W7 Q1

$u = u(x,y)$ $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \left(\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial x}{\partial \theta}\right)\right) + \left(\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial y}{\partial \theta}\right)\right)$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y \partial x}(r\cos\theta)$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial y}\right)\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial^2 u}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \left((u_{yx}\, r\cos\theta - u_{xx}\, r\sin\theta)(-r\sin\theta) + \frac{\partial u}{\partial x}(-r\cos\theta)\right)$$

$$+ \left((u_{xy}(-r\sin\theta) + u_{yy}\, r\cos\theta)\, r\cos\theta + \frac{\partial u}{\partial y}(-r\sin\theta)\right)$$

$$= u_{xx}\, r^2\sin^2\theta - u_{yx}\, r^2\cos\theta\sin\theta - u_x\, r\cos\theta - u_{xy}\, r^2\sin\theta\cos\theta + u_{yy}\, r^2\cos^2\theta - u_y\, r\sin\theta$$