Random Signal Theory

Some signals are described by its fixed mathematical equations. Such type of signals are called <u>deterministic signals</u>. There is one other class of signals, the behaviour of which cannot be predicted. in advance before they actually occur. Such type of signals are called <u>random signals</u>. The example of random signals are the noise interferences in communication systems.

It is possible to analyze the random signals statistically with the help of probability theory.

Basic Definitions related to Probability

<u>Experiment</u> :- An experiment is defined as the process which is conducted to get some results. The theory of prob. is applied to such type of experiments to predict the possibility of a particular output. An experiment is sometimes called <u>trial</u>.

for eg, tossing a coin is an exp. or trial.

<u>Sample Space</u> :- A set of all possible outcomes of an exp. or trial is called sample space (S).

for eg, in an exp. of tossing a coin, sample space is

$$S = \{H, T\}$$

H → Head

T → Tail

If we consider another eg. of tossing three coins simultaneously, then

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

..ent :- The expected subset of the sample space is called an event (E)

As an ex., let us consider an experiment of throwing a cubic die, then

$$S = \{1, 2, 3, 4, 5, 6\}$$

Now, if we want the no. '3' to be an outcome then this subset {3} is an event.

If event E has only one outcome, then it is called an elementary event.

If E = S, then E contains all the outcomes. Such an event is called as certain event.

Probability :- Probability may be defined as the study of random experiments. In any exp., there is always an uncertainity that a particular event will occur or not. As a measure of probability of occurance of an event, a no. b/w 0 to 1 is assigned. If it is sure that an event will occur, then its prob. is '100%' or '1'. If it is not sure whether the event will occur or not, then its probability is b/w 0 and 1. If it is sure that an event will not occur, then its probability is '0%' or '0'.

$$\text{Probability} = \frac{\text{No. of possible favourable outcomes}}{\text{Total no. of outcomes}}$$

Properties of Probability :-

Property 1 → Prob. of a certain event is unity i.e.,

$$P(A) = 1$$

Property 2 - Prob. of any event is always less than or equal to 1 and non-negative.

$$0 \leq P(A) \leq 1$$

Property 3 → If A and B are two mutually exclusive events, then

$$P(A+B) = P(A) + P(B)$$

Property 4 → If A is any event, then prob. of not happeni of A is

$$P(\bar{A}) = 1 - P(A)$$

Property 5 → If A and B are any two events (not mutually exclusive events) then

$$P(A+B) = P(A) + P(B) - P(AB)$$

P(AB) is called the joint probability.

Conditional Probability :-

In an exp., if there are two events A and B. Now the prob. of event B, given that event A has already occured, is represented by $P\left[\frac{B}{A}\right]$ and similarly as $P\left[\frac{A}{B}\right]$.

$P\left[\frac{B}{A}\right]$ & $P\left[\frac{A}{B}\right]$ are conditional Prob.

$$\boxed{P\left[\frac{B}{A}\right] = \frac{P(AB)}{P(A)}} \quad -(1)$$

$$\boxed{P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)}} \quad -(2)$$

Probability of stastically independent events :-

If the occurance of an event does not depend on ~~any~~ the occurance of any other event, then ~~it is~~ these are known as independent events.

for eg., if occurance of event B does not depend on the occurance of event A, then

$$P\left[\frac{B}{A}\right] = P[B]$$

Putting this value in eqⁿ ①

$\therefore$ $P[AB] = P(A)\,P(B)$ for independent events.

Random Variables :-

The Random Variable is used to signify a rule by which a real no. is assigned to each possible outcome of an exp. It is represented by $X(S)$ or simply 'X'.

$X(S)$ is a function which takes on the values $x_1, x_2, \ldots$.

Types of Random Variables :-

1. Discrete Random Variables :-

If the sample space 'S' contains a countable no. of sample points then $X(S)$ will be a discrete random variable. A discrete R.V will thus have a countable no. of distinct values.

For eg., if we toss 3 coins simultaneously ~~th~~ and the desireable outcome is Head, then its R.V can be given as,

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

$X =$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
	3	2	2	2	1	1	1	0

2. Continuous Random Variables :-

A continuous R.V is not restricted to a finite no. of distinct values. Thus continuous R.V has an uncountable of possible values. For eg, a noise Voltage in any signal outcome can be defined as continuous R.V.

Cumulative Distribution Function :-

The CDF of a random variable is defined as the probability that the random variable 'X' takes values less than or equal to x.

$$\text{CDF } F_X(x) = P[X \leq x]$$

Where x is a dummy variable and the CDF is denoted by $F_X(x)$.

Important Properties of CDF :-

Property 1 :- The CDF is always bounded b/w 0 and 1.

$$0 \leq F_X(x) \leq 1$$

As per the definition of CDF, it is a probability function $P[X \leq x]$ and any probability must have a value between 0 and 1. Therefore CDF is always bounded b/w 0 and 1.

Property 2 :- This property states that

$$F_X(\infty) = 1$$

Here $F_X(\infty) = P[X \leq \infty]$. This includes the prob. of all the possible outcomes or events. The random variable $X \leq \infty$ thus, becomes a certain event and therefore has a 100% probability.

Property 3 :- This property states that,

$$F_X(-\infty) = 0$$

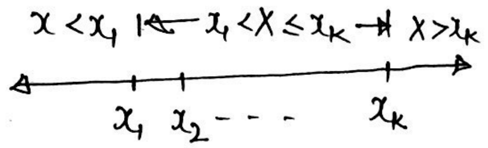
Here $F_X(-\infty) = P[X \leq -\infty]$. The R.V 'X' cannot have any values which is less than or equal to $-\infty$. The, $X \leq -\infty$ is a null event and therefore has a 0% prob.

Property 4 :- This property states that $F_X(x)$ is a monotone non-decreasing fu^n i.e.

$$F_X(x_1) \leq F_X(x_2) \text{ for } x_1 < x_2$$

How to calculate CDF for discrete Random Variables

Let us consider a discrete random variable which is restricted to the distinct values $x_1, x_2, ---x_K$ as shown in fig.



Now let us obtain the CDF of this R.V.

$$F_X(x) = P[X \leq x]$$

$\boxed{1.}$ CDF for $X < x_1$

it is obvious that the values of the R.V are restricte b/w x_1 and x_K. Therefore, there are no possible events for $X < x_1$.

Thus $\quad P[X < x_1] = 0$

$$\therefore F_X[x] = 0 \quad \text{for } X < x_1$$

$\boxed{2.}$ CDF for $x_1 < X > x_K$

~~If $X \geq x_K$~~ All the events $x_1, x_2 --- x_K$ are mutually Exclusive events. $\therefore$ prob. of $x_1 \leq X \leq x_K$ can be written as under:

$$P[x_1 \leq X \leq x_K] = P[X = x_1] + P[X = x_2] + ----- P[X = x_K]$$

$$= \sum_{i=1}^{k} P[X = x_i]$$

of distinct values. Thus

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$$\therefore F_X(x) = \sum_{i=1}^{k} P[X = x_i]$$

3 CDF for $X > x_k$

If $X > X_k$, then, $(X \leq X_k)$ will include all possible values from x_1 to x_2

Thus, $P[X \leq x] = 1$ for $X > X_k$

$\therefore$

$$F_X(x) = \begin{cases} 0 & \text{for } X < X_1 \\ \sum_{i=1}^{k} P(X = X_i) & \text{for } X_1 \leq X \leq X_k \\ 1 & \text{for } X > X_k \end{cases}$$

Probability Density Function :-

The CDF can give useful information about discrete as well as continuous random variables. However PDF is a more convenient way of describing a continuous R.V. The PDF $(f_X(x))$ is defined as the derivative of the CDF.

