Contents

0 2020 Final

0.1 Q1

$\mathbb{R}_3\,(3,1)$ with limited exponent $-1 \leq e \leq 1$

1a) Find smallest number

Smallest is $0.1 \times 3^{-1} = (0.01)_3$

To convert to decimal we have $= 0 \cdot 3^{-1} + 1 \cdot 3^{-2} = \frac{1}{9}$

b) Find largest number

$.222 \times 3^1 = (2.22)_3$

To convert to decimal we have,

$2 \cdot 3^0 + 2 \cdot 3^{-1} + 2 \cdot 3^{-2} = 2.88888$

c)

$\frac{x - Fl(x)}{x} = \delta < b^{(1-t)} = \frac{1}{2}3^{(-2)} = \frac{1}{18} = 0.05555556$

d) $(407)_{10}$ in this system?

This is larger than the largest number in the syustem in b, so we know we can't represent it.

For funsies...

Num	*Den*	*Quot*	*R*
407	3	135	2
135	3	45	0
45	3	15	0
15	3	5	0
5	3	1	2
1	3	0	1

$= 120002$

We can't represent this in our system since the mantissa is not long enough.

e) $(0.567)_{10}$ in this system

mult	*base*	*prod*	*integral*	*fraction*
.567	3	1.701	1	.701
.701	3		2	.103
.103			0	.309
.309			0	.927
.927			2	.781
.781			2	.343

$= (.120 \times 3^0)_3$

f)

.100 .101 .102 .110 .111 .112 .120 .121 .122 .200 .201 .202 .210 .211 .212 .220 .221 .222

18 numbers

Each can be positive or negative

Each can have 3 exponents

So overall 18 * 3 * 2 possible numbers.

0.2 2)

We compute as follows

$d_k...d_0 = d_k \cdot b^k + d_{k-1} \cdot b^{k-1} + \cdots + d_1 \cdot b^1 + d_0$

$= d_0 + b\,(d_1 + b\,(...d_{k-1} + d_k b))$ which leaves k flops

0.3 3 See A2

0.3.1 3c)

Raz:

So the question asks us, what would we do if we had to evaluate it like this, right?

Ax = b L2 P2 L1 P1 Ax = L2 P2 L1 P1 b [2:14 AM]

Yes, I think it's 2n^2 to be exact (on the right side)

And the left side is n^3/3 because the flops analysis is the same as lecture

But doing palu still has the advantage that we can reuse the factorization Unlike this method in the question

0.4 4a)

list both conditions of FPT

0.4.1 b)

$x_{k+1} = x_k + (x_k - x_k^2 y) = x_k - (x_k^2 y - x_k)$ we have the general form for netwons iteration

So $\frac{f(x_k)}{f'(x_k)} = x_k^2 y - x_k$

$x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)}$

If the iteration converges them $x_{k+1} = x_k := t$

So $t = t + \frac{f(x_k)}{f'(x_k)}$ and $\frac{f(x_k)}{f'(x_k)} = 0$ must be true (in the case of convergence) which means $f(x_k) = 0$

if it converges we'll have $x = x - (x^2 y - x) \leftrightarrow x = x - x^2 y + x \leftrightarrow 0 = -x^2 y - x = -x(xy - 1)$

Roots $x = 0, x = \frac{1}{y}$ so $y = \frac{1}{x}$

My understanding:

To determine the purpose of the iteration, we assume it converges and use that fact to make an equation that we can find the roots of. From the roots "it seems?" we can argue that the iterations purpose is to determine those exact roots

Raz's comments on solving this q

1) Find x = 1/y by letting x = 2x - x^2 y and isolating for x

2) Come up with f(x) = y - 1/x because y = 1/x when f(x) = 0

3) Show how we can make xk+1 fit into the netwon definition as follows with the new f(xk) and f'(xk): x_k+1 = xk - f(xk) / f'(xk) = xk - ((y - 1/xk) / (1/xk ^2)) 4) Therefore, the netwon iteration above, supports the claim that xn converges to 1/y as n approaches infinity. So the purpose of the fixed point iteration is to approximate the recipocral of y.

