MTH 372: Assignment I

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Ans 1)

Plotted the graphs using

```r
## Ans1
#a)
rm(list=ls())

set.seed(123) # set seed for reproducibility
sample_size <- 1000 # specify the sample size

data1 = rexp(1000,1)
plot(data1, main="Exp dist , lambda =1", ylab="Y",xlab="X",)

data2 = rexp(1000,2)
plot(data2, main="Exp dist , lambda =2", ylab="Y",xlab="X",)

data3 = rexp(1000,3)
plot(data3, main="Exp dist , lambda =3", ylab="Y",xlab="X",)

data4 = rexp(1000,4)
plot(data4, main="Exp dist , lambda =4", ylab="Y",xlab="X",)
```

R Global Environment

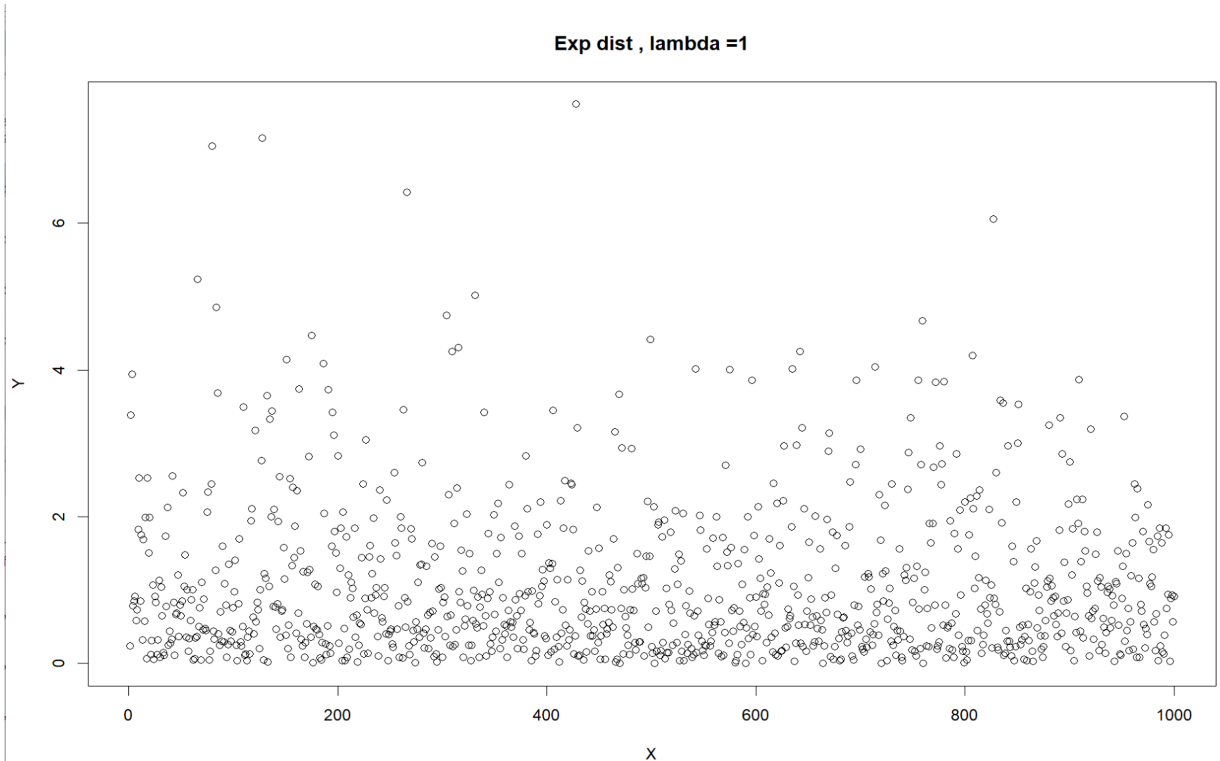
Values	
data1	num [1:1000] 0.8435 0.5766 1.3291 0.0316 0.0562 ...
data2	num [1:1000] 0.1117 0.0588 1.5258 0.2453 1.0191 ...
data3	num [1:1000] 0.68054 0.00159 0.83791 0.22473 0.03614 ...
data4	num [1:1000] 0.22 0.021 0.14 0.195 0.24 ...
sample_size	1000

