Payoff Continuity in Games of Incomplete Information: An Equivalence Result *

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Abstract

Monderer and Samet (1996) and Kajii and Morris (1998) define notions of proximity for countable, common prior information structures that preserve equilibrium payoff continuity. Monderer and Samet (1996) fix a common prior and perturb lists of partitions, while Kajii and Morris (1998) fix a type space and perturb common priors. Due to these differences, the precise relationship between the two notions of proximity has remained an open question. By mapping pairs of partition lists to pairs of common priors, and vice-versa, we establish an equivalence between them, resolving this open question.

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1 Introduction

A game of incomplete information consists of a set of players, a set of actions and payoff functions for each player, and an information structure. How does the set of equilibrium payoffs change as the information structure changes? Rubinstein (1989)'s Email Game illustrates a striking discontinuity; equilibrium payoffs under a common knowledge information structure may not be approximated under an information structure in which there are arbitrarily many, but finite, levels of mutual knowledge. As higher-order beliefs are in principle unobservable, the existence of the discontinuity casts doubt on the robustness of equilibrium predictions.

A natural approach to this problem is to identify what type of precision is required for a modeler to make robust predictions. In the context of countable, common prior games of incomplete information, Monderer and Samet (1996) and Kajii and Morris (1998) define topologies on information structures under which no equilibrium payoff discontinuities arise. While both topologies are based on the same notion of approximate common knowledge, that of common-p belief (Monderer and Samet (1989)), Monderer and Samet (1996) and Kajii and Morris (1998) model the proximity of incomplete information differently. Monderer and Samet (1996) fix a state space and a common prior over it, and consider the differences in beliefs induced by a change in partitions over the state space. Kajii and Morris (1998) fix parameter and type spaces, and consider the differences in beliefs induced by a change in the common prior over its product.

Unfortunately, the precise relationship between the two topologies has remained an open question. As Kajii and Morris (1998) write,

> Our characterization of the proximity of information has a similar flavor to Monderer and Samet's, but we have not been able to establish a direct comparison. By considering a fixed type space, we exogenously determine which types in the information systems correspond to each other. In the Monderer and Samet approach, it is necessary to work out how to identify types in the two information systems. Thus we conjecture that two information systems are close in Monderer and Samet's sense if and only if the types in their construction can be labelled

in such a way that the information systems are close in our sense.

This note exhibits such a labeling and proves Kajii and Morris (1998)'s conjecture true.

The importance of this result is twofold. First, it formally demonstrates the irrelevance of the method of perturbing information structures — either via changing common priors or via changing partitions — in determining the robustness of (Bayesian-Nash) equilibrium predictions. Second, and relatedly, it paves the way for new research identifying an *ex-ante* strategic topology on the Universal Type Space itself, i.e., a notion of proximity of hierarchies of beliefs such that two hierarchies are close-by if and only if equilibrium payoffs across all (bounded) games of incomplete information are close-by in a probabilistic sense. Recently, Gensbittel, Peski, and Renault (2022) have fruitfully applied such a methodology to the study of zero-sum games.

Section 2 introduces the necessary notational preliminaries; Section 3 formally defines labelings and their properties; and Section 4 states the results. All proofs are in the Appendix.

2 Preliminaries

Fix the following primitives: (i) a set of two or more players $\mathcal{N} := \{1, 2..., N\}$, (ii) a countable set of states S, (iii) a countable set of payoff parameters Θ, (iv) a parameter function mapping states to payoff parameters $\phi : S \to \Theta$, and (v) a full-support common prior $P \in \Delta(S)$.

2.1 The MS Topology on Partition Models

Denote by $\mathcal{P}$ the set of all partitions of S and a partition list by $\Pi := (\Pi_1, ..., \Pi_N) \in \mathcal{P}^N$. We define the Monderer and Samet (1996) topology on the set of partition lists $\mathcal{P}^N$.

Denote Player i's partition element containing a state $s \in S$ by $\Pi_i(s)$. At a state $s \in S$, Player i assigns probability $P(E|\Pi_i(s))$ to the event $E \subseteq S$. Player i **p-believes** an event $E \subseteq S$ at a state $s \in S$ if $P(E|\Pi_i(s)) \geq p$. Denote $B^p_{\Pi_i}(E)$ as the set of states at which i p-believes E under the partition Π_i. The set of states at which E is **mutual p-belief** is $B^p_{\Pi}(E) := \cap B^p_{\Pi_i}(E)$. The set of states at which E is **m-level mutual p-belief** is $(B^p_{\Pi})^m(E)$, the m-th iteration of $B^p_{\Pi}(\cdot)$ over the set E. Finally, the set of states at which E is **common**

p-belief is $C^p_\Pi(E) := \cap_{m\geq 1}(B^p_\Pi(E))^m$.

Define $I_{\Pi,\Pi'}(\epsilon)$ as the set of states at which the conditional symmetric difference between each player's partition elements containing that state is less than ϵ,

$$I_{\Pi,\Pi'}(\epsilon) := \bigcap_{i\in\mathcal{N}}\{s \in S : \max\{P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)), P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s))\} < \epsilon\}.$$

Define

$$d^{MS}(\Pi,\Pi') := \max\{d^{MS}_1(\Pi,\Pi'), d^{MS}_1(\Pi',\Pi)\},$$

where $d^{MS}_1(\Pi,\Pi')$ is an ex-ante measure of states for which the event $I_{\Pi,\Pi'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d^{MS}_1(\Pi,\Pi') := \inf\{\epsilon | P(C^{1-\epsilon}_{\Pi'}(I_{\Pi,\Pi'}(\epsilon))) \geq 1-\epsilon\}.$$

While d^{MS} is not the metric defined in Monderer and Samet (1996) (indeed, it does not satisfy the triangle inequality), it follows from Theorem 5.2 in their paper that a sequence converges in their topology if and only if $d^{MS}(\Pi,\Pi_n) \to 0$. Hereafter, we call the topology generated by these convergent sequences the **MS topology**.

