Functions and Precalculus

$$AB = 0 \iff A = 0 \text{ or } B = 0$$

$$|x - c| < \delta$$

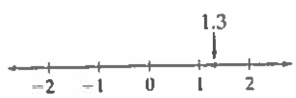
0.1 NUMBERS AND SETS

- Number systems, sets, set membership, and set notation
- The real number line, absolute value, distance, and the Cartesian plane
- Finding unions and intersections of sets on the real number line and in the Cartesian plane

Real Numbers

Mathematics is a language. In order to understand it, we have to learn how to read it and speak it with the correct vocabulary. Since calculus is at its heart the study of functions of real numbers, the universe we will spend most of our time exploring is the set of ***real numbers*** and the relationships between sets of real numbers. Therefore we must begin by setting out the mathematical language that describes numbers and sets of real numbers. Once we all speak the same language, we can start building the theory of functions and calculus.

The real numbers can be represented graphically on a ***number line*** like the one illustrated next. Every real number can be represented as a point on a number line. For example, the real number 1.3 is marked on the number line shown. The arrows at the ends of the number line indicate that the number line continues indefinitely in either direction.



We have different names for different types of real numbers. For example, you probably already know that the positive and negative whole numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

are called the ***integers***. In this course we will be concerned not just with the integers, but with all the real numbers in between. Calculus is fundamentally the study of the entire continuum of real numbers, which we write as $\mathbb{R}$. Because we will work primarily with real numbers in this text, when we say "a number," we will always mean "a *real* number," unless we specify otherwise.

The real numbers split into two important categories of numbers:

DEFINITION 0.1 Rational and Irrational Numbers

(a) A ***rational number*** is a real number that can be written as a quotient of the form $\frac{p}{q}$ for some integers p and q, with $q \neq 0$.

(b) An ***irrational number*** is a real number that cannot be written in the form of a rational number.

In a previous course, you likely thought of rational numbers as terminating or nonrepeating decimals, such as 0.25 or $0.33\overline{3}$. This definition of rational numbers is equivalent to Definition 0.1, but in this course the quotient definition will prove to be far more useful. You should think of rational numbers as quotients of integers and irrational numbers, like π or $\sqrt{2}$, as numbers that cannot be written as quotients of integers.

We can also classify real numbers according to their ***signs***, that is, according to whether they are positive or negative.

DEFINITION [illegible] Positive and Negative Numbers

A real number is said to be

(a) ***positive*** if greater than 0

(b) ***negative*** if less than 0

(c) ***nonnegative*** if greater than or equal to 0

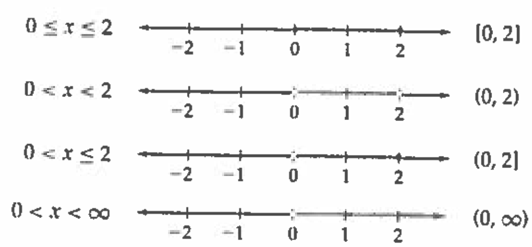
(d) ***nonpositive*** if less than or equal to 0

Here are some examples of real numbers and their classifications into the preceding categories:

- the real number 3 is positive, nonnegative, rational, and an integer;
- the real number $-\frac{11}{7}$ is negative, nonpositive, and rational;
- the real number π is positive, nonnegative, and irrational;
- the real number 0 is nonnegative, nonpositive, rational, and an integer.

Sets and Intervals of Real Numbers

In this course we will often be interested in certain connected subsets of the real number line, such as the set of real numbers greater than 0 or the set of real numbers between 0 and 2. Such subsets are called ***intervals***. Following are four examples of intervals, together with their descriptions in inequality notation on the left and their expressions in ***interval notation*** on the right:



The first interval, $[0, 2]$, includes its endpoints 0 and 2, so we say that this is a ***closed interval***. The second interval, $(0, 2)$, does not include either endpoint 0 or 2, so we say that this is an ***open interval***. The third interval, $(0, 2]$, is neither closed nor open. (We could call this interval half closed, or half open, or—and this is actually true—some people call it a ***clopen interval***.) The fourth interval, $(0, \infty)$, is said to be an ***unbounded interval*** because it contains numbers that are arbitrarily large. Note that we use an open bracket after the ∞ because "∞" is not a real number and therefore is not included in this interval.

Since open and closed intervals will be of particular interest to us in this course, we should state their definitions precisely:

DEFINITION [illegible] Closed and Open Intervals of Real Numbers

An ***interval*** of real numbers is a connected subset of the real number line.

(a) A ***closed interval*** $[a, b]$ is the set of real numbers x with $a \leq x \leq b$.

(b) An ***open interval*** (a, b) is the set of real numbers x with $a < x < b$.

There are other types of intervals besides those just discussed. For example, the entire real number line $\mathbb{R}$ is the interval $(-\infty, \infty)$. As another example, the set $\{2\}$ containing only the number 2 is an interval, since it is a connected subset of the real numbers. Perhaps the strangest subset of real numbers is the set containing no numbers at all. This set is called the ***empty set*** or the ***null set***, and is denoted by the symbol $\emptyset$. For example, the set of real numbers that are both negative and positive is empty, since no real number is both strictly less than and strictly greater than zero.

Intervals are a special type of ***subset*** of the set of real numbers. There are many other types of subsets of real numbers—for example, the set containing the four real numbers 0, 3, 6, and 7. Sets such as this one that consist of a finite collection of real numbers can be described simply by listing the elements in curly brackets:

$$\{0, 3, 6, 7\}.$$

If a set S has an infinite number of elements, or a large number of elements, we can list them all only if they have a recognizable pattern; for example, the set of all positive even numbers can be written as

$$\{2, 4, 6, 8, 10, 12, \ldots\}.$$

Here is a more interesting set of real numbers to consider: The set of real numbers x for which x^2 is greater than 4. The real numbers that satisfy the property $x^2 > 4$ are the numbers x with $x > 2$ and also the numbers x with $x < -2$ (since, for example, $(-3)^2 = 9$ is greater than 4). This set is not an interval, because it is not a connected subset of the real number line. We need a compact way to write such sets in mathematical notation.

In general, we can describe sets compactly by using ***set notation***, which specifies two things: first, a *category* (or larger set) of objects to consider; second, a *test* that describes when an object in that category is in the set.

DEFINITION 0.[illegible] **Set Notation and Elements of a Set**

Schematically, ***set notation*** for a set S is written in the form

$$S = \{x \in \textit{category} \mid \textit{test determining whether } x \textit{ is in the set}\}.$$

An object x is an ***element*** of a set S if it is contained in S, that is, if it passes the test written in the second half of the preceding notation. When this happens, we write $x \in S$.

For example, the set consisting of all real numbers x whose squares are greater than 4 can be written as

$$\{x \in \mathbb{R} \mid x^2 > 4\}.$$

We pronounce this notation as "the set of real numbers x such that x^2 is greater than 4." If we call the set in this example S, then, since $x = 5$ and $x = -3$ pass the test "$x^2 > 4$," we say that $5 \in S$ and $-3 \in S$. In contrast, since $x = 1$ does not pass the test "$x^2 > 4$,", we say that $1 \notin S$.

Considering our earlier discussion, we could also write the preceding set as

$$\{x \in \mathbb{R} \mid x > 2 \text{ or } x < -2\}.$$

As we will see in a moment, with a little bit more notation we can express this set even more compactly as a combination of the intervals $(-\infty, -2)$ and $(2, \infty)$.

Unions and Intersections of Sets

Throughout our study of calculus we will often have to consider combinations of intervals of real numbers. The ***union*** of two sets A and B is the set of elements that are in *either* A or B, and the ***intersection*** of A and B is the set of elements that are in *both* A and B.

DEFINITION 0.5 Union and Intersection

Suppose A and B are sets.

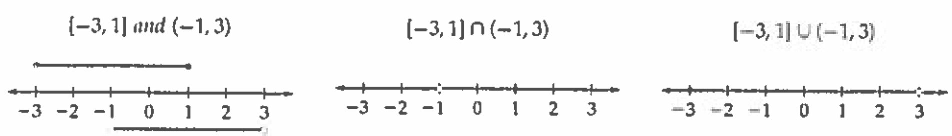
(a) The ***union*** of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

(b) The ***intersection*** of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

It may help you distinguish $\cup$ and $\cap$ if you notice that the symbol $\cup$ looks like "u" for "union" and the symbol $\cap$ looks like lowercase "n" for "intersection." For example, the set of real numbers whose square is greater than 4 is a union of intervals:

$$\{x \in \mathbb{R} \mid x^2 > 4\} = \{x \in \mathbb{R} \mid x > 2 \text{ or } x < -2\} = (-\infty, -2) \cup (2, \infty).$$

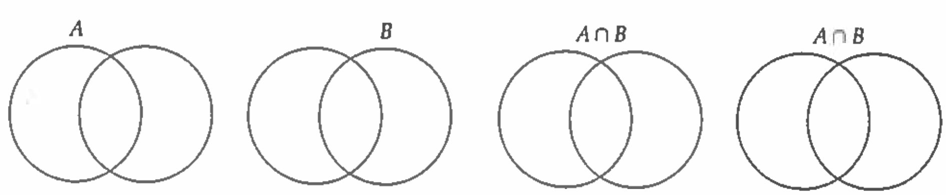
Here is another simple example: Consider the intervals $[-3, 1]$ and $(-1, 3)$ shown next at the left. The intersection $[-3, 1] \cap (-1, 3)$ is the set of real numbers that are contained in both $[-3, 1]$ and $(-1, 3)$. This intersection is the interval $(-1, 1]$ and is shown in the middle figure, which is obtained by looking at where the two original intervals overlap.



On the other hand, the union $[-3, 1] \cup (-1, 3)$ is the set of real numbers that are contained in either $[-3, 1]$ or $(-1, 3)$, or both. The right-hand figure shows this union, $[-3, 3)$, obtained by looking at the real numbers that fall under the shadow of either of the two original intervals in the left-hand figure.

CAUTION In daily language we often use ***exclusive "or,"*** in which "P or Q" is true if either P, or Q, but not both, is true. For example, if Phil says, "I'll meet you in the lobby or at the coffee shop," clearly he means that he will be in one or the other of these places, but not both. In mathematics, however, we always mean ***inclusive "or,"*** where "P or Q" is true if P is true, if Q is true, or if both P and Q are true. For example, "-2 is negative or an integer" is a true statement.

Another way to visualize intersections and unions is with ***Venn diagrams***, by thinking of sets as shaded regions defined by circles. Suppose we represent two sets A and B by circles, as in the first two figures that follow Then the set $A \cap B$ is the intersection of the two circles, and the set $A \cup B$ is their union; see the third and fourth figures.



Absolute Value and Distance

Absolute value and distance give us ways to measure the relative positions of numbers on the real number line. Intuitively, the ***absolute value*** of a real number a is the magnitude, or size, of a and is denoted $|a|$. For example, $|3| = 3$ and $|-3| = 3$. The absolute value of a "makes the number a positive." More precisely, if a is positive or zero, then taking the absolute value should do nothing: In this case $|a| = a$. However, if a is negative, then taking the absolute value should change the sign: In this case $|a| = -a$; for example $|-3| = -(-3) = 3$.

DEFINITION 0.6 Absolute value

The ***absolute value*** of a real number a is $|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$

CAUTION The quantity $-a$ is not necessarily negative; it depends on whether the number a is positive or negative. For example, if $a = -3$, then $-a = -(-3) = 3$ is positive. In Definition 0.6, $|a| = -a$ only if a is negative, in which case $-a$ is a *positive* number.

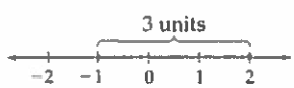
Absolute values behave well with respect to products and quotients; for example, the absolute value of a product is simply the product of the absolute values.

THEOREM 0.7 **Absolute Values Commute with Products and Quotients**

Given any real numbers a and b, $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

For example, $|(-2)(3)| = |-6| = 6$ is equal to $|-2||3| = (2)(3) = 6$. Note that absolute values do *not* behave well with sums; for example $|-2+3| = 1$ is not equal to $|-2|+|3| = 5$.

The absolute value $|a|$ of any real number a is the distance between a and 0 on the real number line. Geometrically, the distance between any two real numbers a and b is the length of the line segment from a to b on the number line. Algebraically, this distance is given by $|b-a|$, regardless of the signs of a and b. For example, the diagram that follows shows that the distance between $a = 2$ and $b = -1$ is 3 units. Algebraically, $|b - a| = |-1-2| = |-3| = 3$. We will take the algebraic formulation as the definition of distance, and in Exercise 14 we will show that this defnition is equivalent to our geometric interpretation.



