MATB42 AMACSS Review Seminar 2023

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Parametrizing Curves and Surfaces

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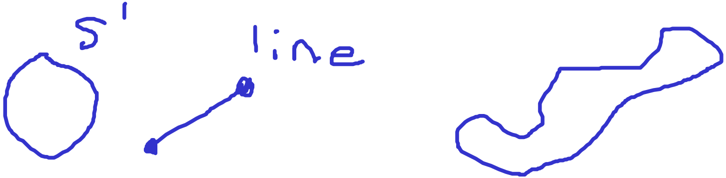

Parametrizing Curves and Surfaces

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Parametrizing a Curve

A parametrization of a curve is a map $c : [a, b] \to \mathbb{R}^n$ (this is commonly called a path).

C - set

path: fcn

$c([a, b]) = C$

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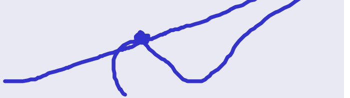
The tangent line to a path at point t_0 is given by

$t \in \mathbb{R}$

$$c(t_0) + (t - t_0)c'(t_0)$$



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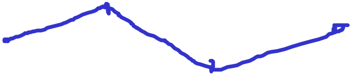

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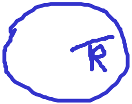

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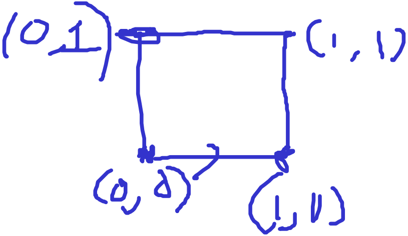
Sample Question

Parametrize the square with corners at $(0,0), (1,0), (1,1), (0,1)$ with the counterclockwise orientation.



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$(0,0)$ $(1,0)$

The first is a line from $(0,0)$ to $(1,0)$. This can be given easily by $(t,0), t \in [0,1]$.

Parametrizing Curves and Surfaces

$c : [0,4] \to \mathbb{R}^2$

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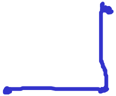
Next comes the line from $(1,0)$ to $(1,1)$. We have to be more careful here. We want to carry on from where we left off from last parametrization, so we will need to adjust the line equation.



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It can be given by $(1, t - 1), t \in [1, 2]$. (Notice that it would be just $(1, t)$ if we were parametrizing using $[0, 1]$).

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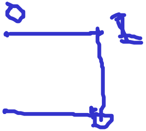
Parametrizing Curves and Surfaces

We then have the line from $(1,1)$ to $(0,1)$, which, again, by adjusting the domain and param is given by $(3-t,1), t \in [2,3]$.



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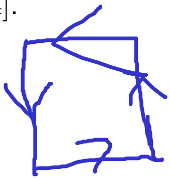
By the same idea, the line from $(0,1)$ to $(0,0)$ is given by $(0, \underline{4-t}), t \in [\underline{3}, 4]$.



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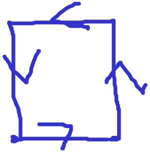
We can wrap this all together into the piecewise parametrization

$$c : \underline{[0,4]} \to \mathbb{R}^2$$

Given by

$$c(t) = \begin{cases} (t,0) & \text{if } t \in [0,1] \\ (1,t-1) & \text{if } t \in [1,2] \\ (3-t,1) & \text{if } t \in [2,3] \\ (0,4-t) & \text{if} t \in [3,4] \end{cases}$$



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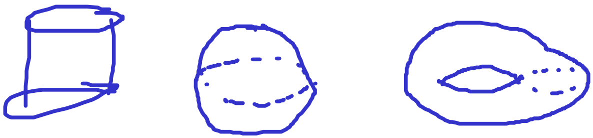
We also have the analogous 2-dimensional version of parametrizing a curve, which is the parametrization of a surface:



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Parametrization of a Surface

A parametrization of a Surface is a function $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. The "Surface" of this parametrization is its image.

$[a, b] \times [c, d]$

$\Phi(D) = S$

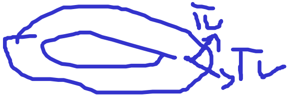
$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

$\frac{\partial}{\partial u}$ $\frac{\partial}{\partial v}$

Analogously, we call Φ C^n if all its component functions are of class C^n wrt u and v.

Parametrizing Curves and Surfaces

T_u and T_v are the component vectors with the partials of Φ in terms of u and v respectively, i.e



Parametrizing Curves and Surfaces

T_u and T_v are the component vectors with the partials of Φ in terms of u and v respectively, i.e

$\Phi(u, v) = (x, y, z)$

$$T_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$$

$$T_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$$

The normal vector to the Surface at point $\Phi(u_0, v_0)$ is given by $n = T_u \times T_v$ at point (u_0, v_0). Its tangent plane at point (x_0, y_0, z_0), if $\Phi(u_0, v_0) = (x_0, y_0, z_0)$, is given by

$$n \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Parametrizing Curves and Surfaces

Common Surface Parametrizations:

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The parametrization of a sphere of radius R centered at the origin can be parametrized by $\Phi : [0, \pi] \overset{D}{\times} [0, 2\pi] \to \mathbb{R}^3$ given by

$$\Phi(u, v) = (Rsin(u)cos(v), Rsin(u)sin(v), Rcos(u))$$

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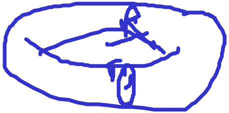
The parametrization of the torus (donut surface) with hole radius R and tube radius r can be parametrized by $\Phi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ given by



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Parametrizing Curves and Surfaces

$r \in [0, \infty)$
$v \in [0, 2\pi]$

$\Phi(r, v) = (r\cos(v), r\sin(v), r)$

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The parametrization of the graph of a scalar function f on $\mathbb{R}^2$ can be parametrized by $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, given by $\Phi(u, v) = (u, v, f(u, v))$.

Parametrizing Curves and Surfaces

Sample Question: Find the Tangent Plane of a Torus

Given a Torus, where $R = 2$ and $r = 1$, find its tangent plane at point $\Phi(0, 0)$

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Now, we compute T_u and T_v

$-sin(u)cos(v)$ $\quad$ $-sin(u) sin(v)$

$$T_u = (\frac{\partial(1 + cos(u))cos(v)}{\partial u}, \frac{\partial(1 + cos(u))sin(v)}{\partial u}, \frac{\partial sin(u)}{\partial u}) = cos(u)$$

Parametrizing Curves and Surfaces

$$T_v = (-cos(v)sin(u), -sin(v)sin(u), cos(u))$$

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$$= (-cos(v)sin(u), -sin(v)sin(u), cos(u))$$

And

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$$= (-cos(v)sin(u), -sin(v)sin(u), cos(u))$$

And

$-2sin(v) - cos(v)sin(v)$

$2cos(v) + cos(v)cos(u)$

$$T_v = (\frac{\partial(2 + cos(u))cos(v)}{\partial v}, \frac{\partial(2 + cos(u))sin(v)}{\partial v}, \frac{\partial sin(u)}{\partial v}) \quad = 0$$

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Now we compute the normal vector $T_u \times T_v$ at $(0, 0)$.

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Parametrizing Curves and Surfaces

Note that at $(0, 0)$, $T_u = (0, 2, 0)$ and $T_v = (1, 0, 0)$, so

$$n = T_u \times T_v = \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

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$$n = T_u \times T_v = \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (0,0,-2)$$

$a\vec{i} + b\vec{j} + c\vec{k}$

$\parallel$

(a, b, c)

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$$n \cdot (x-2, y-0, z-1) = 0(x-2) + 0(y-z) - 2(z-1) = -2z+2$$

at 0, i.e $-2z+2=0$ or $z=-1$. Thus the equation of the tangent plane at $\Phi(0,0)$ is $z=-1$.

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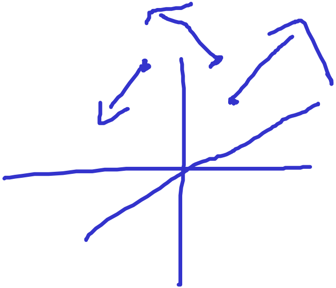
Vector Fields

Vector Field

A Vector Field is a function $f : \mathbb{R}^n \to \mathbb{R}^n$. This is typically visualized as having the vector assigned to the point $\mathbb{R}^n$ pointing out from that point (though length is scaled back in order to avoid cluttering up the plot)



Vector Fields

$F: (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots))$

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Vector Field Operators

$div(F) := \nabla \cdot F = \underline{\frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}}$

Vector Fields

$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right)$

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$div(\nabla F)$

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Important Theorems

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We will discuss another related theorem soon.

Integration

The Path Integral

The path integral or the integral over the path c on a scalar function f in $\mathbb{R}^3$, for $c : [a, b] \to \mathbb{R}^3$, is given by

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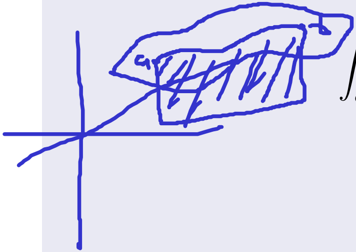

$$\int_c f\,ds = \int_a^b f(x(t), y(t), z(t))||c'(t)||dt$$

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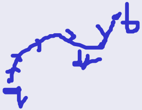

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If c is a plane curve and f is a scalar function on the plane, then

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Integration

Sample Question

Evaluate the ~~line~~ path integral

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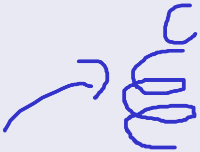

where C is parametrized by $c : [0, \pi] \to \mathbb{R}^3$, $c(t) = (cos(t), sin(t), t)$.