2.2 The KM Topology on Type Models

Fix a countably infinite type space $T := T_1 \times ... \times T_N$, where T_i is a countable set of types for player $i \in \mathcal{N}$. Denote T^S as the set of *type functions* from S to T. Since each function $\tau \in T^S$ induces a measure $\mu \in \Delta(\Theta \times T)$, we define the Kajii and Morris (1998) topology on the function space T^S, placing conditions on the measures they identify. For convenience, call the elements of $\Theta \times T$ **induced states**.

Denote $\mu(t_i)$ as the marginal distribution of μ on T_i evaluated at t_i. If $\mu(t_i) > 0$, the probability of an event $E \subseteq \Theta \times T$ conditional on Player i's type t_i is given by $\mu(E|t_i) := \sum_{(\theta,(t_i,t_{-i}))\in E}\mu(s,t_{-i}|t_i)$. Player i p-**believes** an event $E \subseteq \Theta \times T$ at an induced state $(\theta,t) \in \Theta \times T$ if $\mu(E|\tau_i(s)) \geq p$, where $\tau_i(s)$ is the i-th component of τ. Denote $B^p_{\mu_i}(E)$ as the set of all induced states at which i p-believes E. The set of induced states at which E is **mutual** p**-belief** is $B^p_\mu(E) := \bigcap_{i\in\mathcal{N}} B^p_{\mu_i}(E)$. The set of induced states at which E is m**-level mutual** p**-belief** is $(B^p_\mu(E))^m$, the m-th iteration of $B^p_\mu(\cdot)$ over the set E. The set of induced states at which E is **common** p**-belief** is $C^p_\mu(E) := \cap_{m\geq 1}(B^p_\mu(E))^m$.

Define $A_{\mu,\mu'}(\epsilon)$ as the set of induced states at which each player has conditional beliefs that differ by at most ϵ over any event given their type,

$$A_{\mu,\mu'}(\epsilon) := \{(\theta, t) \in \Theta \times T : \text{for all } i \in \mathcal{N},\ \mu(t_i) > 0, \mu'(t_i) > 0, \text{ and}$$
$$|\mu(E|t_i) - \mu'(E|t_i)| \leq \epsilon \text{ for all } E \subseteq \Theta \times T\}.$$

Define a function mapping pairs of common priors to real numbers:

$$d^{KM}(\mu, \mu') := \max\{d_1^{KM}(\mu, \mu'), d_1^{KM}(\mu', \mu), d_0^{KM}(\mu, \mu')\},$$

where $d_1^{KM}(\mu, \mu')$ is an ex-ante measure of states for which the event $A_{\mu,\mu'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d_1^{KM}(\mu, \mu') := \inf\{\epsilon > 0 : \mu'(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon\},$$

and $d_0^{KM}(\mu, \mu')$ is the distance between μ and μ' in the weak topology,

$$d_0^{KM}(\mu, \mu') := \underset{E \subseteq \Theta \times T}{sup} |\mu(E) - \mu'(E)|.$$

d^{KM} generates a topology on T^S by specifying which nets converge.[1] A net $\{\tau^k : k \in K\}$, where K is a set with partial order $\succ$, converges to τ if for any $\epsilon > 0$, there is a $\bar{k} \in K$ such that $k \succ \bar{k}$ implies that $d^{KM}(\mu^k, \mu) < \epsilon$, where μ^k is the unique probability measure identified by τ^k and μ is the unique probability measure identified by τ. Hereafter, we call the topology generated by these nets the **KM topology**.

3 Labelings

A **partition labeling** is a function from pairs of partition lists to pairs of type functions $L : \mathcal{P}^N \times \mathcal{P}^N \to T^S \times T^S$. We first define and illustrate three properties of partition labelings — consistency, the common support condition, and invariance. We then define the converse labeling from type functions to partition lists. We conclude by discussing the relationship between our labelings and the canonical mappings defined in Werlang and Tan (1992) and Brandenburger and Dekel (1993).

[1] In metric spaces, (or, more generally, in first-countable spaces) one can define a topology by specifying which sequences converge. Since Monderer and Samet (1996) prove their topology is metrizable, we thus define theirs by specifying convergent sequences. Since we have not proved that Kajii and Morris (1998)'s topology is metrizable, we maintain their original definition.

3.1 Consistency

We first define a sense in which a player's type space is consistent with her partitional information.

Definition 1 (Π-Consistency) *A type function τ is Π-**consistent** if,*

$$\tau_i(s) = \tau_i(s') \text{ if and only if } s, s' \in \pi \in \Pi_i.$$

A consistent partition labeling maps pairs of partition lists to pairs of consistent type functions.

Definition 2 (Consistency) *A partition labeling L is **consistent**, or is said to satisfy **COS**, if $L(\Pi, \Pi') = (\tau, \tau')$ implies τ is Π-consistent and τ' is Π'-consistent.*

Notice, consistency places no restrictions on the relationship between the type functions in the co-domain.

