Payoff Continuity in Games of Incomplete Information Across Models of Knowledge*

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Abstract

Monderer and Samet (1996) and Kajii and Morris (1998) define coarse topologies on common prior information structures that preserve Bayesian Nash equilibrium payoff continuity. Monderer and Samet (1996) fix a common prior and perturb lists of partitions, while Kajii and Morris (1998) fix a type space and perturb common priors. Due to these differences, the precise relationship between them has remained an open question. We establish an equivalence between them and resolve it.

Keywords: incomplete information, Bayesian Nash equilibrium, strategic topology

JEL codes: C70, C72, D80.

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1 Introduction

A game of incomplete information consists of a set of players, a set of actions and payoff functions for each player, and an information structure. How does the set of Bayesian Nash equilibrium payoffs change as the information structure changes? Rubinstein (1989)'s Email Game illustrates a striking payoff discontinuity; there can exist an equilibrium under a common knowledge information structure yielding expected payoffs that are not approximated in *any* equilibrium under an information structure in which there are arbitrarily many, but finite, levels of mutual knowledge. As higher-order beliefs are in principle unobservable, the existence of this lower hemidiscontinuity casts doubt on the robustness of equilibrium predictions.

A natural approach to this problem is to identify what type of precision is required for a modeler to make robust predictions. Monderer and Samet (1996) and Kajii and Morris (1998) define coarse topologies on common prior information structures that preserve continuity of ϵ-equilibrium payoffs across all bounded games of incomplete information.[1] While both topologies are based on the same notion of approximate common knowledge, that of common-p belief (Monderer and Samet (1989)), Monderer and Samet (1996) and Kajii and Morris (1998) model the proximity of information structures differently. Monderer and Samet (1996) fix a state space and a common prior over it, and consider the differences in beliefs induced by a change in partitions over the state space. Kajii and Morris (1998) fix parameter and type spaces, and consider the differences in beliefs induced by a change in the common prior over its product.

Unfortunately, the precise relationship between the two topologies has remained an open question. As Kajii and Morris (1998) write,

> Our characterization of the proximity of information has a similar flavor to Monderer and Samet's, but we have not been able to establish a direct comparison. By considering a fixed type space, we exogenously determine which types in the

[1] This exercise is uninformative if one requires that players exactly optimize, i.e., if $\epsilon = 0$. Theorem 5 of Gensbittel, Peski, and Renault (2022) considers a general space of information structures and shows that the topology that preserves payoff continuity of 0-equilibria on this space is the discrete topology.

> information systems correspond to each other. In the Monderer and Samet approach, it is necessary to work out how to identify types in the two information systems. Thus we conjecture that two information systems are close in Monderer and Samet's sense if and only if the types in their construction can be labelled in such a way that the information systems are close in our sense.

We exhibit such a labeling and prove Kajii and Morris (1998)'s conjecture.

2 Preliminaries

Fix the following primitives: (i) a set of two or more players $\mathcal{N} := \{1, 2..., N\}$; (ii) a probability triple (S, Σ, P), where S is a state space, Σ is a σ-algebra of events, and P is a probability measure; (iii) a set of payoff parameters Θ, where $\Theta := S$ and is equipped with the σ-algebra Σ; and (iv) a measurable parameter function mapping states to payoff parameters, $\phi : S \to \Theta$. Notice that the state space S can either be countable, e.g., the set of natural numbers, or uncountable, e.g., the set of real numbers. In what follows, all countable sets are equipped with the discrete σ-algebra and all product sets are equipped with the product σ-algebra.

2.1 The MS Topology on Partition Models

Denote by $\mathcal{P}$ the set of all partitions of S into non-null elements of Σ. Denote by $\Pi := (\Pi_1, ..., \Pi_N) \in \mathcal{P}^N$ the set of all partition lists. We define the Monderer and Samet (1996) topology on the set of partition lists $\mathcal{P}^N$.

Denote player i's partition element containing a state $s \in S$ by $\Pi_i(s)$. At a state $s \in S$, player i assigns probability $P(E|\Pi_i(s))$ to the event $E \subseteq S$. Player i **p-believes** an event $E \subseteq S$ at a state $s \in S$ if $P(E|\Pi_i(s)) \geq p$. Denote $B^p_{\Pi_i}(E)$ as the set of states at which i p-believes E under the partition Π_i. The set of states at which E is **mutual p-belief** is $B^p_{\Pi}(E) := \underset{i\in\mathcal{N}}{\cap} B^p_{\Pi_i}(E)$. The set of states at which E is **m-level mutual p-belief** is $(B^p_{\Pi})^m(E)$, the m-th iteration of $B^p_{\Pi}(\cdot)$ over the set E. Finally, the set of states at which E is **common p-belief** is $C^p_{\Pi}(E) := \underset{m\geq 1}{\cap} (B^p_{\Pi})^m(E)$.

