Solution to Assignment #6

1. Section 1.6: # 4, 8, 12, 14, 18, 20, 34, 36, 38, 42, 43, 44.

   4. Determine if $\{[x, y)\big|x, y \in \mathbb{R}, x, y \geq 0\}$ (the first quadrant of $\mathbb{R}^2$) is a subspace of $\mathbb{R}^2$.

      Answer. Let $W = \{[x, y)\big|x, y \in \mathbb{R}, x, y \geq 0\}$.
      W is not a subspace of $\mathbb{R}^2$ since a scalar multiplication of a nonzero vector from this set with a negative value will yield a vector not in the set.
      i.e. $\mathbf{v} = [2, 3] \in W$ since $2, 3 > 0$ but $-2\mathbf{v} = [-4, -6] \notin W$.
      Therefore does not satisfy closure under scalar multiplication.

   8. Determine if $\{[2x, x + y, y]\big|x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

      Answer: Let $W = \{[2x, x + y, y]\big|x, y \in \mathbb{R}\}$

      (i) Clearly this set is nonempty since for $x = y = 0$ the vector $[0, 0, 0] \in W$ showing W is not empty.

      (ii) If $\mathbf{v} = [2x_1, x_1 + y_1, y_1] \in W$ and $\mathbf{u} = [2x_2, x_2 + y_2, y_2]$ then, $\mathbf{v} + \mathbf{u} = [2x_1 + 2x_2, x_1 + y_1 + x_2 + y_2, y_1 + y_2] = [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), y_1 + y_2] \in W$ showing that there is closure under vector addition.

      (iii) If $\mathbf{v} = [2x, x+y, y] \in W$ and $r \in \mathbb{R}$then, $r\mathbf{v} = r[2x, x+y, y] = [2rx, rx+ry, ry] =\in W$ showing that there is closure under scalar multiplication.

      From (i), (ii), (iii) we see that W is a subspace of $\mathbb{R}^3$.

12. Let a, b, and c be scalars such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$.

Proof: For scalars a, b, c such that $abc \neq 0$, the described plane is the set $W = \{[x, y, z] \in \mathbb{R}^3 \big| ax + by + cz = 0\}$.
(i) Clearly if $x = y = z = 0$ then $ax+by+cz = 0$ and $[0, 0, 0] \in W$, showing W is non empty.
(ii) If $\mathbf{v} = [v_1, v_2, v_3] \in W$ and $\mathbf{u} = [u_1, u_2, u_3] \in W$ then $av_1 + bv_2 + cv_3 = 0$ and $au_1 + bu_2 + cu_3 = 0$.
Therefore $\mathbf{v} + \mathbf{u} = [v_1 + u_1, v_2 + u_2, v_3 + u_3]$ and:

$$\begin{aligned}
&a(v_1 + u_1) + b(v_2 + u_2) + c(v_3 + u_3) \\
&\quad = av_1 + au_1 + bv_2 + bu_2 + cv_3 + cu_3 \\
&\quad = (av_1 + bv_2 + cv_3) + (au_1 + bu_2 + cu_3) \\
&\quad = 0 + 0 = 0
\end{aligned}$$

showing that $\mathbf{v} + \mathbf{u} \in W$.
(iii) If $\mathbf{v} = [v_1, v_2, v_3] \in W$ and $r \in \mathbb{R}$ then $r\mathbf{v} = [rv_1, rv_2, rv_3]$ and $av_1 + bv_2 + cv_3 = 0$. Therefore:

$$a(rv_1) + b(rv_2) + c(rv_3) = r(av_1 + bv_2 + cv_3) = r(0) = 0$$

showing that $r\mathbf{v} \in W$.
(i), (ii), (iii) show that W is a subspace of $\mathbb{R}^3$.

14. Prove that every subspace of $\mathbb{R}^n$ contains the zero vector.

Proof: Let W be a subspace or $\mathbb{R}^n$. Then W is not empty. Let $\mathbf{v} \in W$. Since W is closed under scalar multiplication and $0 \in \mathbb{R}$, then $0\mathbf{v} = \mathbf{0} \in W$.
Thus, every subspace of R^n contains the zero vector.

