Payoff Continuity in Games of Incomplete Information Across Models of Knowledge[*]

Ashwin Kambhampati[†]

May 16, 2024

Abstract

Equilibrium predictions in games of incomplete information are sensitive to the assumed information structure. Monderer and Samet (1996) and Kajii and Morris (1998) define topological notions of proximity for common prior information structures such that two information structures are close if and only if (approximate) equilibrium payoffs are close. However, Monderer and Samet (1996) fix a common prior and define their topology on profiles of partitions over a state space, whereas Kajii and Morris (1998) define their topology on common priors over the product of a state space and a type space. We prove the open conjecture that two partition profiles are close in the Monderer and Samet (1996) topology if and only if there exists a labeling of types such that the associated common priors are close in the Kajii and Morris (1998) topology.

Keywords: incomplete information, Bayesian Nash equilibrium, common knowledge

JEL codes: C70, C72, D80

[*]An associate editor and three anonymous referees provided valuable feedback that improved the clarity and quality of the manuscript. Aislinn Bohren, Eduardo Faingold, Ben Golub, Tetsuya Hoshino, Scott Kaplan, Annie Liang, George Mailath, Stephen Morris, Ricardo Serrano-Padial, Juuso Toikka, and Yuichi Yamamoto provided helpful comments, suggestions, and guidance. The views expressed here are solely my own and do not in any way represent the views of the U.S. Naval Academy, U.S. Navy, or the Department of Defense.

[†]Department of Economics, United States Naval Academy. Email: `kambhamp@usna.edu`. Postal Address: Michelson 361, 572M Holloway Road, Annapolis, MD 21402.

1 Introduction

A game of incomplete information consists of a set of players, an action set and payoff function for each player, and an information structure. How does the set of Bayesian Nash equilibrium payoffs change as the information structure changes? Rubinstein (1989)'s Email Game illustrates a striking payoff discontinuity; there can exist an equilibrium under a common knowledge information structure yielding expected payoffs that are not approximated in *any* equilibrium under an information structure in which there are arbitrarily many, but finite, levels of mutual knowledge.

Monderer and Samet (1996) and Kajii and Morris (1998) identify coarse topologies on common prior information structures that preserve continuity of ϵ- Bayesian Nash equilibrium payoffs across all bounded games of incomplete information.[1] Both topologies demonstrate that common p-belief (Monderer and Samet (1989)) is the appropriate relaxation of common knowledge to preserve continuity of equilibrium payoffs. However, Monderer and Samet (1996) and Kajii and Morris (1998) model the proximity of information structures differently. Monderer and Samet (1996) fix a state space and a common prior over it, and consider the differences in beliefs induced by a change in partitions over the state space. Kajii and Morris (1998) fix state and type spaces, and consider the differences in beliefs induced by a change in the common prior over its product.

The relationship between the two modeling approaches and the resulting topologies has remained an open question. As Kajii and Morris (1998) write,

> Our characterization of the proximity of information has a similar flavor to Monderer and Samet's, but we have not been able to establish a direct comparison. By considering a fixed type space, we exogenously determine which types in the information systems correspond to each other. In the Monderer and Samet approach, it is necessary to work out how to identify types in the two information systems. Thus we conjecture that two information systems are close in Monderer

[1] This exercise is uninformative if one requires that players exactly optimize ($\epsilon = 0$). Theorem 5 of Gensbittel, Peski, and Renault (2022) considers a general space of information structures and shows that the topology that preserves payoff continuity of (exact) Bayesian Nash equilibria on this space is the discrete topology.

and Samet's sense if and only if the types in their construction can be labelled in such a way that the information systems are close in our sense.

In this paper, we prove Kajii and Morris (1998)'s conjecture by constructing distance-preserving maps from partition profiles to common priors.

2 Setting

Fix a set of two or more players $\mathcal{N} := \{1, 2, \dots, N\}$ and a probability triple (S, Σ, P), where S is an infinite state space, Σ is a σ-algebra of events, and P is a probability measure. The state space S can be taken to be countable or uncountable. In what follows, all countable sets are equipped with the discrete σ-algebra and all product sets are equipped with the product σ-algebra.

We introduce here some useful notation. Let X, Y, and Z be measure spaces. For any measurable function $g : X \to Y$, let $g(E)$ denote the image of $E \subseteq X$ under g and $g^{-1}(F)$ denote the pre-image of $F \subseteq Y$ under g. If $h : X \to Z$ is another measurable function, then let $g \times h : X \to Y \times Z$ be the measurable function defined by $(h \times g)(x) = (h(x), g(x))$ for all $x \in X$. Finally, let $\iota : S \to S$ be the identity function on the set of states, i.e., $\iota(s) = s$ for all $s \in S$.

2.1 The MS Topology

Denote by $\mathcal{P}$ the set of all partitions of S into non-null elements of Σ. Denote by $\mathcal{P}^N$ the set of all partition profiles, with typical element denoted by $\Pi = (\Pi_1, \dots, \Pi_N)$. We define the Monderer and Samet (1996) topology on $\mathcal{P}^N$.

Denote by $\Pi_i(s)$ player i's partition element containing a state $s \in S$. At a state $s \in S$, player i assigns probability $P(E|\Pi_i(s))$ to the event $E \subseteq S$. Player i **p-believes** an event $E \subseteq S$ at a state $s \in S$ if $P(E|\Pi_i(s)) \geq p$. Denote by $B^p_{\Pi_i}(E)$ the set of states at which player i p-believes E under the partition Π_i. The set of states at which E is **mutual p-belief** is $B^p_\Pi(E) := \bigcap_{i \in \mathcal{N}} B^p_{\Pi_i}(E)$. The set of states at which E is **m-level mutual p-belief** is $(B^p_\Pi)^m(E)$, the m-th iteration of $B^p_\Pi(\cdot)$ over the set E. Finally, the set of states at which

E is **common p-belief** is $C^p_\Pi(E) := \bigcap_{m\geq 1} (B^p_\Pi)^m(E)$.

