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These solutions are not the only "perfect" or "10/10" solutions to be followed as a model. They are examples of ways to solve these problems. There are many many alternative ways to solve these problems. We hope that these example solutions help you check your work.

Problems

Q1. The position vector of a particle moving along a helix is $\mathbf{c}(t) = (\cos(t), \sin(t), t^2)$.

(a) Find the speed of the particle at time $t_0 = 4\pi$.

Solution: The speed is defined to be $||\mathbf{c}'(t_0)||$. We calculate:

$$\mathbf{c}'(t) = (-\sin(t), \cos(t), 2t)$$

This gives $\mathbf{c}'(4\pi) = (0, 1, 8)$. Thus, the speed is:

$$||\mathbf{c}'(4\pi)|| = \sqrt{0^2 + 1^2 + 8^2} = \sqrt{65}$$

(b) Find the parametrization of the tangent line to $\mathbf{c}(t)$ at $t_0 = 4\pi$.

Solution: The tangent line is parametrized:

$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0)$$

This gives:

$$\ell(t) = (1, 0, 16\pi^2) + (t - 4\pi)(0, 1, 8)$$

(c) Where will the tangent line from $t_0 = 4\pi$ meet the xy-plane?

Solution: To determine where the tangent line meets the xy-plane we note that the xy-plane has equation $z = 0$. Thus, we need that $\ell(t)$ has z-component zero.

$$z = 0 \iff 16\pi^2 + 8(t - 4\pi) = 0 \iff t = \frac{32\pi - 16\pi^2}{8}$$

Thus, the tangent line meets the xy-plane at:

$$\ell\left(\frac{32\pi - 16\pi^2}{8}\right) = (1, 0, 16\pi^2) + \left(\frac{32\pi - 16\pi^2}{8} - 4\pi\right)(0, 1, 8) = \left(1, -2\pi^2, 0\right)$$

Q2. Consider the helix given by $\mathbf{c}(t) = (a\cos(t), a\sin(t), ct)$.
Show that its acceleration vector is always parallel to the xy-plane.

Solution: The acceleration vector is defined to be $\mathbf{c}''(t)$. We calculate:

$$\begin{aligned} \mathbf{c}'(t) &= (-a\sin(t), b\cos(t), c) \\ \mathbf{c}''(t) &= (-a\cos(t), -\sin(t), 0) \end{aligned}$$

Notice that the acceleration vector has **k**-component zero.
Thus, the acceleration vector is always parallel to the xy-plane.

Q3. Prove the dot product rule (M&T p. 218). If $\mathbf{c}_1(t)$ and $\mathbf{c}_2(t)$ are differentiable curves in $\mathbb{R}^n$ then:

$$\frac{d}{dt}\left[\mathbf{c}_1(t)\cdot\mathbf{c}_2(t)\right] = \mathbf{c}_1'(t)\cdot\mathbf{c}_2(t) + \mathbf{c}_1(t)\cdot\mathbf{c}_2'(t)$$

Solution: We write:

$$\mathbf{c}_1(t) = x_{1,1}(t)\mathbf{e}_1 + x_{1,2}(t)\mathbf{e}_2 + \cdots + x_{1,n}(t)\mathbf{e}_n \qquad \mathbf{c}_2(t) = x_{2,1}(t)\mathbf{e}_1 + x_{2,2}(t)\mathbf{e}_2 + \cdots + x_{2,n}(t)\mathbf{e}_n$$

The dot product is given by adding the products of the components:

$$\mathbf{c}_1(t)\cdot\mathbf{c}_2(t) = x_{1,1}(t)x_{1,2}(t) + \cdots + x_{1,n}(t)x_{2,n}(t) = \sum_{i=1}^{n} x_{1,i}(t)x_{2,i}(t)$$

We differentiate and obtain:

$$\begin{aligned} \frac{d}{dt}\left[\mathbf{c}_1(t)\cdot\mathbf{c}_2(t)\right] &= \sum_{i=1}^{n} x_{1,i}'(t)x_{2,i} + x_{1,i}(t)x_{2,i}'(t) \\ &= \sum_{i=1}^{n} x_{1,i}'(t)x_{2,i} + \sum_{i=1}^{n} x_{1,i}(t)x_{2,i}'(t) \\ &= \mathbf{c}_1'(t)\cdot\mathbf{c}_2(t) + \mathbf{c}_1(t)\cdot\mathbf{c}_2'(t) \end{aligned}$$

Q4. The arc-length function $s(t)$ of a given path $\mathbf{c}(x)$ is defined by:

$$s(t) = \int_a^t ||\mathbf{c}'(x)||dx$$

It measures the length of the path $\mathbf{c}(x)$ from $x = a$ to $x = t$.

(a) Compute the arclength function $s(t)$ of $\mathbf{c}_1(t) = (\cos(t), \sin(t), t)$ with $a = 0$.

Solution: First, we compute the speed of the curve.

$$||\mathbf{c}'(t)|| = ||(-\sin(t), \cos(t), 1)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \sqrt{2}$$

Applying the definition of arclength, we get:

$$\begin{aligned} s(t) &= \int_0^t ||\mathbf{c}'(x)||dx \\ &= \int_0^t \sqrt{2}dx \\ &= \sqrt{2}t \end{aligned}$$

(b) Compute the arclength function $s(t)$ of $\mathbf{c}_2(t) = (\cosh(t), \sinh(t), t)$ with $a = 0$.

