Joint Random Variables

PDF/CDF in 2 RVs

- Joint Probability

  $$P(A, B) = \text{Probability of occurances of A and B where they may be dependent or independent}$$

- Conditional Probability

  $$P(A|B) = \text{Probability of A given that B occurs}$$

  $$P(B|A) = \text{Probability of B given that A occurs}$$

- Baye's Theorem

  $$P(A, B) = P(A|B)P(B) = P(B|A)P(A) = P(B, A)$$

Statistical Independence

$$P(A, B) = P(A)P(B)$$

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Joint Random Variables

$$f_{X,Y}(x, y) \qquad F_{X,Y}(x, y)$$

Conditional RVs

$$f_{X|Y}(x), f_{Y|X}(y)$$

$$f_{X,Y}(x, y) = f_{X|Y}(x)f_Y(y) = f_{Y|X}f_X(x)$$

Statistical Independence

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

PDF Properties

1. $f_{X,Y}(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
3. $Pr(x_1 \leq x \leq x_2, y_1 \leq y \leq y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)dxdy$
4. $F_{X,Y}(x,y) = \int_{x}^{\infty}\int_{y}^{\infty} f_{X,Y}(x\prime, y\prime)dx\prime dy\prime$
5. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy; \quad f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

CDF Properties

1. $0 \leq F_{X,Y}(x,y) \leq 1$
2. $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = F_{X,Y}(-\infty, -\infty) = 0$
3. $F_{\infty,Y} = F_Y(y); \quad F_{X,Y}(x, \infty) = F_X(x)$
4. $F_{X,Y}(\infty, \infty) = 1$

Moments

$$E\{g(x)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f_{X,Y}(x,y)dxdy$$

$$g(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)f_{X,Y}(x,y)dxdy$$

$$E\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$$

1. Correlation

$$g(x,y) = xy$$

$$E\{XY\} = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} xyf_{X,Y}dxdy$$

2. Covariance

$$g(x,y) = (x - m_x)(y - m_y)$$

$$\sigma\{XY\} = \int_{-\infty} \infty \int_{\infty}^{\infty} xyf_{X,Y}dxdy$$

3. Special Case: Statistical Independence

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

$$\implies E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy$$

$$\implies E\{XY\} = \left(\int_{-\infty}^{\infty} xf_X(x)dx\right)\left(\int_{-\infty}^{\infty} f_Y(y)dy\right)$$

$$\implies E\{XY\} = E\{X\}\{Y\}$$

$$\implies \sigma_{X,Y} = 0$$

Covariance $\sigma_{XY} = E\{XY\} - m_x m_y = 0$

Orthogonality

$$\hat{a}\cdot\hat{b} = |a||b|cos\theta = 0$$

$$\int_{-\infty}^{\infty} f(x)g(x)dx$$

Correlation is $E\{XY\} = 0$

Correlation coefficient

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = cos\theta$$

Uniform RV

$$f_{XY}(x,y) = \begin{cases} \frac{1}{(x_2-x_1)(y_2-y_1)}, & \forall x_1 \leq x \leq x_2; \\ 0, & otherwise \end{cases}$$

Gaussian

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}e^{-(\frac{(x-\mu_x)^2}{2\sigma_x^2}+\frac{(y-\mu_y)^2}{2\sigma_y^2})}$$