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These solutions are not the only "perfect" or "10/10" solutions to be followed as a model. They are examples of ways to solve these problems. There are many many alternative ways to solve these problems. We hope that these example solutions help you check your work.

# Problems

Q1. Let $n$ and $k$ be integers. Prove the following (almost) orthogonality relations:

(a)

$$\int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx = 0$$

**Solution:** Suppose that $n$ and $k$ are non-zero. Notice that the integrand $\sin(kx)\cos(nx)$ is odd and the interval $[-\pi,\pi]$ is symmetric. We get:

$$\int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx = 0$$

If $n = 0$ and $k \neq 0$ then

$$\int_{-\pi}^{\pi} \sin(kx)\cos(0)dx = \left[-\frac{1}{k}\cos(kx)\right]_{-\pi} \pi = 0$$

If $k = 0$ then $\sin(kx) = 0$ and the integrand vanishes.

(b)

$$\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \begin{cases} 0 & k \neq \pm n \\ \pi & k = \pm n \neq 0 \\ 2\pi & k = n = 0 \end{cases}$$

**Solution:** We consider four cases. If $n = k = 0$ then:

$$\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \int_{-\pi}^{\pi} 1 \cdot 1dx = 2\pi$$

If $k \neq \pm n$ then:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(kx)\cos(nx) &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((k+n)x) + \cos((k-n)x)dx \quad \# \cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b) \\
&= \frac{1}{2}\left[\frac{1}{k+n}\sin((k+n)x) + \frac{1}{k-n}\sin((k-n)x)\right]_{-\pi}^{\pi} \quad \# \text{ need } k \neq \pm n \\
&= 0
\end{aligned}$$

If $k = n \neq 0$ then:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(kx)\cos(nx) &= \int_{-\pi}^{\pi} \cos^2(kx)dx \\
&= \int_{-\pi}^{\pi} \frac{1+\cos(2kx)}{2} \\
&= \left[\frac{1}{2}x + \frac{1}{4x}\sin(2kx)\right]_{-\pi}^{\pi} \quad \# \text{ we need } k \neq 0 \\
&= \frac{1}{2}(2\pi) = \pi
\end{aligned}$$

If $k = -n \neq 0$ then:

$$\int_{-\pi}^{\pi} \cos(kx)\cos(-kx) = \int_{-\pi}^{\pi} \cos^2(kx)dx \;\# \text{ cos is even}$$

This cases reduces to the case when $k = n$. Thus, we have the desired equalities.

$$\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx = \begin{cases} 0 & k \neq \pm n \\ \pi & k = \pm n \neq 0 \\ 2\pi & k = n = 0 \end{cases}$$

Q2. Compute the first three Fourier approximations of the following.
Use a computer to graph the approximations that you obtain.

(a)

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

**Solution:** We calculate:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \;\# \; f(x) \text{ is odd}$$

The cosine terms are:

$$a_k = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\cos(kx)dx = 0 \;\# \; f(x)\cos(kx) \text{ is odd}$$

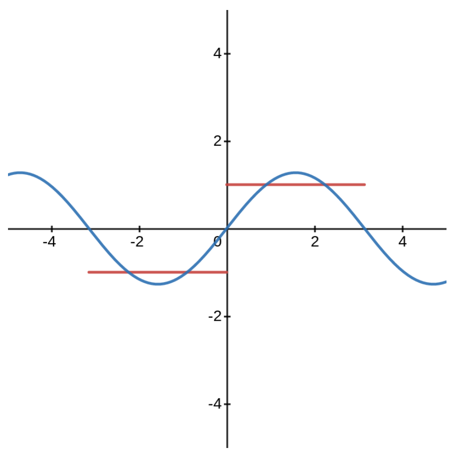
The sine terms are:

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin(kx)dx - \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(kx)dx \\ &= \frac{1}{\pi}\left[-\frac{1}{k}\cos(kx)\right]_0^{\pi} - \frac{1}{\pi}\left[-\frac{1}{k}\cos(kx)\right]_{-\pi}^{0} \\ &= \frac{-1}{\pi k}\left[(-1)^k - 1\right] + \frac{1}{\pi k}\left[1-(-1)^k\right] \\ &= \frac{2}{\pi k}\left[1-(-1)^k\right] \end{aligned}$$

