University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA37 Winter 2020

Assignment # 2

You are may wish to work on this assignment prior to your tutorial during the week of January 20th. You may ask questions about this assignment in that tutorial.

STUDY: Chapter 4, Sections: 4.3 (the rest of this section not covered by A1) and Supplementary material: ONLY up to lower and upper sum definitions and the Darboux definition; *OMIT 'The Integrability Reformulation' (Thm 2 Suppl. Notes) – questions about the Integrability ϵ-Reformulation will appear on A3.*

HOMEWORK:

At the beginning of your TUTORIAL during the week of January 27th you may be asked to either submit the following "Homework" problems or write a quiz based on this assignment and/or related material from the lectures and textbook readings. This part of your assignment will count towards the 20% of your final mark, which is based on weekly assignments / quizzes.

The intent of the following question(s) is to further your comfort with working with the Riemann Sum definition of the definite integral.

1. Express the following limits as a definite integral. Make sure to fully justify your work.

   (a) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right)$

   (b) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{n} f\left(2 + \frac{2i}{n}\right)$ where $f(x)$ is a continuous everywhere function.

2. Textbook Section 4.3 - # 56, 62.

3. Let $a, b \in \mathbb{R}$, $a < b$. Let $c \in \mathbb{R}$. Suppose that f is continuous on $[a, b]$. If $f(x) \geq 0$ on $[a, b]$ then use (only) the Riemann sum definition of the definite integral to prove that $\int_a^b (f(x) + c)dx \geq c(b - a)$.

   The intent of the following question(s) is to get you use to working with the properties of the definite integral and the geometrical meaning of the definite integral.

4. Suppose that f is a continuous everywhere function satisfying $\displaystyle\int_9^1 f(x)\, dx = 10$ and $\displaystyle\int_6^9 f(x)\, dx = -3$. Find the value of $\displaystyle\int_1^6 f(x)\, dx$.

5. Prove that $\displaystyle\int_0^1 e^{-x^2}\, dx \leq 1$.

6. Evaluate the following:

   (a) $\displaystyle\int_0^5 f(x)\, dx$ if $f(x) = \begin{cases} 1, & \text{if } x < 3 \\ x, & \text{if } x \geq 3 \end{cases}$

   (b) $\displaystyle\int_0^9 (2x + \sqrt{81 - x^2} - 3g(x))\, dx$ if $\displaystyle\int_0^9 g(x)\, dx = 16$.

   The intent of the following question(s) is to get you use to the Darboux definition and the ingredients of the Darboux definition of the definite integral.

7. Calculate $L(f, P)$ and $U(f, P)$ (as defined in lecture) for the following. Make sure to justify your work, especially your m_i and M_i computations.

   (a) $f(x) = \pi$, $x \in [1, 2]$; an arbitrary partition $P = \{x_i\}_{i=0}^n$.

   (b) $f(x) = -\sin(x)$, $x \in [0, \pi]$; $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \pi\}$

8. Let function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{, if } x \in \mathbb{Q} \\ 42 & \text{, if } x \notin \mathbb{Q} \end{cases}$

   Use the Darboux definition of the definite integral to prove that $(f(x))^2$ is <u>not</u> integrable on $[-2, -1]$.

CHALLENGE PROBLEMS

These question are not being graded or evaluated in any fashion. These are merely for your interest's sake.

1. Let $a, b \in \mathbb{R}, a < b$. If f is a constant function on $[a, b]$, then use the Darboux definition to prove that f is integral on $[a, b]$. What is the value of this integral? (The answer should be contained in the details of your proof).

2. Let $a, b, c \in \mathbb{R}, a < b$. Use the Darboux definition of the definite integral to prove : If f is integrable on $[a, b]$, then $\int_a^b cf(x)dx = c \int_a^b f(x)dx$.

EXERCISES: You do not need to submit solutions to the following problems but you should make sure that you are able to answer them.

1. Textbook Section 4.3 - # 1(a)-(h), 7, 8, 9, 11, 12, 20, 32, 33, 34, 55, 60, 61

2. Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \cdot \frac{6 + \frac{5i}{n}}{\sqrt{4 + \frac{5i}{n}}}$ as a definite integral, but with :

   (a) $\triangle x = \frac{1}{n}$

   (b) $a = 4$

   (c) $a = 0$.

3. Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ as a definite integral. Make sure to justify your work.

4. Choose the most appropriate answer from the below multiple choices. Make sure to fully justify your answer.

   Let $f(x)$ be a continuous positive function on the interval $[1, 2]$. The area of the region bounded by the graph of f, the x-axis and the lines $x = 1$ and $x = 2$ is:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f(1 + \frac{i-1}{n})$

(b) $\lim_{n\to\infty} \sum_{k=0}^{n-1} f(1 + \frac{k+1}{n}) \frac{1}{n}$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(1+\frac{i}{n})}{n}$

(d) $\int_1^2 f(x)\ dx$

(e) All of the above.

