MATB24 TUTORIAL PROBLEMS 8,

KEY WORDS: Eigenvalue and eigenvectors, diagonalization

RELEVANT SECTIONS IN THE TEXTBOOK: RELEVANT SECTIONS IN THE TEXTBOOK: 7.2 FB or 5.C SA

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms. You may need to consult the material for MATA22 to these definitions.

- Eigenvector of a matrix.
- Eigenvalues of a matrix.
- A diagonalizable matrix.
- Characteristic polynomial of a matrix.
- Orthogonal complement

Recall from MATA22: An $n \times n$ matrix is diagonalizable if it is similar to a diagonal matrix. That is if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Here is how you decide whether a given A is diagonalizable with your MATA22 knowledge:

(1) Find all eigenvalues of A by finding the roots of the characteristic polynomial of A, that is $\det(A - xI) = 0$. For each root, λ, the algebraic multiplicity of λ is the largest number n $(x - \lambda)^n$ appears in factorization of $\det(A - xI)$.
(2) For each eigenvalue λ, find the eigenspace $E_\lambda = Nul(A - \lambda I)$. Then geometric multiplicity of λ is $\dim(E_\lambda)$.
(3) A is diagonalizable if 1) for each λ, algebraic multiplicity of λ is equal to the geometric multiplicity of λ 2) the sum of all geometric multiplicities is equal to n.
(4) The columns of P are basis of E_λ's, and the diagonal entries of D are eigenvalues of A (put in a compatible order)

In this worksheet you develop a new point of view to the concept of diagonalizable matrices using the notion of change of basis, which sheds light on the above procedure. First, you need to generalize the concepts of eigenvalue, eigenvectors and characteristic polynomial from matrices to linear transformations.

A:

DEFINITION 0.1. *Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation. A non-zero vector $\vec{v} \in V$ is said to be an eigenvector of T if*

$$T(\vec{v}) = \lambda\vec{v}$$

for some scalar λ. The scalar λ is called the eigenvalue corresponding to $\vec{v}$.

(1) Show that for any scalar λ, the set $V_\lambda = \{\vec{v} \in V : T(\vec{v}) = \lambda\vec{v}\}$ is a subspace of V. When is this subspace nontrivial (i.e., not equal to $\{\vec{0}\}$)?
(2) For the following transformations, find an eigenvector using any methods you can think of, including basic geometry, if this is possible. What are the corresponding eigenvalues?
   (a) $V = \mathbb{R}^2$, $T =$ reflection over the x-axis.

(b) $V = \mathbb{R}^2$, $T =$ reflection over the line $x = y$.
(c) $V = \mathbb{R}^2$, $T =$ rotation by $90°$.
(d) $V = \mathbb{R}^2$, $T =$ left multiplication by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$
(e) $V = \mathbb{P}_3$ the space of polynomials of degree less than or equal 3 in the variable t, $T(f) = f'$.

B: A SYSTEMATIC WAY TO FIND EIGENVALUES

DEFINITION 0.2. *Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation. Determinant of T, denoted by* $\det(T)$ *is defined to be* $\det([T]_{\mathcal{B}})$*, where $\mathcal{B}$ is any basis for B.*

(1) Is the definition of $\det(T)$ well defined? What may go wrong? Prove that this definition is well-defined.
(2) Let λ be an eigenvalue of $T : V \to V$ and let $\vec{v}$ be an eigenvector with eigenvalue λ. Then
$$(T - \lambda \mathrm{id})\vec{v} = \vec{0}.$$
This means that the linear transformation $T - \lambda\mathrm{id}$ has a nontrivial kernel. Suppose $\mathcal{B}$ is a basis for V. Discuss with your group why we can deduce
$$\det([T - \lambda \mathrm{id}]_{\mathcal{B}}) = 0.$$ [1]
(3) Conversely, reversing the argument shows that if $\det([T - \lambda\mathrm{id}]_{\mathcal{B}}) = 0$, then λ is an eigenvalue of T.
(4) Show that $\det([T - \lambda\mathrm{id}]_{\mathcal{B}})$ does not depend on our choice of basis $\mathcal{B}$. That is choose a different basis $\mathcal{A}$ and show that $\det([T - \lambda\mathrm{id}]_{\mathcal{B}}) = \det([T - \lambda\mathrm{id}]_{\mathcal{A}})$.[2]
(5) Discuss with your group why the following definition is well defined:

DEFINITION 0.3. *(Characteristic polynomial) The characteristic polynomial of the linear transformation T is the polynomial in the variable λ given by* $\det([T - \lambda\mathrm{id}]_{\mathcal{B}})$*, where $\mathcal{B}$ is any basis of V.*

(6) Let A and B be similar matrices and let $r \in F$. Prove that $A - rI$ and $B - rI$ are similar.
(7) Let $\mathcal{B}$ be a basis of V. Show that
$$([T - \lambda\mathrm{id}]_{\mathcal{B}}) = [T]_{\mathcal{B}} - \lambda I$$
Conclude the characteristic polynomial of T is equal to $\det([T]_{\mathcal{B}} - \lambda I)$.
(8) To find all the eigenvalues λ of T, we find roots of $\det([T]_{\mathcal{B}} - \lambda I) = 0$. Use the previous parts to justify this. Discuss with your groups what happens if we choose a different basis.
(9) Choose two transformations in problem A part 2 and find their eigenvalues using this method.
(10) Write down the characteristic polynomials for the linear transformations you chose in previous part.
(11) Suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to *distinct* eigenvalues λ_1 and λ_2. Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.
(12) Extend the statement and proof of the previous problem to r eigenvectors.

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[1] Recall from MATA22 that a matrix have non-zero determinant if and only if it is invertible.
[2] Remember from MATA22 that $\det(AB) = \det(A)\det(B)$.

B: EIGENBASIS

DEFINITION 0.4. *Let V be an n dimensional vector space and let $T : V \to V$ be a linear transformation. T is diagonalizable if there exists a basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}}$ is diagonal.*

DEFINITION 0.5. *Let V be an n dimensional vector space and let $T : V \to V$ be a linear transformation.. A basis $\mathcal{B} = \{b_1, \cdots, b_n\}$ of V is called an eigenbasis if all b_i's are eigenvectors of T.*

(1) Suppose $\mathcal{B} = \{b_1, \cdots, b_n\}$ is an eigenbasis for T with the corresponding eigenvalues $\lambda_1, \cdots, \lambda)n$. Write down $[T]_{\mathcal{B}}$.
(2) Suppose the matrix if T with respect to some basis $\mathcal{A}$ is diagonal. What can we say about the vectors in $\mathcal{A}$?
(3) Prove: T is diagonalizable if and only if V has an eigenbasis for T.
(4) Now read the process of diagonalizing matrices again. Think of A as the standard matrix of a linear transformation from F^n to itself. Explain how the steps 1,2 and 3 in diagonalization of matrices at the top of the worksheet related to finding an eigenbasis? What is D in the context of change of basis? What is another name for matrix P and P^{-1}?