Discrete Math A3

1. Given the set of S natural numbers S = {1, 2, 3, . . . , 200} select one hundred and one numbers from S. Prove that at least one of the numbers you chose is a multiple of another number that you chose.

Place all the odd nu mbers in S inside of their own pigeonhole (100 pigeon holes).

Let the pigeons be the selected integers (101 pigeons).

Let x represent any pigeon (any selected integer):

Place x in the pigeonhole that has the greatest divisor between all the pigeonholes. By PHP this means there will be at least 1 pigeonhole that has at least 2 numbers.

Let a and b represent these two integers.

Let y be the odd integer that and a and b correspond to.

The integers can be re-written like this:

$$a = 2^m y$$

$$b = 2^n y$$

When m, n $\in \mathrm{N}$ and $\mathrm{m} > \mathrm{n}$

Rewrite b to solve for y:

$$y = \frac{b}{2^n}$$

Then sub it into a and use exponent rules:

$$a = 2^m \frac{b}{2^n}$$

$$a = 2^{m-n} b$$

Because m – n is an integer, this statement means that a is a multiple of b.

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2. Linda has 5 weeks to prepare for her CSCA67 final. Her friend has volunteered to tutor her for either 15min or 30min every day until the test but not for more than 15 hours total. Show that during some period of consecutive days, Linda and her tutor will study for exactly 8 3/4 hours.

5 weeks x 7 days/week = 35 days

Let 15 min = 1 unit (for easier calculations)

Let x_i be the number of units Linda's friend has tutored her in total by day i (inclusive).

$1 \leq x_1 < x_2 < \cdots < x_{35} \leq 60$

We are trying to prove that there exists an $x_j = x_i + 35$

Let the pigeonholes be the pairs of numbers which are 35 units apart up to the maximum number of units (i.e. (0, 35), (1, 36), … , (25, 60), (26), (27), … , (34)). There are 35 pigeonholes.

There are 36 different days (including day 0, before any tutoring began), meaning there is 36 pigeons, going into 35 pigeonholes, by PHP there exists two days where the difference in units of tutoring is exactly 35 which corresponds to 8 ¾ hours.

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3. Use the following Venn diagram for this question.

a) Construct a Venn diagram for the following statement: (p → q) → r.

True: 2, 5, 6 ,7, 8

b) Construct an equivalent statement using only ∧, ∨, and/or ¬.

$$(\mathrm{p} \rightarrow \mathrm{q}) \rightarrow \mathrm{r}$$

$$(\neg \mathrm{p} \vee \mathrm{q}) \rightarrow \mathrm{r} \text{ (implication law)}$$

$$\neg(\neg \mathrm{p} \vee \mathrm{q}) \vee \mathrm{r} \text{ (implication law)}$$

$$(\mathrm{p} \wedge \neg \mathrm{q}) \vee \mathrm{r} \text{ (De Morgan's law)}$$

c) Construct a Venn diagram for the following statement: p → (q → r).

True: 1, 2, 4, 5, 6, 7, 8

d) Construct an equivalent statement using only ∧, ∨, and/or ¬.

$$\mathrm{p} \rightarrow (\mathrm{q} \rightarrow \mathrm{r})$$

$$\mathrm{p} \rightarrow (\neg \mathrm{q} \vee \mathrm{r}) \text{ (implication law)}$$

$$\neg \mathrm{p} \vee (\neg \mathrm{q} \vee \mathrm{r}) \text{ (implication law)}$$

$$\neg \mathrm{p} \vee \neg \mathrm{q} \vee \mathrm{r} \text{ (associative law)}$$

e) Is → commutative? associative? Explain your answer.

Implication is neither commutative nor associative. If you evaluate both of the equations, subbing False into all the variables, the one in b) evaluates to False while the one in d) evaluates to True.

4. Prove or disprove the following using equivalence laws:

a) $p \wedge (q \to r) \Longleftrightarrow (p \to q) \to r$

$$p \wedge (q \to r) \Longleftrightarrow (p \to q) \to r$$

$$\text{(implication law) } p \wedge (\neg q \vee r) \Longleftrightarrow (\neg p \vee q) \to r \text{ (implication law)}$$

$$p \wedge (\neg q \vee r) \Longleftrightarrow \neg(\neg p \vee q) \vee r \text{ (implication law)}$$

$$p \wedge (\neg q \vee r) \Longleftrightarrow (p \wedge \neg q) \vee r \text{ (De Morgan's Law)}$$

Let p = False, q = False, r = True

$$F \wedge (\neg F \vee T) \Longleftrightarrow (F \wedge \neg F) \vee T$$

$$F \Longleftrightarrow T$$

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False is not equivalent to True therefore the statements are not equivalent.

