1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.

(a) (3 points) A matrix A with real entries is *orthogonal* if $A^T A = I_n$, with I_n the $(n \times n)$ identity matrix.

A must be nxn

(b) (3 points) A linear transformation $T\colon V \to V$ is *diagonalizable* if $[T]_{\mathcal{B}}$ is diagonal, where $\mathcal{B}$ is a basis for V.

True.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counter example if you think it is false).

   (a) (3 points) Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$. Then $\vec{b}_1$ is the only eigenvector in the basis $\mathcal{B}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} | & & | \\ [T(b_1)]_B & - & [T(b_3)]_B \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ [\lambda_1 b_1]_B & - & [\lambda_3 b_3]_B \\ | & & | \end{bmatrix} \leftarrow \text{if } b_i\text{'s were all eigenvectors}$$

$$= \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$ Only 1st col is in this form

True. b_1 is the only eigen vector in $\mathcal{B}$

   (b) (3 points) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for vectors $\vec{x}, \vec{y}, \vec{z}$ in an inner product space, then $\vec{y} - \vec{z}$ is orthogonal to $\vec{x}$.

$\langle x, y \rangle - \langle x, z \rangle = 0$

$\overline{\langle y, x \rangle} - \overline{\langle z, x \rangle} = 0$ $\quad \therefore x \perp y-z$ True

$\overline{\langle y, x \rangle - \langle z, x \rangle} = 0$

$\overline{\langle y-z, x \rangle} = 0$

$\langle x, y-z \rangle = 0$

(c) (3 points) If A is a (3×4) matrix with real entries, then the matrix $A^T A$ is similar to a diagonal matrix with three or less non-zero entries.

A^TA is symmetric $\quad {}_4[\ ]_3 [\ ]_4^3 = [\ ]_4^4 \quad A^TA$ is 4×4

By real spectral thm, A^TA is diagonalizable

$\therefore A^TA = CDC^{-1} \Rightarrow A^TA$ is similar to a diagonal matrix

But there are 4 diagonal entries so you can't guarentee 3 or less non-zero entries.

False

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cdot & \\ & & & 1 \end{bmatrix}$$

(d) (3 points) The matrices $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are orthonormal in the inner product $\langle A, B\rangle = \text{trace}(A^T B)$ on the vector space $M_{2\times 2}(\mathbb{R})$ of (2×2)-matrices with real entries.

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \langle A, A\rangle = tr(A^TA)$ $\quad$ False

$= tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$= 2 \neq 1$ $\quad \therefore A$ is not unit vector and (A, B) cannot be orthonormal

(e) (3 points) Every unitarily diagonalizable $(n \times n)$-matrix with complex entries is Hermitian.

False.

$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A is normal $\Rightarrow$ unitarily diaganalizable

but A is not Hermitian

$A = A^*$

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist. Include the description in your answer.

   (a) (2 points) A linear map $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ which is an isometry (with respect to the dot product) and a basis $\mathcal{B}$ for $\mathbb{C}^2$ such that $[T]_{\mathcal{B}}$ is not a unitary matrix.

   Let $[T]_{\mathcal{E}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ← This is unitary as shown from 2

   let $\mathcal{B}: \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$ $\mathcal{B}$ must be orthonormal

   (b) (2 points) A possible Jordan canonical form for a matrix with distinct eigenvalues μ and γ with algebraic multiplicities 2 and 3, geometric multiplicities 2 and 1 respectively.

   $$\begin{bmatrix} \mu & & & & \\ & \mu & & & \\ & & \gamma & 1 & \\ & & & \gamma & 1 \\ & & & & \gamma \end{bmatrix}$$

   (c) (2 points) A linear map $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ for which $\ker(T) = \operatorname{im}(T)$.

   Not possible as $\dim(\operatorname{dom}(T)) = 3$

   By rank-nullity $\dim(\operatorname{dom}(T)) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$

   $\Rightarrow \dim(\ker(T)) \neq \dim(\operatorname{im}(T))$

   as 3 is odd

   (d) (2 points) An orthogonal matrix with at least one real eigenvalue.

   $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A - \lambda I) = (1-\lambda)^2$

   $\lambda = 1$

   $A^T A = A A^T = I_n$

   Trivial for $A = I_n$

4. Let $\mathcal{P}_5(\mathbb{R})$ be the vector space of polynomials with real coefficients in a variable x of degree ≤ 5.

(a) (2 points) What is the dimension of $\mathcal{P}_5(\mathbb{R})$?