3.2 The Common Support Condition

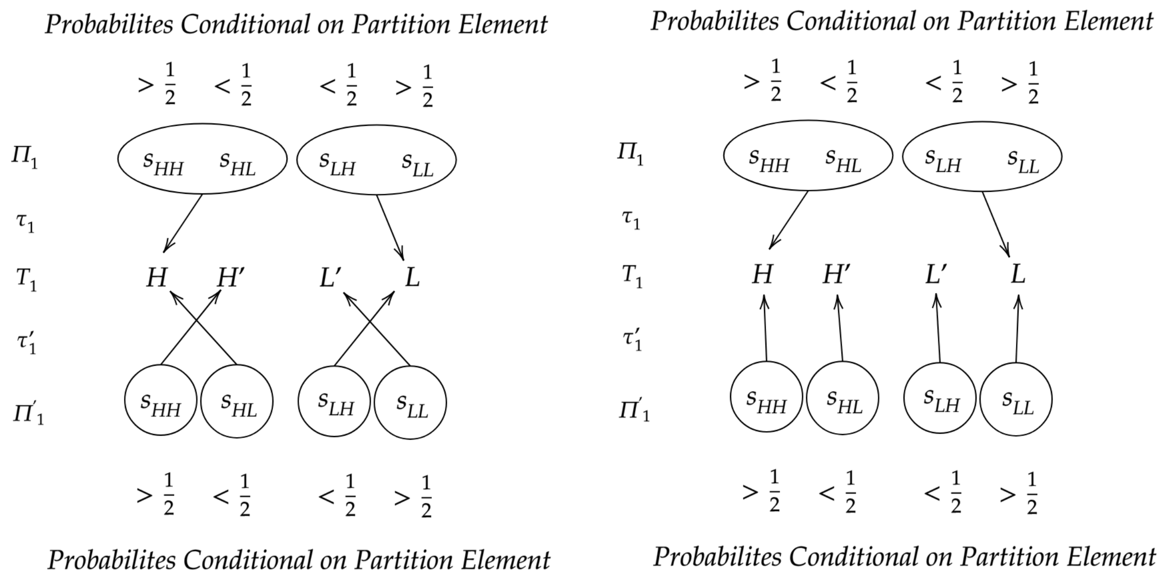
The common support condition *does* place restrictions on the pairs of type functions in the co-domain.

Definition 3 (Common Support Condition) *A partition labeling L satisfies the **common support condition (CSC)** if $L(\Pi, \Pi') = (\tau, \tau')$ and $s \in I_{\Pi,\Pi'}(1/2)$ implies*

$$\tau(s) = \tau'(s).$$

Fixing partition lists Π and Π', any partition labeling satisfying the common support condition must send states to the same type if they are contained in similar enough partition elements across lists, as measured by the conditional symmetric difference.

Example 1 *In Figure 1, the state space is $S := \{s_{HH}, s_{HL}, s_{LH}, s_{LL}\}$ and the prior distribution P places probability $\frac{1}{2} > \epsilon > 0$ on s_{HL} and s_{LH}, and probability $\frac{1}{2} - \epsilon$ on s_{HH} and s_{LL}. Posterior beliefs are written above each state. The Partition Model has $\Pi_1 := \{\{s_{HH}, s_{HL}\}, \{s_{LH}, s_{LL}\}\}$ and $\Pi_1' := \{\{s_{HH}\}, \{s_{HL}\}, \{s_{LH}\}.\{s_{LL}\}\}$.*



(a) COS, but not CSC.

(b) COS and CSC.

Figure 1: Labeling Examples.

While the labeling in Figure 1a does not violate consistency, it does violate the common support condition. To see why, notice that

$$I_{\Pi,\Pi'}(1/2) = \{s_{HH}, s_{LL}\},$$

because

$$P(\Pi_1(s_{HH}) \backslash \Pi_1'(s_{HH}) | \Pi_1(s_{HH})) = P(s_{HL}) < 1/2,$$

and

$$P(\Pi_1(s_{LL}) \backslash \Pi_1'(s_{LL}) | \Pi_1(s_{LL})) = P(s_{LH}) < 1/2.$$

Nevertheless, the type function τ *does not coincide with* τ' *at either* s_{HH} *or* s_{LL}*;* $\tau_1(s_{HH}) = H \neq H' = \tau_1'(s_{HH})$ *and* $\tau_1(s_{LL}) = L \neq L' = \tau_1'(s_{LL})$*. On the other hand, the labeling in Figure 1b satisfies both consistency and the common support condition. This follows because states in* $I_{\Pi,\Pi'}(1/2)$ *are mapped to the same type* ($\tau_1(s_{HH}) = H = \hat{\tau}_1(s_{HH})$ *and* $\tau_1(s_{LL}) = L = \hat{\tau}_1(s_{LL})$*).*

3.3 Invariance

Define $L_i(\Pi, \Pi')$ as the i-th component of the co-domain of L. Our final property, called invariance, requires partition lists in the first coordinate of the domain to be mapped to the same type function independently of the second coordinate.

Definition 4 (Invariance) *A partition labeling L is* ***invariant****, or is said to satisfy* ***INV****, if for all partition lists* $\Pi \in \mathcal{P}^N$ *there exists a type function* $\tau \in T^S$ *such that,*

$$L_1(\Pi, \Pi') = \tau \quad \textit{for any } \Pi' \in \mathcal{P}^N.$$

If a partition labeling L satisfies INV, $L_2(\Pi, \cdot)$ maps the entire space of partition lists $\mathcal{P}^N$ to a subset of the space of type functions T^S. If L also satisfies CSC, so that restrictions are made on pairs of type functions in the co-domain of L, then $L_2(\Pi, \cdot)$ may be regarded as the space of type functions relative to a fixed interpretation τ of a partition list Π. For example, this subset must contain τ, as depicted in Figure 2.

