MATB24 GRADED PROBLEMS 5, DUE Friday Dec 4, 11:59pm

General instructions:

- You should submit your work on Quercus. The only accepted format is PDF.
- Do not wait until last minute to avoid technical difficulties.
- There is a one point penalty for late submissions within 12 hours of the due date.
- You are encouraged to work in groups, ask questions on Piazza, or in office hours, but you should write your homework individually in your own words. You can get help from me, your TA or your peers, but you should write your solution on your own.
- Unless otherwise stated in all questions you should fully justify your answer.
- Your TA will grade a randomly selected subset of the questions in each homework and your grade will be only based on the graded questions.

Reading assignment:

It is assumed that you read at least one of the reading options below

- Sec 9.2, 9.3 FB
- Sec 7.A, 7.B, 7.C SA

Problem 1. *A matrix $A \in M_n(\mathbb{C})$ is normal if $AA^* = A^*A$. In this problem you will prove that a matrix is unitarily diagonalizbale if and only if it is normal.*

(1) *Prove that if A is unitarily diagonalizbale then A is normal.*
(2) *Suppose A is unitarily equivalent to $B \in M_n(\mathbb{C})$, that is $A = UBU^*$, where U is some unitary matrix. Prove that A is normal if and only if B is normal.*
(3) *Prove that an $n \times n$ normal upper triangular matrix B must be diagonal.*
(4) *Prove that any normal matrix is unitarily diagonalizable. Hint (use Schur's diagonalizable lemma and previous parts of the problem)*

Problem 2. *Let V be a complex finite dimensional inner product space. Let $T : V \to V$ be a linear transformation. Recall that in class we proved that if T is self-adjoint then T is unitarily diagonalizable with real eigenvalues.*

(1) *Prove that if T is unitarily diagonalizbale with real eigenvalues then T is self adjoint.*
(2) *Prove that if T is an isometry then the eigenvalues of T are complex numbers with absolute value 1.*
(3) *Prove that if T is unitarily diagonalizable such that all eigenvalues of T are complex numbers with absolute value 1 then T is an isometry.*

Problem 3 (Skew-adjoint linear maps)**.** *Let V be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We studied linear maps $T\colon V \to V$ that satisfy $\langle T(v), w\rangle = \langle v, T(w)\rangle$ for all $v, w \in V$, the* self-adjoint linear maps*. In this problem we study linear maps $S\colon V \to V$ that satisfy $\langle S(v), w\rangle = -\langle v, S(w)\rangle$ for all $v, w \in V$, the* skew-adjoint linear maps*.*

(1) *Give an example of a skew-adjoint linear map $S\colon \mathbb{R}^2 \to \mathbb{R}^2$.*
(2) *If $\mathbb{F} = \mathbb{C}$, prove that S is skew-adjoint if and only if iS is self-adjoint.*
(3) *Explain why part (b) implies the following facts for a linear map $S\colon V \to V$ when $\mathbb{F} = \mathbb{C}$ (your answer should involve citing results from the book or the lectures):*
    (a) *a linear map S is skew-adjoint if and only if $\langle Sv, v\rangle \in i\mathbb{R}$ (i.e. imaginary) for all $v \in V$,*
    (b) *a skew-adjoint linear map S can only have imaginary eigenvalues,*
    (c) *a skew-adjoint linear map S is diagonalizable.*

(d) *if* $\mathcal{B} = (b_1, \ldots, b_n)$ *is an orthonormal basis of* V *and* S *is a skew-adjoint linear map, then its* $\mathcal{B}$*-matrix is* skew-Hermitian, *i.e. satisfies*

$$[T]_{\mathcal{B}}^* = -[T]_{\mathcal{B}}.$$

Problem 4 (The Cayley transform)**.** *If* A *is a* $(n \times n)$*-matrix with real entries such that* $A^T = -A$*, we say it is* skew-symmetric.

(1) *Prove that if* A *is skew-symmetric then* $\mathrm{id}_n + A$ *is invertible.*
(2) *Prove that if* A *is skew-symmetric then* $(\mathrm{id}_n - A)(\mathrm{id}_n + A)^{-1}$ *is orthogonal.*
(3) *Prove that if that* A *is orthogonal and* $(\mathrm{id}_n + A)$ *is invertible, then* $(\mathrm{id}_n - A)(\mathrm{id}_n + A)^{-1}$ *is skew-symmetric.*