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MATB24 TUTORIAL PROBLEMS 9, WEEK OF

KEY WORDS: Orthogonal and unitary linear transformations, orthogonal and unitary matrices, least square solution

RELEVANT SECTIONS IN THE TEXTBOOK: 9.1,9.2,6.3,6.5 FB or 6.C,7.C SA [there is no section in SA dedicated to elementary computations on a complex vector space. If you are not comfortable with complex numbers you should read 9.1 and 9.2 in FB]

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms. You should look up definitions before your tutorial.

- Least square solution to $A\vec{x} = \vec{b}$
- An orthogonal matrix
- An orthogonal linear transformation
- A unitary matrix
- Complex conjugate of a matrix
- A hermitian matrix
- A unitary matrix
- A unitary linear transformation
- An isometry

A: Let $T : \mathbb{R}^n \to \mathbb{R}^n$. Show that the following are equivalent (we proved similar results in class for an operator on any inner product space. Here you specialize to an operator on the Euclidean space $\mathbb{R}^n$ together with dot product. Do NOT assume our results in class and prove the statement from scratch. [1])

(1) T preserves magnitude i.e., $\|T(\vec{v})\| = \|\vec{v}\|$ for all $\vec{v} \in \mathbb{R}^n$.
(2) T preserves distances i.e., $\|T(\vec{v}) - T(\vec{w})\| = \|\vec{v} - \vec{w}\|$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$.
(3) T preserves the dot product that is $T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n$. ($T$ is orthogonal)
(4) T maps any orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.
(5) T maps the standard basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$.
(6) The columns of the standard matrix A of T form an orthonormal basis of $\mathbb{R}^n$.
(7) $A^T A = I_n$.
(8) $AA^T = I_n$.
(9) The (transposes of the) rows of A form an orthonormal basis of $\mathbb{R}^n$.

B:

(1) By Part A, $(3) \rightleftarrows (6)$ and the fact (proved in class) that "a square matrix A is orthogonal iff it has orthonormal columns" implies " $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal if and only if $[T]_{\mathcal{E}}$ is orthogonal". Discuss this implication with your group.
(2) Use part (A) to show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then the matrix of T with respect to any *orthonormal* basis of $\mathbb{R}^n$ is an orthogonal matrix.
(3) Use change of basis matrices to show if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then the matrix of T with respect to any *orthonormal* basis of $\mathbb{R}^n$ is an orthogonal matrix.
(4) Is this true if you replace the phrase "orthonormal basis" by "any basis"? Give examples.

C: COMPLEX ARITHMETIC-REVIEW

[1] Here is a suggested road map $(1) \rightleftarrows (2) \to (3) \to (4) \to (5) \to (6) \to (7) \rightleftarrows (8) \rightleftarrows (9) \to (1)$

(1) Let $z = 1 + 2i$ and $w = 3 - i$. Find $z \pm w$, zw, $\frac{z}{w}$, $\frac{1}{z} = z^{-1}$, $\frac{1}{w} = w^{-1}$, $|z|$, $|w|$, $\arg(z)^2$, $\arg(w)$ and $\bar{z}$. If possible give a geometric interpretation for you calculations.
(2) Write z and w above in in polar form. Find $z \pm w$, zw, $\frac{z}{w}$ in polar form. Give a geometric interpretation.
(3) Suppose two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal. What can we say about θ_1 and θ_2? Note that $\theta_1, \theta_2 \in \mathbb{R}$. Think geometrically.
(4) Let $z = e^{i\pi/4}$. Find $z^2, z^3, z^4, z^5, z^6, z^7, z^8$. Give a geometric interpretation.
(5) Find all the complex forth roots of 81. [3]
(6) (a) Calculus II: A Taylor expansion of a function $f(x)$ about a point a, provided that it exists[4], is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Taylor expansion can be thought of as generalization of approximating a function by a polynomial. If you stop at the second factor you get a linear approximation of f about a, if you stop at the third factor you get a quadratic approximation of f and so forth.
(b) Write the Taylor expansion of e^x, $\sin x$ and $\cos x$ about $x = 0$ (i.e. $a = 0$ in the above formula).
(c) Write the Taylor expansion for $e^{i\theta}$ about $\theta = 0$, for $\theta \in \mathbb{R}$.
(d) Prove the Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$.

D: COMPLEX VECTOR SPACES-REVIEW

(1) True of false? Justify
(a) $\mathbb{C}^2$ is a complex vector space. If true, write down the addition and scalar multiplication. Give a basis for this vector space.
(b) $\mathbb{C}^2$ is a real vector space. If true, write down the addition and scalar multiplication. Give a basis for this vector space.
(c) $\mathbb{R}^2$ is a subset of $\mathbb{C}^2$
(d) $\mathbb{R}^2$ is a subspace of $\mathbb{C}^2$ as a complex vector space.
(e) $\mathbb{R}^2$ is a subspace of $\mathbb{C}^2$ as a real vector space.
(2) Let $\vec{z}_1 = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$ an $\vec{z}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be vectors in the complex vector space $\mathbb{C}^2$.
(a) Is $\{\vec{z}_1, \vec{z}_2\}$ linearly independent?
(b) Is $A = [\vec{z}_1 \; \vec{z}_2]$ invertible? It yes, find A^{-1}.
(c) Are $\vec{z}_1, \vec{z}_2$ parallel, perpendicular or neither?
(d) If $\{\vec{z}_1, \vec{z}_2\}$ is linearly independent, apply Gram Schmidt to construct an orthonormal basis for $\mathbb{C}^2$.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

- An orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.
- An orthogonal linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.
- A basis $\mathcal{B}$ for $\mathbb{R}^2$ and an orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is an orthogonal matrix.

[2] The conversion is to assume $-\pi < \arg(z) \le \pi$

[3] Apply fundamental theorem of algebra to decide how many roots you expect

[4] Note that we are dealing with an infinite series and there is the convergence issue that we need to worry about. The functions you deal with in this question do have Taylor series over their domain.

- A basis $\mathcal{B}$ for $\mathbb{R}^2$ and an orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is NOT an orthogonal matrix.
- A non orthogonal linear transformation that takes an orthogonal basis to an orthogonal basis.
- A system of linear equations that has no least square solution.
- A hermitian matrix that is not in $M_{2\times 2}(\mathbb{R})$
- A hermitian matrix that is in $M_{2\times 2}(\mathbb{R})$
- A unitary matrix that is not in $M_{2\times 2}(\mathbb{R})$
- A unitary matrix that is in $M_{2\times 2}(\mathbb{R})$