DEFINITION 1. *A **binary operation** on a set S is a rule for combining two elements of S to produce a third element of S. More formally, a binary operation is a mapping*

$$S \times S \to S.$$

If the binary operation is called, say $*$, we usually write $s_1 * s_2$ for the third element of S obtained by applying the operation to the pair of elements s_1, s_2 in S.

DEFINITION 2. *Let $*$ be a binary operation on a set S.*

1. *We say $*$ is commutative on S if for all s_1 and s_2 in S we have $s_1 * s_2 = s_2 * s_1$.*
2. *We say $*$ is associative on S if for all s_1, s_2 and s_3 in S we have $(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)$.*
3. *We say $*$ has an identity in S if there exists an element e in S such that $r * e = e * r = r$ for all $r \in S$.*
4. *Suppose $*$ has an identity e in S. We say r in S is invertible if there exists another element s in S such that $r * s = s * r = e$. The element s is called an inverse of r.*

DEFINITION 3. *A* field *is a non-empty set F with two binary operations, denoted "$+$" and "$\times$," such that*

1. *$+$ and $\times$ are associative*
2. *$+$ and $\times$ are commutative*
3. *$+$ has an identity, denote 0_F*
4. *Every element of F has an inverse for the operation $+$ denoted by $-r$*
5. *$\times$ has an identity, denoted 1_F, every $r \neq \{0_F\}$ of F has an inverse for the operation $\times$ denoted by r^{-1}*
6. *The two operations are related by the* distributive properties*: $a \times (b+c) = a \times b + a \times c$ and $(a+b) \times c = a \times c + b \times c$ for all $a, b, c \in F$*

DEFINITION 4. *An* F-vector space *is a set V with two operations: a binary operation called addition, which assigns to each $\vec{v}, \vec{w} \in V$ an element $\vec{v} + \vec{w} \in V$, and a scalar multiplication, which assigns to each $\vec{v} \in V$ and each $c \in F$ an element $c\vec{v} \in V$. These operations must satisfy the following properties.*

A1 *For all $\vec{u}, \vec{v}, \vec{w} \in V$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$. (addition is* associative*)*
A2 *For all $\vec{u}, \vec{v} \in V$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. (addition is* commutative*)*
A3 *There exists an element $\vec{0}$ in V such that for all $\vec{v} \in V$, $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$. ($\exists$ an additive* identity*)*
A4 *For each $\vec{u} \in V$ there is another element $(-\vec{v})$ such that $\vec{v} + (-\vec{v}) = 0$ ($\exists$ additive* inverses*)*
S1 *For all $c \in F$ and $\vec{u}, \vec{v} \in V$, $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.* (distributive *property 1)*
S2 *For all $c, d \in F$ and $\vec{v} \in V$, $(c + d)\vec{v} = c\vec{v} + d\vec{v}$.* (distributive *property 2)*
S3 *For all $c, d \in F$ and $\vec{v} \in V$, $c(d\vec{v}) = (cd)\vec{v}$.*
S4 *For all $\vec{v} \in V$, $1\vec{v} = \vec{v}$.*

DEFINITION 5. *A non-empty subset W of V is a subspace of V if W is itself a vector space with the vector addition and the scalar multiplication in V.*

DEFINITION 6. *A linear combination of vectors $v_1, \cdots v_r$ in V is a vector in V of the form $c_1v_1 + \cdots + c_rv_r$, for some $c_1, \cdots, c_r$ in $\mathbb{R}$. The scalars $c_1, \cdots, c_r$ are called scalar coefficients.*

DEFINITION 7. *Let $S \subseteq V$. The span of S is the set of all linear combinations of vectors in S.*

$$\mathrm{Span}(S) = \mathrm{Sp}(S) = \{c_1v_1 + \cdots + c_rv_r, \mid c_1, \cdots, c_r \in \mathbb{R},\ v_1, \cdots, v_r \in S, r \in \mathbb{N}\}$$

If $S = \{v_1, \cdots v_r\}$ (that is if S is finite) we write $\mathrm{Span}(S) = \mathrm{Span}(v_1, \cdots v_r)$. If $\mathrm{Span}(S) = W$ for a subspace W of V we say S spans W or S generates W or S is a spanning set for W. If W has a finite spanning set, we say W is finitely generated.

DEFINITION 8. *Consider vectors $v_1, \cdots, v_k$ in $X \subset V$. A relation of the form*

$$c_1v_1 + \cdots + c_kv_k = 0, \text{ where } (c_1, \cdots, c_k) \neq (0, \cdots, 0)$$

is called a dependency relation among v_i's or on X.

DEFINITION 9. *A subset X of V is called linearly dependent if there exists a dependency relation among vectors in X. We say X is linearly independent if it is not linearly dependent.*

DEFINITION 10 (alternative). *A set $X \subseteq V$ is linearly independent if for all $v_1, \cdots, v_k$ in X, $k \in \mathbb{N}$ and scalars $c_1, c_2, \cdots, c_k$ in $\mathbb{R}$,*

$$c_1v_1 + \cdots + c_kv_k = 0 \text{ implies } c_1 = c_2 = \cdots = c_k = 0.$$

A set $X \subset V$ is called linearly dependent if it is not linearly independent.