PDF : $\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$

Properties of PDF :-

Property 1 :- The CDF can be derived from PDF by integrating it ie.

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx}$$

Proof :- As, $f_X(x) = \frac{d}{dx} F_X(x)$

Integrating both sides,

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\,dx$$

Here the upper limit is not $+\infty$, but 'x' because $F_X(x)$ has been defined as the probability of $X \leq x$.

$$\therefore \int_{-\infty}^{x} f_X(x)\,dx = \left[F_X(x)\right]_{-\infty}^{x} = F_X(x) - F_X(-\infty)$$

But $F_X(-\infty) = 0$

$$\therefore \int_{-\infty}^{\infty} f_X(x)\,dx = F_X(x) - 0$$

$$= F_X(x)$$

Property 2 :- PDF is a non-negative funⁿ of all values of x i.e.

$$\boxed{f_X(x) \geq 0} \quad \text{for all } x$$

As we know that, CDF is a monotone increasing funⁿ. PDF is the derivative of CDF and the derivative of monotone increasing funⁿ will always be positive.

Property 3 :- The area under PDF curve is always equal to unity.

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

of distinct values. Thus continuous ... has ...

Proof :- As, $f_X(x) = \frac{d}{dx}\{F_X(x)\}$

Integrating both sides,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\,dx$$

$$\Rightarrow [F_X(x)]_{-\infty}^{\infty}$$

$$\Rightarrow F_X(\infty) - F_X(-\infty)$$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1$$

Relation between Probability and PDF :-

The probability of observing X in any interval from x_1 to x_2 is given by the area under the PDF curve over the interval x_1 to x_2.

$$\therefore \quad \boxed{P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\,dx}$$

Proof :- Let us express probability of the event $X \leq x_2$ as the union of two events probabilities as follows :

$$P[X \leq x_2] = P[X \leq x_1] + P[x_1 < X \leq x_2]$$

This is possible because x_1 and x_2 are mutually exclusive events

As $P[X \leq x_2] = F_X(x_2)$

& $P[X \leq x_1] = F_X(x_1)$

Putting the values in eq^n,

$$\therefore \quad F_X(x_2) = F_X(x_1) + P[x_1 < X \leq x_2]$$

$$\boxed{P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)}$$

Now as, $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Using this property, eqⁿ gets modified to,

$$P[x_1 < X \le x_2] = \int_{-\infty}^{x_2} f_X(x)\,dx - \int_{-\infty}^{x_1} f_X(x)\,dx$$

or $$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\,dx$$

Hence Proved

Joint CDF :-

Many a times the outcome of an exp. requires more than one R.V to describe the exp. Let us consider two ~~experiments~~ random variables X and Y. Joint CDF is defined in the same way as we defined CDF for a single R.V.

$$\boxed{F_{XY}(x,y) = P[X \le x, Y \le y]}$$

The joint CDF $F_{XY}(x,y)$ is a probability that the outcome of an exp. will result in a sample point lying inside the quadrant $(-\infty < X \le x, -\infty < Y < y)$, of the joint sample space.

Properties of Joint CDF :-

Property 1 :- The joint CDF i.e. $F_{XY}(x,y)$ is a non negative function.

$$\boxed{F_{XY}(x,y) \geq 0}$$

Property 2 :- The joint CDF is always a continuous fu^n in the xy plane.

Property 3 :- The joint CDF is a non-decreasing fu^n of both x & y

of distinct values. Thus continuous ... has ...

Joint PDF :-

The joint PDF is defined for two or more continuous R.V. which may or may not be statistically independent.

~~the~~ PDF of a single R.V is,

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Similarly, joint PDF of two R.V X & Y as under :

$$\boxed{f_{XY}(x,y) = \frac{\partial^2 [F_{XY}(x,y)]}{\partial x \, \partial y}}$$

The Joint PDF is the partial derivative of the joint CDF with respect to two variables x and y. Partial differentiation is done because two variables are simultaneously involved.

Properties of Joint PDF :-

Property 1 :- The joint PDF is always non-negative.

$$\boxed{f_{XY}(x,y) \geq 0}$$

The derivative of a non-negative fun must be a non-negative function. ~~of x and y~~.

Property 2 :- The total volume under the surface represented by the joint PDF is always 1.

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) \, dx \, dy = 1}$$

Proof :-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) \, dx \, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y} dx \, dy$$

$$\Rightarrow \left[F_{XY}(x,y) \right]_{x=y=-\infty}^{x=y=\infty}$$

As, $F_{XY}(x,y) = P[X \le x, Y \le y]$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = P[X \le x, Y \le y]_{x=y=-\infty}^{x=y=\infty}$$

$$\Rightarrow P[X \le \infty, Y \le \infty] - P[X \le -\infty, Y \le -\infty]$$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1$$

<u>Property 3</u> :- The probability of observing 'X' in any interval from x_1 to x_2 and the probability of observing 'Y' in the interval from y_1 to y_2 is given by the volume under the surface represented by the joint PDF.

$$\boxed{P[x_1 < X \le x_2, y_1 < Y \le y_2] = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy}$$

<u>Proof</u> :-

As, $P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\,dx$

Similarly,

$$P[y_1 < Y \le y_2] = \int_{y_1}^{y_2} f_X(x)\,dx$$

&

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

then,

$$P[x_1 < X \le x_2, y_1 < Y \le y_2] = \int_{y_1}^{y_2} f_Y(y)\,dy \int_{x_1}^{x_2} f_X(x)\,dx = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_X(x)\,dx\, f_Y(y)\,dy$$

$$\Rightarrow \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$$

of distinct values. Thus continuous

Marginal Densities

While dealing with two R.V X & Y, the individual probability densities $f_X(x)$ and $f_Y(y)$ can be obtained from the joint PDF $f_{XY}(x,y)$. Then these individual densities are called as marginal densities or marginal PDFs.

As, $$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

Similarly, $$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

If only the CDF of single R.V 'X' is to be obtained,

i.e. $F_X(x) = P[X \leq x]$

then,

$$F_X(x) = P[X \leq x, -\infty < Y \leq \infty]$$

This means that the value of Y may lie anywhere b/w $-\infty$ & $+\infty$, it will not have any effect on $F_X(x)$.

Thus, $$\boxed{F_X(x) = P[X \leq x, -\infty < Y \leq \infty] = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

Now, to obtain PDF $f_X(x)$,

we have $$f_X(x) = \frac{d}{dx} F_X(x)$$

then

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left[\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy\right] = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$ Similarly, $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

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Conditional PDF :-

If R.V X and Y are not independent, then the dependence of X on Y is expressed by the conditional PDF as under :

$$\boxed{f_X(x/Y=y) = f_X(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

Where, $f_Y(y)$ is the marginal density.

Similarly,

$$f_Y(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Properties of Conditional PDF :-

Property 1 :- The conditional PDF is a non-negative fun.

$$\boxed{f_Y(y/x) \geq 0}$$

This is because the conditional PDF is basically a probability density fun.