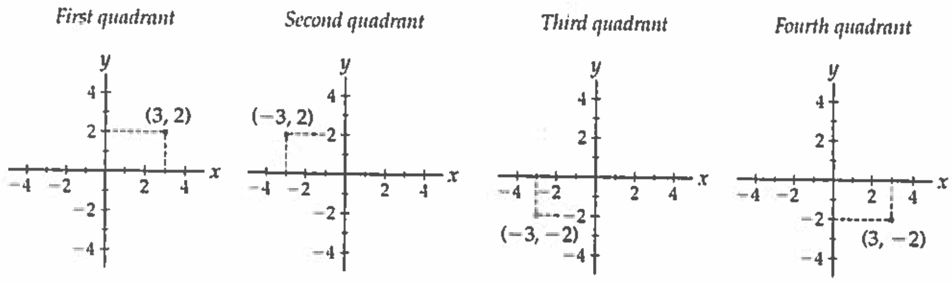
DEFINITION 0.8 Distance Between Two Real Numbers

The ***distance*** between two real numbers a and b is $\text{dist}(a, b) = |b - a|$.

The Cartesian Plane

We now take our knowledge of the one-dimensional real number line and extend it to a two-dimensional plane where points are represented by pairs of real numbers. An ***ordered pair*** of real numbers x and y is denoted (x, y). The order matters; for example, the ordered pair $(2, 5)$ is different from the ordered pair $(5, 2)$. Notice that the notation $(2, 5)$ now has two possible meanings, depending on the context. Earlier we used this notation to represent the open interval from 2 to 5; now we are using it to represent an ordered pair of numbers.

The set of all ordered pairs forms the ***Cartesian plane***, named for the famous mathematician and philosopher René Descartes. This is a two-dimensional plane with two perpendicular axes, one horizontal and one vertical, that intersect at a point called the ***origin***. Every ordered pair (x, y) represents one point in the Cartesian plane, where the first coordinate, x, in the pair represents a horizontal signed distance to the left or right of the origin, and the second coordinate, y, represents a vertical signed distance up or down from the origin. When we denote coordinates with the letters x and y, we refer to the horizontal axis as the ***x-axis***, the vertical axis as the ***y-axis***, and the plane itself as the ***xy-plane***. Shown next are four examples of ordered pairs as points in the Cartesian plane. Notice that the Cartesian plane is naturally divided into four quadrants, depending on the signs of x and y.



Geometrically, the distance between two ordered pairs of real numbers (x_1, y_1) and (x_2, y_2) is the length of the line segment connecting these two points in the plane. Algebraically, this distance is given by the formula that follows. In Section 0.7 we will use the Pythagorean theorem to show that these two interpretations of distance are equivalent.

DEFINITION 0.9 Distance Between Two Points in the Plane

The ***distance*** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is given by the ***distance formula:***

$$\text{dist}(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In the real number line, the midpoint between two points a and b is the midpoint of the line segment from a to b, which algebraically is simply the average of the numbers a and b. In the Cartesian plane, the midpoint between two points (x_1, y_1) and (x_2, y_2) is the point (m_1, m_2), that is, at the midpoint of the line segment connecting the two points. We can find this midpoint algebraically by taking the averages of the x- and y-coordinates of the two points.

DEFINITION 0.10 Midpoint Between Two Points in the Plane

The coordinates of the ***midpoint*** between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane are given by the ***midpoint formula:***

$$\text{midpoint}(P, Q) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Examples and Explorations

EXAMPLE 1 Identifying rational numbers

Which of the numbers that follow are rational numbers? Express each rational number in the form $\frac{p}{q}$ for some integers p and q.

(a) 0.001 **(b)** 5 **(c)** 3.258 **(d)** $-\frac{11}{7}$ **(e)** $0.\overline{333}$

SOLUTION

All of the preceding numbers are rational. The first number is one one-thousandth and is thus represented by the fraction $\frac{1}{1000}$. The number 5 can be written as $\frac{5}{1}$ and is thus a rational number. The third number is the sum of 3 and 0.258, or $\frac{3000}{1000} + \frac{258}{1000} = \frac{3258}{1000}$. The fourth number could be represented in the form $\frac{p}{q}$ either by $\frac{-11}{7}$ or by $\frac{11}{-7}$. Finally, the last number is represented by the fraction $\frac{1}{3}$. (Try dividing 3 into 1 to see why.)

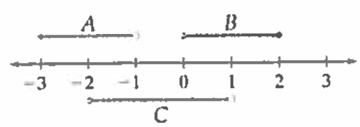
EXAMPLE 2 Unions and intersections of intervals

Consider the intervals $A = [-3, -1)$, $B = [0, 2]$, and $C = [-2, 1)$. Express each of the following in interval notation:

(a) $A \cap B$ **(b)** $A \cup B$ **(c)** $(A \cup B) \cap C$

SOLUTION

The diagram that follows shows the given intervals above and below a real number line. To think about unions and intersections you should think about the "shadows" of these intervals on the real number line and where they overlap.



(a) The intervals A and B do not overlap at all, so their intersection is empty; in other words, $A \cap B = \emptyset$.

(b) The union of A and B is $[-3, -1) \cup [0, 2]$. This is a disconnected subset of the real number line, and there is no simpler way to express $A \cup B$ in interval notation.

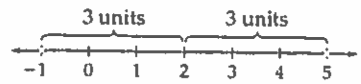
(c) To find $(A \cup B) \cap C$ we must first consider the union of A and B, and then find the subset of those real numbers which are also contained in C. By looking at the number line and where these intervals overlap, we see that $(A \cup B) \cap C = [-2, 1) \cup [0, 1)$.

EXAMPLE 3 Expressing distance statements with absolute values

Suppose x is a real number with the property that the distance between x and 2 is less than 3. Express this statement in mathematical notation with an inequality and absolute value. Then use a number line to find the interval of values of x that satisfy the given property.

SOLUTION

Since the distance between two real numbers a and b is given by $|b - a|$, our real number x satisfies the inequality $|x - 2| < 3$ (or equivalently, $|2 - x| < 3$). The following real number line shows that this means that x must be between -1 and 5, or in other words, x must be in the interval $(-1, 5)$.

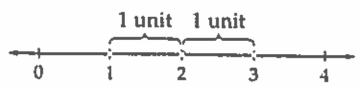


EXAMPLE 4 A punctured interval defined by an absolute value statement

Let A be the set of all real numbers x that satisfy the double inequality $0 < |x - 2| < 1$. Interpret A in terms of distances, sketch A on a real number line, and express A in interval notation.

SOLUTION

The statement $0 < |x - 2| < 1$ says that the distance between x and 2 is greater than 0 and less than 1. This means that x is a real number which is not equal to 2, but is within distance 1 of 2. The set of such values of x is an open interval with a hole in the middle:



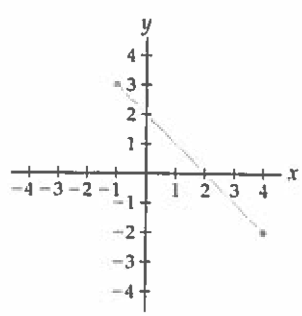
Since this is not a connected subset of the real number line, A is not a single interval. We can, however, express A in interval notation as a union of intervals, as $(1, 2) \cup (2, 3)$. We call this the ***punctured open interval*** of radius 1 around 2.

EXAMPLE 5 Using the distance and midpoint formulas in the Cartesian plane

Plot the points $P = (-1, 3)$ and $Q = (4, -2)$ on a set of axes, and sketch the line segment connecting these two points. Use the distance formula to find the length of this line segment, and use the midpoint formula to find the midpoint of the line segment.

SOLUTION

The points $P = (-1, 3)$ and $Q = (4, -2)$, and the line segment between them, are shown as follows:



The distance between P and Q is

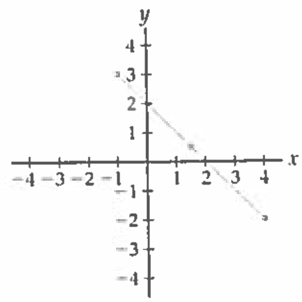
$$\text{dist}(P, Q) = \sqrt{(4 - (-1))^2 + (-2 - 3)^2} = \sqrt{5^2 + (-5)^2} = \sqrt{50} \approx 7.071 \text{ units.}$$

The midpoint of the line segment from P to Q has coordinates

$$\text{midpoint}(P, Q) = \left(\frac{-1+4}{2}, \frac{3+(-2)}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}\right).$$

CHECKING THE ANSWER

We can check whether our answer seems correct by plotting the point $(3/2, 1/2)$ in the plane, as follows:



From the graph it seems right that the midpoint of the line segment is at the point $(3/2, 1/2)$. Moreover, it does not seem unreasonable that this line segment is just over 7 units long.

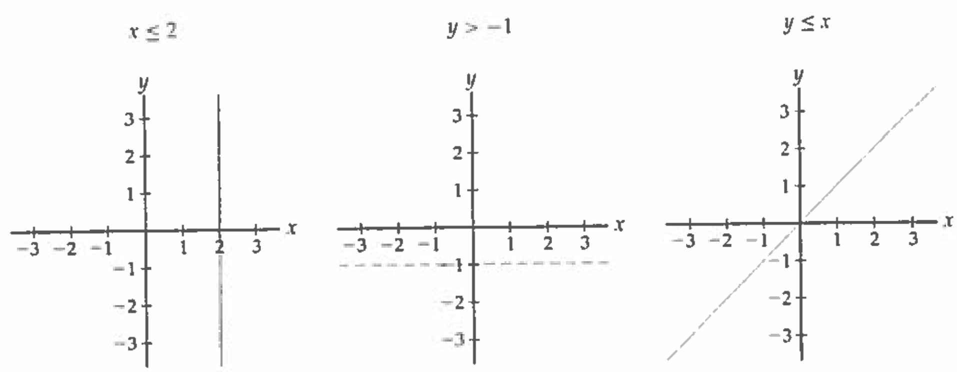
EXAMPLE 6

Intersections of subsets of the Cartesian plane

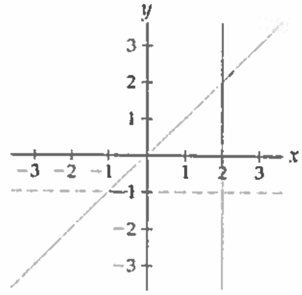
Sketch the points (x, y) in the Cartesian plane that simultaneously satisfy the three inequalities $x \leq 2$, $y > -1$, and $y \leq x$.

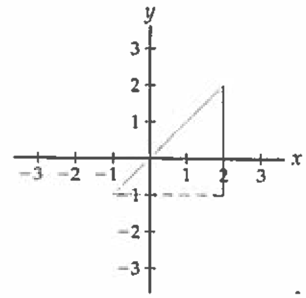
SOLUTION

Each of the three inequalities defines a subset of the plane, as shown in the three figures that follow. For example, the inequality $y \leq x$ defines the set of points (x, y) whose y-coordinates are less than or equal to their x-coordinates. The graphs of the given inequalities are shown in the figures that follow, where in each case we shade in the region of points (x, y) that satisfy the inequality.



We seek the set of points (x, y) contained in all three regions. The figure that follows at the left shows the three boundary lines, and the shaded region shown at the right is the desired intersection. Notice that the $x = -1$ boundary is dashed, which means that the points on that line are not part of the intersection.





? TEST YOUR UNDERSTANDING

- Without referring to decimal notation, what is the condition for a real number to be rational? Irrational?
- What is the algebraic definition of the absolute value of a real number? What does this definition mean in words? Why do we need to consider two cases?
- In what sense does the intersection symbol $\cap$ represent "and"? Is $A \cap B$ the set of all elements of A and all elements of B, or is it the set of all elements that are both in A and in B?
- How do you find the distance and midpoint between two points in the Cartesian plane?
- Define each of the following: open interval, absolute value, second quadrant, union of sets, intersection of sets.

EXERCISES 0.1

Thinking Back

- *Properties of real numbers:* You may remember from a previous math course that the operations of addition and multiplication of real numbers are associative and commutative. What does this mean? Are subtraction and division also associative and/or commutative operations?
- *Properties of integers:* Is the sum of two integers always an integer? What about products and quotients of integers?
- *Sums and products of positive and negative numbers:* If a and b are positive numbers, what can you say about their product and their sum? What if a and b are both negative? What if one is negative and one is positive?

