MATB24 TUTORIAL PROBLEMS 3

KEY WORDS: basis, dimension, linear transformation, image, kernel, onto, one-to-one
RELEVANT SECTIONS IN THE TEXTBOOK: Sec 3.2, 3.4 FB or Sec 3.A, 3.B, 3.D SA

WARM-UP: Write down a complete definition or a complete mathematical characterization for the following terms.

Let V and W be a real vector spaces

- A basis for a subspace W of V
- Dimension of V
- A linear transformation $T : V \to W$
- Image of a subset of V under a linear transformation T
- Inverse image (preimage) of an element under a linear transformation T
- Inverse image (preimage) of a set under a linear transformation T
- Kernel and image of a linear transformation T.

$$\operatorname{img}(T) = \{T(v) \mid v \in V\}, \quad \ker(T) = \{v \in V \mid T(v) = 0_W\}.$$

- An onto or surjective linear transformation
- A 1-1 or injective linear transformation

A: Let V be a vector space with basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(1) Show that $\operatorname{Span}(\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3) = V$
(2) Show that $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ is linearly independent. Deduce that it is a basis for V.
(3) Show that $\operatorname{Span}(\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_1 - \vec{v}_3) \subsetneq V$
(4) Prove or disprove: $\{x, x + x^2, 1 + x + x^2\}$ is a basis for P_2.
(5) Prove or disprove: $\{e_1 + e_2, e_2 + e_3, e_1 - e_3\}$ is a basis for $\mathbb{R}^3$.

B: For each of the functions between vector spaces given below, determine whether or not the function is linear. For each function write down the definition of the kernel and image. When possible, give more explicit descriptions of the image and the kernel.

(1) $F : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $F(p)(x) = x + p(x)$.
(2) $F : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $F(p)(x) = xp(x)$.
(3) $F : \mathbb{P}_2 \to \mathbb{P}_4$ defined by $F(p)(x) = p(x)^2$.
(4) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g) = \frac{d}{dx}(g)$.
(5) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = \displaystyle\int_0^x g(t)\, dt$.[1]
(6) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = |g(x)|$.
(7) $F : C^\infty([0, 1]) \to C^\infty([0, 1])$ defined by $F(g)(x) = g(1 - x)$.

[1] $C^\infty(D)$ or C^∞_D is the set of smooth functions (having derivatives of all orders) with the domain D.

(8) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g)(x) = e^{g(x)}$.

(9) $\det : M_{2\times 2} \to \mathbb{R}$ defined by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

(10) $\operatorname{tr} : M_{2\times 2} \to \mathbb{R}$ defined by $\operatorname{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$.

(11) $F : C^\infty([0,1]) \to C^\infty([0,1])$ defined by $F(g)(x) = \int_0^x g(t)\cos(x-t)dt$.

(12) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$.

(13) $f : \mathbb{R}^3 \to \mathbb{R}^3$ is *dilation by* 2, defined by $f(\vec{v}) = 2\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

(14) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the line $y = x$, defined by $f([x,y]) = [y,x]$ for all $[x,y] \in \mathbb{R}^2$.

(15) $f : \mathbb{R}^2 \to \mathbb{R}$ assigns to every point in the plane its distance from the origin, so that $f([x,y]) = \sqrt{x^2+y^2}$ for all $[x,y] \in \mathbb{R}^2$.

(16) $f : V \to W$ is the *zero transformation* defined by $f(\vec{v}) = \vec{0}_W$ for all $\vec{v} \in V$ (V and W are vector spaces).

(17) $f : V \to V$ is the identity transformation defined by $f(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$ (V is a vector space).

C: Let X, Y and Z be sets. If $f : X \to Y$ and $g : Y \to Z$ are functions (so that the domain of g contains the image of f), then the *composition* of f with g is the function

$$g \circ f : X \to Z$$

defined by $(g \circ f)(x) = g(f(x))$.[2]

(1) Prove that if U, V, and W are vector spaces and if $T : U \to V$ and $S : V \to W$ are linear transformations, then the composite function $S \circ T$ is also a linear transformation from U to W.

(2) Consider the following linear transformations.

$$T_1 : P_3 \to P_2, \text{ and } T_2 : P_2 \to P_3 :$$

We have the following information about T_1 and T_2.

$$T_1(1) = 0, \quad T_1(1+x) = 1, \quad T_1(1+x+x^2) = 1+2x, \quad T_1(1+x+x^2+x^3) = 2+2x$$

$$T_2(x^2) = x^3, \quad T_2(x^2+x) = x^3+x^2 \quad T_2(x^2+x+1) = x^3+x^2$$

(a) Explicitly calculate the value of T_1 on an arbitrary element of P_3.

(b) Explicitly calculate the value of T_2 on an arbitrary element of P_2.

(c) Choose two different bases B and B' for P_3 that contains the vector $x^3 + x^2$, and do the following exercise with each basis. Split the responsibilities in your group and compare your answer at the end.

- Write the vectors in your basis in terms of the vectors whose T_1 is known.
- Find the value of T_1 on your basis.
- Write the image of your basis vectors under T_1 in terms of vectors whose T_2 is known.
- Find the value of $T_2 \circ T_1$ at every vector in your basis.

[2]Notice that we write composition of functions "backwards": $g \circ f$ means first apply f, then apply g. Sometimes, if it is obvious that f and g are functions and that we want to compose them, we can just write "gf" instead of $g \circ f$.

- Give an explicit formula for $T_2 \circ T_1$ for an arbitrary element of P_3.

COOL-OFF: Give an example of the described object or explain why such an example does not exists.

(1) Two different basis for P_n

(2) A 3-dimensional subspace of $\mathcal{F}$.
(3) A linear transformation form $\mathbb{R}^2$ to $\mathbb{R}^3$ with non-trivial kernel
(4) A surjective linear transformation from a 3 dimensional vector space other than $\mathbb{R}^3$ to a 2 dimensional vector space.
(5) A linear transformation $T : \mathcal{F} \to \mathcal{F}$ such that $T(F_5) = 2 \cdot F_1$, where F_c is the constant function c.
(6) A linear map $T : P_3 \to P_2$ such that send every linearly independent set in P_3 to a linearly independent set in P_2.
(7) An invertible linear transformation $T : P_3 \to P_4$.