Techniques of the Calculus of Several Variables II
Final Exam – Vector Calculus
Winter 2022

First Name: ______________________________

Last Name: ______________________________

Signature: ______________________________

Student Number: __________________________

Important Instructions

- Do NOT begin until you are instructed to do so.
- Show ALL your work and justify your answers for each question.
- The exam is 170 minutes long.
- No aids are allowed.
- The examination booklet contains a total of 12 pages.
  Please check to ensure that no pages are missing from your test booklet.
- Do NOT remove any pages from this test booklet.
- Do NOT write on the QR codes.
- The backs of pages will not be graded.
  If you need extra paper there is blank space on page 11.
- Electronic aids of any kind are forbidden.
- Have your student card ready for inspection.
- Show us what you learned in the course. Enjoy writing this exam.

Q1. (10 points) Evaluate the following line integral

$$\int_C \left(y + \cos(x^2)\right) dx + \left(\frac{1}{y} + 3x^2\right) dy$$

along the curve $C = \partial\{(x, y) : 1 \leq x^2 + y^2 \leq 4 \text{ and } 0 \leq x, y\}$ with counter-clockwise orientation.

Q2. (10 points) Solve the wave equation on $[-\pi, \pi]$ with the following intial conditions:

$$\phi(x) = 0 \qquad \psi(x) = \begin{cases} 0 & |x| > \frac{\pi}{2} \\ x & |x| \leq \frac{\pi}{2} \end{cases}$$

Q3. Let S be parametrized by $\mathbf{\Phi}(u, v) = (u\cos v, u\sin v, u^2)$ where $\mathbf{\Phi} : [0, 2] \times [0, 2\pi] \to \mathbb{R}^3$.

(a) (5 points) Find the tangent plane at $\mathbf{\Phi}\left(\sqrt{2}, \frac{\pi}{4}\right)$.

(b) (5 points) Find the surface area of S.

Q4. (10 points) Let S be the portion of the surface $\mathbf{\Phi}(u, v) = (u\cos v, u\sin v, u)$ where $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. Evaluate the following surface integral:

$$\iint_S xyz\ dS$$

Q5. Evaluate $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$ where S is the part of the graph $z = 9 - x^2 - y^2$ above the xy-plane oriented so that $\mathbf{n}(0,0,9) = (0,0,1)$ and $\mathbf{F} = \left(x^3 - y,\ x,\ xz + e^y z\right)$.

Q6. Let $\mathbf{F} = (P, Q, R)$ and define $\omega = P\ dx + Q\ dy + R\ dz$.

(a) (3 points) What statement about $\mathbf{F}$ is equivalent to $d\omega = 0$?

(b) (3 points) What statement about $\mathbf{F}$ is equivalent to $\omega = d\eta$?

(c) (4 points) *Note:* This sub-question is independent of Q6a and Q6b.
Let $\omega = x^2y\ dx + xz^2\ dy + y^3\ dz$ and $\eta = \cos(y)\ dx + e^x\ dy + z\ dz$. Calculate $\omega \wedge \eta$.

Q7. (10 points) Calculate the fourier series of the following function:

$$f(x) = \begin{cases} \pi & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$$

Q8. (10 points) Let S be the unit sphere with outward pointing normal vector and ω be the 2-form:

$$\omega = (x + y + z)\ dxdy + (3x - 2)\ dydz + (x - 5y)\ dzdx$$

Evaluate the integral $\displaystyle\int_S \omega$.

This page is for rough work. By default, it will not be graded.

If any work on this page is to be graded, indicate this **CLEARLY** on the question page.

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If any work on this page is to be graded, indicate this **CLEARLY** on the question page.

This is the end of the exam.