18. Find the solution to the homogeneous system:

$$\begin{array}{rcrcrcrcl}
x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \\
 & & x_2 & + & x_3 & & & = & 0 \\
x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0
\end{array}$$

Solution: The system is equivalent to $A\mathbf{x} = \mathbf{0}$ and we need to find the REF or the RREF of the augmented matrix of this system.

$$[A|\mathbf{0}] = \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 3 & 0 \end{array}\right] \xRightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_3\to R_3-3R_2}\left[\begin{array}{rrrr|r}1&-1&1&-1&0\\0&1&1&0&0\\0&0&-5&4&0\end{array}\right]\xrightarrow{R_1\to R_1+R_2}\left[\begin{array}{rrrr|r}1&0&2&-1&0\\0&1&1&0&0\\0&0&-5&4&0\end{array}\right]$$

$$\xrightarrow{R_3\to -1/5R_3}\left[\begin{array}{rrrr|r}1&0&2&-1&0\\0&1&1&0&0\\0&0&1&-4/5&0\end{array}\right]\xrightarrow{R_2\to R_2-R_3}\left[\begin{array}{rrrr|r}1&0&2&-1&0\\0&1&0&4/5&0\\0&0&1&-4/5&0\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-2R_3}\left[\begin{array}{rrrr|r}1&0&0&3/5&0\\0&1&0&4/5&0\\0&0&1&-4/5&0\end{array}\right]=[H|\mathbf{0}]$$

H is the RREF of A and there is no pivot in the fourth column of H. Therefore we let x_4 be the parameter of the solution: $x_1=-3/5x_4,\ x_2=-4/5x_4,\ x_3=4/5x_4$. Therefore

$$\mathbf{x}=\begin{bmatrix}-3/5x_4\\-4/5x_4\\4/5x_4\\x_4\end{bmatrix}=x_4\begin{bmatrix}-3/5\\-4/5\\4/5\\1\end{bmatrix}.$$

The solution set is $\left\{s\begin{bmatrix}-3/5\\-4/5\\4/5\\1\end{bmatrix}\middle|\, s\in\mathbb{R}\right\}$ with basis the set $\left\{\begin{bmatrix}-3/5\\-4/5\\4/5\\1\end{bmatrix}\right\}$

20. Find the solution to the homogeneous system:

$$\begin{array}{rcrcrcrcl}2x_1&+&x_2&+&x_3&+&x_4&=&0\\3x_1&+&x_2&-&x_3&+&2x_4&=&0\\x_1&+&x_2&+&3x_3&&&=&0\\x_1&-&x_2&-&7x_3&+&2x_4&=&0\end{array}$$

Solution: The system is equivalent to $A\mathbf{x}=\mathbf{0}$ and we need to find the REF or the RREF of the augmented matrix of this system.

$$[A|\mathbf{0}]=\left[\begin{array}{rrrr|r}2&1&1&1&0\\3&1&-1&2&0\\1&1&3&0&0\\1&-1&-7&2&0\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{rrrr|r}1&-1&-7&2&0\\3&1&-1&2&0\\1&1&3&0&0\\2&1&1&1&0\end{array}\right]$$

$$\begin{array}{r}R_2\to R_2-3R_1\\R_3\to R_3-R_1\\R_4\to R_4-2R_1\end{array}\Longrightarrow\left[\begin{array}{rrrr|r}1&-1&-7&2&0\\0&4&20&-4&0\\0&2&10&-2&0\\0&3&15&-3&0\end{array}\right]$$

$$\begin{array}{r} 1/4R_2 \\ R_3 - 1/2R_2 \\ R_4 - 3/4R_2 \end{array} \Longrightarrow \left[\begin{array}{rrrr|r} 1 & -1 & -7 & 2 & 0 \\ 0 & 1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 3 and 4 have no pivots so we use x_3, x_4 as free variables.
The solution is then $x_1 = 2x_3 - x_4, x_2 = -5x_3 + x_4$

$$\mathbf{x} = \begin{bmatrix} 2x_3 - x_4 \\ -5x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The solution set is $\left\{ s \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

with basis the set $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

34. Solve the given linear system $\begin{bmatrix} 1 & -2 & 1 & 5 & | & 7 \end{bmatrix}$ and express the solution set in a form that illustrates Theorem 1.18.

Solution: $x_1 - 2x_2 + x_3 + 5x_4 = 7$ has augmented matrix: $\begin{bmatrix} 1 & -2 & 1 & 5 & | & 7 \end{bmatrix}$ which is already in a RREF with no pivots in columns 2, 3, 4, 5.
So the solution has 4 parameters and it is $x_1 = 7 + 2x_2 - x_3 - 5x_4$
So the solution vector is

$$\mathbf{x} = \begin{bmatrix} 7 + 2x_2 - x_3 - 5x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

36. Solve the given linear system $\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 4 \\ 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ x_1 & - & 7x_2 & - & 6x_3 & + & 2x_4 & = & 6 \end{array}$ and express the solution set in a form that illustrates Theorem 1.18.