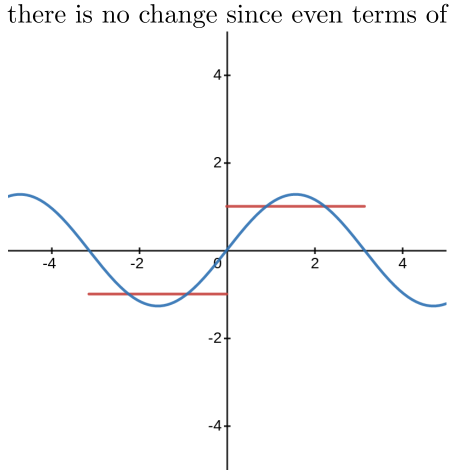
We obtain:

$$a_0 + \sum_{k=1} a_k \sin(kx) + b_k\cos(kx) = 0 + \sum_{k=1} \frac{2}{\pi k}\left[1-(-1)^k\right]\sin(kx)$$

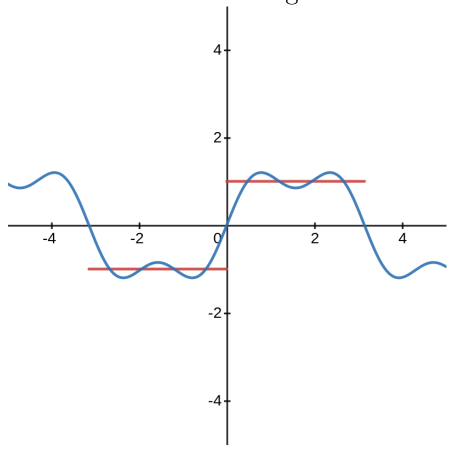
We graph our solutions for $k = 1, 2, 3$. For $k = 1$ we get:




For $k = 2$ we get: (notice there is no change since even terms of the Fourier series vanish)



For $k = 3$ we get:




(b)

$$g(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$$

**Solution:** We calculate:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{2\pi}\left[2 \cdot \frac{1}{2}\pi^2\right] = \frac{\pi}{2} \text{ \# geometry}$$

The cos terms are:

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos(kx)dx \text{ \# even symmetry} \\ &= \frac{2}{\pi}\left[\frac{kx\sin(kx)+\cos(kx)}{k^2}\right]_0^{\pi} \text{ \# integration by parts} \\ &= \frac{2}{\pi}\left[\frac{(-1)^k - 1}{k^2}\right] \end{aligned}$$

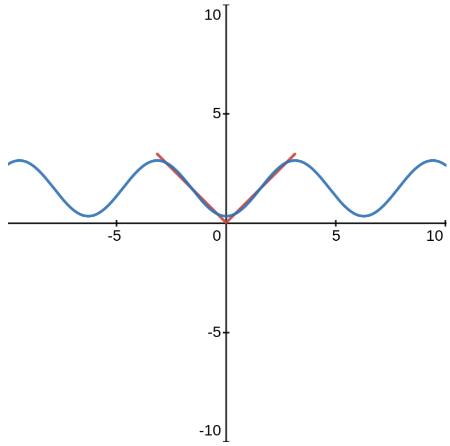
The sine terms are:

$$b_k = \frac{1}{\pi}\int_{\pi}^{\pi} g(x)\sin(kx)dx = 0 \text{ \# } g(x)\sin(kx) \text{ is odd}$$

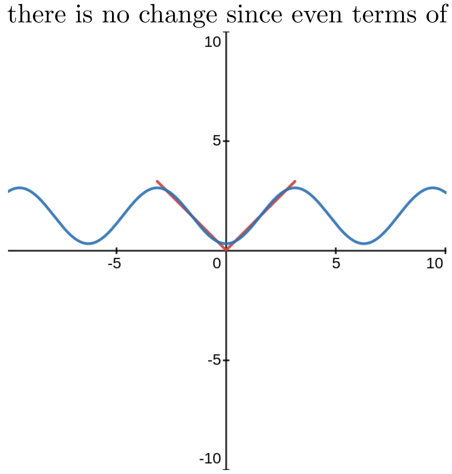
We obtain:

$$a_0 + \sum_{k=1} a_k\sin(kx) + b_k\cos(kx) = \frac{\pi}{2} + \sum_{k=1}\frac{2[(-1)^k - 1]}{\pi k^2}\cos(kx)$$

