Q Find $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ on $S = \{ (x,y,z) \mid x^2+y^2+10z^2=1 \}$

Sol No boundary, by Stokes' thm, integrate to 0

Q2 Show $\int_C (f\nabla g)\cdot d\vec{s} = \iint_S (\nabla f \times \nabla g)\cdot d\vec{S}$

Sol

$$\int_C (f\nabla g)\cdot dS = \iint_S \nabla \times (f\nabla g)\cdot d\vec{S}$$

$f\nabla g = (fg_x, fg_y, fg_z)$

$$= \iint_S (fg_z)_y - (fg_y)_z - \left((fg_z)_x - (fg_x)_z\right) + (fg_y)_x - (fg_x)_y d\vec{S}$$

$$= \iint_S (f_y g_z + \cancel{g_{zy} f}) - (f_z g_y + \cancel{g_{yz} f}) - (f_x g_z + \cancel{g_{zx} f})$$

$$+ (f_z g_x + \cancel{g_{xz} f}) + (f_x g_y + \cancel{g_{yx} f}) - (f_y g_x + \cancel{g_{xy} f})$$

$$= \iint_S (f_y g_z - f_z g_y) - (f_x g_z - f_z g_x) + (f_x g_y - f_y g_x)\cdot d\vec{S}$$

$$= \iint_S (\nabla f \times \nabla g)\cdot d\vec{S}$$

Prop $\nabla \times (f\nabla g) = f(\nabla \times \nabla g) + \nabla f \times \nabla g$ $\quad$ $\text{curl}(fF) = f\,\text{curl}F + \nabla f \times F$

$$= f \times 0 + \nabla f \times \nabla g \qquad \text{curl}(\nabla f) = 0$$

$$= \nabla f \times \nabla g$$

M&T §8.2 Q26b

Question

Suppose that S is a closed and orientable surface with boundary **c**. *Show:*

$$\int_{\mathbf{c}} (f\nabla g + g\nabla f) \cdot d\mathbf{s} = 0$$

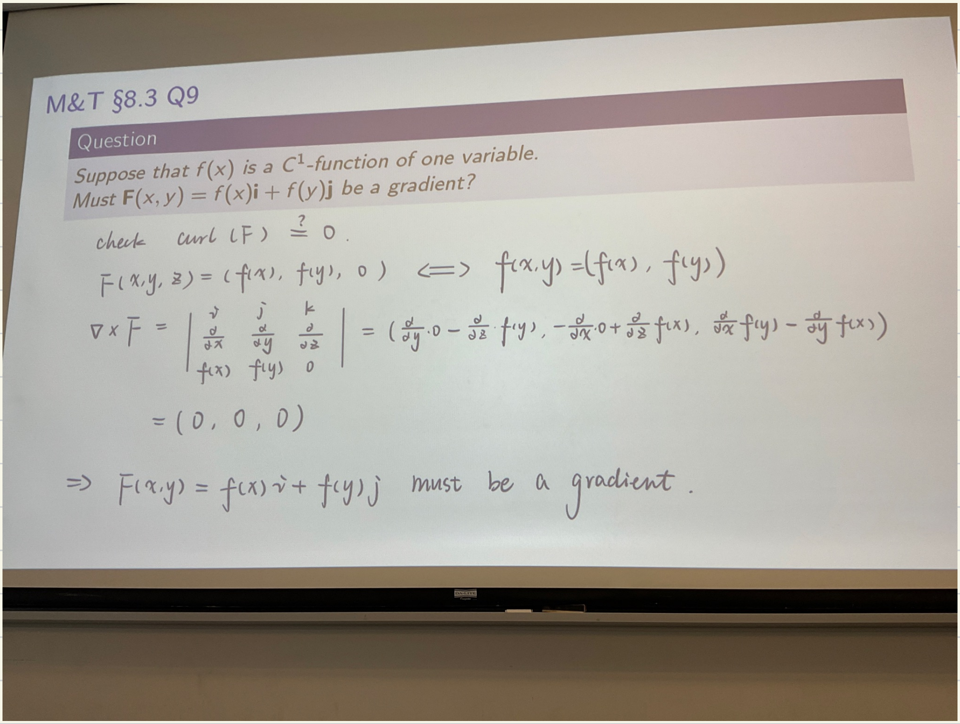
pick parametrization $c: [a,b] \to \mathbb{R}^3$ s.t. $c(b) = c(a)$ b.c S is closed.