Solution:

The system has augmented matrix: $\begin{bmatrix} 1 & -2 & 1 & 1 & | & 4 \\ 2 & 1 & -3 & -1 & | & 6 \\ 1 & -7 & -6 & 2 & | & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & | & 4 \\ 2 & 1 & -3 & -1 & | & 6 \\ 1 & -7 & -6 & 2 & | & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & | & 4 \\ 0 & 5 & -5 & -3 & | & -2 \\ 0 & -5 & -7 & 1 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & | & 4 \\ 0 & 5 & -5 & -3 & | & -2 \\ 0 & 0 & -12 & -2 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & | & 4 \\ 0 & 1 & -1 & -3/5 & | & -2/5 \\ 0 & 0 & 1 & 1/6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 5/6 & | & 4 \\ 0 & 1 & 0 & -13/30 & | & -2/5 \\ 0 & 0 & 1 & 1/6 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1/30 & | & 16/5 \\ 0 & 1 & 0 & -13/30 & | & -2/5 \\ 0 & 0 & 1 & 1/6 & | & 0 \end{bmatrix}$$

The RREF has no pivots in column 4. So the solution has 1 parameter and it is $x_1 = 16/5 + 1/30x_4,\ x_2 = -2 + 13/30x_4,\ x_3 = -1/6x_4$

So the solution vector is

$$\mathbf{x} = \begin{bmatrix} 16/5 + 1/30x_4 \\ -2/5 + 13/30x_4 \\ -1/6x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/30 \\ 13/30 \\ -1/6 \\ 1 \end{bmatrix}$$

38. F T T T F F T T F T

42. Use Theorem 1.13 to explain why a homogeneous system of linear equations has either a unique solution or an infinite number of solutions.

Proof: Clearly if there is a nonzero solution, $\mathbf{v}$, to a homogeneous system $A\mathbf{x} = \mathbf{0}$ then $A\mathbf{v} = \mathbf{0}$ then by the above theorem $r\mathbf{v}$ is also a solution for each $r \in \mathbb{R}$, giving infinitely many solutions.

If there is no nonzero solution to $A\mathbf{x} = \mathbf{0}$ then the zero solution is the only one.

Theorem 1.13: Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous linear system. If $\mathbf{h}_1$ and $\mathbf{h}_2$ are soiutions of $A\mathbf{x} = \mathbf{0}$, then so is the linear combination $r\mathbf{h}_1 + s\mathbf{h}_2$ for any scalars r and s.

43. Use Theorem 1.18 to explain why no system of linear equations can have exactly two solutions.

    Proof: Let $A\mathbf{x} = \mathbf{b}$ represent a linear system. Suppose it is consistent and suppose that there are distinct vectors $\mathbf{v}, \mathbf{u}$ so that $A\mathbf{v} = \mathbf{b}$, $A\mathbf{u} = \mathbf{b}$. Then $\mathbf{v} - \mathbf{u} = \mathbf{h}$ is a nonzero solution to $A\mathbf{x} = \mathbf{0}$ since $A\mathbf{h} = A(\mathbf{v} - \mathbf{u}) = A\mathbf{v} - A\mathbf{u} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

    Since $\mathbf{h}$ is a nonzero solution to $A\mathbf{x} = \mathbf{0}$, then by question 42 $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Therefore, by theorem 1.18, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions .

    Theorem 1.18: Let $Ax = \mathbf{b}$ be a linear system. If $\mathbf{p}$ is any particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{h}$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$, then $\mathbf{p} + \mathbf{h}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Moreover, every solution of $A\mathbf{x} = \mathbf{b}$ has this form $\mathbf{p} + \mathbf{h}$, so that the general solution is $\mathbf{x} = \mathbf{p} + \mathbf{h}$ where $A\mathbf{h} = \mathbf{0}$.

44. Let A be an $m \times n$ matrix such that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
    a. Does it follow that every system $A\mathbf{x} = \mathbf{b}$ is consistent?
    b. Does it follow that every consistent system $A\mathbf{x} = \mathbf{b}$ has a unique solution?

    Answer: a) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution only says that the matrix A has a RREF with a pivot in every column. However, this is not sufficient to say that $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$. If $m > n$ then the RREF of $[A|\mathbf{b}]$ may have a pivot in the column after the partition, in which case the system would be inconsistent.

    A counterexample to (a), is the system

    $$\begin{array}{rcrl} x & + & y = & 0 \\ x & + & 2y = & 0 \\ 2x & + & 3y = & 0 \end{array} \text{ with: } \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

    has only the trivial solution. However

    $$\begin{array}{rcrl} x & + & y = & 1 \\ x & + & 2y = & 1 \\ 2x & + & 3y = & 1 \end{array} \text{ with: } \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

    has no solution.

b) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then if $A \sim H$ and H is in a RREF, then H has a pivot in every column. This shows that the columns of A are linearly independent.

