MATB24 TUTORIAL PROBLEMS 3SOLUTIONS

KEY WORDS: basis, dimension, linear transformation, image, kernel, onto, one-to-one
RELEVANT SECTIONS IN THE TEXTBOOK: Sec 3.4 FB or Sec 3.A, 3.B, 3.D SA

WARM-UP: Write down a complete definition or a complete mathematical characterization for the following terms.

Let V and W be a real vector spaces

- Dimension of V
- A linear transformation $T : V \to W$
- Image of a subset of V under a linear transformation T
- Inverse image (preimage) of an element under a linear transformation T
- Inverse image (preimage) of a set under a linear transformation T
- Kernel and image of a linear transformation T.

$$\operatorname{img}(T) = \{T(v) \mid v \in V\}, \quad \ker(T) = \{v \in V \mid T(v) = 0_W\}.$$

- An onto or surjective linear transformation
- A 1-1 or injective linear transformation

A: For each of the functions between vector spaces given below, determine whether or not the function is linear. For each function write down the definition of the kernel and image. When possible, give more explicit descriptions of the image and the kernel.

(1) $F : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $F(p)(x) = x + p(x)$.
(2) $F : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $F(p)(x) = xp(x)$.
(3) $F : \mathbb{P}_2 \to \mathbb{P}_4$ defined by $F(p)(x) = p(x)^2$.
(4) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g) = \frac{d}{dx}(g)$.
(5) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = \displaystyle\int_0^x g(t)\, dt$.[1]
(6) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = |g(x)|$.
(7) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = g(1 - x)$.
(8) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = e^{g(x)}$.
(9) $\det : M_{2\times2} \to \mathbb{R}$ defined by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.
(10) $\operatorname{tr} : M_{2\times2} \to \mathbb{R}$ defined by $\operatorname{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$.
(11) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = \displaystyle\int_0^x g(t) \cos(x - t) dt$.
(12) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$.
(13) $f : \mathbb{R}^3 \to \mathbb{R}^3$ is *dilation by* 2, defined by $f(\vec{v}) = 2\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

[1] $C^\infty(D)$ or C^∞_D is the set of smooth functions (having derivatives of all orders) with the domain D.

(14) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the line $y = x$, defined by $f([x, y]) = [y, x]$ for all $[x, y] \in \mathbb{R}^2$.
(15) $f : \mathbb{R}^2 \to \mathbb{R}$ assigns to every point in the plane its distance from the origin, so that $f([x, y]) = \sqrt{x^2 + y^2}$ for all $[x, y] \in \mathbb{R}^2$.
(16) $f : V \to W$ is the *zero transformation* defined by $f(\vec{v}) = \vec{0}_W$ for all $\vec{v} \in V$ (V and W are vector spaces).
(17) $f : V \to V$ is the identity transformation defined by $f(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$ (V is a vector space).

Solution.

