CSCC37 A4

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Question 1

a)

Solve using $PA = LU$ factorization

$$Va = y$$

$$\begin{bmatrix} 1 & (-1)^1 & (-1)^2 \\ 1 & (0)^1 & (0)^2 \\ 1 & 1^1 & 1^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},\ L_1V = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ P_2L_1V = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\ L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix},\ L_2P_2L_1V = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix},\ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Va = y \implies PVa = Py \implies LUa = Py \implies Ld = Py, \text{ for} d = Ua$$

Solve $Ld = Py$, for d

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} d = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

$$d_1 = 4$$

$$d_1 + d_2 = 12 \implies d_2 = 8$$

$$d_1 + \frac{1}{2}d_2 + d_3 = 6 \implies d_3 = -2$$

Solve $Ua = d$, for a

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} a = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

$$-a_2 = -2 \implies a_2 = 2$$

$$2a_1 = 8 \implies a_1 = 4$$

$$a_0 - a_1 + a_2 = 4 \implies a_0 = 6$$

Thus $p(x) = 2x^2 + 4x + 6$

b)

$$\text{The data points are represented as } (x_i, y_i)$$
$$\text{Using the formula in class: } l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \in \mathbf{P}_2$$
$$p(x) = 4l_0(x) + 6l_1(x) + 12l_2(x)$$

c)

i	x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$
0	-1	4	$\frac{6-4}{0-(-1)} = 2$	$\frac{6-2}{1-(-1)} = 2$
1	0	6	$\frac{12-6}{1-0} = 6$	
2	1	12		

Thus $p(x) = 4 + 2(x - (-1)) + 2(x - (-1))(x - 0) = 4 + 2(x + 1) + 2x(x + 1)$

d)

We know by the fundamental theory of algebra we know that for any $n + 1$ distinct points there is a unique polynomial of degree n which interpolates these points. Thus we can evaluate these points on each of the derived polynomials to verify they are the same.

$$(x_0,\ y_0) = (-1,\ 4)$$
$$2(-1)^2 + 4(-1) + 6 = 4$$
$$4l_0(x_0) + 6l_1(x_0) + 12l_2(x_0) = 4, \text{ by property of } l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
$$4 + 2(-1 + 1) + 2(-1)(-1 + 1) = 4$$

$$(x_1,\ y_1) = (0,\ 6)$$
$$2(0)^2 + 4(0) + 6 = 6$$
$$4l_0(x_1) + 6l_1(x_1) + 12l_2(x_1) = 6$$
$$4 + 2(0 + 1) + 2(0)(0 + 1) = 6$$

$$(x_2,\ y_2) = (1,\ 12)$$
$$2(1)^2 + 4(1) + 6 = 12$$
$$4l_0(x_2) + 6l_1(x_2) + 12l_2(x_2) = 12$$
$$4 + 2(1 + 1) + 2(1)(1 + 1) = 12$$

All three equations are degree 2 polynomials and interpolate the three distinct points, thus they must all model the same polynomial.

e)

Extend the table of divided difference:

i	x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$	$y[x_{i+3},\ x_{i+2},\ x_{i+1},\ x_i]$
0	-1	4	$\frac{6-4}{0-(-1)} = 2$	$\frac{6-2}{1-(-1)} = 2$	$\frac{-1-2}{2-(-1)} = -1$
1	0	6	$\frac{12-6}{1-0} = 6$	$\frac{4-6}{2-0} = -1$	
2	1	12	$\frac{16-12}{2-1} = 4$		
3	2	16			

$$\text{Thus } p(x) = 4 + 2(x+1) + 2x(x+1) - x(x+1)(x-1)$$

$$\text{In monomial basis: } p(x) = 6 + 5x + 2x^2 - x^3$$

f)

We can generate linear equations between each of the data points to attach them all together since two points can uniquely identify a degree 1 polynomial.

$$\text{Piece-wise linear spline } p(x) = \begin{cases} 2(x+1)+4 & x \in [-1,\ 0] \\ 6x+6 & x \in [0,\ 1] \\ 4(x-1)+12 & x \in [1,\ 2] \end{cases}$$

Question 2

a)

