University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAB52H3 Introduction to Probability

Term Test 2
November 16, 2020

Duration: 60 minutes

Examination aids allowed: Open notes-books

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	20	20	20	20	20	20	20	20	20	20	200
Score:											

1. (20 points) Consider two random variables X and Y that are binary valued, i.e., the set of possible values is $\{0, 1\}$. Let the marginal distribution of X be

$$\mathbb{P}(X = 0) = p, \quad \mathbb{P}(X = 1) = 1 - p.$$

It is given that the conditional distribution of Y given X is

$$\begin{aligned} \mathbb{P}(Y = 0|X = 0) &= \frac{3}{4}, \quad \mathbb{P}(Y = 1|X = 0) = \frac{1}{4}, \\ \mathbb{P}(Y = 1|X = 1) &= \frac{1}{2}, \quad \mathbb{P}(Y = 0|X = 1) = \frac{1}{2}. \end{aligned}$$

(a) (10 points) Find the value of $p \in [0, 1]$ such that the marginal distributions of X and Y are the same.

(b) (10 points) For the value of p given in (a), find the conditional distribution of X given Y.

Solution: (a) By the law of total probability, we have

$$\begin{aligned} \mathbb{P}(Y = 0) &= \mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0) \\ &= \frac{3}{4}p + \frac{1}{2}(1 - p). \end{aligned}$$

We require that X and Y have the same marginal distribution. So

$$\mathbb{P}(Y = 0) = \frac{3}{4}p + \frac{1}{2}(1 - p) = p = \mathbb{P}(X = 0).$$

Solving gives $p = \frac{2}{3}$. Since X and Y are binary valued, if $p = \frac{2}{3}$ then they have the same marginal distributions.

(b) We have

$$\begin{aligned} \mathbb{P}(X = 0|Y = 0) &= \frac{\mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0)}{\mathbb{P}(Y = 0)} = \mathbb{P}(Y = 0|X = 0) = \frac{3}{4}, \\ \mathbb{P}(X = 1|Y = 0) &= 1 - \mathbb{P}(X = 0|Y = 0) = \frac{1}{4}, \\ \mathbb{P}(X = 1|Y = 1) &= \frac{\mathbb{P}(Y = 1|X = 1)\mathbb{P}(X = 1)}{\mathbb{P}(Y = 1)} = \mathbb{P}(Y = 1|X = 1) = \frac{1}{2}, \\ \mathbb{P}(X = 0|Y = 1) &= 1 - \mathbb{P}(X = 1|Y = 1) = \frac{1}{2}. \end{aligned}$$

(Note that the transition probabilities from X to Y are the same as those from Y to X.)

2. (20 points) Consider two random variables X and Y that are binary valued, i.e., the set of possible values is $\{0, 1\}$. Let the marginal distribution of X be

$$\mathbb{P}(X = 0) = p, \quad \mathbb{P}(X = 1) = 1 - p.$$

It is given that the conditional distribution of Y given X is

$$\mathbb{P}(Y = 0|X = 0) = \frac{2}{3}, \quad \mathbb{P}(Y = 1|X = 0) = \frac{1}{3},$$
$$\mathbb{P}(Y = 1|X = 1) = \frac{1}{2}, \quad \mathbb{P}(Y = 0|X = 1) = \frac{1}{2}.$$

(a) (10 points) Find the value of $p \in [0, 1]$ such that the marginal distributions of X and Y are the same.

(b) (10 points) For the value of p given in (a), find the conditional distribution of X given Y.

Solution: (a) By the law of total probability, we have

$$\begin{aligned}\mathbb{P}(Y = 0) &= \mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0) + \mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0)\\ &= \frac{2}{3}p + \frac{1}{2}(1 - p).\end{aligned}$$

We require that X and Y have the same marginal distribution. So

$$\mathbb{P}(Y = 0) = \frac{2}{3}p + \frac{1}{2}(1 - p) = p = \mathbb{P}(X = 0).$$

Solving gives $p = \frac{3}{5}$. Since X and Y are binary valued, if $p = \frac{3}{5}$ then they have the same marginal distributions.

(b) We have

$$\begin{aligned}\mathbb{P}(X = 0|Y = 0) &= \frac{\mathbb{P}(Y = 0|X = 0)\mathbb{P}(X = 0)}{\mathbb{P}(X = 0)} = \mathbb{P}(Y = 0|X = 0) = \frac{2}{3},\\ \mathbb{P}(X = 1|Y = 0) &= 1 - \mathbb{P}(X = 0|Y = 0) = \frac{1}{3},\\ \mathbb{P}(X = 1|Y = 1) &= \frac{\mathbb{P}(Y = 1|X = 1)\mathbb{P}(X = 1)}{\mathbb{P}(Y = 1)} = \mathbb{P}(Y = 1|X = 1) = \frac{1}{2},\\ \mathbb{P}(X = 0|Y = 1) &= 1 - \mathbb{P}(X = 1|Y = 1) = \frac{1}{2}.\end{aligned}$$

(Note that the transition probabilities from X to Y are the same as those from Y to X.)

