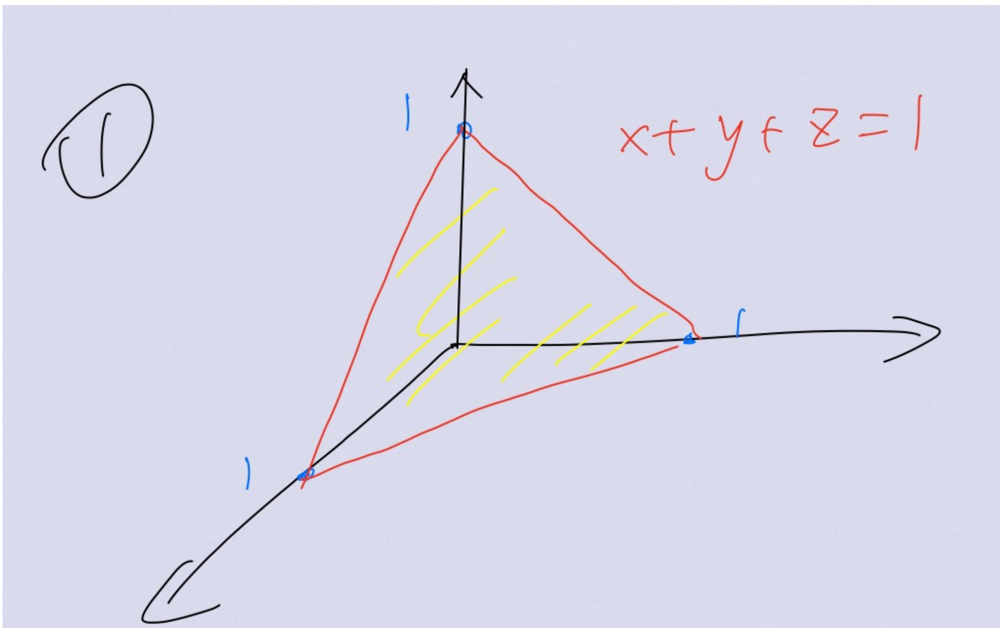
MATB41 HW9

Question 1
Find the volume of the unit tetrahedron defined by the points:
$(0,0,0),(1,0,0),(0,1,0),(0,0,1)$

