CSE 151A - Homework 3

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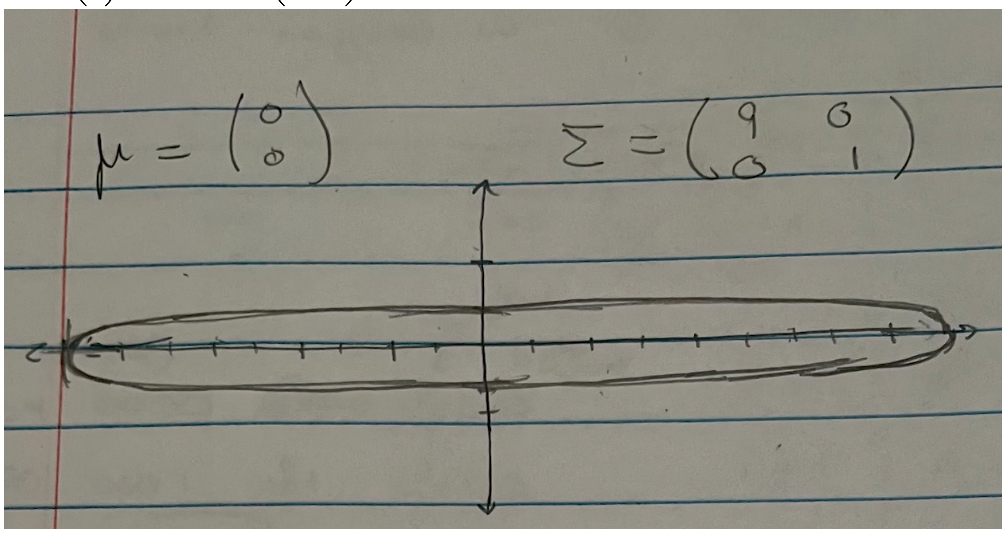
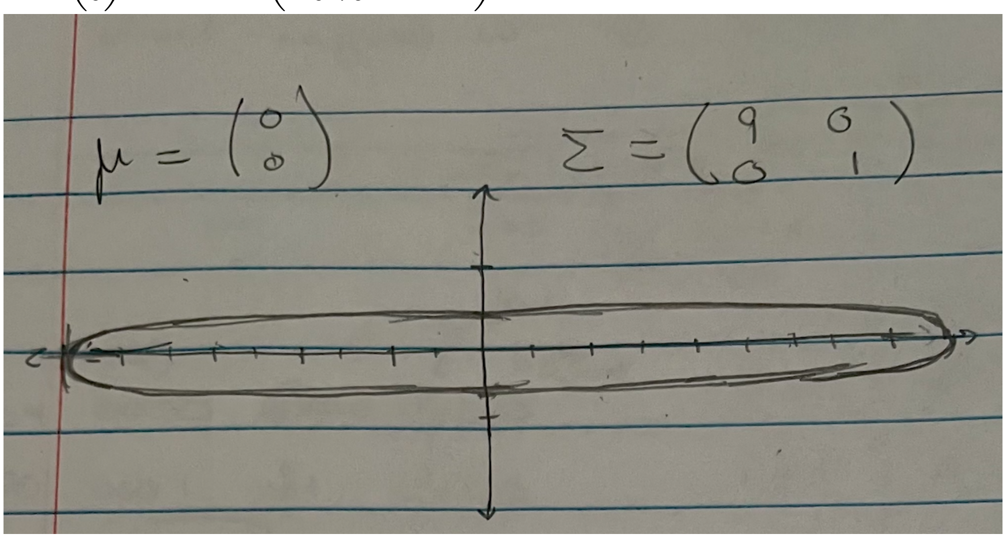
Problem 1. *Gaussian Contours*

Solution.

(a) $\mu = \begin{pmatrix}0\\0\end{pmatrix}$ and $\Sigma = \begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}$



(b) $\mu = \begin{pmatrix}0\\0\end{pmatrix}$ and $\Sigma = \begin{pmatrix}1 & -0.75\\-0.75 & 1\end{pmatrix}$



Problem 2. *Qualitative appraisal of Gaussian parameters*

Solution. Covariance Matrix $\begin{pmatrix} p & q \\ q & r \end{pmatrix}$.

(a) The two variables are negatively correlated.

In the covariance matrix, the q value represents the covariance. For the two variables to be negatively correlated, $q < 0$ and $q > -1$.

(b) The two variables are uncorrelated.

For two variables to be uncorrelated, the covariance must be 0. Therefore, $q = 0$.