Approach a) Since $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ can be expressed as a linear combination of the columns or A. Since these columns are linearly independent, then their linear combination that equals $\mathbf{b}$ must be unique. This shows that $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Approach b) Since H has a pivot in every column then $H = \begin{bmatrix} I \\ O \end{bmatrix}$ where I is the $n \times n$ identity matrix and O is the $(m-n) \times n$ zero matrix.

Since $A\mathbf{x} = \mathbf{b}$ is consistent this means that $\mathbf{c}$ in $[H|\mathbf{c}]$ has no pivot, therefore $H\mathbf{x} = \mathbf{c}$ has a unique solution. ($[A|\mathbf{b}] \sim [H|\mathbf{c}] = \left[\begin{array}{c|c} I & \mathbf{c}' \\ O & \mathbf{0} \end{array}\right]$)

45. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^n$. Prove the following set equalities by showing that each of the spans is contained in the other.
a. $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$
b. $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$

Proof: We can prove the following lemma:
Lemma A: If each of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is an element of $sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)$ then $sp(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \subseteq sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)$.
Proof of lemma A: Since if $\mathbf{x}_i \in sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)$, then:

$$\begin{aligned}
\mathbf{x}_1 &= a_{11}\mathbf{y}_1 + a_{12}\mathbf{y}_2 + \cdots + a_{1r}\mathbf{y}_r \\
\mathbf{x}_2 &= a_{21}\mathbf{y}_1 + a_{22}\mathbf{y}_2 + \cdots + a_{2r}\mathbf{y}_r \\
&\vdots \\
\mathbf{x}_k &= a_{k1}\mathbf{y}_1 + a_{k2}\mathbf{y}_2 + \cdots + a_{kr}\mathbf{y}_r
\end{aligned}$$

therefore

$$\begin{aligned}
s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k &= s_1(a_{11}\mathbf{y}_1 + a_{12}\mathbf{y}_2 + \cdots + a_{1r}\mathbf{y}_r) \\
&+ s_2(a_{21}\mathbf{y}_1 + a_{22}\mathbf{y}_2 + \cdots + a_{2r}\mathbf{y}_r) \\
&\vdots \\
&+ s_k(a_{k1}\mathbf{y}_1 + a_{k2}\mathbf{y}_2 + \cdots + a_{kr}\mathbf{y}_r) \\
&\in sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)
\end{aligned}$$

Thus, every linear combination of the $\mathbf{x}_i$'s is in $sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)$. By Lemma A: $sp(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \subseteq sp(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r)$.

a. Clearly $\mathbf{v}_1 \in sp(\mathbf{v}_1, \mathbf{v}_2)$ and $2\mathbf{v}_1 + \mathbf{v}_2 \in sp(\mathbf{v}_1, \mathbf{v}_2)$
Therefore by Lemma A: $sp(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2) \subseteq sp(\mathbf{v}_1, \mathbf{v}_2)$. $\quad (*)$

Also $\mathbf{v}_1 \in sp(\mathbf{v}_1, 2\mathbf{v}_1+\mathbf{v}_2)$, $\mathbf{v}_2 = (2\mathbf{v}_1+\mathbf{v}_2)-2\mathbf{v}_1 \in sp(\mathbf{v}_1, 2\mathbf{v}_1+\mathbf{v}_2)$,
therefore by Lemma A: $sp(\mathbf{v}_1, \mathbf{v}_2) \subseteq sp(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$. $\quad (**)$
By $(*)$ and $(**)$ we have that $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$.

b. Similarly:
$\mathbf{v}_1 + \mathbf{v}_2 \in sp(\mathbf{v}_1, \mathbf{v}_2)$ and $\mathbf{v}_1 - \mathbf{v}_2 \in sp(\mathbf{v}_1, \mathbf{v}_2)$
therefore by Lemma A: $sp(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) \in sp(\mathbf{v}_1, \mathbf{v}_2)$. $\quad (*)$

Also $\mathbf{v}_1 = (1/2)(\mathbf{v}_1 + \mathbf{v}_2) + (1/2)(\mathbf{v}_1 - \mathbf{v}_2)$ and
$\mathbf{v}_2 = (1/2)(\mathbf{v}_1 + \mathbf{v}_2) - (1/2)(\mathbf{v}_1 - \mathbf{v}_2)$.
Therefore by Lemma A: $sp(\mathbf{v}_1, \mathbf{v}_2) \subseteq sp(\mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1-\mathbf{v}_2)$. $\quad (**)$
By $(*)$ and $(**)$ we have that $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$

2. Let V and W be subspaces of $\mathbb{R}^n$. Prove that the set $V + W = \{\mathbf{v} + \mathbf{w} \mid \mathbf{v} \in V,\ \mathbf{w} \in W\}$ is a subspace of $\mathbb{R}^n$.

