MATB24 Midterm 1

I affirm that I did not give or recieve any unauthorized help on this exam and that all submitted work is my own: Leo Wang

1. a) Consider vectors $\vec{v}_1, \ldots, \vec{v}_k$ in a v.s. V over F. They are l.d. if $\exists c_1, \ldots, c_k \in F$ so that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$

iff

Some c_i, $1 \le i \le k$, doesn't equal 0_F

b) Add T should be bijective linear function

2. a) True. $r_1\vec{u} + r_2\vec{v} = \vec{0}$

$r_1 a\vec{v} + r_2\vec{v} = \vec{0}$ Choose $r_2 = -a$, $r_1 = 1$

$a\vec{v} - a\vec{v} = \vec{0}$ Holds true for some $r \neq 0$ $\therefore$ l.d.

$\vec{0} = \vec{0}$

b) True. Every subspace can be treated like a v.s. and the image of a LT can be its domain if you use the identity map $T: V \to V$, $\vec{v} \mapsto \vec{v}$ where V represents any vector space.

c) True. $\dim(P_3(\mathbb{C})) = 4$, a sample basis can be $\{1, x, x^2, x^3\}$
All bases are the same size and a basis is the smallest spanning set of a v.s. $3 < 4$, $\therefore$ a set of 3 cannot span $P_3(\mathbb{C})$

d) True. By def if $T: V \to W$ is onto, then $\dim(V) \geq \dim(W)$

We know $\dim(\mathbb{R}^5) = 5$ $\therefore$ $\dim(V) \leq 5$

3. a) DNE, Range and ker are indep. of each other and $\therefore$ cannot be equal

b) $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$
$B_2 = \{(i,0,0), (0,i,0), (0,0,i)\}$

c) Let $U = \mathbb{R}^2$. Clearly $U \subseteq \mathbb{C}^2$ and we know $\mathbb{R}^2$ is closed under addition as $\mathbb{R}^2$ is a v.s.
But it is not closed under scalar multiplication. Choose $r = i$
Let $\vec{v} \in \mathbb{R}^2$, $\vec{v} \neq (0,0)$ $i\vec{v} = (iv_1, iv_2) \notin \mathbb{R}^2$

d) Let $V = \mathbb{R}$ $\mathbb{R}$ is clearly a v.s.

Let $S = \{1, 1, 1, \ldots\}$ S is infinite and $sp(S) = \mathbb{R}$

Shown: $sp(S) = r_1(1) + r_2(1) + \ldots$ $r_i \in \mathbb{R}$

4. a) $T(S(e^{5x})) = T(5e^{5x})$ $= 5T(e^{5x})$

T is not defined for the input e^{5x}, so finding $T \circ S(e^{5x})$ isn't possible

e^{5x} cannot be reduced to terms defined for T

b) $S(T(2+3e^{2x})) = S(2T(1) + 3T(e^{2x}))$ bc. T is a LT

$= S(2(0) + 3(x^2))$

$= S(3x^2)$

$= 6x$

c) $T \circ S(f) = 0$ (0 function) Let $f \in C^\infty$

The only way $T(g) = 0$ for some $g \in C^\infty$ is if $g = 1$

Find f st $S(f) = a$, $a \in \mathbb{R}$

$\forall f \in sp(1, x)$, $S(f) = c$, $c \in \mathbb{R}$

$T(c) = cT(1)$ T is a LT

$= c(0)$

$= 0$

$r_1(1) + r_2(x) = 0$

$r_1 + r_2 x = 0$

$\Rightarrow r_1 = r_2 = 0$ so $\{1, x\}$ is indep.

$\dim(sp(1, x)) = 2$

These examples show possible sets $\subseteq ker(T \circ S)$

If $A = \{1, x\}$ then $S \subseteq ker(T \circ S)$

Because $\dim(A) = 2$, $\dim(ker(T \circ S)) \geq 2$

5 a) The zero function maps to 0 everywhere

Let p be the 0 func.

$p(1) = 0 \quad \therefore p \in W$

Addition

Let $f, g \in W$

$(f+g)(1) = f(1) + g(1)$ by func addition

$= 0 + 0$ bc $f, g \in W$

$= 0$

$\therefore f+g \in W$

S. Multiplication

Let $f \in W$, $r \in F$

$(rf)(1) = r f(1)$ by func scalar multi.

$= r(0)$ bc $f \in W$

$= 0$

$\therefore rf \in W$

By subspace test, $W \subseteq_{ss} P_m(F)$

b) No, T is not onto

Let $f(x) = 2$, clearly $f \in P_m(F)$

But there doesn't exist $p \in W$, st $T(p) = f$.

This is b.c. T is an identity mapping, so $dom(T) = range(T)$

Because $f(1) \neq 0$, $f \notin W = dom(T)$ and $\therefore f \notin range(T)$

c) We know there are $m+1$ $p_i \in P_m(F)$ st $p_i(1) = 0$

And all p_i have max degree m

$p_i(x) = (x-1)(x-r_1) \cdots (x-r_{m-1})$

Bc. there are m roots and $m+1$ polynomials, we know from GHW2 that there is overlap between at least 2 p_i

So $\therefore \{p_0, \dots, p_m\}$ is dep.

6. a) $\underline{S}$

$T(\vec{0}) = \vec{0}$ 0 is always mapped to 0

So $\vec{0} \in S$

$\underline{\text{Addition}}$

Let $v, u \in S$

$T(v+u) = T(v) + T(\vec{u})$ bc T is a LT

$= v + u$

$\therefore v+u \in S$

$\underline{\text{s, Multi}}$

Let $v \in S$, $r \in F$

$T(rv) = rT(v)$ bc T is a LT

$= rv$

$\therefore rv \in S$

$\underline{A}$

$T(\vec{0}) = \vec{0} = -\vec{0}$

$\therefore \vec{0} \in A$

$\underline{\text{Addition}}$

Let $v, u \in A$

$T(v+u) = T(v) + T(u)$

$= -v - u$

$= -(v+u)$

$\therefore v+u \in A$

$\underline{\text{s, Multi}}$

Let $v \in A$, $r \in F$

$T(ru) = rT(u)$

$= r(-u)$

$= -ru$

$\therefore ru \in A$

b) Supp. $-1 \neq 1$ in F

We know $T = T^{-1}$

$T(\vec{a} + \vec{s}) = T(\vec{a}) + T(\vec{s})$ T is LT

$= -\vec{a} + \vec{s}$

$T(-\vec{a} + \vec{s}) = T(-\vec{a}) + T(\vec{s})$

$= \vec{a} + \vec{s}$

$T(T(\vec{a} + s)) = \vec{a} + \vec{s}$ by def of T, $\forall \vec{v} \in V$, $\vec{v} = \vec{a} + \vec{s}$

Supp for some $v \in V$, $v = \vec{a}_1 + \vec{s}_1$, $v = \vec{a}_2 + \vec{s}_2$ $a_1 \neq a_2$, $s_1 \neq s_2$

$T(v) = T(a_1 + s_1)$

$= T(a_1) + T(s_1)$

$= -a_1 + s_1$

$T \circ T(v) = T(-a_1 + s_1)$