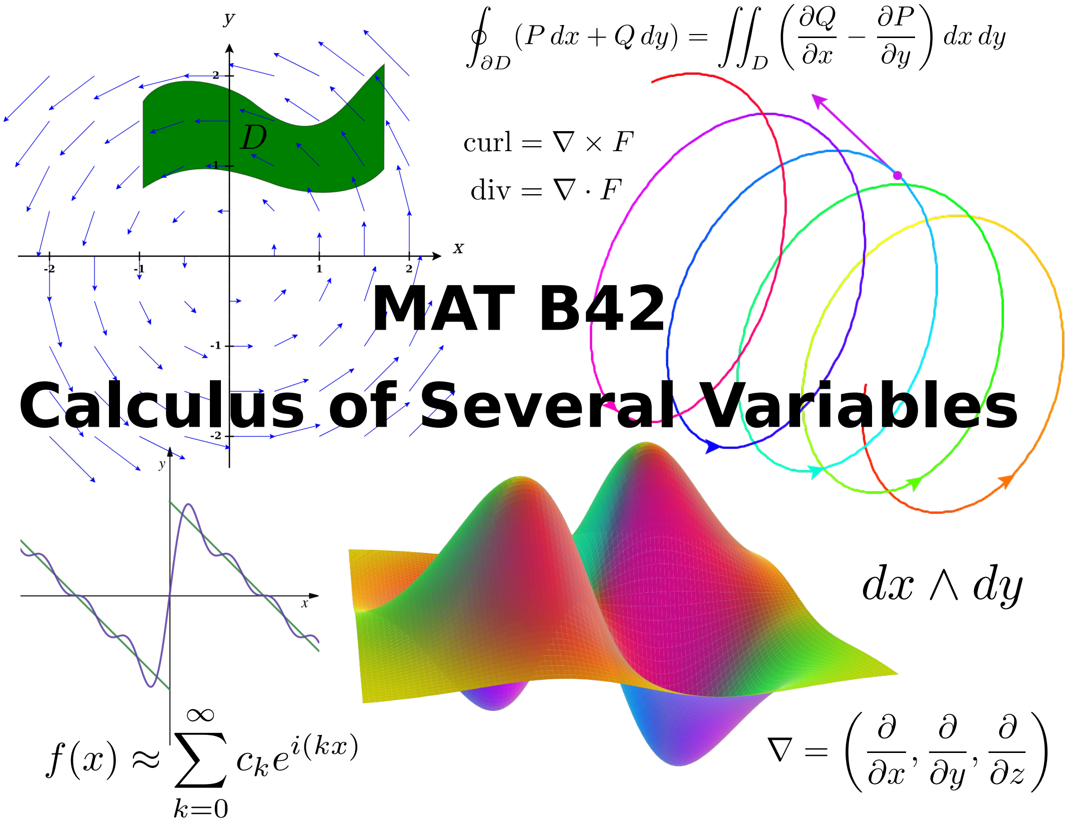
MAT B42
Calculus of Several Variables



Notes

MAT B42: Techniques of the Calculus of Several Variables II (Winter 2023)

Welcome to Week 1 of the course.
Questions? Thoughts? Comments?

News and Reminders:

- Tutorials start next week.
- Homework #1 is due next week on Friday January 20th at 13:59 (EST).

Notes

Warning: More Theoretical than Hughes-Hallett

We are going to take a more high-level and linear algebra heavy approach to Fourier Series than Hughes-Hallett.

There is a lot of new conceptual stuff in this week's material.
Two major areas of math: PDEs and Functional Analysis.

Notes

Partial Differential Equations

Definition

A partial differential equation (PDE) is an equation relating the partial derivatives of a function. To solve a PDE is to find a function that satisfies the constraints of the PDE.

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$ models the height of a string at position x and time t

