GHW4

Problem 1

Let $\mathcal{B}_i$ be a basis for each V_i, by G.S. we know there is a $\mathcal{U}_i$ as a orthonormal basis for each V_i

Consider $\bigcup_{i=1}^{p} \mathcal{U}_i$ for $\mathcal{U}_i = (u_{i1}, \dots, u_{in})$
We know $\forall i, j, k, l$, $i \neq j$ that $u_{ik} \perp u_{jl}$

Thus $\bigcup_{i=1}^{p} \mathcal{U}_i = (u_{11}, \dots, u_{pn})$ is l.i because all u_{ij} are ortho. to each other.
And we know the max length of this list is $\dim(V)$

$$\begin{aligned}
\dim(V_1) + \dots + \dim(V_p) &= \dim(V_1) \oplus \dots \oplus \dim(V_p) \quad \text{as } V_i \cap V_j = \{0\} \text{ for } i \neq j \\
&= \dim(V_1 \cup \dots \cup V_p) \\
&= \dim(\mathcal{U}_1 \cup \dots \cup \mathcal{U}_p) \\
&\leq n
\end{aligned}$$

Problem 2

a) U being orthogonal means T is invertible and an iso.

So we know $\dim(W) = \dim(T(W))$ bc isomorphisms preserve dimensions

Because $T(W) \subseteq W$ and $\dim(W) = \dim(T(W))$
$\Rightarrow T(W) = W$

b) We know $V = W \oplus W^{\perp}$ and $T(W) = W$

So $\forall v \in W^{\perp}, \forall w \in W, \; v \cdot Uw = 0$
$\Rightarrow v^T U w = 0$
$\Rightarrow (U^T v)^T w = 0$
$\Rightarrow (U^{-1} v)^T w = 0$
$\Rightarrow (U^{-1} v) \cdot w = 0$

So $T^{-1}(v) \in W^{\perp}$
$\Rightarrow T^{-1}(W^{\perp}) \subseteq W^{\perp}$
$\Rightarrow W^{\perp} \subseteq T(W^{\perp})$

From a) we know T is an iso. and $\dim(W^{\perp}) = \dim(T(W^{\perp}))$

Since $W^{\perp} \subseteq T(W^{\perp})$, $W^{\perp} = T(W^{\perp})$

Problem 3

a) Let $\mathcal{B} = (b_1, \dots, b_n)$ be a orthonormal basis of V

Let $f, g \in V$ be arbi. $f = \sum_i r_i b_i \quad g = \sum_j s_j b_j$

$$\begin{aligned}\langle f, g\rangle &= \langle \textstyle\sum r_i b_i, \sum s_j b_j\rangle \\ &= \sum_i \sum_j r_i s_j \langle b_i, b_j\rangle \\ &= \sum_i r_i s_i \langle b_i, b_i\rangle \quad \text{bc if } i \neq j,\ \langle b_i, b_j\rangle = 0 \\ &= \sum_i r_i s_i \qquad \text{bc } \langle b_i, b_i\rangle = 1\end{aligned}$$

Let T be the coordinate transformation to basis $\mathcal{B}$

So $T(f) = (r_1, \dots, r_n) \quad T(g) = (s_1, \dots, s_n)$

Thus $\langle f, g\rangle = T(f) \cdot T(g)$

b) From a) we know $\forall f, g \in V$, $\langle f, g\rangle = T(f) \cdot T(g)$ where T is the coordinate iso. to a orthonormal basis.

Let $B \in M_{n \times n}$ be the matrix representation of T

Consider $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
$\quad (v, w) \to v^T A w$

Prove $\langle , \rangle$ is an inner product

Let $u, v, w \in \mathbb{R}^n$, $r \in \mathbb{R}$ be arbi

$$\begin{aligned}\langle u + rv, w\rangle &= (\vec{u} + r\vec{v})^T A w \\ &= (u^T + r v^T) A w \\ &= u^T A w + r v^T A w \\ &= \langle u, w\rangle + r \langle v, w\rangle \quad \therefore \text{linear}\end{aligned}$$

$$\begin{aligned}\langle u, v\rangle &= u^T A v \\ &= (A^T u)^T v \\ &= (v^T A^T u)^T \\ &= (v^T A u)^T \quad \text{bc } A \text{ is sym} \\ &= \langle v, u\rangle \quad \therefore \langle , \rangle \text{ is symmetric}\end{aligned}$$

By def: $u^T A v > 0$ for $u, v \neq \vec{0}$
If u or $v = \vec{0}$, $u^T A v = 0$ is trivial
$\therefore \langle , \rangle$ is positive definite

$$
\begin{aligned}
v^T A w &= \langle v, w \rangle \\
&= T(v)\cdot T(w) \quad \text{by a)} \\
&= Bv \cdot Bw \\
&= v^T B^T B w
\end{aligned}
$$