Property 2 :- The area under a conditional PDF is always equal to 1.

$$\int_{-\infty}^{\infty} f_Y(y/x) = 1 \quad , \quad \int_{-\infty}^{\infty} f_X(x/y) = 1$$

Property 3 :- If the R.V X and Y are statistically independent, then

$$f_Y(y/x) = f_Y(y)$$

$$f_X(x/y) = f_X(x)$$

of distinct values. Thus continuous R.V has ...

This means that the conditional density functions reduce to the marginal density functions.

Proof:-

The conditional PDF is given by,

$$f_Y(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

As X and Y are independent, their joint PDF is given by,

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

then,

$$f_Y(y/x) = \frac{\cancel{f_X(x)} \cdot f_Y(y)}{\cancel{f_X(x)}}$$

$$f_Y(y/x) = f_Y(y)$$

Similarly,

$$f_X\left(\frac{x}{y}\right) = f_X(x)$$

Mean Value / Average Value / Expected Value

Let's consider a discrete random variable X Which has possible values of x_1, x_2 ---- With the probabilities of occurence to be $P(x_1), P(x_2)$ ----. Now, if there ~~are~~ N independent observations, then outcome $X = x_i$ Will occur $NP(x_i)$ times & soon.

Therefore, the arithmetic sum of all the independent observations would be:

$$x_1 P(x_1)N + x_2 P(x_2)N + x_3 P(x_3)N + -----$$

$$\Rightarrow N \sum_{i=1}^{k} x_i P(x_i)$$

Mean value of the R.V is calculated by dividing the sum in eqn by the no. of observations N.

$$\therefore \text{ mean value of } X = \frac{\cancel{N}}{\cancel{N}} \sum_{i=1}^{k} x_i P(x_i)$$

$$\boxed{\text{mean value of } X = m_X = \overline{X} = E[X] = \sum_{i=1}^{k} x_i P(x_i)} \quad -(1)$$

Mean Value of a Continuous Random Variable :-

If R.V 'X' is a continuous R.V, then values x_1, x_2, x_3 ----- comes very close to each other. The summation sign, therefore, gets converted into an integration sign. The integration will be carried out over the entire range from $-\infty$ to $+\infty$.

As,

$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

Substituting $x_2 = x_1 + dx$

$$P[x_1 < X \leq x_1 + dx] = f_X(x)\, dx$$

This is because dx is so small that $f_X(x)$ can be assumed to be constant in the range x_1 to $(x_1 + dx)$.

Substituting x in place of x_1

$$P[x < X \leq x + dx] = f_X(x)\, dx$$

As dx is very small we have

$$P[x < X \leq x + dx] = P(x)$$

of distinct values. Thus continuous ...

then $P(x) = f_X(x)\, dx$

Substitute value of $P(x)$ in eq^n (1.)

$$m_X = \sum_{i=1}^{k} x_i f_X(x)\, dx$$

Replace the summation sign with integration.

$$\boxed{m_X = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx}$$

n^{th} moment of Random Variable X :-

n^{th} moment of X is defined as mean value of X^n.

Thus, n^{th} moment of $X = E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$

1st moment of X (Mean Value) :-

Substituting $n=1$, We get the first moment of X

$$E[X] = \overline{X} = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = m_X$$

Thus the 1st moment of X is always its mean or average value.

2nd moment of X (Mean Square Value) :-

Substituting $n=2$

$$\boxed{E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx}$$

The Central Moment :-

As moment means expected value. The central moment is the expected value of the difference b/w the R.V X and it mean value m_X. Thus n^{th} central moment is defined as,

$$\boxed{n^{th} \text{ Central moment of } X = E[(X-m_X)^n] = \int_{-\infty}^{\infty}(x-m_X)^n f_X(x)\,dx}$$

Variance of Random Variable (σ_X^2) :-

Variance is defined as the second central moment of R.V X.

$$\therefore \text{Variance of } X = E[(X-m_X)^2] = \sigma_X^2 = \int_{-\infty}^{\infty}(x-m_X)^2 f_X(x)\,dx$$

$$\boxed{\sigma_X^2 = E[(X-m_X)^2] = \int_{-\infty}^{\infty}(x-m_X)^2 f_X(x)\,dx}$$

$$\Rightarrow \sigma_X^2 = E[(X-m_X)^2] = E[X^2 + m_X^2 - 2Xm_X]$$

$$\Rightarrow E[X^2] - 2m_X E[X] + m_X^2$$

$$\Rightarrow E[X^2] - 2m_X \cdot m_X + m_X^2$$

$$\boxed{\sigma_X^2 \Rightarrow E[X^2] - m_X^2}$$

Therefore,

Variance = Mean Square Value − Square of mean

of distinct values. Thus continuous R.V has ...

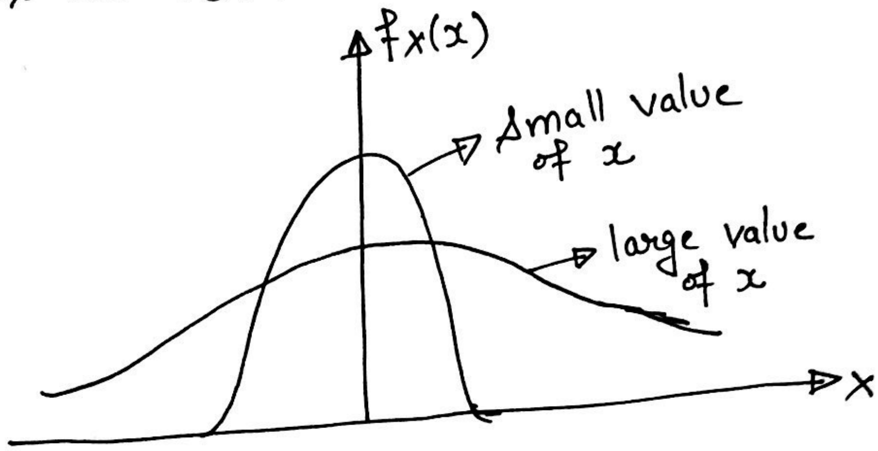
Standard deviation :-

(σ_x) Standard deviation = $\sqrt{\text{Variance}}$

$$\sigma_x = \sqrt{E(X^2) - E(X)^2}$$

$$\boxed{\sigma_x = \sqrt{E[X^2] - m_x^2}}$$

The standard deviation, σ_x, is the measure of the width of its PDF. The larger the value of σ_x, the wider is the PDF.



Significance of Standard deviation :-

The value of standard deviation indicates the deviation in the value of R.V from its mean value. Small values of σ_x shows that all the possible values of a R.V are clustered around its mean value, whereas, for larger values of σ_x, R.V will deviate considerably from its mean value.

Mean and Variance of Sum of R.V :-

Let X & Y be two R.V with mean values of m_x and m_y. Then, it can be proved that the mean of their sum is equal to sum of their means.

$$\boxed{E[X+Y] = m_x + m_y}$$

Similarly,

$$\boxed{\sigma_X^2[X+Y] = \sigma_X^2 + \sigma_Y^2}$$

Proof :-

for mean value of X+Y

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\, f_{XY}(x,y)\,dx\,dy$$

Where, $f_{XY}(x,y)$ is the joint PDF of X and Y.

$$\therefore E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f_{XY}(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\right]dx + \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\right]dy$$

According to marginal PDF,

$$\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = f_X(x)$$

$$\& \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = f_Y(y)$$

$$\therefore E[X+Y] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx + \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

$$\Rightarrow E[X+Y] = E[X] + E[Y]$$

Hence Proved

for Variance

$$\text{Variance of } (X+Y) = E[(X+Y)^2] - [E(X+Y)]^2$$

$$\Rightarrow E[X^2] + E[Y^2] - \{E[X] + E[Y]\}^2$$

of distinct values. Thus continuous ...

