University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAB52H3 Introduction to Probability

Term Test 1
October 17, 2020

Duration: 60 minutes

Examination aids allowed: Open notes/books, scientific calculator.

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (20 points) Consider a probability space and three (jointly) *independent* events A, B, C with probabilities $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/3$, $\mathbb{P}(C) = 1/4$.
   Find the value of $\mathbb{P}((A \cup B) \cap C)$.

2. (20 points) Consider $n \geq 3$ persons, among them are Tom and Ben, who are arranged randomly in a row (say from left to right). What is the probability that Tom and Ben are *NOT* next to each other?

3. (20 points) Consider a medical condition C and two associated symptoms S_1 and S_2. The prevalence of this condition in the population is 10%, and any person with the condition can show none, one, or both symptoms, with probabilities: $\mathbb{P}(S_1|C) = 30\%$, $\mathbb{P}(S_2|C) = 70\%$, $\mathbb{P}(S_1 \cap S_2|C) = 20\%$. The symptoms can also appear in individuals *without* the condition with equal probability $P(S_1|C^c) = P(S_2|C^c) = 5\%$, and in this case the symptoms are conditionally independent, i.e. $P(S_1 \cap S_2|C^c) = P(S_1|C^c) \times P(S_2|C^c)$.
   Find the conditional probability $\mathbb{P}(C|S_1 \cap S_2^c)$, i.e. the probability of having the condition if you only show symptom S_1, but not S_2.

4. (20 points) Consider the experiment of independently flipping a fair coin 4 times. Define the RV X to be the length of the *longest streak of Heads*, i.e. the maximum number of Heads appearing in a row. If there are no Heads in the outcome, then set the value of X equal to 0. Find the probability mass function (PMF) of X.

5. Suppose $X \sim \text{Binomial}(n, p)$.
   (a) (13 points) Show that for $k < n$, we have the identity
   $$\frac{\mathbb{P}(X = k+1)}{\mathbb{P}(X = k)} = \frac{n-k}{k+1}\frac{p}{1-p}.$$
   (b) (7 points) Show that as long as $k > (n+1)p - 1$ we have
   $$\mathbb{P}(X = k) > \mathbb{P}(X = k+1).$$

   (Remark: This method can be used to show that the probability $\mathbb{P}(X = k)$ is maximized when k is near np.)