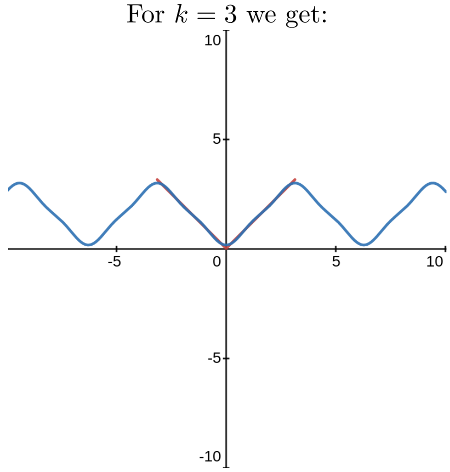
We graph our solutions for $k = 1, 2, 3$. For $k = 1$ we get:




For $k = 2$ we get: (notice there is no change since even terms of the Fourier series vanish)




For $k = 3$ we get:




Q3. Find two fourier expansions for the function $h(x) = \sin(x)$. In one expansion, all the sine terms should have zero as their coefficient. In the other expansion, all the cosine terms should have zero as their coefficient.

**Solution:** We have that $h(x) = \sin(x)$ is a perfectly reasonable Fourier series without any cos terms. Thus, $h(x) = 1\sin(x)$ is our Fourier series without cos terms.

Any Fourier approximation without sin terms will only contain $\cos(kx)$ terms. Thus, it will be an even function. In order to get an approximation of $\sin(x)$ without sin terms we must alter $\sin(x)$ so that it becomes even.

We define:

$$h_e(x) = \begin{cases} \sin(x) & 0 \le x \le \pi \\ -\sin(x) & -\pi \le x \le 0 \end{cases}$$

This function agrees with $h(x)$ on the interval $[0, \pi]$ and so it approximates $\sin(x)$ very well. You might recall from highschool that knowing $\sin(x)$ on the interval $[0, \pi/2]$ is sufficient to calculate any value of $\sin(x)$.

We calculate:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_e(x)dx = \frac{2}{2\pi}\int_0^{\pi} \sin(x)dx = \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{2}{\pi}$$

We compute the cos terms (for $k \neq 1$):

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} h_e(x)\cos(kx)dx \\
&= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(kx)dx \text{ \# integrand is even} \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{\sin((k+1)x) + \sin((1-k)x)}{2}dx \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)x) - \frac{1}{1-k}\cos((1-k)x)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)\pi) - \frac{1}{1-k}\cos((1-k)\pi) + \frac{1}{k+1} + \frac{1}{1-k}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}(-1)^{k+1} - \frac{1}{1-k}(-1)^{1-k} + \frac{1}{k+1} + \frac{1}{1-k}\right] \\
&= \frac{2(1-(-1)^{k+1})}{\pi(k+1)(1-k)} \text{ \# } (-1)^{k+1} = (-1)^{1-k}
\end{aligned}$$

We compute the cos term for $k = 1$:

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} h_e(x)\cos(x)dx \\
&= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(x)dx \text{ \# integrand is even} \\
&= \frac{2}{\pi}\left[\frac{1}{2}\sin^2(x)\right]_0^{\pi} = 0
\end{aligned}$$

We compute the sin terms:

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} h_e(x)\sin(kx)dx \\
&= 0 \text{ \# integrand is odd}
\end{aligned}$$

This gives the following Fourier series for $h_e(x)$:

$$\frac{2}{\pi} + \sum_{k=2} \frac{2(1-(-1)^{k+1})}{\pi(k+1)(1-k)}\cos(kx)$$

Q4. Show that for any positive integer $k$, the function $f(x) = a_k \cos(kx) + b_k \sin(kx)$ has energy $a_k^2 + b_k^2$.

