GHW5

Problem 1

1) Supp. A is unitarily diagonalizable

$A = UDU^*$, U is unitary
D is diagonal

$$
\begin{aligned}
AA^* &= UDU^*(UDU^*)^* \\
&= UDU^*UD^*U^* \\
&= UDD^*U^* && U^*U = I \\
&= UD^*DU^* && \text{dia matrices commute} \\
&= UD^*U^*UDU^* \\
&= (UDU^*)^*UDU^* \\
&= A^*A
\end{aligned}
$$

$\therefore$ A is normal

2) Supp A is normal, $AA^* = A^*A$

$$
\begin{aligned}
BB^* &= (U^*AU)(U^*AU)^* \\
&= U^*AUU^*A^*U && UU^* = I \\
&= U^*AA^*U \\
&= U^*A^*AU && A \text{ is normal} \\
&= U^*A^*UU^*AU \\
&= (U^*AU)^*(U^*AU) \\
&= B^*B
\end{aligned}
$$

$\therefore$ B is normal

Supp B is normal, $BB^* = B^*B$

$$
\begin{aligned}
AA^* &= UBU^*(UBU^*)^* \\
&= UBU^*UB^*U^* \\
&= UBB^*U^* \\
&= UB^*BU^* \\
&= UB^*U^*UBU^* \\
&= (UBU^*)^*(UBU^*) \\
&= A^*A
\end{aligned}
$$

$\therefore$ A is normal

Thus A is normal $\Leftrightarrow$ B is normal

3) Let $B = \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ & \ddots & \vdots \\ 0 & & b_{nn} \end{bmatrix}$ $B^* = \begin{bmatrix} \overline{b_{11}} & & 0 \\ \vdots & \ddots & \\ \overline{b_{n1}} & \cdots & \overline{b_{nn}} \end{bmatrix}$

Let $BB^* = (a_{ij})$, $B^*B = (c_{ij})$ we know $(a_{ij}) = (c_{ij})$

$a_{11} = \sum_{k=1}^{n} b_{k1}\overline{b_{k1}}$ $c_{11} = \overline{b_{11}}\, b_{11}$
$= \|b_{11}\|^2$

$= \sum \|b_{k1}\|^2$

$a_{11} = c_{11} \Rightarrow \sum \|b_{k1}\|^2 = \|b_{11}\|^2$

$\sum_{k=2}^{n} \|b_{k1}\|^2 = 0$

$\Rightarrow \|b_{21}\| = \cdots = \|b_{n1}\| = 0$

$\Rightarrow b_{21} = \cdots = b_{n1} = 0$

$a_{22} = \sum_{k=2}^{n} b_{k2}\overline{b_{k2}}$ $c_{22} = \overline{b_{12}}\, b_{12} + \overline{b_{22}}\, b_{22}$
$= \|b_{12}\|^2 + \|b_{22}\|^2$

$= \sum \|b_{k2}\|^2$

$a_{22} = c_{22} \Rightarrow \sum_{k=2}^{n} \|b_{k2}\|^2 = \|b_{21}\|^2 + \|b_{22}\|^2$
$= \|b_{22}\|^2$

$\sum_{k=3}^{n} \|b_{k2}\|^2 = 0$

$\Rightarrow b_{32} = \cdots = b_{n2} = 0$

Continue the process to see $\forall i,j \in \mathbb{N}$, $1 \le i,j \le n$, $i \ne j$, $b_{ij} = 0$

Thus B is diagonal

4) Supp. A is normal, $A^*A = AA^*$

By Schur's triangulation lemma all $A \in M_n(\mathbb{C})$ are unitarily equivalent to a upper triangular matrix