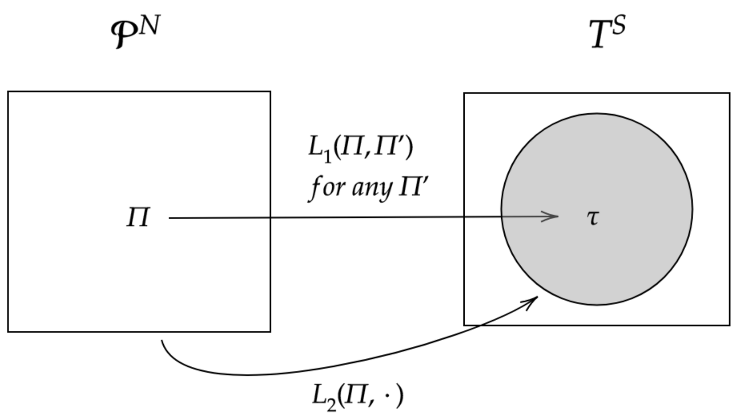


Figure 2: Illustration of INV when CSC is satisfied.

3.4 The Converse Labeling

Given any pair of type functions (τ, τ') we can identify the following pair of partition lists:

$$\Pi := (\Pi_1, ..., \Pi_N) \text{ where } \Pi_i := (\Pi_i(s))_{s\in S} \text{ and } \Pi_i(s) := \{s' \in S : \tau_i(s') = \tau_i(s)\} \quad (1)$$

and

$$\Pi' := (\Pi'_1, ..., \Pi'_N) \text{ where } \Pi'_i := (\Pi'_i(s))_{s\in S} \text{ and } \Pi'_i(s) := \{s' \in S : \tau'_i(s') = \tau'_i(s)\}. \quad (2)$$

Say that Π is **τ-consistent** if it is obtained from Equation 1 and that the pair of partition lists (Π, Π') is **consistent** with (τ, τ') if they are obtained from Equation 1 and 2.

3.5 Relationship to Canonical Mappings and Invertibility

Werlang and Tan (1992) define the canonical mapping from a single Partition Model to a single Type Model. In their construction, as in ours, each player's types are their partition elements. Their mapping identifies each player's entire hierarchy of beliefs at their partitions, i.e. their Universal types. In contrast, we construct a common prior over an ambient set of types, and derive higher-order beliefs from this common prior. Despite this difference, the common prior obtained from any Π-consistent type function τ agrees with the canonical mapping in the sense that each player's beliefs over *induced* states at their partition in the Partition Model coincide with their conditional beliefs at the types to which they are mapped to in the Type Model.

Brandenburger and Dekel (1993) define the canonical mapping from a single Type Model to a single Partition Model. Fixing a space of parameters and types $\Theta \times T$, they define partitions over a state space $S := \Theta \times T$. In our construction, distinct states in S may be mapped by the parameter function ϕ and type function τ to identical elements of $\Theta \times T$.[2] Hence, we may not be able to recover a unique partition of S.

We introduce a new property of type functions called invertibility to address this issue.

Definition 5 (Invertibility) *Fixing a parameter function ϕ, a type function τ is* ***invertible*** *if $(\theta, t) = (\phi(s), \tau(s)) = (\phi(s'), \tau(s'))$ implies $s = s'$.*

Under invertibility, any τ-consistent partition list Π agrees with the one obtained under the canonical mapping. We do not view it as a strong condition for the following reason. Given

[2]To see the problems this causes, suppose that $S = \{s_1, s_2\}$, $P = 1/2 \circ s_1 + 1/2 \circ s_2$, $\phi(s_1) = \phi(s_2) = s$ and $\tau(s_1) = \tau(s_2) = t$. Consider the partition $\Pi = \{\{s_1, s_2\}\}$ and a Π-consistent Type Model τ. If $E = \{s_1\}$, then $\tilde{E}_\tau = \{(s,t)\}$ implying $P(E|\Pi(s_1)) = 1/2 \neq 1 = \mu(\tilde{E}|\tau(s_1))$, where μ is identified by τ.

two states s and s', suppose $(\phi(s), \tau(s)) = (\phi(s'), \tau(s'))$ and τ is Π-consistent. Then, s and s' could have been collapsed into a single state without affecting beliefs over induced states.

4 Main Results

In the Appendix, we construct a labeling satisfying COS, CSC, and INV. Consequently, we obtain the following proposition.

Proposition 1 *There exists a partition labeling satisfying COS, CSC, and INV.*

We now show that any labeling satisfying COS, CSC, and INV sends convergent sequences of partition lists in the MS topology to convergent sequences of type functions in the KM topology.

4.1 Theorem 1: Partition Lists to Type Functions

The set of states induced by an event $E \subseteq S$ is defined by,

$$(\tilde{E})_\tau := \{(\theta, t) \in \Theta \times T : \exists\ s \in E \text{ for which } (\theta, t) = (\phi(s), \tau(s))\}.$$

We first show that if $E \subseteq S$ is common p-belief under the partition list Π, then the set of induced states $(\tilde{E})_\tau$ is common p-belief under any Π-consistent type function τ.[3]

Lemma 1 *Suppose τ is Π-consistent and $E \subseteq S$. If $s \in C^p_\Pi(E)$, then $(\phi(s), \tau(s)) \in C^p_\mu((\tilde{E})_\tau)$, where μ is identified by τ.*

Our next lemma shows that the restrictions placed on conditional beliefs at partitions in the MS topology bound the conditional beliefs of the types to which they are mapped. We require the common support condition so that partitions at which players have similar conditional beliefs are mapped to the same type.

[3]The proof of the lemma is similar to the proof of Theorem 5.3 in Werlang and Tan (1992). Werlang and Tan (1992) show that common knowledge operations are preserved under the canonical mapping from Partition Models to Type Models. As Π-consistent type functions induce a Type Model that coincides with one obtained under the canonical mapping, the lemma may be viewed as a generalization of their theorem to the case of common p-belief.