Integration

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$-sin(t) \quad cos(t)$

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$\|c'(t)\|$

Well,

$$\int_C (xy + z^3)ds = \int_0^{\pi} (cos(t)sin(t) + t^3)\sqrt{sin^2(t) + cos^2(t) + 1}dt$$

plug $c(t)$ in

Integration

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Integration

$$\frac{-\cos^2(\pi)}{2}\sqrt{2}$$

$$= \int_0^{\pi} \sqrt{2}cos(t)sin(t)dt + \int_0^{\pi} \sqrt{2}t^3dt$$

$$\frac{\sqrt{2}\pi^4}{4}$$

Integration

$$= \int_0^{\pi} \sqrt{2}cos(t)sin(t)dt + \int_0^{\pi} \sqrt{2}t^3dt$$

$$= \frac{\sqrt{2}}{2}(-cos^2(\pi) + cos^2(0)) + \frac{\sqrt{2}}{4}(\pi^4 - 0)$$

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Thus

$$\int_C (xy + z^3)ds = \frac{\sqrt{2}\pi^4}{4}$$

Integration

Integral of a Scalar Function over a Surface

If $f : \mathbb{R}^3 \to \mathbb{R}$, and S is a surface parametrized by Φ, then the scalar integral over S is:

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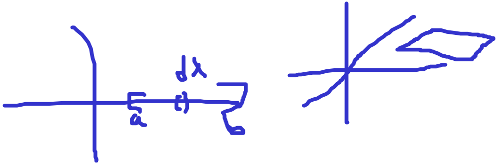
Integration



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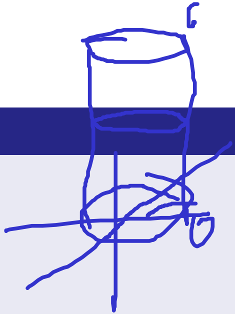
Analogously, if $f = 1$, then the integral gives the surface area of S

$F(x,y,z) = y$



Sample Question

Evaluate the scalar surface integral

$(\sqrt{3}\cos(u))^2 + (\sqrt{3}\sin(u))^2 = 3$

$$\iint y dS$$

where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$

First, we can parametrize this surface through $\Phi(u, v) = (\sqrt{3}cos(u), \sqrt{3}sin(u), v)$, where $0 \leq v \leq 6$ and $0 \leq u \leq 2\pi$.

Integration

Sample Question

Evaluate the scalar surface integral

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$$T_u = \left(\frac{\partial\sqrt{3}cos(u)}{\partial u}, \frac{\partial\sqrt{3}sin(u)}{\partial u}, \overset{0}{\frac{\partial v}{\partial u}}\right) = (-\sqrt{3}sin(u), \sqrt{3}cos(u), 0)$$

$-\sqrt{3}sin(u)$ $\quad$ $\sqrt{3}cos(u)$

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0 0 1

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Integration

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$f_1 i - f_2 j + f_3 k$

$= (f_1, -f_2, f_3)$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -\sqrt{3}sin(u) & \sqrt{3}cos(u) & 0 \end{vmatrix} = (-\sqrt{3}cos(u), -\sqrt{3}sin(u), 0)$$

Warning!

$T_u \times T_v \neq T_v \times T_u$

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Thus,

$y(\phi(u,v))$

$=\sqrt{3}$

$$\iint_S ydS = \iint_D \sqrt{3}sin(u)||T_u \times T_v||dA$$

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Integration

$3\sin(u)$

$$= \int_0^{2\pi} \int_0^6 \sqrt{3} sin(u) \sqrt{3} dv du$$

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Integration

Line Integral (The Important One)

If F is a vector field that is continous on the path $c : [a, b] \to \mathbb{R}^3$, then the line integral of F along c is given by

$$\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) dt$$

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Theorem of Line Integrals on Gradient Vector Fields

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is C^1 and that $c : [a,b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then

$$\nabla f$$

$$\int_C F \cdot ds$$

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

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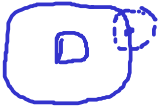

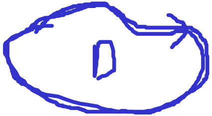

Green's Theorem (Important!)

If D is a region that can be decomposed into simple regions, and its boundary is C, and $F = (P(x,y), Q(x,y))$ is a C^1 vector field, then

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$= \text{curl}(F)$

Integration

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If D is a region that can be decomposed into simple regions, and its boundary is C, and $F = (P(x,y), Q(x,y))$ is a C^1 vector field, then

$$\int_C F \cdot ds = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy$$

We can also express this in the language of the vector field operators, in which case it is

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Sample Question

Evaluate

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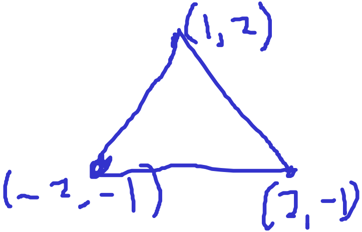
where C is the boundary of the triangle with vertices $(-2,-1),(1,2),(2,-1)$ and $F(x,y) = (2xy + sin(x^2), 3x^2 + 3^y)$.

The parametrization of C seems too difficult. Bring in Green's Theorem.



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6x

$$\frac{\partial Q}{\partial x} = \frac{\partial(3x^2 + \cancel{3^y})}{\partial x} = 6x$$

Integration

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Integration

$$\frac{\partial P}{\partial y} = \frac{\partial(2x\cancel{y} + \cancel{sin(x^2)})}{\partial y} = 2x$$

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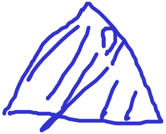

$$\int_C F \cdot ds = \iint_D 6x - 2xdA = \iint_D 4xdA$$

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The x bounds of the triangular region are bounded above by the line segment between $(1, 2)$ and $(2, -1)$, and below by the line segment between $(-2, -1)$ and $(1, 2)$.