Variance of $(X+Y) = \{E[X^2] - E^2[X]\} + \{E[Y^2] - E^2[Y]\}$

$$\sigma^2_{(x+y)} = \sigma_x^2 + \sigma_y^2$$

Hence Proved

Mean of the Product of Independent Random Variables :-

$$\boxed{E[XY] = E[X] \cdot E[Y]}$$

Proof :-

$$E[XY] = \int_{-\infty}^{\infty} xy\, f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$\Rightarrow \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} f_Y(y)\, dy$$

$$\Rightarrow \quad E[XY] = E(X] \cdot E[Y]$$

Hence Proved

Probability Models :-

Practically, it is very difficult to study all these probability distribution functions. Hence the PDF's of all the random variables are approximated to some standard probability density functions :-

1.) Binomial distribution
2.) Poisson distribution

$\rightarrow$ for discrete R.V

3.) Gaussian distribution
4.) Rayleigh distribution
5.) Uniform distribution

$\rightarrow$ for continuous R.V

Binomial Distribution

Total / ^nC_k ·

Binomial distribution is used to describe an integer valued random variable associated with repeated trials of an experiment. Hence this distribution model can be applied to the experiment of tossing a coin repeatedly.

To understand the binomial distribution, let us consider the following example:

(i) Let us assume that binary words, each of length 'n' are being transmitted. Out of these n bits, let us assume that 'k' bits are correctly received and (n-k) bits are in error.

(ii) Let probability of correct reception of a bit = p and probability of error in reception of a bit = 1-p

To understand the Binomial Distribution properly, We use the concept

$$\begin{bmatrix}\text{Probability of receiving an} \\ \text{n bit word with k no. of} \\ \text{correct bits}\end{bmatrix} = \begin{bmatrix}\text{Probability of} \\ \text{having k} \\ \text{correct bits}\end{bmatrix} \times \begin{bmatrix}\text{Prob. of having} \\ \text{(n-k) incorrect} \\ \text{bits}\end{bmatrix}$$

$$\Rightarrow p^k (1-p)^{n-k}$$

(iii) In each word, the 'k' correct bits are placed at different places in the total of n bits per word. Therefore, the total possible no. of such words which contain k correct bits and (n-k) incorrect bits will be,

of distinct values. Thus continuous ...

$$\begin{bmatrix}\text{Total no. of words containing} \\ \text{k correct bits in total n bits}\end{bmatrix} = {}^nC_k$$

(iv) Therefore,

$$\begin{bmatrix}\text{The prob. of n bit words} \\ \text{containing k correct bits}\end{bmatrix} = \begin{bmatrix}\text{No. of words containing} \\ \text{k correct bits}\end{bmatrix} \times \begin{bmatrix}\text{Prob. of having k} \\ \text{correct bits}\end{bmatrix}$$

$$\begin{bmatrix}\text{Probability of n bit} \\ \text{words containing k} \\ \text{correct bits}\end{bmatrix} = {}^nC_k \, p^k (1-p)^{n-k}$$

(v) The L.H.S of this eq^n can be represented as $P[X=k]$ or $P[k]$. Hence, eq^n can be written as under,

$$\boxed{P[X=k] = P[k] = {}^nC_k \, p^k (1-p)^{n-k}}$$

Where, $\underbrace{{}^nC_k}_{\text{binomial coefficient}} = \dfrac{n!}{k!\,(n-k)!}$

PDF of R.V having Binomial Distribution :-

$$f_X(x) = \sum_{k=0}^{n} P[X=k]\,\delta(x-k)$$

$$\Rightarrow \boxed{f_X(x) = \sum_{k=0}^{n} {}^nC_k \, p^k (1-p)^{n-k} \, \delta(x-k)}$$

The term $\delta(x-k)$ has been included in order to show that the PDF is not a continuous fu^n.

CDF of R.V having Binomial distribution :-

The CDF as we know is defined as under :

$$CDF \; F_X(x) = P[X \leq x]$$

$$\Rightarrow \quad \boxed{F_X(x) = \sum_{k=0}^{x} {}^{n}C_k \, p^k (1-p)^{n-k}}$$

Mean Value of binomial distribution :-

$$m_X = E[X] = np$$

Variance of binomial distribution :-

$$\sigma_X^2 = np(1-p)$$

Standard deviation of binomial distribution :-

$$\sigma_X = \sqrt{np(1-p)}$$

Poisson Distribution :-

This is another standard prob. distribution used for the discrete R.V. As the number 'n' increases, the binomial distribution becomes difficult to handle. If 'n' is large, probability 'p' is very small and the mean value 'np' is finite, then the binomial distribution can be approximated by the Poisson Distribution.

The prob. of R.V having Poisson distribution is given by :

$$\boxed{P[X=k] = \frac{m^k e^{-m}}{k!}}$$

of distinct values. Thus continuous R.V has ...

where 'm' is the mean value and $m = np$.

$$\therefore \boxed{P(X=k) = \frac{(np)^k e^{-np}}{k!}}$$

The mean value of poisson distribution is given by :

$$\boxed{m_x = np}$$

and the variance of the Poisson distribution is given by :

$$\boxed{\sigma_x^2 = np}$$

$\therefore$ standard deviation $\boxed{\sigma_x = \sqrt{np}}$

Gaussian distribution (or Normal distribution) :-

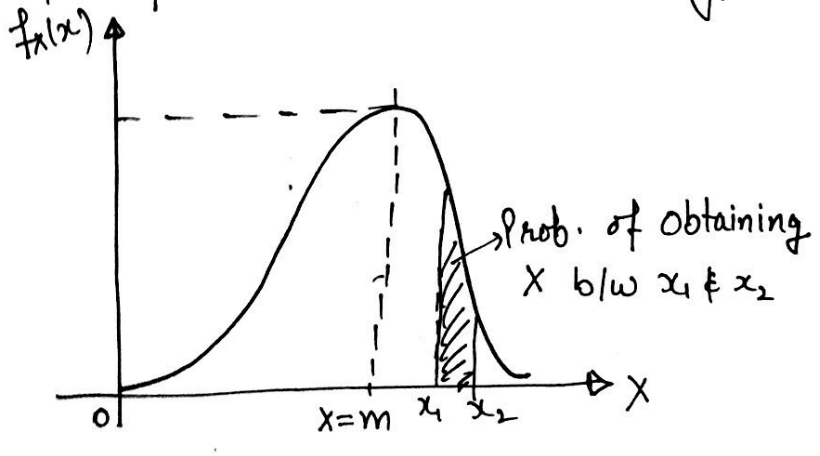
The Gaussian distribution is used for continuous R.V. It is perhaps the most important PDF. The majority of noise processes observed in practice are Gaussian and many naturally occuring experiments are characterized by continuous random variables with Gaussian PDF.

