Techniques of the Calculus of Several Variables II
Week 12 Tutorial Worksheet Winter 2023

Name: ____________________ Student Number: ____________________

There is no new material this week. This worksheet has a mix of review questions from Week 7 to Week 12. It will help to get you ready for the exam. Good luck on all your exams!

Q1. Find the tangent plane to $\mathbf{\Phi}(s,t) = (s^2+t^2, s+2t, t)$ at $(x,y,z) = (10,5,1)$.

Q2. Compute the following line integrals.

(a) $\displaystyle\int_{\mathbf{c}} y\,dx + z^3\,dy + x^2\,dz$ where $\mathbf{c} = (t^2, t, t^3)$ and $0 \le t \le 1$.

(b) $\displaystyle\int_{\mathbf{c}} e^y\,dx + xe^y\,dy$ where $\mathbf{c}$ is the part of $x^3+xy+y^3=1$ joining $(0,1)$ to $(1,0)$.

Q3. Find the area of $x^{2/3}+y^{2/3}=1$ using a line integral.

Q4. Find (as a function of t) the surface area $A(t)$ of the part of the cylinder $x^2+y^2=2tx$ which lies between the xy-plane and the paraboloid $z = 4t^2 - x^2 - y^2$. (Hint: Use a path integral.)

Q5. Let $\mathbf{F} = (xy, x^2y, 2yz)$ and S be the part of the surface $z = x^2+y$ which lies over $[0,1]\times[0,1]$. Compute

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

Q6. Let $\mathbf{F} = (x^3-y, x, xz+e^y z)$. Compute $\displaystyle\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$ where S is the part of $z = 9-x^2-y^2$ above the xy-plane and $\mathbf{n}$ is the normal vector such that $\mathbf{n}(0,0,9) = (0,0,1)$.

Q7. Parametrize the curve of intersection of $2z = x^2+y^2$ and $x+y+z=1$.

Q8. Parametrize the surface obtained by revolved $y = 2x+1$ around the line $y = x$ for $0 \le x \le 1$.

Q9. Suppose $f(x,y,z) = x^2yz$ and $\omega = x^2y\,dx + y\,dy + (x+y+z)\,dz$. Compute:

(a) $d\omega$

(b) $df \wedge \omega$

Week 12 Tutorial

Q1

$$\left.\begin{aligned} x &= s^2+t^2=10 \\ y &= s+2t=5 \\ z &= t=1 \end{aligned}\right\} s=3, t=1$$

$\vec{\Phi}_s(s,t) = \frac{\partial}{\partial s}(s^2+t^2, s+2t, t) \qquad \vec{\Phi}_s(3,1)=(6,1,0)$

$= (2s,1,0)$

$\vec{\Phi}_t(s,t) = \frac{\partial}{\partial t}(s^2+t^2, s+2t, t) \qquad \vec{\Phi}_t(3,1)=(2,2,1)$

$= (2t,2,1)$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix} = (1,-6,10)$$

$\therefore$ tangent plane : $1\cdot(x-10)-6\cdot(y-5)+10\cdot(z-1)=0$

$x-6y+z+19=0$

Q2

a) $\int_{\vec{c}} t\,d(t^2) + t^9 d(t) + t^4 d(t^3)$

$= \int_0^1 (t\cdot 2t + t^9\cdot 1 + t^4\cdot 3t^2)\,dt$

$= \int_0^1 2t^2 + t^9 + 3t^6\,dt$

$= \frac{2}{3}t^3 + \frac{1}{10}t^{10} + \frac{3}{7}t^7 \Big|_0^1$

$= \frac{2}{3} + \frac{1}{10} + \frac{3}{7}$

b) Note : $\nabla(xe^y) = (e^y, xe^y)$

$\int_{\vec{c}} (e^y, xe^y)\cdot(dx,dy)$

$= \int_{\vec{c}} \nabla(xe^y)\cdot d\vec{s}$

$= \int_{(0,1)}^{(1,0)} \nabla(xe^y)\,d\vec{s} = xe^y \Big|_{(0,1)}^{(1,0)} = 1\cdot e^0 - 0\cdot e^1 = 1$

Q3 Parametrize: $(x^{\frac{1}{3}})^2+(y^{\frac{1}{3}})^2=1$

$$\begin{cases} x^{\frac{1}{3}}=\cos(\theta) \\ y^{\frac{1}{3}}=\sin(\theta)\end{cases} \Rightarrow \begin{cases} x=\cos^3(\theta) \\ y=\sin^3(\theta)\end{cases}, \theta\in[0,2\pi]$$