let $A = UDU^*$, U is unitary
D is upper triangular

By part 2) we know A is normal iff D is normal
Thus D is normal

By part 3) we know if D is normal and upper triangular it must be diagonal

Thus $A = UDU^*$, D is diagonal
U is unitary

$\therefore$ Normal matrices are unitarily diagonalizable

Problem 2

1) Supp for some basis $\mathcal{B}$ and LT $T: V \to V$

$[T]_{\mathcal{B}} = UDU^*$, U is unitary
D is diagonal

Let $v, w \in V$ $\mathcal{B}$ be a orthonormal basis of V

$$\begin{aligned}
\langle v, T(w)\rangle &= [T(w)]_{\mathcal{U}}^*[v]_{\mathcal{U}} \\
&= [w]_{\mathcal{U}}^* [T]_{\mathcal{U}}^* [v]_{\mathcal{U}} \\
&= [w]_{\mathcal{U}}^* (UDU^*)^* [v]_{\mathcal{U}} \\
&= [w]_{\mathcal{U}}^* (UD^*U^*)[v]_{\mathcal{U}} \\
&= [w]_{\mathcal{U}}^* (UDU^*)[v]_{\mathcal{U}} \quad D \text{ is real eigenvalues} \\
&= [w]_{\mathcal{U}}^* [T]_{\mathcal{U}} [v]_{\mathcal{U}} \\
&= [w]_{\mathcal{U}}^* [T(v)]_{\mathcal{U}} \\
&= \langle T(v), w\rangle \quad \therefore T \text{ is self-adj}
\end{aligned}$$

2) T is an isometry, $[T]_{\mathcal{U}}$ is unitary, $\mathcal{U}$ is a orthonormal basis

Let v, w be eigenvectors with eigenvalue λ

$$\begin{aligned}
\langle v, w\rangle &= \langle T(v), T(w)\rangle \\
&= \langle \lambda v, \lambda w\rangle \\
&= \lambda\bar{\lambda}\langle v, w\rangle \\
&= |\lambda|^2\langle v, w\rangle \quad v, w \neq \vec{0} \\
&\Rightarrow |\lambda|^2 = 1 \\
&\Rightarrow |\lambda| = 1
\end{aligned}$$

3) $[T]_{\mathcal{U}} = UDU^*$, U is unitary $\quad$ $\mathcal{U}$ is a orthonormal basis of V
D is diagonal

Show $[T]_{\mathcal{U}}$ is unitary

$$\begin{aligned}[T]_{\mathcal{U}}[T]_{\mathcal{U}}^* &= UDU^*(UDU^*)^* \\ &= UDU^*UD^*U^* \\ &= UDD^*U^* \\ &= UU^* \\ &= I_n\end{aligned}$$

$$DD^* = \begin{bmatrix} \lambda_1\bar{\lambda}_1 & & \\ & \ddots & \\ & & \lambda_n\bar{\lambda}_n \end{bmatrix} = \begin{bmatrix} |\lambda_1| & & 0 \\ & \ddots & \\ 0 & & |\lambda_n| \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$$

$\therefore [T]_{\mathcal{U}}$ is unitary and thus T is an isometry

Problem 3

1) $\begin{bmatrix} 0 & -1+i \\ 1+i & 0 \end{bmatrix}$

2) Supp S is skew-adj, $\langle S(v), w\rangle = -\langle v, S(w)\rangle$

$$i\langle S(v), w\rangle = \langle iS(v), w\rangle$$
$$= -i\langle v, S(w)\rangle$$
$$= \langle v, iS(w)\rangle$$
$$\Rightarrow \langle iS(v), w\rangle = \langle v, iS(w)\rangle$$
$\therefore$ iS is self-adj

Supp iS is self-adj, $\langle iS(v), w\rangle = \langle v, iS(w)\rangle$

$$i\langle iS(v), w\rangle = -\langle S(v), w\rangle$$
$$= i\langle v, iS(w)\rangle$$
$$= -i(i)\langle v, S(w)\rangle$$
$$= \langle v, S(w)\rangle$$
$\Rightarrow -\langle S(v), w\rangle = \langle v, S(w)\rangle$ S is skew adj

3)

a) Axler 7.15: T is self-adj $\Leftrightarrow \langle T(v), v\rangle \in \mathbb{R}$

From 2) $\langle iS(v), v\rangle \in \mathbb{R}$ bc iS is self-adj

$$\langle iS(v), v\rangle = i\langle S(v), v\rangle$$
$$\Rightarrow \langle S(v), v\rangle \in i\mathbb{R}$$

b) $\mathcal{B} = (b_1, \dots, b_n)$ is a Orthonormal eigen basis for iS by Spectral thm

