University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2020

Assignment # 1

You are may wish to work on this assignment prior to your tutorial during the week of January 13th. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.1, 4.2 (*omitting* Trapezoid Sums def 4.8 & Upper and Lower Sums def 4.7), 4.3 up to and including pg 343.

HOMEWORK:

At the beginning of your TUTORIAL during the week of January 20th you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

1. Write each of the sums in sigma notation (with no extra terms added, subtracted or multiplied to/from the sum). Identify m, n and a_k in each problem.

   (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{64}$.

   (b) Let $a,\ x \in \mathbb{R}$. $1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-a)^4}{4!} + \cdots + \frac{(x-a)^{666}}{666!}$.

2. Let $k,\ m,\ n \in \mathbb{Z}^{\geq 0}$ such that $m \leq k \leq n$. Suppose that a_k and b_k are real-valued functions of k. Suppose that $b_k \neq 0$ for all k. Prove, or disprove (by providing a counter-example), the following statement :

$$\frac{\sum_{k=m}^{n} -8a_k}{\sum_{k=m}^{n} 2b_k} = -4 \sum_{k=m}^{n} \frac{a_k}{b_k}.$$

x	10	14	18	22	26	30
f(x)	-12	-6	-2	1	3	8

3. The table of values of a function f is shown above. Use the table to find under and over estimates for the exact area A between $y = f(x)$ over interval $[10, 30]$. Round your answers to the nearest integer.

4. Let $n \in \mathbb{Z}^{>0}$. Let $a, b \in \mathbb{R}$ with $a < b$. Let $y = f(x)$ be a continuous real-valued function on $[a, b]$. Let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[a, b]$, i.e., define $\triangle x = \frac{b-a}{n}$ and $x_i = a + i \triangle x,\ i = 0,\ 1,\ \ldots,\ n$. Let L_n and R_n be the left and right Riemann sums for f over $[a, b]$ with n subintervals, respectively. Let M_n denote the Midpoint (Riemann) sum for f over $[a, b]$ with n subintervals.

   (a) Write down a formula for M_n. Express your formula in sigma notation with initial value of the index equal to 1. Make sure to clearly define any expressions involved in your formula.

   (b) Prove : If f is increasing on $[a, b]$ then $L_n \leq M_n \leq R_n$.

   (c) Consider the real-valued function g defined on $[0, 1]$ by $g(x) = 1$ if x is a rational number and $f(x) = 0$ if x is an irrational number. For an arbitrary $n \in \mathbb{Z}^{>0}$, compute each of the following :

   - the left Riemann sum L_n,
   - the right Riemann sum R_n, and
   - the midpoint Riemann sum M_n.

     (*Hint* : *Between any two real numbers lie a rational and an irrational number (i.e., "$\mathbb{Q}$ and $\mathbb{I}$ are dense in $\mathbb{R}$."*)

   (d) Prove that $L_n - R_n = \frac{b-a}{n}\,(f(a) - f(b))$.

5. Textbook Section 4.2 - # 20. Make sure to fully justify your answer.

6. Using $\sum$-notation properties, useful prerequisite sum formulas and limit laws, to evaluate the following. Make sure to justify your work.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{-2i^3 + \pi i^2}{n^3}.$$

7. Let $n \in \mathbb{Z}^{>0}$. Let $a, b \in \mathbb{R}$ with $a < b$. Let $y = f(x)$ be a continuous real-valued function on $[a, b]$. Let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[a, b]$, i.e., define $\triangle\, x = \frac{b-a}{n}$ and $x_i = a + i \triangle x,\ i = 0,\ 1,\ \ldots,\ n$. The **definite integral of f from $x = a$ to $x = b$** is

$$A = \int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \triangle x \text{ any } x_i \in [x_{i-1}, x_i]$$

provided the limit exists. Geometrically, this is the *exact* signed area A between $y = f(x)$ over $[a, b]$.

Consider $f(x) = \sqrt{12}\ x$

(a) Write down a formula L_n and R_n, the left and right Riemann sums for f over $[a, b]$ with n subintervals, respectively. Leave your answers in sigma notation.

(b) Use $\sum$-notation properties, useful prerequisite sum formulas and limit laws, to evaluate $\lim\limits_{n\to\infty} R_n$.

(c) Use $\sum$-notation properties, useful prerequisite sum formulas and limit laws, to evaluate $\lim\limits_{n\to\infty} L_n$.

(We are evaluating ${}_a^b f(x)dx$ by the Riemann definition of the definite integral in parts a) and b) of this question).

8. Evaluate the following integrals by interpreting each of them in terms of areas, i.e., solve "geometrically". Do not use limits and Riemann sums.

(a) $\displaystyle\int_{-3}^{3} |\ x\ |\ \ dx$

(b) $\displaystyle\int_{0.1}^{10} \frac{|\ 20x^2\ | + x}{x}\ dx$

(c) $\displaystyle\int_{0}^{6} \frac{1}{3}\sqrt{9^3 - 9(x-3)^2}\ dx$

EXERCISES: You do not need to submit solutions to the following problems but you should make sure that you are able to answer them.

1. Textbook Section 4.1 - # 1(a)-(h), 2, 7, 21, 23, 41, 45, 46, 47, 49, 51, 58, 59.

2. Textbook Section 4.2 - # 1(a)-(f), 17, 30(to 2 d.p.a - you are allowed a calculator on this qn), 35 (omit the upper, lower and trapezoid sum computations), 37 (omit the upper, lower and trapezoid sum computations), 39, 42, 45, 46(a), 48(a), 51.