(p255, 3) $\nabla(fg) = f\nabla g + g\nabla f$ let $h = f \cdot g$. and $H = \nabla h$, $\underline{\underline{H \text{ is a gradient}}}$

$$\int_c (f\nabla g + g\nabla f) = \int_c \nabla(fg) \cdot ds = \int_c \nabla h \cdot ds = h(c(b)) - h(c(a)) = 0$$

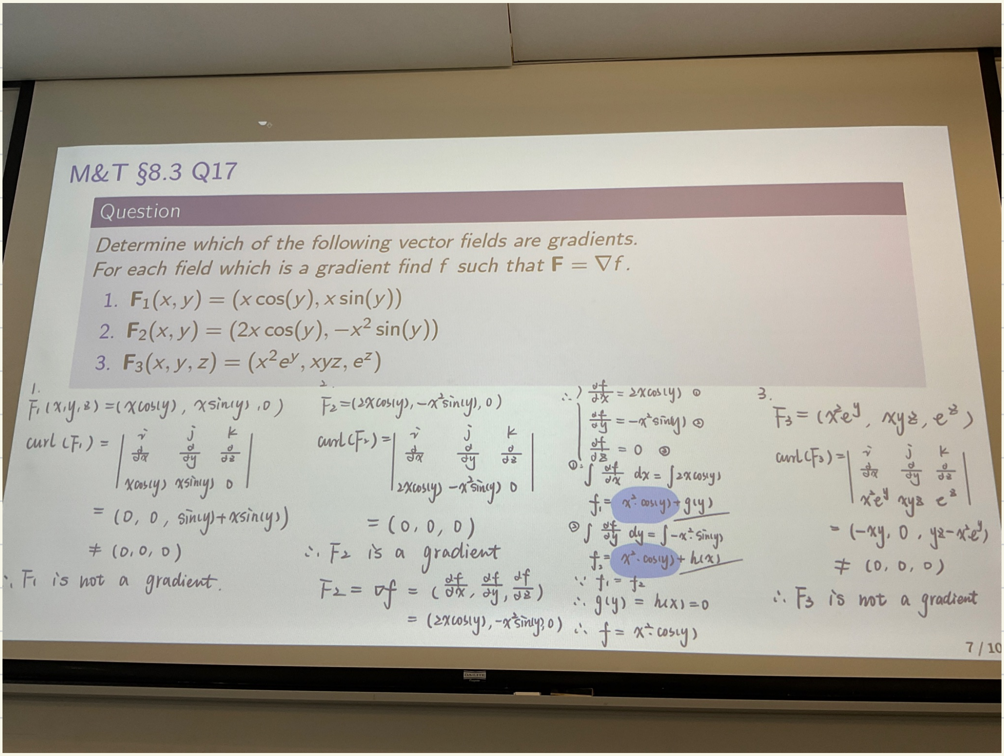
$$\boxed{\text{method 2: } = \iint_S \underbrace{\text{curl}(H)}_{=0} \cdot ds = 0 \quad \text{(By Stoke's Thm)}}$$

Q3



M&T §8.3 Q9
Question
Suppose that $f(x)$ is a C^1-function of one variable.
Must $\mathbf{F}(x, y) = f(x)\mathbf{i} + f(y)\mathbf{j}$ be a gradient?
check curl(F) $\stackrel{?}{=} 0$.
$F(x,y,z) = (f(x), f(y), 0) \iff f(x,y) = (f(x), f(y))$
$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & f(y) & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}\cdot 0 - \frac{\partial}{\partial z}\cdot f(y), -\frac{\partial}{\partial x}\cdot 0 + \frac{\partial}{\partial z} f(x), \frac{\partial}{\partial x} f(y) - \frac{\partial}{\partial y} f(x)\right)$
$= (0, 0, 0)$
$\Rightarrow F(x,y) = f(x)i + f(y)j$ must be a gradient.