Monomial basis evaluation

$$\begin{aligned}
p(x) &= a_0 + a_1 x + \cdots a_n x^n \\
p(x) &= a_0 + x(a_1 + a_2 x + \cdots + a_n x^{n-1}) \\
&\vdots \\
p(x) &= a_0 = x(a_1 + \cdots + x(a_{n-1} + x a_n)\cdots)
\end{aligned}$$

Define f_i starting from the inner most bracket and evaluating outwards

$$\begin{aligned}
f_0(x) &= a_{n-1} + x a_n \\
f_1(x) &= a_{n-2} + x f_0(x) \\
&\vdots \\
f_i(x) &= a_{n-i-1} + x f_{i-1}(x),\ i = 1, 2, \cdots, n \\
f_n(x) &= p(x)
\end{aligned}$$

Using the previous f_i result, every f_i can be calculated in 1 flop.
Thus the monomial basis form for a degree n polynomial can be evaluated in n flops.

b)

Divided difference basis evaluation, assuming all a_i divided differences have been calculated and can be treated as constant

$$\begin{aligned}
p(x) &= a_0 + (x - x_0)a_1 + \cdots + (x - x_0)\cdots(x - x_{n-1})a_n \\
&= a_0 + (x - x_0)\left(a_1 + (x - x_1)a_2 + \cdots + (x - x_1)\cdots(x - x_{n-1})a_n\right) \\
&\vdots \\
&= a_0 + (x - x_0)\left(a_1 + (x - x_1)(a_2 + \cdots (x - x_{n-1})a_n)\cdots\right)
\end{aligned}$$

Evaluate inside out, starting from $y_{n-1} = a_{n-1} + (x - x_{n-1})a_n$ which is 2 flops

Then moving outwards $y_i = a_i + (x - x_i)y_{i-1}$

Thus the divided difference basis form for a degree n polynomial can be evaluated in $2n$ flops.

Question 3

a)

$$\begin{aligned}
p(x) &= \sum_{i=0}^{n} b_i (x - c)^i \\
&= \sum_{i=0}^{n} b_i \sum_{k=0}^{i} \binom{i}{k} x^{i-k}(-c)^k, \text{ binomial expansion} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} b_i \binom{i}{k} x^{i-k}(-c)^k \\
&= \sum_{k=0}^{n} \sum_{i=k}^{n} b_i \binom{i}{i-k} x^{k}(-c)^{i-k}, \text{ swap summations} \\
&= \sum_{k=0}^{n} x^k \sum_{i=k}^{n} b_i \binom{i}{i-k} (-c)^{i-k} \\
a_k &= \sum_{i=k}^{n} b_i \binom{i}{i-k} (-c)^{i-k}
\end{aligned}$$

b)

c	rcond
0	4.2535e-07
0.5	1.9436e-06
1	7.5962e-06
1.5	2.6885e-05
2	5.3226e-05
2.5	0.0001131
3	0.00030227
3.5	0.00016034
4	0.00014415
4.5	3.5049e-05
5	4.8742e-05
5.5	1.9436e-06
6	4.2535e-07

Figure 1: Table of rcond of V matrix with different c values

By experimentation, it seems like the condition of the matrix is minimized at $c = 3$, which makes sense, since it will minimize how large the values can grow towards the right entries as 3 is the average of the x values.

Question 4

x_i	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$	$y[x_{i+3}, \cdots, x_i]$	$y[x_{i+4}, \cdots, x_i]$	$y[x_{i+5}, \cdots, x_i]$	$y[x_{i+6}, \cdots, x_i]$
-1	4	$\frac{7-4}{0-(-1)} = 3$	$\frac{6-3}{0-(-1)} = 3$	$\frac{15-3}{1-(-1)} = 6$	$\frac{20-6}{1-(-1)} = 7$	$\frac{15-7}{1-(-1)} = 4$	$\frac{7-4}{2-(-1)} = 1$
0	7	$\frac{p'(0)}{1!} = 6$	$\frac{21-6}{1-0} = 15$	$\frac{35-15}{1-0} = 20$	$\frac{35-20}{1-0} = 15$	$\frac{29-15}{2-0} = 7$	
0	7	$\frac{28-7}{1-0} = 21$	$\frac{56-21}{1-0} = 35$	$\frac{70-35}{1-0} = 35$	$\frac{93-35}{2-0} = 29$		
1	28	$\frac{p'(1)}{1!} = 56$	$\frac{p''(1)}{2!} = 70$	$\frac{163-70}{2-1} = 93$			
1	28	$\frac{p'(1)}{1!} = 56$	$\frac{219-56}{2-1} = 163$				
1	28	$\frac{247-28}{2-1} = 219$					
2	247						

