Integrals

Previous Years' CBSE Board Questions

7.2 Integration as an Inverse Process of Differentiation

MCQ

1. $\int \frac{\sec x}{\sec x - \tan x} dx$ equals
   (a) $\sec x - \tan x + c$ (b) $\sec x + \tan x + c$
   (c) $\tan x - \sec x + c$ (d) $-(\sec x + \tan x) + c$
   (2023)

VSA (1 mark)

2. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$ (AI 2017)
3. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ (Delhi 2014)
4. Evaluate : $\int \cos^{-1}(\sin x) dx$ (Delhi 2014)
5. Evaluate : $\int \frac{dx}{\sin^2 x \cos^2 x}$
   (Foreign 2014, Delhi 2014 C) U
6. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ (Delhi 2014 C)

SA I (2 marks)

7. Find: $\int \sqrt{1 - \sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$ (Delhi 2019)
8. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$ (2018) Ap

7.3 Methods of Integration

MCQ

9. $\int e^{5 \log x} dx$ is equal to
   (a) $\frac{x^5}{5} + C$ (b) $\frac{x^6}{6} + C$
   (c) $5x^4 + C$ (d) $6x^5 + C$ (2023)
10. $\int x^2 e^{x^3} dx$ equals
    (a) $\frac{1}{3} e^{x^3} + C$ (b) $\frac{1}{3} e^{x^4} + C$
    (c) $\frac{1}{2} e^{x^3} + C$ (d) $\frac{1}{2} e^{x^2} + C$ (2020) Ap
11. $\int \frac{e^x (1+x)}{\cos^2(xe^x)} dx$ is equal to
    (a) $\tan(xe^x) + c$ (b) $\cot(xe^x) + c$
    (c) $\cot(e^x) + c$ (d) $\tan[e^x(1+x)] + c$
    (2020) Ap

VSA (1 mark)

12. Find : $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$ (2020)
13. Find : $\int \frac{\sin^6 x}{\cos^8 x} dx$ (AI 2014 C)

SA I (2 marks)

14. Find : $\int \frac{dx}{\sqrt{4x - x^2}}$ (Term II, 2021-22) Ev

SA II (3 marks)

15. Find : $\int \frac{\sin x}{\sin(x - 2a)} dx$ (Term II, 2021-22 C)
16. Find : $\int \frac{1}{e^x + 1} dx$ (Term II, 2021-22)
17. Find : $\int \frac{2x}{(x^2+1)(x^2+2)} dx$ (Term II, 2021-22)

LA I (4 marks)

18. Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ with respect to x.
    (AI 2019)
19. Find $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$. (Delhi 2016)
20. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)} dx$ (Foreign 2015)
21. Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$ (Delhi 2014)

7.4 Integrals of Some Particular Functions

VSA (1 mark)

22. Find : $\int \frac{dx}{\sqrt{9 - 4x^2}}$ (2020)
23. Find : $\int \frac{dx}{9 + 4x^2}$ (2020)

SA I (2 marks)

24. Find: $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$ (Delhi 2019) Ap
25. Find : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$ (AI 2019)
26. Find : $\int \frac{dx}{x^2 + 4x + 8}$ (Delhi 2017)
27. Find : $\int \frac{dx}{5 - 8x - x^2}$ (AI 2017)

SA II (3 marks)

28. Find : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ (2023)

LA I (4 marks)

29. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$. (Delhi 2016) Ap

30. Find $\int(\sqrt{\tan x}+\sqrt{\cot x})dx$. (AI 2015, 2014) Ap

31. Evaluate : $\int \frac{x+2}{2x^2+6x+5}dx$ (Delhi 2015C)

32. Find $\int \frac{x+3}{\sqrt{5-4x-2x^2}}dx$. (AI 2015C)

33. Evaluate : $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$ (AI 2014)

34. Evaluate : $\int \frac{5x-2}{1+2x+3x^2}dx$ (Delhi 2014C) An

LA II (5/6 marks)

35. Evaluate : $\int \frac{1}{\cos^4 x+\sin^4 x}dx$ (AI 2014)

36. Evaluate : $\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$ (AI 2014) An

7.5 Integration by Partial Fractions

SA I (2 marks)

37. Find $\int \frac{x+1}{(x+2)(x+3)}dx$. (2020)

SA II (3 marks)

38. Find : $\int \frac{x^2}{x^2+6x+12}dx$ (2023)

LA I (4 marks)

39. Find : $\int \frac{x^2}{(x^2+1)(3x^2+4)}dx$ (Term II, 2021-22) An

40. Find : $\int \frac{x^3+1}{x^3-x}dx$ (2020) An

41. Find: $\int \frac{3x+5}{x^2+3x-18}dx$ (Delhi 2019) Ap

42. Evaluate : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ (2019, AI 2015)

43. Find : $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$ (2018)

44. Find : $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$. (Delhi 2017) Ap

45. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$ (AI 2017)

46. Find : $\int \frac{x^2}{x^4+x^2-2}dx$ (AI 2016)

47. Find : $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$ (Foreign 2016) Ap

48. Find : $\int \frac{dx}{\sin x+\sin 2x}$ (Delhi 2015)

49. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$ (Foreign 2015) Ap

50. Find : $\int \frac{x}{(x-1)^2(x+2)}dx$ (Delhi 2015C)

51. Find : $\int \frac{x}{(x^2+1)(x-1)}dx$ (AI 2015C)

52. Evaluate : $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$ (Delhi 2014C) Ap

53. Find : $\int \frac{x^3}{x^4+3x^2+2}dx$ (AI 2014C)

LA II (5/6 marks)

54. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$. (Delhi 2014C)

7.6 Integration by Parts

MCQ

55. $\int e^x\left(\frac{x\log x+1}{x}\right)dx$ is equal to
    (a) $\log(e^x\log x)+c$ (b) $\frac{e^x}{x}+c$
    (c) $x\log x+e^x+c$ (d) $e^x\log x+c$
    (2020) Ap

56. $\int \frac{e^x}{x+1}[1+(x+1)\log(x+1)]dx$ equals
    (a) $\frac{e^x}{x+1}+c$ (b) $e^x\frac{x}{x+1}+c$
    (c) $e^x\log(x+1)+e^x+c$
    (d) $e^x\log(x+1)+c$ (2020C)

VSA (1 mark)

57. Find : $\int x^4\log x\,dx$ (2020)

SA I (2 marks)

58. Find : $\int \frac{\log x-3}{(\log x)^4}dx$. (Term II, 2021-22) An

59. Find: $\int \sin^{-1}(2x)dx$ (Delhi 2019)

60. Find : $\int x\cdot\tan^{-1}x\,dx$ (AI 2019) Ev

SA II (3 marks)

61. Find : $\int e^x \cdot \sin 2x\,dx$ *(Term II, 2021-22)*

LA I (4 marks)

62. Find : $\int \sec^3 x\,dx$ *(2020)*

63. Find : $\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}dx$ *(Foreign 2016)*

64. Integrate the following w.r.t. x : $\frac{x^2-3x+1}{\sqrt{1-x^2}}$ *(Delhi 2015)*

65. Find : $\int \frac{\log x}{(x+1)^2}dx$ *(AI 2015)*

66. Evaluate : $\int e^{2x}\cdot\sin(3x+1)dx$ *(Foreign 2015)*

67. Find : $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$ *(Delhi 2015C)* An

68. Evaluate : $\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}dx$ *(Foreign 2014)*

LA II (5/6 marks)

69. Find : $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ *(AI 2014C)*

70. Find : $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx, x\in[0,1]$ *(AI 2014C)* An

7.8 Fundamental Theorem of Calculus

MCQ

71. $\int_{-1}^{1}\frac{|x-2|}{x-2}dx, x\neq 2$ is equal to

(a) 1 (b) −1 (c) 2 (d) −2 *(2023)*

72. $\int_0^4 (e^{2x}+x)dx$ is equal to

(a) $\frac{15+e^8}{2}$ (b) $\frac{16-e^8}{2}$

(c) $\frac{e^8-15}{2}$ (d) $\frac{-e^8-15}{2}$ *(2023)*

73. $\int_0^{\pi/8}\tan^2(2x)dx$ is equal to

(a) $\frac{4-\pi}{8}$ (b) $\frac{4+\pi}{8}$

(c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$ *(2020)* Ap

74. $\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx$ is equal to

(a) −1 (b) 0

(c) 1 (d) 2 *(2020)* U

VSA (1 mark)

75. Evaluate : $\int_2^3 3^x\,dx$ *(Delhi 2017)*

76. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$ *(Delhi 2014)* Ap

77. Evaluate : $\int_0^{\pi/2} e^x(\sin x-\cos x)dx$ *(Delhi 2014)*

78. If $f(x)=\int_0^x t\sin t\,dt$, then write the value of $f'(x)$. *(AI 2014)* Ap

79. If $\int_0^a \frac{1}{4+x^2}dx=\frac{\pi}{8}$, find the value of a. *(AI 2014)*

80. Evaluate : $\int_0^{\pi/4}\tan x\,dx$ *(Foreign 2014)*

81. Evaluate : $\int_0^{\pi/4}\sin 2x\,dx$ *(Foreign 2014)*

82. Evaluate : $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ *(AI 2014C)*

83. Evaluate : $\int_1^2 \frac{x^3-1}{x^2}dx$ *(AI 2014C)*

SA I (2 marks)

84. Evaluate : $\int_0^1 x^2 e^x\,dx$ *(Term II, 2021-22)*

85. Evaluate $\int_1^2\left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}dx.$ *(2020)* Ap

86. Find the value of $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx.$ *(2020)*

LA I (4 marks)

87. Evaluate : $\int_0^{\pi/2} x^2\sin x\,dx$ *(Delhi 2014C)*

7.9 Evaluation of Definite Integrals by Substitution

VSA (1 mark)

88. Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$ *(AI 2014)* An

89. Evaluate : $\int_e^{e^2}\frac{dx}{x\log x}.$ *(AI 2014)*

90. Evaluate : $\int_0^1 xe^{x^2} dx$ *(Foreign 2014)*

91. Evaluate : $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$ *(AI 2014C)*

SA I (2 marks)

92. Find : $\int_{-\frac{\pi}{4}}^{0} \frac{1+\tan x}{1-\tan x} dx$ *(AI 2019)* Ev

SA II (3 marks)

93. Evaluate : $\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right) dx$ *(2023)*

94. Evaluate : $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x \, dx$ *(2023)*

LA I (4 marks)

95. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$ *(2020C, AI 2014C)* Ev

96. Evaluate $\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4}+x\right) dx$. *(Delhi 2016)*

97. Find : $\int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2\sin 2x}}$ *(AI 2015)*

LA II (5/6 marks)

98. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x} dx$ *(2018)*

99. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x} dx$

(Foreign 2014, Delhi 2014C) Ap

7.10 Some Properties of Definite Integrals

MCQ

100. In the following question, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choice

Assertion (A) : $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}} dx = 3$

Reason (R) : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true, but (R) is not the correct explanation of the (A).

(c) (A) is true and (R) is false.

(d) (A) is false, but (R) is true. *(2023)*

VSA (1 mark)

101. Evaluate : $\int_0^{\pi/2} \frac{1}{1+\cot^{5/2} x} dx$ *(Term II, 2021-22C)*

102. Evaluate : $\int_1^3 |2x-1| dx$ *(2020)*

103. Evaluate : $\int_{-2}^{2} |x| dx$ *(2020)*

104. Find the value of $\int_1^4 |x-5| dx$. *(2020)* An

SA I (2 marks)

105. Evaluate : $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ *(2021C)*

SA II (3 marks)

106. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ *(2023)*

107. Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x} dx$ *(2023)*

108. Evaluate : $\int_1^4 \{|x|+|3-x|\} dx$ *(Term II, 2021-22)*

109. Evaluate : $\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}} dx$ *(Term II, 2021-22)* Ap

LA I (4 marks)

110. Evaluate : $\int_0^{\pi} \frac{x}{9\sin^2 x+16\cos^2 x} dx$ *(Term II, 2021-22)*

111. Evaluate : $\int_{-1}^{2} |x^3-x| dx$

(Term II, 2021-22, 2020, Delhi 2016)

112. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$. *(Delhi 2019)*

OR

Evaluate : $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ *(Delhi 2017)* Ap

113. Prove that $\int_0^a f(x)\, dx = \int_0^a f(a-x) dx$ and hence evaluate $\int_0^1 x^2(1-x)^n \, dx$. *(AI 2019)*

114. Evaluate : $\int_0^{3/2} |x \sin \pi x| \, dx$ *(Delhi 2017)*

115. Evaluate : $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ *(AI 2017, Foreign 2014, Delhi 2014C)*

116. Evaluate : $\int_1^4 \{|x-1|+|x-2|+|x-4|\} dx$ *(AI 2017)*

117. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ *(AI 2016)*

OR

Show that : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$ *(AI 2014C)* Ap

118. Evaluate : $\int_0^{3/2} |x \cos \pi x| dx$ *(AI 2016)*

119. Evaluate : $\int_0^{\pi} \frac{x}{1+\sin\alpha \sin x} dx$ *(Foreign 2016)*

120. Evaluate : $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 \, dx$ *(Delhi 2015)* Ev

121. Evaluate : $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$ *(Foreign 2015)*

122. Evaluate : $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ *(Delhi 2015C)* Ap

123. Evaluate : $\int_0^{\pi/4} \log(1+\tan x) dx$ *(AI 2015C)*

124. Evaluate : $\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$ *(Delhi 2014)*

125. Evaluate : $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$ *(AI 2014)*

LA II (5/6 marks)

126. Evaluate : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ *(Delhi 2014)*

127. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ *(Delhi 2014)*

128. Evaluate : $\int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ *(Foreign 2014)* An

CBSE Sample Questions

7.3 Methods of Integration

SA I (2 marks)

1. Find $\int \frac{1}{\cos^2 x (1-\tan x)^2} dx.$ *(2020-21)*

7.4 Integrals of Some Particular Functions

SA I (2 marks)

2. Find $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}} dx$. *(2020-21)*

7.5 Integration by Partial Fractions

SA I (2 marks)

3. Find : $\int \frac{x+1}{(x^2+1)x} dx$. *(2021-22)* An

SA II (3 marks)

4. Find $\int \frac{(x^3+x+1)}{(x^2-1)} dx$ *(2022- 23)* Ev

5. Evaluate : $\int_0^4 |x-1| dx$ *(2022-23)* Ev

6. Find $\int \frac{x^2+1}{(x^2+2)(x^2+3)} dx.$ *(2020-21)* Ap

7.6 Integration by Parts

VSA (1 mark)

7. Find $\int e^x (1-\cot x + \operatorname{cosec}^2 x) dx.$ *(2020-21)*

SA I (2 marks)

8. Find $\int \frac{\log x}{(1+\log x)^2} dx$. *(2021-22)* Ev

7.10 Some Properties of Definite Integrals

VSA (1 mark)

9. Evaluate : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \sin x dx$ *(2020-21)* Ev

SA I (2 marks)

10. Evaluate : $\int_0^1 x(1-x)^n dx$ *(2020-21)*

LA I (4 marks)

11. Evaluate : $\int_{-1}^{2} |x^3-3x^2+2x| dx$ *(2021-22)* An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. (b): Let $I=\int\frac{\sec x}{\sec x-\tan x}dx$

$=\int\frac{\sec x(\sec x+\tan x)}{(\sec x-\tan x)(\sec x+\tan x)}dx=\int\left(\frac{\sec^2 x+\sec x\tan x}{\sec^2 x-\tan^2 x}\right)dx$

$=\int\sec^2 x\,dx+\int\sec x\tan x\,dx$ $\quad[\because \sec^2 x-\tan^2 x=1]$

$=\tan x+\sec x+c$

2. We have, $\int\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}dx$

$=\int\frac{\sin^2 x}{\sin x\cos x}dx-\int\frac{\cos^2 x}{\sin x\cos x}dx$

$=\int\tan x dx-\int\cot x dx$

$=\ln|\sec x|-\ln|\sin x|+C=\ln\left|\frac{1}{\sin x\cos x}\right|+C$

$=\ln\left|\frac{2}{2\sin x\cos x}\right|+C=\ln|2\operatorname{cosec}2x|+C$

3. The antiderivative of $3\sqrt{x}+\frac{1}{\sqrt{x}}$

$=\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=3\int x^{1/2}dx+\int x^{-1/2}dx$

$=3\cdot\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C=2x\sqrt{x}+2\sqrt{x}+C=2\sqrt{x}(x+1)+C$

4. We have, $\int\cos^{-1}(\sin x)dx=\int\cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$

$=\int\left(\frac{\pi}{2}-x\right)dx=\frac{\pi}{2}x-\frac{x^2}{2}+C$

5. We have, $\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}dx$

$=\int(\sec^2 x+\operatorname{cosec}^2 x)dx=\tan x-\cot x+C$

6. We have, $\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$

$=\int\sec^2 x dx-\int\operatorname{cosec}^2 x\,dx$

$=\tan x+\cot x+C$

7. Let $I=\int(\sqrt{1-\sin 2x})dx$

$=\int\sqrt{\cos^2 x+\sin^2 x-2\sin x\cos x}\,dx=\pm\int(\cos x-\sin x)dx$

Since, $\frac{\pi}{4}<x<\frac{\pi}{2}$, so we get

$I=\int(\sin x-\cos x)dx$

$=-(\cos x+\sin x)+C$

8. Let $I=\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

$=\int\frac{\cos^2 x-\sin^2 x+2\sin^2 x}{\cos^2 x}dx$

$=\int\frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx=\int\frac{1}{\cos^2 x}dx=\int\sec^2 x\,dx$

