in concentration of oxygen which generate sensations such as warmth, thirst or breathlessness.

12.3.3 Homeostasis Control Mechanisms

The control mechanism used for maintaining the consistency of the internal environment is called homeostasis control mechanism. The homeostasis control mechanism is designed to re-establish homeostasis when there is an imbalance in the internal environment due to improper functioning of cells or due to external environment disturbances. The behaviour such as removal of sweater, drinking of water or slowing down the biological process will result in restoration of homeostatic imbalances that are listed in the previous section.

The homeostatic control mechanism should have at least three parts for restoring the homeostasis imbalances. The simple homeostatic control mechanism is shown in Fig. 12.3.

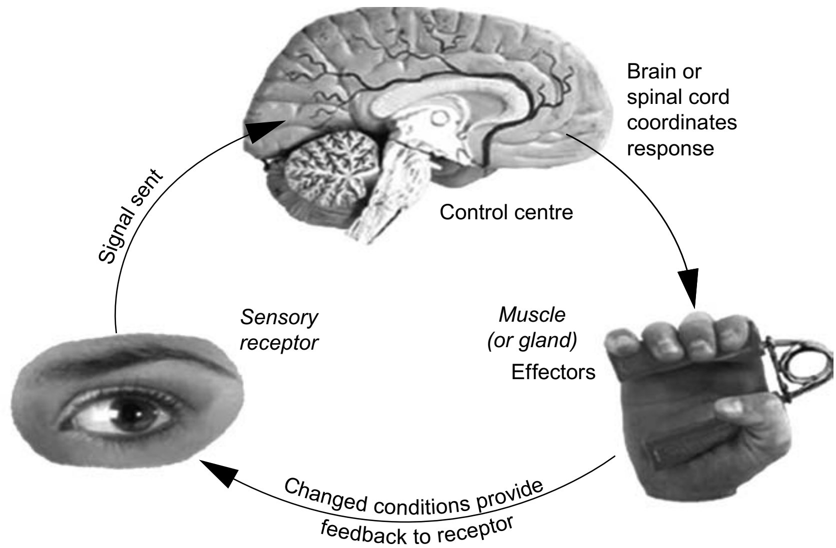


Fig. 12.3 | Homeostatic control mechanism

The three major parts that are available in the control mechanism as shown in Fig. 12.3 alongwith the functions are listed below:

1. Receptor: It is a sensing component that monitors and responds to changes in the environment and sends a stimulus to the control centre.
2. Control centre: It receives the information from receptor and sets the range at which a variable is maintained and sends an appropriate response for the stimulus to the effector.
3. Effector: It receives the signal from the control centre and corrects the homeostasis imbalances.

The homeostatic control mechanism can be clearly understood with the help of an example. If the human body (system) senses hyperglycaemia, i.e., high blood sugar level, it is a homeostasis imbalance and the control mechanism must take necessary actions to balance the blood sugar level. The three major parts required for correcting this imbalance are listed below alongwith their functions.

(a) Pancreas and beta cells—Sensor and control centre: It determines the sugar level and secretes insulin, i.e., stimulus to the blood.
(b) Liver and Muscle cells—Effector: Liver takes glucose from the blood, converts to glycogen and stores it. Muscle cells use this glycogen for contraction. Hence, the blood glucose level gets reduced.

For the above example, sensor and control centre are the same. Hence, only two parts exist in the control mechanism for maintaining the blood sugar level, i.e., homeostasis imbalance.

12.4 Block Diagram of the Physiological Control System

The block diagram representing a typical physiological control system is shown in Fig. 12.4.

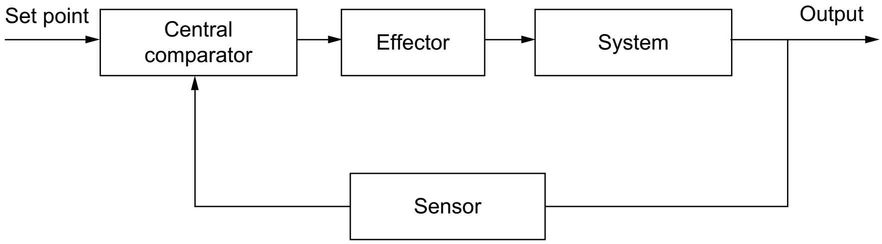


Fig. 12.4 | Block diagram of physiological control system

The set point is the pre-defined and pre-set value and the optimum value of a parameter. The sensor continuously monitors the value of the parameter and sends the real-time value to the control centre. The control centre compares the real time and pre-defined value of the parameter and generates an error signal and in turn generates a response signal, i.e., stimulus. Finally, the effector takes the signal generated from the control centre and makes the parameter value to be maintained at a pre-defined value. The example discussed in Section 12.3.3 can be represented in the form of a block diagram as shown in Fig. 12.5.

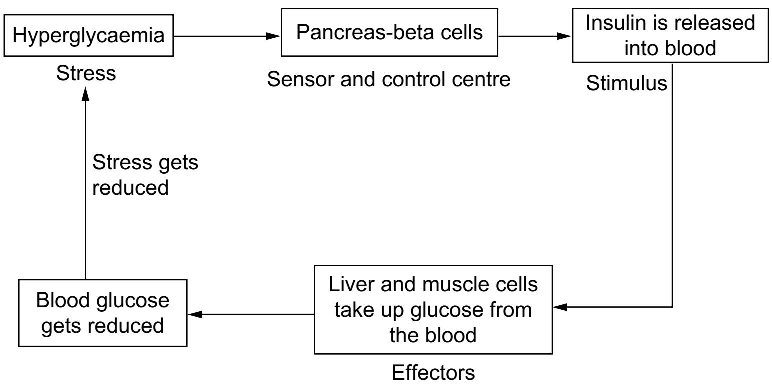


Fig. 12.5 | Physiological control system for maintaining blood sugar level

Any physical system has the natural tendency towards disorderliness and the control mechanism should work in such a way that the orderliness is maintained. Although the physiological control system performs to maintain the constancy of the homeostasis, the main objective is to achieve the steady-state value. The three methods by which the steady state can be achieved are feedback, hierarchical communication and adaptation.

If the external disturbances exist in the physiological control system, then it is called extrinsic; otherwise, it is called intrinsic. If the homeostatic control mechanism fails to restore the homeostasis imbalance in the internal environment when the external disturbance is high, then the living organism will engage in a behaviour such that the homeostasis imbalance is rectified.

12.4.1 Types of Control Mechanism

The control mechanism can be classified into different types based on the feedback mechanism. The two different types of feedback mechanism are positive feedback mechanism and negative feedback mechanism. The detailed description of these two types of mechanism alongwith their advantages and disadvantages is discussed here. In a positive feedback control mechanism, the set point and output values are added. In a negative feedback control mechanism, the set point and output values are subtracted.

Positive feedback mechanism

The positive feedback control mechanism is used by a very few organs in the system. Mostly, this is used for intensifying a change instead of correcting it. In positive feedback mechanism, the system acts in such a way that the imbalance or the perturbation increases. In addition, the positive feedback mechanism leads to a vicious cycle, i.e., when A produces more of B, B produces more of A. Further, the positive feedback mechanism acts in such a way that the stimulus gets intensified. In some cases, it is necessary to increase the stimulus. The general block diagram of positive feedback control mechanism is shown in Fig. 12.6.

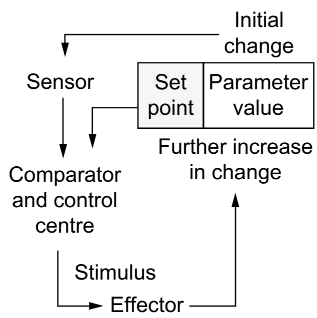


Fig. 12.6 | Positive feedback control system

The example of positive feedback mechanism is shown in Fig. 12.7, which regulates the child birth.

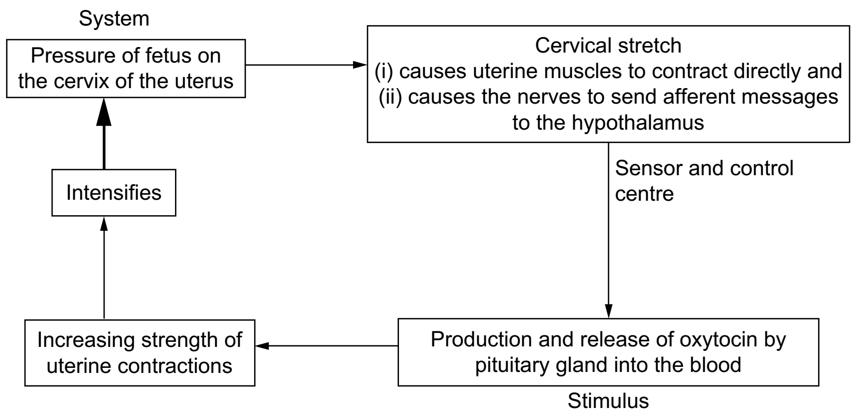


Fig. 12.7 | Positive feedback mechanism

Also, examples for the positive feedback mechanism that leads to harmful effects are shown below:

(i) When the homeostasis imbalance causes fever, a positive feedback in the control mechanism may increase the body temperature continuously. The continuous increase in temperature increases cellular proteins that cause metabolism to stop and results in death.

(ii) When the homeostasis imbalance called chronic hypertension exists in the system, it can favour the atherosclerosis process that causes the lumen of blood vessels to narrow. Then, it intensifies the hypertension and brings on more damage to the walls of blood vessels.

Negative feedback mechanism

The negative feedback mechanism is used in the system for reducing the homeostatic imbalance. Mostly, this is used for making the system self-regulatory so that the stability of the system can be achieved by reducing the effect of fluctuations. In negative feedback mechanism, right amount of corrective measure is applied in the most timely manner, which can make the system very stable, accurate and responsive. The general block diagram of negative feedback control mechanism is shown in Fig. 12.8.

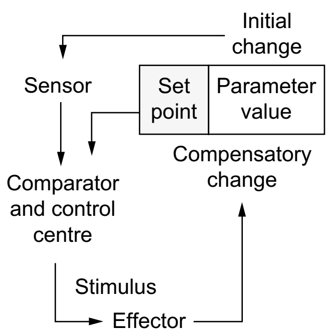


Fig. 12.8 | Negative feedback control system

The simplest and most fundamental of the physiological control system is the muscle stretch reflex. The knee jerk reflex, an example of muscle stretch reflex is used as a routine medical examination in assessing the state of the nervous system. When a sharp tap is given to the patellar tendon in the knee, it leads to an abrupt stretching of the thigh muscle to which patellar tendon is attached. The muscle spindles called stretch receptors get activated due to the stretch in the thigh muscle. The information about the magnitude of the stretch gets encoded by the neural impulses and is sent to the spinal cord of the system through the afferent nerve fibers. The different motor neurons connected to different afferent nerves get activated. This sends the efferent neural impulses back to the same thigh muscle that produces a contraction of the muscle and hence results in extension of knee joint. This concept of reflex is shown in Fig. 12.9.

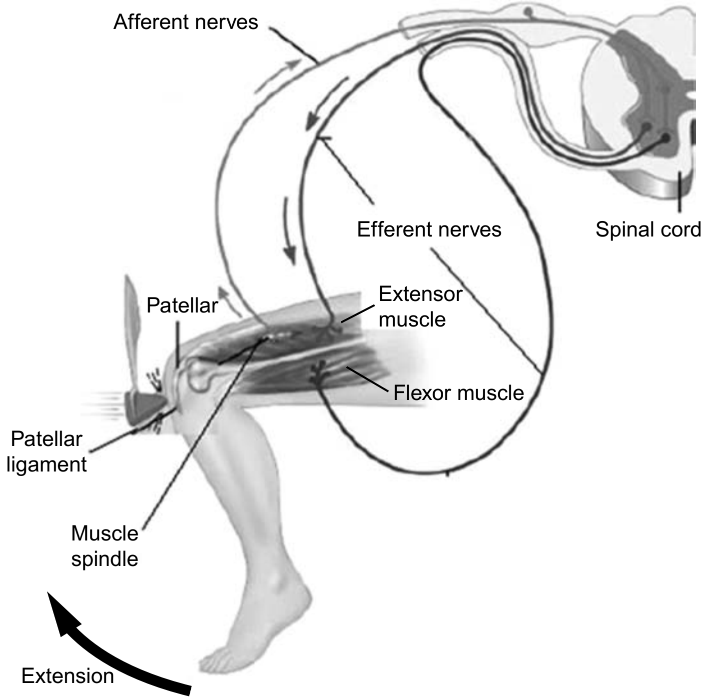


Fig. 12.9 | Concept of reflex

The block diagram representing this type of reflex is shown in Fig. 12.10. In this example, the thigh muscle corresponds to the plant or controlled system. The initial stretch produced by taping the knee is known as disturbance $x(t)$. The input of the feedback sensor will be the amount of stretch $y(t)$, which is produced in proportion to the $x(t)$. The feedback sensor, i.e., muscle spindle, translates $y(t)$ into an increase in afferent neural signal $z(t)$ sent to the

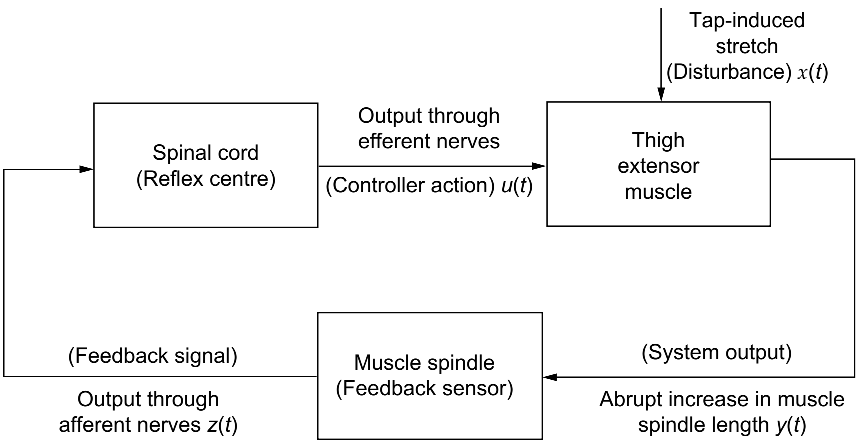


Fig. 12.10 | Block diagram representation of the muscle stretch reflex

reflex centre or the controller. The reflex centre in this system will be the spinal cord. The output of the controller will increase the efferent neural signal $u(t)$, which is directed back to the thigh muscle. This is an example of negative feedback system, since the initial tap-induced disturbance leads to a controller action that reduces the effect of the disturbance.
Some more examples of negative feedback mechanism are:

(i) When the body senses increased temperature in the surrounding, the thermoregulatory centre in the brain situated in hypothalamus makes the sweat glands on the skin to secrete sweat. This causes the body to lose heat and maintain homeostasis. On the other hand, if the body senses cold temperature; then the thermoregulatory centre causes the body to shiver and thus increases the body temperature. This causes the body to generate more heat and maintain homeostasis by negative feedback mechanism.
(ii) When there is increase in blood pressure, there are certain sensors called baro-receptors in the carotid arteries which sense this increase. This information is carried to the centres in medulla oblongata of brain stem. The response is in the form of vasodilatation in the peripheral blood vessels which reduces the peripheral resistance. The result is that the blood pressure is reduced and homeostasis is maintained.

Advantages
The advantages of negative feedback mechanism are:

(i) The negative feedback mechanism is more stable than positive feedback mechanism.
(ii) Most useful control type since it helps in reaching the equilibrium state, whereas positive feedback can lead a system away from an equilibrium, which makes the system unstable.

12.5 Differences Between Engineering and Physiological Control Systems

The differences between engineering and physiological control systems are given in Table 12.1.

Table 12.1 | Comparison of engineering control system with physiological control system

S. No.	Engineering control system	Physiological control system
1.	Single or multiple pre-defined tasks can be achieved	It is versatile and capable of achieving different tasks
2.	Easier to fine-tune the governing parameters to achieve the task	Difficult to tune the governing parameters that help in achieving the task
3.	Characteristics of various components of the system are known	Characteristics of various components of the system are unknown and difficult to analyse
4.	Cross-coupling or interaction between different systems is less	Extensive degree of cross-coupling or interaction between different systems
5.	Engineering control systems are not adaptive in nature	Physiological control systems are adaptive in nature

(Continued)

Table 12.1 (Continued)

S. No.	Engineering control system	Physiological control system
6.	Clear identification between negative and positive feedback mechanisms is possible with the help of comparator circuit	Identification between positive and negative feedback mechanisms is difficult
7.	Engineering control systems may be linear or non-linear	Physiological control systems are non-linear

12.6 System Elements

The different system elements are resistance capacitance and inductance. The definitions of these elements are obtained with the help of electrical circuits and then analogous to different systems are discussed.

12.6.1 Resistance

The Ohm's law used for determining the electrical resistance R is given by

$$V = RI \tag{12.1}$$

where V is the voltage or driving potential across the resistor and I represents the current flowing through it. In physiological control system, V is taken as the "*through*" variable, called a measure of "*effort*" and denoted by ψ and I is taken as the "*through*" variable, called a measure of "*flow*" and denoted by ζ.

Therefore, in physiological control system, resistance is obtained as:

$$\psi = R\zeta \tag{12.2}$$

12.6.2 Capacitance

In electrical system, the generalized system property for storage is taken in the form of capacitance, which is defined as the amount of electrical charge (q) stored in the capacitor per unit voltage (V). Here, the relation between voltage and current is given by

$$V = \frac{1}{C}\int_0^t I \;\, dt \tag{12.3}$$

Thus, in physiological control system, we obtain

$$\Psi = \frac{1}{C}\int_0^t \zeta\, dt \tag{12.4}$$

12.6.3 Inductance

The final property of electrical system that stores kinetic energy is inductance. The relation between the voltage and current relating the inductance is given by

$$V = L\frac{dI}{dt} \tag{12.5}$$

Thus, in physiological control system, we obtain

$$\psi = L\frac{d\zeta}{dt} \tag{12.6}$$

The analogous of physiological system with other systems such as electrical, mechanical, fluid, thermal and chemical are shown in Table 12.2.

Table 12.2 | Analogy of variables in physiological system with other systems

Physiological system	Electrical system	Mechanical system	Fluid system	Thermal system	Chemical system
Effort ψ	Voltage V	Force F	Pressure difference ΔP	Temperature difference $\Delta\theta$	Concentration difference $\Delta\phi$
Flow ζ	Current I	Velocity v	Flow of fluid Q	Flow of heat Q	Flux Q
Generalized resistance R	Electrical resistance R	Mechanical resistance R_M	Fluid resistance R_f	Thermal resistance R_t	Diffusion resistance R_c
Generalized capacitance C	Electrical capacitance C	Compliance $k = \dfrac{1}{\text{Spring constant}}$	Elasticity C_t	Thermal capacitance C_t	Concentration
Charge	Electric Charge q	Displacement x	Volume V	Heat stored q	Volume of the fluid V

12.7 Properties Related to Elements

The Kirchoff's voltage law and current law govern the electrical system. Therefore, by applying Kirchoff's voltage law for physiological system shown in Fig. 12.11, we obtain

$$(\psi_a - \psi_b) + (\psi_b - 0) + (0 - \psi_a) = 0 \tag{12.8}$$

Similarly, applying Kirchoff's current law for physiological system shown in Fig. 12.11, we obtain

$$\zeta_1 + (-\zeta_2) + (-\zeta_3) = 0 \tag{12.9}$$

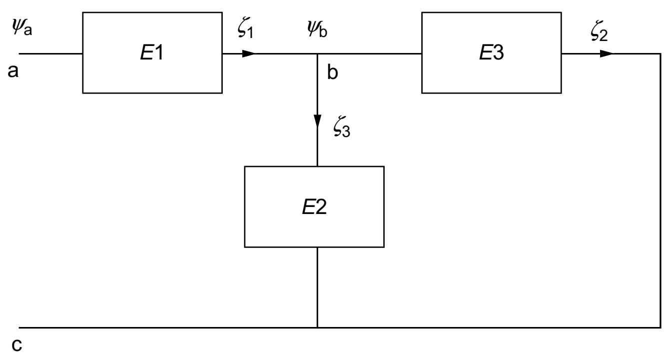


Fig. 12.11 | Generalized system elements network model

The elements connected in series: The resultant or equivalent value of the elements connected in series is calculated similar to the calculation used in electrical system.

Resistors in series: When the resistors are connected in series as shown in Fig. 12.12(a), the equivalent resistance is the summation of individual resistance values, i.e., $R_{eq} = R_1 + R_2$

Capacitors in series: When the capacitors are connected in series as shown in Fig. 12.12(b), the equivalent capacitance is calculated as:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

The elements connected in parallel: The equivalent value of the elements connected in parallel are calculated similar to the calculation used in electrical system.

Resistors in parallel: When the resistors are connected in parallel as shown in Fig. 12.12(c), the equivalent resistance is calculated as:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Capacitors in parallel: When the capacitors are connected in parallel as shown in Fig. 12.12(d), the equivalent capacitance is determined as the summation of individual capacitance values, i.e., $C_{eq} = C_1 + C_2$

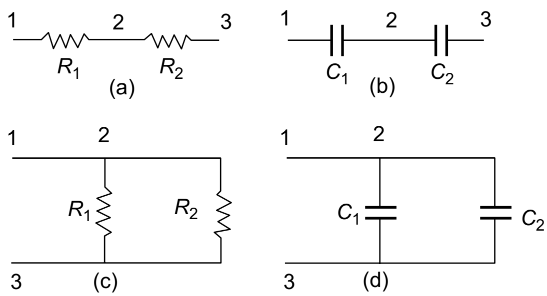


Fig. 12.12 | Model properties of system elements in physiological system: (a) resistors connected in series, (b) capacitors connected in series, (c) resistors connected in parallel and (d) capacitors connected in parallel

12.8 Linear Models of Physiological Systems: Two Examples

The mathematical models are derived in this section, which characterize the input–output properties of two simple physiological models: lung mechanism and skeletal muscle.

12.8.1 Lung Mechanism

The working of lung mechanism should be studied before deriving the mathematical model of the system. The process of modelling the lung mechanism is explained below.

Airways are divided into two categories: the larger or central airways and the smaller or peripheral airways. The fluid mechanical resistances of the two categories of airways are R_C and R_P respectively. The chest wall cavity gets expanded by the volume of air entering the alveoli. The chest wall gets expanded by the same amount of air entering the lungs. This relation is given by the series connection of lung and chest wall compliances represented by C_L and C_W respectively. However, due to the presence of compliance of central airways and gas compressibility, a small fraction of volume of air is shunted away from the alveoli represented by shunt compliance, C_S. In normal circumstances, this shunted volume is very small and if R_P is increased or if C_L or C_W is decreased, it may lead to disease. This effect is taken into consideration by placing C_S in parallel with C_L and C_W. The lung model developed with the help of these variables is shown in Fig. 12.13.

Let the pressures developed at different points of lung model shown in Fig. 12.13 be P_{ao} at the airway opening, P_{aw} at the central airways, P_A in the alveoli and P_{p1} in the pleural space. The reference pressure or the ambient pressure is taken as P_0 which can be set to zero. If the flow rate of air entering the respiratory system is Q, the flow delivered to alveoli is Q_A, then the flow shunted away from alveoli be $Q - Q_A$ and also a mathematical relationship between P_{ao} and Q can be derived by as follows:

Using Kirchhoff's law to the circuit shown in Fig. 12.13,

$$R_P Q_A + \left(\frac{1}{C_L} + \frac{1}{C_W}\right)\int Q_A\,dt = \frac{1}{C_S}\int (Q - Q_A)\,dt \tag{12.10}$$

and

$$P_{ao} = R_C Q + \frac{1}{C_S}\int (Q - Q_A)\,dt \tag{12.11}$$

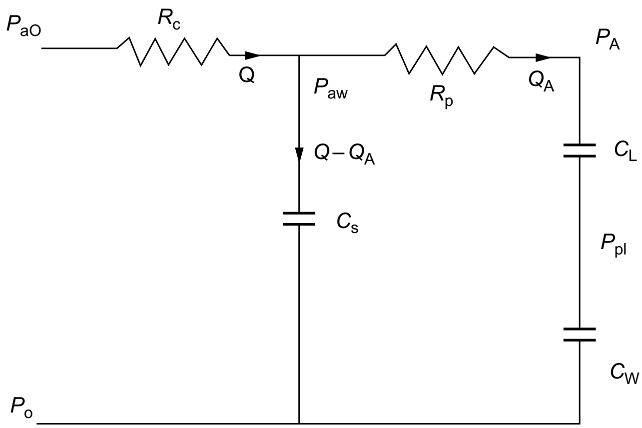


Fig. 12.13 | Linear model of respiratory mechanics

Differentiating Eqn. (12.10) with respect to t, we obtain

$$R_P \frac{dQ_A}{dt} + \left(\frac{1}{C_L} + \frac{1}{C_W}\right) Q_A = \frac{Q}{C_S} - \frac{Q_A}{C_S}$$

Rearranging the above equation, we obtain

$$R_P \frac{dQ_A}{dt} + \left(\frac{1}{C_L} + \frac{1}{C_W} + \frac{1}{C_S}\right) Q_A = \frac{Q}{C_S}$$

Substituting $\frac{1}{C_T} = \frac{1}{C_L} + \frac{1}{C_W} + \frac{1}{C_S}$ in the above equation, we obtain

$$R_P \frac{dQ_A}{dt} + \frac{1}{C_T} Q_A = \frac{Q}{C_S} \tag{12.12}$$

Differentiating Eqn. (12.11) with respect to t, we obtain

$$\frac{dP_{ao}}{dt} = R_C \frac{dQ}{dt} + \frac{1}{C_S}(Q - Q_A) \tag{12.13}$$

Differentiating Eqn. (12.13) with respect to t, we obtain

$$\frac{d^2P_{ao}}{dt^2} = R_C \frac{d^2Q}{dt^2} + \frac{1}{C_S}\left(\frac{dQ}{dt} - \frac{dQ_A}{dt}\right) \tag{12.14}$$

Using Eqn. (12.12), we get

$$\frac{dQ_A}{dt} = \left(\frac{Q}{C_S} - \frac{Q_A}{C_T}\right) \times \frac{1}{R_P} \tag{12.15}$$

Substituting Eqn. (12.15) in Eqn. (12.14), we obtain

$$\frac{d^2P_{ao}}{dt^2} = R_C \frac{d^2Q}{dt^2} + \frac{1}{C_S}\frac{dQ}{dt} - \left(\frac{Q}{C_S} - \frac{Q_A}{C_T}\right) \times \frac{1}{C_S R_P}$$

$$= R_C \frac{d^2Q}{dt^2} + \frac{1}{C_S}\frac{dQ}{dt} - \frac{Q}{C_S{}^2 R_P} + \frac{Q_A}{C_S C_T R_P} \tag{12.16}$$

Using Eqn. (12.13), we obtain

$$Q_A = \left[\frac{-dP_{ao}}{dt} + R_C \frac{dQ}{dt} + \frac{Q}{C_S}\right] C_S \tag{12.17}$$

Substituting Eqn. (12.17) in Eqn. (12.16), we obtain

$$\frac{d^2P_{ao}}{dt^2} = R_C \frac{d^2Q}{dt^2} + \frac{1}{C_S}\frac{dQ}{dt} - \frac{Q}{C_S{}^2 R_P} + \frac{1}{C_T R_P}\left[\frac{-dP_{ao}}{dt} + R_C \frac{dQ}{dt} + \frac{Q}{C_S}\right]$$

Solving the above equation, we obtain

$$\frac{d^2P_{ao}}{dt^2} + \frac{1}{C_T R_P}\frac{dP_{ao}}{dt} = R_C \frac{d^2Q}{dt^2} + \left[\frac{1}{C_S} + \frac{R_C}{C_T R_P}\right]\frac{dQ}{dt} + \frac{1}{C_S R_P}\left[\frac{1}{C_L} + \frac{1}{C_W}\right]Q \tag{12.18}$$

where $\frac{1}{C_T} = \frac{1}{C_L} + \frac{1}{C_W} + \frac{1}{C_S}$

The model of the lung mechanism is represented by Eqn. (12.18).

12.8.2 Skeletal Muscle

The linearized physiological model of skeletal muscle in the human body is shown in Fig. 12.14.

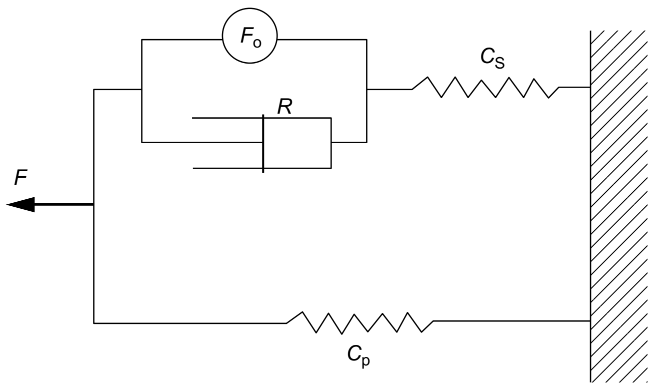


Fig. 12.14 | Model of a skeletal muscle

Let F_0 and F represent the force developed by the active contractile element of the muscle and actual force developed due to mechanical properties of muscle respectively. The viscous damping inherent in the tissue is represented by R and the elastic storage properties of sarcolemma and muscle tendons are represented as C_P and C_S respectively. The different mechanical constraints that exist in the system shown in Fig. 12.14 are:

(i) The entire series combination of R and C_S gets stretched by a displacement $x(t)$ if the element C_P is also stretched with the same displacement.
(ii) The sum of total force applied to both the branches of the configuration is F.
(iii) The individual length extension of elements C_S and R will not be equal, but the summation of the extension will be equal to $x(t)$.

Thus, by assuming the above constraints, if C_S is stretched to a length x_1, the extension in the parallel combination of R and F_0 will be $\left(x(t)-x_1(t)\right)$. The velocity with which the dashpot R extended is given by $\frac{d\left(x(t)-x_1(t)\right)}{dt}$. Also, the force transmitted through the element C_S and through the parallel combination of F_0 and R will be equal. Using this principle, we obtain

$$\frac{x_1(t)}{C_S} = R\left(\frac{dx(t)}{dt} - \frac{dx_1(t)}{dt}\right) + F_0 \tag{12.19}$$

As the summation of total force from both limbs of the parallel combination must be equal to F, we have

$$F = \frac{x_1(t)}{C_S} + \frac{x(t)}{C_P} \tag{12.20}$$

Differentiating Eqn. (12.20), we obtain

$$\frac{dF}{dt}=\frac{1}{C_S}\frac{dx_1(t)}{dt}+\frac{1}{C_P}\frac{dx(t)}{dt} \tag{12.21}$$

Using Eqn. (12.19), we obtain

$$\frac{dx_1(t)}{dt}=\left[-\frac{x_1(t)}{C_S}+R\frac{dx(t)}{dt}+F_0\right]\times\frac{1}{R} \tag{12.22}$$

Substituting Eqn. (12.22) in Eqn. (12.21), we obtain

$$\frac{dF}{dt}=\frac{1}{RC_S}\left[F_0+R\frac{dx(t)}{dt}-\frac{x_1(t)}{C_S}\right]+\frac{1}{C_P}\frac{dx(t)}{dt}$$

Simplifying the above equation, we obtain

$$\frac{dF}{dt}=\frac{F_0}{RC_S}+\left[\frac{1}{C_S}+\frac{1}{C_P}\right]\frac{dx(t)}{dt}-\frac{x_1(t)}{RC_S{}^2} \tag{12.23}$$

Using Eqn. (12.20), we obtain

$$x_1(t)=\left[F-\frac{x(t)}{C_P}\right]C_S \tag{12.24}$$

Substituting Eqn. (12.24) in Eqn. (12.23), we obtain

$$\frac{dF}{dt}=\frac{F_0}{RC_S}+\left[\frac{1}{C_S}+\frac{1}{C_P}\right]\frac{dx(t)}{dt}-\frac{1}{RC_S}\left[F-\frac{x(t)}{C_P}\right] \tag{12.25}$$

Simplifying the above equation, we obtain

$$\frac{dF}{dt}+\frac{F}{RC_S}=\frac{F_0}{RC_S}+\left[\frac{1}{C_S}+\frac{1}{C_P}\right]\frac{dx(t)}{dt}+\frac{1}{RC_SC_P}x(t) \tag{12.26}$$

The above equation represents the mathematical model of the skeletal muscle.

12.9 Simulation—MATLAB and SIMULINK Examples

In order to simplify the mathematical characterization of linear systems, models that provide adequate representations of realistic dynamical behaviour are too complicated to deal with analytically. In such complex situations, the logical approach is to translate the system block representation into a computer model and to solve the corresponding problem numerically. A variety of software tools are available that further simplify the task of model simulation and analysis. SIMULINK is currently used by a large segment of the scientific and engineering community. SIMULINK provides a graphical environment allowing the user to easily convert a block diagram into a network of blocks of mathematical functions. It runs within the interactive, command-based environment called MATLAB, wherein MATLAB and SIMULINK are products of The MathWorks, Inc.

To find out how much tidal volume is delivered to a patient in the intensive care unit when the peak pressure of a ventilator is set at a prescribed level, knowledge of the patient's lung mechanics is needed. Assuming the patient to have a normal mechanics, the values of the various pulmonary parameters are (refer to Fig. 12.13) as follows:

$$R_C = 1\ cmH_2O_sL^{-1},\ R_P = 0.5\ cmH_2O_sL^{-1},$$

$$C_L = 0.2\ LcmH_2O^{-1},\ C_W = 0.2\ LcmH_2O^{-1} \text{ and } C_s = 0.0005\ LcmH_2O^{-1}$$

The first method to solve this problem is to derive the transfer function for the overall system and use it as a single "block" in the SIMULINK program. The differential equation relating total airflow Q to the applied pressure at the airway opening P_{ao} has been derived using Kirchhoff's laws and presented in Eqn. (12.12). Substituting the above parameter values into this differential equation and taking Laplace transform and rearranging, we obtain

$$\frac{Q(s)}{P_{ao}(s)} = \frac{s^2 + 420s}{s^2 + 620s + 4000} \tag{12.27}$$

To implement the above mode, run SIMULINK from within the MATLAB command window (i.e., type simulink at the MATLAB prompt). The SIMULINK main block library will be displayed in a new window.
Steps to follow:

(i) Click the File menu and select *New*. This will open another window named *Untitled*. This is the working window in which we will build our model. The main SIMULINK menu displays several libraries of standard functions selected for model development. In this case, a block representing the transfer function shown in Eqn. (12.27) is chosen.
(ii) Open the *Linear Library* (by double clicking this block): this will open a new window containing a number of block functions in this library.
(iii) Select *Transfer Fun* and drag this block into the working window.
(iv) Double click this block to input the parameters of the transfer function.
(v) Enter the coefficients of the polynomials in s in the numerator and denominator of the transfer function in the form of a row vector in each case. Thus, the coefficients for the numerator will be [1 420 0]. These coefficients are ordered in descending powers of s. Since, the constant term does not appear in the numerator of Eqn. (12.27), it is still necessary to include explicitly as zero the "coefficient" corresponding to this term.
(vi) Once the transfer function block has been set up, the next step is to include a generating source for P_{ao}. This can be found in the *Sources* library.
(vii) Select and drag the *sine wave* function generator into the working window.
(viii) Double click this icon in order to modify the amplitude and frequency of the *sine wave*:
(ix) Set the amplitude to 2.5 cm H_2O (i.e., peak-to-peak swings in P_{ao} will be 5 cmH_2O), and the frequency to 1.57 radian/s, which corresponds to $1.57/(2.\pi) = 0.25H_z$ or 15 breaths per min.

