University of Toronto Scarborough

CSC C63 **Final Examination** **20 August 2015**

NAME: ______________________________

STUDENT NUMBER: ____________________

Do not begin until you are told to do so. In the meantime, put your name and student number on this cover page and read the rest of this page.

Aids allowed: None. .... **Duration:** 3 hours.

There are 11 pages and each is numbered at the bottom. Make sure you have all of them.

Write legibly in the space provided. Use the backs of pages for rough work; they will not be graded.

The last page of this exam contains some facts that you may use without proof, including a list of problems which you may assume to be NP-complete.

1.	____	/ 30
2.	____	/ 10
3.	____	/ 5
4.	____	/ 30
5.	____	/ 10
6.	____	/ 30
7.	____	/ 10
8.	____	/ 5
Total	____	/ 130

1. **[30 marks total; 10 for each part]** For each of the following assertions, state whether it is true or false, and briefly justify your answer.

   (a) If M is a TM such that $L(M) = SAT$, then M is a decider.
   (Here we interpret SAT as a language.)

   (b) Every PSPACE-complete language is mapping reducible to its complement.

   (c) If $TQBF \in \mathrm{P}$, then $TQBF \leq_{\mathrm{p}} HALT$.

2. **[10 marks]** Let f be the polytime reduction from *FORMULA-GAME* to *GG* (generalized geography) as described in class.

   Let ϕ be the formula $\forall x_1 \forall x_2 \exists x_3 \Big[\neg\Big((\overline{x_1} \vee x_2) \wedge \neg(x_1 \vee \overline{x_3})\Big) \vee (x_2 \wedge \overline{x_3})\Big]$.

   Draw $f(\phi)$. Label your graph appropriately and indicate the starting node.

3. **[5 marks]** Suppose A is a co-recognizable language, and x is a string in $\overline{A}$ (the complement of A). Prove that there is an enumerator E such that $L(E) = \overline{A}$ and x is the first string printed by E.

4. **[30 marks total; 15 for each part]** In the following definition of a language, M_1 and M_2 are TMs, both with some common input alphabet Σ.

$$L = \{\langle M_1, M_2\rangle \mid (M_1 \text{ loops on } w \text{ whenever } M_2 \text{ rejects } w) \text{ or } (M_2 \text{ loops on } w \text{ whenever } M_1 \text{ rejects } w)\}.$$

Read the footnote[1] below if you are unsure about the definition of L.

(a) Prove that L is co-recognizable by giving an appropriate deterministic TM.
Briefly explain how your TM works.
Advice: Carefully determine $\overline{L}$ (the complement of L).

[1] "M_1 loops on w whenever M_2 rejects w" means "for any string, if M_2 rejects it, then M_1 loops on it". The <u>or</u> in the definition is an <u>inclusive</u> OR, not exlusive OR.

(b) Use a mapping reduction to prove that L is not recognizable.

For your convenience, the language L is given again here.

$$L = \{\langle M_1, M_2\rangle \mid (M_1 \text{ loops on } w \text{ whenever } M_2 \text{ rejects } w) \text{ or}$$
$$(M_2 \text{ loops on } w \text{ whenever } M_1 \text{ rejects } w)\}.$$

5. **[10 marks]** Let L_1, L_2 be arbitrary languages.

   Let $L_3 = \{w \in L_1 |$ every string in L_2 is longer than w, and some string in L_2 is shorter than $www\}$.

   Prove that L_3 is decidable.

6. **[30 marks total]** _Third Jewel_

   The year is 2017. It is summer. Upon her graduation a year ago with a degree in computer science, Vicky returned to San Francisco, and was employed at the same place where she worked as a student. At first things went well. The work was challenging, but not overwhelming. The city had so much to offer. She rode the streetcars, crossed the iconic bridge, and went whale watching.

   But as time passed, Vicky started to feel a certain malaise, a persistent melancholy. Perhaps she yearned for her friends in Toronto. Or maybe the daily morning fog was getting her down. A famous quote, *"the coldest winter I ever spent was a summer in San Francisco"*, started to haunt her. Who said that? Was it Samuel Langhorne Clemens? Most of all she longed for an opportunity to use the complexity theory she learned in a most enjoyable course at university!

   In her disillusionment, Vicky began roaming the streets at night, and this is where she met that strange couple, Tia and Max. They lived in some sort of commune, yet they seemed very well off. They always wore fantastic, and seemingly expensive, jewelry. Rings, bracelets, necklaces, you name it, they had it. Vicky felt herself inexplicably drawn to Tia and Max. She went out of her way to get to know them better. In time she found out that they were jewel thieves.