By Green's Thm, (a quarter of S)

$$\begin{aligned}
A &= \frac{1}{2}\int_{\partial D} x\,dy-y\,dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^3(\theta)\,d(\sin^3(\theta))-\sin^3(\theta)\,d(\cos^3(\theta)) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}[3\cos^3(\theta)\sin^2(\theta)\cdot\cos(\theta)+3\sin^3(\theta)\cos^2(\theta)\sin(\theta)]\,d\theta \\
&= \frac{3}{2}\int_0^{\frac{\pi}{2}}[\cos^4(\theta)\sin^2(\theta)+\sin^4(\theta)\cos^2(\theta)]\,d\theta \\
&= \frac{3}{2}\int_0^{\frac{\pi}{2}}\sin^2(\theta)\cos^2(\theta)[\cos^2(\theta)+\sin^2(\theta)]\,d\theta \\
&= \frac{3}{2}\int_0^{\frac{\pi}{2}}[\frac{1}{2}\sin(2\theta)]^2\,d\theta \\
&= \frac{3}{2}\int_0^{\frac{\pi}{2}}[\frac{1}{4}\sin(2\theta)]\,d\theta \\
&= \frac{3}{8}\int_0^{\frac{\pi}{2}}\frac{1-\cos(4\theta)}{2}\,d\theta \\
&= \frac{3}{8}\left(\frac{1}{2}\theta\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\cdot\frac{1}{4}\sin(4\theta)\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{3\pi}{32}
\end{aligned}$$

$$A(S)=4\cdot\frac{3\pi}{32}=\frac{3\pi}{8}$$

Q4 $x^2-2tx+y^2=0$ $\qquad \begin{cases} x=t+t\cos(\theta) \\ y=t\sin(\theta)\end{cases}$

$(x-t)^2+y^2=t^2$

$\vec{c}(\theta)=(t+t\cos(\theta), t\sin(\theta))$, $\theta\in[0,2\pi]$

$\vec{c}'(\theta)=(-t\sin(\theta), t\cos(\theta))$

$$\begin{aligned}
\|\vec{c}'(\theta)\| &= \sqrt{[-t\sin(\theta)]^2+[t\cos(\theta)]^2} \\
&= \sqrt{t^2\sin^2(\theta)+t^2\cos^2(\theta)} \\
&= \sqrt{t^2}=t
\end{aligned}$$

$$\begin{aligned}
A(t) &= \int_c f\cdot ds \\
&= \int_a^b f(\vec{c}(t))\cdot\|\vec{c}'(t)\|\,dt \\
&= \int_0^{2\pi}[4t^2-(t+t\cos(\theta))^2-(t\sin(\theta))^2]\cdot t\,d\theta \\
&= \int_0^{2\pi}[2t^2-2t^2\cos(\theta)]\cdot t\,d\theta \\
&= \int_0^{2\pi}2t^3-2t^3\cos(\theta)\,d\theta \\
&= 2t^3\theta\Big|_0^{\pi}-2t^3\sin(\theta)\Big|_0^{2\pi} \\
&= 4\pi t^3-0=4\pi t^3
\end{aligned}$$

Q5 Parametrize: z

$\Phi(u,v) = (u, v, u^2+v)$, $u \in [0,1], v \in [0,1]$

$\Phi_u = (1, 0, 2u)$, $\Phi_v = (0,1,1)$

$\vec{n} = \Phi_u \times \Phi_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 1 \end{vmatrix} = (-2u, -1, 1)$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} \, du dv$

$= \int_0^1 \int_0^1 (uv, u^2v, 2v(u^2+v)) \cdot (-2u, -1, 1) \, du dv$

$= \int_0^1 \int_0^1 -2u^2v - u^2v + 2u^2v + 2v^2 \, du dv$

$= \int_0^1 \int_0^1 -u^2v + 2v^2 \, du dv$

$= \int_0^1 -\frac{1}{3}u^3v \Big|_{u=0}^{u=1} + 2v^2u \Big|_{u=0}^{u=1} dv$

$= \int_0^1 -\frac{1}{3}v + 2v^2 \, dv$

$= -\frac{1}{3} \cdot \frac{1}{2}v^2 \Big|_0^1 + \frac{2}{3}v^3 \Big|_0^1$

$= -\frac{1}{6} + \frac{2}{3} = \frac{1}{2}$

Q6

$\partial S : x^2+y^2=9$ when $z=0$

Parametrize: $\begin{cases} x = 3\cos(\theta) \\ y = 3\sin(\theta) \end{cases}$, $\theta \in [0, 2\pi]$

$\partial S : \vec{c}(\theta) = (3\cos(\theta), 3\sin(\theta), 0)$, $\vec{c}'(\theta) = (-3\sin(\theta), 3\cos(\theta), 0)$

$\iint_S \text{curl}(\vec{F}) \, d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$

$= \int_0^{2\pi} (x-y, x, xz + e^y z) \cdot (-3\sin(\theta), 3\cos(\theta), 0) \, d\theta$

$= \int_0^{2\pi} 3(\cos(\theta) - \sin(\theta)) \cdot (-3\sin(\theta)) + 3\cos(\theta) \cdot 3\cos(\theta) + 0 \, d\theta$

$= \int_0^{2\pi} 9\sin^2(\theta) - 9\sin(\theta)\cos(\theta) + 9\cos^2(\theta) \, d\theta$

