MATB24 Graded HW 1

Problem 1

(3) The empty fails to provide a additie element ($\vec{0}$)
$\therefore$ it cannot be a vector space

(4) We know $\forall x \in \mathbb{R}$, $x + \infty = \infty$ by def. of the set $\mathbb{R} \cup \{-\infty, \infty\}$
We also know $\mathbb{R}$ has a unique additive identity 0 by the def. of a field

Because $x + \infty = \infty$, $x = 0$ by def. of a additive identity
But $x \in \mathbb{R}$ is arbitiary, contradicting the uniqueness of the property.

Problem 2

(1) Suppose $u, v \in U \cup W$, $u \notin W$, $v \notin U$

Suppose to the contrary that $U \cup W \underset{ss}{\subseteq} V$

Bc. $U \cup W$ is a Subspace, $u + v \in U \cup W$ which means $u + v \in U$ or $u + v \in W$

Case 1: $u + v \in U$

Since U is a subspace, $(u+v) - u \in U$ by closure of addition

$$(u+v) - u = u + v - u$$
$$= v$$

But $v \notin U$, contradicting our supposition

Case 2: $u + v \in W$

Since W is a Subspace, $(u+v) - v \in V$ by closure of addition

$$(u+v) - v = u + v - v$$
$$= u$$

But $u \notin W$, contradicting our supposition

$\therefore$ $U \cup W \underset{ss}{\nsubseteq} V$. by contradiction

(2) Let $V=\mathbb{R}^2$, $U=\{(x,0) \mid x\in\mathbb{R}\}$, $W=\{(0,y) \mid y\in\mathbb{R}\}$

Clearly $U\nsubseteq W$ and $W\nsubseteq U$, as the only element they can share is $(0,0)$

Choose $u=(1,0)\in U$, $v=(0,1)\in W$
This means $u,v\in U\cup W$
$u+v=(1,0)+(0,1)$
$=(1,1)$ by vector addition
$(1,1)\notin U$ and $(1,1)\notin W$
$\therefore u+v\notin U\cup W$
$\therefore U\cup W$ is not a subspace of V

(3) Prove $U\cup W \underset{ss}{\subseteq} V \Leftrightarrow U\subseteq W$ or $W\subseteq U$

$U\cup W \underset{ss}{\subseteq} V \Rightarrow U\subseteq W$ or $W\subseteq U$ was proved by the contrapositive in part (1)

Prove $U\subseteq W$ or $W\subseteq U \Rightarrow U\cup W \underset{ss}{\subseteq} V$

Case 1: Suppose $U\subseteq W$ this means $U\cup W=W$
We know W is a subspace,
$\therefore U\cup W$ is a subspace

Case 2: Suppose $W\subseteq U$ this means $U\cup W=U$
We know U is a subspace,
$\therefore U\cup W$ is a subspace

(4) Let $V=\mathbb{P}(\mathbb{R})$ (Set of all polynomial functions with real entries)
$U=\mathbb{P}_2(\mathbb{R})$, $W=\mathbb{P}_1(\mathbb{R})$

Prove $U\cup W \underset{ss}{\subseteq} V \Leftrightarrow U\subseteq W$ or $W\subseteq U$

<u>$U\cup W \underset{ss}{\subseteq} V \Rightarrow U\subseteq W$ or $W\subseteq U$</u>

Suppose $U\cup W \underset{ss}{\subseteq} V$
$U=\{a_0+a_1x+a_2x^2 \mid a_i\in\mathbb{R}\}$
$W=\{a_0+a_1x \mid a_i\in\mathbb{R}\}$
Clearly, W is a subset of U

<u>$U\subseteq W$ or $W\subseteq U \Rightarrow U\cup W \underset{ss}{\subseteq} V$</u>

Suppose $U\subseteq W$ or $W\subseteq U$
You can see that $W\subseteq U \Rightarrow U\cup W=U$
We know U is a subspace, $\therefore U\cup W \underset{ss}{\subseteq} V$

Problem 3

$a \oplus b = (a+b) \bmod 3$ $\quad a \odot b = (a \cdot b) \bmod 3$

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\odot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Let $a, b, c \in \mathbb{F}_3$

