MATB24 A3

Problem 1

1. We know if $S: F^n \to F^m$ is a LT, $\exists A \in M_{m\times n}(F)$ st $S(v) = Av$, $\forall v \in F^n$
and we know $P_3(\mathbb{R}) \cong \mathbb{R}^4$

So $\frac{d}{dx}: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ can be represented as a linear map $T: \mathbb{R}^4 \to \mathbb{R}^4$ which can be represented as a 4×4 matrix.

So $\frac{d}{dx}$ can be represented as a 4×4 matrix

2. $\mathcal{B} = (1, x, x^2, x^3)$

$$\text{Let } A = \begin{bmatrix} | & | & | & | \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ | & | & | & | \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. A^4 represents $\frac{d^4}{dx^4}$ aka taking the fourth derivative
We know taking the 4th derivative of a max 3rd degree polynomial will always be 0.

So $A^4 = [0]$

Problem 2

1. Let $\vec{v} \in V$ be arbi.

$\vec{v} = \vec{v} - \vec{w} + \vec{w}$, $\vec{w} \in sp\{b_1, \dots, b_r\}$ s.t. $T(\vec{v}) = T(\vec{w}) \Rightarrow T(\vec{v} - \vec{w}) = \vec{0}_W$
$\Rightarrow \vec{v} - \vec{w} \in ker(T) = sp\{v_1, \dots, v_d\}$

$\vec{w} = r_1 b_1 + \dots + r_r b_r$
$\vec{v} - \vec{w} = c_1 v_1 + \dots + c_d v_d$

So $\vec{v} = c_1 v_1 + \dots + c_d v_d + r_1 b_1 + \dots + r_r b_r$ so $\vec{v} \in sp\{v_1, \dots, v_d, b_1, \dots, b_r\}$

2. Show if $c_1 v_1 + \dots + c_d v_d + r_1 b_1 + \dots + r_r b_r = 0$, $c_1 = \dots = c_d = r_1 = \dots = r_r = 0$

We know $p_1 T(b_1) + \dots + p_r T(b_r) = 0 \Rightarrow p_1 = \dots = p_r = 0$

$\Rightarrow T(p_1 b_1 + \dots + p_r b_r) \neq 0$ if $p_1 \neq p_r \neq 0$ by linearity and contrapositive

Thus $sp\{b_1, \dots, b_r\} \not\subseteq ker(T)$ so $sp\{b_1, \dots, b_r\} \cap sp\{v_1, \dots, v_d\} = \{\vec{0}\}$

Supp $T(p_1 b_1 + \dots + p_r b_r) = 0$ for some $p_i \neq 0$

Then $p_1 b_1 + \dots + p_r b_r \in ker(T)$, but this contradicts the fact that the intersections of their spans is 0, as no $b_i = 0$.

Thus $\{b_1, \dots, b_r\}$ is l.i. $\Rightarrow \{v_1, \dots v_d, b_1, \dots, b_r\}$ are lin. indep.

3. $dim(ker(T)) = |\{v_1, \dots v_d\}| = d$
$dim(Img(T)) = |\{T(b_1), \dots, T(b_r)\}| = r$
$dim(V) = |\{v_1, \dots, v_d, b_1, \dots, b_r\}| = d + r$

$$\begin{aligned} Rank(T) + Null(T) &= dim(Img(T)) + dim(ker(T)) \\ &= r + d \\ &= dim(V) \end{aligned}$$

Problem 3

We know for any 2 bases A, B of V, and LT $T: V \to V$

$[T]_A$ is similar to $[T]_B$, $\exists C, C^{-1}$ st $[T]_A = C[T]_B C^{-1}$

We also know $[T]_A = [T]_B$ for any A, B

Let $B = (b_1, \dots, b_n)$ be a arbi. basis of V
Let A be an arbi. basis of V

We know $[T]_B = C[T]_A C^{-1}$, where C is the change of basis matrix from $A \to B$

Sub in $[T]_B$, $[T]_B = C[T]_B C^{-1}$
$[T]_B C = C[T]_B$ Multiply right sides by C

Since there are an infinite # of bases, there are an infinite # of C matrices (which are invertible)

We can show that $[T]_B$ must be diagonal

Choose $C = I$, but the ith diagonal entry is -1 rather than 1

$C[T]_B$ yields $[T]_B$ with a negative ith row
$[T]_B C$ yields $[T]_B$ with a negative ith col

So entry $b_{ij} = -b_{ij}$ for $i \neq j$
Thus $b_{ij} = 0$ for $i \neq j$

Now prove all diagonal entries are equal.