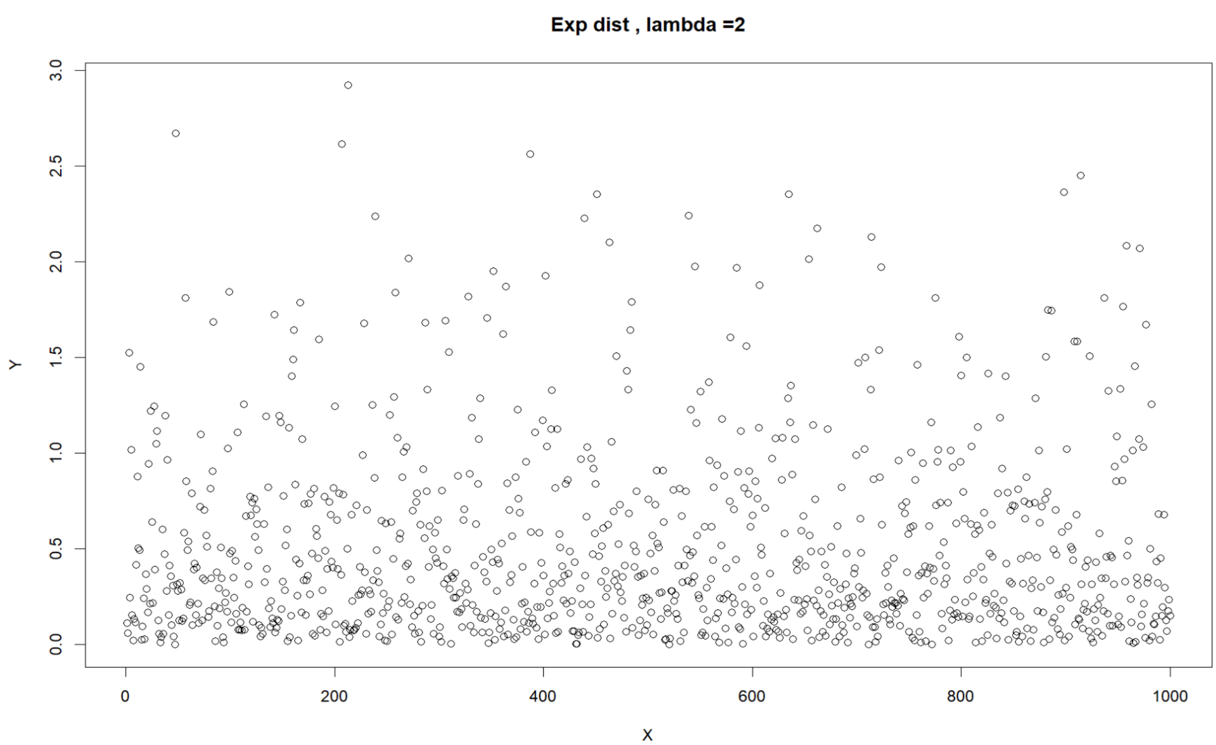
The rexp provides a set of random numbers obtained using the exponential distribution.