**Solution:** We compute the energy of $f(x) = a_k \cos(kx) + b_k \sin(kx)$.

$$\begin{aligned}
E &= \frac{1}{\pi}\langle f, f\rangle \\
&= \frac{1}{\pi}\langle a_k\cos(kx) + b_k\sin(kx), a_k\cos(kx) + b_k\sin(kx)\rangle \\
&= \frac{a_k^2}{\pi}\langle \cos(kx), \cos(kx)\rangle \quad \# \text{ bilinearity of } \langle\cdot,\cdot\rangle \\
&\quad + \frac{2a_kb_k}{\pi}\langle \cos(kx), \sin(kx)\rangle \\
&\quad + \frac{b_k^2}{\pi}\langle \sin(kx), \sin(kx)\rangle \\
&= \frac{1}{\pi}\left[a_k^2\pi + 0 + b_k^2\pi\right] \quad \# \text{ by the results of Q1} \\
&= a_k^2 + b_k^2
\end{aligned}$$

Thus, the energy of $f(x)$ is $a_k^2 + b_k^2$.

Q5. Find the $n$'th Fourier approximation for $f(x) = x$ on the interval $[-\pi, \pi]$. Assume that your approximation converges to $f(x) = x$ when $x \in (-\pi, \pi)$. Use this series to argue that:

$$\sum_{k=1}(-1)^{k+1}\frac{1}{2k-1} = \frac{\pi}{4}$$

**Solution:** We calculate the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0 \quad \# \text{ integrand is odd}$$

We calculate the cos terms:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(kx)dx = 0 \quad \# \text{ integrant is odd}$$

We calculate the sin terms:

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)dx \\
&= \frac{1}{\pi}\left[\frac{\sin(kx) - kx\cos(kx)}{k^2}\right]_{-\pi}^{\pi} \quad \# \text{ integration by parts} \\
&= \frac{1}{\pi}\left[\frac{-k\pi\cos(k\pi) - (-k(-\pi))\cos(-k\pi)}{k^2}\right] \quad \# \ \cos(k\pi) = (-1)^k \\
&= \frac{1}{\pi}\left[\frac{k\pi(-1)^{k+1} + k\pi(-1)^{k+1}}{k^2}\right] = \frac{2(-1)^{k+1}}{k}
\end{aligned}$$

This gives:

$$x \approx \sum_{k=1} \frac{2(-1)^{k+1}}{k} \sin(kx) \iff \frac{x}{2} \approx \sum_{k=1} \frac{(-1)^{k+1}}{k} \sin(kx) \quad (\star)$$

It turns out that equation $(\star)$ is more helpful for evaluating the sum.

To evaluate the sum first note, that at $x = \frac{\pi}{2}$ we have:

$$\sin\left(k\frac{\pi}{2}\right) = \begin{cases} 0 & k \text{ even} \\ (-1)^{n+1} & k = 2n+1 \text{ odd} \end{cases}$$

Evaluating equation $(\star)$ at $x = \frac{\pi}{2}$ we obtain using our hypothesis about convergence:

$$\frac{\left(\frac{\pi}{2}\right)}{2} = \sum_{k=1} \frac{(-1)^{k+1}}{k} \sin\left(k\frac{\pi}{2}\right) = \sum_{n=1} \frac{(-1)^{n+1}}{2n-1} \iff \frac{\pi}{4} = \sum_{n=1} \frac{(-1)^{n+1}}{2n-1}$$

Notice that we re-indexed from $k$ to $n$ because the sum only contains odd terms.

**Comments:** One might wonder how fast this sum converges to $\pi/4 \approx 0.785398163\ldots$

| $N$ | $\sum_{n=1}^{N} \frac{(-1)^{n+1}}{2n-1}$ |
|---|---|
| 10 | 0.7604599047 |
| 100 | 0.7828982258 |
| 1000 | 0.7851481634 |
| 10000 | 0.7853731633 |

To put it politely, the convergence is not rapid. There are faster methods for computing $\pi$.