The definitions of cosh and sinh are as follows:

$$\cosh(t) = \frac{e^x + e^{-x}}{2} \qquad \sinh(t) = \frac{e^x - e^{-x}}{2}$$

They are related to parametrizing the hyperbola $x^2 - y^2 = 1$ in the same way that cos and sin are related to paramatrizing the circle $x^2 + y^2 = 1$. Thus they are called the hyperbolic trigonometric functions.

Solution: Notice that:

$$\frac{d}{dx}[\sinh(x)] = \cosh(x) \qquad \frac{d}{dx}[\cosh(x)] = \sinh(x)$$

We use these identities to compute the speed of the curve.

$$\begin{aligned}
||\mathbf{c}'(t)|| &= ||(\sinh(t), \cosh(t), 1)|| \\
&= \sqrt{(\sinh(t))^2 + (\cosh(t))^2 + 1^2} \\
&= \sqrt{\left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2}\right)^2 + 1} \\
&= \sqrt{\frac{e^{2x} - 2 + e^{-2x}}{4} + \frac{e^{2x} + 2 + e^{-2x}}{4} + 1} \\
&= \sqrt{\frac{e^{2x} + e^{-2x}}{2} + 1} \\
&= \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{2}} \\
&= \sqrt{\frac{(e^x + e^{-x})^2}{2}} \\
&= \frac{e^x + e^{-x}}{\sqrt{2}}
\end{aligned}$$

Applying the definition of arclength, we get:

$$\begin{aligned}
s(t) &= \int_0^t ||\mathbf{c}'(x)|| dx \\
&= \int_0^t \frac{e^x + e^{-x}}{\sqrt{2}} dx \\
&= \int_0^t \sqrt{2}\cosh(x) dx \\
&= \left[\sqrt{2}\sinh(x)\right]_0^t \\
&= \sqrt{2}\sinh(t) - \sqrt{2}\cdot 0 = \sqrt{2}\sinh(t)
\end{aligned}$$

Q5. For this question, we need some definitions.

For a curve $\mathbf{c}(t)$ such that $\mathbf{c}'(t) \neq \mathbf{0}$ for any t, we define: $\mathbf{T}(t) = \mathbf{c}'(t)/||\mathbf{c}'(t)||$ to be its *unit tangent vector*. We say that a curve is *parametrized by arclength* if $||\mathbf{c}'(t)|| = 1$ for all t. If a curve is parametrized by arclength, then its *curvature* at $\mathbf{c}(t)$ is defined to be $\kappa = ||\mathbf{T}'(t)||$.

(a) Check that the helix $\mathbf{c}(t) = \frac{1}{\sqrt{2}}(\cos(t), \sin(t), t)$ is parametrized by arc length.

Solution: We compute the speed of the curve.

$$||\mathbf{c}'(t)|| = ||\frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)|| = \frac{1}{\sqrt{2}}\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} = \frac{1}{\sqrt{2}}\sqrt{2} = 1$$

Because the speed is one, we have that the curve is parametrized by arclength.

(b) Compute the curvature of the helix $\mathbf{c}(t)$.

Solution: We have that the curvature is $\kappa = ||\mathbf{T}'(t)||$. We compute:

$$\mathbf{T}'(t) = \frac{d}{dt}[\mathbf{c}(t)] = \frac{1}{\sqrt{2}}(-\cos(t), -\sin(t), 0)$$

This gives:

$$\kappa = ||\mathbf{T}'(t)|| = \frac{1}{\sqrt{2}}$$

(c) Find a constant c so that $\mathbf{s}(t) = (R\cos(c\,t), R\sin(c\,t))$ has unit speed.
And then, show that the curve $\mathbf{s}(t)$ has constant curvature.

Solution: We compute the speed of $\mathbf{s}(t)$. We have:

$$\mathbf{s}'(t) = (-Rc\sin(c\,t), Rc\cos(c\,t)) \Longrightarrow ||\mathbf{s}'(t)|| = \sqrt{(-Rc\sin(c\,t))^2 + (Rc\cos(c\,t))^2} = Rc$$

We pick $c = \frac{1}{R}$ so that the curve has unit speed. We then calculate:

$$\begin{aligned} \mathbf{s}(t) &= \left(R\cos\left(\frac{1}{R}t\right), R\sin\left(\frac{1}{R}t\right)\right) \\ \mathbf{s}'(t) &= \left(-\sin\left(\frac{1}{R}t\right), \cos\left(\frac{1}{R}t\right)\right) \end{aligned}$$

This allows us to compute the unit tangent vector:

$$\mathbf{T}(t) = \mathbf{s}'(t)/||\mathbf{s}'(t)|| = \mathbf{s}'(t)/1 = \left(-\sin\left(\frac{1}{R}t\right), \cos\left(\frac{1}{R}t\right)\right)$$

To compute $\kappa = ||\mathbf{T}'(t)||$ we take the derivative and get:

$$\mathbf{T}'(t) = \left(-\frac{1}{R}\cos\left(\frac{1}{R}t\right), -\frac{1}{R}\sin\left(\frac{1}{R}t\right)\right)$$

This gives: $||\mathbf{T}'(t)|| = 1/R$. Thus, the curvature of a circle of radius R is $1/R$.