Q4


M&T §8.3 Q17
Question
Determine which of the following vector fields are gradients.
For each field which is a gradient find f such that F = ∇f.
1. F1(x, y) = (x cos(y), x sin(y))
2. F2(x, y) = (2x cos(y), −x^2 sin(y))
3. F3(x, y, z) = (x^2 e^y, xyz, e^z)
1.
F1(x,y,z) = (x cos(y), x sin(y), 0)
curl(F1) =
= (0, 0, sin(y) + x sin(y))
≠ (0, 0, 0)
∴ F1 is not a gradient.
2.
F2 = (2x cos(y), −x^2 sin(y), 0)
curl(F2) =
= (0, 0, 0)
∴ F2 is a gradient
F2 = ∇f = (∂f/∂x, ∂f/∂y, ∂f/∂z)
= (2x cos(y), −x^2 sin(y), 0)
∂f/∂x = 2x cos(y) ①
∂f/∂y = −x^2 sin(y) ②
∂f/∂z = 0 ③
① ∫ ∂f/∂x dx = ∫ 2x cos(y)
f1 = x^2 cos(y) + g(y)
② ∫ ∂f/∂y dy = ∫ −x^2 sin(y)
f2 = x^2 cos(y) + h(x)
∵ f1 = f2
∴ g(y) = h(x) = 0
∴ f = x^2 cos(y)
3.
F3 = (x^2 e^y, xyz, e^z)
curl(F3) =
= (−xy, 0, yz − x^2 e^y)
≠ (0, 0, 0)
∴ F3 is not a gradient
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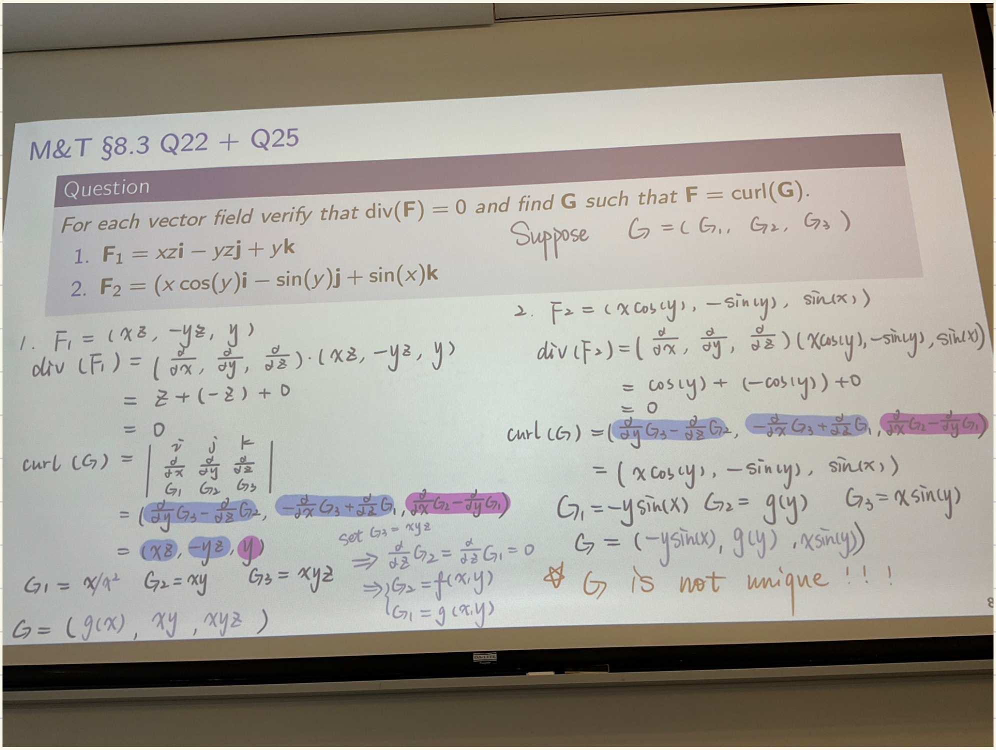
Q5