Let $v = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ - ith entry $\quad w = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ - jth entry $\quad 1 \le i, j \le n$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \quad B^T B = \begin{bmatrix} b_{11} & \cdots & b_{n1} \\ & & \\ b_{1n} & \cdots & b_{nn} \end{bmatrix}$

$$v^T A w = (0, \dots, 1, \dots, 0) \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (0, \dots, 1, \dots, 0) \begin{bmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{bmatrix} = a_{ji}$$

$v^T B^T B w = b_{ji}$ by similar process as A

So $v^T A w = v^T B^T B w \Rightarrow a_{ji} = b_{ji}$ for any $1 \le i, j \le n$

Thus $A = B^T B$

We know B is invertible as its the matrix representation of an iso. and the transpose a an invertible matrix is invertible

Since B and B^T are invertibe, $B^T B$ must be invertble

Choose $C = B^{-1}(B^T)^{-1}$

$B^T B C = B^T B B^{-1} B^T = I_n \therefore \exists C$ st $B^T B C = I_n$ thus $B^T B$ is invertible

Problem 4

a) $T: V \to F$
$v \to \langle v, w \rangle$

Let $v, u \in V$, $r \in F$ be arbi.

$$T(v+ru) = \langle v+ru, w \rangle$$
$$= \langle v, w \rangle + r\langle u, w \rangle \text{ by linearity of inner product}$$
$$= T(v) + rT(u)$$

b) Proof by textbook ☒ jk

Let $\mathcal{B} = (b_1, \dots, b_n)$ be an orthonormal basis of V

Let $v \in V$ be arbi..

$v = \langle v, b_1 \rangle b_1 + \dots + \langle v, b_n \rangle b_n$

$$T(v) = T(\langle v, b_1 \rangle b_1 + \dots + \langle v, b_n \rangle b_n)$$
$$= \langle v, b_1 \rangle T(b_1) + \dots + \langle v, b_n \rangle T(b_n) \text{ by linearity of LT}$$
$$= \overline{\overline{\langle b_1, v \rangle}\,\overline{T(b_1)}} + \dots + \overline{\overline{\langle b_n, v \rangle}\,\overline{T(b_n)}} \text{ by conjugate sym.}$$
$$= \overline{\langle \overline{T(b_1)} b_1, v \rangle} + \dots + \overline{\langle \overline{T(b_n)} b_n, v \rangle} \text{ by linearity of IP}$$
$$= \langle v, \overline{T(b_1)} b_1 \rangle + \dots + \langle v, \overline{T(b_n)} b_n \rangle$$
$$= \langle v, \overline{T(b_1)} b_1 + \dots + \overline{T(b_n)} b_n \rangle \text{ by linearity of IP}$$

Thus w exists and equals $\overline{T(b_1)} b_1 + \dots + \overline{T(b_n)} b_n$

Supp. $T(v) = \langle v, w_1 \rangle$ and $T(v) = \langle v, w_2 \rangle$ for $w_1 \neq w_2$, $\forall v \in V$

$$\langle v, w_1 \rangle = \langle v, w_2 \rangle$$
$$\langle v, w_1 \rangle - \langle v, w_2 \rangle = 0$$
$$\langle v, w_1 - w_2 \rangle = 0 \quad \text{Let } v = w_1 - w_2$$
$$\langle w_1 - w_2, w_1 - w_2 \rangle = 0$$
$$\|w_1 - w_2\|^2 = 0$$
$$w_1 - w_2 = \vec{0}$$
$$w_1 = w_2 \quad \therefore w \text{ is unique by contradiction}$$

c) Let $\langle f,g \rangle = \int_0^1 f(x)g(x)\,dx$

Let $T: P_3(\mathbb{R}) \longrightarrow \mathbb{R}$ be a LT for some $t \in \mathbb{R}$
$f \longmapsto f(t)$

Let $f,g \in P_3(\mathbb{R})$, $r \in \mathbb{R}$ be arbi.

$$\begin{aligned} T(f+rg) &= (f+rg)(t) \\ &= f(t) + rg(t) \quad \text{linearity of functions} \\ &= T(f) + rT(g) \end{aligned}$$

By b) we know $\exists! g \in P_3(\mathbb{R})$ st $T(f) = \langle f, g \rangle$

So taking $g = q_t$ and $T(p) = p(t)$

$\exists! q_t \in P_3(\mathbb{R})$ st $p(t) = \int_0^1 p(x) q_t(x)\,dx$ for any $t \in \mathbb{R}$

d) We know $\{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$

Let $v_1 = 1$ $\quad u_1 = \frac{v_1}{\|v_1\|} = v_1$ $\quad \|v_1\| = \sqrt{\int_0^1 v_1^2\,dx} = \sqrt{1} = 1$

$$\begin{aligned} v_2 &= x - \operatorname{proj}_{sp(u_1)} x \\ &= x - \left(\int_0^1 x\,dx\right) u_1 \\ &= x - \tfrac{1}{2} \end{aligned}$$

