Facts

Fact: Addition and subtraction

Let A and B be matrices of the same size. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$.
   That is, it doesn't matter whether we compute $A+B$ first, or $B + C$ first.
3. $A + 0 = 0 + A = A$
4. $A - A = A + (-1)A = 0$

Let λ and μ (the Greek letter mu, pronounced "mew") be scalars. Then

5. $\lambda(A + B) = \lambda A + \lambda B$
6. $(\lambda + \mu)A = \lambda A + \mu A$
7. $\lambda(\mu A) = \lambda\mu A$
8. $1A = A$

Fact: Transpose

trans Let A and B be matrices, and λ a scalar. Then

1. $(A^T)^T = A$
2. $(\lambda A)^T = \lambda A^T$
3. $(AB)^T = B^T A^T$, when AB and $A^T B^T$ are defined. Notice the order has swapped.
4. $(A + B)^T = A^T + B^T$, when $A + B$ is defined.

Fact: Trace

trace Let A and B be a square matrices of the same size, and λ a scalar. Then

1. $tr(\lambda A + B) = \lambda tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$, even if $AB \neq BA$
3. $tr(A^T) = tr(A)$

Fact: Matrix mulitiplication

mult Let A be an $m \times n$ matrix, B and C be $n \times k$ matrices, and D a $k \times l$ matrix. Then

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$ (notice the order)
3. $A(\lambda B) = (\lambda A)B = \lambda AB$, for λ a scalar
4. $A0 = 0A = 0$, for 0 the appropriate size 0 matrix

Fact: Inverse and product

Let A and B be invertible matrices of the same size. Then the product AB is invertible also, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Note the change of order.

Fact: Inverse and transpose

If A is an invertible matrix, then A^T is invertible also, and

$$(A^T)^{-1} = (A^{-1})^T$$

Fact: Laws of powers

powers Let A be a square matrix and k, $r > 0$ numbers. Then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

If A is invertible then the above identities hold for negative powers also.

Fact

If E is the elementary matrix obtained by applying the elementary row operation O to the identity. Let A be another matrix. The product EA is the matrix obtained from A by applying the elementary row operation O.

Fact

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Fact

Let A be an $n \times n$ matrix. The following are equivalent (that is, either all of them are true, or all of them are false):

1. A is invertible.
2. The matrix equation $A\mathbf{x} = 0$ has the unique solution $\mathbf{x} = 0$ (where 0 is the $n \times 1$ zero matrix).
3. The RREF of A is I_n.
4. A may be represented as a product of elementary matrices. That is $A = E_1E_2\cdots E_n$, where E_i is an elementary matrix.
5. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ (and is therefore consistent).

Fact

Let A and B be square matrices of the same size. Then

1. if $AB = I$, then $B = A^{-1}$
2. if $BA = I$, then $B = A^{-1}$
3. if $B = A^{-1}$, then $A = B^{-1}$ also
4. if AB is invertible then A and B are invertible also

Fact

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then D is invertible if and only if every entry $d_i \neq 0$.
If D is invertible then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Fact

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

be a diagonal matrix. Then

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

for any integer k.

Fact: Properties of triangular matrices

1. The transpose of an upper triangular matrix is lower triangular, and vice versa.
2. The product of two upper triangular matrices is upper triangular. The product of two lower triangular matrices is lower triangular.
3. A triangular matrix is invertible if and only if its diagonal entries are non-zero.
4. The inverse of an upper triangular matrix is upper triangular. The inverse of a lower triangular matrix is lower triangular.

Fact: Products of triangular matrices

Let A and B be upper triangular matrices (or both lower triangular matrices). Then

$$(AB)_{ii} = (BA)_{ii} = (A)_{ii}\,(B)_{ii}$$

Fact: Properties of symmetric matrices

Let A and B be symmetric matrices of the same size, and λ a scalar. Then

1. A^T is symmetric
2. $A+B$ and $A-B$ are symmetric
3. λA is symmetric

Fact

If A is invertible and symmetric, then A^{-1} is symmetric also.

Fact

Let M be a matrix of any size.

1. MM^T and M^TM are symmetric
2. If M is invertible, then MM^T and M^TM are invertible also

Fact: Determinant of a triangular matrix

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be a triangular square matrix (either upper or lower). Then

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

i.e. it is the product of the diagonal entries.

Fact: Determinant of the transpose

trandet Let A be a square matrix. Then

$$det(A) = det(A^T)$$

Fact

Let A be a square matrix. If A has a row or column of 0's, then $det(A) = 0$.