Concepts

0. *Problem Zero:* Read the section and make your own summary of the material.
1. *True/False:* Determine whether each of the statements that follow is true or false. If a statement is true, explain why. If a statement is false, provide a counterexample.
   (a) *True or False:* Every real number is either rational or irrational, but not both.
   (b) *True or False:* Every nonpositive number is less than zero.
   (c) *True or False:* $1.5 \in \{x \in \mathbb{R} \mid x - 1 > 0\}$.
   (d) *True or False:* If $a \in \mathbb{R}$, then $-a$ is negative.
   (e) *True or False:* $|0| = 0$.
   (f) *True or False:* The distance between any point in the coordinate plane and itself is zero.
   (g) *True or False:* If (a, b) is in the second quadrant of the Cartesian plane, then $a > 0$ and $b < 0$.
   (h) *True or False:* If $x \in A \cap B$, then x is an element of A and x is an element of B, but not both.
2. *Examples:* Construct examples of the thing(s) described in the following. Try to find examples that are different than any in the reading.
   (a) A number a that is rational but not an integer, and a number b that is real but not rational.
   (b) An open interval, a closed interval, and an unbounded interval, in both interval notation and as pictures on the real number line.
   (c) Two intervals whose intersection is the empty set, and two intervals whose union is all of $\mathbb{R}$.

3. Is every integer a rational number? Is every rational number a real number? Is every irrational number a real number? Is every real number a complex number?

4. Is it possible to tell with a calculator that the number $\sqrt{2}$ is irrational? Why or why not?

5. Assuming that the pattern continues, is the number $0.6166116661116666111\ldots$ a rational number?

6. Suppose a is a real number whose first 24 decimal places are $a = 0.123123123123123123123123\ldots$. Explain why this information does not necessarily show that a is a rational number.

7. Is $\frac{1.7}{2.3}$ a rational number? If not, why not? If so, write it in the form $\frac{p}{q}$ for some integers p and q.

8. It is an interesting but unusual fact that $0.99\overline{9}$ and 1 represent the same number.
   (a) Use long division to argue that $0.33\overline{3} = \frac{1}{3}$.
   (b) Use part (a) to show that $0.99\overline{9} = 1$.
   (c) What other real numbers have two different decimal representations?

9. What do the following symbols represent?
   (a) $\in$ (b) $\cap$ (c) $\cup$ (d) $\emptyset$

10. List all of the elements of $\{x \in \mathbb{R} \mid x^2 = x\}$.

11. Out of context, the notation $(3, 5)$ could mean two different things. What are those two things?

12. There are 11 possible types of intervals of real numbers. If a and b are any real numbers with $a < b$, these intervals are $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$, $(-\infty, \infty)$, $\{a\}$, and $\emptyset$. Choose two real numbers a and b, and represent each of the 11 types of intervals in set notation, in words, and on a real number line.

13. Is the set $\{x \mid x^2 \geq 4\}$ an interval of real numbers? Why or why not? Include a sketch of this set on the real number line as part of your answer.

14. For each of the values of a and b that follow, sketch a and b on the real number line and visually find the distance between a and b. Then calculate $|b - a|$ and verify that it is equal to the geometric distance.
   (a) $a = 3$, $b = 5$ (b) $a = -3$, $b = 5$
   (c) $a = 3$, $b = -5$ (d) $a = -3$, $b = -5$

15. Verify that $|b - a| = |a - b|$ for each of the pairs of values of a and b given in Exercise 14.

16. Express the punctured interval of radius 3 around 1 in four ways:
   (a) As a picture on the real number line.
   (b) Using interval notation.
   (c) In words, using the concept of distance.
   (d) With a double inequality involving an absolute value.

17. Suppose $P = (u, v)$ and $Q = (x, y)$ are two points in the Cartesian plane. Describe the distance between P and Q in two ways: geometrically and algebraically.

18. Suppose $P = (u, v)$ and $Q = (x, y)$ are two points in the Cartesian plane. Describe the midpoint between P and Q in two ways: geometrically and algebraically.

19. If A and B are any two sets, is $A \cap B$ necessarily a subset of A? Is $A \cap B$ necessarily a subset of $A \cup B$? Is A a subset of $A \cap B$?

20. Use set notation to describe (a) the intersection $A \cap B$ and (b) the union $A \cup B$ of two sets A and B.

21. Is the union of two intervals always an interval? What about the intersection? Why or why not?

22. For each of the following pairs of sets A and B, describe $A \cap B$ and $A \cup B$:
   (a) A = {red pants}, B = {leather pants}.
   (b) A = {cats}, B = {dogs}.
   (c) A = {posters}, B = {Elvis posters}.

Skills

For Exercises 23–32, express each given rational number in the form $\frac{p}{q}$ for some integers p and q.

23. 0.17 24. 0.26
25. 0.00001 26. 5.31
27. -11.7 28. -62.14
29. 0 30. $-\frac{4}{13}$
31. $-0.\overline{66}$ 32. $0.\overline{99}$

Assuming that a is a positive number and b is a negative number, write each of the expressions in Exercises 33–38 without absolute values, if possible. Your answers may involve a and b.

33. $|2b|$ 34. $|8a|$
35. $|ab^3|$ 36. $|b + 2|$
37. $|ab - 1|$ 38. $|b^2 + 1|$

Describe each of the sets in Exercises 39–52, using (a) set notation, (b) interval notation (if possible), and (c) the real number line or the Cartesian plane.

39. The set of real numbers greater than -7 and less than 1.

40. The set of real numbers greater than 1.75 and less than or equal to 3.8.

41. The set of all real numbers whose product with zero is zero.

42. The set of real numbers that are both greater than 5 and less than 1.

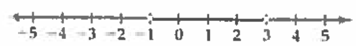
43. The set of real numbers that are either greater than 5 or less than 1.

44. The set containing only -3, 4, 7, and 102.

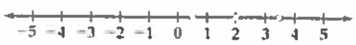
45. The set of all odd negative integers.
46. The set of real numbers whose squares are less than 4.
47. The set of real numbers x with $\text{dist}(x, 3) = 1$.
48. The set of real numbers x with $\text{dist}(x, 3) > 1$.
49. The set of real numbers x that satisfy the double inequality $0 < |x - 3| < 5$.
50. The set of real numbers x that satisfy the double inequality $0 < |x + 2| < 3$.
51. The set of points (x, y) in the Cartesian plane that satisfy $y > x + 1$.
52. The set of points (x, y) in the Cartesian plane that are a distance of 1 from the origin.

Each subset of the real numbers shown in Exercises 53–56 can be interpreted in terms of distance. Express these subsets with absolute value inequalities.

53. 

54. 

55. 

56. 

Sketch each pair of points P and Q in Exercises 57–60 in the Cartesian plane. Calculate the length and midpoint of the line segment from P to Q, and use your picture to check whether your answers are reasonable.

57. $P = (-2, -1)$, $Q = (1, -3)$
58. $P = (-1, 0)$, $Q = (3, -2)$
59. $P = (-5, 2)$, $Q = (2, 5)$
60. $P = (-3, 4)$, $Q = (-3, -6)$

Each of the sets in Exercises 61–72 is some combination of the following six sets:

$$A = [-2, 3) \qquad D = \{x \mid x \geq -1\}$$
$$B = (-\infty, 0] \qquad E = \{x \mid 0 \leq x < 2\}$$
$$C = (-1, 3] \qquad F = \{x \mid x < -2\}$$

Write each set three ways: (a) in set notation; (b) in interval notation; and (c) on a real number line.

61. $C \cap D$
62. $A \cup B$
63. $C \cup D$
64. $E \cup F$
65. $B \cap E$
66. $C \cap F$
67. $C \cup E \cup F$
68. $A \cap B \cap C$
69. $(B \cap C) \cup F$
70. $(E \cup F) \cap B$
71. $(B \cup E) \cap (D \cup F)$
72. $(A \cap C) \cap (D \cap E)$

In Exercises 73–76, sketch the points (x, y) in the Cartesian plane that simultaneously satisfy all the given conditions.

73. $y \leq 3$, $y \geq x$, and $y \geq -x$
74. $y > 1$, $y < x$, and $x < 4$
75. $x \geq 0$ and $\text{dist}((x, y), (0, 0)) \leq 4$
76. $y > 0$, $y > x$, and $\text{dist}((x, y), (0, 0)) < 4$

Applications

77. Suppose a taxi driver's territory is along just one east-west street called Main Street. The taxi dispatch center is at 5th and Main. If the taxi travels from the dispatch center on Main Street to the corners of 9th Street, then 3rd Street, then 12th Street, then 8th Street, then 1st Street, and back to the dispatch center, how many total blocks did the taxi travel? Give not only your final answer, but also an expression of how to solve the problem with absolute values.
78. Starting from your apartment, you drive 6 blocks east to the grocery, then 5 blocks north to the coffee shop, then 17 blocks west to the laundromat, then 9 blocks south to the bakery. How far away are you from your apartment (as the crow flies)?
79. Anne, Bekah, and Calvin are going to lunch. Anne is willing to spend between \$9 and \$17 on her lunch, Bekah between \$3 and \$6, and Calvin between \$5 and \$12. Is there a lunch price that will work for all three people? What lunch prices are acceptable to at least two people?
80. A room at an art gallery measures 20 feet east-west and 15 feet north-south. An artist paints the floor red in a 9×9 square aligned with the southwest corner and blue in a 13×13 square aligned with the northeast corner. What is the area of the purple region where the red and blue squares intersect? *(Recall that the area of a rectangle is found by multiplying its length and width.)*

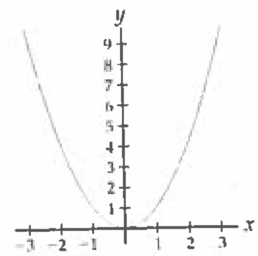
Proofs

81. Argue that all integers are rational numbers. *(Hint: Given an integer n, describe how n can be written in the form $\frac{p}{q}$ for some integers p and q.)*
82. Argue that the sum of two rational numbers is always a rational number. *(Hint: Given two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, what is their sum?)*
83. Argue that the product of two rational numbers is always a rational number. *(Hint: Given two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, what is their product?)*
84. The points (x, y) that lie on a circle of radius 1 centered at the origin satisfy the equation $x^2 + y^2 = 1$. Use the distance formula to show that any point (x, y) satisfying this equation is a distance of 1 from the origin.

Thinking Forward

- *Absolute value inequalities with "less than"*: The absolute value inequality $|x-3| < 7$ describes the real numbers x that are within 7 units of the number 3. Equivalently, this subset of real numbers can be described as those that satisfy both $x - 3 < 7$ and $x - 3 > -7$. Illustrate this equivalence on a real number line.
- *Absolute value inequalities with "greater than"*: The absolute value inequality $|x - 3| > 7$ describes the real numbers x that are more than 7 units from the number 3. Equivalently, this subset of real numbers can be described as those that satisfy either $x - 3 > 7$ or $x - 3 < -7$. Illustrate this equivalence on a real number line.
- *Areas under curves*: The set of points (x, y) that satisfy the equation $y = x^2$ form a curve in the plane, as shown in the figure that follows. Shade the subset of points (x, y) in the plane that satisfy $0 < y < x^2$ and $-2 < x < 3$. In later chapters we will learn how to find the areas of such regions.

Points (x, y) satisfying $y = x^2$



0.2 EQUATIONS

- Solving equations and systems of equations
- Simple factoring techniques and using products to solve equations
- Basic algebra skills involving fractions and dealing with quotients in equations

Equations and Solutions

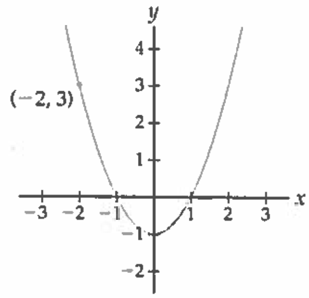
An ***equation*** consists of two mathematical expressions connected by an equals sign. For example,

$$x^2 - 1 = 0, \qquad x^2 - x = y + 2, \qquad \text{and} \qquad 3u + 2v - 8x = 5$$

are equations. We will be concerned primarily with equations involving one or two ***variables.*** Usually we will use letters near the end of the alphabet, such as x, y, z, u, and v, to represent variables and will reserve letters near the beginning of the alphabet, such as a, b, and c, for ***constants.*** A variable represents a changing or unknown quantity, whereas a constant represents a fixed number.

An equation may be true for some values of the variables involved and false for others. For example, the equation $x^2 - 1 = 0$ is clearly not true for all values of x. In fact, this equation is true only when $x = 1$ or when $x = -1$. The set of values for which an equation is true is called the ***solution set*** for that equation (or simply the ***solution*** of that equation). Thus the solution of the equation $x^2 - 1 = 0$ is the set $\{1, -1\}$. A ***system of equations*** is simply a list of two or more equations. A solution of a system of equations is a solution of each of the equations simultaneously.