b) $x \wedge (\neg y \leftrightarrow z) \Longleftrightarrow ((x \to y) \vee \neg z) \to (x \wedge \neg(y \to z))$

$$x \wedge (\neg y \leftrightarrow z) \Longleftrightarrow ((x \to y) \vee \neg z) \to (x \wedge \neg(y \to z))$$

$$\text{(biconditional law) } x \wedge \big((\neg y \to z) \wedge (z \to \neg y)\big) \Longleftrightarrow \big((\neg x \vee y) \vee \neg z\big) \to \big(x \wedge \neg(\neg y \vee z)\big) \text{ (implication law)}$$

$$\text{(implication law) } x \wedge \big((y \vee z) \wedge (\neg z \vee \neg y)\big) \Longleftrightarrow \big((\neg x \vee y) \vee \neg z\big) \to \big(x \wedge (y \wedge \neg z)\big) \text{ (De Morgan's Law)}$$

$$\text{(distributive law) } x \wedge ((y \wedge \neg z) \vee (y \wedge \neg y) \vee (z \wedge \neg z) \vee (z \wedge \neg y)) \Longleftrightarrow ((\neg x \vee y) \vee \neg z) \to (x \wedge y \wedge \neg z) \text{ (associative law)}$$

$$\text{(negation law) } x \wedge ((y \wedge \neg z) \vee F \vee F \vee (z \wedge \neg y)) \Longleftrightarrow \neg(\neg x \vee y \vee \neg z) \vee (x \wedge y \wedge \neg z) \text{ (implication law)}$$

$$\text{(identity law) } x \wedge ((y \wedge \neg z) \vee (z \wedge \neg y)) \Longleftrightarrow (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z) \text{ (De Morgan's Law)}$$

$$\text{(distributive law) } (x \wedge y \wedge \neg z) \vee (x \wedge z \wedge \neg y) \Longleftrightarrow (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z)$$

$$\text{(communitative and associative law) } (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z) \Longleftrightarrow (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z)$$

Therefore, these two statements are equivalent.

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c) $(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow \neg y \to (x \leftrightarrow z)$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow \neg y \to (x \leftrightarrow z)$$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow \neg y \rightarrow ((x \rightarrow z) \wedge (z \rightarrow x)) \text{ (biconditional law)}$$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow \neg y \rightarrow ((\neg x \vee z) \wedge (\neg z \vee x)) \text{ (implication law)}$$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow y \vee ((\neg x \vee z) \wedge (\neg z \vee x)) \text{ (implication law)}$$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Longleftrightarrow (y \vee (\neg x \vee z)) \wedge \big(y \vee (\neg z \vee x)\big) \text{ (distribution law)}$$

$$(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \Leftarrow$$
$$\Rightarrow (x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \text{ (associative and communitative law)}$$

Therefore, the two statements are equivalent.

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5. Let our domain be the natural numbers greater than 1.

   Define: P(x) = “x is prime” Q(x, y) = “x divides y”

   Consider the following statement: For every x that is not prime, there is some prime y that divides it. ”

   a) Write the statement in predicate logic.

   $$\forall x \in N/\{0,1\}, \exists y \in N/\{0,1\}, (\neg P(x) \wedge P(y)) \rightarrow Q(y,x)$$

   b) Negate your statement from part (a).

   $$\exists x \in N/\{0,1\}, \forall y \in N/\{0,1\}, \neg\big(\big(\neg P(x) \wedge P(y)\big) \rightarrow Q(y,x)\big) \text{ (negation law)}$$

   $$\exists x \in N/\{0,1\}, \forall y \in N/\{0,1\}, \neg\big(\neg\big(\neg P(x) \wedge P(y)\big) \vee Q(y,x)\big) \text{ (implication law)}$$

   $$\exists x \in N,/\{0,1\}\ \forall y \in N/\{0,1\}, \big(\neg P(x) \wedge P(y)\big) \wedge \neg Q(y,x)) \text{ (De Morgan's Law)x3}$$

   $$\exists x \in N/\{0,1\}, \forall y \in N/\{0,1\}, \neg P(x) \wedge P(y) \wedge \neg Q(y,x) \text{ (associative law)}$$

   c) Write the English translation of your negated statement. Your statement should sound like English not predicate logic in words.

For every prime value of y, there is some non-prime value x for which y does not divide x.

   d) Write the following statement using the given predicates: “There is exactly one prime number that is even.”

   $$\exists!\, x \in N, P(x) \wedge Q(2,x)$$

6. Give an example of a domain U and predicates P and Q such that

   (∀x ∈ U, P(x)) → (∀x ∈ U, Q(x)) is true and ∀x ∈ U, (P(x) → Q(x)) is false.

Let U the set of integers.

Let Q(x) be x is an odd number.

Let P(x) be x = 2.

Evaluating both statements:

$$(\forall x \in U, P(x)) \rightarrow (\forall x \in U, Q(x))$$

$$F \rightarrow F = T$$

$$\forall x \in U, (P(x) \rightarrow Q(x))$$

Only 1 value of x makes P(x) true, x = 2, and x = 2 does not imply x is an odd number. Therefore the second statement is false.

