Working Principles Of Proof Assistant

And Formalization Of Some Proofs In Agda

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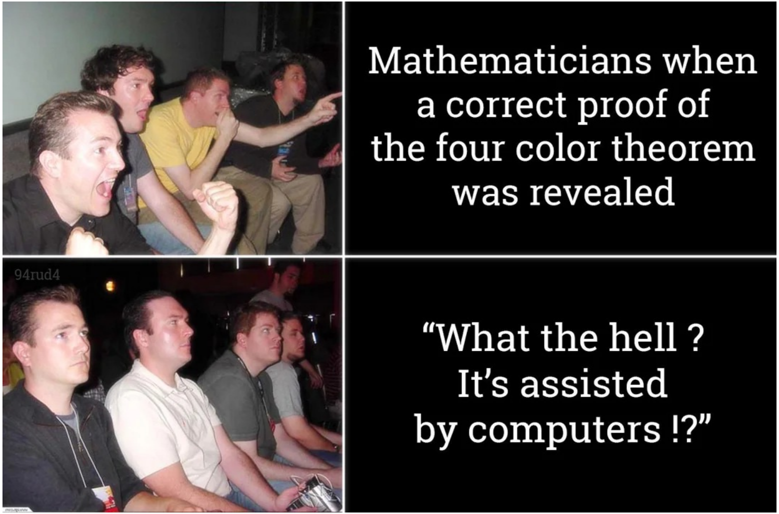
Proof Assistants

What are proof assistant

Proof assistant, are software more specifically a type of programming language thats allows us to formalize mathematical proofs in computer for digital verification.

Need of digital verification

- Fast and Efficient
- Many cases can be explored which would take mathematicians long time
  ex: The Kepler Conjecture's proof , which was so complex that verifying it manually would take 20 person-years, but proof assistants made this verification feasible and fast.
- What if you don't use proof assistants? ABC conjecture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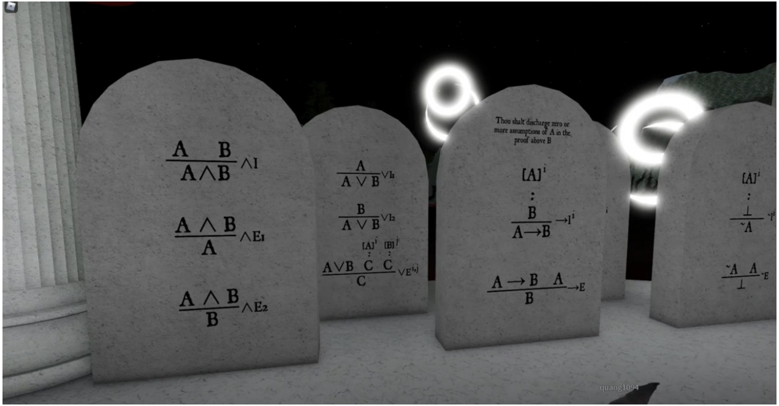
Mathematicians when a correct proof of the four color theorem was revealed
"What the hell ? It's assisted by computers !?"

Foundations

Natural Deduction

- **Natural Deduction** is a rule-based system for deriving conclusions from assumptions in logic.
- Instead of using exhaustive truth tables, proofs are built step-by-step using inference rules.
- Example: Proving from $A \wedge (A \to \perp)$ that $\perp$ (contradiction) can be derived.


Thou shalt discharge zero or more assumptions of A in the proof above B
quang1094

Intuitionistic Logic

- **Intuitionistic Logic** Also called Constructive Logic, reflects principles of constructive mathematics, where a statement is only true if a proof can be constructed.
- Omits some classical logic rules, such as the Law of Excluded Middle.
- Stronger requirement: to prove existence, a method or algorithm must be given.
- Proof assistants leverage this constructive approach for digital verification.

