Differential Equations

Previous Years' CBSE Board Questions

9.2 Basic Concepts

MCQ

1. The sum of the order and the degree of the differential equation $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)$ is

   (a) 2 (b) 3
   (c) 5 (d) 0

   (2023)

2. The order and the degree of the differential equation $\left(1+3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$ respectively are

   (a) $1, \frac{2}{3}$ (b) 3, 1 (c) 3, 3 (d) 1, 2

   (2023)

VSA (1 mark)

3. The degree of the differential equation $1+\left(\frac{dy}{dx}\right)^2 = x$ is ______. *(2020)*

4. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. *(Delhi 2019)* U

5. Write the sum of the order and degree of the following differential equation

   $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0.$ *(AI 2015)*

6. Write the sum of the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0$.

   (Foreign 2015) U

7. Write the sum of the order and degree of the differential equation

   $1+\left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$. *(Delhi 2015C)*

SA I (2 marks)

8. Find the product of the order and the degree of the differential equation $\left[\frac{d}{dx}(xy^2)\right]\cdot\frac{dy}{dx} + y = 0$.

   (2022) U

9. Find the value of $(2a - 3b)$, if a and b represent respectively the order and the degree of the differential equation $x\left[y\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx}\right] = 0$.

   (2022 C)

9.3 General and Particular Solutions of a Differential Equation

MCQ

10. The number of solutions of the differential equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, when $y(1) = 2$, is

    (a) zero (b) one (c) two (d) infinite

    (2023)

11. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)

    (a) 0 (b) 1 (c) 2 (d) 3

    (2020) R

9.4 Methods of Solving First Order, First Degree Differential Equations

MCQ

12. The integrating factor for solving the differential equation $x\frac{dy}{dx} - y = 2x^2$ is

    (a) e^{-y} (b) e^{-x} (c) x (d) $\frac{1}{x}$

    (2023)

13. The integrating factor of the differential equation $(x+3y^2)\frac{dy}{dx} = y$ is

    (a) y (b) $-y$ (c) $\frac{1}{y}$ (d) $-\frac{1}{y}$

    (2020)

VSA (1 mark)

14. The integrating factor of the differential equation $x\frac{dy}{dx} - y = \log x$ is ______. *(2020 C)* U

15. The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is ______. *(2020)*

16. Find the general solution of the differential equation $e^{y-x}\frac{dy}{dx} = 1$. *(2020)*

17. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$.

    (Delhi 2015, AI 2015C) U

18. Write the integrating factor of the following differential equation:

    $(1+y^2) + (2xy - \cot y)\frac{dy}{dx} = 0$ *(AI 2015)*

19. Write the solution of the differential equation $\frac{dy}{dx}=2^{-y}$. *(Foreign 2015)* Ap

20. Find the solution of the differential equation $\frac{dy}{dx}=x^3e^{-2y}$. *(AI 2015C)*

SA I (2 marks)

21. Find the general solution of the differential equation : $\log\left(\frac{dy}{dx}\right)=ax+by$. *(Term II, 2021-22)* Ap

22. Find the general solution of the differential equation $\sec^2x\cdot\tan y\,dx+\sec^2y\cdot\tan x\,dy=0$. *(Term II, 2021-22)*

23. Find the general solution of the following differential equation : $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$ *(Term II, 2021-22)* Ap

24. Find the integrating factor of $x\frac{dy}{dx}+(1+x\cot x)y=x$. *(2021 C)*

25. Solve the following homogeneous differential equation : $x\frac{dy}{dx}=x+y$ *(2020 C)*

26. Solve the following differential equation: $\frac{dy}{dx}+y=\cos x-\sin x$ *(AI 2019)*

SA II (3 marks)

27. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x+y}{x}$, $y(1)=0$. *(2023)*

28. Find the general solution of the differential equation $e^x\tan y\,dx+(1-e^x)\sec^2y\,dy=0$. *(2023)*

29. Find the particular solution of the differential equation $x\frac{dy}{dx}+x\cos^2\left(\frac{y}{x}\right)=y$; given that when $x=1, y=\frac{\pi}{4}$. *(Term II, 2021-22)* Ap

30. Find the general solution of the differential equation $x\frac{dy}{dx}=y(\log y-\log x+1)$. *(Term II, 2021-22)*

31. If the solution of the differential equation $\frac{dy}{dx}=\frac{2xy-y^2}{2x^2}$ is $\frac{ax}{y}$ = $b\log|x|+C$, find the value of a and b. *(2021C)* Ap

LA I (4 marks)

32. **Case study :** An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation A differential equation of the form $\frac{dy}{dx}=F(x,y)$ is said to be homogeneous if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogenous function of degree n if $F(\lambda x, \lambda y) = \lambda^nF(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx}=F(x,y)=g\left(\frac{y}{x}\right)$, we make substitution $y = vx$ and then separate the variables.
Based on the above, answer the following questions.
(i) Show that $(x^2 - y^2)\,dx + 2xy\,dy = 0$ is a differential equation of the type $\frac{dy}{dx}=g\left(\frac{y}{x}\right)$.
(ii) Solve the above equation to find its general solution. *(2023)*