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$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1 - 2}{2 - 1} = -3$$

Integration

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$$m = \frac{-1-2}{2-1} = -3$$

$y - y_1 = m(x - x_1)$

This means the line segment is given by

$$y - 2 = -3(x - 1)$$

, or $x = \frac{y}{-3} + \frac{5}{3}$

$\frac{x}{-3} + \frac{2}{3} = x - 1$

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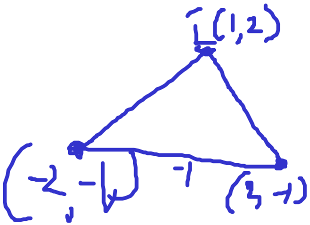

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Thus the x region is $y - 1 \leq x \leq \frac{-y}{3} + \frac{5}{3}$

The y region is given by $-1 \leq y \leq 2$

We can finally evaluate the integral!

Integration

$$\iint_D 4x dA = \int_{-1}^{2} \int_{y-1}^{\frac{-y}{3}+\frac{5}{3}} 4x dx dy$$

Integration

$2x^2$

$$\iint_D 4x\,dA = \int_{-1}^{2}\int_{y-1}^{\frac{-y}{3}+\frac{5}{3}} 4x\,dx\,dy$$

$-x^2 - 2x + 1$

$$= \int_{-1}^{2} 2\left(\frac{-y}{3}+\frac{5}{3}\right)^2 - 2(y-1)^2\,dy$$

$//$

$\frac{x^2}{9} - \frac{x5}{9} - \frac{5x}{9} + \frac{25}{9}$

Integration

$$\iint_D 4x dA = \int_{-1}^{2} \int_{y-1}^{\frac{-y}{3}+\frac{5}{3}} 4x dx dy$$

$$= \int_{-1}^{2} 2(\frac{-y}{3} + \frac{5}{3})^2 - 2(y-1)^2 dy$$

$$= 2\int_{-1}^{2} \frac{y^2}{9} - \frac{10y}{9} + \frac{25}{9} - y^2 + 2y - 1 dy$$

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$$\frac{y^3}{3} \qquad \frac{y^2}{2}$$

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Integration

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$$= 2(4) = 8$$

Integration

$$\int F \cdot ds \qquad \iint_D \text{curl}(F)$$

$$\int F \cdot dS$$

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$$\iint_D \text{curl}(F)$$

$$= 2(4) = 8$$

So $\int_c \cancel{f} ds = 8$.

Integration

Surface Integral over a Vector Field

Let F be a vector field defined over a surface S parametrized by Φ. Then the surface integral over the vector field F is given by

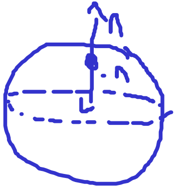
$$\iint_{\Phi} F \cdot dS = \iint_{D} F \cdot (T_u \times T_v) du dv$$

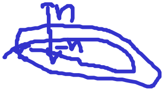
$n = \frac{T_u \times T_v}{\|T_u \times T_v\|}$

$-n$

Where D is the domain of parametrization.

$n(0,0,1) = (0,0,1)$





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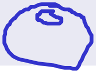
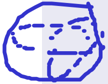


Where D is the domain of parametrization.

Stokes' Theorem

Let S be an oriented surface, given by a one to one parametrization $\Phi : D \subset \mathbb{R}^2 \to S$, where D is a region for which Green's Theorem applies. Let ∂S denote the oriented boundary of S and let F be a C^1 vector field on S. Then



= curl(F)

$$\iint_{S} (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$$

= 0

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Sample Problem

Evaluate

$$\iint_S curl(F) \cdot dS$$

where S is the part of the graph $z = 9 - x^2 - y^2$ above the xy plane oriented so $n(0,0,9) = (0,0,1)$ and $F = (x^3 - y, x, xz + e^y z)$

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$\nabla \times F$

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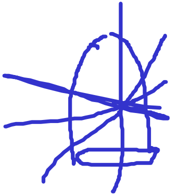

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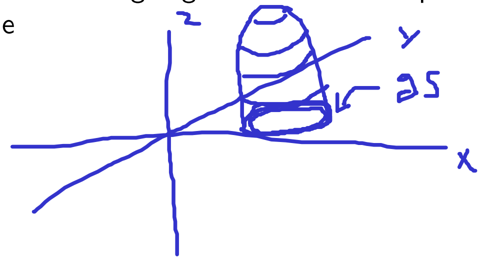
We need to determine ∂S if we are going to do this. ∂S is percisely the part of our surface where $z = 0$, or i.e



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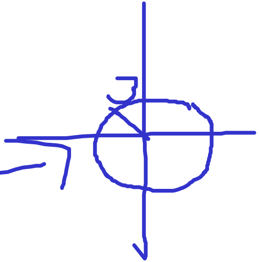
$$0 = 9 - x^2 - y^2$$

$$x^2 + y^2 = 9$$



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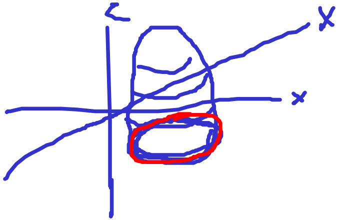

This is the circle of radius 3 in the xy plane.