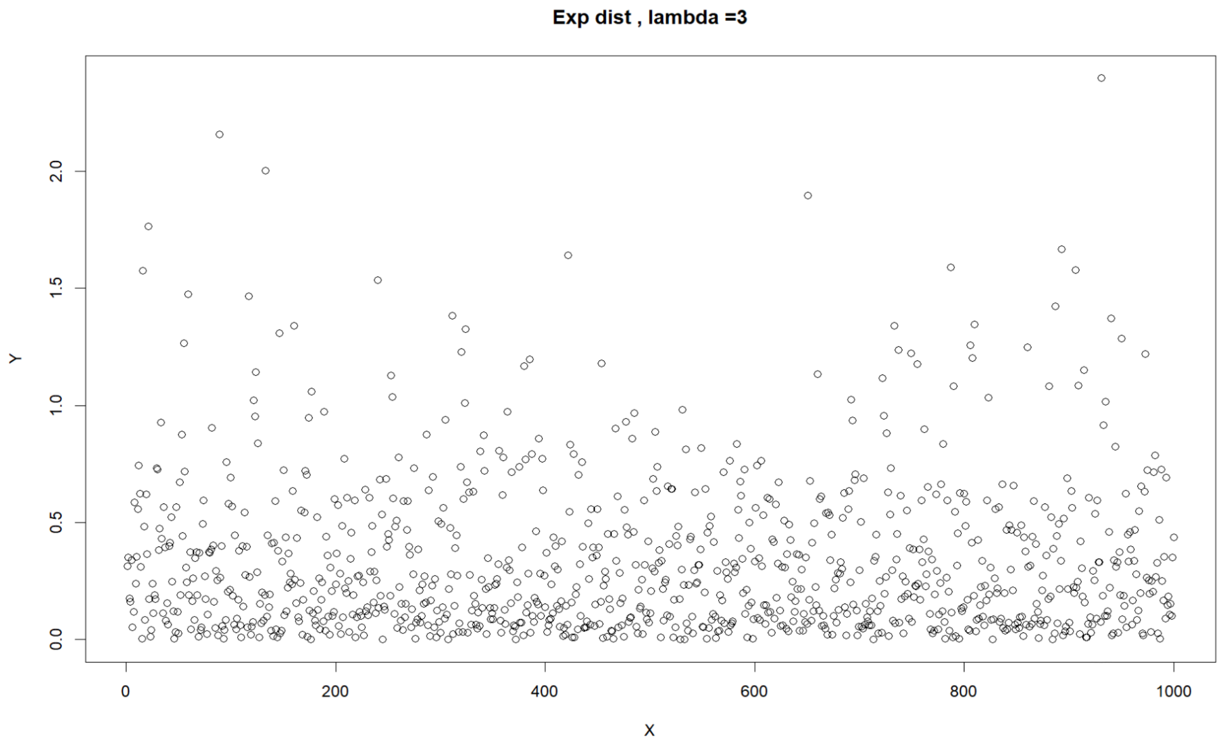
Exponential distribution

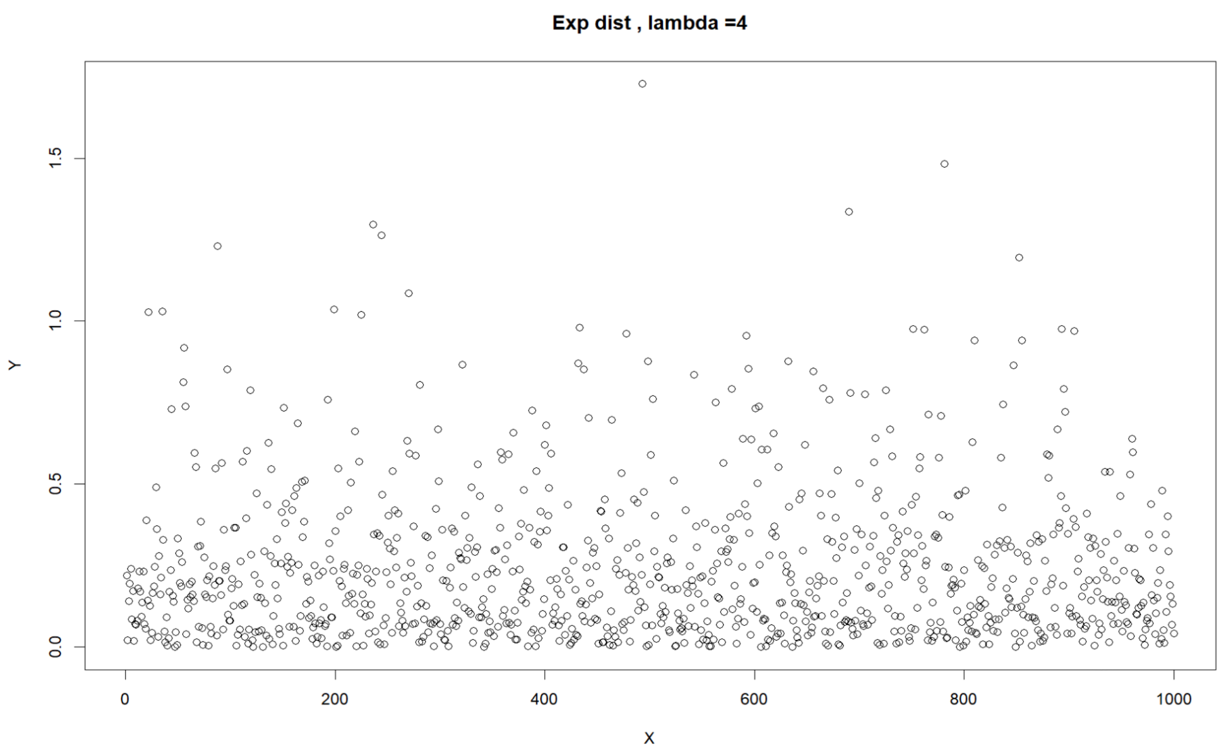
$$f(x_i,\lambda) = \left\{ \lambda e^{-\lambda x} \right\}$$

The graphs that we get are


Exp dist , lambda =1
Y
0
2
4
6
0
200
400
600
800
1000
X


Exp dist , lambda =2
Y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0
200
400
600
800
1000
X