Introduction Rules	Elimination Rules
$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge\mathrm{I}$	$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\wedge\mathrm{E_L} \qquad \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\wedge\mathrm{E_R}$
$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee\mathrm{I_L} \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee\mathrm{I_R}$	$\dfrac{\Gamma \vdash A \vee B \qquad \Gamma, u{:}A \vdash C \qquad \Gamma, w{:}B \vdash C}{\Gamma \vdash C}\vee\mathrm{E}^{u,w}$
$\dfrac{\Gamma, u{:}A \vdash B}{\Gamma \vdash A \supset B}\supset\mathrm{I}^{u}$	$\dfrac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\supset\mathrm{E}$
$\dfrac{\Gamma, u{:}A \vdash p}{\Gamma \vdash \neg A}\neg\mathrm{I}^{p,u}$	$\dfrac{\Gamma \vdash \neg A \qquad \Gamma \vdash A}{\Gamma \vdash C}\neg\mathrm{E}$
$\dfrac{}{\Gamma \vdash \top}\top\mathrm{I}$	*no* $\top$ *elimination*
no $\bot$ *introduction*	$\dfrac{\Gamma \vdash \bot}{\Gamma \vdash C}\bot\mathrm{E}$
$\dfrac{\Gamma \vdash [a/x]A}{\Gamma \vdash \forall x.\ A}\forall\mathrm{I}^{a}$	$\dfrac{\Gamma \vdash \forall x.\ A}{\Gamma \vdash [t/x]A}\forall\mathrm{E}$
$\dfrac{\Gamma \vdash [t/x]A}{\Gamma \vdash \exists x.\ A}\exists\mathrm{I}$	$\dfrac{\Gamma \vdash \exists x.\ A \qquad \Gamma, u{:}[a/x]A \vdash C}{\Gamma \vdash C}\exists\mathrm{E}^{a,u}$

Inference Rules for Intuitionistic Logic

Lambda Calculus

What is Lambda Calculus?

- A formal model of computation by Alonzo Church
- Lambda Terms:

Variables: x, y, z

Abstraction: $\lambda x.E$

Application: $(\lambda x.E)\,F$

Examples

- $\lambda x.x^2$ is a function
- $(\lambda x.x^2)(3) \rightarrow 9$

Type Theory & Simply Typed Lambda Calculus

Type Theory Basics

- Assigns types to terms: $1 : \mathbb{N}$, $+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$
- Typing is decidable

Type Categories

- Base Types: $\mathbb{N}$, Bool, $\bot$
- Arrow Types: $f : A \longrightarrow B$
- Product Types: $\langle a, b \rangle : A \times B$
- Sum Types: $a : A + B$

Typed Lambda Calculus

- $(\lambda x : \mathbb{N}.x^2) : \mathbb{N} \longrightarrow \mathbb{N}$

Dependent Types

- Types depend on values: $Vec\,(n)$
- Indexed types, predicate representation
- $Vec : \mathbb{N} \longrightarrow Type$

Curry–Howard Correspondence

- sometimes referred as Curry Howard Isomorphism
- A connection between logic, computation, and type theory
- Also known as the *proofs-as-programs* and *propositions-as-types* principle

Core Idea

- A **proposition** corresponds to a **type**
- A **proof** of the proposition is a **program** (term) of that type
- Proof checking is equivalent to type checking
- Proof normalization corresponds to program evaluation

Curry–Howard Correspondence

Logic	Type Theory	Programming
Proposition	Type	-
Proof	Term	Program
Implication $A \longrightarrow B$	$A \longrightarrow B$	λ-abstraction
Conjunction $A \wedge B$	$A \times B$	Pair
Disjunction $A \vee B$	$A + B$	Tagged union
Falsehood $\perp$	Empty Type	No term
Universal $\forall x.A(x)$	$\Pi x : A.B(x)$	Function over types
Existential $\exists x.A(x)$	$\Sigma x : A.B(x)$	Dependent pair

Martin-Löf Type Theory

What is it?

- A formal system for constructive mathematics
- Also known as Intuitionistic Type Theory
- Backbone of modern Proof Assistants

Core Types

- (Π-type): Dependent function type, $\forall x : A.B\,(x)$
- (Σ-type): Dependent sum type, $\exists x : A.B\,(x)$
- **Identity Type**: Internal equality between terms

Type Universes

- Types have types: $1 : \mathbb{N}, \mathbb{N} : \mathit{Type}, \mathit{Type} : \cdots$
- Not like sets within sets — avoids paradoxes
- Enables reasoning and abstraction over types themselves

Architecture of proof assistant

Architecture of a Proof Assistant

- **Kernel**: Minimal, trustworthy codebase enforcing logical rules and validating proofs.
- **Tactic Engine**: Helps build and automate proofs step by step.
- **Formal Proof Language**: Rigorously expresses definitions, statements, and proofs.
- **Libraries**: Collections of verified mathematical foundations for reuse.
- **User Interface**: IDEs and plugins for interactive, efficient proof development.

Kernel: The Trusted Core

- The **kernel** is the minimal and most critical part of a proof assistant.
- It enforces the logical rules of the underlying formal system (e.g., type theory).
- Responsible for **validating every proof step** to guarantee correctness.
- Ensures **soundness and trustworthiness**; the rest of the system depends on its integrity.
- Typically very small and rigorously tested or formally verified to avoid bugs.
- Example: Agda's kernel is written in Haskell and integrates normalization to check definitional equality.