If an equation involves two variables, then its solution set consists of ordered pairs of numbers. For example, solutions of the two-variable equation $x^2 - 1 = y$ are ordered pairs (x, y) that make the equation true. In this case, $(x, y) = (1, 2)$ is not a solution (since $1^2 - 1 \neq 2$) and $(x, y) = (-2, 3)$ is a solution (since $(-2)^2 - 1 = 3$). Interestingly, if we draw a point in the Cartesian plane for each solution (x, y) to the equation $x^2 - 1 = y$, we get the curve shown next; note that $(-2, 3)$ is on this curve. Every point (x, y) on the curve is a solution of $x^2 - 1 = y$, and every point (x, y) not on the curve is not a solution.



We'll learn more about how to find graphical representations of solution sets when we talk about functions in Section 0.4. For now, you can use your graphing utility. For example, if you graph the equation $y = x^2 - 1$ with your calculator or graphing utility, you will see the graph of solutions shown.

An equation that is *always* true, for all values of the variables involved, is called an ***identity.*** For example, the equation $(x + 1)^2 = x^2 + 2x + 1$ is true for all real numbers x, and therefore that equation is an identity. Identities will be particularly important when we learn about trigonometric functions. At the other end of the spectrum, some equations have *no* real solutions. For example, the equation $x^2 = -1$ has no real solutions, because there is no real number whose square is -1.

Factoring and Products

Many equations can be solved easily by the method of ***factoring.*** To factor an algebraic expression, you write that expression as a product of simpler expressions. For example, the expression $3x^2 - 6x$ can be factored as $3x(x - 2)$. The reason that factoring can help solve an equation is that a basic property of the real numbers is that a product can be equal to zero only if one of the factors in the product is zero. In the following theorem we use the term ***real expressions,*** which will denote any expressions that when calculated are real numbers.

THEOREM 0.11 **A Product Is Zero If and Only If One of the Factors Is Zero**

For any real expressions A and B, $AB = 0$ if and only if $A = 0$ or $B = 0$.

The phrase "if and only if" is shorthand for saying that the implication goes both ways. The forward direction of this implication says that if AB is zero, then either A or B (or both) is zero. The backwards direction says that if A or B is zero, then AB is also zero. This theorem can be generalized to products involving more than two factors; for example, a product $ABCD = 0$ if and only if one or more of the factors A, B, C, and D is zero.

Theorem 0.11 helps us solve complicated equations by turning them into a series of simpler equations. For example, the equation $3x^2 - 6x = 0$ can be factored as $3x(x - 2) = 0$. This equation is true if and only if one of its factors, either $3x$ or $x - 2$, is zero, so the solutions of the original equation are $x = 0$ and $x = 2$.

CAUTION Theorem 0.11 applies only to equations that are set to zero, that is, equations of the form $AB = 0$. The theorem does *not* say, for example, that $3x(x - 2) = 2$ if and only if $3x = 2$ or $x - 2 = 2$. Why not?

Quadratic equations are equations of the form $ax^2 + bx + c = 0$ for some constants a, b, and c. These equations can often easily be factored by doing the "FOIL" ("First-Outside-Inside-Last") method backwards. For example, the equation $x^2 - 3x - 4 = 0$ can be factored as $(x - 4)(x + 1) = 0$. Once the equations are factored, we can easily find their solutions by applying Theorem 0.11; for example, the solutions of $x^2 - 3x - 4 = 0$ are $x = 4$ and $x = -1$.

Unfortunately, other quadratic equations, such as $3x^2 - 7x + 1 = 0$, are not so easy to factor. For these equations we can apply the ***quadratic formula*** to determine the solutions. (You will prove this theorem in Exercise 80 of Section 0.7.)

THEOREM 0.12 **The Quadratic Formula**

If a, b, and c are real numbers, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are of the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is called the ***discriminant***, and it determines the number of solutions of the quadratic equation $ax^2 + bx + c$. If $b^2 - 4ac$ is positive, then there will be two solutions: one when the $\pm$ symbol is a plus and one when it is a minus. If $b^2 - 4ac$ is zero, then there is only one solution, namely, $x = \frac{-b}{2a}$. If $b^2 - 4ac$ is negative, then its square root is not a real number and thus the equation $ax^2 + bx + c = 0$ has no real solutions. In this case the equation does have complex-number solutions, but in this book we will be concerned only with real numbers, so a quadratic equation with a negative discriminant will be said to have "no solutions."

Differences of squares, or cubes, or any power can be at least partially factored with the use of the following formulas:

THEOREM 0.13 **Formulas for Factoring Differences of Powers**

For all real numbers a and b, and any positive integer n,

$$a^2 - b^2 = (a - b)(a + b)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

The last formula is a generalization of the first two. To prove these formulas, simply multiply out the right-hand sides of the equations; everything will cancel out, and you will have the left-hand sides of the equations. You will do this in Exercises 83 and 84.

The ellipses ("dot-dot-dot") in the last formula in Theorem 0.13 indicate that we are to continue the pattern for as long as necessary. Note that the exponents of each of the terms in the second factor always add up to $n - 1$; for example, the term $a^{n-3}b^2$ has exponents $n - 3$ and 2, which add up to $n - 1$. This formula does *not* necessarily completely factor the expression $a^n - b^n$, but it does pull out one simple factor $a - b$. For example, when $n = 5$, $a = x$, and $b = 2$, the last formula in Theorem 0.13 tells us that

$$\begin{aligned} x^5 - 2^5 &= (x - 2)(x^4 + x^3(2) + x^2(2^2) + x(2^3) + 2^4) \\ &= (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16). \end{aligned}$$

Fractions and Quotients

Many equations we will be interested in will involve fractions. The rules for adding, multiplying, and dividing fractions are given in Theorem 0.14.

THEOREM 0.14 **Algebraic Rules for Fractions**

Let a, b, c, and d be any real numbers. Then (assuming that no denominator in any expression is zero)

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \qquad \left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\frac{ac}{bd} \qquad \frac{(a/b)}{(c/d)}=\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)=\frac{ad}{bc}$$

You can also add fractions by forming the lowest common denominator. As a simple example, note that it is easier to calculate $\frac{1}{6}+\frac{2}{3}=\frac{1}{6}+\frac{4}{6}=\frac{5}{6}$ than it is to calculate $\frac{1}{6}+\frac{2}{3}=\frac{(1)(3)+(6)(2)}{(6)(3)}=\frac{3+12}{18}=\frac{15}{18}=\frac{5}{6}$.

We will accept the rules of fractions in Theorem 0.14 as basic properties of the real numbers. As immediate consequences we obtain the following additional rules (see also Exercises 85 and 86):

THEOREM 0.15 **More Algebraic Rules for Fractions**

Let a, b, c, and d be any real numbers. Then (assuming that no denominator in any expression is zero)

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd} \qquad c\left(\frac{a}{b}\right)=\frac{ac}{b} \qquad \frac{(a/b)}{c}=\frac{a}{bc} \qquad \frac{a}{(b/c)}=\frac{ac}{b}$$

If A is a real number, then $\frac{A}{A}$ is equal to 1 as long as A is not equal to zero. If $A=0$, then the expression $\frac{A}{A}$ is undefined (since we cannot divide zero by zero). This tells us when we can cancel a ***common factor*** in the numerator and the denominator of a fraction. The key point is that if the factor you are canceling involves a variable, then the cancellation is not valid when that factor is equal to zero.

THEOREM 0.16 **Cancellation when Variables Are Involved**

Given any real expression A that depends on a variable x,

$$\frac{A}{A}=\begin{cases}1, & \text{for values of } x \text{ that make } A\neq 0\\ \text{undefined}, & \text{for values of } x \text{ that make } A=0.\end{cases}$$

For example, you can cancel a common factor $x-c$ from the numerator and denominator of a fraction, and the cancelled expression will equal the original expression for all values of x except $x=c$. One illustration of this theorem is the cancellation

$$\frac{x^2-1}{x+1}=\frac{(x-1)(x+1)}{x+1}=\begin{cases}x-1, & \text{if } x\neq -1\\ \text{undefined}, & \text{if } x=-1.\end{cases}$$

Theorem 0.11 told us when a product AB can be equal to zero. The next theorem tells us when a quotient $\frac{A}{B}$ can be equal to zero. This theorem will help us solve equations that can be written as quotients.

THEOREM 0.17 **When Is a Quotient Equal to Zero?**

If A and B are any real expressions, then $\frac{A}{B} = 0$ if and only if $A = 0$ and $B \neq 0$.

For example, the equation $\frac{x^2-1}{x+1} = 0$ is true if and only if $x^2 - 1 = 0$ (which means that $x = 1$ or $x = -1$) but $x + 1 \neq 0$ (which means that $x \neq -1$). Therefore there is only one solution of this equation, and it is $x = 1$.

CAUTION: Note in particular that $B = 0$ does *not* imply that $\frac{A}{B} = 0$. In fact, if $B = 0$, then $\frac{A}{B}$ is undefined. That is why we require not only that $A = 0$ but also that $B \neq 0$. Note also that Theorem 0.17 is valid only for equations that are set to zero.

Examples and Explorations

EXAMPLE 1 Solving equations by factoring

Solve the following equations by using Theorems 0.11–0.13:

(a) $2x^3 - 5x^2 - 3x = 0$ (b) $3x^2 = 7x - 1$ (c) $2x^5 - 32x = 0$

SOLUTION

(a) As we will see in Chapter 4, the number of real number solutions of a polynomial equation is at most the value of the ***degree***, or highest power, of the polynomial. Therefore we can expect the equation $2x^3 - 5x^2 - 3x = 0$ to have at most three real number solutions. The right-hand side of this equation can be easily factored as:

$$2x^3 - 5x^2 - 3x = x(2x^2 - 5x - 3) = x(2x+1)(x-3).$$

By Theorem 0.11, the expression $2x^3 - 5x^2 - 3x$ is zero if and only if one of x, $2x+1$, or $x - 3$ is zero. In other words, the solutions of the equation $2x^3 - 5x^2 - 3x = 0$ are $x = 0$, $x = -\frac{1}{2}$, and $x = 3$.

(b) We first need to write the equation $3x^2 = 7x - 1$ in the general form of a quadratic equation: $3x^2 - 7x + 1 = 0$. This equation cannot be easily factored with the reverse "FOIL" (first-outside-inside-last) method, so we'll apply the quadratic formula. In this example we have $a = 3$, $b = -7$, and $c = 1$, so the solutions of $3x^2 - 7x + 1 = 0$ are

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(1)}}{2(3)} = \frac{7 \pm \sqrt{49 - 12}}{6} = \frac{7 \pm \sqrt{37}}{6}.$$

Therefore, the solutions of $3x^2 - 7x + 1 = 0$ are $x = \frac{1}{6}(7 + \sqrt{37})$ and $x = \frac{1}{6}(7 - \sqrt{37})$. Clearly we could not have easily figured that out by doing the "FOIL" method backwards!

(c) This time the factoring will involve two applications of a formula from Theorem 0.13:

$$\begin{aligned} 2x^5 - 32x &= 0 \\ 2x(x^4 - 16) &= 0 \\ 2x(x^2-4)(x^2+4) &= 0 \quad \leftarrow \text{formula for } a^2 - b^2 \text{ with } a = x^2 \text{ and } b = 4 \\ 2x(x-2)(x+2)(x^2+4) &= 0 \quad \leftarrow \text{formula for } a^2 - b^2 \text{ with } a = x \text{ and } b = 2 \end{aligned}$$

Thus $2x^5 - 32x = 0$ whenever $2x = 0$, $x - 2 = 0$, $x + 2 = 0$, or $x^2 + 4 = 0$. Note that $x^2 + 4 = 0$ has no real solutions, because there is no real number that satisfies $x^2 = -4$. Therefore the real-number solution set of the original equation $2x^5 - 32x = 0$ is $\{-2, 0, 2\}$.

CHECKING THE ANSWER

To check the answers in Example 1, simply substitute each proposed solution into the original equation. Each solution should satisfy the equation. For example, to check that $x = -2$, $x = 0$, and $x = 2$ are solutions in part (c) of the example we note that

$$\begin{aligned} 2(-2)^5 - 32(-2) &= 0, && \leftarrow \text{evaluate equation at } x = -2 \\ 2(0)^5 - 32(0) &= 0, && \leftarrow \text{evaluate equation at } x = 0 \\ 2(2)^5 - 32(2) &= 0. && \leftarrow \text{evaluate equation at } x = 2 \end{aligned}$$

Of course, this will not tell you whether you have *missed* any solutions, but it will tell you whether the solutions you found are correct.

