MATB24 TUTORIAL PROBLEMS 2SOLUTIONS

KEY WORDS: subspace, spanning set, linearly independent, basis
RELEVANT SECTIONS IN THE TEXTBOOK: Sec 3.2 FB or Sec 2.A and Sec 1.C SA

WARM-UP: Write down a complete definition or a complete mathematical characterization for the following terms.

Let V be an F-vector space

- A subspace of V
  A non-empty subset W of V is a *subspace* of V if W is itself a vector space with the vector addition and the scalar multiplication in V.
- A linear combination of $v_1, v_2, \cdots, v_k$ in V
  A linear combination of $v_1, v_2, \cdots, v_k$ in V is an expression of the form $c_1v_1 + c_2v_2 + \cdots + c_kv_k$ where $c_1, c_2, \cdots, c_k$ are in F.
- A dependency relation in a subset X of V
  Consider vectors $v_1, \cdots, v_k$ in $X \subset V$. A relation of the form
  $$c_1v_1 + \cdots + c_kv_k = 0, \text{ where } (c_1, \cdots, c_k) \neq (0, \cdots, 0)$$
  is called a dependency relation among v_i's or on X.
- A spanning set for a subspace W of V
  S is a spanning set of W if span(S) = W
- The span of a subset X of V
  Let $S \subseteq V$. The span of S is the set of all linear combinations of vectors in S.
  $$\mathrm{Span}(S) = \mathrm{Sp}(S) = \{c_1v_1 + \cdots + c_rv_r, \mid c_1, \cdots, c_r \in \mathbb{R},\ v_1, \cdots, v_r \in S, r \in \mathbb{N}\}$$
  If $S = \{v_1, \cdots v_r\}$ (that is if S is finite) we write $\mathrm{Span}(S) = \mathrm{Span}(v_1, \cdots v_r)$. If $\mathrm{Span}(S) = W$ for a subspace W of V we say S *spans* W or S *generates* W or S is a *spanning set* for W.
- A linearly independent subset of V
- A linearly dependent subset of V
  A subset X of V is called linearly dependent if there exists a dependency relation among vectors in X. We say X is linearly independent if it is not linearly dependent.
  A basis for a vector space V is a subset of V that
  (1) spans V.
  (2) is linearly independent.

A: Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Recall that $\mathcal{F}$ is a vector space, with operations $(f+g)(x) = f(x)+g(x)$ and $(kf)(x) = kf(x)$, for all $f, g \in \mathcal{F}$, $k \in \mathbb{R}$, and $x \in \mathbb{R}$. Prove or disprove the following:

(1) The set $\{\sin^2(x), \cos^2(x)\}$ is linearly independent.
(2) The set $\{\sin(x), \cos(2x), \sin(3x)\}$ is linearly independent.
(3) The set $\{1, e^x + e^{-x}, e^x - e^{-x}\}$ is linearly independent.
(4) $\mathrm{Span}(\sin^2(x), \cos^2(x))$ contains all the constant function.
(5) If V is the subspace of $\mathcal{F}$ spanned by $\{1, 2\sin^2(x), 3\cos^2(x)\}$, then $\dim(V) = 2$.

Solution. If you want to prove two functions f, g are linearly independent, you show that $af(x) + bg(x) = 0$ imply $a = 0, b = 0$. Be careful that here the zero on the right hand side is the zero function, not a real number, which means you are not solving specific x, a, b such that the equation holds, but solve a, b such that the equation holds for all x. You can substitute specific value of x to get some equations that a and b must satisfy.