DEFINITION 11. *Let $S \subseteq V$. The span of S is the set of all linear combinations of vectors in S.*

$$\operatorname{Span}(S) = \operatorname{Sp}(S) = \{c_1v_1 + \cdots + c_rv_r, \mid c_1, \cdots, c_r \in \mathbb{R},\ v_1, \cdots, v_r \in S, r \in \mathbb{N}\}$$

If $S = \{v_1, \cdots v_r\}$ (that is if S is finite) we write $\operatorname{Span}(S) = \operatorname{Span}(v_1, \cdots v_r)$. If $\operatorname{Span}(S) = W$ for a subspace W of V we say S spans W or S generates W or S is a spanning set for W. If W has a finite spanning set, we say W is finitely generated.

DEFINITION 12. *A basis for a vector space V is a subset of V that*

(1) *spans V.*
(2) *is linearly independent.*

DEFINITION 13. *The number of vectors in a basis of V is called the dimension of V, denoted by* $\dim V$.

DEFINITION 14. *A map $T : V \to W$ is a linear transformation if*

(1) *For all $v_1, v_2 \in V$ $T(v_1 + v_2) = T(v_1) + T(v_2)$*
(2) *For all $v \in V$ and $r \in F$, $T(rv) = rT(v)$*

DEFINITION 15. *Let $T : V \to W$ be a linear transformation*

(1) *For every v in V, $T(v)$ is called the image of v under T.*
(2) *For every subset U of V, the image of U under T is*
$$T(U) := \{T(u) \mid u \in U\}.$$
(3) *The image of T is defined to be the image of V under T, that is*
$$\operatorname{img}(T) = \{T(v) \mid v \in V\}.$$
(4) *For any $w \in W$, the preimage or inverse image of w under T is*
$$T^{-1}(w) = \{v \in V \mid T(v) = w\}.$$
The inverse image of 0_w is of a special importance and has its own name. It is called the Kernel of T.
(5) *For any subset X of W, the preimage or inverse image of X under T is*
$$T^{-1}(X) = \{v \in V \mid T(v) \in X\}.$$
(6) *T is called one to one or injective if for all $v_1, v_2 \in V$ $T(v_1) = T(v_2)$ implies $v_1 = v_2$.*
(7) *T is called onto or surjective if for all $w \in W$, there is some vector $v \in V$ such that $T(v) = w$.*

DEFINITION 16. *The map $id_V : V \to V$, defined by $id_V(v) = v$ is called the identity map.*

DEFINITION 17. *We say a linear transformation $T : V \to W$ is invertible of there exists a linear transformation $T^{-1} : W \to V$ such that*

$$T \circ T^{-1} = id_W \quad \textit{and} \quad T^{-1} \circ T = id_V$$

If such an T^{-1} exists we call it the inverse of T.

DEFINITION 18. *LeT$_1 : V \to W$ and $T_2 : W \to Y$ be linear transformations. The composition of T_2 and T_1 is defined to be the map $T_2 \circ T_1 : V \to Y$ defined by $T_2 \circ T_1(v) = T_2(T_1(v))$for all $v \in V$.*

DEFINITION 19. *The map $id_V : V \to V$, defined by $id_V(v) = v$ is called the identity map.*

DEFINITION 20. *We say a linear transformation $T : V \to W$ is invertible of there exists a linear transformation $T^{-1} : W \to V$ such that*

$$T \circ T^{-1} = id_W \quad \textit{and} \quad T^{-1} \circ T = id_V$$

If such an T^{-1} exists we call it the inverse of T.

DEFINITION 21. *An invertible linear map $T : V \to W$ is called an isomorphism between V and W. If such a map exists between V and W we say V and W are isomorphic and write $V \cong W$.*

DEFINITION 22. *Two square matrices A are and B are said to be similar if there exists an invertible matrix P such that $A = PBP^{-1}$.*

DEFINITION 23. *An inner product on V is a map*

$$\begin{aligned} V \times V &\to F \\ (v, w) &\to \langle v, w \rangle \end{aligned}$$

that satisfies the following properties

(1) *linear in the first factor:* $\langle v+u, w\rangle = \langle v, w\rangle + \langle u, w\rangle$ *and* $\langle rv, w\rangle = r\langle v, w\rangle$ *for all* $v, u, w \in V$ *and* $r \in F$.
(2) *Positive definite:* $\langle v, v\rangle \geq 0$ *for all* $v \in V$. *The equality occurs only if* $v = 0$.
(3) *Conjugate symmetric :* $\langle v, w\rangle = \overline{\langle w, v\rangle}$ *for all* $v, w \in V$.

A vector space V *together with an inner product is called an inner product space.*

DEFINITION 24. *The magnitude or the norm of a vector* v *in* V *is then defined as*

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

The distance between two vectors v, w *in* V *is defined to be* $\|v - w\|$.