Fact: Determinants via row reduction

erodet Let A and B be square matrices of the same size, related by exactly one elementary row operation.

1. If B is obtained from A by swapping two rows, then
$$det(B) = -det(A)$$
E.g.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
2. If B is obtained from A by multiplying a row by the scalar λ, then
$$det(B) = \lambda det(A)$$
E.g.
$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
3. If B is obtained from A by adding a multiple of one row to another row, then
$$det(B) = det(A)$$
E.g.
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{11} + a_{21} & a_{12} + a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Fact

Let A be a square matrix. If A has a column which is a scalar multiple of another column, or a row which is a scalar multiple of another row. Then $det(A) = 0$.

Fact: Determinants of elementary matrices

Let E be an elementary $n \times n$ matrix. Then

1. If E is obtained from I_n by swapping a row, then $det(E) = -1$.
2. If E is obtained from I_n by multiplying a row by a scalar λ, then $det(E) = \lambda$.
3. If E is obtained from I_n by adding a multiple of one row to another, then $det(E) = 1$.

Fact: Determinant and scalar multiplication

Let A be an $n \times n$ matrix, and λ a scalar. Then $det(\lambda A) = \lambda^n \det(A)$.

Fact

Let A and B be square matrices of the same size. Assume that A is equal to B except in the k-th row.
Let C be the matrix with rows equal to those of A (and B), except in the k-th row: let the k-th row of C be equal to the sum of the k-th row of A and the k-th row of B.
Then $det(C) = det(A) + det(B)$.

Fact: Determinant of a product

Let A and B be square matrices of the same size. Then
$$det(AB) = det(A)det(B).$$

Fact

detinv Let A be a square matrix. Then A is invertible if and only if $det(A) \neq 0$.

Fact

Let A be an invertible matrix. Then
$$det(A^{-1}) = \frac{1}{det A}.$$

Fact: The inverse via the adjoint

Let A be a square matrix. If $det(A) \neq 0$, then
$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Fact: Characteristic equation

Let A be an $n \times n$ matrix. The scalar λ is an eigenvalue of A if and only if it is a solution to the equation
$$det(\lambda I - A) = 0$$
(for I the identity matrix).
The equation
$$det(\lambda I - A) = 0$$
is known as the characteristic equation of A.

Fact: Eigenvalues of a triangular matrix

Let A be a triangular matrix. Then the eigenvalues of A are listed on its main diagonal.

Fact

A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

Fact

diagform Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}_i$ be the eigenvector associated to the eigenvalue λ_i. Let

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

be the matrix of eigenvectors.
Then P diagonalizes A so that $A = PDP^{-1}$, where the diagonal matrix D has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

That is, it is the matrix with the eigenvalues of A on the diagonal, and 0's elsewhere.

Fact

Let A be a square matrix and k a positive integer. If λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k, with associated eigenvector $\mathbf{x}$.

Fact: Trivial solution

Let

$$\mathbf{y}' = A\mathbf{y}$$

be a system of differential equations. Then the vector $\mathbf{y} = 0$ is solution, known as the trivial solution.

Fact

Let z be a complex number and $\overline{z}$ its conjugate. Then

$$z\overline{z} = \overline{z}z$$

is a real number.

Fact

modconj Let z be a complex number. Then

$$|z|^2 = z\overline{z}$$

Fact: Properties of the complex conjugate

Let z and w be complex numbers. Then

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = (\overline{z})(\overline{w})$
- $\overline{\dfrac{z}{w}} = \dfrac{\overline{z}}{\overline{w}}$
- $\overline{\overline{z}} = z$

Fact: Multiplication in polar form

polarmult Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers.
Then

$$zw = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Fact

Let z and w be complex numbers. Then

$$|zw| = |z||w|$$

and

$$Arg(zw) = Arg(z) + Arg(w)$$

Fact: Division in polar form

Let $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$, and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers.
Then

$$\frac{z}{w} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Fact

Let z and w be complex numbers. Then

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

and

$$Arg\left(\frac{z}{w}\right) = Arg(z) - Arg(w)$$

Fact: De Moivre's Formula

Let z be a complex number with $|z| = 1$. Therefore its polar form is

$$z = \cos(\theta) + i\sin(\theta)$$

For any positive integer n, we have

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

This equation is known as De Moivre's Formula.

Fact

complexroots Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number, and n a positive integer.
There are exactly n n-th roots of z, and they are given by

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$.