(1) Suppose $a\sin^2(x)+b\cos^2(x) = 0$, for some scalars a and b. That means the equality holds for all $x \in \mathbb{R}$. In particular, the equality holds for $x = \pi/2$ and $x = 0$. Substituting these values, we have $a + 0 = 0$ and $0 + b = 0$. Therefore $\{\sin^2(x), \cos^2(x)\}$ is linearly independent.
(2) Linearly independent. Suppose $a\sin(x) + b\cos(2x) + c\sin(3x) = 0$ for some $a, b, c \in \mathbb{R}$. Setting $x = 0$ yields $b = 0$, so now $a\sin(x) + c\sin(3x) = 0$. Setting $x = \pi/3$ yields $a = 0$, and hence $c\sin(3x) = 0$ so that $c = 0$ (e.g. set $x = \pi/6$).
(3) Linearly independent. Suppose $a + b(e^x + e^{-x}) + c(e^x - e^{-x}) = 0$. Substituting the values $x = 1, 2, 3$ yields three equations $a + (e^i + e^{-i})b + (e^i - e^{-i})c = 0$ for $i = 1, 2, 3$. The coefficients of these three equations, in variables a, b, c, yields a 3×3 matrix which can be shown to be row equivalent to the identity. This shows that the only solution is $a = b = c = 0$.
(4) Note that $\sin^2(x) + \cos^2(x) = 1$. For any constant $c \in \mathbb{R}$, $c = c\sin^2(x) + c\cos^2(x)$, which is a linear combination of $\sin^2(x), \cos^2(x)$, thus the span of $\sin^2(x), \cos^2(x)$ contain all the constants
(5) Note that $(1/2)2\sin^2(x) + (1/3)3\cos^2(x) = \sin^2(x) + \cos^2(x) = 1$. That is 1 is linear combination of the other vectors in the set, and hence V is spanned by $\{2\sin^2(x), 3\cos^2(x)\}$. Now, suppose $a2\sin^2(x) + b3\cos^2(x) = 0$ for all $x \in \mathbb{R}$ for some $a, b \in \mathbb{R}$. In particular, the equality holds for $x = 0$ and $x = \pi/2$. Substituting these values for x gives us $b = 0$ and $a = 0$. So, $\{2\sin^2(x), 3\cos^2(x)\}$ is a linearly independent spanning set of V, that is a basis. Dimension of V is the number of vectors in a basis for V. Hence, $\dim(V) = 2$.

B: Let V be a real vector space and $\vec{v}_i$ and $\vec{w}_i$ be in V, $i = 1, \ldots, n$.

(1) Give a necessary and sufficient condition for $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$. [1]
(2) Prove that $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ if and only if $\vec{v}_1, \cdots, \vec{v}_n$ are in $\mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ and $\vec{w}_1, \cdots \vec{w}_n$ are in $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$. [2] Is this the same condition that you gave in 1? If not, is it equivalent?
(3) Prove that $\mathrm{Span}(\vec{v}_1, \vec{v}_2) = \mathrm{Span}(\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2)$. You can use the previous parts.
(4) Let $\vec{w} \in V$. Prove or disprove: $\vec{w} \in \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$ if and only if $\mathrm{Span}(\vec{w}, \vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$

Solution.

(1) $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ if and only if $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) \subseteq \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ and $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) \supseteq \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$. In other words if and only if every linear combination of v_i's is a linear combination of w_i's and every linear combination of w_i's is a linear combination of v_i's.
(2) Since $\vec{v}_i$ are elements of $\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$, and likewise for the $\vec{w}_i$, it is immediate that $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ implies $\vec{v}_1, \cdots, \vec{v}_n$ are in $\mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ and $\vec{w}_1, \cdots \vec{w}_n$ are in $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$. Consider the converse. An element of $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$ is of the form $r_1\vec{v}_1 + \cdots + r_n\vec{v}_n$. Since $\vec{v}_i \in \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$, therefore $\vec{v}_i = r_{i1}\vec{w}_1 + \cdots + r_{in}\vec{w}_n$. Substituting each $\vec{v}_i$ with this expression in $r_1\vec{v}_1 + \cdots + r_n\vec{v}_n$, expanding, and collecting terms, it is clear that the result is a linear combination of the $\vec{w}_i$. This shows that $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) \subseteq \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$. Showing $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) \supseteq \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$ is similar, and hence $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$.
(3) Using part two, all we need to show is that $\vec{v}_1, \vec{v}_2$ are in $\mathrm{Span}(\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2)$ and $\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2$ are in $\mathrm{Span}(\vec{v}_1, \vec{v}_2)$. Well. $\vec{v}_1$ is clearly in $\mathrm{Span}(\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2)$, and $\vec{v}_2 = 1/3(2\vec{v}_1 + 3\vec{v}_2) - 2/3\vec{v}_1$ and hence in $\mathrm{Span}(\vec{v}_1, 2\vec{v}_1 + 3\vec{v}_2)$. Conversely, both $\vec{v}_1$, and $2\vec{v}_1 + 3\vec{v}_2$ are linear combination of $\vec{v}_1$ and $\vec{v}_2$, and hence in $\mathrm{Span}(\vec{v}_1, \vec{v}_2)$.
(4) True. If $\vec{w} \in \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$, then it immediately follows from (2) that $\mathrm{Span}(\vec{w}, \vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$. Conversely, if $\mathrm{Span}(\vec{w}, \vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$, we have $\vec{w} \in \mathrm{Span}(\vec{w}, \vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n)$.