Gaussian PDF : $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

m = Mean of R.V

σ^2 = Variance of R.V

The shape of Gaussian PDF is bell type as shown:



⇒ The Gaussian PDF is a bell shaped fun with a peak i.e. $x = m$.

⇒ The Gaussian PDF has an even symmetry about the peak i.e. ~~x. Thus,~~ $\cancel{P[X \leq m]}$

$$f_X(x = m-\sigma) = f_X(x = m+\sigma)$$

⇒ Prob. of obtaining X above and below the mean value is equal i.e. $\frac{1}{2}$.

Thus, $$P[X \leq m] = P[X > m] = \frac{1}{2}$$

⇒ Area under the Gaussian PDF is 1.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

Prob. of observing 'X' between x_1 and x_2 can be obtained by integrating the Gaussian PDF b/w the limits x_1 to x_2. This is shown by the shaded area in figure.

Therefore, $$P[x_1 < X < x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

Gaussian CDF :-

As, $$CDF = \int_{-x}^{x} f_X(x)\, dx$$

Substituting the expression for the Gaussian PDF,

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

Substituting $\frac{m-x}{\sigma\sqrt{2}} = y$, ⇒ $\frac{-dx}{\sigma\sqrt{2}} = dy$

of distinct values. Thus continuous ...

peak i.e.

The limits of integration will be modified as under:

$$x \to -\infty, \; y \to +\infty$$

$$x \to x, \; y \to \frac{m-x}{\sigma\sqrt{2}}$$

Substituting these values, we get,

$$F_X(x) = \int_{\infty}^{\frac{m-x}{\sigma\sqrt{2}}} \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2} \times (-\sigma\sqrt{2}\, dy) = \frac{-1}{\sqrt{\pi}} \int_{\infty}^{\frac{m-x}{\sigma\sqrt{2}}} e^{-y^2} dy$$

By interchanging the limits of integration,

$$F_X(x) = \frac{1}{\sqrt{\pi}} \int_{\frac{m-x}{\sigma\sqrt{2}}}^{\infty} e^{-y^2} dy$$

Rearranging the above eq^n to bring it into a standard form,

$$F_X(x) = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{m-x}{\sigma\sqrt{2}}}^{\infty} e^{-y^2} dy\right]$$

The expression inside the square bracket is known as complementary error function, which is given by,

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} du$$

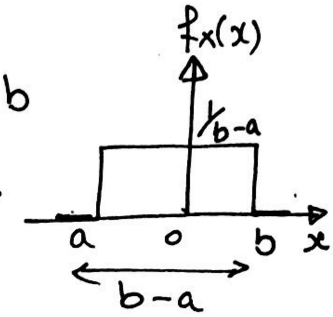
Comparing this expression with eq^n

$$\boxed{F_X(x) = \frac{1}{2}\,\mathrm{erfc}\left[\frac{m-x}{\sigma\sqrt{2}}\right]}$$

Uniform Distribution :-

If a continuous R.V 'X' is equally likely to be observed in a finite range and is likely to have a zero value outside this finite range, then the R.V is said to have a uniform distribution.

Uniform PDF: $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$



Mean of Uniform distribution :-

As R.V 'X' is continuous type,

$$m_X = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

Substituting the expression for $f_X(x)$,

$$m_X = \int_{-\infty}^{\infty} \frac{x}{b-a}\, dx = \frac{1}{(b-a)} \int_a^b x\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$m_X = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{a+b}{2}}$$

Variance of uniform distribution :-

Variance of a R.V. is defined as under :

$$\sigma_X^2 = E[X^2] - m_X^2$$

Where $E[X^2]$ = Mean Square Value of 'X'.

As, $E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$

stituting the expression for $f_X(x)$,

$$E[X^2] = \int_a^b x^2 \left[\frac{1}{b-a}\right] dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$\Rightarrow E[X^2] = \frac{b^2+ab+a^2}{3}$$

$$\therefore \sigma_X^2 = \frac{b^2+ab+a^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$\Rightarrow \frac{b^2+a^2+ab}{3} - \frac{[a^2+2ab+b^2]}{4}$$

$$\boxed{\sigma_X^2 = \frac{(b-a)^2}{12}}$$

Expression for CDF :-

As, $F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$

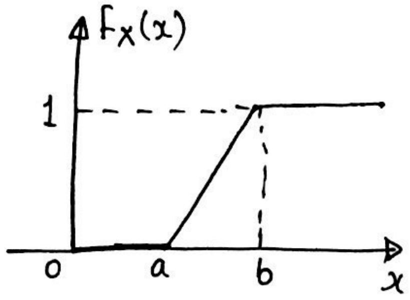
For $x<a, f_X(x) = 0 \quad \therefore \quad F_X(x) = 0$

For $a < x \leq b$,

$$F_X(x) = \int_a^b f_X(x) = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(x-a)$$

$$\therefore \quad \boxed{F_X(x) = \frac{x}{b-a} - \frac{a}{b-a}}$$

For $x > b$, $F_X(x) = 1$

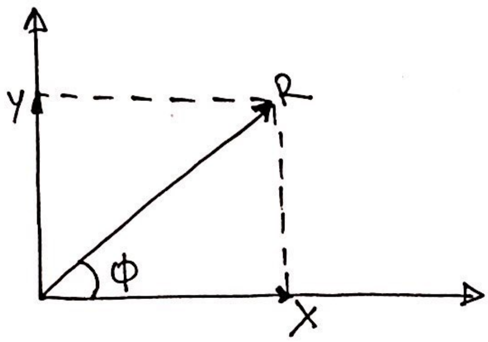


Rayleigh's Distribution :-

The Rayleigh's distribution is used for continuous R.V. It describes a continuous R.V produced from two Gaussian Random Variables. Let X and Y be two Gaussian R.V having

$$m_x = m_y = m \quad \text{and} \quad \sigma_x = \sigma_y = \sigma$$

The Rayleigh's continuous R.V R is related to X & Y by the transformation as shown:



$\therefore$ from figure,

$$R = (X^2 + Y^2)^{1/2} \quad -(1.)$$

& $$\phi = \tan^{-1}(Y/x) \quad -(2.)$$

From eq^n (1.), it is clear that R will always be positive because we are taking the root of addition of two squared quantities. Therefore 'R' will have a non-zero positive mean value, even though X, Y have zero mean values. The value of ϕ will be in the range of 0 to 2π radians.

The expression for Rayleigh's PDF is given by,

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad \text{for } r \geq 0$$

$$= 0 \quad \text{for } r < 0$$

Rayleigh's distribution is specifically used to analyze the narrow band Gaussian noise.

The mean value for Rayleigh's PDF,

$$m_x = \sqrt{\pi/2}\,\sigma$$

The variance of Rayleigh's PDF,

$$\sigma_x^2 = \left[2 - \frac{\pi}{2}\right]\sigma^2$$

Central Limit Theorem :-

The central limit theorem is used when PDF of several independent R.V is to be obtained.

Statement of Central Limit Theorem

Under certain conditions, the probability density function of ~~mean~~ sum of a large no. of independent random variables tends to approach Gaussian PDF, independent of probability densities of the variables.