0.4.2 4c)

This is equivalent to $x^3 = \alpha$, where we solve for x

So $f(x) = x^3$and $x = \alpha$

We use the first form to define our function as $F(x) = x^3 - \alpha$

$F'(x) = 3x^2$

We use newton's iteration to define $g(x) = x - \frac{x^3 - a}{3x^2}$

This function will not converge at $x = 0$ which is clear because of the denominator

Below is a longer and unnecessary? computation of the sink

$g'(x) = \frac{2}{3} - \frac{2\alpha}{3x^3}$ For fpt we need

$g'(x) < 1 \leftrightarrow \left|\frac{2}{3} - \frac{2\alpha}{3x^3}\right| < 1 \leftrightarrow -1 < \frac{2}{3} - \frac{2\alpha}{3x^3} < 1$

...

$\leftrightarrow \frac{-1}{2\alpha} < \frac{1}{x^3} < \frac{5}{2\alpha} \rightarrow x^3 < -2\alpha$ or $x^3 > \frac{2}{5}\alpha$

So $x < \sqrt[3]{-2\alpha}, x > \sqrt[3]{\frac{2}{5}\alpha}$

So our sink is $\left(-\infty, \sqrt[3]{-2\alpha}\right) \cup \left(\sqrt[3]{\frac{2}{5}\alpha}, \infty\right)$

0.5 6d

Wikipedia says that Weiser theorem guarantees the existence of a polynomial with those properties Not that all polynomials will approximate it well . So no contradiction.

Raz: No contradiction. Because if we don't force the condition that p(xi) = yi, there can still exist a polynomial with a max distance epsilion from each point in y(x)"

0.6 7d Analysis

the most efficient is Newton if we add a point

Raz:

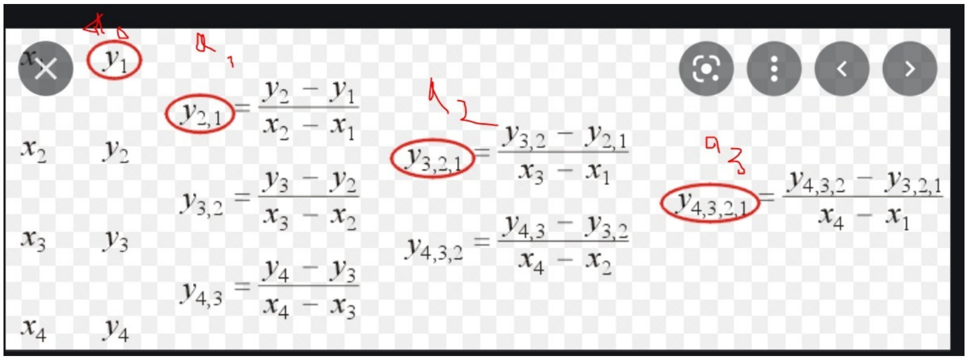
I think for Vandermonde, it's n^3/3 flops because we have to redo PALU for a new matrix .

For Lagrange, it takes at least n^2 flops every time we want to evaluate the polynomial Because evaluating each li(x) takes around 2n flops And there is no nested evaluation trick we can do here

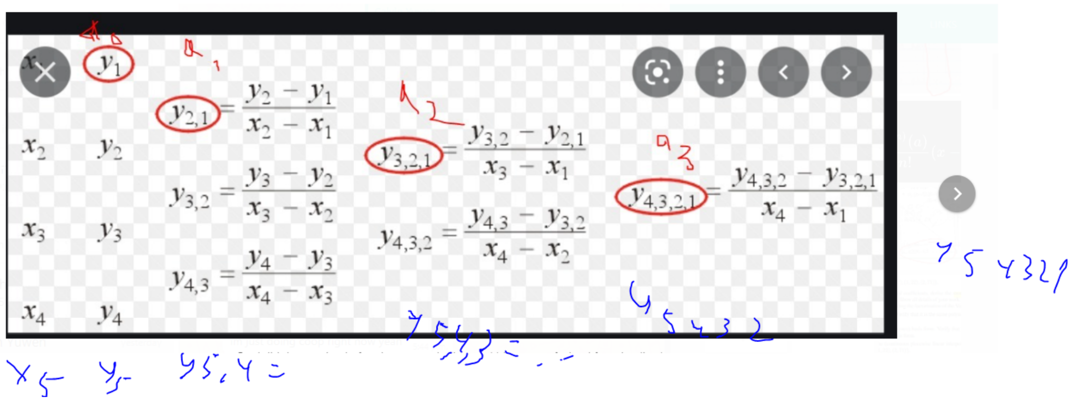
But if we have to do evaluate a polynomial many many times, it ends up not being efficeint at all The Netwon method takes around n^2 flops to find the coefficients of the polynomial (based on the method of divided difference) But only takes n (or maybe 2n) flops to evaluate a polynomial after you found the coefficeints at a specficic