Define $I_{\Pi,\Pi'}(\epsilon)$ to be the set of states at which the conditional symmetric difference between each player's partition elements containing that state is less than ϵ,

$$I_{\Pi,\Pi'}(\epsilon) := \bigcap_{i\in\mathcal{N}} \{s \in S : \max\{P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)), P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s))\} \leq \epsilon\}.$$

Define

$$d^{MS}(\Pi, \Pi') := \max\{d_1^{MS}(\Pi, \Pi'), d_1^{MS}(\Pi', \Pi)\},$$

where $d_1^{MS}(\Pi, \Pi')$ is an ex-ante measure of states in $I_{\Pi,\Pi'}(\epsilon)$ at which the event $I_{\Pi,\Pi'}(\epsilon)$ is common $(1-\epsilon)$-belief:

$$d_1^{MS}(\Pi, \Pi') := \inf\{\epsilon : P(C^{1-\epsilon}_{\Pi'}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)) \geq 1-\epsilon\}.$$

Using d^{MS}, we now define the **MS topology**. For each $\Pi \in \mathcal{P}^N$ and $r > 0$, define the r-neighborhood of Π by

$$B_{d^{MS}}(\Pi, r) := \{\Pi' \in \mathcal{P}^N : d^{MS}(\Pi, \Pi') < r\}.$$

Then, a subset of partition profiles, $O \subseteq \mathcal{P}^N$, is open in the MS topology if and only if, for all $\Pi \in O$, there exists an $r > 0$ such that $B_{d^{MS}}(\Pi, r) \subseteq O$.

We remark here that d^{MS} is not the distance defined in Monderer and Samet (1996). Monderer and Samet (1996) replace the common p-belief operator with the stronger "joint common repeated p-belief" operator and bound $I_{\Pi,\Pi'}(\epsilon)$ by $1/2$ so that their distance satisfies the triangle inequality (and is therefore a pseudo metric). Nevertheless, d^{MS} induces the same topology as the psuedo metric defined in Monderer and Samet (1996) by Theorem 5.2 of their paper.

2.2 The KM Topology

Fix a type space $T := T_1 \times \cdots \times T_N$, where T_i is a countably infinite set of types for player $i \in \mathcal{N}$. We call a pair $(s, t) \in S \times T$ a **KM state**. Denote by $\Delta(S \times T)$ the set of probability measures over KM states, i.e., the set of common priors. Given a measurable function $g : S \to T$, say that a common prior $\mu \in \Delta(S \times T)$ is **g-consistent** if, for any event $E \times F \subseteq S \times T$,

$$\mu(E \times F) = P((\iota \times g)^{-1}(E \times F)). \tag{1}$$

Say that a measurable function $\tau : S \to T$ is a **Π-labeling** if, for any $s, s' \in S$, $\tau_i(s) = \tau_i(s')$ if and only if $\Pi_i(s) = \Pi_i(s')$. Then, a common prior μ is **Π-consistent** if there is a Π-labeling $\tau : S \to T$ such that μ is τ-consistent. We define the Kajii and Morris (1998) topology on the subset of common priors consistent with some partition profile $\Pi \in \mathcal{P}^N$. To ease notation, we henceforth implicitly assume μ is Π-consistent and μ' is Π'-consistent.

Denote by $\mu(t_i)$ the marginal distribution of μ on T_i evaluated at t_i. If $\mu(t_i) > 0$, then the probability of an event $E \subseteq S \times T$ conditional on player i's type t_i is denoted by $\mu(E|t_i) := \mu(E)/\mu(t_i)$. Player i **p-believes** an event $E \subseteq S \times T$ at $(s,t) \in S \times T$ if $\mu(E|t_i) \geq p$ or $\mu(t_i) = 0$. Denote by $B^p_{\mu_i}(E)$ the set of all KM states, (s,t), at which player i p-believes event E. The set of KM states at which E is **mutual p-belief** is $B^p_\mu(E) := \bigcap_{i \in \mathcal{N}} B^p_{\mu_i}(E)$. The set of KM states at which E is **m-level mutual p-belief** is $(B^p_\mu)^m(E)$, the m-th iteration of $B^p_\mu(\cdot)$ over the set E. The set of KM states at which E is **common p-belief** is $C^p_\mu(E) := \bigcap_{m \geq 1} (B^p_\mu)^m(E)$.

We now define the set of KM states at which each player has conditional beliefs that differ by at most ϵ over any event given their type. Specifically, let $A_{\mu,\mu'}(\epsilon)$ be the set of KM states $(s,t) \in S \times T$ such that, for any Π-labeling τ and Π'-labeling τ' for which μ is τ-consistent and μ' is τ'-consistent, we have, for all $i \in \mathcal{N}$,

1. $\tau_i(s) = \tau'_i(s) = t_i$,
2. $\mu(t_i) > 0$ and $\mu'(t_i) > 0$, and
3. $|\mu(E \times F|t_i) - \mu'(E \times F|t_i)| \leq \epsilon$ for all events $E \times F \subseteq S \times T$.

Now, define a function mapping pairs of common priors to real numbers:

$$d^{KM}(\mu, \mu') := \max\{d_1^{KM}(\mu, \mu'), d_1^{KM}(\mu', \mu), d_0^{KM}(\mu, \mu')\},$$

where $d_1^{KM}(\mu, \mu')$ is an ex-ante measure of KM states under which the event $A_{\mu,\mu'}(\epsilon)$ is common $(1-\epsilon)$-belief,

$$d_1^{KM}(\mu, \mu') := \inf\{\epsilon : \mu'(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon\},$$

and $d_0^{KM}(\mu, \mu')$ is the sup-norm distance between μ and μ' over all events,

$$d_0^{KM}(\mu, \mu') := \sup_{E \times F \subseteq S \times T} |\mu(E \times F) - \mu'(E \times F)|.$$

Using d^{KM}, we now define the **KM topology**. For each $\mu \in \Delta(S \times T)$ and $r > 0$, define the r-neighborhood of μ by

$$B_{d^{KM}}(\mu, r) := \{\mu' \in \Delta(S \times T) : d^{KM}(\mu, \mu') < r\}.$$

Then, a subset of common priors, $O \subseteq \Delta(S \times T)$, is open in the KM topology if and only if, for all $\mu \in O$, there exists an $r > 0$ such that $B_{d^{KM}}(\mu, r) \subseteq O$.

We remark here that d^{KM} is not the distance defined in Kajii and Morris (1998). There are two main differences. First, Kajii and Morris (1998) do not place restrictions on the states in S that belong to $A_{\mu,\mu'}(\epsilon)$. Nevertheless, the "infection game" used to establish the necessity of closeness in their distance for closeness in payoffs can be easily used to establish necessity of closeness under d^{KM} for closeness in payoffs (in Proposition 7, simply let $\hat{E} = A_{\mu,\mu'}(\epsilon)$, where $A_{\mu,\mu'}(\epsilon)$ is defined in our sense). Second, in the definition of $d_1^{KM}(\mu, \mu')$, we require

$$\mu'(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon$$

instead of

$$\mu'(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))) \geq 1 - \epsilon.$$

However, Equation (5.11) in Monderer and Samet (1996) (see, also, Section 2.3 of Morris (1999)) establishes

$$\mu'(C_{\mu'}^{1-\epsilon}(E \times F) \cap (E \times F)) \geq (1 - 2\epsilon)\mu'(C_{\mu'}^{1-\epsilon}(E \times F))$$

for any $E \times F \subseteq S \times T$. Hence, d^{KM} induces the topology defined in Kajii and Morris (1998).