3. (20 points) Let (X, Y) be a random vector whose distribution is given by

$$\mathbb{P}((X,Y) = (0,0)) = \mathbb{P}((X,Y) = (0,1)) = \mathbb{P}((X,Y) = (1,1)) = \frac{1}{3}.$$

(a) (4 points) Find the marginal distributions of X and Y, and give their name and parameters.

(b) (16 points) Find the correlation coefficient $\mathrm{Cor}(X, Y) = \rho_{XY}$ between X and Y.

Solution: (a) We have $X \sim \mathrm{Bernoulli}(\frac{1}{3})$ and $Y \sim \mathrm{Bernoulli}(\frac{2}{3})$.

(b) The covariance is given by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

We have

$$\mathbb{E}[XY] = \frac{1}{3}0 \cdot 0 + \frac{1}{3}0 \cdot 1 + \frac{1}{3}1 \cdot 1 = \frac{1}{3},$$

Recall that if $Z \sim \mathrm{Bernoulli}(p)$ then $\mathbb{E}[Z] = p$ and $\mathrm{Var}(Z) = p(1-p)$. So

$$\mathbb{E}[X] = \frac{1}{3}, \quad \mathbb{E}[Y] = \frac{2}{3},$$

and we have

$$\mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{1}{3}\frac{2}{3} = \frac{1}{9}.$$

Also

$$\mathrm{Var}(X) = \frac{1}{2}\frac{2}{3} = \frac{2}{9}, \quad \mathrm{Var}(Y) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}.$$

Thus

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1}{2}.$$

4. (20 points) Let (X, Y) be a random vector whose distribution is given by

$$\mathbb{P}((X,Y) = (0,1)) = \mathbb{P}((X,Y) = (1,0)) = \mathbb{P}((X,Y) = (1,1)) = \frac{1}{3}.$$

(a) (4 points) Find the marginal distributions of X and Y, and give their name and parameters.

(b) (16 points) Find $\mathrm{Var}(X + Y)$.

Solution: (a) We have $X \sim \text{Bernoulli}(\frac{2}{3})$ and $Y \sim \text{Bernoulli}(\frac{2}{3})$.

(b) We have

$$\text{Var}(X+Y) = \text{Var}(X) + 2\text{Cov}(X,Y) + \text{Var}(Y).$$

The covariance is given by

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

We have

$$\mathbb{E}[XY] = \frac{1}{3}0 \cdot 1 + \frac{1}{3}1 \cdot 0 + \frac{1}{3}1 \cdot 1 = \frac{1}{3},$$

Recall that if $Z \sim \text{Bernoulli}(p)$ then $\mathbb{E}[Z] = p$ and $\text{Var}(Z) = p(1-p)$. So

$$\mathbb{E}[X] = \frac{2}{3}, \quad \mathbb{E}[Y] = \frac{2}{3},$$

and we have

$$\text{Cov}(X,Y) = \frac{1}{3} - \frac{2}{3}\frac{2}{3} = \frac{-1}{9}.$$

Also

$$\text{Var}(X) = \text{Var}(Y) = \frac{2}{3}\frac{1}{3} = \frac{2}{9}.$$

Thus

$$\text{Var}(X+Y) = \frac{2}{9} - 2\frac{1}{9} + \frac{2}{9} = \frac{2}{9}.$$

5. (20 points) Consider two *independent* Exponential RVs: $X \sim \text{Exponential}(1)$ and $Y \sim \text{Exponential}(2)$. Find the probability $\mathbb{P}(X > Y)$.

Solution:

The joint PDF is $f_{X,Y}(x,y) = f_X(x)f_Y(y) = e^{-x}2e^{-2y}$, $\forall x, y > 0$. The probability is given by the integral of the joint PDF over the region $R = \{(x,y) \in \mathbb{R} : x > y\} =$

$\{(x, y) \in \mathbb{R} : 0 < x, 0 < y < x\}$

$$\begin{aligned}
\mathbb{P}(X > Y) &= \iint_R f_{X,Y}(x, y) dx dy \\
&= \int_{x=0}^{\infty} \left[\int_{y=0}^{x} 2e^{-x} e^{-2y} dy \right] dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[\int_{y=0}^{x} 2e^{-2y} dy \right] dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[-e^{-2y} \right]_0^x dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[1 - e^{-2x} \right] dx \\
&= \int_{x=0}^{\infty} e^{-x} dx - \int_{x=0}^{\infty} e^{-3x} dx \\
&= 1 - \left[-\tfrac{1}{3} e^{-3x} \right]_{x=0}^{\infty} \\
&= 1 - [0 + \tfrac{1}{3}] = \tfrac{2}{3}.
\end{aligned}$$

6. (20 points) Consider two *independent and identically distributed* Exponential RVs $X, Y \sim$ Exponential(1). Find the probability $\mathbb{P}(X > 2Y)$.