Proof

Let X and Y be two independent variables. Let their addition be equal to another R.V 'Z' i.e.

$$Z = X + Y$$

Let the PDF of Z be $f_Z(z)$. Because $Z = X + Y$

Thus $\quad Y = Z - X$

The prob. that $Z \le z$ is denoted by $P[Z \le z]$ & it is given by,

$$P[Z \le z] = P[(X+Y) \le z]$$

According to definition of CDF, We have $P[Z \leq z] = F_Z(z)$ then,

$$\Rightarrow F_Z(z) = P[Z \leq z] = P[(X+Y) \leq z]$$

If X & Y given,

The event $Z \leq z$ is a joint event $[y \leq z-x$ and x to have any value in the range $(-\infty, \infty)]$

$$\therefore \quad F_Z(z) = P[Z \leq z] = P[X \leq \infty, Y \leq (Z-X)]$$

The joint CDF can be expressed in terms of joint PDF as under :

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy$$

$$\therefore \Rightarrow \boxed{F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{(z-x)} f_{XY}(x,y)\,dx\,dy}$$

$$\Rightarrow \boxed{F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, z-x)\,dx\,dy}$$

To obtain the PDF of Z, We have to differentiate as,

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx \, \frac{d}{dx}\left[\int_{-\infty}^{z-x} dy\right]$$

But X and Y are independent variables and $y = z-x$

$$\therefore \quad dy = dz$$

of distinct values. Thus continuous ...

then, $f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$

If X & Y are independent R.V, then their joint PDF is given by ; $f_{XY}(x,y) = f_X(x) \times f_Y(y)$

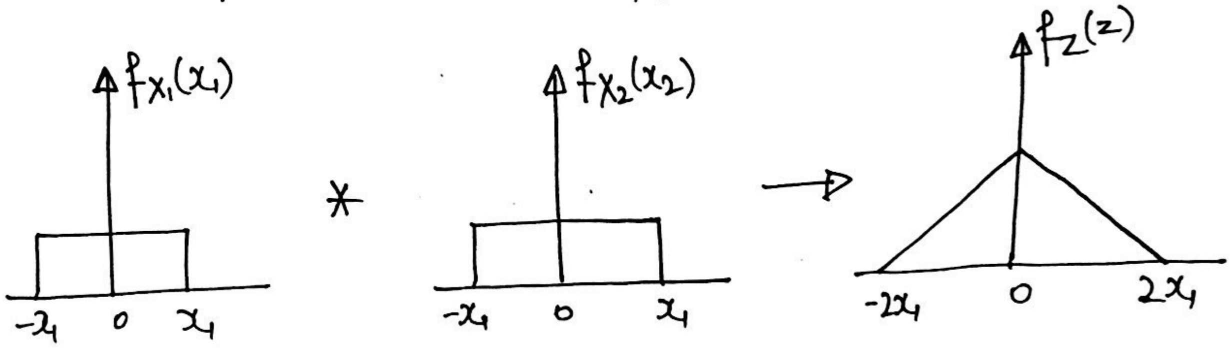
$$\therefore\ f_{XY}(x, z-x) = f_X(x) \cdot f_Y(z-x)$$

$$\Rightarrow \boxed{f_{XY}(x, z-x) = f_X(x) * f_Y(y)}$$

<u>To plot the PDF of Z</u> :-

Let X_1 & X_2 are two R.V & $Z = X_1 + X_2$

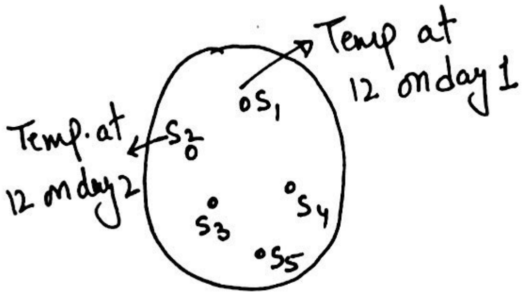
As $f_Z(z) = f_{X_1}(x_1) * f_{X_2}(x_2)$



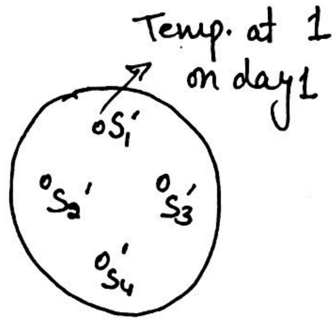
Random Process or Stochastic Process :-

The notion of a random process is an extension of the random variable (RV). Let us consider a R.V 'X' which represents the temp. of a city at 12 O'clock. The temp X is a R.V & can take on different values everyday. Record the values of X on day 1, 2 & so on.

But temp. is not just the f^n of day but also of time. Means temp. at 1 p.m will have a complete distribution from the distribution of temp. at 12 O'clock. Define them in terms of sample points.



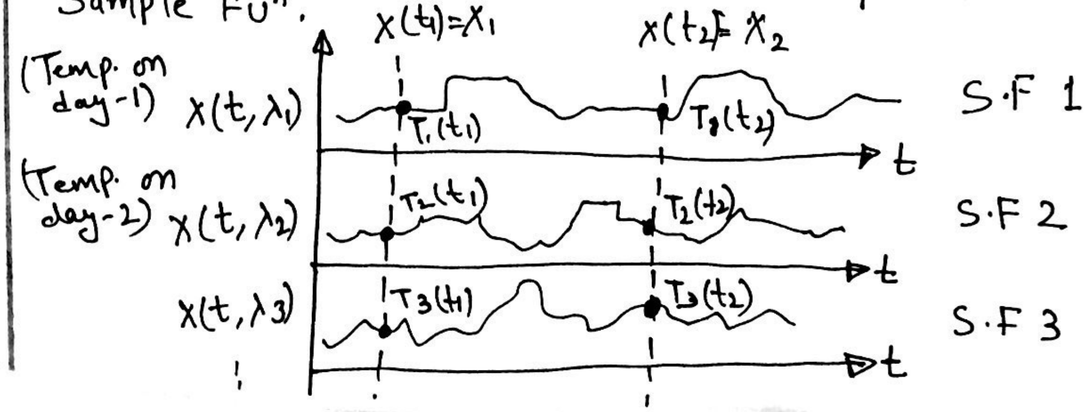
Sample space of a R.V X representing temp. at 12 O'clock



Sample spacet of R.V X representing temp. at 1 pm

Sample Functions :-

Record the city temp. for each value of t (12 O'clock, 1 pm ...) on everyday & plot the waveform $x(t, \lambda_i)$ where λ indicates the day. Each waveform is called as Sample Fu^n.



of distinct values. Thus continuous ...

Ensemble :- Collection of all the possible sample f^n is called as an ensemble.

Similarity b/w ensemble and sample space :-

S.S is the collection of all possible sample points and ensemble is the collection of all possible sample fu^ns.

Definition of Random Process :-

The ensemble comprised of functions of time is called as random or stochastic process.
Each sample fu^n consist of infinite sample points. Hence ensemble of sample fu^n consist of infinite sample points. Since a group of sample points correspond to 1 R.V. Ensemble will correspond to infinite no. of R.V.
R.P is denoted by $X(t)$.
R.P can also be defined as ensemble of R.V. which are function of time.