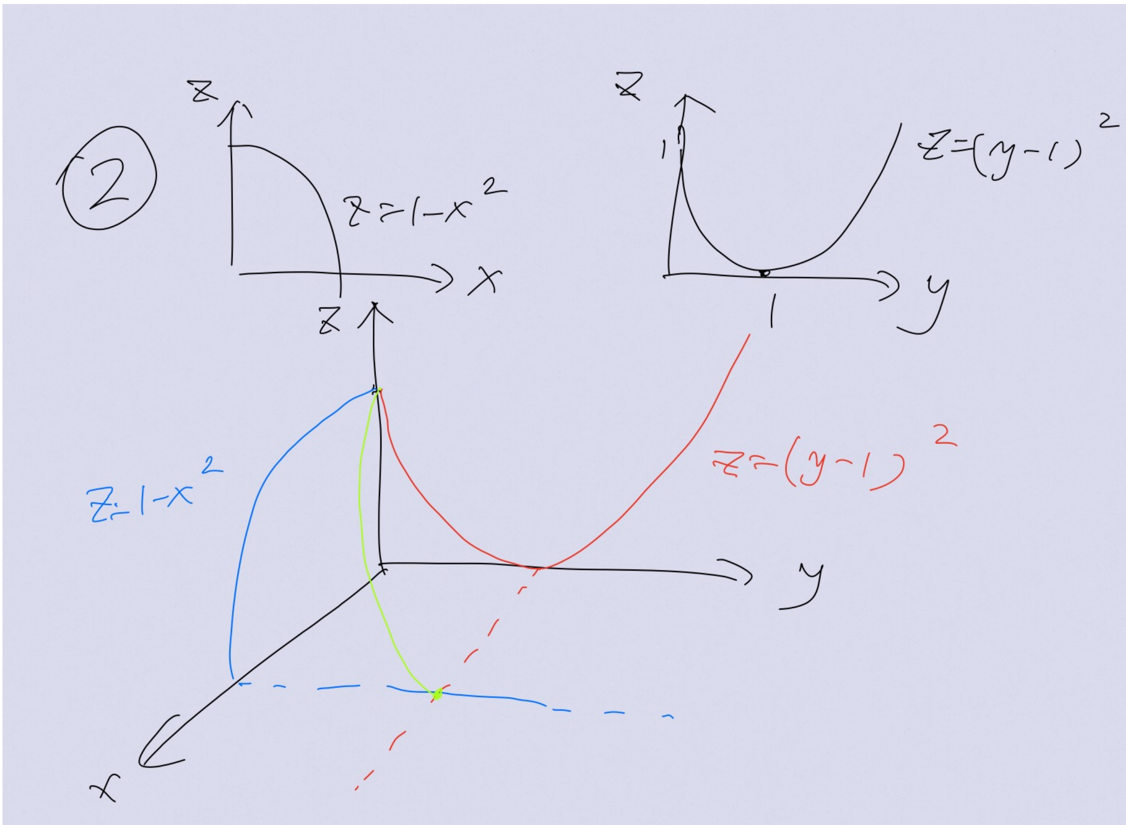


We integrate the function $f = 1$ over the tetrahedron to find the volume. The 2D projection onto the xy-plane is the triangle defined by $x+y+z=1$ and $z=0$, which is $x+y=1$.

$$\int_D 1 = \int_{2D}\int_{z=0}^{z=1-x-y} 1\,dz\,dx\,dy$$
$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} 1-x-y\,dy\,dx$$
$$= \int_{x=0}^{x=1} y - xy - \frac{y^2}{2}\Big|_{y=0}^{y=1-x} dx$$
$$= \int_{x=0}^{x=1} (1-x) - x(1-x) - \frac{(1-x)^2}{2}\,dx$$
$$= \int_{x=0}^{x=1} \frac{x^2}{2} - x + \frac{1}{2}\,dx = \frac{1}{6} - \frac{1}{2} + \frac{1}{2} = \frac{1}{6}$$

Question 2
D is the region in the first octant bounded by:
$z = 1 - x^2$ and $z = (y-1)^2$

Sketch the domain D.
Then, integrate $f(x, y, z)$ over the domain in 6 ways: orderings of dx, dy, dz.



We first sketch the blue curve $z = 1 - x^2$, and extend it toward the y direction, as it does not involve y. We then sketch the red curve $z = (y-1)^2$, and extend it toward the x direction, as it does not involve x. We get the green interesection point at $(1, 1, 0)$, and we smoothly connect the green intersection point with the other intersection at $(0, 0, 1)$ to form the volume in question.

Project onto xz-plane
The projection is defined by $z = 1 - x^2$.
The minimum value of y is 0, and the maximum value of y is defined by the surface on the right, which is $z = (y-1)^2$. This translates into $\pm\sqrt{z} = y - 1$. We must take the negative square root as $y - 1 < 0$ for the left half of the parabola that we are interested in.

$$\int_D f = \int_{2D} \int_{y=0}^{y=-\sqrt{z}+1} f \, dy \, dx \, dz$$

$$= \int_{x=0}^{x=1} \int_{z=0}^{z=1-x^2} \int_{y=0}^{y=-\sqrt{z}+1} f \, dy \, dz \, dx$$

To change the order of integration for the 2D integral, we know that $x^2 = 1 - z$, which translates into $x = \pm\sqrt{1-z}$. We must take the positive square root as $x > 0$ for the left half of the parabola we are interested in.

$$= \int_{z=0}^{z=1} \int_{x=0}^{x=\sqrt{1-z}} \int_{y=0}^{y=-\sqrt{z}+1} f \, dy \, dx \, dz$$