   Proof: (i) Since V and W be subspaces of $\mathbb{R}^n$ then they are non empty. Let $\mathbf{v} \in V$ and $\mathbf{w} \in W$, the $\mathbf{v} + \mathbf{w} \in V + W$ showing that $V + W$ is a nonempty subset of $\mathbb{R}^n$.

   (ii) Let $\mathbf{a}, \mathbf{b} \in V+W$, then there are vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{a}$ and $\mathbf{v}_2 + \mathbf{w}_2 = \mathbf{b}$. Thus:

$$\mathbf{a} + \mathbf{b} = \mathbf{v}_1 + \mathbf{w}_1 + \mathbf{v}_2 + \mathbf{w}_2 = (\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in V + W$$

   since $\mathbf{v}_1 + \mathbf{v}_2 \in V$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$ since both V, W are subspaces of $\mathbb{R}^n$ and therefore closed under addition.

   (iii) Let $\mathbf{a}, \in V + W$, then there are vectors $\mathbf{v}_1 \in V$ and $\mathbf{w}_1 \in W$ such that $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{a}$. For any $r \in \mathbb{R}$, we have:

$$r\mathbf{a} = r(\mathbf{v}_1 + \mathbf{w}_1) = r\mathbf{v}_1 + r\mathbf{w}_1 \in V + W$$

   since $t\mathbf{v}_1 \in V$ and $r\mathbf{w}_1 \in W$ since both V, W are subspaces of $\mathbb{R}^n$ and therefore closed under scalar multiplication.

3. Section 2.1: # 1, 2, 5, 12, 14, 15, 20, 22, 24, 28, 30, 32, 34.

1. Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^2$ to be dependent.

   Answer: Two nonzero vectors $\mathbf{v}$, $\mathbf{u}$ in R^2 are dependent if there is a nonzero solution to $r\mathbf{v} + s\mathbf{u} = \mathbf{0}$ that is $r\mathbf{v} = -s\mathbf{u}$ for some nonzero numbers r and s. This is equivalent to $\mathbf{v} = -s/r\mathbf{u}$ or that the two vectors are parallel.

   On the other hand if the two vectors are parallel then they are also linearly dependent by default.

2) Argue geometrically that any set of three distinct vectors in $\mathbb{R}^2$ is dependent.

   Answer: Since the span of any two nonzero, nonparallel vectors in $\mathbb{R}^2$ is the entire $\mathbb{R}^2$, then a set of 3 distinct vectors will contain a vector which can be written as a linear of the other two.

   If at least one of the vectors is the zero vector, then by definition the set is linearly dependent.

5) Give a geometric criterion for a set of three distinct nonzero vectors in $\mathbb{R}^3$ to be dependent.

   Answer: Three vectors in $\mathbb{R}^3$ are dependent if and only if they all lie in one plane that goes through the origin.

12) Find a basis for the column space of the given matrix A.

   Answer: $$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 \\ 0 & -33 & -9 \\ 0 & -11 & -3 \\ 0 & -44 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 3/11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   The pivots in columns 1 and 2 show that the first two columns vectors of A form a basis for the column space of A.

   $$\text{Basis of } col(A) = \left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 7 \\ -2 \end{bmatrix} \right\}$$

14) Find a basis for the column space of the given matrix A.

   Answer: $$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -6 & -12 \\ 0 & -7 & -7 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   The pivots in columns 1 and 2 show that the first two columns vectors of A form a basis for the column space of A.

   $$\text{Basis of } col(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

15) Find a basis for the row space of the given matrix A (from question 12).

Answer: Since the row space of A is the column space of A^T, we reduce A^T

$$A^T = \begin{bmatrix} 2 & 5 & 1 & 6 \\ 3 & 2 & 7 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 1 & 6 \\ 0 & -1 & 1 & -2 \\ 0 & 3 & -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots in columns 1 and 2 show that the first two columns vectors of A^T form a basis for the row space of A.

Basis of $row(A) = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \right\}$

Equivalently, since $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 \\ 0 & -33 & -9 \\ 0 & -11 & -3 \\ 0 & -44 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 3/11 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = H$

Then a basis for the row space of A is the set of the nonzero rows of H. This is because $row(H) = row(A)$.

