University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAB52H3 Introduction to Probability

Term Test 1
October 17, 2020

Duration: 60 minutes

Examination aids allowed: Non-programmable scientific calculator & formula sheet (provided with exam)

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	20	20	20	20	20	20	20	20	20	20	200
Score:											

1. (20 points) Consider a probability space and three (jointly) *independent* events A, B, C with probabilities $\mathbb{P}(A) = 1/2,\ \mathbb{P}(B) = 1/3,\ \mathbb{P}(C) = 1/4$.
Find the value of $\mathbb{P}((A \cup B) \cap C)$.

Solution:

$$\begin{aligned}
\mathbb{P}((A \cup B) \cap C) &= \mathbb{P}((A \cap C) \cup (B \cap C)) \\
&= \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}((A \cap C) \cap (B \cap C)) \\
&= \mathbb{P}(A)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(C) - \mathbb{P}(A \cap B \cap C) \\
&= \frac{1}{2 \times 4} + \frac{1}{3 \times 4} - \frac{1}{2 \times 3 \times 4} \\
&= \frac{4}{24} = \frac{1}{6}
\end{aligned}$$

2. (20 points) Consider a probability space and three (jointly) *independent* events A, B, C with probabilities $\mathbb{P}(A) = 1/2,\ \mathbb{P}(B) = 1/3,\ \mathbb{P}(C) = 1/4$.
Find the value of $\mathbb{P}((A \cap B)^c \cap C)$.

Solution:

$$\begin{aligned}
\mathbb{P}((A \cap B)^c \cap C) &= \mathbb{P}((A^c \cup B^c) \cap C) \\
&= \mathbb{P}((A^c \cap C) \cup (B^c \cap C)) \\
&= \mathbb{P}(A^c \cap C) + \mathbb{P}(B^c \cap C) - \mathbb{P}((A^c \cap C) \cap (B^c \cap C)) \\
&= \mathbb{P}(A^c)\mathbb{P}(C) + \mathbb{P}(B^c)\mathbb{P}(C) - \mathbb{P}(A^c \cap B^c \cap C) \\
&= \frac{1}{2} \times \frac{1}{4} + \frac{2}{3}\frac{1}{4} - \frac{1}{2}\frac{2}{3}\frac{1}{4} \\
&= \frac{5}{24} = 0.28\bar{3}
\end{aligned}$$

3. (20 points) Consider n persons, among them are Tom and Ben, who are arranged randomly in a row (say from left to right). What is the probability that there are exactly k persons between Tom and Ben? (Assume $n \geq k + 2$.)

Solution: Neglecting the ordering of Tom and Ben, there are $n - k - 1$ generic positions for them to be separated by exactly k persons. Here is an illustration when $n = 9$ and $k = 3$, so that $n - k - 1 = 5$:

OXXXOXXXX, XOXXXOXXX, XXOXXXOXX, XXXOXXXOX, XXXXOXXXO.

(O, O: Tom and Ben; X: other persons.)

For each generic position, there are

$$2 \times (n-2)!$$

ways to arrange the persons. So the required probability is

$$\frac{(n-k-1) \times 2 \times (n-2)!}{n!} = \frac{2(n-k-1)}{n(n-1)}.$$

4. (20 points) Consider $n \geq 3$ persons, among them are Tom and Ben, who are arranged randomly in a row (say from left to right). What is the probability that Tom and Ben are *NOT* next to each other?

Solution: Using the complement rule, we can express the probability as 1 minus the probability of Tom and Ben being next to each other. Neglecting the ordering of Tom and Ben, there are $n-1$ generic positions for them to be next to each other. Below is an illustration for $n=5$, with $n-1=4$ such positions (where O=(Tom/Ben) and X=(other persons)):

OOXXX, XOOXX, XXOOX, XXXOO.

For each generic position, there are $2 \times (n-2)!$ ways to arrange the persons. So the required probability is

$$1 - \frac{(n-1) \times 2 \times (n-2)!}{n!} = 1 - \frac{2(n-1)}{n(n-1)}.$$

5. (20 points) Consider a medical condition C and two associated symptoms S_1 and S_2. The prevalence of this condition in the population is 10%, and any person with the condition can show none, one, or both symptoms, with probabilities: $\mathbb{P}(S_1|C) = 30\%$, $\mathbb{P}(S_2|C) = 70\%$, $\mathbb{P}(S_1 \cap S_2|C) = 20\%$. The symptoms can also appear in individuals *without* the condition with equal probability $P(S_1|C^c) = P(S_2|C^c) = 5\%$, and in this case the symptoms are conditionally independent, i.e. $P(S_1 \cap S_2|C^c) = P(S_1|C^c) \times P(S_2|C^c)$.
Find the conditional probability $\mathbb{P}(C|S_1 \cap S_2^c)$, i.e. the probability of having the condition if you only show symptom S_1, but not S_2.