(1) Not linear: $F(p+q)(x) = x + p(x) + q(x) \neq (x + p(x)) + (x + q(x))$ in general. $\ker(F) = \{p(x) \in \mathbb{P}_2 : x + p(x) = 0\} = \{-x\}$. $\operatorname{im}(F) = \{x + p(x) : p(x) \in \mathbb{P}_2\} = \mathbb{P}_2$.
(2) Linear: $F(p + rq)(x) = x(p(x) + rq(x)) = xp(x) + r(xq(x))$. $\ker(F) = \{p(x) \in \mathbb{P}_2 : xp(x) = 0\} = \{0\}$. $\operatorname{im}(F) = \{xp(x) : p(x) \in \mathbb{P}_2\} = \operatorname{sp}(x, x^2, x^3)$.
(3) Not linear: $F(rp)(x) = r^2 p(x)^2 \neq rp(x)^2$ in general. $\ker(F) = \{p(x) \in \mathbb{P}_2 : p(x)^2 = 0\} = \{0\}$. $\operatorname{im}(F) = \{p(x)^2 : p(x) \in \mathbb{P}_2\}$.
(4) Linear because taking derivatives is linear. $\ker(F) = \{g \in C^\infty([0,1]) : g' = 0\}$, the constant functions. $\operatorname{im}(F) = \{g' : g \in C^\infty([0,1])\} = C^\infty([0,1])$.
(5) Linear because taking integrals is linear. $\ker(F) = \{g \in C^\infty([0,1]) : \int_0^x g(t)dt = 0\} = \{0\}$. $\operatorname{im}(F) = \{\int_0^x g(t)dt : g \in C^\infty([0,1])\} = C^\infty([0,1])$. Both of these follow from the fundamental theorem of calculus.
(6) Not linear: $|g(x) + h(x)| \neq |g(x)| + |h(x)|$ in general. $\ker(F) = \{g \in C^\infty([0,1]) : |g| = 0\} = \{0\}$. $\operatorname{im}(F) = \{|g| : g \in C^\infty([0,1])\}$, the nonnegative functions in $C^\infty([0,1])$.
(7) Linear because its just evaluation at a point. $\ker(F) = \{g \in C^\infty([0,1]) : g(1-x) = 0\} = \{0\}$. $\operatorname{im}(F) = \{g(1-x) : g \in C^\infty([0,1])\} = C^\infty([0,1])$.
(8) Not linear: $e^{rg(x)} \neq re^{g(x)}$ in general. $\ker(F) = \{g \in C^\infty([0,1]) : e^{g(x)} = 0\} = \emptyset$. $\operatorname{im}(F) = \{e^{g(x)} : g \in C^\infty([0,1])\}$, the strictly positive functions in $C^\infty([0,1])$.
(9) Not linear: $\det(2I) = 4 \neq 2 = 2\det(I)$. $\ker(\det) = \{A \in M_{2\times 2} : \det(A) = 0\}$. $\operatorname{im}(\det) = \{\det(A) : A \in M_{2\times 2}\} = \mathbb{R}$.
(10) Linear: Letting the diagonal entries of A be a, b and those of B be c, d, have $\operatorname{tr}(A + rB) = (a+rc)+(b+rd) = \operatorname{tr}(A)+r\operatorname{tr}(B)$. $\ker(\operatorname{tr}) = \{A \in M_{2\times 2} : a_{11}+a_{22} = 0\}$. $\operatorname{im}(\operatorname{tr}) = \{a_{11} + a_{22} : A \in M_{2\times 2}\} = \mathbb{R}$.
(11) Linear because taking integrals is linear. $\ker(F) = \{g \in C^\infty([0,1]) : \int_0^x g(t)\cos(x-t)dt = 0\}$. $\operatorname{im}(F) = \{\int_0^x g(t)\cos(x-t)dt : g \in C^\infty([0,1])\}$.
(12) Not linear: $f(x+y) = x+y+1 \neq (x+1)+(y+1)$ in general. $\ker(f) = \{x \in \mathbb{R} : x+1 = 0\} = \{-1\}$. $\operatorname{im}(f) = \{x+1 : x \in \mathbb{R}\} = \mathbb{R}$.
(13) Linear: $f(\vec{v} + r\vec{w}) = 2\vec{v} + r(2\vec{w})$. $\ker(f) = \{\vec{v} \in \mathbb{R}^3 : 2\vec{v} = \vec{0}\} = \{\vec{0}\}$. $\operatorname{im}(f) = \{2\vec{v} : \vec{v} \in \mathbb{R}^3\} = \mathbb{R}^3$.
(14) Linear: $f([x,y]+r[u,v]) = [y+rv, x+ru]$. $\ker(f) = \{[x,y] \in \mathbb{R}^2 : [y,x] = \vec{0}\} = \{\vec{0}\}$. $\operatorname{im}(f) = \{[y,x] : [x,y] \in \mathbb{R}^2\} = \mathbb{R}^2$.
(15) Not linear: $r\sqrt{x^2+y^2} \neq \sqrt{r^2x^2+r^2y^2}$ in general. $\ker(f) = \{[x,y] \in \mathbb{R}^2 : \sqrt{x^2+y^2} = 0\} = \{\vec{0}\}$. $\operatorname{im}(f) = \{\sqrt{x^2+y^2} : [x,y] \in \mathbb{R}^2\} = [0,\infty)$.
(16) Linear: $f(\vec{v} + r\vec{w}) = \vec{0}_W = f(\vec{v}) + rf(\vec{w})$. $\ker(f) = \{\vec{v} \in V : \vec{0}_W = \vec{0}_W\} = V$. $\operatorname{im}(f) = \{\vec{0}_W : \vec{v} \in V\} = \{\vec{0}_W\}$.

