MATB24 GRADED PROBLEMS 3, DUE Friday Oct 30, 11:59pm

GENERAL INSTRUCTIONS:

- You should submit your work on Quercus. The only accepted format is PDF.
- Do not wait until last minute to avoid technical difficulties.
- There is a one point penalty for late submissions within 12 hours of the due date.
- You are encouraged to work in groups, ask question on piazza or in office hours. But you should write your homework individually in your own words. You can get help from me, your TA or your peers, but you should write your solution on your own.
- Unless otherwise stated in all questions you should fully justify your answer.
- Your TA will grade a randomly selected subset of the questions in each homework and your grade will be only based on the graded questions.

READING ASSIGNMENT:

It is assumed that you read at least one of the reading options below

- Sec 3.3, 3.4, 7.1, 7.2 from Fraleigh–Beauregard
- Sec 3A-D from Axler

Problem 1. *Recall that $\mathcal{P}_3(\mathbb{R})$ denotes the vector space of polynomials in x of degree ≤ 3 with real coefficients. We saw differentiation gives a linear map $\frac{d}{dx}\colon \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$, e.g. $\frac{d}{dx}(1+3x^2) = 6x$. In the following questions you are allowed to use the basic properties of differentiation without proof.*

(1) *How many rows and columns does a matrix for $\frac{d}{dx}$ with respect to a basis have?*
(2) *Determine the matrix A of this linear map with respect to the basis $1, x, x^2, x^3$.*
(3) *Determine A^4 without doing any matrix multiplication.*

Problem 2. *Let V and W be finite-dimensional vector spaces, suppose that $T : V \to W$ is a linear transformation. Let $\vec{v}_1, \dots, \vec{v}_d, \vec{b}_1, \cdots, \vec{b}_r$ be vectors in V such that $(\vec{v}_1, \dots, \vec{v}_d)$ is a basis for the kernel T and $(T(\vec{b}_1), \cdots, T(\vec{b}_r))$ is a basis for the image of T.*

(1) *Prove that $\{\vec{v}_1, \dots, \vec{v}_d, \vec{b}_1, \cdots, \vec{b}_r\}$ is a spanning set for V.*
(2) *Prove that $\{\vec{v}_1, \dots, \vec{v}_d, \vec{b}_1, \cdots \vec{b}_r\}$ is linearly independent.*
(3) *Deduce the Rank-Nullity equation for T.*

Problem 3. *Let V be a finite dimensional vector space. Suppose that $T\colon V \to V$ is a linear transformation that has the same matrix representation with respect to every basis of V. Prove that T must be a scalar multiple of the identity transformation. You can assume that the dimension of V is 3.*

Problem 4. *Let V be a vector space and $\mathcal{B} = (f_1, \dots, f_n)$ be a basis of V.*

(a) *Prove that for every $n \times n$ matrix B, there exists a linear transformation $T : V \to V$ such that B is the $\mathcal{B}$-matrix of T.*

(b) *Prove that the transformation in (a) is unique.*

(c) *Let $\mathcal{L}(V, V)$ be the vector space consisting of all linear transformations from V to V. Use part (a) and (b) to prove that $\mathcal{L}(V, V)$ is isomorphic to $\mathbb{R}^{n\times n}$.*

(d) *Prove that for any transformation $T : V \to V$, there exists a number m and $a_0, \dots, a_m$ not all zero such that*

$$a_0 I + a_1 T + \ldots + a_m T^m = 0.$$

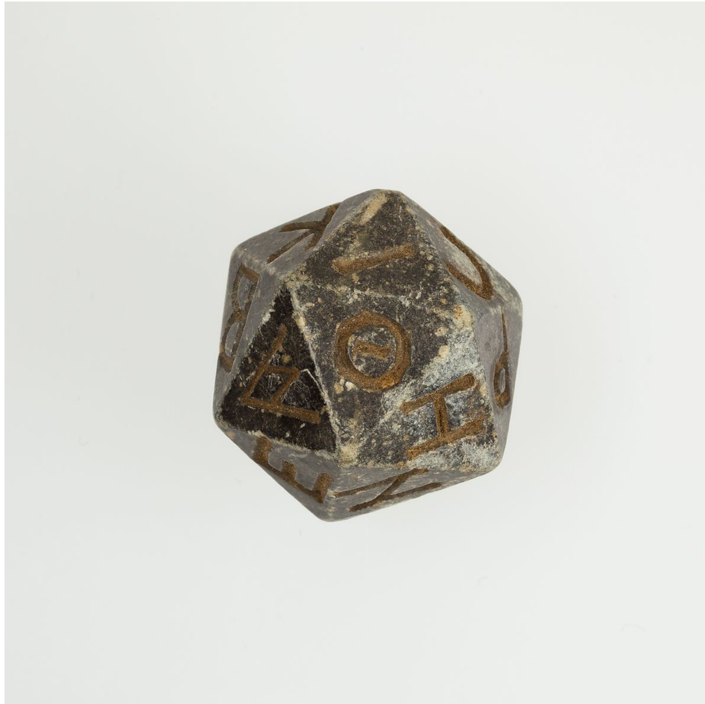

FIGURE 0.1. Twenty-sided die (icosahedron) with faces inscribed with Greek letters 2nd century B.C.4th century A.D. From Egypt

Here, $I : V \to V$ denotes the identity transformation defined as $I(f) = f$ for all $f \in V$.

Problem 5. *Recall that the angle between two lines in $\mathbb{R}^3$ is given by the (smallest) angle between them in the plane that they span. An* equiangular *set of lines in $\mathbb{R}^3$ is a collection $L_1, \dots, L_n$ of lines through the origin such that all angles $\angle(L_i, L_j)$ for $i \neq j$ are equal. For example, the coordinate axes give three equiangular lines, the angle between them being $90°$.*

(1) *Use the cube and icosahedron to give a collection of four and six equiangular lines. What about the dodecahedron? You do not need to give full proofs.*
(2) *Prove that for all $\theta \in (0, \pi/2]$ and $n \geq 1$ the matrix $(1 - \cos(\theta)^2)\mathrm{id}_n + \cos(\theta)^2 J_n$ is invertible, where J_n is the $(n \times n)$-matrix with all entries equal to 1.*
(3) *Let v_i be a unit vector along L_i, then $L_1, \dots, L_n$ are equiangular if there exists an angle $\theta \in (0, \pi/2]$ such that $|\langle v_i, v_j \rangle| = \cos(\theta)$ for all $i \neq j$. Prove that if $L_1, \dots, L_n$ are equiangular, then the (3×3)-matrices $v_i v_i^t$ for $1 \leq i \leq n$ are linearly independent.*
(4) *Recall that $\mathbb{R}^{3,3}$ is the 9-dimensional vector space of (3×3)-matrices. What is the dimension of the subspace of symmetric matrices, i.e. those A that satisfy $A^t = A$? Conclude that the largest number of equiangular lines in $\mathbb{R}^3$ is 6.*
(5) *(Bonus 1 point) Prove that there cannot be more than $\binom{n+1}{2}$ equiangular lines in $\mathbb{R}^n$.*