But by adding additional data point with the divided difference table , you just need 2n additional flops for n additional calculations to get the remaining new coefficeint

each calculation takes 2 flops right (one for the numerator, one for the denominator). In the first column we have n -1 calculations, in the second row n - 2, ..., in the last column we have 1 calculation So if you sum it all up it will be around n^2



So adding the additional blue data point looks like this



1 2015 Final

a)

$\sqrt{1+x}-\sqrt{1-x}=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2x}{\sqrt{1+x}+\sqrt{1-x}}$

$x=0$

2a)

$$
\begin{aligned}
L_4P_4L_3P_3L_2P_2L_1P_1A &\leftrightarrow L_4P_4L_3P_3L_2\left(P_2L_1P_2\right)P_2P_1A\\
&\leftrightarrow L_4P_4L_3P_3L_2\left(P_2L_1P_2\right)P_3P_3P_2P_1A\\
&\leftrightarrow L_4P_4L_3\left(P_3L_2P_3\right)\left(P_3P_2L_1P_2P_3\right)P_3P_2P_1A\\
&\leftrightarrow L_4P_4L_3\left(P_3L_2P_3\right)\left(P_3P_2L_1P_2P_3\right)P_4P_4P_3P_2P_1A\\
&\leftrightarrow L_4P_4L_3\left(P_3L_2P_3\right)P_4P_4\left(P_3P_2L_1P_2P_3\right)P_4P_4P_3P_2P_1A\\
&\leftrightarrow L_4\left(P_4L_3P_4\right)\left(P_4P_3L_2P_3P_4\right)\left(P_4P_3P_2L_1P_2P_3P_4\right)P_4P_3P_2P_1A
\end{aligned}
$$

$L_1^{-1} = \left(P_4P_3P_2L_1P_2P_3P_4\right)^{-1} = \left(P_4P_3P_2L_1^{-1}P_2P_3P_4\right)$

$P_1 = P_{14}$

$P_2 = P_{25}$

$P_3 = P_{34}$

$P_4 = P_{45}$

Final form after applying perms (part b or c)

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0\\
-m_{51} & 1 & 0 & 0 & 0\\
-m_{41} & 0 & 1 & 0 & 0\\
-m_{21} & 0 & 0 & 1 & 0\\
-m_{31} & 0 & 0 & 0 & 1
\end{bmatrix}
$$

q4

We just can't use variation two since $g\left(\tilde{x}\right) \neq \tilde{x}$, ie the iteration doesn't yield the root when evaluated at $\sqrt{2}$

b)

$\left(x - x^2 + 2\right)' = 1 - 2x$

$|1 - 2x| < 1 \leftrightarrow -1 < 1 - 2x < 1 \leftrightarrow 2 > 2x > 0 \leftrightarrow 1 > x > 0$

variation 2

$\left(x - x^2 + 4\right)' = 1 - 2x$, this will have the same sink.

Scratch the above.

b)

We can discard the first 3 variations because their derivatives evaluated at $\sqrt{2}$ are greater than 1 or less than -1

c)

Our current variation converges linearly

We Use NM to get quadratic convergence

so we take $x - \frac{f(x)}{f'(x)} = x - \frac{x^2-2}{2x}$

5a)

Points $\{(-1,4),(0,6),(1,12)\}$

$$\begin{bmatrix} x & y[x_i] & y[x_{i+1},x_i] & y[x_{i+2},...,x_i] \\ -1 & 4 & & \\ & & \frac{6-4}{1} = 2 & \\ 0 & 6 & & \frac{6-2}{1--1} = 2 \\ & & \frac{12-6}{1} = 6 & \\ 1 & 12 & & \end{bmatrix}$$

$p(x) = y[x_0] + (x - x_0)\, y[x_1, x_0] + (x - x_0)(x - x_1)\, y[x_2, x_1, x_0]$

$= 4 + 2(x+1) + 2(x+1)(x)$

$= 4 + 2x + 2 + 2(x^2 + x) = 4 + 2x + 2 + 2x^2 + 2x$

$= 6 + 4x + 2x^2$

b)

$Points\,\{(-1,4),(0,6),(1,12)\}$

We want to calculate $p(x) = \sum_{i=0}^{n} l_i(x)\, y_i$ where $l_i(x) = \Pi_{i=0, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}$