33. Find the particular solution of the differential equation $(1+x^2)\frac{dy}{dx}+2xy=\tan x$, given $y(0) = 1$. *(Term II, 2021-22C)*

34. Find the particular solution of the differential equation $(1+\sin x)\frac{dy}{dx}=-x-y\cos x$, given $y(0) = 1$. *(Term II, 2021-22C)*

35. Find the particular solution of the differential equation $x\frac{dy}{dx}+2y=x^2\log x$, given $y(1) = 1$. *(Term II, 2021-22C)*

36. Find the particular solution of the differential equation $x\frac{dy}{dx}+y+\frac{1}{1+x^2}=0$, given that $y(1) = 0$. *(Term II, 2021-22)*

37. Find the general solution of the differential equation $x(y^3 + x^3)\,dy = (2y^4 + 5x^3y)dx$. *(Term II, 2021-22)*

38. Solve the following differential equation : $(y - \sin^2x)dx + \tan x\,dy = 0$ *(Term II, 2021-22)*

39. Find the general solution of the differential equation : $(x^3 + y^3)dy = x^2y\,dx$ *(Term II, 2021-22)* Ap

OR

Find the general solution of the differential equation $x^2y\,dx -(x^3 + y^3)\,dy = 0$. *(2020)*

40. Find the general solution of the differential equation $ye^y\,dx = (y^3 + 2x\,e^y)dy$. *(2020)*

41. Solve the following differential equation : $(1+e^{y/x})dy+e^{y/x}\left(1-\frac{y}{x}\right)dx=0\,(x\neq0)$. *(2020)*

42. Find the particular solution of the differential equation $x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$, given that $y=\frac{\pi}{4}$ at $x = 1$. *(2020)* Ap

43. Find the particular solution of the differential equation $\cos y\,dx + (1 + e^{-x})\sin y\,dy = 0$ given that $y=\frac{\pi}{4}$ when $x = 0$. *(2020)*

44. Find the general solution of the differential equation $ye^{x/y}\,dx = (xe^{x/y} + y^2)dy, y\neq 0$ *(2020)* An

45. Solve the differential equation : $x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$, given that $y = 0$ when $x = 1$. *(Delhi 2019)*

46. Solve the differential equation : $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$, subject to the initial condition $y(0) = 0$. *(Delhi 2019)* Ap

47. Solve the differential equation : $\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that $y = 1$ when $x = 0$. *(AI 2019)*

48. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$, given that $y = 1$ when $x = 0$. *(AI 2019, Delhi 2015)* Ev

49. Solve the following differential equation : $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x;\ x\neq 0$. *(AI 2019C, 2014C)* Ev

50. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y=\frac{\pi}{4}$ when $x = 0$. *(2018)*

51. Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$, given that $y = 0$ when $x=\frac{\pi}{3}$. *(2018, Foreign 2014)* Ev

52. Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)\, dy$, where C is a parameter. *(NCERT, Delhi 2017)*

53. Solve the differential equation $(\tan^{-1}x - y)dx = (1 + x^2)dy$. *(AI 2017)* Ev

54. Find the general solution of the following differential equation :
$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$
(NCERT Exemplar, Delhi 2016)

55. Find the particular solution of the differential equation $(1 - y^2)(1 + \log x)dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. *(Delhi 2016)* An

56. Solve the differential equation : $y+x\frac{dy}{dx}=x-y\frac{dy}{dx}$ *(AI 2016)*

57. Solve the following differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$ *(NCERT, Exemplar, Foreign 2016)*

58. Solve the following differential equation $(\cot^{-1} y + x)dy = (1 + y^2)dx$ *(Foreign 2016)* Ap

59. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$, given that $y=\frac{\pi}{2}$, when $x = 1$. *(Delhi 2014)* Ev

60. Solve the following differential equation : $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}, |x|\neq 1$ *(Delhi 2014)*

61. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx+\frac{y}{x}dy=0$ given that $y = 1$ when $x = 0$. *(Delhi 2014)* An

62. Solve the following differential equation : $\operatorname{cosec} x \log y\frac{dy}{dx}+x^2y^2=0$. *(Delhi 2014)*

63. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$, given that $y = 0$ when $x = 1$ *(AI 2014)* An

64. Solve the differential equation $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$ *(AI 2014)*

65. Find the particular solution of the differential equation $x(1 + y^2)\, dx - y(1 + x^2)dy = 0$, given that $y = 1$ when $x = 0$. *(AI 2014)* Ap

66. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right)=3x+4y$, given that $y = 0$ when $x = 0$. *(NCERT, AI 2014)*

67. Solve the differential equation $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$, given that $y = 1$ when $x = 1$. *(Foreign 2014)* Ap