Associativity of Addition

Prove $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$(a \oplus b) \oplus c = ((a+b) \bmod 3 + c) \bmod 3$
$= ((a+b)+c) \bmod 3$ def. of modulo addition
$= (a+(b+c)) \bmod 3$ associativity of addition
$= (a + (b+c) \bmod 3) \bmod 3$
$= a \oplus (b \oplus c)$

Commutivity of Addition

Prove $a \oplus b = b \oplus a$

$a \oplus b = (a+b) \bmod 3$
$= (b+a) \bmod 3$ commutivity of addition
$= b \oplus a$

Additive Identity

Prove $a \oplus 0 = a = 0 \oplus a$

$a \oplus 0 = (a+0) \bmod 3$
$= (a) \bmod 3$
$= a$

$a \oplus 0 = 0 \oplus a$ by commutivity property

Additive Inverse

Prove $\exists a' \in \mathbb{F}_3$ st $a' \oplus a = 0 = a \oplus a'$

$a' \oplus a = a \oplus a'$ by commutivity property

Case 1: $a = 0$

Let $a' = 0$

$a' \oplus a = (a' + a) \bmod 3$
$= (0 + 0) \bmod 3$
$= 0$

Case 2: $a \neq 0$

Let $a' = 3 - a$

$a' \oplus a = ((3-a)+a) \bmod 3$
$= (3) \bmod 3$
$= 0$

Associativity of Multiplication

Prove $(a \odot b) \odot c = a \odot (b \odot c)$

$$
\begin{aligned}
(a \odot b) \odot c &= ((ab) \bmod 3 \cdot c) \bmod 3 \\
&= ((ab)c) \bmod 3 \\
&= (a(bc)) \bmod 3 \\
&= (a(bc) \bmod 3) \bmod 3 \\
&= a \odot (b \odot c)
\end{aligned}
$$

Commutivity of Multiplication

Prove $a \odot b = b \odot a$

$$
\begin{aligned}
a \odot b &= (ab) \bmod 3 \\
&= (ba) \bmod 3 \\
&= b \odot a
\end{aligned}
$$

Multiplicative Identity

Prove $a \odot 1 = a = 1 \odot a$

$$
\begin{aligned}
a \odot 1 &= (a(1)) \bmod 3 \\
&= (a) \bmod 3 \\
&= a
\end{aligned}
$$

$a \odot 1 = 1 \odot a$ by commutivity property

Multiplicative Inverse

Prove $\exists a' \in \mathbb{F}_3$ st $a \odot a' = 1 = a' \odot a$ (except $a = 0$)

From the table, for $a = 1$, choose $a' = 1$
for $a = 2$, choose $a' = 2$

$a \odot a' = a' \odot a$ by commutivity property

Distributive Property

Prove $a \odot (b \oplus c) = a \odot b \oplus a \odot c$

$$
\begin{aligned}
a \odot (b \oplus c) &= (a \cdot (b+c) \bmod 3) \bmod 3 \\
&= (a \cdot (b+c)) \bmod 3 \\
&= (a \cdot b + a \cdot c) \bmod 3 \\
&= ((ab) \bmod 3 + (ac) \bmod 3) \bmod 3 \\
&= a \odot b \oplus a \odot c
\end{aligned}
$$

Prove $(a \oplus b) \odot c = a \odot c \oplus b \odot c$

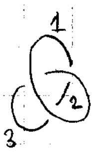
$$
\begin{aligned}
(a \oplus b) \odot c &= ((a+b) \bmod 3 \cdot c) \bmod 3 \\
&= ((a+b)c) \bmod 3 \\
&= (ac + bc) \bmod 3 \\
&= ((ac) \bmod 3 + (bc) \bmod 3) \bmod 3 \\
&= a \odot c \oplus b \odot c
\end{aligned}
$$

Problem 4

(1) Every intersection is made of 3 different line segments. Since there are only 3 line segments, the possible tricolourings are the permutations of (red, green, blue) and the 3 trivial cases of all the same colours.

