MATA22 Assignment 5
Due Date: April 3rd

1. Let $A, B \in M_{n\times n}$, where n is an integer greater than 1. Is true that

$$det(A+B) = det(A) + det(B)$$

   If so, then give a proof. If not, then give a counterexample.

2. Let $A \in M_{3\times 3}$ and $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and they are linearly independent. Suppose that we have

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, A\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A\vec{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Then find the value of the determinant of the matrix A.

   Solution: Let $B \in M_{3\times 3}$ whose column vectors are $\vec{x}.\vec{y}$ and $\vec{z}$. Then you may have the following identity:

$$AB = \begin{bmatrix} | & | & | \\ A\vec{x} & A\vec{x} & A\vec{x} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   Therefore:

$$det(A)det(B) = det(AB) = det(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}) = 0$$

   while you should notice that $det(B) \neq 0$. So $det(A) = 0$.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 9 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Then find the value of

$$det(A^2B^{-1}A^{-2}B^2).$$

4. Prove that the determinant of an $n \times n$ skew matrix is zero if n is odd number.

   Solution: Let A be an $n \times n$ skew-symmetric matrix and n is odd. Check the following identity:

   $$det(A) = det(A^T) = det(-A) = (-1)^n det(A) = -det(A).$$

   By $det(A) = -det(A)$, you know $det(A)$ must be zero.

This question helps you understand Eigenvalue and Eigenvector

1. Let $A = [a_{ij}] \in M_{n\times n}$ with the following properties.

   (a) $a_{ij} \geq 0$,

   (b) $a_{i1} + a_{i2} + ... + a_{in} = 1$

   for $1 \leq i, j \leq n$.
   1) show that A has an eigenvalue 1.

   Solution: Just notice that $A = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix}$.(Recall the definition of eigenvalue λ and eigenvector $\vec{v}$, $A\vec{v} = \lambda\vec{v}$.) Therefore the matrix A has an eigenvalue 1.

   2) The absolute value of any eigenvalue of A is less than or equal to 1.

   Solution Let λ be an eigenvalue of $A = [a_{ij}]$ correspond to eigenvector $\vec{v} = [v_1, v_2...v_n]$.

   $$A\vec{v} = \lambda\vec{v}$$

Check the matrix multiplication of left hand side of above equation. For each row, you may notice the following identity:

$$a_{i1}v_1 + a_{i2}v_2 + ... + a_{in}v_n = \lambda v_i$$

for $i = 1, 2...n$.
Let

$$|v_k| = max\{|v_1|, |v_2|...|v_n|\},$$

Note that $|v_k| > 0$ since otherwise $\vec{v} = 0$. For kth row of matrix A, check the following inequality:

$$|\lambda||v_k| = |a_{k1}v_1 + a_{k2}v_2 + ... + a_{kn}v_n|$$

$$\leq a_{k1}|v_1| + a_{k2}|v_2| + ... + a_{kn}|v_n|$$

$$\leq a_{k1}|v_k| + a_{k2}|v_k| + ... + a_{kn}|v_k|$$

$$= (a_{k1}...a_{kn})|v_k| = |v_k|$$

It follows that

$$\lambda \leq 1.$$

2. Let A be a square matrix. Prove that the eigenvalues of the A^T are the same as the eigenvalues of A.

3. Let $A \in M_{n\times n}(\mathbb{C})$. Its only eigenvalues are 6,69,666,999, possibly with multiplicities. what's the rank of the matrix $A + I_{n\times n}$.

4. Suppose that $A \in M_{n\times n}(\mathbb{R})$ and A is singular. Prove that for sufficiently small $\delta > 0$, the matrix $A - \delta I_{n\times n}$ is non-singular.

Solution

Let $p(x) = \Pi_i^n(\lambda_i - x)$ be the characteristic polynomial for A, where λ_i is a root of $p(x)$. (By definition, we may also call λ_i as eigenvalue for A.)
Let λ_{min} be the non-zero eigenvalue of A of the smallest absolute value. Then for any $0 < \delta < |\lambda_{min}|$, we have $det(A - \delta I_{n\times n}) = p(x = \delta) \neq 0$. (ie $x = \delta$ is not the root of $p(x)$.)
Therefore, $A - \delta I_{n\times n}$ is non-singular.

This question helps you understand Eigenvalue and Eigenvector

1. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

   1) Diagonalize the matrix A.

   Solution You just need to find the eigenvalues and corresponding eigenvectors for A.

$$\lambda_1 = -1, \vec{v}_1 = [1, -1], \lambda_2 = 5, \vec{v}_2 = [1, 2].$$

   Then we may diagonalize A as:

$$S^{-1}AS = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

   where $S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$.

   2) Diagonalize the matrix $A^3 - 5A^2 + 3A + I_{2\times 2}$.

   Solution You just need to notice the following facts:

   (a) $A^2 = SD^2S^{-1}$ where $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

   (b) $A^3 = SD^3S^{-1}$

   (c) $A^3 - 5A^2 + 3A + I_{2\times 2} = S(D^3 - 5D^2 + 3D + I)S^{-1}$

   By above facts, we will have the following diagonalization.

$$S^{-1}(A^3 - 5A^2 + 3A + I)S = D^3 - 5D^2 + 3D + I = \begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}.$$

   3) Compute A^{100}.

Solution

Just using $A^k = SD^kS^{-1}$, then you may compute A^{100} as following.

$$A^{100} = SD^{100}S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} (\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix})^{-1}$$

4) Compute $(A^3 - 5A^2 + 3A + I_{2\times 2})^{100}$. Let $w = 2^{100}$. Express your solution in terms of w.

Solution

use part 2) and repeat part 3).

2. Prove that if $A \in M_{n\times n}$ is a diagonalizable matrix with the following property.
$$\exists k \in \mathbb{Z}_{>0}, A^k = 0,$$
then A must be zero matrix.

3. Each eigenvalue of the real skew symmetric matrix A is either 0 or a purely imaginary.

Solution

Using $Av = \lambda v$. and $A\bar{v} = \bar{\lambda}\bar{v}$ (note that $\bar{A} = A$ since A is real. Check the following equality.

$$\bar{v}^T Av = \lambda \bar{v}^T \vec{v} = \lambda |v|^2$$

on the other hand.

$$\bar{v}^T Av = (Av)^T \bar{v} = v^T A^T \bar{v} = -v^T A\bar{v}$$
$$= -v^T \bar{\lambda}\bar{v} = -\bar{\lambda}|v|^2.$$

Therefore

$$-\bar{\lambda}|v|^2 = \lambda |v|^2.$$

implies

$$-\bar{\lambda} = \lambda$$

So λ is either 0 or a purely imaginary.

4. If $A_{n\times n}(\mathbb{C})$ with $A^2 = A$, then A is diagonalizable.