

W7 Q1

$$u = u(x, y) \quad x = r \cos \theta \quad y = r \sin \theta$$

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta) \end{aligned}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \left(\frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right) \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial \theta} \left(\frac{\partial x}{\partial \theta} \right) \right) + \left(\frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial y} \right) \frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial \theta} \left(\frac{\partial y}{\partial \theta} \right) \right)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right) &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) \frac{\partial y}{\partial \theta} \\ &= \frac{\partial^2 u}{\partial x^2} (-r \sin \theta) + \frac{\partial^2 u}{\partial y \partial x} (r \cos \theta) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial y} \right) &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) \frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \frac{\partial y}{\partial \theta} \\ &= \frac{\partial^2 u}{\partial x \partial y} (-r \sin \theta) + \frac{\partial^2 u}{\partial y^2} (r \cos \theta) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= \left((u_{yx} r \cos \theta - u_{xx} r \sin \theta) (-r \sin \theta) + \frac{\partial u}{\partial x} (-r \cos \theta) \right) \\ &\quad + \left((u_{xy} (-r \sin \theta) + u_{yy} r \cos \theta) r \cos \theta + \frac{\partial u}{\partial y} (-r \sin \theta) \right) \end{aligned}$$

$$= u_{xx} r^2 \sin^2 \theta - u_{yx} r^2 \cos \theta \sin \theta - u_x r \cos \theta - u_{xy} r^2 \sin \theta \cos \theta + u_{yy} r^2 \cos^2 \theta - u_y r \sin \theta$$