Ensemble Mean/ Average :-

Ensemble Mean is taken over the collection of waveforms at a fixed instant of time.
for example, Ensemble mean value taken at $t=t_1$ will consist of all the values taken at $t=t_1$ i.e. $T_1(t_1), T_2(t_1)$ - - -

$$\boxed{\overline{X(t)} \text{ or } m_x = \int_{-\infty}^{\infty} x \cdot f_X(x;t)\, dx}$$

In given eq^n, 't' is treated as const. because we are obtaining ensemble at fixed time instant. Hence, the only variable is 'x'.
The value of ensemble mean at diff. time instants will be different from each other. ∴ Ensemble mean is a f^n of time

The Average :-

Ensemble averages are obtained at constant values of
Whereas time avg. are obtained by changing time t. Time avg. of R.P is defined as statistical avg. obtained by considering time t as variable.

$$m_X(T) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt$$

The time avg. of S.F 1 need not to be identical to that of S.F 2 & 3 & so on.

Classification of Random Process :-

1.) Stationary & Non Stationary
2.) Wide Sense Stationary
3.) Ergodic Process

Stationary & Non Stationary R.P :-

A Random Process whose statistical characterstics do not change with time is known as stationary R.P. Hence, the shift of time origin will not have any effect on the stationary R.P. As in the previous example of temp., The Random Variables are denonted as $X_1, X_2 ---- X_n$. Hence the CDF of this set of R.V is denoted by

$$F_{X(t_1), X(t_2) \dots X(t_n)}(x_1, x_2 --- x_n)$$

Now if each each instant of observation i.e. $t_1, t_2 ---$ is shifted by a fixed amt. τ. The joint CDF of this new set of R.V is given as,

$$F_{X(t_1+\tau) \dots}(x_1, x_2 --- x_n)$$

of distinct values. Thus continuous ...

R... if ... that of ... to that of τ. ... $F_{X(t_1)}$...

The R.P $X(t)$ is said to be stationary in the strict sense if the joint CDF of original set of R.V is equal to that of new set of R.V obtained after a time shift of τ.

$$F_{X(t_1),X(t_2),\dots X(t_n)}(x_1,x_2 \dots x_n) = F_{X(t_1+\tau),X(t_2+\tau)\dots X(t_n+\tau)}(x_1,x_2 \dots)$$

Properties of Strictly Stationary Process :-

Property 1 :- For $n=1$,

$$F_{X(t_1)}(x_1) = F_{X(t_1+\tau)}(x_1) = f_X(x)$$

Hence, the 1st property says that the 1st order distribution fun of a stationary R.P is independent of time.

Property 2 :- for $n=2$ and $\tau = -t_1$,

$$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(t_1-t_1)\,X(t_2-t_1)}(x_1,x_2)$$

$$\Rightarrow F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0)X(t_2-t_1)}(x_1,x_2)$$

The 2nd property says that 2nd distribution fun of stationary R.P depends only on the time difference b/w the observation times (t_2-t_1) and not on the particular time instants.

Mean of a Stationary R.P :-

The mean value of R.P is given by

$$m_X(t) = \text{Expected value of } X(t)$$

$$\Rightarrow E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx.$$

Where, $f_{X(t)}(x)$ is the 1st order PDF of R.P.

AutoCovariance Fun of a Stationary Process :-

The autocovariance fun of a stationary R.P is given as :

$$C_X(t_2-t_1) = E[(X(t_1)-m_X)(X(t_2)-m_X)] = R_X(t_2-t_1) - m_X^2$$

Autocovariance fun of X(t) is dependent only on the time difference (t_2-t_1).

Wide Sense Stationary Process :-

A process may not be stationary in the strict sense, still it may have mean value $m_X(t)$ and an autocorrelation fun which are independent of the shift of time origin.

$$m_X(t) = \text{Constant and } R_X(t_1, t_2) = R_X(t_2-t_1)$$

Such a process is known as Wide Sense Stationary Process or Weakly Stationary Process. All the stationary processes are wide-sense stationary but every wide-sense stationary process may not be strictly stationary.

Ergodicity :-

Let $x(t)$ is a sample fun $x(t)$ of a wide-sense stationary process $X(t)$. Let the observation interval be $-T < t \leq T$. Then the time avg. of sample fun $x(t)$ is given by :

$$m_X(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

The value of time average $m_X(T)$ depends on the observation interval and which sample fun $x(t)$ of the R.P $X(t)$ is being chosen. Hence $m_X(T)$ is itself a R.V. The mean value

of distinct values. Thus continuous ...

of the time avg. $m_x(T)$ is given by,

$$E[m_x(T)] = E\left[\frac{1}{2T}\int_{-T}^{T} x(t)\,dt\right]$$

Interchanging the operation of expectation & integration, we get

$$E[m_x(T)] = E\,\frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt$$

The term $E[x(t)]$ represents the mean value of $x(t)$ and when we integrate it over the interval $-T \leq t \leq T$ and divide it by $2T$, the mean of the R.P $X(t)$ i.e. m_x is obtained.

$$\therefore \quad E[m_X(T)] = m_X$$

Conditions for Ergodicity in mean

(i.) The time avg. $m_x(T)$ approaches the ensemble avg. m_x with the observation interval T tending to infinity.

$$\boxed{\lim_{T\to\infty} m_X(T) = m_X}$$

Ensemble avg.

Time avg.

(ii) The variance of $m_x(T)$ which is treated as a R.V., approaches zero with the observation interval T tending to infinity.

$$\boxed{\lim_{T\to\infty} Var[m_x(T)] = 0}$$

Conditions for ergodicity in the autocorrelation function:

(i) $\lim_{T\to\infty} R_x(\tau, T) = R_x(\tau)$

This means that the time averaged autocorrelation fun approaches the autocorrelation fun of the system observation interval T tending to ∞.

(ii) $\lim_{T \to \infty} Var[R_x(\tau, T)] = 0$

This means that the variance of $R_x(\tau,T)$ which is a R.V. approaches zero with the observation interval tending to ∞.

Conclusion on Ergodicity of a process

When all the ensemble averages equal the corresponding time averages, the process is called ergodic process.

Correlation Function

The correlation fun provides a measure of similarity or coherence b/w a given signal and a replica of the signal by a variable amount.

Autocorrelation Function :-

Autocorrelation fun may be defined as a measure of similarity b/w a signal or process and its replica by a variable amount.

The autocorrelation fun of a stationary process X(t) may be defined as

$$R_X(t_j - t_i) = E[X(t_j)X(t_i)]$$

Where X_{tj} & X_{ti} are the R.V obtained by observing the process X(t) at times t_j and t_i respectively.

Using $\tau = t_j - t_i$

$$R_X(\tau) = E[X(t)\,X(t-\tau)]$$

The variable τ is known as time lag or time-delay parameter.

of distinct values. Thus continuous ...