Choose $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$ $\quad [T]_B C = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} b_{11} & 0 & 0 \\ b_{11} & b_{22} & 0 \\ 0 & b_{22} & b_{33} \end{bmatrix}$ $\quad = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{22} & b_{22} & 0 \\ 0 & b_{33} & b_{33} \end{bmatrix}$

Thus $b_{11} = b_{22}$ and $b_{22} = b_{33}$
$\therefore$ All diagonal entries are equal

So $[T]_B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$ where $c = b_{11} = b_{22} = b_{33} \in F$

$= c \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus $[T]_B$ is a scalar multiple of the identity transformation and all basis representations of T are the same so $T = c \cdot id_V$

Problem 4

1. Let $B \in M_{n\times n}$ be arbi.
Prove $\exists$ LT T, st. $B = [T]_{\mathcal{B}}$, $\mathcal{B} = (f_1, \dots, f_n)$ $\quad B = \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix}$

Let S be the coordinate iso. to $\mathcal{B}$
$S: V \to F^n$
$\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$

Let $T(\vec{v}) = S^{-1}(B[\vec{v}]_{\mathcal{B}})$

$$[T(f_i)]_{\mathcal{B}} = S(S^{-1}(B[f_i]_{\mathcal{B}})) \qquad [f_i]_{\mathcal{B}} = (0, \dots, 1, \dots, 0)$$
1 at position i
$$= B[f_i]_{\mathcal{B}}$$
$$= \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} b_{i1} \\ \vdots \\ b_{in} \end{pmatrix}$$

So $[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(f_1)]_{\mathcal{B}} & \cdots & [T(f_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} = B$

$\therefore$ B is the matrix representation of T in basis $\mathcal{B}$

2. Supp. $\exists T, S$ st $[T]_{\mathcal{B}} = B$ and $[S]_{\mathcal{B}} = B$, $T \neq S$ From pt 1: $B \in M_{n\times n}$

Let $v \in V$ be arbi.

$[T(v)]_{\mathcal{B}} = Bv$ $\qquad$ So $[T(v)]_{\mathcal{B}} = [S(v)]_{\mathcal{B}}$ for all $v \in V$
$= [S(v)]_{\mathcal{B}}$ $\qquad$ So $T = S$ $\therefore$ contradiction

Thus $\exists!$ LT T st $B = [T]_{\mathcal{B}}$

3. Let $S: \mathcal{L}(V,V) \to \mathbb{R}^{n \times n}$, transform a LT to its matrix representation in some basis

By part a, we know $\forall A \in \mathbb{R}^{n\times n}, \exists T \in \mathcal{L}(V,V)$, st $[T]_{\mathcal{B}} = A$, $\mathcal{B}$ is some basis of V

In other words $\forall A \in \mathbb{R}^{n\times n}, \exists T \in \mathcal{L}(V,V)$ st $S(T) = A$, so $S(T)$ is onto

By part b, $\forall A, B \in \mathcal{L}(V,V)$, $A \neq B \Rightarrow S(A) \neq S(B)$

In other words, S is one-to-one

$\therefore$ S is an isomorphism from $\mathcal{L}(V,V) \to \mathbb{R}^{n\times n}$, so $\mathcal{L}(V,V)$ is isomorphic to $\mathbb{R}^{n\times n}$

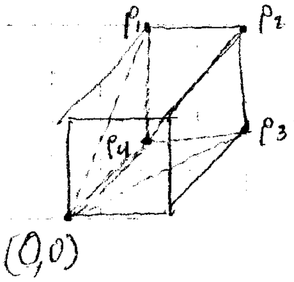
4. From c, $\dim(\mathbb{R}^{n\times n}) = \dim(\mathcal{L}(V,V)) = n^2$

If $m = n^2$, then $\{I, T^1, .., T^m\}$ contains n^2+1 vectors in $\mathcal{L}(V,V)$, which means $\{I, T^1, .., T^m\}$ is a l.d. set, as the max # of mutually l.i. vectors is equal to $\dim(\mathcal{L}(V,V)) = n^2$

Thus $\exists a_i \neq 0$ st $a_0 I + a_1 T + \cdots + a_m T^m = 0$

Problem 5

1. The lines on a cube and icosahedron are made like this:



Its clear to see the angles between all p_i are 45°
Similar concept for icosahedron, set a vertex as the origin, connect 6 points on the other half

However, because of the symmetry of the dodecahedron, this doesn't work, the max equal angular lines that can be made is 3, by taking any point as the origin and using the adjacent points to make 3 lines.

2. Supp. $\theta \in (0, \frac{\pi}{2}], n \geq 1 \quad \cos^2\theta \in [0,1]$

$$(1-(\cos\theta)^2)id_n + \cos(\theta)^2 J_n = M_{n \times n} \text{ st } \forall \text{rows } i, \text{ cols } j$$

$$M = \begin{cases} (1-\cos^2\theta)+\cos^2\theta & \text{if } i=j \\ \cos^2\theta & \text{if } i \neq j \end{cases}$$

Show if $M\vec{v}=0$ then $\vec{v}=0 \quad \vec{v}=(v_1, \dots, v_n)$

$$M\vec{v}=0 \Rightarrow \begin{matrix} v_1 + \cos^2\theta\, v_2 + \dots + \cos^2\theta\, v_n = 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \cos^2\theta\, v_1 + \dots\dots + v_n = 0 \end{matrix}$$

