MATB24 TUTORIAL PROBLEMS 2

KEY WORDS: subspace, spanning set, linearly independent, basis
RELEVANT SECTIONS IN THE TEXTBOOK: Sec 3.2 FB or Sec 2.A and Sec 1.C SA

WARM-UP: Write down a complete definition or a complete mathematical characterization for the following terms.

Let V be an F-vector space

- A subspace of V
- A linear combination of $v_1, v_2, \cdots, v_k$ in V
- A dependency relation in a subset X of V
- A spanning set for a subspace W of V
- The span of a subset X of V
- A linearly independent subset of V
- A linearly dependent subset of V

A: Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Recall that $\mathcal{F}$ is a vector space, with operations $(f+g)(x) = f(x)+g(x)$ and $(kf)(x) = kf(x)$, for all $f, g \in \mathcal{F}$, $k \in \mathbb{R}$, and $x \in \mathbb{R}$. Prove or disprove the following:

(1) The set $\{\sin^2(x), \cos^2(x)\}$ is linearly independent.
(2) The set $\{\sin(x), \cos(2x), \sin(3x)\}$ is linearly independent.
(3) The set $\{1, e^x + e^{-x}, e^x - e^{-x}\}$ is linearly independent.
(4) $\mathrm{Span}(\sin^2(x), \cos^2(x))$ contains all the constant function.
(5) If V is the subspace of $\mathcal{F}$ spanned by $\{1, 2\sin^2(x), 3\cos^2(x)\}$, then $\dim(V) = 2$.

B: Let V be a real vector space and $\vec{v}_i$ and $\vec{w}_i$ be in V, $i = 1, \ldots, n$.

(1) Give a necessary and sufficient condition for $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$. [1]
(2) Prove that $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ if and only if $\vec{v}_1, \cdots, \vec{v}_n$ are in $\mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ and $\vec{w}_1, \cdots \vec{w}_n$ are in $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$. [2] Is this the same condition that you gave in 1? If not, is it equivalent?
(3) Prove that $\mathrm{Span}(\vec{v}_1, \vec{v}_2) = \mathrm{Span}(\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2)$. You can use the previous parts.
(4) Let $\vec{w} \in V$. Prove or disprove: $\vec{w} \in \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$ if and only if $\mathrm{Span}(\vec{w}, \vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$

[1] Means fill in the blank for ..... if and only if $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$.

[2] Remember how we proved two sets are equal in class. You need to show they are subsets of one another.

C: Let P denote the vector space of all polynomials (that is, functions of the form $f(x) = a_n x^n + \cdots + a_1 x + a_0$ where each $a_i \in \mathbb{R}$ and n is any non-negative integer).

(1) Give two spanning sets for P.
(2) What do you think it means for a vector space to be finitely generated? Write down the definition explicitly. Check your definition with your TA.
(3) Suppose V is a finitely generated subspace of P, and S is a finite spanning set for V. Let $m = \max\{\deg(f(x)) \mid f \in S\}$. Discuss with your group why m is a well defined integer. One way to think about this is, if you were to program a computer to find m what would your algorithm be, and does that algorithm end.
(4) Construct an element g in P of degree $m+1$. Prove that g is not in $\mathrm{Span}(S)$.
(5) Prove that P is not finitely generated.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) A finitely generated vector space.

(2) A vector space that is not finitely generated

(3) A linearly dependent subset of $\mathcal{F}$ and a dependency relation among its vectors

(4) A linearly independent subset of $\mathcal{F}$.