Sample Problem

Evaluate

$$\iint_S curl(F) \cdot dS$$

$\Phi(u,v) = (u, v, 9-u^2-v^2)$

where S is the part of the graph $z = 9 - x^2 - y^2$ above the xy plane oriented so $n(0,0,9) = (0,0,1)$ and $F = (x^3 - y, x, xz + e^y z)$

$\Phi(0,0) = (0,0,9)$

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$T_u = (1, 0, -2u)$

$T_v = (0, 1, -2v)$

$$\frac{(2(0), 2(0), 1)}{\sqrt{2(0)^2 + 2(0)^2 + 1^2}} = (0,0,1)$$

$$0 = 9 - x^2 - y^2$$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix}$$

$$x^2 + y^2 = 9$$

$= 2u\, i + 2v\, j + k = (2u, 2v, 1)$

This is the circle of radius 3 in the xy plane.

Integration

We can parametrize this by $c(t) = (3cos(t), 3sin(t), 0)$, $t \in [0, 2\pi]$

$$(3\cos(t))^2 + (3\sin(t))^2 = 9$$

Integration

$-3\sin(t)$ $3\cos(t)$

We can parametrize this by $c(t) = (3cos(t), 3sin(t), 0)$, $t \in [0, 2\pi]$

Note then that $c'(t) = (3sin(t), -3cos(t), 0)$, and that
$F(c(t)) = (27cos^3(t) - 3sin(t), 3cos(t), 0)$

Integration

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$$= \int_0^{2\pi} F(c(t)) \cdot c'(t)dt$$

Integration

We can parametrize this by $c(t) = (3cos(t), 3sin(t), 0)$, $t \in [0, 2\pi]$

Note then that $c'(t) = (-3sin(t), +3cos(t), 0)$, and that $F(c(t)) = (27cos^3(t) - 3sin(t), 3cos(t), 0)$

$$\iint_S curl(F) \cdot dS = \int_{\partial S} F \cdot dS$$

$$\begin{bmatrix} -3sin(t) \\ 3cos(t) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 27cos^3(t) - 3sin(t) \\ 3cos(t) \\ 0 \end{bmatrix}$$

$$= \int_0^{2\pi} F(c(t)) \cdot c'(t)dt$$

$$= -81 sin(t) cos^3(t) + 9 sin^2(t) + 9 cos^2(t)$$

$$= \int_0^{2\pi} -81cos^3(t)sin(t) + 9sin^2(t) + 9cos^2(t)dt$$

Integration

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$$\iint_S curl(F) \cdot dS = \int_{\partial S} F \cdot dS$$

$$= \int_0^{2\pi} F(c(t)) \cdot c'(t)dt$$

$$= \int_0^{2\pi} 81cos^3(t)sin(t) - 9sin^2(t) - 9cos^2(t)dt$$

$$\frac{81\cos^4(t)}{4} = 0$$

$$= \int_0^{2\pi} -81cos^3(t)sin(t)dt + \int_0^{2\pi} -9(cos^2(t) + sin^2(t))dt$$

Integration

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$$= \int_0^{2\pi} 81cos^3(t)sin(t) dt + \int_0^{2\pi} -9(cos^2(t) + sin^2(t)) dt$$

Integration

$$= (\frac{-cos^4(2\pi)81}{4} + \frac{cos^4(0)81}{4}) + \cancel{-}18\pi$$

$$= 0 \cancel{-} 18\pi$$

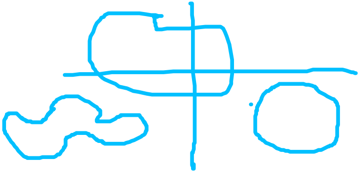
$$= \cancel{-}18\pi$$

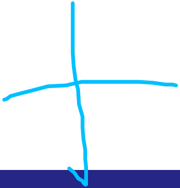
Thus

$$\iint_S curl(F) \cdot dS = \cancel{-}18\pi$$

Integration

$\nabla \times F = 0$

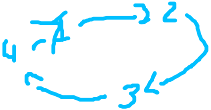




Conservative Vector Fields

If F is a C^1 vector field on $\mathbb{R}^3$, except at a finite number of points, then F is called "conservative" if one and hence all of the following equivalent conditions hold:

1. For any oriented simple closed curve, $\int_C F \cdot dR = 0$
2. For any 2 oriented simple curves with the same endpoints, $\int_{C_1} F \cdot dR = \int_{C_2} F \cdot dS$
3. F is the gradient of some function f (we call the function f the potential of F)
4. $\nabla \times F = 0$



Integration

Sample Question

Given $F(x, y, z) = (2xy^2 + 5, 2x^2y - 3y^2, 0)$, compute

$$\int_C F \cdot dr$$

where C is the ellipse given by $\frac{(x-4)^2}{5} + \frac{y^2}{4} = 1$

Integration

Sample Question

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Yikes. This looks too nasty to do directly. However, what if the vector field was conservative? Let's check.