$=\tan x+C$

9. (b): Let $I=\int e^{5\log x}dx$

$=\int e^{\log x^5}dx=\int x^5 dx$ $\quad[\because e^{\log x}=x]$

$=\frac{x^6}{6}+C$

10. (a): Let $I=\int x^2e^{x^3}dx$

Put $x^3=t \Rightarrow 3x^2dx=dt$

$\therefore\quad I=\int e^t\frac{dt}{3}=\frac{1}{3}e^t+C=\frac{1}{3}e^{x^3}+C$

11. (a): Let $I=\int\frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $xe^x=t$

$\Rightarrow\quad (xe^x+e^x)\,dx=dt$

$\Rightarrow\quad e^x(x+1)\,dx=dt$

$\therefore\quad I=\int\frac{dt}{\cos^2 t}=\int\sec^2 t\,dt=\tan t+c=\tan(xe^x)+c$

12. Let $I=\int\frac{2x}{\sqrt[3]{x^2+1}}dx$

Put $x^2+1=z\Rightarrow 2x\,dx=dz$

$\therefore\quad I=\int\frac{dz}{(z)^{1/3}}=\int z^{-1/3}dz=\frac{z^{(-1/3)+1}}{(-1/3)+1}+C$

$=\frac{3}{2}(x^2+1)^{2/3}+C$

13. Let $I=\int\frac{\sin^6 x}{\cos^8 x}dx=\int\frac{\sin^6 x}{\cos^6 x\cdot\cos^2 x}dx$

$=\int\tan^6 x\sec^2 x dx$

Put $\tan x=t\Rightarrow \sec^2 x\,dx=dt$

$\therefore\quad I=\int t^6 dt=\frac{t^7}{7}+C=\frac{1}{7}\tan^7 x+C$

14. $\int\frac{dx}{\sqrt{4x-x^2}}=\int\frac{dx}{\sqrt{4-(x-2)^2}}$

Put $x-2=2\sin\theta$ ...(i)

$\Rightarrow\quad dx=2\cos\theta\,d\theta$

$=\int\frac{2\cos\theta}{\sqrt{4-(2\sin\theta)^2}}d\theta$

Now, $4-(2\sin\theta)^2=4-4\sin^2\theta=4(1-\sin^2\theta)=4\cos^2\theta$

$=\int\frac{2\cos\theta\,d\theta}{\sqrt{4\cos^2\theta}}=\int\frac{2\cos\theta\,d\theta}{2\cos\theta}=\theta+c$

$=\sin^{-1}\left(\frac{x-2}{2}\right)+c,$ $\quad\left[\text{From (i)}, \theta=\sin^{-1}\left(\frac{x-2}{2}\right)\right]$

where c is an arbitrary constant.

15. Let $I=\int\frac{\sin x}{\sin(x-2a)}dx$

Put $x-2a=t$

$\Rightarrow \quad x = 2a + t \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin(t+2a)}{\sin t} dx$

$= \int \frac{(\sin t \cos 2a + \cos t \sin 2a)}{\sin t} dx = \int (\cos 2a + \cot t \cdot \sin 2a) dx$

$= t \cos 2a + \sin 2a \log|\sin t| + c$

$= (x - 2a) \cos 2a + \sin 2a \log|\sin(x - 2a)| + c$

16. Let $I = \int \frac{1}{e^x + 1} dx$

Put $e^x + 1 = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t-1}$

$\therefore \quad I = \int \frac{dt}{t(t-1)} = \int \frac{t-(t-1)}{t(t-1)} dt$

$= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt = \log(t-1) - \log t + C$

$= \log e^x - \log(e^x + 1) + c = \log \frac{e^x}{e^x + 1} + c$

17. $\int \frac{2x}{(x^2+1)(x^2+2)} dx$

Let $x^2 + 2 = t \Rightarrow 2x\, dx = dt$

$\Rightarrow \quad \int \frac{2x}{(x^2+1)(x^2+2)} dx = \int \frac{dt}{(t-1)t} = \int \frac{(1-t+t)\,dt}{(t-1)t}$

$= -\int \frac{(t-1)dt}{(t-1)t} + \int \frac{t}{(t-1)t} dt = -\int \frac{1}{t} dt + \int \frac{1}{(t-1)} dt$

$= -\log|t| + \log|t-1| + \log C$

$= \log\left|\frac{t-1}{t}\right| + \log C = \log\left|\left(\frac{x^2+1}{x^2+2}\right)C\right|,$

where C is an arbitrary constant.

18. We have, $\int \frac{\cos(x+a)}{\sin(x+b)} dx = \int \frac{\cos(x+b+a-b)}{\sin(x+b)} dx$

$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)} dx$

$= \cos(a-b) \int \frac{\cos(x+b)}{\sin(x+b)} dx - \sin(a-b) \int dx$

$= \cos(a - b) \log \sin(x + b) - x \sin(a - b) + C$

Commonly Made Mistake

- Using formula for $\cos(A + B) = \cos A \cos B - \sin A \sin B$ and $\cos(A - B) = \cos A \cos B + \sin A + \sin B$

19. Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

(Put $\cos^2\theta = 1 - \sin^2\theta$)

$= 3\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta - 2\int \frac{\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta$

$= 3I_1 - 2I_2$ (say) ...(i)

Now, $I_1 = \int \frac{\sin\theta\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta\, d\theta = dt$

$\therefore \quad I_1 = \frac{1}{2}\int \frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2}\int \frac{dt}{(\sqrt{t}-2)^2}$

Put $\sqrt{t} - 2 = u \Rightarrow \sqrt{t} = u + 2$

$\Rightarrow \quad \frac{1}{2\sqrt{t}} dt = du \Rightarrow dt = 2(u+2)du$

$\therefore \quad I_1 = \int \frac{(u+2)}{u^2} du = \int \frac{du}{u} + 2\int \frac{du}{u^2}$

$= \log u - \frac{2}{u} + C_1 = \log(\sqrt{t} - 2) - \frac{2}{\sqrt{t}-2} + C_1$

$= \log(\sin\theta - 2) - \frac{2}{\sin\theta - 2} + C_1$ ...(ii)

Also, $I_2 = \int \frac{\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta$

Put $\sin\theta = m \Rightarrow \cos\theta\, d\theta = dm$

$\therefore \quad I_2 = \int \frac{dm}{4+m^2-4m} = \int \frac{dm}{(m-2)^2}$

$= \frac{-1}{m-2} + C_2 = \frac{-1}{\sin\theta - 2} + C_2$ ...(iii)

From (i), (ii) and (iii), we get

$I = 3\log(\sin\theta - 2) - \frac{6}{\sin\theta - 2} + \frac{2}{\sin\theta - 2} + C,$

where $C = 3C_1 - 2C_2$

$\Rightarrow \quad I = 3\log(\sin\theta - 2) - \frac{4}{\sin\theta - 2} + C$

20. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx = \int \frac{\sin(x+a-2a)}{\sin(x+a)} dx$

$= \int \left(\frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)} \right) dx$

$\Rightarrow \quad I = \cos 2a \int dx - \sin 2a \int \frac{\cos(x+a)}{\sin(x+a)} dx$

Put $\sin(x + a) = t \Rightarrow \cos(x + a)dx = dt$

$\Rightarrow \quad I = \cos 2a \int dx - \sin 2a \int \frac{dt}{t}$

$\Rightarrow \quad I = x \cos 2a - \sin 2a \log|t| + C$

$\Rightarrow \quad I = x \cos 2a - \sin 2a \log|\sin(x + a)| + C$

21. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^6 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^4 x}{\cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x} dx$

$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x} dx + \int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x} dx$

$= \int \tan^2 x\, dx - \int \sin^2 x\, dx + \int \cot^2 x\, dx - \int \cos^2 x\, dx$

$= \int (\sec^2 x - 1)dx - \int \sin^2 x\, dx + \int (\mathrm{cosec}^2 x - 1)\, dx - \int (1 - \sin^2 x)\, dx$

$= \tan x - x + (-\cot x) - x - x + C = \tan x - \cot x - 3x + C$

22. Let $I = \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4} - x^2}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$

23. Let $I = \int \frac{dx}{9+4x^2} = \frac{1}{4}\int \frac{dx}{x^2 + \frac{9}{4}} = \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2}$

$$= \frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

24. Let $I=\int\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore\ I=\int\frac{dt}{\sqrt{t^2+4}}=\log|t+\sqrt{t^2+4}|+C$$

$$=\log|\tan x+\sqrt{\tan^2 x+4}|+C$$

25. Let $I=\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{7}{2}-1-2x-x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right)+C=\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$$

26. We have, $\int\frac{dx}{x^2+4x+8}=\int\frac{dx}{x^2+4x+4+4}$

$$=\int\frac{dx}{(x+2)^2+(2)^2}=\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$$

27. Let $I=\int\frac{dx}{5-8x-x^2}$

$$=\int\frac{dx}{5+16-16-8x-x^2}=\int\frac{dx}{21-(x+4)^2}$$

$$=\int\frac{dx}{(\sqrt{21})^2-(x+4)^2}=\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$$

$$\left[\because\ \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$$

28. Let $I=\int\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Putting $e^x = t \Rightarrow e^x\,dx = dt$, we get

$$I=\int\frac{dt}{\sqrt{5-4t-t^2}}=\int\frac{dt}{\sqrt{9-(t^2+4t+4)}}=\int\frac{dt}{\sqrt{3^2-(t+2)^2}}$$

$$=\sin^{-1}\left(\frac{t+2}{3}\right)+C=\sin^{-1}\left(\frac{e^x+2}{3}\right)+C$$

29. Let $I=\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

Put $x^{3/2}=t \Rightarrow \frac{3}{2}x^{1/2}dx=dt$

$$\therefore\quad I=\frac{2}{3}\int\frac{dt}{\sqrt{a^3-t^2}}=\frac{2}{3}\int\frac{dt}{\sqrt{(a^{3/2})^2-t^2}}$$

$$=\frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right]+C=\frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right]+C$$

$$=\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+C$$

30. Let $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx$

$$=\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x+1-1}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}}dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

$$\therefore\quad I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\sin^{-1}t+C$$

$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

31. Let $I=\int\frac{x+2}{2x^2+6x+5}dx$

Let $x+2=A\left[\frac{d}{dx}(2x^2+6x+5)\right]+B$

$\Rightarrow\quad x+2 = A(4x+6)+B$

Equating coefficients of x and constant terms, we get

$4A = 1 \Rightarrow A = 1/4$ and $6A + B = 2 \Rightarrow B = 1/2$

$$\therefore\quad I=\frac{1}{4}\int\frac{4x+6}{2x^2+6x+5}dx+\frac{1}{2}\int\frac{dx}{2x^2+6x+5}$$

$$=\frac{1}{4}\log\left|2x^2+6x+5\right|+\frac{1}{4}\int\frac{dx}{x^2+3x+\frac{5}{2}}$$

$$=\frac{1}{4}\log\left|2x^2+6x+5\right|+\frac{1}{4}\int\frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}}$$

$$=\frac{1}{4}\log\left|2x^2+6x+5\right|+\frac{1}{4}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$=\frac{1}{4}\log\left|2x^2+6x+5\right|+\frac{1}{2}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right)+C$$

$$=\frac{1}{4}\log\left|2x^2+6x+5\right|+\frac{1}{2}\tan^{-1}(2x+3)+C$$

32. Let $I=\int\frac{x+3}{\sqrt{5-4x-2x^2}}dx$

Let $x+3=A\frac{d}{dx}(5-4x-2x^2)+B=A(-4-4x)+B$

On comparing the coefficients of like term, we get

$-4A=1\Rightarrow A=-\frac{1}{4}$ and $-4A+B=3\Rightarrow B=2$

$x+3=-\frac{1}{4}(-4-4x)+2$

$$\Rightarrow\quad I=\int\frac{-\frac{1}{4}(-4-4x)+2}{\sqrt{5-4x-2x^2}}dx$$

$$= \frac{-1}{4}\int\frac{-4-4x}{\sqrt{5-4x-2x^2}}dx+2\cdot\int\frac{1}{\sqrt{5-4x-2x^2}}dx$$

$$= -\frac{1}{4}I_1+2I_2 \quad \text{...(i)}$$

where $I_1=\int\frac{-4-4x}{\sqrt{5-4x-2x^2}}dx$

Put $5-4x-2x^2=t \Rightarrow (-4-4x)dx=dt$

$$\therefore \quad I_1=\int\frac{dt}{\sqrt{t}}=\int t^{-1/2}dt=2\sqrt{t}+C_1$$

$$=2\sqrt{5-4x-2x^2}+C_1 \quad \text{...(ii)}$$

and $I_2=\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right)+C_2 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$I=-\frac{1}{4}\cdot2\sqrt{5-4x-2x^2}+2\cdot\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$$

where $C=C_1+C_2$

$$\Rightarrow \quad I=-\frac{1}{2}\sqrt{5-4x-2x^2}+\sqrt{2}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$$

33. Let $I=\int\frac{x+2}{\sqrt{x^2+5x+6}}dx=\int\frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx$

$$=\frac{1}{2}\int(x^2+5x+6)^{-1/2}(2x+5)dx-\frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+6}}$$

Put $x^2+5x+6=t \Rightarrow (2x+5)\,dx=dt$

$$\Rightarrow \quad I=\frac{1}{2}\int t^{-1/2}dt-\frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}+C$$

$$=\frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\sqrt{x^2+5x+6}-\frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right|+C$$

Concept Applied

- In $\int\frac{px+q}{ax^2+bx+c}$

  substitute $px+q=A\frac{d}{dx}(ax^2+bx+c)+B.$

34. Let $I=\int\frac{5x-2}{1+2x+3x^2}dx$

Let $5x-2=A\frac{d}{dx}(1+2x+3x^2)+B=A(2+6x)+B$

On comparing the coefficients of like terms, we get

$$6A=5\Rightarrow A=\frac{5}{6}\text{ and }2A+B=-2\Rightarrow B=\frac{-11}{3}$$

$$\therefore \quad I=\int\frac{\frac{5}{6}\frac{d}{dx}(1+2x+3x^2)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int\frac{\frac{d}{dx}(1+2x+3x^2)}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{dx}{1+2x+3x^2}$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{3\cdot3}\int\frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{9}\int\frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{9}\cdot\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right)+C$$

$$=\frac{5}{6}\log|1+2x+3x^2|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

35. Let $I=\int\frac{1}{\cos^4x+\sin^4x}dx=\int\frac{\sec^4x}{1+\tan^4x}dx$

$$=\int\frac{(\tan^2x+1)\sec^2x}{1+\tan^4x}dx$$

Put $\tan x=t\Rightarrow\sec^2x\,dx=dt$

$$\therefore \quad I=\int\frac{t^2+1}{t^4+1}dt=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Put $t-\frac{1}{t}=y\Rightarrow\left(1+\frac{1}{t^2}\right)dt=dy$

$$\therefore \quad I=\int\frac{dy}{y^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{2}}\right)+C$$

36. Let $I=\int\frac{1}{\sin^4x+\sin^2x\cos^2x+\cos^4x}dx$

$$=\int\frac{\sec^4x\,dx}{\tan^4x+\tan^2x+1}=\int\frac{(1+\tan^2x)\sec^2x\,dx}{\tan^4x+\tan^2x+1}$$

Put $\tan x=t\Rightarrow\sec^2x\,dx=dt$

$$\therefore \quad I=\int\frac{1+t^2}{t^4+t^2+1}dt=\int\frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt$$

Put $t-\frac{1}{t}=y\Rightarrow\left(1+\frac{1}{t^2}\right)dt=dy$

Thus, $I=\int\frac{dy}{y^2+(\sqrt{3})^2}=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)+C$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{3}}+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{3}}\right)+C$$

37. Let $I=\int\frac{(x+1)}{(x+2)(x+3)}dx$

Also let, $\frac{(x+1)}{(x+2)(x+3)}=\frac{A}{(x+2)}+\frac{B}{(x+3)}$

$$\Rightarrow \quad x+1=A(x+3)+B(x+2) \quad \text{...(i)}$$

Putting $x = -3$ in (i), we get
$-B = -3 + 1 = -2 \Rightarrow B = 2$
Putting $x = -2$ in (i), we get
$A = -2 + 1 = -1$

$$\therefore \quad I = \int \frac{-1}{(x+2)} dx + 2\int \frac{1}{(x+3)} dx$$
$$= -\log(x+2) + 2\log(x+3) + C$$

Answer Tips

- Form of rational fraction to the partial fraction.
$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