Project onto yz-plane
The projection is defined by $z = (y-1)^2$.
The minimum value of x is 0, and the maximum value of x is defined by the surface facing toward us, which is $z = 1 - x^2$. This translates into $x = \pm\sqrt{1-z}$. We must take the positive square root as $x > 0$ for the left half of the parabola we are interested in.

$$\int_D f = \int_{2D} \int_{x=0}^{x=\sqrt{1-z}} f \, dx \, dy \, dz$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=(y-1)^2} \int_{x=0}^{x=\sqrt{1-z}} f \, dx \, dz \, dy$$

To change the order of integration for the 2D integral, we know that $z = (y-1)^2$, which translates into $\pm\sqrt{z} = y - 1$. We must take the negative square root as $y - 1 < 0$ for the left half of the parabola that we are interested in.

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=-\sqrt{z}+1} \int_{x=0}^{x=\sqrt{1-z}} f \, dx \, dy \, dz$$

Project onto xy-plane

The projection is the square defined by $(1, 1, 0)$.

However, the green connection curve also get projected onto the xy-plane. It is the intersection of 2 surfaces , and it is given by $z = 1 - x^2 = (y - 1)^2$, which is a portion of a circle shifted upward by 1 unit toward the y axis. This translates into $x^2 = 2y - y^2$.

The minimum value of z is 0, but the maximum value of z depends on whether (x, y) is to 1 side or the other side of this shifted circle. We need to split the square into 2 portions D_1 and D_2 and integrate the 2 portions separately. Let D_1 be the portion that is to the top left of the circle on the xy-plane, and D_2 be the portion that is to the bottom right of the circle on the xy-plane. In D_1, the maximum value of z is defined by the surface on the right, which is $z = (y - 1)^2$. In D_2, the maximum value of z is defined by the surface facing toward us, which is $z = 1 - x^2$.

$$\int_D f = \int_{D_1} \int_{z=0}^{z=(y-1)^2} f \, dz \, dx \, dy + \int_{D_2} \int_{z=0}^{z=1-x^2} f \, dz \, dx \, dy$$

The circle translates into $x = \pm\sqrt{2y - y^2}$. We must take the positive square root as $x > 0$ for the portion of the curve that we are interested in. For D_1, the minimum value of x is 0, while for D_2, the maximum value of x is 1.

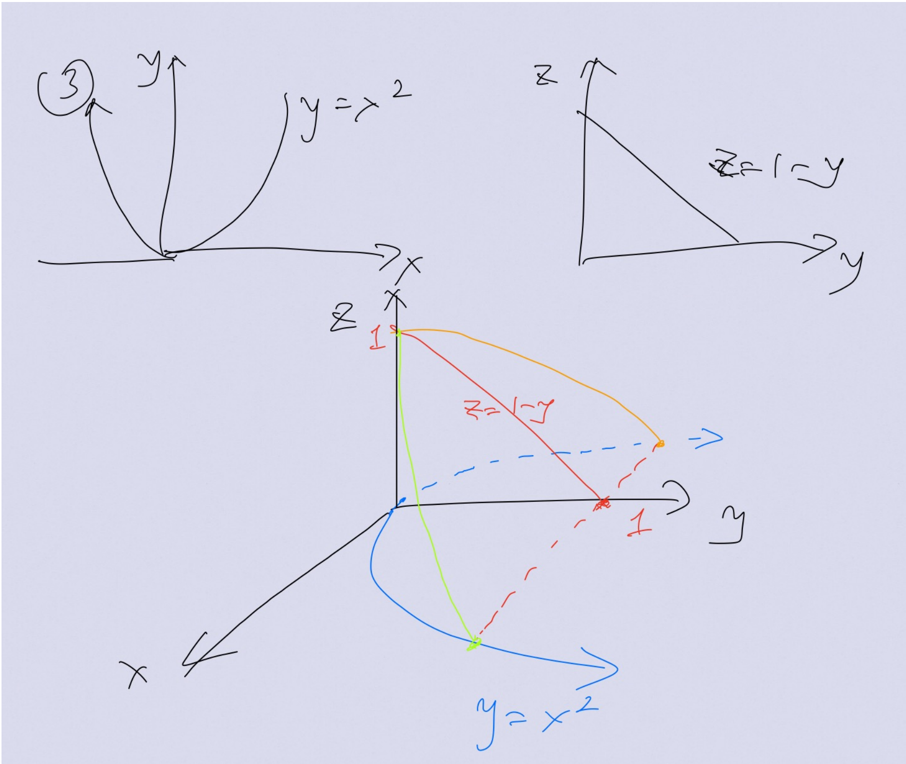
$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{2y-y^2}} \int_{z=0}^{z=(y-1)^2} f \, dz \, dx \, dy + \int_{y=0}^{y=1} \int_{x=\sqrt{2y-y^2}}^{x=1} \int_{z=0}^{z=1-x^2} f \, dz \, dx \, dy$$

