Week 7 Tutorial

7.3 Q22 a) Let $a<b$ and $D=[0,\pi]\times[0,2\pi]$

$$\vec{\Phi}(u,v)=(a\sin(u)\cos(v),\ b\sin(u)\sin(v),\ c\cos(u))$$

Show that $\vec{\Phi}(D)\subseteq S$ where:

$$S=\{(x,y,z):\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}$$

b) Let $a<b$ and $D=[0,\pi]\times[0,2\pi]$

$$\vec{\Phi}(u,v)=(a\sin(u)\cos(v),\ b\sin(u)\sin(v),\ c\cos(u))$$

Show that $\vec{\Phi}$ is regular everywhere.

7.3 Q18 a) We have that:

$$\vec{\Phi}(u,v)=(2\sin(u)\cos(v),\ 2\sin(u)\sin(v),\ 2\cos(u))$$

parametrizes a sphere S of radius $r=2$ around the origin. Find the tangent plane at $(1,1,\sqrt{2})$ by calculating $\vec{n}=\vec{T}_u\times\vec{T}_v$.

b) The sphere S can also be seen as the graph of:

$$z=\sqrt{4-x^2-y^2}$$

Find the tangent plane to S at $(1,1,\sqrt{2})$ by calculating z_x and z_y.

Q3 You want to make a beautiful spherical napkin ring. Suppose that a sphere has a hole drilled through it, s.t. the cylindrical hole has height h. Find the remaining volume of the sphere.

Q4 Let $\vec{\Phi}: D\subset\mathbb{R}^2\to\mathbb{R}^3$ be a nice parametrization. We introduce the following notation:

$$E=\|\vec{T}_u\|^2 \qquad F=\vec{T}_u\cdot\vec{T}_v \qquad G=\|T_v\|^2$$

Show that $\|\vec{T}_u\times\vec{T}_v\|=\sqrt{FG-E^2}$. This gives an alternative form of area.

$$A(S)=\iint_S\sqrt{FG-E^2}\,dS$$

What happens to this formula if $\vec{T}_u$ and $\vec{T}_v$ are orthogonal?

Q5 The sphere of radius R is:

$$\vec{\Phi}(u,v) = (R\sin(u)\cos(v), R\sin(u)\sin(v), R\cos(u))$$

Use

$$A(S) = \iint_S \sqrt{FG - E^2}\, dS$$

to find its area.

Q6 If $\vec{\Phi}: D \to \mathbb{R}^3$ is a nice parametrization we define:

$$J(\vec{\Phi}) = \frac{1}{2}\iint_D \left(\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2\right) du\,dv$$

Use the arithmetic-geometric mean inequality to show: $A(\vec{\Phi}) \leq J(\vec{\Phi})$.

Why define this weird thing $J(\vec{\Phi})$?

1. $J(\vec{\Phi})$ is a nicer formula than $A(\vec{\Phi})$.
2. If $J(\vec{\Phi}) = A(\vec{\Phi})$ then $\vec{\Phi}$ is a conformal mapping.
   $\|\vec{T}_u\|^2 = \|\vec{T}_v\|^2 \quad \vec{T}_u \cdot \vec{T}_v = 0$
3. J is important in complex analysis.

1. Parametrization of a surface = function $\Phi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$\vec{\Phi}(u,v) = (x(u,v), y(u,v), z(u,v))$$

x, y, z are C^1 function $\Rightarrow \vec{\Phi}$ is also C^1

2. Parametrization of a plane = $\Phi(u,v) = \vec{v}_0 + \vec{v}_1 u + \vec{v}_2 v$

v_0 = a point in the plane $\quad$ $\vec{v}_1$ and $\vec{v}_2$ cannot be parallel ($\vec{\Phi}$ = line o.w.)