$\therefore$ Total $= 3P3 + 3 = 6 + 3 = 9$

(2)



Intersections: $(1,1,2), (1,1,3), (1,2,3)$

For any tricolouring to work, the colour of $1=2$ for the 1st intersection and $1=3$ for the 2nd, $\therefore 1=2=3$ and only the 3 trivial cases apply where all lines are the same colour.

(3) We know from problem 3 that $\mathbb{F}_3$ is a field. We can say $\mathbb{F}_3^n$ is a vector space over $\mathbb{F}_3$ using the field's properties and binary operations $(\oplus, \odot)$

Let $V \subseteq \mathbb{F}_3^n$, $V = \{(a_1, \dots, a_n) \mid \forall i \in \mathbb{N}, 1 \le i \le n, a_i = (c_1, c_2, c_3), \forall j \in \mathbb{N}, 1 \le j \le 3, c_j \in \mathbb{F}_3, c_1 \oplus c_2 \oplus c_3 = 0\}$

We can treat $\mathbb{F}_3$ as the colours for the knot, $\{0: \text{red}, 1: \text{blue}, 2: \text{green}\}$

If we show V is a subspace of $\mathbb{F}_3^n$, it must be a vector space.

Let $u, v \in V$, $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$

<u>Non-empty</u>

We know the 3 trivial cases still apply for a arbitrary amount of intersections.

$\forall a \in V, \forall a_i \in a, \forall c \in a_i, c = 0$ or $c = 1$ or $c = 2$

<u>Closed under Addition</u>

$$
\begin{aligned}
u \oplus v &= (u_1, \dots, u_n) \oplus (v_1, \dots, v_n) \\
&= (u_1 \oplus v_1, \dots, u_n \oplus v_n) \\
&= ((u_{11}, u_{12}, u_{13}) \oplus (v_{11}, v_{12}, v_{13}), \dots, (u_{n1}, u_{n2}, u_{n3}) \oplus (v_{n1}, v_{n2}, v_{n3})) \\
&= ((u_{11} \oplus v_{11}, u_{12} \oplus v_{12}, u_{13} \oplus v_{13}), \dots, (u_{n1} \oplus v_{n1}, u_{n2} \oplus v_{n2}, u_{n3} \oplus v_{n3}))
\end{aligned}
$$

$\forall u_i, v_i$ check if in S.S. $(u_{i1} \oplus v_{i1}) \oplus (u_{i2} \oplus v_{i2}) \oplus (u_{i3} \oplus v_{i3}) = (u_{i1} \oplus u_{i2} \oplus u_{i3}) \oplus (v_{i1} \oplus v_{i2} \oplus v_{i3})$

$= 0 \oplus 0$

$= 0 \quad \therefore u \oplus v \in V$

Closed Under Scalar Multiplication

Let $r \in \mathbb{F}_3$

$r \odot u = r \odot (u_1, \ldots, u_n)$

$= r \odot ((u_{11}, u_{12}, u_{13}), \ldots, (u_{n1}, u_{n2}, u_{n3}))$

$= ((r \odot u_{11}, r \odot u_{12}, r \odot u_{13}), \ldots, (r \odot u_{n1}, r \odot u_{n2}, r \odot u_{n3}))$

$\forall u_i$: check if in S.S $\quad (r \odot u_{i1}) \oplus (r \odot u_{i2}) \oplus (r \odot u_{i3})$

$= r \odot (u_{i1} \oplus u_{i2} \oplus u_{i3})$ distributivity of $\mathbb{F}_3$

$= r \odot (0)$ by property of U

$= 0$

$\therefore r \odot u \in V$

Bc V is a subspace of $\mathbb{F}_3^n$, V is a vector space

(4) Conjecture: If the ends of every line segment ends with an intersection
The number of of tricolourings is between 3 and 9 inclusive.

Thought process

The number of tricolourings is trivially greater than 3, as you can easily make all lines the same colour.

Suppose there is a tricolouring that isn't trivial this means there must be at least 1 line of every colour.
If you permute the colours {red, blue, green}
You get $_3P_3 = 3! = 6$ possible colourings that are not all the same colour
$\therefore$ Max tricolourings = 3 trivial cases + 6 non-trivial cases