MATB24 – Midterm I –Fall 2020

Professors: Camelia Karimianpour, Sander Kupers

Solutions

1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.

(a) (3 points) Consider vectors $\vec{v}_1, \cdots, \vec{v}_k$ in a vector space V over $\mathbb{F}$. They are *linearly dependent* if there exist $c_1, \ldots, c_k \in \mathbb{F}$ so that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$.

> **Solution:** incorrect Consider vectors $\vec{v}_1, \cdots, \vec{v}_k$ in a vector space V over $\mathbb{F}$. They are *linearly dependent* if there exist $c_1, \ldots, c_k \in \mathbb{F}$, **not all zero**, so that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$.

(b) (3 points) An *isomorphism* between vector spaces V and W over $\mathbb{F}$ is a bijective function $T\colon V \to W$.

> **Solution:** Incorrect. An *isomorphism* between vector spaces V and W over $\mathbb{F}$ is a bijective **linear** function $T\colon V \to W$.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counter example if you think it is false).

   (a) (3 points) If $\vec{u}, \vec{v}$ are non-zero linearly dependent vectors in a vector space V over $\mathbb{F}$, then there exists an $a \in \mathbb{F}$ such that $\vec{u} = a\vec{v}$.

   Solution: True, There exists a dependency relation $a_1\vec{v} + a_2\vec{u} = \vec{0}$. $\vec{u}, \vec{v}$ are non-zero implies a_1 and a_2 are nonzero. Hence $\vec{u} = \frac{a_1}{a_2}\vec{v}$.

   (b) (3 points) Every subspace is the image of a linear transformation.[1]

   Solution: True. Let V be a subspace of W. Then the map $i : V \to W, \vec{v} \mapsto \vec{v}$ is linear and $ImgT = V$.

   (c) (3 points) No list of three polynomials spans $\mathcal{P}_3(\mathbb{C})$, the vector space over $\mathbb{C}$ of polynomials of degree ≤ 3.

   Solution: True, $\dim \mathcal{P}_3(\mathbb{C}) = 4$. That is every spanning set of $\mathcal{P}_3(\mathbb{C})$ has at least 4 vectors.

   (d) (3 points) If $T\colon \mathbb{R}^5 \to V$ is a surjective linear transformation,then $\dim(V) \leq 5$.

   Solution: True. By the rank nullity equation for T, we have $\dim(\mathbb{R}^5) = \dim\ker T + \dim\operatorname{im} T$, so $5 = \dim\ker T + \dim V$, that is $\dim V = 5 - \dim\ker T$.

[1] A "linear transformation" is the same as a "linear map."

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist. Include the description in your answer.

   (a) (2 points) A linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{im}(T) = \ker(T)$.

   Solution: Consider $T : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a-b \\ a-b \end{bmatrix}$

   (b) (2 points) Two different bases of $\mathbb{C}^3$.

   Solution: $\{\vec{e}_1, \vec{e}_2, \vec{e}_2\}$ and $\{\vec{e}_1 + \vec{e}_2, \vec{e}_2, \vec{e}_2\}$

   (c) (2 points) A non-empty subset U of $\mathbb{C}^2$ which is closed under addition (so $\vec{u}+\vec{v} \in U$ for all $\vec{u}, \vec{v} \in U$) and taking additive inverses (so $-\vec{u} \in U$ for all $\vec{u} \in U$), but which is not a subspace of $\mathbb{C}^2$.

   Solution: $\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, k\begin{bmatrix} i \\ i \end{bmatrix}, k \in \mathbb{Z}\}$

   (d) (2 points) An infinite spanning set for a finite dimensional vector space.

   Solution: $P_1(\mathbb{R}) = \{kx \mid k \in \mathbb{R}\}$

4. Let $\mathcal{C}^\infty$ denote the vector space of functions $f\colon \mathbb{R} \to \mathbb{R}$ which have derivatives of all orders.[2] Let $S\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$, defined by $S(f) = f'$, and let $T\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$ be a linear transformation such that $T(e^{2x}) = x^2$, $T(e^{3x}) = \cos x$, and $T(1) = 0$.

   (a) (4 points) Find $T \circ S(e^{5x})$ or explain why it is not possible to do so with the given information.

   Solution: Not possible!

   $$T \circ S(e^{5x}) = T(S(e^{5x})) = T(5e^{5x}) = 5T(e^{5x})$$

   In order to find $T(e^{5x})$ we need to write it as linear combination of e^{2x} and e^{3x}. But $\{e^{2x}, 3^{3x}, e^{5x}\}$ is linearly independent (needs a proof)

   (b) (4 points) Find $S \circ T(2 + 3e^{2x})$ or explain why it is not possible to do so with the given information.

   Solution:

   $$S \circ T(2 + 3e^{2x}) = S(T(2 + 3e^{2x})) = S(2T(1) + 3T(e^{2x})) = S(3x^2) = 6x$$

   (c) (4 points) Prove that $\dim(\ker(T \circ S)) \geq 2$.

   Solution: We find two vectors in this kernel: $T \circ S(1) = T(0) = 0$ and $T \circ S(x) = T(1) = 0$. Note that 1 and x are linearly independent.

[2] That is, the i-th order derivative of f exists for all $i \in \mathbb{N}$. These are also called "smooth" functions.

