MATB24 TUTORIAL PROBLEMS 5, WEEK OF Oct19-23

KEY WORDS: isomorphism, invertible linear transformation, change of coordinate matrix
RELEVANT SECTIONS IN THE TEXTBOOK:Sec 3.3, 7.1,7.2 FB or 3C, 3D SA

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms.

- Let $\mathfrak{B}$ and $\mathcal{A}$ be a ordered basses for finite dimensional vector spaces W and V respectively. A matrix representation of a linear transformation $T : V \to W$ with respect to $\mathcal{A}$ and $\mathcal{B}$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Define the $\mathfrak{B}$-coordinates of a vector $\vec{v} \in W$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Give an isomorphism between W and $\mathbb{R}^{\dim W}$

A: Consider the basis $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix})$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T([\vec{v}]_\mathcal{E}) = \begin{bmatrix}1 & 2\\0 & -1\end{bmatrix} [\vec{v}]_\mathcal{E}$.

(1) Find the change of basis matrices $C_{\mathcal{B}\to\mathcal{E}}$ and $C_{\mathcal{E}\to\mathcal{B}}$ and explain what they do and how they are related.
(2) Let $T_{\mathcal{B}\to\mathcal{E}}$ and $T_{\mathcal{E}\to\mathcal{B}}$ be linear transformations corresponding to $C_{\mathcal{B}\to\mathcal{E}}$ and $C_{\mathcal{E}\to\mathcal{B}}$ respectively. Use composition of maps to construct a linear transformation that takes in $[\vec{v}]_\mathcal{B}$ as input and gives $[T\vec{v}]_\mathcal{B}$ as output.
(3) Compute a matrix for your answer in previous part.
(4) Compare your answer to the matrix

$$[[T(\begin{bmatrix}1\\1\end{bmatrix})]_\mathcal{B}, [T(\begin{bmatrix}-1\\1\end{bmatrix})]_\mathcal{B}].$$

B: Let V be the vector space of polynomials of degree less than or equal to 2 in the variable t. Let $\mathcal{B}$ be the basis of V given by $(1, t, t^2)$.

(1) Let $\vec{v}_1 = 1 + t$, $\vec{v}_2 = t^2 - t$, $\vec{v}_3 = 1 - t + t^2$. Show that $\mathcal{C} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for V.
(2) Describe explicitly the coordinate isomorphism $V \cong \mathbb{R}^3$ given by $\mathcal{C}$. Where does this isomorphism send the element $a + bt + ct^2$ of V ? (Here a, b, c are scalars.)
(3) Let $\vec{w}_1 = 1 - t$, $\vec{w}_2 = 1 + t^2$, $\vec{w}_3 = t$. Show that $\mathcal{D} = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$ is also a basis of V.
(4) Describe explicitly the coordinate isomorphism $V \cong \mathbb{R}^3$ given by $\mathcal{D}$. Where does this isomorphism send the element $a + bt + ct^2$ of V ? (Here a, b, c are scalars.)
(5) Let A be the matrix

$$A = \begin{bmatrix} [\vec{w}_1]_\mathcal{C} & [\vec{w}_2]_\mathcal{C} & [\vec{w}_3]_\mathcal{C} \end{bmatrix}$$

Let B be the matrix

$$B = \begin{bmatrix} [\vec{v}_1]_{\mathcal{D}} & [\vec{v}_2]_{\mathcal{D}} & [\vec{v}_3]_{\mathcal{D}} \end{bmatrix}$$

If an element $\vec{v}$ in V has coordinates $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in the basis $\mathcal{C}$, how would you compute its coordinates in the basis $\mathcal{D}$? Now do the same question with $\mathcal{C}$ and $\mathcal{D}$ interchanged.

(6) Are the matrices A and B related in a simple way?

C: Let V and W be n and m dimensional F-vectorspaces and let $\mathcal{B}$ and $\mathcal{A}$ be bases for V and W respectively. Let $T_{\mathcal{B}} : V \to F^n$ and $T_{\mathcal{A}} : W \to F^m$ denote the coordinate isomorphisms. Let $S : V \to W$ be a linear transformation.

(1) Prove that $\mathrm{Null}[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{B}}(\ker S)$ and $\mathrm{C}ol[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{A}}(\mathrm{Img} S)$.
(2) Prove that $\mathrm{Null}[S]_{\mathcal{B},\mathcal{A}} \cong \ker S$ and $\mathrm{C}ol[S]_{\mathcal{B},\mathcal{A}} \cong \mathrm{Img} S$
(3) Rank-nullity theorem for S states that $\dim \ker S + \dim \mathrm{Img} S = \dim V$. Use (1) and (2) to prove the rank-nullity theorem for S. You can use your MATA22 knowledge on how to find rank and nullity of a matrix.
(4) Let $\mathcal{B} = (\cos x, \sin x, 1)$ and let $V = \mathrm{Span}(B)$ $S : V \to V$be a linear transformation with $[S]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & -1 \end{bmatrix}$. Find the kernel and the image of S. Justify your computation using the results you proved in this problem.

COOL-OFF:

(1) Give an example of a linear transformation whose matrix representation doesn't depend on the choice of basis
(2) Give an example of a linear transformation T and two different matrix representation of T.