3. Parametrization of a torus = $u, v \in [0, 2\pi)$ $\quad \Phi = [0, 2\pi)^2 \rightarrow \mathbb{R}^3$:

$$\Phi(u,v) = ((R + r\cos(u))\sin(v), (R + r\cos(u))\cos(v), r\sin(u))$$

4. Tangent Vector = If Φ is a C^1-surface and (u_0, v_0) is a point on the surface.

Directional derivatives of Φ:

u-direction: $\vec{T}_u = \frac{\partial \Phi}{\partial u} = \left(\frac{\partial x}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial y}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial z}{\partial u}\Big|_{(u_0,v_0)} \right)$

v-direction: $\vec{T}_v = \frac{\partial \Phi}{\partial v} = \left(\frac{\partial x}{\partial v}\Big|_{(u_0,v_0)}, \frac{\partial y}{\partial v}\Big|_{(u_0,v_0)}, \frac{\partial z}{\partial v}\Big|_{(u_0,v_0)} \right)$

5. Regular Surface = A surface is regular at $\Phi(u_0, v_0)$ if $\vec{T}_u \times \vec{T}_v \neq 0$ (i.e. u_0 and v_0 are non-parallel)

6. Tangent Plane: If a surface Φ is differentiable and regular at (u_0, v_0), the normal of Φ is: $n = \vec{T}_u \times \vec{T}_v$.

For $(x_0, y_0, z_0) = \Phi(u_0, v_0)$, tangent plane: $(x - x_0, y - y_0, z - z_0) \cdot n = 0$.

7. Surface Area: $\text{Area}(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\,dv = \iint_D \|\vec{n}\| \, du\,dv$

Solutions

7.3 Q22

Check $\vec{\Phi}(D)$ satisfies the parametrization from S.

$$\frac{[a\sin(u)\cos(v)]^2}{a^2} + \frac{[b\sin(u)\sin(v)]^2}{b^2} + \frac{[c\cos(u)]^2}{c^2}$$

$$= \sin^2(u)\cos^2(v) + \sin^2(u)\sin^2(v) + \cos^2(u)$$

$$= 1$$

$$\vec{T}_u = (a\cos(u)\cos(v),\ b\cos(u)\sin(v),\ -c\sin(u))$$

$$\vec{T}_v = (-a\sin(u)\sin(v),\ b\sin(u)\cos(v),\ 0)$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} i & j & k \\ a\cos(u)\cos(v) & b\cos(u)\sin(v) & -c\sin(u) \\ -a\sin(u)\sin(v) & b\sin(u)\cos(v) & 0 \end{vmatrix}$$

$$= (bc\sin^2(u)\cos(v),\ ac\sin^2(u)\sin(v),\ ab\sin(u)\cos(u)) \neq (0,0,0)$$

if $bc\sin^2(u)\cos(v) = 0$

$u = 0$ or π

$v = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$ac\sin^2(u)\sin(\frac{\pi}{2}) \neq 0$

$ac\sin^2(u)\sin(\frac{3\pi}{2}) \neq 0$

7.3 Q18

$$\vec{T}_u = (2\cos(u)\cos(v),\ 2\cos(u)\sin(v),\ -2\sin(u))$$

$$\vec{T}_v = (-2\sin(u)\sin(v),\ 2\sin(u)\cos(v),\ 0)$$

$$\vec{n} = \vec{T}_u \times \vec{T}_v = (4\sin^2(u)\cos(v),\ 4\sin^2(u)\sin(v),\ 4\sin(u)\cos(u))$$

at $(1,1,\sqrt{2})$ $\Rightarrow \begin{cases} 2\sin(u)\cos(v) = 1 \\ 2\sin(u)\sin(v) = 1 \\ 2\cos(u) = \sqrt{2} \end{cases} \Rightarrow \vec{n} = (\sqrt{2}, \sqrt{2}, 2)$

$u \in [0, \pi]$

$v \in [0, 2\pi]$

$\therefore$ Tangent Plane: $(x-1, y-1, z-\sqrt{2}) \cdot (\sqrt{2}, \sqrt{2}, 2) = 0 \iff x+y=\sqrt{2}z=4$

$$z_x = \frac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}} \cdot (-2x) = -x(4-x^2-y^2)^{-\frac{1}{2}}$$

$$z_y = \frac{1}{2}(4-x^2-y^2)^{-\frac{1}{2}} \cdot (-2y) = -y(4-x^2-y^2)^{-\frac{1}{2}}$$

at $x=1, y=1, z=\sqrt{2}$, $z_x = -\frac{\sqrt{2}}{2}$ $z_y = -\frac{\sqrt{2}}{2}$

