1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.

   (a) (3 points) Let V be an $\mathbb{F}$-vector space, with $\mathbb{F}$ given by $\mathbb{R}$ or $\mathbb{C}$. A map $\langle , \rangle : V \times V \to \mathbb{F}$ is an *inner product* on V if it is positive definite, symmetric, and linear in the first entry.

   Should be conjugate symmetric

   (b) (3 points) A set of unit vectors $\{\vec{b}_1, \cdots, \vec{b}_k\}$ in an inner product space is *orthonormal* if the $\vec{b}_i$'s are mutually orthogonal.

   The set would we considered orthogonal

   but to be orthonormal, all b_i must be unit vectors

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counter example if you think it is false).

   (a) (3 points) Let W be a subspace of a finite dimensional inner product space V. Suppose the linear map $\mathrm{T}: V \to V$ is defined by $\vec{x} \mapsto \mathrm{Proj}_W \vec{x}$ where Proj_W denotes the orthogonal projection onto W. Then $\dim \mathrm{im}(T) = \dim V - \dim W^{\perp}$.

True.

We know $\dim(V) = \dim(W) + \dim(W^{\perp})$, so prove $\dim(Im(T)) = \dim(W)$

$\forall w \in W$, show $\exists v \in V$ st $T(v) = w$

We know $\forall v$, $v = v_w + v_{w^\perp}$; so $\exists \vec{v} \in V$, st $w = v - v_{w^\perp}$

$$T(w) = T(v - v_{w^\perp})$$
$$w = T(v) - T(v_{w^\perp})$$
$$= T(v) \quad \blacksquare$$

   (b) (3 points) Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be nonzero vectors in $\mathbb{R}^3$. Then $\vec{v}_1, \vec{v}_2 - \mathrm{Proj}_{\mathrm{Span}(\vec{v}_1)}\vec{v}_2$ and $\vec{v}_3 - \mathrm{Proj}_{\mathrm{Span}(\vec{v}_1,\vec{v}_2)}\vec{v}_3$ is an orthogonal basis for $\mathbb{R}^3$.

Let $v_1 = [1,0,0]$ $\quad u_1 = v_1$

$v_2 = [1,1,0]$

$$u_2 = [1,1,0] - \frac{\langle v_2, u_1\rangle}{\langle v_2, v_2\rangle} v_1$$
$$= [1,1,0] - \frac{1}{2}[1,0,0]$$
$$= [\tfrac{1}{2},1,0]$$

$u_1 \cdot u_2 = [1,0,0]\cdot[\frac{1}{2},1,0]$

$= \frac{1}{2}$.

Thus u_1 is not ortho with u_2 and cannot make a ortho basis

(c) (3 points) Recall that the trace of a matrix is the sum of its diagonal entries. Let $M_{n\times n}(\mathbb{C})$ be the complex vector space of $(n \times n)$-matrices, then $\langle A, B\rangle =$ trace$(A+B)$ is an inner product on $M_{n\times n}(\mathbb{C})$.

Let $A, B, C \in M_{n\times n}(\mathbb{C})$, $r \in \mathbb{C}$ be arbi.

$$\langle A+rB, C\rangle = tr(A+rB+C)$$
$$= X$$

Let $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $B=\begin{bmatrix}2&1\\1&2\end{bmatrix}$, $C=\begin{bmatrix}3&2\\2&3\end{bmatrix}$

$$\langle A+B, C\rangle = tr\left(\begin{bmatrix}6&3\\3&6\end{bmatrix}\right) = 12 \qquad \langle A,C\rangle + \langle A,B\rangle = tr\left(\begin{bmatrix}4&2\\2&4\end{bmatrix}\right) + tr\left(\begin{bmatrix}3&1\\1&3\end{bmatrix}\right) = 8+6 = 14$$

So $\langle A,B\rangle$ is not linear in the first entry and thus not an inner product