2.3 Payoff Continuity Results

We point out here that Monderer and Samet (1996) and Kajii and Morris (1998) prove payoff continuity in their respective topologies with respect to different classes of games. Specifically, let A_i be an action set for player i that is finite and has more than two elements. Let $A := A_1 \times \ldots \times A_n$ be the set of all action profiles. Theorem 5.4 of Monderer and Samet (1996) establishes the upper- and lower- hemicontinuity of ϵ- Bayesian Nash equilibrium payoffs in their topology across all bounded games in which each player i has a utility function $u_i : A \times S \to \mathbb{R}$. In contrast, Proposition 8 of Kajii and Morris (1998), whose proof is corrected by Rothschild (2005), establishes the upper- and lower- hemicontinuity of ϵ- Bayesian Nash equilibrium payoffs across all bounded games in which each player i has a utility function $u_i : A \times S \times T \to \mathbb{R}$. It is thus immediate that convergence of common priors in the KM topology implies convergence of any "belief consistent" sequence of partition profiles in the MS topology. Our main result shows that the converse also holds when types with similar beliefs across partition profiles are mapped to the same element of T. The crucial property of the mapping ensures that when a player's "belief type" is perturbed in the MS topology, her "payoff type" is *not* perturbed. Hence, differences in payoffs cannot be artificially generated through the utility function in the Kajii and Morris (1998) setup.

3 Main Result

The main result, Theorem 1, establishes the existence of a distance-preserving map with respect to every partition profile, Π, in the MS topology. Specifically, the map sends every partition profile Π' to a Π'-consistent common prior μ'. In addition, partition profile labelings are defined so that each partition element in Π'_i that is "close" to a partition element in Π_i is mapped to the same type, t_i. This property, which we call the common support condition, ensures that partition profiles Π and Π' are close in the MS topology if and only if the associated common priors μ and μ' are close in the KM topology.

Theorem 1

1. *Fix a partition profile $\Pi \in \mathcal{P}^N$. There exists a function $f_\Pi : \mathcal{P}^N \to \Delta(S \times T)$ such that, for every $\Pi' \in \mathcal{P}^N$, $f_\Pi(\Pi')$ is Π'-consistent, and, for any $\epsilon \in (0, 1/2)$,*

   (a) if $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $d^{KM}(f_\Pi(\Pi), f_\Pi(\Pi')) \leq \epsilon$, and

   (b) if $d^{KM}(f_\Pi(\Pi), f_\Pi(\Pi')) \leq \epsilon$, then $d^{MS}(\Pi, \Pi') \leq \epsilon$.

2. *For any $\epsilon \in (0, 1/2)$, there exists a Π-consistent common prior μ and a Π'-consistent common prior μ' such that $d^{KM}(\mu, \mu') \leq \epsilon$ only if $d^{MS}(\Pi, \Pi') \leq \epsilon$.*

The first part of Theorem 1 establishes that both f_Π and its inverse f_Π^{-1} are Lipschitz continuous at Π (with Lipschitz constant of one) under the distances d^{MS} and d^{KM}. As an immediate corollary, we observe that convergent sequences of partition profiles in the MS topology are mapped to convergent sequences of common priors in the KM topology by f_Π. A converse statement holds for sequences of common priors that belong to $f_\Pi(\mathcal{P}^N)$.

Corollary 1 (Convergence under f_Π) *If a sequence of partition profiles (Π_n) converges to Π in the MS topology, then the sequence of common priors $(f_\Pi(\Pi_n))$ converges to $f_\Pi(\Pi)$ in the KM topology. Moreover, if a sequence of common priors $(f_\Pi(\Pi_n))$ converges to $f_\Pi(\Pi)$ in the KM topology, then the sequence of partition profiles (Π_n) converges to Π in the MS topology.*

The first part of Theorem 1 also immediately establishes one direction of Kajii and Morris (1998)'s conjecture: for any $\epsilon \in (0, 1/2)$, if two partition profiles are in an ϵ-neighborhood in the MS topology, then f_Π labels them so that the associated common priors are in an ϵ-neighborhood in the KM topology. The second part of Theorem 1 formalizes and establishes the other direction: for any $\epsilon \in (0, 1/2)$, if it is possible to label partition profiles in such a way that the associated common priors are in an ϵ-neighborhood in the KM topology, then these partition profiles must be in an ϵ-neighborhood in the MS topology. We isolate Kajii and Morris (1998)'s conjecture as a corollary of Theorem 1.

Corollary 2 (Kajii and Morris (1998) Conjecture) *There exists a Π-consistent common prior μ and a Π'-consistent common prior μ' such that $d^{KM}(\mu, \mu') \leq \epsilon$ if and only if $d^{MS}(\Pi, \Pi') \leq \epsilon$.*

4 Proof of Main Result

The outline of the proof of Theorem 1 is as follows. We first construct a function $f_\Pi : \mathcal{P}^N \to \Delta(S \times T)$ such that, for every $\Pi' \in \mathcal{P}^N$, $f_\Pi(\Pi')$ is Π'-consistent (Section 4.1). We then establish part 1(a) of Theorem 1 using this function (Section 4.2). Finally, we jointly establish parts 1(b) and 2 of Theorem 1 (Section 4.3).

4.1 Construction of f_Π and the Common Support Condition

Fix a partition profile $\Pi \in \mathcal{P}^N$. We first define $f_\Pi(\Pi) \in \Delta(S \times T)$. For each i, partition T_i into countably infinite sets X_i and Y_i. Let $\alpha_i : \Pi_i \to X_i$ be a one-to-one function. Such a function exists because any non-null partition is countable and X_i is countably infinite. Now, define $\tau : S \to T$ to be the Π-labeling

$$\tau(s) := (\alpha_1(\Pi_1(s)), \ldots, \alpha_n(\Pi_n(s))).$$

Then, set $f_\Pi(\Pi)$ equal to the unique τ-consistent common prior μ.