38. Let $I = \int \frac{x^2}{x^2+6x+12} dx$

$$= \int \frac{x^2+6x+12-(6x+12)}{x^2+6x+12} dx = \int dx - \int \frac{6x+12}{x^2+6x+12} dx$$
$$= x - 3\int \frac{2x}{x^2+6x+12} dx - 12\int \frac{dx}{x^2+6x+12}$$
$$= x - 3\int \frac{2x+6-6}{x^2+6x+12} dx - 12\int \frac{dx}{x^2+6x+12}$$
$$= x - 3\int \frac{2x+6}{x^2+6x+12} dx - 18\int \frac{dx}{x^2+6x+12}$$
$= x - 3I_1 - 18I_2$

Consider, $I_1 = \int \frac{2x+6}{x^2+6x+12} dx$

Put $x^2 + 6x + 12 = t \Rightarrow (2x+6)dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| + C = \log|x^2+6x+12| + C_1$$

and $I_2 = \int \frac{dx}{x^2+6x+12}$

$$= \int \frac{dx}{(x+3)^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C_2$$

$$\therefore \quad I = x - 3\log|x^2+6x+12| + 6\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C$$

39. Let $I = \int \frac{x^2}{(x^2+1)(3x^2+4)} dx$

Let $x^2 = y$

So, $\frac{x^2}{(x^2+1)(3x^2+4)} = \frac{y}{(y+1)(3y+4)}$
$$= \frac{A}{y+1} + \frac{B}{3y+4} = \frac{A(3y+4)+B(y+1)}{(y+1)(3y+4)}$$
$\Rightarrow \quad y = A(3y+4) + B(y+1)$
$\Rightarrow \quad y = (3A+B)y + (4A+B)$

Equating the like coefficients, we get
$3A + B = 1$ and $4A + B = 0$

On solving we get $A = -1, B = 4$

So, $I = \int \left[-\frac{1}{x^2+1} + \frac{4}{3x^2+4}\right] dx$

$$= -\int \frac{1}{1+x^2} dx + \frac{4}{3}\int \frac{1}{x^2+\frac{4}{3}} dx$$
$$= -\tan^{-1}x + \frac{4}{3} \times \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$$
$$= -\tan^{-1}x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$$

40. Let $I = \int \frac{x^3+1}{x^3-x} dx = \int \frac{(x+1)(x^2-x+1)}{x(x^2-1)} dx$

$$= \int \frac{(x^2-x+1)}{x(x-1)} dx = \int \left[1 + \frac{1}{x(x-1)}\right] dx \quad \text{...(i)}$$

Consider, $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$\Rightarrow \quad \frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)}$

$\Rightarrow \quad 1 = A(x-1) + Bx$

On solving, we get $A = -1, B = 1$

$\Rightarrow \quad \frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$

$$\therefore \quad I = \int \left(1 + \frac{1}{x-1} - \frac{1}{x}\right) dx = x + \log|x-1| - \log|x| + c$$

41. Let $I = \int \frac{3x+5}{x^2+3x-18} dx = \int \frac{3x+5}{(x+6)(x-3)} dx$

Let $\frac{3x+5}{(x+6)(x-3)} = \frac{A}{(x+6)} + \frac{B}{(x-3)}$

$\Rightarrow \quad 3x + 5 = A(x-3) + B(x+6)$ ...(i)

Putting $x = 3$ in (i), we get $9B = 14 \Rightarrow B = \frac{14}{9}$

Putting $x = -6$ in (i), we get $-9A = -13 \Rightarrow A = \frac{13}{9}$

$$\therefore \quad I = \frac{13}{9}\int \frac{1}{(x+6)} dx + \frac{14}{9}\int \frac{1}{(x-3)} dx$$
$$= \frac{13}{9}\log(x+6) + \frac{14}{9}\log(x-3) + C$$

42. Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)} dx$ ...(i)

Let $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$ ...(ii)

$\Rightarrow \quad x^2 + x + 1 = (Ax+B)(x+2) + C(x^2+1)$ ...(iii)

Put $x = 0, 1$ and -2 in (iii), We get
$1 = 2B + C$; $3 = 3(A+B) + 2C$ and $3 = 5C$

$\Rightarrow \quad C = \frac{3}{5}, B = \frac{1}{5}$ and $A = \frac{2}{5}$

From (ii), we get

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\left(\frac{2}{5}x + \frac{1}{5}\right)}{x^2+1} + \frac{\frac{3}{5}}{x+2}$$
$$= \frac{1}{5} \cdot \frac{2x+1}{x^2+1} + \frac{3}{5} \cdot \frac{1}{x+2}$$

$$\therefore \quad I = \frac{1}{5}\int \frac{2x+1}{x^2+1} dx + \frac{3}{5}\int \frac{1}{x+2} dx$$

$$=\frac{1}{5}\left[\int\frac{2x}{x^2+1}dx+\int\frac{dx}{x^2+1}\right]+\frac{3}{5}\int\frac{dx}{x+2}$$

$$=\frac{1}{5}[\log|x^2+1|+\tan^{-1}x]+\frac{3}{5}\log|x+2|+C_1$$

43. Let $I=\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$\therefore\ I=2\int\frac{1}{(1-t)(1+t^2)}dt$ ...(i)

Now, $\frac{1}{(1-t)(1+t^2)}=\frac{A}{(1-t)}+\frac{Bt+C}{(1+t^2)}$

$\Rightarrow\ 1 = A(1+t^2) + (Bt+C)(1-t)$

$\Rightarrow\ 1 = A + At^2 + Bt - Bt^2 + C - Ct$

$\Rightarrow\ 1 = (A+C) + t^2(A-B) + t(B-C)$

On equating coefficients of like terms on both sides, we get

$A + C = 1, A - B = 0 \Rightarrow A = B$

$B - C = 0 \Rightarrow B = C$

$\therefore\ A = B = C$

$\therefore\ A=\frac{1}{2}, B=\frac{1}{2}, C=\frac{1}{2}$

$\therefore\ \frac{1}{(1-t)(1+t^2)}=\frac{1}{2(1-t)}+\frac{1}{2}\frac{t}{(1+t^2)}+\frac{1}{2(1+t^2)}$

Put above equation in (i), we get

$$I=2\left[\int\frac{1}{2(1-t)}dt+\int\frac{1}{2}\frac{t}{(1+t^2)}dt+\int\frac{1}{2(1+t^2)}dt\right]$$

$$=\int\frac{dt}{(1-t)}+\int\frac{t}{(1+t^2)}dt+\int\frac{dt}{(1+t^2)}$$

$$=-\log(1-t)+\frac{1}{2}\log(1+t^2)+\tan^{-1}(t)+C$$

$$=-\log(1-\sin x)+\frac{1}{2}\log(1+\sin^2 x)+\tan^{-1}(\sin x)+C$$

$$=-\log(1-\sin x)+\log(1+\sin^2 x)^{1/2}+\tan^{-1}(\sin x)+C$$

$$=\log\frac{\sqrt{1+\sin^2 x}}{1-\sin x}+\tan^{-1}(\sin x)+C$$

44. Let $I=\int\frac{2x}{(x^2+1)(x^2+2)^2}dx$

Put $x^2 = y \Rightarrow 2x\,dx = dy$

$\therefore\ I=\int\frac{dy}{(y+1)(y+2)^2}$

Let $\frac{1}{(y+1)(y+2)^2}=\frac{A}{y+1}+\frac{B}{y+2}+\frac{C}{(y+2)^2}$

$\Rightarrow\ 1 = A(y+2)^2 + B(y+1)(y+2) + C(y+1)$ ...(i)

Putting $y = -1$, in (i), we get $1 = A$

Putting $y = -2$, in (i), we get $1 = -C \Rightarrow C = -1$

Putting $y = 0$, in (i), we get $1 = 4A + 2B + C$

$\Rightarrow\ B=\frac{1-4+1}{2}=-1$

$\therefore\ I=\int\left[\frac{1}{y+1}-\frac{1}{y+2}-\frac{1}{(y+2)^2}\right]dy$

$$=\log(y+1)-\log(y+2)+\frac{1}{y+2}+c$$

$$=\log\left(\frac{y+1}{y+2}\right)+\frac{1}{y+2}+c$$

$$=\log\left(\frac{x^2+1}{x^2+2}\right)+\frac{1}{x^2+2}+c \qquad [\because y=x^2]$$

45. $I=\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

$$=\int\frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta\,d\theta = dt$

$\therefore\ I=\int\frac{1}{(4+t^2)(1+4t^2)}dt$

Consider, $\frac{1}{(4+t^2)(1+4t^2)}=\frac{At+B}{4+t^2}+\frac{Ct+D}{1+4t^2}$

(Using partial fractions)

$1 = (At+B)(1+4t^2) + (Ct+D)(4+t^2)$

$= At + B + 4At^3 + 4Bt^2 + 4Ct + Ct^3 + 4D + Dt^2$

$= (4A+C)t^3 + (4B+D)t^2 + (A+4C)t + (B+4D)$

On comparing the coefficients of like terms, we get

$4A + C = 0$ ...(i)

$4B + D = 0$ ...(ii)

$A + 4C = 0$ ...(iii)

$B + 4D = 1$ ...(iv)

Solving (i) & (iii), we get

$A = 0$ and $C = 0$

Solving (ii) & (iv), we get

$B=\frac{-1}{15}$ and $D=\frac{4}{15}$

$\therefore\ \frac{1}{(4+t^2)(1+4t^2)}=\frac{-1/15}{4+t^2}+\frac{4/15}{1+4t^2}$

$\therefore\ I=-\frac{1}{15}\int\frac{1}{4+t^2}dt+\frac{4}{15}\times\frac{1}{4}\int\frac{1}{\frac{1}{4}+t^2}dt$

$$=-\frac{1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+\frac{1}{15}\times\frac{1}{1/2}\tan^{-1}\left(\frac{t}{1/2}\right)+C$$

$$=-\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right)+\frac{2}{15}\tan^{-1}(2t)+C$$

$$=\frac{2}{15}\tan^{-1}(2\sin\theta)-\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right)+C$$

46. Let $I=\int\frac{x^2}{x^4+x^2-2}dx=\int\frac{x^2}{(x^2-1)(x^2+2)}dx$

Let $x^2 = z$

$\therefore\ \frac{x^2}{(x^2-1)(x^2+2)}=\frac{z}{(z-1)(z+2)}$

Using partial fractions, we have

$\frac{z}{(z-1)(z+2)}=\frac{A}{z-1}+\frac{B}{z+2}$

$\Rightarrow\ z = A(z+2) + B(z-1)$

when $z = 1$, we get $A=\frac{1}{3}$

and when $z = -2$, we get $B=\frac{2}{3}$

$\therefore\ I=\int\frac{x^2}{(x^2-1)(x^2+2)}dx$

$$=\int \frac{1/3}{(x^2-1)}dx+\int \frac{2/3}{(x^2+2)}dx$$

$$=\frac{1}{3}\int \frac{1}{x^2-1}dx+\frac{2}{3}\int \frac{1}{x^2+(\sqrt{2})^2}dx$$

$$=\frac{1}{3}\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+\frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

$$=\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

Concept Applied

➲ $\int \frac{1}{x^2-a^2}\cdot dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

47. Let $I=\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Let $x^2 = t$

$$\therefore \quad \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}=\frac{(t+1)(t+4)}{(t+3)(t-5)}$$

$$=\frac{t^2+5t+4}{(t+3)(t-5)}=1+\frac{7t+19}{(t+3)(t-5)}$$

Let $\frac{7t+19}{(t+3)(t-5)}=\frac{A}{t+3}+\frac{B}{t-5}$

$\Rightarrow \quad 7t+19 = A(t-5)+B(t+3)$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$$\therefore \quad \frac{t^2+5t+4}{(t+3)(t-5)}=1+\frac{1}{4(t+3)}+\frac{27}{4(t-5)}$$

$$\Rightarrow \quad I=\int\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx=\int dx+\frac{1}{4}\int\frac{1}{(x^2+3)}dx+\frac{27}{4}\int\frac{1}{(x^2-5)}dx$$

$$=x+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{27}{4}\times\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$$

$$=x+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$$

48. Let $I=\int\frac{1}{\sin x+\sin 2x}dx$

$$=\int\frac{1}{\sin x+2\sin x\cos x}dx=\int\frac{1}{\sin x(1+2\cos x)}dx$$

$$=\int\frac{\sin x}{\sin^2 x(1+2\cos x)}dx$$

Let $u=\cos x \Rightarrow du=-\sin x dx$

Also, $\sin^2 x = 1-\cos^2 x = 1-u^2$

$$\therefore\ I=\int\frac{-1}{(1-u^2)(1+2u)}du$$

$$=\int\frac{-1}{(1+u)(1-u)(1+2u)}du$$

Using partial fractions, we have

$$\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{A}{(1+u)}+\frac{B}{(1-u)}+\frac{C}{(1+2u)}$$

$\Rightarrow \quad -1 = A(1-u)(1+2u)+B(1+u)(1+2u)+C(1+u)(1-u)$

Putting $u = 1$, we get $B = -1/6$

Putting $u = -1$, we get $A = 1/2$

Put $u=-\frac{1}{2}$, we get $C=\frac{-4}{3}$

So, $\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}$

$$\Rightarrow\ I=\int\left[\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}\right]du$$

$$=\frac{1}{2}\log(1+u)+\frac{1}{6}\log(1-u)-\frac{4}{3\times2}\log(1+2u)+C_1$$

$$=\frac{1}{2}\log(1+\cos x)+\frac{1}{6}\log(1-\cos x)-\frac{2}{3}\log(1+2\cos x)+C_1$$

49. Let $I=\int\frac{x^2}{(x^2+4)(x^2+9)}dx$

Put $x^2 = y$. Then $\frac{x^2}{(x^2+4)(x^2+9)}=\frac{y}{(y+4)(y+9)}$

Let $\frac{y}{(y+4)(y+9)}=\frac{A}{y+4}+\frac{B}{y+9}$ ...(i)

$\Rightarrow \quad y = A(y+9)+B(y+4)$ ...(ii)

Putting $y = -4$ and $y = -9$ successively in (ii), we get $A=\frac{-4}{5}$ and $B=\frac{9}{5}$

Substituting the values of A and B in (i), we get

$$\frac{y}{(y+4)(y+9)}=\frac{-4/5}{(y+4)}+\frac{9/5}{(y+9)}$$

$$\Rightarrow \quad \frac{x^2}{(x^2+4)(x^2+9)}=\frac{-4}{5(x^2+4)}+\frac{9}{5(x^2+9)}$$

$$\therefore \quad I=\int\frac{x^2}{(x^2+4)(x^2+9)}dx$$

$$=\frac{-4}{5}\int\frac{1}{(x^2+4)}dx+\frac{9}{5}\int\frac{1}{(x^2+9)}dx$$

$$=\frac{-4}{5}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{9}{5}\times\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$=\frac{-2}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right)+C$$

50. Let $I=\int\frac{x}{(x-1)^2(x+2)}dx$. ...(i)

Using partial fraction, we have

$$\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2} \quad ...(ii)$$

$\Rightarrow \quad x = A(x-1)(x+2)+B(x+2)+C(x-1)^2$ ...(iii)

On comparing the coefficients of x^2, x and constant term in (iii), we get $0 = A+C$, $1 = A+B-2C$, $0 = -2A+2B+C$

Solving these, we get

$A=\frac{2}{9}$, $B=\frac{1}{3}$, $C=-\frac{2}{9}$

From (ii), we get $\frac{x}{(x-1)^2(x+2)}=\frac{2}{9}\cdot\frac{1}{x-1}+\frac{1}{3}\cdot\frac{1}{(x-1)^2}-\frac{2}{9}\cdot\frac{1}{x+2}$

$\therefore \quad I=\frac{2}{9}\int\frac{1}{x-1}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{x+2}dx$

$\Rightarrow \quad I=\frac{2}{9}\log|x-1|-\frac{1}{3}\cdot\frac{1}{x-1}\frac{-2}{9}\log|x+2|$

$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3}\cdot\frac{1}{x-1}+C_1$

51. Let $I=\int\frac{x}{(x^2+1)(x-1)}dx$

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$ ...(i)

$\Rightarrow \quad x=(Ax+B)(x-1)+C(x^2+1)$ ...(ii)

Comparing coefficients of x^2, x and constant terms, we get $0=A+C$, $1=B-A$, $0=-B+C$

Solving these, we get

$A=-\frac{1}{2}, C=\frac{1}{2}$ and $B=\frac{1}{2}$

$\therefore$ From (i), we get

$\frac{x}{(x^2+1)(x-1)}=\frac{-\frac{1}{2}(x-1)}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x-1}$

$=-\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x-1}$

$\therefore \quad I=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{dx}{x^2+1}+\frac{1}{2}\int\frac{dx}{x-1}$

$\Rightarrow I=-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C_1$

52. Let $I=\int\frac{x^2}{(x^2+1)(x^2+4)}dx$

Put $x^2=y$

So, $\frac{x^2}{(x^2+1)(x^2+4)}=\frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)}=\frac{A}{y+1}+\frac{B}{y+4}$ ...(i)

$\Rightarrow \quad y=A(y+4)+B(y+1)$ ...(ii)

Putting $y=-4$ and $y=-1$, successively in (ii), we get

$A=\frac{-1}{3}$ and $B=\frac{4}{3}$

From (i), we get

$\frac{y}{(y+1)(y+4)}=\frac{-1}{3(y+1)}+\frac{4}{3(y+4)}$

$\Rightarrow \quad \frac{x^2}{(x^2+1)(x^2+4)}=\frac{-1}{3(x^2+1)}+\frac{4}{3(x^2+4)}$

