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These solutions are not the only "perfect" or "10/10" solutions to be followed as a model. They are examples of ways to solve these problems. There are many many alternative ways to solve these problems. We hope that these example solutions help you check your work.

Problems

Q1. Suppose that f is a C^2-function on the plane. Let $\mathbf{c}(t)$ be a flowline of $\mathbf{F} = -\nabla f$.
Prove that $f(\mathbf{c}(t))$ is a decreasing function of t.

Solution: Using techniques from MAT B41, we calculate the directional derivative along the flowline. We write $D_{\mathbf{v}} f$ for the directional derivative of f in the direction $\mathbf{v}$. Recall that $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f$. The flowline equation gives $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)) = -\nabla f$. We obtain:

$$D_{\mathbf{c}'(t)} f = \mathbf{c}'(t) \cdot \nabla f = \mathbf{F}(\mathbf{c}(t)) \cdot \nabla f = -\nabla f \cdot \nabla f = -\|\nabla f\| < 0$$

Thus, f is decreasing in the direction $\mathbf{c}'(t)$. It follows that f is decreasing as a function of t.

Q2. Consider the following vector field.

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

(a) Check that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Solution: We check the equality of the partial derivatives.

$$\begin{aligned} P &= \frac{-y}{x^2+y^2} \\ \frac{\partial P}{\partial y} &= \frac{-x^2+y^2}{(x^2+y^2)^2} \\ Q &= \frac{x}{x^2+y^2} \\ \frac{\partial Q}{\partial x} &= \frac{-x^2+y^2}{(x^2+y^2)^2} \end{aligned}$$

(b) Show that $\displaystyle\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ is NOT path-independent.
(For the definition of path-independent see M&T p. 453).

Solution: Consider the paths $\mathbf{c}_1$ and $\mathbf{c}_2$ from $(1,0)$ to $(-1,0)$ given by: $\mathbf{c}_1(t) = (\cos(t), \sin(t))$ for $0 \le t \le \pi$ and $\mathbf{c}_2(t) = (\cos(t), -\sin(t))$ for $0 \le t \le \pi$. Notice that $\mathbf{c}_1$ passes above $(0,0)$

and $\mathbf{c}_2$ passes below $(0,0)$. We calculate:

$$\begin{aligned}
\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} &= \int_0^\pi \mathbf{F}(\mathbf{c}_1(t))\cdot \mathbf{c}_1'(t)dt \\
&= \int_0^\pi \left(\frac{-\sin(t)\mathbf{i}+\cos(t)\mathbf{j}}{\cos^2(t)+\sin^2(t)}\right)\cdot(-\sin(t)\mathbf{i}+\cos(t)\mathbf{j})\,dt \\
&= \int_0^\pi \frac{\sin^2(t)+\cos^2(t)}{\cos^2(t)+\sin^2(t)}dt \\
&= \int_0^\pi 1\,dt = \pi
\end{aligned}$$

And now we do this same calculation for $\mathbf{c}_2$:

$$\begin{aligned}
\int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} &= \int_0^\pi \mathbf{F}(\mathbf{c}_2(t))\cdot \mathbf{c}_2'(t)dt \\
&= \int_0^\pi \left(\frac{\sin(t)\mathbf{i}+\cos(t)\mathbf{j}}{\cos^2(t)+\sin^2(t)}\right)\cdot(-\sin(t)\mathbf{i}-\cos(t)\mathbf{j})\,dt \\
&= \int_0^\pi \frac{-\sin^2(t)-\cos^2(t)}{\cos^2(t)+\sin^2(t)}dt \\
&= \int_0^\pi -1\,dt = -\pi
\end{aligned}$$

Comments: In class, we shows that all gradient vector fields are path independent. For us, a good sign that $\mathbf{F}=(P,Q)$ is a gradient is that $P_y = Q_x$. The calculation in this question is interesting because it suggests that equality of the partial derivatives $P_y = Q_x$ is not enough to guarantee that something is a gradient and hence path-independence. In this case, the vector field is discontinuous at the origin.

(c) We say that a simple closed curve *surrounds the origin* if the origin is contained in the interior of the curve. For example, $\mathbf{c}(t) = (\cos(t),\sin(t))$ surrounds the origin. Use Green's Theorem to prove that:

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \pm 2\pi$$

for any simple closed curve $\mathbf{c}$ which surrounds the origin. You may assume that the distance from the origin to the curve $\mathbf{c}$ is at least $\epsilon > 0$. (For the definition of simple closed curve see M&T p. 368).

