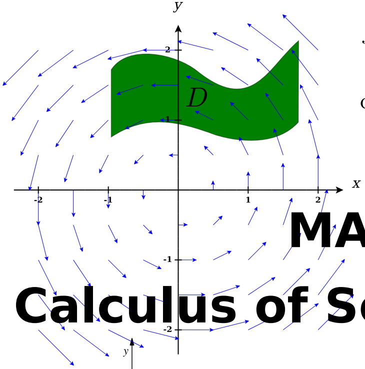
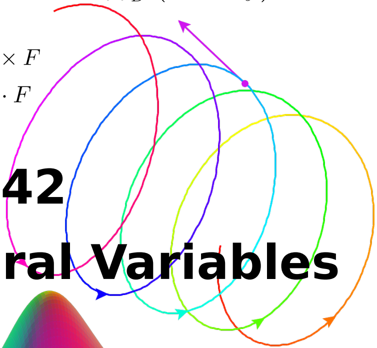


$$\oint_{\partial D} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

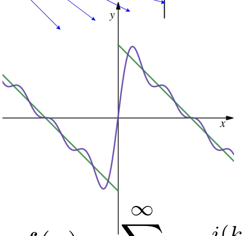
$$\text{curl} = \nabla \times F$$

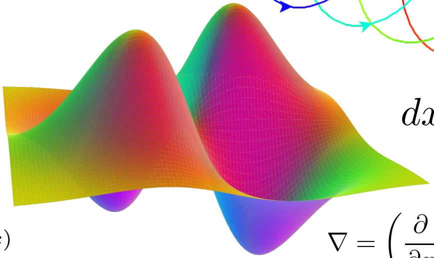
$$\text{div} = \nabla \cdot F$$



MAT B42

Calculus of Several Variables





$$dx \wedge dy$$

$$f(x) \approx \sum_{k=0}^{\infty} c_k e^{i(kx)}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

MAT B42: Techniques of the Calculus of Several Variables II (Winter 2023)

Welcome to the Week 2 Tutorial.
Questions? Thoughts? Comments?

News and Reminders:

- Parker holds office hours
- The FSGs (Facilitated Study Groups) are a great way to practice for the term test.

Fourier Polynomials

Definition (Hughes-Hallett p. 566)

The *n*th Fourier polynomial $F_n(x)$ of $f(x)$ has the form:

$$F_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$

where:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

Symmetry and Fourier

Question

Suppose $f(x)$ is even on $[-\pi, \pi]$.
What can you conclude about b_k for $k \geq 1$?

Notice that the integrand is odd over a symmetric interval.

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = 0$$

Question

Suppose $f(x)$ is odd on $[-\pi, \pi]$.
What can you conclude about a_k for $k \geq 1$?

Notice that the integrand is odd over a symmetric interval.

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx = 0$$

A Helpful Lemma

Lemma

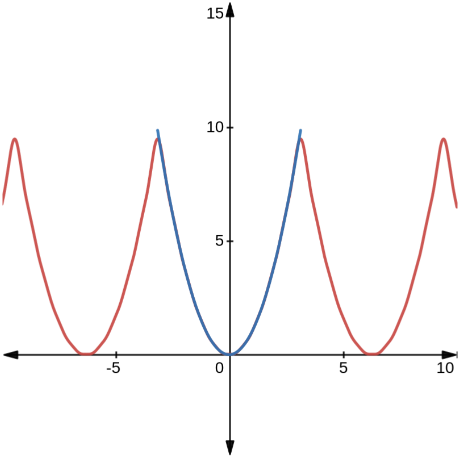
$\cos(k\pi) = (-1)^k$ *and* $\sin(k\pi) = 0$ *for all* $k \in \mathbb{Z}$.

Finding a Fourier Polynomial (for a Parabola)

Question

Find F_n for the parabola $f(x) = x^2$ when $-\pi \leq x \leq \pi$.