Define $I_{\Pi,\Pi'}(\epsilon)$ as the set of states at which the conditional symmetric difference between each player's partition elements containing that state is less than ϵ,

$$I_{\Pi,\Pi'}(\epsilon) := \bigcap_{i\in\mathcal{N}} \{s \in S : \max\{P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)), P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s))\} < \epsilon\}.$$

Define

$$d^{MS}(\Pi, \Pi') := \max\{d_1^{MS}(\Pi, \Pi'), d_1^{MS}(\Pi', \Pi)\},$$

where $d_1^{MS}(\Pi, \Pi')$ is an ex-ante measure of states at which the event $I_{\Pi,\Pi'}(\epsilon)$ is common $(1-\epsilon)$-belief:

$$d_1^{MS}(\Pi, \Pi') := \inf\{\epsilon | P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1-\epsilon\}.$$

While d^{MS} is not the metric defined in Monderer and Samet (1996) (it does not satisfy the triangle inequality), it follows from Theorem 5.2 in their paper that a sequence converges in their topology if and only if $d^{MS}(\Pi, \Pi_n) \to 0$. Hereafter, we call the topology generated by these convergent sequences the **MS topology**.[2]

2.2 The KM Topology on Type Models

Fix a countably infinite type space $T := T_1 \times ... \times T_N$, where T_i is a countable set of types for player $i \in \mathcal{N}$. Denote T^S as the set of *type functions* from S to T with the property that the pre-image of any subset $E \subseteq T$ is either a non-null element of Σ or the empty set. Since each function $\tau \in T^S$ induces a measure $\mu \in \Delta(\Theta \times T)$, we define the Kajii and Morris (1998) topology on the function space T^S, placing conditions on the measures they identify. For convenience, call the elements of $\Theta \times T$ **induced states**.

Denote $\mu(t_i)$ as the marginal distribution of μ on T_i evaluated at t_i. If $\mu(t_i) > 0$, the probability of an event $E \subseteq \Theta \times T$ conditional on player i's type t_i is given by $\mu(E|t_i) := \mu(E)/\mu(t_i)$. Player i p**-believes** an event $E \subseteq \Theta \times T$ at an induced state $(\theta, t) \in \Theta \times T$ if $\mu(E|t_i) \geq p$. Denote $B_{\mu_i}^p(E)$ as the set of all induced states at which i p-believes E. The set of induced states at which E is **mutual** p**-belief** is $B_\mu^p(E) := \bigcap_{i\in\mathcal{N}} B_{\mu_i}^p(E)$. The set of

[2] Theorem 5.4 of Monderer and Samet (1996) establishes the upper- and lower- hemicontinuity of ϵ-Bayesian Nash equilibrium payoffs in this topology. The reader is referred to the original paper for the formal definitions and details.

induced states at which E is m**-level mutual** p**-belief** is $(B^p_\mu)^m(E)$, the m-th iteration of $B^p_\mu(\cdot)$ over the set E. The set of induced states at which E is **common** p**-belief** is $C^p_\mu(E) := \bigcap_{m\geq 1}(B^p_\mu)^m(E)$.

Define $A_{\mu,\mu'}(\epsilon)$ as the set of induced states at which each player has conditional beliefs that differ by at most ϵ over any event given their type,

$$A_{\mu,\mu'}(\epsilon) := \{(\theta,t)\in\Theta\times T : \text{for all } i\in\mathcal{N},\ \mu(t_i)>0, \mu'(t_i)>0, \text{ and}$$

$$|\mu(E|t_i)-\mu'(E|t_i)|\leq\epsilon \ \text{ for all events } E\subseteq\Theta\times T\}.$$

Define a function mapping pairs of common priors to real numbers:

$$d^{KM}(\mu,\mu') := \max\{d_1^{KM}(\mu,\mu'), d_1^{KM}(\mu',\mu), d_0^{KM}(\mu,\mu')\},$$

where $d_1^{KM}(\mu,\mu')$ is an ex-ante measure of states for which the event $A_{\mu,\mu'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d_1^{KM}(\mu,\mu') := \inf\{\epsilon>0 : \mu'(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon))\geq 1-\epsilon\},$$

and $d_0^{KM}(\mu,\mu')$ is the sup-norm distance between μ and μ' over all events,

$$d_0^{KM}(\mu,\mu') := \sup_{E\subseteq\Theta\times T}|\mu(E)-\mu'(E)|.$$

d^{KM} generates a topology on T^S by specifying which nets converge.[3] A net $\{\tau^k : k\in K\}$, where K is a set with partial order $\succ$, converges to τ if for any $\epsilon>0$, there is a $\bar{k}\in K$ such that $k\succ\bar{k}$ implies that $d^{KM}(\mu^k,\mu)<\epsilon$, where μ^k is the unique probability measure identified by τ^k and μ is the unique probability measure identified by τ. Hereafter, we call the topology generated by these nets the **KM topology**.[4]

[3] In metric spaces, (or, more generally, in first-countable spaces) one can define a topology by specifying which sequences converge. Since Monderer and Samet (1996) prove their topology is metrizable, we thus define theirs by specifying convergent sequences. Since we have not proved that Kajii and Morris (1998)'s topology is metrizable (or, first-countable), we maintain their original definition.

[4] Proposition 8 of Kajii and Morris (1998), whose proof is corrected by Rothschild (2005), establishes the upper- and lower- hemicontinuity of ϵ- Bayesian Nash equilibrium payoffs in this topology. The reader is referred to the original papers for the formal definitions and details. We remark that Kajii and Morris (1998) and Rothschild (2005) allow only for a countable set of payoff parameters, whereas we have allowed for the possibility that it is uncountable. Nevertheless, this restriction plays no substantial role in their proofs.