Exp dist , lambda =3
Y
0.0
0.5
1.0
1.5
2.0
X
0
200
400
600
800
1000


Exp dist , lambda =4
Y
0.0
0.5
1.0
1.5
X
0
200
400
600
800
1000

1 b)
Now, we have to find the maximum likelihood estimate of the unknown parameter.
The function we have used

```
le.exp<-function(lambda,data){
  data_sum <- sum(data)
  val = 1000*(log(lambda)) - lambda*data_sum
  return(val)
}
```

Here our function is likelihood function and we have to use the result to calculate maximum likelihood.
Here, we are returning the log-likelihood function as given the pages attached , here n =1000
Where μ is the mean of the data.
Since we are using nlimb, which gives the minimum value, we will multiply the result of function with (-1) and then pass it though nlimb to get the estimate.

Exponential distribution

$$f(x_i, \lambda) = \left\{ \lambda e^{-\lambda x} \right\}$$

Likelihood function

$$L = \prod_{i=1}^{n} f(x_i ; \lambda)$$

$$L = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ln L = \ln \left(\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \right) \quad \text{[log likelihood]}$$

$$= n \ln \lambda + \left(-\lambda \sum_{i=1}^{n} x_i \right)$$

To find Maximum Likelihood estimate

$$\frac{dL}{d\lambda} = 0 \quad \text{or} \quad \frac{d \ln L}{d\lambda} = 0$$

$$\frac{d}{d\lambda} \left(n \ln \lambda - \lambda \sum_{i=1}^{n} x_i \right) = 0$$

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\lambda = \frac{n}{\sum_{i=1}^{n} x_i} \rightarrow \lambda = \frac{1}{\text{mean}}$$

According to the above proof, the value of estimate of parameter should be equal to 1/ mean(data)

For Lambda =1 , we are getting estimate = 0.9708938 in all 3 cases

```
> ## Lambda = 1
> lambda_mom = 1/mean(data1)
> cat("Parameter using Method of Moments (MOM) when lambda = 1 is " , lambda_mom)
Parameter using Method of Moments (MOM) when lambda = 1 is  0.9708933>
>
> mlle.exp <-nlminb(1,objective=le.exp_nlminb,hessian=T,lower=0,upper=1, data=data1)
>
> # Print the MLE estimate
> cat("MLE estimate: for value-> 1 and lambda=1 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> 1 and lambda=1 is   0.9708938
>
> mlle.exp <-nlminb(0.5,objective=le.exp_nlminb,hessian=T,lower=0,upper=1, data=data1)
> cat("MLE estimate: for value-> 0.5 and lambda=1 is  ", (mlle.exp)$par, "\n" )
MLE estimate: for value-> 0.5 and lambda=1 is   0.9708938
>
> mlle.exp <-nlminb(lambda_mom,objective=le.exp_nlminb,hessian=T,lower=0,upper=1, data=data1)
> cat("MLE estimate: for value-> MOM and lambda=1 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> MOM and lambda=1 is   0.9708933
```

For Lambda = 2 , we are getting estimate = 2.012931 in all 3 cases

```
> #### Lambda = 2
> lambda_mom = 1/mean(data2)
> cat("Parameter using Method of Moments (MOM) when lambda = 2 is " , lambda_mom)
Parameter using Method of Moments (MOM) when lambda = 2 is  2.012931>
> mlle.exp <-nlminb(1,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data2)
>
> # Print the MLE estimate
> cat("MLE estimate: for value-> 1 and lambda=2 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> 1 and lambda=2 is   2.012931
>
> mlle.exp <-nlminb(0.5,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data2)
> cat("MLE estimate: for value-> 0.5 and lambda=2 is  ", (mlle.exp)$par, "\n" )
MLE estimate: for value-> 0.5 and lambda=2 is   2.012932
>
> mlle.exp <-nlminb(lambda_mom,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data2)
> cat("MLE estimate: for value-> MOM and lambda=2 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> MOM and lambda=2 is   2.012931
```

