GHW2

Problem 1

1. True. Because $m = \dim(V)$ means there are m elements in any basis of V. We know a basis is a l.i. spanning set of V, and therefore the largest l.i. subset of V. So all l.i. sets must be less than or equal to size $\dim(V)$

2. False. Let $T: \mathbb{R}^2 \to \mathbb{R}$
$(a,b) \to a$

Let $v = (1,0)$
$T(v) = 1$
$\{T(v)\}$ clearly spans $\mathbb{R}$
but $\{(1,0)\}$ does not span $\mathbb{R}^2$

3. True. For $\{v_1, \ldots, v_k\}$ to be l.i. there cannot be a v_i in the set that can be written as a l.c. of the other elements in the set.
$\therefore v_k \notin \text{span}\{v_1, \ldots, v_{k-1}\}$. $= r_1 v_1 + \ldots + r_{k-1} v_{k-1}$ } L.C. of the other elements.

4. True.

$W = r_1 x + r_2 x^2 + \ldots + r_n x^n$
$= sp(x, x^2, \ldots, x^n)$

$V = s_1 x + s_2(x^2 + x) + \ldots + s_n(x^n + \ldots + x)$
$= s_1 x + s_2 x^2 + s_2 x + \ldots + s_n x^n + \ldots + s_n x$
$= (s_1 + \ldots + s_n)x + \ldots + s_n x^n$
$= sp(x, x^2, \ldots, x^n)$

Problem 2

1. Non-empty

$\vec{0} \in U, \vec{0} \in W$ bc U and W are subspaces
So $\vec{0} + \vec{0} \in U + W$
$\vec{0} + \vec{0} = \vec{0}$

Closed under addition

Let $u, v \in U + W$
$u = u_1 + v_1 \quad u_1, u_2 \in U$
$v = u_2 + v_2 \quad v_1, v_2 \in W$

$u + v = (u_1 + v_1) + (u_2 + v_2)$
$= (u_1 + u_2) + (v_1 + v_2) \quad u_1 + u_2 \in U$
$\in U + W \quad v_1 + v_2 \in W$
bc. U and W are subspaces

Closed under scalar multl.

Let $u \in U + W$, $r \in F$
$u = u_1 + v_1$, $u_1 \in U$, $v_1 \in W$
$ru = ru_1 + rv_1$, $ru_1 \in U$, $rv_1 \in W$
$\in U + W$ bc U & W are subspaces

2. We know $U = sp(u_1, \dots, u_r)$ $W = sp(w_1, \dots, w_s)$ $U \cup W = sp(u_1, \dots, u_r, w_1, \dots, w_s)$

$U + W = \{u + w, u \in U, w \in W\}$
$\forall u \in U, w \in W$, $u + w = (r_1 u_1 + \dots + r_r u_r) + (s_1 w_1 + \dots + s_s w_s) \in U \cup W$

$\therefore U + W = sp(U \cup W)$

3. Prove if $r_1 u_1 + \dots + r_r u_r + s_1 w_1 + \dots + s_s w_s = \vec{0}$ then $r_i = s_j = 0$
Supp there is a LC of $\mathcal{U} \cup \mathcal{W}$ st it isn't trivial and equals $\vec{0}$
Since $\mathcal{U}$ and $\mathcal{W}$ are l.i. $\forall \vec{u} \in \mathcal{U}, \vec{w} \in \mathcal{W}$, $\vec{u} \neq \vec{0}, \vec{w} \neq \vec{0}$, so there must exist a L.C. of $\mathcal{U}$ that equals some $w \in W$ or L.C. of $\mathcal{W}$ that equals some $u \in U$
But we know $U \cap W = \{\vec{0}\}$ so no $u \in U$ can equal $w \in W$ (except $\vec{0}$)
Thus contradicting our supposition
$\therefore \mathcal{U} \cup \mathcal{W}$ is l.i. ∎

4. We know $\mathcal{U} \cup \mathcal{W}$ is l.i. and a spanning set of $U + W$, so it is a basis for $U + W$
$\dim(U) = |\mathcal{U}| = r$ $\dim(W) = |\mathcal{W}| = s$ $\dim(U+W) = |\mathcal{U} \cup \mathcal{W}| = r + s$

