UNIVERSITY OF TORONTO
Department of Computer Science, Mathematics Statistics

MATA22S (winter) Final exam
Duration: 180 minutes

Examiners : Stanislav Balchev, Kaidi Ye

April,16,2020

Name: ______________________________

Student Number: ______________________________

---

1. (20 points) (a) (10 points) Let A, B be $n \times n$ matrices which share a common eigenvector. Show that $\det(AB - BA) = 0$.

   Solution:

   Let v be the common eigenvector, then $(AB - BA)v = ABv - BAv = A\lambda_1 v - B\lambda_2 v = \lambda_1\lambda_2 - \lambda_2\lambda_1 = 0$ hence v is a non-zero solution hence $AB - BA$ has determinant 0.

   (b) (10 points) Let A be an $n \times n$ matrix. Prove that for sufficiently small $\epsilon > 0$ the matrix $A - \epsilon I$ is invertible.

   Solution:

   Let $p(x)$ be the characteristic polynomial of A. It is of degree n hence it has at most n distinct roots. Let $b = \min\{|\lambda| \ : \ \det A - \lambda I = 0, \lambda \neq 0\}$. Then for any $0 < \epsilon < b$, $P(\epsilon) \neq 0$ that is $\det(A - \epsilon I) \neq 0$ hence $A - \epsilon I$ is invertible.

2. (20 points) Suppose that A is an $n \times n$ matrix such that the algebraic multiplicity of its eigenvalues equals their geometric multiplicities. If $A^3 = A^2 + A - I$ show that A is invertible and that A is its own inverse.

Solution 1:
Let λ be any eigenvalue of A. Then for any corresponding eigenvector v_λ we have that

$$A^3 v_\lambda = A^2 v_\lambda + A v_\lambda - I v_\lambda$$

that is

$$(\lambda^3 - \lambda^2 - \lambda + 1) v_\lambda = 0$$

and therefore any eigenvalue satisfies $\lambda^3 - \lambda^2 - \lambda + 1 = 0$. Since this is equivalent to $(\lambda - 1)^2(\lambda + 1) = 0$ we conclude that the only eigenvalues are ± 1. Now let n_1, n_{-1} be their algebraic multiplicities. Since $n_1 + n_{-1} = n$ and their geometric multiplicities are the same we get that we have n linearly independent eigenvectors, hence A is diagonalizable. Then $A = P^{-1} D_A P$ for some $D_A = [e_1, , , e_{n_1}, -e_{n_1+1}, ..., -e_n]$. Then $A^2 = P^{-1} D_A P P^{-1} D_A P = P^{-1} D_A D_A P = I$ since $D_A D_A = [e_1, ..., e_n] = I$. This shows that A is diagonalizable and is its own inverse. Note: Since the characteristic polynomial of A is of degree n one cannot argue trivially that since 0 is not a root of the degree 3 polynomial $\lambda^3 - \lambda^2 - \lambda + 1$ then 0 cannot be an eigenvalue of A hence A is invertible.

Solution 2:
Rewrite $A^3 = A^2 + A - I$ as $(A - I)^2 (A - I) = 0$. This implies the eigenvalues of A just 1 and/or -1. If you want to use the fact that we covered in class " Every symmetric matrix is diagonalizable.",then there is no defective eigenvalue so this means that $(A - I)(A + I) = 0$, this writes $A^2 = I$.

3. (20 points) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined by

$$T(x_1, x_2, ...x_n) = (x_2 - x_1, x_3 - x_2, ..., x_1 - x_n)$$

(a) (6 points) Is T invertible ? Explain.

Solution:
It's not invertible. It sends all constant vector $v = [c, c, ...c]$ to $\vec{0}$. Thus, T has a nontrivial null space and is not invertible.

(b) (14 points) Find the eigenvalues of T.

Solution:
Let A be the matrix representation of T with respect to the standard basis of $\mathbb{R}^n$. The eigenvalues satisfy $Av = \lambda v$. Writing this relation in terms of components gives

$$u_2 - u_1 = \lambda u_1$$

or

$$u_2 = (1 + \lambda)u_1.$$

In general, $u_{j+1} = (1 + \lambda)u_j, j = 1, ..., n - 1$, and $u_1 = (1 + \lambda)u_n$. Thus,

$$u_1 = (1 + \lambda)u_n = (1 + \lambda)^2 u_{n-1} = ... = (1 + \lambda)^n u_1.$$

This implies that $(1 + \lambda)^n = 1$. Thus, the eigenvalues have the form $\lambda = \mu - 1$ where μ is any of the nth roots of unity.