$u_2 = \frac{v_2}{\|v_2\|} = \sqrt{12}\left(x - \frac{1}{2}\right)$

$$\begin{aligned} \|v_2\|^2 &= \int_0^1 v_2^2\,dx \\ &= \int_0^1 x^2 - x + \tfrac{1}{4}\,dx \\ &= \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4} \\ &= \tfrac{1}{12} \end{aligned}$$

$$\begin{aligned} v_3 &= x^2 - \operatorname{proj}_{sp(u_1,u_2)} x^2 \\ &= x^2 - \left(\int_0^1 x^2 u_1\,dx\right) u_1 - \left(\int_0^1 x^2 u_2\,dx\right) u_2 \\ &= x^2 - \tfrac{1}{3} - \sqrt{12} \int_0^1 x^3 - \tfrac{1}{2}x^2\,dx\ u_2 \\ &= x^2 - \tfrac{1}{3} - \left(x - \tfrac{1}{2}\right) \\ &= x^2 - x + \tfrac{1}{6} \end{aligned}$$

$u_3 = \frac{v_3}{\|v_3\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$

$$\begin{aligned} \|v_3\|^2 &= \int_0^1 v_3^2\,dx \\ &= \int_0^1 \left(x^2 - x + \tfrac{1}{6}\right)^2 dx \\ &= \frac{1}{180} \end{aligned}$$

$$\begin{aligned} v_4 &= x^3 - \operatorname{proj}_{sp(u_1,u_2,u_3)} x^3 \\ &= x^3 - \int_0^1 x^3 u_1\,dx\ u_1 - \int_0^1 x^3 u_2\,dx\ u_2 - \int_0^1 x^3 u_3\,dx\ u_3 \\ &= x^3 - \tfrac{1}{4} - 3^{3/2}/20\left(\sqrt{12}\left(x - \tfrac{1}{2}\right)\right) - \tfrac{1}{4\sqrt{5}}\, 6\sqrt{5}\left(x^2 - x + \tfrac{1}{6}\right) \\ &= x^3 - \tfrac{1}{4} - \tfrac{9}{10}\left(x - \tfrac{1}{2}\right) - \tfrac{3}{2}\left(x^2 - x + \tfrac{1}{6}\right) \\ &= x^3 - \tfrac{3}{2}x^2 + \tfrac{3}{5}x - \tfrac{1}{20} \end{aligned}$$

$$\|v_4\|^2 = \int_0^1 v_4^2\,dx = \frac{1}{2800}$$

$$u_4 = \frac{v_4}{\|v_4\|} = 20\sqrt{7}\left(x^3 - \tfrac{3}{2}x^2 + \tfrac{3}{5}x - \tfrac{1}{20}\right)$$

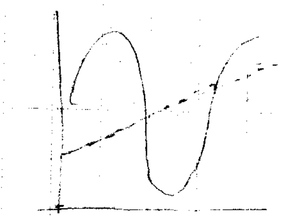
Let $T(f) = f(\frac{1}{2})$

From b) we know for $T(f) = \langle f, q_{\frac{1}{2}} \rangle$

$$\begin{aligned} q_{\frac{1}{2}} &= \overline{T(u_1)}\, u_1 + \dots + \overline{T(u_4)}\, u_4 \\ &= 1(1) + \sqrt{12}\left(\tfrac{1}{2} - \tfrac{1}{2}\right) u_2 + 6\sqrt{5}\left(\tfrac{1}{2}^2 - \tfrac{1}{2} + \tfrac{1}{6}\right) u_3 + 20\sqrt{7}\left(\tfrac{1}{2}^3 - \tfrac{3}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{3}{5}\left(\tfrac{1}{2}\right) - \tfrac{1}{20}\right) u_4 \\ &= 1 + (-15)\left(x^2 - x + \tfrac{1}{6}\right) \\ &= -15x^2 + 15x - \tfrac{3}{2} \end{aligned}$$

Problem 5

a) Linear regression gives a line of ~~bast~~ best fit



This clearly doesn't model the graph as the data provided doesn't closely resemble a straight line

This is modelled by the sum of squared errors

$$\sum_{i=1}^{n}(y_i - ax_i - b)^2 = \|\vec{y} - A\vec{c}\|^2 \quad A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Check: $\|y - Ac\|^2 = \left\| \begin{pmatrix} y_1 - ax_1 - b \\ \vdots \\ y_n - ax_n - b \end{pmatrix} \right\|^2 = \sum_{i=1}^{n}(y_i - ax_i - b)^2$

b) let $A = \begin{bmatrix} \sin(x_1) & 1 \\ \vdots & \vdots \\ \sin(x_n) & 1 \end{bmatrix}$ $c = \begin{bmatrix} a \\ b \end{bmatrix}$