$\therefore$ Tangent Plane: $(-\frac{\sqrt{2}}{2})(x-1) + (-\frac{\sqrt{2}}{2})(y-1) = z - \sqrt{2} \Rightarrow x + y + \sqrt{2}z = 4$

$F(x,y,z) = \sqrt{4-x^2-y^2} - z$

$F_x = z_x$, $F_y = z_y$, $F_z = -1$

Tangent Plane: $(F_x, F_y, F_z)(x-x_0, y-y_0, z-z_0) = 0$

Q3

$V = V_{\text{sphere}} - V_{\text{cylindrical hole}} - 2V_{\text{spherical caps}}$

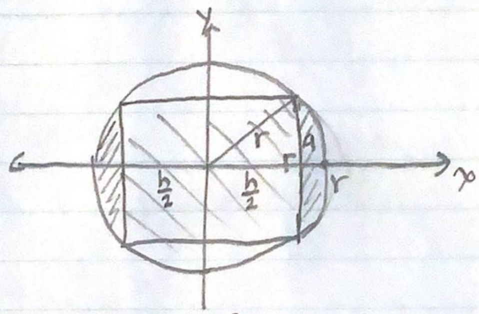
$= \frac{4}{3}\pi r^3 - \pi a^2 h - 2 \cdot \int_{\frac{h}{2}}^{r} \pi (r^2 - x^2)\,dx$

$= \frac{4}{3}\pi r^3 - \pi a^2 h - 2\pi \cdot (r^2 x - \frac{1}{3}x^3)\Big|_{\frac{h}{2}}^{r}$

$= \frac{4}{3}\pi r^3 - \pi a^2 h - 2\pi(r^3 - \frac{r^2}{2}h - \frac{1}{3}r^3 + \frac{1}{3}\cdot\frac{h^3}{8})$

$= -\pi h(r^2 - \frac{1}{4}h^2) - 2\pi(-\frac{r^2}{2}h + \frac{1}{3}\cdot\frac{h^3}{8})$

$= \frac{1}{6}\pi h^3 = \frac{4}{3}\pi(\frac{h}{2})^3 \Rightarrow$ no relationship with r



$a^2 + (\frac{1}{2}h)^2 = r^2$

$\Rightarrow a^2 = r^2 - \frac{1}{4}h^2$

Q4

$\vec{T}_u = \frac{\partial \vec{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial y}{\partial u}\Big|_{(u_0,v_0)}, \frac{\partial z}{\partial u}\Big|_{(u_0,v_0)} \right) = (x_u, y_u, z_u)$

$\vec{T}_v = \frac{\partial \vec{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}\Big|_{(u_0,v_0)}, \frac{\partial y}{\partial v}\Big|_{(u_0,v_0)}, \frac{\partial z}{\partial v}\Big|_{(u_0,v_0)} \right) = (x_v, y_v, z_v)$

$F = x_u^2 + y_u^2 + z_u^2$

$E = x_u x_v + y_u y_v + z_u z_v$

$G = x_v^2 + y_v^2 + z_v^2$

$\|\vec{T}_u \times \vec{T}_v\| = \left\| \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \right\| = \|(y_u z_v - z_u v_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v)\|$

$= \sqrt{y_u^2 z_v^2 + z_u^2 y_v^2 - 2y_u z_v z_u y_v + z_u^2 x_v^2 + x_u z_v^2 - 2z_u x_v x_u z_v + x_u^2 y_v^2 + y_u^2 x_v^2 - 2x_u y_v y_u x_v}$

$FG = (x_u^2+y_u^2+z_u^2)(x_v^2+y_v^2+z_v^2) = x_u^2x_v^2 + x_u^2y_v^2 + x_u^2z_v^2 + y_u^2x_v^2 + y_u^2y_v^2 + y_u^2z_v^2 + z_u^2x_v^2 + z_u^2y_v^2 + z_u^2z_v^2$

