UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer and Mathematical Sciences
Midterm Test, July 2020

STAB52 Introduction to Probability
Duration: One hour and fifty minutes

Last Name:________________________ First Name: ________________________

Student number: ________________________

All your work must be presented clearly in order to get credit. Answer alone (even though correct) will only qualify for **ZERO** credit. For questions that require numerical answers, you should provide numerical answers to a reasonable degree of accuracy. Just explaining how do them or just coping down the method of solving them from the class notes/book will not qualify for credit. Please show your work in the space provided; you may use the back of the pages, if necessary, but you MUST remain organized. Show your work and answer in the space provided.

Note: Please note that academic integrity is fundamental to learning and scholarship. The work you submit should be your own. If I or the TAs feel suspicious of your work (e.g. if your work doesn't appear to be consistent with what we have discussed in class), I will not grade your exam. Instead, I will ask you to present your work in an individual quercus session and your grade will be determined based on your presentation.

The are 7 questions and 10 pages including this page. Please check to see you have all the pages.

Good Luck!

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	10	10	10	10	10	70
Score:								

1. A and B are two events in a sample space such that $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$.

   (a) (3 points) Find $P(A^c \cup B^c)$.

   Solution: $P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 0.2 = 0.8$ ■

   (b) (3 points) Find $P(A^c \cap B)$.

   Solution: $P(A^c \cap B) + P(A \cap B) = P(B)) \implies P(A^c \cup B) = P(B) - P(A \cap B) = P(B)) = 0.5 - 0.2 = 0.3$ ■

   (c) (4 points) Find $P(A^c \cap B^c)$.

   Solution: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.2 = 0.9$.
   $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.9 = 0.1$ ■

2. The continuous random variable X has p.d.f. give by

$$f_X(x) = \begin{cases} cx^2e^{-4x^3}, & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 points) Find the value of the constant c.

Solution: $\int_0^\infty cx^2e^{-4x^3}dx = c\int_0^\infty \left[-\frac{e^{-4x^3}}{12}\right]' dx = c\left[-\frac{e^{-4x^3}}{12}\right]_0^\infty = c\frac{e^0}{12}$ and $c\frac{e^0}{12} = 1 \implies c = 12$ ■

(b) (3 points) Calculate the probability $P(0.5 < X \leq 2)$.

Solution: $P(0.5 < X \leq 2) = \int_{0.5}^2 f_X(x)dx = \int_{0.5}^2 cx^2e^{-4x^3}dx = c\left[-\frac{e^{-4x^3}}{12}\right]_{0.5}^2 = c\left[\frac{e^{-4\times 0.5^3}}{12} - \frac{e^{-4\times 2^3}}{12}\right] = c\left[\frac{e^{-0.5}}{12} - \frac{e^{-32}}{12}\right]$ ■.

TA: For this part give full credit for $\left[\frac{e^{-0.5}}{12} - \frac{e^{-32}}{12}\right]$

(c) (3 points) Find the value x_0 such that $F_X(x_0) = 0.5$. (F_X is the c.d.f. of X)

Solution: $F(x_0) = \int_0^{x0} cx^2e^{-4x^3}dxc\left[-\frac{e^{-4x^3}}{12}\right]_0^{x0} = c\times\frac{1-e^{-4x_0^3}}{12} = 1 - e^{-4x_0^3}$ and $F(x_0) = 0.5 \implies 1 - e^{-4x_0^3} = 0.5 \implies x_0 = \left(\frac{\ln(2)}{4}\right)^{1/3}$ ■

Question 2 continues on the next page. . .

TA: If the value of c in part (a) is incorrect, the points should be deducted in part (a) but in this part, the answer must be assesed assuming that as the correct value of c.

3. A, B and C are three events defined in some sample space. Assume $P(A) = 0.3, P(B|A) = 0.75, P(B|A^c)) = 0.20, P(C|A \cap B) = 0.20, P(C|A^c \cap B) = 0.15, P(C|A \cap B^c) = 0.80$, and $P(C|A^c \cap B^c) = 0.90$.

   (a) (3 points) Find $P(A \cap B \cap C)$.

   Solution: $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B) = 0.3 \times 0.75 \times 0.2 = 0.045$ (General multiplication rule) ■

   (b) (3 points) Find $P(B^c \cap C)$.

   Solution: $P(B^c \cap C) = P(A \cap B^c \cap C) + P(A^c \cap B^c \cap C)$ (Law of total probability)
   $= P(A)P(B^c|A)P(C|A \cap B^c) + P(A^c)P(B^c|A^c)P(C|A^c \cap B^c)$ (Multiplication rule)
   $= 0.3 \times (1 - 0.75) \times 0.80 + (1 - 0.3) \times (1 - 0.2) \times 0.90 = 0.564$ ■

   (c) (4 points) Find $P(C)$.

   Solution: Again using the law of total probability and mulstiplication rule,
   $P(C) = P(A \cap B \cap C) + P(A^c \cap B \cap C) + P(A \cap B^c \cap C) + P(A^c \cap B^c \cap C)$
   $= P(A)P(B|A)P(C|A \cap B) + P(A^c)P(B|A^c)P(C|A^c \cap B) + P(A)P(B^c|A)P(C|A \cap B^c) + P(A^c)P(B^c|A^c)P(C|A^c \cap B^c) = 0.3 \times 0.75 \times 0.2 + (1 - 0.3) \times 0.20 \times 0.15 + 0.3 \times (1 - 0.75) \times 0.8 + (1 - 0.3) \times (1 - 0.20) \times 0.90 = 0.63$ ■

4. A box contains 4 white balls and 6 black balls.

   (a) Five balls are drawn, one by one with replacement (i.e. you put the ball back in the box before you draw the next ball).

      i. (2 points) Let X be the number of white balls in the five balls selected. Write down the probability mass function of X.