Solution: We first explain our thought process for this question. If the claim is true for all curves $\mathbf{c}$ which surround the origin, then it will be true for circles which surround the origin. A quick calculation shows a counter-clockwise circle around the origin integrates to 2π and a clockwise circle integrates to -2π. The difficulty is to show that all other curves $\mathbf{c}$ which surround the origin give the same value. This requires a clever choice of region and an application of Green's Theorem.

Let $\mathbf{c} = \partial D$ for some region D which contains the origin. Consider the region R obtained by deleting disk of radius ϵ around the origin.

$$R = \{x : x \in D \text{ and } d(x,(0,0)) > \epsilon\}$$

The region R has two boundary components: a circle $\mathbf{c}_\epsilon$ of radius ϵ around the origin and $\mathbf{c}$. The vector field $\mathbf{F}$ is continuous on R because R does not contain the origin. Moreover, on the region R we have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Putting the pieces together we get:

$$\int_{\mathbf{c}_\epsilon} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial R} \mathbf{F} \cdot d\mathbf{s} = \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dxdy = 0$$

Therefore, we have for any curve $\mathbf{c}$ surrounding the origin:

$$\int_{\mathbf{c}_\epsilon} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

We now check the value of the integral around a circle:

$$\begin{aligned} \int_{\mathbf{c}_\epsilon} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}_\epsilon(t)) \cdot \mathbf{c}_\epsilon'(t) dt \\ &= \int_0^{2\pi} \left(\frac{\epsilon \sin(t)\mathbf{i} + \epsilon \cos(t)\mathbf{j}}{\epsilon^2 \cos^2(t) + \epsilon^2 sin^2(t)} \right) \cdot (\epsilon \sin(t)\mathbf{i} + \epsilon \cos(t)\mathbf{j})\, dt \\ &= \int_0^{2\pi} \frac{\epsilon^2(1)}{\epsilon^2(1)} dt = 2\pi \end{aligned}$$

It follows that, accounting for orientation, we have:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{c}_\epsilon} \mathbf{F} \cdot d\mathbf{s} = \pm 2\pi$$

Comments : As a technical remark, we could use topology to show that any closed curve which surrounds the origin (but does not pass through the origin) must be seperated from the origin by some positive distance $\epsilon > 0$. To prove this, we note that the image $\mathbf{c}$ is compact, and so: $d(t) = d((0,0), \mathbf{c}(t))$ attains some minimum value by the single variable extreme value theorem.

Q3. Evaluate the integral

$$\int_{\mathbf{c}} xe^y\,dx + x^2y\,dy$$

along the path $\mathbf{c}(t) = (e^t, e^t)$ for $-1 \le t \le 1$.

Solution: Surprisingly, this is not a gradient vector field! We grind out the calculation.

$$\begin{aligned}
\int_{\mathbf{c}} xe^y\,dx + x^2y\,dy &= \int_{-1}^{1} (e^t)e^{e^t}\,d(e^t) + (e^t)^2(e^t)\,d(e^t) \\
&= \int_{-1}^{1} (e^t)e^{e^t}e^t dt + (e^t)^2(e^t)e^t dt \\
&= \int_{-1}^{1} e^te^{e^t}e^t dt + \int_{-1}^{1} e^{4t} dt \\
&= \int_{-1}^{1} e^te^{e^t}e^t dt + \frac{1}{4}\left[e^{4t}\right]_{-1}^{1} \\
&= \int_{e^{-1}}^{e^1} ue^u du + \frac{1}{4}\left[e^{4t}\right]_{-1}^{1} \\
&= [(u-1)e^u]_{e^{-1}}^{e^1} + \frac{1}{4}\left[e^{4t}\right]_{-1}^{1} \\
&= (e-1)e^e - (e^{-1}-1)e^{e^{-1}} + \frac{1}{4}\left[e^4 - e^{-4}\right] \approx 40.597
\end{aligned}$$

Q4. Consider the vector field $\mathbf{F} = (2xy^2, 2yx^2)$. Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ from $(1,2)$ to $(5,2)$ are shown on the diagram. Show that the line integrals along both curves are equal and compute their values.

