Solution to Assignment #7

1. Section 2.3: # 2, 4, 10, 12, 14, 16, 21, 23, 24, 29, 30, 32, 34

2. Is $T([x_1, x_2, x_3]) = [0, 0, 0, 0]$ a linear transformation of $\mathbf{R}^3$ into $\mathbf{R}^4$? Why or why not?

   Answer: Let $\mathbf{u} = [u_1, u_2, u_3], \mathbf{v} = [v_1, v_2, v_3] \in \mathbf{R}^3, r \in \mathbf{R}$, then :

   $T(\mathbf{u})+T(\mathbf{v}) = T([u_1, u_2, u_3])+T([v_1, v_2, v_3]) = [0, 0, 0, 0]+[0, 0, 0, 0] = [0, 0, 0, 0] = T([u_1 + v_1, u_2 + v_2, u_3 + v_3]) = T(\mathbf{u} + \mathbf{v})$

   and

   $T(r\mathbf{u}) = T([ru_1, ru_2, ru_3]) = [0, 0, 0, 0] = r[0, 0, 0, 0] = rT(\mathbf{u})$

   Thus T is a linear transformation.

4. Is $T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$ a linear transformation of $\mathbf{R}^2$ into $\mathbf{R}^3$? Why or why not?

   Answer: No, because $T([0, 0]) = [0, 1, 0] \neq [0, 0, 0]$. This is a violation of the property of linear transformations that they map the zero vector into a zero vector.

10. If $T([-1, 1]) = [2, 1, 4]$ and $T([1, 1]) = [-6, 3, 2]$, find $T([x, y])$.

    Solution: Since $[-1, 1], [1, 1]$ are two nonzero, nonparallel vector in $\mathbf{R}^2$, they form a basis for $\mathbf{R}^2$. Thus $[x, y]$ can be written as a unique linear combination of $[-1, 1], [1, 1]$. Let $a[-1, 1] + b[1, 1] = [x, y]$ this is equivalent to

    $$\left[\begin{array}{rr|c} -1 & 1 & x \\ 1 & 1 & y \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & -x \\ 0 & 2 & x+y \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & (-x+y)/2 \\ 0 & 1 & (x+y)/2 \end{array}\right]$$

    Thus:

    $$[x, y] = \tfrac{(-x+y)}{2}[-1, 1] + \tfrac{(x+y)}{2}[1, 1] \qquad \text{and}$$

    $$\begin{aligned} T([x, y]) &= \tfrac{(-x+y)}{2}T([-1, 1]) + \tfrac{(x+y)}{2}T([1, 1]) \\ &= \tfrac{(-x+y)}{2}[2, 1, 4] + \tfrac{(x+y)}{2}[-6, 3, 2] \\ &= [-4x - 2y, x + 2y, -x + 3y] \end{aligned}$$

12. If $T([2,3,0]) = 8, T([1,2,-1]) = -5$ and $T([4,5,1]) = 17$, find $T([-3,11,-4])$.

Solution:

We need to write the vector $[-3,11,-4]$ as a linear combination of the vectors $[2,3,0],[1,2,-1],[4,5,1]$. So let $[-3,11,-4] = a[2,3,0]+b[1,2,-1]+c[4,5,1]$ which has a an augmented matrix the matrix:

$$\left[\begin{array}{rrr|r} 2 & 1 & 4 & -3 \\ 3 & 2 & 5 & 11 \\ 0 & -1 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 1 & 4 & -3 \\ 1 & 1 & 1 & 14 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 2 & 1 & 4 & -3 \\ 0 & -1 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 0 & -1 & 2 & -31 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 0 & 1 & -2 & 31 \\ 0 & 0 & -1 & 27 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & 41 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & -27 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 64 \\ 0 & 1 & 0 & -23 \\ 0 & 0 & 1 & -27 \end{array}\right]$$

Thus $[-3,11,-4] = 64[2,3,0]-23[1,2,-1]-27[4,5,1]$ and

$$\begin{aligned} T([-3,11,-4]) &= 64T([2,3,0]) - 23T([1,2,-1]) - 27T([4,5,1]) \\ &= 64(8) - 23(-5) - 27(17) = 168 \end{aligned}$$

14. Give the standard matrix representation of the given linear transformation, $T([x_1,x_2]) = [2x_1 - x_2, x_1 + x_2, x_1 + 3x_2]$.