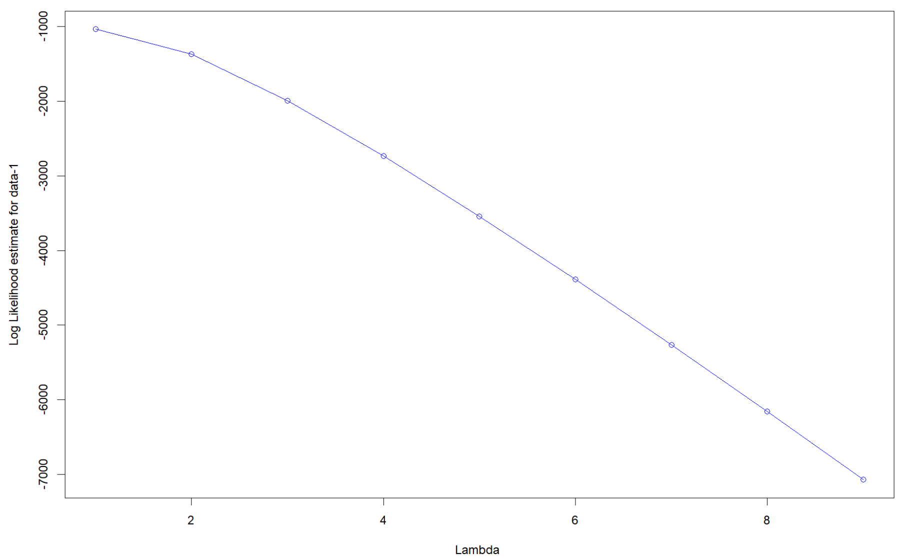
For lambda = 3 , we are getting estimate = 2.92968 in all 3 cases

```
> #### Lambda = 3
> lambda_mom = 1/mean(data3)
> cat("Parameter using Method of Moments (MOM) when lambda = 3 is " , lambda_mom)
Parameter using Method of Moments (MOM) when lambda = 3 is  2.929678>
> mlle.exp <-nlminb(1,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data3)
>
> # Print the MLE estimate
> cat("MLE estimate: for value-> 1 and lambda=3 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> 1 and lambda=3 is   2.92968
>
> mlle.exp <-nlminb(0.5,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data3)
> cat("MLE estimate: for value-> 0.5 and lambda=3 is  ", (mlle.exp)$par, "\n" )
MLE estimate: for value-> 0.5 and lambda=3 is   2.929678
>
> mlle.exp <-nlminb(lambda_mom,objective=le.exp_nlminb,hessian=T,lower=0,upper=4, data=data3)
> cat("MLE estimate: for value-> MOM and lambda=3 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> MOM and lambda=3 is   2.929678
```

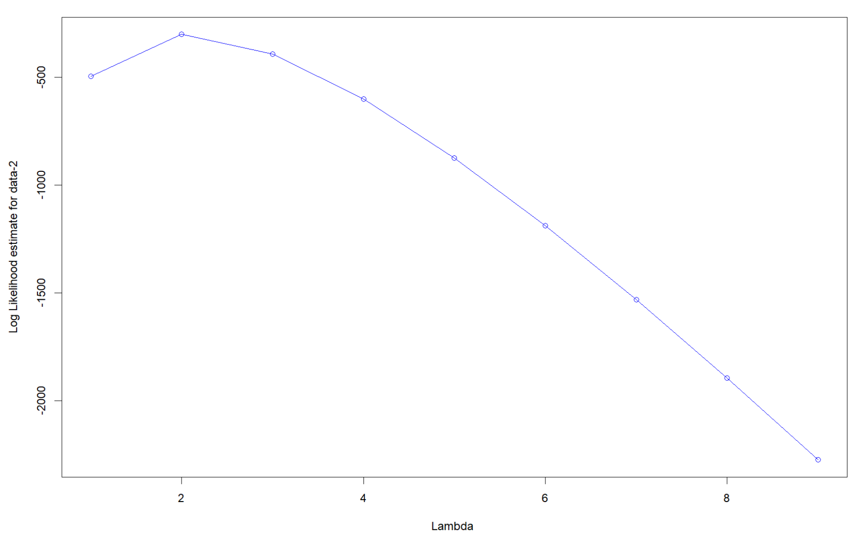
For lambda = 4 , we are getting estimate = 4.200171 in all 3 cases

```
> #### Lambda = 4
> lambda_mom = 1/mean(data4)
> cat("Parameter using Method of Moments (MOM) when lambda = 4 is " , lambda_mom)
Parameter using Method of Moments (MOM) when lambda = 4 is  4.200171>
> mlle.exp <-nlminb(1,objective=le.exp_nlminb,hessian=T,lower=0,upper=5, data=data4)
>
> # Print the MLE estimate
> cat("MLE estimate: for value-> 1 and lambda=4 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> 1 and lambda=4 is   4.200171
>
> mlle.exp <-nlminb(0.5,objective=le.exp_nlminb,hessian=T,lower=0,upper=5, data=data4)
> cat("MLE estimate: for value-> 0.5 and lambda=4 is  ", (mlle.exp)$par, "\n" )
MLE estimate: for value-> 0.5 and lambda=4 is   4.200171
>
> mlle.exp <-nlminb(lambda_mom,objective=le.exp_nlminb,hessian=T,lower=0,upper=5, data=data4)
> cat("MLE estimate: for value-> MOM and lambda=4 is  ", (mlle.exp)$par, "\n")
MLE estimate: for value-> MOM and lambda=4 is   4.200171
```