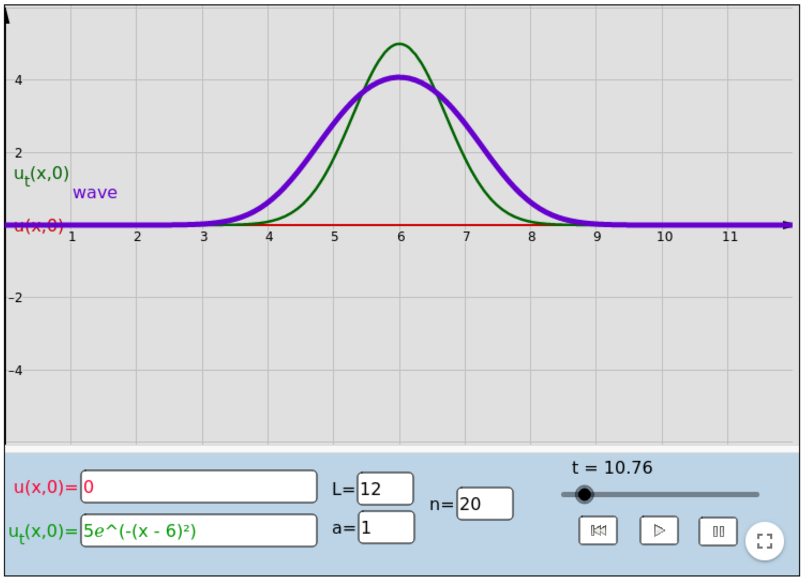
The Heat Equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$ models the heat of a rod at position x and time t

Notes

A Wave Equation Applet



Explore on GeoGebra

Notes

The Wave Equation is Linear

Theorem (Superposition Principle)

If u_1 and u_2 are solutions of the wave equation, then so is $u_1 + u_2$.

Notes

Eigenvalues and Eigenvectors of Double Differentiation

Example

$\sin(kx)$ and $\cos(kx)$ are eigenvectors of double differentiation

Notes

Fourier Polynomials

Definition (Hughes-Hallett p. 566)

The nth Fourier polynomial $F_n(x)$ of $f(x)$ has the form:

$$F_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$

where:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)dx \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)dx$$

Question

Why do this? And, why on earth would you pick these coefficients?!

Notes

Two Forms of Approximation: Taylor versus Fourier

$$f(x) \approx \sum_{k=0}^{\infty} c_k x^k$$

Pros: Simple convergence tests. Good local approximations for all functions.
Cons: Unbounded terms. Not eigenvectors of $\frac{\partial^2}{\partial x^2}$.

$$f(x) \approx \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Pros: Good global approximations for periodic functions.
Formed from eigenvectors of $\frac{\partial^2}{\partial x^2}$.
Cons: No simple convergence tests.

Notes

These coefficients come from orthogonal projection!

Notes

An Inner Product Space of Functions

Definition

Define $\langle f(x), g(x) \rangle = \int\limits_{-\pi}^{\pi} f(x)g(x)dx$. This is a symmetric bilinear inner product.

The corresponding norm is the L^2-norm:

$$\|f\|_2^2 = \int_{-\pi}^{\pi} [f(x)]^2 \, dx$$

Notes

The Orthogonal Basis Projection Lemma

Lemma (MAT B24)

Suppose that you have an inner product space $(V, \langle\cdot,\cdot\rangle)$.
If $\{\vec{v}_0, \ldots, \vec{v}_n\}$ *is an orthogonal set of vectors and* $\vec{x} \in \mathrm{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ *then:*

$$\vec{x} = \frac{\langle\vec{x}, \vec{v}_0\rangle}{\langle\vec{v}_0, \vec{v}_0\rangle}\vec{v}_0 + \frac{\langle\vec{x}, \vec{v}_1\rangle}{\langle\vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 + \cdots + \frac{\langle\vec{x}, \vec{v}_n\rangle}{\langle\vec{v}_n, \vec{v}_n\rangle}\vec{v}_n$$

Alternatively,

$$\vec{x} = \mathrm{proj}_{\vec{v}_1}(\vec{x}) + \cdots + \mathrm{proj}_{\vec{v}_n}(\vec{x})$$

Notes

An (Almost) Orthonormal Basis

Problem

$$\langle \sin(kx), \cos(nx) \rangle = \int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx = 0$$

$$\langle \cos(kx), \cos(nx) \rangle = \int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \begin{cases} 0 & k \neq \pm n \\ \pi & k = \pm n \neq 0 \\ 2\pi & k = n = 0 \end{cases}$$

Exercise: What it is corresponding statement for $\langle \sin(kx), \sin(nx) \rangle$*?*

Definition

The Fourier basis for $L^2([-\pi, \pi])$ is $\{1\} \cup \bigcup_{n \in \mathbb{N}} \{\cos(nx), \sin(nx)\}$.