We now define $f_\Pi(\Pi')$ for each partition profile $\Pi' \neq \Pi$. For this purpose, let $\tilde{\Pi}_i := \{\pi \in \Pi_i\colon \pi \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$. We claim that the function $\beta_i : \tilde{\Pi}'_i \to \tilde{\Pi}_i$ with $\beta_i(\Pi'_i(s)) = \Pi_i(s)$ for all $s \in I_{\Pi,\Pi'}(1/2)$ is one-to-one. To prove β_i is one-to-one, take two distinct partition elements $\pi_1, \pi_2 \in \tilde{\Pi}'_i$. Suppose, towards contradiction, that $\beta_i(\pi_1) = \beta_i(\pi_2) := \pi \in \tilde{\Pi}_i$. Since $P(\pi \backslash \pi_1|\pi) + P(\pi \cap \pi_1|\pi) = 1$ and $\pi \in \tilde{\Pi}_i$ implies $P(\pi \backslash \pi_1|\pi) < 1/2$, $P(\pi \cap \pi_1|\pi) > 1/2$. Similarly, $P(\pi \cap \pi_2|\pi) > 1/2$. But then we have disjoint events $E_1 := \pi \cap \pi_1$ and $E_2 := \pi \cap \pi_2$ such that $P(E_1|\pi) + P(E_2|\pi) > 1$, a contradiction. Now, let $B_i := \{s \in S : \Pi'_i(s) \cap I_{\Pi,\Pi'}(1/2) \neq \emptyset\}$ and $\alpha'_i : \Pi'_i \backslash \tilde{\Pi}'_i \to Y_i$ be a one-to-one function, where such a function α'_i once again exists because Π'_i is countable and Y_i is countably infinite. Define the Π'-labeling $\tau' : S \to T$ by

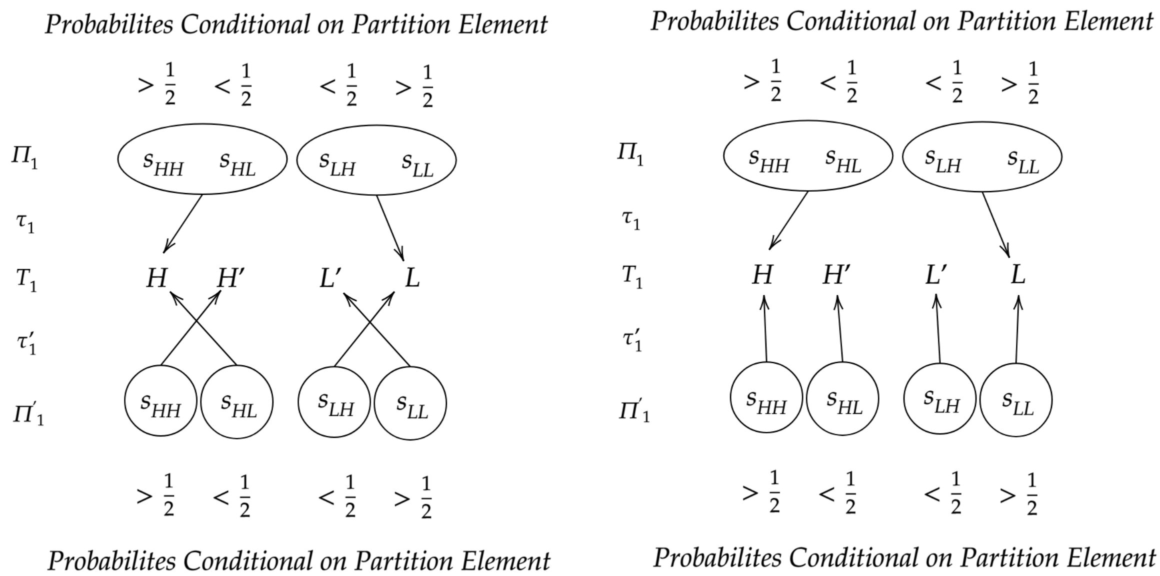
$$\tau'_i(s) := \begin{cases} \alpha_i(\beta_i(\Pi'_i(s))) & \text{if } s \in B_i \\ \alpha'_i(\Pi'_i(s)) & \text{otherwise.} \end{cases}$$

Then, set $f_\Pi(\Pi')$ equal to the unique τ'-consistent common prior μ'.

From the preceding paragraphs, we have obtained a function f_Π such that, for every $\Pi' \in \mathcal{P}^N$, $f_\Pi(\Pi')$ is Π'-consistent. This function also possesses a crucial property called the **common support condition (CSC)**: if $\mu = f_\Pi(\Pi)$ is τ-consistent with Π-labeling τ and $\mu' = f_\Pi(\Pi')$ is τ'-consistent with Π'-labeling τ', then $s \in I_{\Pi,\Pi'}(1/2)$ implies $\tau(s) = \tau'(s)$. A simple example illustrates the condition.

Example 1 *In Figure 1, the state space is finite for simplicity of exposition:*

$$S := \{s_{HH}, s_{HL}, s_{LH}, s_{LL}\}.$$



(a) $f_\Pi(\cdot)$ violates CSC.

(b) $f_\Pi(\cdot)$ does not violate CSC.

Figure 1: Illustration of common support condition.

The prior distribution P places probability $\epsilon \in (0, \frac{1}{2})$ on s_{HL} and s_{LH}, and probability $\frac{1}{2} - \epsilon$ on s_{HH} and s_{LL}. Posterior beliefs are written above each state. The partitions are $\Pi_1 := \{\{s_{HH}, s_{HL}\}, \{s_{LH}, s_{LL}\}\}$ and $\Pi_1' := \{\{s_{HH}\}, \{s_{HL}\}, \{s_{LH}\}.\{s_{LL}\}\}$.

Let $\mu = f_\Pi(\Pi)$ be τ-consistent with some Π-labeling τ and $\mu' = f_\Pi(\Pi')$ be τ'-consistent with some Π'-labeling τ'. In Figure 1a, τ' is chosen in a way that causes f_Π to violate the common support condition. To see why, notice that

$$I_{\Pi,\Pi'}(1/2) = \{s_{HH}, s_{LL}\}$$

because

$$P(\Pi_1(s_{HH}) \backslash \Pi_1'(s_{HH}) | \Pi_1(s_{HH})) = P(s_{HL}) < 1/2$$

and

$$P(\Pi_1(s_{LL}) \backslash \Pi_1'(s_{LL}) | \Pi_1(s_{LL})) = P(s_{LH}) < 1/2.$$

Nevertheless, the Π-labeling τ does not coincide with the Π'-labeling τ' at either s_{HH} or s_{LL}: $\tau_1(s_{HH}) = H \neq H' = \tau_1'(s_{HH})$ and $\tau_1(s_{LL}) = L \neq L' = \tau_1'(s_{LL})$. On the other hand, the labeling in Figure 1b does not violate the common support condition because states in $I_{\Pi,\Pi'}(1/2)$ are mapped to the same type: $\tau_1(s_{HH}) = H = \tau_1'(s_{HH})$ and $\tau_1(s_{LL}) = L = \tau_1'(s_{LL})$.