3 Labelings

A **partition labeling** is a function from pairs of partition lists to pairs of type functions $L : \mathcal{P}^N \times \mathcal{P}^N \to T^S \times T^S$. We first define and illustrate three properties of partition labelings — consistency, the common support condition, and invariance. We then define the converse labeling from type functions to partition lists. We conclude the section by discussing the relationship between our labelings and the canonical mappings defined in Werlang and Tan (1992) and Brandenburger and Dekel (1993).

3.1 Consistency

We first define a sense in which a player's type space is consistent with her partitional information.

Definition 1 (Π-Consistency) *A type function τ is Π-**consistent** if it satisfies the following property: for all i, $\tau_i(s) = \tau_i(s')$ if and only if $s, s' \in \pi \in \Pi_i$.*

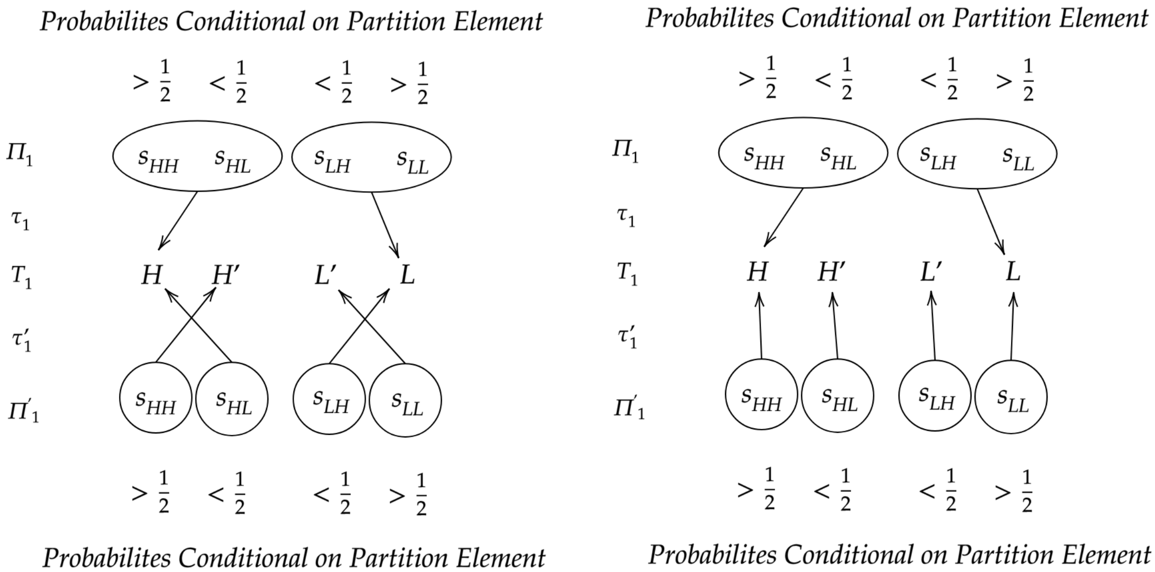
A consistent partition labeling maps pairs of partition lists to pairs of consistent type functions.

Definition 2 (Consistency) *A partition labeling L is **consistent**, or is said to satisfy **COS**, if $L(\Pi, \Pi') = (\tau, \tau')$ implies τ is Π-consistent and τ' is Π'-consistent.*

3.2 The Common Support Condition

Consistency places no restrictions on the relationship between the type functions in the codomain of a partition labeling. The following condition *does*, and is necessary to ensure that the "types" in the Partition Model correspond to those in the Type Model.

Definition 3 (Common Support Condition) *A partition labeling L satisfies the **common support condition (CSC)** if $L(\Pi, \Pi') = (\tau, \tau')$ and $s \in I_{\Pi,\Pi'}(1/2)$ implies $\tau(s) = \tau'(s)$.*



(a) COS, but not CSC.

(b) COS and CSC.

Figure 1: Labeling Examples.

Fixing partition lists Π and Π', any partition labeling satisfying the common support condition must send states to the same type if they are contained in similar enough partition elements across lists, as measured by the conditional symmetric difference.

Example 1 *In Figure 1, the state space is* $S := \{s_{HH}, s_{HL}, s_{LH}, s_{LL}\}$. *The prior distribution* P *places probability* $\epsilon \in (0, \frac{1}{2})$ *on* s_{HL} *and* s_{LH}, *and probability* $\frac{1}{2} - \epsilon$ *on* s_{HH} *and* s_{LL}. *Posterior beliefs are written above each state. The Partition Model has* $\Pi_1 := \{\{s_{HH}, s_{HL}\}, \{s_{LH}, s_{LL}\}\}$ *and* $\Pi_1' := \{\{s_{HH}\}, \{s_{HL}\}, \{s_{LH}\}.\{s_{LL}\}\}$.