(x) Connect the output port of the sine wave block to the input port of the transfer function block with a line.
(xi) To view the resulting output Q, open the *Sinks* library and drag a *Scope* block into the working window.
(xii) Double click the Scope block and enter the ranges of the horizontal (time) and vertical axes. It is always useful to view the input simultaneously.
(xiii) Drag another Scope block into the working window and connect the input of this block to the line that "transmits" P_{ao}. The model is now complete and one can proceed to run the simulation. However, it is useful to add a couple of features. The results in terms of volume delivered to the patient can be viewed. This is achieved by integrating Q.
(xiv) From the *Linear library*, select the *integrator* block. Send Q into the input of this block and direct the output (Vol) of the block to a third *Scope*. Save the results of each simulation run into a data file that can be examined later.
(xv) From the *Sinks* library, select and drag the *To File* block into the working window. This output file will contain a matrix of numbers. Assign a name to this output file as well as the matrix variable. Name the file as *respm1.mat* and the matrix variable as *respm1*. Note that *mat* files are the standard (binary) format in which MATLAB variables and results are saved. The first row of matrix *respm1* will contain the times that correspond to each iteration step of the simulation procedure. Save the time histories of P_{ao}, Q and Vol.
(xvi) Thus, the *To File* block will have to be adjusted to accommodate three inputs. Since this block expects the input to be in the form of a vector, the three scalar variables will have to be transformed into a three-element vector prior to being sent to the *To File* block.
(xvii) This is achieved with the use of the *Mux* block, found in the connections library. After dragging it into the working window, double click the *Mux* block and enter 3 for the number of inputs. Upon exiting the Mux dialog block, note that there are now three input ports on the Mux block.
(xviii) Connect the Mux output with the input of the *To File* block.
(xix) Connect the input ports of the *Mux* block to P_{ao}, Q and Vol. The block diagram of the completed SIMULINK model is shown in Fig. 12.15.
(xx) The final step is to run the simulation. Go to the *Simulation* menu and select *Parameters*. This allows the user to specify the duration over which the simulation will be conducted, as well as the minimum and maximum step sizes for each computational step. The latter will depend on the dynamics of the system in question. In this case, choose both time steps as 0.01 s. The user can also select the algorithm for performing integration. Again, the particular choice depends on the problem. The user is encouraged to experiment with different algorithms for a given problem. Some algorithms may produce numerically unstable solutions, while others may not.
(xxi) Finally, double click *Start* in the *Simulation* menu to proceed with the simulation.

Figure 12.16 shows sample simulation results produced by the model implementations. The ventilator generates a sinusoidal P_{ao} waveform of amplitude 2.5 cmH_2O.

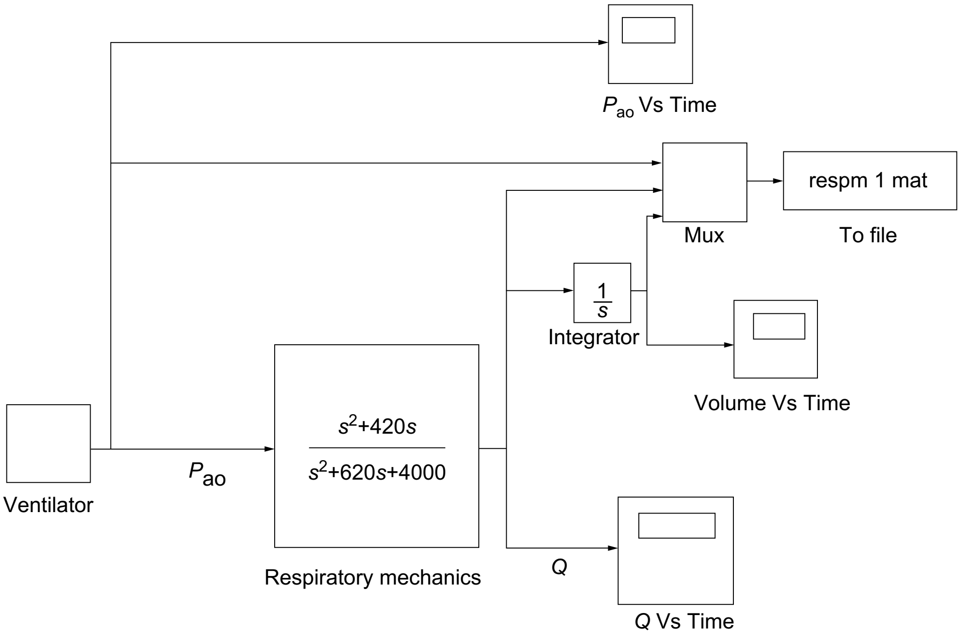


Fig. 12.15 | SIMULINK model of lung mechanics

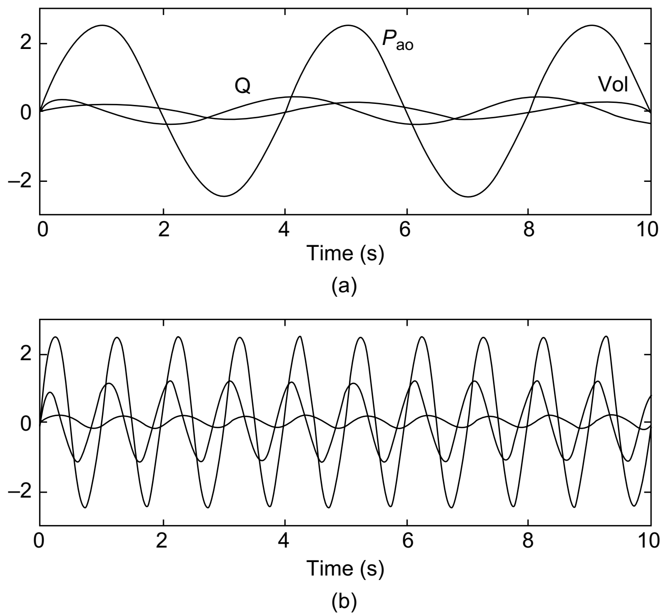


Fig. 12.16 | Simulation results from SIMULINK implementation of lung mechanics model: (a) Predicted dynamics of airflow Q and volume at 15 breaths per min and (b) predicted dynamics of airflow Q and volume at 60 breaths per min.

The ventilator frequency is initially set at 15 breaths per min (Fig. 12.16(a)), which is approximately the normal frequency of breathing at rest. At this relatively low frequency, the volume waveform is more in phase with P_{ao}. The airflow shows a substantial phase lead relative to P_{ao}. This demonstrates that lung mechanics are dominated by compliance effects at such low frequencies. The peak-to-peak change in volume (i.e., tidal volume) is approximately 0.5L, while peak $Qis \approx 0.4Ls^{-1}$. When the ventilator frequency is increased fourfold to 60 breaths per min (Fig. 12.16(b)) with amplitude kept unchanged, peak Q clearly increases ($\approx 1.2Ls^{-1}$), whereas tidal volume is decreased ($\approx 0.4L$). Now, Q has become more in phase with P_{ao}, whereas volume displays a significant lag. Thus, resistive effects have become more dominant at the higher frequency. The changes in peak Q and tidal volume with frequency demonstrate frequency dependence of pulmonary resistance and compliance, i.e., the lungs appear stiffer and less resistive as frequency increases from normal breathing.

Review Questions

1. Distinguish between positive and negative feedback control systems.
2. List out the differences between engineering and physiological control systems.
3. How is the system storage property represented in electrical, fluid, thermal and chemical systems?
4. Draw the linear model of muscle mechanics.
5. What is the need for linearization?
6. Draw the block diagram representing the muscle stretch reflex and explain the adaptive characteristics.
7. Draw a simple model consisting of a network of a generalized system element and discuss its properties.
8. Explain the linear model of respiratory mechanics with neat diagram and also derive the differential equation.

13

STATE-VARIABLE ANALYSIS

13.1 Introduction

The systems discussed in previous chapters were of Single-Input Single-Output (SISO) type. But the real world systems are of Multi-Input Multi-Output (MIMO) type. To analyse MIMO type of systems, state-variable technique can be effectively used by which the complexity of mathematical equations governing the systems can be reduced. The state-variable technique provides a convenient formulation procedure for modelling such MIMO systems. While the conventional approach is based on input-output relationship or transfer function, the modern approach is based on the description of system equations in terms of differential equations. State-variable technique uses this modern approach to represent a system. The state-variable technique is applicable to linear and non-linear time-invariant time-varying systems . This technique also facilitates the determination of the internal behaviour of a system very easily. The state of a system at time t_0 is the minimum information necessary to completely specify the condition of the system at time t_0 in continuous-time systems and n_0 in discrete-time systems. It allows determination of the system outputs at any time $t > t_0$ ($or\ n > n_0$), when inputs upto time t are specified. The state of a system at time t_0 ($or\ n_0$) is a set of values, at time t_0 or n_0 of set variables. The information-bearing variables of a system are called *state variables*. The set of all state variables is called a system's *state*. State variables contain sufficient information so that all future states and outputs can be computed if the past history, input/output relationships and future inputs of a system are known.

The state variables are chosen such that they correspond to physically measurable quantities. State-variable technique employing state variables can be extended to non-linear and time-varying systems also. It is also convenient to consider an N-dimensional space in which each coordinate is defined by one of the state variables $x_1(t)_, x_2(t)_{, \cdots ,} x_n(t)$, where n is the number of state variables of a system. This N-dimensional space is called *state-space*. The state vector is defined as an N-dimensional vector $x(n)$, whose elements are the state variables. The state vector defines a point in the state-space at any time t. As the time changes, the system state changes and a set of points, which is nothing but the locus of the tip of the state vector as time progresses, is called a *trajectory* of a system.

An alternate time-domain representation of a causal LTI discrete-time system is by means of the state-space equation. They can be obtained by reducing the N^{th} order difference equation to a system of N-first-order equations.

Conventional control method using Bode and Nyquist plots that are frequency domain approach requires Laplace transform for continuous-time systems and z-transform for discrete-time systems. But for both continuous and discrete-time systems, vector matrix form of state-space representation greatly simplifies system representation and gives accuracy of system performance.

13.1.1 Advantages of State-Variable Analysis

The advantages of state-variable analysis are:

(i) The state-variable analysis includes the effect of all initial conditions.
(ii) It is useful to determine the time-domain response of non-linear systems effectively.
(iii) State equations involving matrix algebra are highly compatible for simulation on digital computers.
(iv) It simplifies the mathematical representation of a system.
(v) MIMO systems can be easily represented and analysed using state variables.
(vi) The simulation diagram for an equation can be obtained directly.

13.2 State-Space Representation of Continuous-Time LTI Systems

A system with n state variables with m inputs and k outputs can be represented by n first-order differential equations and k output equations as shown below.

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= a_{11}x_1(t)+a_{12}x_2(t)+\cdots+a_{1n}x_n(t)+b_{11}u_1(t)+b_{12}u_2(t)+\cdots+b_{1m}u_m(t)\\
\frac{dx_2(t)}{dt} &= a_{21}x_1(t)+a_{22}x_2(t)+\cdots+a_{2n}x_n(t)+b_{21}u_1(t)+b_{22}u_2(t)+\cdots+b_{2m}u_m(t)\\
&\vdots\\
\frac{dx_n(t)}{dt} &= a_{n1}x_1(t)+a_{n2}x_2(t)+\cdots+a_{nn}x_n(t)+b_{n1}u_1(t)+b_{n2}u_2(t)+\cdots+b_{nm}u_m(t)
\end{aligned} \tag{13.1}$$

and

$$\begin{aligned}
y_1(t) &= c_{11}x_1(t)+c_{12}x_2(t)+\cdots+c_{1n}x_n(t)+d_{11}u_1(t)+d_{12}u_2(t)+\cdots+d_{1m}u_m(t)\\
y_2(t) &= c_{21}x_1(t)+c_{22}x_2(t)+\cdots+c_{2n}x_n(t)+d_{21}u_1(t)+d_{22}u_2(t)+\cdots+d_{2m}u_m(t)\\
&\vdots\\
y_k(t) &= c_{k1}x_1(t)+c_{k2}x_2(t)+\cdots+c_{kn}x_n(t)+d_{k1}u_1(t)+d_{k2}u_2(t)+\cdots+d_{km}u_m(t)
\end{aligned} \tag{13.2}$$

where $u_i(t)$ for $\boldsymbol{i}=1,2,3,\cdots,m$ are the system inputs, $x_i(t)$ for $i=1,2,3,\cdots,n$ are the state variables, $y_i(t)$ for $i=1,2,3,\cdots,k$ are the system outputs and a_{nn}, b_{nm}, c_{kn} and d_{km} are the coefficients.

All the equations present in Eqn. (13.1) are called *state equations* and all the equations present in Eqn. (13.2) are called *output equations*, which together constitute the state-space model of a system. It will be very difficult to find the solution of such a set of time-varying state equations. If a system is time-invariant, then it will be easy to find the solution of the state equations.

Also, a compact matrix notation can be used for the state-space model and using the laws of linear algebra, the state equations can easily be manipulated. The vectors and matrices are defined below.

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},\ u(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix},\ y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_k(t) \end{bmatrix} \tag{13.3}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \tag{13.4}$$

$$C=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{bmatrix} \text{ and } D=\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{k1} & d_{k2} & \cdots & d_{km} \end{bmatrix}$$

Now the state equation and output equation given in Eqs. (13.1) and (13.2) respectively can be compactly written as

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{13.5}$$

$$y(t)=Cx(t)+Du(t) \tag{13.6}$$

where $\dot{x}(t)=\dfrac{dx(t)}{dt}$, A is a matrix of order $n\times n$, B is a matrix of order $n\times m$, C is a matrix of order $k\times n$, D is a matrix of order $k\times m$, $x(t)$ is a matrix of order $n\times 1$, $u(t)$ is a matrix of order $n\times n$ and $y(t)$ is a matrix of order $k\times 1$.

The different state-space models of both time-variant and invariant types represented in different domains are given in Table 13.1.

Table 13.1 | Different state-space models

System Type	State-space Model
Continuous time-invariant	$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t) + Du(t)$
Continuous time-variant	$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ $y(t) = C(t)x(t) + D(t)u(t)$
Discrete time-invariant	$x(k+1) = Ax(k) + Bu(k)$ $y(k) = Cx(k) + Du(k)$
Discrete time-variant	$x(k+1) = A(k)x(k) + B(k)u(k)$ $y(k) = C(k)x(k) + D(k)u(k)$
Laplace domain of continuous time-invariant	$sX(s) = AX(s) + BU(s)$ $Y(s) = CX(s) + DU(s)$
z-domain of discrete time-invariant	$zX(z) = AX(z) + BU(z)$ $Y(z) = CX(z) + DU(z)$

13.3 Block Diagram and SFG Representation of a Continuous State-Space Model

The basic element for drawing the block diagram from the state-space model of a system is the integrator. The procedure to obtain the block diagram of a state-space model is given below:

Step 1: The state-space model of a system is obtained by using one of the methods which is to be discussed in the following sections.

Step 2: The individual state equation and output equation are written by using the obtained state-space model.

Step 3: The block diagrams for the individual state equation and output equation are drawn.

Step 4: The number of integrators used for the given state-space model is n (order of matrix A).

Step 5: The individual block diagrams obtained in Step 3 can be interconnected in an appropriate way and by using the n integrators, the block diagram for the given state-space model can be obtained.

The schematic illustration of a system given by Eqs. (13.5) and (13.6) is shown in Fig. 13.1. The double lines indicate a multiple-variable signal flow path. The blocks represent matrix multiplication of the vectors and matrices.

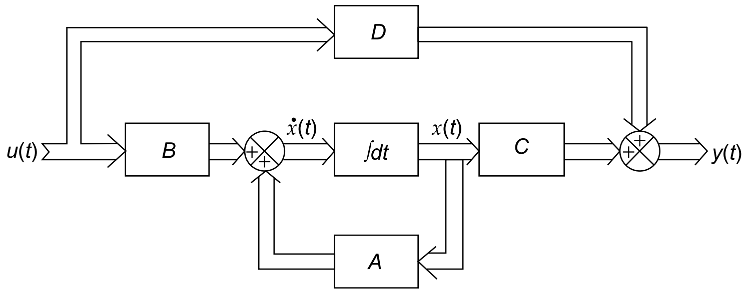


Fig. 13.1 | Block diagram of the state-variable model

Step 6: Once the block diagram of the state-space model is obtained using the steps mentioned in Chapter 4, the SFG for the model can be obtained.

The SFG representation for the block diagram of a system shown in Fig. 13.1 is shown in Fig. 13.2.

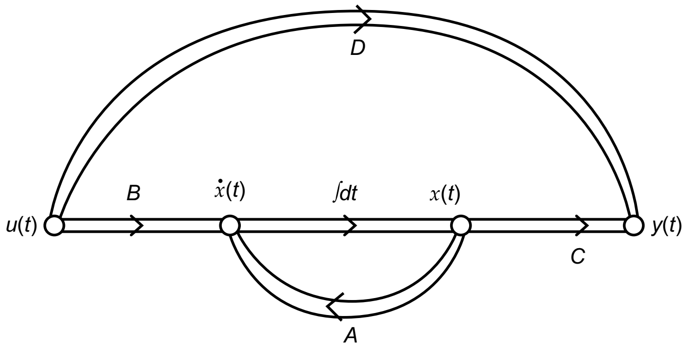


Fig. 13.2 | SFG representation of the state-variable model

13.4 State-Space Representation

The state-space model of a system can be obtained in three different ways when differential equations or transfer function is given. The different ways of representing a system in state-space model are shown in Fig. 13.3.

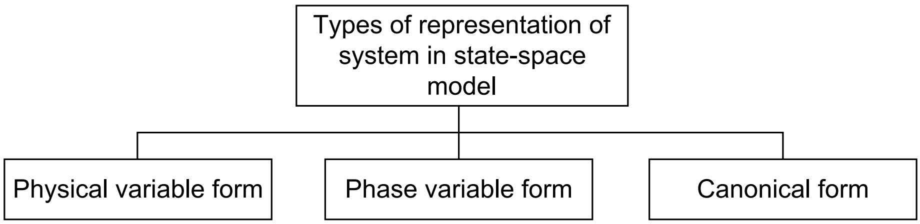


Fig. 13.3 | Different ways of representing a system in state-space model

13.5 State-Space Representation of Differential Equations in Physical Variable Form

The state-space model of electrical, mechanical (translational and rotational) and electro-mechanical systems can be obtained using physical variables. The physical variables which are considered as state variables differ from system to system. In an electrical system, the voltages across the resistance, inductance and capacitance and/or currents flowing through the resistance, inductance and capacitance are taken as state variables. In translational mechanical system, linear displacement, linear velocity and linear acceleration are taken as state variables and in rotational mechanical system, angular displacement, angular velocity and angular acceleration are taken as state variables.

13.5.1 Advantages of Physical Variable Representation

The advantages of physical variable representation are:

(i) The values of the physical quantities involving the physical variables are measurable.
(ii) Implementation is simple.
(iii) The performance of a system can be analysed as the behaviour of the physical variables with time is determined.
(iv) The feedback design for a system can be done as the information about state variables and output variables are available from the feedback.

13.5.2 Disadvantages of Physical Variable Representation

The disadvantages of physical variable representation are:

(i) The method of obtaining solution is tedious.
(ii) The state-space model of a system can be obtained if and only if the differential equation relating physical variables exists.
(iii) It is difficult to determine the differential equation of a complex system.

13.6 State-Space Model Representation for Electric Circuits

The state-space model representation of an electric circuit is done as follows:

Step 1: For a given electric circuit, there exists p number of state variables and q number of output variables.

Step 2: Determine the number of input signals m for a system.

Step 3: If the state variables n and the output variables k are given for an electric circuit, then these variables are assumed as the state variables and output variables. Otherwise, n number of state variables $1 < n \leq p$ and k number of output variables, $1 < k \leq q$ are assumed.

Step 4: Using Kirchhoff's current and/or voltage laws, obtain the differential equation of a system.

Step 5: Using the chosen state variables, the differential equations of a system obtained in the previous step are modified to obtain its first-order derivatives.

Step 6: Determine m number of output equations relating n number of state variables.

Step 7: Finally, by defining the matrices, the state-space model of the electrical circuit is obtained.

Example 13.1: **Determine the state-space model for the electrical system shown in Fig. E13.1.**

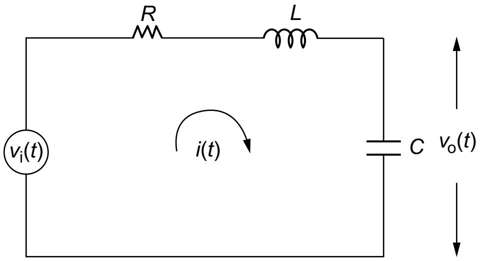


Fig. E13.1

Solution: As the state variables for the given electric circuit are not given, we choose the two state variables as the current through inductor $i(t)$ and voltage across capacitor $v_o(t)$.

Therefore, $x_1(t) = i(t)$, $x_2(t) = v_o(t)$ and $u(t) = v_i(t)$.

Applying Kirchhoff's voltage law to the circuit shown in Fig. E13.1,

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_o(t)$$

Therefore,
$$\frac{di(t)}{dt} = \frac{1}{L}v_i(t) - \frac{R}{L}i(t) - \frac{1}{L}v_o(t) \tag{1}$$

Also,
$$v_o(t) = \frac{1}{C}\int i(t)\,dt$$

Therefore,
$$\frac{dv_o(t)}{dt} = \frac{1}{C}i(t) \tag{2}$$

Substituting $\frac{di(t)}{dt} = \dot{x}_1(t)$ and $\frac{dv_o(t)}{dt} = \dot{x}_2(t)$ in Eqs. (1) and (2), we obtain

$$\dot{x}_1(t) = -\frac{R}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{1}{L}u(t) \tag{3}$$

$$\dot{x}_2(t) = \frac{1}{C}x_1(t) \tag{4}$$

Representing Eqs. (3) and (4) in matrix form, we obtain the state equation as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} u(t) \tag{5}$$

The output variable $y(t) = v_o(t) = x_2(t)$

Therefore, the output equation in matrix form can be obtained as

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{6}$$

The Eqs. (5) and (6) together represent the state-space model of a system.

Example 13.2: Determine the state equation of the electrical circuit shown in Fig. E13.2(a).

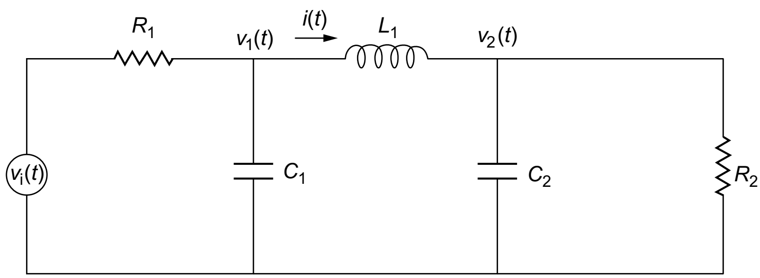


Fig. E13.2(a)

Solution: The circuit of Fig. E13.2(a) is redrawn, with the various branch currents and voltages as shown in the Fig. E13.2(b).

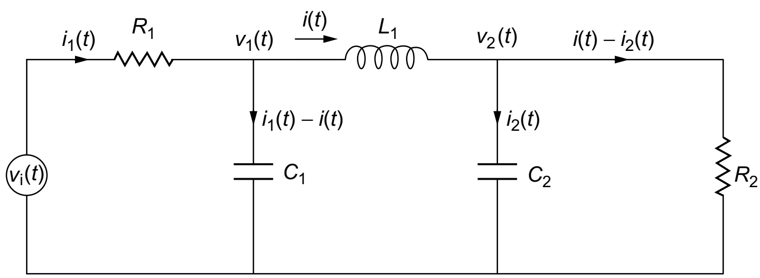


Fig. E13.2(b)

As the state variables for the given electric circuit are not given, we choose the state variables as $x_1(t) = v_1(t)$, $x_2(t) = i(t)$, $x_3(t) = v_2(t)$ and $u(t) = v_i(t)$, where $v_1(t)$ and $v_2(t)$ are the node voltages and $i(t)$ is the current flowing through the inductor L_1.

Applying Kirchhoff's voltage law for the loops in the above circuit, we obtain

$$v_i(t) = i_1(t)R_1 + v_1(t) \tag{1}$$

$$v_1(t) = L_1 \frac{di(t)}{dt} + v_2(t) \tag{2}$$

$$v_2(t) = \left(i(t) - i_2(t)\right)R_2 \tag{3}$$

Applying Kirchhoff's current law to the node at which voltage is $v_1(t)$, we obtain

$$v_1(t) = \frac{1}{C_1}\int\left(i_1(t) - i(t)\right)dt$$

Differentiating with respect to t, we obtain

$$\frac{dv_1(t)}{dt} = \frac{1}{C_1}\left(i_1(t) - i(t)\right) \tag{4}$$

Simplifying Eqn. (1), we obtain

$$i_1(t) = \frac{v_i(t) - v_1(t)}{R_1}$$

Substituting in Eqn. (4), we obtain

$$\frac{dv_1(t)}{dt} = \frac{1}{C_1}\left[\frac{v_i(t) - v_1(t)}{R_1}\right] - \frac{1}{C_1}i(t)$$

Substituting $\dot{x}_1(t) = \dfrac{dv_1(t)}{dt}$, $v_1(t) = x_1(t)$, $u(t) = v_i(t)$ and $i(t) = x_2(t)$, we obtain

$$\dot{x}_1(t) = \frac{1}{R_1C_1}u(t) - \frac{1}{R_1C_1}x_1(t) - \frac{1}{C_1}x_2(t)$$

$$= -\frac{1}{R_1C_1}x_1(t) - \frac{1}{C_1}x_2(t) + \frac{1}{R_1C_1}u(t) \tag{5}$$

Simplifying Eqn. (2), we obtain

$$\frac{di(t)}{dt} = \frac{1}{L_1}v_1(t) - \frac{1}{L_1}v_2(t)$$

Substituting $\dot{x}_2(t) = \frac{di(t)}{dt}$, $v_1(t) = x_1(t)$ and $v_2(t) = x_3(t)$ in the above equation, we obtain

$$\dot{x}_2(t) = \frac{1}{L_1} x_1(t) - \frac{1}{L_1} x_3(t) \tag{6}$$

Also, $\quad i_2(t) = C_2 \frac{dv_2(t)}{dt}$

Simplifying, we obtain

$$\frac{dv_2(t)}{dt} = \frac{1}{C_2} i_2(t) \tag{7}$$

From Eqn. (3), we obtain

$$i_2(t) = i(t) - \frac{v_2(t)}{R_2}$$

Substituting the above equation in Eqn. (7), we obtain

$$\frac{dv_2(t)}{dt} = \frac{1}{C_2}\left[i(t) - \frac{v_2(t)}{R_2}\right]$$

Substituting $\dot{x}_3(t) = \frac{dv_2(t)}{dt}$, $i(t) = x_2(t)$ and $v_2(t) = x_3(t)$ in the above equation, we obtain

$$\dot{x}_3(t) = \frac{1}{C_2} x_2(t) - \frac{1}{R_2 C_2} x_3(t) \tag{8}$$

Representing Eqs. (5), (6) and (8) in matrix form, we get the state equation of a system as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ \frac{1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & \frac{1}{C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

13.7 State-Space Model Representation for Mechanical System

A mechanical system is subdivided into translational mechanical system and rotational mechanical system. The state-space model representations for the two systems are obtained below.

13.7.1 State-Space Model Representation of Translational / Rotational Mechanical System

The state-space model representation of a translational / rotational mechanical system is done by the following steps:

(i) For a given translational / rotational mechanical system, there exists p number of state variables and q number of output variables.
(ii) Determine the number of input signals m for a system.
(iii) If the state variables n and the output variables k are given for a translational/ rotational mechanical system, then these variables are assumed as the state variables and output variables. Otherwise, n number of state variables $1 < n \le p$ and k number of output variables $1 < k \le q$ are assumed.
(iv) Using D'Alembert's principle, differential equation of a system is obtained.
(v) Using the chosen state variables, the differential equation of a system obtained in the previous step is modified so that first-order derivative of the chosen state variables is obtained.
(vi) Determine m number of output equations relating n number of state variables.
(vii) Finally, by defining the matrices, the state-space model of a translational/rotational mechanical system is obtained.

Example 13.3: Obtain the state equation and output equation of the translational mechanical system shown in Fig. E13.3.

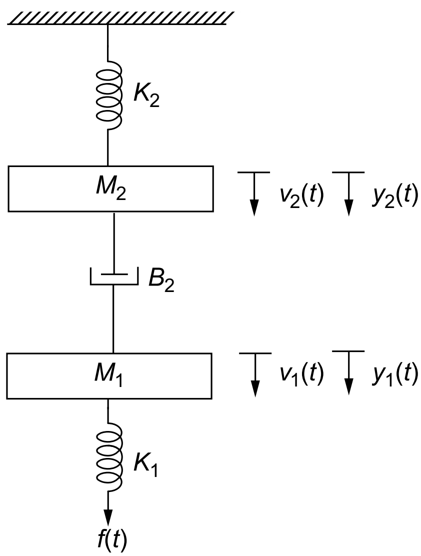


Fig. E13.3

Solution: For a translational mechanical system shown in Fig. E13.3, let $y_1(t)$, $y_2(t)$ be the displacements and $v_1(t)$, $v_2(t)$ be the velocities.

Applying D'Alembert's principle to both the masses M_1 and M_2, we obtain

$$f(t) = M_1 \frac{d^2 y_1(t)}{dt^2} + B_2 \frac{d\left(y_1(t) - y_2(t)\right)}{dt} + K_1 y_1(t) \tag{1}$$

$$0 = M_2 \frac{d^2 y_2(t)}{dt^2} + B_2 \frac{d\left(y_2(t) - y_1(t)\right)}{dt} + K_2 y_2(t) \tag{2}$$

Since the state variables for the given system are not specified, we choose the following variables as state variables.

$x_1(t) = y_1(t)$, $x_2(t) = y_2(t)$, $x_3(t) = \dfrac{dy_1(t)}{dt} = v_1(t)$, $x_4(t) = \dfrac{dy_2(t)}{dt} = v_2(t)$, $\dot{x}_3(t) = \dfrac{d^2 y_1(t)}{dt^2}$,

$\dot{x}_4(t) = \dfrac{d^2 y_2(t)}{dt^2}$ and input variable $f(t) = u(t)$.

Substituting the chosen state variables in Eqs. (1) and (2), we obtain

$$u(t) = M_1 \dot{x}_3(t) + B_2\left(x_3(t) - x_4(t)\right) + K_1 x_1(t)$$

$$0 = M_2 \dot{x}_4(t) + B_2\left(x_4(t) - x_3(t)\right) + K_2 x_2(t)$$

Simplifying and rearranging the terms of the above equations, we obtain

$$\dot{x}_3(t) = \frac{-K_1}{M_1} x_1(t) - \frac{B_2}{M_1} x_3(t) + \frac{B_2}{M_1} x_4(t) + \frac{1}{M_1} u(t) \tag{3}$$

$$\dot{x}_4(t) = -\frac{K_2}{M_2} x_2(t) + \frac{B_2}{M_2} x_3(t) - \frac{B_2}{M_2} x_4(t) \tag{4}$$

Representing the chosen state variables, Eqs. (3) and (4) in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-K_1}{M_1} & 0 & \dfrac{B_2}{M_1} & \dfrac{-B_2}{M_1} \\ 0 & \dfrac{-K_2}{M_1} & \dfrac{B_2}{M_2} & \dfrac{-B_2}{M_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{M_1} \\ 0 \end{bmatrix} u(t)$$

Considering the displacements $y_1(t)$ and $y_2(t)$, the output equation is given by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

Example 13.4: Obtain the state equation and output equation of the rotational mechanical system shown in Fig. E13.4.

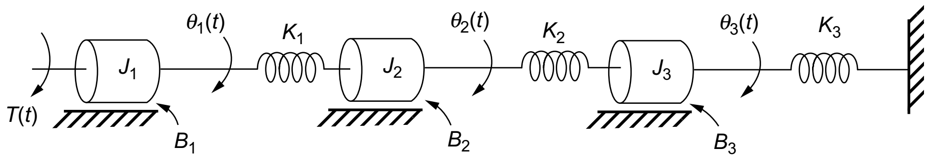


Fig. E13.4

Solution: For the rotational mechanical system shown in Fig. E13.4, let $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ be the angular displacements and $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$ be the angular velocities.

Applying D'Alembert's principle to the given system, we obtain

$$J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + K_1\left(\theta_1(t) - \theta_2(t)\right) = T(t) \tag{1}$$

$$J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\left(\theta_2(t)\right)}{dt} + K_2\left(\theta_2(t) - \theta_3(t)\right) + K_1\left(\theta_2(t) - \theta_1(t)\right) = 0 \tag{2}$$

$$J_3 \frac{d^2\theta_3(t)}{dt^2} + B_3 \frac{d\left(\theta_3(t)\right)}{dt} + K_2\left(\theta_3(t) - \theta_2(t)\right) + K_3\theta_3(t) = 0 \tag{3}$$

Since the state variables are not specified for the given system, we choose the following variables as state variables.

$$x_1(t) = \theta_1(t), \quad x_2(t) = \theta_2(t), \quad x_3(t) = \theta_3(t), \quad x_4(t) = \frac{d\theta_1(t)}{dt} = \omega_1(t), \quad x_5(t) = \frac{d\theta_2(t)}{dt} = \omega_2(t)$$

$x_6(t) = \dfrac{d\theta_3(t)}{dt} = \omega_3(t)$, $\dot{x}_4(t) = \dfrac{d^2\theta_1(t)}{dt^2}$, $\dot{x}_5(t) = \dfrac{d^2\theta_2(t)}{dt^2}$, $\dot{x}_6(t) = \dfrac{d^2\theta_3(t)}{dt^2}$ and input variable $T(t) = u(t)$.

Substituting the chosen state variables in Eqs. (1), (2) and (3), we obtain

$$u(t) = J_1\dot{x}_4(t) + B_1 x_4(t) + K_1\left(x_1(t) - x_2(t)\right)$$

$$J_2\dot{x}_5(t) + B_2\, x_5(t) + K_2\left(x_2(t) - x_3(t)\right) + K_1\left(x_2(t) - x_1(t)\right) = 0$$

$$J_3\dot{x}_6(t) + B_3\, x_6(t) + K_2\left(x_3(t) - x_2(t)\right) + K_3\, x_3(t) = 0$$

Simplifying and rearranging the above equations, we obtain

$$\dot{x}_4(t) = -\frac{K_1}{J_1}x_1(t) + \frac{K_1}{J_1}x_2(t) - \frac{B_1}{J_1}x_4(t) + \frac{1}{J_1}u(t) \tag{4}$$

$$\dot{x}_5(t) = \frac{K_1 x_1(t)}{J_2} - \left(\frac{K_1}{J_2} + \frac{K_2}{J_2}\right)x_2(t) + \frac{K_2}{J_2}x_3(t) - \frac{B_2}{J_2}x_5(t) \tag{5}$$

$$\dot{x}_6(t) = \frac{K_2\, x_2(t)}{J_3} - \frac{(K_2 + K_3)x_3(t)}{J_3} - \frac{B_3\, x_6(t)}{J_3} \tag{6}$$

Representing the chosen state variable, Eqs. (4), (5) and (6) in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \\ \dot{x}_6(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{-K_1}{J_1} & \dfrac{K_1}{J_1} & 0 & \dfrac{-B_1}{J_1} & 0 & 0 \\ \dfrac{K_1}{J_2} & \dfrac{-(K_1 + K_2)}{J_2} & \dfrac{K_2}{J_2} & 0 & 0 & \dfrac{-B_2}{J_2} \\ 0 & \dfrac{-K_2}{J_3} & \dfrac{-(K_2 + K_3)}{J_3} & 0 & 0 & \dfrac{-B_3}{J_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

Considering the angular displacements $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ as the output of the systems, the output equation is given by

$$\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}$$

13.8 State-Space Model Representation of Electromechanical System

An electromechanical system is the combination of electrical system and mechanical system. An example for an electromechanical system is DC motor. The armature and field controls of DC motor are discussed below.