Solution. To find the standard matrix representation we need the images of the vectors in the standard basis of $\mathbf{R}^2$:

$T(\mathbf{e}_1) = T([1,0]) = [2,1,1], T(\mathbf{e}_2) = T([0,1]) = [-1,1,3]$ and

$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$ is the standard matrix representation of T.

16. Give the standard matrix representation of the given linear transformation, $T([x_1,x_2,x_3]) = [2x_1 + x_2 + x_3, x_1 + x_2 + 3x_3]$.

Solution. To find the standard matrix representation we need the images of the vectors in the standard basis of $\mathbf{R}^3$:

$T(\mathbf{e}_1) = T([1,0,0]) = [2,1], T(\mathbf{e}_2) = T([0,1,0]) = [1,1]$ and $T(\mathbf{e}_3) = T([0,0,1]) = [1,3]$, therefore:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ is the standard matrix representation of T.

21. Determine whether the linear transformation $T([x_1, x_2]) = [x_1 + x_2, x_1 - 3x_2]$ is invertible. If it is, find a formula for $T^{-1}(\mathbf{x})$ in row notation. If it is not, explain why it is not.

Solution. The standard matrix representation of T is the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$.

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & -1/4 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3/4 & 1/4 \\ 0 & 1 & 1/4 & -1/4 \end{array}\right]$$

Therefore T is invertible and $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \left[\ \frac{1}{4}(3x_1 + x_2),\ \ \frac{1}{4}(x_1 - x_2)\right]$.

23. Determine whether the linear transformation $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$ is invertible. If it is, find a formula for $T^{-1}(\mathbf{x})$ in row notation. If it is not, explain why it is not.

Solution. The standard matrix representation of T is the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

Therefore T is invertible and $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = [x_3, x_2 - x_3, x_1 - x_2]$.

24. Determine whether the linear transformation $T([x_1, x_2, x_3]) = [2x_1 + x_2 + x_3, x_1 + x_2 + 3x_3]$ is invertible. If it is, find a formula for $T^{-1}(\mathbf{x})$ in row notation. If it is not, explain why it is not.

Solution. The standard matrix representation of T is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ which is nor a square matrix, therefore not invertible.

Therefore T is not invertible.

29. T F T T T F T T F T

30. Verify that $T^{-1}(T(\mathbf{x})) = \mathbf{x}$ for the linear transformation $T([x_1, x_2, x_3]) = [x_1 - 2x_2 + x_3, x_2 - x_3, 2x_2 - 3x_3]$ in Example 9 of the text.

    Answer: From example 9 in this section we see that T is invertible and $T^{-1}([x_1, x_2, x_3]) = [x_1 + 4x_2 - x_3, 3x_2 - x_3, 2x_2 - x_3]$.

    Let $\mathbf{x} = [x_1, x_2, x_3]$, then

$$\begin{aligned} T^{-1}(T(\mathbf{x})) &= T^{-1}(T([x_1, x_2, x_3])) = T^{-1}([x_1 - 2x_2 + x_3, x_2 - x_3, 2x_2 - 3x_3]) \\ &= [x_1 - 2x_2 + x_3 + 4(x_2 - x_3) - (2x_2 - 3x_3), \\ &\quad 3(x_2 - x_3) - (2x_2 - 3x_3), 2(x_2 - x_3) - (2x_2 - 3x_3)] \\ &= [x_1, x_2, x_3] \end{aligned}$$

32. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Prove from Definition 2.3 that $T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and all scalars r and s.

