MATB24 TUTORIAL PROBLEMS 6, WEEK OF

KEY WORDS: inner product, inner product space, dot product, orthogonal complement, orthogonal projection

RELEVANT SECTIONS IN THE TEXTBOOK: 6.1, 6.2, 3.5FB or 6.A, 6.B SA

WARM-UP:

Write down a complete definition or a complete mathematical characterization for the following terms.

(1) An inner product on a vector apace V
(2) An inner product space
(3) Orthogonal decomposition of a vector $\vec{u}$ in an inner product space V onto another vector v
(4) An orthonormal set of vectors
(5) An orthogonal set of vectors
(6) An orthogonal basis for a vector space W
(7) An orthonormal basis for a vector space W

A: Which of the following are examples of inner product spaces? Explain why or why not.

(1) V= the space of continuous functions from $[0,1]$ to $\mathbb{R}$.
$$\langle f,g\rangle = \int_0^1 f(t)g(t)dt.$$
(2) V= the space of polynomials in the variable t of degree less than or equal to 3.
$$\langle f,g\rangle = \int_0^1 f(t)g(t)dt.$$
(3) V= the space of infinite sequences $a_1, a_2, \ldots$.
$$\langle f,g\rangle = \sum_i a_i b_i$$
(4) V= the space of infinite *bounded* sequences $a_1, a_2, \ldots$.
$$\langle f,g\rangle = \sum_i \frac{a_i b_i}{2^i}.$$
(5) $V = \mathbb{R}^{m\times n}$ = the space of all $m \times n$ matrices.
$$\langle A,B\rangle = \text{tr}(A+B).$$
(6) $V = \mathbb{R}^{m\times n}$ = the space of all $m \times n$ matrices.
$$\langle A,B\rangle = \text{tr}(A^T B).$$

B: Let $\vec{v}, \vec{w}$ be in $\mathbb{R}^2$. Consider

$$\langle \vec{v}, \vec{w}\rangle = \vec{v}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \vec{w} \tag{1}$$

(1) Find an explicit formula for (1) in terms of the components of $\vec{v}$ and $\vec{w}$.
(2) Show that (1) is an inner product on $\mathbb{R}^2$ which is different from the dot product.
(3) Can you come up with other such examples ?

C: In this question, you will prove the following theorem

Theorem 1. *Let V be an inner product space vector space and let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be an orthonormal basis of V. Then for every $v \in V$,*

$$v = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \cdots + \langle v, u_n\rangle u_n$$

(1) Suppose $v \in V$ and $v = \sum_{i=1}^{n} c_i u_i$. Compute $\langle v, u_1\rangle$.
(2) Use your answer in part one to compute c_1.
(3) Prove the theorem.
(4) Restate the theorem with the word orthonormal replaced with orthogonal. Make any other necessary changes in order to get a correct statement. X

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

- An inner product on $\mathbb{R}^2$ other than the dot product.
- An inner product on $\mathbb{R}^3$ other than the dot product.
- Two vectors in $\mathbb{R}^3$ that are orthogonal with respect to dot product but not with respect to your example of inner product.
- A vector in $\mathbb{R}^3$ with length one with respect to dot product and a different length with respect to your example of inner product.
- Two different orthogonal bases of $\mathbb{R}^2$.
- A vector in an inner product space V that is orthogonal to every other vector.
- 4 mutually orthogonal vectors in $\mathbb{R}^3$.