Fact

Let $z = r(\cos(\theta) + i\sin(\theta))$ be a complex number.
Then

$$z = re^{i\theta}$$

Fact

Let $z = re^{i\theta}$ be a complex number. Then

$$\mathrm{Re}(z) = r\cos(\theta)$$
$$\mathrm{Im}(z) = r\sin(\theta)$$

Fact

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and k, m scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
6. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
7. $k(m\mathbf{u}) = (km)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Fact: Properties of the norm

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$, and k a scalar. Then

- $||\mathbf{u}|| \geq 0$
- $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- $||k\mathbf{u}|| = |k|\,||\mathbf{u}||$

Fact: Distance between two points

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}n$. The distance between the endpoints of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$||\mathbf{u} - \mathbf{v}|| \tag{1}$$

Fact

Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{u} = ||\mathbf{u}||^2$$

Fact

cos Let $\mathbf{u}$ and $\mathbf{v}$ be vectors and θ the angle between them. Then

$$\mathbf{u} \bullet \mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos(\theta)$$

Fact: Properties of the dot product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and k a scalar. Then

- $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
- $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
- $k(\mathbf{u} \bullet \mathbf{v}) = (k\mathbf{u}) \bullet \mathbf{v}$
- $\mathbf{u} \bullet \mathbf{u} \geq 0$, and $\mathbf{u} \bullet \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Fact: The Cauchy-Schwarz Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$|\mathbf{u} \bullet \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$$

Fact: The Triangle Inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Fact: Parallelogram Rule

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right)$$

Fact

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then

$$\mathbf{u} \bullet \mathbf{v} = \frac{1}{4}||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4}||\mathbf{u} - \mathbf{v}||^2$$

Fact

Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbb{R}^n$. Then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Fact: Calculating distance

Let $ax + by + c = 0$ be a line in $\mathbb{R}^2$, and (x_0, y_0) a point. The shortest distance from (x_0, y_0) to the line is given by

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{2}$$

If $x + by + cz + d = 0$ is a plane in $\mathbb{R}^3$, and (x_0, y_0, z_0) is a point, then he shortest distance from (x_0, y_0, z_0) to the plane is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{3}$$

Fact

Let A be a an $m \times n$ matrix. An $n \times 1$ vector $\mathbf{x}$ is a solution to the equation

$$A\mathbf{x} = \mathbf{0}$$

if and only if $\mathbf{x}$ is orthogonal to every row vector of A.

Fact: Properties of the cross product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
4. $k\,(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
5. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Fact: Relation to other operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. That is
$$\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$$
2. $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - (\mathbf{u} \bullet \mathbf{v})^2$
3. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\,\mathbf{w}$

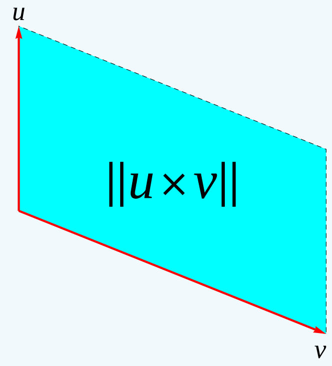
Fact

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. Their cross product may be computed

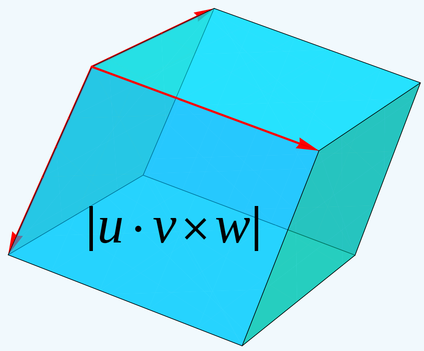
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Fact

1. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then $||\mathbf{u} \times \mathbf{v}||$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.



2. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\,|$ is the area of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.



Fact

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. If $|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\,| = 0$ then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie in a plane.

Fact

Let V be a vector space, $\mathbf{u} \in V$ and k a scalar. Then

1. $0\mathbf{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\,\mathbf{u} = -\mathbf{u}$
4. If $k\mathbf{u} = \mathbf{0}$ then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Fact: When is a subset a subspace?

Let V be a vector space. The subset $W \subset V$ is a subspace if and only if

1. W is closed under addition: if $\mathbf{u} \in W$ and $\mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$
2. W is closed under scalar multiplication: if $\mathbf{u} \in W$ then $k\mathbf{u} \in W$ for all scalars k

Fact: Spans are subspaces

Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$. Then span$(S)$ is a subspace of V.

Fact

Let A be an $m \times n$ matrix. The set of solutions to the equation

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $\mathbb{R}^n$.

Fact

Let V be a vector space and W_1, W_2 subspaces. Then $W_1 \cap W_2$ is a subspace.