$E^2 = (x_ux_v+y_uy_v+z_uz_v)^2 = x_u^2x_v^2 + y_u^2y_v^2 + 2x_ux_vy_uy_v + z_u^2z_v^2 + 2x_ux_vz_uz_v + 2y_uy_vz_uz_v$

$\sqrt{FG-E^2} = \sqrt{x_u^2y_v^2 + x_u^2z_v^2 + y_u^2x_v^2 + z_u^2x_v^2 + z_u^2y_v^2 - 2x_ux_vy_uy_v - 2x_ux_vz_uz_v - 2y_uy_vz_uz_v}$

$\therefore \|\vec{T}_u \times \vec{T}_v\| = \sqrt{FG-E^2}$

If $\vec{T}_u$ and $\vec{T}_v$ are orthogonal, then $\theta = \frac{\pi}{2}$.

$E = \vec{T}_u \cdot \vec{T}_v = \|\vec{T}_u\| \cdot \|\vec{T}_v\| \cdot \cos(\frac{\pi}{2}) = 0$

$\therefore E^2 = 0$

$FG = \|\vec{T}_u\|^2 \cdot \|\vec{T}_v\|^2$

$\vec{T}_u \times \vec{T}_v = \|\vec{T}_u\| \cdot \|\vec{T}_v\| \cdot \sin(\theta)$
$= \|\vec{T}_u\| \cdot \|\vec{T}_v\|$

$A(S) = \iint_S \sqrt{FG-E^2}\, ds = \iint_S \sqrt{FG}\, ds = \iint_S \|\vec{T}_u\| \cdot \|\vec{T}_v\|\, ds$

Q5

$F = \|\vec{T}_u\|^2 = \|(R\cos(u)\cos(v), R\cos(u)\sin(v), -R\sin(u)\|^2$
$= R^2\cos^2(u)\cos^2(v) + R^2\cos^2(u)\sin^2(v) + R^2\sin^2(u) = R^2$

$G = \|\vec{T}_v\|^2 = \|(-R\sin(u)\sin(v), R\sin(u)\cos(v), 0)\|^2 = R^2\sin^2(u)\sin^2(v) + R\sin^2(u)\cos^2(v)$
$= R^2\sin^2(u)$

$E = \vec{T}_u \cdot \vec{T}_v = -R^2\cos(u)\cos(v)\sin(u)\sin(v) + R^2\cos(u)\sin(v)\sin(u)\cos(v) = 0$

$\therefore A(S) = \iint_S \sqrt{FG-E^2}\, ds = \iint_S \sqrt{FG}\, ds = \int_0^{2\pi}\int_0^{\pi} \sqrt{R^2 \cdot R^2\sin^2(u)}\, du\, dv$

$= \int_0^{2\pi}\int_0^{\pi} R^2\sin(u)\, du\, dv = R^2\int_0^{2\pi} -\cos(u)\Big|_0^{\pi} dv = 4\pi R^2$

$\because u \in [0,\pi]$
$\therefore \sin(u) \geq 0$
$\therefore \sqrt{\sin^2(u)} = \sin(u)$

Q6

WTP: $\sqrt{\|\vec{T}_u\|^2 \cdot \|\vec{T}_v\|^2 - (\vec{T}_u \cdot \vec{T}_v)^2} \leq \frac{1}{2}(\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2)$

$LHS = \sqrt{\|\vec{T}_u\|^2 \cdot \|\vec{T}_v\|^2 - \|\vec{T}_u\|^2 \cdot \|\vec{T}_v\|^2 \cdot \cos^2(\theta)}$
$= \|\vec{T}_u\| \cdot \|\vec{T}_v\| \cdot \sin(\theta)$
$\leq \frac{\|\vec{T}_u\|^2 + \|\vec{T}_v\|^2}{2} \cdot 1 = RHS$

$\therefore A(\Phi) \leq J(\Phi)$ ∎