DEFINITION 25. *We say* $v, w \in V$ *are orthogonal if* $\langle v, w\rangle = 0$.

DEFINITION 26. *A set of vectors* $\{v_1, \cdots, v_k\}$ *in* V *is called orthogonal if* v_i*'s are mutually orthogonal. That is if* $\langle v_i, v_j\rangle = 0$ *for* $i \neq j$, $i, j \in \{1, \cdots, k\}$.[1] *If an orthogonal set is a basis for a subspace, we call it an orthogonal basis.*

DEFINITION 27. *A set of vectors* $\{v_1, \cdots, v_k\}$ *in* V *is called orthonormal if* v_i*'s are mutually orthogonal and* $\|v_i\| = 1$, *for all* $i \in \{1, \cdots, k\}$. *That is for* $i, j \in \{1, \cdots, k\}$

$$\langle v_i, v_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

If an orthonormal set is a basis for a subspace, we call it an orthonormal basis.

DEFINITION 28. *For a subspace* W *of* V *the orthogonal complement of* W, *denoted by* $W^{\perp}$, *is*

$$\{v \in V \mid \langle v, w\rangle = 0, \text{for all } w \in W\}$$

Theorem 0.1. *Every* $v \in V$ *can be uniquely expressed as* $v = v_W + v_{W^\perp}$, *where* $v_W \in W$ *and* $v_{W^\perp} \in W^\perp$.

DEFINITION 29. *The decomposition* $v = v_W + v_{W^\perp}$ *is Theorem 0.1 is called the orthogonal decomposition of* v *with respect to* W.

DEFINITION 30. *The vector* v_W *in Theorem 0.1 is called the orthogonal projection of* v *on* W *and is denoted by* $\mathrm{Proj}_W(v)$.

DEFINITION 31. *The linear transformation* $T : V \to V$ *is an isometry if it preserves the inner product of* V. *That is for all* v *and* w,

$$\langle T(v), T(w)\rangle = \langle v, w\rangle$$

DEFINITION 32. *A square matrix* $A \in M_n(\mathbb{R})$ *is orthogonal if* $A^T A = I_n$.

DEFINITION 33. *Let* A *be a complex matrix whose* ij*-th entry is* $a_{ij} \in \mathbb{C}$

(1) *The conjugate of* A *is a matrix whose* ij*-th entry is* $\bar{a_{ij}}$, *and is denote by* $\bar{A}$.
(2) *The conjugate transpose of* A *is a matrix whose* ij*-th entry is* $\bar{a_{ji}}$, *and is denote by* A^*.

DEFINITION 34. *A matrix* A *with complex entries is called Hermitian if* $A = A^*$.

DEFINITION 35. *A square matrix* U *with complex entries is called unitary if* $U^*U = I_n$

DEFINITION 36. *let* V *be an inner product space. A linear transformation* $T : V \to V$ *is self-adjoint if for all* v *and* w,

$$\langle T(v), w\rangle = \langle v, T(w)\rangle$$

DEFINITION 37. *A matrix* A *in* $M_n(\mathbb{R})$ *is orthogonally diagonalizbale if there is an orthogonal* $U \in M_n(\mathbb{R})$ *and a diagonal* $D \in M_n(\mathbb{R})$ *such that* $A = UDU^T$.

DEFINITION 38. *A matrix* A *in* $M_n(\mathbb{C})$ *is unitarily diagonalizable if there exists a diagonal matrix* $D \in M_n(\mathbb{C})$ *and a unitary matrix* $U \in M_n(\mathbb{C})$ *such that* $A = UDU^*$.

DEFINITION 39. *let* V *be an inner product space.* $T : V \to V$ *is orthogonally (unitarily) diagonalizbale if there exists an orthonormal basis* $\mathcal{U}$ *of* V *such that* $[T]_{\mathcal{U}}$ *is diagonal.*

DEFINITION 40. *A matrix* $A \in M_n(\mathbb{C})$ *is normal if* $AA^\star = A^\star A$.

[1] Warning: the word orthogonal in linear algebra is used in different context and in each context it has a different meaning. this is the first of three different sense in which the word orthogonal is used.

DEFINITION 41. *We say A is unitarily equivalent to B if for some unitary matrix U we have $A = UBU^{-1} = UBU^*$. We denote this relation by $A \underset{U.E}{\sim} B$*

DEFINITION 42. *For $A \in M_n(\mathbb{C})$ we define the equivalence class of A to be*

$$E_A = \{B \in M_n(\mathbb{C}) \mid B \underset{U.E}{\sim} A\}.$$

DEFINITION 43. *A Jordan block of size k is a $k \times k$ matrix in the form*

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \dots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

.

DEFINITION 44. *An $n \times n$ matrix J is said to be in Jordan canonical form if it consists of Jordan blocks located corner to corner on the main diagonal, and zero everywhere else. That is*

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix} \tag{1}$$

where J_i's are Jordan blocks.