Q10

1.

a) RV $X \sim N(\mu_X = 0, \sigma_X^2 = 4)$ $\quad X \perp W$

$W \sim N(\mu_W = 2, \sigma_W^2 = 2)$

$V = X + W$

$$\begin{pmatrix} X \\ W \end{pmatrix} = N\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\right)$$ We know $V \sim N(\mu, \sigma^2)$

$$\begin{aligned} E(V) &= E(X+W) \\ &= E(X) + E(W) \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} Var(V) &= Var(X+W) \\ &= Var(X) + Var(W) + 2Cov(X,W) \\ &= 4 + 2 + 0 \\ &= 6 \end{aligned}$$

$V \sim Normal(\mu = 2, \sigma^2 = 6)$

b) $$Z_V = \frac{V - \mu}{\sigma} = \frac{V-2}{\sqrt{6}}$$

$$Z_X = \frac{X - \mu_X}{\sigma_X} = \frac{X - 0}{2} = \frac{X}{2}$$

$$\begin{aligned} P(|X| > 3) &= P(X > 3) + P(X < -3) \\ &= P(\mu_X + \sigma_X Z_X > 3) + P(\mu_X + \sigma_X Z_X < -8) \\ &= P\left(Z_X > \frac{3-0}{2}\right) + P\left(Z_X < \frac{-3-0}{2}\right) \\ &= 1 - P(Z_X \leq \tfrac{3}{2}) + P(Z_X < -\tfrac{3}{2}) \\ &= 1 - 0.93319 + 0.06681 \\ &= 0.13362 \end{aligned}$$

$$\begin{aligned} P(|V| > 3) &= P(V > 3) + P(V < -3) \\ &= P(2 + \sqrt{6} Z_V > 3) + P(2 + \sqrt{6} Z_V < -3) \\ &= P\left(Z_V > \frac{3-2}{\sqrt{6}}\right) + P\left(Z_V < \frac{-3-2}{\sqrt{6}}\right) \\ &= 1 - P(Z_V \leq 0.41) + P(Z_V < -2.04) \\ &= 1 - 0.65910 + 0.02068 \\ &= 0.36158 \end{aligned}$$

2. RV $X \sim N(0,2)$, $Y = X^2$

$P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$ for $y > 0$

$= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y})$

$= \int_{-\infty}^{\sqrt{y}} \frac{1}{2\sqrt{\pi}} e^{-\frac{y}{4}} dy - \int_{-\infty}^{-\sqrt{y}} \frac{1}{2\sqrt{\pi}} e^{-\frac{y}{4}} dy$

$= \frac{1}{2\sqrt{\pi}} \left(\left[-4e^{-\frac{y}{4}} \right]_{-\infty}^{\sqrt{y}} - \left[-4e^{-\frac{y}{4}} \right]_{-\infty}^{-\sqrt{y}} \right)$

$= \frac{1}{2\sqrt{\pi}} \left(-4e^{-\frac{\sqrt{y}}{4}} - \lim_{A \to -\infty} -4e^{-\frac{A}{4}} - \left(-4e^{\frac{\sqrt{y}}{4}} - \lim_{B \to -\infty} -4e^{-\frac{B}{4}} \right) \right)$

$= \frac{1}{2\sqrt{\pi}} \left(-4e^{-\frac{\sqrt{y}}{4}} + 4e^{\frac{\sqrt{y}}{4}} \right)$

$P(Y = y) = \frac{d}{dy} P(Y \leq y)$

$= \frac{1}{2\sqrt{\pi}} \left(-4e^{-\frac{\sqrt{y}}{4}} \left(-\frac{1}{4} \frac{1}{2\sqrt{y}} \right) + 4e^{\frac{\sqrt{y}}{4}} \left(\frac{1}{4} \frac{1}{2\sqrt{y}} \right) \right)$

$= \frac{1}{2\sqrt{\pi}} \left(e^{-\frac{\sqrt{y}}{4}} \cdot \frac{1}{2\sqrt{y}} + e^{\frac{\sqrt{y}}{4}} \cdot \frac{1}{2\sqrt{y}} \right)$

$= \frac{1}{4\sqrt{\pi y}} \left(e^{-\frac{\sqrt{y}}{4}} + e^{\frac{\sqrt{y}}{4}} \right)$