Subtract two arbi. eq. $1 \leq i,j \leq n$

$$(v_i - \cos^2\theta v_i) + (\cos^2\theta v_j - v_j) = 0$$
$$(1-\cos^2\theta)v_i - (1-\cos^2\theta)v_j = 0$$
$$(1-\cos^2\theta)(v_i - v_j) = 0$$
$$1-\cos^2\theta \neq 0 \text{ when } \theta \in (0, \tfrac{\pi}{2}]$$
$$v_i - v_j = 0$$
$$v_i = v_j$$

So $\vec{v} = c\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Subbing $\vec{v}$ back in $C\vec{v}$
we can see that $C\vec{v}=0$ iff $c=0$
$\Rightarrow \vec{v}=\vec{0}$

3. Supp. $L_1, \dots, L_n$ is equal angular

We know $\forall v_i$, unit vector on L_i, $\exists \theta \in (0, \frac{\pi}{2}]$ st $|\langle v_i, v_j \rangle| = \cos\theta$

Show if $c_1 v_1 v_1^T + \dots + c_n v_n v_n^T = \vec{0}$, then all $c_i = 0$

$$\sum_{i=1}^{n} c_i v_i v_i^T = \vec{0}$$

$$\Rightarrow v_j^T \left(\sum_{i=1}^{n} c_i v_i v_i^T \right) v_j = \vec{0}$$

$$\Rightarrow \sum_{i=1}^{n} c_i v_j^T v_i v_i^T v_j = \vec{0}$$

Note: $v_j^T v_i = \langle v_j, v_i \rangle = \langle v_i, v_j \rangle$ bc the field is $\mathbb{R}$

$$\Rightarrow \sum c_i \langle v_j, v_i \rangle \langle v_i, v_j \rangle = \vec{0}$$

$$\Rightarrow \sum c_i \cos^2\theta = \vec{0}$$

bc $\langle v_j, v_i \rangle \langle v_i, v_j \rangle = \langle v_i, v_j \rangle^2 = |\langle v_i, v_j \rangle|^2 = \cos^2\theta$

Except when $j = i$ $\langle v_j, v_i \rangle \langle v_i, v_j \rangle = (\|v_j\| \|v_j\|)^2 = (1)(1) = 1$

So we have $c_1 \cos^2\theta + \dots + c_j \|v_j\|^2 \|v_j\|^2 + \dots + c_n \cos^2\theta$

$\Rightarrow c_1 \cos^2\theta + \dots + c_j + \dots + c_n \cos^2\theta$

Construct a matrix with the equations

$$\begin{bmatrix} 1 & & \cos\theta \\ & \ddots & \\ \cos\theta & & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

From part 2, we know the sol. for this particular matrix is $c_1 = \dots = c_n = 0$

Thus $\{v_1 v_1^T, \dots, v_n v_n^T\}$ are l.i $\blacksquare$

4. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in sp([a_{11}],[a_{12}],[a_{13}],[a_{22}],[a_{23}],[a_{33}])$

where $[a_{ji}]$ is the matrix with 1's on entries a_{ji} and a_{ij}, and 0 everywhere else

Thus dimension of the space of 3x3 symmetric matrices is 6

We know $v_i v_i^T = \begin{bmatrix} v_{i1}^2 & v_{i1} \cdot v_{i2} & v_{i1} \cdot v_{i3} \\ v_{i2} \cdot v_{i1} & v_{i2}^2 & v_{i2} \cdot v_{i3} \\ v_{i3} \cdot v_{i1} & v_{i3} \cdot v_{i2} & v_{i3}^2 \end{bmatrix}$ by matrix multiplication

We know if $L_1, \dots, L_n$ are equalangular $\exists \theta \in (0, \frac{\pi}{2}]$ st $|\langle v_i, v_j \rangle| = \cos\theta$

From part 3, we know all $v_i v_i^T$ are l.i. so this means the max # of v_i's is 6, as $v_i v_i^T$ is always symmetric.

Thus the max # of L_i's is 6 in $\mathbb{R}^3$.

5. $\binom{n+1}{2} = \frac{(n+1)!}{2!(n+1-2)!} = \frac{(n+1)!}{2!(n-1)!} = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i$ Ie: Sum from 1 to n

Notice this is the generalized # of dimensions of a space made from nxn symmetric matrices, as it adds up the # of upper/lower triangle entries.

By part 4, we know the max # of equalangular lines in $\mathbb{R}^3$ is equal to the dimension of the space of symmetric 3x3 matrices.

We also know the conclusion of part 4 lies within part 3

If we constructed part 3 for $\mathbb{R}^n$ rather than $\mathbb{R}^3$ (clearly not hard, just let $\vec{v} \in \mathbb{R}^n$)

We can construct the same proof in part 4, but generalized. (Notice: $\binom{3+1}{2} = 6$)

By part 4, the max # of equalangular lines in $\mathbb{R}^n$ is equal to the dimension of the space of nxn symmetric matrices, which shown above is the sum of 1 to n, which is $\binom{n+1}{2}$ ∎