1. Confirm, complete, or correct the following definitions of the italicized term. Copy the given definition in your answer sheet. To confirm clearly write "confirmed", to correct clearly cross out the incorrect part, and to complete clearly circle what you add.
   (a) (3 points) A matrix A with real entries is *orthogonal* if $A^T A = I_n$, with I_n the $(n \times n)$ identity matrix.
   (b) (3 points) A linear transformation $T\colon V \to V$ is *diagonalizable* if $[T]_{\mathcal{B}}$ is diagonal, where $\mathcal{B}$ is a basis for V.

2. State whether each statement is true or false and provide a short justification for your claim (a short proof if you think the statement is true or a counterexample if you think it is false).
   (a) (3 points) Suppose $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of a transformation $V \xrightarrow{T} V$ with respect to some basis $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$. Then $\vec{b}_1$ is the only eigenvector in the basis $\mathcal{B}$.
   (b) (3 points) If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for vectors $\vec{x}, \vec{y}, \vec{z}$ in an inner product space, then $\vec{y} - \vec{z}$ is orthogonal to $\vec{x}$.
   (c) (3 points) If A is a (3×4) matrix with real entries, then the matrix $A^T A$ is similar to a diagonal matrix with three or less non-zero entries.
   (d) (3 points) The matrices $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are orthonormal in the inner product $\langle A, B \rangle = \mathrm{trace}(A^T B)$ on the vector space $M_{2\times 2}(\mathbb{R})$ of (2×2)-matrices with real entries.
   (e) (3 points) Every unitarily diagonalizable $(n \times n)$-matrix with complex entries is Hermitian.

3. In each part, give an **explicit** example of the mathematical object described or explain why such an object doesn't exist.
   (a) (2 points) A linear map $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ which is an isometry (with respect to the dot product) and a basis $\mathcal{B}$ for $\mathbb{C}^2$ such that $[T]_{\mathcal{B}}$ is not a unitary matrix.
   (b) (2 points) A possible Jordan canonical form for a matrix with distinct eigenvalues μ and γ with algebraic multiplicities 2 and 3, geometric multiplicities 2 and 1 respectively.
   (c) (2 points) A linear map $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ for which $\ker(T) = \mathrm{im}(T)$.
   (d) (2 points) An orthogonal matrix with at least one real eigenvalue.

4. Let $\mathcal{P}_5(\mathbb{R})$ be the vector space of polynomials with real coefficients in a variable x of degree ≤ 5.
   (a) (2 points) What is the dimension of $\mathcal{P}_5(\mathbb{R})$?
   (b) (4 points) Let $\mathcal{B} = \{x^5, x^4, x^3, x^2, x, 1\}$ be the standard basis of $\mathcal{P}_5(\mathbb{R})$. For the linear map

$$\begin{aligned} T\colon \mathcal{P}_5(\mathbb{R}) &\longrightarrow \mathcal{P}_5(\mathbb{R}) \\ p(x) &\longmapsto \frac{dp(x)}{dx} + p(x) \end{aligned}$$

   give the matrix $[T]_{\mathcal{B}}$.
   (c) (4 points) Find all the eigenvalues of T and their geometric multiplicities.
   (d) (5 points) Prove that T is *not* diagonalizable.

5. Suppose that A is an $(n \times n)$-matrix with real entries.
   (a) (6 points) Prove that $\ker(A^T A) = \ker(A)$.
   (b) (5 points) Recall that the *rank* of a matrix is the dimension of its image. Prove that $\mathrm{rank}(A^T A) = \mathrm{rank}(A)$.

6. Let A be a symmetric $(n \times n)$-matrix with real entries. We say A is *strictly positive* if $\vec{v}^T A \vec{v} > 0$ for all non-zero $\vec{v} \in \mathbb{R}^n$.
   (a) (7 points) Prove A is strictly positive if and only if all eigenvalues of A are positive.
   (b) (8 points) Prove that A is strictly positive if and only if there exists an invertible $(n \times n)$-matrix P such that $A = P^T P$ (Hint: use the spectral theorem).

Please attach a signed copy of the statement below in your handwriting. Your signature under the statement is required for your exam to be graded.

"I affirm that I did not give or receive any unauthorized help on this exam and that all submitted work is my own."