MATB24 TUTORIAL PROBLEMS 5, SOLUTIONS

KEY WORDS: isomorphism, invertible linear transformation, change of coordinate matrix
RELEVANT SECTIONS IN THE TEXTBOOK:Sec 3.3, 7.1,7.2 FB or 3C, 3D SA

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms.

- Let $\mathfrak{B}$ and $\mathcal{A}$ be a ordered basses for finite dimensional vector spaces W and V respectively. A matrix representation of a linear transformation $T : V \to W$ with respect to $\mathcal{A}$ and $\mathcal{B}$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Define the $\mathfrak{B}$-coordinates of a vector $\vec{v} \in W$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Give an isomorphism between W and $\mathbb{R}^{\dim W}$

A: Consider the basis $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix})$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T([\vec{v}]_{\mathcal{E}}) = \begin{bmatrix}1 & 2\\0 & -1\end{bmatrix} [\vec{v}]_{\mathcal{E}}$.

(1) Find the change of basis matrices $C_{\mathcal{B}\to\mathcal{E}}$ and $C_{\mathcal{E}\to\mathcal{B}}$ and explain what they do and how they are related.
(2) Let $T_{\mathcal{B}\to\mathcal{E}}$ and $T_{\mathcal{E}\to\mathcal{B}}$ be linear transformations corresponding to $C_{\mathcal{B}\to\mathcal{E}}$ and $C_{\mathcal{E}\to\mathcal{B}}$ respectively. Use composition of maps to construct a linear transformation that takes in $[\vec{v}]_{\mathcal{B}}$ as input and gives $[T\vec{v}]_{\mathcal{B}}$ as output.
(3) Compute a matrix for your answer in previous part.
(4) Compare your answer to the matrix

$$[[T(\begin{bmatrix}1\\1\end{bmatrix})]_{\mathcal{B}}, [T(\begin{bmatrix}-1\\1\end{bmatrix})]_{\mathcal{B}}].$$

Solution.

(1) $C_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$ and $C_{\mathcal{E}\to\mathcal{B}} = \begin{bmatrix}1/2 & 1/2\\-1/2 & 1/2\end{bmatrix}$. These matrices change coordinates from one basis to another, and are inverses of each other.
(2) The composition is $C_{\mathcal{E}\to\mathcal{B}} \circ T \circ T_{\mathcal{B}\to\mathcal{E}}$.
(3) $\begin{bmatrix}1/2 & 1/2\\-1/2 & 1/2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\-2 & -1\end{bmatrix}$
(4) It is the same matrix: The first column is computed as $C_{\mathcal{E}\to\mathcal{B}}T([1,1]) = [1,-2]$, and the second column is $C_{\mathcal{E}\to\mathcal{B}}T([-1,1]) = [0,-1]$.

B: Let V be the vector space of polynomials of degree less than or equal to 2 in the variable t. Let $\mathcal{B}$ be the basis of V given by $(1, t, t^2)$.

(1) Let $\vec{v}_1 = 1 + t$, $\vec{v}_2 = t^2 - t$, $\vec{v}_3 = 1 - t + t^2$. Show that $\mathcal{C} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for V.

(2) Describe explicitly the coordinate isomorphism $V \cong \mathbb{R}^3$ given by $\mathcal{C}$. Where does this isomorphism send the element $a + bt + ct^2$ of V ? (Here a, b, c are scalars.)
(3) Let $\vec{w}_1 = 1 - t$, $\vec{w}_2 = 1 + t^2$, $\vec{w}_3 = t$. Show that $\mathcal{D} = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$ is also a basis of V.
(4) Describe explicitly the coordinate isomorphism $V \cong \mathbb{R}^3$ given by $\mathcal{D}$. Where does this isomorphism send the element $a + bt + ct^2$ of V ? (Here a, b, c are scalars.)
(5) Let A be the matrix

$$A = \begin{bmatrix} [\vec{w}_1]_{\mathcal{C}} & [\vec{w}_2]_{\mathcal{C}} & [\vec{w}_3]_{\mathcal{C}} \end{bmatrix}$$

Let B be the matrix

$$B = \begin{bmatrix} [\vec{v}_1]_{\mathcal{D}} & [\vec{v}_2]_{\mathcal{D}} & [\vec{v}_3]_{\mathcal{D}} \end{bmatrix}$$

If an element $\vec{v}$ in V has coordinates $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in the basis $\mathcal{C}$, how would you compute its coordinates in the basis $\mathcal{D}$? Now do the same question with $\mathcal{C}$ and $\mathcal{D}$ interchanged.
(6) Are the matrices A and B related in a simple way?

