CSCB36 TT2 Q1

$\forall n \in \mathbb{N}, n \geq 18$ Define $Q(n)$:
If n is a natural #, $n \geq 18$, program terminates and returns (m, p), $m, p \in \mathbb{N}$ st $4m + 7p = n$

Basis: $Q(18)$
Lines 1-2 exec bc $n = 18$
returns $(1, 2)$
$4(1) + 7(2) = 18$
$= n$

$Q(19)$
Lines 3-4 exec bc $n = 19$
returns $(3, 1)$
$4(3) + 7(1) = 19$
$= n$

Thus basis holds for $n = 18, 19$

Induction:
For $k \in \mathbb{N}$, $18 \leq k+1 < n$, Supp $Q(k)$ holds [IH]
Prove $Q(n)$

$n - 2 < n$ $\therefore Q(n-2)$ holds
Lines 1-4 don't exec bc $n > k+1$
≥ 19
So $n > 19$

$(q, r) = Stamps(n-2)$ [line 5]
$q, r \in \mathbb{N}$ st $4q + 7r = n - 2$ by IH $*$

Case 1: $r \geq 2$
returns $(q+4, r-2)$ [line 7]
$4(q+4) + 7(r-2) = 4q + 16 + 7r - 14$
$= n - 2 + 2$ by $*$
$= n$

$r \geq 2 \Rightarrow r - 2 \geq 0$
$\therefore r - 2 \in \mathbb{N}$
$q \in \mathbb{N} \Rightarrow q + 4 \in \mathbb{N}$
($r, q \in \mathbb{N}$ by $*$)

Thus $(q+4, r-2)$ satisfies $4(q+4) + 7(r-2) = n$, $(q+4), (r-2) \in \mathbb{N}$

Case 2: $r < 2$
returns $(q-3, r+2)$ [line 9]
$4(q-3) + 7(r+2) = 4q + 7r - 12 + 14$
$= n - 2 + 2$ by $*$
$= n$

We know $n \geq 20$ and $r < 2$ $(r \leq 1)$
So $7(r+2) \leq 21$

$r \in \mathbb{N}$ by $*$
$\Rightarrow r + 2 \in \mathbb{N}$

Then we can say $4(q-3) + 7(r+2) < 4(q-3) + 28$

$20 \leq n$
$= 4(q-3) + 7(r-2)$
$\leq 4(q-3) + 21$
$= 4q - 12 + 21$
$= 4q + 9$

$\Rightarrow 20 \leq 4q + 9$
$\Rightarrow 11 \leq 4q$ sub 9 both sides
$\Rightarrow \frac{11}{4} \leq q$ div 4 both sides
q is a natural # by $*$ $\therefore q \geq \lceil \frac{11}{4} \rceil = 3$
$\therefore q - 3 \geq 0 \Rightarrow q - 3 \in \mathbb{N}$

Thus $(q-3, r+2)$ satisfies $4(q-3)+7(r+2)=n$ and $(q-3),(r+2) \in \mathbb{N}$

Hence $Q(n)$ holds, as wanted