University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2020

Assignment # 3

You are expected to work on this assignment prior to your tutorial during the week of January 27th. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.4; 4.5, and Supplementary material: 'The Integrability Reformulation' (Thm 2 Suppl. Notes).

HOMEWORK:

At the beginning of your TUTORIAL during the week of February 3rd you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

1. Let $a, b \in \mathbb{R}, a < b$. Let function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ -\frac{1}{2} & \text{, if } x \notin \mathbb{Q} \end{cases}$$

   Use the Integrability ϵ-Reformulation definition to prove that $2f(x) + 1$ is not integrable on $[a, b]$.

2. Prove, or disprove (by providing a counter-example), the following statements :

   (a) Let functions f be continuous everywhere. $\displaystyle\int (f(x))^2\,dx = \left(\int f(x)dx\right)^2$

(b) $\displaystyle\sum_{n=4}^{666} \frac{1}{n(n+1)} = \frac{221}{666}$

(c) Let functions g be (non-zero) continuous everywhere. $\dfrac{1}{\int g(x)dx} = \int (g(x))^{-1} \; dx$

3. Evaluate the following integrals. You may have to use algebra, educated guess-and-check, and/or recognize an integrand as the result of a product or quotient calculation. Note, when using FTOC part I, you do <u>not</u> need to check that the hypothesis is true before using the consequent; This is the **only** theorem where we will be allowed to do this 'omission of checking hypothesis'.

(a) $\displaystyle\int_{-1}^{0} \frac{2}{5+\sqrt{2}x} \; dx$

(b) $\displaystyle\int \cos^2(x+1)dx$

(c) $\displaystyle\int_0^1 e^{x-1}e^{2x+1}dx$

(d) $\displaystyle\int (2u-1)(u^2+u)^2du$

(e) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(t)dt$

(f) $\displaystyle\int \frac{\pi}{1+4x^2} \; dx$

(g) Let $a, b \in \mathbb{R}^+$, $\displaystyle\int a^{-x}(1+b^{-x})dx$

(h) $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\cos(x)\sin^5(x)dx$

(i) $\displaystyle\int \frac{2x\ln(x)-x}{(\ln(x))^2} \; dx$

(j) $\displaystyle\int xe^{3x^2+5} \; dx$

(k) $\displaystyle\int \frac{1}{(2x+1)^{\frac{1}{2}}} \; dx$

(l) $\displaystyle\int_0^4 \mid x-1 \mid\mid x+2 \mid dx$

CHALLENGE PROBLEMS

These question are <u>not</u> being graded or evaluated in any fashion. These are only for your interest's sake.

1. Let f be defined on $[-1, 4]$ by

$$f(x) = \begin{cases} 1 & \text{, if } x < 1 \\ -8 & \text{, if } x > 1 \end{cases}$$

   Use the Integrability ϵ-Reformulation (Theorem 2 of Chapter 13 of Suppl. Notes) to prove that f is integrable on $[0, 2]$.

2. Let f be defined on $[0, 2]$ by $f(x) = 1$ if $x \neq 1$ and $f(1) = 0$. Use the Integrability Reformulation (Theorem 2 of Chapter 13 of Suppl. Notes) to prove that f is integrable on $[0, 2]$.

EXERCISES: You do not need to submit solutions to the following problems but you should make sure that you are able to answer them.

1. Textbook Section 4.4 - # 1(a)-(h), 2, 8, 13, 16, 17, ODD numbered qns in 21-60 omit any involving hyperbolic trig functions, 63, 65, 67, 71, 73, 75, 77. – **You get better at integrating by practicing!**

2. Textbook Section 4.5 - # 1(a)-(f), 2, 8, 15, 16, ODD numbered qns in 19-63 omit any involving hyperbolic trig functions. – **You get better at integrating by practicing!**, 73, 75.

3. Let $a, b \in \mathbb{R}, a < b$. Let function $f \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ -1 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

   Use the Integrability ϵ-Reformulation definition to prove that $f(x)$ is <u>not</u> integrable on $[a, b]$.

4. Let $a, b \in \mathbb{R}, a < b$. Let function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & , \text{if } x \in \mathbb{Q} \\ -\pi & , \text{if } x \notin \mathbb{Q} \end{cases}$$

Use the Integrability ϵ-Reformulation to prove that $\int_a^b \cos(f(x))\,dx$ does not exist.

5. Let $k \in \mathbb{R}$. Prove : If f is continuous everywhere then $\int kf(x)\ dx = k \int f(x)\ dx$

6. Find the values of the following sums. *Hint : In the proof of FTOC I we encountered a telescoping sum. Are these telescoping (possibly 'in disguise', i.e. after equivalently re-writing the general term)?*

(a) $\displaystyle\sum_{n=2}^{500} \ln\left(1 + \frac{1}{n}\right)$

(b) $\displaystyle\sum_{n=3}^{100} \frac{3}{(n+1)(n+2)}$

(c) $\displaystyle\sum_{k=2}^{999} \left(\frac{1}{k-1} - \frac{1}{k+1}\right)$

7. Evaluate the following integrals. You may have to use algebra, educated guess-and-check, and/or recognize an integrand as the result of a product or quotient calculation.

(a) $\displaystyle\int x(\sqrt[3]{x} + \sqrt[4]{x})dx.$

(b) $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{t^2-1}{t^4-1}dt.$

(c) $\displaystyle\int \frac{\sqrt{2}}{\sqrt{x^{21}}}\ dx.$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} \sec(u)\tan(u)du.$

(e) $\displaystyle\int \sin(x)\cos(x)dx.$

(f) $\displaystyle\int_0^2 \frac{\sqrt{x}(2x+1)}{x^2}\ dx.$

(g) $\displaystyle\int a^x(2+b^x)dx$ where $a, b \in \mathbb{R}^+ - \{1\}$.

(h) $\displaystyle\int_{-1}^{1} f(x)dx$ where $f(x) = \begin{cases} \cos(x) & , \text{if } x \geq 0 \\ 1 - x^{\frac{5}{3}} & , \text{if } x < 0 \end{cases}$.

(i) $\displaystyle\int \frac{a}{1+(bx)^2}\, dx$, where $a, b \in \mathbb{R} - \{0\}$.

(j) $\displaystyle\int_0^1 \frac{t}{4t^2+1} dt$.

(k) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}} dx$.

(l) $\displaystyle\int \frac{\sec(x)\left(\sec(x)+\tan(x)\right)}{\sec(x)+\tan(x)} dx$.

(m) $\displaystyle\int \frac{1}{a^2+x^2} dx$ where $a \in \mathbb{R}$.

(n) $\displaystyle\int_0^1 (e^{x-1})^2 dx$

(o) $\displaystyle\int_{-1}^{3} \mid 4 - x^2 \mid dx$.

8. Let $a, b \in \mathbb{R}$, $a < b$. Let f be a function such that f' is continuous on $[a, b]$. Prove that $\displaystyle\int_a^b f(t)f'(t)dt = \frac{1}{2}\left(f^2(b) - f^2(a)\right)$.

Mathematics is the music of reason. – *James Joseph Sylvester*