$B\vec{w} = b$
$A\vec{v} = a$
$C\vec{u} = c$
$C^{-1}\vec{c} = \vec{u}$

Question 1

$n \times n$ $n \times 1$ $n \times 1$ $1 \times n$

[10 marks]

Consider $A, B, C \in \mathcal{R}^{n \times n}$ and $x, y, z \in \mathcal{R}^n$. Let A, B and C be non-singular (invertible) matrices, and let $I \in \mathcal{R}^{n \times n}$ represent the $n \times n$ identity matrix.

$A\vec{x} = \vec{b}$
$A \; \vec{x} \, \vec{b}^T$

a. Show how to compute

$$z = B^{-1}(2A + I)(C^{-1} + A)x$$

without explicitly inverting either B or C. You may use vector addition, matrix-vector multiplication, and/or a routine for solving a system of linear equations. (You may assume the existence of such a routine—you do **not** need to give the details of Gaussian elimination and the $PA = LU$ factorization; i.e., you may assume such a factorization exists and can be computed.)

Z can be calculated by doing the following:
assume $B\vec{w} = \vec{b}$, & $C\vec{u} = \vec{c}$ for some $n \times 1$ vectors $\vec{w}, \vec{u}, \vec{b}, \vec{c}$. then $B^{-1} = \vec{b}^T\vec{w}$ and $C^{-1} = \vec{c}^T\vec{u}$

then $Z = \vec{b}^T\vec{w}(2A+I)(\vec{c}^T\vec{u} + A)x$

since $\vec{w}, \vec{u}$ are $n \times 1$ then $\vec{b}^T, \vec{c}^T$ are $1 \times n$.

Thus, to solve for $\vec{z}$ you first multiply each element in A by 2 and add 1 to each element in the diagonal to get $2A+I$. Second you calculate $\vec{c}^T\vec{u}$ to get an $n \times n$ matrix that we add to A to get $C^{-1}+A$
Finally you multiply $\vec{b}\vec{w}^T$ by $2A+I$ which you then multiply $(C^{-1}+A)$ to get a new $n \times n$ matrix which you multiply with $\vec{x}$ to get $\vec{z}$

CONTINUED ...

$A\vec{u} = \vec{a}$
$A^{-1} = \vec{u}\,\vec{a}^{-1}$

b. Show how to compute $y = A^{-6}x$ *without* explicitly inverting A. (Note: $A^{-6} = (A^{-1})^6$.) The same assumptions as in **(a)** apply.

Assume $A\vec{x} = \vec{b}$ For some $\vec{b} \in \mathbb{R}^n$

then $\vec{x} = A^{-1}\vec{b}$

$\therefore y = \vec{x} \cdot \vec{x} \cdot \vec{x} \cdot \vec{x} \cdot \vec{x} \cdot \vec{x} \cdot \vec{x}$
$= A^{-1}\vec{b} \cdot A^{-1}\vec{b} \cdot A^{-1}\vec{b} \cdot A^{-1}\vec{b} \cdot A^{-1}\vec{b} \cdot A^{-1}\vec{b} \cdot x$

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Question 2

[15 marks]

Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 4 & 4 \\ 1 & 5 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 40 \\ 24 \\ 26 \end{bmatrix}.$$

a. Compute the $PA = LU$ factorization of A. Use exact arithmetic. Show all intermediate calculations, including Gauss transforms and permutation matrices.

$$P_1 A = \begin{bmatrix} 4 & 4 & 4 \\ 3 & 5 & 9 \\ 1 & 5 & 5 \end{bmatrix} \quad P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}$$

$$L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 2 & 6 \\ 0 & 4 & 4 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad P_2 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 2 & 6 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \quad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$L_2 P_2 L_1 P_1 A = U$$
$$\Leftrightarrow L_2 P_2 L_1 P_2 P_2 P_1 A = U$$
$$\Leftrightarrow L_2 \tilde{L}_1 P_2 P_1 A = U$$
$$\Leftrightarrow P_2 P_1 A = \tilde{L}_1^{-1} L_2^{-1} U$$
$$\Leftrightarrow PA = LU$$

$$\tilde{L}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 0 & 1 \end{bmatrix} \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$P = P_2 P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \checkmark$$

$$L = \tilde{L}_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/2 & 1 \end{bmatrix} \checkmark$$

$$U = \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \checkmark$$

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CONTINUED ...