4.2 Proof of Theorem 1: part 1(a)

We now show that if Π and Π' are close in the MS topology, then $f_\Pi(\Pi)$ and $f_\Pi(\Pi')$ are close in the KM topology. Formally, for $\epsilon \in (0, 1/2)$, $d^{MS}(\Pi, \Pi') \leq \epsilon$ implies $d^{KM}(f_\Pi(\Pi), f_\Pi(\Pi')) \leq \epsilon$.

A preliminary Lemma establishes the precise sense under which first-order beliefs of each type cohere with those in an associated partition model.

Lemma 1 *For any event $E \subseteq S$, Π'-labeling τ', and τ'-consistent common prior μ', if $s \in S$, then*

$$P(E|\Pi_i'(s)) = \mu'((\iota \times \tau')(E)|\tau_i'(s)).$$

Proof See Appendix A.1. ■

Moreover, under the common support condition, the restrictions placed on conditional beliefs at partition elements in the MS topology bound the conditional beliefs of the types to which they are mapped.

Lemma 2 *Suppose $\mu = f_\Pi(\Pi)$ is τ-consistent with the Π-labeling τ and $\mu' = f_\Pi(\Pi')$ is τ'-consistent with the Π'-labeling τ'. For any $0 < \epsilon < 1/2$, if $s \in I_{\Pi,\Pi'}(\epsilon)$, then $(s, \tau'(s)) \in A_{\mu,\mu'}(\epsilon)$.*

Proof See Appendix A.2. ■

The next Lemma concerns higher-order beliefs. Specifically, if $E \subseteq S$ is common p-belief under the partition profile Π, then $(\iota \times \tau')(E)$ is common p-belief under the common prior obtained from any Π'-consistent type function τ'.

Lemma 3 *For any event $E \subseteq S$, Π'-labeling τ', and τ'-consistent common prior μ', if $s \in C^p_{\Pi'}(E)$, then $(s, \tau'(s)) \in C^p_{\mu'}((\iota \times \tau')(E))$.*

Proof See Appendix A.3. ■

Using the preceding lemmas, we finally show that if two partition lists are close in the MS topology and are mapped to common priors consistent with a pair of type functions satisfying the common support condition, then the common priors are close in the KM topology.

Lemma 4 *Suppose $\mu = f_\Pi(\Pi)$ is τ-consistent with the Π-labeling τ and $\mu' = f_\Pi(\Pi')$ is τ'-consistent with the Π'-labeling τ'. For any $0 < \epsilon < 1/2$, if $d^{MS}(\Pi, \Pi') \le \epsilon$, then $d^{KM}(\mu, \mu') \le \epsilon$.*

Proof See Appendix A.4. ■

Theorem 1 part 1(a) follows directly from Lemma 4.

4.3 Theorem 1: part 1(b) and 2

We now show that if $f_\Pi(\Pi)$ and $f_\Pi(\Pi')$ are close in the KM topology, then Π and Π' are close in the MS topology. Formally, for $\epsilon \in (0, 1/2)$, $d^{KM}(f_\Pi(\Pi), f_\Pi(\Pi')) \le \epsilon$ implies $d^{MS}(\Pi, \Pi') \le \epsilon$. The proof mirrors the proof of part 1(a).

A preliminary Lemma establishes the sense in which first-order beliefs at a partition element cohere with those at the type to which the partition element is mapped.

Lemma 5 *For any event $E \times F \subseteq S \times T$, Π'-labeling τ', and τ'-consistent common prior μ', if $s \in S$, then*

$$\mu'(E \times F | \tau'_i(s)) = P((\iota \times \tau')^{-1}(E \times F) | \Pi'_i(s)).$$

Proof See Appendix A.5. ■

Moreover, the restrictions placed on conditional beliefs on types in the KM topology bound the conditional beliefs at the partition elements to which they correspond.

Lemma 6 *Suppose μ is τ-consistent with the Π-labeling τ and μ' is τ'-consistent with the Π'-labeling τ'. For any $0 < \epsilon < 1/2$, if $s \in (\iota \times \tau')^{-1}(A_{\mu,\mu'}(\epsilon))$, then $s \in I_{\Pi,\Pi'}(\epsilon)$.*

Proof See Appendix A.6. ■

The next Lemma concerns higher-order beliefs. If $E \subseteq S \times T$ is common p-belief at a KM state (s, t') under μ', where μ' is obtained from a Π'-consistent type function τ', then $(\iota \times \tau')^{-1}(E)$ is common p-belief under Π'.

Lemma 7 *Suppose $E \subseteq S \times T$, τ' is a Π'-labeling, and μ' is τ'-consistent. If $(s, t') \in C^p_{\mu'}(E)$, and $\tau'(s) = t'$, then $s \in C^p_{\Pi'}((\iota \times \tau')^{-1}(E))$.*

Proof See Appendix A.7. ■

Using the preceding lemmas, we show that if two common priors are close in the MS topology, then the associated partition profiles are close in the KM topology.

Lemma 8 *Suppose μ is τ-consistent with the Π-labeling τ and μ' is τ'-consistent with the Π'-labeling τ'. For any $0 < \epsilon < 1/2$, if $d^{KM}(\mu, \mu') \leq \epsilon$, then $d^{MS}(\Pi, \Pi') \leq \epsilon$.*

Proof See Appendix A.8. ■

We now argue that Lemma 8 immediately establishes Theorem 1 parts 1(b) and 2. Fix $\epsilon \in (0, 1/2)$. Observe that $f_\Pi(\Pi')$ is Π'-consistent for all $\Pi' \in \mathcal{P}^N$. Hence, for any $\Pi, \Pi' \in \mathcal{P}^N$, there exists a Π-labeling τ and a Π'-labeling τ' such that $f_\Pi(\Pi)$ is τ-consistent and $f_\Pi(\Pi')$ is τ'-consistent. Then, from Lemma 8, $d^{KM}(f_\Pi(\Pi), f_\Pi(\Pi')) \leq \epsilon$ implies $d^{MS}(\Pi, \Pi') \leq \epsilon$. Lemma 8 also establishes Theorem 1 part 2 because its proof does not rely on any other properties of f_Π, e.g., the common support condition. Specifically, if μ is Π-consistent and μ' is Π'-consistent, then there exists a Π-labeling τ and a Π'-labeling τ' such that μ is τ-consistent and μ is τ'-consistent. Then, from Lemma 8, $d^{KM}(\mu, \mu') \leq \epsilon$ implies $d^{MS}(\Pi, \Pi') \leq \epsilon$.