$\therefore \quad I=\int\frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{-1}{3}\int\frac{dx}{x^2+1}+\frac{4}{3}\int\frac{dx}{x^2+4}$

$=\frac{-1}{3}\times\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

$=\frac{-1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$

53. Let $I=\int\frac{x^3}{x^4+3x^2+2}dx$

Put $x^2=t \Rightarrow xdx=\frac{1}{2}dt$

$\therefore \quad I=\frac{1}{2}\int\frac{t}{t^2+3t+2}dt=\frac{1}{2}\int\frac{t}{(t+2)(t+1)}dt$ ...(i)

Let $\frac{t}{(t+2)(t+1)}=\frac{A}{(t+2)}+\frac{B}{(t+1)}$

$\Rightarrow \quad t=A(t+1)+B(t+2)$

Put $t=-1, -2$ in it, we get $A=2$, $B=-1$

$\therefore \quad \frac{t}{(t+2)(t+1)}=\frac{2}{t+2}-\frac{1}{t+1}$ ...(ii)

From (i) and (ii), we get $I=\frac{1}{2}\int\left[\frac{2}{(t+2)}-\frac{1}{(t+1)}\right]dt$

$=\frac{1}{2}[2\log|t+2|-\log|t+1|]+C=\frac{1}{2}[2\log|x^2+2|-\log|x^2+1|]+C$

Concept Applied

➲ $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

54. Let $I=\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$ ...(i)

$\Rightarrow \quad x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$

Put $x=-1, -2, 0$ successively in it, we get

$B=1; C=3; A=-2$

From (i), we get

$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}$

Integrating both sides w.r.t. x, we get

$I=\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

$=-2\int\frac{1}{x+1}dx+\int\frac{1}{(x+1)^2}dx+3\int\frac{1}{x+2}dx$

$=-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C_1$

55. (d) : Let $I=\int e^x\left(\log x+\frac{1}{x}\right)dx$

$\Rightarrow \quad I=e^x\log x+c \quad \left(\because \int e^x[f(x)+f'(x)]dx=e^xf(x)+c\right)$

56. (d) : Let $I=\int\frac{e^x}{x+1}[1+(x+1)\log(x+1)]dx$

$=\int e^x\left[\frac{1}{x+1}+\log(x+1)\right]dx$

It is of the form $\int e^x[f(x)+f'(x)dx]$,

where $f(x)=\log(x+1)$ and $f'(x)=\frac{1}{x+1}$

So, $I=e^x\log(x+1)+C$

Concept Applied

➲ $\int e^x[f(x)+f'(x)]dx=e^xf(x)+c$

57. Let $I = \int x^4 \log x\, dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$

$= \frac{x^5}{5}\log x - \frac{1}{5}\int x^4 dx = \frac{1}{5}x^5 \log x - \frac{x^5}{25} + C$

58. Let $I = \int \frac{\log x - 3}{(\log x)^4} dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$\therefore \quad I = \int \left[\frac{t-3}{t^4}\right] e^t dt = \int e^t t^{-3} dt - 3\int t^{-4} e^t dt$

$= t^{-3}e^t + 3\int t^{-4}e^t dt - 3\int t^{-4}e^t dt + c$

$= t^{-3}e^t + c = (\log x)^{-3} x + c$

59. Let $I = \int \sin^{-1}(2x) dx = \int 1 \cdot \sin^{-1}(2x) dx$

$= \sin^{-1}(2x)x - \int\left(\frac{1}{\sqrt{1-4x^2}} \frac{d}{dx}(2x) \cdot x\right)dx$

$= x\sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} dx$

$= x\sin^{-1}(2x) + \int \frac{dt}{4\sqrt{t}}$ (Putting $1-4x^2 = t \Rightarrow -8x\, dx = dt$)

$= x\sin^{-1}(2x) + \frac{2}{4}(t)^{1/2} + C = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

60. Let $I = \int x \cdot \tan^{-1} x\, dx$

On integrating by parts w.r.t. x, we get

$I = \tan^{-1} x \int x dx - \int\left\{\frac{d}{dx}(\tan^{-1} x)\int x dx\right\}dx$

$= (\tan^{-1} x)\frac{x^2}{2} - \int \frac{1}{(1+x^2)} \frac{x^2}{2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$

$= \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{x}{2} + C$

61. Let $I = \int e^x \sin 2x\, dx$ ...(i)

$= \left[\sin 2x\, e^x - 2\int \cos 2x\, e^x dx\right]$

$= \left[e^x \sin 2x - 2\int e^x \cos 2x\, dx\right]$

$= \left[e^x \sin 2x - 2\left[\cos 2x e^x + 2\int \sin 2x e^x dx\right]\right]$

$\Rightarrow \quad I = e^x \sin 2x - 2e^x \cos 2x - 4I + C$ (From (i))

$\Rightarrow \quad 5I = e^x \sin 2x - 2e^x \cos 2x + C$

$\Rightarrow \quad I = \frac{1}{5}(e^x \sin 2x - 2e^x \cos 2x) + \frac{C}{5}$

$= \frac{e^x}{5}(\sin 2x - 2\cos 2x) + C_1$, where $C_1 = \frac{C}{5}$ is an arbitrary constant.

Answer Tips

- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\ [a\sin bx - b\cos bx] + C$

62. Let $I = \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$

$= \sec x \tan x - \int \sec x \tan x \cdot \tan x\, dx$

(Applying integration by parts)

$= \sec x \tan x - \int \sec x(\sec^2 x - 1) dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$= \sec x \tan x - I + \ln|\sec x + \tan x| + C_1$

$\Rightarrow \quad 2I = \sec x \tan x + \ln|\sec x + \tan x| + C_1$

$\therefore \quad I = \frac{1}{2}(\sec x \tan x) + \frac{1}{2}\ln|\sec x + \tan x| + C \quad \left(\text{Where, } C = \frac{C_1}{2}\right)$

Key Points

- $\int \sec x\, dx = \log|\sec x + \tan x| + C$

63. Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

$= \sin^{-1} x \int \frac{x}{\sqrt{1-x^2}} dx - \int\left[\frac{d}{dx}(\sin^{-1} x)\int \frac{x}{\sqrt{1-x^2}} dx\right]dx$

(Applying integration by parts)

Firstly, let us evaluate the integral $\int \frac{x}{\sqrt{1-x^2}} dx$

Put $t = 1 - x^2 \Rightarrow dt = -2x\, dx$.

$\therefore \quad \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$

$\therefore \quad I = \sin^{-1} x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$

$= -\sqrt{1-x^2}\sin^{-1} x + \int dx = -\sqrt{1-x^2}\sin^{-1} x + x + C$

64. Let $I = \int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx$

$= -\int \frac{(-x^2 + 3x - 1)}{\sqrt{1-x^2}} dx = -\int \frac{(1-x^2) + 3x - 2}{\sqrt{1-x^2}} dx$

i.e., $I = -\int \sqrt{1-x^2}\, dx + \int \frac{-3x+2}{\sqrt{1-x^2}} dx$

$= -\int \sqrt{1-x^2}\, dx + \frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}} dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$

$= -\int \sqrt{1-x^2}\, dx + \frac{3 \times 2}{2}\sqrt{1-x^2} + 2\sin^{-1} x + C$

$= -\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + 3\sqrt{1-x^2} + 2\sin^{-1} x + C$

$= -\frac{x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + 3\sqrt{1-x^2} + C$

Commonly Made Mistake

- Remember the formula for $\int \sqrt{a^2 - x^2}\, dx$ and $\int \frac{1}{\sqrt{a^2 - x^2}} dx$.

65. Let $I=\int\frac{\log x}{(x+1)^2}dx=\int(x+1)^{-2}\cdot\log x\,dx$

On integrating by parts, taking log x as first function, we have

$$I=\frac{(x+1)^{-1}}{-1}\cdot\log x-\int\frac{(x+1)^{-1}}{-1}\cdot\frac{1}{x}\,dx$$

$$=\frac{-\log x}{x+1}+\int\frac{dx}{x(x+1)}=\frac{-\log x}{x+1}+\int\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$=-\frac{\log x}{x+1}+\log x-\log(x+1)+C$$

$$=-\frac{\log x}{x+1}+\log\left(\frac{x}{x+1}\right)+C$$

66. Let $I=\int e^{2x}\sin(3x+1)dx$

On integrating by parts, taking e^x as first function, we have

$$=e^{2x}\int\sin(3x+1)dx-\int\left(\frac{d(e^{2x})}{dx}\cdot\int\sin(3x+1)dx\right)dx$$

$$=e^{2x}\frac{[-\cos(3x+1)]}{3}-\int 2e^{2x}\cdot\frac{[-\cos(3x+1)]}{3}dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\int e^{2x}\cos(3x+1)dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\left[e^{2x}\int\cos(3x+1)dx-\int\left(\frac{d}{dx}(e^{2x})\cdot\int\cos(3x+1)dx\right)dx\right]$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)\ -\frac{4}{9}\int e^{2x}\sin(3x+1)dx$$

$$\Rightarrow\quad I+\frac{4}{9}I=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow\quad \frac{13I}{9}=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow\quad I=\frac{9}{13}\left[\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1\right]$$

$$=\frac{9}{13}e^{2x}\left[\frac{2\sin(3x+1)-3\cos(3x+1)}{9}\right]+\frac{9}{13}C_1$$

$$=\frac{1}{13}e^{2x}[2\sin(3x+1)-3\cos(3x+1)]+C\qquad\left(\text{Where, }C=\frac{9C_1}{13}\right)$$

67. Let $I=\int\frac{x^2+1}{(x+1)^2}e^x dx$

$$=\int e^x\cdot\left[\frac{(x^2-1)+2}{(x+1)^2}\right]dx$$

$$=\int e^x\cdot\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]dx$$

$$=\int e^x\cdot\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]dx\qquad\ldots(i)$$

Take $f(x)=\frac{x-1}{x+1}$

$$\Rightarrow\quad f'(x)=\frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}=\frac{2}{(x+1)^2}$$

By using the formula, we get

$$I=\int e^x[f(x)+f'(x)]dx\ =e^x f(x)+C$$

$$I=e^x\cdot\left[\frac{x-1}{x+1}\right]+C$$

68. Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x=\theta\Rightarrow x=\cos\theta\Rightarrow dx=-\sin\theta\,d\theta$

$$\Rightarrow\quad I=\int\frac{\cos\theta(\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)d\theta$$

$$\Rightarrow\quad I=-\int\theta\cos\theta d\theta$$

$$\Rightarrow\quad -I=\theta\int\cos\theta\,d\theta-\int\left(\frac{d}{d\theta}\theta\int\cos\theta d\theta\right)d\theta$$

(Applying integration by parts)

$$\Rightarrow\quad -I=\theta\sin\theta-\int\sin\theta d\theta$$

$$\Rightarrow\quad -I=\theta\sin\theta+\cos\theta+C$$

$$\Rightarrow\quad I=-[\theta\sqrt{1-\cos^2\theta}+\cos\theta]+C$$

$$\Rightarrow\quad I=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Key Points

- The value of $\sin(-\theta)$ is –ve and value of $\cos(-\theta)$ is +ve.

69. Let $I=\int\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$

$$=\int\frac{\sqrt{x^2+1}\log\left(\frac{x^2+1}{x^2}\right)}{x^4}dx$$

$$\Rightarrow\quad 2x\,dx=\frac{-1}{(t-1)^2}dt\ \Rightarrow\ dx=-\frac{1}{2x}\cdot\frac{1}{(t-1)^2}dt$$

$$=-\frac{1}{2}\cdot\sqrt{t-1}\cdot\frac{1}{(t-1)^2}dt\ =-\frac{dt}{2(t-1)^{3/2}}$$

Also, $\sqrt{x^2+1}=\sqrt{\frac{1}{t-1}+1}=\sqrt{\frac{t}{t-1}}$

$$\therefore\quad I=\int\sqrt{\frac{t}{t-1}}\cdot\log t\cdot\frac{1}{1/(t-1)^2}\times\frac{-dt}{2(t-1)^{3/2}}\ =-\frac{1}{2}\int\sqrt{t}\cdot\log t\,dt$$

On integrating by parts, taking log t as first function, we have

$$=-\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\cdot\log t-\int\frac{t^{3/2}}{3/2}\cdot\frac{1}{t}dt\right]+C$$

$$=-\frac{1}{3}\left[t^{3/2}\log t-\int t^{1/2}dt\right]+C$$

$$=\frac{-1}{3}\left[t^{3/2}\log t-\frac{2}{3}t^{3/2}\right]+C$$

$$=\frac{-1}{3}\left[\left(\frac{x^2+1}{x^2}\right)^{3/2}\log\left(\frac{x^2+1}{x^2}\right)-\frac{2}{3}\left(\frac{x^2+1}{x^2}\right)^{3/2}\right]+C$$

70. Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx, x\in[0,1]$

We know that $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\sqrt{x}=\frac{\pi}{2}-\cos^{-1}\sqrt{x}$$

$$\therefore \quad I=\int\frac{\frac{\pi}{2}-2\cos^{-1}\sqrt{x}}{\pi/2}dx$$

$$=\int 1\cdot dx-\frac{4}{\pi}\int 1\cdot\cos^{-1}\sqrt{x}\,dx$$

$$=x-\frac{4}{\pi}\left[x\cdot\cos^{-1}\sqrt{x}-\int x\cdot\frac{-1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}dx\right]+C$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}+\frac{1}{2}\int\sqrt{\frac{x}{1-x}}\,dx\right]+C$$

Put $x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta\, d\theta$

$$\therefore I=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta d\theta+C$$

$$=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}\int\frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta d\theta+C$$

$$=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}\int(1-\cos 2\theta)d\theta+C$$

$$=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}\left[\theta-\frac{\sin 2\theta}{2}\right]+C$$

$$=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}[\theta-\sin\theta\cos\theta]+C$$

$$=x-\frac{4}{\pi}x\cos^{-1}\sqrt{x}-\frac{2}{\pi}[\sin^{-1}\sqrt{x}-\sqrt{x}\sqrt{1-x}]+C$$

Concept Applied

- $\cos 2\theta = 1-2\sin^2\theta$
  $\Rightarrow 2\sin^2\theta = 1-\cos 2\theta$

71. (d): Let $I=\int_{-1}^{1}\frac{|x-2|}{x-2}dx$

$$=\int_{-1}^{1}\frac{-(x-2)}{x-2}dx=\int_{-1}^{1}-1\cdot dx=[-x]_{-1}^{1}=-[1-(-1)]=-2$$

72. (a): Let $I=\int_0^4(e^{2x}+x)dx=\left[\frac{e^{2x}}{2}+\frac{x^2}{2}\right]_0^4$

$$=\frac{e^8}{2}+\frac{16}{2}-\frac{e^0}{2}-0=\frac{e^8}{2}+\frac{16}{2}-\frac{1}{2}=\frac{e^8+15}{2}$$

73. (a): Let $I=\int_0^{\pi/8}\tan^2(2x)dx=\int_0^{\pi/8}(\sec^2(2x)-1)dx$

$$=\left(\frac{1}{2}\tan 2x-x\right)_0^{\pi/8}=\frac{1}{2}\tan 2\left(\frac{\pi}{8}\right)-\frac{\pi}{8}=\frac{1}{2}\tan\frac{\pi}{4}-\frac{\pi}{8}$$

$$=\frac{1}{2}-\frac{\pi}{8}=\frac{4-\pi}{8}$$

74. (d): Let $I=\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx=[\tan x]_{-\pi/4}^{\pi/4}$

$$=\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)=1+1=2$$

75. We have, $\int_2^3 3^x dx=\left[\frac{3^x}{\log 3}\right]_2^3=\frac{3^3-3^2}{\log 3}=\frac{18}{\log 3}$

76. $\int\frac{dx}{9+x^2}=\int\frac{dx}{x^2+3^2}=\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]=F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have

$$\int_0^3\frac{dx}{9+x^2}=F(3)-F(0)=\frac{1}{3}[\tan^{-1}1-\tan^{-1}0]=\frac{1}{3}\left[\frac{\pi}{4}-0\right]=\frac{\pi}{12}$$

Answer Tips

- $\int\frac{1}{a^2+b^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

77. Let $I=\int_0^{\pi/2}e^x(\sin x-\cos x)dx$

$$=\underset{\text{I}}{\int_0^{\pi/2}e^x\sin x\,dx}-\underset{\text{II}}{\int_0^{\pi/2}e^x\cos x\,dx}$$

On integrating II integral by parts, we get

$$I=\int_0^{\pi/2}e^x\sin x\,dx-\left[\{e^x\cos x\}_0^{\pi/2}-\int_0^{\pi/2}e^x(-\sin x)dx\right]$$

$$=\int_0^{\pi/2}e^x\sin x\,dx-[e^{\pi/2}\cdot 0-e^0\cdot 1]-\int_0^{\pi/2}e^x\sin x\,dx=1$$