$\therefore \dim(U+W) = \dim(U) + \dim(W)$

Problem 3

1. Supp. V is a V.S. over F is finite dimensional
This means a basis of V is finite
Let $B = \{v_1, \dots, v_n\}$ be a basis of V
$\forall \vec{v} \in V$, $\vec{v} = r_1 v_1 + \dots + r_n v_n$, $r_i \in F$
We know there is a finite number of possible r_i's bc. F is finite
So there is a finite number of L.C. of B.
$\therefore$ There is a finite # of $\vec{v} \in V$

Supp. there is a finite # of $\vec{v} \in V$
Supp. to the contrary that V is infinite dimensional
This means any basis of V is infinite
Let a basis $B = \{v_1, v_2, \dots\}$
This means $\forall \vec{v} \in V$, $\vec{v} = r_1 v_1 + r_2 v_2 + \dots$
Since there are an infinite # of L.C. of B, there are an infinite # of $\vec{v} \in V$, but this contradicts the supp. that there are a finite # of elements in V.
$\therefore$ V is finite dimensional ∎

2. Supp. V is a vector space over F and is finite
Prove V has $(\#F)^{\dim(V)}$ elements

Let $m = \#$ of elements in F and $B = \{v_1, \ldots, v_n\}$ be a basis of V
So $\forall \vec{v} \in V$, $\vec{v} = r_1 v_1 + \cdots + r_n v_n$, $r_i \in F$, $1 \leq i \leq n$
Using permutations, the # of different $\vec{v}$'s is m^n
We also know $\dim(V) = n$, as there are n elements in a basis of V
So $m^n = (\#F)^{\dim(V)}$ ∎

Note: The permutations of $\vec{v}$ is the different # of elements in F multiplied n times (different # of r_i's)
So $\underbrace{\#F \cdot \ldots \cdot \#F}_{n \text{ times}} = m^n$

3. We know F_3 is a field with a finite # of elements

From asn 1 F_3^n is a vector space over F_3 representing the # of tricolourings for a knot. From part 1, we know F_3^n is finite dimensional.
Let $\dim(F_3^n) = n$, $n \geq 1$ bc. we know there is always 3 trivial cases for a knot.
We know the # of elements in F_3 is 3 $(0, 1, 2)$
From part 2, we know the # of elements in F_3^n is $(\#F_3)^{\dim(F_3^n)}$

$\therefore$ The # of elements in $F_3^n = 3^n$, $n \geq 1$ ∎

Problem 4

1.
$\vec{c} = [1, 2, 1]$ $\quad T_{\vec{c}}(f) = \begin{bmatrix} f(1) \\ f(2) \\ f(1) \end{bmatrix}$

Let $f, g \in P_2$, $r \in F$

$$T_{\vec{c}}(f + rg) = \begin{bmatrix} (f+rg)(1) \\ (f+rg)(2) \\ (f+rg)(1) \end{bmatrix} = \begin{bmatrix} f(1) + rg(1) \\ f(2) + rg(2) \\ f(1) + rf(1) \end{bmatrix} = \begin{bmatrix} f(1) \\ f(2) \\ f(1) \end{bmatrix} + r \begin{bmatrix} g(1) \\ g(2) \\ g(1) \end{bmatrix} = T_{\vec{c}}(f) + rT_{\vec{c}}(g)$$

by properties of func. addition and scalar multi.