(d) (3 points) Use the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ on the real vector space $C^0[0,1]$ of real-valued continuous functions on the domain $[0,1]$. The orthogonal projection of x^3 onto the subspace Span$\{1, x\}$ is given by $\left(\int_0^1 x^3\,dx\right) 1 + \left(\int_0^1 x^4\,dx\right) x \in C^0[0,1]$.

$$proj_{sp(1,x)} x^3 = \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle}(1) + \frac{\langle x^3, x\rangle}{\langle x, x\rangle} x$$
$$= \frac{\int_0^1 x^3\,dx}{\int_0^1 1\,dx}(1) + \frac{\int_0^1 x^4\,dx}{\int_0^1 x^2\,dx} x$$
$$= \int_0^1 x^3 dx\,(1) + \frac{\int_0^1 x^4 dx}{\frac{1}{3}}(x) \neq \int_0^1 x^3 dx\,(1) + \int_0^1 x^4\,dx\,(x)$$

bc $\langle x, x\rangle \neq 1$ $\therefore$ False

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist. Include the description in your answer.

(a) (2 points) A non-identity matrix $A \in M_n(\mathbb{R})$, such that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A \vec{w}$ for $\vec{v}, \vec{w} \in \mathbb{R}^n$, is an inner product on $\mathbb{R}^n$.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ A just has to be diagonal to satisfy the inner product properties

(b) (2 points) Two different inner products $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ on $\mathbb{C}^2$.

$\langle v, w \rangle = v^T w \qquad \langle v, w \rangle = 2v^T w$

(c) (2 points) Mutually orthogonal vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ with respect to the Euclidean (complex) inner product in $\mathbb{C}^3$.

$v_1 = [1, 0, 0] \quad v_2 = [0, 1, 0] \quad v_3 = [0, 0, 1] \quad v_4 = \vec{0}$ if the vectors need not form a basis or orthogonal set

Otherwise you can only have max 3 mutually ortho. non-zero vectors in $\mathbb{C}^3$ bc $\dim(\mathbb{C}^3) = 3$

(d) (2 points) A vector in the orthogonal complement of $\text{Span}\{[1 \ \ 2 \ \ -1]^T, [-1 \ \ 3 \ \ 2]^T\} \subset \mathbb{R}^3$.

Find $Nul(A^T)$: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & 0 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right]$

$x_1 + 7x_2 = 0$ Let $x_2 = s \in \mathbb{R}^3$

$5x_2 + x_3 = 0$

So $\begin{pmatrix} -7s \\ s \\ -5s \end{pmatrix} = s\begin{pmatrix} -7 \\ 1 \\ -5 \end{pmatrix}$

$x_1 = -7s$

$x_3 = -5s$

$Nul(A^T) = Sp\begin{pmatrix} -7 \\ 1 \\ -5 \end{pmatrix}$

Choose $\begin{bmatrix} -7 \\ 1 \\ -5 \end{bmatrix}$ in the ortho. comp.

4. Let $U = \{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{R} \}$ be the vector space of all upper triangular 2×2 matrices. Let $\mathcal{B} = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right] \right)$ and $\mathcal{E} = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right)$ be two bases of U, and let $D \colon U \to U$ be the linear transformation defined by

$$D \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+b & b+2c \\ 0 & c \end{pmatrix}.$$

(a) (5 points) Find the matrix $[D]_{\mathcal{B}}$ of D with respect to the basis $\mathcal{B}$.

$$[D]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [D([d_1]_{\mathcal{B}})]_{\mathcal{B}} & \cdots & [D([d_3]_{\mathcal{B}})]_{\mathcal{B}} \\ | & & | \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix}$$

where $d_i = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$d_i \in \mathcal{B}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$D(b_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$D(b_3) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$D(b_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(b) (5 points) Find the matrix $[D]_{\mathcal{E}}$ of D with respect to the basis $\mathcal{E}$.

$$[D]_{\mathcal{E}} = \begin{bmatrix} | & & | \\ D([d_1]_{\mathcal{E}}) & \cdots & D([d_3]_{\mathcal{E}}) \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(c) (5 points) Find an invertible matrix S such that $S[D]_{\mathcal{B}} = S^{-1}[D]_{\mathcal{E}}S$.

