MATB24 TUTORIAL PROBLEMS 4, SOLUTIONS

KEY WORDS: isomorphism, invertible linear transformation, change of coordinate matrix
RELEVANT SECTIONS IN THE TEXTBOOK:Sec 3.3, 3.4, 7.1 FB or 3C,3D SA

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms.

- An invertible linear transformation
- Give an equivalent condition for a linear transformation being invertible in terms of injectivity and surjectivity.
- Give an equivalent condition for a linear transformation T being invertible in terms of the Kernel and image of T.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Define the $\mathfrak{B}$-coordinates of a vector $\vec{v} \in W$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Give an isomorphism between W and $\mathbb{R}^{\dim W}$

A: In class we said (or will say) a linear transformation respects the structure of a vector space, for instance it maps a subspace to a subspace, the zero vector to zero vector, and so on. In this question you investigate how a linear transformation treats a linear independent set and a spanning set. We use the following result

Lemma 0.1. *Let $T : V \to W$ be a linear transformation.*

(1) *T is one-to-one if and only if $\ker T = \{0_V\}$.*
(2) *T is onto if and only if $\operatorname{img}(T) = W$*

(1) (a) Consider $I = \{e^x, e^{2x}, e^{3x}\}$ in $\mathcal{F}$. I is linearly independent (why?). Let $V = \operatorname{Span}(I)$. Let $T : V \to \mathcal{F}$ be a linear transformation, and suppose

$$T(e^x) = 1, \quad T(e^{2x}) = \cos^2 x, \quad T(e^{3x}) = \sin^2 x$$

Write down a formula for T of an arbitrary element of V

(b) Show that $T(I)$ is not linearly independent.
(c) Prove that T is not one-to one.[1]
(d) Show that $T(I)$ is not a spanning set for $\mathcal{F}$.
(e) Prove that T is not onto.

(2) Let $T : V \to W$ be a linear transformation. Prove that $T(I)$ is a linearly independent subset of W for every linearly independent subset I of V if and only if T is one to one.[2]
(3) Let $T : V \to W$ be a linear transformation. Let S be a spanning set for V. Prove T is onto if and only of $T(S)$ is a spanning set for W.
(4) Prove that the finite-dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

[1]You can find a nonzero vector in the kernel of T

[2]For every linearly independent subset I is a key information in one of the two directions (which one?)

Solution.

(1) (a) Yes, I is linearly independent. Solve system $c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$ at $x \in \{1, 2, 3\}$ or equivalent. Next, calculate $T(c_1e^x+c_2e^{2x}+c_3e^{3x}) = c_1T(e^x)+c_2T(e^{2x}) + c_3T(e^{3x}) = c_1 + c_2\cos^2(x) + c_3\sin^2(x)$.
(b) Observe dependency $T(e^{2x}) + T(e^{3x}) = T(e^x)$.
(c) Observe that $T(e^{2x} + e^{3x} - e^x) = T(e^{2x}) + T(e^{3x}) - T(e^x) = 0 \implies e^{2x} + e^{3x} - e^x \in \ker T$. Then apply Lemma 1(1).
(d) Calculate $\mathrm{sp}(T(I)) = \mathrm{sp}\{1, \cos^2(x)\} \subsetneq \mathcal{F}$.
(e) Observe $\mathrm{im}(T) = \mathrm{sp}(T(I)) = \mathrm{sp}\{1, \cos^2(x)\} \subsetneq \mathcal{F}$ and apply Lemma 1(2).

(2) Suppose T is one-to-one. Take image $T(I)$ of linearly independent set $I = \{v_j\}$ and observe $\sum_j c_jT(v_j) = T(\sum_j c_jv_j) = 0$ implies $\sum_j c_jv_j = 0$ since $\ker T = \{0_V\}$. Then, since $\{v_j\}$ is linearly independent, $c_j = 0\,\forall j$ and so $\{T(v_j)\}$ is also linearly independent. On the other hand, suppose $T(I)$ is always linearly independent. Take basis $B = \{v_j\}$, let $v := \sum_j c_jv_j \in \ker T$ and observe $0_W = T(v) = T(\sum_j c_jv_j) = \sum_j c_jT(v_j) \iff c_j = 0\,\forall j$ since $\{T(v_j)\}$ is linearly independent. Hence $v = 0$, $\ker T = \{0_V\}$ and T is one-to-one.

(3) Suppose T is onto. Let $S = \{v_j\}$ be a spanning set for V. Then $\forall w \in W, \exists v = \sum_j c_jv_j$ such that $T(v) = w$. In particular, $w = T(\sum_j c_jv_j) = \sum_j c_jT(v_j)$ and so $T(S)$ is a spanning set for W. On the other hand, suppose $T(S) = \{T(v_j)\}$ is a spanning set for W. Then $w = \sum_j c_jT(v_j) = T(\sum_j c_jv_j)$ for some c_j. In particular, $w \in \mathrm{im}(V)$ and T is onto.