Thus $p(x) = 4 + 3(x+1) + 3x(x+1) + 6x^2(x+1) + 7x^2(x+1)(x-1) + 4x^2(x+1)(x-1)^2 + x^2(x+1)(x-1)^3$

The interp values are evaluated below :)

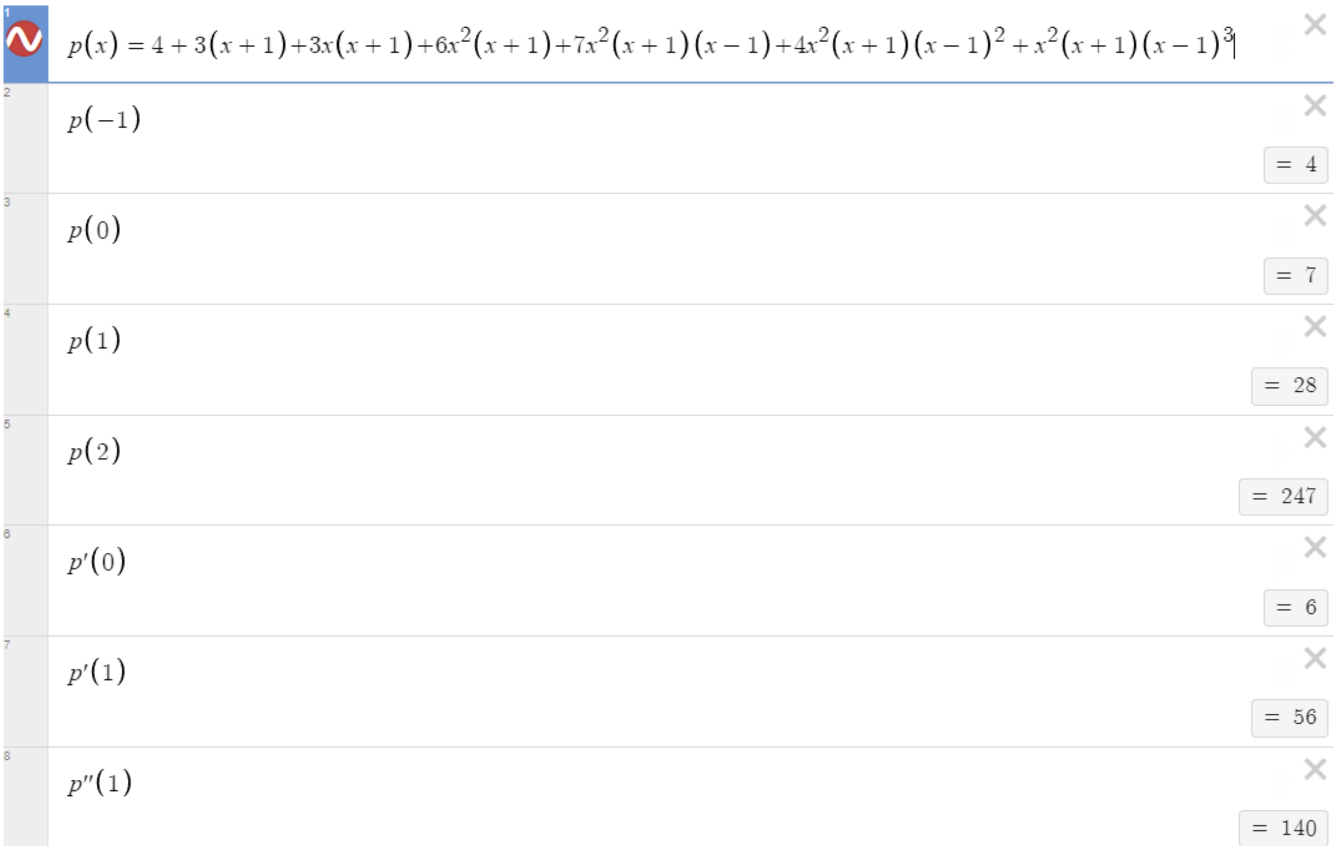

Figure 2: Desmos evaluation of $p(x)$