Properties of Autocorrelation fu^n of Random Process

The autocorrelation fu^n $R_X(\tau)$ of a WSS process has the following properties :

(i) The autocorrelation fu^n of WSS process $X(t)$ is always an even fu^n of time lag.

$$R_X(\tau) = R_X(-\tau)$$

<u>Proof</u> :-

$$\text{As} \quad R_X(\tau) = E[X(t)\,X(t-\tau)]$$

Substituting $\tau = -\tau$,

$$R_X(-\tau) = E[X(t)\,X(t+\tau)]$$

For WSS process,

$$\cancel{E}\; X(t)X(t-\tau) = X(t)\,X(t+\tau)$$

$$\therefore R_X(\tau) = R_X(-\tau)$$

(ii) The mean sq. value of WSS process is equal to the autocorrelation fu^n of the R.P for zero time lag.

$$R_X(0) = R_X(\tau)\Big|_{\tau=0} = E[X(t)\,X(t-\tau)]_{\tau=0}$$

$$R_X(0) = E[X(t)\,X(t-0)] = E[X^2(t)]$$

$$\Rightarrow R_X(0) = E[X^2(t)]$$

(iii) The autocorrelation fu^n of a WSS process has the max. magnitude at zero time lag $(\tau = 0)$.

$$|R_X(\tau)| = \text{magnitude of } R_X(\tau) = R_X(0)$$

<u>Proof</u> :- As the mean sq. value of th diff. b/w $X(t)$ and $X(t-\tau)$ is always non-negative i.e.

$$E[\{X(t) - X(t-\tau)\}^2] \geq 0$$

$$\Rightarrow E[X^2(t) - 2X(t)X(t-\tau) + X^2(t-\tau)] \geq 0$$

The like an symmetry

As E is a linear operation,

$$E[X^2(t)] - 2E[X(t)X(t-\tau)] + E[X^2(t-\tau)] \geq 0$$

for WSS process,

$$E[X^2(t)] = E[X^2(t-\tau)] = R_X(0)$$

$$\& \quad E[X(t)X(t-\tau)] = R_X(\tau)$$

$$\therefore |R_X(\tau)| \leq R_X(0)$$

Cross-Correlation Function

An autocorrelation fun is determined for single R.P but cross-correlation fun is determined for two R.P. Let two R.P $X(t)$ & $Y(t)$ with autocorrelation fun $R_X(t,u)$ and $R_Y(t,u)$ are there. There will be two cross-correlation fun b/w two R.P $X(t)$ & $Y(t)$.

These cross-correlations may be defined as,

$$R_{XY}(t,u) = E[X(t)\,Y(u)]$$

$$R_{YX}(t,u) = E[Y(t)\,X(u)]$$

$R(t,u)$ is called the correlation matrix of R.P $X(t)$ & $Y(t)$

Eqn may also be written as,

$$R(t,u) = \begin{bmatrix} R_X(t,u) & R_{XY}(t,u) \\ R_{YX}(t,u) & R_Y(t,u) \end{bmatrix}$$

or

$$R(t,u) = \begin{bmatrix} R_X(t-u) & R_{XY}(t-u) \\ R_{YX}(t-u) & R_Y(t-u) \end{bmatrix}$$

$R(t,u)$ is called the correlation matrix of R.P $X(t)$ & $Y(t)$.

of distinct values. Thus continuous R.P ...

→ The cross-correlation fu^n is not an even fu^n of τ like an autocorrelation fu^n. It is having following symmetry relation:

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

→ Cross-correlation does not have a max. at the origin like the autocorrelation fu^n. The two R.P $X(t)$ & $Y(t)$ are said to be incoherent or orthogonal if cross-correlation fu^n of $X(t)$ & $Y(t)$ is zero.

$$R_{XY}(\tau) = 0$$

→ The two R.P are said to be non-correlated, if their cross-correlation fu^n $R_{XY}(\tau)$ are equal to the multiplication of mean values.

$$R_{XY}(\tau) = E[X(t)]\,E[Y(t)]$$

The incoherent or orthogonal processes are non-correlated processes with $E[X(t)]$ and or $E[Y(t)] = 0$.

Power Spectral Density

Let the impulse response of a LTI filter is equal to the IFT of the transfer function $H(f)$.

$$\therefore\ h(\tau_1) = \int_{-\infty}^{\infty} H(f).e^{j2\pi f\tau_1}\,df$$

As,

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

Substituting $h(\tau_1)$ in eq^n

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} H(f)e^{j2\pi f\tau_1}df\right]h\tau_2\,R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

Rearranging,

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f)\,df \int_{-\infty}^{\infty} h(\tau_2)\,d\tau_2 \int_{-\infty}^{\infty} R_X(\tau_2-\tau_1)\,e^{j2\pi f\tau_1}\,d\tau_1$$

Let $\tau = (\tau_2 - \tau_1)$

$\Rightarrow \tau_1 = \tau_2 - \tau$ & $d\tau_1 = -d\tau$

$\therefore$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df\, H(f) \int_{-\infty}^{\infty} h(\tau_2)\,d\tau_2 \int_{-\infty}^{\infty} R_X(\tau)\,e^{j2\pi f(\tau_2-\tau)}\,d\tau$$

$$\Rightarrow \int_{-\infty}^{\infty} df\, H(f) \int_{-\infty}^{\infty} h(\tau_2)\,e^{j2\pi f\tau_2}\,d\tau_2 \int_{-\infty}^{\infty} R_X(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

But, $\int_{-\infty}^{\infty} h(\tau_2)\,e^{j2\pi f\tau_2}\,d\tau_2 = H^*(f)$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} H(f)\,H^*(f)\,df \int_{-\infty}^{\infty} R_X(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

But, $H(f)\,H^*(f) = |(H(f)|^2$

$$\therefore E[Y^2(t)] = \int_{-\infty}^{\infty} df\, |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

In this expression, the last term is the F.T of the transformation of the autocorrelation fun $R_X(\tau)$ of the input R.P $X(t)$.

Hence, $X(t) \xleftrightarrow{F} \int_{-\infty}^{\infty} R_X(\tau).\,e^{-j2\pi f\tau}\,d\tau$

Let's introduce a new parameter for the RHS of this expression as $S_X(f)$ as under:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

The Parameter $S_X(f)$ is known as the <u>power spectral density</u> or <u>power spectrum</u> of WSS process $X(t)$

of distinct values. Thus continuous ...

Properties

$$\therefore E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$$

Properties of PSD :-

Property 1 :- The value of PSD of a wide sense stationary process at $f=0$ is equal to the total area under the curve of autocorrelation function.

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

$$\boxed{\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) . e^{-j2\pi f\tau}\, d\tau \\ R_X(\tau) &= \int_{-\infty}^{\infty} S_X(f)\, e^{-j2\pi f\tau}\, d\tau \end{aligned}} \Rightarrow \text{Einstein-Weiner Khintchin Relations}$$

Proof :-

As, $\quad S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi\tau}\, d\tau$

Putting $f = 0$

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{0}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

Property 2 :- The mean sq. value of a WSS process is equal to the total area under the curve of power spectral density. This means that

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df$$

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Proof:- As, $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, d\tau$

Substituting $\tau = 0$ into this expression, we get,

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, e^{0}\, df = \int_{-\infty}^{\infty} S_X(f)\, df$$

and $E[X^2(t)] = R_X(0)$

Hence, $E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df$

Property 3:- The PSD of a WSS process will always be non-negative for all the values of f.

$$S_X(f) \geq 0 \quad \text{for all } f$$

Property 4:- The PSD of a real valued random process is an even f^n of frequency.

$$S_X(-f) = S_X(f)$$

Proof:-

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

substituting $-f$ for f to get

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(-\tau)\, e^{-j2\pi f\tau}\, d\tau$$

Now substitute $-\tau$ for τ, to obtain

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(-\tau)\, e^{-j2\pi f\tau}\, d\tau$$

But, $R_X(-\tau) = R_X(\tau)$

$$\therefore \quad S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau = S_X(f)$$

Hence Proved

Property 5:- If the psd is approximately normalized, then it ... prob. density f^n (PDF).