University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2020

Assignment # 4

You are expected to work on this assignment prior to your tutorial during the week of February 3rd. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.7 (OMIT Thm 4.35); Chapter 5: Section 5.1 (OMIT Thm 5.4 - we never mix variables;*if we perform a u-subst. to a definite integral then our u integrand* ***must*** *have corresponding u integration limits if we keep our integral in definite form*; OMIT Example 4 for now - u-subst. "with algebra" will appear on A5).

HOMEWORK:

At the beginning of your TUTORIAL during the week of February 10th you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

1. Let $a, b \in \mathbb{R}$, $a < b$. Provide a complete and detailed proof of the following statement :

   If f is continuous on $[a, b]$ and define $F(x) = \int_a^x f(t)dt$, any $x \in [a, b]$, then $F'_-(b) = f(b)$. That is that the left-hand derivative of F at $x = b$ equals $f(b)$.

   Do not use FTOCI.

2. Textbook Section 4.7 - # 28, 32, 48 (note $x > 0$ for # 48).

3. Prove, or disprove (by providing a counter-example), the following statements :

(a) Suppose that f is continuous everywhere. Then $\frac{d}{dx}\left(\int_4^x f(t)dt\right) = \frac{d}{dx}\left(\int_{-2}^x f(t)dt\right)$.

(b) Suppose that f is continuous everywhere. Let $a \in \mathbb{R}$. Then $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \frac{d}{dx}\left(\int_a^x f(u)du\right)$.

(c) Let function g be continuous everywhere. Suppose that c is some constant. Then $\displaystyle\int_c^x g(x)dx$ equals an infinite family of antiderivatives of g.

4. For all $x \in \mathbb{R}^+$. Prove that there exists some constant c between $\ln(x)$ and x^2. That is, prove $\forall x \in \mathbb{R}^+,\ \exists c \in \mathbb{R}$ such that $c \in [\ln(x),\ x^2]$ or $c \in [x^2,\ \ln(x)]$.

5. Find the indicated derivatives of the following functions without integrating. Make sure to fully justify your work!

(a) Let $x > 0$. Define $H(x) = \displaystyle\int_{\ln(x)}^1 \sin^3(u)\ du$; $H''(x)$

(b) $\displaystyle\frac{d}{dx}\left(\int \frac{1}{4+t^2}\ dt\right)$

(c) Let $x > 0$. Define $G(x) = \displaystyle\int_{\sqrt{x}}^{2x} t\arctan(t)\ dt$; $G'(x)$

6. Evaluate the following integrals.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\ dx$ with the subst. $u = \sin(x)$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\ dx$ with the subst. $u = \cos(x)$

(c) $\displaystyle\int \frac{x}{\sqrt{3x^2+1}}dx$

(d) $\displaystyle\int \frac{e^x}{4-e^x}dx$

(e) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

(f) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{(\sin(x)+1)^{\frac{3}{2}}}{sec(x)}dx$

(g) Textbook Section 5.1 - # 18.

EXERCISES: You do not need to submit solutions to the following problems but you should make sure that you are able to answer them.

1. Textbook Section 4.7 - # 1(a)-(h), 17, 29, 37, 39, 40, 46, 41, 46, 43, 47, 50, 69 (MVT for Integrals).

2. Textbook Section 5.1 - # 1(c)(d)(f)-(h), 21-37, 39, 41, 43, 45 — You get better at integrating by practicing!

3. Find the indicated derivatives of the following functions. Make sure to fully justify your work.

   (a) $H(x) = \displaystyle\int_{\sqrt{2}}^{x} \frac{1}{1+t^2+e^t}\, dt$; $H''(x)$

   (b) Let $x > 0$. $H(x) = \displaystyle\int_{\sqrt{x}}^{2x} \arctan(t)\, dt$; $H'(x)$

   (c) Let $a, b \in \mathbb{R}$. $H(x) = \displaystyle\int_{a}^{b} \frac{x}{1+t^6}\, dt$; $H'(x)$

4. Let $x > 0$. Prove that the value of the following expression does <u>not</u> depend on x: $\displaystyle\int_0^x \frac{1}{1+t^4} dt + \frac{1}{3}\int_0^{\frac{1}{x^3}} \frac{1}{1+t^{\frac{4}{3}}} dt.$ Do NOT evaluate the integrals.

5. Let $a \in \mathbb{R}$. Suppose that f is continuous on $[-a, a]$. Prove the following statements. Use **only** the subst. rule and integration properties. Do not use FTOC I.