    Proof: Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, then by

$$\begin{aligned} T(r\mathbf{u} + s\mathbf{v}) &= T(r\mathbf{u}) + T(s\mathbf{v}) && \text{by preservation of addition} \\ &= rT(\mathbf{u}) + sT(\mathbf{v}) && \text{by preservation of scalar multiplication} \end{aligned}$$

34. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and let U be a subspace of $\mathbf{R}^m$. Prove that the inverse image $T^{-1}[U]$ is a subspace of $\mathbf{R}^n$..

    Proof: (i) Now suppose that U is a subspace of $\mathbf{R}^m$. This means that $\mathbf{0}' \in U$ and since $T(\mathbf{0}) = \mathbf{0}'$, then $\mathbf{0} \in T^{-1}[U]$, so $T^{-1}[U]$ is non empty.

    (ii) Let $\mathbf{u}, \mathbf{v} \in T^{-1}[U]$. This means that $T(\mathbf{u}), T(\mathbf{v}) \in U$, therefore $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \in U$, since U is a subspace. Thus $\mathbf{u} + \mathbf{v} \in T^{-1}[U]$.

    (iii) Let $r \in \mathbf{R}$ and $\mathbf{u}, \in T^{-1}[U]$. This means that $T(\mathbf{u}), \in U$, therefore $T(r\mathbf{u}) = rT(\mathbf{u}) \in U$, since U is a subspace. Thus $r\mathbf{u} \in T^{-1}[U]$.

    (i), (ii) and (iii) prove that $T^{-1}[U]$ is a subspace of $\mathbf{R}^n$.

2. Let $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be a linear transformation and W be a subspace of $\mathbf{R}^n$. Prove that $T[W]$ is a subspace of $\mathbf{R}^m$,

Proof:

(i) Since W is a subspace of $\mathbf{R}^n$, then it is nonempty. Let $\mathbf{v}$ be in W, then $T(\mathbf{v}) \in T[W]$ showing that $T[W]$ is nonempty.

(ii) If $\mathbf{u}, \mathbf{v} \in T[W]$, then there exist vectors $\mathbf{x}, \mathbf{y} \in W$ so that $T(\mathbf{x}) = \mathbf{u}$ and $T(\mathbf{y}) = \mathbf{v}$.

Furthermore, $\mathbf{x} + \mathbf{y} \in W$ since W is a subspace of $\mathbf{R}^n$ giving that $T(\mathbf{x} + \mathbf{y}) \in T[W]$. Thus, $\mathbf{u} + \mathbf{v} = T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y}) \in T[W]$.

(iii) If $\mathbf{u}, \in T[W]$, then there exist vectors $\mathbf{x} \in W$ so that $T(\mathbf{x}) = \mathbf{u}$.

For any $r \in \mathbf{R}$, $r\mathbf{x} \in W$ since W is a subspace of $\mathbf{R}^n$ giving that $T(r\mathbf{x}) \in T[W]$. Thus, $r\mathbf{u} = rT(\mathbf{x}) = T(r\mathbf{x}) \in T[W]$.

(i), (ii) and (iii) show that $T[W]$ is a subspace of $\mathbf{R}^m$.

3. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbf{R}^3$. Assume $T(\mathbf{v}_1) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $T(\mathbf{v}_3) = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

a) Determine whether $\mathbf{w} = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$ is in the $range(T)$.

b) Find a basis for $range(T)$.

c) Find the $nullity(T)$.

Solution:

Since $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{R}^3$, then

$$range(T) = sp\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

a) $\mathbf{w} = \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}$ is in the $range(T)$ if $\mathbf{w} \in col(A)$ where $A = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$$[A|\mathbf{w}] = \left[\begin{array}{rrr|r} -2 & 0 & -2 & -6 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 2 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & -2 & -2 & -5 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 2 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

This shows that $A\mathbf{x} = \mathbf{w}$ is inconsistent. Thus $\mathbf{w}$ is not in the range of T.

b) Basis for the range of T is the nullspace of A which is $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

c) $A\mathbf{x} = \mathbf{0}$ if

$2x_2 + 2x_3 = 0$ or $x_2 = -x_3$ and

$x_1 + x_2 + 2x_3 = 0$. Thus, $x_1 - x_3 + 2x_3 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

Thus $Ker(T)$ is $= \left\{ \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} \right\} = \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$.

