1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.

   (a) (3 points) Let V be an $\mathbb{F}$-vector space, with $\mathbb{F}$ given by $\mathbb{R}$ or $\mathbb{C}$. A map $\langle\ ,\ \rangle\colon V \times V \to \mathbb{F}$ is an *inner product* on V if it is positive definite, symmetric, and linear in the first entry.

   (b) (3 points) A set of unit vectors $\{\vec{b}_1, \cdots, \vec{b}_k\}$ in an inner product space is *orthonormal* if the $\vec{b}_i$'s are mutually orthogonal.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counterexample if you think it is false).

   (a) (3 points) Let W be a subspace of a finite dimensional inner product space V. Suppose the linear map $\mathrm{T}\colon V \to V$ is defined by $\vec{x} \mapsto \mathrm{Proj}_W \vec{x}$ where Proj_W denotes the orthogonal projection onto W. Then $\dim \mathrm{im}(T) = \dim V - \dim W^{\perp}$.

   (b) (3 points) Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be nonzero vectors in $\mathbb{R}^3$. Then $\vec{v}_1, \vec{v}_2 - \mathrm{Proj}_{\mathrm{Span}(\vec{v}_1)}\vec{v}_2$ and $\vec{v}_3 - \mathrm{Proj}_{\mathrm{Span}(\vec{v}_1,\vec{v}_2)}\vec{v}_3$ is an orthogonal basis for $\mathbb{R}^3$.

   (c) (3 points) Recall that the trace of a matrix is the sum of its diagonal entries. Let $M_{n\times n}(\mathbb{C})$ be the complex vector space of $(n \times n)$-matrices, then $\langle A, B\rangle = \mathrm{trace}(A + B)$ is an inner product on $M_{n\times n}(\mathbb{C})$.

   (d) (3 points) Use the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ on the real vector space $C^0[0,1]$ of real-valued continuous functions on the domain $[0,1]$. The orthogonal projection of x^3 onto the subspace $\mathrm{Span}\{1, x\}$ is given by $\left(\int_0^1 x^3\,dx\right) 1 + \left(\int_0^1 x^4\,dx\right) x \in C^0[0,1]$.

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object doesn't exist.

   (a) (2 points) A non-identity matrix $A \in M_n(\mathbb{R})$, such that the function $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w}$ for $\vec{v}, \vec{w} \in \mathbb{R}^n$, is an inner product on $\mathbb{R}^n$.

   (b) (2 points) Two different inner products $\langle -,-\rangle_1$ and $\langle -,-\rangle_2$ on $\mathbb{C}^2$.

   (c) (2 points) Mutually orthogonal vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ with respect to the Euclidean (complex) inner product in $\mathbb{C}^3$.

   (d) (2 points) A vector in the orthogonal complement of $\mathrm{Span}\{\begin{bmatrix}1 & 2 & -1\end{bmatrix}^T, \begin{bmatrix}-1 & 3 & 2\end{bmatrix}^T\} \subset \mathbb{R}^3$.

4. Let $U = \left\{\left(\begin{smallmatrix} a & b \\ 0 & c\end{smallmatrix}\right) \,\middle|\, a, b, c \in \mathbb{R}\right\}$ be the vector space of all upper triangular 2×2 matrices. Let $\mathcal{B} = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 0 & 1\end{smallmatrix}\right]\right)$ and $\mathcal{E}$ $\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right]\right)$ be two bases of U, and let $D\colon U \to U$ be the linear transformation defined by

$$D\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} = \begin{pmatrix} a+b & b+2c \\ 0 & c\end{pmatrix}.$$

   (a) (5 points) Find the matrix $[D]_{\mathcal{B}}$ of D with respect to the basis $\mathcal{B}$.

   (b) (5 points) Find the matrix $[D]_{\mathcal{E}}$ of D with respect to the basis $\mathcal{E}$.

   (c) (5 points) Find an invertible matrix S such that $S[D]_{\mathcal{B}} = [D]_{\mathcal{E}}S$.

5. Recall that the trace of a matrix is the sum of its diagonal entries. Let $M_{2\times 2}(\mathbb{R})$ be the real vector space of (2×2)-matrices with inner product $\langle A, B\rangle = \frac{1}{2}\mathrm{trace}(A^T B)$. Parts (a)–(c) below refer to this inner product.

   (a) (4 points) Find $\|Q\|$, where Q is any orthogonal (2×2)-matrix (that is, the columns of Q are orthonormal).

   (b) (4 points) For any angle θ, let $R_\theta = \left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{smallmatrix}\right]$ be the standard matrix of the counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. Given angles $\alpha, \beta \in \mathbb{R}$, find the inner product $\langle R_\alpha, R_\beta\rangle$ in terms of α and β.

   (c) (4 points) Given $\alpha, \beta \in \mathbb{R}$, find the angle φ between R_α and R_β in $M_{2\times 2}(\mathbb{R})$.

6. Suppose that V is an n-dimensional vector space and that $T\colon V \to V$ satisfies $T^2 = T$. You can use the fact that every vector $\vec{v} \in V$ can be written uniquely as a sum $\vec{v} = \vec{u} + \vec{w}$ with $\vec{u} \in \mathrm{im}(T)$ and $\vec{w} \in \mathrm{im}(\mathrm{id} - T)$ without proof.

   (a) (7 points) Prove that there exists a basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix with each of its diagonal entries equal to either 1 or 0. (Hint: pick a basis of $\mathrm{im}(T)$ and a basis of $\mathrm{im}(\mathrm{id} - T)$.)

Please attach a signed copy of the statement below in your handwriting. Your signature under the statement is required for your exam to be graded.

"I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own."