The circle also translates into $\pm\sqrt{1 - x^2} = y - 1$. We must take the negative square root as $y - 1 < 0$ for the portion of the curve that we are interested in. For D_1, the maximum value of y is 1, while for D_2, the minimum value of y is 0.

$$= \int_{x=0}^{x=1} \int_{y=-\sqrt{1-x^2}+1}^{y=1} \int_{z=0}^{z=(y-1)^2} f \, dz \, dy \, dx + \int_{x=0}^{1=1} \int_{y=0}^{y=-\sqrt{1-x^2}+1} \int_{z=0}^{z=1-x^2} f \, dz \, dy \, dx$$

Question 3
D is the region bounded by:
$y = x^2$, $z = 1 - y$, $z = 0$
(not necessarily in the first octant)

Sketch the domain D.
Then, integrate $f(x, y, z)$ over the domain in 6 ways: orderings of dx, dy, dz.



We first sketch the blue curve $y = x^2$, and extend it toward the z direction, as it does not involve z. We then sketch the red curve $z = 1 - y$, and extend it toward the x direction, as it does not involve x. We get the green interesection point at $(1, 1, 0)$, and we smoothly connect the green intersection point with the other intersection at $(0, 0, 1)$. We also get an orange intersection point toward the back at $(-1, 1, 0)$, and we smoothly connect the orange intersection point with the other intersection at $(0, 0, 1)$, to form the volume in question.

Project onto xy-plane
The projection is the region above $y = x^2$ on the xy-plane.
The minimum value of z is 0, and the maximum value of z is defined by the surface on the top, which is $z = 1 - y$.

$$\int_D f = \int_{2D} \int_{z=0}^{z=1-y} f \, dz \, dx \, dy$$

$$= \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} \int_{z=0}^{z=1-y} f \, dz \, dy \, dx$$

To change the order of integration for the 2D integral, we know that $y = x^2$, which translates into $x = \pm\sqrt{y}$. We must take both the positive and negative square roots as the minimum value of x is the negative square root, and the maximum value of x is the positive square root.

$$= \int_{y=0}^{y=1} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} \int_{z=0}^{z=1-y} f \, dz \, dx \, dy$$

Project onto yz-plane
The projection is defined by $z = 1 - y$.
The minimum value of x is $-\sqrt{y}$, and the maximum value of x is $\sqrt{y}$.

$$\int_D f = \int_{2D} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} f \, dx \, dy \, dz$$

$$= \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} f \, dx \, dz \, dy$$

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=1-z} \int_{x=-\sqrt{y}}^{x=\sqrt{y}} f \, dx \, dy \, dz$$

Project onto xz-plane
The projection is defined by the projection of the green curve and the orange curve. They are the intersection of the 2 surfaces, and it is given by $y = 1 - z = x^2$, which is a portion of a parabola shifted upward by 1 unit toward the z axis. The projection is below the curve $z = 1 - x^2$, on the xz-plane.
The minimum value of y is **not** 0, as the volume is not touching the xz-plane. Rather, the volume starts at surface to the left, $y = x^2$, which is the minimum value of y. The maximum value value of y is defined by the surface to the right, $y = 1 - z$.