A Proofs of Lemmas

A.1 Lemma 1

Observe that $\Pi'_i(s) = (\tau')^{-1}(\{\tau'_i(s)\} \times T_{-i})$ because τ' is a Π'-labeling. Hence, for an arbitrary event $E \subseteq S$,

$$P(E \cap \Pi'_i(s)) = \mu'(E \times \{\tau'_i(s)\} \times T_{-i})$$

because μ' is τ'-consistent and τ' is a Π'-labeling. Moreover,

$$\mu'(E \times \{\tau'_i(s)\} \times T_{-i}) = \mu'(E \times \{\tau'_i(s)\} \times \tau'_{-i}(E))$$

because

$$\mu'(E \times \{\tau'_i(s)\} \times (T_{-i} \backslash \tau'_{-i}(E))) = P(E \cap (\tau')^{-1}(\{\tau'_i(s)\} \times T_{-i} \backslash \tau'_{-i}(E))) = P(E \cap \emptyset) = 0.$$

It follows that

$$P(E|\Pi_i'(s)) = \frac{P(E \cap \Pi_i'(s))}{P(S \cap \Pi_i'(s))} = \frac{\mu'(E \times \{\tau_i'(s)\} \times \tau_{-i}'(E))}{\mu'(S \times \{\tau_i'(s)\} \times T_{-i})} = \mu((\iota \times \tau')(E)|\tau_i'(s)).$$

A.2 Lemma 2

If $(s, \tau'(s)) \in I_{\Pi,\Pi'}(\epsilon) \times \tau'(I_{\Pi,\Pi'}(\epsilon))$, then $\tau_i(s) = \tau_i'(s) = t_i$ for all i by $0 < \epsilon < 1/2$ and the common support condition. Suppose, towards contradiction, that $|\mu(E \times F|t_i) - \mu'(E \times F|t_i)| > \epsilon$ for some event $E \times F \subseteq S \times T$ and some player i. Then, for $G = E \cap (\tau')^{-1}(F)$, it must be that

$$|P(G|\Pi_i(s)) - P(G|\Pi_i'(s))| > \epsilon$$

because τ is a Π-labeling and τ' is a Π'-labeling. We show that this cannot occur and hence $(s, \tau'(s)) \in A_{\mu,\mu'}(\epsilon)$.

Towards contradiction, observe that

$$\begin{aligned} P(G|\Pi_i(s)) - P(G|\Pi_i'(s)) = \quad & P((\Pi_i(s)\backslash\Pi_i'(s)) \cap G|\Pi_i(s)) - P((\Pi_i'(s)\backslash\Pi_i(s)) \cap G|\Pi_i'(s)) + \\ & P((\Pi_i'(s) \cap \Pi_i(s)) \cap G|\Pi_i(s)) - P((\Pi_i'(s) \cap \Pi_i(s)) \cap G|\Pi_i'(s)) \\ \leq \quad & P(\Pi_i(s)\backslash\Pi_i'(s) \cap G|\Pi_i(s)) + \\ & P((\Pi_i'(s) \cap \Pi_i(s)) \cap G|\Pi_i(s)) - P((\Pi_i'(s) \cap \Pi_i(s)) \cap G|\Pi_i'(s)), \end{aligned}$$

where the inequality follows from $P(\Pi_i'(s)\backslash\Pi_i(s) \cap G|\Pi_i'(s)) \geq 0$. Moreover,

$$\begin{aligned} |P((\Pi_i'(s) \cap \Pi(s)) \cap G|\Pi_i(s)) - P((\Pi_i'(s) \cap \Pi_i(s)) \cap G|\Pi_i'(s))| = \\ |\frac{(P(\Pi_i'(s)) - P(\Pi_i(s)))P(\Pi_i'(s) \cap \Pi_i(s) \cap G)}{P(\Pi_i(s))P(\Pi_i'(s))}| \leq \\ |\frac{(P(\Pi_i'(s)) - P(\Pi_i(s)))P(\Pi_i'(s) \cap \Pi_i(s))}{P(\Pi_i(s))P(\Pi_i'(s))}| = \\ |P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s))|. \end{aligned}$$

Hence,

$$\begin{aligned} P(G|\Pi_i(s)) - P(G|\Pi_i'(s)) \leq \\ P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s)) + \max\{P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i(s)) - P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s)), 0\} = \\ \max\{1 - P(\Pi_i'(s) \cap \Pi_i(s)|\Pi_i'(s)), P(\Pi_i(s)\backslash\Pi_i'(s)|\Pi_i(s))\}. \end{aligned}$$

From $s \in I_{\Pi,\Pi'}(\epsilon)$, we have $P(\Pi_i(s)\backslash\Pi'_i(s)|\Pi_i(s)) \leq \epsilon$ and $P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s)) \leq \epsilon$. Moreover, $P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s))+P(\Pi'_i(s)\cap\Pi_i(s)|\Pi'_i(s)) = 1$ implies $P(\Pi'_i(s)\cap\Pi_i(s)|\Pi'_i(s)) \geq 1-\epsilon$ by $P(\Pi'_i(s)\backslash\Pi_i(s)|\Pi'_i(s)) \leq \epsilon$. So,

$$P(G|\Pi_i(s)) - P(G|\Pi'_i(s)) \leq \epsilon.$$

A symmetric argument ensures

$$P(G|\Pi'_i(s)) - P(G|\Pi_i(s)) \leq \epsilon.$$

Hence,

$$|P(G|\Pi_i(s)) - P(G|\Pi'_i(s))| \leq \epsilon,$$

our desired contradiction.

A.3 Lemma 3

We show by induction that $s \in \cap_{m\geq 1}(B^p_{\Pi'})^m(E) = C^p_{\Pi'}(E)$ implies $(s, \tau'(s)) \in \cap_{m\geq 1}(B^p_{\mu'})^m(E\times \tau'(E)) = C^p_{\mu'}(E \times \tau'(E)) = C^p_{\mu'}((\iota \times \tau')(E))$. For the base case, take $s \in B^p_{\Pi'}(E)$. Then, for all i,

$$\mu(E \times \tau'(E)|\tau'_i(s)) = P(E|\Pi'_i(s)) \geq p,$$

where the first equality is from Lemma 1 and the inequality is from the definition of $B^p_{\Pi'}(E)$. Hence, $(s, \tau'(s)) \in (B^p_{\mu'})(E \times \tau'(E))$.