3. Textbook Section 4.3 - # 1(a)(b), 3, 4, 5, 7, 9, 11, 12, 20, 21, 23, 27, 41, 43.

4. Write each of the sums in sigma notation (with no extra terms added, subtracted or multiplied to/from the sum). Identify m, n and a_k in each problem.

   (a) $1+0-1+0+1+0-1+0+1+0-1+0+1.$

   (b) $5+8+11+14+17+\ldots+32.$

   (c) $1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}.$

   (d) $-3+2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}.$

5. Let $k,\ m,\ n \in \mathbb{Z}^{\geq 0}$ such that $m \leq k \leq n$. Suppose that a_k and b_k are real-valued functions of k. Prove, or disprove (by providing a counter-example), the following statements :

   (a) $\displaystyle\sum_{k=m}^{n}\left(\ln(\sqrt{2})a_k-\frac{b_k}{2}\right)=\ln(\sqrt{2})\sum_{k=m}^{n}a_k-\frac{1}{2}\sum_{k=m}^{n}b_k.$
   Do <u>not</u> use sigma-notation properties in your justification.

   (b) $\left(\sum_{k=m}^{n}a_k\right)\left(\sum_{k=m}^{n}b_k\right)=\sum_{k=m}^{n}a_kb_k.$

6. Using $\sum$-notation properties, useful sum formulas and limit laws, to evaluate the following. Make sure to justify your work.

   (a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2+i-2}{n^4}$

   (b) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^2}{n^2-1}$

7. Suppose that f is negative, decreasing and continuous on $[0, 1]$. Let L_n and R_n be the left and right Riemann sums for f on $[0, 1]$ with $n \in \mathbb{N}$ subintervals, respectively.

   (a) Prove that $R_n \leq L_n$. Make sure to fully justify your arguments.

   (b) Provide an example of a function f that satisfies the criteria of this question such that $L_n = R_n$. Prove that your f satisfies $L_n = R_n$.

8. Determine whether the following statements are true or false. If true prove it otherwise provide a counterexample to disprove it.

   (a) Let $a, b \in \mathbb{R}$, $a < b$. Let $n \in \mathbb{Z}^{>0}$. If f is any constant function on $[a, b]$, then the left Riemann sum L_n and right Riemann sum R_n give the exact value same value.

   (b) Let $a, b \in \mathbb{R}$, $a < b$. Let $n \in \mathbb{Z}^{>0}$. If f is any constant function on $[a, b]$, then the limit as $n \to$ of the left Riemann sum L_n and the limit as $n \to \infty$ of the right Riemann sum R_n give the exact value same value.

9. Let $I = \displaystyle\int_1^{4.2} f(x)dx$. Use the Midpoint (Riemann) Sum and the given data

x	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8	4.2
f(x)	7.1	8.6	8.4	7.6	7.4	6.3	8.3	6.5	7.1

   to estimate the value of the integral I.

10. Consider the definite integral $I = \displaystyle\int_9^{14} e^{-x^4} dx$. Suppose that the approximations L_{1200}, L_{20} and R_{1200} are computed for I. Identify which of these Riemann sum approximations correspond to which of the following numbers 0.33575, 0.36814 and 0.36735. Make sure to justify your choices. *(Hint : do not compute these approximations. The numbers provided are fictitious.)*

11. Let $a, b \in \mathbb{R}$ with $a < b$. Let A be the exact area under the graph of an *increasing* positive continuous function f from a to b. Let L_n and R_n be

the left and right Riemann sum approximations to A with n subintervals, respectively.

(a) How are A, L_n, and R_n related? Draw a sample diagram to support your answer. (You do not need to rigorously justify or prove your claim).

(b) Is the "positive" assumption necessary?

12. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n}\left[\left(\frac{3k}{n}\right)^2-6\left(\frac{3k}{n}\right)\right].$$

Make sure to fully justify your answer.

13. Let $n \in \mathbb{N}$.

(a) Write the sum

$$\frac{1}{n}\left[\left(\frac{1}{n}+1\right)+\left(\frac{2}{n}+1\right)+\cdots+\left(\frac{n}{n}+1\right)\right]$$

in sigma notation (with no extra terms added or subtracted from the sum). Identify m, n and a_k in each problem.

(b) Use sigma-notation properties and useful prerequisite formulas to help evaluate the limit as n approaches infinity of your answer to part (a).

14. Let $n \in \mathbb{Z}^{>0}$.

(a) Express $\int_0^3 (x-5)\,dx$ as a limit of a (general) Riemann sum.

(b) Express $\int_0^3 x^3 dx$ as a limit of a right Riemann sum.

(c) Using $\sum$-notation properties, useful sum formulas and limit laws, evaluate your answer to part (a). That is, evaluate the integral in part (a) be the Riemann definition of the definite integral.

15. Evaluate the following integrals by interpreting each of them in terms of areas, i.e., solve "geometrically". Do not use limits.

(a) $\int_0^5 (1-t)\, dt$

(b) $\int_{-5}^0 (1-t)\, dt$

(c) $\int_{-1}^3 \sqrt{4-(x-1)^2}\, dx$

(d) $\int_0^1 \frac{x+2}{4x+8}\, dx$

(e) Let $r \in \mathbb{R}^+$. $\int_{-r}^{r} -\sqrt{r^2-x^2}\, dx$

(f) $\int_0^5 f(x)\, dx$ where $f(x) = \begin{cases} 2 & \text{if } 0 \leq x < 1 \\ 4-2x & \text{if } 1 \leq x < 2 \\ 5x-10 & \text{if } 2 \leq x \leq 3 \end{cases}$

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. — *David Hilbert*