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These solutions are not the only "perfect" or "10/10" solutions to be followed as a model. They are examples of ways to solve these problems. There are many many alternative ways to solve these problems. We hope that these example solutions help you check your work.

Problems

Q1. Let $\mathbf{\Phi} : D \to \mathbb{R}^3$ be defined by $\mathbf{\Phi}(u,v) = (u-v, u+v, uv)$ for $D = \{(u,v) : u^2+v^2 \leq 1\}$.

(a) Show that $\mathbf{\Phi}$ parametrizes part of the surface $S : 4z = y^2 - x^2$.

Solution: We check that $\mathbf{\Phi}$ parametrizes part of the surface $4z = y^2 - x^2$. Subtituting the given values for (x, y, z) we have:

$$\begin{aligned} y^2 - x^2 &= (u+v)^2 - (u-v)^2 \\ &= u^2 + 2uv + v^2 - (u^2 - 2uv + v^2) \\ &= 4uv = 4z \end{aligned}$$

Thus, $\mathbf{\Phi}(D) \subset S$ is contained in the surface S

(b) Find the surface area of $\mathbf{\Phi}(D)$.

Solution: We find the area of the surface S. We have:

$$A(S) = \iint_S 1 dS = \iint_S \|\mathbf{T}_u \times \mathbf{T}_v\| \, dS$$

So, we calculate the tangent components.

$$\mathbf{T}_u = (1, 1, v) \qquad \mathbf{T}_v = (-1, 1, u)$$

This gives:

$$\mathbf{T}_u \times \mathbf{T}_v = (u - v, -(u+v), 2)$$

Taking norms, we get:

$$\begin{aligned} \|\mathbf{T}_u \times \mathbf{T}_v\| &= \sqrt{(u-v)^2 + [-(u+v)]^2 + 2^2} \\ &= \sqrt{u^2 - 2uv + v^2 + u^2 + 2uv + v^2 + 4} \\ &= \sqrt{2(u^2+v^2) + 4} \end{aligned}$$

Notice that this contains the term u^2+v^2. To integrate $\iint_S \|\mathbf{T}_u \times \mathbf{T}_v\|\, dS$ we switch to polar coordinates. The disc D in polar coordinates is $D=[0,2\pi]\times[0,1]$. We obtain:

$$\begin{aligned}
\iint_S \|\mathbf{T}_u \times \mathbf{T}_v\|\, ds &= \iint_S \sqrt{2(u^2+v^2)+4} dS \\
&= \int_0^{2\pi}\int_0^1 \sqrt{2(r^2)+4}\; r dr d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_4^6 d\theta \text{ \# Substitute } x=2r^2+4 \text{ and } dx = 4r\ dr \\
&= \frac{1}{4}\cdot\frac{2}{3}\int_0^{2\pi}\left[6^{\frac{3}{2}}-4^{\frac{3}{2}}\right] d\theta \\
&= \frac{1}{4}\cdot\frac{2}{3}\int_0^{2\pi}\left[6^{\frac{3}{2}}-4^{\frac{3}{2}}\right] d\theta \\
&= 2\pi\frac{1}{4}\cdot\frac{2}{3}\left[6^{\frac{3}{2}}-4^{\frac{3}{2}}\right] \text{ \# integrand is constant in } \theta \\
&= \frac{\pi}{3}\left(6^{\frac{3}{2}}-8\right)
\end{aligned}$$

Thus, the area of the surface is $A(S)=\frac{\pi}{3}\left(6^{\frac{3}{2}}-8\right)\approx 7.013$.

Q2. Consider the S surface defined by

$$z=\frac{1}{\sqrt{x^2+y^2}} \qquad 1\le z<\infty$$

(a) Find a parametrization of S by polar coordinates (r,θ) where $0<r<1$.

Solution: We parametrize the surface using polar coordinates.

$$\begin{aligned}
\mathbf{\Phi}(r,\theta) &= \left(r,\theta,\frac{1}{\sqrt{x^2+y^2}}\right) \\
&= \left(r,\theta,\frac{1}{\sqrt{r^2}}\right) \text{ \# } r^2=x^2+y^2 \text{ in polar coordinates} \\
&= \left(r,\theta,\frac{1}{r}\right) \text{ \# } r\ge 0
\end{aligned}$$

(b) Find the volume bounded by S and the plane $z=1$.

