CSCC37 A3

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Question 1

a)

Fixed Point Theorem

If there is an interval $[a, b]$ st $\forall x \in [a, b],\ g(x) \in [a, b]$ and $|g'(x)| \leq L < 1$

Then we claim: $g(x)$ has a unique fixed point in $[a, b]$

b)

Suppose there is an interval $[a, b]$ s.t. $\forall x \in [a, b],\ g(x) \in [a, b]$ and $|g'(x)| \leq L < 1$
Start with $x_0 \in [a, b]$ and iterate with $x_{k+1} = g(x_k)$
This ensures all $x_i \in [a, b]$

Moreover,

$$\begin{aligned} x_{k+1} - x_k &= g(x_k) - g(x_{k-1}) \\ &= g'(\xi_k)(x_k - x_{k-1}), \text{ by MVT for some } \xi_k \in [x_{k+1}, x_k] \subseteq [a, b] \end{aligned}$$

$$\begin{aligned} \text{Then } |x_{k+1} - x_k| &\leq L|x_k - x_{k-1}|, \text{ for } |g'(\xi_k)| \leq L < 1 \\ &\leq L\left(L|x_{k-1} - x_{k-2}|\right) \\ &\leq \cdots \\ &\leq L^k|x_1 - x_0| \end{aligned}$$

$$\text{Since } L < 1,\ |x_{k+1} - x_k| \longrightarrow 0, \text{ as } k \to \infty$$
$$\implies x_k \text{ converges to some point } \tilde{x} \in [a, b]$$

c)

Prove $\tilde{x}$ is a fixed point

Consider: $f(x) = g(x) - x$

Prove: $f(\tilde{x}) = 0$

$$
\begin{aligned}
f(\tilde{x}) &= \lim_{k\to\infty} f(x_k) \\
&= \lim_{k\to\infty} g(x_k) - x_k \\
&= \lim_{k\to\infty} g(x_k) - g(x_{k-1}) \\
&= \lim_{k\to\infty} g(\xi_k)(x_k - x_{k-1}), \text{ by MVT} \\
&= \lim_{k\to\infty} g(\xi_k) \times \lim_{k\to\infty} (x_k - x_{k-1}) \\
&= \lim_{k\to\infty} g(\xi_k) \times 0 \\
&= 0
\end{aligned}
$$

$$f(\tilde{x}) = 0 \iff g(\tilde{x}) = \tilde{x}$$

Thus $\tilde{x}$ is a fixed point.

d)

Suppose there are 2 distinct fixed points $\tilde{x}_1,\ \tilde{x}_2 \in [a,b]$

$$
\begin{aligned}
\tilde{x}_1 - \tilde{x}_2 &= g(\tilde{x}_1) - g(\tilde{x}_2) \\
&= g'(\xi)(\tilde{x}_1 - \tilde{x}_2), \text{ by MVT} \\
\iff g'(\xi) &= \frac{\tilde{x}_1 - \tilde{x}_2}{\tilde{x}_1 - \tilde{x}_2} \\
&= 1
\end{aligned}
$$

Thus a contradiction to the condition $|g'(x)| \leq L < 1$

$\therefore \tilde{x}$ is unique

Question 2

a)

f has one root at $x = \dfrac{1}{2}$, this is fairly easy to show:

$$1 = \frac{1}{2x} \implies x = \frac{1}{2}$$

Testing the root on g :

$$\begin{aligned} g(\frac{1}{2}) &= 2(\frac{1}{2})(1 - \frac{1}{2}) \\ &= \frac{1}{2}, \text{ thus } \frac{1}{2} \text{ is a fixed point in g} \end{aligned}$$

Note that $g(x)$ is quadratic so $g'(x)$ is linear, and thus only has one non-overlapping interval which $|g'(x)| < 1$ (and this interval cannot be expanded while this property holds true).

b)

$$\begin{aligned} g'(x) &= \frac{d}{dx}(2x - 2x^2) \\ &= 2 - 4x \end{aligned}$$

Find the interval which $|g'(x)| < 1$

$$\begin{gathered} |g'(x)| < 1 \\ |2 - 4x| < 1 \\ -1 < 2 - 4x < 1 \\ -3 < -4x < -1 \\ \frac{1}{4} < x < \frac{3}{4} \end{gathered}$$

By inspection, we know g(x) increases in $x \in (-\infty,\ \frac{1}{2})$ and decreases in $x \in (\frac{1}{2},\ \infty)$ with a peak of $g(\frac{1}{2}) = \frac{1}{2}$.