Integration

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Let us compute the curl.

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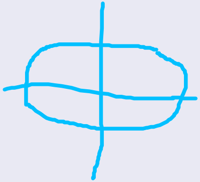


where C is the ellipse given by $\frac{(x-4)^2}{5} + \frac{y^2}{4} = 1$

Yikes. This looks too nasty to do directly. However, what if the vector field was conservative? Let's check.

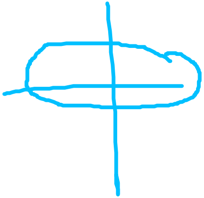
Let us compute the curl.

$$curl(F) = \underline{\nabla \times F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + \cancel{5} & 2x^2y - \cancel{3y^2} & 0 \end{vmatrix} = (0, 0, \underline{4xy - 4xy}) = (0,0,0)$$

Integration

So the vector field IS conservative! Thus

Integration



simple closed curve!

So the vector field IS conservative! Thus

$$\int_C F \cdot dr = 0$$

Integration

So the vector field IS conservative! Thus

$$\int_C F \cdot dr = 0$$

Since an ellipse is a closed simple curve.

Integration

So the vector field IS conservative! Thus

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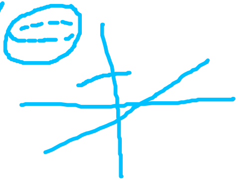
Since an ellipse is a closed simple curve.

Integration

little change on inside = big change on outside

W

∂W



Gauss'/The Divergence Theorem

Let W be a symmetric elementary region in space, and ∂W be the surface that bounds it. Then, if F is C^1, then

$$\iiint_W div(F)dV = \iint_{\partial W} F \cdot dS$$

$S = \partial W$

Filled in (yummy!)

hollow surface as the boundary

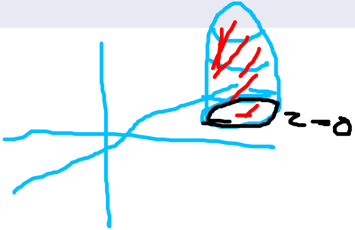
Integration

Sample Questions

Evaluate

$$\iint_S F \cdot dS$$

where $F = (2xz, 1 - 4xy^2, 2z - z^2)$ where S is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$



Integration

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Yeesh. Too hard to evaluate directly. Let's use the divergence theorem.

Integration

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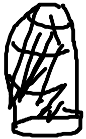
$$\iint_S F \cdot dS = \iiint_E div(F)dV$$



.

Integration

Stokes: $\nabla \times F$

Gauss: F

Sample Questions

Evaluate

$-2(x^2+y^2) = -2r^2$

$$\iint_S F \cdot dS$$

where $F = (2xz, 1 - 4xy^2, 2z - z^2)$ where S is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$

$z = 6 - r^2$

Yeesh. Too hard to evaluate directly. Let's use the divergence theorem. So, we have

$$\iint_S F \cdot dS = \iiint_E div(F)dV$$

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Where E is the region S bounds. Let's set it up in cylindrical coordinates.

Integration

Sample Questions

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where $F = (2xz, 1 - 4xy^2, 2z - z^2)$ where S is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$

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So, we have

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.

Where E is the region S bounds. Let's set it up in cylindrical coordinates.

Integration

$z = 6 - 2r^2$

$0 \le z \le 6 - 2r^2$

$6 - 2(\sqrt{3})^2$

$= 6 - 6 = 0$

Notice that in cylindrical coordinates, the region is $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$, $0 \le z \le 6 - 2r^2$

$6 - 2x^2 - 2y^2$

$= 6 - 2(x^2 + y^2) = 6 - 2r^2$

Integration

z

Notice that in cylindrical coordinates, the region is $0 \leq r \leq \sqrt{3}$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 6 - 2r^2$

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$$div(F) = \underline{\frac{\partial \overset{F_1}{2xz}}{\partial x}} + \frac{\partial \overset{F_2}{1 - 4xy^2}}{\partial y} + \frac{\partial \overset{F_3}{2z - z^2}}{\partial z} = \cancel{2z} - 8xy + 2 - \cancel{2z} = 2 - 8xy$$

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Next, we compute the divergence. It is

$$div(F) = \frac{\partial 2xz}{\partial x} + \frac{\partial 1 - 4xy^2}{\partial y} + \frac{\partial 2z - z^2}{\partial z} = 2z - 8xy + 2 - 2z = 2 - 8xy$$

At last, we can compute the integral, remembering to pick up an r since we converted E to polar coordinates.