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7. Consider the definition of the limit of a sequence in calculus. Negate this:

$$\forall \varepsilon \in R^{+}, \exists N \in Z, \forall n \in Z, n > N \rightarrow L - \varepsilon < a_n < L + \varepsilon$$

Negation:

$$\exists \varepsilon \in R^{+}, \forall N \in Z, \exists n \in Z, \neg(n > N \rightarrow L - \varepsilon < a_n < L + \varepsilon) \text{ (negation law)}$$

$$\exists \varepsilon \in R^{+}, \forall N \in Z, \exists n \in Z, \neg(\neg(n > N) \vee (L - \varepsilon < a_n < L + \varepsilon)) \text{ (implication law)}$$

$$\exists \varepsilon \in R^{+}, \forall N \in Z, \exists n \in Z, (n > N) \wedge \neg(L - \varepsilon < a_n < L + \varepsilon)) \text{ (De Morgan's Law)}$$

$$\exists \varepsilon \in R^{+}, \forall N \in Z, \exists n \in Z, (n > N) \wedge ((L - \varepsilon \leq a_n) \vee (a_n \geq L + \varepsilon)) \text{ (distributive law)}$$

8. Prove that for all integers a, b and c, if a divides b and a divides c then a divides (b + c). You may use the notation n | x for n divides x. Recall that n divides x means there exists q such that x = nq.

Given: a | b and a | c

Prove: a | (b + c)

Proof:

b = na, n ∈ Z

c = ka, k ∈ Z

a | (b + c) = b + c = qa, q ∈ Z

Sub in the previously shown values of a and b

na + ka = qa

a(n + k) = qa

Let n + k = q, because the sum of two integers is an integer.

aq = qa ■

9. Prove that for all integers n and m, if n − m is even then n^3 – m^3 is even.

Given: (n – m) = 2p, p ∈ Z

Prove: (n^3 – m^3) = 2q, q ∈ Z

Proof:

$$n^3 - m^3 = 2q$$

$$(n-m)(n^2 + nm + m^2) = 2q$$

$$2p(n^2 + nm + m^2) = 2q$$

Let $2p(n^2 + nm + m^2)$ = q, because the sum and product of integers is an integer

$$2q = 2q$$

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10. Prove ∀n ∈ Z, 4 ∤ (n^2 + 5). a ∤ b means a does not divide b. Hint: Consider when n is even and when n is odd.

<u>N is odd</u>

Let n = 2k + 1, k ∈ Z

Suppose 4 | (n^2 + 5):

$$\mathrm{n}^2 + 5 = 4\mathrm{p}, \mathrm{p} \in \mathrm{Z}$$

$$4k^2 + 4\mathrm{k} + 1 + 5 = 4\mathrm{p}$$

$$4k^2 + 4\mathrm{k} + 6 = 4\mathrm{p}$$

$$2k^2 + 2\mathrm{k} + 3 = 2\mathrm{p}$$

$$2k(\mathrm{k} + 2) + 3 = 2\mathrm{p}$$

Let m = k + 2, because the sum of an integer and 2 is a random integer.

$$2km + 3 = 2\mathrm{p}$$

Let n = km, because the product of two integers in a random integer.

$$2n + 3 = 2\mathrm{p}$$

LS evaluates to an odd integer (the sum of odd and even is odd)

RS evaluates to any even integer (two times an integer in an even integer).

Therefore, LS doesn't equal RS and the supposition is false.

<u>N is even</u>

Let n = 2k, k ∈ Z

Suppose 4 | (n^2 + 5):

$$n^2 + 5 = 4p, p \in Z$$

$$4k^2 + 5 = 4p$$

LS evaluates to an odd value (4 times an integer is even, the sum of even and odd is odd).

RS evaluates to an even value (the product of an even number and an integer is even).

Therefore, LS doesn't equal RS, and the supposition is false.

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11. . Two numbers are consecutive if the second number is one greater than the first number. A natural number x is a perfect square if there exists natural number y such that x = y^2.

Prove that the product of any four consecutive natural numbers is equal to some perfect square minus 1.

a) Rewrite the claim in predicate logic.

$$\forall n \in N, \exists x \in N, (n(n+1)(n+2)(n+3) = x^2 - 1)$$

b) Prove the claim.

$$n(n+1)(n+2)(n+3) = x^2 - 1$$

$$(n^2 + 3n)(n^2 + 3n + 2) = x^2 - 1$$

Let m = $n^2 + 3n$, the sum of two arbitrary natural numbers is a natural number.

$$m(m+2) = x^2 - 1$$

$$m^2 + 2m + 1 = x^2$$

$$(m+1)^2 = x^2$$

Let x = m +1, because the sum of an arbitrary and 1 is an arbitrary integer.

$$x^2 = x^2$$

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