Solution: We find the volume bounded by S and the plane $z=1$. Notice that $z(r,\theta)=\frac{1}{r}$. Thus,

$$z\ge 1 \Longrightarrow \frac{1}{r}\ge z \Longrightarrow r\le 1$$

The domain of integration is $D=[0,2\pi]\times[0,1]$ in polar coordinates.

Using techniques from MAT B41, we integrate the height of $\Phi(r,\theta)$ above $z=1$ to get the volume.

$$\begin{aligned} V &= \iint_D \Phi(r,\theta) - 1 dA \\ &= \iint_D \left(\frac{1}{r} - 1\right) r dr d\theta \text{ \# } dA = r dr d\theta \text{ in polar coordinates} \\ &= \int_0^{2\pi} \int_0^1 (1-r) dr d\theta \\ &= \int_0^{2\pi} \left[r - \frac{1}{2}r^2\right]_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{2} d\theta \\ &= \frac{1}{2} \cdot 2\pi = \pi \text{ \# integrand is constant in } \theta \end{aligned}$$

Thus, the volume bounded by S and $z=1$ is π cubic units. Notice that this volume is FINITE.

(c) Find the surface area of S.

Solution: We calculate the surface area. We have:

$$A(S) = \iint_S 1 dS = \iint_D \|\mathbf{T}_r \times \mathbf{T}_\theta\| dS$$

Thus, we calculate the tangent components.

$$\mathbf{T}_r = \left(1, 0, -\frac{1}{r^2}\right) \qquad \mathbf{T}_\theta = (0,1,0)$$

Taking the cross-product, we get:

$$\mathbf{T}_r \times \mathbf{T}_\theta = \left(\frac{1}{r^2}, 0, 1\right)$$

Taking norms, we get:

$$\|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{\frac{1}{r^4} + 1}$$

The surface area integral becomes:

$$\begin{aligned} \iint_S \|\mathbf{T}_r \times \mathbf{T}_\theta\| ds &= \iint_D \sqrt{\frac{1}{r^4} + 1} dr d\theta \\ &= \iint_D \frac{1}{r^2}\sqrt{1+r^4} dr d\theta \\ &\geq \iint_D \frac{1}{r^2} dr d\theta \text{ \# we lower bound the integral: } \sqrt{1+r^4} \geq 1 \end{aligned}$$

This lower bound greatly simiplifies evaluating the integral. However, the resulting integral representing the lower bound is improper at $r = 0$. To handle this, we will take a limit as $r = t \to 0^+$.

$$\begin{aligned}
\iint_D \frac{1}{r^2} dr d\theta &= \lim_{t\to 0^+} \int_0^{2\pi} \int_t^1 \frac{1}{r^2} dr d\theta \\
&= \lim_{t\to 0^+} \int_0^{2\pi} \left[-\frac{1}{r}\right]_t^1 d\theta \\
&= \lim_{t\to 0^+} \int_0^{2\pi} \frac{1}{t} - 1 d\theta \\
&= \lim_{t\to 0^+} 2\pi\left(\frac{1}{t} - 1\right) \quad \text{\# integrand is constant in } \theta
\end{aligned}$$

Notice that the lower bound on area goes to infinity as $r = t \to 0^+$.
This means that S has INFINITE surface area.

(d) Briefly explain why the answers to (b) and (c) are paradoxial.

Solution: The results of Q2a and Q2b are paradoxical because S has finite volume and infinite surface area. This is *very* strange.

To make things more concrete: Imagine trying to paint or apply wallpaper to S. If one put π units of paint in to the region bounded by S then it would fill up, and the surface would be covered with paint. On the other hand, it would take an infinite amount of paper to cover the surface. For more information about this weird surface, check out: Gabriel's Horn on Wikipedia.

Q3. Evaluate the following surface integrals.

(a) $\iint_S \sqrt{z} dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$ above $z = 1$.