$P_5(R) = sp\{1, x, x^2, \dots, x^5\}$ ← clearly l.i. $\therefore$ basis length 6

$\Rightarrow \dim(P_5(R)) = 6$

(b) (4 points) Let $\mathcal{B} = \{x^5, x^4, x^3, x^2, x, 1\}$ be the standard basis of $\mathcal{P}_5(\mathbb{R})$. For the linear map

$$\begin{aligned} T\colon \mathcal{P}_5(\mathbb{R}) &\longrightarrow \mathcal{P}_5(\mathbb{R}) \\ p(x) &\longmapsto \frac{dp(x)}{dx} + p(x) \end{aligned}$$

give the matrix $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(x^5)]_{\mathcal{B}} & \cdots & [T(1)]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1 & & & & & \\ 5 & 1 & & & & \\ & 4 & 1 & & & \\ & & 3 & 1 & & \\ & & & 2 & 1 & \\ & & & & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{array}$$

$T(x^5) = 5x^4 + x^5$

$T(x^4) = 4x^3 + x^4$

$T(x^3) = 3x^2 + x^3$

$\vdots$

$T(1) = 0 + 1$

(c) (4 points) Find all the eigenvalues of T and their geometric multiplicities.

$$\det(A-\lambda I) = (1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda)$$
$$= (1-\lambda)^6 \to \text{alg multi } 6$$
$$\lambda = 1$$

$$\text{Nul}(A-I) = \left[\begin{array}{cccc|c} 0 & & & & \\ 5 & 4 & & & \\ & & \ddots & & \\ & & & 1 \quad 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 5 & 4 & 0 & & \\ 0 & & \ddots & 1 \quad 0 & 0 \\ 0 & \cdots & & 0 & \end{array}\right] \quad \begin{array}{l} v_6 = t \in \mathbb{R} \\ v_1, \ldots, v_5 = 0 \end{array}$$

$$\text{Nul}(A-I) = \text{Sp}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \text{geo multi } 1 \qquad \begin{pmatrix} 0 \\ \vdots \\ s \end{pmatrix} = s\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

(d) (5 points) Prove that T is *not* diagonalizable.

We know for eigen value $\lambda = 1$: alg-m $= 6$
geo-m $= 1$

And if alg $\neq$ geo for any eigenvalue we know the transformation is not diagonalizable ∎

5. Suppose that A is an $(n \times n)$-matrix with real entries.

   (a) (6 points) Prove that $\ker(A^T A) = \ker(A)$.

let $v \in \ker(A^TA)$

$A^TAv = 0$

$v^TA^TAv = 0$

$(Av)^TAv = 0$

$\langle Av, Av \rangle = 0$

$Av = 0$ by positive-definite property

$\Rightarrow v \in \ker(A)$

$\Rightarrow \ker(A^TA) = \ker(A)$

let $v \in \ker(A)$

$Av = 0$

$A^TAv = A^T0$

$= 0$

$\Rightarrow v \in \ker(A^TA)$

   (b) (5 points) Recall that the *rank* of a matrix is the dimension of its image. Prove that $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$.

Using result from a) if $\ker(A^TA) = \ker(A) \Rightarrow Nul(A^TA) = Nul(A)$

A and A^TA are $n \times n$

$$\begin{cases} n = Nul(A) + rank(A) \\ n = Nul(A^TA) + rank(A^TA) \end{cases}$$

$\Rightarrow \cancel{Nul(A)} + rank(A) = Nul(A^TA) + rank(A^TA)$ by part a)

$rank(A) = rank(A^TA)$

$\blacksquare$

6. Let A be a symmetric $(n \times n)$-matrix with real entries. We say A is *strictly positive* if $\vec{v}^T A\vec{v} > 0$ for all non-zero $\vec{v} \in \mathbb{R}^n$.

(a) (7 points) Prove A is strictly positive if and only if all eigenvalues of A are positive.

Supp A is strictly positive

Let v be an eigenvector of A w. value λ

$v^T A v > 0$

$v^T \lambda v > 0$

$\lambda v^T v > 0$

$\lambda \|v\|^2 > 0$ $\quad \|v\|^2 > 0$

$\Rightarrow \lambda > 0$

Supp all λ are positive

$A = CDC^{-1}$, C is orthogonal and D entries are positive, by spectral thm

Let $v \in \mathbb{R}^n$ be non-zero

$v^T A v = v^T C D C^{-1} v$

$= (C^T v)^T D (C^T v) = \left[(C^T v)_i^2 \lambda_i \right] > 0$ as λ_i are positive

(b) (8 points) Prove that A is strictly positive if and only if there exists an invertible $(n \times n)$-matrix P such that $A = P^T P$ (Hint: use the spectral theorem).

Supp A is strictly positive $\quad$ C is ortho

$A = CDC^{-1}$ by spectral thm, D has positive entries by a)

$v^T A v = v^T C D C^{-1} v$ $\quad$ $\sqrt{D}$ exists as it has positive entries

$= v^T C \sqrt{D} \sqrt{D} C^T v$

$= v^T (\sqrt{D} C^T)^T (\sqrt{D} C^T) v$ $\Rightarrow$ by choosing values of v this can show $A = (\sqrt{D}C^T)^T(\sqrt{D}C^T)$ let $P = \sqrt{D} C$

$= P^T P$

C^T is invertible as the col are l.i.

$\sqrt{D} C^T$ the col are still l.i. just scaled

$\Rightarrow P$ is invertible

Supp $\exists P$ st $A = P^T P$

let $v \in \mathbb{R}^n$ $v \neq \vec{0}$

$v^T A v = v^T P^T P v$

$= (Pv)^T P v$

$= \langle Pv, Pv \rangle > 0$ by inner product properties

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