78. By the first fundamental theorem of integral calculus

$f'(x)=\frac{d}{dx}\int_0^x t\sin t\,dt=x\sin x.$

79. Given, $\int_0^a\frac{1}{4+x^2}dx=\frac{\pi}{8}$

$$\Rightarrow \int_0^a\frac{1}{x^2+2^2}dx=\frac{\pi}{8}\Rightarrow\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a=\frac{\pi}{8}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{a}{2}=\frac{\pi}{8}\Rightarrow\tan^{-1}\frac{a}{2}=\frac{\pi}{4}$$

$$\Rightarrow \frac{a}{2}=\tan\frac{\pi}{4}=1\Rightarrow a=2$$

80. We have, $\int_0^{\pi/4}\tan x\,dx=[\log|\sec x|]_0^{\pi/4}$

$$=\log\left|\sec\frac{\pi}{4}\right|-\log|\sec 0|=\log|\sqrt{2}|-\log|1|$$

$$\therefore \int_0^{\pi/4}\tan x\,dx=\frac{1}{2}\log 2$$

81. We have, $\int\sin 2x\,dx=-\frac{1}{2}[\cos 2x]=F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have $\int_0^{\pi/4}\sin 2x\,dx=F\left(\frac{\pi}{4}\right)-F(0)$

$$\Rightarrow \int F(\sin 2x)dx=-\frac{1}{2}[\cos(\pi/2)-\cos(0)]=\frac{1}{2}$$

82. We have, $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx=[\sin^{-1}x]_0^1$

$=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}-0=\frac{\pi}{2}$

83. Here, $\int_1^2 \frac{x^3-1}{x^2}dx=\int_1^2 (x-x^{-2})dx$

$=\left[\frac{x^2}{2}-\frac{x^{-1}}{-1}\right]_1^2=\left[\frac{x^2}{2}+\frac{1}{x}\right]_1^2$

$=\left[\frac{4}{2}+\frac{1}{2}\right]-\left[\frac{1}{2}+1\right]=\frac{5}{2}-\frac{3}{2}=1$

84. $\int_0^1 x^2e^x dx=[x^2e^x]_0^1-2\int_0^1 xe^x dx$

$=e-2[xe^x-e^x]_0^1=e-2(0-(-1))=e-2$

85. Let $I=\int_1^2\left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}dx$

Putting $2x=y \Rightarrow 2dx=dy$
When $x\to 1$, then $y\to 2$
and when $x\to 2$, then $y\to 4$

$\therefore \quad I=\frac{1}{2}\int_2^4\left[\frac{2}{y}-\frac{2}{y^2}\right]e^y dy=\int_2^4\left[\frac{1}{y}-\frac{1}{y^2}\right]e^y dy$

$\Rightarrow \quad I=\left[e^y\cdot\frac{1}{y}\right]_2^4=\frac{1}{4}e^4-\frac{1}{2}e^2=\frac{e^2}{2}\left(\frac{e^2}{2}-1\right)$

86. Let $I=\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$

$=\int_0^1 \tan^{-1}\left[\frac{(1-x)-x}{1+x(1-x)}\right]dx$

$I=\int_0^1 [\tan^{-1}(1-x)-\tan^{-1}x]dx$ ...(i)

$I=\int_0^1 [\tan^{-1}x-\tan^{-1}(1-x)]dx$ ...(ii)

$\left[\text{Using properly }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I=\int_0^1 [\tan^{-1}(1-x)-\tan^{-1}x+\tan^{-1}x-\tan^{-1}(1-x)]dx=0$

$\Rightarrow \quad I=0$

87. Let $I=\int_0^{\pi/2} x^2\sin x\,dx$

On integrating by parts, we have

$I=[x^2(-\cos x)]_0^{\pi/2}-\int_0^{\pi/2} 2x(-\cos x)dx$

$=-\frac{\pi^2}{4}.0+0+2\int_0^{\pi/2} x\cos x dx=2\int_0^{\pi/2} x\cos x dx$

Again integrating by parts, we have

$I=2\left[[x\sin x]_0^{\pi/2}-\int_0^{\pi/2} 1\cdot\sin x dx\right]$

$=2\left\{\frac{\pi}{2}\cdot 1-0-[-\cos x]_0^{\pi/2}\right\}=2\left[\frac{\pi}{2}+(0-1)\right]=\pi-2$

88. Let $I=\int_2^4 \frac{x}{x^2+1}dx$

Put $x^2+1=t \Rightarrow xdx=\frac{1}{2}dt$

When $x=2$, then $t=5$ and when $x=4$, then $t=17$

$\therefore \quad I=\frac{1}{2}\int_5^{17}\frac{dt}{t}=\frac{1}{2}[\log t]_5^{17}=\frac{1}{2}[\log 17-\log 5]=\frac{1}{2}\log\left(\frac{17}{5}\right)$

89. Let $I=\int_e^{e^2}\frac{dx}{x\log x}$

Put $\log x=t \Rightarrow \frac{1}{x}dx=dt$

When $x=e$, then $t=\log e=1$
and when $x=e^2$, then $t=\log e^2=2\log e=2$

$\therefore \quad I=\int_1^2\frac{dt}{t}=[\log t]_1^2=\log 2-\log 1=\log 2$

90. Let $I=\int_0^1 xe^{x^2}dx$

Put $x^2=t\Rightarrow xdx=\frac{1}{2}dt.$

When $x=0$, then $t=0$ and when $x=1$, then $t=1$.

$\therefore \quad I=\frac{1}{2}\int_0^1 e^t dt\Rightarrow I=\frac{1}{2}[e^t]_0^1=\frac{1}{2}(e-1)$

91. Let $I=\int_0^1\frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x=t\Rightarrow \frac{1}{1+x^2}dx=dt$

When $x=0$, then $t=0$ and when $x=1$, then $t=\frac{\pi}{4}$

$\therefore \quad I=\int_0^{\pi/4} t dt=\left[\frac{1}{2}t^2\right]_0^{\pi/4}=\frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2-0\right]=\frac{\pi^2}{32}$

92. Let $I=\int_{-\frac{\pi}{4}}^0\frac{(1+\tan x)}{(1-\tan x)}dx=\int_{-\frac{\pi}{4}}^0\frac{\left(1+\frac{\sin x}{\cos x}\right)}{\left(1-\frac{\sin x}{\cos x}\right)}dx$

$=\int_{-\frac{\pi}{4}}^0\frac{\cos x+\sin x}{\cos x-\sin x}dx$

Put $\cos x-\sin x=t\Rightarrow -(\sin x+\cos x)\,dx=dt$

When $x=0$, then $t=1$, when $x=\frac{-\pi}{4}$, then $t=\sqrt{2}$

$$\therefore\ I=\int_{\sqrt{2}}^{1}-\frac{dt}{t}=\int_{1}^{\sqrt{2}}\frac{dt}{t}=[\log t]_1^{\sqrt{2}}=\log\sqrt{2}-\log 1=\frac{1}{2}\log 2$$

Concept Applied

➲ $\int_b^a f(x)dx=-\int_b^a f(x)dx$

93. Let $I=\int_{\pi/4}^{\pi/2}e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

Put $2x=t \Rightarrow dx=\frac{1}{2}dt$

When $x=\frac{\pi}{4}$, $t=\frac{\pi}{2}$; When $x=\frac{\pi}{2}$, $t=\pi$

$$\therefore\ I=\frac{1}{2}\int_{\pi/2}^{\pi}e^t\left(\frac{1-\sin t}{1-\cos t}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi}e^t\left(\frac{1-2\sin t/2\cos t/2}{2\sin^2 t/2}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi}e^t\left(\frac{1}{2}\operatorname{cosec}^2\frac{t}{2}-\cot\frac{t}{2}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi}e^t\left(-\cot\frac{t}{2}+\frac{1}{2}\operatorname{cosec}^2\frac{t}{2}\right)dt$$

$$\Rightarrow I=\left[\frac{1}{2}e^t\left(-\cot\frac{t}{2}\right)\right]_{\pi/2}^{\pi}\ \left(\because \int e^x[f(x)+f'(x)]dx=e^xf(x)+C\right)$$

$$\Rightarrow I=\frac{1}{2}\left[e^{\pi}\left(-\cot\frac{\pi}{2}\right)-e^{\pi/2}\left(-\cot\frac{\pi}{4}\right)\right]=\frac{1}{2}(0+e^{\pi/2})=\frac{e^{\pi/2}}{2}$$

94. Let $I=\int_0^{\pi/2}\sqrt{\sin x}\cos^5 x\,dx$

$$\Rightarrow I=\int_0^{\pi/2}\sqrt{\sin x}\,(\cos^2 x)^2\cos x\,dx$$

$$\Rightarrow I=\int_0^{\pi/2}\sqrt{\sin x}\,(1-\sin^2 x)^2\cos x\,dx$$

Put $\sin x=t \Rightarrow \cos x\,dx=dt$
When $x=0$, $t=\sin 0=0$
When $x=\pi/2$, $t=\sin\pi/2=1$

$$\therefore\ I=\int_0^1\sqrt{t}\,(1-t^2)^2\,dt=\int_0^1\sqrt{t}\,(1+t^4-2t^2)\,dt$$

$$=\int_0^1(\sqrt{t}+t^{9/2}-2t^{5/2})\,dt=\left[\frac{t^{3/2}}{3/2}+\frac{t^{11/2}}{11/2}-\frac{2t^{7/2}}{7/2}\right]_0^1$$

$$=\frac{2}{3}+\frac{2}{11}-\frac{4}{7}=\frac{154+42-132}{231}=\frac{64}{231}$$

95. Let $I=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx$

$$=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Put $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

When $x=\frac{\pi}{3}$, then $t=\frac{\sqrt{3}-1}{2}=\alpha$

and when, $x=\frac{\pi}{6}$, then $t=\frac{1-\sqrt{3}}{2}=-\alpha$

$$\therefore\ I=\int_{-\alpha}^{\alpha}\frac{dt}{\sqrt{1-t^2}}=[\sin^{-1}t]_{-\alpha}^{\alpha}=2\sin^{-1}\alpha=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

96. Let $I=\int_0^{\pi}e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx$

Put $\frac{\pi}{4}+x=t\Rightarrow x=t-\frac{\pi}{4}\quad\Rightarrow dx=dt$

When $x=0$, $t=\frac{\pi}{4}$ and when $x=\pi$, $t=\frac{5\pi}{4}$

$$\therefore\ I=\int_{\pi/4}^{5\pi/4}e^{2\left(t-\frac{\pi}{4}\right)}\sin t\,dt=e^{-\pi/2}\int_{\pi/4}^{5\pi/4}e^{2t}\sin t\,dt$$

$$=e^{-\pi/2}\left[\left(\sin t\frac{e^{2t}}{2}\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\cos t\frac{e^{2t}}{2}dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(e^{5\pi/2}\sin\frac{5\pi}{4}-e^{\pi/2}\sin\frac{\pi}{4}\right)-\left(\frac{e^{2t}}{4}\cos t\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\frac{e^{2t}}{4}\sin t\,dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(\frac{-1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right)-\frac{1}{4}\left(-\frac{1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right)\right]-\frac{I}{4}$$

$$\Rightarrow\quad I+\frac{1}{4}I=-\frac{1}{2\sqrt{2}}[e^{2\pi}+1]+\frac{1}{4\sqrt{2}}[e^{2\pi}+1]$$

$$\Rightarrow\quad \frac{5}{4}I=\frac{(e^{2\pi}+1)}{2\sqrt{2}}\left[\frac{1}{2}-1\right]=-\frac{1}{4\sqrt{2}}[e^{2\pi}+1]\Rightarrow I=\frac{-1}{5\sqrt{2}}(1+e^{2\pi})$$

97. Let $I=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\cdot 2\sin x\cos x}}$

$$=\frac{1}{2}\int\frac{dx}{\cos^{\left(3+\frac{1}{2}\right)}x\cdot\sin^{\frac{1}{2}}x}=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^{\frac{7}{2}}x\cdot\tan^{\frac{1}{2}}x\cdot\cos^{\frac{1}{2}}x}$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^4 x\sqrt{\tan x}}=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\sqrt{\tan x}}dx$$

Put $\tan x=t\Rightarrow \sec^2 x dx=dt$

When $x=0$, then $t=0$ and when $x=\frac{\pi}{4}$, then $t=1$

$$\therefore\quad I=\frac{1}{2}\int_0^1\frac{(1+t^2)dt}{\sqrt{t}}=\frac{1}{2}\int_0^1(t^{-\frac{1}{2}}+t^{\frac{3}{2}})dt$$

$$=\frac{1}{2}\left[\frac{t^{1/2}}{1/2}+\frac{t^{5/2}}{5/2}\right]_0^1=\frac{1}{2}\left[2+\frac{2}{5}\right]=1+\frac{1}{5}=\frac{6}{5}$$

98. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

Put $\sin x-\cos x=t$

$\Rightarrow$ $(\cos x+\sin x)\,dx=dt$

and $1-2\sin x\cos x=t^2 \Rightarrow 1-\sin 2x=t^2$

When $x=\frac{\pi}{4}, t=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$

When $x=0$, $t=\sin 0-\cos 0=-1$

$\therefore \int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx=\int_{-1}^{0}\frac{dt}{16+9(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{25-9t^2}=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{3}\right)^2-t^2}\quad=\frac{1}{9}\cdot\frac{1}{2\times\frac{5}{3}}\left[\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^{0}$

$=\frac{1}{30}[\log 1-(\log 1-\log 4)]=\frac{1}{30}\log 4$

Key Points

➲ $\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

99. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Put $\sin x-\cos x=t$

$\Rightarrow$ $(\cos x+\sin x)\,dx=dt$

Also, $1-2\sin x\cos x=t^2 == 1-\sin 2x=t^2$

When $x=\frac{\pi}{4}$, $t=0$ and when $x=0$, $t=-1$

$\therefore\ I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$

$=\frac{1}{16}\cdot\frac{4}{2\times 5}\left[\log\left|\frac{5/4+t}{5/4-t}\right|\right]_{-1}^{0}\quad=\frac{1}{40}\left[\log 1-\log\left(\frac{1}{9}\right)\right]$

$=\frac{1}{40}[\log 1-\log 1+\log 9]=\frac{1}{40}\log 9$

100. (a): Let $I=\int_2^8\frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ... (i)

$=\int_2^8\frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx\ \left(\because\ \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$

$=\int_2^8\frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$ ... (ii)

Adding (i) and (ii), we get

$2I=\int_2^8\frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}dx=\int_2^8 1\,dx=[x]_2^8$

$\Rightarrow\ I=\frac{1}{2}(8-2)=\frac{6}{2}=3$

Hence, both assertion and reason are true and reason is the correct explanation of assertion.

101. Let $I=\int_0^{\pi/2}\frac{1}{1+\cot^{5/2}x}dx$

$\Rightarrow\ I=\int_0^{\pi/2}\frac{\sin^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$

$\Rightarrow\ I=\int_0^{\pi/2}\frac{\sin^{5/2}\left(\frac{\pi}{2}-x\right)}{\sin^{5/2}\left(\frac{\pi}{2}-x\right)+\cos^{5/2}\left(\frac{\pi}{2}-x\right)}dx$ ...(i)

$\left[\because\ \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\Rightarrow\ I=\int_0^{\pi/2}\frac{\cos^{5/2}x}{\cos^{5/2}x+\sin^{5/2}x}dx$ ...(ii)

Adding (i) and (ii), we get

$2I=\int_0^{\pi/2}\frac{\sin^{5/2}x+\cos^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$

$\Rightarrow\ 2I=\int_0^{\pi/2}1.\,dx=(x)_0^{\pi/2}=\frac{\pi}{2}\Rightarrow 2I=\ \frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$

102. Let $I=\int_1^3|2x-1|dx$

$=\int_1^3(2x-1)dx=\left[\frac{2x^2}{2}-x\right]_1^3$

$=[(3^2-3)-(1^2-1)]=[(9-3)-(1-1)]=6$

103. Let $I=\int_{-2}^{2}|x|dx$

$\therefore\ I=\int_{-2}^{0}(-x)dx+\int_0^2 x\,dx=\left[-\frac{x^2}{2}\right]_{-2}^{0}+\left[\frac{x^2}{2}\right]_0^2$

$=2+2=4$

104. Let $I=\int_1^4|x-5|\,dx$

$=-\int_1^4(x-5)\,dx=\left[-\frac{x^2}{2}+5x\right]_1^4$

$=-\frac{16}{2}+5(4)+\frac{1}{2}-5=-8+20-5+\frac{1}{2}=7+\frac{1}{2}=\frac{15}{2}$

105. Let $I=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ ...(i)

$\Rightarrow\ I=\int_0^{\pi/2}\log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$

$\left[\text{Using property }\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\Rightarrow\ I=\int_0^{\pi/2}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx+\int_0^{\pi/2}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)\cdot\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\pi/2}\log 1\,dx=0\Rightarrow I=0$$

