MATA22 Assignment 6
Due Date: April 7th

1. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation and let A be the matrix representation of T with respect to the standard basis of $\mathbb{R}^2$.
   Prove that the following two statements are equivalent.
   (1) There are exactly two distinct lines L_1, L_2 in $\mathbb{R}^2$ passing through the origin that are mapped into themselves:
   $$T(L_1) = L_1, T(L_2) = L_2.$$
   (2) The matrix A has two distinct nonzero real eigenvalues.

Question 1 max: 20 points

Each direction worth 10 points.

For (1) $\implies$ (2), students must carefully check the 3 cases for λ_i. Missing each case, reduce by 2 points.

Solution:

(1) $\implies$ (2)

Let L_1 =span($\vec{v_1}$) and L_2 =span($\vec{v_2}$) where $\vec{v_1}, \vec{v_2}$ are some nonzero vectors in $\mathbb{R}^2$. (You may realize L_1, L_2 as one-dimensional subspaces of $\mathbb{R}^2$.)

we have

$$T(\vec{v_i}) \subset L_i, i = 1, 2 \tag{1}$$

Since $T(L_1) = L_1, T(L_2) = L_2$ and $\vec{v_i} \subset L_i, i = 1, 2$.

Using matrix representation of T, we may rewrite equation (1) as following.

$$A(\vec{v_i}) = \lambda_i(\vec{v_i}), i = 1, 2 \tag{2}$$

where $\lambda_i \in \mathbb{R}$. You may note that $\lambda_i(\vec{v_i})$ just represent another vector on L_i.

There are 3 main points we need to check for λ_i.

- (a) $\lambda_i \neq 0$. Otherwise, we have $A(\vec{v_i}) = \lambda_i(\vec{v_i}) = (0)(\vec{v_i}) = 0$.
  $\forall \vec{w} \in L_i$ and we may represent $\vec{w} = \alpha_i(\vec{v_i}), \alpha \in \mathbb{R}$ since $\vec{v_i}$ can serve as a basis of L_i.

$$A(\vec{w}) = A(\alpha_i(\vec{v_i})) = \alpha_i A(\vec{v_i}) = 0, i = 1, 2 \tag{3}$$

Equation (3) implies T maps L_i to $\{0\}$, a contradiction.

(b) $\lambda_1 \neq \lambda_2$. Assume on the contrary that $\lambda = \lambda_1 = \lambda_2$. For all $\vec{w} \in \mathbb{R}$ can be written as a linear combination

$$\vec{w} = a\vec{v_1} + b\vec{v_2}, a, b \in \mathbb{R}$$

since $\vec{v_1}, \vec{v_1}$ can serve as a basis of $\mathbb{R}^2$. Then we have

$$A(\vec{w}) = A(a\vec{v_1} + b\vec{v_2}) = a\lambda(\vec{v_1}) + b\lambda(\vec{v_2}) = \lambda(a\vec{v_1} + b\vec{v_2}) = \lambda\vec{w} \tag{4}$$

The equation (4) implies that any line spanned by a nonzero vector $\vec{w}$ is mapped onto itself by the linear transformation T. This contradicts our assumption that $L_i, i = 1, 2$ are the only two such lines.

(c) λ_i must be real number. Otherwise, it will contradict equation (4) (ie: real number= some complex number with nonzero imaginary part.)

- (2) $\implies$ (1)

We want to show that no other lines. passing through the origin are mapped onto themselves by T. Assume that we have $L_3 = sp(\vec{w})$ such that $T(L_3) = L_3$. By similar argument as L_i, you may have the following.

$$A(\vec{w}) = \mu(\vec{w}), \mu \in \mathbb{R} \tag{5}$$

and

$$A(\vec{w}) = a\vec{v_1} + b\vec{v_2}, a, b \in \mathbb{R} \tag{6}$$

since $\vec{v_1}, \vec{v_1}$ can serve as a basis of $\mathbb{R}^2$.
(5),(6) imply that:

$$A(\vec{w}) = A(a\vec{v_1}+b\vec{v_2}) = a\lambda_1(\vec{v_1})+b\lambda_2(\vec{v_2}) = \mu(\vec{w}) = a\mu\vec{v_1}+b\mu\vec{v_2} \tag{7}$$

(7) implies

$$a\lambda_1(\vec{v_1}) + b\lambda_2(\vec{v_2}) = a\mu\vec{v_1} + b\mu\vec{v_2} \tag{8}$$

or

$$a(\lambda_1 - \mu)(\vec{v_1}) + b(\lambda_2 - \mu)(\vec{v_2}) = 0 \tag{9}$$

Recall that $\vec{v_1}, \vec{v_1}$ are linearly independent, thus (9) implies

$$a(\lambda_1 - \mu) = 0, b(\lambda_2 - \mu) = 0 \tag{10}$$

$a(\lambda_1 - \mu)(\vec{v_1}) = 0$ implies either $a = 0$ or $(\lambda_1 - \mu) = 0$. If $a = 0$, then we have $\vec{w} = b\vec{v_2}$, and hence $L_3 = L_2$.