   Vicky's discovery only made her want to be more like Tia and Max. She joined them on their adventurous heists. Since there were two of them and just one of Vicky, they decided that each haul should be divided so that Vicky gets one third of the total value, and Tia and Max the remaining two thirds. It makes no sense to divide individual pieces of jewelry, hence it may not be possible for Vicky to get exactly one third. So they agreed to the procedure where Vicky may choose as many jewelry pieces from each heist as she wants, as long as her portion does not exceed one third of the total value. Vicky was thrilled. *She finally found a crunchy NP-complete problem!*

Here is Vicky's crunchy problem along with a related non-decision problem.

THIRD-JEWEL

Input: Numbers $d_1, \cdots, d_n$ (dollar values of jewelry pieces from a heist), and a number k.

Question: Is there a subset V of $\{1, \cdots, n\}$ such that $\sum\limits_{1 \le i \le n} d_i - k \le 3 \sum\limits_{i \in V} d_i \le \sum\limits_{1 \le i \le n} d_i$?

I.e., can 3 times Vicky's share be within k dollars of the total value without exceeding it?

MAX-THIRD-JEWEL

Input: Numbers $d_1, \cdots, d_n$ (dollar values of jewelry pieces from a heist).

Question: Find the maximum value of $\sum\limits_{i \in V} d_i$ subject to $3 \sum\limits_{i \in V} d_i \le \sum\limits_{1 \le i \le n} d_i$.

I.e., how big (in dollars) can Vicky's share be without exceeding one third of the total value?

(a) **[5 marks]** Assuming *MAX-THIRD-JEWEL* can be solved in polytime, give a polytime algorithm to solve *THIRD-JEWEL*. Briefly explain how your algorithm works.

(b) **[10 marks]** Assuming *THIRD-JEWEL* can be solved in polytime, give a polytime algorithm to solve *MAX-THIRD-JEWEL*. Briefly explain how your algorithm works.

(c) [**15 marks**] Prove that *THIRD-JEWEL* is NP-complete.
Reminder: Remember the full definition of NP-complete.

7. **[10 marks total; 5 for each part]** For each of the following problems, (i) prove that it is in P, **or** (ii) prove that it is NP-hard. *Do not prove both.*

   (a) *DOUBLE-FALSIFIABLE*

   Input: A boolean formula F.

   Question: Does F have two (or more) falsifying truth assignments?

   (b) *TRIPLE-DISJOINT-VERTEX-COVER*

   Input: A graph G and a number k.

   Question: Does G contain three (or more) mutually disjoint vertex covers of size k?
   (By "mutually disjoint", we mean the intersection of any two such vertex covers is empty.)

8. [**5 marks**] Write an interesting story about a UofT student and a York student, then define an NP-complete problem that is based on your story. No proof is required.

What you may assume

◊ *You may assume these problems are NP-complete.*

SAT
Input: A boolean formula F.
Question: Is F satisfiable?

3SAT
Input: A CNF formula F where each clause has exactly 3 literals.
Question: Is F satisfiable?

CLIQUE
Input: A graph G and a number k.
Question: Does G contain a clique of size k?
(A clique is a set of pairwise adjacent nodes, or a complete subgraph.)

VERTEX-COVER
Input: A graph G and a number k.
Question: Does G contain a vertex cover of size k?
(A vertex cover is a set of nodes such that every edge is covered by some node in the set.)

HAMPATH
Input: A directed graph G and 2 nodes s, t.
Question: Does G contain a Hamilton path from s to t?
(A Hamilton path is a path that visits every node in the graph exactly once.)

UHAMPATH
Input: A (undirected) graph G and 2 nodes s, t.
Question: Does G contain a Hamilton path from s to t?
(A Hamilton path is a path that visits every node in the graph exactly once.)

SUBSET-SUM
Input: A multiset S of numbers and a number t.
Question: Does S contain a subset A such that $\sum\limits_{x \in A} x = t$?

3-COL
Input: A graph G.
Question: Is G 3-colourable? (I.e., does G have a 3-colouring? Or put another way, is there a way to colour the nodes of G with at most 3 colours so that no adjacent nodes have the same colour?)

◊ *You may assume these facts.*

- $HALT = \{\langle M, w\rangle \mid M$ is a TM that halts on input string $w\}$ is recognizable but not co-recognizable.
- $E_{\mathsf{TM}} = \{\langle M\rangle \mid M$ is a TM and $L(M) = \emptyset\}$ is co-recognizable but not recognizable.
- $EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2$ are TMs and $L(M_1) = L(M_2)\}$ is neither recognizable nor co-recognizable.
- *TQBF*, *FORMULA-GAME* and *GG* (generalized geography) are PSPACE-complete.