MATB24 – Midterm I –Summer 2020

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Time: 110 mins.

1. You may use any statement that proved in class, tutorial and graded homework unless you are specifically asked to prove that statement. Should you need to use a result proves in class, tutorials or homework, you should clearly state it.
2. Remember to show all your work.
3. No calculators, notes, or other outside assistance allowed.

1. Confirm, complete or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part and to complete clearly circle what you add.

   (a) (2 points) Let V be an F vector space and $S \subseteq V$. The *span* of S is

   $$\{a_1 s_1 + a_2 s_2 + \cdots + a_n s_n | s_1, \cdots, s_n \in S, a_1, \cdots, a_n \in F\}$$

   .

   (b) (2 points) An *isomorphism* between two vector spaces V and W is an invertible map $T : V \to W$.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counter example if you think it is false). Please include the statement in your answer.

   (a) (2 points) Let $V = \mathbb{R}$. For $u, v \in V$ and $a \in \mathbb{R}$ define vector addition by $u \boxplus v := u + v + 2$ and scalar multiplication by $a \boxdot u := au + 2$. $(V, \boxplus, \boxdot)$ is a vector space over the scalar field $\mathbb{R}$.

   (b) (2 points) Let W be the set of all polynomials of the form $p(t) = at^2$, where a is in $\mathbb{R}$. Then W is a subspace of the set P_2 of all polynomials with degree less than or equal to 2 with coefficients in $\mathbb{R}$.

   (c) (2 points) The space of all 3×3 lower triangular matrices is isomorphic to $\mathbb{R}^6$.

   (d) (2 points) Every subspace of an infinity generated vector space is infinity generated.

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist. Include the description in your answer.
   (a) (2 points) A 2-dimensional subspace of $\mathcal{C}^1$, the vector space of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.
   (b) (2 points) An invertible linear transformation $T : V \to W$ and a basis of B of V such that the image of B does not span W.
   (c) (2 points) A nonzero vector in a vector space V that is it's own inverse.
   (d) (2 points) Two different ordered bases A and B for P_3 and a change of basis matrix C that changes A-coordinate of a vector $\vec{v} \in P_3$ into B-coordinates of $\vec{v}$.
   (e) (2 points) An isomorphism between P_2 and $V = \{A \in M_{2\times 2}(\mathbb{R}) | \operatorname{tr}(A) = 0\}$. [1].
   (f) (2 points) Let V be a finite dimensional vector space. A linear transformation $T : V \to V$ whose matrix representation with respect to all bases of V is the same.

[1] tr stands for trace. Trace of a matrix is the sum ot its diagonal entries

4. Let V be the subspace of $\mathcal{F}$ spanned by $\mathcal{B} = \{f_1(x) = e^x, f_2(x) = x^2e^x\}$. Recall that given $f_1, \cdots, f_k$ in $\mathcal{C}^{k-1}_{(0,1)}$, $W(f_1, \cdots, f_k)(x) := \det \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ f_1''(x) & f_2''(x) & \cdots & f_k''(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{k-1}(x) & f_2^{k-1}(x) & \cdots & f_k^{k-1}(x) \end{bmatrix}$.

   (a) (4 points) Show that there exists an $x \in \mathbb{R}$ such that $W(f_1, f_2)(x) \neq 0$.

   (b) (3 points) Prove that $\mathcal{B}$ is a basis for V.

   (c) (4 points) Let $T : V \to V$ be a linear transformation such that $T(e^x) = x^2e^x$ and $T(x^2e^x) = (x^2 + 1)e^x$. Find the matrix representation $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

   (d) (3 points) Prove that T is an isomorphism.

5. Let $S = \{A \in M_2(\mathbb{R}) | A = A^T\}$. Let $\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$, $\mathcal{A} = \left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

(a) (3 points) Prove that S is a subspace of $M_2(\mathbb{R})$

(b) (3 points) $\mathcal{B}$ is an ordered basis for S (you don't need to prove this). Prove that $\mathcal{A}$ is an ordered basis for S.

(c) (3 points) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that $[M]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find a, b and c.

(d) (3 points) Let C be a matrix such that $C[M]_{\mathcal{A}} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$. Find C^{-1}.

6. A MATB24 student proves the following statement.

   Let $V = \text{Span}(-1-x, -1+x^2)$ and $W = \text{Span}(1+x+x^2)$. Then $P_2 = V \oplus W$.

   Proof. Step 1 It is enough to show that every polynomial $p \in P_2$ can be uniquely written as $v + w$ where $v \in V$ and $w \in W$.

   Step 2 Note that $\{-1-x, -1+x^2\}$ is a basis for V and $\{1+x+x^2\}$ a basis of W and $B = \{v_1 = -1-x, v_2 = -1+x^2, w_1 = 1+x+x^2\}$ is a basis P_2.

   Step 3 Because B is a basis for P_2, for every $p \in P_2$, $p = a_1v_1 + a_2v_2 + a_3w_1$ uniquely, for $a_i \in \mathbb{R}$.

   Step 4 Note that $a_1v_1 + a_2v_2 \in V$ and $a_3w_1 \in W$ hence $p = v + w$ where $v = a_1v_1 + a_2v_2 \in V$ and $w = a_3w_1 \in W$ in a unique way. We are done. $\square$

   (a) (3 points) Either confirm the student's proof or find at least one flaw in their proof.

   (b) (3 points) Justify step one by connecting what the student proves with the definition of a direct sum. You can refer to any statement proved in our course material.

   (c) (4 points) Justify step 4 by explicitly proving that the decomposition $p = v + w$ is unique.

Please attach a signed copy of the statement below in your handwriting. Your signature under the statement is required for your exam to be graded.

"I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own."