68. Solve the differential equation $\frac{dy}{dx}+y\cot x=2\cos x$, given that $y = 0$ when $x=\frac{\pi}{2}$. *(Foreign 2014)*

69. Solve the differential equation $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$. *(NCERT, Foreign 2014)* Ev

70. If $y(x)$ is a solution of the differential equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$ and $y(0)=1$, then find the value of $y\left(\frac{\pi}{2}\right)$. *(Delhi 2014C)* Ev

71. Find the general solution of the differential equation $(x-y)\frac{dy}{dx}=x+2y$. *(Delhi 2014C)*

72. Find the particular solution of the differential equation $x\frac{dy}{dx}-y+x\operatorname{cosec}\left(\frac{y}{x}\right)=0$; given that $y = 0$ when $x = 1$. *(AI 2014C)*

73. Solve the differential equation $x\frac{dy}{dx}+y=x\cos x+\sin x$, given $y\left(\frac{\pi}{2}\right)=1$. *(AI 2014C)*

LA II (5 / 6 marks)

74. Solve the differential equation $x\frac{dy}{dx}+y = x\cos x + \sin x$, given that $y = 1$ when $x=\frac{\pi}{2}$. *(Delhi 2017)*

75. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=(x+2y)$, given that $y = 0$ when $x = 1$. (AI 2017) Ap

76. Solve the differential equation : $(\tan^{-1}y - x)dy = (1 + y^2)dx$. (NCERT, Delhi 2015)

77. Show that the differential equation $\frac{dy}{dx}=\frac{y^2}{xy-x^2}$ is homogeneous and also solve it. (AI 2015)

78. Find the particular solution of the differential equation $(\tan^{-1}y-x)\, dy = (1+y^2)dx$, given that $x = 1$ when $y = 0$. (NCERT, AI 2015)

79. Solve the following differential equation :

$$\left[y-x\cos\left(\frac{y}{x}\right)\right]dy+\left[y\cos\left(\frac{y}{x}\right)-2x\sin\left(\frac{y}{x}\right)\right]dx=0$$

(Foreign 2015) Ap

80. Solve the following differential equation :

$$\left(\sqrt{1+x^2+y^2+x^2y^2}\right)dx+xy\,dy=0$$ (Foreign 2015)

81. Find the particular solution of the differential equation $x\frac{dy}{dx}+y-x+xy\cot x=0;\ x\neq 0$, given that when $x=\frac{\pi}{2}, y=0$. (NCERT, Delhi 2015C) An

82. Solve the differential equation $x^2\, dy + (xy+y^2)dx = 0$ given $y = 1$, when $x = 1$ (Delhi 2015C)

83. Solve the differential equation

$$\left(x\sin^2\left(\frac{y}{x}\right)-y\right)dx+xdy=0 \text{ given } y=\frac{\pi}{4} \text{ when } x = 1$$

(AI 2015C, 2014C) An

84. Solve the differential equation $\frac{dy}{dx}-3y\cot x = \sin 2x$ given $y = 2$ when $x=\frac{\pi}{2}$. (AI 2015C)

CBSE Sample Questions

9.2 Basic Concepts

MCQ

1. If m and n, respectively, are the order and the degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4=0$, then $m+n=$

(a) 1 (b) 2 (c) 3 (d) 4

(2022-23) U

VSA (1 mark)

2. For what value of n is the following a homogeneous differential equation: $\frac{dy}{dx}=\frac{x^3-y^n}{x^2y+xy^2}$? (2020-21)

SA I (2 marks)

3. Write the sum of the order and the degree of the following differential equation $\frac{d}{dx}\left(\frac{dy}{dx}\right)=5$.

(Term II, 2021-22) Ap

9.3 General and Particular Solutions of a Differential Equation

VSA (1 mark)

4. How many arbitrary constants are there in the particular solution of the differential equation $\frac{dy}{dx}=-4xy^2; y(0)=1$? (2020-21)

9.4 Methods of Solving First order, First Degree Differential Equations

SA I (2 marks)

5. Solve the following differential equation:

$\frac{dy}{dx}=x^3\operatorname{cosec} y$, given that $y(0) = 0$ (2020-21)

SA II (3 marks)

6. Solve the differential equation : $ydx + (x - y^2)dy = 0$

(2022-23) Ap

7. Solve the differential equation :

$xdy-ydx=\sqrt{x^2+y^2}\,dx$ (2022-23) Ap

8. Find the general solution of the following differential equation.

$x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$ (Term II, 2021-22)

9. Find the particular solution of the following differential equation, given that $y=0$ when $x=\frac{\pi}{4}$.

$\frac{dy}{dx}+y\cot x=\frac{2}{1+\sin x}$ (Term II, 2021-22) Ev

10. Find the general solution of the following differential equation:

$xdy - (y + 2x^2)dx = 0$ (2020-21) An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. (b) : [There is error in question, the given differential equation should be $\frac{d}{dx}\left(\frac{dy}{dx}\right)^3=0$.]