(4)

B:Let $M_{n\times n}$ be the vector space of $n \times n$ matrices.

(1) Let $P \in M_{n\times n}$. Define the function $T_P : M_{n\times n} \to M_{n\times n}$ by $T_P(A) = PA$ for all $A \in M_{n\times n}$. Is T_P always linear? If so, is T_P ever an isomorphism?
(2) Let P be an invertible $n \times n$ matrix. Prove that the function $A \mapsto PAP^{-1}$ from $M_{n\times n}$ to $M_{n\times n}$ is an isomorphism. (This transformation is called *conjugation by* P).

Solution.

(1) Always linear: $T_P(A+rB) = P(A+rB) = PA+r(PB) = T_P(A)+rT_P(B)$. It is an isomorphism when P is invertible: $PA = O \implies A = P^{-1}O = O$ so injective, and given A, $P(P^{-1}A) = A$ so surjective.
(2) Let this function be called T. Linear: $T(A + rB) = P(A + rB)P^{-1} = (PA + +rPB)P^{-1} = PAP^{-1} + rPBP^{-1}$. Injective: $PAP^{-1} = O \implies A = P^{-1}OP = O$. Surjective: Given A, $P(P^{-1}AP)P^{-1} = A$.

C: Let $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$, and consider the bases

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

of the vector space $M_{2\times 2}$ of 2×2 matrices.

(1) Find $[I_2]_{\mathcal{E}}$ and $[A]_{\mathcal{E}}$. (Recall, for example, $[I_2]_{\mathcal{E}}$ is the coordinate vector of I_2 relative to the ordered basis $\mathcal{E}$ for $M_{2\times 2}$.)
(2) Find $[I_2]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.
(3) Find a basis $\mathcal{C}$ of $M_{2\times 2}$ such that $[A]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.
(4) Find a matrix C such that $C[B]_{\mathcal{B}} = [B]_{\mathcal{C}}$ for all B in $M_{2\times 2}$.
(5) Find a matrix D such that $D[B]_{\mathcal{C}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.
(6) Find a matrix F such that $F[B]_{\mathcal{B}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.
(7) Draw a diagram relating the linear transformations corresponding to the matrices F, C and D.

Solution.

(1) Since $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, therefore $[I_2]_{\mathcal{E}} = [1, 0, 0, 1]$. Similarly, $[A]_{\mathcal{E}} = [3, 1, -1, -1]$.

(2) $[I_2]_{\mathcal{B}} = [1, 0, 0, 0]$. For A, write $\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = r_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + r_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + r_4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} r_1 + r_2 & r_3 - r_4 \\ r_3 + r_4 & r_1 - r_2 \end{bmatrix}$. This implies $2r_1 = 2$ so $r_1 = 1$ and hence $1 + r_2 = 3$ so $r_2 = 2$, and $2r_3 = 0$ so $r_3 = 0$ and hence $0 + r_4 = -1$ so $r_4 = -1$. Thus, $[A]_{\mathcal{B}} = [1, 2, 0, -1]$.

(3) $\mathcal{C} = \left\{ \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. Since this ordered set is just a scaled version of the $\mathcal{E}$ basis, it is seen that it is a basis.

(4) Note that we are finding the change-of-basis matrix $C = C_{\mathcal{B}\to\mathcal{C}}$. We use Equation (9) on page 393 in the textbook, as follows. Let $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$. $[B_1]_{\mathcal{C}} = [1/3, 0, 0, -1]$, $[B_2]_{\mathcal{C}} = [1/3, 0, 0, 1]$, $[B_3]_{\mathcal{C}} = [0, 1, -1, 0]$, and $[B_4]_{\mathcal{C}} = [0, -1, -1, 0]$. Therefore, $C = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$.

(5) Now we want to find $D = D_{\mathcal{C}\to\mathcal{E}}$. Write $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$. Have $[C_1]_{\mathcal{E}} = [3, 0, 0, 0]$, $[C_2]_{\mathcal{E}} = [0, 1, 0, 0]$, $[C_3]_{\mathcal{E}} = [0, 0, -1, 0]$, and $[C_4]_{\mathcal{E}} = [0, 0, 0, -1]$. So,

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(6) Now we want $F = F_{\mathcal{B}\to\mathcal{E}}$. But, $F = D_{\mathcal{C}\to\mathcal{E}} C_{\mathcal{B}\to\mathcal{C}} = DC = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$.

COOL-OFF:

(1) Give three different isomorphism between P_n and $\mathbb{R}^{n+1}$.
(2) Give three different isomorphism between $M_{n\times m}(\mathbb{R})$ and $\mathbb{R}^{n\times m}$.
(3) Let V ad W be F- vector spaces and let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ be bases for V and W respectively. Construct three different isomorphism between V and W.