EXAMPLE 2

A simple system of equations

Solve the system of equations $\begin{cases} x^2 - 1 = 0 \\ x^2 - x - 2 = 0. \end{cases}$

SOLUTION

The first equation can be written as $x^2 = 1$, so its solutions are $x = \pm\sqrt{1}$, that is, $x = 1$ and $x = -1$. The second equation can be factored as $(x - 2)(x + 1) = 0$, and thus its solutions are $x = 2$ and $x = -1$. The only number that is a solution of *both* equations is $x = -1$, so the solution set for the system of equations is $\{-1\}$.

EXAMPLE 3

Solving a two-variable system of equations

Solve the systems of equations $\begin{cases} x^2 + y = 3 \\ 2x + y = 0. \end{cases}$

SOLUTION

In solving a system of equations, sometimes it is a good strategy to use one of the equations to solve for one variable in terms of the other variables and then use the information obtained to simplify the remaining equations. We'll start by using the second equation to solve for y in terms of x. Since $2x + y = 0$, it follows that $y = -2x$. Now we can substitute this into the first equation (note that the resulting equation has only one variable) and solve:

$$\begin{aligned} x^2 + y &= 3 && \leftarrow \text{first equation} \\ x^2 - 2x &= 3 && \leftarrow \text{substitute } y = -2x \\ x^2 - 2x - 3 &= 0 && \leftarrow \text{collect terms to left side} \\ (x - 3)(x + 1) &= 0 && \leftarrow \text{factor} \end{aligned}$$

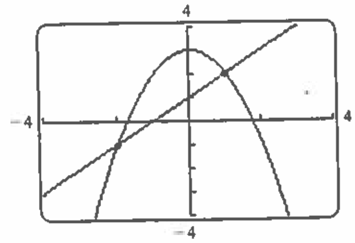
The last equation clearly has solutions $x = 3$ and $x = -1$, so *any* solution (x, y) of the original system has the property that $x = 3$ or $x = -1$. Keeping in mind the fact that $y = -2x$, we can use these values of x to solve for the corresponding values of y. If $x = 3$, then $y = -2(3) = -6$. If $x = -1$, then $y = -2(-1) = 2$. Thus the solutions of the system of equations are $(x, y) = (3, -6)$ and $(x, y) = (-1, 2)$.

CHECKING THE ANSWER

To check our solutions, we need only substitute them into each equation in the system of equations to see whether they work. For example, $(x, y) = (3, -6)$ satisfies both of the equations in the system, because $3^2 + (-6) = 3$ and $2(3) + (-6) = 0$. In a similar fashion we can verify that $(x, y) = (-1, 2)$ is a solution to the system. This won't tell us if we have missed any solutions, but it will verify that the solutions we found are, in fact, correct.

We can also do a graphical check, by graphing the equations in the system and looking at where their solution sets intersect. If we solve the equations in our system for y we obtain $y = 3 - x^2$ and $y = -2x$ as shown in the following figure. Note that there are two intersection points, namely, at $(3, -6)$ and at $(-1, 2)$.

$y = 3 - x^2$ *and* $y = -2x$



EXAMPLE 4

Solving a system by adding one equation to another

Solve the system of equations $\begin{cases} a + 2b - c &= 0 \\ a + b &= c - 1 \\ a - 2b + c &= 2. \end{cases}$

SOLUTION

This system of equations has three equations and three variables, a, b, and c. The solutions of the system will be the triples (a, b, c) that are solutions of all three equations.

Since the first and last equations look somewhat similar, we'll try adding them together; in other words, we will add the left-hand side of the last equation to the left-hand side of the first equation and, similarly, add the right-hand side of the last equation to the right-hand side of the first equation. (Why is this valid?) If we do so, we get the new equation

$$\begin{aligned} (a + 2b - c) + (a - 2b + c) &= 0 + 2 && \leftarrow \text{add first and third equations} \\ 2a &= 2 && \leftarrow \text{simplify} \\ a &= 1. \end{aligned}$$

This calculation immediately tells us that any solution (a, b, c) of our original system must have $a = 1$. Replacing a with 1 in the second equation gives us $1 + b = c - 1$, which implies that $b = c - 2$. Using $a = 1$ and $b = c - 2$, we can rewrite the first equation so that it involves only one variable:

$$\begin{aligned} a + 2b - c &= 0 && \leftarrow \text{first equation} \\ 1 + 2(c - 2) - c &= 0 && \leftarrow \text{substitute } a = 1 \text{ and } b = c - 2 \\ c - 3 &= 0 && \leftarrow \text{simplify} \\ c &= 3. \end{aligned}$$

Now, because $b = c - 2$, we know that $b = 3 - 2 = 1$. There is only one solution of the system of equations, namely, $(a, b, c) = (1, 1, 3)$.

EXAMPLE 5

Writing an expression as a simple fraction

Write the following expression in the form of a ***simple fraction***, that is, in the form $\frac{A}{B}$ for some expressions A and B that do not themselves involve fractions and do not have any common factors:

$$\frac{\frac{3}{x+2} - \frac{1}{x}}{x - 1}$$

SOLUTION

Using the rules for fractions, we obtain

$$\frac{\frac{3}{x+2}-\frac{1}{x}}{x-1} = \frac{\frac{3x-1(x+2)}{(x+2)(x)}}{x-1} \qquad \leftarrow \text{combine numerator using common denominator}$$

$$= \frac{\frac{2x-2}{x(x+2)}}{x-1} \qquad \leftarrow \text{simplify numerator of numerator}$$

$$= \frac{2x-2}{x(x+2)(x-1)} \qquad \leftarrow \tfrac{A}{B} \text{ divided by } C \text{ is } \tfrac{A}{BC}$$

$$= \frac{2(x-1)}{x(x+2)(x-1)} \qquad \leftarrow \text{factor numerator}$$

$$= \frac{2}{x(x+2)}. \qquad \leftarrow \text{cancel common factor of } x-1$$

The cancellation in the last equality is true only for $x \neq 1$. Can you list each of the algebraic rules used in the calculation?

There will often be more than one correct way to simplify an expression. In this example we could have instead used the technique of ***clearing denominators***, in which we multiply the numerator and denominator of the original expression by something that cancels out all the denominators in the sub-expressions. Since the sub-expressions have denominators of $x+2$ and x in this example, we multiply numerator and denominator by $x(x+2)$:

$$\frac{\frac{3}{x+2}-\frac{1}{x}}{x-1} \cdot \frac{x(x+2)}{x(x+2)} = \frac{(3x)-(x+2)}{x(x+2)(x-1)} \qquad \leftarrow \text{multiply by } \frac{x(x+2)}{x(x+2)} = 1$$

$$= \frac{2x-2}{x(x+2)(x-1)} \qquad \leftarrow \text{simplify numerator}$$

$$= \frac{2(x-1)}{x(x+2)(x-1)} \qquad \leftarrow \text{factor}$$

$$= \frac{2}{x(x+2)}. \qquad \leftarrow \text{cancel common factor of } x-1$$

Once again the cancellation in the last step is true only for $x \neq 1$, and we obtain the same answer as before.

EXAMPLE 6 Solving equations involving quotients

Find the solution sets of the following equations:

(a) $\dfrac{x+1}{x-1} = 4$ **(b)** $\dfrac{x^3+x^2-6x}{x^2-3x+2} = 0$

SOLUTION

(a) One way to solve this equation is to rewrite it so that one side of the equation is zero. We could do that by subtracting the 4 from both sides of the equation and simplifying, but it is easier to clear denominators first by multiplying both sides of the equation by $x-1$. As long as $x \neq -1$, this will produce an equivalent equation that can be solved easily:

$$x+1 = 4(x-1) \qquad \leftarrow \text{multiply both sides by } x = 1$$

$$x+1 = 4x-4 \qquad \leftarrow \text{expand}$$

$$5 = 3x \qquad \leftarrow \text{collect } x \text{ terms to right-hand side}$$

$$x = \frac{5}{3}. \qquad \leftarrow \text{solve for } x$$

Thus the solution set of the equation $\frac{x+1}{x-1} = 4$ is $\left\{\frac{5}{3}\right\}$.

(b) We will use Theorem 0.17 to solve this equation, but first we need to factor the numerator and the denominator:

$$\frac{x^3+x^2-6x}{x^2-3x+2} = \frac{x(x^2+x-6)}{x^2-3x+2} = \frac{x(x-2)(x+3)}{(x-2)(x-1)}.$$

This expression is equal to zero only when the numerator is equal to zero (and the denominator is *not* zero). The numerator of the expression is zero when $x = 2$, but since the denominator is also zero when $x = 2$, the entire expression is undefined when $x = 2$. The solution of the original equation consists only of those values of x for which the numerator is zero while the denominator is nonzero, namely, $x = 0$ and $x = -3$.

? TEST YOUR UNDERSTANDING

- What do we mean when we say that $x = 2$ and $x = -2$ are "solutions" to the equation $x^2 - 4 = 0$?
- How can factoring help us solve an equation? Why is it often easier to solve a factored equation than a non-factored equation?
- What is the quadratic formula for? What does it help you find?
- Explain why the expression $\frac{(x-2)(x+1)}{x-2}$ is not equivalent to the expression $x+1$ for all values of x. For which value(s) of x do they differ, and why?
- Define each of the following: one-variable equation, identity, factoring, quadratic equation, discriminant.

EXERCISES 0.2

Thinking Back

Finding a common denominator: Compute each of the following quantities by finding a common denominator.

- $\frac{1}{8} + \frac{3}{4}$
- $\frac{3}{5} - \frac{11}{2}$
- $7 + \frac{3}{7}$

Common algebra mistakes: Each of the false statements that follow illustrates a common algebra mistake. For each false statement, find a number x for which the right-hand side is not equal to the left-hand side.

- *False:* $\frac{1}{x} + \frac{x}{2} = \frac{1+x}{x+2}$
- *False:* $\frac{2x+1}{2x-1} = \frac{x+1}{x-1}$
- *False:* $\frac{3x}{x^2-1} = \frac{3}{x-1}$

Factoring simple quadratics: Factor each of the quadratic expressions that follow. Consult your old algebra textbook or an online resource if you need help remembering how to do this.

- $x^2 + 2x - 15$
- $x^2 - 9x + 18$
- $2x - 48 + x^2$
- $3x^2 - 11x - 4$
- $6x^2 - x - 2$
- $-3 - 5x + 12x^2$

Using a graphing utility: If a graphing utility is allowed in your course, then you should be familiar with using it. Practice by using a graphing utility to graph each of the following equations in the given graphing window.

- $y = 2 - x + x^4$, in window $-4 \le x \le 4$, $0 \le y \le 10$
- $y = x(x^2-1)$, in window $-3 \le x \le 3$, $-3 \le y \le 3$
- $y - 3x = 6 - x^2$, in window $-3 \le x \le 5$, $-3 \le y \le 10$

Concepts

0. *Problem Zero:* Read the section and make your own summary of the material.

1. *True/False:* Determine whether each of the statements that follow is true or false. If a statement is true, explain why. If a statement is false, provide a counterexample.

(a) *True or False:* If $AB = 0$, then $A = 0$ and $B = 0$.

(b) *True or False:* $\frac{A}{B} = 0$ if and only if $A = 0$.

(c) *True or False:* $\frac{A}{A} = 1$ for every real number A.

(d) *True or False:* Every quadratic equation has exactly two real solutions.

(e) *True or False:* An identity is an equation that is always true.

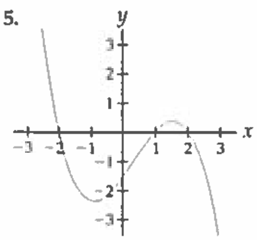
(f) *True or False:* For all x, $\dfrac{(x+1)(x+2)}{x+1} = x+2$.

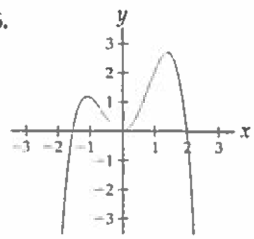
(g) *True or False:* If $b, d \neq 0$, then $3\left(\dfrac{a}{b}+\dfrac{c}{d}\right) = \dfrac{3ad+3bc}{bd}$.

(h) *True or False:* If $a, b \neq 0$, then $\dfrac{12}{(a/b)} = \dfrac{12b}{a}$.

2. *Examples:* Construct examples of the thing(s) described in the following. Try to find examples that are different than any in the reading.
   (a) A two-variable equation.
   (b) An equation that is an identity.
   (c) A three-variable system of equations.
3. Suppose you solve an equation and want to check your answers. Describe two ways (one algebraic, one graphical) that you can check your work.
4. Suppose you solve a system of equations and want to check your answers. Describe two ways (one algebraic, one graphical) that you can check your work.

In Exercises 5 and 6, a graph describes the points (x, y) that are solutions of some equation involving x and y. Use the graph to estimate three points (x, y) that are solutions and three points that are not.

5. 

6. 