The given differential equation is,

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)=0 \Rightarrow 3\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right)=0$$

$\therefore$ Order = 2 and degree = 1

So, required sum = 2 + 1 = 3

2. (b) : We have, $\left(1+3\frac{dy}{dx}\right)^2=4\frac{d^3y}{dx^3}$

Here, order = 3 as highest order derivative is $\frac{d^3y}{dx^3}$.

And degree = 1, as power of highest order derivative *i.e.*, $\frac{d^3y}{dx^3}$ is 1.

3. The degree of the differential equation

$1+\left(\frac{dy}{dx}\right)^2=x$ is 2.

4. The given differential equation is

$x^2\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^4$ $\therefore$ Its order is 2 and degree is 1.

5. The given differential equation is

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\}=0 \Rightarrow 3.\left(\frac{dy}{dx}\right)^2.\frac{d^2y}{dx^2}=0$$

Order = 2 and Degree = 1 $\therefore$ Order + Degree = 2 + 1 = 3

Concept Applied

- Highest order derivative appearing in a differential equation is called order of the differential equation.

6. Order = 2, Degree = 2 $\therefore$ Order + Degree = 2 + 2 = 4

7. Order = 2, Degree = 3

$\therefore$ Order + Degree = 2 + 3 = 5

8. The given differential equation is $\left[\frac{d}{dx}(xy^2)\right]\cdot\frac{dy}{dx}+y=0$

$$\Rightarrow \left[x\cdot 2y\frac{dy}{dx}+y^2\cdot 1\right]\frac{dy}{dx}+y=0 \Rightarrow 2xy\left(\frac{dy}{dx}\right)^2+y^2\left(\frac{dy}{dx}\right)+y=0$$

$\therefore$ Its order is 1 and degree is 2.

$\therefore$ Required product = 1 × 2 = 2

Concept Applied

- The degree of a differential equation is the power of the highest ordered derivative, when differential coefficients are made free from radicals and fractions.

9. We have, $x\left[y\left(\frac{d^2y}{dx^2}\right)^3+x\left(\frac{dy}{dx}\right)^2-\frac{y}{x}\frac{dy}{dx}\right]=0$

Its order is 2 and degree is 3.

$\therefore$ $a = 2, b = 3$

$2a - 3b = 2 \times 2 - 3 \times 3 = 4 - 9 = -5$

10. (b): Given that; $\frac{dy}{dx}=\frac{y+1}{x-1} \Rightarrow \frac{dy}{y+1}=\frac{dx}{x-1}$

On integrating both sides, we get $\int\frac{dy}{y+1}=\int\frac{dx}{x-1}$

$\Rightarrow$ $\log(y + 1) = \log(x - 1) - \log C$

$\Rightarrow$ $\log(y + 1) + \log C = \log(x - 1) \Rightarrow C=\frac{x-1}{y+1}$

Now, $y(1) = 2 \Rightarrow C=\frac{1-1}{2+1}=0$

$\therefore$ Required solution is $x - 1 = 0$

Hence, only one solution exist.

11. (a): In the particular solution of a differential equation of any order, there is no arbitrary constant because in the particular solution of any differential equation, we remove all the arbitrary constant by substituting some particular values.

12. (d): We have, $x\frac{dy}{dx}-y=2x^2$

i.e., $\frac{dy}{dx}-\frac{y}{x}=2x$ $\therefore$ I.F. $=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=e^{\ln x^{-1}}=\frac{1}{x}$

$\therefore$ Integrating factor is $\frac{1}{x}$

13. (c) : We have, $(x+3y^2)\frac{dy}{dx}=y$

$$\Rightarrow \frac{x+3y^2}{y}=\frac{dx}{dy} \Rightarrow \frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation.

$\therefore$ I.F. $=e^{-\int\frac{dy}{y}}=e^{-\log y}=e^{\log y^{-1}}=\frac{1}{y}$

14. We have, $x\frac{dy}{dx}-y=\log x \Rightarrow \frac{dy}{dx}-\frac{y}{x}=\frac{\log x}{x}$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$

$\therefore$ I.F. $=e^{\int-\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$

15. We have, $x\frac{dy}{dx}+2y=x^2 \Rightarrow \frac{dy}{dx}+2\frac{y}{x}=x$

$\therefore$ I.F. $=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

16. We have, $e^{y-x}\frac{dy}{dx}=1 \Rightarrow e^y\cdot e^{-x}\frac{dy}{dx}=1 \Rightarrow e^y dy=e^x dx$

Integrating both sides, we get

$e^y = e^x + c,\ y = \log(e^x + c)$

17. We have, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$ or $\frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore$ I.F. $= e^{\int P dx} \Rightarrow$ I.F. $= e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