14

14 51-25=26

b. Use the factorization computed in (**a**) to solve the system.

$A\vec{x}=b \iff LU\vec{x}=\vec{b} \quad U\vec{x}=d$

Solve $L\vec{d}=\vec{b}$

$\iff \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 24 \\ 26 \end{bmatrix}$

$\iff \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 24 \\ 26 \end{bmatrix}$

$d_1=40$

$\frac{1}{4}d_1+d_2=24$

$10+d_2=24$

$d_2=14$

$\frac{3}{4}d_1+\frac{1}{2}d_2+d_3=26$

$30+7+d_3=26$

$d_3=-11$

2.5/5

Solve $U\vec{x}=d$

$\iff \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 14 \\ -11 \end{bmatrix}$

$4x_3=-11$

$x_3=\frac{-11}{4}$ ✗

$4x_2+4x_3=14$

$4x_2+11=14$

$4x_2=25$ ✓

$x_2=\frac{25}{4}$

$4x_1+4x_2+4x_3=40$

$4x_1+25-11=40$

$4x_1=26 \quad x_1=\frac{26}{4}=\frac{13}{2}$

c. Why is Gaussian Elimination usually implemented as in this question (i.e., $PA = LU$ is computed separately, and then the factorization is used to solve $Ax = b$)?

So that the entire process can be done in $\Theta(n^3)$ time.

✗

CONTINUED ...

Question 3

[10 marks]

In lecture we saw that Gaussian elimination with partial pivoting usually, but not always, leads to a stable factorization of $A \in \mathcal{R}^{n\times n}$. A stable factorization is guaranteed if we use *full* pivoting, which employs both row and column interchanges before the k-th stage of the elimination to ensure that the largest element in magnitude in the $(n-k)\times(n-k)$ submatrix finds its way to the pivot position.

Full pivoting leads to a $PAQ = LU$ factorization, where P and Q are permutation matrices. Show how this factorization can be used to solve $Ax = b$.

as you reduce A to get U you will pivot A to get the largest value into the current pivot position (the diagonal)

At most you need to swap one row and one column creating P_i and Q_i respectively (i = the step you are on while triangulating A)

when you have correctly triangulize A you will have something like this

$$L_{n-1}P_{n-1}L_{n-2}P_{n-2}\cdots L_1P_1AQ_1\cdots Q_{n-2}Q_{n-1} = U$$

Then you shift the L's over to the other side in the same way you do with partial pivoting to get $PAQ = LU$

Where $P = P_{n-1}P_{n-2}\cdots P_1$ $Q = Q_1Q_2\cdots Q_{n-1}$ $L = \tilde{L}_1^{-1}\tilde{L}_2^{-1}\cdots L_{n-1}$

then $A = P^{-1}LUQ^{-1}$, you solve $A\vec{x} = b$

$P^{-1}L\vec{d} = \vec{b}$ and $UQ^{-1}\vec{x} = d$

4/10

CONTINUED ...

$3(18+1) = 57 / 18+3 = 21$ $\frac{18+6}{2} = 12$ $\frac{18+2}{2/3}$ $\frac{3(20)}{2} = 30$

Question 4

[15 marks]

$\frac{-4}{2\epsilon} = \frac{-2}{\epsilon}$ $\frac{2}{2\epsilon} = \frac{1}{\epsilon}$ $= \frac{-1}{\epsilon}$

Consider the linear system $Ax = b$ where

$-4+2 = \frac{2}{2\epsilon} = \frac{1}{}$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2+\epsilon \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{1}$$

$4+2\epsilon-4$

and $0 \le \epsilon < 1$.

