Q1

We know $R \equiv R^*$ iff $\mathcal{L}(R) = \mathcal{L}(R^*)$

and $R \equiv R + \varepsilon + \emptyset$ iff $\mathcal{L}(R) = \mathcal{L}(R + \varepsilon + \emptyset)$

So it is sufficient to show $\mathcal{L}(R) = \mathcal{L}(R^*) \Rightarrow \mathcal{L}(R) = \mathcal{L}(R + \varepsilon + \emptyset)$

Note: $\mathcal{L}(R^*) = \mathcal{L}(R)^*$ by lang props.

$\mathcal{L}(R + \varepsilon + \emptyset) = \mathcal{L}(R) \cup \mathcal{L}(\varepsilon) \cup \mathcal{L}(\emptyset)$ also by lang. props.

Supp $R \equiv R^*$ $(\mathcal{L}(R) = \mathcal{L}(R)^*)$

Prove $\mathcal{L}(R) = \mathcal{L}(R) \cup \mathcal{L}(\varepsilon) \cup \mathcal{L}(\emptyset)$

$$\begin{aligned}
\mathcal{L}(R) \cup \mathcal{L}(\varepsilon) \cup \mathcal{L}(\emptyset) &= \mathcal{L}(R)^* \cup \mathcal{L}(\varepsilon) \cup \mathcal{L}(\emptyset) && \text{by } \mathcal{L}(R) = \mathcal{L}(R)^* \\
&= \mathcal{L}(R)^* \cup \{\varepsilon\} \cup \emptyset && \mathcal{L}(\varepsilon) = \{\varepsilon\} \text{ and } \mathcal{L}(\emptyset) = \emptyset \\
&= \mathcal{L}(R)^* \cup \{\varepsilon\} && \text{property of empty set}
\end{aligned}$$

Show $\mathcal{L}(R)^* \cup \{\varepsilon\} = \mathcal{L}(R)^*$

We know $L^0 = \{\varepsilon\}$ for reg lang. L

So that means $\mathcal{L}(R)^0 = \{\varepsilon\}$

By Kleene star: $\mathcal{L}(R)^* = \bigcup_{k \in \mathbb{N}} \mathcal{L}(R)^k$

so $\mathcal{L}(R)^0 \in \mathcal{L}(R)^*$

Thus $\{\varepsilon\} \in \mathcal{L}(R)^*$ and $\mathcal{L}(R)^* \cup \{\varepsilon\} = \mathcal{L}(R)^*$

Thus $\mathcal{L}(R) \cup \mathcal{L}(\varepsilon) \cup \mathcal{L}(\emptyset) = \mathcal{L}(R)^*$

$= \mathcal{L}(R)$ by supp.

So $R \equiv R^* \Rightarrow R \equiv R + \varepsilon + \emptyset$ $\blacksquare$