MATB24 TUTORIAL PROBLEMS 6SOLUTIONS

KEY WORDS: inner product, inner product space, dot product, orthogonal complement, orthogonal projection

RELEVANT SECTIONS IN THE TEXTBOOK: 6.1, 6.2, 3.5FB or 6.A, 6.B SA

WARM-UP:

Write down a complete definition or a complete mathematical characterization for the following terms.

(1) An inner product on a vector apace V
(2) An inner product space
(3) Orthogonal decomposition of a vector $\vec{u}$ in an inner product space V onto another vector v
(4) An orthonormal set of vectors
(5) An orthogonal set of vectors
(6) An orthogonal basis for a vector space W
(7) An orthonormal basis for a vector space W

A: Which of the following are examples of inner product spaces? Explain why or why not.

(1) V= the space of continuous functions from $[0, 1]$ to $\mathbb{R}$.
$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$
(2) V= the space of polynomials in the variable t of degree less than or equal to 3.
$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$
(3) V= the space of infinite sequences $a_1, a_2, \ldots$.
$$\langle f, g\rangle = \sum_i a_i b_i$$
(4) V= the space of infinite *bounded* sequences $a_1, a_2, \ldots$.
$$\langle f, g\rangle = \sum_i \frac{a_i b_i}{2^i}.$$
(5) $V = \mathbb{R}^{m\times n}$ = the space of all $m \times n$ matrices.
$$\langle A, B\rangle = \text{tr}(A + B).$$
(6) $V = \mathbb{R}^{m\times n}$ = the space of all $m \times n$ matrices.
$$\langle A, B\rangle = \text{tr}(A^T B).$$

Solution.

(1) Properties of integration make most of the properties easy to verify. Suppose $\int_0^1 f(t)^2 dt = 0$. Since $f(t)^2 \geq 0$, this is true iff $f = 0$. This verifies positive-definiteness.
(2) This is a subspace of (1) so it inherits being an inner product space.

(3) Note that with $a_i = i$ and $b_i = 1$, $\sum_i a_i b_i = \sum_i i = \infty$ which is not a real number, so this is not an inner product space.
(4) Since $\sum_i \frac{a_i b_i}{2^i} \leq M \sum_i \frac{1}{2^i} \in \mathbb{R}$ for some constant M, therefore this operation is well-defined. Since there is always convergence, most of the properties are easy to verify. For positive-definiteness, $\sum_i \frac{a_i^2}{2^i} = 0$ implies $a_i = 0$ for all i.
(5) Positive-definiteness fails: Choose A to have zeros on its diagonal and ones everywhere else, then $A \neq O$ and $\langle A, A \rangle = \mathrm{tr}(2A) = 0$.
(6) Using linearity of taking transpose and trace most of the properties are easy to verify. For positive-definiteness, $\langle A, A \rangle = \mathrm{tr}(A^T A) = \sum_{i,j} a_{ij}^2 = 0$ implies $a_{ij} = 0$ and so $A = O$.

B: Let $\vec{v}, \vec{w}$ be in $\mathbb{R}^2$. Consider

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{w} \tag{1}$$

(1) Find an explicit formula for (1) in terms of the components of $\vec{v}$ and $\vec{w}$.
(2) Show that (1) is an inner product on $\mathbb{R}^2$ which is different from the dot product.
(3) Can you come up with other such examples ?

Solution.
(1) Let $\vec{v} = [x, y]$ and $\vec{w} = [u, v]$. Then, $\langle \vec{v}, \vec{w} \rangle = [x, y]^T [u + v, u + 2v] = x(u + v) + y(u + 2v)$.
(2) From (1) it is seen that this is not the same as the dot product. To check positive-definiteness, suppose $0 = \langle \vec{v}, \vec{v} \rangle = x(x+y)+y(x+2y) = x^2+2xy+2y^2$. Viewing this as an equation in x, we find the discriminate $(2y)^2 - 4(2y^2) = -4y^2$. Since we know x is a real solution to $0 = x^2 + 2xy + 2y^2$, we are forced to have $-4y^2 = 0$ which implies $y = 0$. Substituting this into our equation yields $0 = x^2$ and hence $x = 0$.
(3) Any symmetric positive definite matrix works. For example,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{2}$$

C: In this question, you will prove the following theorem

Theorem 1. *Let V be an inner product space vector space and let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be an orthonormal basis of V. Then for every $v \in V$,*

$$v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \cdots + \langle v, u_n \rangle u_n$$

(1) Suppose $v \in V$ and $v = \sum_{i=1}^n c_i u_i$. Compute $\langle v, u_1 \rangle$.
(2) Use your answer in part one to compute c_1.
(3) Prove the theorem.
(4) Restate the theorem with the word orthonormal replaced with orthogonal. Make any other necessary changes in order to get a correct statement. X

Solution.
(1) $\langle v, u_1 \rangle = \langle \sum_{i=1}^n c_i u_i, u_1 \rangle = \sum_{i-1}^n c_i \langle u_i, u_1 \rangle = c_1 \langle u_1, u_1 \rangle = c_1$

(2) $c_1 = \frac{\langle v, u_1 \rangle}{1}$
(3) Similar computation shows $c_i = \frac{\langle v, u_i \rangle}{1}$, thus since the coefficient of left hand side and right hand side are the same, so these two vectors are equal.

(4) Let V be an inner product space vector space and let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be an orthonormal basis of V. Then for every $v \in V$,

$$v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 + \cdots + \frac{\langle v, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

- An inner product on $\mathbb{R}^2$ other than the dot product.
- An inner product on $\mathbb{R}^3$ other than the dot product.
- Two vectors in $\mathbb{R}^3$ that are orthogonal with respect to dot product but not with respect to your example of inner product.
- A vector in $\mathbb{R}^3$ with length one with respect to dot product and a different length with respect to your example of inner product.
- Two different orthogonal bases of $\mathbb{R}^2$.
- A vector in an inner product space V that is orthogonal to every other vector.
- 4 mutually orthogonal vectors in $\mathbb{R}^3$.

Solution.

(1) $(x_1, x_2)\dot{(}y_1, y_2) = 2x_1y_1 + 2x_2y_2$
(2) let $x = (x_1, x_2, x_3)$ $y = (y_1, y_2, y_3)$ be coloumn vectors, let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, then $x \cdot y = x^T A y$ is an innner product.
(3) $x = [1, 2, 0], y = [1, 0, 0]$ are orthogonal in the inner product define above but not orthogonal in standard inner product.
(4) $[1, 0, 0]$ has length 1 under standard inner product, but has length 2 under the inner product I defined.
(5) $\{[1, 0], [0, 1]\}, \{[-1, 0], [0, -1]\}$
(6) The only element that is orthogonal to every other vectors if and only if this element is zero vector
(7) Such example does not exist. Since 4 mutually orthogonal vectors must be independent, but there cannot exist a set of 4 independent vectors in $\mathbb{R}^3$