Therefore $nullity(T) = nullity(A) = dim(nullspace(A)) = 1$.

4. Let $\mathbf{a} = [a_1, a_2]$ be a nonzero vector in $\mathbf{R}^2$ and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(\mathbf{x}) = \left(\dfrac{\mathbf{x} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right) \mathbf{a}$.

(a) Find $T([x, y])$. (b) Prove the T is a linear transformation.
(c) Find the standard matrix representation of T.

Proof: (a) $T(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}$, therefore:

$T([x, y]) = \left(\frac{[x,y] \cdot [a_1,a_2]}{a_1^2 + a_2^2} \right) [a_1, a_2] = \left(\frac{xa_1 + ya_2}{a_1^2 + a_2^2} \right) [a_1, a_2]$

(b) (i) Let $\mathbf{u} = [u_1, u_2], \mathbf{v} = [v_1, v_2] \in \mathbf{R}^2$, then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T([u_1 + v_1, u_2 + v_2]) = \left(\frac{(u_1 + v_1)a_1 + (u_2 + v_2)a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] \\ &= \left(\frac{u_1a_1 + v_1a_1 + u_2a_2 + v_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] \\ &= \left(\frac{u_1a_1 + u_2a_2}{a_1^2 + a_2^2} + \frac{v_1a_1 + v_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] \\ &= \left(\frac{u_1a_1 + u_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] + \left(\frac{v_1a_1 + v_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(ii) Let $\mathbf{u} = [u_1, u_2] \in \mathbf{R}^2$ and $r \in \mathbf{R}$, then

$$T(r\mathbf{u}) = T([ru_1, ru_2]) = \left(\frac{ru_1a_1 + ru_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] = r \left(\frac{u_1a_1 + u_2a_2}{a_1^2 + a_2^2} \right) [a_1, a_2] = rT(\mathbf{u})$$

(i) and (ii) show that T is a linear transformation.

(c) $T(\mathbf{e}_1) = T([1,0]) = \left(\dfrac{[1,0]\cdot[a_1,a_2]}{a_1^2+a_2^2}\right)[a_1,a_2] = \left(\dfrac{a_1}{a_1^2+a_2^2}\right)[a_1,a_2]$

$= \left(\dfrac{1}{a_1^2+a_2^2}\right)[a_1^2, a_1a_2]$

$T(\mathbf{e}_2) = T([0,1]) = \left(\dfrac{[0,1]\cdot[a_1,a_2]}{a_1^2+a_2^2}\right)[a_1,a_2] = \left(\dfrac{a_2}{a_1^2+a_2^2}\right)[a_1,a_2]$

$= \left(\dfrac{1}{a_1^2+a_2^2}\right)[a_1a_2, a_2^2]$

The standard matrix representation of T is

$A = \left(\dfrac{1}{a_1^2+a_2^2}\right)\begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix}$

5. Using linear transformation methods show that matrix multiplication is not commutative.

Proof: If T_1, T_2 are linear transformations such that $T_1 : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ and $T_2 : \mathbf{R}^k \longrightarrow \mathbf{R}^n$ where $m \neq k$, then $T_1 \circ T_2$ is defined but $T_2 \circ T_1$ is not defined since the $range(T_1) \subset \mathbf{R}^m \not\subset domain(T_2) = \mathbf{R}_k$.

Choose linear transformations $T_1 : \mathbf{R}^n \longrightarrow \mathbf{R}^k$ and $T_2 : \mathbf{R}^k \longrightarrow \mathbf{R}^n$ so that $T_1 \circ T_2 \neq T_2 \circ T_1$. We know this is possible.

(One such example is $T_1 : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ defined by $T_1([x,y]) = [x+y, x+2y]$ $T_2 : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ defined by $T_2([x,y]) = [2x-y, x+y]$ Then $T_1 \circ T_2([x,y]) = T_1(T_2([x,y])) = T_1([[2x-y, x+y]) = [3x, 4x+y]$ where $T_2 \circ T_1([x,y]) = T_2(T_1([x,y])) = T_2([[x+y, x+2y]) = [x, 2x+3y]$. $T_1 \circ T_2([x,y]) \neq T_2 \circ T_1([x,y])$.)