For each equation in Exercises 7 and 8, (a) determine whether $x = 1$ and $x = 2$ are solutions and explain why or why not, and then (b) use a graphing utility to estimate all of the solutions of the equation.

7. $x^3 - 4x^2 + x + 6 = 0$
8. $x^3 - 3x^2 - 6x + 8 = 0$

For each equation in Exercises 9 and 10, (a) determine whether $(x, y) = (1, 4)$ and $(x, y) = (2, 1)$ are solutions and explain why or why not, (b) use the equation to find three additional solutions, and then (c) use a graphing utility to graph the equation and verify your solutions.

9. $y = 2 - x + 3x^5$
10. $y - 1 = (x-2)(x-3)$

For each system of equations in Exercises 11 and 12, (a) determine whether $(1, -1)$ and $(-2, -1)$ are solutions and explain why or why not, and then (b) use a graphing utility to approximate all of the solutions of the system.

11. $\begin{cases} x - y = -1 \\ y + x^2 = 3 \end{cases}$
12. $\begin{cases} y + x^3 = 0 \\ y - x^2 + 2x = 0 \end{cases}$

13. The equation $9x^2 + 3x - 2 = 0$ can be rewritten as $3x(3x+1) - 2 = 0$. Would the rewritten equation be considered factored? Why or why not?
14. The equation $9x^2 + 3x - 2 = 0$ can be rewritten as $3x(3x+1) = 2$. Can Theorem 0.11 be applied to this rewritten equation? Why or why not?
15. If a quadratic equation has two distinct roots r_1 and r_2 given by the quadratic formula, what are the possible factored forms of the equation? What if there is only one root r_1 given by the quadratic formula?
16. Why does the discriminant of a quadratic equation determine the number of real solutions? Think about square roots and the cases when the discriminant is positive, negative, and zero.
17. What is the value of $\dfrac{x+2}{x^2-4}$ when $x = -2$? After factoring and cancelling, this expression becomes $\dfrac{1}{x-2}$. What is the value of the cancelled expression when $x = -2$? Is it the same as the value of the original expression when $x = -2$? Why or why not?
18. The numerator of $\dfrac{x+2}{x^2-4}$ is zero when $x = -2$. Does this mean that the entire expression is equal to zero when $x = -2$? Why or why not?

Skills

Factor each of the expressions in Exercises 19–32 as much as possible.

19. $6x^3 + 2x^2 - 20x$
20. $8x^2 - 50$
21. $4x^2 - 81$
22. $x^2 + 16$
23. $81x^4 - 1$
24. $16 - x^4$
25. $x^3 + 3x^2 + 3x + 1$
26. $x^4 + 81x$
27. $27 - 64x^3$
28. $27x^3 - 8$
29. $x^5 - 1$
30. $x^5 - 243$
31. $t^7 - x^7$
32. $x^8 - a^8$

Write each of the expressions in Exercises 33–38 in the form of a simple fraction. For which values of x, if any, does your simplified expression differ from the original expression?

33. $\dfrac{x^2-1}{\dfrac{1}{x+1} - \dfrac{1}{x}}$
34. $\dfrac{\dfrac{x+1}{x} - \dfrac{1}{x-1}}{x+1}$
35. $\dfrac{\left(\dfrac{x^2-4}{x+1}\right)}{\left(\dfrac{x+2}{x-1}\right)}$
36. $\dfrac{x^2}{x-2}\left(\dfrac{x+1}{x-2} + \dfrac{3}{x+1}\right)$
37. $3x\left(\dfrac{2-x}{x^2-1}\right) - \dfrac{x}{x-1}$
38. $\dfrac{\dfrac{1}{x^2-4} - \dfrac{3}{x}}{\dfrac{x-2}{x^2-x}}$

Find the solution set of each equation in Exercises 39–68. When possible, solve by factoring and using Theorem 0.11 and Theorem 0.17. Some equations may not factor easily and will require the quadratic formula.

39. $6 - 7x + x^2 = 0$

40. $x^2 + 2x - 8 = 0$

41. $4x(x^2 - 1) = 6x^2$

42. $3x^2 + 5x = -2$

43. $2x^2 - x - 3 = 0$

44. $2x^2 + 4x + 2 = 0$

45. $4x^2 - 12x + 9 = 0$

46. $x^2 + 2x + 3 = 0$

47. $10x = 32x^2 - 6x^3$

48. $6x^2 + 3 = -11x$

49. $5 - 2x - x^2 = 0$

50. $x - 2 - 3x^2 = 0$

51. $2x(x - 1) = 3$

52. $x^2 - 2x - 4 = 0$

53. $(x - 2)^2 = 2$

54. $4x^2 - 12x = -9$

55. $x^3 - 9x = 0$

56. $8 - x^3 = 0$

57. $16x^4 - 81 = 0$

58. $x^3 - 3x^2 + 3x = 1$

59. $4x^3 - 9x = 0$

60. $\dfrac{x^2 - 1}{x^2 - 4} = 0$

61. $\dfrac{x - 1}{x + 1} = 3$

62. $\dfrac{2x - 1}{x - 3} = \dfrac{x - 2}{1 + x}$

63. $\dfrac{x^3 + x^2 - 2x}{x^2 - 4x + 3} = 0$

64. $\dfrac{4x^3 - 1}{2x^2 + x - 1} = 0$

65. $\dfrac{2}{x^2 - x} = \dfrac{1}{1 - x}$

66. $\dfrac{(x - 2)^2}{3x - 2} = x - 5$

67. $\dfrac{1}{x} = \dfrac{1}{x + 1}$

68. $\dfrac{1}{3x^2 - 2x - 1} = 2x - 1$

Find the solution sets of each of the systems of equations in Exercises 69–78.

69. $\begin{cases} 2x + y = 4 \\ 3x + 2y = 0 \end{cases}$

70. $\begin{cases} x^2 + y = 0 \\ y - x = 2 \end{cases}$

71. $\begin{cases} 3a + b = -1 \\ a + 5b = 3 \end{cases}$

72. $\begin{cases} r^2 - s = 0 \\ r^2 + s = 1 \end{cases}$

73. $\begin{cases} 3t - u^2 = -3 \\ u - 1 = t \end{cases}$

74. $\begin{cases} 3x + y - z = 2 \\ x - y + 2z = 0 \end{cases}$

75. $\begin{cases} 2a + 3b - c = 4 \\ a - b - c = 3 \\ 2a + b - c = 2 \end{cases}$

76. $\begin{cases} r + s + t = 1 \\ r - s - t = 0 \\ r - s + t = 2 \end{cases}$

77. $\begin{cases} x + 2y = 1 \\ 2y = 1 - 3x \\ 3x - y = 2 \end{cases}$

78. $\begin{cases} x - 2y + z = 0 \\ x + y - 3z = 1 \\ 3x - 2y + z = 2 \end{cases}$

Applications

79. Len works for a company that produces cans that are 6 inches tall and vary in radius. The cost of producing a can with height 6 inches and radius r is $C = 10\pi r^2 + 24\pi r$ cents. If Len wants to spend 300 cents on each can, how large should he make the radius? *(Hint: Set up and solve an equation; you will need the quadratic formula.)*

80. Suppose x and y are positive real numbers whose sum is 25 and whose product is 136. Construct and solve a system of equations to find x and y.

81. Three real numbers have the property that when you add them in pairs, you get 19, 30, and 35. Construct and solve a system of equations to find these three real numbers.

82. A sheet of paper with height h and width w is to have 2-inch margins on all four sides. Suppose the perimeter of the paper is 38 inches and the area of the printable part of the paper (inside the margins) is 45 square inches. Construct and solve a system of equations to find h and w.

Proofs

83. The second part of Theorem 0.13 says that for any real numbers a and b,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Prove this identity by multiplying out the right-hand side and then showing that it is equal to the left-hand side.

84. The third part of Theorem 0.13 says that for any real numbers a and b, and any positive integer n,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Prove this identity by multiplying out the right-hand side and then showing that it is equal to the left-hand side.

85. The second part of Theorem 0.15 says that for any real numbers a, b, and c with $b \neq 0$,

$$c\left(\frac{a}{b}\right) = \frac{ac}{b}.$$

Show that this statement is true by using the algebraic rules for fractions from Theorem 0.14. *(Hint: How can you write the real number c as a fraction?)*

86. The third part of Theorem 0.15 says that for any real numbers a, b, and c with b and c nonzero,

$$\frac{(a/b)}{c} = \frac{a}{bc}.$$

Show that this statement is true by using the algebraic rules for fractions from Theorem 0.14.

Thinking Forward

- *Polynomials, roots, and factors:* Quadratic equations are examples of ***polynomials***, which are expressions built by adding up terms of the form ax^k where k is a nonnegative integer; for example, $-x^7+2x^3-5x+4$ is a polynomial. Write down a polynomial that has roots $x = -2$, $x = -1$, $x = 3$, $x = 5$, and $x = 11$. *(Hint: Write the polynomial in factored form.)*
- *Difficulties with solving inequalities:* Consider the inequality $\frac{5}{x-2} < 3$. If this were an equation with an equals sign instead of a less-than sign, our first step in solving the equation would be to multiply both sides by $x - 2$. Unfortunately, in the next section we will see that this method will not work with an inequality. To illustrate, find a value of x for which $\frac{5}{x-2} < 3$ is true but $5 < 3(x-2)$ is false.
- *Finding where an expression has roots or does not exist:* In later chapters we will be interested in finding the values (called ***roots***) that make an expression equal to zero and the values for which an expression does not exist. Find these values for each of the following expressions:
  - $\dfrac{4x^2-1}{x^3-x^2-2x}$
  - $\dfrac{3x^2-x-2}{x^2-x}$
  - $\dfrac{1}{x^4-5x^2+4}$
  - $\dfrac{x^3+x^2}{x^2-1}$

0.3 INEQUALITIES

- Algebraic rule for solving inequalities
- Solving inequalities by examining signs of factors
- Turning absolute value inequalities into pairs of simpler inequalities

Basic Properties of Inequalities

An ***inequality*** consists of two mathematical expressions related by a greater-than ($>$), less-than ($<$), greater-than-or-equal-to ($\geq$), or less-than-or-equal-to ($\leq$) sign. The ***solution set*** of an inequality is the set of all values that make the inequality true. Whereas the solution set of an equation often consists of a finite number of points, the solution set of an inequality is often an *interval* of real numbers.

The next theorem describes what happens to inequalities when they are inverted or multiplied by a nonzero constant. This information will be useful for solving inequalities.

THEOREM 0.18 **Algebraic Rules for Inequalities**

Suppose a, b, and c are nonzero real numbers.

(a) If $a < b$ and $c > 0$, then $ac < bc$. **(b)** If $a < b$ and $c < 0$, then $ac > bc$.

(c) If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$. **(d)** If $a < b$, then $a + c < b + c$.

In particular, notice that part (b) implies that multiplying an inequality by a negative number "flips" the inequality; for example, if $a < b$, then $-a > -b$. Part (d) illustrates that we may add any number to, or subtract any number from, both sides of an inequality without changing the direction of the inequality. Theorem 0.18 is written in terms of ***strict inequalities*** $>$ and $<$, but similar results apply for the inequalities $\geq$ and $\leq$.

CAUTION

Since multiplying both sides of an inequality by a negative number "flips" the inequality, you cannot multiply both sides of an inequality by a quantity whose sign you do not know. For example, you can't multiply both sides of the inequality $\frac{3}{x} > 2$ by x, because x may be positive or negative. Therefore, you don't know whether or not multiplying both sides of an inequality by x will change the direction of the inequality.

A key property of the real numbers is that a product of two real numbers is positive if the numbers have the same sign and negative if the numbers have opposite signs. This property suggests the following theorem, which can help us solve certain factored inequalities:

THEOREM 0.19 **The Sign of a Product Depends on the Signs of Its Factors**

If A and B are any real expressions, then

(a) $AB > 0$ if and only if ($A > 0$ and $B > 0$) or ($A < 0$ and $B < 0$).

(b) $AB < 0$ if and only if ($A > 0$ and $B < 0$) or ($A < 0$ and $B > 0$).

An easy corollary of this theorem is that $AB \geq 0$ if and only if either A and B have the same sign or one of A and B is zero. Theorem 0.19 also holds for quotients: $\frac{A}{B}$ is greater than zero if A and B have the same sign and is less than zero if A and B have opposite signs. We can also generalize this theorem to products with more than two factors:

THEOREM 0.20 **The Sign of a Product Depends on the Number of Its Negative Factors**

A product or quotient of nonzero real numbers is positive if and only if it has an even number of negative factors and is negative if and only if it has an odd number of negative factors.

For example, the product $(-2)(-1)(2)(-6)$ is negative because it has an odd number of negative factors. In contrast, $\frac{(-3)(-2)(-8)}{(7)(-4)}$ is positive because it has an even number of negative factors.

