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Problem 1

a) Consider vectors $\vec{v}_1 \cdots \vec{v}_k$ is a vectorspace V over $\mathbb{F}$. They are linearly dependent if there exists $c_1 \cdots c_k \in \mathbb{F}$ so that

$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ and $(c_1, c_2, \cdots, c_k) \neq (0, 0, \cdots, 0)$, $k \in \mathbb{N}$

b) An isomorphism between vector space V and W over $\mathbb{F}$ is a

bijective ~~function~~ linear transformation $T : V \longrightarrow W$

I affirm that I did not give / receive any unauthorized help on this exam and that all submitted work is my own

Problem 2

a) True if $\{\vec{u}, \vec{v}\}$ are dependent, then $\exists\ x, y \in \mathbb{F}$ s.t ($x \neq 0$ or $y \neq 0$) and $x\vec{u} + y\vec{v} = \vec{0}$

Then choose $a = -\frac{y}{x}$, and $\vec{u} = a\vec{v}$

b) True Subspaces are still vector space, for any subspace V, define LT $T: V \to V$

$$\vec{v} \mapsto \vec{v}$$

Then the subspace is the image of T

c) True $\dim(P_3(\mathbb{C})) = 4$, means its basis has 4 elements. All spanning set thus need to have at least 4 elements (proved in class), so no list of 3 polynomial can span $P_3(\mathbb{C})$

d) True By rank-nullity theorem, $\dim(\mathbb{R}^5) = \dim(\ker(T)) + \dim(\mathrm{Img}(T))$

Since T is surjective: $\dim(\mathbb{R}^5) = 5 = \dim(\ker(T)) + \dim(V)$

Since $\dim(\ker(T)) \geq 0$: $\dim(V) = 5 - \dim(\ker(T))$

$$\leq 5 - 0 = 5$$

Problem 3

a) Define $T\ \mathbb{R}^2 \to \mathbb{R}^2$

$[a,b] \longmapsto [a-b, a-b]$

Then $im(T) = span([1,1])$

$ker(T) = span([1,1])$

b) Suppose $\mathbb{C}^3$ is over the field $\mathbb{C}$

Then $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ are both

basis of $\mathbb{C}^3$, and they are clearly not equal

c) Choose $U \subseteq \mathbb{C}^2$ where $U = \{ [a,0] \mid a \in \mathbb{Z} \text{ (integer)} \}$

Then $\forall \vec{u}, \vec{v} \in U$ $\vec{u} = [a,0]$ $\vec{v} = [b,0]$

$\vec{u} + \vec{v} = [a+b, 0] \in U$ $(a+b \in \mathbb{Z})$

and choose $-\vec{u} = [-a, 0] \in U$

$\vec{u} + -\vec{u} = \vec{0}$

But U is not closed under scalar multiple since $\forall \vec{u} \in U$ and $r \in \mathbb{C} \setminus \mathbb{Z}$, $r\vec{u} \notin U$

d) $\mathbb{R}$ is a vectorspace of dimension 1 over $\mathbb{R}$

$span(\mathbb{R}) = \mathbb{R}$, and $|\mathbb{R}| = \infty$

So $\mathbb{R}$ is an infinite spanning set of a finite dimension vector space $\mathbb{R}$

Problem 4

a) $T \circ S(e^{5x}) = T(S(e^{5x})) = T(5e^{5x})$

We cannot calculate $T(5e^{5x})$ because $5e^{5x} \notin \mathrm{span}(e^{2x}, e^{3x}, 1)$

So we cannot break dow $T(5e^{5x})$ into a LC of $T(e^{2x})$, $T(e^{3x})$ and $T(1)$, thus we don't have enough information

b) $S \circ T(2+3e^{2x}) = S(T(2+3e^{2x})) = S(2T(1) + 3T(e^{2x}))$ [linearity]

$= S(2(0) + 3(x^2))$ [substitud]

$= S(3x^2)$

$= 6x$ [by def of $S(f) = f'$]

c) We know T maps all constant function and function of 0 to $\vec{0}$.

And $\forall f(x) = \alpha x$ where α is a constant, $S(f) = \alpha$ (a constant)

$\forall g(x) = b$ where b is a constant, $S(g) = 0$

So choose $f(x) = x$, $g(x) = 1$, then $\mathrm{span}(f, g) = P_1$ $T \circ S(\mathrm{span}(f,g)) = \{\vec{0}\}$

since $\dim(P_1) = 2$, so $\dim(\ker(T \circ S))$ is at least 2

Since $\mathrm{span}(e^{2x}, e^{3x}, 1) \neq C^\infty$, so there could exist more elements in C^∞ that gets mapped to 0 (example, if we map $T \circ S(e^{5x})$ to 0)

So $\dim(\ker(T \circ S)) \geq 2$

Problem 5

a) ① WTS $W \neq \emptyset$

The function $p(x) = 0$ is in all possible vector spaces of functions,
and $p(x) = 0 \rightarrow p(1) = 0$, so the 0 function is in W
Thus $W \neq \emptyset$

② WTS W is closed under addition
Let $f, g \in W$ be arbitrary
By def of W, $f(1) = g(1) = 0$
$(f+g)(1) = f(1) + g(1)$ [def of f^n +]
$= 0 + 0$
$= 0$
So $(f+g)$ is in W by def
So W is closed under +