[1] Means fill in the blank for ..... if and only if $\mathrm{Span}(\vec{v}_1, \cdots, \vec{v}_n) = \mathrm{Span}(\vec{w}_1, \cdots \vec{w}_n)$.

[2] Remember how we proved two sets are equal in class. You need to show they are subsets of one another.

C: Let P denote the vector space of all polynomials (that is, functions of the form $f(x) = a_n x^n + \cdots + a_1 x + a_0$ where each $a_i \in \mathbb{R}$ and n is any non-negative integer).

(1) Give two spanning sets for P.
(2) What do you think it means for a vector space to be finitely generated? Write down the definition explicitly. Check your definition with your TA.
(3) Suppose V is a finitely generated subspace of P, and S is a finite spanning set for V. Let $m = \max\{\deg(f(x)) \mid f \in S\}$. Discuss with your group why m is a well defined integer. One way to think about this is, if you were to program a computer to find m what would your algorithm be, and does that algorithm end.
(4) Construct an element g in P of degree $m+1$. Prove that g is not in $\mathrm{Span}(S)$.
(5) Prove that P is not finitely generated.

Solution.

(1) $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3, \ldots\}$ and $\{1, x, x^2, x^3 \ldots\}$.
(2) A vector space V is finitely generated if there exists a finite subset S of V such that $\mathrm{Span}(S) = V$.
(3) "Well defined" means that according to the definition, there is no ambiguities in finding the elements and the elements always exists. In this question, the question defined m using $\max$ function, and the $\max$ of a infinite set can possibly be not exist. But here since S is finite, therefore m is a maximum of a finite set and hence maximum always exist.
(4) An example of g is $g(x) = x^{m+1}$. To see that g is not in $\mathrm{Span}(S)$, note that the largest degree polynomial in $\mathrm{Span}(S)$ is degree m: This is true for S, and $\mathrm{Span}(S)$ is obtained from S by taking linear combinations of elements of S, and linear combinations of polynomials has the degree given by those polynomials. So, $g \notin \mathrm{Span}(S)$ because its degree is bigger than m.
(5) If P were finitely generated, then by (3) there would exist some m such that P would not contain polynomials of degree greater than m, which is false.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) A finitely generated vector space.

(2) A vector space that is not finitely generated

(3) A linearly dependent subset of $\mathcal{F}$ and a dependency relation among its vectors

(4) A linearly independent subset of $\mathcal{F}$.

Solution.

(1) $\mathbb{R}^n$ and any vector space that "looks similar"(you will have a precise definition of what does looks similar mean) to $\mathbb{R}^n$ is finitely generated.
(2) The vector space consists of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. (Think about it why the function space cannot be finitely generated) or the vector space of all polynomials $\mathbb{P}$.
(3) Recall that $\mathcal{F}$ is the set of functions from $\mathbb{R}$ to $\mathbb{R}$. The subset $\{\sin^2(x), \cos^2(x), 1\}$ is a linear dependent set since $\sin^2(x) + \cos^2(x) = 1$
(4) $\{$1, ln x$\}$ is a linear independent subset of $\mathcal{F}$