While the labeling in Figure 1a does not violate consistency, it does violate the common support condition. To see why, notice that

$$I_{\Pi,\Pi'}(1/2) = \{s_{HH}, s_{LL}\},$$

because

$$P(\Pi_1(s_{HH}) \backslash \Pi_1'(s_{HH}) | \Pi_1(s_{HH})) = P(s_{HL}) < 1/2$$

and

$$P(\Pi_1(s_{LL}) \backslash \Pi_1'(s_{LL}) | \Pi_1(s_{LL})) = P(s_{LH}) < 1/2.$$

Nevertheless, the type function τ *does not coincide with* τ' *at either* s_{HH} *or* s_{LL}*;* $\tau_1(s_{HH}) = H \neq H' = \tau_1'(s_{HH})$ *and* $\tau_1(s_{LL}) = L \neq L' = \tau_1'(s_{LL})$*. On the other hand, the labeling in Figure 1b satisfies both consistency and the common support condition. This follows because states in* $I_{\Pi,\Pi'}(1/2)$ *are mapped to the same type (*$\tau_1(s_{HH}) = H = \hat{\tau}_1(s_{HH})$ *and* $\tau_1(s_{LL}) = L = \hat{\tau}_1(s_{LL})$*).*

3.3 Invariance

Define $L_i(\Pi, \Pi')$ as the i-th component of the co-domain of L. Our final property requires partition lists in the first coordinate of the domain to be mapped to the same type function independently of the second coordinate.

Definition 4 (Invariance) *A partition labeling* L *is* ***invariant****, or is said to satisfy* ***INV****, if for all partition lists* $\Pi \in \mathcal{P}^N$ *there exists a type function* $\tau \in T^S$ *such that*

$$L_1(\Pi, \Pi') = \tau \quad \textit{for any } \Pi' \in \mathcal{P}^N.$$

Theorem 2 will map convergent sequences of partition lists (Π_n) to convergent sequences of type functions (τ_n). INV ensures that the limit of (τ_n) coheres with the limit of (Π_n).

3.4 The Converse Labeling

Given any pair of type functions (τ, τ'), we can identify the following pair of partition lists:

$$\Pi := (\Pi_1, ..., \Pi_N), \quad \text{where} \quad \Pi_i(s) := \{s' \in S : \tau_i(s') = \tau_i(s)\}, \tag{1}$$

and

$$\Pi' := (\Pi_1', ..., \Pi_N'), \quad \text{where} \quad \Pi_i'(s) := \{s' \in S : \tau_i'(s') = \tau_i'(s)\}. \tag{2}$$

Say that Π is **τ-consistent** if it is obtained from (1) and that the pair of partition lists (Π, Π') is **consistent** with (τ, τ') if they are obtained from (1) and (2).

3.5 Relationship to Canonical Mappings

Werlang and Tan (1992) define the canonical mapping from a single Partition Model to a single Type Model. In their construction, as in ours, each player's types are their partition elements. Their mapping identifies each player's entire hierarchy of beliefs at their partitions, i.e., their "universal" types. In contrast, we construct a common prior over an ambient set of types, and derive higher-order beliefs from this common prior. Despite this difference, the common prior obtained from any Π-consistent type function τ agrees with the canonical mapping in the sense that each player's beliefs over *induced* states at their partition in the Partition Model coincide with their conditional beliefs at the types to which they are mapped in the Type Model.

Brandenburger and Dekel (1993) define the canonical mapping from a single Type Model to a single Partition Model. Fixing a space of parameters and types $\Theta \times T$, they define partitions over the state space $\Theta \times T$. While distinct states in S may be mapped by the parameter function ϕ and type function τ to identical elements of $\Theta \times T$, there is a unique τ-consistent partition of S. Conditional beliefs in a Partition Model in which all partitions are τ-consistent thus cohere with those in the associated Type Model.

4 Results

We now present our results. First, we establish the existence of a partition labeling satisfying COS, CSC, and INV. Second, we show that any labeling satisfying COS, CSC, and INV sends convergent sequences in the MS topology to convergent sequences in the KM topology. Finally, we establish a converse result.

4.1 Existence

We first construct a labeling satisfying COS, CSC, and INV.

Theorem 1 *There exists a partition labeling satisfying COS, CSC, and INV.*

Proof

The proof is constructive. For each i, partition T_i into countably infinite sets X_i and Y_i. For each $\Pi \in \mathcal{P}^n$, we first define $\tau = L_1(\Pi, \cdot)$. Let $\alpha_i : \Pi_i \to X_i$ be a one-to-one function. Such a function exists because any non-null partition is countable and X_i is countably infinite.

Now, define $\tau \in T^S$ by

$$\tau(s) := (\alpha_1(\Pi_1(s)), ..., \alpha_n(\Pi_n(s))).$$

Hence, L satisfies INV and τ is Π-consistent.

We next define $\tau' = L_2(\Pi, \Pi')$ for each pair of partition lists $\Pi, \Pi' \in \mathcal{P}^N$ so that L satisfies CSC and COS. For this purpose, let $\tilde{\Pi}_i := \{\pi \in \Pi_i\colon \pi \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$. We claim that the function $\beta_i : \tilde{\Pi}'_i \to \tilde{\Pi}_i$ with $\beta_i(\Pi'_i(s)) = \Pi_i(s)$ for all $s \in I_{\Pi,\Pi'}(1/2)$ is one-to-one. To prove β_i is one-to-one, take two distinct partition elements $\pi_1, \pi_2 \in \tilde{\Pi}'_i$. Suppose, towards contradiction, that $\beta_i(\pi_1) = \beta_i(\pi_2) := \pi \in \tilde{\Pi}_i$. Since $P(\pi \backslash \pi_1|\pi) + P(\pi \cap \pi_1|\pi) = 1$ and $\pi \in \tilde{\Pi}_i$ implies $P(\pi \backslash \pi_1|\pi) < 1/2$, $P(\pi \cap \pi_1|\pi) > 1/2$. Similarly, $P(\pi \cap \pi_2|\pi) > 1/2$. But then we have disjoint events $E_1 := \pi \cap \pi_1$ and $E_2 := \pi \cap \pi_2$ such that $P(E_1|\pi) + P(E_2|\pi) > 1$, a contradiction.