(f) None of the above.

5. Prove or disprove the following statement. That is, if it is true then prove it. If it is false, then provide a counter-example. Make sure to fully justify your work.

   - Suppose that f and g are continuous functions on $[0, 2]$. Then $\int_0^2 f(x)g(x)\ dx = f(x) \int_0^2 g(x)\ dx$.

6. Let $a, b \in \mathbb{R},\ a < b$. Let f be integrable on $[a, b]$. Give an example of a function f such that $f(x) \leq 0$ for all x, and $f(x) < 0$ for some $x \in [a, b]$, and $\int_a^b f(x)\ dx = 0$. Make sure to justify why your example satisfies the desired criteria.

7. Re-write the integral $\int_0^{\frac{5\pi}{4}} |\cos(x) - \sin(x)|\ dx$ into an equivalent sum of integrals without the absolute value in the integrand.

8. Let $a, b \in \mathbb{R}$, $a < b$. Let $c \in \mathbb{R}$. Suppose that f is continuous on $[a, b]$. If $f(x) \geq 0$ on $[a, b]$ then use (only) the Riemann sum definition of the definite integral to prove that $\int_a^b (f(x) + c)dx \geq c(b - a)$.

9. Write as a single integral of the form $\int_a^b f(x)dx$:

$$\int_{-2}^{2} f(x)dx + \int_{2}^{5} f(x)dx - \int_{-2}^{-1} f(x)dx.$$

10. Evaluate $\displaystyle\int_7^7 e^{x^2}dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}dx$.

11. If $\displaystyle\int_1^4 f(x)dx = 6$, $\displaystyle\int_2^4 f(x)dx = 4$, and $\displaystyle\int_1^3 f(x)dx = 2$, find $\displaystyle\int_2^3 f(x)dx$.

12. Prove $\displaystyle\int_0^4 (x^2 - 4x + 4 + e^{x^2})dx \geq 0$. Make sure to fully justify your work.

13. Given that $\int_1^4 f(x)dx = 5$, $\int_3^4 f(x)dx = 7$, $\int_1^8 f(x)dx = 11$. Find the following. Make sure to justify your work :

    (a) $\int_4^8 f(x)dx$

    (b) $\int_4^3 f(x)dx$

    (c) $\int_1^3 f(x)dx$

    (d) $\int_3^8 f(x)dx$

    (e) $\int_4^4 f(x)dx$

14. Let $a, b \in \mathbb{R}$, $a < b$. Suppose that f is continuous on $[a, b]$. In lecture, we stated the following property $(\star)$:

    If $\exists m, M \in \mathbb{R}^+$ such that $m \leq f(x) \leq M \ \forall x \in [a, b]$
    then $m(b-a) \leq \displaystyle\int_a^b f(x)dx \leq M(b-a)$.

    (a) Provide a sample diagram to illustrate this property. Make sure to clearly label m, M, your interval of interval of interest $[a, b]$ and $y = f(x)$ in your diagram.

    (b) Use your integration properties together with "geometrically evaluating" some of the integrals involved, to prove $(\star)$.

15. Let a, $b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, show that

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

    Hint : $-|f(x)| \leq f(x) \leq |f(x)|$ holds for all $x \in dom(f)$.

16. Calculate $L(f,P)$ and $U(f,P)$ (as defined in lecture) for the following. Make sure to justify your work, especially your m_i and M_i computations.

    (a) $f(x) = 2x$, $x \in [0,1]$; $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$

    (b) $f(x) = \begin{cases} 3 & \text{if } x \in \mathbb{Q} \\ -2 & \text{if } x \notin \mathbb{Q} \end{cases}$, $x \in [0,1]$; any partition $P = \{x_i\}_{i=0}^n$.

    (c) $f(x) = -x^2$, $x \in [0,1]$; an arbitrary partition $P = \{x_i\}_{i=0}^n$. Do <u>not</u> simplify your upper and lower sums.

17. Let $a, b \in \mathbb{R}$, $a < b$. Suppose that f is bounded on $[a,b]$. Prove that $L(f,P) \leq U(f,P)$ for any partition P of $[a,b]$.

18. Let function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{, if } x \in \mathbb{Q} \\ -1 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

    Use the Darboux definition to prove that f is <u>not</u> integrable on $[0,1]$.

19. Let function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -\frac{\pi}{2} & \text{, if } x \in \mathbb{Q} \\ 0 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

    Use the Darboux definition to prove that $\int_0^1 e^{\sin(f(x))}dx$ does not exist.

20. Prove or disprove the following statement. That is, if it is true then prove it. If it is false, then provide a counter-example. Make sure to fully justify your work.

    - If f^2 is integrable on $[-1,0]$, then f is integrable on $[-1,0]$.

A mind is not a vessel to be filled but a fire to be kindled. – *Plutarch*