Tactic Engine: Proof Construction Assistant

- The **tactic engine** supports users in constructing proofs interactively.
- It breaks complex proof goals into simpler subgoals using **proof strategies** called tactics.
- Provides **automation** for common proof patterns, speeding up proof development.
- Enables both **forward** and **backward** reasoning approaches.
- Even fully automated tactics rely on the kernel for final verification.
- Varies among assistants (Agda has minimal/no tactics, Coq and Lean have powerful tactic systems).

Formal Proof Language: Expressing Proofs Precisely

- This language allows expressing **definitions, propositions, and proofs** rigorously.
- Typically a **dependently typed language** so logical properties can be encoded as types.
- Provides **syntax and semantics** suitable for formal reasoning and machine checking.
- Enables users to write **human-readable yet unambiguous** formal proofs.
- Integrates smoothly with tactics and type checker to maintain correctness.
- Example languages: Agda's core language, Coq's Gallina, Lean's dependent type language.

Libraries: Reusable Verified Foundations

- Extensive collections of **formalized mathematics and algorithms** supporting new developments.
- Include **basic theories** such as arithmetic, algebra, logic, and set theory.
- Enable users to **build on existing verified results** without re-proving foundations.
- Libraries evolve and grow, fostering **collaboration and community sharing**.
- Well-maintained libraries reduce duplication and improve proof assistant adoption.
- Examples include Coq's Standard Library, Agda Standard Library, Lean's mathlib.

User Interface: Proof Development Environment

- Provides **interactive tools** like IDEs, editor plugins, or command line interfaces.
- Features include **syntax highlighting, error reporting, real-time proof state visualization,** and **auto-completion**.
- Enhances **usability and productivity** for proof authors.
- Supports **integration with tactics and proof language** for seamless workflow.
- Examples: CoqIDE, Proof General, Emacs-mode for Agda, VS Code extensions.
- A good interface lowers the learning curve and makes formalization more accessible.

Comparative Study

Comparative Table: Agda, Rocq (Coq), and Lean

Component	Agda	Rocq (Coq)	Lean
Proof Style	Explicit term-based, manual proof writing	Tactic-based, automated backward reasoning	Both tactic-based and term-style
Kernel	Minimal, written in Haskell, tight integration with normalization	Based on Calculus of Inductive Constructions (CIC), written in Coq (extracted to OCaml)	CIC-based, written in C++/C
Type Checking	Bidirectional, transparent, normalization by evaluation	Bidirectional, heavy conversion, strong automation	Bidirectional, smart elaboration (coercion, backtracking, overloading)
Automation	Limited (no tactics, minimal automation)	Extensive tactic engine and proof search	Advanced, seamless tactic/term mixing, smart elaborator
Use Cases	Foundations, education, dependently typed programming	Large/complex formalizations, industrial-scale proofs	Research, education, combinatorial/mathematical formalizations

Formalization Of Some Proofs

Eg: Defining Natural Numbers

```
data N : Set where
     Zero : N
     suc : N -> N
```

Eg: Some mathematical properities

Equality and Transitivity:

```
data _==_ { A : Set } ( x : A ) : A -> Set where
    refl : x == x

trans : { A : Set } { x y z : A}
    -> x == y
    -> y == z
    -> x == z
    trans refl refl = refl
```

Formalization Of Some Proofs

Formalization Of DeMorgan's Law

DeMorgan's Law in agda

```
DeMorgan's Law
open import Agda.Primitive using (Level; lzero)
open import Data.Product using (_×_; _,_)
open import Data.Sum using (_⊎_; inj1; inj2)
open import Relation.Nullary using (Dec; yes; no)
open import Data.Empty using (⊥; ⊥-elim)
open import Relation.Nullary.Negation using (¬_)
--One Direction
deMorganOneWay : ∀ {ℓ} {P Q : Set ℓ} → (¬ P) ⊎ (¬ Q) → ¬ (P × Q)
deMorganOneWay (inj1 np) (p , q) = np p
deMorganOneWay (inj2 nq) (p , q) = nq q
```

```
Converse, Requires Non Constructive Assumptions
deMorganOtherWay :
∀ {ℓ} {P Q : Set ℓ}
→ Dec P
→ Dec Q
→ ¬ (P × Q)
→ (¬ P) ⊎ (¬ Q)
deMorganOtherWay (yes p) _ notPQ = inj2 (λ q → notPQ (p , q))
deMorganOtherWay (no np) (yes q) _ = inj1 np
deMorganOtherWay (no np) (no nq) _ = inj1 np -- or inj2 nq
```

Limitations

- Issue while implementing Real Numbers Construcitvely.
- Bidirectional Typechecking Algorithm isn't discussed rigorously.

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Thank you!