Solving Inequalities That Involve Products or Quotients

Theorem 0.19 reduces the problem of solving a factored inequality to the problem of solving a collection of related simpler inequalities and using intersections and unions as appropriate. For example, the inequality $x^2 - x < 0$ factors as $x(x-1) < 0$. The product of x and $x-1$ is negative exactly when x and $x-1$ have opposite signs, which means that the solution of the original inequality is the solution of:

$$(x > 0 \text{ and } x - 1 < 0) \text{ or } (x < 0 \text{ and } x - 1 > 0).$$

In interval notation, and using intersections for "and" and unions for "or," this expression is equal to

$$((0, \infty) \cap (-\infty, 1)) \cup ((-\infty, 0) \cap (1, \infty)).$$

This expression in turn is equal to $(0, 1) \cup \emptyset = (0, 1)$, so the solution of the original inequality $x(x-1) < 0$ is the interval $(0, 1)$.

Although the method of breaking up a factored inequality into a union and intersection of a lot of simple inequalities is not difficult, it can become tedious when there are more than two factors involved. The next theorem will allow us a quicker way to solve inequalities that feature ***polynomials***. A polynomial is an expression that can be written as a sum of terms of the form ax^k, where a is a real number and k is a nonnegative integer. For example, $x^4 + x^2 - 2$ is a polynomial, that, in factored form, can be written as $(x^2 + 2)(x + 1)(x - 1)$. We will say that an expression ***changes sign*** at $x = c$ if, as we consider x varying from just less than c to just greater than c, the expression changes sign from positive to negative or from negative to positive. For example, the expression $x - 5$ changes sign (from negative to positive) at $x = 5$.

THEOREM 0.21 **When Can a Polynomial or Quotient of Polynomials Change Sign?**

A polynomial can change sign only at values where it is equal to zero. A quotient of polynomials can change sign only at values where its numerator or denominator is equal to zero.

This theorem is a special case of the Intermediate Value Theorem, which we will prove in Chapter 1.

As an illustration of the power of Theorem 0.21, consider the problem of solving the inequality $(x^2 + 2)(x + 1)(x - 1) < 0$. We could examine all the possible cases when an odd number of the three factors are negative, but this would be quite tedious. Instead, let's apply Theorem 0.21. Clearly the expression $(x^2 + 2)(x + 1)(x - 1)$ is zero only when $x = -1$ or $x = 1$. (Note that $x^2 + 2$ is never zero.) Therefore these are the only two x-values at which the expression can change sign. That means that the expression has the same sign on the entire interval $(-\infty, -1)$, and the same sign on all of $(-1, 1)$, and the same sign on all of $(1, \infty)$. To determine what the sign is in each case we need only check the sign of the expression at one point in each interval; for example, at $x = -2$, $x = 0$, and $x = 2$:

$$\begin{aligned}
((-2)^2 + 2)(-2 + 1)(-2 - 1) &= (\text{pos})(\text{neg})(\text{neg}) = \text{positive},\\
(0^2 + 2)(0 + 1)(0 - 1) &= (\text{pos})(\text{pos})(\text{neg}) = \text{negative},\\
(2^2 + 2)(2 + 1)(2 - 1) &= (\text{pos})(\text{pos})(\text{pos}) = \text{positive}.
\end{aligned}$$

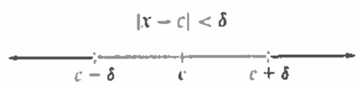
We can mark this information on a number line with a ***sign chart*** by making tick marks at each x-value that made the expression zero and writing + or − above each interval we tested, as shown in the sign chart that follows. Note that we include tick marks *only* at the potential sign-changing points.

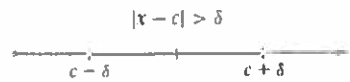
Finally, we can conclude that $(x^2+2)(x+1)(x-1)$ is less than zero exactly when $x \in (-1, 1)$, and we have solved the inequality.

CAUTION The sign charts in this text will follow the convention that tick marks are only marked at the locations where the quantity being measured is zero or fails to exist. Do not make additional tick marks on your number lines if you follow this convention. Or, if you prefer to use additional tick marks as reference points, then you must label all of the relevant tick marks with "0" or "DNE" as needed.

Inequalities and Absolute Values

In Section 0.1 we saw that the distance between two real numbers x and c is by definition $|x - c|$. Therefore, the solutions of inequalities of the form $|x - c| < \delta$ and $|x - c| > \delta$ describe how close or far away x is from c. (The symbol δ is the Greek letter ***delta*** and is one of the traditional letters used to represent a distance.) On the real number line, we can graphically represent the solution sets to $|x - c| < \delta$ and $|x - c| > \delta$ as shown in the two figures that follow. Remember that the blue shaded areas are the locations on the real number line where a number x satisfies the inequality.





The preceding discussion suggests the following theorem, which will allow us to immediately solve certain simple absolute value inequalities:

THEOREM 0.22 **Expressing Distances with Inequalities and Intervals**

Suppose x and c are any real numbers and δ is any positive real number. Then

(a) The following statements are equivalent:

- the distance between x and c is less than δ;
- $|x - c| < \delta$;
- x satisfies both $x > c - \delta$ and $x < c + \delta$;
- $x \in (c - \delta, c + \delta)$.

(b) The following statements are equivalent:

- the distance between x and c is greater than δ;
- $|x - c| > \delta$;
- x satisfies either $x < c - \delta$ or $x > c + \delta$;
- $x \in (-\infty, c - \delta) \cup (c + \delta, \infty)$.

Theorem 0.22 also holds for "greater than or equal to" and "less than or equal to," with appropriate changes. With this theorem we can immediately solve simple absolute value inequalities involving $|x - c|$. For example, if x satisfies the inequality $|x - 2| \geq 5$, then the distance between x and 2 is greater than or equal to 5. This means that x is greater than or equal to $2 + 5 = 7$ and less than or equal to $2 - 5 = -3$. Therefore $x \in (-\infty, -3] \cup [7, \infty)$.

For more complicated inequalities, such as $|2x - 5| > 3$ and $|x^2 - 4| < 12$, the following theorem is helpful:

THEOREM 0.23 **Replacing an Absolute Value Inequality with Two Simpler Inequalities**

Suppose δ is any positive real number and A is any real expression. Then

(a) $|A| < \delta$ if and only if $A < \delta$ and $A > -\delta$ (that is, $-\delta < A < \delta$);

(b) $|A| > \delta$ if and only if $A > \delta$ or $A < -\delta$.

The words "and" and "or" in the theorem are essential. The "and" represents an intersection, whereas the "or" represents a union. As usual, this theorem also holds when $<$ and $>$ are replaced by $\leq$ and $\geq$. We will prove Theorem 0.23 in Section 0.7.

Theorem 0.23 allows us to trade an absolute value inequality for a pair of simple inequalities. For example, the absolute value inequality $|2x + 5| < 3$ becomes a pair of inequalities that do not involve absolute values, namely, $2x+5 < 3$ and $2x+5 > -3$. These inequalities are easy to solve, and the intersection of their solutions, $(-\infty, -1) \cap (-4, \infty) = (-4, -1)$, is the solution of the original inequality.

Examples and Explorations

EXAMPLE 1

Solving a simple inequality

Find the solution set of the inequality $3 - 2x > 7$.

SOLUTION

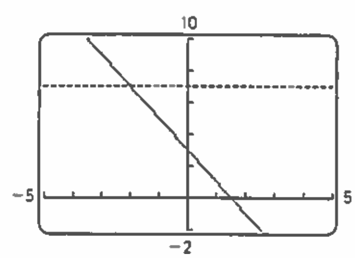
This is a simple calculation that makes use of parts (b) and (d) of Theorem 0.18. Note the inequality "flip" in the last step:

$$3 - 2x > 7 \quad \Longrightarrow \quad -2x > 7 - 3 \quad \Longrightarrow \quad -2x > 4 \quad \Longrightarrow \quad x < -2.$$

(The symbol $\Longrightarrow$ used in the calculation above denotes that the inequality to the left of the symbol logically implies the inequality to the right of the symbol.) Therefore the solution set of the original inequality is $(-\infty, -2)$.

CHECKING THE ANSWER

You can use a graphing utility to check the solution we just found by graphing $y = 3 - 2x$ (the diagonal line) and $y = 7$ (the dashed line), as shown next. The values of x for which the line described by $3 - 2x$ has a y-coordinate greater than 7 are those to the left of $x = -2$.



EXAMPLE 2

Solving an inequality by examining signs of factors

Solve the inequality $\frac{3}{x+1} > 2$ by applying Theorem 0.19.

SOLUTION

Although it is tempting, we can't multiply both sides of the inequality by $x + 1$, because $x+1$ may be positive or may be negative (depending on the value of x). Instead, we will use some algebra to turn this inequality into one that involves a product or quotient of factors on one side and zero on the other:

$$\frac{3}{x+1} > 2 \qquad \leftarrow \text{original inequality}$$

$$\frac{3}{x+1} - 2 > 0 \qquad \leftarrow \text{collect terms so 0 is on right-hand side}$$

$$\frac{3}{x+1} - \frac{2(x+1)}{x+1} > 0 \qquad \leftarrow \text{find common denominator}$$

$$\frac{3-2(x+1)}{x+1} > 0 \qquad \leftarrow \text{combine fractions}$$

$$\frac{3-2x-2}{x+1} > 0 \qquad \leftarrow \text{simplify numerator}$$

$$\frac{1-2x}{x+1} > 0 \qquad \leftarrow \text{simplify numerator some more}$$

The expression $\frac{1-2x}{x+1}$ is positive if $1-2x$ and $x+1$ both have the same sign; in other words, the solution set of this inequality consists of all values x for which

$$(1-2x > 0 \text{ and } x+1 > 0) \quad \text{or} \quad (1-2x < 0 \text{ and } x+1 < 0).$$

Solving these four simple inequalities, we get

$$(x < 1/2 \text{ and } x > -1) \quad \text{or} \quad (x > 1/2 \text{ and } x < -1).$$

Therefore the solution of the original inequality is

$$((-\infty, 1/2) \cap (-1, \infty)) \cup ((1/2, \infty) \cap (-\infty, -1)),$$

which is equal to $(-1, 1/2) \cup \emptyset = (-1, 1/2)$.

EXAMPLE 3 Solving an inequality by testing points between sign changes

Solve the inequality $\frac{3}{x+1} > 2$ again, this time by using the "point-testing" method allowed by Theorem 0.21.

SOLUTION

We begin the same way as we did in Example 2, by writing the inequality in factored form with zero on one side:

$$\frac{1-2x}{x+1} > 0.$$

By Theorem 0.21, the expression $\frac{1-2x}{x+1}$ can change sign only at $x = 1/2$ and $x = -1$. We now find the sign of the expression at representative points $x = -2$, $x = 0$, and $x = 1$ between each of these potential sign-changing points:

$$\frac{1-2(-2)}{-2+1} = \frac{\text{pos}}{\text{neg}} = \text{negative},$$

$$\frac{1-2(0)}{0+1} = \frac{\text{pos}}{\text{pos}} = \text{positive},$$

$$\frac{1-2(1)}{1+1} = \frac{\text{neg}}{\text{pos}} = \text{negative}.$$

We can record this information on a number line as follows:

This means that $\frac{1-2x}{x+1}$ is greater than zero only on the interval $(-1, 1/2)$.

EXAMPLE 4 Solving an inequality that involves three factors

Find the solution set of the inequality $\frac{1}{x-2} \leq \frac{1}{6-x}$.

SOLUTION

Be careful; we cannot cross multiply here, because we do not know whether $x-2$ and $6-x$ are positive or negative. Instead we rewrite:

$$\frac{1}{x-2} \leq \frac{1}{6-x} \qquad \leftarrow \text{original inequality}$$

$$\frac{1}{x-2} - \frac{1}{6-x} \leq 0 \qquad \leftarrow \text{collect terms so 0 is on right-hand side}$$

$$\frac{(6-x)-(x-2)}{(x-2)(6-x)} \leq 0 \qquad \leftarrow \text{combine fractions using common denominator}$$

$$\frac{8-2x}{(x-2)(6-x)} \leq 0 \qquad \leftarrow \text{simplify numerator}$$

$$\frac{2(4-x)}{(x-2)(6-x)} \leq 0. \qquad \leftarrow \text{simplify numerator some more}$$

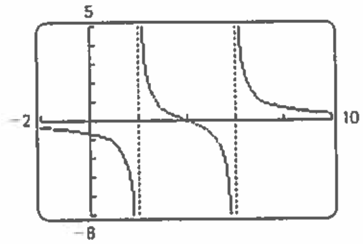
There are three factors in the numerator and denominator of the last expression, namely, $4-x$, $x-2$, and $6-x$. We could solve this inequality by examining all the cases where an odd number of these factors is negative, but it will be faster to use the "point-testing" method of the previous example.