M&T §8.3 Q22 + Q25
Question
For each vector field verify that div(F) = 0 and find G such that F = curl(G).
Suppose G = (G1, G2, G3)
1. F1 = xzi − yzj + yk
2. F2 = (x cos(y)i − sin(y)j + sin(x)k
1. F1 = (xz, −yz, y)
div (F1) = (∂/∂x, ∂/∂y, ∂/∂z) · (xz, −yz, y)
= z + (−z) + 0
= 0
curl (G) = | i j k ; ∂/∂x ∂/∂y ∂/∂z ; G1 G2 G3 |
= (∂/∂y G3 − ∂/∂z G2, −∂/∂x G3 + ∂/∂z G1, ∂/∂x G2 − ∂/∂y G1)
= (xz, −yz, y)
set G3 = xyz
⇒ ∂/∂z G2 = ∂/∂z G1 = 0
⇒ G2 = f(x, y), G1 = g(x, y)
G1 = x/a² G2 = xy G3 = xyz
G = (g(x), xy, xyz)
2. F2 = (x cos(y), −sin(y), sin(x))
div (F2) = (∂/∂x, ∂/∂y, ∂/∂z) (x cos(y), −sin(y), sin(x))
= cos(y) + (−cos(y)) + 0
= 0
curl (G) = (∂/∂y G3 − ∂/∂z G2, −∂/∂x G3 + ∂/∂z G1, ∂/∂x G2 − ∂/∂y G1)
= (x cos(y), −sin(y), sin(x))
G1 = −y sin(x) G2 = g(y) G3 = x sin(y)
G = (−y sin(x), g(y), x sin(y))
G is not unique !!!

Building a Path

set $G_2 = xy$
$\Rightarrow \frac{\partial}{\partial y} G_1 = 0$
$\therefore G_1 = g(x)$

Question

In the proof of the Classification of Conservative Vector Fields (M&T Thm 7. p. 453) there is an elaborate construction of a path from $(0,0,0)$ to (x,y,z) such that: $\partial f/\partial z = F_3$. In lecture, we gave a path for $\partial f/\partial y = F_2$.

Find the corresponding path for $\partial f/\partial x = F_1$.

Suppose that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ only depends on the endpoints of $\mathbf{c}$.
We must construct a function $\mathbf{F}$ such that $\mathbf{F} = \nabla f$.
Pick a path $\mathbf{c}$ from $(0,0,0)$ to (x,y,z).
We define $f(x,y,z) = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.
One particular path is:

- Travel from $(0,0,0)$ to $(x,0,0)$ parallel to the x-axis.
- Travel from $(x,0,0)$ to $(x,0,z)$ parallel to the z-axis.
- Travel from $(x,0,z)$ to (x,y,z) parallel to the y-axis.

This gives the following:

$$f = \int_0^x F_1(t,0,0)dt + \int_0^z F_3(x,0,t)dt + \int_0^y F_2(x,t,z)dt$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\int_0^x F_1(t,0,0)\,dt + \int_0^z F_3(x,0,t)\,dt + \int_0^y F_2(x,t,z)\,dt\right]$$

$$\overset{\text{FTC}}{=} F_1$$

TA Reference: Building a Path

Suppose that $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ only depends on the endpoints of $\mathbf{c}$.
We must construct a function $\mathbf{F}$ such that $\mathbf{F} = \nabla f$.
Pick a path $\mathbf{c}$ from $(0,0,0)$ to (x,y,z).
We define $f(x,y,z) = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.
One particular path is:

- Travel from $(0,0,0)$ to $(x,0,0)$ parallel to the x-axis.
- Travel from $(x,0,0)$ to $(x,0,z)$ parallel to the z-axis.
- Travel from $(x,0,z)$ to (x,y,z) parallel to the y-axis.

This gives the following:

$$f = \int_0^x F_1(t,0,0)dt + \int_0^z F_3(x,0,t)dt + \int_0^y F_2(x,t,z)dt$$

We obtain:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\overbrace{\int_0^x F_1(t,0,0)dt}^{0} + \overbrace{\int_0^z F_3(x,0,t)dt}^{0} + \int_0^y F_2(x,t,z)dt\right] = F_2$$