18. The given differential equation is

$$(1+y^2)+(2xy-\cot y)\frac{dy}{dx}=0$$

$$\Rightarrow (1+y^2)\frac{dx}{dy}+2xy-\cot y=0 \Rightarrow \frac{dx}{dy}+\frac{2y}{1+y^2}\cdot x=\frac{\cot y}{1+y^2}$$

This is a linear differential equation of the form

$\frac{dx}{dy}+Px=Q$, where, $P=\frac{2y}{1+y^2}$ and $Q=\frac{\cot y}{1+y^2}$

$\therefore$ I.F. $=e^{\int P dy}=e^{\int \frac{2y}{1+y^2}dy} = e^{\log(1+y^2)} = 1+y^2.$

19. We have, $\frac{dy}{dx}=2^{-y} \Rightarrow \frac{dy}{dx}=\frac{1}{2^y} \Rightarrow 2^y dy = dx$ ...(i)

Integrating both sides of (i), we get $\frac{2^y}{\log 2}=x+C$

$\Rightarrow \quad 2^y=(C+x)\log 2$

Taking log on both sides to the base 2, we get

$\log_2 2^y = \log_2[(C+x)\log 2]$

$\Rightarrow \quad y=\log_2[(C+x)\log 2]$

This is the required solution.

Answer Tips

- $\int a^x dx = \frac{a^x}{\ln(a)}+C$, where C is arbitrary constant.

20. We have, $\frac{dy}{dx}=x^3e^{-2y} \Rightarrow e^{2y}dy = x^3 dx$

Integrating both sides, we get $\frac{e^{2y}}{2}=\frac{x^4}{4}+C'$

$\Rightarrow \quad 2e^{2y}=x^4+C$, where $C=4C'$

21.

$$\log\left(\frac{dy}{dx}\right) = ax+by$$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by}$$

$$\Rightarrow \frac{dy}{dx} = e^{ax}\cdot e^{by} \qquad [e^{a+b}=e^a\cdot e^b]$$

$$\Rightarrow \frac{dy}{e^{by}} = e^{ax}dx$$

$$\Rightarrow e^{-by}dy = e^{ax}dx$$

On integrating both sides.

$$\int e^{-by}dy = \int e^{ax}dx$$

$$\Rightarrow \frac{-e^{-by}}{b} = \frac{e^{ax}}{a}+c$$

$$\frac{e^{ax}}{a}+\frac{e^{-by}}{b} = c' \qquad [c'=-c]$$

where c & c' are constants.

$$\boxed{\text{Answer: } \frac{e^{ax}}{a}+\frac{e^{-by}}{b} = c}$$

[Topper's Answer, 2022]

22. We have, $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\Rightarrow \quad \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy \quad \Rightarrow \quad \frac{d(\tan x)}{\tan x} = -\frac{d(\tan y)}{\tan y}$

$\Rightarrow \quad \log(\tan x) = -\log(\tan y) + \log c$ (integrating on both sides)

$\Rightarrow \quad \tan x \tan y = c$

23. We have, $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \quad dy = \frac{(e^x + x^2)}{e^y} dx \Rightarrow e^y dy = (e^x + x^2) dx$

Integrating on both sides, we get

$\int e^y\, dy = \int (e^x + x^2)\, dx$

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$, which is required solution.

24. We have, $x\frac{dy}{dx} + (1 + x\cot x)y = x$

$\Rightarrow \quad \frac{dy}{dx} + \frac{(1 + x\cot x)}{x} y = 1$

Clearly it is a linear differential equation of the form, $\frac{dy}{dx} + Py = Q$, where $P = \frac{1 + x\cot x}{x}$ and $Q = 1$

$\therefore \quad$ I.F. $= e^{\int \left(\frac{1}{x} + \cot x\right) dx}$

$= e^{\log x + \log \sin x} = e^{\log(x \sin x)} = x \sin x.$

25. We have, $x\frac{dy}{dx} = x + y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}$

This is a homogeneous differential equation

Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \Rightarrow v + x\frac{dv}{dx} = 1 + v$

$\Rightarrow \quad x\frac{dv}{dx} = 1 \Rightarrow dv = \frac{dx}{x}$

Integrating both sides, we get

$v = \log x + \log c$

$v = \log xc$

$\Rightarrow \quad y = x \log cx$

26. We have, $\frac{dy}{dx} + y = \cos x - \sin x$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 1$, $Q = \cos x - \sin x$

$\therefore \quad$ I.F. $= e^{\int dx} = e^x$

The solution of given differential equation is

$ye^x = \int e^x (\cos x - \sin x) dx + C$

$\Rightarrow \quad ye^x = e^x \cos x + C$

$\Rightarrow \quad y = \cos x + Ce^{-x}$

27. Given differential equation is $\frac{dy}{dx} = \frac{x+y}{x}$

$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 1$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}$, $Q = 1$

I.F. $= e^{\int \frac{-1}{x} dx} = e^{-\log x} = \frac{1}{x}$

$\therefore \quad$ Solution is given by $y \cdot \frac{1}{x} = \int 1 \cdot \frac{1}{x} dx + C$