Now if A, B are the standard matrix representations of T_1, T_2 respectively then

$(AB)\mathbf{x} = A(T_2\mathbf{x}) = T_1(T_2(\mathbf{x})) = T_1 \circ T_2(\mathbf{x}) \neq T_2 \circ T_1(\mathbf{x}) = T_2(T_1(\mathbf{x})) = B(T_1(\mathbf{x})) = B(A\mathbf{x}) = BA\mathbf{x}$

Thus $AB \neq BA$.

6. Using linear transformation methods show that matrix multiplication is associative.

Proof: Let A be an $m \times n$ matrix, B be an $n \times k$ matrix and C be a $k \times q$ matrix. Using linear transformation methods we will show that matrix multiplication is associative, i.e we will prove that $A(BC) = (AB)C$.

Since A be an $m \times n$ matrix, B be an $n \times k$ matrix and C be a $k \times q$ matrix. Then $A(BC)$ and $(AB)C$ are both defined and are both $m \times q$ matrices. Now we need to show they are equal.

A, B, C can be thought of as standard matrix representations of the linear transformations T_1, T_2, T_3 respectively so that for $\mathbf{x}'' \in \mathbf{R}^n$, $A\mathbf{x}'' = T_1(\mathbf{x}'')$, for $\mathbf{x}' \in \mathbf{R}^k$, $B\mathbf{x}' = T_2(\mathbf{x}')$ and for $\mathbf{x} \in \mathbf{R}^q$, $C\mathbf{x} = T_3(\mathbf{x})$

Then $T_1 : \mathbf{R}^n \to \mathbf{R}^m$, $T_2 : \mathbf{R}^k \to \mathbf{R}^n$ and $T_3 : \mathbf{R}^q \to \mathbf{R}^k$.

Let $\mathbf{x}$ be an arbitrary vector in $\mathbf{R}^q$. Then for some $\mathbf{y} \in \mathbf{R}^k, \mathbf{z} \in \mathbf{R}^n, \mathbf{w} \in \mathbf{R}^m$ we have that $T_3(\mathbf{x}) = \mathbf{y}$, $T_2(\mathbf{y}) = \mathbf{z}$ and $T_1(\mathbf{z}) = \mathbf{w}$

$T_1 \circ (T_2 \circ T_3)(\mathbf{x}) = T_1((T_2 \circ T_3)(\mathbf{x})) = T_1(T_2(T_3(\mathbf{x}))) = T_1(T_2(\mathbf{y})) = T_1(\mathbf{z}) = \mathbf{w}$

$((T_1 \circ T_2) \circ T_3)(\mathbf{x}) = ((T_1 \circ T_2)(T_3(\mathbf{x})) = (T_1 \circ T_2)(\mathbf{y}) T_1(T_2(\mathbf{y})) = T_1(\mathbf{z}) = \mathbf{w}$.

Since $T_1 \circ (T_2 \circ T_3)(\mathbf{x}) = ((T_1 \circ T_2) \circ T_3)(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^q$, then $T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$.

Furthermore it was shown that "*A composition of two linear transformations T and T' yields a linear transformation $T' \circ T$ having as its associated matrix the product of the matrices associated with T' and T, in that order* (page 150).

Applying this we know that $T_1 \circ (T_2 \circ T_3)$ is a linear transformation with matrix representation $A(BC)$ and $(T_1 \circ T_2) \circ T_3$ is a linear transformation with matrix representation $(AB)C$.

Since for any $\mathbf{x} \in \mathbf{R}^q$ we have that

$(AB)C\mathbf{x} = ((T_1 \circ T_2) \circ T_3))(\mathbf{x}) = (T_1 \circ (T_2 \circ T_3))(\mathbf{x}) = A(BC)\mathbf{x}$ then

$(AB)C = A(BC)$.