Solution:

$$
\begin{aligned}
\mathbb{P}(C|S_1 \cap S_2^c) &= \frac{\mathbb{P}(C \cap S_1 \cap S_2^c)}{\mathbb{P}(S_1 \cap S_2^c)} \\
&= \frac{\mathbb{P}(C \cap S_1 \cap S_2^c)}{\mathbb{P}(S_1 \cap S_2^c \cap C) + \mathbb{P}(S_1 \cap S_2^c \cap C^c)} \\
&= \frac{\mathbb{P}(S_1 \cap S_2^c|C)P(C)}{\mathbb{P}(S_1 \cap S_2^c|C)P(C) + \mathbb{P}(S_1 \cap S_2^c|C^c)P(C^c)} \\
&\left\{ \begin{array}{c} \mathbb{P}(S_1 \cap S_2^c|C) = \mathbb{P}(S_1|C) - \mathbb{P}(S_1 \cap S_2|C) = .3 - .2 = .1 \\ \mathbb{P}(S_1 \cap S_2^c|C^c) = \mathbb{P}(S_1|C^c) - \mathbb{P}(S_1 \cap S_2|C^c) = .05 - (.05)^2 = 0.0475 \end{array} \right\} \\
&= \frac{.1 \times .1}{.1 \times .1 + 0.0475 \times .9} = \frac{40}{211} \approx 18.95735\%
\end{aligned}
$$

6. (20 points) Consider a medical condition C and two associated symptoms S_1 and S_2. The prevalence of this condition in the population is 10%, and any person with the condition can show none, one, or both symptoms, with probabilities: $\mathbb{P}(S_1|C) = 30\%$, $\mathbb{P}(S_2|C) = 70\%$, $\mathbb{P}(S_1 \cap S_2|C) = 20\%$. The symptoms can also appear in individuals *without* the condition with equal probability $P(S_1|C^c) = P(S_2|C^c) = 5\%$, and in this case the symptoms are conditionally independent, i.e. $P(S_1 \cap S_2|C^c) = P(S_1|C^c) \times P(S_2|C^c)$.
Find the conditional probability $\mathbb{P}(C|S_1^c \cap S_2)$, i.e. the probability of having the condition if you only show symptom S_2, but not S_1.

Solution:

$$
\begin{aligned}
\mathbb{P}(C|S_1^c \cap S_2) &= \frac{\mathbb{P}(C \cap S_1^c \cap S_2)}{\mathbb{P}(S_1^c \cap S_2)} \\
&= \frac{\mathbb{P}(C \cap S_1^c \cap S_2)}{\mathbb{P}(S_1^c \cap S_2 \cap C) + \mathbb{P}(S_1^c \cap S_2 \cap C^c)} \\
&= \frac{\mathbb{P}(S_1^c \cap S_2|C)P(C)}{\mathbb{P}(S_1^c \cap S_2|C)P(C) + \mathbb{P}(S_1^c \cap S_2|C^c)P(C^c)} \\
&\left\{ \begin{array}{c} \mathbb{P}(S_1^c \cap S_2|C) = \mathbb{P}(S_2|C) - \mathbb{P}(S_1 \cap S_2|C) = .7 - .2 = .5 \\ \mathbb{P}(S_1^c \cap S_2|C^c) = \mathbb{P}(S_2|C^c) - \mathbb{P}(S_1 \cap S_2|C^c) = .05 - (.05)^2 = 0.0475 \end{array} \right\} \\
&= \frac{.5 \times .1}{.5 \times .1 + 0.0475 \times .9} = \frac{200}{371} \approx 53.90836\%
\end{aligned}
$$

7. (20 points) Consider the experiment of independently flipping a fair coin 4 times. Define the RV X to be the length of the *longest streak of Heads*, i.e. the maximum number of Heads appearing in a row. If there are no Heads in the outcome, then set the value of X equal to 0. Find the probability mass function (PMF) of X.

Solution: The sample space contains $2^4 = 16$ outcomes:

$$S = \begin{Bmatrix} (H,H,H,H) & (H,H,H,T) & (H,H,T,H) & (H,H,T,T) \\ (H,T,H,H) & (H,T,H,T) & (H,T,T,H) & (H,T,T,T) \\ (T,H,H,H) & (T,H,H,T) & (T,H,T,H) & (T,H,T,T) \\ (T,T,H,H) & (T,T,H,T) & (T,T,T,H) & (T,T,T,T) \end{Bmatrix}$$

The RV X maps these outcomes to the values:

$$X(S) \rightarrow \begin{Bmatrix} 4 & 3 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{Bmatrix}$$

Since this is a discrete uniform sample space, each outcome has probability 1/16. The PMF of X is found by adding the probabilities of all outcomes that map to these values:

$$p_X(x) = P(X = x) = \begin{cases} 1/16, & x = 0 \\ 7/16, & x = 1 \\ 5/16, & x = 2 \\ 2/16, & x = 3 \\ 1/16, & x = 4 \end{cases}$$

8. (20 points) Consider the experiment of independently flipping a fair coin 4 times. Define the RV X to be the length of the *longest streak of the same result*, i.e. the maximum number of Heads or Tails appearing in a row. Find the probability mass function (PMF) of X.