Solution.

(1) Since $\dim V = 3$, it is enough to show linear independence. If $r_1\vec{v}_1 + r_2\vec{v}_2 + r_3\vec{v}_3 = \vec{0}$, then $r_1 + r_3 = 0$, $r_1 - r_2 - r_3 = 0$, and $r_2 + r_3 = 0$. Forming the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, it is seen that it is row equivalent to the identity, and hence they are independent.
(2) This is the linear transformation $T : V \to \mathbb{R}^3$ given by $T(\vec{v}) = [\vec{v}]_{\mathcal{C}}$. Writing $a + bt + ct^2 = r_1\vec{v}_1 + r_2\vec{v}_2 + r_3\vec{v}_3 = (r_1 + r_3) + (r_1 - r_2 - r_3)t + (r_2 + r_3)t^2$. Solving the matrix from (1) augmented by $[a, b, c]$ yields the solution $[r_1, r_2, r_3] = [b+c, -a+b+2c, a-b-c]$. Hence, $T(a+bt+ct^2) = [b+c, -a+b+2c, a-b-c]$.
(3) Same argument as in (1).
(4) Let the transformation be S. As in (1) and (2), one write $r_1(1-t)+r_2(1+t^2)+r_3t = (r_1 + r_2) + (-r_1 + r_3)t + r_2t^2$, and obtains the matrix $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Augmenting it by $[a, b, c]$ yields the solution $S(a + bt + ct^2) = [a - c, c, a + b - c]$.
(5) Note that A is the change-of-coordiantes matrix from $\mathcal{D}$ to $\mathcal{C}$, and B is the change-of-coordinates matrix $\mathcal{C}$ to $\mathcal{D}$. Hence, to compute $[\vec{v}]_{\mathcal{D}}$, we compute $B \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Likewise, given $[\vec{v}]_{\mathcal{D}} = [a, b, c]$, compute $[\vec{v}]_{\mathcal{C}} = A \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
(6) Have $AB = I_3 = BA$, or that they are inverses.

C: Let V and W be n and m dimensional F-vectorspaces and let $\mathcal{B}$ and $\mathcal{A}$ be bases for V and W respectively. Let $T_{\mathcal{B}} : V \to F^n$ and $T_{\mathcal{A}} : W \to F^m$ denote the coordinate isomorphisms. Let $S : V \to W$ be a linear transformation.

(1) Prove that $\text{Null}[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{B}}(\ker S)$ and $\text{C}ol[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{A}}(\text{Img}S)$.
(2) Prove that $\text{Null}[S]_{\mathcal{B},\mathcal{A}} \cong \ker S$ and $\text{C}ol[S]_{\mathcal{B},\mathcal{A}} \cong \text{Img}S$
(3) Rank-nullity theorem for S states that $\dim\ker S + \dim \text{Img}S = \dim V$. Use (1) and (2) to prove the rank-nullity theorem for S. You can use your MATA22 knowledge on how to find rank and nullity of a matrix.
(4) Let $\mathcal{B} = (\cos x, \sin x, 1)$ and let $V = \text{Span}(B)$ $S : V \to V$be a linear transformation with $[S]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & -1 \end{bmatrix}$. Find the kernel and the image of S. Justify your computation using the results you proved in this problem.

Solution.

(1) Let us first show that $\text{Null}[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{B}}(\ker S)$. Let $x \in \text{Null}[S]_{\mathcal{B},\mathcal{A}} \subseteq F^n$. Note that since $T_{\mathcal{B}}$ is an isomorphism, it is in particular onto. Hence there exists an $\alpha \in V$ such that $T_{\mathcal{B}}(\alpha) = [\alpha]_{\mathcal{B}} = x$.

$$x \in \text{Null}[S]_{\mathcal{B},\mathcal{A}} \Rightarrow 0 = [S]_{\mathcal{B},\mathcal{A}}x = [S]_{\mathcal{B},\mathcal{A}}[\alpha]_{\mathcal{B}} = [S(\alpha)]_{\mathcal{A}}$$

Because $T_{\mathcal{A}}$ is an isomorphism and in particular 1-1. So $S(\alpha) = 0_V$ that is $\alpha \in \ker(S)$. Hence $T_{\mathcal{B}}(\alpha) = x$ shows that $x \in T_{\mathcal{B}}(\ker S)$.

