◇ **Best before:** term test 1.
Where necessary, refer to assignment 1 for the definition of a computable function.

1. Let $\mathcal{F}$ be the set of functions mapping from $\mathbb{N}$ (set of natural numbers) to $\mathbb{N}$.
   (a) Use diagonalization to prove that $\mathcal{F}$ is uncountable, then explain why this means some functions in $\mathcal{F}$ are not computable.
   (b) Give an example of an uncomputable function in $\mathcal{F}$.

2. **(Goldbach Conjecture)** In 1742, Prussian mathematician Christian Goldbach conjectured that *every even number greater than two can be expressed as the sum of two prime numbers.*[1] Although much work has been done on it, Goldbach's conjecture remains unvalidated, making it one of the oldest (and perhaps most famous) unsolved mysteries in mathematics today.[2]

   Discuss the recognizability and decidability of the following languages and their complements under the assumption that Goldbach's conjecture is true and under the assumption that Goldbach's conjecture is false. Assume that all variables used to describe these sets are natural numbers.

   $G_1 = \{\langle n\rangle \mid \text{ for all } j > n, \text{ there exist prime numbers } p, q \text{ such that } 2j = p + q\}.$

   $G_2 = \{\langle n\rangle \mid \text{ for all } j > 0, \text{ there exist prime numbers } p, q \text{ such that } 2nj = p + q\}.$

3. (a) Let $D_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a Turing machine and } M \text{ is a decider}\}$.
   Prove that D_{TM} is neither recognizable nor co-recognizable.
   (b) Let $D'_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a Turing machine and } L(M) \text{ is decidable}\}$.
   Is D'_{TM} recognizable? Is D'_{TM} co-recognizable?

4. Let f be a function of the form $f(M, w) = n$, where M is a Turing machine, w is an input string, and n is an integer. Suppose that for arbitrary Turing machine M and string w,

   M halts on input w within $f(M, w)$ steps $\Longleftrightarrow$ M halts on input w.

   Prove that f cannot be computable.

   Hint: Show that the computability of f can be used to solve the Halting Problem.

5. (a) Prove that every recognizable language has infinitely many Turing machines that recognize it.
   (b) Let T be the set of Turing machine descriptions and let $g : T \to T$ be a function such that $g(\langle M\rangle) = \langle M'\rangle$ means M' is the Turing machine with the smallest description (by shortlex order) that is equivalent to M. Prove that g cannot be computable.
   Hint: Show that the computability of g can be used to decide EQ_{TM}.

6. Consider the problem of determining whether a TM ever writes a blank symbol over a nonblank symbol during the course of its computation on any input. Formulate this problem as a language and show that it is undecidable. Is it recognizable?

7. Consider the problem of whether a Turing machine M on an input w ever moves its head to the right of the right-most cell occupied by w. Formulate this problem as a language and show that it is decidable.
   Hint: Consider the number of configurations there can be when M runs on w without moving its head beyond w.

---

[1] Well ... not exactly. In a letter to Leonhard Euler, Goldbach suggested that *every number greater than five can be expressed as the sum of three prime numbers*, and Euler replied that this is equivalent to saying *every even number greater than two can be expressed as the sum of two prime numbers.*

[2] Some publisher offered a million dollar prize for proving (or disproving) Goldbach's conjecture in 2000, but the time limit of two years for that prize expired without anyone claiming it.

8. (a) Find every language in chapter 5 of Sipser for which Rice's theorem applies and use Rice's theorem to prove that it is undecidable.

   (b) Find an undecidable language for which Rice's theorem cannot be used to prove it undecidable. Explain your answer.