W3 Q2

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0} \dfrac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

$= \lim_{h \to 0} \dfrac{f(ha, hb) - 0}{h}$

$= \lim_{h \to 0} \dfrac{1}{h} \cdot \dfrac{(ha)(hb)^2}{(ha)^2 + (hb)^2}$

$= \lim_{h \to 0} \dfrac{h^2 ab^2}{h^2(a^2+b^2)}$

$= \dfrac{ab^2}{a^2+b^2}$

$= ab^2$

b) $\lim_{\vec{h} \to \vec{0}} \dfrac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a}) \cdot \vec{h}}{|\vec{h}|}$

$\frac{\partial f}{\partial x}$, $\vec{u} = (1,0)$ $\quad (1)(0)^2 = 0$

$\frac{\partial f}{\partial y}$, $\vec{u} = (0,1)$ $\quad (0)(1)^2 = 0$

$= \lim_{\vec{h} \to \vec{0}} \dfrac{f(\vec{a}+\vec{h}) - 0 - (0,0) \cdot \vec{h}}{|\vec{h}|}$

$\nabla f(\vec{a}) = (0,0)$

$= \lim_{(x,y) \to (0,0)} \dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{xy^2}{x^2+y^2}$

$\vec{h} = (x,y)$

$= \lim_{(x,y) \to (0,0)} \dfrac{xy^2}{(x^2+y^2)^{3/2}}$

Let $y = 0$

$\dfrac{x(0)^2}{(x^2+0^2)^{3/2}} = \dfrac{0}{x^3} = 0$

Let $y = x$

$\dfrac{x(x^2)}{(x^2+x^2)^{3/2}} = \dfrac{x^3}{(2x^2)^{3/2}} = \dfrac{x^3}{2x^3} = \dfrac{1}{2}$ limits don't equal

$\therefore \lim_{(x,y) \to (0,0)} \dfrac{xy^2}{(x^2+y^2)^{3/2}}$ DNE

$\Rightarrow$ f is not diff at $(0,0)$