$\Rightarrow \quad \frac{y}{x} = \log x + C \qquad$ ...(i)

We have $y(1) = 0$

When $x = 1, y = 0$

$\therefore \quad 0 = 0 + C \Rightarrow C = 0$

$\therefore \quad$ From (i) $\frac{y}{x} = \log x \Rightarrow y = x \log x$

28. We have, $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 1) \sec^2 y\, dy$

$\Rightarrow \quad \frac{e^x}{e^x - 1} dx = \frac{\sec^2 y}{\tan y} dy$

Integrating both sides, we get

$$\int \frac{e^x}{e^x - 1} dx = \int \frac{\sec^2 y}{\tan y} dy \qquad \text{... (i)}$$

Put $e^x - 1 = u \Rightarrow e^x dx = du$

and $\tan y = v \Rightarrow \sec^2 y\, dy = dv$

$\therefore \quad$ From (i) $\int \frac{du}{u} = \int \frac{dv}{v} \Rightarrow \log(u) = \log(v) + \log C$

$\Rightarrow \quad \log(e^x - 1) = \log(\tan y) + \log C$

$\Rightarrow \quad \log(e^x - 1) = \log(C \tan y)$

$\Rightarrow \quad e^x - 1 = C \tan y$

29. We have, $x\frac{dy}{dx} + x\cos^2\left(\frac{y}{x}\right) = y$

$\Rightarrow \quad \frac{dy}{dx} + \cos^2\left(\frac{y}{x}\right) = \frac{y}{x}$

This is a homogeneous differential equation.

Now, put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} + \cos^2 v = v \Rightarrow \frac{x dv}{dx} = -\cos^2 v$

$\Rightarrow \quad \sec^2 v\, dv = -\frac{dx}{x}$

Integrating both sides, we get

$\tan v = -\log x + \log c$

$\Rightarrow \quad \tan v = \log\left|\frac{c}{x}\right| \Rightarrow \tan\frac{y}{x} = \log\left|\frac{c}{x}\right|$

When $x = 1, y = \frac{\pi}{4}$

$\therefore \quad \tan\frac{\pi}{4} = \log\frac{c}{1} \Rightarrow \log c = 1$

$\Rightarrow \quad c = e$

Particular solution is $\tan\frac{y}{x} = \log\frac{e}{x}$

$\Rightarrow \quad \tan\frac{y}{x} = 1 - \log x$

30.

$x\frac{dy}{dx} = y(\log y - \log x + 1)$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$ $\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

on putting $x = \lambda x$, $y = \lambda y$.

$f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x}\left(\log\left(\frac{\lambda y}{\lambda x}\right) + 1\right)$

$f(\lambda x, \lambda y) = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$

$= f(x, y)$

Thus, this equation is homogeneous equation

Let $\frac{y}{x} = t$ or $y = tx$

on differentiating with respect to x

$\frac{dy}{dx} = t + x\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$

$t + \frac{x\,dt}{dx} = t(\log t + 1)$ $\left[\because \frac{y}{x} = t\right]$

$t + \frac{x\,dt}{dx} = t\log t + t$

$\frac{x\,dt}{dx} = t\log t$

$\frac{dt}{t\log t} = \frac{dx}{x}$

on integrating both sides.

$\int \frac{dt}{t\log t} = \int \frac{dx}{x}$

$\int \frac{dt}{t\log t} = \log_e x + c$

let $\log t = u$

on differentiating,

$\frac{1}{t}\,dt = du$

$\int \frac{du}{u} = \ln x + c$ $[\log_e x = \ln x]$

$\ln u = \ln x + c$ [c is integration constant]

$\ln(\ln t) = \ln x + c$ $[u = \log_e t = \ln t]$

$\ln\left(\ln\left(\frac{y}{x}\right)\right) = \ln x + c$

$\ln\left(\ln\left(\frac{y}{x}\right)\right) - \ln x = c$

$\ln\left(\frac{\ln(y/x)}{x}\right) = c$

$\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

$\boxed{\text{Answer: } \ln\left(\frac{\ln(y/x)}{x}\right) = c}$

[where $\ln x = \log_e x$]

[Topper's Answer, 2022]

31. We have, $\frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$ ...(i)

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2$

This is a homogeneous differential equation.