$$\int_D f = \int_{2D} \int_{y=x^2}^{y=1-z} f \, dy \, dx \, dz$$

$$= \int_{x=-1}^{x=1} \int_{z=0}^{z=1-x^2} \int_{y=x^2}^{y=1-z} f \, dy \, dz \, dx$$

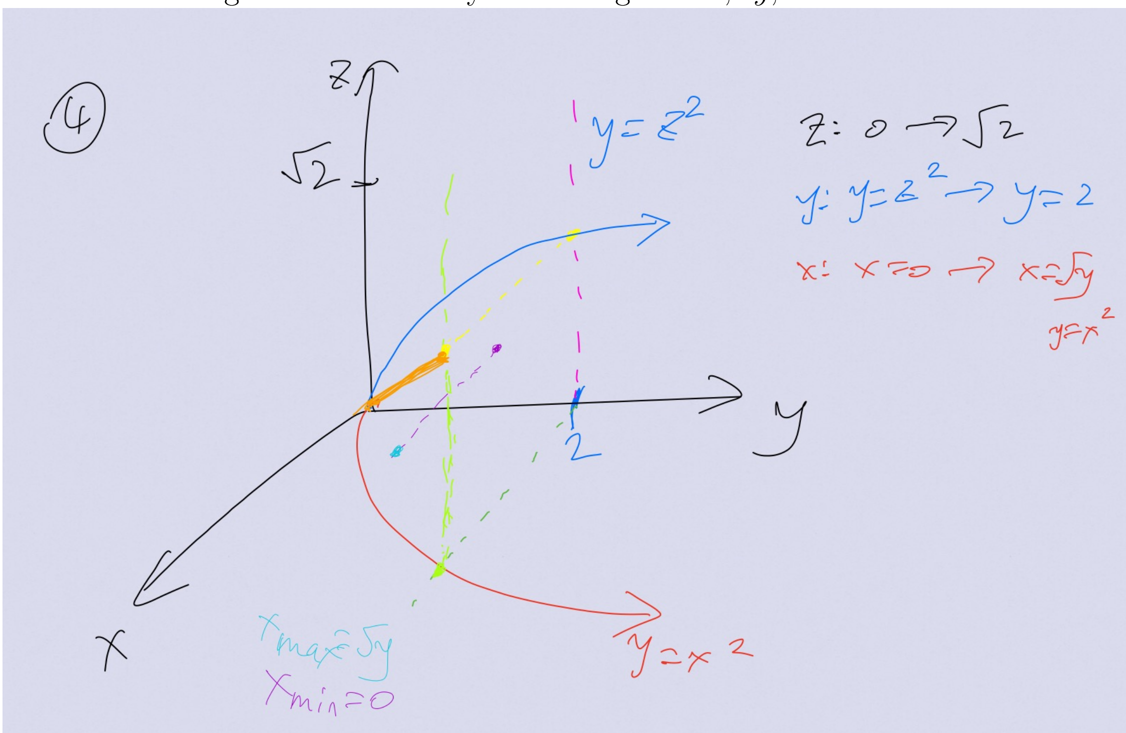
To change the order of integration for the 2D integral, we know that $z = 1 - x^2$, which translates into $x = \pm\sqrt{1-z}$. We must take both the positive and negative square roots as the minimum value of x is the negative square root, and the maximum value of x is the positive square root.

$$= \int_{z=0}^{z=1} \int_{x=-\sqrt{1-z}}^{z=\sqrt{1-z}} \int_{y=x^2}^{y=1-z} f \, dy \, dx \, dz$$

Question 4
Sketch the domain D defined by the following integral.

$$\int_0^{\sqrt{2}} \int_{z^2}^{2} \int_0^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

Rewrite the integral in 5 other ways: orderings of dx, dy, dz.