Now, let $B_i := \{s \in S : \Pi'_i(s) \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$ and $\alpha'_i : \Pi'_i \backslash \tilde{\Pi}'_i \to Y_i$ be a one-to-one function, where such a function α'_i once again exists because Π'_i is countable and Y_i is countably infinite. For all i, define

$$\tau'_i(s) := \begin{cases} \alpha_i(\beta_i(\Pi'_i(s))) & \text{if } s \in B_i \\ \alpha'_i(\Pi'_i(s)) & \text{otherwise.} \end{cases}$$

Then, L satisfies CSC and τ' is Π'-consistent by construction. Because τ is Π-consistent, L thus satisfies COS. ■

We next show that any labeling satisfying COS, CSC, and INV sends convergent sequences of partition lists in the MS topology to convergent sequences of type functions in the KM topology.

4.2 Partition Lists to Type Functions

The set of states induced by an event $E \subseteq S$ is

$$(\tilde{E})_\tau := \{(\theta, t) \in \Theta \times T : \exists\ s \in E \text{ for which } (\theta, t) = (\phi(s), \tau(s))\}.$$

We first show that if $E \subseteq S$ is common p-belief under the partition list Π, then the set of induced states $(\tilde{E})_\tau$ is common p-belief under any Π-consistent type function τ.[5]

Lemma 1 *Suppose τ is Π-consistent and $E \subseteq S$ is an event. If $s \in C^p_\Pi(E)$, then $(\phi(s), \tau(s)) \in C^p_\mu((\tilde{E})_\tau)$, where μ is identified by τ.*

Proof See Appendix A.1. ■

Our next lemma shows that the restrictions placed on conditional beliefs at partitions in the MS topology bound the conditional beliefs of the types to which they are mapped. We require the common support condition so that partitions at which players have similar conditional beliefs are mapped to the same type.

Lemma 2 *Fix $0 < \epsilon < 1/2$. Suppose L satisfies COS and CSC. If $L(\Pi, \Pi') = (\tau, \tau')$, then $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau \subseteq A_{\mu,\mu'}(2\epsilon)$, where μ is identified by τ and μ' is identified by τ'.*

Proof See Appendix A.2. ■

Using the two lemmas, we then show that if two partition lists are close in the MS topology and are mapped to a pair of type fuctions by a labeling satisfying consistency and the common support condition, then the type functions are close in the KM topology. If the labeling also satisfies invariance, we can fix a limit partition list and the unique limit type function it identifies. As a corollary of the pairwise result, it follows that sequences of partition lists approaching the limit partition list are sent to sequences of type functions approaching the limit type function. We thus obtain Theorem 2, illustrated in Figure 2.

Theorem 2 *Fix a partition labeling L satisfying COS, CSC, and INV. If a sequence of partition lists (Π_n) converges to Π in the MS topology and $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, then the sequence of type functions (τ_n) converges to τ in the KM topology.*

Proof See Appendix A.3. ■

[5] The proof of the lemma is similar to the proof of Theorem 5.3 in Werlang and Tan (1992). Werlang and Tan (1992) show that common knowledge operations are preserved under the canonical mapping from Partition Models to Type Models. As Π-consistent type functions induce a Type Model that coincides with one obtained under the canonical mapping, the lemma may be viewed as a generalization of their theorem to the case of common p-belief.

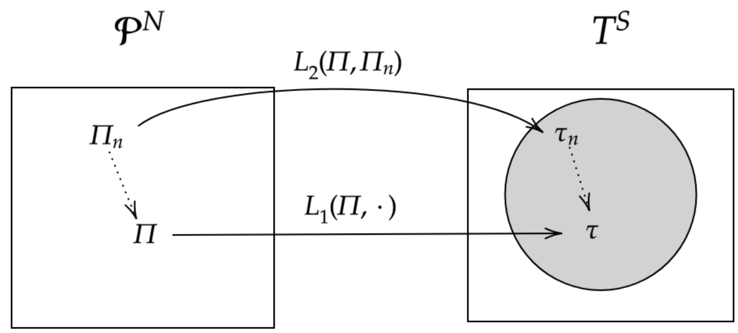


Figure 2: Illustration of Theorem 2.

4.3 Type Functions to Partition Lists

The proof of the converse mirrors the proof of Theorem 2. For any event $E \subseteq \Theta \times T$, define the set of states sent to some element of E,

$$(\tilde{E})_\Pi := \{s \in S : \exists(\theta, t) \in E \text{ for which } (\phi(s), \tau(s)) = (\theta, t)\}.$$

We first show that if E is common p-belief under a type function τ, then $(\tilde{E})_\Pi$ is common p-belief under a τ-consistent partition list.

