MATB24 TUTORIAL PROBLEMS 4, WEEK OF Oct5-9

KEY WORDS: isomorphism, invertible linear transformation, change of coordinate matrix
RELEVANT SECTIONS IN THE TEXTBOOK:Sec 3.3, 3.4, 7.1 FB or 3C,3D SA

WARM-UP: As usual, write down a complete definition or a complete mathematical characterization for the following terms.

- An invertible linear transformation
- Give an equivalent condition for a linear transformation being invertible in terms of injectivity and surjectivity.
- Give an equivalent condition for a linear transformation T being invertible in terms of the Kernel and image of T.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Define the $\mathfrak{B}$-coordinates of a vector $\vec{v} \in W$.
- Let $\mathfrak{B}$ be a an ordered basis for a vector space W. Give an isomorphism between W and $\mathbb{R}^{\dim W}$

A: In class we said (or will say) a linear transformation respects the structure of a vector space, for instance it maps a subspace to a subspace, the zero vector to zero vector, and so on. In this question you investigate how a linear transformation treats a linear independent set and a spanning set. We use the following result

Lemma 0.1. *Let $T : V \to W$ be a linear transformation.*

(1) *T is one-to-one if and only if* $\ker T = \{0_V\}$.
(2) *T is onto if and only if* $\operatorname{img}(T) = W$

(1) (a) Consider $I = \{e^x, e^{2x}, e^{3x}\}$ in $\mathcal{F}$. I is linearly independent (why?). Let $V = \operatorname{Span}(I)$. Let $T : V \to \mathcal{F}$ be a linear transformation, and suppose

$$T(e^x) = 1, \quad T(e^{2x}) = \cos^2 x, \quad T(e^{3x}) = \sin^2 x$$

Write down a formula for T of an arbitrary element of V

(b) Show that $T(I)$ is not linearly independent.
(c) Prove that T is not one-to one.[1]
(d) Show that $T(I)$ is not a spanning set for $\mathcal{F}$.
(e) Prove that T is not onto.

(2) Let $T : V \to W$ be a linear transformation. Prove that $T(I)$ is a linearly independent subset of W for every linearly independent subset I of V if and only if T is one to one.[2]
(3) Let $T : V \to W$ be a linear transformation. Let S be a spanning set for V. Prove T is onto if and only of $T(S)$ is a spanning set for W.
(4) Prove that the finite-dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

[1] You can find a nonzero vector in the kernel of T

[2] For every linearly independent subset I is a key information in one of the two directions (which one?)

B:Let $M_{n\times n}$ be the vector space of $n \times n$ matrices.

(1) Let $P \in M_{n\times n}$. Define the function $T_P : M_{n\times n} \to M_{n\times n}$ by $T_P(A) = PA$ for all $A \in M_{n\times n}$. Is T_P always linear? If so, is T_P ever an isomorphism?
(2) Let P be an invertible $n \times n$ matrix. Prove that the function $A \mapsto PAP^{-1}$ from $M_{n\times n}$ to $M_{n\times n}$ is an isomorphism. (This transformation is called *conjugation by* P).

C: Let $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$, and consider the bases

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

of the vector space $M_{2\times 2}$ of 2×2 matrices.

(1) Find $[I_2]_{\mathcal{E}}$ and $[A]_{\mathcal{E}}$. (Recall, for example, $[I_2]_{\mathcal{E}}$ is the coordinate vector of I_2 relative to the ordered basis $\mathcal{E}$ for $M_{2\times 2}$.)
(2) Find $[I_2]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.
(3) Find a basis $\mathcal{C}$ of $M_{2\times 2}$ such that $[A]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.
(4) Find a matrix C such that $C[B]_{\mathcal{B}} = [B]_{\mathcal{C}}$ for all B in $M_{2\times 2}$.
(5) Find a matrix D such that $D[B]_{\mathcal{C}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.
(6) Find a matrix F such that $F[B]_{\mathcal{B}} = [B]_{\mathcal{E}}$ for all B in $M_{2\times 2}$.
(7) Draw a diagram relating the linear transformations corresponding to the matrices F, C and D.

COOL-OFF:

(1) Give three different isomorphism between P_n and $\mathbb{R}^{n+1}$.
(2) Give three different isomorphism between $M_{n\times m}(\mathbb{R})$ and $\mathbb{R}^{n\times m}$.
(3) Let V ad W be F- vector spaces and let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ be bases for V and W respectively. Construct three different isomorphism between V and W.