(17) Linear: $f(\vec{v}+r\vec{w})=\vec{v}+r\vec{w}$. $\ker(f)=\{\vec{v}\in V:\vec{v}=\vec{0}\}=\{\vec{0}\}$. $\mathrm{im}(f)=\{\vec{v}:\vec{v}\in V\}=V$.

B: In class we said (or will say) a linear transformation respects the structure of a vector space, for instance it maps a subspace to a subspace, the zero vector to zero vector, and so on. In this question you investigate how a linear transformation treats a linear independent set and a spanning set. We use the following result

Lemma 1. *Let $T:V\to W$ be a linear transformation.*

(1) *T is one-to-one if and only if $\ker T=\{0_V\}$.*
(2) *T is onto if and only if $\mathrm{img}(T)=W$*

(1) (a) Consider $I=\{e^x,e^{2x},e^{3x}\}$ in $\mathcal{F}$. I is linearly independent (why?). Let $V=\mathrm{Span}(I)$. Let $T:V\to\mathcal{F}$ be a linear transformation, and suppose
$$T(e^x)=1,\quad T(e^{2x})=\cos^2 x,\quad T(e^{3x})=\sin^2 x$$
Write down a formula for T of an arbitrary element of V
 (b) Show that $T(I)$ is not linearly independent.
 (c) Prove that T is not one-to one.[2]
 (d) Show that $T(I)$ is not a spanning set for $\mathcal{F}$.
 (e) Prove that T is not onto.
(2) Let $T:V\to W$ be a linear transformation. Prove that $T(I)$ is a linearly independent subset of W for every linearly independent subset I of V if and only if T is one to one.[3]
(3) Let $T:V\to W$ be a linear transformation. Let S be a spanning set for V. Prove T is onto if and only of $T(S)$ is a spanning set for W.

Solution.

(1) (a) Yes, I is linearly independent. Solve system $c_1e^x+c_2e^{2x}+c_3e^{3x}=0$ at $x\in\{1,2,3\}$ or equivalent. Next, calculate $T(c_1e^x+c_2e^{2x}+c_3e^{3x})=c_1T(e^x)+c_2T(e^{2x})+c_3T(e^{3x})=c_1+c_2\cos^2(x)+c_3\sin^2(x)$.
 (b) Observe dependency $T(e^{2x})+T(e^{3x})=T(e^x)$.
 (c) Observe that $T(e^{2x}+e^{3x}-e^x)=T(e^{2x})+T(e^{3x})-T(e^x)=0\implies e^{2x}+e^{3x}-e^x\in\ker T$. Then apply Lemma 1(1).
 (d) Calculate $\mathrm{sp}(T(I))=\mathrm{sp}\{1,\cos^2(x)\}\subsetneq\mathcal{F}$.
 (e) Observe $\mathrm{im}(T)=\mathrm{sp}(T(I))=\mathrm{sp}\{1,\cos^2(x)\}\subsetneq\mathcal{F}$ and apply Lemma 1(2).
(2) Suppose T is one-to-one. Take image $T(I)$ of linearly independent set $I=\{v_j\}$ and observe $\sum_j c_jT(v_j)=T(\sum_j c_jv_j)=0$ implies $\sum_j c_jv_j=0$ since $\ker T=\{0_V\}$. Then, since $\{v_j\}$ is linearly independent, $c_j=0\,\forall j$ and so $\{T(v_j)\}$ is also linearly independent. On the other hand, suppose $T(I)$ is always linearly independent. Take basis $B=\{v_j\}$, let $v:=\sum_j c_jv_j\in\ker T$ and observe $0_W=T(v)=T(\sum_j c_jv_j)=$