We first see that z is from 0 to $\sqrt{2}$, and y is from $y = z^2$ to $y = 2$. We sketch the blue curve $y = z^2$. We set the end to be $y = 2$, which corresponds to $z = \sqrt{2}$ exactly. Thus the projection onto the yz-plane is given by the region below $y = z^2$. We extend $y = z^2$ toward the x direction, as it does not involve x. We then see that x is from $x = 0$ to $x = \sqrt{y}$. We sketch the red curve $y = x^2$. We have the purple dot indicating the minimum value of x is zero, and the cyan dot indicating the maximum value of x is $\sqrt{y}$. We extend $y = x^2$ toward the z direction, as it does not involve z. We get the green interesection point at $(\sqrt{2}, 2, 0)$, from the surface $y = 2$ and $y = x^2$. We also get the orange inersection curve from $y = z^2$ and $y = x^2$, which ends at the yellow point $(\sqrt{2}, 2, \sqrt{2})$.

Project onto yz-plane
To change the order of integration for the 2D integral, we know that $y = z^2$, which translates into $z = \pm\sqrt{y}$. We must take the positive square root as $z > 0$ for the portion of the curve that we are interested in.

$$\int_D f = \int_{2D} \int_{x=0}^{x=\sqrt{y}} f \, dx \, dy \, dz$$

$$= \int_{z=0}^{z=\sqrt{2}} \int_{y=z^2}^{y=2} \int_{x=0}^{x=\sqrt{y}} f \, dx \, dy \, dz$$

$$= \int_{y=0}^{y=2} \int_{z=0}^{z=\sqrt{y}} \int_{x=0}^{x=\sqrt{y}} f \, dx \, dy \, dz$$

Project onto xy-plane
The projection is defined by $y = x^2$ on the xy-plane.
The minimum value of z is 0, the maximum value of z is defined by the surface on top, $y = z^2$, which translates into $z = \pm\sqrt{y}$. We must take the positive square root as $z > 0$ for the portion of the curve that we are interested in.

$$\int_D f = \int_{2D} \int_{z=0}^{z=\sqrt{y}} f \, dz \, dx \, dy$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=x^2}^{y=2} \int_{z=0}^{z=\sqrt{y}} f \, dz \, dy \, dx$$

$$= \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{y}} \int_{z=0}^{z=\sqrt{y}} f \, dz \, dx \, dy$$

Project onto xz-plane
The projection is the square , defined by the yellow intersection point $(\sqrt{2}, 2, \sqrt{2})$, which becomes $(\sqrt{2}, 0, \sqrt{2})$ after projection. However, the orange connection curve also get projected onto the xy-plane. It is the intersection of 2 surfaces , and it is given by $y = z^2 = x^2$. This translates into $z = \pm x$. We must take the positive sign, as both x and z are positive. The maximum value of y is 2, but the minimum value of y depends on whether (x, z) is to 1 side or the other side of the projected curve $z = x$. We need to split the square into 2 portions D_1 and D_2 and integrate the 2 portions separately. Let D_1 be the portion that is to the top left of the line $z = x$ on the xz-plane, and D_2 be the portion that is to the bottom right of the line $z = x$ on the xz-plane. In D_1, the minimum value of y is defined by the surface on top, which is $y = z^2$. In D_2, the minimum value of y is defined by the surface facing toward us, which is $y = x^2$.

$$\int_D f = \int_{D_1} \int_{y=z^2}^{y=2} f\, dy\, dx\, dz + \int_{D_2} \int_{y=x^2}^{y=2} f\, dy\, dx\, dz$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{z=x}^{z=\sqrt{2}} \int_{y=z^2}^{y=2} f\, dy\, dz\, dx + \int_{x=0}^{x=\sqrt{2}} \int_{z=0}^{z=x} \int_{y=x^2}^{y=2} f\, dy\, dz\, dx$$

$$= \int_{z=0}^{z=\sqrt{2}} \int_{x=0}^{x=z} \int_{y=z^2}^{y=2} f\, dy\, dx\, dz + \int_{z=0}^{z=\sqrt{2}} \int_{x=z}^{x=\sqrt{2}} \int_{y=x^2}^{y=2} f\, dy\, dx\, dz$$