Lemma 2 *Fix $0 < \epsilon < 1/2$. Suppose L satisfies COS and CSC. If $L(\Pi, \Pi') = (\tau, \tau')$, then $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau \subseteq A_{\mu,\mu'}(2\epsilon)$, where μ is identified by τ and μ' is identified by τ'.*

Using the two lemmas, we may then show that if two partition lists are close in the MS topology and are mapped to a pair of type fuctions by a labeling satisfying consistency and the common support condition, then the type functions are close in the KM topology. If the labeling also satisfies invariance, we can fix a limit partition list and the unique limit type function it identifies. As a corollary of the pairwise result, it follows that sequences of partition lists approaching the limit partition list are sent to sequences of type functions approaching the limit type function. We thus obtain Theorem 1, illustrated in Figure 3.

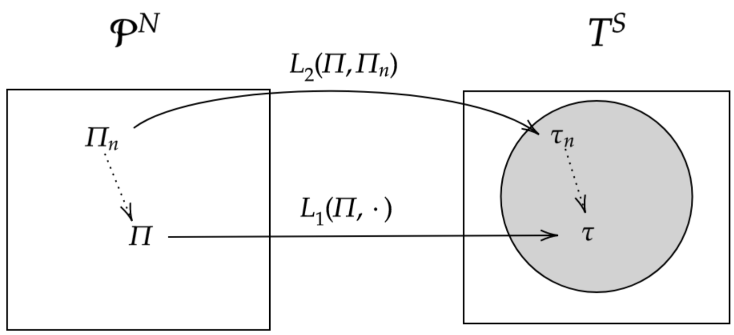


Figure 3: Illustration of Theorem 1.

Theorem 1 *Fix a partition labeling L satisfying COS, CSC, and INV. If a sequence of partition lists (Π_n) converges to Π in the MS topology and $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, then the sequence of type functions (τ_n) converges to τ in the KM topology.*

4.2 Theorem 2: Type Functions to Partition Lists

Given invertibility, the proof of the converse mirrors the proof of Theorem 1. For any event $E \subseteq \Theta \times T$, define the set of states sent to some element of E,

$$(\tilde{E})_\Pi := \{s \in S : \exists(\theta, t) \in E \text{ for which } (\phi(s), \tau(s)) = (\theta, t)\}.$$

We first show that if E is common p-belief under an invertible type function τ, then $(\tilde{E})_\Pi$ is common p-belief under a τ-consistent partition list.

Lemma 3 *If $(\theta, t) \in C_{\mu}^{1-\epsilon}(E)$ and τ is invertible, then $s \in C_{\Pi}^{1-\epsilon}((\tilde{E})_{\Pi})$ if Π is τ-consistent.*

We next show that if conditional beliefs are close under τ and τ' at some induced state (θ, t), the state s mapped to (θ, t) must be contained in partitions having a small conditional symmetric difference.

Lemma 4 *Fix $0 < \epsilon < 1/2$ and suppose the partition lists (Π, Π') are consistent with invertible type functions (τ, τ'). Then, $(\tilde{A}_{\mu,\mu'}(\epsilon))_{\Pi} \subseteq I_{\Pi,\Pi'}(\epsilon)$, where μ is identified by τ and μ' is identified by τ'.*

Finally, fixing an invertible type function τ and a τ-consistent partition list Π, we prove that if a net of invertible type functions converges to τ and is sent to a consistent net of partition lists, then the net of partition lists must converge to Π.

Theorem 2 *Suppose an invertible net of type functions (τ_n) converges to an invertible type function τ in the KM topology. If (Π_n, Π) is consistent with (τ_n, τ) for all n, then the net of partition lists (Π_n) converges to Π in the MS topology.*

Consider a type function τ and a partition list $\Pi = L_1^{-1}(\tau, \cdot)$, where L satisfies COS, CSC, and INV. An immediate corollary of Theorem 2 is that any net of invertible type functions contained in the range of $L_2(\Pi, \cdot)$ converging to τ, must be sent by $L_2^{-1}(\tau, \cdot)$ to a net of partition lists converging to Π. Figure 4 illustrates.

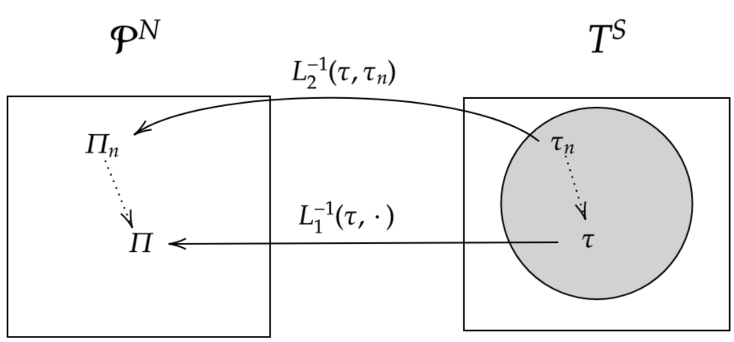


Figure 4: Illustration of Theorem 2.

Proofs

Proposition 1

We construct an invariant, consistent partition labeling L satisfying the common support condition. Begin by indexing S by the natural numbers. Denote its elements $s_1, s_2, ...$[4]

We first define $\tau = L_1(\Pi, \cdot)$ for all $\Pi \in \mathcal{P}^N$ so that L satisfies invariance. Given $\Pi := \Pi_1 \times ... \times \Pi_N$, order the partition elements of Π_i so that $\pi_i^1 \in \Pi_i$ contains s_1 and $\pi_i^k \in \Pi_i$ contains the smallest $s \in S \backslash (\underset{j<k}{\cup} \pi_i^j)$ for $k > 1$. Define $\tau_i(s) := t_i^j$ if $s \in \pi_i^j$. Notice, by construction, τ is consistent with Π.