The only places where $\frac{2(4-x)}{(x-2)(6-x)}$ can change sign are at $x=4$, $x=2$, and $x=6$. Testing the sign of this expression at representative x-values between these potential sign-changing points (say, at $x=0$, $x=3$, and $x=5$, for example) results in the following sign chart:

From this information we can see exactly what happens in the intervals between the points $x=2$, $x=4$, and $x=6$. But we have to be careful about the value of the expression *at* those points. We are looking for the places where $\frac{2(4-x)}{(x-2)(6-x)}$ is greater than *or equal to* zero, so we must include the point $x=4$ in our solution. We also must be careful to exclude the points $x=2$ and $x=6$ from our solution, for at these values of x the expression is not defined at all. Putting this all together with our sign chart, we see that the solution is $(-\infty, 2) \cup [4, 6)$.

CHECKING THE ANSWER

To test whether the solution we found is reasonable, try evaluating the inequality at a few values in the solution set and at a few values outside the solution set. For example, the inequality should be true for $x = 0$ and $x = 5$ (these are in the solution set) but false for $x = 3$ and for $x = 8$. You can also check the answer by using a graphing utility to graph the equation $y = \frac{1}{x-2} - \frac{1}{6-x}$. The graph should have y-values that are less than or equal to zero for $x \in (-\infty, 2) \cup [4, 6)$. For all other values of x, the y-values should be positive. Verify that this is the case for the following graph (the graph of the equation has three components, and the dashed lines at $x = 2$ and $x = 6$ are not part of the graph):



EXAMPLE 5 Using cases to solve inequalities that involve absolute values

Solve the following inequalities:

(a) $|2x - 5| > 3$ **(b)** $|x^2 - 4| \leq 12$

SOLUTION

(a) By Theorem 0.23, $|2x - 5| > 3$ if and only if

$$\begin{aligned} 2x - 5 &> 3 \\ 2x &> 8 \\ x &> 4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x - 5 &< -3 \\ 2x &< 2 \\ x &< 1 \end{aligned}$$

The solution set of the original inequality is the union of the two solutions just found: $(-\infty, 1) \cup (4, \infty)$.

(b) By Theorem 0.23, $|x^2 - 4| \leq 12$ if and only if

$$\begin{aligned} x^2 - 4 &\leq 12 \\ x^2 &\leq 16 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x^2 - 4 &\geq -12 \\ x^2 &\geq -8 \end{aligned}$$

The solution set of the original inequality is the intersection of the solutions of these two inequalities. The first inequality, $x^2 \leq 16$, is true for values of x that are inclusively between -4 and 4. The second inequality, $x^2 \geq -8$, is *always* true. Therefore the solution of the original inequality is $[-4, 4] \cap (-\infty, \infty) = [-4, 4]$.

? TEST YOUR UNDERSTANDING

- Why you can't solve the inequality $\frac{2}{x+1} < 7$ by multiplying both sides of the inequality by $x + 1$?
- What are the possible cases to consider when solving an inequality of the form $AB > 0$ or $AB < 0$?
- What are the possible cases to consider when solving an inequality of the form $\frac{A}{B} \geq 0$ or $\frac{A}{B} \leq 0$?
- What are the possible cases to consider when solving an inequality of the form $|A| \leq \delta$ or $|A| \geq \delta$?
- Define each of the following: strict inequality, sign of a real number, factor, polynomial.

EXERCISES 0.3

Thinking Back

Algebra review of fractions: Write each of the following expressions as a simple fraction.

▷ $x - \dfrac{1}{x+5}$

▷ $\dfrac{1}{x-1} - \dfrac{1}{x+1}$

▷ $3 - \dfrac{1}{x^2-4}$

▷ $\dfrac{1}{x-2} - \dfrac{x-2}{x-1}$

Algebra review of factoring: Factor each expression to identify the values of x for which the expression is zero or does not exist.

▷ $\dfrac{4x^2-9}{x-1}$

▷ $\dfrac{1}{x^4-16}$

▷ $\dfrac{x^2+x-2}{x^2+2x-3}$

▷ $\dfrac{6x^2-11x+3}{x^3-7x^2+10x}$

Concepts

0. *Problem Zero:* Read the section and make your own summary of the material.

1. *True/False:* Determine whether each of the statements that follow is true or false. If a statement is true, explain why. If a statement is false, provide a counterexample.

(a) *True or False:* If $a \le b$, then $-3a \le -3b$.

(b) *True or False:* If $a < b$, then $-4a < b$.

(c) *True or False:* If $x < -2$, then $\frac{1}{x} < -\frac{1}{2}$.

(d) *True or False:* If a, b, and c are negative real numbers, then $-2abc$ is negative.

(e) *True or False:* $|x - c| > \delta$ if and only if $x < c - \delta$ and $x > c + \delta$.

(f) *True or False:* The set $\{x \mid x > 4 \text{ and } x < 6\}$ is equal to the set $(4, \infty) \cup (-\infty, 6)$.

(g) *True or False:* For any real numbers a and b, $|a + b| \le |a| + |b|$.

(h) *True or False:* For any real numbers a and b, $|a - b| \le |a| - |b|$.

2. *Examples:* Construct examples of the thing(s) described in the following. Try to find examples that are different than any in the reading.

(a) A polynomial that changes sign at $x = -2$ and $x = 5$.

(b) A real number x for which $|x - 7| > 2$ is true but $x - 7 > 2$ is not, and a real number x for which $x - 7 > 2$ is true but $|x - 7| > 2$ is not.

(c) Real numbers a and b for which $|a + b|$ is strictly less than $|a| + |b|$.

3. Answer the following two questions without solving any inequalities or using a graphing utility: Is $x = 2$ a solution of the inequality $x^4 \le 11x - 1$? What about $x = 3$? Explain your answers.

4. Consider the inequality $(x-2)^2(x-1) > 0$. The left-hand side of this inequality has three factors. If you were to solve the inequality by examining the signs of the factors, would you need to look at the signs of *all* of the factors? Why or why not? *(Hint: Think about the sign of $(x-2)^2$.)*

5. Explain the difference between the sets $\{x \mid x < 2 \text{ or } x > 0\}$ and $\{x \mid x < 2 \text{ and } x > 0\}$. Use a number line and the words "union" and "intersection" in your explanation.

6. Can you solve the inequality $x^2 > 4$ by taking the square root of both sides? Why or why not? What is the solution of this inequality?

In Exercises 7–12, A, B, C and D are real expressions and $\delta > 0$. For each inequality, write a list of simpler inequalities with "and" and "or" that together are equivalent to the original inequality.

7. $AB \ge 0$

8. $AB \le 0$

9. $ABC < 0$

10. $ABCD > 0$

11. $|A| \ge \delta$

12. $|A| \le \delta$

13. Suppose A, B, C, and D are real numbers and that $A \le B < C = D$. What can we say about how A is related to D and why? What if we knew instead that $A < B = C \ge D$?

Exercises 14–18 each involve something that is either (a) an absolute value inequality, (b) a combination of simple inequalities, (c) a set in interval notation, (d) a statement involving distance, or (e) a picture on the real number line. Whichever way the statement is given, express the statement in the remaining four ways listed.

14. $|x - 2| < 0.1$

15. $x \ge -1$ and $x \le 4$

16. $(-\infty, 4] \cup [3, \infty)$

17. The distance between x and -3 is greater than 2.

18.

Skills

Find the solution sets of the inequalities in Exercises 19–40 by testing signs between potential sign-changing points. Illustrate your results on a labeled number line. When you are done, check your answers with a graphing utility.

19. $2x + 1 \ge 4$

20. $1 - 5x > 11$

21. $0 \le \dfrac{1}{x} \le 3$

22. $(x+1)(x-2) > 0$

23. $2x^2 - 7x + 3 \le 0$

24. $x^3 - x \ge 0$

25. $x^3 \le 4x$

26. $6x^2 > x^3 + 9x$

27. $x^4 > 16$

28. $x^5 > 32$

29. $\dfrac{x-1}{x+2} \geq 0$

30. $\dfrac{x-3}{x+5} \geq 0$

31. $\dfrac{5}{x-2} \leq 1$

32. $\dfrac{2}{x+1} \geq 3$

33. $\dfrac{x^2-1}{x-3} < 0$

34. $\dfrac{x(x-1)}{x^2-4} \geq 0$

35. $\dfrac{3x^2-2x-1}{x^2+x+6} \leq 0$

36. $\dfrac{x^2+7x+10}{4x^2+6x+2}$

37. $x+2 \leq \dfrac{1}{x+2}$

38. $\dfrac{1}{1+x} < \dfrac{1}{x+x^2}$

39. $\dfrac{1}{x-1} < \dfrac{2}{x+2}$

40. $\dfrac{1}{x-1} \geq \dfrac{x-1}{x+5}$

Find the solution sets of the absolute value inequalities in Exercises 41–52.

41. $|x+2| > 4$

42. $|x-3| < 8$

43. $|x+8| \leq 1$

44. $|3x+1| \geq 5$

45. $|4-2x| > 6$

46. $|3x+1| \leq 4$

47. $|x^8-3x+1| \geq -2$

48. $|x(x-1)| < 8$

49. $|x^2-1| \geq 3$

50. $|x^2-4| \geq 5$

51. $|x^3+x^2-3x| \leq 3x$

52. $|x^2+4x-1| > 4$

Solve each of the inequalities in Exercises 53–58 by factoring it and then reducing it to a system of simpler inequalities combined with unions and intersections.

53. $x^2 \geq 16$

54. $x^2-7 \leq 0$

55. $3x^2-5x-2 < 0$

56. $6x^2-5x > -1$

57. $x^3-x^2-6x \leq 0$

58. $x^3-4x^2-5x \geq 0$

Use a graphing utility to approximate the solution sets of the inequalities in Exercises 59–62.

59. $x^5 \leq 2x^3+5$

60. $x^3-4x^2 > 1$

61. $3x^4-1 \geq x^5$

62. $\dfrac{x-1}{x^2+5} < \dfrac{x}{x+1}$

Applications

63. The formula for converting degrees Fahrenheit (F) to degrees Celsius (C) is $C = \frac{5}{9}(F-32)$.

(a) For water to be between its freezing and boiling points, its temperature must be between 32° and 212° Fahrenheit. What is the corresponding temperature range in degrees Celsius?

(b) Ella says that tomorrow's temperature will be between 20 degrees Celsius and 25 degrees Celsius. What is the corresponding range of temperatures in degrees Fahrenheit?

64. The instructions that came with Byron's stereo claim that the optimal volume range v for his speakers satisfies the inequality $|v-85| < 35$, where v is measured in decibels. What is the lowest optimal volume for Byron's speakers? The highest?

65. For his summer job, Frank fills large buckets with presliced pickles. The label on each bucket indicates that it contains 20 pounds of pickles. His boss says it is okay to be over or under by as much as half a pound. Use an absolute value inequality to express the acceptable weight W of a full bucket of pickles.

66. The cylindrical cans that Len's company produces are 6 inches tall and vary in radius. The cost of producing a can with height 6 inches and radius r is $C = 10\pi r^2 + 24\pi r$ cents.

(a) Len's boss wants him to construct the cans so that the cost is between 250 and 450 cents. Write this condition as an absolute value inequality.

(b) Given the cost restrictions in part (a), what is the acceptable range of values for r?

Proofs

67. Prove that for any $c > 0$, the solution set of the inequality $x^2 < c^2$ is $(-c, c)$.

68. Prove that for any $c > 0$, the solution set of the inequality $x^2 > c^2$ is $(-\infty, -c) \cup (c, \infty)$.

Thinking Forward

Conditional inequalities: In Chapter 1 we will be interested in how the range of values of a one-variable expression depends on a range of values for its variable. Each of the problems that follow involves two related absolute value inequalities. State these absolute value inequalities and then use them to solve the problems.

- Show that if x is within a distance of 1 from 2, then $3x+1$ is within a distance of 3 from 7.

- Show that if x is within a distance of 0.25 from 1, then $4x - 2$ is within distance 1 of 2.
- If we want to require that x^2 is within a distance of 3 from 0, how close must x be to 0?
- If we want to require that $1 - 2x$ is within a distance of 0.5 from -2, how close must x be to 3?
- Use the following graph of $y = 1 - x^2$ to illustrate that $1 - x^2$ is within a distance of 4 from 1 if and only if x is within a distance of 2 from 0:

Points (x, y) satisfying $y = 1 - x^2$