Basis of $row(A) = \{[1, 7, 2], [0, 1, 3/11]\}$

20) Determine if the set of vectors $\{[2, 1], [-6, -3], [1, 4]\}$ is dependent or independent.

Answer: From question 2 above we know that any set of 3 vectors in $\mathbb{R}^2$ is dependent, so the set $\{[2, 1], [-6, -3], [1, 4]\}$ is dependent.

22) Determine if the set of vectors $\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\}$ is dependent or independent.

Answer: Putting the vectors as columns of a matrix and using row reduction we get

$$\begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 5 \end{bmatrix}$$

The row reduced echelon form of the matrix has a pivot in each column so the set of vectors is independent.

24) Determine if the set of vectors$\{[1, 4, -1, 3], [-1, 5, 6, 2], [1, 13, 4, 7]\}$ in $\mathbb{R}^4$ linearly dependent.

Answer: Putting the vectors as columns of a matrix and using row reduction we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 5 & 13 \\ -1 & 6 & 4 \\ 3 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 9 & 9 \\ 0 & 5 & 5 \\ 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The row reduced echelon form of the matrix has a pivot in each column so the set of vectors is independent.

28) T F T F T T F T T T T T F T

30) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be independent vectors in $\mathbb{R}^n$. Prove that $\mathbf{w}_1 = 3\mathbf{v}_1, \mathbf{w}_2 = 2\mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_3$ are also independent.

Proof: Suppose that $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 = \mathbf{0}$, then

$$r_1(3\mathbf{v}_1) + r_2(2\mathbf{v}_1 - \mathbf{v}_2) + r_3(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0},$$

giving $(3r_1 + 2r_2 + r_3)\mathbf{v}_1 + (-r_2)\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

Since we are given that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are independent we conclude that

$3r_1 + 2r_2 + r_3 = 0$, $-r_2 = 0$, $r_3 = 0$

With the only solution $r_1 = r_2 = r_3 = 0$, thus showing that the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent.

32) Find all scalars s, if any exist, such that $[1,0,1], [2,s,3], [2,3,1]$ are independent.

Solution: To find the answer to this question we consider the solution to a linear combination of the vectors that equals to the zero vector. Letting $r_1[1,0,1] + r_2[2,s,3] + r_3[2,3,1] = [0,0,0]$ we look for the solution:

$$[r_1 + 2r_2 + 2r_3, sr_2 + 3r_3, r_1 + 3r_2 + r_3] = [0,0,0],$$

giving the system $\begin{array}{rrrcl} r_1+ & 2r_2+ & 2r_3 & = & 0 \\ & sr_2+ & 3r_3 & = & 0 \\ r_1+ & 3r_2+ & r_3 & = & 0 \end{array}$ with augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & s & 3 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & s & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3+s & 0 \end{array}\right]$$

There is a dependence relation if and only if $s = -3$. Thus the given vectors are independent for all $s \neq -3$.

34) Let $\mathbf{v}$ and $\mathbf{w}$ be independent column vectors in $\mathbb{R}^3$, and let A be an invertible 3×3 matrix. Prove that the vectors $A\mathbf{v}$ and $A\mathbf{w}$ are independent.

Proof: Let $s(A\mathbf{v}) + r(A\mathbf{w}) = \mathbf{0}$, then

$A(s\mathbf{v} + r\mathbf{w}) = \mathbf{0}$ by left distributive property of matrix multiplication.

$A^{-1}A(s\mathbf{v} + r\mathbf{w}) = A^{-1}\mathbf{0}$, since A is invertible.

$I(s\mathbf{v} + r\mathbf{w}) = \mathbf{0}$ Since $A^{-1}A = I$ and $A^{-1}\mathbf{0} = \mathbf{0}$.

Since $s\mathbf{v} + r\mathbf{w} = \mathbf{0}$ and $\mathbf{v}$, $\mathbf{w}$ are independent vectors, then $r = s = 0$ is the only solution.

Thus if $s(A\mathbf{v}) + r(A\mathbf{w}) = \mathbf{0}$, then $r = s = 0$ is the only solution.

This proves that the vectors $A\mathbf{v}$ and $A\mathbf{w}$ are independent.

4. Section 2.2: # 4, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.

4. Find the rank of the given matrix, a basis for the row space, a basis for the column space and a basis for the nullspace.