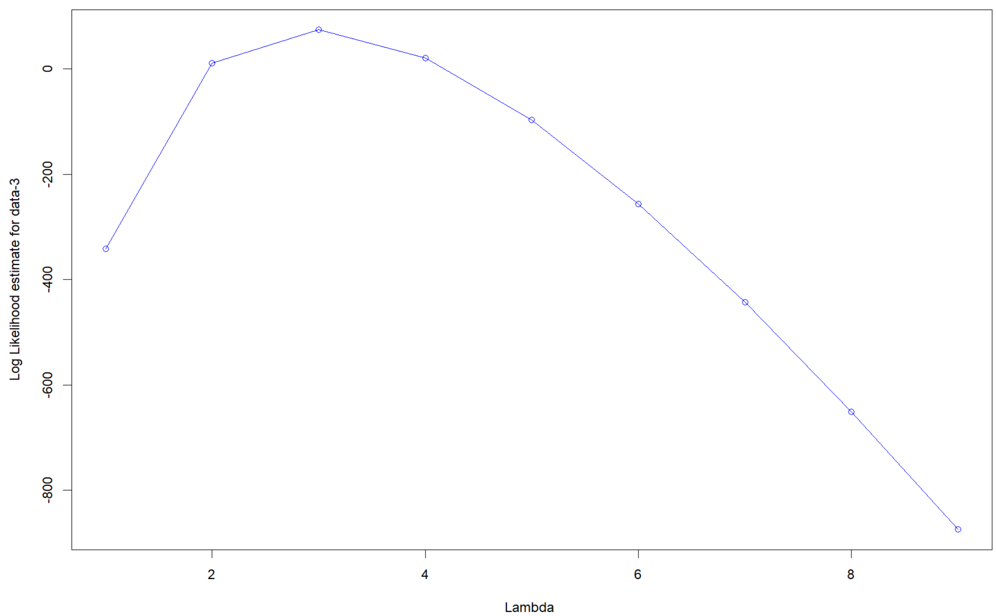
1c) Verifying the above results graphically
Here, as we can see that, for data generated by lambda = 1, we are getting maximum value for lambda = 1.



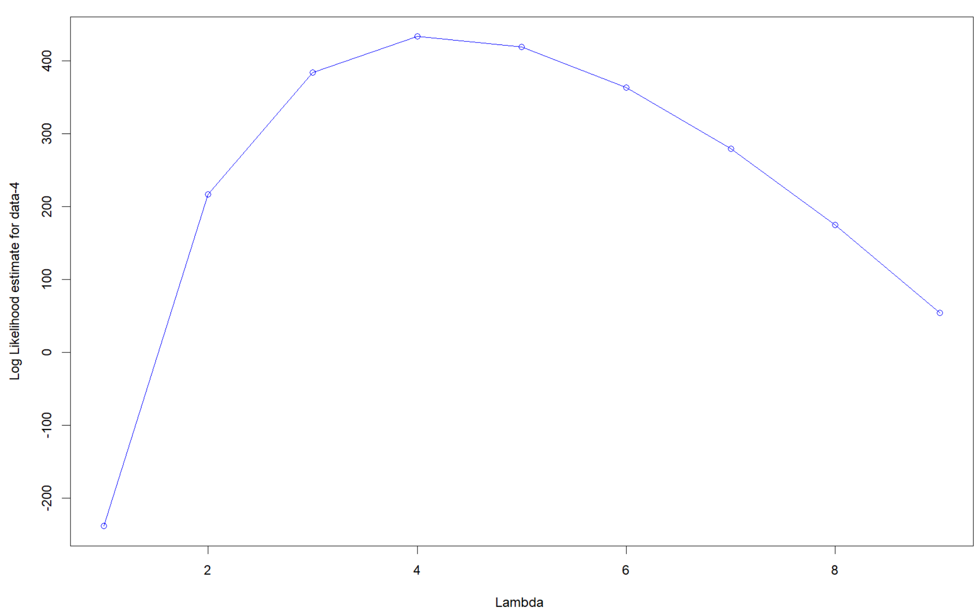
Here, as we can see that, for data generated by lambda = 2, we are getting maximum value for lambda = 2.