③ WTS W is closed under scalar multiple
Let $f \in W$, $r \in \mathbb{F}$ be arbitrary
$(rf)(1) = r \cdot f(1)$ [by def of function scalar multiple]
$= r \cdot 0$
$= 0$ [0 identity property]
So (rf) is in W by def
So W is closed under scalar multi

W passed subspace test and is a subspace of $P_{(m)}\mathbb{F}$

b) It is not onto, since $q(x) = 1$, $q \in P_m$, but $q \notin W$ since it doesn't have root at $x = 1$
So there does not exist $p \in W$, where $T(p) = p = q$, so T is not onto

Problem 5

c) So we have LC $c_0 p_0 + c_1 p_1 + \quad + c_m p_m = 0$

WTS $\exists (c_0 \cdots c_m) \neq (0, \cdots, 0)$

List sys of equations that if $i<j$, degree $(p_i) \leq$ degree (p_j)

$$
\begin{aligned}
c_0 p_0'(1) + c_1 p_1'(1) + \cdots + c_m p_m'(1) &= 0 \\
\vdots \qquad\qquad\qquad & \;\; \vdots \\
c_0 p_0^{m}(1) + c_1 p_1^{m}(1) + \cdots + c_m p_m^{m}(1) &= 0 \\
c_0 p_0(1) + c_1 p_1(1) + \cdots + c_m p_m(1) &= 0
\end{aligned}
$$

$$\downarrow$$

$$
\begin{bmatrix}
p_0'(1) & p_1'(1) & \cdots & p_m'(1) \\
\vdots & & & \\
p_0^{m}(1) & p_1^{m}(1) & \cdots & p_m^{m}(1) \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
$$

The matrix have m row m column, but 1 row is all 0, so it is equivalent to matrix of $m-1$ row and m column. From MATA22, we know such sys of equation has infinite solution, which means $\exists (c_0 \cdots c_m) \neq (0, \cdots, 0)$ where $\sum_{i=0}^{m} c_i p_i = 0$ and so $\{p_i\}_{i=0}^{m}$ is linearly dependent

Problem 6

a) S:

$T(\vec{0}) = \vec{0}$ so $\vec{0} \in S$ and $S \neq \emptyset$

$\forall \vec{v}, \vec{w} \in S$, $r \in \mathbb{F}$

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{v}+\vec{w}$ so $\vec{v}+\vec{w} \in V$

[linearity]

$T(r\vec{v}) = rT(\vec{v}) = r\vec{v}$ so $r\vec{v} \in V$

[linearity]

So S passed subspaec test and is a subspace

A.

$T(\vec{0}) = \vec{0} = -\vec{0}$ so $\vec{0} \in A$ and $A \neq \emptyset$

$\forall \vec{v}, \vec{w} \in A$ $r \in \mathbb{F}$

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = -\vec{v} + -\vec{w} = -(\vec{v}+\vec{w})$ so $\vec{v}+\vec{w} \in A$

$T(r\vec{v}) = rT(\vec{v}) = r(-\vec{v}) = -(r\vec{v})$ so $r\vec{v} \in A$

[both linearity]

So A passes subspace test and is a subspace also

b) Let $\vec{v} \in V$

Choose $\vec{u} = \vec{v} + T(\vec{v})$, then $\vec{u} \in S$ since $T(\vec{u}) = T(\vec{v} + T(\vec{v}))$

$= T(\vec{v}) + T(T(\vec{v}))$

$= \vec{v} + T(\vec{v}) = \vec{u}$

$\vec{w} = \vec{v} - T(\vec{v})$ then $\vec{w} \in A$ since $T(\vec{w}) = T(\vec{v} - T(\vec{v}))$

$= T(\vec{v}) - T(T(\vec{v}))$

$= -\vec{v} - (-T(\vec{v}))$

$= -\vec{v} + T(\vec{v}) = -\vec{w}$

Since S, A are s.s, choose $\vec{x} = \frac{1}{2}\vec{u}$, $y = \frac{1}{2}\vec{w}$, and $\vec{x} \in S$, $\vec{y} \in A$

$\vec{x} + \vec{y} = \frac{1}{2}(\vec{v} + T(\vec{v})) + \frac{1}{2}(\vec{v} - T(\vec{v})) = \frac{1}{2}(2\vec{v} + 2T(\vec{v})) = \vec{v}$

Since $1 \neq -1$, $-\vec{v} \neq \vec{v}$, and there is no $\vec{v} \in V$ s.t $\vec{v} \in S \wedge \vec{v} \in A$

So $\vec{x} + \vec{y} = \vec{v}$ is unique

c) Define T on tricolour to be reflection of the knot on one axis.
Define $SY = \{ \vec{k} \in K \mid T(\vec{k}) = \vec{k} \}$
SY then describe all tricolouring that are symmetric

Remark, T is linear since 2 symmetric knot layered on top is still symmetric

And SY is a subspace from part (a)

SY is clearly a finite subspace of all tricolouring. so it is also a finite vector space over finite field.
By same reason from GHW2 Problem 4.3, $|SY| = |\mathbb{F}|^m = 3^m$ for some $m \geq 1$

d) Define T_2 to be reflection over same axis as T, but then reflect another time at axis perpendicular to the axis used in T, then $DSY = \{ \vec{k} \in K \mid T_2(\vec{k}) = -\vec{k} \}$ is similar to A from part a.

By part b, it means every $\vec{k} \in K$ can be described as a unique sum of element from SY and DSY

Or, every tricolouring can be represented as 2 other tricolouring, where 1 is symmetric, and 1 is not symmetric, but looks the same after 2 perpendicular reflections.