Next, we define $L(\Pi, \Pi') = (\tau, \tau')$ for arbitrary partitions $\Pi, \Pi' \in \mathcal{P}^N$. τ is pinned down by the previous step so we need only choose τ'. Define $\tau'(s) := \tau(s)$ for all $s \in I_{\Pi,\Pi'}(1/2)$ so that L satisfies CSC.

If $s \in I_{\Pi,\Pi'}(1/2)$ and $s' \in \Pi_i(s)$, then either $\tau'(s') = \tau'(s)$ or $\tau'(s')$ has not yet been specified. In other words, two states in the same partition element cannot have been mapped to distinct types. Hence, if $s' \in \Pi'_i(s)$ for some $s \in I_{\Pi,\Pi'}(1/2)$, we may define $\tau'_i(s') := \tau_i(s)$ so that τ' is consistent with all partition elements containing a state in $I_{\Pi,\Pi'}(1/2)$. Denoting $\tilde{\Pi}_i := \{\pi \in \Pi_i \colon \pi \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$, this property follows as a consequence of the following claim.

Claim 1 *The mapping $b_i : \tilde{\Pi}_i \to \tilde{\Pi}'_i$ where $b_i(\Pi_i(s)) = \Pi'_i(s)$ is a bijection.*

Proof To prove b_i is one-to-one take two distinct partition elements $\pi_1, \pi_2 \in \tilde{\Pi}_i$. Suppose towards contradiction that $b_i(\pi_1) = b_i(\pi_2) := \pi' \in \tilde{\Pi}'_i$. Since $P(\pi' \backslash \pi_1 | \pi') + P(\pi' \cap \pi_1 | \pi') = 1$ and $\pi' \in \tilde{\Pi}'_i$ implies $P(\pi' \backslash \pi_1 | \pi') < 1/2$, $P(\pi' \cap \pi_1 | \pi') > 1/2$. Similarly, $P(\pi' \cap \pi_2 | \pi') > 1/2$. But then we have disjoint events $E_1 = \pi' \cap \pi_1$ and $E_2 = \pi' \cap \pi_2$ such that $P(E_1|\pi') + P(E_2|\pi') > 1$, a contradiction.

To prove b_i is onto take a partition element $\pi' \in \tilde{\Pi}'_i$. We claim $b_i(\Pi_i(s)) = \pi'$ for some

[4] Indexing S is without loss of generality. We may think of S as an arbitrary countable set together with an injective function $f : S \to \mathbb{N}$. By the well-ordering principle, we may order the elements of S so that s_1 is the element of S mapped to the smallest natural number in the range of f, s_2 is the element of S mapped to the second smallest natural number in the range of f, and so on.

$s \in \pi'$. By the definition of $\tilde{\Pi}_i'$, there exists an $s \in \pi' \cap I_{\Pi,\Pi'}$. If $s \in \pi' \cap I_{\Pi,\Pi'}$, then $s \in \Pi_i(s) \cap I_{\Pi,\Pi'}$ and so $b_i(\Pi_i(s)) = \pi'$. ■

Finally, for all i, we define τ_i' on the domain $S \backslash B_i$, where $B_i := \{s' \in S : s' \in \pi_i' \cap I_{\Pi,\Pi'}$ for some $\pi_i' \in \Pi_i\}$. If for all i, (i) $\tau_i'(s) = \tau_i'(s')$ if and only if $s, s' \in \pi_i' \subset S \backslash B_i$ and (ii) for every $s \in S \backslash B_i$, $\tau_i'(s) \neq \tau_i'(s')$ for any $s' \in B_i$, τ' is Π'-consistent. There is at least one, and may be many, functions satisfying these conditions.

Lemma 1

Suppose τ is Π-consistent and $E \subseteq S$. We show that, for any $m \geq 1$ and $i \in \mathcal{N}$, if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. If this is true, then $s \in \cap_{m \geq 1}(B^p_\Pi)^m(E) = C^p_\Pi(E)$ implies $(\phi(s), \tau(s)) \in \cap_{m \geq 1}(B^p_\mu)^m((\tilde{E})_\tau) = C^p_\mu((\tilde{E})_\tau)$.

The proof is by induction. For the base case, take $s \in B^p_{\Pi_i}(E)$ and consider $(\phi(s), \tau(s))$. Then,

$$\mu((\tilde{E})_\tau | \tau_i(s)) \geq P(E|\Pi_i(s)) \geq p,$$

where the first inequality follows because μ is identified by τ and τ is Π-consistent, and the second because $s \in B^p_{\Pi_i}(E)$.[5]

The induction hypothesis is that if $s \in (B^p_{\Pi_i})^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_{\mu_i})^m((\tilde{E})_\tau)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

By the induction hypothesis, if $s \in (B^p_\Pi)^m(E)$, then $(\phi(s), \tau(s)) \in (B^p_\mu)^m((\tilde{E})_\tau)$. Since μ is identified by τ and τ is Π-consistent,

$$\mu((B^p_\Pi)^m((\tilde{E})_\tau)|\tau_i(s)) \geq P((B^p_\Pi)^m(E)|\Pi_i(s)) \geq p.$$

Lemma 2

Since $0 < \epsilon < 1/2$, by the common support condition, if $(\theta, t) \in (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau$, then for some $s \in I_{\Pi,\Pi'}(\epsilon)$, $\tau(s) = \tau'(s) = t := (t_1, ..., t_N)$ and $\phi(s) = \theta$. Suppose, towards contradiction

[5] Without invertibility, the first inequality cannot be made an equality because distinct states may be mapped to the same induced state.

and without loss of generality, that $\mu(E|t_i) - \mu'(E|t_i) > 2\epsilon$ for some $E \subseteq \Theta \times T$ and some player i. By consistency, then, there must be a collection of states $F \subseteq \Pi_i(s)$ for which $(\tilde{F})_\tau \subseteq E$ and for which $P(F|\Pi_i(s)) - P(F|\Pi_i'(s)) > 2\epsilon$. We show that this cannot occur and hence $(\theta, t) \in A_{\mu,\mu'}(2\epsilon)$.