The induction hypothesis is that if $s \in (B^p_{\Pi'})^m(E)$, then $(s, \tau'(s)) \in (B^p_{\mu'})^m(E \times \tau'(E))$. Take $s \in (B^p_{\Pi'})^{m+1}(E)$. Then, for all i,

$$\mu'((B^p_{\Pi'})^m(E) \times \tau'((B^p_{\Pi'})^m(E))) = P((B^p_{\Pi'})^m(E)|\Pi'_i(s)) \geq p,$$

where the equality again follows from Lemma 1. By the induction hypothesis, if $s \in (B^p_{\Pi'})^m(E)$, then $(s, \tau'(s)) \in (B^p_{\mu'})^m(E \times \tau'(E))$. It follows that

$$(B^p_{\mu'})^m(E \times \tau'(E)) \supseteq (B^p_{\Pi'})^m(E) \times \tau'((B^p_{\Pi'})^m(E))$$

and, therefore, for all i,

$$\mu'((B^p_{\mu'})^m(E \times \tau'(E))|\tau'_i(s)) \geq \mu((B^p_{\Pi'})^m(E) \times \tau'((B^p_{\Pi'})^m(E))|\tau'_i(s)) \geq p.$$

Hence, $(s, \tau'(s)) \in (B^p_{\mu'})^{m+1}(E \times \tau'(E))$.

A.4 Lemma 4

If $d^{MS}(\Pi, \Pi') \leq \epsilon$, then $d_1^{MS}(\Pi, \Pi') \leq \epsilon$ and

$$P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)) \geq 1 - \epsilon.$$

Because μ' is τ'-consistent,

$$\mu'(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon))) = P((\iota \times \tau')^{-1}(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon))).$$

Hence, it suffices to show

$$(\iota \times \tau')^{-1}(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \supseteq C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon) \tag{2}$$

to establish $d_1^{KM}(\mu, \mu') \leq \epsilon$, i.e.,

$$\mu'(C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon.$$

To prove (2), observe from Lemma 2 that if $s \in I_{\Pi,\Pi'}(\epsilon)$, then $(s, \tau'(s)) \in A_{\mu,\mu'}(\epsilon)$. We claim, also, that if $s \in C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$, then $(s, \tau'(s)) \in C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))$. Take $s \in C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon))$. Then, by Lemma 3, $(s, \tau'(s)) \in C_{\mu'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon) \times \tau'(I_{\Pi,\Pi'}(\epsilon)))$. By Lemma 2, $I_{\Pi,\Pi'}(\epsilon) \times \tau'(I_{\Pi,\Pi'}(\epsilon)) \subseteq A_{\mu,\mu'}(\epsilon)$ and hence $(s, \tau'(s)) \in C_{\mu'}^{1-\epsilon}(A_{\mu,\mu'}(\epsilon))$ because $C_{\mu'}^{1-\epsilon}(E) \subseteq C_{\mu'}^{1-\epsilon}(F)$ for any events $E \subseteq F$. We have thus established (2) and, therefore, $d_1^{KM}(\mu, \mu') \leq \epsilon$. A symmetric argument establishes $d_1^{KM}(\mu', \mu) \leq \epsilon$.

It remains to show that $d_0^{KM}(\mu, \mu') \leq \epsilon$. Let $E \times F \subseteq S \times T$ be an arbitrary event. Then,

$$|\mu(E \times F) - \mu'(E \times F)| = |P(E \cap \tau^{-1}(F)) - P(E \cap (\tau')^{-1}(F))|$$

because μ is τ-consistent and μ' is τ'-consistent. By the triangle inequality, we have

$$\begin{aligned}|P(E \cap \tau^{-1}(F)) - P(E \cap (\tau')^{-1}(F))| \leq &\underbrace{|P(E \cap I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)) - P(E \cap I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F))|}_{(i)} \\ &+ \underbrace{|P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)) - P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F))|}_{(ii)}.\end{aligned}$$

Term (i) equals zero by the common support condition: for every $s \in I_{\Pi,\Pi'}(\epsilon)$, we have $\tau(s) = \tau'(s)$. Hence,

$$\underbrace{\{s \in E \cap I_{\Pi,\Pi'}(\epsilon) : \tau(s) \in F\}}_{= E \cap I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)} = \underbrace{\{s \in E \cap I_{\Pi,\Pi'}(\epsilon) : \tau'(s) \in F\}}_{= E \cap I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F)}.$$

If $d^{MS}(\Pi, \Pi') \leq \epsilon$, then term (ii) is less than ϵ because

$$\max\{P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)), P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F))\} \leq P(S \backslash I_{\Pi,\Pi'}(\epsilon))$$

and

$$1 - P(S \backslash I_{\Pi,\Pi'}(\epsilon)) = P(I_{\Pi,\Pi'}(\epsilon)) \geq P(C_{\Pi'}^{1-\epsilon}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)) \geq 1 - \epsilon.$$

So,

$$P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)) \leq \epsilon$$

and

$$P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F)) \leq \epsilon.$$

Hence,

$$|P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap \tau^{-1}(F)) - P(E \backslash I_{\Pi,\Pi'}(\epsilon) \cap (\tau')^{-1}(F))| \leq \epsilon.$$

We have thus shown

$$|\mu(E \times F) - \mu'(E \times F)| = |P(E \cap \tau^{-1}(F)) - P(E \cap (\tau')^{-1}(F))| \leq \epsilon.$$

Moreover, because $E \times F \subseteq S \times T$ was an arbitrary event, we have established

$$d_0^{KM}(\mu, \mu') = \sup_{E \times F \subseteq S \times T} |\mu(E \times F) - \mu'(E \times F)| \leq \epsilon.$$

A.5 Lemma 5

Because μ' is τ'-consistent and τ' is a Π'-labeling,

$$\mu'((E \times F) \cap (S \times \{\tau_i'(s)\} \times T_{-i})) = P((\iota \times \tau')^{-1}(E \times F) \cap \Pi_i'(s))$$

and

$$\mu'(S \times \{\tau_i'(s)\} \times T_{-i}) = P(S \cap (\tau_i')^{-1}(s)) = P(\Pi_i'(s)).$$

It follows from the definition of conditional probability that

$$\begin{aligned}\mu'(E \times F | \tau_i'(s)) &= \frac{\mu'((E \times F) \cap (S \times \{\tau_i'(s)\} \times T_{-i}))}{\mu'(S \times \{\tau_i'(s)\} \times T_{-i})} \\ &= \frac{P((\iota \times \tau')^{-1}(E \times F) \cap \Pi_i'(s))}{P(\Pi_i'(s))} \\ &= P((\iota \times \tau')^{-1}(E \times F) | \Pi_i'(s)).\end{aligned}$$

A.6 Lemma 6

We prove the contrapositive of the statement of the Lemma. Suppose $s \notin I_{\Pi,\Pi'}(\epsilon)$. Then, without loss of generality, there exists a player i and a state $s \in S$ under which

$$P(\Pi_i(s) \backslash \Pi_i'(s) | \Pi_i(s)) > \epsilon.$$

But,

$$\mu(\Pi_i(s) \backslash \Pi_i'(s) \times T | \tau_i(s)) = P(\Pi_i(s) \backslash \Pi_i'(s) | \Pi_i(s))$$

and

$$\mu'(\Pi_i(s) \backslash \Pi_i'(s) \times T | \tau_i'(s)) = 0.$$

So, if $\tau_i'(s) = \tau_i(s) = t_i$, then

$$|\mu(\Pi_i(s) \backslash \Pi_i'(s) \times T | t_i) - \mu'(\Pi_i(s) \backslash \Pi_i'(s) \times T | t_i)| > \epsilon.$$

Hence, $(s, \tau'(s)) \notin A_{\mu,\mu'}(\epsilon)$. If $\tau_i'(s) \neq \tau_i(s)$, on the other hand, then $(s, \tau'(s)) \notin A_{\mu,\mu'}(\epsilon)$ by definition.

