MATB24 TUTORIAL PROBLEMS 1SOLUTIONS

KEY WORDS: binary operation, vector addition, scalar multiplication, vector space.
READING: Sec 3.1 FB or Sec 1.B SA[1]

WARM-UP: You should know the definitions of the following terms word by word

(1) A binary operation
(2) An identity element for a binary operation
(3) An invertible element for a binary operation
(4) An inverse of an invertible element
(5) A vector space

A :Below are tables for four different binary operations on the set $S = \{a, b, c, d\}$, called $\clubsuit$, $\diamondsuit$, $\heartsuit$ and $\spadesuit$.

(1) Discuss with your groupmates a **convention** for interpreting the tables, so that you do not confuse $a \star b$ with $b \star a$. Make sure your convention is consistent with mine: I want $a \diamondsuit b = d$.
(2) Which of the operations are commutative? Discuss with your group mates.
(3) Which have an identity? Identify it. Recall: An identity for a binary operation $\star$ on a set S is an element e such that $s \star e = e \star s = s$ for all $s \in S$.
(4) TRUE or FALSE: if there exists some e such that $x \star e = x$ for all $x \in S$, then e is an identity for $\star$.
(5) Investigate which elements of $\heartsuit$ have inverses.

$\clubsuit$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

$\diamondsuit$	a	b	c	d
a	a	d	c	b
b	b	a	d	c
c	c	b	a	d
d	d	c	b	a

$\heartsuit$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

$\spadesuit$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

[1]FB: Fraleigh Beauregard, SA: Sheldon Axler

(1) To define $a \star b$, you have two options: Either you start with a as a column (on top of the table), and then find b as a row (and the left side of the table), and get your answer, or start with a on a row and then find b as a column. You have to decide which one of these ways your are going to always use when defining $a \star b$ (this is your convention). (2) All but $\diamondsuit$ are commutative (e.g. tables which are commutative are symmetric along their diagonals), and (3) all but $\diamondsuit$ have an identity ($e_{\heartsuit} = b$, $e_{\clubsuit} = a$, $e_{\spadesuit} = a$). (4) It is FALSE that we only need to check an identity "on one side": $\diamondsuit$ gives a counterexample with $e = a$. For a real-life counterexample, you can take the operation of subtraction on $\mathbb{Z}$: here 0 is a right-identity since $n - 0 = n$ for all $n \in \mathbb{Z}$, but not an identity, since $0 - n \neq n$ in general. (5) For the operation $\heartsuit$, only b and d have inverses, and those inverses are themselves.

B:Let n be a non-negative integer. Consider the set of all polynomials of degree less than or equal to n

$$P_n = \{a_0x^0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, \cdots, a_n \in \mathbb{R}\}$$

(1) Explicitly explain the (standard) addition and the (standard) scalar multiplication on P_n.
(2) Show that your vector addition in the previous part satisfies A1-A4.
(3) Show that your scalar multiplication satisfies S1-S4.
(4) Conclude that P_n is a vector space.
(5) We said n is a non-negative integer. What is P_n if $n = 0$?
(6) Let $n \neq m$ be integers. Is $P_n \cup P_m$ a vector space? Explain why.
(7) Define $P := \bigcup_{n\in\mathbb{N}} P_n$. Explain in words what this set is. Is it a vector space? Explain why?(I don't need you to write a formal proof)

(1) For two arbitrary element $p(x) = a_0x^0 + a_1x + a_2x^2 + \cdots a_nx^n$ and $q(x) = b_0x^0 + b_1x + b_2x^2 + \cdots b_nx^n$ in P_n and $r \in \mathbb{R}$

$$p(x) + q(x) = (a_0 + b_0)x^0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots (a_n + b_n)x^n$$

and

$$rp(x) = ra_0x^0 + ra_1x + ra_2x^2 + \cdots ra_nx^n$$

Note that both operations take values in P_n and hence are closed operations on P_n.

(2) Follows from properties of arithmetic over polynomials (it is a good practice to write the down explicitly). For example, the properties A1-A4 hold for P_n because they hold for $\mathbb{R}$ which are where the coefficients are from, and addition and scalar multiplication for polynomials has to do with addition and scalar multiplication of the coefficients. In particular, let's check existence of additive inverses. Let $p(x) = a_0x^0 + \cdots + a_nx^n$. Let $q(x) = (-a_0)x^0 + \cdots + (-a_n)x^n$. Then, $p(x)+q(x) = (a_0-a_0)x^0+\cdots+(a_n-a_n)x^n = 0x^0+\cdots+0x^n$, where we see the last polynomial as the 0 element of P_n as a vector space. Likewise, $q(x) + p(x) = 0$.

(3) Follows from properties of arithmetic over polynomials (e.g. see (2))

(4) Follows from the definition of a vector space and the previous parts.