106. Let $I=\int_0^{2\pi}\frac{1}{1+e^{\sin x}}dx$ ...(i)

$$\Rightarrow I=\int_0^{2\pi}\frac{1}{1+e^{\sin(2\pi-x)}}dx \qquad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{2\pi}\frac{1}{1+e^{-\sin x}}dx$$

$$=\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+1}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{2\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx=\int_0^{2\pi}1\cdot dx$$

$$\Rightarrow I=\frac{1}{2}[x]_0^{2\pi}=\frac{1}{2}\times 2\pi=\pi$$

107. Let $I=\int_{-2}^{2}\frac{x^2}{1+5^x}dx$ ...(i)

Using property : $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$\therefore I=\int_{-2}^{2}\frac{(-2+2-x)^2}{1+5^{-2+2-x}}dx \Rightarrow I=\int_{-2}^{2}\frac{x^2}{1+5^{-x}}dx$$

$$\Rightarrow I=\int_{-2}^{2}\frac{5^x\cdot x^2}{5^x+1}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{-2}^{2}\frac{5^x x^2+x^2}{5^x+1}dx$$

$$\Rightarrow 2I=\int_{-2}^{2}x^2dx=\left[\frac{x^3}{3}\right]_{-2}^{2} \Rightarrow I=\frac{1}{6}(8+8)=\frac{16}{6}=\frac{8}{3}$$

108. Let $I=\int_1^4\{|x|+|3-x|\}dx$

$$|x|+|3-x|=\begin{cases}x+3-x, & 1\le x<2\\ x+3-x, & 2\le x<3\\ x+x-3, & 3\le x<4\end{cases}=\begin{cases}3, & 1\le x<3\\ 2x-3, & 3\le x<4\end{cases}$$

$$\therefore I=\int_1^3 3dx+\int_3^4(2x-3)dx=[3x]_1^3+\left[\frac{2x^2}{2}-3x\right]_3^4$$

$=(9-3)+(16-12-9+9)$
$=6+4=10$

109. Let $I=\int_1^3\frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx$ ...(i)

$$I=\int_1^3\frac{\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}}dx \quad \text{...(ii)} \qquad \left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

Adding (i) and (ii), we get

$$2I=\int_1^3\frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx$$

$$\Rightarrow 2I=\int_1^3 1\,dx \Rightarrow 2I=[x]_1^3$$

$$\Rightarrow 2I=2\Rightarrow I=1$$

110. Let $I=\int_0^{\pi}\frac{x}{9\sin^2x+16\cos^2x}dx$ ...(i)

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)dx}{9\sin^2(\pi-x)+16\cos^2(\pi-x)}$$

$$\left(\text{Using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)dx}{9\sin^2x+16\cos^2x} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi dx}{9\sin^2x+16\cos^2x}$$

Consider $f(x)=\frac{1}{9\sin^2x+16\cos^2x}$

$$f(\pi-x)=\frac{1}{9\sin^2(\pi-x)+16\cos^2(\pi-x)}=\frac{1}{9\sin^2x+16\cos^2x}=f(x)$$

$$\therefore I=\pi\int_0^{\pi/2}\frac{dx}{9\sin^2x+16\cos^2x}$$

$$\left(\text{Using }\int_0^{2a}f(x)dx=2\int_0^a f(x)dx,\text{ if } f(2a-x)=f(x)\right)$$

$$\Rightarrow I=\pi\int_0^{\pi/2}\frac{\sec^2x\,dx}{9\tan^2x+16}$$ (Dividing N^r & D^r by $\cos^2x$)

Put $\tan x=t\Rightarrow \sec^2x\,dx=dt$

When $x=0, t=0; x=\pi/2, t=\infty$

$$\Rightarrow I=\pi\int_0^{\infty}\frac{dt}{9t^2+16}=\frac{\pi}{9}\int_0^{\infty}\frac{dt}{t^2+\frac{16}{9}}$$

$$=\frac{\pi}{9}\cdot\frac{3}{4}\left[\tan^{-1}\frac{3t}{4}\right]_0^{\infty}=\frac{\pi}{12}\left(\tan^{-1}\infty-\tan^{-1}0\right)=\frac{\pi^2}{24}$$

111. Let $I=\int_{-1}^{2}|x^3-x|dx=\int_{-1}^{2}|x(x-1)(x+1)|dx$

$$=\int_{-1}^{0}(x^3-x)dx-\int_0^1(x^3-x)dx+\int_1^2(x^3-x)dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}-\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_0^1+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2=\frac{-3}{4}+\frac{3}{2}+2=\frac{11}{4}$$

112. In the R.H.S. integral, put $(a-x)=t$, so that $dx=-dt$.
Now, when $x=0$, then $t=a$
and when $x=a$, then $t=0$

$$\therefore \int_0^a f(a-x)dx=-\int_a^0 f(t)dt=\int_0^a f(t)dt=\int_0^a f(x)dx$$

Hence, $\int_0^a f(x)dx=\int_0^a f(a-x)\,dx$ ...(i)

Let $I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$ ... (ii)

$$=\int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+[\cos(\pi-x)]^2}dx \quad \text{[Using (i)]}$$

$$\Rightarrow \quad I=\int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad \text{...(iii)}$$

Adding (ii) and (iii), we get

$$2I=\int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$$

Put $\cos x=t \Rightarrow -\sin x\,dx=dt$
When $x=0$, then $t=1$ and when $x=\pi$, then $t=-1$

$$\therefore\ 2I=\int_1^{-1}\frac{-\pi dt}{1+t^2} \Rightarrow I=\frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2}$$

$$\therefore\ I=\frac{\pi}{2}[\tan^{-1}t]_{-1}^{1}=\frac{\pi}{2}[\tan^{-1}(1)-\tan^{-1}(-1)]$$

$$=\frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi^2}{4}.$$

113. In the R.H.S. integral, put $(a-x)=t$, so that $dx=-dt$.
Now, when $x=0$, then $t=a$
and when $x=a$, then $t=0$

$$\therefore \int_0^a f(a-x)dx=-\int_a^0 f(t)dt=\int_0^a f(t)dt=\int_0^a f(x)dx$$

Hence, $\int_0^a f(x)dx=\int_0^a f(a-x)\,dx$ ...(i)

Now, $\int_0^1 x^2(1-x)^n dx=\int_0^1 (1-x)^2(1-(1-x))^n dx$ [using (i)]

$$=\int_0^1 (1-x)^2 x^n dx=\int_0^1 (1+x^2-2x)x^n dx=\int_0^1 (x^n+x^{n+2}-2x^{n+1})dx$$

$$=\int_0^1 x^n dx+\int_0^1 x^{n+2}dx-2\int_0^1 x^{n+1}dx=\left[\frac{x^{n+1}}{n+1}\right]_0^1+\left[\frac{x^{n+3}}{n+3}\right]_0^1-2\left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$=\frac{1}{n+1}+\frac{1}{n+3}-\frac{2}{n+2}.$$

114. Let $I=\int_0^{3/2}|x\sin\pi x|dx$

When $0<x<1 \Rightarrow 0<\pi x<\pi \Rightarrow \sin\pi x>0$

When $1<x<\frac{3}{2} \Rightarrow \pi<\pi x<\frac{3\pi}{2} \Rightarrow \sin\pi x<0$

$$\therefore\ |x\sin\pi x|=\begin{cases} x\sin\pi x, \text{if } 0<x<1 \\ -x\sin\pi x, \text{if } 1<x<\frac{3}{2}\end{cases}$$

$$\therefore\ I=\int_0^1 x\sin\pi x\,dx-\int_1^{3/2} x\sin\pi x\,dx$$

$$=\left[\frac{-x\cos\pi x}{\pi}\right]_0^1+\int_0^1\frac{\cos\pi x}{\pi}dx+\left[\frac{x\cos\pi x}{\pi}\right]_1^{3/2}-\int_1^{3/2}\frac{\cos\pi x}{\pi}dx$$

$$=\frac{1}{\pi}-0+\left[\frac{\sin\pi x}{\pi^2}\right]_0^1+0-\frac{(-1)}{\pi}-\left[\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$=\frac{2}{\pi}+0-\frac{(-1)}{\pi^2}+0=\frac{2}{\pi}+\frac{1}{\pi^2}$$

115. Let $I=\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

$$=\int_0^\pi \frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx=\int_0^\pi\frac{x\sin x}{1+\sin x}dx \quad \text{...(i)}$$

$$=\int_0^\pi\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$$

$$\left[\text{Using the property} \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow \quad I=\int_0^\pi\frac{(\pi-x)\sin x}{1+\sin x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^\pi\frac{\pi\sin x}{1+\sin x}dx$$

$$\Rightarrow \quad I=\frac{\pi}{2}\int_0^\pi\frac{\sin x}{1+\sin x}dx=\frac{\pi}{2}\int_0^\pi\frac{1+\sin x-1}{1+\sin x}dx$$

$$=\frac{\pi}{2}\int_0^\pi\left(1-\frac{1}{1+\sin x}\right)dx$$

$$=\frac{\pi}{2}\left[\int_0^\pi dx-\int_0^\pi\frac{1}{1+\sin x}dx\right]=\frac{\pi}{2}\left[[x]_0^\pi-\int_0^\pi\frac{1-\sin x}{\cos^2 x}dx\right]$$

$$=\frac{\pi}{2}\left[[x]_0^\pi-\int_0^\pi\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx\right]$$

$$=\frac{\pi}{2}\left[(\pi-0)-\int_0^\pi(\sec^2 x-\tan x\cdot\sec x)dx\right]$$

$$=\frac{\pi}{2}\left[\pi-(\tan x-\sec x)_0^\pi\right]=\frac{\pi}{2}[\pi-[(0-(-1))-(0-1)]]=\frac{\pi}{2}[\pi-2]$$

116. Let $I=\int_1^4(|x-1|+|x-2|+|x-4|)dx$

Also, let $f(x)=|x-1|+|x-2|+|x-4|$
We have three critical points $x=1, 2$ and 4.

$$f(x)=\begin{cases}(x-1)-(x-2)-(x-4) \text{ if } 1\le x<2\\ (x-1)+(x-2)-(x-4) \text{ if } 2\le x<4\end{cases}$$

$$\therefore\ f(x)=\begin{cases}-x+5 \text{ if } 1\le x<2\\ x+1 \text{ if } 2\le x<4\end{cases}$$

$$\therefore\ I=\int_1^4 f(x)dx=\int_1^2 f(x)dx+\int_2^4 f(x)dx$$

$$=\int_1^2(-x+5)dx+\int_2^4(x+1)dx=\left[-\frac{x^2}{2}+5x\right]_1^2+\left[\frac{x^2}{2}+x\right]_2^4$$

$$=\left(-\frac{4}{2}+10\right)-\left(-\frac{1}{2}+5\right)+\left(\frac{16}{2}+4\right)-\left(\frac{4}{2}+2\right)$$

$$=8-\frac{9}{2}+12-4=16-\frac{9}{2}=\frac{23}{2}$$

117. Let $I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$ ...(i)

$$\Rightarrow\ I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow\ I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\left(\frac{\sin^2 x}{\sin x+\cos x}+\frac{\cos^2 x}{\sin x+\cos x}\right)dx\Rightarrow 2I=\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx$$

$$\Rightarrow\ 2I=\int_0^{\pi/2}\frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}dx$$

$$\Rightarrow\ 2I=\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$\Rightarrow\ 2I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t\Rightarrow\sec^2\frac{x}{2}\cdot\frac{1}{2}dx=dt$

When $x=0$, then $t=0$ and when $x=\frac{\pi}{2}$, then $t=1$

$$\therefore\ 2I=\int_0^1\frac{2dt}{2t+1-t^2}=2\int_0^1\frac{1}{(\sqrt{2})^2-(t-1)^2}dt$$

$$\Rightarrow\ 2I=2\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\left\{0-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\Rightarrow\ I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

Key Points

- $\sin 2A=\frac{2\tan A}{1+\tan^2 A}$ and $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$

118. Let $I=\int_0^{3/2}|x\cos\pi x|dx$

When $0<x<\frac{1}{2}\Rightarrow 0<\pi x<\frac{\pi}{2}\Rightarrow\cos\pi x>0$

When $\frac{1}{2}<x<\frac{3}{2}\Rightarrow\frac{\pi}{2}<\pi x<\frac{3\pi}{2}\Rightarrow\cos\pi x<0$

$$\therefore\ |x\cos\pi x|=\begin{cases}x\cos\pi x,\text{ if }0<x<\frac{1}{2}\\-x\cos\pi x,\text{ if }\frac{1}{2}<x<\frac{3}{2}\end{cases}$$

$$\therefore\ I=\int_0^{1/2}x\cos\pi x\,dx+\int_{1/2}^{3/2}-(x\cos\pi x)dx$$

$$\Rightarrow\ I=\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_0^{1/2}-\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_{1/2}^{3/2}$$

(Applying integration by parts)

$$=\left[\frac{1}{\pi}\left(\frac{1}{2}-0\right)+\frac{1}{\pi^2}(0-1)\right]-\left[\frac{1}{\pi}\left(\frac{3}{2}(-1)-\frac{1}{2}(1)\right)+\frac{1}{\pi^2}(0-0)\right]$$

$$=\left(\frac{1}{2\pi}-\frac{1}{\pi^2}\right)-\left(\frac{-2}{\pi}\right)=\left(\frac{5\pi-2}{2\pi^2}\right)$$

119. Let $I=\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi-x}{1+\sin\alpha\sin(\pi-x)}dx\qquad\left[\because\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-I\ \Rightarrow\ 2I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{1}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{1}{1+\sin\alpha\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)}dx$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

$$\therefore\ I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

Put $\tan\frac{x}{2}=t\Rightarrow\sec^2\frac{x}{2}dx=2dt$

Also, when $x\to 0$, $t\to\tan 0=0$;

when $x\to\pi$, $t\to\tan\frac{\pi}{2}=\infty$

$$\therefore\ I=\frac{\pi}{2}\int_0^{\infty}\frac{2dt}{t^2+2t\sin\alpha+1}$$

$$\Rightarrow \quad I=\pi\int_0^{\infty}\frac{1}{(t+\sin\alpha)^2+\cos^2\alpha}\,dt$$

$$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\infty-\tan^{-1}(\tan\alpha)\right]\Rightarrow I=\frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)$$

120. Let $I=\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2\,dx$

$$=\int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx-2\cos ax\sin bx)dx$$

$$=\int_{-\pi}^{\pi}\cos^2 ax\,dx+\int_{-\pi}^{\pi}\sin^2 bx\,dx-2\int_{-\pi}^{\pi}\cos ax\sin bx\,dx$$

$$=2\left[\int_0^{\pi}\cos^2 ax\,dx+\int_0^{\pi}\sin^2 bx\,dx\right]$$

$$\left[\because \int_{-a}^{a}f(x)dx=\begin{cases}0, \text{ if } f(x) \text{ is odd}\\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even}\end{cases}\right]$$

$$=2\left[\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx+\int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx\right]$$

$$=\int_0^{\pi}(1+\cos 2ax)dx+\int_0^{\pi}(1-\cos 2bx)dx$$

$$=2\int_0^{\pi}1\cdot dx+\int_0^{\pi}\cos 2a\,x\,dx-\int_0^{\pi}\cos 2b\,x\,dx$$

$$=(2x)_0^{\pi}+\frac{1}{2a}(\sin 2ax)_0^{\pi}-\frac{1}{2b}(\sin 2bx)_0^{\pi}$$

121. Let $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^{-x}+1}e^{-x}dx$

$$=\int_{-\pi/2}^{\pi/2}\frac{\cos x(e^{-x}+1-1)}{e^{-x}+1}dx=\int_{-\pi/2}^{\pi/2}\cos x\,dx-\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^{-x}+1}dx$$

Now, put $x=-z$ in 2nd integral,

$\therefore \quad dx=-dz$

Also, when $x=\frac{-\pi}{2}$, then $z=\frac{\pi}{2}$ and when $x=\frac{\pi}{2}$, then $z=\frac{-\pi}{2}$

$$\therefore \quad I=\int_{-\pi/2}^{\pi/2}\cos x\,dx+\int_{\pi/2}^{-\pi/2}\frac{\cos z}{e^z+1}dz$$

$$\Rightarrow \quad I=[\sin x]_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^x+1}dx$$

$$\Rightarrow \quad I=\left[\sin\frac{\pi}{2}+\sin\frac{\pi}{2}\right]-I\Rightarrow 2I=2\Rightarrow I=1$$

122. Let $I=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$

$$\Rightarrow \quad I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \text{...(i)}$$

By the property, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$$I=\int_0^{\pi/2}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx$$

$$=\int_0^{\pi/2}1\cdot dx=[x]_0^{\pi/2}=\frac{\pi}{2}\quad\Rightarrow\quad I=\frac{\pi}{4}$$

123. Let $I=\int_0^{\pi/4}\log(1+\tan x)dx$

By using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$$I=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$=\int_0^{\pi/4}[\log 2-\log(1+\tan x)]dx$$

$$\Rightarrow \quad 2I=\log 2\left[\frac{\pi}{4}-0\right]\Rightarrow I=\frac{\pi}{8}\log 2$$

Key Points

- $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

124. Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x\,\mathrm{cosec}\,x}dx \qquad \text{...(i)}$

Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$$I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\mathrm{cosec}(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x\,\mathrm{cosec}\,x} \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\left[\frac{x\tan x}{\sec x\,\mathrm{cosec}\,x}+\frac{(\pi-x)\tan x}{\sec x\,\mathrm{cosec}\,x}\right]dx$$

$$=\pi\int_0^{\pi}\frac{\tan x}{\sec x\,\mathrm{cosec}\,x}dx=\pi\int_0^{\pi}\frac{\sin x/\cos x}{\frac{1}{\cos x}\cdot\frac{1}{\sin x}}dx$$

$$=\pi\int_0^{\pi}\sin^2 x dx=\pi\int_0^{\pi}\frac{1-\cos 2x}{2}dx=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{\pi^2}{2}\Rightarrow I=\frac{\pi^2}{4}$$

125. Let $I=\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}dx$ ... (i)

$$I=\int_0^{\pi}\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx \quad \left[\text{By using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow I=\int_0^{\pi}\frac{4(\pi-x)\sin x}{1+\cos^2 x}dx \quad ... (ii)$$

Adding (i) and (ii), we get

Hence, $2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x = t \Rightarrow \sin x\, dx = -dt$

Also, when $x = 0$, then $t = 1$ and when $x = \pi$, when $t = -1$

$$\therefore \quad I=2\pi\int_1^{-1}\frac{-dt}{1+t^2}=2\pi\int_{-1}^{1}\frac{dt}{t^2+1^2}=2\pi[\tan^{-1}t]_{-1}^{1}$$

$$=2\pi[\tan^{-1}1-\tan^{-1}(-1)]=2\pi\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\pi^2$$

126. Let $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ ...(i)

$$\therefore \quad I=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x\cos x dx}{\cos^4 x+\sin^4 x} \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x dx}{\sin^4 x+\cos^4 x}$$

Dividing numerator and denominator by $\cos^4 x$, we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\tan x\sec^2 x dx}{1+\tan^4 x}$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = \infty$

$$\therefore \quad I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}=\frac{\pi}{8}[\tan^{-1}t]_0^{\infty}=\frac{\pi}{8}\times\frac{\pi}{2}=\frac{\pi^2}{16}$$

127. Let $I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ...(i)

$$\Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)\right]^{1/2}}{[\sin(\pi/3+\pi/6-x)]^{1/2}+[\cos(\pi/3+\pi/6-x)]^{1/2}}$$

$$\Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{[\sin(\pi/2-x)]^{1/2}}{[\sin(\pi/2-x)]^{1/2}+[\cos(\pi/2-x)]^{1/2}}dx$$

$$\Rightarrow I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\cos x}\right)dx$$

$$\Rightarrow 2I=\int_{\pi/6}^{\pi/3}1\cdot dx=[x]_{\pi/6}^{\pi/3}\Rightarrow I=\frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]\Rightarrow I=\frac{\pi}{12}$$

Concept Applied

- $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

128. Let $I=\int_0^{\pi}\frac{x dx}{a^2\cos^2 x+b^2\sin^2 x}$ ...(i)

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$

$$\left[\text{Using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2 x+b^2\sin^2 x} \quad ...(ii)$$

Adding (i) and (ii), we get $I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$

Let $f(x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}$

$$\Rightarrow f(\pi-x)=\frac{1}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$

$$\Rightarrow f(\pi-x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}=f(x)$$

$$\left[\therefore \text{By using }\int_0^{2a}f(x)dx=2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x)\right]$$

$$\therefore \quad I=\frac{\pi}{2}\left(2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\right)\Rightarrow I=\pi\int_0^{\pi/2}\frac{\sec^2 x dx}{a^2+b^2\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt.$

Also, when $x = 0$, then $t = \tan 0 = 0.$

and when $x=\frac{\pi}{2}$, then $t=\tan\frac{\pi}{2}=\infty$

$$\therefore \quad I=\int_0^{\infty}\frac{dt}{a^2+b^2t^2}\Rightarrow I=\frac{\pi}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}$$

$$\Rightarrow I=\frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}\Rightarrow I=\frac{\pi}{ab}[\tan^{-1}\infty-\tan^{-1}0]=\frac{\pi^2}{2ab}$$

CBSE Sample Questions

1. Let $I=\int\frac{1}{\cos^2 x(1-\tan x)^2}dx$

Put $1-\tan x = t \Rightarrow -\sec^2 x\, dx = dt$ **(1)**

$$\therefore I=-\int\frac{dt}{t^2}=-\int t^{-2}dt=\frac{1}{t}+C=\frac{1}{1-\tan x}+C \quad \textbf{(1)}$$

2. Let $I=\int\frac{\sin 2x}{\sqrt{9-\cos^4 x}}dx$

Put $\cos^2 x = t$

$\Rightarrow \quad -2\cos x \sin x\, dx = dt \Rightarrow \sin 2x\, dx = -dt$ **(1)**

$\therefore \quad I = -\int \frac{dt}{\sqrt{3^2 - t^2}} = -\sin^{-1}\frac{t}{3} + c = -\sin^{-1}\left(\frac{\cos^2 x}{3}\right) + c$ **(1)**

3. Let $\frac{x+1}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x} = \frac{(Ax+B)x + C(x^2+1)}{(x^2+1)x}$ **(1/2)**

$\Rightarrow \quad x + 1 = (Ax + B)x + C(x^2 + 1)$

By equating the like coefficients, we get

$B = 1, C = 1, A + C = 0$

Hence, $A = -1$, $B = 1$ and $C = 1$ **(1/2)**

$\therefore$ The given integral $= \int \frac{-x+1}{x^2+1} dx + \int \frac{1}{x} dx$

$= \frac{-1}{2}\int \frac{2x-2}{x^2+1} dx + \int \frac{1}{x} dx \quad = \frac{-1}{2}\int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx + \int \frac{1}{x} dx$ **(1/2)**

$= \frac{-1}{2}\log(x^2+1) + \tan^{-1} x + \log|x| + c$ **(1/2)**

4. Let $I = \int \frac{(x^3+x+1)}{(x^2-1)} dx = \int\left(x + \frac{2x+1}{(x-1)(x+1)}\right) dx$ **(1/2)**

Now resolving $\frac{2x+1}{(x-1)(x+1)}$ into partial fractions as

$\frac{2x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{x(A+B)+(A-B)}{(x-1)(x+1)}$ **(1/2)**

Calculating A and B we get,

$\frac{2x+1}{(x-1)(x+1)} = \frac{3}{2(x-1)} + \frac{1}{2(x+1)}$

Now, $I = \int \frac{(x^3+x+1)}{(x^2-1)} dx = \int\left(x + \frac{2x+1}{(x-1)(x+1)}\right) dx$

$= \int\left(x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}\right) dx$ **(1)**

$= \frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$

$= \frac{x^2}{2} + \frac{1}{2}\log|(x-1)^3(x+1)| + C$ **(1)**

Concept Applied

- $\frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

5. We have, $\int_0^4 |x-1| dx = \int_0^1 (1-x) dx + \int_1^4 (x-1) dx$ **(1)**

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$ **(1)**

$= \left(1 - \frac{1}{2}\right) + (8-4) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} + 4 + \frac{1}{2} = 5$ **(1)**

6. Let $I = \int \frac{x^2+1}{(x^2+2)(x^2+3)} dx$

Now, put $x^2 = y$ to make partial fractions.

i.e., $\frac{x^2+1}{(x^2+2)(x^2+3)} = \frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2} + \frac{B}{y+3}$ **(1/2)**

$\Rightarrow \quad y + 1 = A(y + 3) + B(y + 2) \quad \ldots$ (i) **(1/2)**

Comparing coefficients of y and constant terms on both sides of (i), we get

$A + B = 1$ and $3A + 2B = 1$

Solving, we get $A = -1, B = 2$ **(1)**

$\therefore \quad I = \int \frac{x^2+1}{(x^2+2)(x^2+3)} dx = \int \frac{-1}{x^2+2} dx + 2\int \frac{1}{x^2+3} dx$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$ **(1)**

7. Let $I = \int e^x(1 - \cot x + \operatorname{cosec}^2 x) dx$

$= \int e^x dx + \int e^x((-\cot x) + \operatorname{cosec}^2 x) dx$

$= e^x + e^x(-\cot x) + C$

$= e^x(1 - \cot x) + C$ **(1)**

8. We have, $\int \frac{\log x}{(1+\log x)^2} dx = \int \frac{\log x + 1 - 1}{(1+\log x)^2} dx$

$= \int \frac{1}{1+\log x} dx - \int \frac{1}{(1+\log x)^2} dx$ **(1/2)**

$= \frac{1}{1+\log x} \times x - \int \frac{-1}{(1+\log x)^2} \times \frac{1}{x} \times x dx$

$- \int \frac{1}{(1+\log x)^2} dx + c = \frac{x}{1+\log x} + c$ **(1½)**

9. $\because$ $f(x) = x^2 \sin x$ is an odd function.

$\therefore \quad \int_{-\pi/2}^{\pi/2} x^2 \sin x dx = 0$ **(1)**

10. Let $I = \int_0^1 x(1-x)^n dx$

$\Rightarrow \quad I = \int_0^1 (1-x)[1-(1-x)]^n dx \quad \left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx\right)$ **(1/2)**

$\Rightarrow \quad I = \int_0^1 (1-x)x^n dx = \int_0^1 (x^n - x^{n+1}) dx \Rightarrow I = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$ **(1)**

$\Rightarrow \quad I = \left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - 0\right] = \frac{1}{(n+1)(n+2)}$ **(1/2)**

11. The given definite integral

$= \int_{-1}^{2} |x(x-1)(x-2)| dx$

$= \int_{-1}^{0} |x(x-1)(x-2)| dx + \int_{0}^{1} |x(x-1)(x-2)| dx + \int_{1}^{2} |x(x-1)(x-2)| dx$ **(1½)**

$= -\int_{-1}^{0} (x^3 - 3x^2 + 2x) dx + \int_{0}^{1} (x^3 - 3x^2 + 2x) dx - \int_{1}^{2} (x^3 - 3x^2 + 2x) dx$ **(1/2)**

$= -\left[\frac{x^4}{4} - x^3 + x^2\right]_{-1}^{0} + \left[\frac{x^4}{4} - x^3 + x^2\right]_{0}^{1} - \left[\frac{x^4}{4} - x^3 + x^2\right]_{1}^{2}$

$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$ **(2)**

Self Assessment

Case Based Objective Questions (4 marks)

1. An Integration is the process of finding the anti-derivative of a function. In this process, we are provided with the derivative of a function and asked to find out the function (*i.e.*, Primitive)

Integration is the inverse process of differentiation. Let $f(x)$ be a function of x. If there is a function $g(x)$, such that $\frac{d}{dx}(g(x)) = f(x)$, then $g(x)$ is called an integral of $f(x)$ w.r.t. x and is denoted by $\int f(x)dx = g(x) + c$, where c is constant of integration.

Also, the given integral $\int f(x)dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$

Consider, $I = \int f(x)dx = \int f(g(t))\, g'(t)\, dt$

Based on the above information, answer the following questions.

(i) Evaluate : $\int \frac{4x+6}{x^2+3x}dx$

(a) $3\log|x+3x^2|+C$ (b) $3\log|x^2+3x|+C$
(c) $2\log|x^2+3x|+C$ (d) $\log|4x+6|+C$

(ii) Evaluate : $\int \frac{1+\cos x}{x+\sin x}dx$

(a) $\log|x+\operatorname{cosec} x|$ (b) $\log|x+\sec x|+C$
(c) $\log|x+\cos x|+C$ (d) $\log|x+\sin x|+C$

(iii) Evaluate : $\int \frac{(x+1)^2}{x(x^2+1)}dx$

(a) $\log|x|+2\tan^{-1}x+C$ (b) $2\tan^{-1}x-\log|x|+C$
(c) $\log|x|-2\tan^{-1}x+C$ (d) None of these

(iv) Evaluate : $\int \tan^2 x\, dx$

(a) $\tan x+x+C$ (b) $\tan x-x+C$
(c) $\tan x+x^2+C$ (d) None of these

(v) Evaluate : $\int \frac{dx}{\sin^2 x\cos^2 x}$

(a) $-2\cot 2x+C$ (b) $2\cot 2x+C$
(c) $\cot 2x+C$ (d) None of these

Multiple Choice Questions (1 mark)

2. $\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

3. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

(a) $\frac{\pi^2}{32}$ (b) $\frac{\pi^2}{32}$ (c) $\frac{\pi}{16}$ (d) $\frac{\pi^2}{16}$

OR

Evaluate : $\int_0^1\left\{e^x+\sin\frac{\pi x}{4}\right\}dx$

(a) $1-\frac{4}{\pi}+\frac{2\sqrt{2}}{\pi}$ (b) $1+\frac{2}{\pi}-\frac{2\sqrt{2}}{\pi}$
(c) $1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$ (d) None of these

4. $\int_0^{\pi/2}\cos x\, e^{\sin x}dx$ is equal to ________.

(a) $e+1$ (b) $e-1$
(c) e^2-1 (d) e^2+1

5. $\int \frac{x^3}{x+1}dx$ is equal to

(a) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$

(b) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

(c) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$

(d) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

6. If $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$, then

(a) $a=\frac{-1}{10}, b=\frac{-2}{5}$ (b) $a=\frac{1}{10}, b=-\frac{2}{5}$
(c) $a=\frac{-1}{10}, b=\frac{2}{5}$ (d) $a=\frac{1}{10}, b=\frac{2}{5}$

7. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to

(a) $\frac{e^x}{1+x^2}+C$ (b) $\frac{-e^x}{1+x^2}+C$
(c) $\frac{e^x}{(1+x^2)^2}+C$ (d) $\frac{-e^x}{(1+x^2)^2}+C$

VSA Type Questions (1 mark)

8. Evaluate : $\int \cos^3 x\, e^{\log \sin x}dx$

9. Evaluate : $\int_1^3 x^2\log x\, dx$

10. Evaluate : $\int \sin^{-1}x\, dx$

OR

Evaluate : $\int \frac{x}{x^4-1}dx$

11. Evaluate : $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

12. Evaluate : $\int_0^\pi \frac{x}{1+\sin x}dx$

SA I Type Questions (2 marks)

13. Evaluate : $\int \frac{\sin x}{3+4\cos^2 x}dx$

14. Evaluate : $\int \frac{1+\cos x}{1-\cos x}dx$

15. Evaluate : $\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$

16. Evaluate : $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

OR

If $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$, then find the value of a.

SA II Type Questions (3 marks)

17. Evaluate : $\int \frac{2x+1}{\sqrt{x^2+4x+3}}dx$

18. Evaluate : $\int_0^1 x\log(1+2x)dx$

19. Evaluate : $\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$

20. Evaluate : $\int [\sin(\log x)+\cos(\log x)]dx$

OR

Evaluate : $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$

21. Evaluate : $\int \frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}dx$

Case Based Questions (4 marks)

22. Let f be a continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$

This result is very useful as it gives us a method of calculating the definite integral easily. Here, we have no need to write integration constant c because if, we will write $F(x) + c$, instead of $f(x)$, we get

$\int_a^b f(x)dx = [F(x)+c]_a^b = F(b)+c-F(a)-c = F(b)-F(a)$

Based on the above information, answer the following questions.