Lemma 3 *Suppose* Π *is* τ*-consistent and* $E \subseteq \Theta \times T$ *is an event. If* $(\theta, t) \in C_\mu^{1-\epsilon}(E)$*, where* μ *is identified by* τ*, then* $s \in C_\Pi^{1-\epsilon}((\tilde{E})_\Pi)$*.*

Proof See Appendix A.4. ■

We next show that if conditional beliefs are close under τ and τ' at some induced state (θ, t), then the state s mapped to (θ, t) must be contained in partitions having a small conditional symmetric difference.

Lemma 4 *Fix* $0 < \epsilon < 1/2$ *and suppose the partition lists* (Π, Π') *are consistent with the type functions* (τ, τ')*. Then,* $(\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi \subseteq I_{\Pi,\Pi'}(\epsilon)$*, where* μ *is identified by* τ *and* μ' *is identified by* τ'*.*

Proof See Appendix A.5. ■

Finally, fixing a type function τ and a τ-consistent partition list Π, we prove that if a net of type functions converges to τ and is sent to a consistent net of partition lists, then the net of partition lists must converge to Π.

Theorem 3 *Suppose a net of type functions* (τ_n) *converges to a type function* τ *in the KM topology. If* (Π_n, Π) *is consistent with* (τ_n, τ) *for all* n*, then the net of partition lists* (Π_n) *converges to* Π *in the MS topology.*

Proof See Appendix A.6. ■

Consider a type function τ and a partition list $\Pi = L_1^{-1}(\tau, \cdot)$, where L satisfies COS, CSC, and INV. An immediate corollary of Theorem 3 is that any net of invertible type functions contained in the range of $L_2(\Pi, \cdot)$ converging to τ must be sent by $L_2^{-1}(\tau, \cdot)$ to a net of partition lists converging to Π. Figure 3 illustrates.

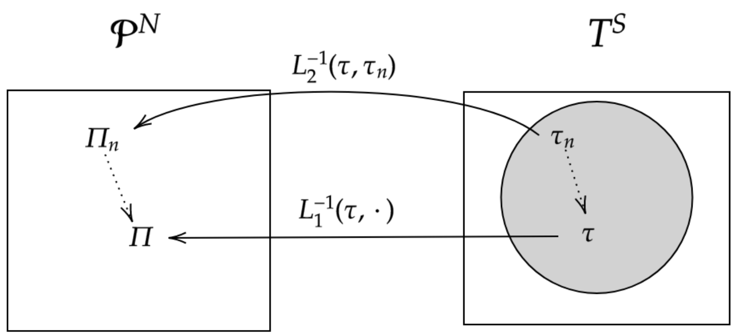


Figure 3: Illustration of Theorem 3.

A Proofs

A.1 Lemma 1

Suppose τ is Π-consistent and $E \subseteq S$ is an event. We show that, for any $m \geq 1$ and $i \in \mathcal{N}$, if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. If this is true, then $s \in \cap_{m\geq 1}(B^p_{\Pi})^m(E) = C^p_{\Pi}(E)$ implies $(\phi(s), \tau(s)) \in \cap_{m\geq 1}(B^p_{\mu})^m((\tilde{E})_\tau) = C^p_{\mu}((\tilde{E})_\tau)$.

The proof is by induction. For the base case, take $s \in B^p_{\Pi_i}(E)$ and consider $(\phi(s), \tau(s))$. Then,

$$\mu((\tilde{E})_\tau | \tau_i(s)) = P(E|\Pi_i(s)) \geq p,$$

where the first inequality follows because μ is identified by τ and τ is Π-consistent, and the second because $s \in B^p_{\Pi_i}(E)$.

The induction hypothesis is that if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

By the induction hypothesis, if $s \in (B^p_\Pi)^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_\mu)^m((\tilde{E})_\tau)$. Since μ is identified by τ and τ is Π-consistent,

$$\mu((B^p_\Pi)^m((\tilde{E})_\tau)|\tau_i(s)) = P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

A.2 Lemma 2

Since $0 < \epsilon < 1/2$, by the common support condition, if $(\theta, t) \in (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau$, then for some $s \in I_{\Pi,\Pi'}(\epsilon)$, $\tau(s) = \tau'(s) := t = (t_1, ..., t_N)$ and $\phi(s) = \theta$. Suppose, towards contradiction and without loss of generality, that $\mu(E|t_i) - \mu'(E|t_i) > 2\epsilon$ for some event $E \subseteq \Theta \times T$ and some player i. By consistency, then, there must be an event $F \subseteq \Pi_i(s)$ for which $(\tilde{F})_\tau \subseteq E$ and for which $P(F|\Pi_i(s)) - P(F|\Pi'_i(s)) > 2\epsilon$. We show that this cannot occur and hence $(\theta, t) \in A_{\mu,\mu'}(2\epsilon)$.

By the triangle inequality, for any event $F \subseteq S$, $|P(F|\Pi_i(s)) - P(F|\Pi'_i(s))|$ is less than or equal to the sum of the following three terms:

1. $|P((\Pi'_i(s) \backslash \Pi_i(s)) \cap F)|\Pi_i(s)) - P((\Pi_i(s) \backslash \Pi'_i(s)) \cap F)|\Pi'_i(s))|$.