13.8.1 Armature-Controlled DC Motor

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux. The armature winding and the field winding forms an electrical system. The rotating part of the motor and load connected to the shaft of the motor form a mechanical system. In armature-controlled DC motor, armature voltage is varied to attain the desired speed and the field voltage is kept constant. An armature-controlled DC motor is shown in Fig. 13.4.

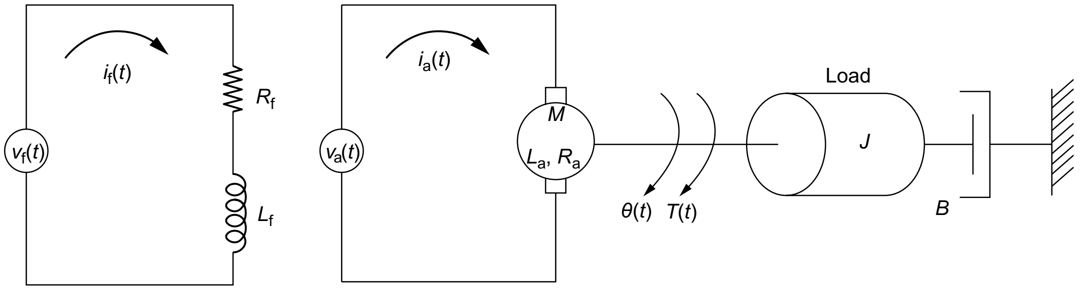


Fig. 13.4 | Armature-controlled DC motor

The DC machine parameters are : R_a is the armature resistance in Ω, L_a is the armature inductance in H, $i_a(t)$ is the armature current in A, $v_a(t)$ is the armature voltage in V, $e_b(t)$ is the back emf in V, $T(t)$ is the torque developed by motor in N-m, $\theta(t)$ is the angular displacement of shaft in rad, $\omega(t)$ is the angular velocity of the shaft in rad/sec, J is the moment of inertia of motor and load in kg-m²/rad and B is the frictional coefficient of motor and load in N-m/(rad/sec).

The equivalent circuit of armature is shown in Fig. 13.5.

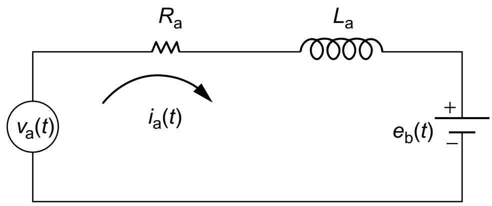


Fig. 13.5 | Electrical equivalent of armature

Applying Kirchoff's Voltage Law to the above circuit, we obtain

$$v_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_b(t) \tag{13.7}$$

The torque output of the DC motor is proportional to the product of flux and current. Since flux is constant in this system (by keeping the field voltage as constant), the torque is proportional to armature current $(\text{i.e., } T(t) \propto i_a(t))$

Therefore, Torque, $T(t) = K_t i_a(t)$ (13.8)

where K_t is the torque constant in N-m/A.

The mechanical system of the DC motor is shown in Fig. 13.6.

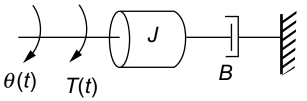


Fig. 13.6 | Mechanical system of DC motor

The differential equation governing a rotational mechanical system of motor is given by

$$J\frac{d^2\theta(t)}{dt^2}+B\frac{d\theta(t)}{dt}=T(t) \tag{13.9}$$

Substituting Eqn. (13.8) in Eqn. (13.9), we obtain

$$J\frac{d^2\theta(t)}{dt^2}+B\frac{d\theta(t)}{dt}=K_t i_a(t) \tag{13.10}$$

The back emf of the DC motor is proportional to speed (angular velocity) of the shaft i.e., $e_b(t)\propto\frac{d\theta(t)}{dt}$

Therefore, the back emf, $e_b(t)=K_b\frac{d\theta(t)}{dt}$ (13.11)

where K_b is the back emf constant in V/(rad/sec).
Substituting Eqn. (13.11) in Eqn. (13.7), we obtain

$$R_a i_a(t)+L_a\frac{di_a(t)}{dt}+K_b\frac{d\theta(t)}{dt}=v_a(t) \tag{13.12}$$

The Eqs. (13.10) and (13.11) are the differential equations governing the armature-controlled DC motor. The chosen state variables are

$$x_1(t)=i_a(t),\ x_2(t)=\omega(t)=\frac{d\theta(t)}{dt} \text{ and } x_3(t)=\theta(t).$$

The input variable is armature voltage, $v_a(t)=u(t)$.

Substituting the chosen state variables for the physical variables in Eqn. (13.12), we obtain

$$R_a x_1(t)+L_a\frac{dx_1(t)}{dt}+K_b x_2(t)=u(t)$$

or

$$R_a x_1(t)+L_a\dot{x}_1(t)+K_b x_2(t)=u(t)$$

Simplifying, we obtain

$$\dot{x}_1(t)=\frac{R_a}{L_a}x_1(t)-\frac{K_b}{L_a}x_2(t)+\frac{1}{L_a}u(t) \tag{13.13}$$

Substituting the chosen state variables for the physical variables in Eqn. (13.10), we obtain

$$J\frac{d^2x_2(t)}{dt^2}+B\frac{dx_2(t)}{dt}=K_t x_1(t)$$

or $$J\dot{x}_2(t) + Bx_2(t) = K_t x_1(t)$$

Simplifying, we obtain

$$\dot{x}_2(t) = \frac{K_t}{J}x_1(t) - \frac{B}{J}x_2(t) \tag{13.14}$$

and $$\dot{x}_3(t) = x_2(t) \tag{13.15}$$

Representing Eqs. (13.13), (13.14) and (13.15) in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 \\ \frac{K_t}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} u(t) \tag{13.16}$$

The output variables are $y_1(t) = i_a(t)$, $y_2(t) = \omega(t) = \frac{d\theta(t)}{dt}$ and $y_3(t) = \theta(t)$.

Relating the output variables to state variables, we obtain

$$y_1(t) = x_1(t);\ y_2(t) = x_2(t);\ y_3(t) = x_3(t)$$

Representing the above equations in matrix form, we obtain

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{13.17}$$

The state equation given by Eqn. (13.16) and the output equation given by Eqn. (13.17) together constitute the state-space model of the armature-controlled DC motor.

The block diagram representation obtained by combining the state variables given by Eqs. (13.13), (13.14) and (13.15) and the output variables is shown in Fig. 13.7.

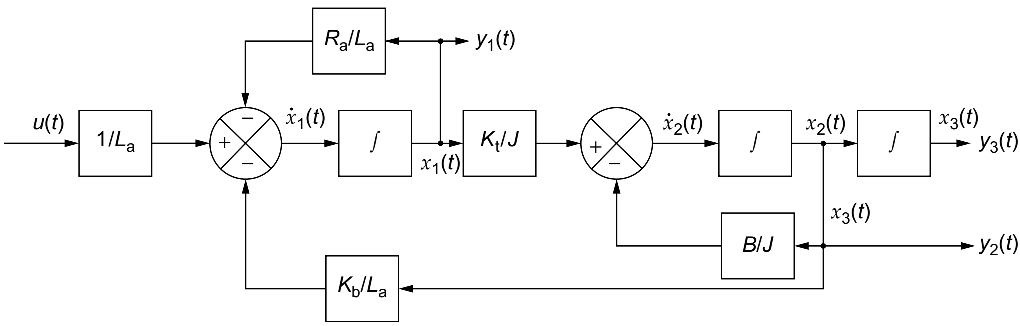


Fig. 13.7 | Block diagram representation of the state-space model of an armature-controlled DC motor

13.8.2 Field-Controlled DC Motor

The speed of DC motor is directly proportional to armature voltage and inversely proportional to flux. The armature winding and the field winding form an electrical system. An electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The rotating part of the motor and load connected to the shaft of the motor form a mechanical system. In field-controlled DC motor, the field voltage is varied to attain the desired speed since the field current which is proportional to the flux can be controlled and the armature voltage is kept constant. The field-controlled DC motor is shown in Fig. 13.8.

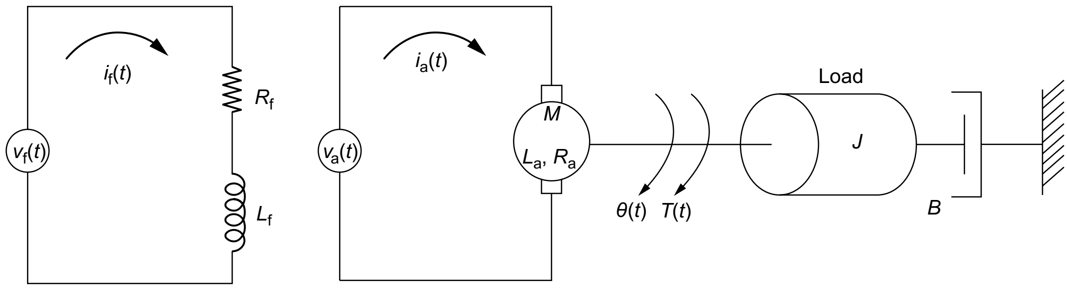


Fig. 13.8 | Field-controlled DC motor

The parameters of DC machine are : R_f is the field resistance in Ω, L_f is the field inductance in H, $i_f(t)$ is the field current in A, $v_f(t)$ is the field voltage in V, $e_b(t)$ is the back emf in V, $T(t)$ is the torque developed by motor in N-m, $\theta(t)$ is the angular displacement of shaft in rad, $\omega(t)$ is the angular velocity of the shaft in rad/sec, J is the moment of inertia of motor and load in kg-m^2/rad and B is the frictional coefficient of motor and load in N-m/(rad/sec).

An equivalent circuit of field is shown in Fig. 13.9.

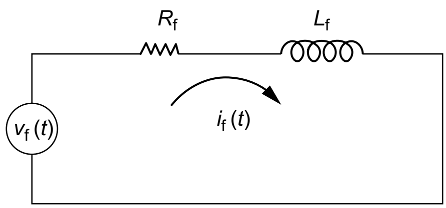


Fig. 13.9 | Electrical equivalent circuit of field

Applying Kirchhoff's voltage law to the circuit shown in Fig.13.9, we obtain

$$v_f(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt} \tag{13.18}$$

The torque output of DC motor is proportional to the product of flux and armature current. Since armature current is constant in the system (by keeping the armature voltage constant), the torque is proportional to flux, which is proportional to field current i.e., $T(t) \propto i_f(t)$. Therefore, Torque, $T(t) = K_{tf} i_f(t)$ (13.19)

where K_{tf} is the torque constant in N-m/A.
A mechanical system of the DC motor is shown in Fig. 13.10.

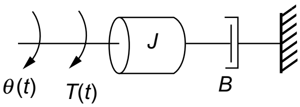


Fig. 13.10 | Mechanical system of DC motor

The differential equation governing a rotational mechanical system of motor is given by

$$J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} = T(t) \tag{13.20}$$

Substituting Eqn. (13.19) in Eqn. (13.20), we obtain

$$J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} = K_{tf}\, i_f(t)$$

The state variables chosen are $x_1(t) = i_f(t)$, $x_2(t) = \omega(t) = \dfrac{d\theta(t)}{dt}$ and $x_3(t) = \theta(t)$. The input variable is armature voltage, $v_f(t) = u(t)$.

Substituting the state variables and input variable in Eqn. (13.18), we obtain

$$R_f\, x_1(t) + L_f\frac{dx_1(t)}{dt} = u(t)$$

or $$R_f x_1(t) + L_f \dot{x}_1(t) = u(t)$$

Simplifying, we obtain

$$\dot{x}_1(t) = -\frac{R_f}{L_f}x_1(t) + \frac{1}{L_f}u(t) \tag{13.21}$$

Substituting the state variables in Eqn. (13.20), we obtain

$$J\dot{x}_2(t) + Bx_2(t) = K_{tf}x_1(t)$$

Simplifying, we obtain

$$\dot{x}_2(t) = \frac{K_{tf}}{J}x_1(t) - \frac{B}{J}x_2(t) \tag{13.22}$$

Also, $$\dot{x}_3(t) = x_2(t) \tag{13.23}$$

Representing the Eqs. (13.21), (13.22) and (13.23) in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R_f}{L_f} & 0 & 0 \\ \dfrac{K_{tf}}{J} & -\dfrac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_f} \\ 0 \\ 0 \end{bmatrix} u(t) \tag{13.24}$$

The output variables are $y_1(t) = \omega(t)$ and $y_2(t) = \theta(t)$.

Relating the output variables to state variables, we obtain

$$y_1(t) = x_2(t);\ y_2(t) = x_3(t)$$

Representing the above equations in matrix form, we obtain

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{13.25}$$

The state equation given by Eqn. (13.24) and the output equation given by Eqn. (13.25) together constitute the state-space model of the field-controlled DC motor shown in Fig. 13.11.

The block diagram representation obtained by combining the state variables given by Eqs. (13.21), (13.22) and (13.23) and the output variables is shown in Fig. 13.11.

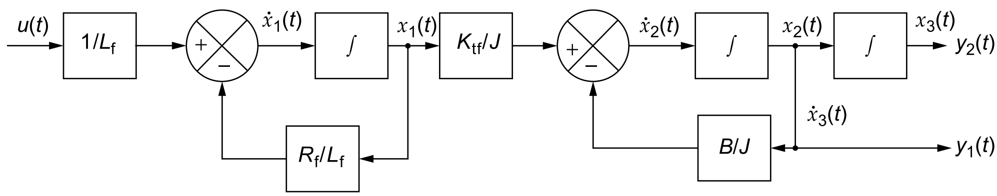


Fig. 13.11 | Block diagram representation of the state-space model field-controlled DC motor

13.9 State-Space Representation of a System Governed by Differential Equations

When a differential equation of a system is provided, the state equation and output equation are obtained by the selection of state variables and output variables which can be clearly understood with the following examples.

Example 13.5: Obtain state-space representation for the system represented by

$$\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = u(t).$$

Solution: We choose the state variables as

$$x_1(t) = y(t),\ x_2(t) = \frac{dy(t)}{dt} = \dot{x}_1(t) \text{ and } x_3(t) = \frac{d^2y(t)}{dt^2} = \dot{x}_2(t).$$

Representing the given differential equation using the chosen state variables, we obtain

$$\dot{x}_3(t) = -8x_3(t) - 11x_2(t) - 6x_1(t) + u(t) = -6x_1(t) - 11x_2(t) - 8x_3(t) + u(t)$$

Therefore, the chosen state variables and above equation can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

which is the state equation for the given system.

The output equation in matrix form is represented as

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Example 13.6: Find state-space representation for the system $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 2y(t) = 0$.

Solution: We chose the state variables as $x_1(t) = y(t)$ and $x_2(t) = \frac{dy(t)}{dt} = \dot{x}_1(t)$.

Representing the given differential equation using the chosen state variables, we obtain

$$\dot{x}_2(t) = -2x_1(t) - 6x_2(t)$$

Representing the chosen state variables and above equation in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which is the state equation for the given system.

The output equation $y(t) = x_1(t)$ in matrix form is represented as

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

13.10 State-Space Representation of Transfer Function in Phase Variable Forms

The phase variables are the state variables which are obtained by assuming one of the system variables as a state variable and other state variables as the derivatives of the selected system variable. In most cases, the output variable which is one of the system variables is considered as the state variable. The state-space model of the system using phase variables can be obtained if and only if the differential equation of the system or the system transfer function is known. The state-space model of the system using phase variables can be obtained using three methods which are discussed below.

13.10.1 Method 1

Consider the n^{th} order differential equation of a system as

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt}(t) + a_n y(t) = bu(t) \tag{13.26}$$

where $y(t)$ is the output and $x(t)$ is the input.

Expressing the state variables in terms of the output $y(t)$, we obtain

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} = \dot{y}(t) = y'(t) = \dot{x}_1(t) \\ x_3(t) &= \frac{d^2 y(t)}{dt^2} = \ddot{y}(t) = y''(t) = \dot{x}_2(t) \\ &\vdots \\ x_n(t) &= \frac{d^{n-1} y(t)}{dt^n} = \dot{x}_{n-1}(t) \end{aligned} \tag{13.27}$$

and $$\dot{x}_n(t) = \frac{d^n y(t)}{dt^n}$$

Substituting all the equations of Eqn. (13.27) in Eqn. (13.26), we obtain

$$\dot{x}_n(t) + a_1 x_n(t) + \cdots + a_{n-1} x_2(t) + a_n x_1(t) = bu(t) \tag{13.28}$$

Simplifying Eqn. (13.28), we obtain

$$\dot{x}_n(t) = -a_1 x_n(t) - \cdots - a_{n-1} x_2(t) - a_n x_1(t) + bu(t) \tag{13.29}$$

Representing Eqn. (13.27) and Eqn. (13.29) in matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t) \tag{13.30}$$

Using Eqn. (13.27), we obtain the output equation as

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \tag{13.31}$$

Therefore, if the differential equation of a system is given by Eqn. (13.26), then the state-space model of such a system is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ and } y(t) = C\,x(t)$$

where $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$.

If a matrix is of the form as given by matrix A, then it is called the bush or companion form.

13.10.2 Method 2

Consider the n^{th} order differential equation of a system as

$$\frac{dy^n(t)}{dt^n} + a_1 \frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{du^m(t)}{dt^m} + b_1 \frac{du^{m-1}(t)}{dt^{m-1}} + \cdots + b_{m-1} \frac{du(t)}{dt} + b_m u(t) \tag{13.32}$$

To represent the system given by Eqn. (13.32) by state-space model, we consider $n = m = 3$. Therefore, Eqn. (13.32) will be simplified as

$$\frac{dy^3(t)}{dt^3} + a_1 \frac{dy^2(t)}{dt^2} + a_2 \frac{dy(t)}{dt} + a_3 y(t) = b_0 \frac{du^3(t)}{dt^3} + b_1 \frac{du^2(t)}{dt^2} + b_2 \frac{du(t)}{dt} + b_3 u(t) \tag{13.33}$$

Taking Laplace transform and simplifying, we obtain

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3} \right)} \tag{13.34}$$

The SFG of Eqn. (13.34) is shown in Fig. 13.12.

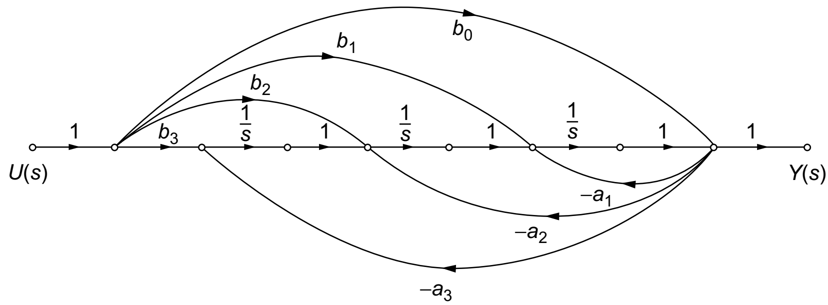


Fig. 13.12

Let us consider the output of each integrator as the state variable. The number of state variables of the system is three as there are three integrators in the system as shown in Fig. 13.12.

Considering the state variables as $x_1(t)$, $x_2(t)$ and $x_3(t)$, the SFG will be modified as shown in Fig. 13.13.

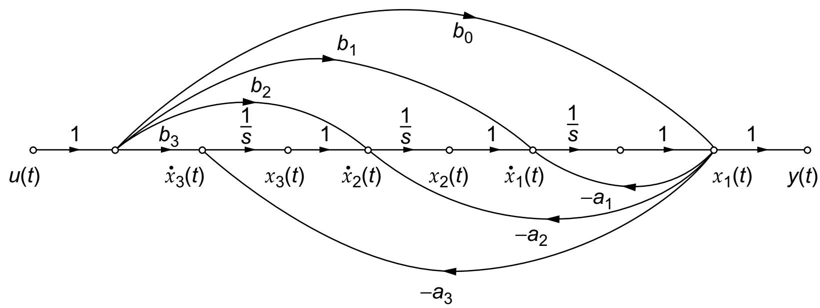


Fig. 13.13

From Fig. 13.13, we obtain

$$\dot{x}_1(t) = -a_1\left(x_1(t) + b_0 u(t)\right) + x_2(t) + b_1 u(t)$$

$$= -a_1 x_1(t) + x_2(t) + (b_1 - a_1 b_0) u(t) \tag{13.35}$$

$$\dot{x}_2(t) = -a_2\left(x_1(t) + b_0 u(t)\right) + x_3(t) + b_2 u(t)$$

$$= -a_2 x_1(t) + x_3(t) + (b_2 - a_2 b_0) u(t) \tag{13.36}$$

$$\dot{x}_3(t) = -a_3 (x_1(t) + b_0 u(t)) + b_3 u(t)$$

$$= -a_3 x_1(t) + (b_3 - a_3 b_0) u(t) \tag{13.37}$$

$$y(t) = x_1(t) + b_0 u(t) \tag{13.38}$$

Representing Eqs. (13.35), (13.36) and (13.37) in matrix form, we obtain the state equation as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u(t) \tag{13.39}$$

Representing Eqn. (13.38) in matrix form, we obtain the output equation as

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + b_0 u(t) \tag{13.40}$$

Therefore, for an n^{th} order system, the state equation and output equation are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + b_0 u(t)$$

13.10.3 Method 3

There exists another method to determine the state-space equations for the system represented by Eqn. (13.32).

To understand this method of state-space representation, we assume $n = m = 3$. Therefore, Eqn. (13.32) will be simplified as

$$\frac{dy^3(t)}{dt^3} + a_1 \frac{dy^2(t)}{dt^2} + a_2 \frac{dy(t)}{dt} + a_3 y(t) = b_0 \frac{du^3(t)}{dt^3} + b_1 \frac{du^2(t)}{dt^2} + b_2 \frac{du(t)}{dt} + b_3 u(t) \tag{13.41}$$

Taking Laplace transform and simplifying, we obtain

$$\frac{Y(s)}{U(s)}=\frac{b_0s^3+b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3} \tag{13.42}$$

Let
$$\frac{Y(s)}{U(s)}=\frac{Y_1(s)}{U(s)}\cdot\frac{Y(s)}{Y_1(s)}$$

where
$$\frac{Y_1(s)}{U(s)}=\frac{1}{s^3+a_1s^2+a_2s+a_3} \tag{13.43}$$

and
$$\frac{Y(s)}{Y_1(s)}=b_0s^3+b_1s^2+b_2s+b_3 \tag{13.44}$$

Simplifying Eqn. (13.43), we obtain

$$s^3Y_1(s)+a_1s^2Y_1(s)+a_2sY_1(s)+a_3Y_1(s)=U(s) \tag{13.45}$$

Taking inverse Laplace transform, we obtain

$$\dddot{y}_1(t)+a_1\ddot{y}_1(t)+a_2\dot{y}_1(t)+a_3y_1(t)=u(t) \tag{13.46}$$

Let the state variables be

$$x_1(t)=y_1(t),\ x_2(t)=\dot{x}_1(t)=\dot{y}_1(t)\text{ and } x_3(t)=\dot{x}_2(t)=\ddot{y}_1(t) \tag{13.47}$$

Substituting the above state variables in Eqn. (13.46), we obtain

$$\dot{x}_3(t)+a_1x_3(t)+a_2x_2(t)+a_3x_1(t)=u(t)$$

or
$$\dot{x}_3(t)=-a_1x_3(t)-a_2x_2(t)-a_3x_1(t)+u(t) \tag{13.48}$$

Simplifying Eqn. (13.44), we obtain

$$Y(s)=b_0s^3Y_1(s)+b_1s^2Y_1(s)+b_2sY_1(s)+b_3Y_1(s) \tag{13.49}$$

Taking inverse Laplace transform, we obtain

$$y(t)=b_0\dddot{y}_1(t)+b_1\ddot{y}_1(t)+b_2\dot{y}_1(t)+b_3y_1(t) \tag{13.50}$$

Substituting the state variables given by Eqn. (13.47) in the above equation, we obtain

$$y(t)=b_0\dot{x}_3(t)+b_1x_3(t)+b_2x_2(t)+b_3x_1(t) \tag{13.51}$$

Substituting Eqn. (13.48) in the above equation, we obtain

$$y(t)=b_0\left(-a_3x_1(t)-a_2x_2(t)-a_1x_3(t)+u(t)\right)+b_1x_3(t)+b_2x_2(t)+b_3x_1(t)$$

or
$$y(t)=(b_3-a_3b_0)x_1(t)+(b_2-a_2b_0)x_2(t)+(b_1-a_1b_0)x_3(t)+b_0u(t) \tag{13.52}$$

Representing Eqs. (13.48) and (13.52) in matrix form, we get the state and output equations as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{13.53}$$

$$y(t) = \left[(b_3 - a_3 b_0)\ (b_2 - a_2 b_0)\ (b_1 - a_1 b_0) \right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + b_0 u(t) \tag{13.54}$$

Therefore, for an n^{th} order system, the state equation and output equation are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t) \tag{13.55}$$

$$y(t) = \left[(b_n - a_n b_0)(b_{n-1} - a_{n-1} b_0) \cdots (b_2 - a_2 b_0)(b_1 - a_1 b_0) \right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + b_0 u(t) \tag{13.56}$$

13.10.4 Advantages of Phase-Variable Representation

The advantages of phase-variable representation are:

(i) The implementation is simple.
(ii) The state and output equations can be obtained by inspecting the differential equations constituting the system.
(iii) The transfer function design and time-domain design can be related easily by using the phase variables.
(iv) It is a powerful method for obtaining the mathematical model.
(v) It is not necessary to consider the phase variables as the physical variables.

13.10.5 Disadvantages of the Phase-Variable Representation

The disadvantages of phase-variable representation are:

(i) As the phase variables are not physical variables, the significance in measurement and control is lost for practical considerations.
(ii) It is difficult to obtain higher order derivatives of output.
(iii) It does not provide practical mathematical information.

Example 13.7: **Find the state equation and output equation for the system given by** $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3+4s^2+3s+3}$.

Solution: Given $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3+4s^2+3s+3}$

$$\left[s^3+4s^2+3s+3\right]Y(s) = U(s)$$

Taking inverse Laplace transform, we obtain

$$\dddot{y}(t)+4\ddot{y}(t)+3\dot{y}(t)+3y(t) = u(t) \tag{1}$$

Let the chosen state variables be

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t) = \dot{x}_1(t) \tag{2}$$

and $$x_3(t) = \ddot{y}(t) = \dot{x}_2(t)$$

Substituting Eqn. (2) in Eqn. (1), we obtain

$$\dot{x}_3(t) = -3x_1(t) - 3x_2(t) - 4x_3(t) + u(t) \tag{3}$$

Representing Eqs. (2) and (3) in matrix form, we obtain the state equation as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

The output equation $y(t) = x_1(t)$ in matrix form is given by

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Example 13.8: **Determine the state representation of a continuous-time LTI system with system function** $G(s) = \dfrac{3s+7}{(s+1)(s+2)(s+5)}$.

Solution: The transfer function is converted to the form

$$G(s) = \frac{3s+7}{s^3+8s^2+17s+10} = \frac{\frac{3}{s^2}+\frac{7}{s^3}}{1-\left(-\frac{8}{s}-\frac{17}{s^2}-\frac{10}{s^3}\right)}$$

The SFG of the above equation is shown in Fig. E13.8(a).

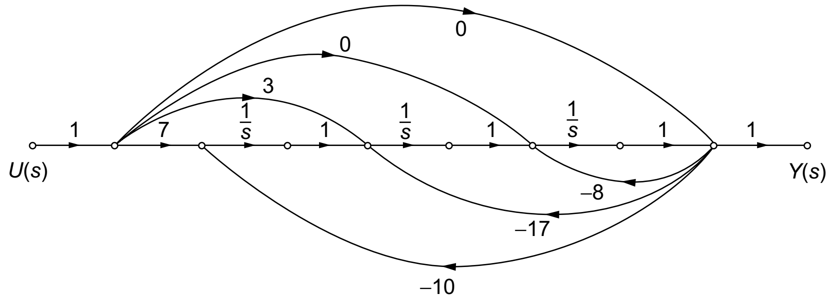


Fig. E13.8(a)

Let the output of each integrator be the state variable. The number of state variables of the system is three i.e., there are three integrators in the system as shown in Fig. 13.1(b). Considering the state variables as $x_1(t)$, $x_2(t)$ and $x_3(t)$, the SFG will be modified as shown in Fig. E13.8(b).

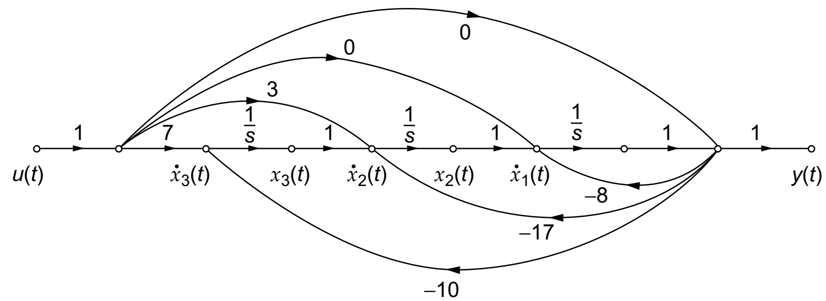


Fig. E13.8(b)

From Fig. E13.8(b), the state equations are obtained as

$$\dot{x}_1(t) = -8\big(x_1(t)+0.u(t)\big)+x_2(t)+0\big(u(t)\big) = -8x_1(t)+x_2(t)$$

$$\dot{x}_2(t) = -17\big(x_1(t)+0.u(t)\big)+x_3(t)+3u(t) = -17x_1(t)+x_3(t)+3u(t)$$

$$\dot{x}_3(t) = -10\big(x_1(t)+0.u(t)\big)+7u(t) = -10x_1(t)+7u(t)$$

Representing the above three equations in matrix form, we get the state equation as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} u(t)$$

The output equation $y(t) = x_1(t)$ in matrix form is given by

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Example 13.9: Find the state equation and output equation for the system given by $\frac{Y(s)}{U(s)} = \frac{s^3+5s^2+6s+1}{s^3+4s^2+3s+3}$.

Solution: We know that $\frac{Y(s)}{U(s)} = \frac{Y_1(s)}{U(s)} \cdot \frac{Y(s)}{Y_1(s)} = \frac{s^3+5s^2+6s+1}{s^3+4s^2+3s+3}$

Let $\quad \frac{Y_1(s)}{U(s)} \cdot \frac{Y(s)}{Y_1(s)} = \frac{s^3+5s^2+6s+1}{s^3+4s^2+3s+3}$

where $\quad \frac{Y_1(s)}{U(s)} = \frac{1}{s^3+4s^2+3s+3} \quad (1)$

and $\quad \frac{Y(s)}{Y_1(s)} = s^3+5s^2+6s+1 \quad (2)$

Simplifying Eqn. (1), we obtain

$$s^3 Y_1(s) + 4s^2 Y_1(s) + 3s Y_1(s) + 3Y_1(s) = U(s)$$

Taking inverse Laplace transform, we obtain

$$\dddot{y}_1(t) + 4\ddot{y}_1(t) + 3\dot{y}_1(t) + 3y_1(t) = u(t) \quad (3)$$

Let the state variables for the system be

$$x_1(t) = y_1(t),$$

$$x_2(t) = \dot{y}_1(t) = \dot{x}_1(t) \quad (4)$$

and $\quad x_3(t) = \ddot{y}_1(t) = \dot{x}_2(t)$

Substituting Eqn. (4) in Eqn. (3), we obtain

$$\dot{x}_3(t)+4x_3(t)+3x_2(t)+3x_1(t)+x_1(t)=u(t) \tag{5}$$

or
$$\dot{x}_3(t)=-3x_1(t)-3x_2(t)-4x_3(t)+u(t) \tag{6}$$

Simplifying Eqn. (2), we obtain

$$Y(s)=s^3Y_1(s)+5s^2Y_1(s)+6sY_1(s)+Y_1(s)$$

Taking inverse Laplace transform, we obtain

$$y(t)=\dddot{y}_1(t)+5\ddot{y}_1(t)+6\dot{y}_1(t)+y_1(t) \tag{7}$$

Substituting Eqn. (4) in the above equation, we obtain

$$y(t)=\dot{x}_3(t)+5x_3(t)+6x_2(t)+x_1(t) \tag{8}$$

Substituting Eqn. (6) in Eqn. (8), we obtain

$$y(t)=-3x_1(t)-3x_2(t)-4x_3(t)+u(t)+5x_3(t)+6x_2(t)+x_1(t)$$

$$=-2x_1(t)+3x_2(t)+x_3(t)+u(t) \tag{9}$$

Representing Eqn. (4) and Eqn. (6) in matrix form, we get the state equations as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -3 & -4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t) \tag{10}$$

Representing Eqn. (9) in matrix form, the output equation is

$$y(t)=\begin{bmatrix}-2 & 3 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t).$$

13.11 State-Space Representation of Transfer Function in Canonical Forms

Consider a system defined by

$$a_0\frac{dy^n(t)}{dt^n}+a_1\frac{dy^{n-1}(t)}{dt^{n-1}}+\cdots+a_{n-1}\frac{dy(t)}{dt}+a_n y(t)=b_0\frac{du^n(t)}{dt^n}+b_1\frac{du^{n-1}(t)}{dt^{n-1}}+\cdots+b_{n-1}\frac{du(t)}{dt}+b_n u(t)$$

where $u(t)$ is the input and $y(t)$ is the output.

Taking Laplace transform and simplifying, we obtain

$$\frac{Y(s)}{U(s)}=\frac{b_0s^n+b_1s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

13.11.1 Controllable Canonical Form

The transfer function for state-space representation in controllable canonical form is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{13.57}$$

The state and output equation for the above equation is obtained for (i) $b_0 \neq 0$ and (ii) $b_0 = 0$. Rewriting Eqn. (13.57), we obtain

$$\frac{Y(s)}{U(s)} = b_0 + \frac{b_1' s^{n-1} + b_2' s^{n-2} + \cdots + b_{n-1}' s + b_n'}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{13.58}$$

where $b_i' = b_i - a_i b_0$ for $i = 0, 1, \cdots, n$

Therefore, the state equation is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

The output equation is given by

$$y(t) = \left[b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_0 u(t)$$

Substituting $b_0 = 0$ in Eqn. (13.58), the transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Therefore, the state equation is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

The output equation is given by

$$y(t) = \begin{bmatrix} b_n \, b_{n-1} \cdots b_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

The controllable canonical form is important in discussing the pole-placement approach to the control systems design.

13.11.2 Observable Canonical Form

The transfer function for state-space representation in observable canonical form is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \cdots + \frac{b_{n\text{-}1}}{s^{n\text{-}1}} + \frac{b_n}{s^n}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_{n\text{-}1}}{s^{n\text{-}1}} + \frac{a_n}{s^n}}$$

Therefore,

$$Y(s) = b_0 U(s) + \frac{1}{s}\left[\left[\left(b_1 U(s) - a_1 Y(s)\right)\right] + \frac{1}{s}\left[b_2 U(s) - a_2 Y(s)\right]\right] + \cdots$$
$$+ \frac{1}{s^{n-1}}\left[\left(b_{n-1}U(s) - a_{n-1}Y(s) + \frac{1}{s^n}\left[b_n U(s) - a_n Y(s)\right]\right)\right]$$

Rewriting the above equation, we obtain

$$Y(s) = b_0 U(s) + X_n(s)$$

where

$$X_n(s) = \frac{1}{s}\left[\left[\left(b_1 U(s) - a_1 Y(s)\right)\right] + \frac{1}{s}\left[b_2 U(s) - a_2 Y(s)\right]\right] + \cdots + \frac{1}{s^{n-1}}\left[\left(b_{n-1}U(s) - a_{n-1}Y(s)\right.\right.$$
$$\left.\left. + \frac{1}{s^n}\left[b_n U(s) - a_n Y(s)\right]\right)\right]$$

with

$$\dot{X}_1(s) = \frac{1}{s}\left[b_n U(s) - a_n b_0 Y(s) - a_n X_1(s)\right]$$

$$\dot{X}_2(s) = \frac{1}{s}\left[b_{n-1}U(s) - a_{n-1}b_0 U(s) - a_{n-1}X_2(s)\right]$$

$$\dot{X}_n(s) = \frac{1}{s}\left[b_1 U(s) - a_1 b_0 U(s) + X_n(s)\right]$$

Taking inverse Laplace transform and representing them in matrix form, the state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u(t) \tag{13.59}$$

The output equation is given by

$$y(t) = \begin{bmatrix} 0 & 0 & \cdots 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + b_0 u(t) \tag{13.60}$$

Substituting $b_0 = 0$ in Eqn. (13.58), the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Following similar procedure as done with $b_0 \neq 0$, the state equation is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix} u(t) \tag{13.61}$$

The output equation is

$$y(t) = \begin{bmatrix} 0 & 0 & \cdots 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \tag{13.62}$$

13.11.3 Diagonal Canonical Form

The transfer function for state-space representation in diagonal canonical form is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= c_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n} \tag{13.63}$$

where c_1, c_2, ... c_n are residues and p_1, p_2, ... , p_n are roots of denominator polynomial (or poles of the system).