Solution:

The joint PDF is $f_{X,Y}(x, y) = f_X(x) f_Y(y) = e^{-x} e^{-y}$, $\forall x, y > 0$. The probability is given by the integral of the joint PDF over the region $R = \{(x, y) \in \mathbb{R} : x > 2y\} =$

$\{(x,y) \in \mathbb{R} : 0 < x, 0 < y < x/2\}$

$$\begin{aligned}
\mathbb{P}(X > 2Y) &= \iint_R f_{X,Y}(x,y)dxdy \\
&= \int_{x=0}^{\infty} \left[\int_{y=0}^{x} e^{-x}e^{-y}dy \right] dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[\int_{y=0}^{x/2} e^{-y}dy \right] dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[-e^{-y} \right]_0^{x/2} dx \\
&= \int_{x=0}^{\infty} e^{-x} \left[1 - e^{-x/2} \right] dx \\
&= \int_{x=0}^{\infty} e^{-x}dx - \int_{x=0}^{\infty} e^{-3x/2}dx \\
&= 1 - \left[-\tfrac{2}{3}e^{-3x/2} \right]_{x=0}^{\infty} \\
&= 1 - [0 + \tfrac{2}{3}] = \tfrac{1}{3}.
\end{aligned}$$

7. Let the RV X follow Uniform$(0,1)$ distribution, and define the new RV $Y = \frac{1}{1+X}$.
   (a) (3 points) What is the range of possible values of Y?
   (b) (10 points) Find the CDF of Y.
   (c) (7 points) Find the PDF of Y.

Solution:

(a) Since X ranges in $[0,1]$, then $Y = \frac{1}{1+X}$ will take values between $\frac{1}{1+0} = 1$ and $\frac{1}{1+1} = \frac{1}{2}$, i.e. $Y \in [\frac{1}{2}, 1]$

(b) The CDF of the Uniform(0,1) is the identity function, $F_X(x) = x$, $\forall x \in (0,1)$. We have

$$\begin{aligned}
F_Y(y) = \mathbb{P}(Y \le y) &= \mathbb{P}\left(\frac{1}{1+X} \le y \right) \\
&= \mathbb{P}\left(1 + X \ge \frac{1}{y} \right) = \mathbb{P}\left(X \ge \frac{1}{y} - 1 \right) \\
&= 1 - P\left(1 + X < \frac{1}{y} \right) = 1 - F_X\left(\frac{1}{y} - 1 \right) \\
&= 1 - \left(\frac{1}{y} - 1 \right) = 2 - \frac{1}{y}, \quad \forall y \in [\tfrac{1}{2}, 1]
\end{aligned}$$

(c) There are two ways to find the PDF of Y:
a) Differentiate the CDF of Y form the previous part:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(2 - y^{-1}) = y^{-2} = \frac{1}{y^2}, \quad \forall y \in [\tfrac{1}{2}, 1]$$

b) Use the PDF method:

$$h(x) = (1+x)^{-1} \Rightarrow \begin{cases} h^{-1}(y) = y^{-1} - 1 \\ h'(x) = -(1+x)^{-2} \end{cases}$$

$$\begin{aligned} f_Y(y) &= \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} = \frac{1}{|-(1+h^{-1}(y))^{-2}|} \\ &= \frac{1}{(1+y^{-1}-1)^{-2}} = \frac{1}{y^2}, \quad \forall y \in [\tfrac{1}{2}, 1] \end{aligned}$$

8. Let the RV X follow Uniform$(0,1)$ distribution, and define the new RV $Y = 1 - X^2$.
   (a) (3 points) What is the range of possible values of Y?
   (b) (10 points) Find the CDF of Y.
   (c) (7 points) Find the PDF of Y.