Here, as we can see that, for data generated by lambda = 3, we are getting maximum value for lambda = 3.



Here, as we can see that, for data generated by lambda = 4, we are getting maximum value for lambda = 4.



Ans 2)

Here we have μ and σ^2 as our Maximum parameters.

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Here our parameters are μ and σ^2

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}(x-\mu)^2/\sigma^2}$$

Liklihood function

$$L(\theta) = \prod_{i=1}^{n} f(x_i) \qquad \text{(iid variables)}$$

$$(\text{where } \theta = \mu, \sigma^2)$$

$$= \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{\frac{-1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 + n\mu^2 - 2\mu\sum_{i=1}^{n} x_i\right)}$$

log liklihood function for easier Calulations

$$\ln(L(\theta)) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 + n\mu^2 - 2\mu\sum_{i=1}^{n} x_i\right)$$

To Maximize

$$\frac{d(\ln(L(\theta))}{d\mu} = 0 \quad \text{and} \quad \frac{d(\ln(L(\theta)))}{d(\sigma^2)} = 0$$

Now we have
1st condition

$$\frac{d(\ln(L(\theta)))}{d\mu} = \sum_{i=1}^{n} \frac{x_i}{\sigma^2} - \frac{2n\mu}{2\sigma^2} = 0$$

$$\Rightarrow n\mu = \Sigma x_i$$

$$\mu = \frac{\Sigma x_i}{n}$$ → This is sample mean

2nd condition

$$\frac{d(\ln(L(\theta)))}{d\sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\left(\sum_{i=1}^{n} x_i^2 + n\mu^2 - 2\mu\sum_{i=1}^{n} x_i\right) = 0$$

$$n = \sum_{i=1}^{n} \frac{(x_i - \mu)}{\sigma^2}$$

$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)}{n}$$

Here μ is the sample mean
σ^2 is the sample Variance.

Now to obtain MLE of the parameters,
we will make use of Hessian Matrix

$$H_{f(x,y)} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

In our case It will be

Considering

$$\ln(L(\theta)) = \ell(\theta)$$
where $\theta = \{\mu, \sigma^2\}$

$$H = \begin{bmatrix} \frac{\partial^2 \ell(\theta)}{\partial \mu^2} & \frac{\partial^2 \ell(\theta)}{\partial \mu \, \partial \sigma^2} \\ \frac{\partial^2 \ell(\theta)}{\partial \sigma^2 \, \partial \mu} & \frac{\partial^2 \ell(\theta)}{\partial (\sigma^2)^2} \end{bmatrix}$$

$$\frac{\partial^2 \ell(\theta)}{\partial \mu^2} = \frac{-n}{\sigma^2}$$

$$\frac{\partial^2 \ell(\theta)}{\partial (\sigma^2)^2} = \frac{+n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$\frac{\partial^2 (l(\theta))}{\partial \sigma^2 \partial \mu} = \frac{-1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu)$$

$$\frac{\partial^2 (l(\theta)}{\partial \mu \partial \sigma^2} = -\frac{1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu)$$

So we get Hessian Matrix

$$H = \begin{bmatrix} \frac{-n}{\sigma^2} & \frac{-1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu) \\ \frac{-1}{(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu) & \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^{n} (x_i - \mu)^2 \end{bmatrix}$$

Since we want Maxima (MLE) of parameters

Double derivative should be -ve

$$\frac{\partial^2 l(\theta)}{\partial \mu^2} < 0 \Rightarrow \frac{-n}{\sigma^2} < 0$$

$\frac{-n}{\sigma^2}$ is always -ve, as n and σ^2 are +ve

Similiarly for

$\frac{\partial \left[l(\theta) \right.}{\partial(\sigma^2)^2} < 0$ should be True

$\frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^{n}(x_i - \mu)^2$ should be -ve

Now we have $\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n} = \sigma^2$

$\Rightarrow \frac{n}{2\sigma^4} - \frac{n\sigma^2}{\sigma^6}$ should be -ve

$\Rightarrow \frac{n}{2\sigma^4} - \frac{n}{\sigma^4}$ should be -ve

which is true as $\frac{n}{2\sigma^4}\left(\frac{-1}{2}\right)$ is -ve.