By the triangle inequality, for any $F \subseteq S$, $|P(F|\Pi_i(s)) - P(F|\Pi_i'(s))|$ is less than or equal to the sum of the following three terms:

1. $|P((\Pi_i'(s)\backslash\Pi_i(s)) \cap F)|\Pi_i(s)) - P((\Pi_i(s)\backslash\Pi_i'(s)) \cap F)|\Pi_i'(s))|$.

2. $|P((\Pi_i(s)\backslash\Pi_i'(s)) \cap F|\Pi_i(s)) - P(\Pi_i'(s)\backslash\Pi_i(s)) \cap F|\Pi_i'(s))|$.

3. $|P((\Pi_i'(s) \cup \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi_i'(s) \cup \Pi_i(s)) \cap F|\Pi_i'(s))|$.

The first term is zero because any states out of one's partition element are believed to have occurred with probability zero. The second term is less than ϵ since, by the definition of $I_{\Pi,\Pi'}(\epsilon)$,

$$\max\{P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s)), P(\Pi_i'(s)\backslash\Pi_i(s)|\Pi_i'(s))\} \leq \epsilon.$$

We bound the third term as follows:

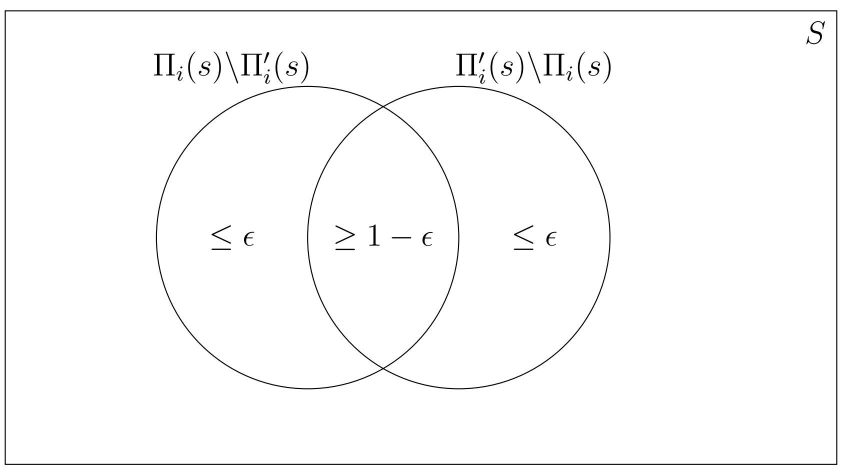


Figure 5: Visual Aid for the Proof of Lemma 2.

$$|P((\Pi_i'(s) \cap \Pi_i(s)) \cap F|\Pi_i(s)) - P((\Pi_i'(s) \cap \Pi_i(s)) \cap F|\Pi_i'(s))| =$$

$$|\frac{(P(\Pi_i(s)) - P(\Pi_i'(s)))P(\Pi_i(s) \cap \Pi_i'(s) \cap F)}{P(\Pi_i(s))P(\Pi_i'(s))}| \leq |\frac{(P(\Pi_i(s)) - P(\Pi_i'(s)))P(\Pi_i(s) \cap \Pi_i'(s))}{P(\Pi_i(s))P(\Pi_i'(s))}| =$$

$$|P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s))|.$$

Since $P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)) + P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) = 1$ and $P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)) \leq \epsilon$, $P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) \geq 1-\epsilon$. Similarly, $P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s)) \geq 1-\epsilon$. Hence, $|P(\Pi'_i(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi'_i(s) \cap \Pi_i(s)|\Pi'_i(s))| \leq \epsilon$.

Theorem 1

If (Π_n) converges to Π in the MS topology, then as $n \to \infty$, $d^{MS}(\Pi, \Pi_n) \to 0$. Suppose $L(\Pi, \Pi_n) = (\tau, \tau_n)$ for all n, where τ is the invariant function to which L maps Π. Fixing $0 < \epsilon < 1/2$, we show that if $L(\Pi, \Pi') = (\tau, \tau')$ and $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $d^{KM}(\mu, \mu') \leq 2\epsilon$, where μ is identified by τ and μ' is identified by τ'. Hence, for the sequence of measures (μ_n) identified by the sequence of Type Models (τ_n), $d^{KM}(\mu, \mu_n) \to 0$. As any sequence is a net, the result follows.

We now prove that if $L(\Pi, \Pi') = (\tau, \tau')$ and $d^{MS}(\Pi, \Pi') \leq \epsilon$, then for μ identified by τ and μ' identified by τ', $d^{MS}(\mu, \mu') \leq 2\epsilon$. We first show that $\max\{d_1^{KM}(\mu', \mu), d_1^{KM}(\mu, \mu')\} < 2\epsilon$. If $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1-\epsilon$. We claim that if $s \in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$, then $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. Take $s \in C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. Then, by Lemma 1, $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)$. As L satisfies CSC, by Lemma 2, $(\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau \subseteq A_{\mu,\mu'}(2\epsilon)$ and hence $(\phi(s), \tau(s)) \in C_\mu^{1-\epsilon}((\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) \subseteq C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))$. It follows that,

$$1-2\epsilon \leq 1-\epsilon \leq P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \leq \mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(2\epsilon))) \leq \mu(C_\mu^{1-2\epsilon}(A_{\mu,\mu'}(2\epsilon))).$$

Then, by definition, $d_1^{KM}(\mu', \mu) \leq 2\epsilon$. Since $d^{MS}(\Pi, \Pi') \leq \epsilon$ also implies $P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))) \geq 1-\epsilon$, a symmetric argument ensures $d_1^{KM}(\mu, \mu') \leq 2\epsilon$.

