Q5

1. A set A of vectors is orthogonal on V iff $\forall v,u \in A;\ v \neq u \Rightarrow \langle v,u \rangle = 0$ using inner product defined on V

2. This isn't possible, we know orthogonal vectors are l.i. of each other, and the max # of l.i. vectors in $V = \dim(V)$

So $\dim(\mathbb{R}^4) = 4$ is the max # of l.i. vectors in $\mathbb{R}^4$.
$\therefore$ There isn't a orthogonal set of 5 vectors in $\mathbb{R}^4$

3. $V = \mathbb{R}^{m\times n}$ $\langle A,B \rangle = tr(A^T B)$

Prove $\langle A,A \rangle \geq 0$ and $= 0$ only if $A = \vec{0}$

$$\langle A,A \rangle = tr(A^T A)$$
$$= \sum_{i=1}^{m} b_i \cdot b_i \qquad \text{Let } b_i \text{ be the ith col of } A$$
$$= \sum \|b_i\| \geq 0 \quad \text{bc} \quad b_i \cdot b_i \geq 0$$

If $A = 0$ matrix, $tr(A^T A) = tr([0]) = 0$
If $tr(A^T A) = 0$, we know $\sum_{i=1}^{n} \|b_i\| = 0$, and $b_i \cdot b_i = 0$ iff $b_i = \vec{0}$
So all $b_i = \vec{0}$, thus $A = [0]$

Prove $\langle A+rB, C \rangle = \langle A,C \rangle + r\langle B,C \rangle$

$$\langle A+rB, C \rangle = tr((A+rB)^T C)$$
$$= tr(A^T C + rB^T C)$$
$$= tr(A^T C) + rt(B^T C)$$
$$= \langle A,C \rangle + r\langle B,C \rangle$$

Prove $\langle A,B \rangle = \langle B,A \rangle$

$$\langle A,B \rangle = tr(A^T B) \quad \text{By trace and transpose properties}$$
$$= tr((A^T B)^T) \quad \text{Thus } V \text{ is an inner product space}$$
$$= tr(B^T A)$$
$$= \langle B,A \rangle$$