$= \int_0^{2\pi} 9 - 9\sin(\theta)\cos(\theta) \, d\theta$

$= 9\theta \Big|_0^{2\pi} - \frac{9}{2} \int_0^{2\pi} \sin(2\theta) \, d\theta$

$= 18\pi + \frac{9}{2} \cdot \frac{\cos(2\theta)}{2} \Big|_0^{2\pi}$

$= 18\pi + \frac{9}{4}(1-1) = 18\pi$

Q7

$\frac{x^2+y^2}{2} = 1 - x - y$

$x^2 + 2x + y^2 + 2y = 2$

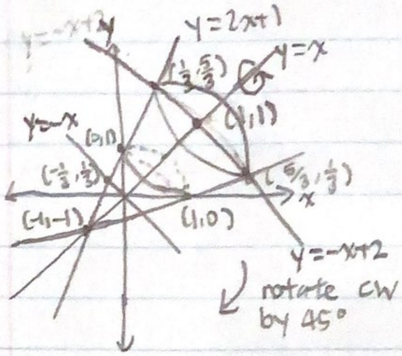
$(x+1)^2 + (y+1)^2 = 4$

$\begin{cases} x = -1 + 2\cos(\theta) \\ y = -1 + 2\sin(\theta) \end{cases}$

$z = 1 - x - y = 3 - 2\cos(\theta) - 2\sin(\theta)$

$\vec{c}(\theta) = (-1 + 2\cos(\theta), -1 + 2\sin(\theta), 3 - 2\cos(\theta) - 2\sin(\theta))$, $\theta \in [0, 2\pi]$

Q8



Segment from $(0,0)$ to $(1,1)$ has length $\sqrt{2}$.
The closest point to $(0,0)$ on $y=2x+1$ is $(-\frac{1}{3},\frac{1}{3})$.
The closest point to $(1,1)$ on $y=2x+1$ is $(\frac{1}{3},\frac{5}{3})$.
Regard $\vec{r}(t)$ as the radius function of the surface.

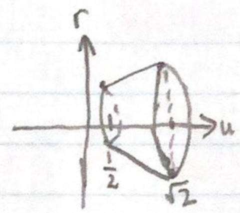
$$\vec{r}(0)=\sqrt{(-\tfrac{1}{3})^2+(\tfrac{1}{3})^2}=\frac{\sqrt{2}}{3}$$

$$\vec{r}(\sqrt{2})=\sqrt{(\tfrac{1}{3}-1)^2+(\tfrac{5}{3}-1)^2}=\frac{2\sqrt{2}}{3}$$

$$\therefore \vec{r}(t)=\frac{\frac{2\sqrt{2}}{3}-\frac{\sqrt{2}}{3}}{\sqrt{2}}t+\frac{\sqrt{2}}{3}=\frac{1}{3}t+\frac{\sqrt{2}}{3},\ u\in[0,\sqrt{2}]$$
$$v\in[0,2\pi]$$

$$\Phi(u,v)=(u,r(u)\cos(v),r(u)\sin(v))$$

conical fustrum



Let x be the parameter u.

Q9 a)
$$d\omega=d(x^2y\,dx+y\,dy+(x+y+z)\,dz)$$
$$=d(x^2y)\wedge dx+(x^2y)\wedge\underbrace{d(dx)}_{=0}+\underbrace{dy\wedge dy}_{=0}+y\wedge\underbrace{d(dy)}_{=0}+d(x+y+z)\wedge dz+(x+y+z)\wedge\underbrace{d(dz)}_{=0}$$
$$=\left(\frac{\partial(x^2y)}{\partial x}dx+\frac{\partial(x^2y)}{\partial y}dy\right)\wedge dx+\left(\frac{\partial(x+y+z)}{\partial x}dx+\frac{\partial(x+y+z)}{\partial y}dy+\frac{\partial(x+y+z)}{\partial z}dz\right)\wedge dz$$
$$=2xy\,\overbrace{dx\wedge dx}^{=0}+x^2dy\wedge dx+dx\wedge dz+dy\wedge dz+\underbrace{dz\wedge dz}_{=0}$$
$$=x^2\,dydx+dxdz+dydz$$

b)
$$df=d(x^2yz)$$
$$=\frac{\partial(x^2yz)}{\partial x}dx+\frac{\partial(x^2yz)}{\partial y}dy+\frac{\partial(x^2yz)}{\partial z}dz$$
$$=2xyz\,dx+x^2z\,dy+x^2y\,dz$$
$$df\wedge\omega=(2xyz\,dx+x^2y\,dy+x^2y\,dz)\wedge(x^2y\,dx+y\,dy+(x+y+z)\,dz)$$
$$=2x^3y^2z\,\overbrace{dx\wedge dx}^{=0}+2xy^2dx\wedge dy+2xyz(x+y+z)dx\wedge dz+x^4yz\,dy\wedge dx$$
$$+x^2y^2dy\wedge dz+x^2y(x+y+z)\overbrace{dz\wedge dz}^{=0}$$
$$=(2xy^2-x^4yz)\,dxdy+(2xyz(x+y+z)-x^4y^2)\,dxdz+(x^2z(x+y+z)-x^2y^2)\,dydz$$