Solution:

(a) Since X ranges in $[0,1]$, then $Y = 1 - X^2$ will take values between $1 - 0^2 = 1$ and $1 - 1^2 = 0$, i.e. $Y \in [0,1]$

(b) The CDF of the Uniform(0,1) is the identity function, $F_X(x) = x$, $\forall x \in (0,1)$. We have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}\left(1 - X^2 \le y\right) \\ &= \mathbb{P}\left(X^2 \ge 1 - y\right) = \mathbb{P}\left(X \le \sqrt{1-y}\right) \\ &= 1 - P\left(X < \sqrt{1-y}\right) = 1 - F_X\left(\sqrt{1-y}\right) \\ &= 1 - \sqrt{1-y}, \quad \forall y \in [0,1] \end{aligned}$$

(c) There are two ways to find the PDF of Y:
a) Differentiate the CDF of Y form the previous part:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - \sqrt{1-y}\right) = \frac{d}{dy}\left(1 - (1-y)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}(1-y)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1-y}}, \quad \forall y \in [0,1] \end{aligned}$$

b) Use the PDF method:

$$h(x) = 1 - x^2 \Rightarrow \begin{cases} h^{-1}(y) = \sqrt{1-y} \\ h'(x) = -2x \end{cases}$$

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} = \frac{1}{|-2(\sqrt{1-y})|}$$
$$= \frac{1}{2(\sqrt{1-y})}, \quad \forall y \in [0,1]$$

9. Consider two RVs X, Y where X has marginal PDF $f_X(x) = \begin{cases} x/2, & x \in (0,2) \\ 0, & \text{otherwise} \end{cases}$ and Y has conditional PDF $f_{Y|X}(y|x) = \begin{cases} 1/x, & y \in (0,x) \\ 0, & \text{otherwise} \end{cases}$ (i.e. Y given X is uniformly distributed in $(0, X)$.)
   (a) (10 points) Find the marginal PDF of Y; are X and Y independent? (justify your answer).
   (b) (10 points) Find the conditional PDF of X given Y, and identify its distribution (i.e. give its name and parameters).

Solution:

(a) The joint PDF of X, Y is given by

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} \frac{x}{2x} = \frac{1}{2}, & 0 < y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

The marginal PDF of Y is found by integrating out X from the joint PDF

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_{x=y}^{2} \frac{1}{2} dx = \frac{2-y}{2}, & y \in (0,2) \\ 0, & \text{otherwise} \end{cases}$$

The RVs are *NOT* independent, since the conditional and marginal PDFs of Y are different (moreover, the conditional range of Y depends on X).

(b) The conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1/2}{(2-y)/2} = \frac{1}{2-y}, & x \in (y,2) \\ 0, & \text{otherwise} \end{cases}$$

Note that the conditional PDF of X does not depend on X (is flat), and is actually Uniform$(y, 2)$.

10. Consider two RVs X, Y where X has marginal PDF $f_X(x) = \begin{cases} 2x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$ and Y has conditional PDF $f_{Y|X}(y|x) = \begin{cases} 1/x, & y \in (0,x) \\ 0, & \text{otherwise} \end{cases}$ (i.e. Y given X is uniformly distributed in $(0, X)$.)

(a) (10 points) Find the marginal PDF of Y; are X and Y independent? (justify your answer).

(b) (10 points) Find the value of the conditional probability $\mathbb{P}(X > \frac{3}{4} | Y = \frac{1}{2})$.

Solution:

(a) The joint PDF of X, Y is given by

$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} 2x/x = 2, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal PDF of Y is found by integrating out X from the joint PDF

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) dx = \begin{cases} \int_{x=y}^{1} 2dx = 2(1-y), & y \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

The RVs are *NOT* independent, since the conditional and marginal PDFs of Y are different (moreover, the conditional range of Y depends on X).

(b) Note that the joint distribution is Uniform over the triangle $0 < y < x < 1$. If you condition on $Y = 1/2$, then X will range uniformly over $(1/2, 1)$; hence, the conditional probability of $X > 3/4$ is $P(X > \frac{3}{4} | Y = \frac{1}{2}) = 1/2$. You can verify this more formally using the conditional PDF of X given Y

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{2(1-y)} = \frac{1}{1-y}, & x \in (y,1) \\ 0, & \text{otherwise} \end{cases}$$

Note that the conditional PDF of X does not depend on X (is flat), and is actually Uniform$(y, 1)$. For $Y = 1/2$, we have $f_{X|Y}(x|1/2) = \frac{1}{1-1/2} = 2, \forall x \in (1/2, 1)$.

$$\begin{aligned} \mathbb{P}(X > \tfrac{3}{4} | Y = \tfrac{1}{2}) &= \int_{3/4}^{1} f_{X|Y}(x|1/2) dx \\ &= \int_{3/4}^{1} 2dx = 2(1 - 3/4) = 1/2 \end{aligned}$$