Solution: We evaluate $\iint_S \sqrt{z} dS$ where $S = \{(x, y, z) : x^2 + y^2 + z^2 = 4 \text{ and } z \geq 1\}$. Note that S is a spherical region, so we write it in spherical coordinates.

$$\mathbf{\Phi}(\phi, \theta) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$$

The radius is $r = 2$ and so the condition $z \geq 1$ becomes:

$$\begin{aligned}
z \geq 1 &\iff 2\cos\phi \geq 1 \\
&\iff \cos\phi \geq \frac{1}{2} \\
&\iff \phi \leq \frac{\pi}{3}
\end{aligned}$$

Thus the domain of the parametrization is $D = [0, 2\pi] \times \left[0, \frac{\pi}{3}\right]$. We obtain:

$$\begin{aligned}
\iint\limits_S \sqrt{z} dS &= \iint\limits_D \sqrt{2\cos\phi}\, \|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\, d\theta d\phi \\
&= \iint\limits_D \sqrt{2\cos\phi}\, \|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\, d\theta d\phi \\
&= \iint\limits_D \sqrt{2\cos\phi}(2^2 \sin\phi) d\theta d\phi \text{ \# from class } \|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \rho^2 |\sin\phi| d\theta d\phi. \\
&= 2^{\frac{5}{2}} \iint\limits_D \sqrt{\cos\phi}(\sin\phi) d\phi d\theta \text{ \# Fubini} \\
&= -2^{\frac{5}{2}} \int_0^{2\pi} \int_1^{1/2} \sqrt{u} du d\theta \text{ \# } u = \cos\phi \\
&= -2^{\frac{5}{2}} \int_0^{2\pi} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^{1/2} d\theta \\
&= 2^{\frac{5}{2}} \int_0^{2\pi} \left[1 - \left(\frac{1}{2}\right)^{\frac{3}{2}}\right] d\theta \\
&= 2^{\frac{5}{2}} \left[1 - \left(\frac{1}{2}\right)^{\frac{3}{2}}\right] (2\pi) \text{ \# integrand is constant in } \theta \\
&= \frac{8\pi}{3} \left[2^{\frac{3}{2}} - 1\right]
\end{aligned}$$

Thus, the integral $\iint_S \sqrt{z} dS = \frac{8\pi}{3}\left[2^{\frac{3}{2}} - 1\right] \approx 15.3$.

(b) $\iint_S (x^2 z + y^2 z) dS$ where S is the portion of the plane $z = 4 + x + y$ inside the cylinder $x^2 + y^2 = 4$.

Solution: We first parametrize the surface.

$$\Phi(u, v) = (u, v, 4 - u - v)$$

So, we calculate the tangent components.

$$\mathbf{T}_u = (1, 0, -1) \qquad \mathbf{T}_v = (0, 1, -1)$$

This gives:

$$\mathbf{T}_u \times \mathbf{T}_v = (1, 1, 1)$$

Taking norms, we get:

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{3}$$

This gives:

$$\begin{aligned}
\iint\limits_S x^2z + y^2zdS &= \iint\limits_S z(x^2+y^2)dS \\
&= \iint\limits_D (4-u-v)(u^2+v^2)\left\|\mathbf{T}_u \times \mathbf{T}_v\right\| dS \\
&= \sqrt{3}\iint\limits_D (4-u-v)(u^2+v^2)dS
\end{aligned}$$

We notice the term u^2+v^2 and switch to polar coordinates. The domain becomes $D = [0,2\pi]\times[0,2]$. We calculate:

$$\begin{aligned}
\sqrt{3}\iint\limits_D (4-u-v)(u^2+v^2)dS &= \sqrt{3}\int_0^{2\pi}\int_0^2 (4-r\cos\theta - r\sin\theta)(r^2)rdrd\theta \\
&= \sqrt{3}\int_0^{2\pi}\int_0^2 4r^3 - r^4\cos\theta - r^4\sin\theta drd\theta \\
&= \sqrt{3}\int_0^{2\pi}\int_0^2 \left[r^4 - \frac{1}{5}r^5\cos\theta - \frac{1}{5}r^5\sin\theta\right]_0^2 d\theta \\
&= \sqrt{3}\int_0^{2\pi} 2^4 - \frac{2^5}{5}\cos\theta - \frac{2^5}{5}\sin\theta d\theta \\
&= \sqrt{3}(2\pi)2^4 = 2^5\sqrt{3}\pi \ \# \text{ cos and sin are } 2\pi\text{-periodic}
\end{aligned}$$

Thus, the integral $\iint_S (x^2z+y^2z)dS = 2^5\sqrt{3}\pi \approx 174.125$.