$\lambda_i b_i = iS(b_i) = i(S(b_i))$

$\Rightarrow S(b_i) = -i\lambda_i b_i$

$$\langle S(b_i), b_i\rangle$$
$$= \langle -i\lambda_i b_i, b_i\rangle$$
$$= -i\lambda_i\langle b_i, b_i\rangle$$
$$= -i\lambda_i \in i\mathbb{R} \quad \text{as } \lambda_i \text{ are real}$$

Thus S has only imaginary eigen values

c) iS is self-adj $\Leftrightarrow$ $[iS]_{\mathcal{U}}$ is unitarily diagonalizable
iS has real eigenvalues

$$[iS]_{\mathcal{U}} = UDU^*$$

$$\begin{aligned} i[S]_{\mathcal{U}} &= UDU^* \\ [S]_{\mathcal{U}} &= -iUDU^* \\ &= U(-iD)U^* \quad \text{let } D' = -iD \\ &= UD'U^* \end{aligned}$$

Thus S is diagonalizable

d) $$\begin{aligned} ([iS]_{\mathcal{B}})^* &= (iI_n[S]_{\mathcal{B}})^* \\ &= -[S]_{\mathcal{B}}^*(iI_n) \\ &= -[S]_{\mathcal{B}}^*(iI_n) \quad \text{bc } S \end{aligned}$$

$$\begin{aligned} ([iS]_{\mathcal{B}})^* &= [iS]_{\mathcal{B}} \\ &= (iI_n)[S]_{\mathcal{B}} \end{aligned}$$

$$\begin{aligned} \Rightarrow -[S]_{\mathcal{B}}^*(iI_n) &= (iI_n)[S]_{\mathcal{B}} \\ -[S]_{\mathcal{B}}^* &= [S]_{\mathcal{B}} \\ [S]_{\mathcal{B}}^* &= -[S]_{\mathcal{B}} \end{aligned}$$

$\therefore$ S is skew-Hermitian

Problem 4

1) Supp $A^T = -A$

Let $v \in Nul(I+A)$

$$\begin{aligned} (I+A)v &= 0 \\ v + Av &= 0 \\ v^T + v^T A^T &= 0 \\ v^T - v^T A &= 0 \\ v^T &= v^T A \end{aligned}$$

$$\begin{aligned} (I+A)v &= 0 \\ v^T(I+A)v &= 0 \\ v^T v + v^T A v &= 0 \\ v^T v + v^T v &= 0 \\ 2\|v\|^2 &= 0 \\ v &= \vec{0} \end{aligned}$$

$\therefore$ $I + A$ is invertible

2) Supp $A^T = -A$

Let $B = (I_n - A)(I_n + A)^{-1}$

$$\begin{aligned} B^T &= ((I_n - A)(I_n + A)^{-1})^T \\ &= (I + A^T)^{-1}(I - A^T) \\ &= (I - A)^{-1}(I + A) \end{aligned}$$

$$\begin{aligned} B^T B &= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\ &= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\ &= I \cdot I \\ &= I \end{aligned}$$

Note: $(I-A)(I+A) = I - A + A - A^2 = I - A^2$
$(I+A)(I-A) = I + A - A - A^2 = I - A^2$

3) Supp A is ortho and $(I+A)$ is invertible

Let $B = (I-A)(I+A)^{-1}$, $-B = (A-I)(A+I)^{-1}$

$$\begin{aligned} B^T &= (I + A^T)^{-1}(I - A^T) \\ &= (I + A^T)^{-1} A^T (I - A^T) A \\ &= (I + A^T)^{-1} A^{-1} (I - A^T) A \\ &= ((I + A^T)A)^{-1} (A - I) \\ &= (A + I)^{-1}(A - I) \\ &= (A+I)^{-1}(A-I)(A+I)(A+I)^{-1} \\ &= (A+I)^{-1}(A+I)(A-I)(A+I)^{-1} \\ &= (A-I)(A+I)^{-1} \\ &= -B \quad \therefore B \text{ is skew-sym} \end{aligned}$$

$$\begin{aligned} (I - A^T) &= A^T(A - I) \\ &= A^T(I - A^T)A \end{aligned}$$