Let S be the change of basis matrix $C_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $C_{\mathcal{B}\to\mathcal{E}} \sim I_3$ thus invertibl

$$S[D]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad [D]_{\mathcal{E}} S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

5. Recall that the trace of a matrix is the sum of its diagonal entries. Let $M_{2\times 2}(\mathbb{R})$ be the real vector space of (2×2)-matrices with inner product $\langle A, B\rangle = \frac{1}{2}\,\text{trace}(A^T B)$. Parts (a)–(c) below refer to this inner product.

   (a) (4 points) Find $\|Q\|$, where Q is any orthogonal (2×2)-matrix (that is, the columns of Q are orthonormal).

$$\|Q\|^2 = \langle Q, Q\rangle = \frac{1}{2}\,tr(Q^T Q)$$
$$= \frac{1}{2}\,tr(I_2) \quad \text{bc } Q \text{ is orthonormal}$$
$$= \frac{1}{2}(2)$$
$$= 1$$
$$\Rightarrow \|Q\|^2 = 1 \Rightarrow \|Q\| = 1$$

   (b) (4 points) For any angle θ, let $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ be the standard matrix of the counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. Given angles $\alpha, \beta \in \mathbb{R}$, find the inner product $\langle R_\alpha, R_\beta\rangle$ in terms of α and β.

$$\langle R_\alpha, R_\beta\rangle = \frac{1}{2}\,tr(R_\alpha^T R_\beta)$$
$$= \frac{1}{2}(2\cos(\alpha-\beta))$$
$$= \cos(\alpha-\beta)$$

$$R_\alpha^T R_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$
$$\begin{bmatrix} \cos(\alpha-\beta) & \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta - \sin\alpha\cos\beta & \cos(\alpha-\beta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$$

   (c) (4 points) Given $\alpha, \beta \in \mathbb{R}$, find the angle φ between R_α and R_β in $M_{2\times 2}(\mathbb{R})$.

$$\varphi = \arccos\left(\frac{|\langle R_\alpha, R_\beta\rangle|}{\|R_\alpha\|\|R_\beta\|}\right)$$
$$= \arccos(|\cos(\alpha-\beta)|)$$
$$= \alpha-\beta$$

$$\langle R_\alpha, R_\alpha\rangle = \frac{1}{2}\,tr(R_\alpha^T R_\alpha)$$
$$= \frac{1}{2}(1+1)$$
$$= 1$$

similar with R_β

6. Suppose that V is an n-dimensional vector space and that $T\colon V \to V$ satisfies $T^2 = T$. You can use the fact that every vector $\vec{v} \in V$ can be written uniquely as a sum $\vec{v} = \vec{u} + \vec{w}$ with $\vec{u} \in \operatorname{im}(T)$ and $\vec{w} \in \operatorname{im}(\mathrm{id} - T)$ without proof.

   (a) (7 points) Prove that there exists a basis $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix with each of its diagonal entries equal to either 1 or 0. (Hint: pick a basis of $\operatorname{im}(T)$ and a basis of $\operatorname{im}(\mathrm{id} - T)$.)

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(b_1)]_{\mathcal{B}} & \cdots & [T(b_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

Let $\mathcal{A} = (a_1, \ldots, a_m)$ be a basis for $\operatorname{im}(id - T)$

$\forall v \in V$ = some l.c. of $\mathcal{B}$ and $\mathcal{A}$

$v = c_1 b_1 + \cdots + c_n b_n + k_1 a_1 + \cdots + k_m a_m$

$T(v) = T(\text{———})$

$= c_1 T(b_1) + \cdots + k_m T(a_m)$

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"I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own."

Leo Wang

I affirm that I did not give or recieve any unauthorized help on this exam and that all submitted work is my own