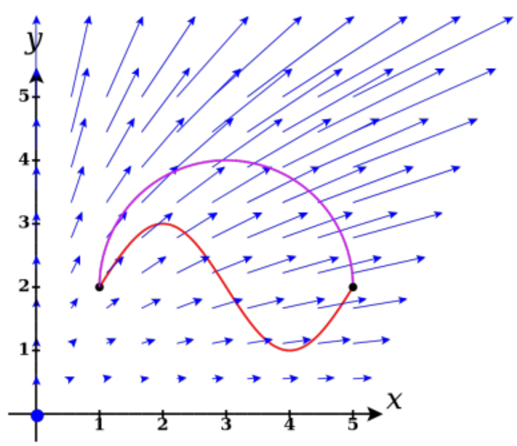

Figure 1: The curves $\mathbf{c}_1$ (top purple) and $\mathbf{c}_2$ (bottom red) in Question 4.

Solution: Notice that $\mathbb{F}$ is the gradient of $f(x,y) = x^2y^2$. We have the following theorem: If

$\mathbf{F} = \nabla f$ is a gradient vector field and $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is a C^1-curve then $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$. Applying the theorem to this vector field and these curves we get for $i = 1, 2$:

$$\int_{\mathbf{c}_i} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}_i(b)) - f(\mathbf{c}_i(a)) = f(5,2) - f(1,2) = 5^2 \cdot 2^2 - 1^2 \cdot 2^2 = 96$$

Q5. Suppose that $\mathbf{c}$ is the line segment from (a, b) to (c, d).

(a) Show that:

$$\int_{\mathbf{c}} x\,dy - y\,dx = ad - bc$$

Briefly describe what this integral represents geometrically.
Hint: Think back to linear algebra. What do determinants measure?

Solution: We parametrize the line segment for $0 \leq t \leq 1$ by:

$$\mathbf{c}(t) = (a, b) + t((c, d) - (a, b)) = (a + t(c - a), b + t(d - b))$$

This allows us to evaluate the line integral:

$$\begin{aligned}
\int_{\mathbf{c}} x\,dy - y\,dx &= \int_0^1 (a + t(c-a))d(b + t(d-b)) - (b + t(d-b))d(a + t(c-a)) \\
&= \int_0^1 (a + t(c-a))(d-b)dt - (b + t(d-b))(c-a)dt \\
&= \int_0^1 \left[(a + t(c-a))(d-b) - (b + t(d-b))(c-a)\right] dt \\
&= \left[\left(at + \frac{1}{2}t^2(c-a)\right)(d-b) - \left(bt + \frac{1}{2}t^2(d-b)\right)(c-a)\right]_0^1 \\
&= \left(a + \frac{1}{2}(c-a)\right)(d-b) - \left(b + \frac{1}{2}(d-b)\right)(c-a) \\
&= ad - bc
\end{aligned}$$

This integrals represents the signed area formed by the parallelogram with vertices $(0, 0), (a, b), (c, d), (a + c, b + d)$. This is the usual interpretation of a determinant in terms of signed areas.

Commentary: Notice that the area is signed according to whether or not one procreeds counter-clockwise or clockwise from (a, b) to (c, d). The signed area of the "parallelogram from $(1, 0)$ to $(0, 1)$" is positive, but the signed area of the "parallelogram from $(0, 1)$ to $(1, 0)$" is negative.

(b) Suppose that a polygon P has vertices $\{(x_i, y_i)\}_{i=1}^n$ in counter-clockwise order.
Use Green's theorem to show that P has area:

$$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right]$$

Solution: We use the first part of the question to find the signed area of a triangle with vertices $(0,0),(x_i,y_i),(x_{i+1},y_{i+1})$. To obtain the signed area of the triangle with vertices $(0,0),(x_i,y_i),(x_{i+1},y_{i+1})$ we must divide the signed area of the parallelogram by two. Thus,

$$A = \frac{1}{2}\left(x_i y_{i+1} - x_{i+1} y_i\right)$$

We parametrize the border of P as a piecewise linear curve $\mathbf{c}(t)$. We define the line segment $\mathbf{c}_i$ to pass from vertex (x_i,y_i) to vertex (x_{i+1},y_{i+1}) for $i = 1,2,\ldots,n-1$. The last line segment $\mathbf{c}_n$ passes from (x_n,y_n) to (x_1,y_1). Now we calculate the total signed area of P.

$$\begin{aligned} A &= \frac{1}{2}\int_{\mathbf{c}} x\,dy - y\,dx = \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbf{c}_i} x\,dy - y\,dx \\ &= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right] \end{aligned}$$