MATB24 TUTORIAL PROBLEMS 7

KEY WORDS: inner product space, orthogonal decomposition, orthogonal projection, orthogonal complement

RELEVANT SECTIONS IN THE TEXTBOOK: 6.2 6.3 6.5 FB or 6.B 6.C SA

WARM-UP:

Write down a complete definition or a complete mathematical characterization

(1) Orthogonal decomposition of a vector $\vec{u}$ in an inner product space V onto a vector v
(2) orthogonal complement of a subspace W of an inner product space V
(3) An orthogonal basis for a vector space W
(4) An orthonormal basis for a vector space W

A:

(1) Let V_1 be the subspace of $\mathbb{R}^4$ given by

$$x_1 - x_2 - 2x_3 = 0,$$
$$x_2 + x_3 - 2x_4 = 0.$$

Find an orthonormal basis of V_1.

(2) Let V_2 be the subspace of $\mathbb{R}^4$ given by

$$x_1 + x_2 - x_3 - 2x_4 = 0.$$

Find an orthonormal basis of V_2.

(3) Let $\vec{w}$ be the vector

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \in \mathbb{R}^4.$$

Find the orthogonal projections of $\vec{w}$ onto the subspaces V_1, V_2.

B: Let W be a subspace of an inner product space V. We proved (or will prove) in class that for every $\vec{x} \in V$ there is a unique vector $\text{proj}_W\vec{x} \in V$ such that $\vec{x} - \text{proj}_W\vec{x}$ is orthogonal to W. The vector $\text{proj}_V\vec{x}$ is called the *orthogonal projection* of $\vec{x}$ onto W, and we found a formula for it: if $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis of W, then

$$\text{proj}_V(\vec{x}) = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \vec{u}_i.$$

(1) Prove that the transformation $\text{proj}_V : V \to V$ is linear.
(2) What is the kernel of proj_V?
(3) Write the rank nullity equation for this transformation. Is it familiar?
(4) Suppose $V = \mathbb{R}^m$. Since the orthogonal projection $\text{proj}_W : \mathbb{R}^m \to \mathbb{R}^m$ is linear, is there an $m \times m$ matrix P such that $\text{proj}_V(\vec{x}) = P\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$? [1]

C: Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

[1] we will do this in class

(1) Find a basis for $V^\perp$.
(2) Find the standard matrix of proj_V.
(3) Find the kernel of proj_V. How does this compare to $V^\perp$?
(4) What is $\dim V^\perp + \dim V$? How can you see this an instance of rank-nullity?
(5) Explain why the union of an orthonormal basis for V and one for $V^\perp$ are a basis for $\mathbb{R}^3$. What is the matrix of π_V in this basis?

D:

Let V and W be subspaces of coordinate vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, and suppose that $T : V \to W$ is an isomorphism.

(1) If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_r\}$ is a basis of V, is the set $\{T(\vec{b}_1), \dots, T(\vec{b}_r)\}$ necessarily a basis of W?
(2) If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_r\}$ is an orthonormal basis of V, is the set $\{T(\vec{b}_1), \dots, T(\vec{b}_r)\}$ necessarily an orthonormal basis of W?
(3) Which $n \times n$ matrices A do you expect to have the property that for every orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_n\}$ of $\mathbb{R}^n$, the set $\{A\vec{u}_1, \dots, A\vec{u}_n\}$ is also an orthonormal basis of $\mathbb{R}^n$? Can you come up with a few examples that works and a few that doesn't? [2]

[2]we will revisit this question in class soon