[2] You can find a nonzero vector in the kernel of T

[3] For every linearly independent subset I is a key information in one of the two directions (which one?)

$\sum_j c_j T(v_j) \iff c_j = 0\,\forall j$ since $\{T(v_j)\}$ is linearly independent. Hence $v = 0$, $\ker T = \{0_V\}$ and T is one-to-one.

(3) Suppose T is onto. Let $S = \{v_j\}$ be a spanning set for V. Then $\forall w \in W, \exists v = \sum_j c_j v_j$ such that $T(v) = w$. In particular, $w = T(\sum_j c_j v_j) = \sum_j c_j T(v_j)$ and so $T(S)$ is a spanning set for W. On the other hand, suppose $T(S) = \{T(v_j)\}$ is a spanning set for W. Then $w = \sum_j c_j T(v_j) = T(\sum_j c_j v_j)$ for some c_j. In particular, $w \in \mathrm{im}(V)$ and T is onto.

C: Let X, Y and Z be sets. If $f : X \to Y$ and $g : Y \to Z$ are functions (so that the domain of g contains the image of f), then the *composition* of f with g is the function

$$g \circ f : X \to Z$$

defined by $(g \circ f)(x) = g(f(x))$.[4]

(1) Prove that if U, V, and W are vector spaces and if $T : U \to V$ and $S : V \to W$ are linear transformations, then the composite function $S \circ T$ is also a linear transformation from U to W.

(2) Consider the following linear transformations.

$$T_1 : P_3 \to P_2, \text{ and } T_2 : P_2 \to P_3 :$$

We have the following information about T_1 and T_2.

$$T_1(1) = 0, \quad T_1(1+x) = 1, \quad T_1(1+x+x^2) = 1+2x, \quad T_1(1+x+x^2+x^3) = 2+2x$$

$$T_2(x^2) = x^3, \quad T_2(x^2+x) = x^3+x^2 \quad T_2(x^2+x+1) = x^3+x^2$$

(a) Explicitly calculate the value of T_1 on an arbitrary element of P_3.
(b) Explicitly calculate the value of T_2 on an arbitrary element of P_2.
(c) Choose two different bases B and B' for P_3 that contains the vector $x^3 + x^2$, and do the following exercise with each basis. Split the responsibilities in your group and compare your answer at the end.
- Write the vectors in your basis in terms of the vectors whose T_1 is known.
- Find the value of T_1 on your basis.
- Write the image of your basis vectors under T_1 in terms of vectors whose T_2 is known.
- Find the value of $T_2 \circ T_1$ at every vector in your basis.
- Give an explicit formula for $T_2 \circ T_1$ for an arbitrary element of P_3.

Solution.

(1) For $\vec{v}, \vec{w} \in U$, and $r \in \mathbb{R}$, $(S \circ T)(\vec{v} + r\vec{w}) = S(T(\vec{v}) + rT(\vec{w})) = (S \circ T)(\vec{v}) + r(S \circ T)(\vec{w})$.

(2) (a) Note $T_1(x) = T_1((1+x) - 1) = T_1(1+x) - T(1) = 1$, $T_1(x^2) = T_1(1+x+x^2) - T_1(1+x) = 2x$, and $T_1(x^3) = T_1(1+x+x^2+x^3) - T_1(1+x+x^2) = 1$. Hence, $T(a + bx + cx^2 + dx^3) = a \cdot 0 + b \cdot 1 + c \cdot (2x) + d \cdot 1 = b + d + 2cx$.