$\therefore \frac{\partial (l(\theta))}{\partial \mu^2} < 0$ and $\frac{\partial (l(\theta))}{\partial (\sigma^2)^2} < 0$

$\therefore$ μ and σ^2 are our ~~parameters~~
Maximum parameters

Now we have to find μ and σ^2 using R.

Here, we will construct a function.

```
le.normal_pos <- function(theta_param, data){
  mean <- theta_param[1]
  var <- theta_param[2]
  ll <- -(0.5)*length(data)*log(2*pi*var) - (0.5/var)*sum((data - mean)^2)
  return(ll)
}
```

Here, I have used the log-likelihood function that I have written on the pages attached above.

This provides the log likelihood of estimates of parameters.
When passing in the optim function, we will multiply the result of this function by (-1) to find the maximum value of the parameters.

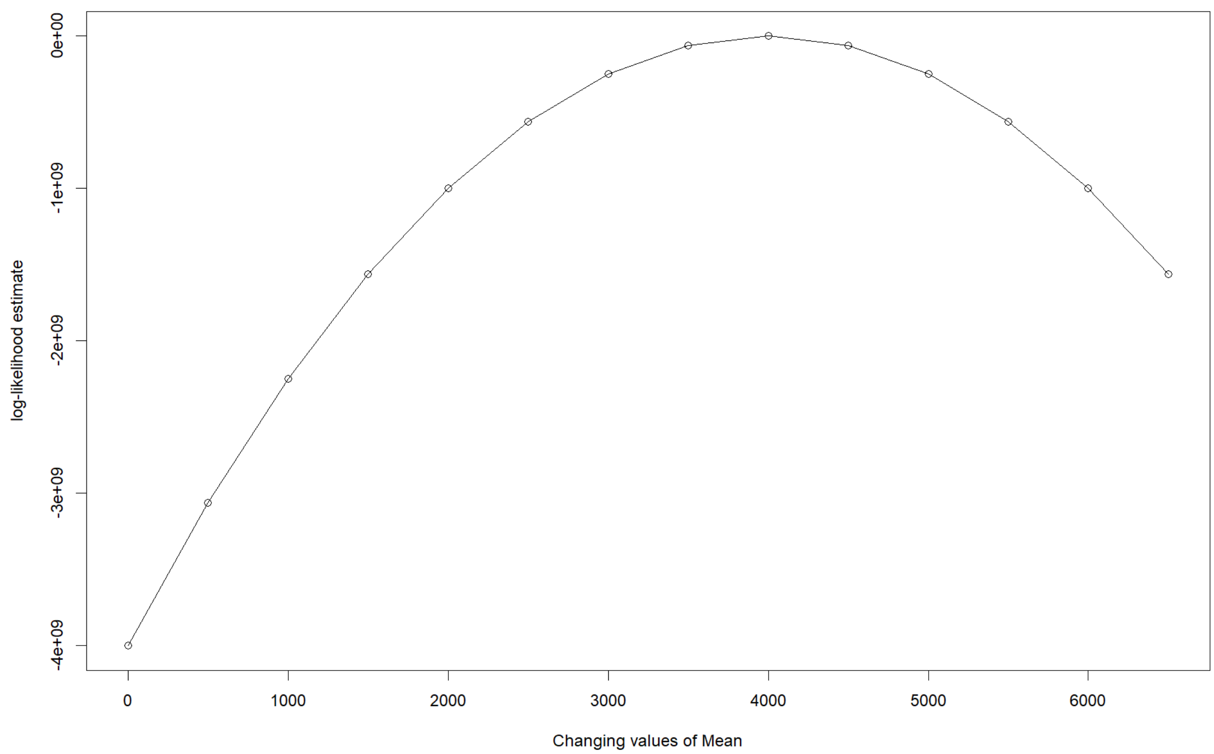
```
> mean_data <- theta_param[1]
> sigma2 <- theta_param[2]
>
>
> cat( "Mean is " , mean_data )
Mean is  4000.044> cat ("variance is " , sigma2 )
variance is  15.55781>
```

Our Mean is 4000.044, and variance is 15.55781.

B)

Here, I changed the mean value in each iteration of the log-likelihood function while keeping the variance constant $\sigma^2 = 1$. As we can see that we are getting the , maximum value of likelihood in case of

$\mu = 4000.044$



C).

Now we have to find ML estimate of exp(−μ).

This will be the ML estimate of exp(-4000.044) = $e^{(-4000.044)}$.

Here using the invariance property, as we have μ as a MLE, then $f_2(\mu)$ will also be MLE. Here $f_2(\mu) = e^{(-4000.044)}$.