Then $\|y - Ac\|^2 = \sum_{i=1}^{n}(y_i - a\sin(x_i) - b)^2$

So changing the first col of A to the function of choice expressed in terms of x_i gives the modified linear regression

c) let $D = \begin{bmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{bmatrix}$ let $\langle v, u \rangle = v^T D u$

$$\begin{aligned} \langle v + rw, u \rangle &= (v + rw)^T Du \\ &= v^T Du + rw^T Du \\ &= \langle v, u \rangle + r\langle w, u \rangle \end{aligned}$$

$$\begin{aligned} \langle v, v \rangle &= v^T D v \quad v \neq \vec{0} \\ &= Dv^T v \\ &> D(0) = 0 \end{aligned}$$

$$\begin{aligned} \langle v, u \rangle &= v^T D u \\ &= (u^T D^T v)^T \\ &= (u^T D v)^T \\ &= \langle u, v \rangle \end{aligned}$$

$$\begin{aligned} \langle v, v \rangle &= v^T D v \quad v = \vec{0} \\ &= 0 \end{aligned}$$

$\therefore \langle v, u \rangle = v^T D u$ is ~~a~~ an inner product

Prove $\text{proj}_{Col(A)} v = A(A^TDA)^{-1}A^TDv$

Show (A^TDA) is invertible

Let $c \in Nul(A^TDA)$

$$A^TDAc = \vec{0}$$
$$c^TA^TDAc = \vec{0}$$
$$(Ac)^TDAc = \vec{0}$$
$$\|Ac\|^2 = 0$$
$$Ac = \vec{0}$$
$$c = \vec{0}$$

Since $Nul(A^TDA) = \{\vec{0}\}$, A^TDA is invertible

Let $v \in Col(A)$, so $v = Ac$ for some c, show $\text{proj}_{Col(A)} v = A(A^TDA)^{-1}A^TD$

$$A(A^TDA)^{-1}A^TDv = A(A^TDA)^{-1}A^TDAc$$
$$= Ac$$
$$= v$$

Let $v \in Col(A)^\perp$

$$\text{proj}_{Col(A)} v = 0$$
$$A(A^TDA)^{-1}A^TDv = 0$$

Let $P = A(A^TDA)^{-1}A^TD$

For $v \in \mathbb{R}^n$

$v = v_{Col(A)} + v_{Col(A)^\perp}$, let $W = Col(A)$

$$Pv = Pv_W + Pv_{W^\perp}$$
$$= v_W + 0$$
$$= \text{proj}_W v$$

$A\vec{c} = \text{proj}_{Col(A)} \vec{y} = A(A^TDA)^{-1}A^TD\vec{y}$, let $P = A(A^TDA)^{-1}A^TD$

$$\|y - \text{proj}_{Col(A)} y\|^2 = \langle y - \text{proj}_{Col(A)} y, y - \text{proj}_{Col(A)} y \rangle$$
$$= (y - Py)^TD(y - Py)$$
$$= (y^T - y^TP^T)D(y - Py)$$
$$= D(y^Ty - y^TP^Ty - y^TPy + y^TP^TPy)$$
$$= D(y^Ty - y^TPy - y^TPy + y^TPy)$$
$$= D(y^Ty - y^TPy)$$
$$= \sum_{i=1}^{n} u_i(y_i - ax_i - b)^2$$

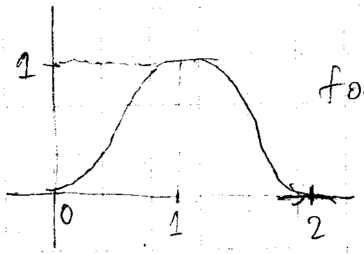
d) From c) we have $A\vec{c} = \text{proj}_{Col(A)}\, y$

$$\Rightarrow A\vec{c} = A(A^TDA)^{-1}A^TDy$$
$$\Rightarrow \vec{c} = (A^TDA)^{-1}A^TDy \quad \text{with } D = \begin{bmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{bmatrix}$$

all as defined in c)

e) Likely related to locally weighed scatterplot smoothing

The graph for $u_i(x)$ seems to resemble a sort of probability dist:



for $c=1$, $x_i=1$

change in x_i transform the graph sideways

Change in c stretches the graph, and flips for $c<0$.

The graph probably represents a range of weights to choose for the local area around x_i, and choosing a specific "c" will widen or shrink the range, though c should be chosen greater than 0, as for $c \leq 0$, the graph no longer becomes concave down.