Solution:

$$\begin{bmatrix} 3 & 1 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \leftrightarrow R_1]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1, R_4-3R_1]{R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 7 & -1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 7 & -1 \end{bmatrix} \xrightarrow[R_4-R_2]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} \xrightarrow[1/5R_4]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow[R_4+2R_3]{} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2+R_4]{R_1-R_4} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-2R_3]{R_1+R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) The rank of the matrix is 4.

b) The row space has basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. c) The column space has basis $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

d) The nullspace of the matrix $= \{\mathbf{0}\}$ therefore the basis of the nullspace is the empty set.

6. Find the rank of the given matrix, a basis for the row space, a basis for the column space and a basis for the nullspace.

Solution:

$$\begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -4 & 4 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_4 - R_1]{-R_1} \begin{bmatrix} 4 & -4 & -1 & -4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ 0 & -4 & 0 & -2 \end{bmatrix} \xrightarrow{1/4R_1}$$

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ 0 & -4 & 0 & -2 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{} \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix} \xrightarrow[R_4 - 2R_2]{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 5/4 & -1/2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\xrightarrow[1/6R_4]{-4/25R_3} \begin{bmatrix} 1 & 0 & 5/4 & -1/2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_4 - R_3]{R_1 - 5/4R_3,\ R_2 - 3R_3} \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) The rank of the matrix is 3.

b) The row space has basis $\{[1, 0, 0, -1/2], [0, 2, 0, 1], [0, 0, 1, 0]\}$.

c) The column space has basis $\left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

d) For the nullspace of the matrix we see that $x_3 = 0$, $2x_2 + x_4 = 0$ so $x_2 = -x_4/2$ and $x_1 - 1/2x_4 = 0$ giving $x_1 = 1/2x_4$.

The basis of the nullspace is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

10. Determine whether the given matrix is invertible by finding its rank.

Solution: $$\begin{bmatrix} 3 & 0 & -1 & 2 \\ 4 & 2 & 1 & 8 \\ 1 & 4 & 0 & 1 \\ 2 & 6 & -3 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 4 & 2 & 1 & 8 \\ 3 & 0 & -1 & 2 \\ 2 & 6 & -3 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_1, R_4 - 2R_1]{R_2 - 4R_1}$$

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -14 & 1 & 4 \\ 0 & -12 & -1 & -1 \\ 0 & -2 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & -12 & -1 & -1 \\ 0 & -14 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 - 6R_2,\ R_4 - 7R_2]{-R_2} \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 17 & 5 \\ 0 & 0 & 22 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & -7/2 \end{bmatrix}$$

The rank is 4 so the matrix is invertible.

11. T F T T F F T F F T
12. Prove that, if A is a square matrix, the nullity of A is the same as the nullity of A^T.

    Proof: Let A be an $n \times n$ matrix. Then

    $$null(A) = n - rank(A) = n - rank(A^T) = null(A^T).$$

14. Prove that the column space of AC is contained in the column space of A.

    Proof: Let C be an $m \times n$ matrix. Every vector in the column space of AC is a linear combination of the columns of A, therefore it is of the form $\mathbf{v} = (AC)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{v} = A(C\mathbf{x})$ which is a vector in the column space of A. Thus $colspace(AC) \subseteq colspace(A)$.

15. Is it true that the column space of AC is contained in the column space of C? Explain.

    Solution. No it is not true that the column space of AC is contained in the column space of C. Here is a counterexample. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, then the column space of $AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the column space of C is the span of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So clearly $colspace(AC) \not\subseteq colspace(C)$.

16. The row space of AC is contained in the row space of C.
17. Give an example of a 3×3 matrix A such that $rank(A) = 2$ and $rank(A^3) = 0$.

    Example: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then A^3 is the zero matrix. So $rank(A) = 2$ and $rank(A^3) = 0$.

18. Prove that $rank(AC) \leq rank(A)$.

    Proof: By question 14 above we have $dim(colspace(AC) \leq dim(colspace(A))$. That is $rank(AC) \leq rank(A)$.

19. Give an example where $rank(AC) < rank(A)$.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $AC = C$, $rank(A) = 2$ and $rank(AC) = 1$ thus $rank(AC) < rank(A)$.

(a) Prove that any set $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of non-zero pairwise-orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$. (pairwise orthogonal means $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.)

Solution. We need to prove that

$$r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_n\mathbf{u}_n = \mathbf{0}$$

has only the trivial solution.

