MATB24 TUTORIAL PROBLEMS 1

KEY WORDS: binary operation, vector addition, scalar multiplication, vector space.
READING: Sec 3.1 FB or Sec 1.B SA[1]

WARM-UP: You should know the definitions of the following terms word by word

(1) A binary operation
(2) An identity element for a binary operation
(3) An invertible element for a binary operation
(4) An inverse of an invertible element
(5) A vector space

A :Below are tables for four different binary operations on the set $S = \{a, b, c, d\}$, called $\clubsuit$, $\diamondsuit$, $\heartsuit$ and $\spadesuit$.

(1) Discuss with your groupmates a **convention** for interpreting the tables, so that you do not confuse $a \star b$ with $b \star a$. Make sure your convention is consistent with mine: I want $a \diamondsuit b = d$.
(2) Which of the operations are commutative? Discuss with your group mates.
(3) Which have an identity? Identify it. Recall: An identity for a binary operation $\star$ on a set S is an element e such that $s \star e = e \star s = s$ for all $s \in S$.
(4) TRUE or FALSE: if there exists some e such that $x \star e = x$ for all $x \in S$, then e is an identity for $\star$.
(5) Investigate which elements of $\heartsuit$ have inverses.

$\clubsuit$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

$\diamondsuit$	a	b	c	d
a	a	d	c	b
b	b	a	d	c
c	c	b	a	d
d	d	c	b	a

$\heartsuit$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

$\spadesuit$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

[1]FB: Fraleigh Beauregard, SA: Sheldon Axler

B:Let n be a non-negative integer. Consider the set of all polynomials of degree less than or equal to n

$$P_n = \{a_0x^0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, \cdots, a_n \in \mathbb{R}\}$$

(1) Explicitly explain the (standard) addition and the (standard) scalar multiplication on P_n.
(2) Show that your vector addition in the previous part satisfies A1-A4.
(3) Show that your scalar multiplication satisfies S1-S4.
(4) Conclude that P_n is a vector space.
(5) We said n is a non-negative integer. What is P_n if $n = 0$?
(6) Let $n \neq m$ be integers. Is $P_n \cup P_m$ a vector space? Explain why.
(7) Define $P := \bigcup_{n\in\mathbb{N}} P_n$. Explain in words what this set is. Is it a vector space? Explain why?(I don't need you to write a formal proof)

C: Let $\mathcal{S}$ be the set of all sequences of real numbers. An example of an element in $\mathcal{S}$ might be $\{1, 2, 3, 5, 8, 13, \dots\}$.

(1) Is there a way to define addition and scalar multiplication to make this a vector space?
(2) What is the zero vector?
(3) Is the set of sequences of rational numbers a subspace?
(4) What about sequences of integers?
(5) What about the subset of sequences that are "eventually constant"?

D PROOF PRACTICE
For each item, explain a natural way to define addition and real scalar multiplication on the given set, and then choose two axioms from the definition of a vector space

(ideally different axioms for each part) and carefully prove that your operations satisfy the axioms. All the given sets are indeed real vector spaces.

(1) Complex numbers $\mathbb{C}$.
(2) The set $\mathcal{C}^0$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
(3) The set $\mathcal{C}^r$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that have r-th derivative. [2]
(4) The set V of functions f in C^4 such that the fourth derivative of f equals f.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) A real vector space other than $\mathbb{R}^n$.
(2) A set with a binary operation that has an identity.
(3) An invertible element in a set with a binary operation that has an identity.
(4) An invertible element in a set with a binary operation without an identity.
(5) A binary operation with two distinct identity elements.

[2]Some books use C for continuous functions, and D for differentiable functions