Solution: The sample space contains $2^4 = 16$ outcomes:

$$S = \begin{Bmatrix} (H,H,H,H) & (H,H,H,T) & (H,H,T,H) & (H,H,T,T) \\ (H,T,H,H) & (H,T,H,T) & (H,T,T,H) & (H,T,T,T) \\ (T,H,H,H) & (T,H,H,T) & (T,H,T,H) & (T,H,T,T) \\ (T,T,H,H) & (T,T,H,T) & (T,T,T,H) & (T,T,T,T) \end{Bmatrix}$$

The RV X maps these outcomes to the values:

$$X(S) \to \begin{Bmatrix} 4 & 3 & 2 & 2 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 2 & 2 & 3 & 4 \end{Bmatrix}$$

Since this is a discrete uniform sample space, each outcome has probability $1/16$. The PMF of X is found by adding the probabilities of all outcomes that map to these values:

$$p_X(x) = P(X = x) = \begin{cases} 2/16, & x = 1 \\ 8/16, & x = 2 \\ 4/16, & x = 3 \\ 2/16, & x = 4 \end{cases}$$

9. Suppose $X \sim \text{Binomial}(n, p)$.

   (a) (13 points) Show that for $k < n$, we have the identity

   $$\frac{\mathbb{P}(X = k+1)}{\mathbb{P}(X = k)} = \frac{n-k}{k+1}\frac{p}{1-p}.$$

   (b) (7 points) Show that as long as $k < (n+1)p - 1$ we have

   $$\mathbb{P}(X = k) < \mathbb{P}(X = k+1).$$

(Remark: This method can be used to show that the probability $\mathbb{P}(X = k)$ is maximized when k is near np.)

Solution:

(a) We have

$$\begin{aligned} \binom{n}{k+1} &= \frac{n!}{(n-(k+1))!(k+1)!} \\ &= \frac{n!}{\frac{(n-k)!}{(n-k)}(k+1)k!} \\ &= \frac{n-k}{k+1}\binom{n}{k}. \end{aligned}$$

We have

$$\mathbb{P}(X = k) = \binom{n}{k}(1-p)^{n-k}p^k, \quad \mathbb{P}(X = k+1) = \binom{n}{k+1}(1-p)^{n-k-1}p^{k+1}.$$

Taking the ratio, we have

$$\begin{aligned}\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} &= \frac{\binom{n}{k+1}(1-p)^{n-k-1}p^{k+1}}{\binom{n}{k}(1-p)^{n-k}p^k} \\ &= \frac{n-k}{k+1}\frac{p}{1-p}.\end{aligned}$$

(b) Suppose $k < (n+1)p-1$. Claim: $\mathbb{P}(X=k) < \mathbb{P}(X=k+1)$, i.e., $\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} > 1$. We have

$$\begin{aligned}(n-k)p-(k+1)(1-p) &= np-kp-(k+1)+kp+p \\ &= np-k-1+p \\ &> np-((n+1)p-1)-1+p \\ &= 0.\end{aligned}$$

So $\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} = \frac{(n-k)p}{(k+1)(1-p)} > 1$ as desired.

10. Suppose $X \sim \text{Binomial}(n,p)$.

(a) (13 points) Show that for $k < n$, we have the identity

$$\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} = \frac{n-k}{k+1}\frac{p}{1-p}.$$

(b) (7 points) Show that as long as $k > (n+1)p-1$ we have

$$\mathbb{P}(X=k) > \mathbb{P}(X=k+1).$$

(Remark: This method can be used to show that the probability $\mathbb{P}(X=k)$ is maximized when k is near np.)

Solution:

(a) We have

$$\begin{aligned}\binom{n}{k+1} &= \frac{n!}{(n-(k+1))!(k+1)!} \\ &= \frac{n!}{\frac{(n-k)!}{(n-k)}(k+1)k!} \\ &= \frac{n-k}{k+1}\binom{n}{k}.\end{aligned}$$

We have

$$\mathbb{P}(X = k) = \binom{n}{k}(1-p)^{n-k}p^k, \quad \mathbb{P}(X = k+1) = \binom{n}{k+1}(1-p)^{n-k-1}p^{k+1}.$$

Taking the ratio, we have

$$\begin{aligned}\frac{\mathbb{P}(X = k+1)}{\mathbb{P}(X = k)} &= \frac{\binom{n}{k+1}(1-p)^{n-k-1}p^{k+1}}{\binom{n}{k}(1-p)^{n-k}p^k} \\ &= \frac{n-k}{k+1}\frac{p}{1-p}.\end{aligned}$$

(b) Suppose $k > (n+1)p - 1$. Claim: $\mathbb{P}(X = k) > \mathbb{P}(X = k+1)$, i.e., $\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} < 1$. We have

$$\begin{aligned}(n-k)p - (k+1)(1-p) &= np - kp - (k+1) + kp + p \\ &= np - k - 1 + p \\ &< np - ((n+1)p - 1) - 1 + p \\ &= 0.\end{aligned}$$

So $\frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} = \frac{(n-k)p}{(k+1)(1-p)} < 1$ as desired.