2. $|P((\Pi_i(s) \backslash \Pi'_i(s)) \cap F|\Pi_i(s)) - P(\Pi'_i(s) \backslash \Pi_i(s)) \cap F|\Pi'_i(s))|$.

3. $|P((\Pi'_i(s) \cap \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi'_i(s) \cap \Pi_i(s)) \cap F|\Pi'_i(s))|$.

The first term is zero because any states out of one's partition element are believed to have occurred with probability zero. The second term is less than ϵ since, by the definition of

$I_{\Pi,\Pi'}(\epsilon)$,

$$\max\{P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)), P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s))\} \leq \epsilon.$$

We bound the third term as follows:

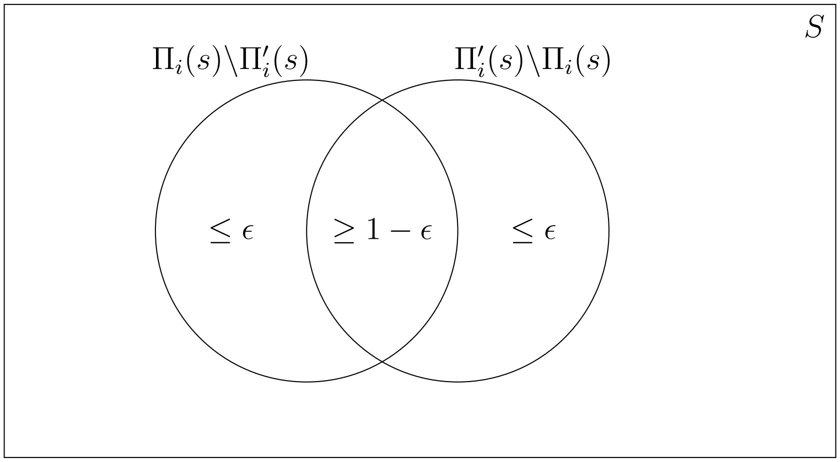


Figure 4: Visual Aid for the Proof of Lemma 2.

$$|P((\Pi'_i(s) \cap \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi'_i(s) \cap \Pi_i(s)) \cap F|\Pi'_i(s))| =$$

$$|\frac{(P(\Pi_i(s)) - P(\Pi'_i(s)))P(\Pi_i(s) \cap \Pi'_i(s) \cap F)}{P(\Pi_i(s))P(\Pi'_i(s))}| \leq |\frac{(P(\Pi_i(s)) - P(\Pi'_i(s)))P(\Pi_i(s) \cap \Pi'_i(s))}{P(\Pi_i(s))P(\Pi'_i(s))}| =$$

$$|P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s))|.$$

Since $P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)) + P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) = 1$ and $P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)) \leq \epsilon$, $P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) \geq 1 - \epsilon$. Similarly, $P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s)) \geq 1 - \epsilon$. Hence, $|P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s))| \leq \epsilon$.

A.3 Theorem 2

If (Π_n) converges to Π in the MS topology, then as $n \to \infty$, $d^{MS}(\Pi, \Pi_n) \to 0$. Suppose $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, where τ is the invariant function to which L maps Π. Fixing $0 < \epsilon < 1/2$, we show that if $L(\Pi, \Pi') = (\tau, \tau')$ and $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $d^{KM}(\mu, \mu') \leq 2\epsilon$, where μ is identified by τ and μ' is identified by τ'. Hence, for the sequence of measures (μ_n) identified by the sequence of Type Models (τ_n), $d^{KM}(\mu, \mu_n) \to 0$. As any sequence is a net, the result follows.

We now prove that if $L(\Pi,\Pi')=(\tau,\tau')$ and $d^{MS}(\Pi,\Pi')\leq\epsilon$, then for μ identified by τ and μ' identified by τ', $d^{KM}(\mu,\mu')\leq 2\epsilon$. We first show that $\max\{d_1^{KM}(\mu',\mu),d_1^{KM}(\mu,\mu')\}<2\epsilon$. If $d^{MS}(\Pi,\Pi')\leq\epsilon$, then $P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)))\geq 1-\epsilon$. We claim that if $s\in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$, then $(\phi(s),\tau(s))\in C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. Take $s\in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. Then, by Lemma 1, $(\phi(s),\tau(s))\in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)$. As L satisfies CSC, by Lemma 2, $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau\subseteq A_{\mu,\mu'}(2\epsilon)$ and hence $(\phi(s),\tau(s))\in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)\subseteq C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. It follows that

$$1-2\epsilon\leq 1-\epsilon\leq P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)))\leq\mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon)))\leq\mu(C_\mu^{1-2\epsilon}(A_{\mu,\mu'}(2\epsilon))).$$

Then, by definition, $d_1^{KM}(\mu',\mu)\leq 2\epsilon$. Since $d^{MS}(\Pi,\Pi')\leq\epsilon$ also implies $P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)))\geq 1-\epsilon$, a symmetric argument ensures $d_1^{KM}(\mu,\mu')\leq 2\epsilon$.