The Eqn. (13.63) can be rearranged as

$$\frac{Y(s)}{U(s)} = c_0 + \frac{c_1}{s\left(1+\frac{p_1}{s}\right)} + \frac{c_2}{s\left(1+\frac{p_2}{s}\right)} + \cdots + \frac{c_n}{s\left(1+\frac{p_n}{s}\right)} \tag{13.64}$$

$$= c_0 + \frac{\frac{c_1}{s}}{1+\frac{p_1}{s}} + \frac{\frac{c_2}{s}}{1+\frac{p_2}{s}} + \cdots + \frac{\frac{c_n}{s}}{1+\frac{p_n}{s}} \tag{13.65}$$

Therefore,

$$Y(s) = c_0 U(s) + \left[\frac{\frac{1}{s}}{1+\left(\frac{1}{s}\right)p_1} c_1\right] U(s) + \left[\frac{\frac{1}{s}}{1+\left(\frac{1}{s}\right)p_2} c_2\right] U(s) + \cdots + \left[\frac{\frac{1}{s}}{1+\left(\frac{1}{s}\right)p_n} c_n\right] U(s) \tag{13.66}$$

The Eqn. (13.66) can be represented by block diagram as shown in Fig . 13.14.

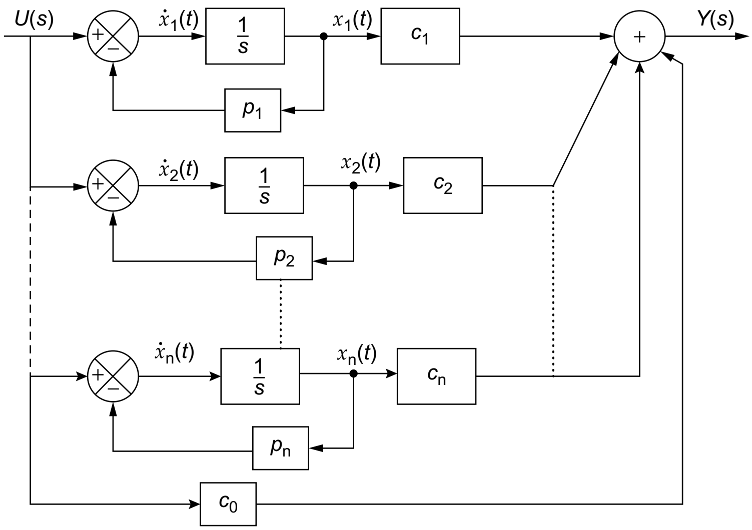


Fig. 13.14

Assuming the output of each $1/s$ block as a state variable, we get the state equations as

$$\begin{aligned}\dot{x}_1(t) &= -p_1 x_1(t) + u(t) \\ \dot{x}_2(t) &= -p_2 x_2(t) + u(t) \\ &\vdots \\ \dot{x}_n(t) &= -p_n x_n(t) + u(t)\end{aligned} \tag{13.67}$$

The output equation is $y(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + c_0 u(t)$ (13.68)

The state equation is given by

$$\begin{bmatrix}\dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t)\end{bmatrix} = \begin{bmatrix}-p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -p_n\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t)\end{bmatrix} + \begin{bmatrix}1 \\ 1 \\ \vdots \\ 1\end{bmatrix}u(t) \tag{13.69}$$

The output equation is given by

$$y(t) = \begin{bmatrix}c_1\, c_2 \cdots c_n\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t)\end{bmatrix} + c_0 u(t) \tag{13.70}$$

13.11.4 Jordan Canonical Form

Consider the case where the denominator polynomial involves multiple roots. Here, the preceding diagonal canonical form must be modified into the Jordan canonical form. Suppose, for example, that the roots of the denominator polynomial are different from one another, except for the first m roots (i.e., $p_1 = p_2 = p_3 = \cdots = p_m$), then the transfer function for state-space representation in Jordan canonical form is

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)^m (s+p_{m+1})\cdots(s+p_n)}$$

The partial-fraction expansion of the above equation becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s+p_1)^m} + \frac{c_2}{(s+p_1)^{m-1}} + \frac{c_3}{(s+p_1)^{m-2}} + \cdots + \frac{c_m}{(s+p_1)^1} + \frac{c_{m+1}}{s+p_{m+1}} + \cdots + \frac{c_n}{s+p_n}$$

To represent the above equation, let us assume that m = 3. Therefore,

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \cdots + \frac{c_n}{s+p_n}$$

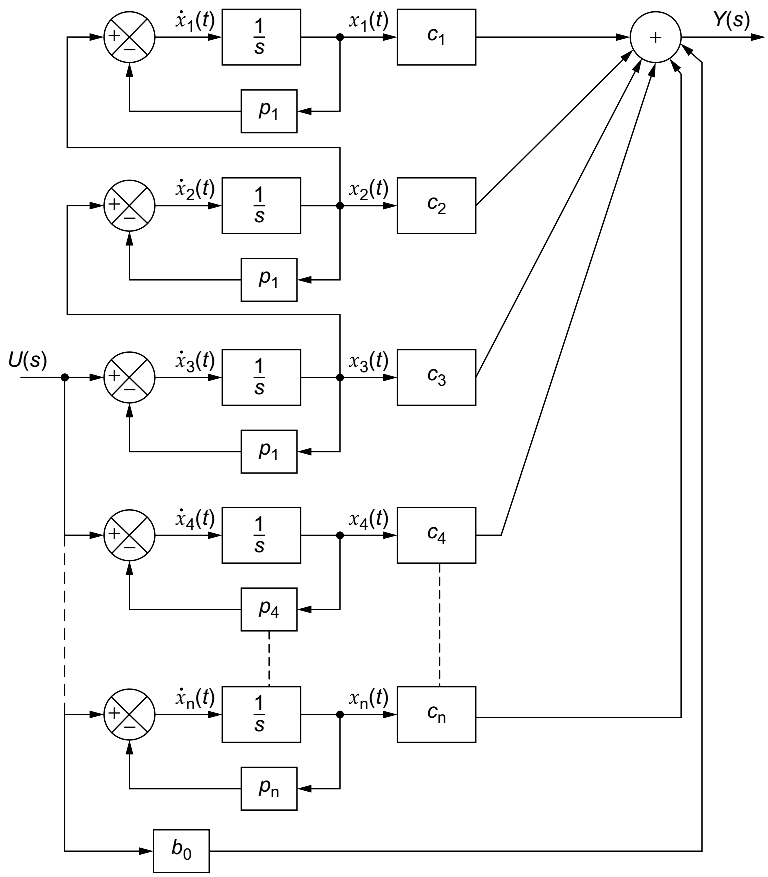


Fig. 13.15

The state equations from the block diagram shown in Fig. 13.15 are

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) - p_1 x_1(t) \\
\dot{x}_2(t) &= x_3(t) - p_1 x_2(t) \\
\dot{x}_3(t) &= -p_1 x_3(t) + u(t) \\
\dot{x}_4(t) &= -p_4 x_4(t) + u(t) \\
&\vdots \\
\dot{x}_n(t) &= -p_n x_n(t) + u(t)
\end{aligned}
$$

The output equation from the block diagram is

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + C_n x_n(t) + b_0 u(t)$$

The state equation in matrix form is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & : & \cdots & : \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -p_4 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -p_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \tag{13.71}$$

The output equation is given by

$$y(t) = \begin{bmatrix} c_1\, c_2 \cdots c_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_0 u(t) \tag{13.72}$$

Therefore, in general for m number of equal roots, the state equation and the output equation in matrix form can be written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_m(t) \\ \dot{x}_{m+1}(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & 0 & 0 & & 0 \\ 0 & 0 & \ddots & 1 & 0 & & \vdots \\ 0 & \cdots & 0 & -p_1 & & & 0 \\ 0 & \cdots & 0 & 0 & -p_{m+1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & -p_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \\ x_{m+1}(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

The output equation is given by

$$y(t) = \begin{bmatrix} c_1\, c_2 \cdots c_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_0 u(t)$$

Example 13.10: **Determine the state representation of a continuous-time LTI system with system function $G(s) = \dfrac{s+1}{(s+5)(s+2)}$ in controllable canonical form.**

Solution: Given $G(s) = \dfrac{s+1}{s^2+7s+10}$

Comparing the above equation with the standard form as given in Eqn. (13.57), we find $n = 2$ and the values for the controllable canonical form are

$$a_1 = 7,\ a_2 = 10,\ b_0 = 0,\ b_1 = 1 \text{ and } b_2 = 1.$$

Using these values, the state-space model of the given system can be obtained as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example 13.11: **Determine the state representation of a continuous-time LTI system with system function $G(s) = \dfrac{s+7}{(s+2)(s+3)}$ in observable canonical form.**

Solution: Given $G(s) = \dfrac{s+7}{s^2+5s+6}$

Comparing the above equation with the standard form as given in Eqn. (13.57), we find $n = 2$ and the values for the observable canonical form are

$$a_1 = 5,\ a_2 = 6,\ b_0 = 0,\ b_1 = 1 \text{ and } b_2 = 7.$$

Using these values, the state-space model of the given system can be obtained as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example 13.12: **Determine the state representation of a continuous-time LTI system with system function $G(s) = \dfrac{(s+5)}{(s+1)(s+3)}$ in diagonal canonical form.**

Solution: Given $G(s) = \dfrac{s+5}{(s+1)(s+3)}$

Using partial-fraction expansion,

$$\frac{s+5}{(s+1)(s+3)} = \frac{A_1}{s+1} + \frac{A_2}{s+3}$$

Here, $s+5 = A_1(s+3) + A_2(s+1)$ (1)

Equating coefficients of s on both sides, we obtain

$$1 = A_1 + A_2 \tag{2}$$

Equating coefficients of constants on both sides of Eqn. (1), we obtain

$$5 = 3A_1 + A_2 \tag{3}$$

Solving Eqs. (2) and (3), we obtain

$$A_1 = 2 \text{ and } A_2 = -1$$

Therefore, $$\frac{s+5}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{1}{s+3}$$

Comparing the above equation with the standard diagonal canonical form, the coefficient values for the diagonal canonical form are

$$p_1 = -1,\ p_2 = -3,\ c_0 = 0,\ c_1 = 2 \text{ and } c_2 = -1.$$

Using these values, the state-space model of the given system can be obtained as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

and $$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example 13.13: Determine the state representation of a continuous-time LTI system with system function $\frac{1}{(s+3)^2(s+2)}$ in Jordan canonical form.

Solution:

Given $G(s) = \frac{1}{(s+3)^2(s+2)}$

Using partial-fraction expansion,

$$\frac{1}{(s+3)^2(s+2)} = \frac{A_1}{(s+3)^2} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+2)}$$

$$1 = A_1(s+2) + A_2(s+3)(s+2) + A_3(s+3)^2 \tag{1}$$

Equating coefficients of s^2 on both sides, we obtain

$$0 = A_2 + A_3$$

Equating coefficients of s on both sides of the Eqn. (1), we obtain

$$0 = A_1 + 5A_2 + 6A_3 \tag{2}$$

Equating coefficients of constants on both sides of the Eqn. (1), we obtain

$$1 = 2A_1 + 6A_2 + 9A_3 \tag{3}$$

Solving Eqs. (1), (2) and (3), we obtain

$$A_1 = -1,\ A_2 = -1 \text{ and } A_3 = 1$$

$$T(s) = -\frac{1}{(s+3)^2} - \frac{1}{(s+3)} + \frac{1}{(s+2)}$$

Comparing the above equation with the standard Jordan canonical form, the coefficient values for the diagonal canonical form are

$$c_1 = 0,\ c_2 = -1,\ c_3 = -1,\ c_4 = 1,\ p_1 = 3 \text{ and } p_4 = 2$$

Using these values, the state-space model of the given system can be obtained as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

and

$$y(t) = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

13.12 Transfer Function from State-Space Model

Let the state-space model of the system be

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{13.73}$$

The Laplace transforms of the equations are

$$sX(s) = AX(s) + BU(s) \tag{13.74}$$

$$Y(s) = CX(s) + DU(s) \tag{13.75}$$

Rewriting Eqn. (13.74), we obtain

$$[sI - A]X(s) = BU(s)$$

Therefore,

$$X(s) = [sI - A]^{-1} BU(s) \tag{13.76}$$

where I is an identity matrix.

Substituting Eqn. (13.76) in Eqn. (13.75), we obtain

$$Y(s) = C[sI - A]^{-1} BU(s) + DU(s)$$

Therefore, the transfer function

$$\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D \tag{13.77}$$

Here $\frac{Y(s)}{U(s)}$ must have $m \times n$ dimensionality and thus has mn elements. Therefore, for every input, there are m transfer functions with one for each output which is the reason that the state-space representation can easily be the preferred choice for MIMO systems.

When the output and input are not directly connected, the matrix D will be a null matrix.

Therefore, the transfer function $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B$ (13.78)

Example 13.14: Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solution: Since two columns exist in the B matrix, given system has two inputs $u_1(t)$, $u_2(t)$. Also, as two rows exist in the C matrix, given system has two outputs $y_1(t)$, $y_2(t)$. Therefore, four transfer functions exist for the given system which are given by

$$\frac{Y_1(s)}{U_1(s)}, \frac{Y_1(s)}{U_2(s)}, \frac{Y_2(s)}{U_1(s)} \text{ and } \frac{Y_2(s)}{U_2(s)}$$

From the given state-space model, we obtain

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = 0$$

Transfer function, $\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D$

$$[sI-A]=\begin{bmatrix} s+1 & -1 & 1 \\ 0 & s+2 & -1 \\ 0 & 0 & s+3 \end{bmatrix}$$

and $$[sI-A]^{-1}=\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{-1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}$$

Hence, transfer function $$=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{-1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)} & \frac{1}{(s+3)} \\ 0 & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} \frac{2(s+2)}{(s+1)(s+3)} & \frac{1}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$$

Therefore, $\dfrac{Y_1(s)}{U_1(s)}=\dfrac{2(s+2)}{(s+1)(s+3)}$, $\dfrac{Y_1(s)}{U_2(s)}=\dfrac{1}{s+1}$, $\dfrac{Y_2(s)}{U_1(s)}=\dfrac{1}{s+2}$ and $\dfrac{Y_2(s)}{U_2(s)}=\dfrac{1}{s+2}$

Example 13.15: Obtain the transfer function for the state-space representation of a system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}=\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solution: From the given model,

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \text{ and } D = [0]$$

Transfer function is given by

$$\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D$$

$$\left[sI - A\right] = \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix}$$

$$\left[sI - A\right]^{-1} = \frac{1}{s^3 + 10^2 + 35s + 41} \begin{bmatrix} s^2 + 8s + 19 & s+5 & 1 \\ -3 & s^2 + 7s + 10 & s+2 \\ -3s - 9 & -4s - 11 & s^2 + 5s + 6 \end{bmatrix}$$

Therefore, the transfer function is

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \frac{1}{s^3 + 10^2 + 35s + 41} \begin{bmatrix} s^2 + 8s + 19 & s+5 & 1 \\ -3 & s^2 + 7s + 10 & s+2 \\ -3s - 9 & -4s - 11 & s^2 + 5s + 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^3 + 10^2 + 35s + 41} \begin{bmatrix} -3 & s^2 + 7s + 10 & s+2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{s+2}{s^3 + 10s^2 + 35s + 41}$$

Example 13.16: Determine the transfer function for the parameters $A = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $D = 0$

Solution: The transfer matrix is given by

$$\frac{Y(s)}{U(s)} = C\left[sI - A\right]^{-1} B + D$$

$$\left[sI - A\right] = \begin{bmatrix} s+3 & -1 \\ 0 & s+1 \end{bmatrix}$$

Therefore, $\left[sI-A\right]^{-1}=\begin{bmatrix}\dfrac{1}{s+3} & \dfrac{1}{(s+1)(s+3)}\\ 0 & \dfrac{1}{s+1}\end{bmatrix}$

Transfer function, $\dfrac{Y(s)}{U(s)}=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\dfrac{1}{s+3} & \dfrac{1}{(s+1)(s+3)}\\ 0 & \dfrac{1}{s+1}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$

$$=\begin{bmatrix}\dfrac{1}{s+3} & \dfrac{s+4}{(s+1)(s+3)}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

Therefore, $\dfrac{Y(s)}{U(s)}=\dfrac{(2s+5)}{(s+1)(s+3)}$

13.13 Solution of State Equation for Continuous Time Systems

Consider a system with state equation as given by

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{13.79}$$

The above state equation can be of homogenous or non-homogenous type.

13.13.1 Solution of Homogenous-Type State Equation

State equation is said to be of homogenous type when the system is free running (i.e., with zero input forces). Then the state equation becomes

$$\dot{x}(t)=Ax(t) \tag{13.80}$$

The procedure for obtaining the solution for the above equation is discussed as follows. Consider a differential equation as given by

$$\frac{dx(t)}{dt}=ax(t) \tag{13.81}$$

The above equation is a homogeneous equation with zero input vector and with the initial condition $x(0)=x_0$.

The solution of Eqn. (13.81) is assumed to be given by

$$x(t)=b_0+b_1t+b_2t^2+\cdots+b_kt^k \tag{13.82}$$

where $b_0, b_1, b_2, \cdots, b_k$ are constants.
In the above equation, at $t=0$, $x(0)=x_0=b_0$.
Substituting Eqn. (13.82) in Eqn. (13.81), we obtain

$$\frac{d}{dt}\left[b_0+b_1t+b_2t^2+\cdots+b_kt^k\right]=a\left[b_0+b_1t+b_2t^2+\cdots+b_kt^k\right]$$

Simplifying, we obtain

$$b_1+2b_2t+\cdots+kb_kt^{k-1}=ab_0+ab_1t+\cdots+ab_{k-1}t^{k-1}+ab_kt^k \tag{13.83}$$

Equating the coefficients of constants and time, t^i for $i=0, 1,\ 2,\ 3 \ldots (k-1)$ in the above equation, we obtain

$$\left.\begin{aligned} b_1 &= ab_0 \\ 2b_2 &= ab_1 \\ &\vdots \\ kb_k &= ab_{k-1} \end{aligned}\right\} \tag{13.84}$$

Therefore,

$$\left.\begin{aligned} &b_1 = ab_0 \\ &b_2 = \frac{1}{2}ab_1 = \frac{1}{2}a^2b_0 = \frac{1}{2!}a^2b_0 \\ &\vdots \\ &b_k = \frac{1}{k}ab_{k-1} = \frac{1}{k!}a^kb_0 \end{aligned}\right\} \tag{13.85}$$

Substituting Eqn. (13.85) in Eqn. (13.82), we obtain

$$x(t)=b_0+ab_0t+\frac{1}{2!}a^2b_0t^2+\cdots+\frac{1}{k!}a^kb_0t^k \tag{13.86}$$

$$=b_0\left[1+at+\frac{1}{2!}a^2t^2+\cdots+\frac{1}{k!}a^kt^k\right] \tag{13.87}$$

Substituting $b_0=x(0)=x_0$ in the above equation, we obtain

$$x(t)=\left[1+at+\frac{1}{2!}a^2t^2+\cdots+\frac{1}{k!}a^kt^k\right]x_0 \tag{13.88}$$

In the above equation, $1+at+\frac{1}{2!}a^2t^2+\cdots+\frac{1}{k!}a^kt^k$ represents an exponential series which is represented as e^{at}.

Therefore, Eqn. (13.88) can be written as

$$x(t) = e^{at}x_0$$

The above equation is the solution of the homogenous equation in scalar form.

Therefore, for the state equation given by Eqn. (13.80), we have

$$\dot{x}(t) = Ax(t)$$

The solution for the above equation is

$$x(t) = e^{At}x_0 \tag{13.89}$$

where e^{At} is not a scalar, but a matrix termed as State Transition Matrix (STM) of order $n \times n$, which will be discussed in the forthcoming sections.

13.13.2 Solution of Non-Homogenous Type State Equation

State equation is said to be of non-homogenous type when the system is with input forces. Then the state equation remains the same as given in Eqn. (13.79) which is given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

or

$$\dot{x}(t) - Ax(t) = Bu(t) \tag{13.90}$$

Pre-multiplying both sides by e^{-At}, we obtain

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At}Bu(t)$$

The above equation can be written as

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)\left[\sin ce\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\left[\dot{x}(t) - Ax(t)\right]\right] \tag{13.91}$$

Integrating Eqn. (13.91) with respect to time with limits 0 and t, we obtain

$$e^{-At}\,x(t)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Therefore, $\quad e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$

Pre-multiplying both sides by e^{At}, we obtain

$$e^{At}e^{-At}x(t) - e^{At}x(0) = e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Therefore,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{13.92}$$

This is the solution for the state equation given by Eqn. (13.90).

From Eqn. (13.92), it is observed that the solution is divided into two different parts. The first part $e^{At}x(0)$ is the solution of homogenous-type state equation and it is termed as Zero Input Response (ZIR).

The second part $\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ is the solution due to the application of input $u(t)$ from time 0 to t. Therefore, the solution is termed as forced solution or Zero State Response (ZSR).

Thus, Eqn. (13.92) can be represented as

$$x(t) = \underbrace{e^{At}x(0)}_{ZIR} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{ZSR}$$

If the initial time is $t = t_0$, then the solution is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

13.13.3 State Transition Matrix

For obtaining the solution of the homogeneous and non-homogeneous state equations, it is necessary to determine the State Transition Matrix (STM).

The STM represented as $\phi(t)$ will be defined and derived as follows:

For homogenous-type state equation,

$$\dot{x}(t) = Ax(t)$$

Taking Laplace transform, we obtain

$$sX(s) - x(0) = AX(s)$$

or $$[sI - A]X(s) = x(0)$$

Therefore, $$X(s) = [sI - A]^{-1}x(0) \tag{13.93}$$

Taking inverse Laplace transform, we obtain

$$x(t) = \mathcal{L}^{-1}\left\{[sI - A]^{-1}x(0)\right\} = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}x(0) \tag{13.94}$$

Comparing the above equation with the solution of homogenous-type state equation, we obtain

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}$$

Therefore, $\mathcal{L}\{\phi(t)\} = \mathcal{L}\{e^{At}\} = \phi(s) = [sI - A]^{-1}$

where $\phi(s)$ is called as the resolvent matrix, for which the inverse Laplace transform yields $\phi(t) = e^{At}$.

For non-homogenous-type state equation,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Taking Laplace transform, we obtain

$$sX(s)-x(0)= \quad AX(s)+BU(s)$$

or $[sI-A]X(s)=x(0)+BU(s)$

Multiplying $[sI-A]^{-1}$ on both sides, we obtain

$$X(s)=[sI-A]^{-1}x(0)+[sI-A]^{-1}BU(s) \tag{13.95}$$

Taking inverse Laplace transform, we obtain

$$x(t)=\mathcal{L}^{-1}\left\{[sI-A]^{-1}x(0)\right\}+\mathcal{L}^{-1}\left\{[sI-A]^{-1}BU(s)\right\}$$

$$=\mathcal{L}^{-1}\left\{[sI-A]^{-1}\right\}x(0)+\mathcal{L}^{-1}\left\{[sI-A]^{-1}BU(s)\right\} \tag{13.96}$$

Comparing the above equation with the solution of non-homogenous-type state equation, we obtain

$$\mathcal{L}\{\phi(t)\}=\mathcal{L}\left\{e^{At}\right\}=\phi(s)=[sI-A]^{-1}$$

where $\phi(s)$ is called as the resolvent matrix for which the inverse Laplace transform yields $\phi(t)=e^{At}$.

To prove that $\phi(t) = e^{At}$:

We know that,

$$\phi(s)=(sI-A)^{-1}$$

$$=\frac{1}{s}\left(I-\frac{A}{s}\right)^{-1}$$

$$=\frac{1}{s}\left(I+\frac{A}{s}+\frac{A^2}{s^2}+\cdots\right)$$

Therefore,

$$\phi(s)=\frac{1}{s}+\frac{A}{s^2}+\frac{A^2}{s^3}+\cdots \tag{13.97}$$

Taking inverse Laplace transform, we get

$$\phi(t)=I+At+\frac{A^2t^2}{2!}+\frac{A^3t^3}{3!}+\cdots$$

$$=e^{At} \tag{13.98}$$

Various procedures to determine the STM are discussed below:

Inverse Laplace Transform Method

We know that the resolvent matrix $\phi(s)$ is given by

$$\phi(s) = [sI - A]^{-1} = \frac{Adj[sI - A]}{|sI - A|}$$

Taking inverse Laplace transform, we obtain

$$\phi(t) = \mathcal{L}^{-1}\{\phi(s)\} \tag{13.99}$$

Series Summation method

We know that, $\phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!}$

The series represented by the above equation is used to determine $\phi(t)$. The series summation method is better suited for digital computation. Assuming, $D = At$ and substituting in the above equation, we obtain

$$\phi(t) = I + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots + \frac{D^n}{n!}$$

$$= I + D + \frac{D(D)}{2} + \frac{D}{3}\left[\frac{D(D)}{2}\right] + \cdots + \frac{D}{n}\left[\frac{D^{n-1}}{n-1}\right] \tag{13.100}$$

From the above equation, it is observed that each term in the above series contains the preceding term and a multiplier in a regular order. If the terms of the series are denoted as $P_1, P_2, P_3, \ldots, P_n$, then each term can be represented as

$$\left.\begin{aligned} P_1 &= I \\ P_2 &= D(I) = \frac{D}{2-1}(P_1) \\ P_3 &= \frac{D}{2}(D) = \frac{D}{3-1}(P_2) \\ &\vdots \\ P_n &= \frac{D}{n-1}(P_{n-1}) \end{aligned}\right\} \tag{13.101}$$

The series would converge quickly if exponential terms are present in it.

λ-matrix Method

By λ-matrix method, a non-singular matrix M, called the *modal matrix* is to be determined such that the transformation of state variable is given by

$$x(t) = Mz(t) \tag{13.102}$$

Differentiating, we obtain

$$\dot{x}(t) = M\dot{z}(t) \tag{13.102}$$

Therefore, substituting Eqs. (13.101) and (13.102) in state equation of a system, we obtain

$$M\dot{z}(t) = AMz(t) + Bu(t) \tag{13.103}$$

Multiplying Eqn. (13.103) by M^{-1} on both sides, we obtain

$$M^{-1}M\,\dot{z}(t) = \dot{z}(t) = M^{-1}AMz(t) + M^{-1}Bu(t)$$

$$\equiv \lambda z(t) + \Gamma u(t) \tag{13.104}$$

where $\lambda \equiv M^{-1}AM$

and $$\Gamma \equiv M^{-1}B \tag{13.105}$$

The transformation done in the above equations is called *similarity transformation*.

Proceeding further, it is required that eigen values and eigen vectors for a matrix are defined.

If A is a matrix with order $n \times n$, X is a non-zero vector and λ is a scalar such that

$$AX = \lambda X \tag{13.106}$$

Therefore, X is the eigen vector and λ is an eigen value of A, since eigen values and eigen vectors are interdependent (i.e., X is the eigen vector corresponding to eigen value λ and λ is an eigen value corresponding to the eigen vector X).

By similarity transformation, matrix A is diagonalised with its diagonal elements being the eigen values.

That is, $$\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \tag{13.107}$$

The modal matrix M can be obtained by the eigen vectors R_i of A as

$$M = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}$$

where R_i is the eigen vector corresponding to the eigen value λ_i.
Therefore, for the eigen vector R_1, Eqn. (13.106) becomes

$$AR_1 = \lambda_1 R_1 \tag{13.108}$$

Representing the above equation in matrix form, we obtain

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{bmatrix} = \lambda_1 \begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{bmatrix} \quad (13.109)$$

Extending the above equation from eigenvector R_1 to eigenvector R_n, we obtain

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} = \begin{bmatrix} \lambda_1 r_{11} & \lambda_2 r_{12} & \cdots & \lambda_n r_{1n} \\ \lambda_1 r_{21} & \lambda_2 r_{22} & \cdots & \lambda_n r_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 r_{n1} & \lambda_2 r_{n2} & \cdots & \lambda_n r_{nn} \end{bmatrix} \quad (13.110)$$

or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad (13.111)$$

Equation (13.111) can be represented as

$$AM = \lambda M \quad (13.112)$$

Multiplying the above equation by M^{-1} on both sides and rearranging the terms, we obtain

$$M^{-1}M\lambda = M^{-1}AM$$

Therefore, $\lambda = M^{-1}AM$ (13.113)

From the above equation, it is observed that by similarity transformation, modal matrix M formed by the eigen vectors of A diagonalises it.

However, if the matrix A is of the phase-variable canonical form, the modal matrix may be given by

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \quad (13.114)$$

where $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigen values of A.

By modal matrix, the STM is obtained from the similarity transformation of $e^{\lambda t}$ from Eqn. (13.113) as

$$\phi(t) = e^{At} = Me^{\lambda t}M^{-1} \tag{13.115}$$

From Eqn.(13.113), we have

$$\left.\begin{aligned} A &= M\lambda M^{-1} \\ A^2 &= \left(M\lambda\, M^{-1}\right)\left(M\lambda\, M^{-1}\right) \\ &= M\lambda\left(M^{-1}M\right)\lambda M^{-1} \\ &= M\lambda^2 M^{-1} \\ &\vdots \\ A^n &= M\lambda^n M^{-1} \end{aligned}\right\} \tag{13.116}$$

Therefore,

$$\left.\begin{aligned} \phi(t) = e^{At} &= I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots + \frac{A^n t^n}{n!} \\ &= I + M\lambda M^{-1}t + \frac{M\lambda^2M^{-1}t^2}{2!} + \cdots + \frac{M\lambda^n M^{-1}t^n}{n!} \\ &= I + M(\lambda\, t)M^{-1} + M\left[\frac{(\lambda t)^2}{2!}\right]M^{-1} + \cdots + M\left[\frac{(\lambda t)^n}{n!}\right]M^{-1} \\ &= Me^{\lambda t}M^{-1} \end{aligned}\right\} \tag{13.117}$$

13.13.4 Properties of State Transition Matrix

The properties of STM $\phi(t)$ are

(i) $\phi(0) = I$ (13.118)

(ii) $\phi^{-1}(t) = \phi(-t)$ (13.119)

Proof: Post multiplying both sides of Eqn. (13.98) by e^{-At}, we obtain

$$\phi(t)e^{-At} = e^{At}e^{-At} = I \tag{13.120}$$

Then, premultiplying both sides of Eqn. (13.98) by $\phi^{-1}(t)$, we obtain

$$\phi^{-1}(t)\phi(t) = \phi^{-1}(t)e^{At}$$

$$I = \phi^{-1}(t)e^{At}$$

Therefore,

$$e^{-At} = \phi^{-1}(t) \tag{13.121}$$

Thus, $\phi(-t) = \phi^{-1}(t) = e^{-At}$ (13.122)

An interesting result from this property, of $\phi(t)$ is,

$$x(0) = \phi(-t)x(t) \tag{13.123}$$

Which means that the state-transition process can be considered as bilateral in time, i.e., the transition in time can take place in either direction.

(iii) $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0)$ for any t_0, t_1, t_2 (13.124)

Proof

$$\phi(t_2 - t_1)\phi(t_1 - t_0) = e^{A(t_2 - t_1)}e^{A(t_1 - t_0)}$$

$$= e^{A(t_2 - t_0)} = \phi(t_2 - t_0) \tag{13.125}$$

This property of the STM is important since it implies that a state-transition process can be divided into a number of sequential transitions.

(iv) $\left[\phi(t)\right]^k = \phi(kt)$ for k-positive integer (13.126)

Proof

$$\left[\phi(t)\right]^k = e^{At}e^{At}\ldots e^{At} \quad (k\ terms) \tag{13.127}$$

$$= e^{kAt} = \phi(kt)$$

The important properties of the STM are listed in Table 13.2.

Table 13.2 | Important properties of the STM

$\phi(0) = e^{A(0)} = I$
$\phi(t) = e^{At} = e^{-(-At)^{-1}} = \left[\phi(-t)\right]^{-1}$
$\phi^{-1}(t) = \left(e^{At}\right)^{-1} = e^{-At} = \phi(-t)$
$\phi(t_1 + t_2) = e^{At_1}.e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$
$\left[\phi(t)\right]^k = \left(e^{At}\right)^k = e^{A(kt)} = \phi(kt)$
$\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$

Example 13.17: **Find the STM for a system described by $\dot{x}(t) = Ax(t) + Bu(t)$ where $A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, by inverse Laplace transform method and by series summation method. Also, determine the solution of the system, that is, State Vector $x(t)$ with input $u(t) = 1$(unit step function) for $t \geq 0$ and initial vector $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.**

Solution: (a) To determine the STM

(i) Inverse Laplace Transform Method

Given $A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$

Therefore, $[sI - A] = \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}$

$$\phi(s) = [sI - A]^{-1} = \begin{bmatrix} \dfrac{1}{s-2} & 0 \\ \dfrac{-1}{(s-2)(s+1)} & \dfrac{1}{s+1} \end{bmatrix}$$

Taking inverse Laplace transform of the individual matrix elements,

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t},\ \mathcal{L}^{-1}\{0\} = 0,\ \mathcal{L}^{-1}\left\{\frac{-1}{(s-2)(s+1)}\right\} = -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\text{Hence, } \phi(t) = \mathcal{L}^{-1}\{\phi(s)\} = \begin{bmatrix} e^{2t} & 0 \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & e^{-t} \end{bmatrix} \tag{1}$$

(ii) Series Summation Method

Given $A = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$

Therefore, $A^2 = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$

Also, $A^3 = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -3 & -1 \end{bmatrix}$

By series summation method,

$$\phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!}$$

Neglecting higher order terms, considering till n = 3 and substituting A, A^2 and A^3 in the above equation, we obtain

$$\phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 8 & 0 \\ -3 & -1 \end{bmatrix} \frac{t^3}{6} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ -t & -t \end{bmatrix} + \begin{bmatrix} 2t^2 & 0 \\ -\frac{t^2}{2} & \frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} \frac{4t^3}{3} & 0 \\ -\frac{t^3}{2} & -\frac{t^3}{6} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + 2t + 2t^2 + \frac{4t^3}{3} + \cdots & 0 \\ -t - \frac{t^2}{2} - \frac{t^3}{2} + \cdots & 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & e^{-t} \end{bmatrix} \tag{2}$$

(b) Solution of the System

As the system is given by $\dot{x}(t) = Ax(t) + Bu(t)$, the solution of the system has two parts as given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{3}$$

Substituting $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u(t) = 1$ and STM $e^{At} = \phi(t) = \begin{bmatrix} e^{2t} & 0 \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & e^{-t} \end{bmatrix}$ in Eqn. (3), we obtain

$$x(t) = \begin{bmatrix} e^{2t} & 0 \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{2(t-\tau)} & 0 \\ -\frac{1}{3}e^{2(t-\tau)} + \frac{1}{3}e^{-(t-\tau)} & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{2t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{2(t-\tau)} \\ -\frac{1}{3}e^{2(t-\tau)} + \frac{2}{3}e^{-(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{2t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \end{bmatrix} + \begin{bmatrix} -\frac{e^{2(t-\tau)}}{2} \\ \frac{1}{6}e^{2(t-\tau)} + \frac{2}{3}e^{-(t-\tau)} \end{bmatrix}_0^t$$

$$= \begin{bmatrix} e^{2t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} + \frac{e^{2t}}{2} \\ \frac{1}{6} - \frac{1}{6}e^{2t} + \frac{2}{3} - \frac{2}{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} + \frac{e^{2t}}{2} \\ \frac{5}{6} - \frac{1}{6}e^{2t} - \frac{2}{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} + \frac{3e^{2t}}{2} \\ \frac{5}{6} - \frac{1}{2}e^{2t} - \frac{1}{3}e^{-t} \end{bmatrix}$$

Therefore, the solution with State Vectors $x_1(t)$ and $x_2(t)$ are

$$x_1(t) = -\frac{1}{2} + \frac{3e^{2t}}{2}$$

and

$$x_2(t) = \frac{5}{6} - \frac{1}{2}e^{2t} - \frac{1}{3}e^{-t}$$

Example 13.18: The state equation of a system is described by $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t)$. Find eigen values, eigen vectors and STM by λ matrix method. Also, determine the solution of the system, that is, state vector $x(t)$ with input $u(t) = 1$ (unit step function) for $t \geq 0$ and initial vector $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: To determine STM :

Given $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Therefore, $|\lambda I - A| = \left|\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}\right|$

$$= \begin{vmatrix}\lambda & -1\\2 & \lambda+3\end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2$$

Solving the equation $\lambda^2 + 3\lambda + 2 = 0$, the eigen values of matrix A are $\lambda_1 = -1$ and $\lambda_2 = -2$.