3 $6+\epsilon$

2ϵ $\frac{2}{\epsilon} + \frac{1}{\epsilon} = \frac{3}{\epsilon}$ $\frac{4}{2\epsilon} + \frac{2}{2\epsilon} =$

a. Derive a formula for $\text{cond}_1(A)$, the 1-norm condition number of A. What is $\lim_{\epsilon \to 0} \text{cond}_1(A)$?

$\text{cond}_1(A) = \|A\|_1 \|A^{-1}\|_1$

$A^{-1} = \frac{1}{2(2+\epsilon) - 4(1)} \begin{bmatrix} 2+\epsilon & -4 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{2\epsilon} \begin{bmatrix} 2+\epsilon & -4 \\ -1 & 2 \end{bmatrix}$

$\|A\|_1 =$ max absolute column sum $=$

$= (6+\epsilon) \cdot \left(\frac{3}{\epsilon}\right)$

$= \frac{18}{\epsilon} + 3$

$= \frac{18+3\epsilon}{\epsilon}$

$\|A^{-1}\|_1 = \max\left(\frac{3+\epsilon}{2\epsilon}, \frac{3}{\epsilon}\right)$

$= \frac{3}{\epsilon}$

$\lim_{\epsilon \to 0} \text{cond}_1(A) = \infty$

positive infinity

System is poorly conditioned

4/4

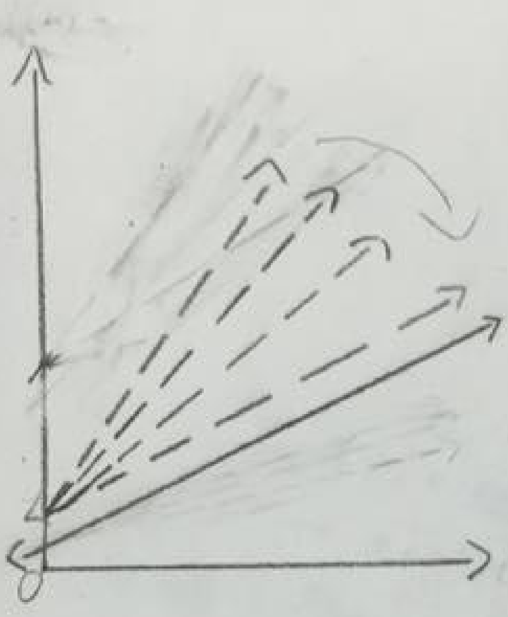
$y = -\frac{2x}{4} + \frac{b_1}{4}$

$2x + 4y = b_1$

$x + 2y + \epsilon y = b_2$

$2 + \epsilon y = x + b_2$

b. Sketch a graph illustrating the general trend of (1) as $\epsilon \to 0$. (Since you are not given specific values for the right-hand side b, you cannot pin down exact x and y intercepts.) Also show on the graph the potential effect(s) of small perturbations in the coefficients of A, such as those introduced when (1) is solved on a computer using Gaussian Elimination with partial pivoting.



$\longleftrightarrow\ -\frac{1}{2}x + \frac{b_1}{4}$

$\dashleftarrow\dashrightarrow\ -\frac{1}{2+\epsilon}x + \frac{b_2}{2+\epsilon}$

+ve

as $\epsilon \to 0$ the dotted line will slowly become more and more parallel with the solid line. At the same time the system will get a bigger and bigger condition number

6/7

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \frac{1}{-2} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$4 + \frac{3}{2}$$

$$-2 - \frac{1}{2}$$

$$= \begin{bmatrix} \frac{11}{2} & -\frac{3}{2} \\ \frac{-21}{4} & 7/4 \end{bmatrix}$$

$$\frac{6}{4} + \frac{1}{4} = \frac{7}{4}$$

$$\frac{-24}{4} - \frac{3}{4}$$