(b) Note $T_2(x) = T_2(x^2+x) - T_2(x^2) = x^2$ and $T_2(1) = T_2(x^2+x+1) - T_2(x^2+x) = 0$. Hence, $T_2(a + bx + cx^2) = a \cdot 0 + b \cdot x^2 + c \cdot x^3 = bx^2 + cx^3$.

(c) Consider the basis $B = \{1, x, x^2, x^3 + x^2\}$. Have $1 = 1$, $x = (1+x) - 1$, $x^2 = (1+x+x^2) - (1+x)$, and $x^3 + x^2 = (1+x+x^2+x^3) - (1+x)$. So, for our basis, $T_1(1) = 0$, $T_1(x) = 1 - 0 = 1$, $T_1(x^2) = (1+2x) - 1 = 2x$, and $T_1(x^2+x^3) = (2+2x) - 1 = 2x+1$. Then decompose $1 = (x^2+x+1) - (x^2+x)$, $2x = 2(x^2+x) - 2x^2$, and $2x + 1 = 2(x^2+x) - 2x^2 + ((x^2+x+1) - (x^2+x)) = (x^2+x) - 2x^2 + (x^2+x+1)$. Then, $(T_2 \circ T_1)(1) = T_2(0) = 0$, $(T_2 \circ T_1)(x) = T_2(x^2+x+1) - T_2(x^2+x) = 0$, $(T_2 \circ T_1)(x^2) = 2T_2(x^2+x) - 2T_2(x^2) = 2x^2$, and $(T_2 \circ T_1)(x^3) = T_2(x^2+x) - 2T_2(x^2) + T_2(x^2+x+1) = (x^3+x^2) - 2(x^3) + (x^3+x^2) = 2x^2$. Finally, $(T_2 \circ T_1)(a + bx + cx^2 + dx^3) = a \cdot 0 + b \cdot 0 + c \cdot (2x^2) + d \cdot (2x^2) = 2(c+d)x^2$.

[4] Notice that we write composition of functions "backwards": $g \circ f$ means first apply f, then apply g. Sometimes, if it is obvious that f and g are functions and that we want to compose them, we can just write "gf" instead of $g \circ f$.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) A linear transformation form $\mathbb{R}^2$ to $\mathbb{R}^3$ with non-trivial kernel
(2) A surjective linear transformation from a 3 dimensional vector space other than $\mathbb{R}^3$ to a 2 dimensional vector space.
(3) A linear transformation $T : \mathcal{F} \to \mathcal{F}$ such that $T(F_5) = 2 \cdot F_1$, where F_c is the constant function c.
(4) A linear map $T : P_3 \to P_2$ such that send every linearly independent set in P_3 to a linearly independent set in P_2.
(5) An invertible linear transformation $T : P_3 \to P_4$.

Solution.

(1) There are many examples such as $f(x, y) = [0, 0, 0]$ with $\ker(f) = \mathbb{R}^2$.
(2) $f : X \to Y$ is surjective means for any $y \in Y, \exists x \in X$ such that $f(x) = y$, or equivalently, $Imf = Y$. There are many examples, let $f : p_2 \to \mathbb{R}^2, f(ax^2 + bx + c) = [b, c]$.
(3) No such map exists. Since $F_5 = 5F_1$, if T is a linear transformation, we must have $T(F_5) = 5T(F_1)$.
(4) No such map exists. Take the spanning set $\{T(1), T(x), T(x^2), T(x^3)\}$ for P_2. Observe that it is not a basis since $\dim(P_2) = 3 < 4$ and thus cannot be linearly independent.
(5) None exists, as an invertible map preserves dimension and $\dim(P_3) = 4$ while $\dim(P_4) = 5$.