If R_1 is the eigen vector corresponding to the eigen value $\lambda_1 = -1$, then the equation $[\lambda_1 I - A]R_1 = 0$ must be satisfied.

i.e., $\left[(-1)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\-2 & 3\end{bmatrix}\right] - \begin{bmatrix}p_{11}\\p_{21}\end{bmatrix} = 0$

Since $\begin{bmatrix}1 & 1\\-2 & -2\end{bmatrix}\begin{bmatrix}p_{11}\\p_{21}\end{bmatrix} = 0,$

the equations obtained from the above matrix are

$$p_{11} + p_{21} = 0 \tag{1}$$

$$-2p_{11} - 2p_{21} = 0 \tag{2}$$

Solving Eqs. (1) and (2), we obtain $p_{11} = -p_{21}$.

Assuming $p_{11} = p_{21} = 1$, the eigen vector associated with the eigen value $\lambda_1 = -1$ is

$$p_1 = \begin{bmatrix}1\\-1\end{bmatrix}$$

If R_2 is the eigen vector corresponding to the eigen value $\lambda_1 = -2$, then the equation $[\lambda_1 I - A]R_2 = 0$ must be satisfied.

i.e., $\left[-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}\right]\begin{bmatrix}p_{12}\\p_{22}\end{bmatrix} = 0$

$$\begin{bmatrix}2 & 1\\-2 & -1\end{bmatrix}\begin{bmatrix}p_{12}\\p_{22}\end{bmatrix} = 0$$

The equations obtained from the above matrix are

$$2p_{12} + p_{22} = 0$$

$$-2p_{12} - p_{22} = 0$$

Solving Eqs. (3) and (4), we obtain $p_{22} = -2p_{12}$.

Assuming $p_{12} = 1$ and $p_{22} = -2$, the eigen vector associated with the eigen value $\lambda_2 = -2$ is

$$P_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore, the modal matrix is

$$M = \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

Taking inverse for the above matrix, we obtain

$$M^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

It is noted that if modal matrix M is correctly formed, it should diagonalise A through similarity transformation with the diagonal elements remaining the same as the determined eigen values. It is observed that,

$$M^{-1}AM = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \lambda \tag{3}$$

The above matrix shows that the determined modal matrix is correct since diagonal elements are same as the determined eigen values.

Since $\quad \lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad e^{\lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

Therefore, STM, $\phi(t) = Me^{\lambda t}M^{-1}$

$$\phi(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \tag{4}$$

Solution of the System

As the system is of homogenous type (i.e., $\dot{x}(t) = Ax(t)$), solution of the system is given by

$$x(t) = e^{At}x(0) \tag{5}$$

Substituting $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $u(t) = 1$ and STM $e^{At} = \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$ in Eqn. (5), we obtain

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix}$$

Therefore, the solutions with state vectors $x_1(t)$ and $x_2(t)$ are

$$x_1(t) = 2e^{-t} - e^{-2t}$$

and $$x_2(t) = 2e^{-t} - 2e^{-2t}$$

13.14 Controllability and Observability

A system is said to be controllable, if there exists some finite control vector $u(t)$ that will bring the system from any initial state $u(t)$ at $t = t_0$ to any specific desired state $x(t)$ in the state-space, within a specified finite time interval given by $t_0 \le t \le t_f$.

Observability is the dual of controllability, by which it is possible to construct an input vector $u(t)$, which is unconstrained, transferring an initial output $y(t_0)$ to a final output $y(t_f)$ within a specified finite time interval $t_0 \le t \le t_f$. Thus, the system is said to be observable if the outputs $y(t)$ can be measured by identifying every state $x(t_0)$ within a specified finite time interval.

13.14.1 Criteria for Controllability

A system with state equation, $\dot{x}(t) = Ax(t) + Bu(t)$ and output equation, $y(t) = Cx(t) + Du(t)$ is said to be controllable if the controllability matrix Q_c of order $n \times n$ given by

$$Q_c = \left[B : AB : A^2B : \ldots : A^{n-1}B \right] \text{ has rank } n.$$

13.14.2 Criteria for Observability

A system with state equation, $\dot{x}(t) = Ax(t) + Bu(t)$ and output equation, $y(t) = Cx(t) + Du(t)$ is said to be observable, if the observability matrix Q_o of order $n \times n$ given by

$$Q_o = \left[C^T : A^T C^T : \left(A^T\right)^2 C^T : \ldots : \left(A^T\right)^{n-1} C^T\right] \text{ has rank } n.$$

Example 13.19: A system is given by the state equation

$$x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u(t) \text{ and output equation } y(t) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \text{ Check}$$

whether the system is controllable and observable.

Solution: Comparing the standard state-space model of the system with the given state-space model, we obtain

$$A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \mathrm{D} = 0$$

Controllable

To check whether the system is controllable or not, the controllability matrix Q_c is to be determined using the state-space model of the system.

Since the order of matrix A is 3, the controllability matrix is given by $Q_c = \left[B : AB : A^2 B\right]$.

The controllability matrix for the given state-space model of the system is formed as follows:

Step 1: $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$

Step 2: $AB = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$

Step 3: $A^2 B = [A][AB] = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}$

Step 4: Therefore, $Q_c = \begin{bmatrix} 0 & 1 & 2 & -2 & -2 & 7 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$

Step 5: Since Q_c is not a square matrix, it is necessary to check all the possibility of higher order square matrix that can be obtained from Q_c. In this case, the higher order matrix that can be obtained from Q_c is 3×3. The number of 3×3 matrix that can be obtained from Q_c is 20. Therefore, if the determinant value of any one of 20 matrices is non-zero, then the system is said to be completely controllable. For the obtained Q_c, the determinant value of all the 3×3 matrix is non-zero, therefore the rank of $Q_c = 3$.

Step 6: Since the rank of controllability matrix and order of matrix A are same, the given system is completely controllable.

Observable

To check whether the system is observable or not, the observability matrix Q_o is to be determined using the state-space model of the system.

Since the order of matrix A is 3, the observability matrix is given by $Q_o = \left[C^T : A^T C^T : A^{T^2} C^T\right]$.

The observability matrix for the given state-space model of the system is formed as:

Step 1: $C^T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Step 2: $A^T C^T = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

Step 3: $A^{T^2} C^T = A^T\left[A^T C^T\right] = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 11 \\ 0 & -4 \\ 1 & -1 \end{bmatrix}$

Step 4: Therefore, $Q_o = \begin{bmatrix} 0 & 1 & 0 & -4 & 0 & 11 \\ 0 & 1 & 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 & 1 & -1 \end{bmatrix}$

Step 5: $|Q_o| = \begin{vmatrix} -4 & 0 & 11 \\ 1 & 0 & -4 \\ 2 & 1 & -1 \end{vmatrix} = -5 \neq 0$. Therefore, rank of $Q_o = 3$.

Step 6: Since the rank of observability matrix and order of matrix A are same, the given system is completely observable.

Therefore, the given system is completely controllable and observable.

Example 13.20: **Write the state equation for the block diagram of the system shown below in Fig . 13.21(a), in which $x_1(t)$, $x_2(t)$ and $x_3(t)$ constitute the state vector. Also, determine whether the system is completely controllable and observable.**

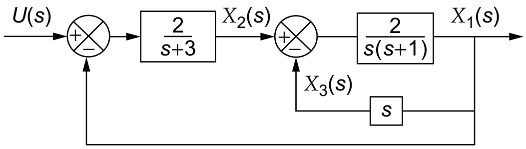


Fig. E13.21(a)

Solution: The state equations are obtained by considering the blocks shown in Fig. 13.21(a). Consider the first block as shown in Fig. E13.21(b).

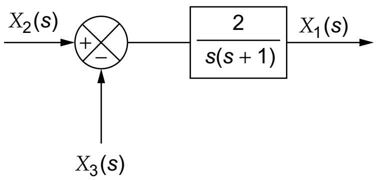


Fig. E13.21(b)

From Fig. 13.21(b), it is observed that

$$X_1(s) = \frac{2}{s(s+1)}\left[X_2(s) - X_3(s)\right]$$

Therefore, $\left[s^2 + s\right]X_1(s) = 2X_2(s) - 2X_3(s)$

Taking inverse Laplace transform, we obtain

$$\ddot{x}_1(t) + \dot{x}_1(t) = 2x_2(t) - 2x_3(t) \tag{1}$$

Consider the second block as shown in Fig. E13.21(c).

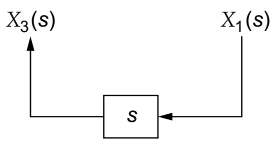


Fig. E13.21(c)

From Fig. 13.21(c), it is observed that

$$X_3(s) = sX_1(s)$$

Taking inverse Laplace transform, we obtain

$$\dot{x}_1(t) = x_3(t) \tag{2}$$

Consider the third block as shown in Fig . E13.21(d).

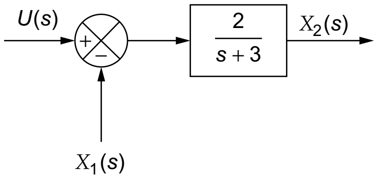


Fig. E13.21(d)

From Fig. E13.21(d), it is observed that

$$X_2(s) = \left(\frac{2}{s+3}\right)\left[U(s) - X_1(s)\right]$$

Simplifying, we obtain

$$(s+3)X_2(s) = 2U(s) - 2X_1(s)$$

$$sX_2(s) = -2X_1(s) - 3X_2(s) + 2U(s)$$

Taking inverse Laplace transform, we obtain

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + 2u(t) \tag{3}$$

Differentiating Eqn. (2) with respect to t, we obtain

$$\ddot{x}_1(t) = \dot{x}_3(t)$$

Substituting the above equation and Eqn. (2) in Eqn. (1) and simplifying, we obtain

$$\dot{x}_3(t) = 2x_2(t) - 3x_3(t) \tag{4}$$

Let the output $y(t) = x_1(t)$ (5)

Using Eqs. (2), (3) , (4) and Eqn. (5), the required state-space model of the system is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u(t) \text{ (state equation)} \tag{6}$$

$$y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} \text{ (output equation)} \tag{7}$$

From Eqs. (6) and (7), we obtain

$$A=\begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix},\ B=\begin{bmatrix}0\\ 2\\ 0\end{bmatrix} \text{ and } C=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

Controllable

To check whether the system is controllable or not, the controllability matrix Q_c is to be determined using the state-space model of the system.

Since the order of matrix A is 3, the controllability matrix is given by $Q_c=\left[B:AB:A^2B\right]$.

The controllability matrix for the given state-space model of the system is formed as g:

Step 1: $B=\begin{bmatrix}0\\ 2\\ 0\end{bmatrix}$

Step 2: $AB=\begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix}\times\begin{bmatrix}0\\ 2\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -6\\ 4\end{bmatrix}$

Step 3: $A^2B=A(AB)=\begin{bmatrix}0 & 0 & 1\\ -2 & -3 & 0\\ 0 & 2 & -3\end{bmatrix}\times\begin{bmatrix}0\\ -6\\ 4\end{bmatrix}=\begin{bmatrix}4\\ 18\\ -24\end{bmatrix}$

Step 4: Therefore, $Q_c=\begin{bmatrix}0 & 0 & 4\\ 2 & -6 & 18\\ 0 & 4 & -24\end{bmatrix}$

Step 5: $|Q_c|=\begin{vmatrix}0 & 0 & 4\\ 2 & -6 & 18\\ 0 & 4 & -24\end{vmatrix}=32\neq 0$. Therefore, rank of $Q_c=3$.

Step 6: Since the rank of controllability matrix and order of matrix A are same, the given system is completely controllable.

Observable

To check whether the system is observable or not, the observability matrix Q_o is to be determined using the state-space model of the system.

Since the order of matrix A is 3, the observability matrix is given by $Q_o=\left[C^T:A^TC^T:A^{T^2}C^T\right]$.

The observability matrix for the given state-space model of the system is formed as follows:

Step 1: $C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Step 2: $A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Step 3: $\left(A^T\right)^2 C^T = A^T\left[A^T C^T\right] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

Step 4: Therefore, $Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

Step 5: $|Q_o| \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} = -2 \neq 0$. Therefore, rank of $Q_o = 3$.

Step 6: Since the rank of observability matrix and order of matrix A are same, the given system is completely observable.

Therefore, the given system is completely controllable and observable.

13.15 State-Space Representation of Discrete-Time LTI Systems

Consider a SISO discrete-time LTI system which is described by an Nth-order difference equation

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = u[n] \tag{13.128}$$

Here, if $u[n]$ is given for $n \geq 0$, Eqn. (13.128) requires N initial conditions $y[-1], y[-2], \ldots, y[-N]$ to uniquely determine the complete solution for $n > 0$. Thus, N values are required to specify the state of the system at any time.

Then, N state variables $x_1[n], x_2[n], \ldots, x_N[n]$ are defined as

$$\begin{aligned}
x_1[n] &= y[n-N] \\
x_2[n] &= y[n-(N-1)] = y[n-N+1] = x_1[n+1] \\
x_3[n] &= y[n-(N-2)] = y[n-N+2] = x_2[n+1] \\
&\vdots \\
x_{N-1}[n] &= y[n-2] = x_{N-2}[n+1] \\
x_N[n] &= y[n-1]
\end{aligned} \tag{13.129}$$

Then from Eqs. (13.128) and (13.129), we have

$$x_N[n+1] = -a_N x_1[n] - a_{N-1}x_2[n] - \cdots - a_1 x_N[n] + u[n]$$

and
$$y[n] = -a_N x_1[n] - a_{N-1}x_2[n] - \cdots - a_1 x_N[n] + u[n]$$

In matrix form, the above equations can be expressed as

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ \vdots \\ x_N[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u[n] \tag{13.130}$$

$$y[n] = \begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} + [1]u[n] \tag{13.131}$$

Equations (13.130) and (13.131) can be rewritten compactly as

$$x[n+1] = Ax[n] + Bu[n] \tag{13.132}$$

$$y[n] = Cx[n] + Du[n] \tag{13.133}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_1 \end{bmatrix}$;

$D = 1$ and $x[n]$ is the $N \times 1$ matrix (or N-dimensional vector) state vector which is given by

$x[n] = \begin{bmatrix} x_1[n] & x_2[n] & x_3[n] & \ldots & x_N[n] \end{bmatrix}$

Equations (13.132) and (13.133) which represent the state equation and output equation are called as N-dimensional state-space representation of the system and the $N \times N$ matrix A is called the system matrix.

If a discrete-time LTI system has m inputs and p outputs and N state variables, then a state-space representation of the system can be represented as

$$x[n+1] = Ax[n] + Bu[n]$$
$$y[n] = Cx[n] + Du[n]$$

where $x[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix}$, $u[n] = \begin{bmatrix} u_1[n] \\ u_2[n] \\ \vdots \\ u_m[n] \end{bmatrix}$ and $y[n] = \begin{bmatrix} y_1[n] \\ y_2[n] \\ \vdots \\ y_p[n] \end{bmatrix}$

and matrices $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{Nm} \end{bmatrix}$

$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & a_{p2} & \cdots & c_{pN} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$

where A is a matrix of order $N \times N$, B is a matrix of order $N \times m$, C is a matrix of order $p \times N$, D is a matrix of order $p \times m$, $x[n]$ is a matrix of order $N \times 1$, $u[n]$ is a matrix of order $n \times n$ and $y[n]$ is a matrix of order $p \times 1$.

13.15.1 Block Diagram and SFG of Discrete State-Space Model

The basic block which is to be used in representing the discrete state-space model using block diagram technique is the delay unit (in continuous state-space model it is integrator block). The steps to be followed in representing the discrete state-space model using block diagram is similar to the steps discussed in Section 13.3. The block diagram representation of the discrete state-space model is shown in Fig. 13.16.

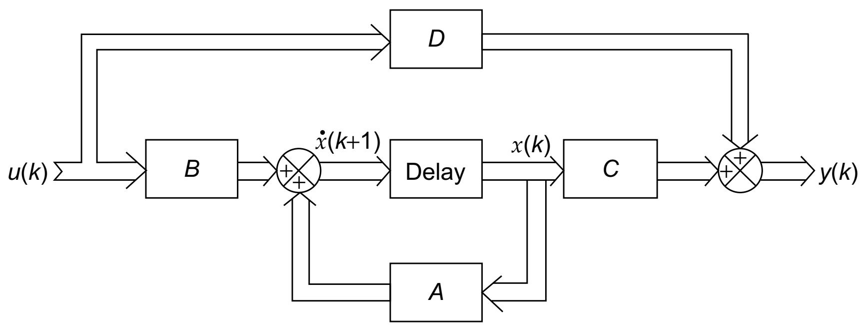


Fig. 13.16 | Block diagram of the state-variable model

The SFG representation for the block diagram of the system shown in Fig. 13.16 is shown in Fig. 13.17.

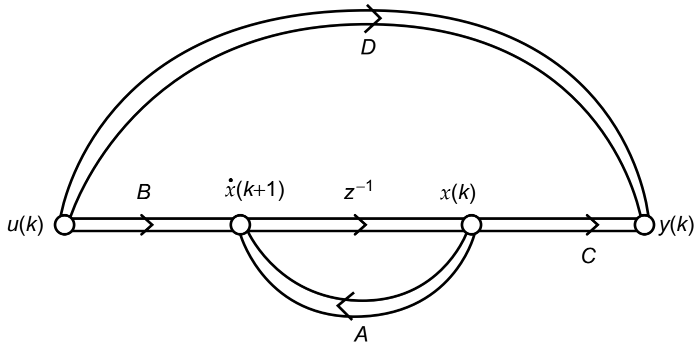


Fig. 13.17 | SFG representation of the state-variable model

13.16 Solutions of State Equations for Discrete-Time LTI Systems

Consider an N-dimensional state representation

$$x[n+1] = Ax[n] + Bu[n] \tag{13.134}$$

$$y[n] = Cx[n] + Du[n] \tag{13.135}$$

where A, B, C and D are $N \times N$, $N \times 1$, $1 \times N$ and 1×1 matrices, respectively.

Taking the z-transform of Eqs. (13.134) and (13.135) and using time-shifting property of z-transform, we obtain

$$zX(z) - zx(0) = AX(z) + BU(z) \tag{13.136}$$

$$Y(z) = CX(z) + DU(z) \tag{13.137}$$

where $X(z) = Z\{x[n]\}$, $Y(z) = Z\{y[n]\}$ and

$$X(z) = Z\{x[n]\} = \begin{bmatrix} X_1(z) \\ X_2(z) \\ \vdots \\ X_N(z) \end{bmatrix} \text{ where } X_k(z) = Z\{x_k[n]\}$$

Rearranging Eqn. (13.136), we have

$$(zI - A)X(z) = zx(0) + BU(z) \tag{13.138}$$

Premultiplying both sides by $(zI - A)^{-1}$, we obtain

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(z) \tag{13.139}$$

Taking inverse z-transform, we obtain

$$x[n] = Z^{-1}\left\{(zI - A)^{-1} z\right\} x(0) + Z^{-1}\left\{(zI - A)^{-1} BU(z)\right\} \tag{13.140}$$

Substituting Eqn. (13.140) in Eqn. (13.135), we obtain

$$y[n] = CZ^{-1}\left\{(zI - A)^{-1} z\right\} x(0) + CZ^{-1}\left\{(zI - A)^{-1} BU(z)\right\} + Du[n] \tag{13.141}$$

13.16.1 System Function $H(z)$

The system function $H(z)$ of a discrete-time LTI system is defined by $H(z) = Y(z)/U(z)$ with zero initial conditions. Thus, setting $x[0] = 0$ in Eqn. (13.140), we have

$$X(z) = (zI - A)^{-1} BU(z) \tag{13.142}$$

Substituting Eqn. (13.142) in Eqn. (13.137), we obtain

$$Y(z) = \left[C\,(zI - A)^{-1} B + D\right] U(z)$$

Thus, $H(z) = \dfrac{Y(z)}{U(z)} = \left[C(zI - A)^{-1} B + D\right]$

13.17 Representation of Discrete LTI System

The methods used for representing continuous linear time-invariant systems which have been discussed in Sections 13.10 and 13.11 are also applicable to the discrete linear time-invariant systems except for the fact that s is replaced by z.

Example 13.21 **Determine the state equations of a discrete-time LTI system with system function** $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$.

Solution: Comparing the given system function with Eqn. (13.34), we can get the state and output equations as

$$x[n+1] = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x[n] + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \end{bmatrix} u[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[n] + b_0 u[n]$$

Example 13.22: Given a discrete-time LTI system with system function $H(z)=\frac{z}{2z^2-3z+1}$, find a state representation of the system.

Solution: Given $H(z)=\frac{z}{2z^2-3z+1}$

Therefore, $H(z)=\dfrac{z}{2z^2\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}=\dfrac{\frac{1}{2}z^{-1}}{\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}$

$$H(z)=\frac{Y(z)}{U(z)}=\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}$$

Therefore, $x[n+1]=\begin{bmatrix}\frac{3}{2} & 1\\ -\frac{1}{2} & 0\end{bmatrix}x[n]+\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}u[n]$

and $y[n]=\begin{bmatrix}1 & 0\end{bmatrix}x(n)$

Example 13.23: Sketch a block diagram of a discrete-time system, $x[n+1]=\begin{bmatrix}0 & 1\\ 2 & 5\end{bmatrix}x[n]+\begin{bmatrix}0\\ 1\end{bmatrix}u[n]$ and $y[n]=[8 \quad -7]x[n]$ with the state-space representation.

Solution: The given equation can be expressed as

$$x_1[n+1]=x_2[n]$$

$$x_2[n+1]=2x_1[n]+5x_2[n]+u[n]$$

$$y[n]=8x_1[n]-7x_2[n]$$

Therefore, the block diagram representation for the above equations is shown in Fig. E13.23.

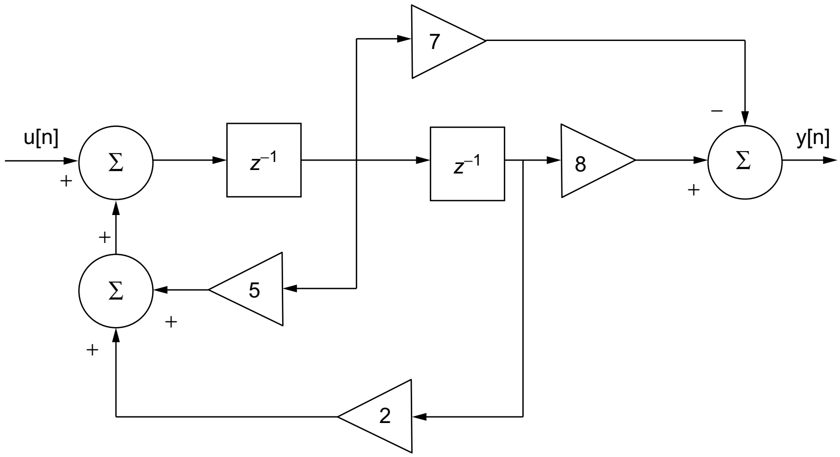


Fig. E13.23

13.18 Sampling

Let $x(t)$ be a continuous-time varying signal. The signal $x(t)$ which is shown in Fig . 13.18(a) is sampled at regular intervals of time with sampling period T as shown in Fig. 13.18(b). The sample signal $x(nT)$ is given by

$$x(nT) = x(t)\big|_{t=nT}, \quad -\infty < n < \infty \tag{13.143}$$

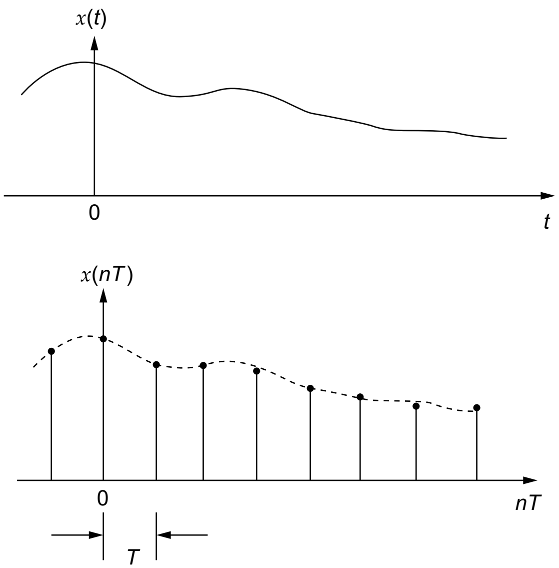


Fig. 13.18 | (a) Continuous-time signal and (b) sampling of a continuous-time signal

A sampling process can be interpreted as a modulation or multiplication process, as shown in Fig. 13.19.

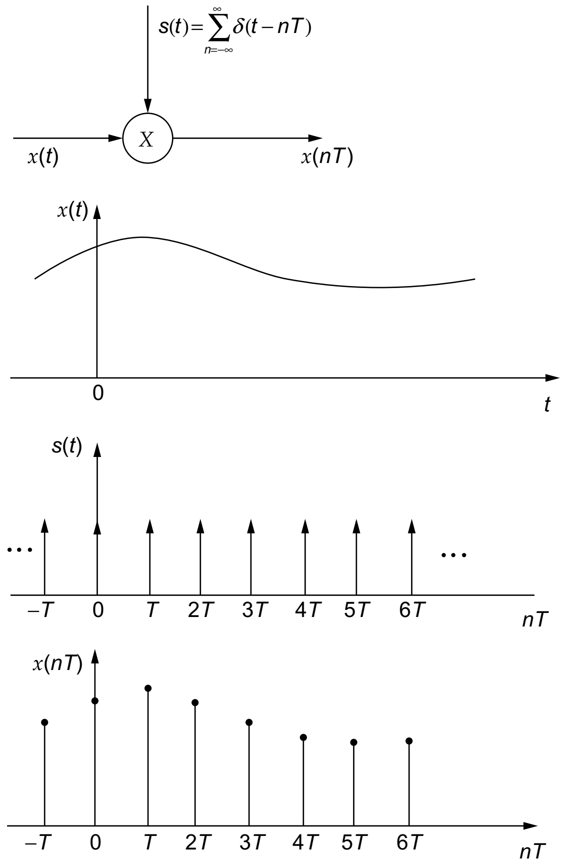


Fig. 13.19 | Periodic sampling of $x(t)$

The continuous-time signal $x(t)$ is multiplied by the sampling function $s(t)$ which is a series of impulses (periodic impulse train); the resultant signal is a discrete-time signal $x(nT)$.

$$x(nT) = x(t) \sum_{n=-\infty} \delta(t - nT) \tag{13.144}$$

13.18.1 Sampling Theorem

The sampling theorem states that a band-limited signal $x(t)$ having finite energy which has no frequency components higher than f_h Hz can be completely reconstructed from its samples taken at the rate of $2f_h$ samples per second (i.e., $f_s \geq 2f_h$ where f_s is the sampling frequency and f_h is the highest signal frequency.

The sampling rate of $2f_h$ samples per second is the Nyquist rate and its reciprocal $\dfrac{1}{2f_h}$ is the Nyquist period. For simplicity, $x(nT)$ is denoted as $x(n)$.

13.18.2 High Speed Sample-and-Hold Circuit

Figure 13.20(a) shows a sample-and-hold circuit for high speed of operation. The MOS transistor M shown is an analog switch is capable of switching by logic levels, such as that from TTL. It alternately connects and disconnects the capacitor C_1 to the output of op-amp A_1. Diodes D_1 and D_2 are inverse-parallel connected. They prevent op-amp A_1 from getting into saturation when the transistor M is OFF. This makes the operation of the circuit faster. Hence, the output of op-amp A_1 will be $v_o'(t) \approx v_i(t) - 0.7V$ when $v_i(t) < v_o(t)$ and $v_o'(t) \approx v_i(t) + 0.7V$ when $v_i(t) > v_o(t)$.

When transistor M is ON, the op-amps A_1 and A_2 act as voltage followers. The waveforms shown in Fig . 13.20(b) illustrate the operation of the circuit. The transistor M is alternately switched ON and OFF by the control voltage V_S at its gate terminal. Note that the voltage V_S must be higher than the threshold voltage of the FET. When the transistor switch M is ON for a short interval of time, the capacitor C_1 quickly charges or discharges to the value of the analog signal at that instant. In other words, when input v_i is larger than capacitor voltage v_C and the transistor M is OFF, it rapidly charges to the level of v_i the instant M switches ON. Similarly, if v_C is initially greater than v_i, then C_1 rapidly discharges to the level of v_i when M becomes ON.

When M is OFF, only the input bias current of op-amp A_2 and the gate-source reverse leakage current of FET are effective in discharging the capacitor. Hence, the sampled voltage is held constant by C_1 until the next *sampling instant* or *acquisition time*. Figure 13.20(c) shows the *sampling* or *acquisition* time t_1 and *holding* time t_2. During the sampling time t_1, C_1 is charged through the FET channel resistance $R_{DS(ON)}$ and the charging time $t_1 = 5R_{DS(ON)}C_1$ when the capacitor charges to 0.993 of input voltage. During the hold time t_2, the capacitor partially discharges. This is called *hold-mode droop*. To avoid this, the op-amp A_2 must have very low input bias current, the capacitor should have a low leakage dielectric and M must have very low reverse leakage current between its gate and source terminals. The low channel resistance $R_{DS(ON)}$ is desirable for the FET to achieve faster charging and discharging of C_1.

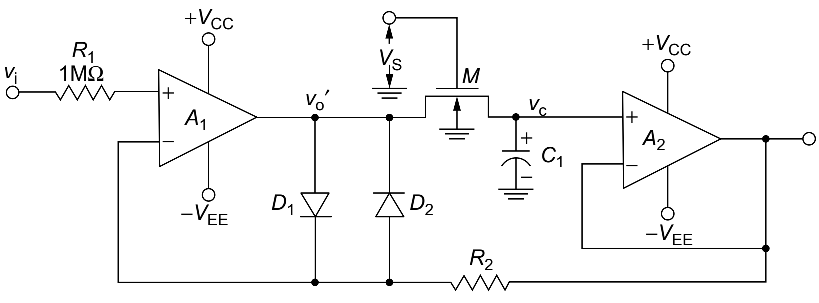


Fig. 13.20(a) | Sample-and-hold circuit

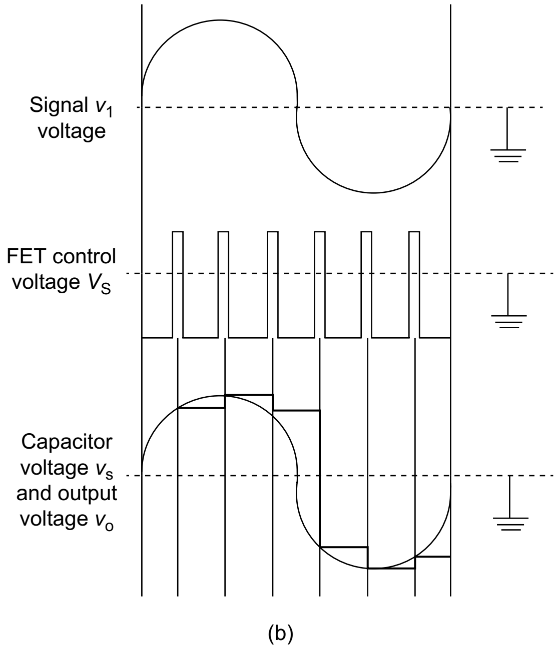


Fig. 13.20(b) | Signal voltage, control voltage and output voltage waveforms

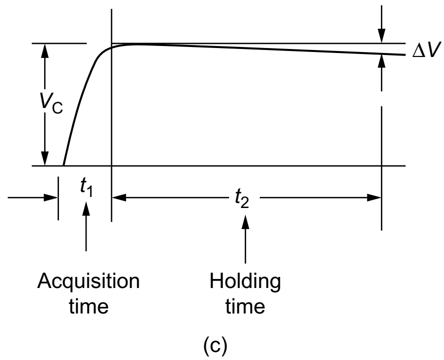


Fig. 13.20(c) | Capacitor voltage waveform

13.18.2.1 Continuous-Data Control Systems

In continuous-data control system, the signals at various parts of the system are *l* functions of the continuous-time variable *t*. Examples for continuous-data control system include AC control system and DC control system.

In DC control system, the signals are unmodulated. The schematic diagram of a closed-loop DC control system with waveforms of the signals in response to a step-function input are also shown in Fig. 13.21. Typical components of a DC control system are DC tachometers, potentiometers, DC motors, etc.

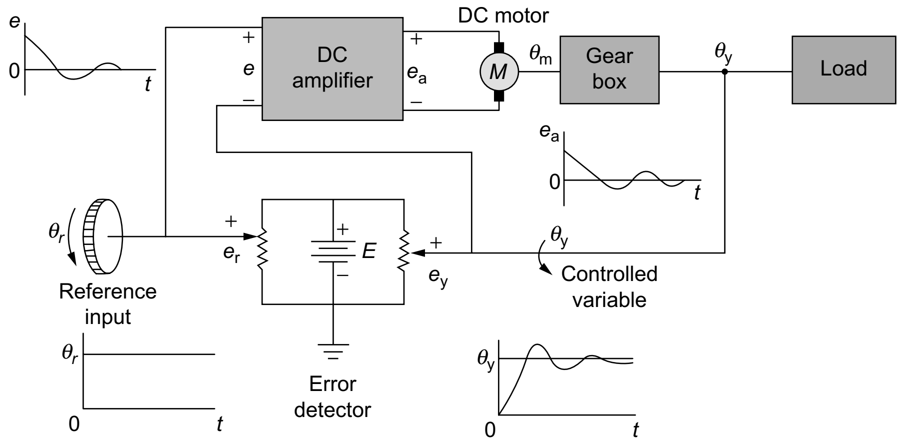


Fig. 13.21 | Schematic diagram of a typical DC closed-loop control system

In AC control system, the signals are modulated. The schematic diagram of a typical AC control system is shown in Fig. 13.22. The modulated information signal transmitted by an AC carrier signal is demodulated by the low-pass characteristics of the AC motor. AC control systems are used extensively in aircraft and missile control systems, wherein noise and disturbance often cause problems. By using modulated AC control systems with carrier frequencies of 400 Hz or higher, the system will be less vulnerable to low-frequency noise. Typical components of an AC control system are synchros, AC amplifiers, gyroscopes, accelerometers, AC motors, etc.

