1 Useful Formulas

1.1 General

- Arithmetic Series: $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

- Geometric Series: $\begin{cases} \sum_{i=0}^{n} a^i = \dfrac{1 - a^{n+1}}{1 - a}, & \forall a \in \mathbb{R} \\ \sum_{i=0}^{\infty} a^i = \dfrac{1}{1 - a}, & \forall |a| < 1 \end{cases}$

- Binomial Thm: $\begin{cases} (x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \\ \forall \, x, y \in \mathbb{R}, \ n \in \mathbb{N} \end{cases}$

- Exponential Fn: $\sum_{i=0}^{\infty} \dfrac{a^i}{i!} = e^a, \ \ \forall a \in \mathbb{R}$

- Gamma Fn: $\begin{cases} \Gamma(a) = \int_{0}^{\infty} x^{a-1} e^{-x} dx, \ a > 0 \\ \Gamma(n) = (n-1)!, \ n \in \mathbb{N} \end{cases}$

1.2 Probability

- Law of Total Probability: $\forall$ partition $\{A_i\}_{i=1}^{n}$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

- Bayes Rule: $\forall$ partition $\{A_i\}_{i=1}^{n}$

$$P(A_k | B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

- Multiplication Rule: $P\left(\bigcap_{i=1}^{n} A_i \right) =$

$$P(A_1)P(A_2|A_1) \times \cdots \times P(A_n|A_1, \ldots, A_{n-1})$$

- Inclusion-Exclusion Principle:

$$P\left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) +$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_j)$$

$$\vdots$$

$$+ (-1)^{n-1} P\left(A_1 \cap \cdots \cap A_n \right)$$

1.3 Calculus

- Change of Variables:

$$\int_{a}^{b} f(x) dx \overset{(x = h(u))}{=} \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u)) h'(u) du$$

- Fundamental Theorem of Calculus:

$$F'(x) = f(x) \Rightarrow \int_{a}^{b} f(x) dx = F(b) - F(a)$$

$$F(x) = \int_{a}^{x} f(u) du \Rightarrow F'(x) = f(x)$$

- Integration by Parts:

$$\int_{a}^{b} f(x) g'(x) dx = [f(x)g(x)]_{a}^{b} - \int_{a}^{b} f'(x) g(x) dx$$

1.4 Multivariate Distributions

- Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

- Conditional PDF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

- 2D PDF method, for $(Z, W) = h(X, Y) \Leftrightarrow$

$$\Leftrightarrow (X, Y) = h^{-1}(Z, W)$$

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(h^{-1}(z, w))}{|J(h^{-1}(z, w))|},$$

$$\text{where } |J| = \begin{vmatrix} dz/dx & dz/dy \\ dw/dx & dw/dy \end{vmatrix}$$

- Convolution method, for $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x) dx$$

- Order Statistics, for $X_i \sim^{iid} F(x)$

$$P(X_{(i)} \le x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}$$

1.5 Expectations

- Mean of Cont./Discr. RV X

$$E(X) = \mu = \int_{-\infty}^{\infty} x f_X(x) dx \bigg/ = \sum_{x} x p_X(x)$$

- Mean of function h of RV X

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x) dx \bigg/ = \sum_{x} h(x) p_X(x)$$

- Variance of RV

$$V(X) = \sigma^2 = E[(X - \mu)^2] = E(X^2) - [E(X)]^2$$

- Moment Generating Function of RV X

$$m_X(t) = E(e^{tX})$$

$$E(X^k) = m_X^{(k)}(0)$$

- Covariance of two RVs

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

- Expectation of Linear Function of RVs

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

- Variance of Linear Function of RVs

$$V\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 V(X_i) +$$
$$+ 2 \sum_{1 \le i < j \le n} a_i a_j Cov(X_i, X_j)$$

- Law of Total Expectation/Variance

$$E(X) = E(E(X|Y))$$
$$V(X) = E(V(X|Y)) + V(E(X|Y))$$

1.6 Inequalities

- Markov Inequality, for $X > 0$

$$P(X \ge a) \le E(X)/a$$

- Chebyshev Inequality

$$P(|X - \mu| \ge a) \le V(X)/a^2$$

- Cauchy-Schwarz Inequality

$$(E(XY))^2 \le E(X^2)E(Y^2)$$

- Jensen Inequality, for convex $g(\cdot)$

$$E(g(X)) \ge g(E(X))$$

1.7 Limit Theorems

- Weak Law of Large Numbers for i.i.d. X_i

$$(X_1 + \cdots + X_n)/n = \bar{X}_n \to^P \mu$$

- Central Limit Theorem for i.i.d. X_i

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \to^D N(0, 1)$$

1.8 Important Distributions

Name	PMF/PDF	CDF	Mean	Variance	MGF
Discrete Distributions					
Binomial(n, p)	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, \ldots, n$		np	$np(1-p)$	$(1 - p + pe^t)^n$
Geometric(p) (# trials)	$p(1-p)^{x-1}$ $x = 1, \ldots, \infty$	$1 - (1-p)^x$	$1/p$	$(1-p)/p^2$	$\frac{pe^t}{1-(1-p)e^t}$ $t < -\ln(1-p)$
Poisson(λ)	$e^{-\lambda} \lambda^x / x!$ $x = 0, \ldots, \infty$		λ	λ	$\exp\{\lambda(e^t - 1)\}$
NegBinom(r, p)	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$ $x = r, r+1, \ldots, \infty$		r/p	$r(1-p)/p^2$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$ $t < -\ln(1-p)$
HyperGeom $(n, M, N,)$	$\binom{M}{x}\binom{N-M}{n-x}/\binom{N}{n}$ $x = 0, \ldots, n,\ x \le M,$ $x \ge n - N + M$		nM/N	$n\frac{M}{N}\frac{(N-M)}{N}\frac{(N-n)}{(N-1)}$	
Continuous Distributions					
Uniform(ℓ, u)	$1/(u-\ell),\ \ x \in [\ell, u]$	$\frac{x-\ell}{u-\ell}$	$\frac{u+\ell}{2}$	$\frac{(u-\ell)^2}{12}$	$\frac{e^{tu} - e^{t\ell}}{t(u-\ell)}$
Exponential(λ)	$\lambda e^{-\lambda x},\ \ x \ge 0$	$1 - e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda - t},\ t < \lambda$
Normal(μ, σ^2)	$\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$ $x \in \mathbb{R}$		μ	σ^2	$\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$
Gamma(a, λ)	$\frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x},\ \ x \ge 0$		a/λ	a/λ^2	$\left(\frac{\lambda}{\lambda - t}\right)^a,\ t < \lambda$