Fact

If S is a set of n linearly independent vectors in $\mathbb{R}^n$ then $\text{span}(S) = \mathbb{R}^n$.

Fact: A basis of P_n

Let P_n denote the vector space of polynomials in x of degree at most n. The set

$$S = \{1, x, x^2, \ldots, x^n\}$$

is a basis for P_n. The set S is known as the standard basis of P_n.

Fact

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a basis. Every vector $\mathbf{v} \in V$ may be expressed as a linear combination of the basis vectors in exactly one way

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$$

Fact: Orthogonal implies linearly independent

Let V be a vector space and S a set of vectors in $\mathbb{R}^n$. If S is orthogonal then it is linearly independent.

Fact: A useful property of an orthonormal basis

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. Given a vector $\mathbf{u}$

1. if S is orthogonal then

$$\mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \ldots + \frac{\mathbf{u} \bullet \mathbf{v}_n}{||\mathbf{v}_n||^2}\mathbf{v}_n$$

and the co-ordinate vector of $\mathbf{u}$ in terms of S is

$$\mathbf{u} = \left(\frac{\mathbf{u} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}, \ldots \frac{\mathbf{u} \bullet \mathbf{v}_n}{||\mathbf{v}_n||^2}\right)_S$$

2. if S is orthonormal then

$$\mathbf{u} = (\mathbf{u} \bullet \mathbf{v}_1)\,\mathbf{v}_1 + \ldots + (\mathbf{u} \bullet \mathbf{v}_n)\,\mathbf{v}_n$$

and the co-ordinate vector of $\mathbf{u}$ in terms of S is

$$\mathbf{u} = (\mathbf{u} \bullet \mathbf{v}_1, \ldots, \mathbf{u} \bullet \mathbf{v}_n)_S$$

Fact: Orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$. Given a vector $\mathbf{v} \in \mathbb{R}^n$, we have

$$\mathbf{v} = \text{proj}_W(\mathbf{v}) + \text{proj}_{W^\perp}(\mathbf{v})$$

where $\text{proj}_W(\mathbf{v}) \in W$ and $\text{proj}_{W^\perp}(\mathbf{v})$ is orthogonal to W.

Fact: Finding the orthogonal decomposition

Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$$

Given a vector $\mathbf{v} \in \mathbb{R}^n$, we have

$$\text{proj}_W(\mathbf{v}) = \frac{\mathbf{v} \bullet \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{v} \bullet \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \cdots + \frac{\mathbf{v} \bullet \mathbf{v}_m}{||\mathbf{v}_m||^2}\mathbf{v}_m$$

If the basis S is orthonormal then

$$\text{proj}_W(\mathbf{v}) = (\mathbf{v} \bullet \mathbf{v}_1)\,\mathbf{v}_1 + (\mathbf{v} \bullet \mathbf{v}_2)\,\mathbf{v}_2 + \cdots (\mathbf{v} \bullet \mathbf{v}_m)\,\mathbf{v}_m$$

The component of $\mathbf{v}$ orthogonal to W is given by

$$\text{proj}_{W^\perp}(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

Fact

Let V be a vector space and S_1 and S_2 be bases for V. Then S_1 contains the same number of vectors as S_2.

Fact

Let V be a vector space and W a subspace of V. Then

$$\dim(W) \leq \dim(V)$$

Fact: Adding and removing vectors

Let V be a vector space. Then

1. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. If $\mathbf{u} \notin \text{span}(S)$, then the set

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\}$$

is linearly independent also.

That is, adding a vector which does lie in $\text{span}(S)$ does not break linear independence.

2. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly dependent set of vectors in V. If $\mathbf{v}_k$ may be written as a linear combination of the other vectors in S, then

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}) = \text{span}(S)$$

That is, removing vectors which can be written as linear combinations of the other vectors does not change the span.

3. let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent also.

That is, removing any number of vectors from a linearly independent set does not break linear independence.

Fact

Let V be a finite dimensional vector space, and S a finite set of vectors in V. Then

1. if S is not linearly independent but $\mathrm{span}(S) = V$, then S can upgraded to a basis of V by removing vectors.
2. if S is linearly independent but $\mathrm{span}(S) \neq V$, then S can be upgraded to a basis of V by adding vectors which are not in $\mathrm{span}(S)$.

Fact

Let A be an $n \times m$ matrix. The equation

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution if and only if $\mathbf{b} \in \mathrm{col}(A)$.

Fact

Let A and B be matrices related by a sequence of elementary row operations. Then $\mathrm{row}(A) = \mathrm{row}(B)$ and $\mathrm{null}(A) = \mathrm{null}(B)$.