Q9

1.

a) $E(\bar{X}_n) = E(\frac{1}{n}(X_1 + \dots + X_n))$

$= \frac{1}{n}(E(X_1) + \dots + E(X_n))$ by linearity of E

$= \frac{1}{n}(n)$

$= 1$

$P(\bar{X}_n \geq 2) \leq \frac{E(\bar{X}_n)}{2}$ by Markov Inq

$= \frac{1}{2}$ ▨

b) $Cov(X_i, X_j) = 0$ for $i \neq j$ bc X_i's are indep.

So $Var(\bar{X}_n) = Var(\frac{1}{n}(E(X_1) + \dots + E(X_n)))$

$= \frac{1}{n^2} Var(E(X_1) + \dots + E(X_n))$

$= \frac{1}{n^2}(n) = \frac{1}{n}$

$P(\bar{X}_n \geq 2) = P(\bar{X}_n - 1 \geq 1)$

$\leq P(|\bar{X}_n - 1| \geq 1)$

$\leq \frac{\frac{1}{n}}{1}$ by Chebyshev's inq

$= \frac{1}{n}$ ▨

c) $V(\bar{X}_n) = V(\frac{1}{n}\sum X_i)$

$= \frac{1}{n^2} V(\sum X_i)$

$= \frac{1}{n^2}(\sum V(X_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2Cov(X_i, X_j))$

$= \frac{1}{n^2}(n + \sum_{i=1}^{n-1} i)$

$= \frac{1}{n^2}\left(n + \frac{n(n-1)}{2}\right)$

$= \frac{1}{n} + \frac{n-1}{2n}$

$= \frac{n+1}{2n}$

For any $\varepsilon > 0$

$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{V(\bar{X}_n)}{\varepsilon^2}$ by Chebyshev

$= \frac{n+1}{2n\varepsilon^2}$

$\lim_{n \to \infty} \frac{n+1}{2n\varepsilon^2} = \lim_{n \to \infty} \frac{0}{2\varepsilon^2}$ L'Hopital's rule

$= 0$