(c) $f(x,y,z) = x - y$ where S is given by $z = \frac{1}{2}y^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

Solution: We integrate $\iint\limits_S (x-y)dS$ where

$$S = \{(x,y,z) : 0 \le x, y \le 1 \text{ and } z = \frac{1}{2}y^2\}$$

We parametrize the surface as: $\mathbf{\Phi}(u,v) = \left(u, v, \frac{1}{2}v^2\right)$ where $0 \le u, v \le 1$. We obtain:

$$\mathbf{T}_u = (1,0,0) \qquad \mathbf{T}_v = (0,1,v)$$

This gives the cross product: $\mathbf{T}_u \times \mathbf{T}_v = (0, -v, 1)$. And now we compute the surface integral:

$$\begin{aligned}
\iint_S (x-y)dS &= \iint_D (u-v)\left\|\mathbf{T}_u \times \mathbf{T}_v\right\| dudv \\
&= \iint_D (u-v)\sqrt{v^2+1}dudv \\
&= \int_0^1 \int_0^1 (u-v)\sqrt{v^2+1}dudv \\
&= \int_0^1 \left[\left(\frac{1}{2}u^2 - uv\right)\sqrt{v^2+1}\right]_0^1 dudv \\
&= \int_0^1 \left(\frac{1}{2} - v\right)\sqrt{v^2+1}dv \\
&= \int_0^1 \frac{1}{2}\sqrt{v^2+1}dv - \int_0^1 v\sqrt{v^2+1}dv
\end{aligned}$$

At this point in the calculation we pause to note that $\int \sqrt{v^2+1}dv$ is a hard integral to calculate. It usually requires a $v = \sinh(x)$ subsitution, which is rarely taught in first year calculus classes despite appearing in many popular textbooks. There is also an approach to integrating $\int \sqrt{v^2+1}dv$ via $v = \tan(x)$ which works well. We will not dwell on this integral, but note that a table of integrals or WolframAlpha give:

$$\int \sqrt{1+v^2}dv = \frac{1}{2}v\sqrt{1+v^2} + \frac{1}{2}\ln\left|v + \sqrt{1+v^2}\right|$$

Using this indefinite integral, we obtain:

$$\int_0^1 \frac{1}{2}\sqrt{v^2+1}dv = \frac{1}{2}\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left|1+\sqrt{2}\right|\right) = \frac{1}{4}\left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)$$

The second term in our surface integral is much easier to integrate:

$$\int_0^1 v\sqrt{v^2+1}dv = \frac{1}{2}\int_1^2 \sqrt{x}dx = \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^2 = \frac{1}{3}\left[2^{\frac{3}{2}} - 1\right]$$

This gives our final answer:

$$\iint_S (x-y)dS = \left(\frac{1}{4}\left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right) - \left(\frac{1}{3}\left[2^{\frac{3}{2}} - 1\right]\right)$$

Q4. Suppose that $0 < r < R$. Consider the torus:

$$\mathbf{\Phi}(u,v) = ((R + r\cos(u))\sin(v), (R + r\cos(u))\cos(v), r\sin(u))$$

(a) Find the surface area of the torus.

Solution: We use the given parametrization to calculate the surface area of the torus.

$$\mathbf{T}_u = (-r\sin(u)\sin(v), -r\sin(u)\cos(v), r\cos(u))$$

$$\mathbf{T}_v = ((R + r\cos(u))\cos(v), -(R + r\cos(u))\sin(v), 0)$$

This gives the following cross-product:

$$\begin{aligned}
\mathbf{T}_u \times \mathbf{T}_v &= -(R + r\cos(u))r\cos(u)\sin(v)\mathbf{i} \\
&\quad +(R + r\cos(u))r\cos(u)\cos(v)\mathbf{j} \\
&\quad + \left((R + r\cos(u))r\sin(u)\sin^2(v) + (R + r\cos(u))r\sin(u)\cos^2(v)\right)\mathbf{k} \\
&= (R + r\cos(u))r\left(-\cos(u)\sin(v)\mathbf{i} + \cos(u)\cos(v)\mathbf{j} + 1\sin(u)\mathbf{k}\right)
\end{aligned}$$