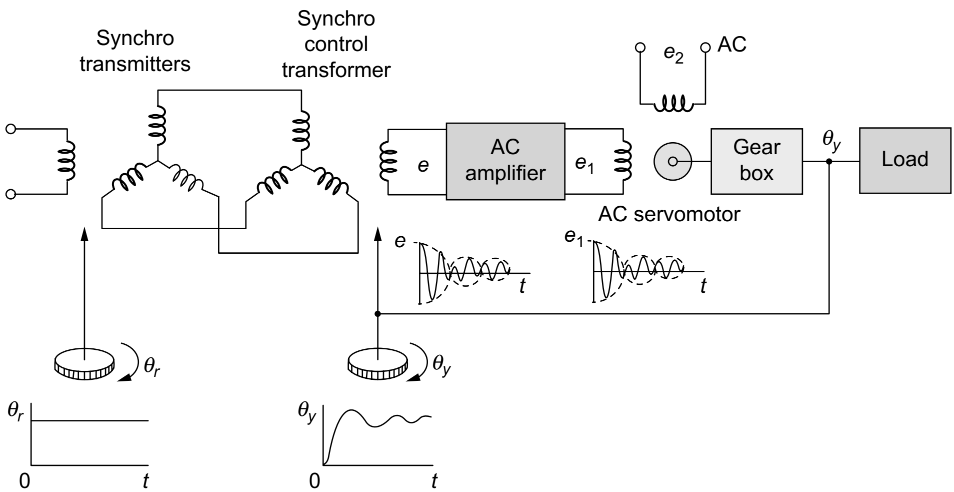


Fig. 13.22 | Schematic diagram of a typical AC closed-loop control system

13.18.2.2 Discrete-Data Control Systems

In discrete-date control systems, the signals provided to the system are in the form of either train of pulses or a digital code. Discrete-data control systems are divided into sampled-data and digital control systems. In sampled-data control systems the signals are in the form of pulse data. In digital-data control system digital computer or controller is used in the system so that the signals are digitally coded.

In general, a sampled-data system receives data or information intermittently at specific instants of time. For example, the error signal in the sampled-data control system can be supplied only in the form of pulses, which has information about the error signal during the periods between two consecutive pulses.

Figure 13.23 shows the block diagram of sampled-data control system. A continuous-data input signal $r(t)$, applied to the system is compared with the output signal $y(t)$ and the error signal $e(t)$ is sent to the sampler, from which a sequence of pulses are obtained as output. The sampling rate of the sampler can be uniform or non-uniform. The advantages of the sampling operation is that expensive equipment used in the system may be utilized effectively as it would be time-shared among several other control equipments and noise cancellation is effectively achieved.

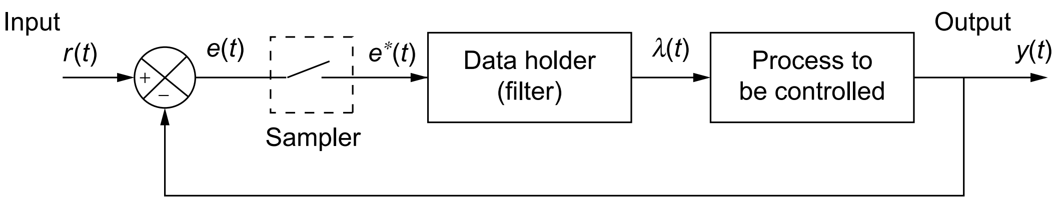


Fig. 13.23 | Block diagram of sampled-data control system

Digital computers provide many advantages such as reduced size and increased flexibility. Hence, computer control is a popular technique employed in recent times. Figure 13.24 shows the block diagram of a digital-data control system which is used as autopilot for guided missile control.

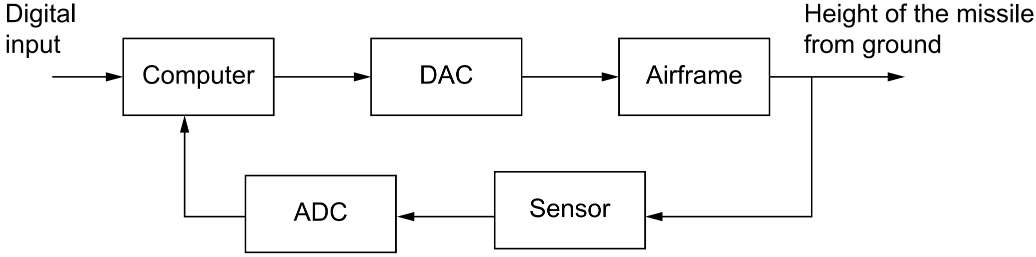


Fig. 13.24 | Example for digital-data control system (a guided missile)

Review Questions

1. Explain the concept of state.
2. Define a) state variables b) state vector and c) state-space.
3. What are the advantages of state-space analysis?
4. What do you mean by homogenous and non-homogenous state equations?
5. Define state transition matrix.
6. What are the properties of state transition matrix?
7. What are the advantages and disadvantages of representing the system using phase variables?
8. What are the advantages and disadvantages of representing the system using canonical variables?
9. What are the different methods available in representing the system using state-space model?
10. Define controllability and observability.
11. What is the difference between continuous-time signal and discrete-time signal?
12. What is a sampler?
13. Explain the sampling process.
14. Check the following systems for controllability and observability:

(a) $A=\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C=\begin{bmatrix} 1 & -1 \end{bmatrix}$

(b) $A=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

(c) $A=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix}$, $B=\begin{bmatrix} 40 \\ 10 \\ 0 \end{bmatrix}$, $C=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

15. The state equation for a system is $\dot{X}(t)=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}U(t)$. Determine the unit step response of the system when $X(0)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
16. The state equation for a homogenous system is $\dot{X}(t)=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$. Determine the solution of the system when $X(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
17. Determine the system matrix A for a certain system when $X(0)=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, we have $X(t)=\begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$ and when $X(0)=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $X(t)=\begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$.

18. The state-space model of a system is given by $\dot{X}(t)=\begin{bmatrix}0 & 3\\ -2 & -5\end{bmatrix}\begin{bmatrix}X_1(t)\\ X_2(t)\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}U(t)$ and $Y(t)=\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}X_1(t)\\ X_2(t)\end{bmatrix}$. Determine the transfer function of the system.

19. The state-space model of a system is given by $\dot{X}(t)=\begin{bmatrix}0 & 3\\ -2 & -5\end{bmatrix}\begin{bmatrix}X_1(t)\\ X_2(t)\end{bmatrix}+\begin{bmatrix}3\\ 5\end{bmatrix}U(t)$ and $Y(t)=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}X_1(t)\\ X_2(t)\end{bmatrix}$. Determine the transfer function of the system.

20. Obtain the state-space model of the system whose transfer function is given by $\frac{C(s)}{R(s)}=\frac{1}{(s+2)^2(s+1)}$ in Jordan cananonical form.

21. The transfer function of the system is given by $\frac{C(s)}{R(s)}=\frac{s^2+3s+3}{s^3+2s^2+3s+1}$. Determine the state-space model of the system using phase variables.

22. The differential equation of a system is given by $\frac{d^3Y(t)}{dt^3}+4\frac{d^2Y(t)}{dt^2}+7\frac{dY(t)}{dt}+2Y(t)$ $=5U(t)$. Determine the state-space model of the system.

23. Determine the STM for a system whose A matrix is given by $\begin{bmatrix}0 & -1\\ 2 & -3\end{bmatrix}$.

24. The difference equation of a system is given by $6y(n+3)+y(n+2)+5y(n+1)+6y(n)$ $=u(n)$. Determine the state-space model of the system.

This Page is intentionally left blank.

14

MATLAB PROGRAMS

14.1 Introduction

MATLAB stands for MATrixLABoratory. It is a technical computing environment for high-performance numeric computation and visualization. It integrates numerical analysis, matrix computation, signal processing and graphics in an easy-to-use environment, where problems and solutions are mathematically expressed instead of traditional programming. MATLAB allows us to express the entire algorithm in a few dozen lines, to compute the solution with great accuracy in a few minutes on a computer and to readily manipulate a three-dimensional display of the result in colour.

MATLAB is an interactive system whose basic data element is a matrix that does not require dimensioning. It enables us to solve many numerical problems in a fraction of the time than it would take to write a program and execute in a language such as FORTRAN, BASIC or C. It also features a family of application-specific solutions called toolboxes. Areas in which toolboxes are available include control system design, image processing, signal processing, dynamic system simulation, system identification, neural networks, wavelength communication and others. It can handle linear, non-linear, continuous-time, discrete-time, multivariable and multirate systems. This chapter gives simple programmes to solve specific problems that are included in the previous chapters.

The system models in a control system which are developed by applying laws such as Kirchhoff's current law, Kirchhoff's voltage law, Newton's laws and so on lead to linear, non-linear and ordinary differential equations. The standard input signals are applied to the system model to obtain the response of a control system. The compensation system are used along with the system to improve and optimize the system if the response of the system is unsatisfactory.

There exists two approach for the analysis and design of control systems. They are classical or frequency domain approach and state–variable approach. In classical approach, differential or difference equations are converted into transfer functions by relating the input and output using Laplace or z-transforms and then the transfer functions are used for analysing the system.

14.2 MATLAB in Control Systems

14.2.1 Laplace Transform

Laplace transform converts the differential equations of a dynamic system into algebraic equations of a complex variable *s*. The syntax for using the Laplace transform in MATLAB is **laplace(function_in_time_domain)**.

The commands which are to be used in getting the Laplace transform of a system are:

```
syms t
f = function_in_t
laplace(f)
```

14.2.2 Inverse Laplace Transform

The inverse Laplace transform helps in determining the system equations in time domain *t* from the equations in *s* domain. The syntax for using the inverse Laplace transform in MATLAB is

ilaplace(function_in_s_domain).

The commands which are to be used in getting the system equation in time domain are:

```
syms s,t
f = function_in_s
ilaplace(f)
```

Example for the usage of **laplace** and **ilaplace** command is given in Table 14.1

Table 14.1 | Laplace and inverse Laplace transform in MATLAB

Code	Output
syms s, t f = t; f1 = laplace(f) f2 = 1/s; f3 = ilaplace(f3)	f1 = 1/s^2 f3 = 1

14.2.3 Partial Fraction Expansion

Let us consider a polynomial $P(s)=\dfrac{N(s)}{D(s)}=\dfrac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_0}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_0}$ where b_i and a_i are coefficients of s^i in the numerator and denominator respectively. The above polynomial can be simplified using partial fraction expansion that is a tedious process. MATLAB provides a simplified way for converting a polynomial into a partial fractions. The command in MATLAB for partial fraction conversion is residue (numerator, denominator).

The commands which are to used in getting the partial fraction of a polynomial is given below:

n = [b_m b_{m-1} $\cdots$ b_1 b_0] % coefficients of numerator polynomial
d = [a_n a_{n-1} $\cdots$ a_1 a_0] % coefficients of denominator polynomial
[r, p, k] = residue (n, d)

where r is the column vector that contains the residues, **p** is the column vector that contains the poles of the system and **k** is the row vector which contains the direct terms.

Therefore, the partial fraction of the $P(s)$ is

$$P(s)=\frac{r(1)}{s-p(1)}+\frac{r(2)}{s-p(2)}+\cdots+\frac{r(n)}{s-p(n)}+k(s)$$

If all the values in the column vector p are same, then

$$P(s)=\frac{r(1)}{(s-p)}+\frac{r(2)}{(s-p)^2}+\cdots+\frac{r(n)}{(s-p)^{\mathrm{n}}}+k(s)$$

Example for the usage of residue command is given in Table 14.2.

Consider a polynomial $P(s)=\dfrac{(s+1)}{(s^3+6s^2+12s+8)}$

Table 14.2 | Partial fraction in MATLAB

Code	Output
n = [1 1]; **d = [1 6 12 8];** **[r, p, k] = residue (n, d)**	**r =** **0** **1.0000** **−1.0000** **p =** **−2.0000** **−2.0000** **−2.0000** **k =** **[]**

Therefore, $P(s)=\dfrac{0}{s+2}+\dfrac{1}{(s+2)^2}+\dfrac{-1}{(s+2)^3}$

Similarly, to obtain polynomials of the numerator and the denominator, we can use the same residue command as follows:

[n, d] = residue(r, p, k)

Example for the usage of residue command for getting the coefficients of polynomial is given in Table 14.3.

Table 14.3 | Usage of residue command in MATLAB

Code	Output
r = [0; 1; −1]; **p = [−2; −2; −2];** **k = [];** **[n, d] = residue (r, p, k)**	**n =** **0 1 1** **d =** **1 6 12 8**

14.2.4 Transfer Function Representation

The time-domain analysis, frequency domain analysis and stability of the system can be analysed easily when the model of the system is represented in the form of transfer function. Let the transfer function of a system be

$$G(s)=\frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s}$$

The above transfer function can be represented in MATLAB, if the coefficients of numerator polynomial and denominator polynomial are known. These coefficients are known by taking the Laplace transform of the equation in time domain which has been explained in the previous section.

Consider the transfer function of a system be

$$G(s)=\frac{N(s)}{D(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_0}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_0}$$

In MATLAB, this transfer function can be represented using the following syntax:

n = [b_m b_{m-1} ⋯ b_1 b_0]
d = [a_n a_{n-1} ⋯ a_1 a_0]
sys = tf (n, d)

Similarly, if the transfer function of the system is obtained in MATLAB, the coefficients of the numerator and denominator polynomials can be obtained using the command **tfdata**. The syntax for using the command **tfdata** is

[n, d] = tfdata(sytem_transfer_function, 'v')

where the variable **v** ensures that the output variables **n** and **d** are polynomials of transfer function and return the values as a row vector. The usage of commands **tf** and **tfdata** for getting the transfer function of a system and getting the coefficients of the polynomial is given in Table 14.4.

Table 14.4 | Usage of tf and tfdata commands in MATLAB

Code	Output
n = [1 2]; d = [1 10 15 150]; sys = tf (n,d) [n1, d1] = tfdata (sys, 'v')	sys = s + 2 ------------------------------ s^3 + 10 s^2 + 15 s + 150 Continuous-time transfer function n1 = 0 0 1 2 d1 = 1 10 15 150

14.2.5 Zeros and Poles of a Transfer Function

The gain, zeros and poles of a transfer function can be obtained in MATLAB using the command **tf2zp**. The syntax for using the command is

[z, p, k] = tf2zp (numerator_poly, denominator_poly)

where **z** is the zeros of the transfer function, **p** is the poles of the transfer function and **k** is the gain of the system.
Example for the usage of **tf2zp** for getting the gain, zeros and poles of a transfer function is given in Table 14.5.

Table 14.5 | Zeros and poles from a transfer function in MATLAB

Code	Output
n = [1 2]; d = [1 10 15 150]; [z, p, k] = tf2zp (n, d)	z = −2 p = −10.0000 −0.0000 + 3.8730i −0.0000 − 3.8730i k = 1

Also, from the poles, zeros and gain of a system, the coefficients of the polynomial of a transfer function can be obtained using the command **zp2tf**. The syntax for using the command is

[n, d] = zp2tf (z, p, k)

Example for the usage of **zp2tf** for getting the coefficients of polynomial of a transfer function is given in Table 14.6.

Table 14.6 | Coefficients of polynomial from zeros and poles of a system

Code	Output
z = [−2]; p = [0 −1 −4]; k = 10; [n, d] = zp2tf(z, p, k) printsys (n, d, 's')	n = 0 0 10 20 d = 1 5 4 0 num/den = 10 s + 20 ------------------------ s ^ 3 + 5 s ^ 2 + 4 s

The command **printsys** used in the above code will be help in printing the transfer function of the system as a ratio of two polynomials in the variable mentioned within the quotes.

It is to be noted that if the commands **tf2zp** and **zp2tf** are used alone (i.e., without any output variable), the output will show only the zeros and coefficient of the numerator polynomial respectively. An example to explain the above concept is given in Table 14.7.

Table 14.7 | Usage of tf2zp and zp2tf in MATLAB

Code	Output
n = [1 2 5 6]; d = [3 4 8 9 10]; tf2zp(n, d)	ans = −0.2836 + 2.0266i −0.2836 − 2.0266i −1.4329
z = −0.5; p = [0 0 − 4.5]; k = 2; zp2tf(z, p, k)	ans = 0 0 2 1

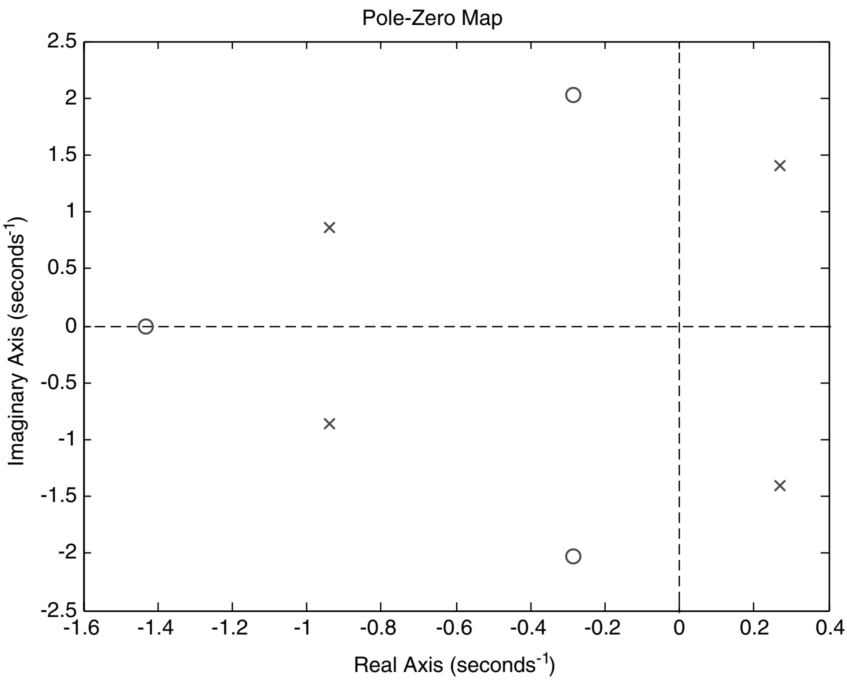
14.2.6 Pole-Zero Map of a Transfer Function

The poles and zeros of a transfer function can be plotted in a graph in MATLAB using the command **pzmap**. The syntax for using the command is

pzmap(n, d)

Example for the usage of **pzmap** for plotting the poles and zeros of a transfer function is given in Table 14.8.

Table 14.8 | Pole-zero map in MATLAB

Code	Output
n = [1 2 5 6]; d = [3 4 8 9 10]; pzmap(n, d)	

14.2.7 State-Space Representation of a Dynamic System

The state-space representation of a continuous dynamic system is described by

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

where $\dot{X}$ is the velocity vector or the derivative of the state vector of order (n × 1)
X is the state vector of order (n × 1)
U is the input or control signal vector of order (m × 1)
Y is the output vector of order (p × 1)
A is the system or state matrix of order (n × n)
B is the input matrix of order (n × m)
C is the output matrix of order (p × n)
D is the feed-through or feed-forward matrix of order (p × m)

If the matrices A, B, C and D are known, these matrices can be converted into a state-space model in MATLAB with the help of the command **ss**. The syntax for using the command is

variable_name = ss (A, B, C, D)

Example for the usage of **ss** for creating the state-space model for a system is given in Table 14.9.

Table 14.9 | Creation of state-space model for a system

Code	Output
A = [1 1; 1 2]; **B = [1 ; 1];** **C = [−1 1];** **D = [0];** **system = ss (A, B, C, D);**	**system =** **a =** **x1 x2** **x1 1 1** **x2 1 2** **b =** **u1** **x1 1** **x2 1** **c =** **x1 x2** **y1 −1 1** **d =** **u1** **y1 0** **Continuous-time state-space model.**

14.2.8 Phase Variable Canonical Form

The phase variable canonical form of a system can be obtained from the state-space model of a system using the similarity transformation matrix P such that

$$AP = P^{-1}AP$$
$$BP = P^{-1}B$$
$$CP = C * P$$
$$DP = D$$

where AP, BP, CP and DP are the system matrices in phase variable canonical form.

Consider a state-space matrices of a system as $A = \begin{bmatrix} -5 & -7 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 \end{bmatrix}$. Obtain the phase variable canonical form of the system using MATLAB with the help of the similarity transformation matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

The code which is used for determining the phase variable canonical form of a system is given below:

Code	Output
A = [–5 –7 –10; 1 0 0; 0 1 0]; B = [1; 0 ; 0]; C = [1 1 1]; D = [0]; P = [0 0 1; 0 1 0; 1 0 0]; AP = inv(P) * A * P BP = inv(P) * B CP = C * P DP = D	AP = 0 1 0 0 0 1 –10 –7 –5 BP = 0 0 1 CP = 1 1 1 DP = 0

14.2.9 Transfer Function to State-Space Conversion

The transfer function of a system can be converted into a state-space model using the command **tf2ss**. The syntax for using the command is

[A, B, C, D] = tf2ss (n, d)

where **n** is the row vector that contains the coefficient of numerator polynomial and **d** is the row vector which contains the coefficient of denominator polynomial

14.2.10 State-Space to Transfer Function Conversion

The state-space model of a system can be converted into a transfer function model using the command **ss2tf**. The syntax for using the command is

[n, d] = ss2tf (A, B, C, D, k)

where the parameter **k** is required if the system has more than one input and the value of **k** specifies the particular input.

An example for the usage of **ss2tf** to obtain transfer function of a system whose state-space model is given by $A=\begin{bmatrix}0 & 1\\ -2 & -1\end{bmatrix}$, $B=\begin{bmatrix}0\\1\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\end{bmatrix}$ and $D=\begin{bmatrix}0\end{bmatrix}$ and the usage of **tf2ss** to obtain state-space model of a system whose transfer function is given by $G(s)=\dfrac{(s^3+6s^2+7s+9)}{(s^3+10s^2+15s+100)}$ is given in Table 14.10.

Table 14.10 | Usage of tf2ss and ss2tf in MATLAB for a system

Code	Output
n = [1 6 7 9]; **d = [1 10 15 100];** **[A, B, C, D] = tf2ss(n, d)** **A = [0 1; –2 –1];** **B = [0; 1];** **C = [1 0];** **D = [0];** **[n, d] = ss2tf (A, B, C, D)**	**A =** **–10 –15 –100** **1 0 0** **0 1 0** **B =** **1** **0** **0** **C =** **–4 –8 –91** **D =** **1** **n =** **0 0 1.0000** **d =** **1.0000 1.0000 2.0000**

14.2.11 Series/Cascade, Parallel and Feedback Connections

If there exists two transfer functions $G_1(s)$ and $G_2(s)$ which are connected either in series (cascade), parallel or in feedback connection as given in Table 14.11, the resultant transfer function of the system in MATLAB is obtained using the functions mentioned below.

Consider two systems with the transfer functions $G_1(s)=\dfrac{n1}{d1}$ and $G_2(s)=\dfrac{n2}{d2}$ where $n1$ and $n2$ are the row matrices that contain the coefficients of numerator polynomials of transfer function $G_1(s)$ and $G_2(s)$ respectively and $d1$ and $d2$ are the row matrices that contain the coefficients of denominator polynomials of transfer function $G_1(s)$ and $G_2(s)$ respectively.

The syntax for determining the resultant of the system for different type of connection is given in Table 14.11.

Table 14.11 | Syntax for Series, Parallel and Cascade connections in MATLAB

Series connection	**[ns, ds] = series (n1, d1, n2, d2)**
Parallel connection	**[np, dp] = parallel (n1, d1, n2, d2)**
Feedback connection	**[nf, df] = feedback(n1, d1, n2, d2)**

where ***ns***, ***np*** and ***nf*** are the row matrices which contains the coefficients of numerator polynomials of series, parallel and feedback connection respectively. ***ds***, ***dp*** and ***df*** are the row matrices which contains the coefficients of numerator polynomials of series, parallel and feedback connection respectively.
An example to explain the concept of series, parallel and feedback connection is given in Table 14.12.

Table 14.12 | Series, parallel and feedback connections using MATLAB

Code	Output
n1 = [1 12 5]; d1 = [1 2 3 4]; n2 = [1 0 4]; d2 = [1 3 10 16 5]; [ns, ds] = series (n1, d1, n2, d2); printsys (ns, ds, 's') [np, dp] = parallel (n1, d1, n2, d2); printsys (nf, df, 's') [nf, df] = feedback (n1, d1, n2, d2); printsys (nf, df, 's')	**num/den = s ^ 4 + 12 s ^ 3 + 9 s ^ 2 + 48 s + 20 -- s ^ 7 + 5 s ^ 6 + 19 s ^ 5 + 49 s ^ 4 + 79 s ^ 3 + 98 s ^ 2 + 79 s + 20 num/den = s ^ 6 + 15 s ^ 5 + 51 s ^ 4 + 151 s ^ 3 + 247 s ^ 2 + 140 s + 25 -- s ^ 7 + 5 s ^ 6 + 19 s ^ 5 + 50 s ^ 4 + 91 s ^ 3 + 107 s ^ 2 + 127 s + 40 num/den = s^6 + 15 s^5 + 51 s^4 + 151 s^3 + 247 s^2 + 140 s + 25 -- s ^ 7 + 5 s ^ 6 + 19 s ^ 5 + 50 s ^ 4 + 91 s ^ 3 + 107 s ^ 2 + 127 s + 40**

14.2.12 Time Response of Control System

The detailed study about the time-domain analysis of a system is described in Chapter 5. In this section, we will be discussing how the time-domain analysis can be analysed using MATLAB.

14.2.12.1 Step response of a system: The step response of a system can be obtained if the system is represented either by state-space model or transfer function model. The different syntax for determining the step response of a systems are given below:

Case 1: If the system is described using the transfer function:

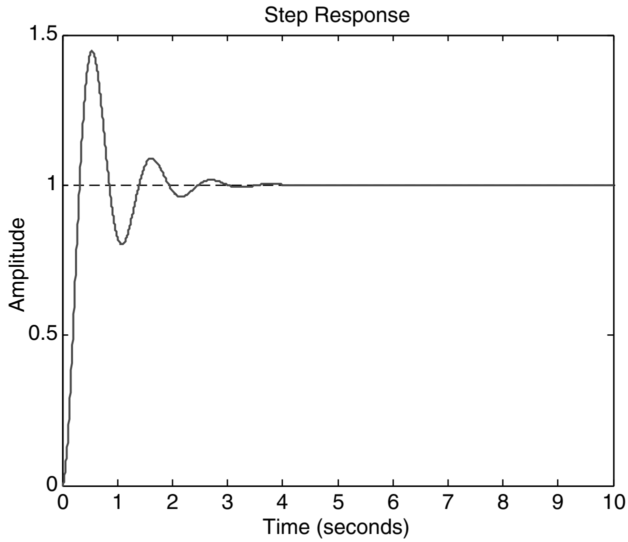
- **(i) step (n, d)**: This function will plot the time response of the system based on the default values of time **t**.
- **(ii) step (n, d, t)**: This function will plot the time response of the system based on the user-defined values of time **t**.

Examples for using the command is given in Table 14.13.

Case 2: If the system is described using state-space model:

- **(i) step (A, B, C, D)**: This function will plot the time response of the system based on the default values of time **t**.
- **(ii) step (A, B, C, D, 1, t)**: This function will plot the time response of the system based on the user defined values of time **t**.
- **(iii) step (A, B, C, D, i, t)**: This function will plot the time response of the multi-input–multi-output (MIMO) system based on the input specified by **i** for the user defined values of time **t**.

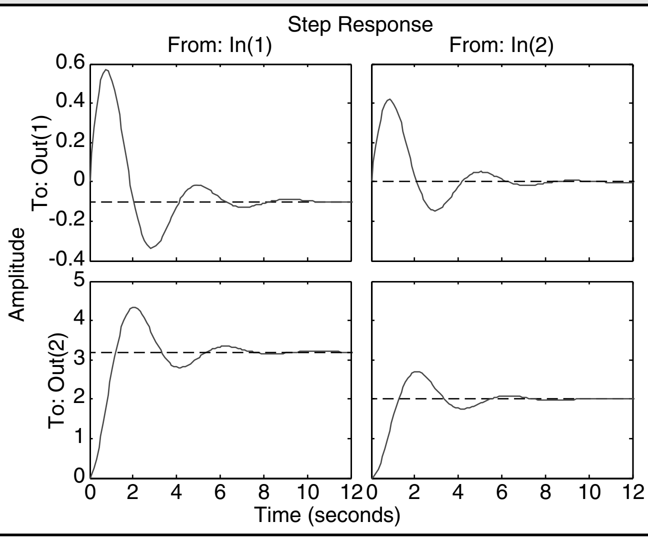
Table 14.13 | Step response from transfer function

Code	Output
1. Using user-defined values of time t **n = [36];** **d = [1 3 36];** **t = 0: 0.005: 10;** **step (n, d, t)**	

If i = 0 for the MIMO system, the number of time response graphs which will be displayed is (n * n) where n is the number of input to the system or number of output from the system.

Example for using the command is given in Table 14.14.

Table 14.14 | Step response from state-space model

Code	Output
A = [−1 −0.5; 5 0]; **B = [1.5 1; 0.5 0];** **C = [1 0; 0 1];** **D = [0 0; 0 0];** **step (A, B, C, D)**	

In addition, the step response of the system can be obtained using the plot command for a system when it is represented either by means of transfer function or by state-space model. The syntax for the usage of plot command given below:

Case 1: If the system is described by the transfer function

(i) **[y, x, t] = step (n, d)**; (based on the default values of time t)
plot (t, y)
(ii) **[y, x, t] = step (n, d, t)**; (based on the user defined values of time t)
plot (t, y)

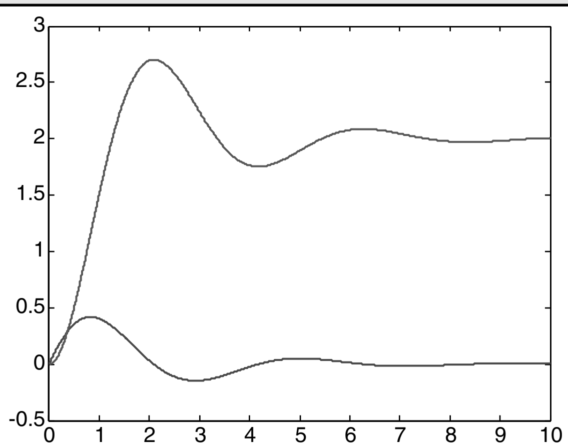
Case 2: If the system is described by the state-space model

(i) **[y, x, t] = step (A, B, C, D)**; (based on the default values of time t)
plot (t, y)
(ii) **[y, x, t] = step (A, B, C, D, 1, t)**; (based on the user defined values of time t)
plot (t, y)
(iii) MIMO system

(i) Based on the default values of time t
(a) **[y, x, t] = step (A, B, C, D)**; (when i = 0)
plot (t, y)
(b) **[y, x, t] = step (A, B, C, D, i)**; (when i > 0)
plot (t, y)
(ii) Based on the user defined values of time t
(c) **[y, x, t] = step (A, B, C, D, i, t)**; (when i > 0)
plot (t, y)

An example to show the usage of plot command in plotting the time response of the system is given in Table 14.15.

Table 14.15 | Step response of a system using plot command

Code	Output
A = [−1 −0.5; 5 0]; **B = [1.5 1; 0.5 0];** **C = [1 0; 0 1];** **D = [0 0; 0 0];** **t = 0: 0.005: 10;** **[y, x, t] = step (A, B, C, D, 2, t);** **plot(t, y)**	

14.2.12.2 Impulse Response of a System: The different MATLAB commands used for determining the time response of a system when a system is subjected to impulse input is given in Table 14.16.

Table 14.16 | Impulse response of a system in MATLAB

Syntax	Time t	Description
impulse (n, d)	Default values of time t	Determines the impulse response of the system when the system is represented by the transfer function
[y, x, t] = impulse (n, d); plot (t, y)		
impulse (n, d, t)	User-defined values of time t	
[y, x, t] = impulse (n, d, t); plot(t, y)		
impulse (A, B, C, D)	Default values of time t	Determines the impulse response of the system when the system is represented by the state-space model (single input–single output) system
[y, x, t] = impulse (A, B, C, D); plot (t, y)		
impulse (A, B, C, D, 1, t)	User-defined values of time t	
[y, x, t] = impulse (A, B, C, D, 1, t); plot (t, y)		
impulse (A, B, C, D), i = 0	Default values of time t	Determines the impulse response of the system when the system is represented by the state-space model (MINO) system
[y, x, t] = impulse (A, B, C, D), i = 0 plot (t, y)		
impulse (A, B, C, D, i), i > 0		
[y, x, t] = impulse (A, B, C, D, i); plot (t, y)		
impulse (A, B, C, D, i, t)	User-defined values of time t	
[y, x, t] = impulse (A, B, C, D, i, t), plot (t, y)		

14.2.12.3 Ramp Response of a System: Similar to the step and impulse response of a system, it is not so easy to determine the ramp response of a system as ramp command does not exists in MATLAB. But by using the default command such as step and impulse it is possible to determine the ramp response of a system.

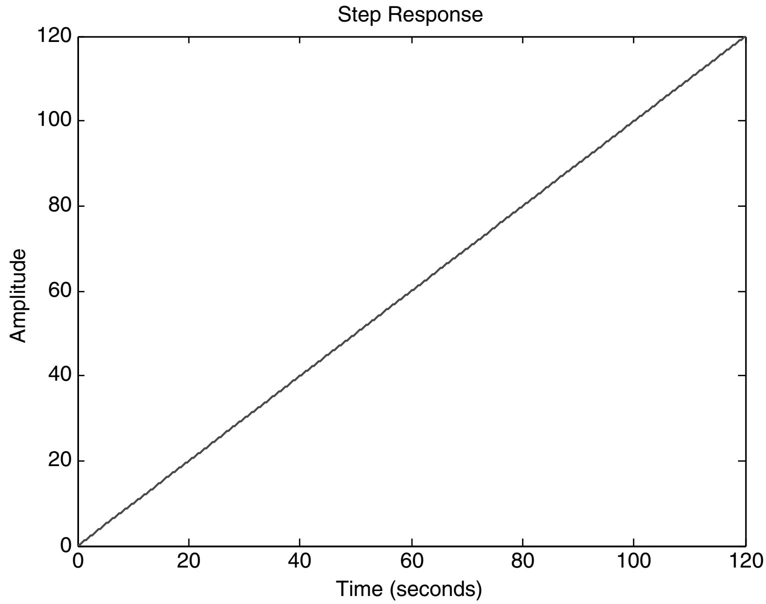
(i) With the help of **step** command: If the transfer function of the system is divided by s, then the **step** command can be used to determine the time response of the system when it is subjected to ramp input.

An example to determine the ramp response of a system using the **step** command is given in Table 14.17.

(ii) With the help of **impulse** command

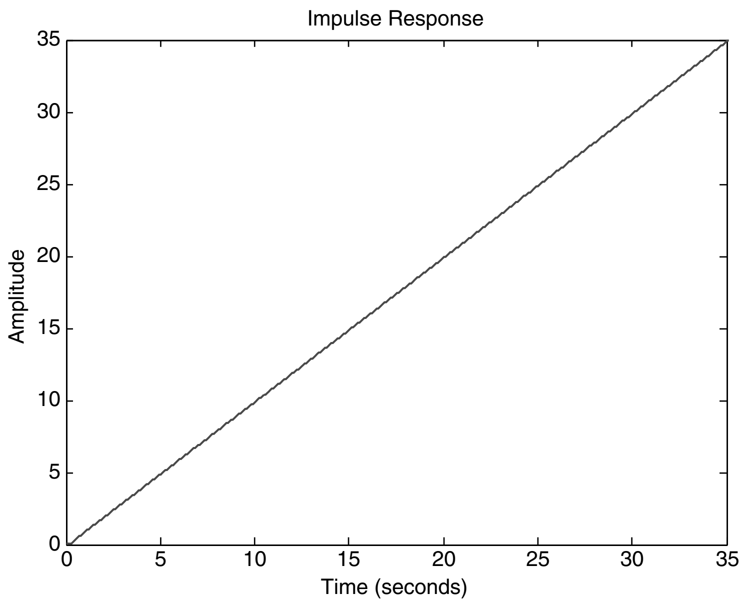
If the transfer function of the system is divided by s^2, then the **impulse** command can be used to determine the time response of the system when it is subjected to ramp input.