Now, put $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = v - \frac{1}{2}v^2$

$\Rightarrow x\frac{dv}{dx} = -\frac{1}{2}v^2 \Rightarrow -\frac{1}{v^2}dv = \frac{1}{2x}dx$

Integrating both sides, we get

$\frac{1}{v} = \frac{1}{2}\log|x| + C \Rightarrow \frac{x}{y} = \frac{1}{2}\log|x| + C$

Given, solution of (i) is $\frac{ax}{y} = b\log|x| + C$

On comparing, $a = 1, b = \frac{1}{2}$

32. (i) We have, $(x^2 - y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy} = \frac{y^2 - x^2}{2xy}$

Now, putting $\frac{dy}{dx} = F(x, y)$ and find $F(\lambda x, \lambda y)$,

$\Rightarrow F(x, y) = \frac{y^2 - x^2}{2xy}$

$\therefore F(\lambda x, \lambda y) = \frac{\lambda^2 y^2 - \lambda^2 x^2}{2\lambda^2 xy} = \frac{y^2 - x^2}{2xy} = F(x, y)$

So, $F(x, y)$ is a homogeneous function and the given differential equation is of the type $g\left(\frac{y}{x}\right)$

(ii) We have, $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx} = \frac{v^2 - 1}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2 + 1}{2v}\right) \Rightarrow \int \frac{2v}{v^2 + 1}dv = -\int \frac{dx}{x}$

$\Rightarrow \int \frac{2v}{v^2 + 1}dv + \int \frac{dx}{x} = \log C$

$\Rightarrow \log|v^2 + 1| + \log x = \log C$

$$\Rightarrow \log\left|\left(\frac{y^2+x^2}{x^2}\right)\times x\right|=\log C$$

$$\Rightarrow \frac{y^2+x^2}{x}=C \Rightarrow x^2+y^2=Cx$$

is the required general solution.

33. We have, $(1+x^2)\frac{dy}{dx}+2xy=\tan x$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\tan x}{1+x^2} \qquad \text{...(i)}$$

Clearly, it is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, Where $P=\frac{2x}{1+x^2}$ and $Q=\frac{\tan x}{1+x^2}$

$\therefore$ I.F. $= e^{\int\frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=1+x^2$

$\therefore$ Solution of (i) is

$$y(1+x^2)=\int(1+x^2)\frac{\tan x}{(1+x^2)}dx+C$$

$$\Rightarrow y(1+x^2)=\int\tan x\,dx+C$$

$$\Rightarrow y(1+x^2)=\log|\sec x|+C$$

Also given, $y(0) = 1$

$\therefore$ $1 = \log\sec 0 + C \Rightarrow C = 1$

Particular solution is

$y(1+x^2) = \log|\sec x| + 1$

34. We have, $(1+\sin x)\frac{dy}{dx}=-x-y\cos x$

$$\Rightarrow (1+\sin x)\frac{dy}{dx}+y\cos x=-x$$

$$\Rightarrow \frac{dy}{dx}+\frac{\cos x}{1+\sin x}y=\frac{-x}{1+\sin x}$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$,

where, $P=\frac{\cos x}{1+\sin x}$, $Q=\frac{-x}{1+\sin x}$

I.F. $=e^{\int\frac{\cos x}{1+\sin x}dx}=e^{\log(1+\sin x)}=1+\sin x$

The solution of given differential equation is

$$y\cdot(1+\sin x)=\int\frac{-x}{(1+\sin x)}(1+\sin x)dx+C$$

$$\Rightarrow y\cdot(1+\sin x)=\frac{-x^2}{2}+C$$

Also given $y(0) = 1$

$\therefore$ $1(1+\sin 0) = 0 + C \Rightarrow C = 1$

$\therefore$ Particular solution is $y(1+\sin x)=\frac{-x^2}{2}+1$

35. We have, $\frac{xdy}{dx}+2y=x^2\log x$

$$\Rightarrow \frac{dy}{dx}+\frac{2y}{x}=x\log x \qquad \text{...(i)}$$

Clearly, it is a linear differential equation of the form,

$\frac{dy}{dx}+Py=Q$, where $P=\frac{2}{x}$ and $Q = x\log x$.

I.F. $=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ Solution of (i) is

$$y\cdot x=\int x^2\cdot x\log x\,dx+C \Rightarrow y\cdot x^2=\int x^3\cdot\log x\,dx+C$$

$$y\cdot x^2=\log x\cdot\frac{x^4}{4}-\int\frac{1}{x}\cdot\frac{x^4}{4}dx+C \Rightarrow y\cdot x^2=\frac{x^4}{4}\log x-\frac{x^4}{16}+C$$

$$\Rightarrow y=\frac{x^2}{4}\log x-\frac{x^2}{16}+\frac{C}{x^2}$$

Also, given $y(1) = 1$

$$\therefore 1=0-\frac{1}{16}+C \Rightarrow C=\frac{17}{16}$$

$\therefore$ Particular solution is $y=\frac{x^2}{4}\log x-\frac{x^2}{16}+\frac{17}{16x^2}$

36. We have, $x\frac{dy}{dx}+y+\frac{1}{1+x^2}=0 \qquad \text{...(i)}$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\frac{-1}{(1+x^2)x}$$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{x}$ and $Q=\frac{-1}{x(1+x^2)}$

I.F. $=e^{\int\frac{1}{x}dx}=e^{\log x}=x$.