(i) Evaluate : $\int_0^1 xe^x dx$ (ii) Evaluate : $\int_0^{\pi/4} 2\tan^3 x\,dx$

LA Type Questions (4 / 6 marks)

23. Evaluate : $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

24. Evaluate : $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

25. Evaluate : $\int_0^3 (|x|+|x-1|+|x-2|)dx$

OR

Evaluate : $\int \frac{xe^{2x}}{(2x+1)^2}dx$

Detailed SOLUTIONS

1. (i) (c) : Let $I = \int \frac{4x+6}{x^2+3x}dx$

$= 2\int \frac{2x+3}{x^2+3x}dx$

Put $x^2 + 3x = t$

$\Rightarrow (2x+3)\,dx = dt$

$\therefore I = \int 2\frac{dt}{t} = 2\log|t| + C$

$= 2\log|x^2+3x| + C$

(ii) (d): Let $I = \int \frac{1+\cos x}{x+\sin x}dx$

Put $x + \sin x = t$

$\Rightarrow (1+\cos x)\,dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|x+\sin x| + C$

(iii) (a): Let $I = \int \frac{(x+1)^2}{x(x^2+1)}dx = \int \frac{x^2+1+2x}{x(x^2+1)}dx$

$= \int \left(\frac{1}{x}+\frac{2}{x^2+1}\right)dx = \log|x| + 2\tan^{-1}x + c$

(iv) (b): $\int \tan^2 x\,dx = \int (\sec^2 x-1)\,dx$

$= \tan x - x + c$

(v) (a): Let $I = \int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{4}{4\sin^2 x\cos^2 x}dx$

$= 4\int \text{cosec}^2 2x\,dx = -2\cot 2x + c$

2. (a) : Let $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x} = \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x}$

$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx=\int_0^{\pi/4}\sec^2 x\,dx$

$\left[\text{Using property for even function } f(x), \int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx\right]$

$=[\tan x]_0^{\pi/4}=\left[\tan\frac{\pi}{4}-\tan 0\right]=1$

3. (b) : We have, $I=\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$

When $x = 0$, $t = 0$ and when $x = 1$, $t=\frac{\pi}{4}$

$\therefore\ I=\int_0^1\frac{\tan^{-1}x}{1+x^2}dx=\int_0^{\pi/4} t\,dt=\left[\frac{t^2}{2}\right]_0^{\pi/4}=\frac{\pi^2}{32}$

OR

(d) : We have, $I=\int_0^1\left\{e^x+\sin\frac{\pi x}{4}\right\}dx$

$=[e^x]_0^1+\frac{4}{\pi}\left[-\cos\frac{\pi}{4}x\right]_0^1=e-1-\frac{4}{\sqrt{2}\pi}+\frac{4}{\pi}$

4. (b) : Let $I=\int_0^{\pi/2}\cos x\, e^{\sin x}dx$

Substitute $\sin x = t \Rightarrow \cos x\, dx = dt$

$x\to 0 \Rightarrow t\to 0$

and $x\to\pi/2 \Rightarrow t\to 1$

$\therefore\ I=\int_0^1 e^t dt=[e^t]_0^1=e^1-e^0=e-1$

5. (d) : Let $I=\int\frac{x^3}{x+1}dx$

$=\int\left((x^2-x+1)-\frac{1}{(x+1)}\right)dx=\int(x^2-x+1)\,dx-\int\frac{dx}{x+1}$

$=\frac{x^3}{3}-\frac{x^2}{2}+x-\log|x+1|+C$

6. (c) : We have given

$\int\frac{dx}{(x+2)(x^2+1)}=a\log|1+x^2|+b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$

Taking, $I=\int\frac{dx}{(x+2)(x^2+1)}$

$\frac{1}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$

[using method of partial fraction]

$\Rightarrow\ 1 = A(x^2 + 1) + (Bx + C)(x + 2)$

$\Rightarrow\ 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$\Rightarrow\ 1 = (A + B)x^2 + (2B + C)x + A + 2C$

$\Rightarrow\ A + B = 0, A + 2C = 1, 2B + C = 0$

By solving all three, we get

$A=\frac{1}{5}, B=-\frac{1}{5}$ and $C=\frac{2}{5}$

$\therefore\int\frac{dx}{(x+2)(x^2+1)}=\frac{1}{5}\int\frac{1}{x+2}dx+\int\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}dx$

$=\frac{1}{5}\int\frac{1}{x+2}dx-\frac{1}{5}\int\frac{x}{1+x^2}dx+\frac{1}{5}\int\frac{2}{1+x^2}dx$

$=\frac{1}{5}\log|x+2|-\frac{1}{10}\log|1+x^2|+\frac{2}{5}\tan^{-1}x+C$

$\therefore\ b=\frac{2}{5}$ and $a=-\frac{1}{10}$ [By comparing]

7. (a) : We have, $I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$

$=\int e^x\left(\frac{1+x^2-2x}{(1+x^2)^2}\right)dx=\int e^x\left(\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right)dx$

Above integral is of the type $\int e^x(f(x)+f'(x))dx$

$\therefore$ Solution is $e^x f(x) + C$

$=e^x\left(\frac{1}{1+x^2}\right)+C$

8. We have, $I=\int\cos^3 x\, e^{\log\sin x}dx=\int\cos^3 x\sin x\,dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow\ I=-\int t^3dt=-\frac{t^4}{4}+C=-\frac{\cos^4 x}{4}+C$

9. $\int_1^3 x^2\log x\,dx$

$=\left[(\log x)\left(\frac{x^3}{3}\right)\right]_1^3-\int_1^3\frac{1}{x}\cdot\frac{x^3}{3}dx$

$=9\log 3-0-\frac{1}{3}\left[\frac{x^3}{3}\right]_1^3=9\log 3-\frac{26}{9}$

10. Let $I=\int 1\cdot\sin^{-1}x\,dx$

$=(\sin^{-1}x)x-\int\frac{1}{\sqrt{1-x^2}}\cdot x\,dx+C'$

Put $1-x^2=t^2 \Rightarrow -2x\,dx=2t\,dt$

$=x\sin^{-1}x-\int\frac{(-t\,dt)}{t}+C=x\sin^{-1}x+\sqrt{1-x^2}+C$

OR

Let $I=\int\frac{x}{x^4-1}dx$

Substitute $x^2=t \Rightarrow 2x\,dx=dt \Rightarrow x\,dx=\frac{1}{2}dt$

$\Rightarrow\ I=\frac{1}{2}\int\frac{dt}{t^2-1}=\frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+C$

$\left[\text{Using }\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right]$

$=\frac{1}{4}[\log|x^2-1|-\log|x^2+1|]+C$

11. Let $I=\int_0^1\frac{x}{\sqrt{1+x^2}}dx$

Substitute $1 + x^2 = t^2$

$\Rightarrow \quad 2xdx = 2tdt \quad \Rightarrow xdx = tdt$

$$\Rightarrow I = \int_1^{\sqrt{2}} \frac{tdt}{t} = [t]_1^{\sqrt{2}} = \sqrt{2} - 1$$

12. Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$ ... (i)

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \pi \int_0^{\pi} \frac{(1-\sin x)dx}{\cos^2 x} \quad [\because \cos^2 x = 1 - \sin^2 x]$$

$$= \pi \int_0^{\pi} (\sec^2 x - \tan x \cdot \sec x) dx = \pi[\tan x]_0^{\pi} - \pi[\sec x]_0^{\pi}$$

$= \pi[\tan \pi - \sec \pi - \tan 0 + \sec 0]$

$\Rightarrow \quad 2I = \pi[0 + 1 - 0 + 1] = 2\pi$

$\therefore \quad I = \pi$

13. Let $I = \int \frac{\sin x}{3+4\cos^2 x} dx$

Substitute $\cos x = t \Rightarrow -\sin x \, dx = dt$

$$\therefore \quad I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4} \int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= -\frac{1}{4} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t}{\sqrt{3}} + C \quad \left[\because \int \frac{1dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= -\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

14. Let $I = \int \frac{1+\cos x}{1-\cos x} dx$

$$\Rightarrow I = \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx = \int \cot^2 \frac{x}{2} dx$$

$$\Rightarrow I = \int \left(\text{cosec}^2 \frac{x}{2} - 1\right) dx = -2\cot \frac{x}{2} - x + C$$

15. Let $I = \int_0^{\pi/2} \sqrt{1 - \sin 2x} \, dx$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2} dx$$

$$\left[\begin{array}{l} \because \text{When } 0 < x < \frac{\pi}{4}, \cos x > \sin x \text{ and} \\ \text{when } \frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x \end{array}\right]$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

16. Let $I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x dx$

Now, let $f(x) = x^3 \sin^4 x$, then

$f(-x) = (-x)^3 (\sin(-x))^4 = -x^3 \sin^4 x = -f(x)$

So, $f(x)$ is an odd function.

Hence, $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x dx = 0$

OR

We have, $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$

Taking $\int_0^a \frac{1}{4\left(\frac{1}{4} + x^2\right)} dx = \frac{1}{4} \int_0^a \frac{1}{\left(x^2 + \left(\frac{1}{2}\right)^2\right)} dx$

$$= \frac{2}{4}[\tan^{-1} 2x]_0^a = \frac{1}{2} \tan^{-1} 2a - 0 = \frac{1}{2} \tan^{-1} 2a$$

Since, $\frac{1}{2} \tan^{-1} 2a = \frac{\pi}{8} \Rightarrow \tan^{-1} 2a = \frac{\pi}{4}$

$\Rightarrow \quad 2a = \tan \frac{\pi}{4} = 1 \Rightarrow a = 1/2$

17. Let $I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$

Let $2x + 1 = A\left(\frac{d}{dx}(x^2+4x+3)\right) + B = A(2x+4) + B$

Equating the coefficients of x and constant terms, we get $2A = 2$ and $4A + B = 1 \Rightarrow A = 1$ and $B = -3$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+3}} dx - \int \frac{3}{\sqrt{x^2+4x+3}} dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+3}} dx - 3\int \frac{dx}{\sqrt{(x+2)^2 - (1)^2}}$$

$$= 2\sqrt{x^2+4x+3} - 3\log\left|(x+2) + \sqrt{x^2+4x+3}\right| + C$$

18. Let $I = \int_0^1 x \log(1+2x) dx$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+2x} \cdot 2 \cdot \frac{x^2}{2} dx$$

[Integration by parts]

$$=\frac{1}{2}[x^2\log(1+2x)]_0^1-\int_0^1\frac{x^2}{1+2x}dx$$

$$=\frac{1}{2}[1\log3-0]-\left[\int_0^1\left(\frac{x}{2}-\frac{\frac{x}{2}}{1+2x}\right)dx\right]$$

$$=\frac{1}{2}\log3-\frac{1}{2}\int_0^1 x\,dx+\frac{1}{2}\int_0^1\frac{x}{1+2x}dx$$

$$=\frac{1}{2}\log3-\frac{1}{2}\left[\frac{x^2}{2}\right]_0^1+\frac{1}{2}\int_0^1\frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$$

$$=\frac{1}{2}\log3-\frac{1}{2}\left[\frac{1}{2}-0\right]+\frac{1}{4}\int_0^1 dx-\frac{1}{4}\int_0^1\frac{1}{1+2x}dx$$

$$=\frac{1}{2}\log3-\frac{1}{4}+\frac{1}{4}[x]_0^1-\frac{1}{8}[\log|(1+2x)|]_0^1$$

$$=\frac{1}{2}\log3-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}[\log3-\log1]$$

$$=\frac{1}{2}\log3-\frac{1}{8}\log3=\frac{3}{8}\log3$$

19. Let $I=\int\frac{\sin(\tan^{-1}x)}{1+x^2}dx$

Put $\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$

$\therefore \quad I=\int\frac{\sin(\tan^{-1}x)}{1+x^2}dx=\int\sin t\,dt=-\cos t+C$

$\Rightarrow \quad I=-\cos(\tan^{-1}x)+C$

20. We have, $I=\int[\sin(\log x)+\cos(\log x)]dx$

Put $\log x = t$, $x = e^t \Rightarrow dx = e^t dt$

$\therefore \quad I=\int(\sin t+\cos t)e^t\,dt$

Consider, $f(t) = \sin t$

$\Rightarrow \quad f'(t) = \cos t$

$\therefore \quad$ Integrand is in the form $e^t(f(t) + f'(t))$

$\therefore \quad I=\int e^t(\sin t+\cos t)dt=e^t\sin t+C=x\sin(\log x)+C$

OR

Let $I=\int\frac{\cos x-\cos 2x}{1-\cos x}dx$

$$=\int\frac{2\sin\frac{3x}{2}\cdot\sin\frac{x}{2}}{1-1+2\sin^2\frac{x}{2}}dx \quad \left[\begin{array}{l}\text{Using } \cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\\ \text{and } \cos A=1-2\sin^2\frac{A}{2}\end{array}\right]$$

$$=\int\frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}dx$$

$$=\int\frac{3\sin\frac{x}{2}-4\sin^3\frac{x}{2}}{\sin\frac{x}{2}}dx \qquad [\because \sin3A = 3\sin A - 4\sin^3A]$$

$$=3\int dx-4\int\sin^2\frac{x}{2}dx$$

$$=3\int dx-4\int\frac{1-\cos x}{2}dx$$

$$=3\int dx-2\int dx+2\int\cos x\,dx$$

$$=\int dx+2\int\cos x\,dx=x+2\sin x+C$$

21. We have, $I=\int\frac{\sin2x\cos2x}{\sqrt{9-\cos^4 2x}}dx$

Put $\cos^2 2x = t \Rightarrow -4\sin2x\cos2x\,dx = dt$

$\therefore \quad I=\frac{-1}{4}\int\frac{dt}{\sqrt{9-t^2}}=\frac{-1}{4}\sin^{-1}\frac{t}{3}+C$

$$=\frac{-1}{4}\sin^{-1}\left(\frac{\cos^2 2x}{3}\right)+C$$

22. (i) Here, $\int xe^x dx=x\int e^x\cdot dx-\int\left(\frac{d}{dx}(x)\cdot\int e^x dx\right)dx$

$=xe^x-\int1\cdot e^x dx$

$= xe^x - e^x = e^x(x - 1)$

$= F(x)$

Now, $\int_0^1 xe^x dx=F(1)-F(0)$

$= e(1 - 1) - e^0(0 - 1) = 0 + 1 = 1$

(ii) We have, $\int 2\tan^3 x\,dx$

$=\int2\tan x\tan^2 x\,dx$

$=\int2\tan x(\sec^2 x-1)dx$

$=2\int\tan x\sec^2 x\,dx-2\int\tan x\,dx$

$=2\left[\frac{\tan^2 x}{2}\right]-2[-\log|\cos x|] = \tan^2 x + 2\log|\cos x|$

Now, $\int_0^{\pi/4}2\tan^3 x=F(\pi/4)-F(0)$

$$=\left(\tan^2\frac{\pi}{4}+2\log\left|\cos\frac{\pi}{4}\right|\right)-(\tan^2 0+2\log|\cos 0|)$$

$$=\left(1+2\log\frac{1}{\sqrt2}\right)-(0+2\log1)=1+2\left(\frac{-1}{2}\log2\right)-0$$

$= 1 - \log 2$

23. Let $I=\int\frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$

By using partial fraction, we get

$$\frac{2x-1}{(x-1)(x+2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-3)}$$

$\Rightarrow \quad 2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$

Substitute $x = 1$, we get

$2 - 1 = A(1 + 2)(1 - 3)$

$\Rightarrow \quad 1=-6A \Rightarrow A=-\frac{1}{6}$

Substitute $x = 3$, we get

$6 - 1 = C(3 - 1)(3 + 2)$

$\Rightarrow \ 5 = 10C \ \Rightarrow \ C = \frac{1}{2}$

Now, substitute $x = -2$, we get

$-4 - 1 = B(-2 - 1)(-2 - 3)$

$\Rightarrow \ -5 = 15B \ \Rightarrow \ B = -\frac{1}{3}$

$\therefore \ I = -\frac{1}{6}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx + \frac{1}{2}\int \frac{1}{x-3}dx$

$= -\frac{1}{6}\log|(x-1)| - \frac{1}{3}\log|(x+2)| + \frac{1}{2}\log|(x-3)| + C$

$= -\log|(x-1)|^{1/6} - \log|(x+2)|^{1/3} + \log\left|\sqrt{(x-3)}\right| + C$

$= \log\left|\sqrt{x-3}\right| - \log\left|(x-1)^{1/6}(x+2)^{1/3}\right| + C$

$= \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$

24. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}dx$

$= \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x}dx$

$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)}{\sin^2 x \cdot \cos^2 x}dx$

$\left[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2) \text{ and } \sin^2 x + \cos^2 x = 1\right]$

$= \int \frac{\sin^4 x}{\sin^2 x\cos^2 x}dx + \int \frac{\cos^4 x}{\sin^2 x \cdot \cos^2 x}dx - \int \frac{\sin^2 x\cos^2 x}{\sin^2 x \cdot \cos^2 x}dx$

$= \int \tan^2 x dx + \int \cot^2 x dx - \int 1 dx$

$= \int (\sec^2 x - 1)dx + \int (\text{cosec}^2 x - 1)dx - \int 1dx$

$= \int \sec^2 x dx + \int \text{cosec}^2 x dx - 3\int dx$

$\Rightarrow \ I = \tan x - \cot x - 3x + C$

25. Let $I = \int_0^3 (|x| + |x-1| + |x-2|)\, dx$

$f(x) = |x| + |x - 1| + |x - 2|$

When $0 \le x < 1$, then

$f(x) = x - (x - 1) - (x - 2) = x - x + 1 - x + 2 = -x + 3$

When $1 \le x < 2$, then

$f(x) = x + (x - 1) - (x - 2) = x + x - 1 - x + 2 = x + 1$

When $2 \le x < 3$, then

$f(x) = x + (x - 1) + (x - 2) = 3x - 3$

$$\therefore \ f(x) = \begin{cases} -x+3, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ 3x-3, & 2 \le x < 3 \end{cases}$$

$\therefore \ I = \int_0^3 (|x| + |x-1| + |x-2|)dx$

$\Rightarrow \ I = \int_0^1 (-x+3)dx + \int_1^2 (x+1)dx + \int_2^3 (3x-3)dx$

$= \left[\frac{-x^2}{2} + 3x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2 + \left[\frac{3x^2}{2} - 3x\right]_2^3$

$= \left(\frac{-1}{2} + 3\right) + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} + 1\right)\right] + \left[\left(\frac{27}{2} - 9\right) - \left(\frac{12}{2} - 6\right)\right]$

$= \frac{5}{2} + \left[4 - \frac{3}{2}\right] + \left[\frac{9}{2} - 0\right] = \frac{5}{2} + 4 - \frac{3}{2} + \frac{9}{2} = \frac{19}{2}$

OR

Let $I = \int \frac{xe^{2x}}{(2x+1)^2}dx$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$\therefore \ I = \frac{1}{4}\int \frac{te^t}{(t+1)^2}dt = \frac{1}{4}\int e^t\left[\frac{t+1}{(t+1)^2} - \frac{1}{(t+1)^2}\right]dt$

$= \frac{1}{4}\int e^t\left(\frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt$

If $f(t) = \frac{1}{t+1} \Rightarrow f'(t) = \frac{-1}{(t+1)^2}$

So, $I = \frac{1}{4}\frac{e^{2x}}{(2x+1)} + C$