MATB24 TUTORIAL PROBLEMS 7SOLUTIONS

KEY WORDS: inner product space, orthogonal decomposition, orthogonal projection, orthogonal complement

RELEVANT SECTIONS IN THE TEXTBOOK: 6.2 6.3 6.5 FB or 6.B 6.C SA

WARM-UP:

Write down a complete definition or a complete mathematical characterization

(1) Orthogonal decomposition of a vector $\vec{u}$ in an inner product space V onto a vector v
(2) orthogonal complement of a subspace W of an inner product space V
(3) An orthogonal basis for a vector space W
(4) An orthonormal basis for a vector space W

A:

(1) Let V_1 be the subspace of $\mathbb{R}^4$ given by

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 0, \\ x_2 + x_3 - 2x_4 &= 0. \end{aligned}$$

Find an orthonormal basis of V_1.

(2) Let V_2 be the subspace of $\mathbb{R}^4$ given by

$$x_1 + x_2 - x_3 - 2x_4 = 0.$$

Find an orthonormal basis of V_2.

(3) Let $\vec{w}$ be the vector

$$\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \in \mathbb{R}^4.$$

Find the orthogonal projections of $\vec{w}$ onto the subspaces V_1, V_2.

Solution.

(1) Writing the equations as rows yields the matrix $\begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$. Letting the last two columns be the free variables r, s, the nullspace is given by $[(-s+2r)+2s, -s+2r, s, r]$, which is spanned by the following two vectors:

$$(\vec{u}, \vec{v}) = \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right),$$

These are already orthogonal, so normalizing gives the orthonormal basis $\left(\frac{1}{\sqrt{3}}\vec{u}, \frac{1}{3}\vec{v}\right)$.

(2) By inspection, $\vec{u} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}0\\2\\0\\1\end{bmatrix}$ are orthogonal vectors in V_2, and $\vec{z} = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \in V_2$ is independent of $\vec{u}$ and $\vec{v}$ (e.g. the zeros in components do not match) and orthogonal to $\vec{u}$. So

$$(\vec{u}, \vec{v}, \vec{z} - \text{proj}_{\vec{v}}(\vec{z})) = \left(\vec{u}, \vec{v}, \vec{z} - \frac{\vec{z}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right) = \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-\frac{2}{5}\\-1\\\frac{4}{5}\end{bmatrix}\right)$$

is an orthogonal set in V_2 (e.g. $\vec{z}_{\vec{v}^\perp} = z - \text{proj}_{\vec{v}}(\vec{z})$, so this vector is orthogonal to $\vec{v}$, and since $\text{proj}_{\vec{v}}(\vec{z})$ is parallel to $\vec{v}$ which is orthogonal to $\vec{u}$, and since $\vec{z}$ is orthogonal to $\vec{u}$, therefore $\vec{z}_{\vec{v}^\perp}\cdot\vec{u} = \vec{z}\cdot\vec{u} - \text{proj}_{\vec{v}}(\vec{z})\cdot u = 0 - 0 = 0$), and by normalizing we get an orthonormal basis of V_2.

(3) Letting $\vec{u}, \vec{v}$ be as in the solution to Problem 1, we have

$$\text{proj}_{V_1}(\vec{w}) = \left(\vec{w}\cdot\frac{1}{\sqrt{3}}\vec{u}\right)\frac{1}{\sqrt{3}}\vec{u} + \left(\vec{w}\cdot\frac{1}{3}\vec{v}\right)\frac{1}{3}\vec{v} = \frac{1}{9}\begin{bmatrix}10\\22\\-6\\8\end{bmatrix}.$$

By a similar computation using Problem (2),

$$\text{proj}_{V_2}(\vec{w}) = \begin{bmatrix}1\\2\\-1\\2\end{bmatrix}.$$

(4) This is the Gram-Schmidt process. Convert the basis $(\vec{b}_1, \ldots, \vec{b}_r)$ of V into the orthonormal basis $(\vec{u}_1, \ldots, \vec{u}_r)$ of V by inductively defining $\vec{u}_k$ to be the normalization of

$$\vec{b}_k - \sum_{i=1}^{k-1}\frac{\vec{b}_k\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\vec{u}_i.(\vec{b}_k\cdot\vec{u}_1)\vec{u}_1 - \cdots - (\vec{b}_k\cdot\vec{u}_{k-1})\vec{u}_{k-1}.$$

B: Let W be a subspace of an inner product space V. We proved (or will prove) in class that for every $\vec{x} \in V$ there is a unique vector $\text{proj}_W\vec{x} \in V$ such that $\vec{x} - \text{proj}_W\vec{x}$ is orthogonal to W. The vector $\text{proj}_V\vec{x}$ is called the *orthogonal projection* of $\vec{x}$ onto W, and we found a formula for it: if $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis of W, then

$$\text{proj}_V(\vec{x}) = \sum_{i=1}^{n}\langle\vec{x}, \vec{u}_i\rangle\vec{u}_i.$$

(1) Prove that the transformation $\text{proj}_V : V \to V$ is linear.
(2) What is the kernel of proj_V?
(3) Write the rank nullity equation for this transformation. Is it familiar?

(4) Suppose $V = \mathbb{R}^m$. Since the orthogonal projection $\text{proj}_W : \mathbb{R}^m \to \mathbb{R}^m$ is linear, is there an $m \times m$ matrix P such that $\text{proj}_V(\vec{x}) = P\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$? [1]

Solution.

(1) Using the formula $\text{proj}_V(\vec{x}) = \sum_{i=1}^n (\vec{x} \cdot \vec{u}_i)\vec{u}_i$, this follows from the linearity of the dot product.

(2) $V^\perp$. For example, if $\text{proj}_V(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{x} - \text{proj}(\vec{x})$ is orthogonal to V, i.e. $\vec{x} \in V^\perp$.

