MATB24 GRADED PROBLEMS 2, DUE Thursday Oct 8, 11:59am

General instructions:

- You should submit your work on Quercus. The only accepted format is PDF.
- Do not wait until last minute to avoid technical difficulties.
- There is a one point penalty for late submissions within 12 hours of the due date.
- You are encouraged to work in groups, ask question on piazza or in office hours. But you should write your homework individually in your own words. You can get help from me, your TA or your peers, but you should write your solution on your own.
- Unless otherwise stated in all questions you should fully justify your answer.
- Your TA will grade a randomly selected subset of the questions in each homework and your grade will be only based on the graded questions.

Reading assignment:

It is assumed that you read at least one of the reading options below

- Sec 3.2,3.4 from Fraleigh–Beauregard
- Sec 2.A-C, 3.A-B from Axler

Problem 1. *True or False? Justify your answer.*

(1) *Let V be a vector space, with $\dim(V) = m$. If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a linearly independent set in V, then $n \leq m$.*

(2) *Let V and W be vector spaces, and suppose that $T : V \to W$ is a linear transformation. If there are vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in V such that the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)$ span W, then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span V.*

(3) *Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in a vector space V. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a linearly independent set in V then $\vec{v}_k \notin \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{k-1}\}$.*

(4) *Let $\mathcal{P}$ denote the vector space over $\mathbb{F}$ given by all polynomials in a variable x with coefficients in $\mathbb{F}$. Let $W \subseteq \mathcal{P}$ be the span of $\{x^n, \cdots, x^3, x^2, x\}$, and let $V \subseteq \mathcal{P}$ be the span of $\{x, x^2 + x, x^3 + x^2 + x, \cdots, x^n + \cdots + x^3 + x^2 + x\}$. Then $V = W$.*[1]

Problem 2. *Let U and W be subspaces of a finite dimensional vector space V such that $U \cap W = \{\vec{0}\}$. Define their sum $U + W := \{\boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U, \boldsymbol{w} \in W\}$.*

(1) *Prove that $U + W$ is a subspace of V.*

(2) *Let $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$ and $\mathcal{W} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_s\}$ be bases of U and W respectively. Prove that $\mathcal{U} \cup \mathcal{W}$ is a spanning set for $U + W$.*

(3) *Prove that $\mathcal{U} \cup \mathcal{W}$ is linearly independent.*

(4) *Conclude that $\dim(U + W) = \dim U + \dim W$.*

Problem 3. *Let $\mathbb{F}$ be a field with finitely many elements.*

(1) *Prove that a vector space V over $\mathbb{F}$ is finite-dimensional if and only if it has finitely many elements.*

(2) *Prove that if V is a finite-dimensional vector space over $\mathbb{F}$, it has $(\#\mathbb{F})^{\dim(V)}$ elements.*

(3) *Prove that every knot diagram has 3^n tricolorings for some $n \geq 1$. (You can use results from Graded Homework 1 without proof.)*

[1] TUT 2 part B is helpful.

Problem 4. *Let $\mathcal{P}_n$ be the vector space over $\mathbb{F}$ given by all polynomials in a variable x with coefficients in $\mathbb{F}$ of degree at most n. For each $\vec{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^{n+1}$, define the mapping $T_{\vec{c}} : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by*

$$T_{\vec{c}}(f) = \begin{bmatrix} f(c_0) \\ \vdots \\ f(c_n) \end{bmatrix}.$$

(1) *Show that $T_{\vec{c}}$ is linear for $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.*

(2) *Find $\ker(T_{\vec{c}})$ for $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.*

(3) *Describe the set $C = \{\vec{c} \in \mathbb{R}^{n+1} : T_{\vec{c}}$ is an isomorphism$\}$, and prove that your characterization is correct.* [2]

Problem 5 (Bonus:1 point added to the total of problem set for a complete answer)**.** *Find the flaws in the following inductive proof*[3] *(there are at least two flaws):*

Ridiculous Claim: *Any set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ of non-zero vectors in $\mathbb{R}^m$ is linearly independent.*

Proof: *We proceed by induction on the size of the set. Consider the base case in which we have a single vector $\boldsymbol{v}$. Since $\boldsymbol{v} \neq \boldsymbol{0}$, the set $\{\boldsymbol{v}\}$ is linearly independent. Now assume that for a fixed $n \in \mathbb{N}$, any set of n vectors is linearly independent. Consider any set of $n+1$ vectors, say $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1}\}$. By the inductive hypothesis, the sets $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and $\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_{n+1}\}$ are linearly independent. Therefore, the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1}\}$ is also linearly independent. By induction, the claim thus holds for a set of non-zero vectors of any size.*

[2] Hint: What happens if $c_i = c_j$?

[3] See the notes in *Writing mathematics* page on Quercus if you need a refresher on induction.