Table 14.17 | Ramp response of a system using step command in MATLAB

Code	Output
n = [30]; **d = [1 5 30];** **n1 = [30];** **d1 = [1 5 30 0];** **step(n1, d1)**	

An example to determine the ramp response of a system using the **impulse** command is given in Table 14.18.

Table 14.18 | Ramp response of a system using impulse command in MATLAB

Code	Output
n = [30]; **d = [1 5 30];** **n1 = [30];** **d1 = [1 5 30 0 0];** **impulse(n1, d1)**	

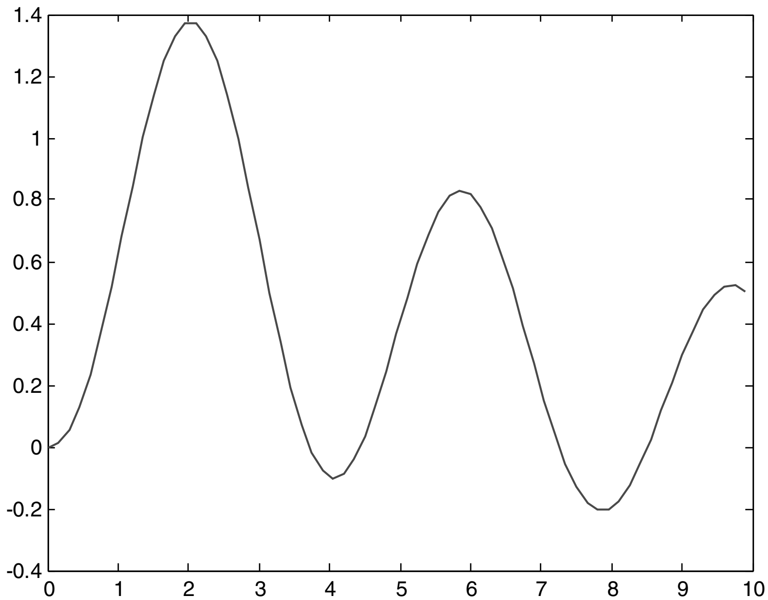
14.2.12.4 Response to an Arbitrary Input: The MATLAB command **lsim** is used to determine the time response of a system when the system is subjected to any random input other than the standard inputs. The main difference between the other commands (**step, impulse**) is that the time values should be mentioned in **lsim** command. The different syntax used for determining the time response of a system when it is subjected to an input u is given below:

(A) When a system is represented in the form of transfer function
  (i) **lsim (n, d, u, t)**
  (ii) **y = lsim (n, d, u, t)**
  plot (t, y)

(B) When a system is represented in the form of state-space model
  (i) **lsim (A, B, C, D, u, t)**: when a system is represented in the form of state-space model
  (ii) **y = lsim (A, B, C, D, u, t)**
  plot (t, y)

An example for the usage of **lsim** command in MATLAB is given in Table 14.19.

Table 14.19 | Response of a system for an arbitrary input

Code	Output
n = [1 5]; **d = [1 2 3 5];** **t = 0: 0.15: 10;** **u = exp (–0.2 * t);** **y = lsim (n, d, u, t);** **plot (t, y)**	

14.2.13 Performance Indices from the Response of a System

The different performance indices which can be obtained from the time response of a system are

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Maximum peak overshoot M_p
5. Settling time t_s

As the formulae for determining the performance indices varies for different inputs, it is necessary to write individual programmes for different inputs. But if the formulae are not used to determine the performance indices, a single programme will be enough to determine the performance indices for all the inputs.

A programme to determine the performance indices of a system for Step input is given in Table 14.20.

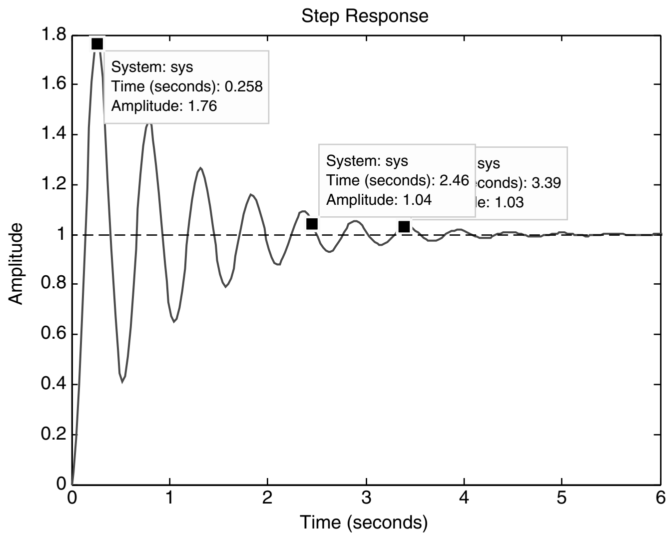
Table 14.20 | Code to determine the performance indices of a system

```
clear all
close all
clc
%Enter the transfer function G(s)
n = [144];
d = [1 2 144];
sys_tf = tf(n, d);
step(n,d)
% Step response of the system
[y, x, t] = step(n, d);
% To determine peak time
v = max(y);
for i = 1:length(t)
    if v == y(i)
        k = i;
    end
end
t_peak = t(k);
%To determine rise time
k1 = 1;
while (y(k1) < 0.1)
    k1 = k1 + 1;
    break
end
t1 = t(k1);
while (y(k1) < 0.9)
    k1 = k1 + 1;
    break
end
t2 = t(k1);
t_rise = t2 - t1;
% Maximum peak overshoot in percentage
perc_overshoot = (y(k) - 1)*100;
% Settling time for 2 percent tolerance
for k2 = length(t):-1:1
    if ((y(k2) - 1) > = 0.02)
        s_t = k2;
        break
    end
end
```

```
set_time_2_per = t(s_t);
% Settling time for 5 percent tolerance
for k3 = length(t):-1:1
    if ((y(k3) - 1) > = 0.05)
        s_t1 = k3;
        break
    end
end
set_time_5_per = t(s_t1);
fprintf('The rise time of the system is %f sec\n', t_rise);
fprintf('The maximum peak overshoot of the system is %f percent\n’,
perc_overshoot);
fprintf('The peak time of the system is %f sec\n', t_peak);
fprintf('The settling time of the system for 2 percent tolerance is
%f sec\n', set_time_2_per);
fprintf(‘The settling time of the system for 5 percent tolerance is
%f sec\n', set_time_5_per);
```

The output of the code given in Table 14.20 is given in Table 14.21.

Table 14.21 | Output of the code

Output in command window	Plot
The rise time of the system is 0.026180 sec. The maximum peak overshoot of the system is 76.891616 percent. The peak time of the system is 0.261799 sec. The settling time of the system for 2 percent tolerance is 3.481932 sec. The settling time of the system for 5 percent tolerance is 2.905973 sec.	

14.2.14 Steady State Error from the Transfer Function of a System

The steady-state error of a system is determined with the help of the error constants (position, velocity and acceleration error constants). In addition, the steady-state error and the error constants depends on both TYPE of the system and type of the input applied to the system.

The relationship among the TYPE of the system, static error constant and steady-state error of the system is given in Table 5.5.

A program to determine the steady-state error of a system for any input and any TYPE of a system is given in Table 14.22

Table 14.22 | Code to determine the steady-state error of a system

```
clear all
close all
clc
%Enter the numerator and denominator of G(s)
n1 = [10 20];
d1 = [1 7 12 0 0];
%Enter the numerator and denominator of H(s)
n2 = [1];
d2 = [1];
%To determine the loop transfer function G(s)H(s)
[n3, d3] = series(n1, d1, n2, d2);
[z, p, k] = tf2zp(n3, d3);
k = 0;
for i = 1:length(p)
    if (p(i) == 0)
        k = k + 1;
    end
end
K = n3(length(n3))/d3(length(d3) - k);
switch k
    case 0
        Position_const = K; Velocity_const = 0; Acceleration_const = 0;
        Step_steady_error = 1/(1 + Position_const);
        Ramp_steady_error = 1/Velocity_const;
        Parabolic_steady_error = 1/Acceleration_const;
    case 1
        Position_const = inf ; Velocity_const = K; Acceleration_const = 0;
        Step_steady_error = 1/(1 + Position_const);
        Ramp_steady_error = 1/Velocity_const;
        Parabolic_steady_error = 1/Acceleration_const;
    case 2
        Position_const = inf; Velocity_const = inf; Acceleration_
        const = K;
        Step_steady_error = 1/(1 + Position_const);
        Ramp_steady_error = 1/Velocity_const;
        Parabolic_steady_error = 1/Acceleration_const;
    otherwise
        Position_const = inf; Velocity_const = inf; Acceleration_
        const = inf;
```

```
        Step_steady_error = 1/(1 + Position_const);
        Ramp_steady_error = 1/Velocity_const;
        Parabolic_steady_error = 1/Acceleration_const;
end
fprintf('The given system is TYPE %d system\n', k);
fprintf('The error constants of the system are \n');
fprintf('Kp = %f \nKv = %f \nKa = %f\n\n',Position_const,Velocity_
const,Acceleration_const);
fprintf('The steady state error of the system for \n');
fprintf('Unit step input = %f \nUnit ramp input = %f \nUnit para-
bolic input = %f \n\n',Step_steady_error,Ramp_steady_error,Parabolic_
steady_error);
```

The output for the code given in Table 14.22 is given below:
The given system is TYPE 2 system
The error constants of the system are
Kp = Inf
Kv = Inf
Ka = 1.666667
The steady state error of the system for
Unit step input = 0.000000
Unit ramp input = 0.000000
Unit parabolic input = 0.600000

14.2.15 Routh–Hurwitz Criterion

The Routh–Hurwitz criterion is one of the method for determining the stability of the system. The characteristic equation of a system is to be known for determining the stability of a system based on Routh–Hurwitz criterion. The detailed description about the Routh–Hurwitz criterion is discussed in Chapter 6. A programme to determine the stability of a system based on Routh–Hurwitz criterion can be coded.

14.2.16 Root Locus Technique

The root locus technique helps in determining the stability of the system based on the location of closed-loop poles in the *s*-plane. The step-by-step procedure for plotting the root locus of a system is explained in Chapter 7. In MATLAB, two default command exist in plotting the root locus of a system.

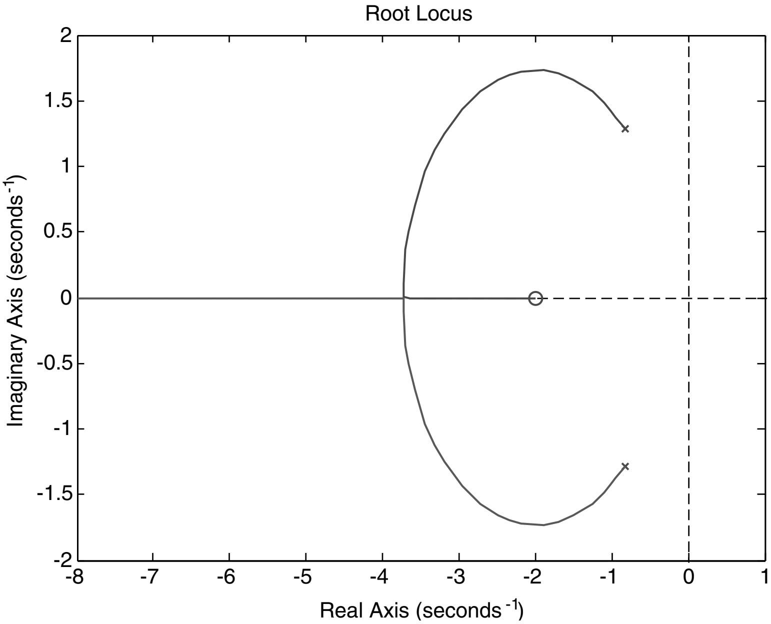
Case 1: Using the command **rlocus**
The default command **rlocus** is used to plot the root locus of a system. The syntax for the command **rlocus** is given below:

rlocus (n, d)

where **n** is the row vector which contains the coefficient of numerator polynomial and **d** is the row vector which contains the coefficient of denominator polynomial

An example to show the usage of **rlocus** command in MATLAB is given in Table 14.23.

Table 14.23 | Root locus of a system

Code	Output
n = [1 2]; **d = [3 5 7];** **rlocus (n, d)**	

In the above command, the value of gain K is automatically taken from 0 to ∞. But if the user defines certain value of gain K, the command to be used in MATLAB for plotting the root locus of a system is

rlocus (n, d, K)

where **K** is the user-defined values of gain K.

Case 2: Using the command **plot**
The **plot** command can also be used to plot the root locus of a system. The syntax for using the **plot** command are given below:

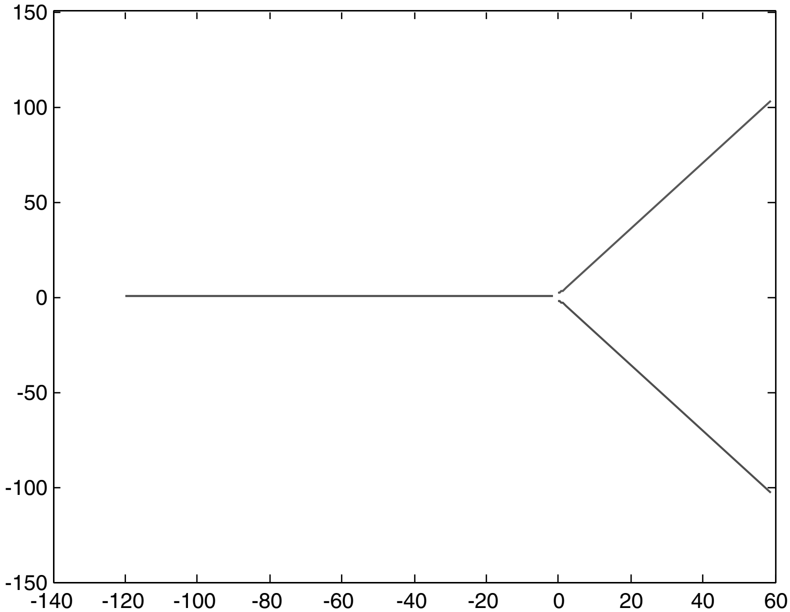
(a) For default gain values K:
[r, K] = rlocus (n, d);
plot (r)

(b) For user defined gain values K:
[r, K] = rlocus (n, d, K);
plot (r)

(c) **r = rlocus (n, d) or**
r = rlocus (n, d, K);
plot (r)

An example to show the usage of plot command to plot the root locus of a system is given in Table 14.24.

Table 14.24 | Root locus of a system

Code	Output
n = [0.4]; **d = [1 2 4 5];** **[r, K] = rlocus (n, d);** **plot (r)**	

It is to be noted that if a point is clicked in the root locus plot of a system, it will display the information such as frequency (rad/s), overshoot (%), damping and gain at that point.

14.2.16.1 Determination of Gain Value *K* at a Given Point in the Root Locus

The gain value K at any point on the root locus of a system in MATLAB is determined using the command **rlocfind**. The different syntax for using the command **rlocfind** are given below:

(i) **[K, r] = rlocfind (n, d)**
(ii) **K = rlocfind (n, d)**
(iii) **rlocfind (n, d)**

where K is the gain value at a particular point and r is the column vector that gives the location of roots at gain K.

In the commands shown above, the first will return both the gain and location of roots while the second and third command when used in MATLAB, will return only the gain value K.

The step-by-step procedure to determine the gain value K at a given point are given below:

(a) Plot the root locus of a system using the command mentioned earlier.
(b) Use the command **rlocfind** in any one of the format mentioned above.
(c) Once the command **rlocfind** is executed a message '**select a point in the figure window**' will be displayed in the command window.
(d) A cursor will be also available in root locus plot to select an exact point in the plot.
(e) If we select a point in the root locus plot using the cursor, the gain and the roots at that particular point will be displayed in the command window.

An example to determine the value of gain K from the root locus using MATLAB is given in Table 14.25.

Table 14.25 | Gain of a system from root locus in MATLAB

Code	Output
n = [1]; **d = [1 2 3 0];** **rlocus (n, d);** **[K, r] = rlocfind (n, d)**	**Select a point in the graphics window** **selected_point = −5.2725 + 1.2174i** **K = 116.1355** **r = −5.4115** **1.7057 + 4.3071i** **1.7057 − 4.3071i**

14.2.16.2 Root Locus Plot Using State-Space Representation of a System

The root locus plot for a system which is represented using the state-space model can be plotted with the help of the command **rlocus**. The syntax for using the command are given below:

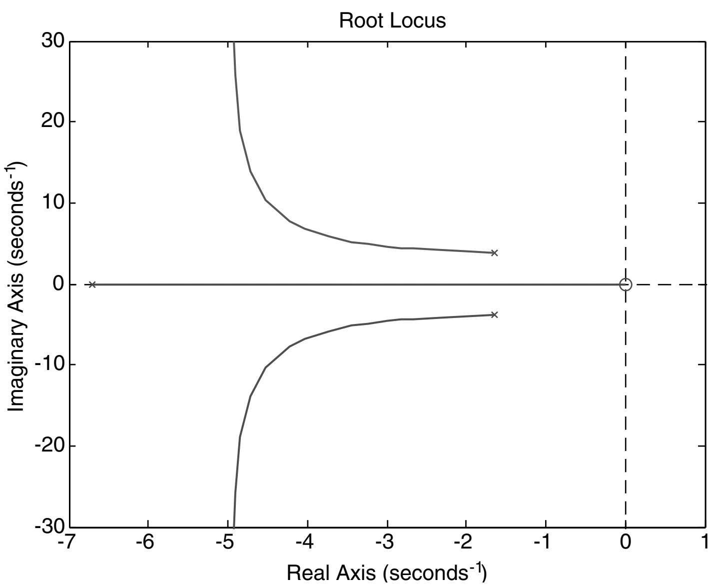
(i) **rlocus (A, B, C, D)**: gain value K ranging from 0 to ∞
(ii) **rlocus (A, B, C, D, K)**: user-defined values of gain K

Usage of plot command

(iii) **[r, K] = rlocus (A, B, C, D)** – gain value K ranging from 0 to ∞
plot (r)
(iv) **[r, K] = rlocus (A, B, C, D, K)** user-defined values of gain K
plot (r)

An example to show the usage of **rlocus** plot using state-space model is given in Table 14.26.

Table 14.26 | Root locus of a state – space model of a system

Code	Output
A = [0 1 0; 0 0 1; −120 −40 −10]; **B = [0 ; 1; −10];** **C = [1 0 0];** **D = [0];** **rlocus (A, B, C, D)**	

14.2.17 Bode Plot

The Bode plot is one of the frequency-response technique which is used to analyse the stability of the system. The stability of the system is analysed based on the value of the frequency domain specifications (gain margin, phase margin, gain crossover frequency and phase crossover frequency). The manual determination of frequency domain specification from the Bode plot is discussed in Chapter 8. In this section, plotting of Bode plot in MATLAB and determination of frequency domain specification from Bode plot in MATLAB are discussed.

Plotting Bode plot for system

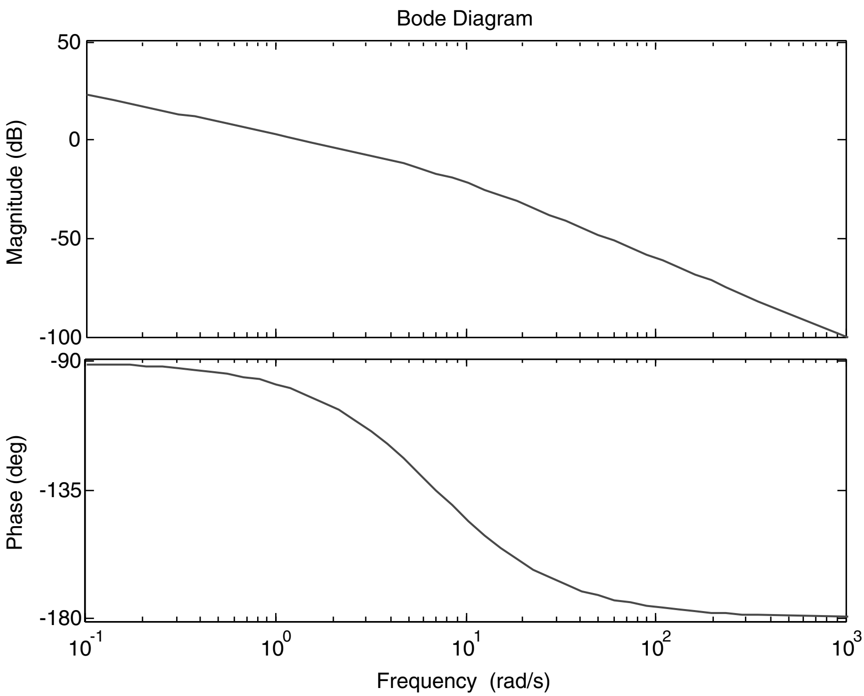
The default command in MATLAB which is used for plotting the Bode plot of a system is **bode**. The syntax for using the command **bode** for plotting the Bode plot for a system are given below:

Case A: If the system is represented in the transfer function model

1. If the frequency range for plotting the Bode plot is automatically selected, the syntax for plotting the Bode plot will be
   bode (n, d) or
   bode (sys_tf)
   where **n** is the row vector containing the coefficients of numerator polynomial, **d** is the row vector containing the coefficients of denominator polynomial and **sys_tf** is the system transfer function obtained using the command **tf**.

An example for plotting the Bode plot is given in Table 14.27.

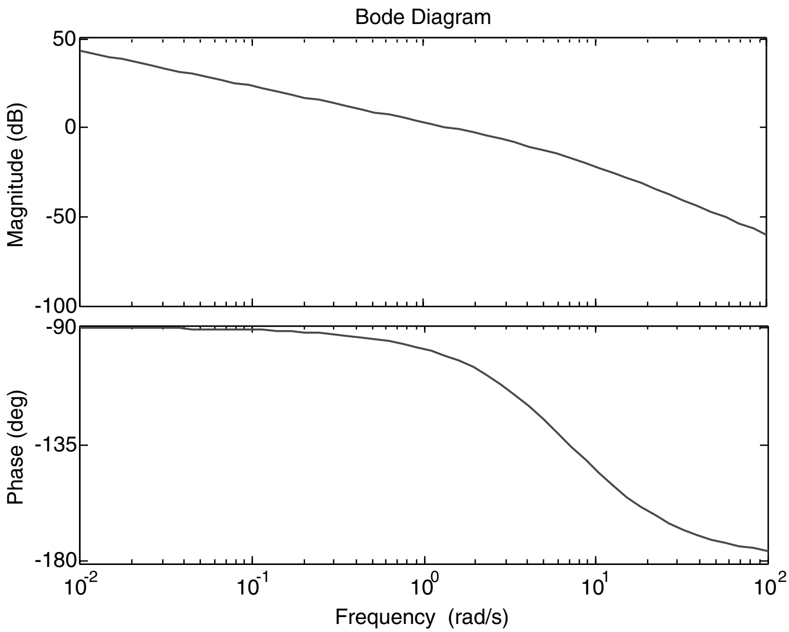
Table 14.27 | Bode plot of a system

Code	Output
n = [10]; **d = [1 7 0];** **bode (n, d)** **sys_tf = tf (n, d);** **bode (sys_tf)**	

2. If the frequency range for plotting the Bode plot is user defined, the syntax for plotting the Bode plot will be
   bode (n, d, w) or
   bode (sys_tf, w)
   where **w** is the range of frequency for plotting the Bode plot.
   As the Bode plot is plotted in the logarithmic scale of the frequency w the user-defined values should also be in logarithmic scale. The MATLAB command which is used to obtain the frequency value in logarithmic scale is
   w = logspace (a, b).
   where 10^a is the starting point of frequency range and 10^b is the ending point of frequency range

An example for plotting the Bode plot is given in Table 14.28.

Table 14.28 | Bode plot of a system

Code	Output
n = [10]; **d = [1 7 0];** **a = −2;** **b = 2;** **w = logspace (a, b);** **bode (n, d, w)** **sys_tf = tf (n, d);** **bode (sys_tf, w)**	

3. When the magnitude and phase values corresponding to different frequencies are to be stored, the syntax for getting the values will be
   (a) For default values of frequency w
   [mag, phase, w] = bode (n, d) or
   [mag, phase, w] = bode (sys_tf)
   (b) For user defined values of frequency w
   [mag, phase, w] = bode (n, d, w) or
   [mag, phase, w] = bode (sys_tf, w)

where **mag** and **phase** are the column vectors that contains the magnitude and phase values of the system corresponding to the different frequencies respectively.

Here, **mag** value can be converted into decibels with the help of **20 *log10(mag)**.

An example for obtaining the magnitude and phase values corresponding to different frequencies is given in Table 14.29.

Table 14.29 | Bode plot of a system

Code	Output
n = [10]; **d = [1 7 0];** **[mag, phase, w]= bode (n, d);**	The value of **mag**, **phase** and **w** can be obtained by typing **mag** or **phase** or **w** respectively, in the command window. The output is not shown here as the length of the column vectors **mag, phase** and **w** are large.

It is to be noted that at a particular frequency, the magnitude and phase angles can be obtained by clicking at the plot which is shown in Fig. 14.1.

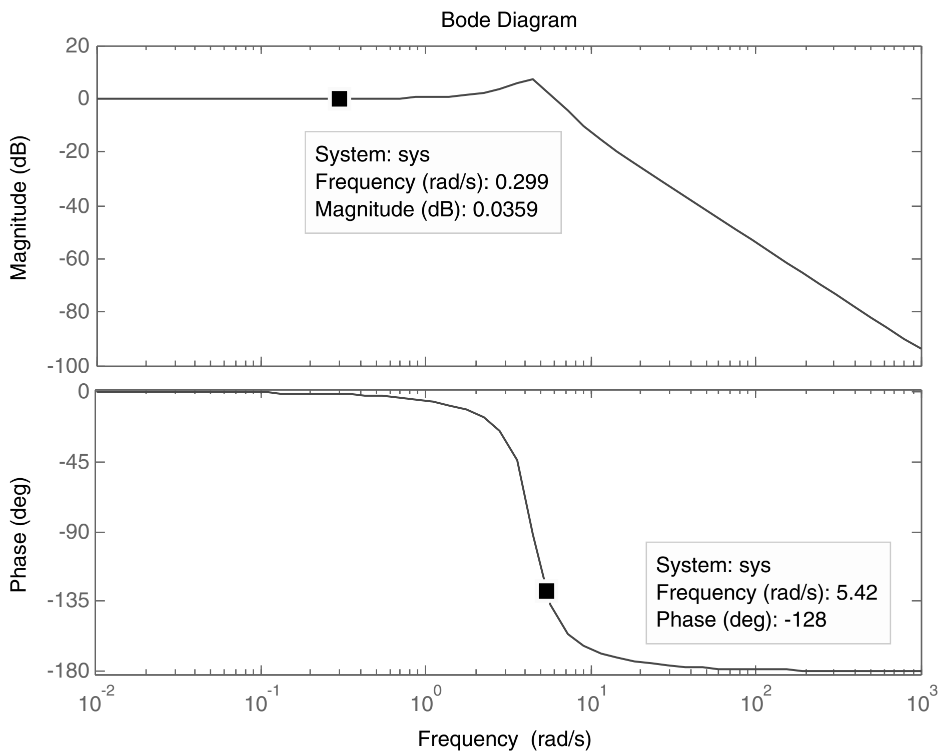


Fig. 14.1 | Bode plot with the magnitude and phase angle at a frequency

14.2.17.1 Determination of frequency domain specification from Bode plot

The frequency domain specifications which can be obtained from Bode plot are gain margin, phase margin, gain crossover frequency and phase crossover frequency. The definition of each frequency domain specifications and its manual calculation is discussed in Chapter 8. In this section, determination of frequency domain specification values from Bode plot using MATLAB is discussed.

The default command used in MATLAB for determining the frequency domain specification is margin. The syntax for using the command in MATLAB are given below:

(i) **margin (*n*, *d*)** displays only the gain margin and phase margin of the system.
or **[Gm, Pm, pcf, gcf] = margin (*n*, *d*)** displays all the frequency domain specifications.

(ii) **margin (sys_tf)** displays only the gain margin and phase margin of the system.
or **[Gm, Pm, pcf, gcf] = margin (sys_tf)** displays all the frequency domain specifications.

where **Gm** is the gain margin of the system, **Pm** is the phase margin of the system**, pcf** is the phase crossover frequency of the system and **gcf** is the gain crossover frequency of the system.

An example to show the usage of the command is given in Table 14.30.

Table 14.30 | Frequency domain specification of a system using bode plot

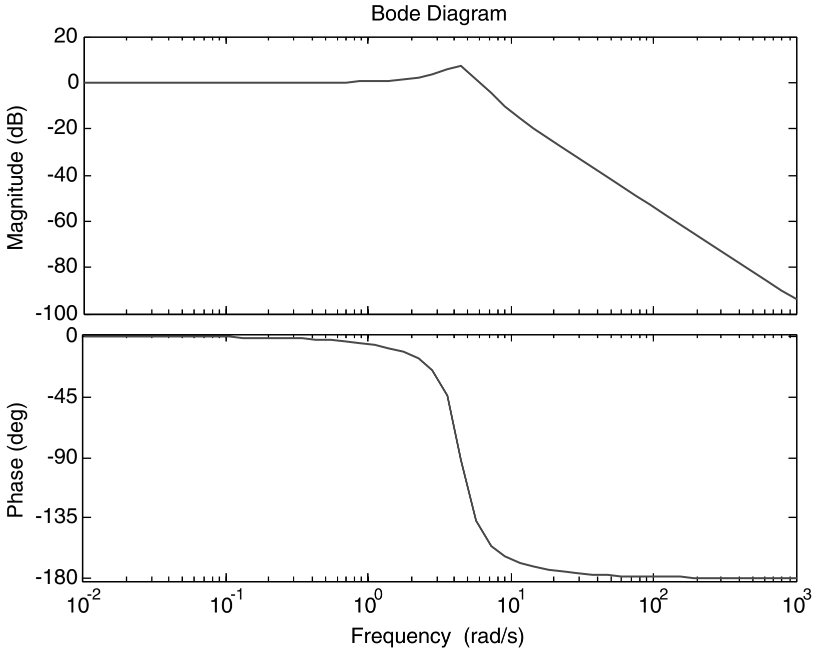
Code	Output
n = [10]; **d = [1 7 0];** **[Gm, Pm, pcf, gcf] = margin (n, d)**	Gm = Inf Pm = 78.6851 pcf = Inf gcf = 1.4006

Case B: If the system is represented using state-space model

1. System with one input
   (a) If the frequency range for plotting the Bode plot is automatically selected, the syntax for plotting the Bode plot will be
   bode (A, B, C, D)
   where **A, B, C, D** are the state-space matrix of a system.
   (b) If the frequency range for plotting the Bode plot is user defined, the syntax for plotting the Bode plot will be
   bode (A, B, C, D, 1, w)
   where **w** is the user-defined frequency range obtained using the default command logspace.

An example for the usage of the command bode is given in Table 14.31.

Table 14.31 | Bode plot of a system

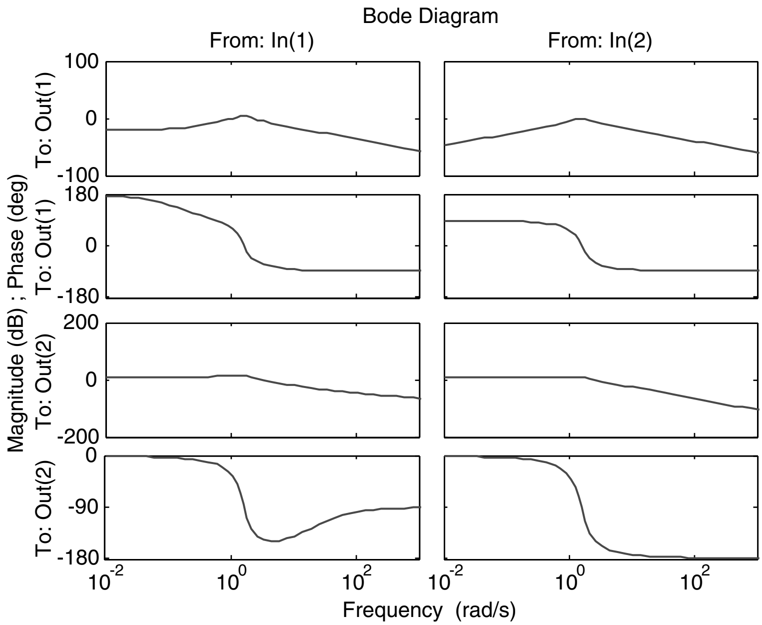
Code	Output
A = [0 1; −20 −2]; **B = [0; 20];** **C = [1 0];** **D = [0];** **a = −2;** **b = 3;** **w = logspace (a, b);** **bode (A, B, C, D, 1, w)**	

2. System with more than one input and more than one output.
   (c) If the input to which the Bode plot is to be plotted is mentioned. The syntax will be
   bode (A, B, C, D, i)
   where **i** is the input for which the Bode plot has to be plotted.
   (d) If the input to which the Bode plot is to be plotted is not mentioned. The syntax will be
   bode (A, B, C, D)
   If we use this command, the number of bode plots generated will be $(m * n)$ where m is the number of inputs and n is the number of outputs.
   If the system has two inputs and two outputs, then the number of Bode plot generated by using the second command will be 4 (2 * 2).

An example for the usage of this command is given in Table 14.32.

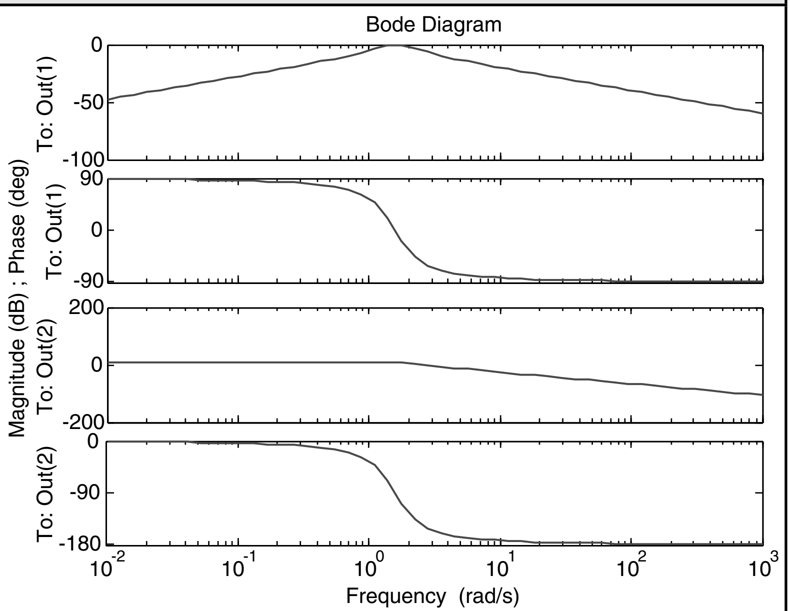
   (e) If the frequency range for plotting the Bode plot is user defined, the syntax for plotting the Bode plot will be
   bode (A, B, C, D, i, w)

Table 14.32 | Bode plot of a system

Code	Output
B = [1.5 1; 0.5 0]; **C = [1 0; 0 1];** **A = [–1 –0.5; 5 0];** **D = [0 0; 0 0];** **bode(A, B, C, D)**	

An example for the usage of this command is given in Table 14.33.