5. Let $\mathcal{P}_m(\mathbb{F})$ be the vector space of polynomials of degree $\leq m$ with coefficients in $\mathbb{F}$. Let $W := \{p(x) \in \mathcal{P}_m(\mathbb{F}) \mid p(1) = 0\}$.

   (a) (3 points) Prove that W is a subspace of $\mathcal{P}_m(\mathbb{F})$.

   Solution: For all $p(x)$ and $q(X)$ in $\mathcal{P}_m(\mathbb{F})$ and $k \in \mathbb{F}$. Then $p + q(1) = p(1) + q(1) = 0 + 0 = 0$ and $kp(1) = k(p(1)) = 0$. Note that zero polynomial in W. Subspace test implies W is a subspace of $\mathcal{P}_m(\mathbb{F})$.

   (b) (3 points) Consider the linear transformation $T : W \to \mathcal{P}_m(\mathbb{F})$ given by $p(x) \mapsto p(x)$. Is T onto? Justify.

   Solution: No. For instance the constant polynomial 1 is not in W and hence not in th image of T. [note that the counter example you give should work for any field including $\mathbb{F}_2$ or $\mathbb{F}_3$].

(c) (5 points) Suppose that $p_0, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(1) = 0$ for each j. Prove that $p_0, \ldots, p_m$ are linearly dependent.

Solution: From (b) we know that W is a strict subspace of $\mathcal{P}_m(\mathbb{F})$, hence $\dim W < m$. The number of Every linearly independent set in W s less than or equal to $m = \dim(\mathcal{P}_m(\mathbb{F}))$. $p_0, \ldots, p_m$ is a collection of $m+1$ vectors in W and hence linearly dependent.

6. Suppose that V is a vector space over $\mathbb{F}$.

(a) (4 points) Let $T\colon V \to V$ be a linear map such that $T \circ T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. Prove that $S = \{\vec{v} \in V \mid T(\vec{v}) = \vec{v}\} \subset V$ and $A = \{\vec{v} \in V \mid T(\vec{v}) = -\vec{v}\} \subset V$ are subspaces.

Solution: Note that $\vec{0}$ is in S and A.

Given $\vec{v}, \vec{w} \in S$, and $k \in \mathbb{F}$. Since T is linear, $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{v} + \vec{w}$. Similarly $T(k\vec{v}) = kT(\vec{v}) = k\vec{v}$. Hence $\vec{v} + \vec{w}$ and $k\vec{v}$ are in S. S is a subspace of V.

Similarly, $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = -\vec{v} + -\vec{w} = -(\vec{v}+\vec{w})$. Similarly $T(k\vec{v}) = kT(\vec{v}) = k(-\vec{v}) = -k\vec{v}$. Hence $\vec{v} + \vec{w}$ and $k\vec{v}$ are in A. A is a subspace of V.

(b) (4 points) Prove that if $1 \neq -1$ in $\mathbb{F}$, then every $\vec{v}$ of V can be written uniquely as a sum $\vec{v} = \vec{s} + \vec{a}$ of a vector $\vec{s} \in S$ and a vector $\vec{a} \in A$.

Solution: Since $1 \neq -1$, $2 := 1 + 1 \neq 0$ and hence 2 is invertible in $\mathbb{F}$. Given $\vec{v} \in V$, write $\vec{v} = 2^{-1}(\vec{v} + T(\vec{v})) + 2^{-1}(\vec{v} - T(\vec{v}))$. Note that

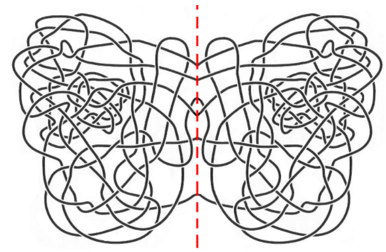
$$T(2^{-1}(\vec{v} + T(\vec{v}))) = 2^{-1}(T(\vec{v}) + T^2(\vec{v})) = 2^{-1}(T(\vec{v}) + \vec{v}),$$

and

$$T(2^{-1}(\vec{v} - T(\vec{v}))) = 2^{-1}(T(\vec{v}) - T^2(\vec{v})) = 2^{-1}(T(\vec{v}) - \vec{v}) = -2^{-1}(\vec{v} - T(\vec{v}))$$

Hence $2^{-1}(\vec{v} + T(\vec{v})) \in S$ and $2^{-1}(\vec{v} - T(\vec{\ }))\in A$.

(c) (2 points) Let K be the knot diagram pictured to the right. Explain why part (a) of this problem implies that the number of tricolorings of K which are symmetric with respect to reflection in the dotted line is equal to 3^m for some $m \geq 1$. You do not need to give a proof.



Solution: Let V be the vector space of tricolorings of K. Let T be the reflection with respect to the dotted line. T is a linear transformation with property $T \circ T = id_V$. By (a) tricolorings of K that are symmetric with respect to reflection is a subspace of V, precisely the subspace S. We saw in GHW1 that, after making certain choices, we can identify V with a subspace of the finite dimensional vector space $\mathbb{F}_3^n$, where n is the number of line segments in K. Hence S is a subspace of $\mathbb{F}_3^n$ and hence finite dimensional vector space over $\mathbb{F}_3$. We proved in GHW2 that such vector spaces have 3^m vectors, for some $m \geq 1$.

(d) (1 point) Explain what part (b) means for tricolorings of K. You do not need to give a proof.

Solution: Every tricoloring of K can be written as a sum of a symmetric tricoloring and an anti-symmetric tricoloring with respect to the dotted line.

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”I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own.”