(c) One variable is a linear function of the other.

If one variable is a linear function of the other, the two would be perfectly correlated. Therefore, $|q| = 1$

(d) The second variable is a constant.

If the second variable is constant, the variable's variance must be 0. Furthermore, the covariance must be 0 as well. So, $r = q = 0$.

Problem 3. *Unit vector in the same direction as $x = (1, 2, 3)$.*

Solution.

$$\hat{x} = \frac{x}{||x||}$$

$$\hat{x} = \frac{(1,2,3)}{\sqrt{1^2 + 2^2 + 3^2}}$$

$$\hat{x} = \frac{(1,2,3)}{\sqrt{1 + 4 + 9}}$$

$$\hat{x} = \frac{(1,2,3)}{\sqrt{14}}$$

$$\hat{x} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$$

Problem 4. *Find all unit vectors in* $\mathbb{R}^2$ *orthogonal to* $x = (1, 1)$.

Solution.

$$\hat{x} \cdot \hat{y} = 0$$
$$(1, 1) \cdot (a, b) = 0$$
$$a + b = 0$$
$$a = -b$$

All unit vectors orthogonal to $(1, 1)$ **can be represented by the vector** $(a, -a)$.

Problem 5. $f(x) = 2x_1 - x_2 + 6x_3$ *written as* $w \cdot x$.

Solution.

$$w \cdot x = 2x_1 - x_2 + 6x_3$$
$$w \cdot x = (2, -1, 6)(x_1, x_2, x_3)$$
$$w = (2, -1, 6)$$

Problem 6. *Dimensions of* A *and* B.

Solution. AB has dimensions 10×20
A has dimensions $x_a \times 30$.
B has dimensions $30 \times y_b$.
AB has dimensions $x_a \times y_b$
Therefore, A **has dimensions** 10×30 **and** B **has dimensions** 30×20**.**

Problem 7. n *data points in a matrix* X.

Solution.

(a) What is a dimension of X? One row with d columns.

$$1 \times d$$

(b) What is a dimension of XX^T?

$$d \times d$$

(c)

$$\text{Entry } (i, j) = X_i \times X_j$$

Problem 8. $x = (1, 3, 5)$ *find* $x^T x$ *and* xx^T

Solution.

$$x = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \qquad x^T = (1, 3, 5)$$

$$x^T x = (1, 3, 5) \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$x^T x = 1 * 1 + 3 * 3 + 5 * 5$$

$$x^T x = 35$$

$$xx^T = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \cdot (1, 3, 5)$$

$$xx^T = \begin{pmatrix} 1*1 & 1*3 & 1*5 \\ 3*1 & 3*3 & 3*5 \\ 5*1 & 5*3 & 5*5 \end{pmatrix}$$

$$xx^T = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{pmatrix}$$

Problem 9. *Symmetric matrix M from Quadratic Function*

Solution. $f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$

$$x = (x_1, x_2, x_3)$$

$$x^T M x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$

$$x = (x_1, x_2, x_3)$$

$$x^T M x = \begin{bmatrix} \_x_1 & \_x_1 & \_x_1 \\ \_x_2 & \_x_2 & \_x_2 \\ \_x_3 & \_x_3 & \_x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$

$$x^T M x = \begin{bmatrix} 3x_1 & 2x_1 & -4x_1 \\ 0 & 0 & 0 \\ 0 & 0 & 6x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$

$$M = \begin{bmatrix} 3 & 2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Problem 10. $A = diag(1, 2, 3, 4, 5, 6, 7, 8)$

Solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

(a) $|A|$

$$|A| = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$$

$$|A| = 8!$$

$$|A| = 40320$$

(b) A^{-1}

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$$

Problem 11. *Let U be the $d \times d$ matrix with rows u_i.*

Solution. U is orthogonal.

(a) $UU^T = I_d$

(b) $U^{-1} = U^T$

Problem 12. *Matrix A is singular.*

Solution. A matrix is singular iff its determinant $= 0$.