Conversely, suppose $x \in T_{\mathcal{B}}(\ker S)$. That is $x = [\alpha]_{\mathcal{B}}$ for some $\alpha \in \ker(S)$. So $S(\alpha) = 0$. This implies $0_{F^m} = [0_V]_{\mathcal{A}} = [S(\alpha)]_{\mathcal{A}} = [S]_{\mathcal{B},\mathcal{A}}[\alpha]_{\mathcal{B}} = [S]_{\mathcal{B},\mathcal{A}}x$. Hence $x \in \text{Null}[S]_{\mathcal{B},\mathcal{A}}$.

Now let's show that $\text{C}ol[S]_{\mathcal{B},\mathcal{A}} = T_{\mathcal{A}}(\text{Img}S)$. Let $y \in \text{C}ol[S]_{\mathcal{B},\mathcal{A}} \subseteq F^m$. First note that $T_{\mathcal{A}}$ is an isomorphism and in particular onto, there exists a $\beta \in W$ such that $y = [\beta]_{\mathcal{A}} = T_{\mathcal{A}}(\beta)$. Now $y \in \text{Col}[S]_{\mathcal{B},\mathcal{A}}$ implies that there exists an $x \in F^n$ such that $[S]_{\mathcal{B},\mathcal{A}}x = y$. Similarity, since $T_{\mathcal{B}}$ is an isomorphism $x = [\alpha]_{\mathcal{B}}$ for some α in V.

$$[S]_{\mathcal{B},\mathcal{A}}x = y \Rightarrow [S]_{\mathcal{B},\mathcal{A}}[\alpha]_{\mathcal{B}} = [\beta]_{\mathcal{A}} \Rightarrow \beta \in \text{Img}S$$

Hence $y = T_{\mathcal{A}}(\beta)$ shows that $y \in T_{\mathcal{A}}(\text{Img}S)$.

Conversely, suppose $y \in T_{\mathcal{A}}(\text{Img}S)$. That is $y = [\beta]_{\mathcal{A}}$ for some $\beta \in \text{Img}S$. Since $\beta \in \text{Img}S$ there exists an $\alpha \in V$ such that $S(\alpha) = \beta$. This implies $[S]_{\mathcal{B},\mathcal{A}}[\alpha]_{\mathcal{B}} = [\beta]_{\mathcal{A}}$. Hence $y = [\beta]_{\mathcal{A}} \in \text{C}ol[S]_{\mathcal{B},\mathcal{A}}$.

(2) Consider the maps $T'_{\mathcal{B}} : \ker S \to \text{Null}[S]_{\mathcal{B},\mathcal{A}}$, $v \mapsto [v]_{\mathcal{B}}$ and $T'_{\mathcal{A}} : \Im \to \text{C}ol[S]_{\mathcal{B},\mathcal{A}}$, $v \mapsto [v]_{\mathcal{A}}$. $T'_{\mathcal{B}}$ and $T'_{\mathcal{A}}$ are 1-1 linear maps since T_A and T_B are 1-1 linear maps. They are well-defined onto by part (1). Hence $\text{Null}[S]_{\mathcal{B},\mathcal{A}} \cong \ker S$ and $\text{C}ol[S]_{\mathcal{B},\mathcal{A}} \cong \text{Img}S$.

(3) From MATA22, we know that for the matrix $[S]_{\mathcal{B},\mathcal{A}}$, we have the following results: $\dim \text{Null}[S]_{\mathcal{B},\mathcal{A}} + \dim \text{Col}[S]_{\mathcal{B},\mathcal{A}} = n$, using the results from (1) and (2) we have $\dim\ker S + \dim \text{Img}S = \dim V$.

(4) We compute $\mathrm{Nul}[S]_{\mathcal{B}} = \mathrm{Span}\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\}$ and $\mathrm{Col}[S]_{\mathcal{B}} = \mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\}$. By part (1), $\ker S = T_{\mathcal{B}}^{-1}(\mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\}) = \mathrm{Span}\{-5\cos x + \sin x + 1\}$ and $\mathrm{Img} S = T_{\mathcal{B}^{-1}}(\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\}) = \mathrm{Span}\{\cos x + 2\sin x, 2\cos x + 4\sin x + 1\}$.

COOL-OFF:

(1) Give an example of a linear transformation whose matrix representation doesn't depend on the choice of basis
(2) Give an example of a linear transformation T and two different matrix representation of T.

Solution.

(1) Let T be the 0 transformation, then under any choices of basis, the matrix representation will be the zero matrix
(2) Many examples exist in previous questions.