(3) Since $V = \text{Im}(\text{proj}_V)$ and $V^\perp = \ker(\text{proj}_V)$, the rank-nullity theorem says $\dim(V) + \dim(V^\perp) = m$

(4) If A is the $m \times n$ matrix with columns $\vec{u}_1, \ldots, \vec{u}_n$, then $\text{proj}(\vec{x}) = A(A^TA)^{-1}A^T\vec{x}$. The following is a brief explanation. Since $\text{proj}(\vec{x}) \in V$, and the column space of A is precisely V, therefore $\text{proj}(\vec{x}) = A\vec{r}$ for some $\vec{r} \in \mathbb{R}^n$. Then, since $\vec{x} - A\vec{r}$ is orthogonal to V, therefore for any $\vec{y} \in \mathbb{R}^n$, $(\vec{x} - A\vec{r}) \cdot A\vec{y} = 0$, i.e. $(A\vec{y})^T(\vec{x} - A\vec{r}) = \vec{y}^T(A^T\vec{x} - A^TA\vec{r}) = 0$. Since this holds for all such $\vec{y}$, it follows that $A^T\vec{x} - A^TA\vec{r} = 0$. Since A is invertible due to its columns being independent, therefore A^TA is invertible, and hence $\vec{r} = (A^TA)^{-1}A^T\vec{x}$. Then, $\text{proj}_V(\vec{x}) = A\vec{r} = A(A^TA)^{-1}A^T\vec{x}$.

C: Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(1) Find a basis for $V^\perp$.

Solution. We are looking for vectors $[a \quad b \quad c]^T$ that have dot product zero with both $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Dotting with the second vector, we see $b = 0$. Dotting with the first, we see $a = -c$. So we get $V^\perp$ is the one dimensional subspace with basis $[1 \quad 0 \quad -1]^T$.

(2) Find an orthonormal basis for V. Use it to compute the projection of $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane V. Now write $\vec{w} = \vec{w}_1 + \vec{w}_2$ where $w_1 \in V$ and $\vec{w}_2 \in V^\perp$. Is this expression unique?

Solution. We can take $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as the required orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ in the theorem. Then the projection is

$$(\vec{w} \cdot \vec{u}_1)\vec{u}_1 + (\vec{w} \cdot \vec{u}_2)\vec{u}_2 = 2\sqrt{2}\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

[1] we will do this in class

Now we can take this projection to be $\vec{w}_1$, and $\vec{w}_2$ therefore must be $\vec{w} - \vec{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, which is in $V^\perp$. So the decomposition of $\vec{w}$ into its components in V and $V^\perp$ is

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is unique.

(3) Find the standard matrix of proj_V.

Solution. $A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$.

(4) Find the kernel of proj_V. How does this compare to $V^\perp$?

Solution. The kernel is the solutions of $A\vec{x} = \vec{0}$. Solving, we see the kernel is the one dimensional space with basis $[1 \ \ 0 \ \ -1]^T$. This is the same as $V^\perp$! This makes sense, since the line through the origin perpendicular to the plane consists exactly of the vectors which map to zero under the projection to the plane.

(5) What is $\dim V^\perp + \dim V$? How can you see this an instance of rank-nullity?

Solution. We see $1 + 2 = 3$. This is rank nullity for π_V, since V is the image of π_V, $V^\perp$ is the kernel of π_V, and $\mathbb{R}^3$ is the source of $\mathbb{R}^3$.

(6) Explain why the union of an orthonormal basis for V and one for $V^\perp$ are a basis for $\mathbb{R}^3$. What is the matrix of π_V in this basis?

Solution. V and $V^\perp$ have only the zero vector in common. So a basis for V and a basis for $V^\perp$ will combine to get a basis for $V + V^\perp = \mathbb{R}^n$. To see why it is an orthonormal basis: we already know all have unit length. We also know that anything in V has dot product against anything in $V^\perp$. And the dot product of two elements both in the orthonormal basis for V or both in the orthonormal basis $V^\perp$ also produces zero.

D:

Let V and W be subspaces of coordinate vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, and suppose that $T : V \to W$ is an isomorphism.

(1) If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_r\}$ is a basis of V, is the set $\{T(\vec{b}_1), \dots, T(\vec{b}_r)\}$ necessarily a basis of W?
(2) If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_r\}$ is an orthonormal basis of V, is the set $\{T(\vec{b}_1), \dots, T(\vec{b}_r)\}$ necessarily an orthonormal basis of W?

(3) Which $n \times n$ matrices A do you expect to have the property that for every orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_n\}$ of $\mathbb{R}^n$, the set $\{A\vec{u}_1, \dots, A\vec{u}_n\}$ is also an orthonormal basis of $\mathbb{R}^n$? Can you come up with a few examples that works and a few that doesn't? [2]

Solution.

(1) Yes, because an isomorphism always map basis to basis.
(2) No! For instance, dilating $\mathbb{R}^2$ by a factor of 2 (i.e. $T(\vec{x}) = 2\vec{x}$) converts the orthonormal basis $(\vec{e}_1, \vec{e}_2)$ of $\mathbb{R}^2$ into the non-orthonormal basis $(2\vec{e}_1, 2\vec{e}_2)$.
(3) By taking $(\vec{u}_1, \dots, \vec{u}_n)$ to be the standard basis of $\mathbb{R}^n$, we see that the columns of A must form an orthonormal basis of $\mathbb{R}^n$. In fact, this is also a sufficient condition, since a product of orthogonal $n \times n$ matrices is itself an orthogonal matrix. We will be better equipped to understand this after defining *orthogonal* transformations and matrices on the next week.

[2] we will revisit this question in class soon