$\therefore T_{\vec{c}}$ is a LT

2. $f(c_1)=0$ $\quad a(1)^2+b(1)+c=0$
$f(c_2)=0$ $\quad a(2)^2+b(2)+c=0$
$f(c_3)=0$ $\quad a(1)^2+b(1)+c=0$

$$\Rightarrow \left[\begin{array}{ccc|c} a & b & c & 0 \\ 4a & 2b & c & 0 \\ a & b & c & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \end{array} \underset{R_3-R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2-R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1-R_2}{\sim} \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-2a+c=0$
$3a+b=0$
$0a=0$
Let $a=s$, $s\in F$

$\Rightarrow$ $c=2s$, $b=-3s$, $a=s$ $\Rightarrow sp\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right)$ $\quad ker(T_{\vec{c}})=sp(2-3x+x^2)$

3. Prove $T_{\vec{c}}$ is a L.T., $\vec{c}=[c_0,\dots,c_n]$, no c_i is equal to another
Let $f,g\in P_n$, $r\in F$

$$T_{\vec{c}}(f+rg)=\begin{bmatrix} (f+rg)(c_0) \\ \vdots \\ (f+rg)(c_n) \end{bmatrix} = \begin{bmatrix} f(c_0) \\ \vdots \\ f(c_n) \end{bmatrix} + r\begin{bmatrix} g(c_0) \\ \vdots \\ g(c_n) \end{bmatrix}$$ Same process as step 1

$$= T_{\vec{c}}(f)+rT_{\vec{c}}(g)$$
$\therefore T_{\vec{c}}$ is a LT

Prove $C=\{$set of $\vec{c}\in\mathbb{R}^{n+1}$ where no c_i is equal to another$\}$
We know $C=\{\vec{c}\in\mathbb{R}^{n+1} : T_{\vec{c}}$ is an iso.$\}$
Prove $T_{\vec{c}}$ is injective for $\vec{c}\in\mathbb{R}^{n+1}$, where no c_i is equal to another
Find $ker(T_{\vec{c}})$, let $f\in ker(T_{\vec{c}})$ be arbi.

$$T_{\vec{c}}(f)=\begin{bmatrix} f(c_0) \\ \vdots \\ f(c_n) \end{bmatrix}=\vec{0}$$ Since all c_i are different, this means there are $n+1$ distinct roots of f.

By def f is a n degree polynomial, meaning it can have a max of n roots, unless f is the 0 function.
$\therefore ker(T_{\vec{c}})=\{0$ func$\} \Leftrightarrow T_{\vec{c}}$ is injective

Since $\dim(P_n)=n+1=\dim(\mathbb{R}^{n+1})$, we can say injective $\Leftrightarrow$ surjective
Injective (1-1): $\forall f\in P_n, \exists! n\in\mathbb{R}^{n+1}$ st $T_{\vec{c}}(f)=n$
Since $\dim(P_n)=\dim(\mathbb{R}^{n+1})$ this also means
$\forall n\in\mathbb{R}^{n+1}, \exists! f\in P_n$ st $T_{\vec{c}}(f)=n$, which is the def of onto
$\therefore T_{\vec{c}}$ is surjective
Thus $T_{\vec{c}}$ is iso. iff $\vec{c}\in\mathbb{R}^{n+1}$ st no c_i is equal to another ∎

Problem 5

2 flaws are in the induction step

First: Arbitrary $n \geq m$

The induction step assumes for any $n \in \mathbb{N}$, a set of length n is l.i.
However for $n > m$ ($m = \dim(\mathbb{R}^m)$), this is clearly impossible as a l.i. set of length m is already a spanning set of $\mathbb{R}^m$. Any $v \in \mathbb{R}^m$ is dependant on a basis (l.i. spanning set).

Similar case with $n = m$, except it deals with the outcome of the IH. A set of length $n+1 > m$ cannot be l.i.
So the induction hypothesis does not hold

Second: $n < m$

Lets say 2 sets $\{v_1, \dots, v_n\}$, $\{v_2, \dots, v_{n+1}\}$ are l.i. this still doesn't make $\{v_1, \dots, v_{n+1}\}$ l.i.

Example. Let $m = 3$

$\{(1,0,0), (0,1,0)\}$, $\{(0,1,0), (2,0,0)\} \subseteq \mathbb{R}^3$
These sets are clearly l.i. and differ by one element.
$v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (2,0,0)$

$\{v_1, v_2\}$ and $\{v_2, v_3\}$ are l.i. but $\{v_1, v_2, v_3\}$ clearly isn't bc. $v_3 = 2v_1$
So the conclusion of the induction step is also invalid.