Consider the term:

$$-\cos(u)\sin(v)\mathbf{i} + \cos(u)\cos(v)\mathbf{j} + \sin(u)\mathbf{k}$$

It is very surprising that this has norm one. Parker would love to know a geometric explanation of why this term of norm one arises in this calculation for the torus. We check:

$$\begin{aligned}
(-\cos(u)\sin(v))^2 + (\cos(u)\cos(v))^2 + \sin^2(u) &= 1\cos^2(u) + \sin^2(u) \\
&= 1
\end{aligned}$$

This gives: $\|\mathbf{T}_u \times \mathbf{T}_v\| = |(R + r\cos(u))r| = (R + r\cos(u))r$ since $R > r > 0$.
From here, we can calculate the surface area:

$$\begin{aligned}
A(S) &= \iint_S (R + r\cos(u))r \\
&= \int_0^{2\pi}\int_0^{2\pi} (R + r\cos(u))r\,du\,dv \\
&= \int_0^{2\pi} 2\pi Rr\,dv \quad \text{\# cos is } 2\pi\text{-periodic} \\
&= 4\pi^2 Rr
\end{aligned}$$

Thus, the sruface area of a torus of inner radius r and outer radius R is $4\pi^2 Rr$.

(b) Find the volume of the torus.

Solution: We find the volume of the torus using the "washer method" from first year calculus. Consider the cross-sections of the torus in the xz-plane. At height z we have:

$$(x - R)^2 + z^2 = r^2 \implies x = R \pm \sqrt{r^2 - z^2}$$

The inner radius of the washer is $x = R - \sqrt{r^2 - z^2}$ and the outer radius is $x = R + \sqrt{r^2 - z^2}$. Our bounds of integration are $z = -r$ and $z = r$. This gives the following calculation:

$$\begin{aligned}
V &= \int_{-r}^{r} \pi\left(R + \sqrt{r^2 - z^2}\right)^2 - \pi\left(R - \sqrt{r^2 - z^2}\right)^2 dz \\
&= \int_{-r}^{r} 4\pi R\sqrt{r^2 - z^2}\,dz
\end{aligned}$$

The integral $\displaystyle\int_{-r}^{r} \sqrt{r^2 - z^2}\,dz$ calculates the area of a semi-circle of radius r. Thus, we have:

$$\int_{-r}^{r} \sqrt{r^2 - z^2}\,dz = \frac{1}{2}\pi r^2$$

Substituting this in to our original integral for the volume we get:

$$V = 4\pi R\left(\frac{1}{2}\pi r^2\right) = 2\pi^2 R r^2$$

(c) Compare the volume of a torus of inner radius R and outer radius r with the volume of a cylinder $C(R, r)$ of length $2\pi R$ and radius r.

Solution: From highschool geometry, we know that the volume of $C(R, r)$ is:

$$V = (\text{base area})(\text{height}) = \pi r^2(2\pi R) = 2\pi^2 R r$$

This volume exactly matches our calculation for the torus! The result is quite suprising because the region $C(R, r)$ gets deformed when it is bent to form the torus. As the calculations show, the volume is preserved under this bending.

Q5. Parametrize a Möbius band as $\Phi : D \to \mathbb{R}^3$ where $D = [0, 2\pi) \times [-1, 1]$.

(a) Explain how your parametrization is constructed.
As a model, use the explanations given in class for the cycloid and torus.

(b) Use CalcPlot3D's "Parametric Surface" function to plot your Möbius band.

(c) Calculate the normal vector $\mathbf{n}$ of your parametrization.

Solution: This question is too open-ended to have a single concrete answer. Different people will construct their Möbius bands in different ways and arrive at different parametrizations and normal vectors. One parametrization is given by:

$$\mathbf{\Phi}(u, v) = \left(\left(2 + u\cos\left(\frac{1}{2}v\right)\right)\cos(v), \left(2 + u\cos\left(\frac{1}{2}v\right)\right)\sin(v), u\sin\left(\frac{1}{2}v\right)\right)$$

CalcPlot3D gives the following visualization:

./graphics/mobius.png

Click here for the CalcPlot3D visualization