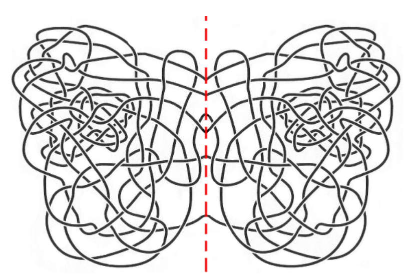
1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.
   (a) (3 points) Consider vectors $\vec{v}_1, \cdots, \vec{v}_k$ in a vector space V over $\mathbb{F}$. They are *linearly dependent* if there exist $c_1, \ldots, c_k \in \mathbb{F}$ so that $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$.
   (b) (3 points) An *isomorphism* between vector spaces V and W over $\mathbb{F}$ is a bijective function $T\colon V \to W$.
2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counter example if you think it is false).
   (a) (3 points) If $\vec{u}, \vec{v}$ are non-zero linearly dependent vectors in a vector space V over $\mathbb{F}$, then there exists an $a \in \mathbb{F}$ such that $\vec{u} = a\vec{v}$.
   (b) (3 points) Every subspace is the image of a linear transformation.[1]
   (c) (3 points) No list of three polynomials spans $\mathcal{P}_3(\mathbb{C})$, the vector space over $\mathbb{C}$ of polynomials of degree ≤ 3.
   (d) (3 points) If $T\colon \mathbb{R}^5 \to V$ is a surjective linear transformation, then $\dim(V) \leq 5$.
3. In each part, give an **explicit** example of the mathematical object described or explain why such an object doesn't exist.
   (a) (2 points) A linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{im}(T) = \ker(T)$.
   (b) (2 points) Two different bases of $\mathbb{C}^3$.
   (c) (2 points) A non-empty subset U of $\mathbb{C}^2$ which is closed under addition (so $\vec{u} + \vec{v} \in U$ for all $\vec{u}, \vec{v} \in U$) and taking additive inverses (so $-\vec{u} \in U$ for all $\vec{u} \in U$), but which is not a subspace of $\mathbb{C}^2$.
   (d) (2 points) An infinite spanning set for a finite dimensional vector space.
4. Let $\mathcal{C}^\infty$ denote the vector space of functions $f\colon \mathbb{R} \to \mathbb{R}$ which have derivatives of all orders.[2] Let $S\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$, defined by $S(f) = f'$, and let $T\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$ be a linear transformation such that $T(e^{2x}) = x^2$, $T(e^{3x}) = \cos x$, and $T(1) = 0$.
   (a) (4 points) Find $T \circ S(e^{5x})$ or explain why it is not possible to do so with the given information.
   (b) (4 points) Find $S \circ T(2 + 3e^{2x})$ or explain why it is not possible to do so with the given information.
   (c) (4 points) Prove that $\dim(\ker(T \circ S)) \geq 2$.
5. Let $\mathcal{P}_m(\mathbb{F})$ be the vector space of polynomials of degree $\leq m$ with coefficients in $\mathbb{F}$. Let $W := \{p(x) \in \mathcal{P}_m(\mathbb{F}) \mid p(1) = 0\}$.
   (a) (3 points) Prove that W is a subspace of $\mathcal{P}_m(\mathbb{F})$.
   (b) (3 points) Consider the linear transformation $T\colon W \to \mathcal{P}_m(\mathbb{F})$ given by $p(x) \mapsto p(x)$. Is T onto? Justify.
   (c) (5 points) Suppose that $p_0, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(1) = 0$ for each j. Prove that $p_0, \ldots, p_m$ are linearly dependent.
6. Suppose that V is a vector space over $\mathbb{F}$.
   (a) (4 points) Let $T\colon V \to V$ be a linear map such that $T \circ T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. Prove that $S = \{\vec{v} \in V \mid T(\vec{v}) = \vec{v}\} \subset V$ and $A = \{\vec{v} \in V \mid T(\vec{v}) = -\vec{v}\} \subset V$ are subspaces.
   (b) (4 points) Prove that if $1 \neq -1$ in $\mathbb{F}$, then every $\vec{v}$ of V can be written uniquely as a sum $\vec{v} = \vec{s} + \vec{a}$ of a vector $\vec{s} \in S$ and a vector $\vec{a} \in A$.
   (c) (2 points) Let K be the knot diagram pictured to the right. Explain why part (a) of this problem implies that the number of tricolorings of K which are symmetric with respect to reflection in the dotted line is equal to 3^m for some $m \geq 1$. You do not need to give a proof.



   (d) (1 point) Explain what part (b) means for tricolorings of K. You do not need to give a proof.

Please attach a signed copy of the statement below in your handwriting. Your signature under the statement is required for your exam to be graded.

"I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own."

[1] A "linear transformation" is the same as a "linear map."

[2] That is, the i-th order derivative of f exists for all $i \in \mathbb{N}$. These are also called "smooth" functions.