$$\begin{aligned} g\left(\frac{1}{4}\right) &= 2\left(\frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{3}{8} \end{aligned}$$

$$\begin{aligned} g\left(\frac{3}{4}\right) &= 2\left(\frac{3}{4}\right)\left(1 - \frac{3}{4}\right) \\ &= \frac{3}{8} \end{aligned}$$

Thus for $x \in \left(\frac{1}{4},\ \frac{3}{4}\right)$, $g(x) \in \left(\frac{1}{4},\ \frac{3}{4}\right)$, FPT indicates we need to use an closed interval for x
Including the endpoints in the interval will still work for convergence because even if x_0 is one of the endpoints as $g(x)$ where x is one of the endpoints ends up in $\left(\frac{1}{4},\ \frac{3}{4}\right)$.
Thus using the interval $x \in \left[\frac{1}{4},\ \frac{3}{4}\right]$, $x_{k+1} = g(x_k)$ is guaranteed to converge

Question 3

a)

$$1.\ -ln(x) = x - (x + ln(x)),\ \text{first form}$$

$$2.\ e^{-x} = x - (x + ln(x))\underbrace{\frac{1}{x+ln(x)}(x - e^{-x})}_{h(x)},\ \text{second form}$$

$$3.\ \frac{x+e^{-x}}{2} = x - (x + ln(x))\underbrace{\frac{1}{x+ln(x)}(\frac{e^{-x}-x}{2})}_{h(x)},\ \text{second form}$$

All three equations can be used to solve $f(x) = x + ln(x) = 0$

b)

Using an interval of $x \in [0.1,\ 1]$

1. $\frac{d}{dx}(-ln(x)) = -\frac{1}{x}$, monotonically increasing
2. $\frac{d}{dx}\left(e^{-x}\right) = -e^{-x}$, monotonically increasing
3. $\frac{d}{dx}\left(\frac{x+e^{-x}}{2}\right) = \frac{1-e^{-x}}{2}$, monotonically increasing

Check if $|g'(x)| \leq 1$

1. $-\frac{1}{0.1} = 10,\ \left|\frac{1}{x}\right| \nleq 1$, equation 1 cannot use FPT
2. $-e^{-0.1} \approx -0.9048,\ -e^{-1} \approx -0.3679 \implies |-e^{-x}| \leq 1$
3. $\frac{1-e^{-0.1}}{2} \approx 0.0476,\ \frac{1-e^{-1}}{2} \approx 0.3161 \implies \left|\frac{1-e^{-x}}{2}\right| \leq 1$

Check if $\forall x \in [0.1,\ 1],\ g(x) \in [0.1,\ 1]$

2. $e^{-0.1} \approx 0.9048,\ e^{-1} \approx 0.3679$, we know e^{-x} is monotonically decreasing, thus $g(x) \in [0.3679,\ 0.9048]$
3. $\frac{0.1+e^{-0.1}}{2} \approx 0.5024,\ \frac{1+e^{-1}}{2} \approx 0.6839$, we know $\frac{x+e^{-x}}{2}$ is monotonically increasing for $x > 0$

Thus $g(x) \in [0.5024,\ 0.6839]$

$\forall x \in [0.1,\ 1],\ g(x),\ e^{-x}$ and $\frac{x+e^{-x}}{2} \in [0.1,\ 1]$

Thus equations 2 and 3 fit the criteria for FPT using the interval $[0.1, 1]$
For $x = 0.5671$ which is approximately the root of $f(x) = x + ln(x)$ we can apply RCT

$$2.\ -e^{-0.5671} = -0.5671 \neq 0,\ \text{thus has 1st order convergence}$$

$$3.\ \frac{1-e^{-0.5671}}{2} = 0.2164 \neq 0,\ \text{thus has 1st order convergence}$$

The absolute "C" value for equation 3 is smaller thus the convergence will be faster for equation 3 since 3 and 2 have the same order of convergence.
Thus we should use equation 3.

2.

c)

Using Newton's Method, we can get a quadratically converging iteration

$$g(x) = x - \frac{x + ln(x)}{1 + \frac{1}{x}},\ g'(x) = -\frac{x + ln(x)}{(x+1)^2},$$

Test FPT on $x \in [0.4,\ 0.6]$

$g'(x)$ decreases on this interval, and $g(0.4) \approx 0.263$, $g(0.6) \approx 0.034$, showing $|g'(x)| < 1$

We can also see $g(x)$ increases on approximately $(0,\ 0.567)$ and decreases after that since the derivative is negative.