(5)

$$P_0 = \{a_0x^0 \mid a_0 \in \mathbb{R}\} = \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}$$

(6) Well, first we need to make sure the addition and scalar multiplication are closed on $P_n \cup P_m$. For addition, if you pick two arbitrary polynomials in $P_n \cup P_m$ if they are both in P_n or both in P_m, the result will end up in P_n or respectively P_m. If one belongs to P_n and the other P_m, WLOG assume $m \geq n$, hence the sum ends up in $P_m \subset P_n \cup P_m$. Also, if your scalar multiply an arbitrary polynomial in $P_n \cup P_m$ you will end up the $P_n \cup P_m$ again. Showing the properties is similar to the case for P_n.

(7) This is the set of all polynomials, no restriction in degree. Yes! The reasoning is similar to why $P_n \cup P_m$ in (6) was a vector space.

C: Let $\mathcal{S}$ be the set of all sequences of real numbers. An example of an element in $\mathcal{S}$ might be $\{1, 2, 3, 5, 8, 13, \dots\}$.

(1) Is there a way to define addition and scalar multiplication to make this a vector space?
(2) What is the zero vector?
(3) Is the set of sequences of rational numbers a subspace?
(4) What about sequences of integers?
(5) What about the subset of sequences that are "eventually constant"?

(1) Add $\{a_1, a_2, a_3, a_4, a_5, \dots\}+\{b_1, b_2, b_3, b_4, b_5, \dots\}$ "slot-wise": $\{a_1+b_1, a_2+b_2, a_3+b_3, a_4 + b_4, \dots\}$. Similarly $\lambda\{a_1, a_2, a_3, a_4, a_5, \dots\} = \{\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \lambda a_5, \dots\}$ for $\lambda \in \mathbb{R}$. This is a real vector space. (2) The zero element is the sequence $\{0, 0, 0, 0, 0, \dots\}$. (3) The subset of sequences of rational numbers is NOT a subspace, since it is not closed

under scalar multiplication (say, by π). (4) Ditto for sequences of integers. (5) But the "eventually constant" sequences do form a subspace. For example, let $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ be two eventually constant sequences. This means there exists N and some $c \in \mathbb{R}$ such that for all $n \geq N$, $a_n = c$. Likewise, there exists M and d for the b_i sequence. Without loss of generality, assume $N \geq M$. Then, for all $n \geq N$, $a_n = c$ and $b_n = d$, and so $a_n + b_n = c + d$. This shows that the sum of the two sequences is eventually cosntant (with constant value $c + d$). Since the zero sequence is eventually constant and one can similarly show that a scalar multiple of an eventually constant sequence is eventually constant, it is seen that this subset is indeed a subspace.

D PROOF PRACTICE

For each item, explain a natural way to define addition and real scalar multiplication on the given set, and then choose two axioms from the definition of a vector space

(ideally different axioms for each part) and carefully prove that your operations satisfy the axioms. All the given sets are indeed real vector spaces.

(1) Complex numbers $\mathbb{C}$.
(2) The set $\mathcal{C}^0$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
(3) The set $\mathcal{C}^r$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that have r-th derivative. [2]
(4) The set V of functions f in C^4 such that the fourth derivative of f equals f.

The standard way of adding and scalar multiplying complex numbers makes $\mathbb{C}$ into a real vector space. For all the given function spaces, the standard function addition and scalar multiplication make them into a vector space. For the following we will check a some axioms. (1) Let's check commutativity. Given two complex numbers $z = x+iy$, $w = u+iv$, we have $z+w = (x+u)+i(y+v) = (u+x)+i(v+y) = w+z$, using commutativity of $\mathbb{R}$. (2) The zero vector is the zero function $f(x) = 0$ for all x because it is continuous and for any continuous function g, for all x, $f(x)+g(x) = 0+g(x) = g(x) = g(x)+f(x)$. (3) Note that C^r is a subset of C^0 and it is closed under function addition and scalar multiplication, so all the axioms hold for C^r because they hold for C^0. (4) Similarly to (3), we see that V is a subset of C^r when $r = 4$ closed under function addition and scalar multiplication.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) A real vector space other than $\mathbb{R}^n$.
(2) A set with a binary operation that has an identity.
(3) An invertible element in a set with a binary operation that has an identity.
(4) An invertible element in a set with a binary operation without an identity.
(5) A binary operation with two distinct identity elements.

[2]Some books use C for continuous functions, and D for differentiable functions

(1) The set of complex numbers $\mathbb{C}$, you can easily check that it satisfy all the requirement.
(2) You saw a lot of examples in the previous sections.
(3) You saw a lot of examples in the previous sections.
(4) It is impossible to have a invertible element without having an identity. Since an invertible element cannot be defined without the existence of the identity element.
(5) Identity element is unique. If not, assume we have e_1, e_2 both are identity elements. Then we have $e_1 = e_1 \cdot e_2 = e_2$, which show that they are the same element.