We calculate $l_i(x)$ for each i as follows,

$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x)(x-1)}{(-1)(-2)} = \frac{x^2-x}{2}$

$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(1)(-1)} = \frac{x^2-1}{-1}$

$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x)}{(2)(1)} = \frac{x^2+x}{2}$

Now we compute $p(x) = \sum_{i=0}^{n} l_i\left(x\right) y_i$, we have,

$\frac{x^2-x}{2}\left(4\right) + \frac{x^2-1}{-1}\left(6\right) + \frac{x^2+x}{2}\left(12\right) = 2x^2 - 2x - 6x^2 + 6 + 6x^2 + 6x$

$= 2x^2 + 4x + 6$

d)

Points $\left\{\left(-1,4\right),\left(0,6\right),\left(1,12\right),\left(2,16\right)\right\}$

$$\begin{bmatrix} x & y\left[x_i\right] & y\left[x_{i+1},x_i\right] & y\left[x_{i+2},...,x_i\right] & \\ -1 & 4 & & & \\ & & \frac{6-4}{1}=2 & & \\ 0 & 6 & & \frac{6-2}{1--1}=2 & \\ & & \frac{12-6}{1}=6 & & \frac{-1-2}{2--1}=-1 \\ 1 & 12 & & \frac{4-6}{2-0}=-1 & \\ & & \frac{16-12}{2-1}=4 & & \\ 2 & 16 & & & \end{bmatrix}$$

$p\left(x\right) = y\left[x_0\right] + \left(x-x_0\right) y\left[x_1,x_0\right] + \left(x-x_0\right)\left(x-x_1\right) y\left[x_2,x_1,x_0\right] + \left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right) y\left[x_3,x_2,x_1,x_0\right]$

$= 4 + 2\left(x+1\right) + 2\left(x+1\right)\left(x\right) + \left(x+1\right)\left(x\right)\left(x-1\right)\left(-1\right)$

$= 4 + 2x + 2 + 2\left(x^2+x\right) + \left(-x^3+1\right) = 4 + 2x + 2 + 2x^2 + 2x + 1 - x^3$

$= 7 + 4x + 2x^2 + x^3$

e)

$y = \frac{x_i - x}{x_i - x_{i-1}} y_{i-1} + \frac{x - x_{i-1}}{x_i - x_{i-1}} y_i$

$\left\{\left(-1,4\right),\left(0,6\right),\left(1,12\right),\left(2,16\right)\right\}$

$y_1 = \frac{-x}{1}\cdot 4 + \frac{x+1}{1} 6 = -4x + 6x + 6 = 2x + 6$

$y_2 = 6\frac{1-x}{1} + 12\frac{x}{1} = 6 - 6x + 12x = 6 + 6x$

$y_3 = 12\frac{2-x}{1} + 16\frac{x-1}{1} = 24 - 12x + 16x - 16 = 8 + 4x$

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 1 & 0 & 0 & 6 \\ 1 & 1 & 1 & 12 \end{bmatrix}$$

Part I

Simpson's rule practice from end of last lec, deriving precision

$\frac{b-a}{6}\left(f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right)$

$$\begin{array}{cccc}
x^0=1 & \frac{b-a}{6}\left(1+4+1\right) & & =b-a=I\left(1\right) \\
x^1 & \frac{b-a}{6}\left(a+2a+2b+b\right) & & =\frac{b-a}{2}\left(a+b\right)=\frac{b^2-a^2}{2}=I\left(x^1\right) \\
x^2 & \frac{b-a}{6}\left(a^2+a^2+2ab+b^2+b^2\right) & & =\frac{b-a}{3}\left(a^2+2ab+b^2\right)=\frac{b^3-a^3}{3}=I\left(x^2\right) \\
x^3 & \frac{(b+a)(b-a)\left(b^2+a^2\right)}{4} & & =\frac{b^4-a^4}{4}=I\left(x^3\right) \\
x^4 & \frac{(b-a)\left(5b^4+4b^3a+6b^2a^2+4ba^3+5a^4\right)}{24} & & \neq\frac{b^5-a^5}{5}
\end{array}$$

So Simpson's rule has precision $m=4$

Fall 2015 Q 3

$$\left.\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 3 \\
1 & 1 & 1 & 1 & 7 \\
1 & 2 & 4 & 8 & 37 \\
1 & 3 & 9 & 27 & 141
\end{array}\right.$$