Integration

$$\iiint_E div(F)dV = \iiint_E 2 - 8xydV$$

Integration

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$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} r(2 - 8r^2cos(\theta)sin(\theta))dzdrd\theta$$

Jacobian

Integration

$$\iiint_E div(F)dV = \iiint_E 2 - 8xydV$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{6-2r^2} r(2 - 8r^2 cos(\theta) sin(\theta)) dzdrd\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{6-2r^2} 2r - 8r^3 cos(\theta) sin(\theta) dzdrd\theta$$

Integration

$$\iiint_E div(F)dV = \iiint_E 2 - 8xydV$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{6-2r^2} (2 - 8r^2 cos(\theta)sin(\theta))dzdrd\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{6-2r^2} 2r - 8r^3 cos(\theta)sin(\theta)dzdrd\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r - 8r^3 cos(\theta)sin(\theta)(6 - 2r^2)drd\theta$$

Integration

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 - (48r^3 - 16r^5) cos(\theta) sin(\theta) dr d\theta$$

Integration

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$6r^2$ $\quad r^4 \quad$ $12r^4$ $\quad \frac{16r^6}{6}$

$$= \int_0^{2\pi} 6(\sqrt{3})^2 - (\sqrt{3})^4 - (12(\sqrt{3})^4 - \frac{8}{3}(\sqrt{3})^6)cos(\theta)sin(\theta)d\theta$$

Integration

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 12r - 4r^3 - (48r^3 - 16r^5)cos(\theta)sin(\theta)drd\theta$$

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$$= \int_0^{2\pi} 9 - 36cos(\theta)sin(\theta)d\theta$$

Integration

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$$= \int_0^{2\pi} 9 - 36cos(\theta)sin(\theta)d\theta$$

$18cos^2(\theta)$

$$= 9(2\pi) + 18\cancel{cos^2(4\pi)} - 9(0) - 1\cancel{8cos^2}(0) = 18\pi$$

Integration

Thus,

$$\iint_S F \cdot dS = 18\pi$$

Differential Forms

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Differential Forms

Scalars: $F: \mathbb{R}^3 \to \mathbb{R}$

Basic 2-forms $dx \wedge dy$ $\quad dx \wedge dz \quad dy \wedge dz$

Space of K forms

The space of k forms on $\mathbb{R}^3$ is denoted $\bigwedge^k(\mathbb{R}^3)$. Some important facts:

1. $\bigwedge^k(\mathbb{R}^3)$ is a vector space with functions as scalars and will consist of all linear combinations of basic k forms. For $\bigwedge^1(\mathbb{R}^3)$, it's $f_1dx + f_2dy + f_3dz$, for $\bigwedge^2(\mathbb{R}^3)$, it's $f_1dx \wedge dy + f_2dx \wedge dz + f_3dy \wedge dz$, and for $\bigwedge^3(\mathbb{R}^3)$, it's $f_1dx \wedge dy \wedge dz$

Basic 3-form is just $dx\,dy\,dz$

Differential Forms

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2. As a sanity check, $\bigwedge^k(\mathbb{R}^3)$ should always be $C(3,k)$ dimensional.
3. $\bigwedge^0(\mathbb{R}^3)$ is just the space of scalar functions $f : \mathbb{R}^3 \to \mathbb{R}$.

$\Lambda^k(\mathbb{R}^3) \cong \{0\}$ if $k > 3$

Differential Forms

The Wedge Product

The wedge product takes a k form and an l form and makes a $k+l$ form. It is determined by the following rules:

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- $dx \wedge dx = 0$

Differential Forms

F is a 0-form

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- $dx \wedge dx = 0$
- $\omega \wedge 0 = 0$ for any k form ω

$\Lambda^k(\mathbb{R}^5)$ ← has a 0-form

Differential Forms

The Exterior Derivative

The natural generalization of the derivative to differential forms. Generalizes the curl, grad and div. It is a map $d : \underline{\bigwedge^k(\mathbb{R}^3)} \to \underline{\bigwedge^{k+1}(\mathbb{R}^3)}$ determined by the following rules:

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$d(\omega + \Omega) = d\omega + d\Omega$

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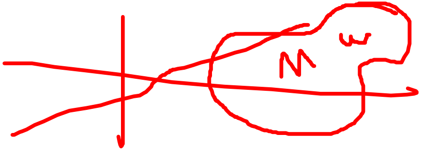
- For any 0 form f, $d(f) = \underline{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz} \rightarrow \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$

Differential Forms

The Generalized Stokes' Theorem

This is the magnum opus of Vector/Manifold Calculus. It generalizes Green's, Gauss' and Stokes' Theorems with the appropriate identifications.

Differential Forms



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Differential Forms

Sample Question

Let $\omega = x^2ydx + xz^2dy + y^3dz$ and $\eta = cos(y)dx + e^xdy + zdz$. Calculate $\omega \wedge \eta$

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$$\omega \wedge \eta = (x^2ydx + xz^2dy + y^3dz) \wedge (cos(y)dx + e^xdy + zdz)$$

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$$\omega \wedge \eta = (x^2ydx + xz^2dy + y^3dz) \wedge (cos(y)dx + e^xdy + zdz)$$

Distributive Property of $\wedge$

$$x^2ydx \wedge (cos(y)dx + e^xdy + zdz) + xz^2dy \wedge (cos(y)dx + e^xdy + zdz) + y^3dz \wedge (cos(y)dx + e^xdy + zd$$

Differential Forms

Sample Question

Let $\omega = x^2ydx + xz^2dy + y^3dz$ and $\eta = cos(y)dx + e^xdy + zdz$. Calculate $\omega \wedge \eta$