      Solution: Note that for sampling with replacement X has a Binomial ($n = 5$, $p = 4/10 = 0.4$) distribution and so $p_X(x) = \binom{5}{x} \times 0.4^x \times 0.6^{5-x}$ for $x = 0, 1, 2, 3, 4, 5$ and zero otherwise ■

      ii. (4 points) Find the probability that there will be *at least* one (i.e. one or more) white ball among the five balls drawn.

      Solution: $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{5}{0} \times 0.4^0 \times 0.6^{5-0} = 0.92224$ ■

   (b) (4 points) What is the probability that there will be *at least* one white ball among the five balls drawn if the five balls were drawn without replacement.

   Solution: Letting X be the number of white balls, we again need $P(X \geq 1) = 1 - P(X = 0)$ and $X = 0$ means all balls selected are black and so $P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{6}{5}}{\binom{10}{5}}$ ■

5. Five people, designated as A, B, C, D, E, are arranged in a line. Assuming that each possible order is equally likely, what is the probability that

   (a) (6 points) there is exactly one person between A and B?

   Solution: Ex 44 p53 Sheldon Ross, First Course in Probability
   If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $(2\times3\times3!)/5! = 3/10$.

   (b) (4 points) there are exactly two people between A and B?

   Solution: $(2 \times 2 \times 3!)/5! = 1/5$

6. The two parts (a and b) of this question are not exactly related but there are some significant similarities and so I am stating them as two of the same question.

   (a) (4 points) The random variable X has p.d.f

$$f(x) = \begin{cases} kx^6e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Find the value of k that makes this a p.d.f.

   Solution: Solution:
   This is a gamma distribution with $\alpha = 7$ and $\lambda = 2$ and so $k = \frac{\lambda^\alpha}{\Gamma(\alpha)} = \frac{2^7}{\Gamma(7)} = \frac{2^7}{(7-1)!} = \frac{128}{720} = \frac{8}{45}$ ■

   (b) (6 points) The random variable X has p.d.f

$$f(x) = \begin{cases} kx^{17}e^{-x^3} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Find the value of k that makes this a p.d.f.
   Hint: For the integral involved, a suitable substitution will be helpful.

   Solution: Solution:
   $\int_0^\infty f(x)dx = 1 \implies k\int_0^\infty x^{17}e^{-x^3}dx = 1.$
   Substitute $t = x^3$, then $dt = 3x^2dx$ and $x = t^{\frac{1}{3}}$ and so
   $\int_0^\infty x^{17}e^{-x^3}dx = \int_0^\infty t^{\frac{17}{3}}e^{-t}\frac{1}{3}t^{\frac{-2}{3}}dt = \frac{1}{3}\int_0^\infty t^5e^{-t}dt = \frac{1}{3}\int_0^\infty t^{6-1}e^{-t}dt = \frac{1}{3}\Gamma(6) = \frac{1}{3} \times 5! = \frac{1}{3} \times 120 =$ and so $k = \frac{3}{120} = \frac{1}{40}$ ■

7. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be a sample space of equally likely outcomes, i.e. $P(\{s\}) = \frac{1}{10}, \forall s \in S$. Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{2, 3, 5, 6\}$ and I_A, I_B, and I_C be their associated indicator functions respectively. Calculate the following probabilities.
Hint: First express each event in terms of the three original events, and their unions, intersections and complements etc. E.g. $\{I_A.I_B = 1\} = A \cap B$.

(a) (3 points) $P(\{I_A + I_B + I_C = 0\})$

Solution: $P(\{I_A + I_B + I_C = 0\}) = P(\{I_A = 0\} \cap \{I_B = 0\} \cap \{I_A = 0\}) = P(A^c \cap B^c \cap C^c) = P(\{(A \cup B \cup C)^c\}) = 1 - P(\{A \cup B \cup C\}) = P(\{1, 2, 3.4.5, 6\}) = 1 - 0.6 = 0.4$ ■

(b) (4 points) $P(\{I_A + I_B + I_C = 1\})$

Solution: $P(\{I_A + I_B + I_C = 1\}) = P(\{I_A = 1\} \cap \{I_B = 0\} \cap \{I_C = 0\} \cup \{I_A = 0\} \cap \{I_B = 1\} \cap \{I_C = 0\} \cup \{I_A = 0\} \cap \{I_B = 0\} \cap \{I_C = 1\}) = P((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)) = P(\{1, 4, 6\}) = 0.3$ ■

(c) (3 points) $P(\{I_A.I_B.I_C = 0\})$

Solution: $P(\{I_A.I_B.I_C = 0\}) = P(\{I_A.I_B.I_C = 1\}^c) = P((A \cap B \cap C)^c) = 1 - P(A \cap B \cap C) = 1 - P(\{3\}) = 1 - 0.1 = 0.9$ ■

Question 7 continues on the next page…

END OF TEST