If $a(\lambda_1 - \mu) = 0$, then $\mu = \lambda_1$. Then we have $\vec{w} = a\vec{v_2}$, and hence $L_3 = L_1$.
Same argument for case $b(\lambda_2 - \mu) = 0$. Both cases, L_3 must be L_1 or L_2. This completes the proof.

2. A real symmetric $n \times n$ matrix A with property

$$\vec{x}^T A\vec{x} > 0$$

for all **nonzero vectors** $\vec{x} \in \mathbb{R}^n$. Prove that all the eigenvalues of matrix A are positive.

Question 2 max: 5 points
Without mention "$\|\vec{v}\| \neq 0$ since $\vec{v} \neq 0$ as it's an eigenvector " ,reduce 1 point.

Solution:
Let λ be a real eigenvalue of A (Recall that the eigenvalues of a real symmetric matrix are real.) and Let $\vec{v}$ be a corresponding real ((why?) eigenvector.

$$A\vec{v} = \lambda\vec{v}$$

We may multiply $\vec{v}^T$ on the left and obtain

$$\vec{v}^T A\vec{v} = \lambda\vec{v}^T\vec{v}$$

$$= \lambda\|\vec{v}\|^2$$

Recall that $\vec{x}^T A\vec{x} > 0$ and $\|\vec{v}\|^2 > 0$ (why $\|\vec{v}\| \neq 0$?), hence λ must be positive. This completes the proof.

3. Let A, B be two $n \times n$ diagonalizable matrices, which share the same set of eigenvectors. Show that $AB = BA$.

   Question 3 max: 5 points

   Solution:
   Since they both are diagonalizable and share the same set of eigenvectors, there exist an invertible P such that $A = P^{-1}D_AP$ and $B = P^{-1}D_BP$. Then

$$BA = P^{-1}D_BPP^{-1}D_AP = P^{-1}D_BD_AP$$

$$= P^{-1}D_AD_BP = P^{-1}D_APP^{-1}D_BP = AB$$

4. Let $A^2 = I$ be an $n \times n$ matrix and $A \neq I$. Show that A is diagonalizable.

   Question 4 max: 5 points
   If students found the only eigenvalues of A are ± 1, then award them 2 points.
   Students must carefully show that why we have n linearly independent vectors in order to receive additional 3 points.

   Solution:
   The only eigenvalues of A are ± 1 since

$$Av = \lambda v \implies A^2v = v = \lambda^2 v \implies \lambda = \pm 1.$$

   Now let E_1 be the eigenspace of A corresponding to the eigenvalue 1 and E_{-1} be the eigenspace corresponding to the eigenvalue -1.
   Note that $E_1 = \ker(A - I)$. Now since $A \neq I$ exists $v \in \mathbb{R}^n$ such that $0 \neq (A - I)v \in Range(A - I)$.
   Then we have $A((A - I)v) = -(A - I)v$ and therefore $(A - I)v \in E_{-1}$. This shows that $Range(A - I) \subseteq E_{-1}$.
   Now we use Fund. Th. Lin. Maps to obtain

$$n = \dim E_1 + \dim Range(A - I) \leq \dim E_1 + \dim E_2$$

   and we further have that $E_1 \bigcap E_{-1} = \{0\}$ because $Cv = \lambda_1 v$ and $Cv = \lambda_2 v$ for $v \neq 0$ implies that $\lambda_1 = \lambda_2$. Therefore we get

$\mathbb{R}^n = E_1 \bigoplus E_{-1}$, which shows that we can obtain $\dim E_1$ independent vectors corresponding to 1 and $\dim E_{-1}$ independent vectors corresponding to -1 and obtain a diagonalization matrix P s.t. $A = P^{-1} D_A P$.

5. Let $A \in M_{n\times n}(\mathbb{C})$ with characteristic polynomial $p(x) = cx^a \Pi_{i=1}^{k}(\lambda_i - x)$ and $\lambda_i \neq 0, \forall i, a \in \mathbb{Z}_{>0}$. Show that if $dim(ker(A)) + k = n$, then $A = C^2$ for some complex matrix C.

   Question 5 max: 5 points

   Students must carefully check the dimension of the kernel and argue linearly independence of eigenvectors rigorously, otherwise, deduct at least 3 points.

   Solution:

   The k distinct eigenvalues correspond to k linearly independent eigenvectors. The $\dim \ker A = n - k$ tells us that there is $n - k$ linearly independent eigenvectors corresponding to the eigenvalue 0. Therefore we obtain n linearly independent eigenvectors, and we can diagonalize $A = P^{-1}DP$.

   Now since our field is $\mathbb{C}$ we can take the square root of each entry and write $D = \sqrt{D}\sqrt{D}$ and therefore write

$$A = P^{-1}DP = P^{-1}\sqrt{D}\sqrt{D}P = P^{-1}\sqrt{D}PP^{-1}\sqrt{D}P = C^2$$

for $C = P^{-1}\sqrt{D}P$.