Notes

Computing the a_0 Fourier Coefficient

Question

Why is $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)dx$*?*

Notes

Computing the a_k Fourier Coefficient

Question

Why is $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$*?*

Notes

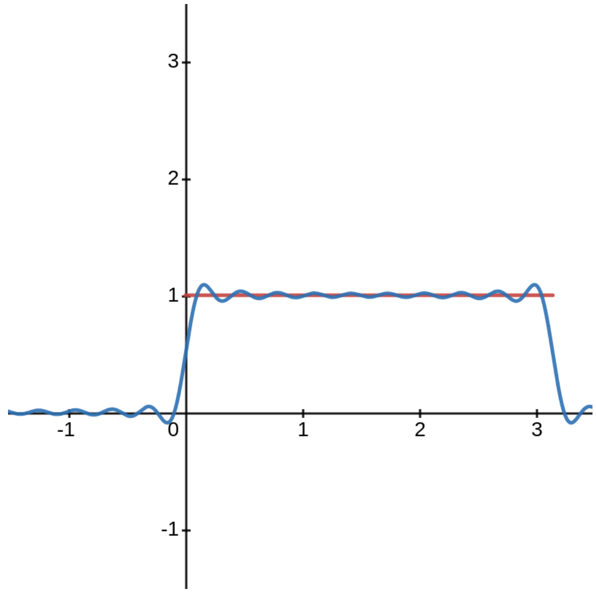
An Example

Question

Compute F_n the nth Fourier polynomial for

$$f(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ 1 & 0 < x < \pi \end{cases}$$

Notes



Explore on Desmos

Notes

Harmonics and Energy

Definition

The kth harmonic of $f(x)$ is the term $a_k \cos(kx) + b_k \sin(x)$ in its Fourier polynomial. (A musical heuristic: this is the kth "note" in the "chord" $f(x)$.)

Definition

The energy of $f(x)$ is:

$$E = \frac{1}{\pi} \int_{-1}^{1} (f(x))^2 dx$$

Alternatively, $E = \|f\|_2^2$.

Notes

The Energy Theorem

Theorem

Define $A_0 = \sqrt{2}a_0$ *and* $A_k = \sqrt{a_k^2 + b_k^2}$. *One has the following:*

$$E = A_0^2 + A_1^2 + A_2^2 + \cdots = 2a_0^2 + (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + \ldots$$

From the homework, the energy of a harmonic is:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(a_k\cos(kx) + b_k\sin(kx))^2 dx = a_k^2 + b_k^2 = A_k^2$$

Thus, the energy of f is the sum of the energies of the harmonics of f.

Notes

The Pythagorean Theorem

Lemma (MAT A22)

Suppose that you have an inner product space $(V, \langle \cdot, \cdot \rangle)$ *with norm* $\|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle$*.*
If $\{\vec{v}_0, \ldots, \vec{v}_n\}$ *is an orthogonal set of vectors and* $\vec{x} = \vec{v}_1 + \cdots + \vec{v}_n$ *then:*

$$\|\vec{x}\|^2 = \|\vec{v}_0\|^2 + \|\vec{v}_1\|^2 + \cdots + \|\vec{v}_n\|^2$$

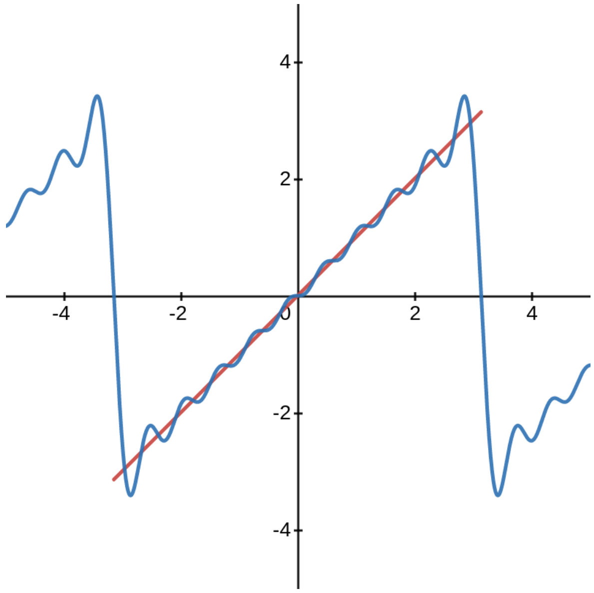
Notes

An Example of Energy

Question

Find the total energy of the first three harmonics $k = 0, 1, 2$ of $f(x) = x$.

Notes



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Notes