Finally, we show that $d_0^{KM}(\mu,\mu') < 2\epsilon$. For an arbitrary event $E \subseteq \Theta \times T$,

$$
\begin{aligned}
|\mu(E) - \mu'(E)| \leq & |\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| + \\
& |\mu(E \backslash (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \backslash (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| \\
\leq & |\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau)| + \epsilon \\
= & |\sum_{t \in \tilde{I}_{\Pi,\Pi'}} \Big(\mu(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau | t) - \mu'(E \cap (\tilde{I}_{\Pi,\Pi'}(\epsilon))_\tau | t)\Big) \mu(t)| + \epsilon \\
\leq & \epsilon \sum_{t \in \tilde{I}_{\Pi,\Pi'}} \mu(t) + \epsilon \\
\leq & 2\epsilon,
\end{aligned}
$$

where the first inequality follows from the triangle inequality, the second inequality because $P(I_{\Pi,\Pi'}(\epsilon)) > 1 - \epsilon$, the equality from the law of total probability, the third inequality from CSC and COS, and the final inequality because μ is a probability measure.

Lemma 3

Suppose τ is Π-consistent and $E \subseteq \Theta \times T$. We show that, for any $m \geq 1$ and $i \in \mathcal{N}$, if $(\theta, t) \in (B^p_{\mu_i})^m(E)$ and $(\phi(s), \tau(s)) = (\theta, t)$, then $s \in (B^p_{\Pi_i})^m((\tilde{E})_\Pi)$. If this is true, then $(\theta, t) \in \cap_{m \geq 1} (B^p_\mu)^m(E) = C^p_\mu(E)$ implies $s \in \cap_{m \geq 1} (B^p_\Pi)^m(E) = C^p_\Pi((\tilde{E})_\Pi)$.

The proof is by induction. For the base case, take $(\theta, t) \in B^p_{\mu_i}(E)$ and any s for which $(\phi(s), \tau(s)) = (\theta, t)$. Then,

$$P((\tilde{E})_\Pi | \Pi_i(s)) = \mu(E | \tau_i(s)) \geq p,$$

where the equality follows because μ is identified by τ and τ is Π-consistent, and the second because of invertibility.

The induction hypothesis is that if $(\theta, t) \in (B^p_{\mu_i})^m(E)$, then $s \in (B^p_{\Pi_i})^m((\tilde{E})_\Pi)$. Take $s \in (B^p_{\Pi_i})^{m+1}(E)$. Then,

$$\mu((B^p_\mu)^m(E) | \tau_i(s)) \geq p.$$

By the induction hypothesis, if $(\theta, t) \in (B^p_\mu)^m(E)$, then $s \in (B^p_\Pi)^m((\tilde{E})_\Pi)$. Since μ is identified by τ and τ is Π-consistent,

$$P((B^p_\Pi)^m((\tilde{E})_\Pi) | \Pi_i(s)) \geq \mu((B^p_\mu)^m(E) | \tau_i(s)) \geq p.$$

Lemma 4

If $s \in (\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi$, then there is an induced state $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$ for which $(\phi(s), \tau(s)) = (\theta, t)$. If $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$, then, for all $i \in \mathcal{N}$, $\mu(t_i) > 0$ and $\mu'(t_i) > 0$. Hence, there is a state s for which $\tau(s) = \tau'(s) = t := (t_1, ..., t_N)$. Further, defining $E := \Pi(s) \backslash \Pi'_i(s)$, for all $i \in \mathcal{N}$,

$$\epsilon \geq |\mu((\tilde{E})_\Pi | t_i) - \mu'((\tilde{E})_\Pi | t_i)| = |P(E|\Pi_i(s)) - P(E|\Pi'_i(s)|)| = P(E|\Pi_i(s)),$$

where the inequality follows from the definition of $A_{\mu,\mu'}(\epsilon)$, the first equality follows from invertibility, and the second equality because states outside of one's partition are believed to have occurred with probability zero. Setting $F := \Pi'(s) \backslash \Pi_i(s)$ and repeating the previous steps shows $s \in I_{\Pi,\Pi'}(\epsilon)$.

Theorem 2

If (τ_n) converges to τ in the KM topology, then $d^{KM}(\mu, \mu_n) \to 0$, where μ is identified by τ and μ_n by τ_n. We show that if $d^{KM}(\mu, \mu') \leq \epsilon$, where μ is identified by τ and μ' by τ', then $d^{MS}(\Pi, \Pi') \leq \epsilon$, where (Π, Π') are consistent with τ and τ'. The result follows.

If $d^{KM}(\mu, \mu') \leq \epsilon$, then $\mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))) \geq 1 - \epsilon$. Suppose $(\theta, t) \in C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))$ and consider any s for which $(\phi(s), \tau(s)) = (\theta, t)$ for some $(\theta, t) \in A_{\mu,\mu'}(\epsilon)$. Then, from Lemma 3, $s \in C_\Pi^{1-\epsilon}((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi)$. By Lemma 4, $(\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi \subseteq I_{\Pi,\Pi'}(\epsilon)$. Hence, $s \in C_\Pi^{1-\epsilon}((\tilde{A}_{\mu,\mu'}(\epsilon))_\Pi) \subseteq C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. It follows from invertibility that,

$$1 - \epsilon \leq \mu(C_\mu^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))) = P(C_\Pi^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))).$$

By definition, then, $d_1^{MS}(\Pi', \Pi) \leq \epsilon$. A symmetric argument ensures $d_1^{MS}(\Pi, \Pi') \leq \epsilon$.

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