Finally, we show that $d_0^{KM}(\mu,\mu')<2\epsilon$. For an arbitrary event $E\subseteq\Theta\times T$,

$$\begin{aligned}
|\mu(E)-\mu'(E)|\leq&|\mu(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)-\mu'(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)|+\\
&|\mu(E\backslash(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)-\mu'(E\backslash(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)|\\
\leq&|\mu(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)-\mu'(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)|+\epsilon\\
=&|\sum_{t\in\mathrm{marg}_T(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau}\Big(\mu(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau|t)-\mu'(E\cap(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau|t)\Big)\mu(t)|+\epsilon\\
\leq&\epsilon\sum_{t\in\mathrm{marg}_T(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau}\mu(t)+\epsilon\\
\leq&2\epsilon,
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second inequality because $P(I_{\Pi,\Pi'}(\epsilon))>1-\epsilon$, the equality from the law of total probability, the third inequality from CSC and COS, and the final inequality because μ is a probability measure.

A.4 Lemma 3

Suppose τ is Π-consistent and $E\subseteq\Theta\times T$ is an event. We show that, for any $m\geq 1$ and $i\in\mathcal{N}$, if $(\theta,t)\in(B_{\mu_i}^p)^m(E)$ and $(\phi(s),\tau(s))=(\theta,t)$, then $s\in(B_{\Pi_i}^p)^m((\tilde{E})_\Pi)$. If this is true, then $(\theta,t)\in\cap_{m\geq 1}(B_\mu^p)^m(E)=C_\mu^p(E)$ implies $s\in\cap_{m\geq 1}(B_\Pi^p)^m(E)=C_\Pi^p((\tilde{E})_\Pi)$.

The proof is by induction. For the base case, take $(\theta,t)\in B_{\mu_i}^p(E)$ and any s for which $(\phi(s),\tau(s))=(\theta,t)$. Then,

$$P((\tilde{E})_\Pi|\Pi_i(s))=\mu(E|\tau_i(s))\geq p,$$

where the equality follows because μ is identified by τ and Π is τ-consistent.

The induction hypothesis is that if $(\theta, t) \in (B^p_{\mu_i})^m(E)$, then $s \in (B^p_{\Pi_i})^m((\tilde{E})_\Pi)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$\mu((B^p_\mu)^m(E)|\tau_i(s)) \geq p.$$

By the induction hypothesis, if $(\theta, t) \in (B^p_\mu)^m(E)$, then $s \in (B^p_\Pi)^m((\tilde{E})_\Pi)$. Since μ is identified by τ and Π is τ-consistent,

$$P((B^p_\Pi)^m((\tilde{E})_\Pi)|\Pi_i(s)) = \mu((B^p_\mu)^m(E)|\tau_i(s)) \geq p.$$

A.5 Lemma 4

If $s \in (\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi$, then there is an induced state $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$ for which $(\phi(s), \tau(s)) = (\theta, t)$. If $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$, then, for all $i \in \mathcal{N}$, $\mu(t_i) > 0$ and $\mu'(t_i) > 0$. Hence, there is a state s for which $\tau(s) = \tau'(s) := t = (t_1, ..., t_N)$. Further, defining the event $E := \Pi(s)\backslash\Pi'_i(s)$, for all $i \in \mathcal{N}$,

$$\epsilon \geq |\mu((\tilde{E})_\Pi|t_i) - \mu'((\tilde{E})_\Pi|t_i)| = |P(E|\Pi_i(s)) - P(E|\Pi'_i(s)|)| = P(E|\Pi_i(s)),$$

where the inequality follows from the definition of $A_{\mu,\mu'}(\epsilon)$, the first equality follows from consistency, and the second equality because states outside of one's partition are believed to have occurred with probability zero. Setting $F := \Pi'(s)\backslash\Pi_i(s)$ and repeating the previous steps shows $s \in I_{\Pi,\Pi'}(\epsilon)$.

A.6 Theorem 3

If (τ_n) converges to τ in the KM topology, then $d^{KM}(\mu, \mu_n) \to 0$, where μ is identified by τ and μ_n by τ_n. We show that if $d^{KM}(\mu, \mu') \leq \epsilon$, where μ is identified by τ and μ' by τ', then $d^{MS}(\Pi, \Pi') \leq \epsilon$, where (Π, Π') are consistent with τ and τ'. The result follows.

If $d^{KM}(\mu, \mu') \leq \epsilon$, then $\mu(C^{1-\epsilon}_\mu(A_{\mu,\mu'}(\epsilon))) \geq 1 - \epsilon$. Suppose $(\theta, t) \in C^{1-\epsilon}_\mu(A_{\mu,\mu'}(\epsilon))$ and consider any s for which $(\phi(s), \tau(s)) = (\theta, t)$ for some $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$. Then, from Lemma 3, $s \in C^{1-\epsilon}_\Pi((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi)$. By Lemma 4, $(\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi \subseteq I_{\Pi,\Pi'}(\epsilon)$. Hence, $s \in C^{1-\epsilon}_\Pi((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi) \subseteq C^{1-\epsilon}_\Pi(I_{\Pi,\Pi'}(\epsilon))$. It follows from consistency that

$$1 - \epsilon \leq \mu(C^{1-\epsilon}_\mu(A_{\mu,\mu'}(\epsilon))) = P(C^{1-\epsilon}_\Pi(I_{\Pi,\Pi'}(\epsilon))).$$

By definition, then, $d_1^{MS}(\Pi', \Pi) \leq \epsilon$. A symmetric argument ensures $d_1^{MS}(\Pi, \Pi') \leq \epsilon$.

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