$$|A| = (1)(z) - (2)(3)$$

$$0 = z - 6$$

$$z = 6$$

Problem 13. *Classifying MNIST digits using generative modeling.*

Solution. Code

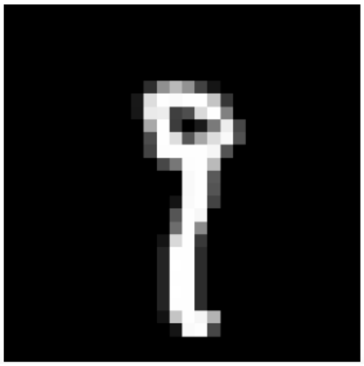
```
[7]: def fit_generative_model(x,y):
         k = 10  # labels 0,1,...,k-1
         d = (x.shape)[1]  # number of features
         mu = np.zeros((k,d))
         sigma = np.zeros((k,d,d))
         pi = np.zeros(k)

         ###
         ### Your code goes here
         for label in range(10):
             dig = training_set[training_labels == label]

             c = 3000
             mu[label] = np.mean(dig,axis = 0)
             sigma[label] = np.cov(dig, rowvar=False) + (c * np.identity(784))

         for l in training_labels:
             pi[l] += 1
         pi = [prob / len(training_labels) for prob in pi]


         ###
         # Halt and return parameters
         return mu, sigma, pi
```

```
[9]: # Compute log Pr(label|image) for each [test image,label] pair.
     score = np.zeros((len(test_labels),10))
     for label in range(0,10):
         rv = multivariate_normal(mean=mu[label], cov=sigma[label], allow_singular=True)
         for i in range(0,len(test_labels)):
            score[i,label] = np.log(pi[label]) + rv.logpdf(test_data[i,:])
     predictions = np.argmax(score, axis=1)
     # Finally, tally up score
     errors = np.sum(predictions != test_labels)
     print("Your model makes " + str(errors) + " errors out of 10000")
     print("Error rate: " + str(errors/len(test_data)))

     Your model makes 426 errors out of 10000
     Error rate: 0.0426
```

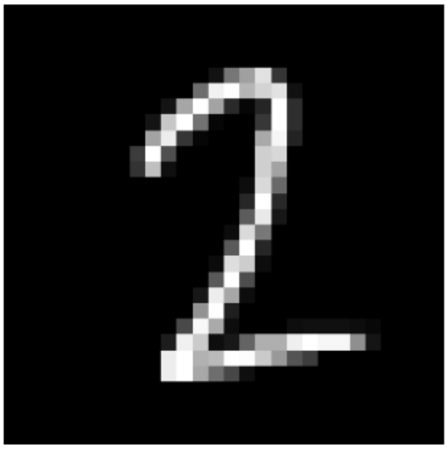
- $c = 1$: Error Rate = 0.187
- $c = 1000$: Error Rate = 0.048
- $c = 2000$: Error Rate = 0.0439
- **c = 3000: Error Rate = 0.0426**
- $c = 4000$: Error Rate = 0.0429

```
[19]: displayed = 0
rng = np.array(range(0, 1000))
np.random.shuffle(rng)
for i in rng:
    if(predictions[i] != test_labels[i]):

        print("Predicted Label: " + str(predictions[i]) + " - Actual Label: " + str(test_labels[i]))
        displaychar(test_data[i])

        displayed+=1
    if(displayed == 5):
        break
```

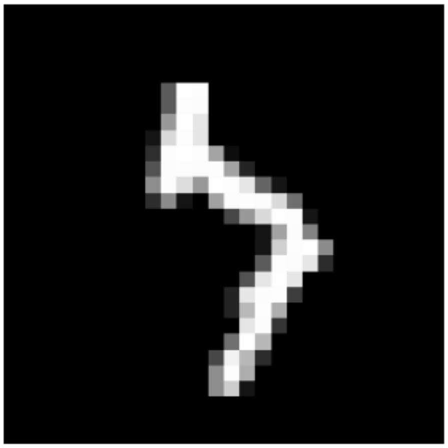
```
Predicted Label: 1 - Actual Label: 9
```



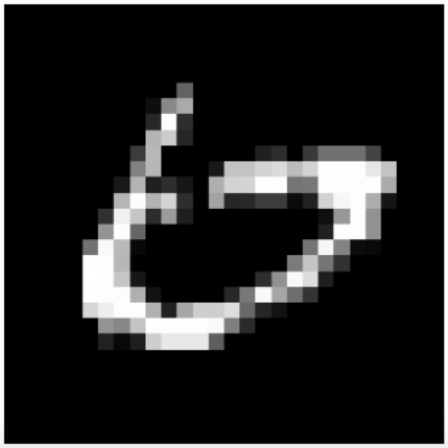
```
Predicted Label: 1 - Actual Label: 2
```



```
Predicted Label: 5 - Actual Label: 7
```



```
Predicted Label: 0 - Actual Label: 6
```



```
Predicted Label: 4 - Actual Label: 8
```