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anti - Commutivity
OF $\wedge$

$$-(cos(y)dx + e^xdy + zdz) \wedge x^2ydx - (cos(y)dx + e^xdy + zdz) \wedge xz^2dy - (cos(y)dx + e^xdy + zdz) \wedge y$$

Differential Forms

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Differential Forms

Distributive Property + Homogeneity of Fcns

$$= -e^x x^2 y dy \wedge dx - x^2 yz dz \wedge dx - cos(y) xz^2 dx \wedge dy - xz^3 dz \wedge dy - cos(y) y^3 dx \wedge dz - e^x y^3 dy \wedge dz$$

Differential Forms

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anti-symmetry

$$= e^x x^2 y \underline{dx \wedge dy} + x^2 yz dx \wedge dz - cos(y) xz^2 dx \wedge dy + xz^3 dy \wedge dz - cos(y) y^3 \underline{dx \wedge dz} - e^x y^3 \underline{dy \wedge dz}$$

Differential Forms

$$= -e^x x^2 y dy \wedge dx - x^2 yz dz \wedge dx - cos(y) xz^2 dx \wedge dy - xz^3 dz \wedge dy - cos(y) y^3 dx \wedge dz - e^x y^3 dy \wedge dz$$

$$= e^x x^2 y dx \wedge dy + x^2 yz dx \wedge dz - cos(y) xz^2 dx \wedge dy + xz^3 dy \wedge dz - cos(y) y^3 dx \wedge dz - e^x y^3 dy \wedge dz$$

because $\Lambda^2(\mathbb{R}^3)$ is a vector space

$$(e^x x^2 y - cos(y) xz^2) \underline{dx \wedge} dy + (x^2 yz - cos(y) y^3) dx \wedge dz + (xz^3 - e^x y^3) dy \wedge dz$$

Differential Forms

$$= -e^x x^2 y dy \wedge dx - x^2 yz dz \wedge dx - cos(y) xz^2 dx \wedge dy - xz^3 dz \wedge dy - cos(y) y^3 dx \wedge dz - e^x y^3 dy \wedge dz$$

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Thus
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Differential Forms

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$-dz \wedge dx$

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Differential Forms

Sample Question

Let S be the unit sphere with outward pointing normal, and ω be the 2 form
$(x+y+z)dxdy + (3x-2)dydz + (x-5y)dzdx$

Differential Forms

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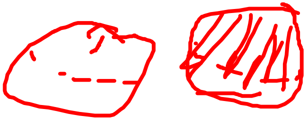
Yeesh. Too hard to evaluate directly once again. Time to bring out the generalized stokes' Theorem.

Differential Forms



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Differential Forms

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Differential Forms

Next, we need to compute $d\omega$. Note:

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Differential Forms

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$$d\omega = d((x+y+z)dxdy + (3x-2)dydz + (x-5y)dzdx)$$

$d(dx dy) = 0$

$$= d((x+y+z)dxdy) + d((3x-2)dydz) + d((x-5y)dzdx)$$

product rule wrt $\wedge$

$$= d(x+y+z)dxdy+(x+y+z)\overset{=0}{d(dxdy)}+d(3x-2)dydz+(3x-2)\overset{=0}{d(dydz)}+d(x-5y)dzdx+(x-5$$

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$$= d(x+y+z)dxdy + (x+y+z)d(dxdy) + d(3x-2)dydz + (3x-2)d(dydz) + d(x-5y)dzdx + (x-$$

$$= dzdxdy + 3dxdydz - 5dydzdx$$

Differential Forms

anti-symmetry
of $\wedge$

$$= dxdydz + 3dxdydz - 5dxdydz$$

Differential Forms

$$= dxdydz + 3dxdydz - 5dxdydz$$

$$= -dxdydz$$

Differential Forms

$$= dxdydz + 3dxdydz - 5dxdydz$$

$$-dxdydz$$

Thus, by the generalized Stokes' Theorem, we have

$S = \partial B$

$$\int_S \omega = \int_{\partial B} \omega$$

$$= \int_B d(\omega)$$

$$= \iiint_B -1dxdydz$$

Differential Forms

This is the negative volume of the unit ball, which is

$$\frac{4\pi r^3}{3}$$

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Fourier Series

A Fourier approximation is a way of approximating a symmetric function on a certain interval.

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Fourier Series

The fourier coefficients of a function defined on $[-\pi, \pi]$ are given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)cos(kx)dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)sin(kx)dx$$

Fourier Series

$a_0 = \frac{3\pi}{4}$

$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx.$

Sample Question

Calculate the fourier series of the following function:

$$f(x) = \left\{ \begin{array}{ll} \pi, & \text{if } -\pi \le x \le 0 \\ x, & \text{if } 0 \le x \le \pi \end{array} \right\}$$

$\frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\cos(kx)\,dx + \int_0^{\pi} x\cos(kx)\,dx\right)$

$\frac{1}{\pi}\left(\left(\frac{\pi\sin(kx)}{k}\right)_{-\pi}^{0} + \left(\int \right)_0^{\pi}\right.$

Differential Forms

That's all for the review seminar! Any questions?

Differential Forms

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