   (a) If f is an even function on $[-a, a]$ then $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$

   (b) If f is an odd function on $[-a, a]$ then $\displaystyle\int_{-a}^{a} f(x)dx = 0.$

   (c) Use the above properties to evaluate $\displaystyle\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4} dx.$

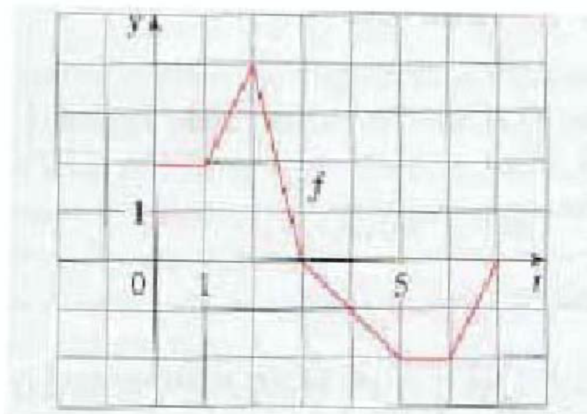
6. Let $a, b \in \mathbb{R}$, $a < b$. Let f be a function such that f' is continuous on $[a, b]$. Prove that $\displaystyle\int_a^b f(t)f'(t)dt = \frac{1}{2}\left(f^2(b) - f^2(a)\right).$

7. Let $a, b, c \in \mathbb{R}$. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(x)dx.$$

8. Evaluate $\displaystyle\int_0^1 \left(\frac{d}{dx}\left(\int_0^1 e^{x^2}dx\right)\right)dx.$

9. Let $g(x) = \int_0^x f(t)dt$ where f is the function whose graph is shown below.



(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$ and $g(6)$.

(b) On what interval is g increasing?

(c) Where does g have a maximum value?

(d) Sketch a rough graph of g.

10. Find $h'(2)$ for $h(x) = \left(\displaystyle\int_1^x \frac{1}{2+\sin^2(t)}dt\right)^3$. Make sure to justify your work. *(Hint : Do not evaluate these integrals.)*

11. Find the derivative of $H(x) = \displaystyle\int_{15}^x \left(\int_8^u \frac{1}{t^4+1}\,dt\right)\,du.$

12. Suppose that g is continuous everywhere. Suppose that g satisfies the equation

$$\int_0^x e^t g(t)dt = \frac{x}{x^2+1}.$$

Find an explicit formula for $g(x)$. Make sure to fully justify your answer.

13. Prove that the value of the $\displaystyle\int_{-\cos(x)}^{\sin(x)} \frac{1}{\sqrt{1-t^2}}dt$, $x \in (0, \frac{\pi}{2})$ does not depend on x.

14. Suppose that f is a continuous function and that for $x > 0$,

$$\int_0^x tf(t)dt = x\sin(x) + \cos(x) - 1.$$

   (a) Find $f(\pi)$.

   (b) Calculate $f'(x)$.

15. On what interval is the curve $y = \displaystyle\int_0^x \frac{t^2}{t^2+t+2}dt$ concave down?

16. The natural logarithm may be defined as an area accumulation function. Namely, for $x > 0$ the natural logarithm function is defined by $\ln(x) = \displaystyle\int_1^x \frac{1}{t}dt$. Prove each of the following from Section 4.7 of your textbook using this new definition of $\ln(x)$. # 71-74.

17. Evaluate the following :

   (a) $\displaystyle\int \frac{x}{1+x^4}dx.$

   (b) $\displaystyle\int \sqrt{\cot(x)}\csc^2(x)dx.$

   (c) $\displaystyle\int_0^1 \frac{1+x}{3+x^2}dx.$

   (d) $\int_{-3}^{-1}\left((2x+5)^8 + \frac{2^x}{2^x+3}\right)dx.$

   (e) $\displaystyle\int \frac{x+e^{2x}}{x^2+e^{2x}}dx.$

   (f) $\displaystyle\int x\sin^3(x^2)\cos(x^2)dx.$

   (g) $\displaystyle\int \frac{g(x)g'(x)}{\sqrt{1+g^2(x)}}dx$, where $g'(x)$ is continuous.

   (h) $\int_0^{\frac{\pi}{2}} \cos(x)\sin^3(x)dx$. Use algebra to rewrite the integrand and use the u-substitution $u = \cos(x)$

   (i) $\displaystyle\int (6x^2+4x)(x^3+x^2)^{\frac{3}{2}}dx.$

   (j) $\displaystyle\int_0^2 \frac{3e^{3x}}{1+e^{3x}}dx.$

   (k) $\displaystyle\int \frac{e^{\sqrt{5x}}}{\sqrt{3x}}dx.$

(l) $\displaystyle\int_0^{\pi} \cos^2(x)\sin^5(x)dx.$

(m) $\displaystyle\int_3^{9} (y-6)^{301}dx.$

(n) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}dx.$

(o) $\displaystyle\int \tan^3(x)\sec^2(x)dx.$

(p) $\displaystyle\int \sqrt{\frac{1-x}{1+x}}dx.$

All things are difficult before they are easy. — *Thomas Fuller*
