# MATB24 – Midterm –Fall 2019

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**Time: 110 mins.**

Name: ______________________________ TUT number : ____________

1. Answer each question in the space provided. If you require more space you may use the blank page at the end of the exam. ***You must clearly indicate, in the provided answer space. You may not use any paper not provided with this exam.***
2. Remember to show all your work.
3. No calculators, notes, or other outside assistance allowed.

1. Finish each of following sentences by giving complete, precise definitions for, or precise mathematical characterizations of, the italicized terms.

   (a) (3 points) A *subspace* $W$ of a real vector space $V$.

   (b) (3 points) A *linear transformation* $T : V \to W$, where $V$ and $W$ are real vector spaces.

   (c) (3 points) A *linearly dependent* subset $S$ (finite or infinite) of a vector space $V$.

   (d) (3 points) An *isomorphism* between two vector spaces $V$ and $W$.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof is you think the statement is true or a counter example if you think it is false).

   (a) (3 points) The orthogonal complement of Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3$ is the plane $x+2y+3z=0$ in $\mathbb{R}^3$.

   (b) (3 points) Let $U$ and $W$ be non-trivial subspaces of $V$. Then $U \cup W$ is a subspace of $V$.

   (c) (3 points) If $V$ is not finitely generated then every non-trivial subspace of $V$ is also not finitely generated.

(d) (3 points) The set $\{e^x, e^x + e^{2x}, e^x + e^{2x} + e^{3x}\}$ is a basis for $\mathrm{Span}\{e^x, e^{2x}, e^{3x}\}$.

(e) (3 points) The space of all upper triangular $3 \times 3$ matrices is isomorphic to $P_5$.

(f) (3 points) Let $V$ and $W$ be vector spaces, and suppose that $T : V \to W$ is an injective linear transformation. If there are vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $V$ such that the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)$ span $W$, then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span $V$.

3. In each part, give an **explicit** example of the mathematical object described or explain why such object does not exist.

   (a) (3 points) A 4-dimensional subspace of $\mathcal{F}$, the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

   (b) (3 points) A surjective and non-injective map from a 3-dimensional vector space $V$ other than $\mathbb{R}^3$ to a 3-dimensional vector space $W$ other than $\mathbb{R}^3$.

   (c) (3 points) Two different bases for $M_{2\times 2}(\mathbb{R})$, the vector space of all $2 \times 2$ matrices with entries in $\mathbb{R}$.

(d) (3 points) An isomorphism between $M_{2\times 2}(\mathbb{R})$, the vector space of all $2 \times 2$ matrices with entries in $\mathbb{R}$, and $P_3$, the vector space of all cubic polynomials with coefficients in $\mathbb{R}$.

(e) (3 points) Two different bases $\beta$ and $\alpha$ for $\mathbb{R}^2$ and a change of basis matrix $C$ that changes $\beta$-coordinate of a vector $\vec{v} \in \mathbb{R}$ into $\alpha$-coordinates of $\vec{v}$.

(f) (3 points) An isomorphism between $\mathbb{R}^3$ and $V = \{A \in M_{2\times 2}(\mathbb{R}) | \operatorname{tr}(A) = 0\}$, that is the vector space of all $2 \times 2$ matrices with entries in $\mathbb{R}$ that have zero trace[1].

[1] trace of a matrix is the sum ot its diagonal entries

4. Let $P_n$ be the vector space of all polynomials with degree less than or equal to $n$. Consider the map $D : P_n \to P_n$, given by $D(p(x)) = p'(x)$.

   (a) (4 points) Show that $D$ is a linear transformation.

   (b) (4 points) Prove that $\{a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \mid a_i \in \mathbb{R}\}$ is a subspace of $P_n$.

   (c) (5 points) Suppose $n = 2$. Let $\mathcal{A} = \{1, x, x^2\}$ and $\mathcal{B} = \{1 + x + x^2, 1 - x + x^2, 1 + x - x^2\}$ be bases for $P_2$ (you don't need to prove this). Suppose $B$ is a matrix such that $B[p(x)]_{\mathcal{B}} = [p'(x)]_{\mathcal{B}}$. Find $B$.

(d) (6 points) Let $C$ be a matrix such that $(C^{-1}BC)[p(x)]_{\mathcal{A}} = [p'(x)]_{\mathcal{A}}$. Find $C^{-1}$.

(e) (6 points) Suppose $p(x) = a + bx + cx^2 \in P_2$ is such that $[p'(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. Find $b$ and $c$.

5. Let $U = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$ be the vector space of all upper triangular $2 \times 2$ matrices.
Let

$$\mathcal{B} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

be a basis of $U$.
Let $T : U \to \mathbb{R}^2$ be the linear transformation defined by

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a \\ b + 2c \end{pmatrix}.$$

(a) (5 points) Find a basis for the kernel of $T$.

(b) (5 points) Is $T$ surjective? Justify.

(c) (5 points) Show that $\mathrm{Null}([T]_{\mathcal{B}})$ is 1-dimensional.

6. Let $V$ and $W$ be $n$ and $m$-dimensional vector spaces respectively. Let $k$ be a fixed integer between 1 and $n$. Suppose $T : V \to W$ is a linear transformation. Let $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ be a basis for $V$ and $\{\vec{v}_k, \cdots, \vec{v}_n\}$ be a subset of $V$. Let $[T]_{\mathcal{B}}$ be the matrix of $T$ with respect to $\mathcal{B}$. Assume the first $k$ columns of $[T]_{\mathcal{B}}$ are zero and all other columns are linearly independent.

   (a) (6 points) Carefully prove that $\ker(T) = \mathrm{Span}\{\vec{b}_1, \cdots, \vec{b}_k\}$.

(b) (6 points) Suppose $[\vec{v}_j]_{\mathcal{B}} = \vec{e}_j + \vec{e}_k$ for $k \leq j \leq n$, where $\vec{e}_j$'s are standard vectors in $\mathbb{R}^n$. Prove that $\mathrm{Img}(T) = \mathrm{Span}\{\vec{v}_k, \cdots, \vec{v}_n\}$.

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