$g(0.4) \approx 0.548$, $g(0.6) \approx 0.567$, thus $g(x) \in [0.4,\ 0.6]\ for\ x \in [0.4,\ 0.6]$

So we can use FPT and say this will converge to a unique fixed point with quadratic rate of convergence.

Question 4

$$\begin{aligned} g(x) &= 2x - x^2y \\ &= x - (x^2y - x) \end{aligned}$$

Using Newton's Method

$$\begin{aligned} \frac{f(x)}{f'(x)} &= x^2y - x \\ \int \frac{f(x)}{f'(x)}\ dx &= \int \left(x^2y - x\right) dx \\ \int \frac{f'(x)}{f(x)}\ dx &= \int \left(\frac{1}{x^2y - x}\right) dx \\ ln|f(x)| &= \int \left(-\frac{1}{x} + \frac{y}{xy-1}\right) dx \\ ln|f(x)| &= -\int \frac{1}{x}\ dx + \int \frac{y}{xy-1}\ dx \\ ln|f(x)| &= -ln|x| + ln|xy-1| \\ ln|f(x)| &= ln\left|\frac{xy-1}{x}\right| \\ f(x) &= \frac{xy-1}{x} \end{aligned}$$

This iteration is to approximate roots of $f(x) = \frac{xy-1}{x}$ which are at $\frac{1}{y}$ using Newton's Method, below shows that $\frac{1}{y}$ is a fixed point in $g(x)$

$$\begin{aligned} g\left(\frac{1}{y}\right) &= 2\left(\frac{1}{y}\right) - \left(\frac{1}{y}\right)^2 y \\ &= 2\left(\frac{1}{y}\right) - \frac{1}{y} \\ &= \frac{1}{y} \end{aligned}$$

Question 5

a)

$$\begin{aligned} g(x) &= x - \frac{0^2}{f(x+0) - 0}, \text{ for } f(x) = 0 \\ &= x - \frac{0}{f(x)} \\ &= x - \frac{0}{0} \end{aligned}$$

Thus roots of f are not defined on g

$$g(x) = x \iff \frac{f(x)^2}{f(x + f(x)) - f(x)} = 0$$

Since the expression of $f(x)$ has asymptotic behaviour at the roots of f, g will not have defined fixed points.

b)

Recall the limit definition of a derivative:

$$\frac{f(x+h) - f(x)}{h}$$

Use $h = f(x)$, step size between the last x_k point and some point x + h, which is bound by $f'(x)$
Use this approximation in Newton's Method

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{f(x_k)}{\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}} \\ &= x_k - \frac{f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)} \end{aligned}$$

c)

Newton's Method

$$\begin{aligned} g(x) &= x - \frac{x^2 - 10x + 24}{2x - 10} \\ g'(x) &= \frac{(x-4)(x-6)}{2(x-5)^2} \end{aligned}$$

Clearly $g'(x)$ as roots at $x = 4,\ 6$
Thus by RCT, $g(x)$ has at least 2nd order convergence

Steffensen's Method

$$g(x) = x - \frac{f(x)^2}{f(x+f(x)) - f(x)}, \text{ for } f(x) = x^2 - 10x + 24$$

$$\begin{aligned} f(x+f(x)) - f(x) &= (x+f(x))^2 - 10(x+f(x)) + 24 - (x^2 - 10x + 24) \\ &= 2xf(x) + f(x)^2 - 10f(x) \\ &= f(x)(f(x) + 2x - 10) \end{aligned}$$

$$\begin{aligned} g(x) &= x - \frac{f(x)^2}{f(x)\,(f(x) + 2x - 10)} \\ &= x - \frac{f(x)}{(f(x) + 2x - 10)} \\ g'(x) &= 1 - \frac{f'(x)(f(x) + 2x - 10) - (f'(x) + 2)f(x)}{(f(x) + 2x - 10)^2} \\ g'(4) &= 1 - \frac{-2(0 + 2(4) - 10) - (-2 + 2)(0)}{(0 + 2(4) - 10)^2} = 0 \\ g'(6) &= 1 - \frac{2(0 + 2(6) - 10) - (2 + 2)(0)}{(0 + 2(6) - 10)^2} = 0 \end{aligned}$$

$g'(x)$ has roots at $x = 4,\ 6$

Thus by RCT, $g(x)$ has at least 2nd order convergence

d)

Steffensen's Method converges just as fast as Newton's without having to evaluate the function's derivative, which is useful when the derivative is very complicated.