A.7 Lemma 7

We show by induction that $(s, t') \in \cap_{m \geq 1} (B^p_{\mu'})^m(E) = C^p_{\mu'}(E)$ and $\tau'(s) = t'$ implies $s \in \cap_{m \geq 1} (B^p_{\Pi'})^m((\iota \times \tau')^{-1}(E)) = C^p_{\Pi'}((\iota \times \tau')^{-1}(E))$. For the base case, take $(s, t') \in B^p_{\mu'}(E)$ and $\tau'(s) = t'$. Then, for all i,

$$P((\iota \times \tau')^{-1}(E) | \Pi_i'(s)) = \mu'(E | \tau_i'(s)) \geq p,$$

where the equality follows from Lemma 5. It follows that $(s, t') \in B^p_{\Pi'}((\iota \times \tau')^{-1}(E))$.

The induction hypothesis is that if $(s, t') \in (B^p_{\mu'})^m(E)$ and $\tau'(s) = t'$, then $s \in (B^p_{\Pi'})^m((\iota \times \tau')^{-1}(E))$. Take $(s, t') \in (B^p_{\mu'})^{m+1}(E)$. Then, for all i,

$$P((\iota \times \tau')^{-1}((B^p_{\mu'})^m(E)) | \Pi_i'(s)) = \mu'((B^p_{\mu'})^m(E) | \tau_i'(s)) \geq p,$$

where the equality again follows from Lemma 5. By the induction hypothesis, if $(s, t') \in (B^p_{\mu'})^m(E)$, then $s \in (B^p_{\Pi'})^m((\iota \times \tau')^{-1}(E))$. It follows that

$$(B^p_{\Pi'})^m((\iota \times \tau')^{-1}(E)) \supseteq (\iota \times \tau')^{-1}((B^p_{\mu'})^m(E))$$

and, therefore,

$$P((B^p_{\Pi'})^m((\iota \times \tau')^{-1}(E))|\Pi'_i(s)) \geq P((\iota \times \tau')^{-1}((B^p_{\mu'})^m(E))|\Pi'_i(s)) \geq p.$$

Hence, $s \in (B^p_{\Pi'})^{m+1}((\iota \times \tau')^{-1}(E))$.

A.8 Lemma 8

If $d^{KM}(\mu, \mu') \leq \epsilon$, then $\mu'(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon$. Because μ' is τ'-consistent,

$$P((\iota \times \tau')^{-1}(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon))) = \mu'(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon)) \geq 1 - \epsilon.$$

To show that

$$P(C^{1-\epsilon}_{\Pi'}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)) \geq 1 - \epsilon,$$

i.e., $d^{MS}_1(\Pi, \Pi') \leq \epsilon$, it thus suffices to show that $s \in (\iota \times \tau')^{-1}(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon))$ implies $s \in C^{1-\epsilon}_{\Pi'}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)$. By Lemma 6, if $s \in (\iota \times \tau')^{-1}(A_{\mu,\mu'}(\epsilon))$, then $s \in I_{\Pi,\Pi'}(\epsilon)$. Moreover, Lemma 7 establishes that if $(s, t') \in C^p_{\mu'}(A_{\mu,\mu'}(\epsilon))$ and $\tau'(s) = t'$, then $s \in C^p_{\Pi'}((\iota \times \tau')^{-1}(A_{\mu,\mu'}(\epsilon)))$. Because $(\iota \times \tau')^{-1}(A_{\mu,\mu'}(\epsilon)) \subseteq I_{\Pi,\Pi'}(\epsilon)$ from Lemma 6 and $C^p_{\Pi'}(E) \subseteq C^p_{\Pi'}(F)$ for any events $E \subseteq F$, we thus have $s \in C^p_{\Pi'}(I_{\Pi,\Pi'}(\epsilon))$. It follows from the definition of $(\iota \times \tau')^{-1}$ that $s \in (\iota \times \tau')^{-1}(C^{1-\epsilon}_{\mu'}(A_{\mu,\mu'}(\epsilon)) \cap A_{\mu,\mu'}(\epsilon))$ implies $s \in C^{1-\epsilon}_{\Pi'}(I_{\Pi,\Pi'}(\epsilon)) \cap I_{\Pi,\Pi'}(\epsilon)$. Finally, a symmetric argument ensures $d^{MS}_1(\Pi', \Pi) \leq \epsilon$. Thus, $d^{MS}(\Pi, \Pi') = \max\{d^{MS}_1(\Pi, \Pi'), d^{MS}_1(\Pi', \Pi)\} \leq \epsilon$.

References

Gensbittel, F., Peski, M., and Renault, J. "Value-Based Distance Between Information Structures." *Theoretical Economics*, Vol. 17 (2022), pp. 1225–1267.

Kajii, A. and Morris, S. "Payoff continuity in incomplete information games." *Journal of Economic Theory*, Vol. 82 (1998), pp. 267–276.

Monderer, D. and Samet, D. "Approximating common knowledge with common beliefs." *Games and Economic Behavior*, Vol. 1 (1989), pp. 170–190.

Monderer, D. and Samet, D. “Proximity of Information in Games with Incomplete Information.” *Mathematics of Operations Research*, Vol. 21 (1996), pp. 707–725.

Morris, S. “Approximate common knowledge revisited.” *International Journal of Game Theory*, Vol. 28 (1999), pp. 385–408.

Rothschild, C.G. “Payoff continuity in incomplete information games: A comment.” *Journal of Economic Theory*, Vol. 120 (2005), pp. 270–274.

Rubinstein, A. “The Electronic Mail Game: Strategic Behavior Under “Almost Common Knowledge”.” *The American Economic Review*, Vol. 79 (1989), pp. 385–391.