Let $j \in \{1, 2, \cdots, n\}$ and lets take the dot product of $\mathbf{u}_j$ with both sides of the above equation: :

$$\begin{aligned} \mathbf{u}_j \cdot (\ \textstyle\sum_{i=1}^{n} r_i \mathbf{u}_i) &= \mathbf{u}_j \cdot \mathbf{0} \\ \textstyle\sum_{i=1}^{n} r_i(\mathbf{u}_j \cdot \mathbf{u}_i) &= 0 \\ r_j \mathbf{u}_j \cdot \mathbf{u}_j &= 0 \text{ since } \mathbf{u}_j \cdot \mathbf{u}_i = 0 if i \neq j \\ r_j \|\mathbf{u}_j\|^2 &= 0 \end{aligned}$$

SInce $\mathbf{u}_j \neq \mathbf{0}$ then $\|\mathbf{u}_j\| \neq 0$ proving that $r_j = 0$.

Thus $U = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a linearly independent set.

Since $dim(\mathbb{R}^n) = n$ and U has n linearly independent vectors from $\mathbb{R}^n$, then U must be a basis of $\mathbb{R}^n$.

5. Let $\mathbf{u}$ be a nonzero column vector $\in \mathbb{R}^m$ and $\mathbf{v}$ a column vector in $\mathbb{R}^n$. Define $A = \mathbf{u}\mathbf{v}^T$.
(a) Show that $col(A) = span(\mathbf{u})$ and $row(A) = span(\mathbf{v})$.
(b) Find the nullity of A.
(c) Show the nullspace of A contains all the vectors orthogonal to $\mathbf{v}$.

Proof: a) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_1 \\ \vdots \\ u_m \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_1 \\ \vdots \\ v_n \end{bmatrix}$.

Then $A = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 \\ u_1 \\ \vdots \\ u_m \end{bmatrix} [v_1, v_2, \dots, v_n] = \begin{bmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_n \\ u_2v_1 & u_2v_2 & \dots & u_2v_n \\ \vdots & \vdots & \vdots & \vdots \\ u_mv_1 & u_mv_2 & \dots & u_mv_n \end{bmatrix} =$

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ v_1\mathbf{u} & v_2\mathbf{u} & \dots & v_n\mathbf{u} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \dots & u_1\mathbf{v} & \dots \\ \dots & u_2\mathbf{v} & \dots \\ \vdots & \vdots & \vdots \\ \dots & u_m\mathbf{v} & \dots \end{bmatrix}.$$

Since every column of A is a scalar multiple of $\mathbf{u}$, then $col(A) = span(\mathbf{u})$. Similarly every row of A is a scalar multiple of $\mathbf{v}$ showing that $row(A) = span(\mathbf{v})$.

b) From a) we see that $rank(A) = 1$ (since $\mathbf{u}$ is nonzero and $col(A) = span(\mathbf{u})$). Thus from the rank equation we get that $nullity(A) = n-1$.

c) From (a) we see that $A\mathbf{x} = \begin{bmatrix} \dots & u_1\mathbf{v} & \dots \\ \dots & u_2\mathbf{v} & \dots \\ \vdots & \vdots & \vdots \\ \dots & u_m\mathbf{v} & \dots \end{bmatrix} \mathbf{x} = \begin{bmatrix} u_1\mathbf{v}\cdot\mathbf{x} \\ u_2\mathbf{v}\cdot\mathbf{x} \\ \vdots \\ u_m\mathbf{v}\cdot\mathbf{x} \end{bmatrix}$.

Thus:

$$null(A) = \{\mathbf{x} \in \mathbb{R}^n \big| A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} \big| \mathbf{u_i}\mathbf{v}\cdot\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} \big| \mathbf{v}\cdot\mathbf{x} = \mathbf{0}\}$$

6. Prove that any set $U = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ of non-zero pairwise-orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$. (pairwise orthogonal means $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.)

Proof: Let $A = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}$, then$A^T = \begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \mathbf{u}_2 & -- \\ & \vdots & \\ -- & \mathbf{u}_n & --| \end{bmatrix}$.

Thus:

$$A^T A = \begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \mathbf{u}_2 & -- \\ & \vdots & \\ -- & \mathbf{u}_n & --| \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \mathbf{u}_1 \cdot \mathbf{u}_2 & \cdots & \mathbf{u}_2 \cdot \mathbf{u}_n \\ \mathbf{u}_2 \cdot \mathbf{u}_1 & \mathbf{u}_2 \cdot \mathbf{u}_2 & \cdots & \mathbf{u}_2 \cdot \mathbf{u}_n \\ & & \vdots & \\ \mathbf{u}_n \cdot \mathbf{u}_1 & \mathbf{u}_n \cdot \mathbf{u}_2 & \cdots & \mathbf{u}_n \cdot \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & & 1 \end{bmatrix} = I$$

Thus $A^T = A^{-1}$ showing that A is invertible and therefore the columns of A are linearly independent.