4. (20 points) Let $A, B \in M_{n\times n}(\mathbb{R})$ such that A is real symmetric with

$$\vec{x}^T A\vec{x} > 0, \vec{x} \in \{\mathbb{R}^n - 0\}$$

and B is skew symmetric matrix.

(a) (10 points) Show that $det(A + B) \neq 0$.

Solution:
For B, let $x \in \mathbb{R}^n$, notice that

$$x^T Bx = (x^T Bx)^T = (Bx)^T x = x^T(-B)x = -x^T Bx.$$

Thus, $x^T Bx = 0$.
Consider

$$(A + B)x = 0$$

Then, multiplying on the left by x^T, this implies

$$x^T(A + B)x = x^T Ax + x^T Bx = x^T Ax = 0.$$

Since A is positive definite, $x^T Ax = 0$ implies $x = 0$. We conclude that $A + B$ is invertible.

(b) (10 points) Show that A^{-1} has the property

$$\vec{x}^T A^{-1}\vec{x} > 0, \vec{x} \in \{\mathbb{R}^n - 0\}$$

Solution:
All eigenvalues of A^{-1} are of the form $\frac{1}{\lambda}$, where λ is an eigenvalue of A. Since each eigenvalues of A is positive, then $\frac{1}{\lambda}$ is positive as well. Thus A^{-1} is positive definite.

5. (20 points) Let V be a real vector space of all real sequences $a_{i}{}_{i=1}^{\infty} = (a_1, ...)$. Let U be a subspace of V defined by

$$U = \{a_{i}{}_{i=1}^{\infty} \in V | a_{n+2} = 2a_{n+1} + 3a_n, n = 1, 2, ...\}.$$

Let T be the linear transformation from U to U defined by

$$T((a_1, a_2, ...)) = (a_2, a_3, ...).$$

Find any one of its eigenvalues and it's corresponding eigenvector of T.

Solution:
Note that each sequence $a_{i}{}_{i=1}^{\infty}$ in U is determined by the first two number a_1, a_2 since the rests can be obtained from the linear recurrence relation $a_{n+2} = 2a_{n+1} + 3a_n$. Thus, for example, we take $(a_1, a_2) = (1, 0), (0, 1)$ and obtain a basis $B = u_1, u_2$ of U, where

$$u_1 = (1, 0, 3, 6, 21, ...)$$

$$u_2 = (0, 1, 2, 7, 20, ...).$$

We find the matrix A of T with respect to the basis B. We have

$$T(u_1) = (0, 3, 6, 21, ...) = 0(u_1) + 3(u_2)$$

$$T(u_1) = (1, 2, 7, 20, ...) = 1(u_1) + 2(u_2).$$

Thus, the matrix representation of T is

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

The characteristic polynomial for A is

$$p(t) = det(A - tI) = (t + 1)(t - 3).$$

Thus the eigenvalues are $t = -1, 3$. Let's find the eigenvectors corresponding to $t = -1$. Like question 3, let v be the eigenvector for $t = -1$, we have

$$T(v) = (-1)v$$

$$T((a_1, a_2, ...)) = -(a_1, a_2, ...)$$

or

$$(a_2, a_3, ...) = -(a_1, a_2, ...).$$

This yields the relation

$$a_{n+1} = -a_n$$

for $n = 1, 2, ...$ Hence the sequence $a_{i}{}_{i=1}^{\infty}$ is a geometric progression with initial value a_1 and ratio -1. Therefore, the sequence $a_{i}{}_{i=1}^{\infty}$ such that

$$a_i = (-1)^{i-1} a_1$$

are eigenvectors corresponding to the eigenvalue -1.
Similarly, eigenvectors corresponding to the eigenvalue $t = 3$ are geometric progression with initial value a_1 and ratio 3, thus

$$a_i = (3)^{i-1} a_1.$$