Table 14.33 | Bode plot of a system

Code	Output
B = [1.5 1; 0.5 0]; **C = [1 0; 0 1];** **A = [–1 –0.5; 5 0];** **D = [0 0; 0 0];** **a = -2;** **b = 3;** **w = logspace (a, b);** **i = 2;** **bode(A, B, C, D, i, w)**	

It is to be noted that the determination of frequency domain specification using the default command margin in MATLAB is not possible. To use the default command **margin**, the system must be a SISO system.

14.2.18 Nyquist Plot

The frequency response technique that analyze the stability of the closed-loop system based on the open-loop frequency response is known as Nyquist plot. The detailed analysis of the Nyquist plot is discussed in Chapter 9. The default command which is used to plot the Nyquist plot of a system in MATLAB is **nyquist**. The syntax for using the command in MATLAB are discussed below for different cases.

Case A: If the system is represented in the transfer function model

1. If the frequency range for plotting the Nyquist plot is automatically selected, the syntax for plotting the Nyquist plot will be
   nyquist (n, d) or
   nyquist (sys_tf)
   where **n** is the row vector containing the coefficients of numerator polynomial, **d** is the row vector containing the coefficients of denominator polynomial and **sys_tf** is the system transfer function obtained using the command **tf**.
2. If the frequency range for plotting the Nyquist plot is user defined, the syntax for plotting the Nyquist plot will be
   nyquist (n, d, w) or
   nyquist (sys_tf, w)
   where **w** is the range of frequency for plotting the Nyquist plot.
   The range of frequency is generated with the help of the command **linspace**. The syntax for usage of **linspace** is
   w = linspace (a, b, c);
   where **a** is the starting point of the frequency range, **b** is the ending point of the frequency range and **c** is the number of points to be taken between **a** and **b**.

An example for plotting the Nyquist plot is given in Table 14.34.

3. When the real and imaginary values corresponding to different frequencies are to be stored, the syntax for getting the values will be
   (a) For default values of frequency w
   [re, im, w] = nyquist (n, d) or
   [re, im, w] = nyquist (sys_tf)
   (b) For user defined values of frequency, **w**
   [re, im, w] = nyquist (n, d, w) or
   [re, im, w] = nyquist (sys_tf, w)
   where **re** and **im** are the column vectors that contain the real and imaginary values of the system corresponding to the different frequencies respectively.

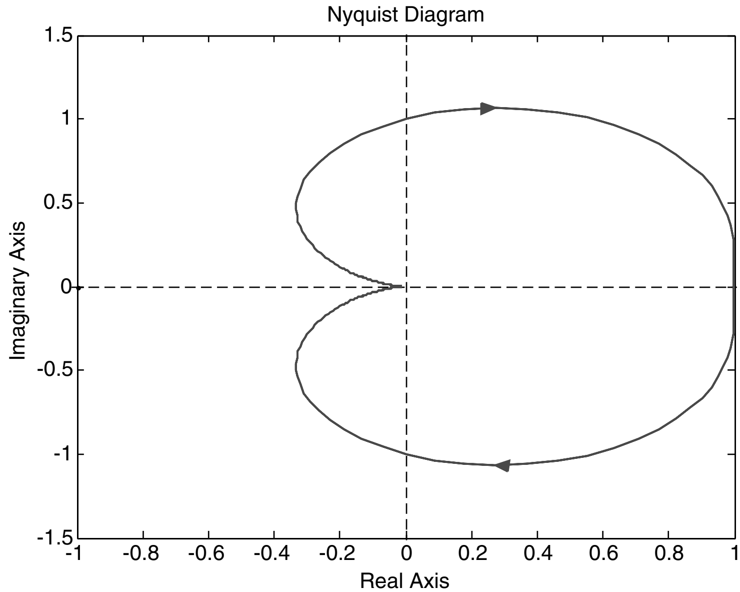
An example for obtaining the real and imaginary values corresponding to different frequencies is given in Table 14.35.

Table 14.34 | Nyquist plot of a system

<table>
<tr><th>Code</th><th>Output</th></tr>
<tr><td>

```
n = [1];
d = [1 1 1];
w = linspace (–10, 10, 512);
nyquist (n, d, w)
sys_tf = tf (n, d);
nyquist (sys_tf, w)
```

</td><td>



</td></tr>
</table>

Table 14.35 | Nyquist plot data of a system

Code	Output
n = [10]; **d = [170];** **[re, im, w]= nyquist (n, d);**	The value of **re, im** and **w** can be obtained by typing **mag** or **phase** or **w** respectively in the command window. The output is not shown here as the length of the column vectors **re**, **im** and **w** are large.

It is to be noted that at a particular frequency, the real value, imaginary value and its corresponding frequency can be obtained by clicking at the plot which is shown in Fig. 14.2.

Case B: If the system is represented using state-space model

1. System with one input
   (a) If the frequency range for plotting the Nyquist plot is automatically selected, the syntax for plotting the Nyquist plot will be
   nyquist (A, B, C, D)
   where **A, B, C, D** are the state-space matrix of a system.
   (b) If the frequency range for plotting the Nyquist plot is user defined, the syntax for plotting the Nyquist plot will be
   nyquist (A, B, C, D, 1, w)
   where **w** is the user-defined frequency range obtained using the default command **logspace**.

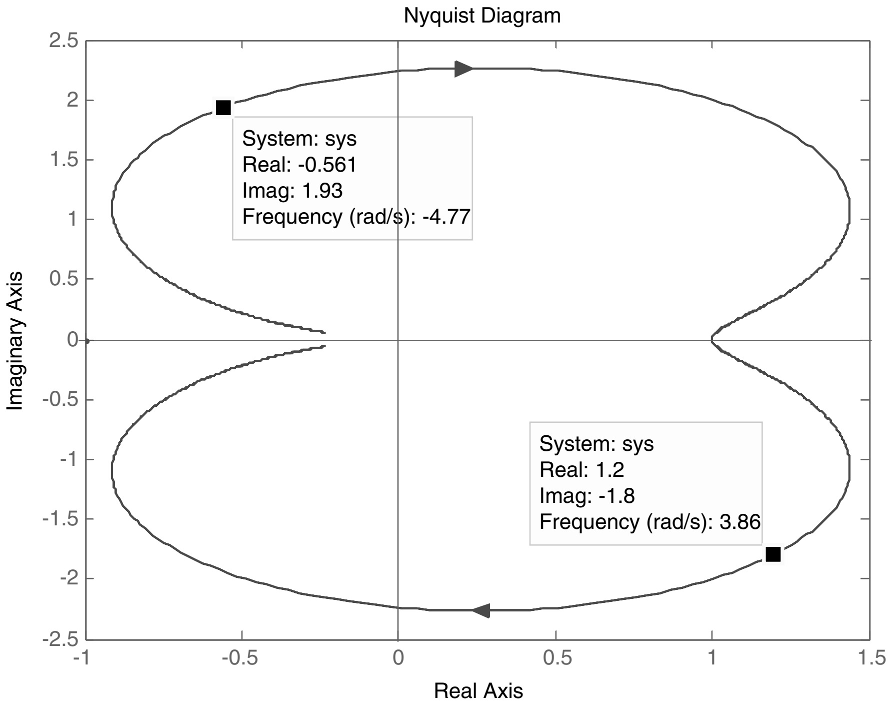
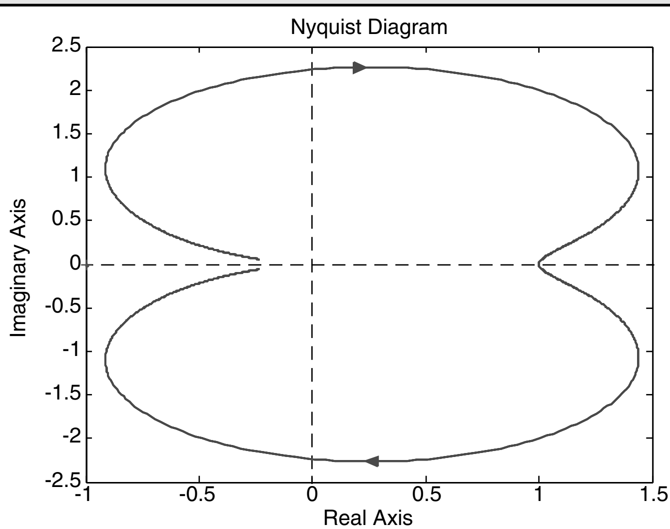


Fig. 14.2 | Nyquist plot with the real and imaginary value at a frequency

An example for the usage of the command **nyquist** is given in Table 14.36

Table 14.36 | Nyquist plot

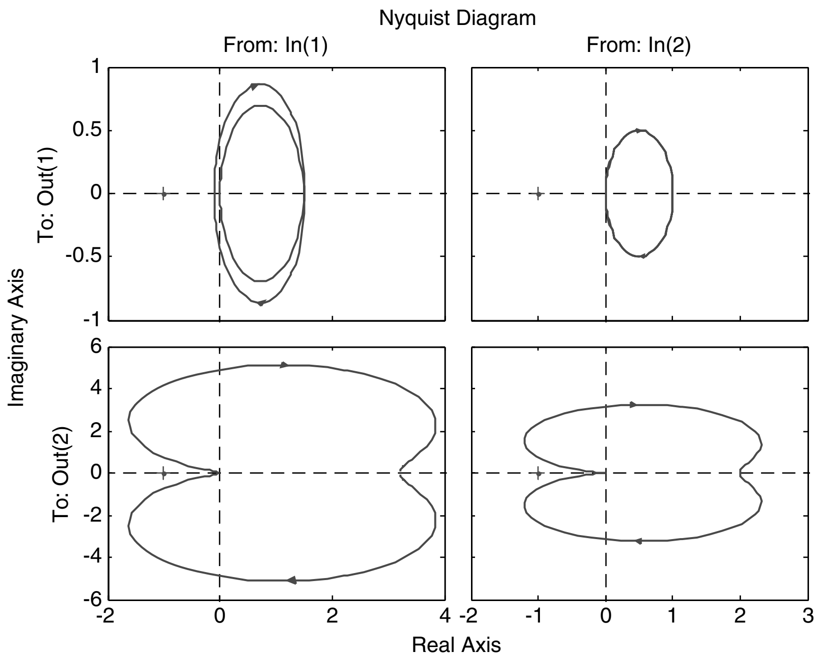
Code	Output
A = [0 1; −20 −2]; **B = [0; 20];** **C = [1 0];** **D = [0];** **w = linspace (−10, 10, 1024);** **nyquist (A, B, C, D, 1, w)**	

2. System with more than one input and more than one output.

   (c) If the frequency range for plotting the Nyquist plot is automatically selected, the syntax for plotting the Nyquist plot will be
   (i) If the input to which the Nyquist plot of a system is to be plotted is mentioned, the syntax will be
   nyquist (A, B, C, D, i)
   where **i** is the input for which the Nyquist plot has to be plotted.
   (ii) If the input to which the Nyquist plot of a system is to be plotted is not mentioned, the syntax will be
   nyquist (A, B, C, D)
   If we use this command, the number of Nyquist plots generated will be $(m * n)$.
   where **m** is the number of inputs and **n** is the number of outputs.
   If the system has two inputs and two outputs, then the number of Nyquist plot generated by using the second command will be 4 (2 * 2).

An example for the usage of this command is given in Table 14.37.

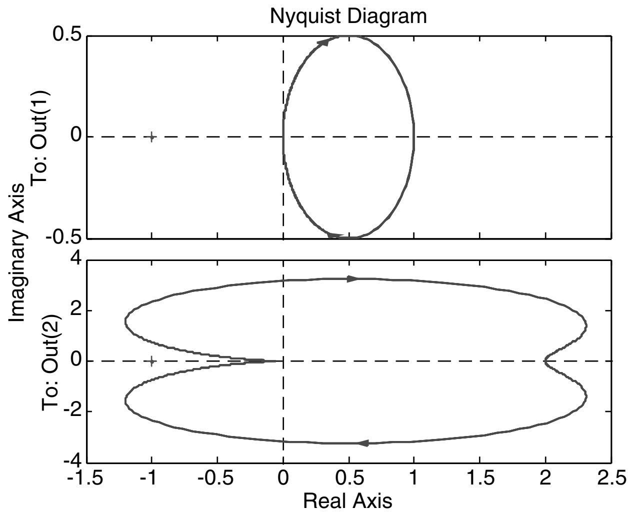
Table 14.37 | Nyquist plot for a state-space model

Code	Output
B = [1.5 1; 0.5 0]; **C = [1 0; 0 1];** **A = [−1 −0.5; 5 0];** **D = [0 0; 0 0];** **nyquist (A, B, C, D)**	

   (d) If the frequency range for plotting the Bode plot is user defined, the syntax for plotting the Bode plot will be
   nyquist (A, B, C, D, i, w)

An example for the usage of this command is given in Table 14.38.

Table 14.38 | Nyquist plot for state-space model with user-defined frequency

Code	Output
B = [1.5 1; 0.5 0]; **C = [1 0; 0 1];** **A = [−1 −0.5; 5 0];** **D = [0 0; 0 0];** **a = −10;** **b = 10;** **w = linspace (a, b, 1024);** **i = 2;** **nyquist(A, B, C, D, i, w)**	

14.2.19 Design of Compensators Using MATLAB

The detailed analysis of the design of compensators is discussed in Chapter 11. There is no default command which can be used for designing the compensators in MATLAB. In this section, we will be analyzing the design of compensators in MATLAB with the help of an example.

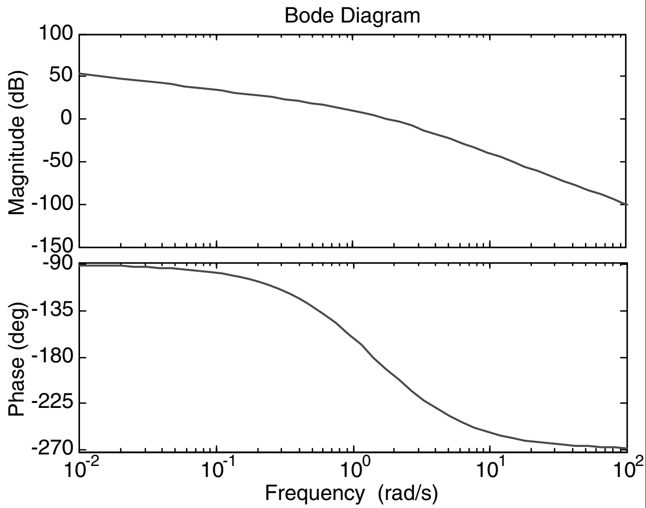
14.2.19.1 Lag compensator: Consider an uncompensated system with the open-loop transfer function as $G(s)=\dfrac{K}{s(s+1)(0.5s+1)}$. Design a lag compensator for the system such that the compensated system has static velocity error constant $K_v = 5\,\text{sec}^{-1}$, phase margin $p_m = 40^\circ$ and the gain margin $g_m = 10\ dB$.

Solution

As discussed in Chapter 10, step-by-step procedure are followed to determine the transfer function of the compensator.

1. The value of K is determined as 5.
2. The bode plot for the system $G(s)=\dfrac{5}{s(s+1)(0.5s+1)}$ is plotted and the frequency domain specifications are determined in MATLAB as given in Table 14.39.
3. Adding tolerance of 5°, the desire phase margin will be $\left((p_m)_d\right)_{new} = 45^\circ$.
4. To have the phase margin of 45°, the phase angle of the system should be -135°. Therefore, for -135°, the frequency of the system is 0.56 rad/s. Hence, $\left(\omega_{gc}\right)_{new}$ = 0.56 rad/s.

Table 14.39 | Bode plot for uncompensated system

Code	Output in command window	Bode plot
n = [5]; d = [0.5 1.5 1 0]; bode(n, d) [Gm,Pm,pcf,gcf] = margin(n, d)	Gm = 0.6000 Pm = −12.9919 pcf = 1.4142 gcf = 1.8020	

5. The magnitude value A in dB corresponding to $\left(\omega_{gc}\right)_{new}$ is 17.5 dB. The magnitude value from bode plot corresponding to desired phase margin is shown in Fig. 14.3.

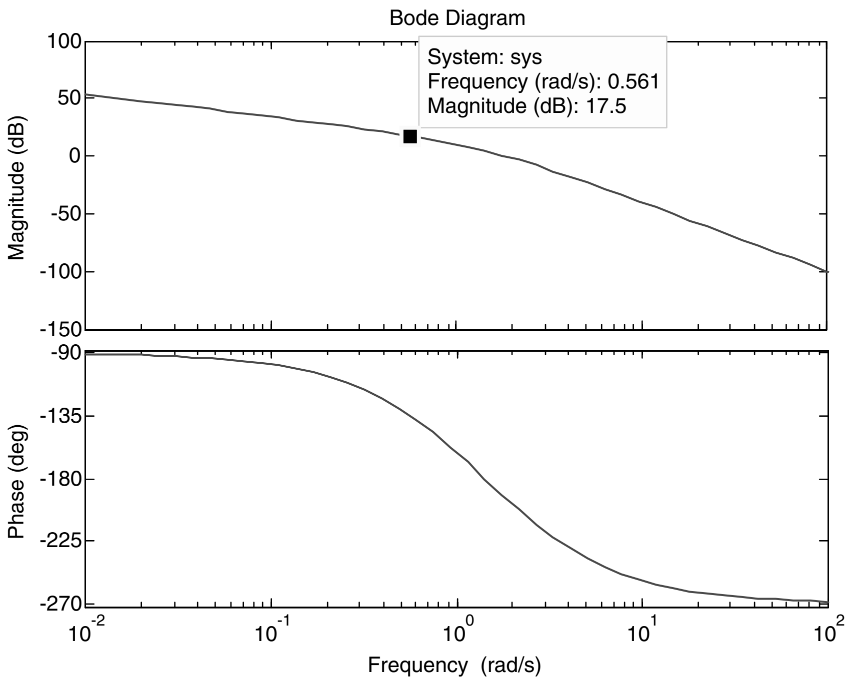


Fig. 14.3 | Magnitude value corresponding to desired phase margin

6. The value of β is determined using the formula $-20\log\beta = -17.5$ and the value is determined as $\beta = 7.49$.
7. The value of T is determined using the formula $\frac{1}{T} = \frac{(\omega_{gc})_{new}}{10}$ and the value is determined as $T = 17.85$.
8. Therefore, the transfer function of the lag compensator will be

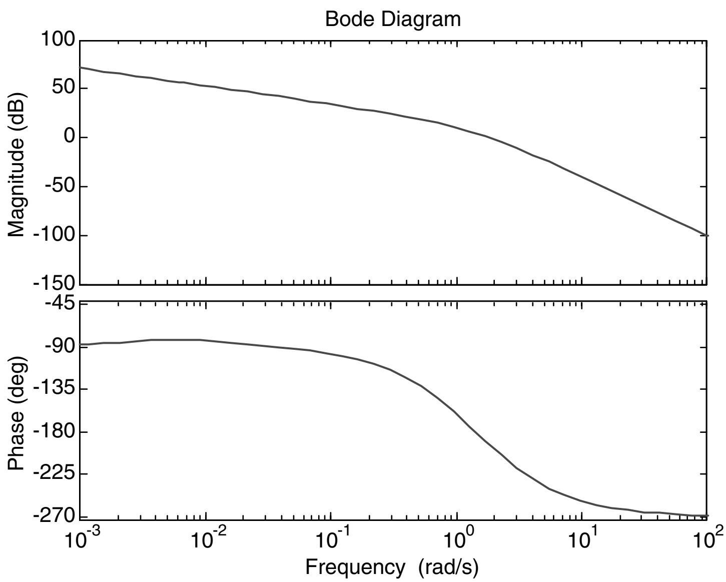
$$G_{la}(s) = \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\beta T}\right)} = \frac{(s+0.056)}{(s+0.00747)}$$

9. Thus, the transfer function of the compensated system will be

$$G_c(s) = G(s)\,G_{la}(s) = \frac{5(s+0.056)}{s(s+1)(0.5s+1)(s+0.00747)}$$

10. The bode plot of the compensated system is plotted in MATLAB and the plot is given in Table 14.40.

Table 14.40 | Bode plot of compensated system

Code	**Bode plot**
% Enter the uncompensated system G(s) **n1 = [5];** **d1= [0.5 1.5 1 0];** **% Enter the compensator system Gla(s)** **n2 = [1 0.0056]** **d2 = [1 0.00747];** **% Coefficients of compensated system Gc(s)** **[n3, d3] = series (n1, d1, n2, d2);** **bode(n3, d3)**	

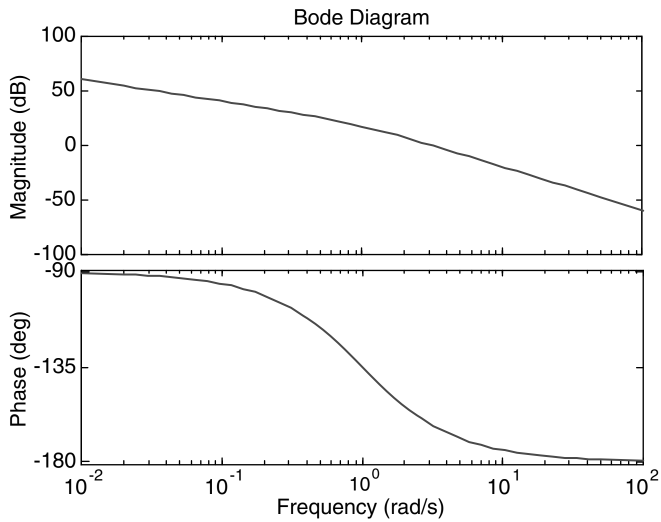
14.2.19.2 Lead compensator: Consider an uncompensated system with the open-loop transfer function as $G(s) = \frac{K}{s(s+2)}$. Design a lead compensator for the system such that the compensated system has static velocity error constant $K_v = 20\,\text{sec}^{-1}$, phase margin $p_m = 50^\circ$ and the gain margin $g_m = 10\ dB$.

Solution

As discussed in Chapter 10, step-by-step procedure are followed to determine the transfer function of the compensator.

1. The value of K is determined as 10.
2. The bode plot for the system $G(s)=\dfrac{10}{s(s+2)}$ is plotted and the frequency domain specifications are determined in MATLAB as given in Table 14.41.

Table 14.41 | Bode plot for uncompensated system

Code	Output in command window	Bode plot
n = [10]; **d = [1 1 0];** **bode(n, d)** [Gm, Pm, pcf, gcf] = **margin(n, d)**	**Gm =** **Inf** **Pm =** **17.9642** **pcf =** **Inf** **gcf =** **3.0842**	

3. By adding tolerance of $5°$, $\left((p_{\mathrm{m}})_{\mathrm{d}}\right)_{\mathrm{new}}=(p_{\mathrm{m}})_{\mathrm{d}}-p_{\mathrm{m}}+\varepsilon$ is calculated as $38°$.
4. The value of α is determined using $\sin\left(\left(p_{\mathrm{m}}\right)_{\mathrm{d}}\right)_{\mathrm{new}}=\dfrac{1-\alpha}{1+\alpha}$ and the obtained value of α is 0.24.
5. Determine $-10\log\left(\dfrac{1}{\alpha}\right)$ dB which is −6.2 dB. Let $A = -6.2$dB.
6. The frequency corresponding to AdB is 4.48 rad/s which is determined from magnitude plot of the system as shown in Fig. 14.4. Let that frequency be $(\omega_{\mathrm{gc}})_{\mathrm{new}}$.
7. Using the formula $(\omega_{\mathrm{gc}})_{\mathrm{new}}=\dfrac{1}{T\sqrt{\alpha}}$, the value of T is determined as 0.455.
8. The value of K_c is determined using the formula $K = K_{\mathrm{c}}\alpha$ as 41.66.
9. Therefore, the transfer function of the lead compensator will be

$$G_{\mathrm{le}}(s)=K_{\mathrm{c}}\frac{\left(s+\dfrac{1}{T}\right)}{\left(s+\dfrac{1}{\alpha T}\right)}=K_{\mathrm{c}}\alpha\frac{(Ts+1)}{(\alpha Ts+1)}=10\frac{(0.455s+1)}{(0.1092s+1)}$$

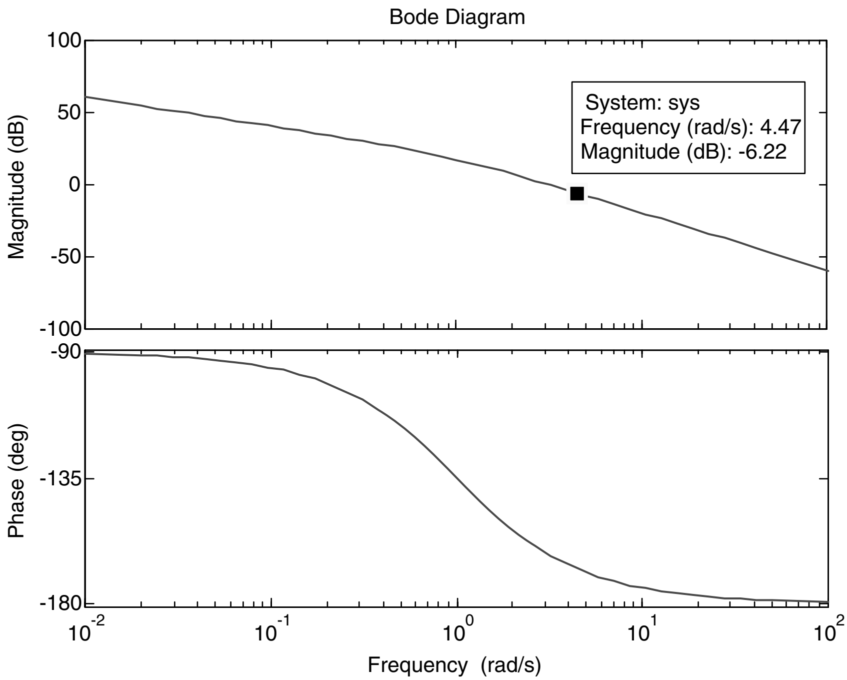


Fig. 14.4 | Determination of frequency corresponding to AdB

10. Thus, the transfer function of the compensated system will be

$$G_c(s) = G(s)\,G_{le}(s) = \frac{417(s+2.19)}{s(s+2)(s+9.15)}$$

11. The bode plot of the compensated system is plotted in MATLAB and the plot is given in Table 14.42.

14.2.19.3 Lag–Lead compensator: Consider an uncompensated system with the open-loop transfer function as $G(s) = \dfrac{K}{s(s+1)(s+2)}$. Design a lag–lead compensator for the system such that the compensated system has static velocity error constant $K_v = 10\,\text{sec}^{-1}$, phase margin $p_m = 50^\circ$ and the gain margin $g_m = 10\,\text{dB}$.

Solution

As discussed in Chapter 10, step-by-step procedure are followed to determine the transfer function of the compensator.

1. The value of K is determined as 20.
2. The bode plot for the system $G(s) = \dfrac{20}{s(s+1)(s+2)}$ is plotted and the frequency domain specifications are determined in MATLAB as given in Table 14.43.

Table 14.42 | Bode plot of compensated system

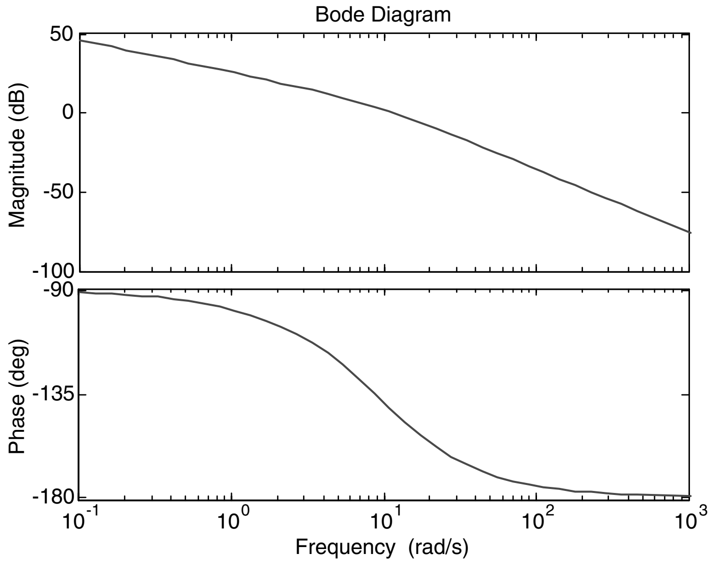
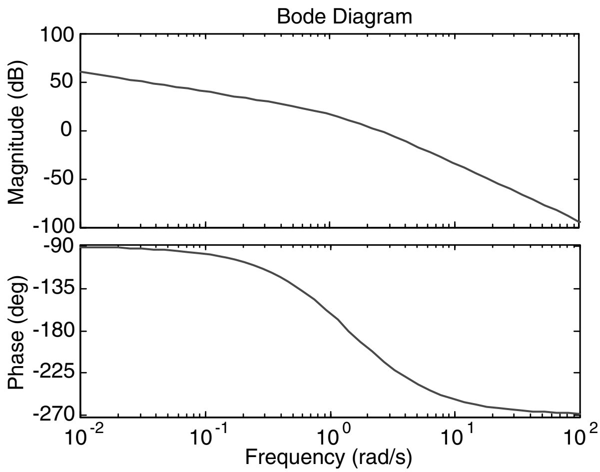
Code	Bode plot
% Enter the uncompensated system G(s) n1 = [417]; d1 = [1 2 0]; % Enter the compensator system Gle(s) n2 = [1 2.19]; d2 = [1 9.15]; % Coefficients of compensated system Gc(s) [n3, d3] = series(n1, d1, n2, d2); bode(n3, d3)	

Table 14.43 | Bode plot for uncompensated system

Code	Output in command window	Bode plot
n = [20]; d = [1 3 2 0]; bode(n, d) [Gm, Pm, pcf, gcf] = margin(n,d)	Gm = 0.3000 Pm = –28.0814 pcf = 1.4142 gcf = 2.4253	

3. Let $(p_m)_d = 50°$
4. Using the formulae $\sin(p_m)_d = \dfrac{1-\alpha}{1+\alpha}$ and $\alpha\beta = 1$, the value of β is determined as 7.54.
5. Let $(\omega_{gc})_{new} = pcf = 1.4 \text{rad/s}$.

6. The time constant of lag compensator is determined as:

$$T_2 = \frac{10}{(\omega_{gc})_{new}} = \frac{10}{1.4} = 7.142\text{rad/s}.$$

7. Therefore, the transfer function of lag compensator is

$$G_{la}(s) = \beta \frac{(1+T_2 s)}{(1+\beta T_2 s)} = 7.54 \frac{(1+7.142s)}{(1+53.85s)}.$$

8. The magnitude of the system from the magnitude plot at $(\omega_{gc})_{new}$, $A = 10.7\text{dB}$. The determination of magnitude of the system from magnitude plot is shown in Fig. 14.5.

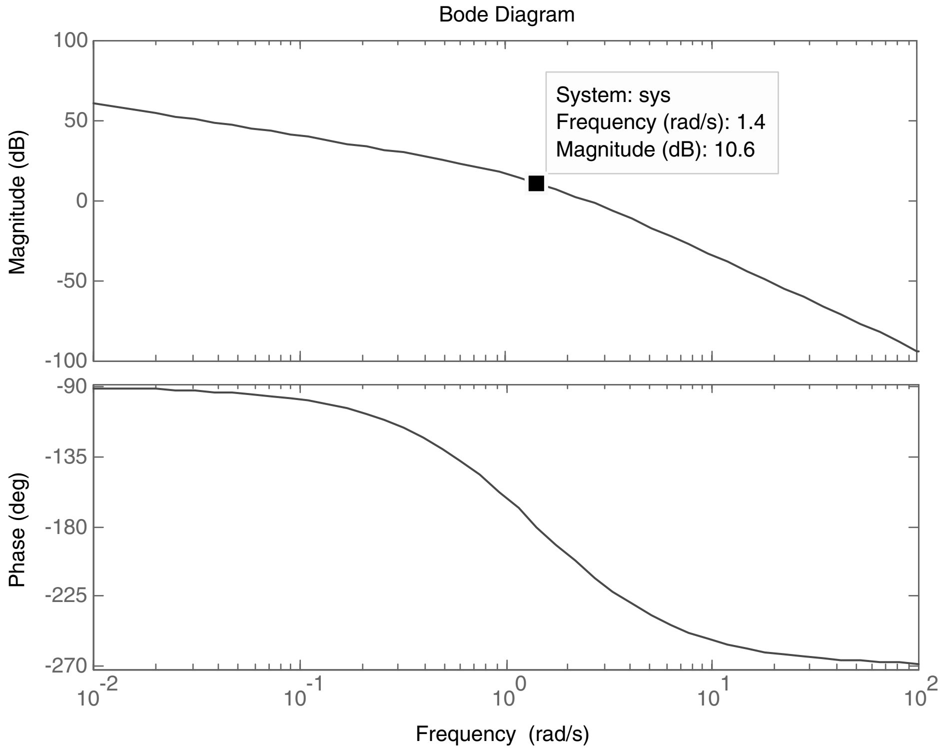


Fig. 14.5 | Determination of magnitude A at $(\omega_{gc})_{new}$

9. If the slope of +20 dB/decade is drawn from (1.4 rad/s, −10.7 dB), the slope cuts the 0dB line and −20 dB line at $\omega = 7.5$rad/s and at $\omega = 0.75$rad/s respectively.
10. Thus, the transfer function of the lead portion of the lag–lead compensator is

$$G_{le}(s) = \frac{s+0.75}{s+7.5} = \frac{1}{10}\left(\frac{1+1.333s}{1+0.1333s}\right)$$

12. Therefore, the transfer function of the lag–lead compensator is

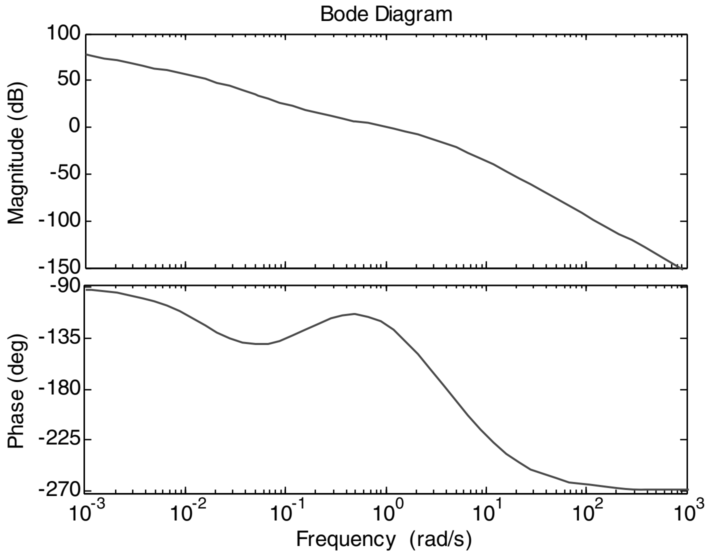
$$G_{\text{la-le}}(s) = G_{\text{la}}(s)G_{\text{le}}(s) = 0.754\frac{(1+1.333s)(1+7.142s)}{(1+0.1333s)(1+53.85s)}$$

13. Thus, the transfer function of the compensated system is

$$G_{\text{c}}(s) = G_{\text{la-le}}(s)G(s) = 0.754\frac{(1+1.333s)(1+7.142s)}{(1+0.1333s)(1+53.85s)}\times\frac{20}{s(s+1)(s+2)}$$

14. The bode plot of the compensated system is plotted in MATLAB and the plot is given in Table 14.44.

Table 14.44 | Bode plot of compensated system

Code	**Bode plot**
% Enter the uncompensated system G(s) **n1 = [20];** **d1= [1 3 2 0];** **% Enter the compensator system Gla- le(s)** **n2 = [1 0.754];** **d2 = [0.1333 1];** **n3 = [7.142 1];** **d3 = [53.85 1];** **[n4, d4] = series(n2, d2, n3, d3);** **% Coefficients of compensated system Gc(s)** **[n5, d5] = series(n1, d1, n4, d4);** **bode(n5, d5)**	

Index

D

E

F

G

H

I

S

x

Z