MATB24 GRADED PROBLEMS 1, DUE Friday Sep 25, 11:59pm

General instructions:

- you should submit your work on Quercus. The only accepted format is pdf.
- don't wait until last minute to avoid technical difficulties
- there is a one point penalty for late submissions within 12 hours of the due date.
- you are encouraged to work in groups, ask question on piazza or in office hours. But you should write your homework individually in your own words. You can get help from me, your TA or your peers, but you should write the final solution on your own.
- unless otherwise stated in all questions you should fully justify your answer.
- your TA will grade a randomly selected subset of the questions in each homework and your grade will be only based on the graded questions.

Reading assignment:

It is assumed that you read at least one of the reading options below

- Sec 3.1,3.2 from Fraleigh Beauregard
- Sec 1.B, 1.C, 2.A from Sheldon Axler

Problem 1. *Which of the following is* not *a vector space over $\mathbb{R}$? If it is not, explain why.*

(1) $\mathbb{R}$ *itself, with its ordinary addition and scalar multiplication.*
(2) *The set $\{0\}$, with the only possible addition and scalar multiplication.*
(3) *The empty set $\varnothing$ (which has no elements).*
(4) *The set $\mathbb{R}\cup\{\infty,-\infty\}$ with elements given by the real numbers together with two additional elements ∞ and $-\infty$. The addition and scalar multiplication is as usual on the subset of $\mathbb{R}$ of $\mathbb{R}\cup\{\infty,-\infty\}$. We extend addition to all elements of $(\mathbb{R}\cup\{\infty,-\infty\})\times(\mathbb{R}\cup\{\infty,-\infty\})$ by*

$$\begin{aligned} x+\infty &= \infty = \infty + x && \text{for all } x\in\mathbb{R} \\ x+(-\infty) &= -\infty = (-\infty)+x && \text{for all } x\in\mathbb{R} \\ \infty+\infty &= \infty \\ (-\infty)+(-\infty) &= -\infty \\ \infty+(-\infty) &= 0 = (-\infty)+\infty, \end{aligned}$$

and extend scalar multiplication to all elements of $\mathbb{R}\times(\mathbb{R}\cup\{\infty,-\infty\})$ by

$$a\infty = \begin{cases} \infty & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -\infty & \text{if } a<0, \end{cases} \qquad a(-\infty) = \begin{cases} -\infty & \text{if } a>0 \\ 0 & \text{if } a=0 \\ \infty & \text{if } a<0. \end{cases}$$

Problem 2. *Let V be a vector space and let U and W be two subspaces of V. Let*

$$U\cup W = \{\vec{v}\in V \mid \vec{v}\in U \text{ or } \vec{v}\in W\}.[1]$$

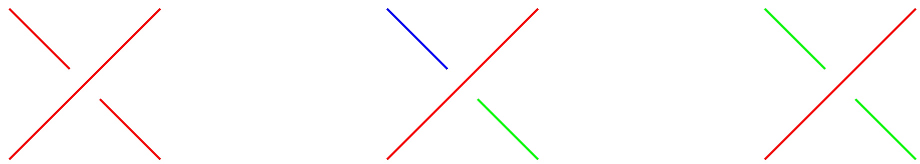
(1) *Prove that if $U\not\subseteq W$ and $W\not\subseteq U$ then $U\cup W$ is not a subspace of V.*
(2) *Give an example of V, U and W such that $U\not\subseteq W$ and $W\not\subseteq U$. Explicitly verify the implication of the statement in part (1).*
(3) *Prove that $U\cup W$ is a subspace of V if and only if $U\subseteq W$ or $W\subseteq U$.*[2]
(4) *Give an example of V, U and W such that $U\subseteq W$ or $W\subseteq U$. Explicitly verify the implication of the statement in part (3). Choose a different vector space V than the one you chose in part (2).*

Problem 3. *Consider the set $\mathbb{F}_3 := \{0,1,2\}$. Define and addition $\oplus$ and multiplication $\odot$ on $\mathbb{F}_3$ so that $(\mathbb{F}_3,\oplus,\odot)$ is a field. You can define the operators by giving operation tables*[3]*, or by defining explicit formulas. You should verify all the axioms in the definition of a field.*

[1] I suggest that you read the math hygiene note before doing this question. Specially the sections on "connectives", "negation" and "contra-positive"

[2] To prove "P if and only if Q" you should show "P implies Q" and "Q implies P" separately. Usually one side is easier. In this question you already proved the contra-positive of the difficult side.

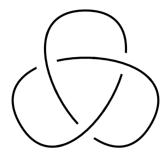
[3] similar to TUT 1



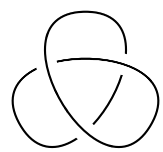
one color is allowed *three colors are allowed* *two colors is* not *allowed*

Problem 4. *Given a diagram of a knot and the three colors* $\{red, blue, green\}$*, a* tricoloring *is an assignment of a color to each line segment such that the three strands coming into each crossing either have all three colors, or just a single color.*

(1) *Explain why the trefoil diagram below has* 9 *tricolorings:*



(2) *How many tricolorings does the following slightly different knot diagram have?*



(3) *Define an addition and scalar multiplication on the set of tricolorings of a knot diagram so that it becomes a vector space over* $\mathbb{F}_3$*. (Hint: it will depend on some choices.)*

(4) *A* conjecture *is a statement that you think is true, but don't have a proof for (yet). It may turn out to be false. Come up with a conjecture about the number of tricolorings of a knot diagram.*