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(kx) dx = (-1)^k \frac{4}{k^2} \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(kx) dx = 0 \ \text{(symmetry)} \\
F_n(x) &= \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) \cos(kx)
\end{aligned}
$$



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Use the Fourier Series of a Parabola

Theorem

The parabola $f(x) = x^2$ has Fourier polynomial

$$F_n(x) = \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) \cos(kx)$$

Assume convergence $\lim_{n\to\infty} F_n(x) = f(x)$.

Question

Use this convergence to calculate the following sums.

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad B = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Sum for A

$$
\begin{aligned}
\pi^2 &= f(\pi) \\
&= \lim_{n\to\infty} F_n(\pi) \\
&= \lim_{n\to\infty} \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) \cos(k\pi) \\
&= \lim_{n\to\infty} \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) (-1)^k \\
&= \lim_{n\to\infty} \frac{\pi^2}{3} + \sum_{k=1}^{n} \left(\frac{4}{k^2}\right) \\
\frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) &= \frac{\pi^2}{6} = \sum_{k=1} \frac{1}{k^2} = A
\end{aligned}
$$

Sum for B

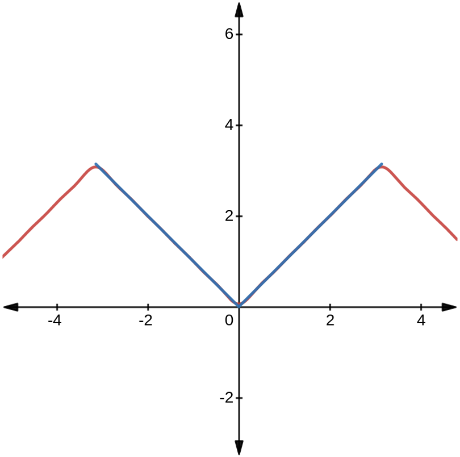
$$\begin{aligned}
0 &= 0^2 = f(0) \\
&= \lim_{n\to\infty} F_n(0) \\
&= \lim_{n\to\infty} \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) \cos(0) \\
&= \lim_{n\to\infty} \frac{\pi^2}{3} + \sum_{k=1}^{n} (-1)^k \left(\frac{4}{k^2}\right) \\
\frac{\pi^2}{12} &= \sum_{k=1} \frac{(-1)^{k+1}}{k^2} = B
\end{aligned}$$

Finding a Fourier Polynomial (for an Absolute Value)

Question

Find F_n for the absolute value $f(x) = |x|$ when $-\pi \leq x \leq \pi$.

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{\pi^2}{2\pi} = \frac{\pi}{2} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx) dx = \frac{2((-1)^k - 1)}{\pi k^2} \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(kx) dx = 0 \ (\text{symmetry}) \\
F_n(x) &= \frac{\pi}{2} + \sum_{k=1}^{n} \frac{2((-1)^k - 1)}{\pi k^2} \cos(kx)
\end{aligned}
$$



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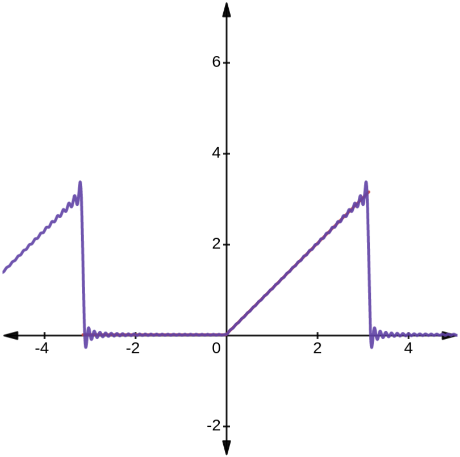
Finding a Fourier Polynomial (for a Saw)

Question

Find F_n for the function:

$$h(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)dx = \frac{1}{2\pi}\left(\frac{1}{2}\pi^2\right) = \frac{\pi}{4} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)dx = \frac{((-1)^k - 1)}{\pi k^2} \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} x\sin(kx)dx = \frac{(-1)^{k+1}}{k} \\
F_n(x) &= \frac{\pi}{4} + \sum_{k=1}^{n} \frac{((-1)^k - 1)}{\pi k^2}\cos(kx) + \frac{(-1)^{k+1}}{k}\sin(kx)
\end{aligned}$$



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