$\therefore$ Solution of (i) is

$$y\cdot x=\int\frac{-1}{x(1+x^2)}\cdot x\,dx+C$$

$$\Rightarrow yx=\int\frac{-1}{1+x^2}dx+C \Rightarrow yx=-\tan^{-1}x+C$$

Also, given $y(1) = 0$

$$\therefore 0\cdot 1=-\tan^{-1}1+C \Rightarrow C=\tan^{-1}1=\frac{\pi}{4}$$

$\therefore$ Particular solution of given differential equation is

$$yx=-\tan^{-1}x+\frac{\pi}{4}$$

37. We have, $x(y^3+x^3)\,dy = (2y^4+5x^3y)dx$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^4+5x^3y}{xy^3+x^4} \Rightarrow \frac{dy}{dx}=\frac{2(y/x)^4+5(y/x)}{(y/x)^3+1}$$

Put $v=\frac{y}{x} \Rightarrow y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore v+x\frac{dv}{dx}=\frac{2v^4+5v}{v^3+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v^4+5v}{v^3+1}-v=\frac{v^4+4v}{v^3+1}$$

$$\Rightarrow \frac{v^3+1}{v^4+4v}dv=\frac{dx}{x} \Rightarrow \int\frac{v^3+1}{v^4+4v}dv=\int\frac{dx}{x}$$

Putting $v^4+4v=t \Rightarrow (4v^3+4)\,dv=dt$

$\therefore \quad \frac{1}{4}\int \frac{dt}{t} = \int \frac{dx}{x}$

$\Rightarrow \frac{1}{4}\log(v^4+4v) = \log x + \log C$

$\Rightarrow \frac{1}{4}\log(v^4+4v) = \log x + \log C \Rightarrow \frac{y^4+4yx^3}{x^8} = C$

38. $(y - \sin^2 x)\, dx + \tan x\, dy = 0$

$\Rightarrow \quad (y - \sin^2 x)dx = -\tan x\, dy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y - \sin^2 x}{-\tan x} \Rightarrow \frac{dy}{dx} = \frac{\sin^2 x}{\tan x} - \frac{y}{\tan x}$

$\Rightarrow \quad \frac{dy}{dx} = \sin x \cos x - y \cot x$

$\Rightarrow \quad \frac{dy}{dx} + y\cot x = \sin x \cos x$

So, it is a linear differential equation, where
$P = \cot x$, $Q = \cos x \sin x$

I.F. $= e^{\int P dx} = e^{\int \cot x\, dx}$
$= e^{\log_e |\sin x|} = \sin x$

General solution. : y (I.F.) $= \int Q(\text{I.F.})dx$

$\Rightarrow \quad y \cdot \sin x = \int \cos x \sin x \cdot \sin x\, dx$

$\Rightarrow \quad y \cdot \sin x = \int \cos x \cdot \sin^2 x\, dx$

$= \int t^2\, dt = \frac{t^3}{3} + c \qquad [\because \text{ Let } \sin x = t \Rightarrow dx = \frac{dt}{\cos x}]$

$\Rightarrow \quad y \sin x = \frac{\sin^3 x}{3} + c \Rightarrow y = \frac{\sin^2 x}{3} + \frac{c}{\sin x}$

39. $(x^3 + y^3)\, dy = x^2 y\, dx$ is rearranged as $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$

Let $\frac{y}{x} = v \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3} \qquad \left[\because \frac{dy}{dx} = \frac{y/x}{1+(y/x)^3}\right]$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v \Rightarrow x\frac{dv}{dx} = \frac{v - v - v^4}{1+v^3}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

$\Rightarrow \quad \int \frac{1+v^3}{v^4} dv = -\int \frac{dx}{x} \qquad$ [Integrating on both sides]

$\Rightarrow \quad \int \left(\frac{1}{v^4} + \frac{1}{v}\right) dv = -\int \frac{dx}{x}$

$\Rightarrow \quad \frac{-1}{3v^3} + \log|v| = -\log|x| + c$

$\Rightarrow \quad \frac{-x^3}{3y^3} + \log\left|\frac{y}{x}\right| = -\log|x| + c \Rightarrow \frac{-x^3}{3y^3} + \log|y| = c$

40. We have, $ye^y dx = (y^3 + 2xe^y)dy$

$\Rightarrow \quad \frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y} \Rightarrow \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y} \qquad$ ...(i)

This is a linear D.E. of the form $\frac{dx}{dy} + Px = Q$

Where $P = -\frac{2}{y}$ and $Q = y^2 e^{-y}$

$\therefore \quad$ I.F. $= e^{-\int \frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$

So, the solution of (i) is $x \cdot \frac{1}{y^2} = \int \frac{1}{y^2} \cdot y^2 e^{-y} dy$

$\Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C \Rightarrow x = -y^2 e^{-y} + Cy^2$

41. We have,

$(1 + e^{y/x})dy + e^{y/x}\left(1 - \frac{y}{x}\right)dx = 0, x \neq 0$

$\Rightarrow \quad \frac{dy}{dx} + \frac{e^{y/x}}{(1+e^{y/x})}\left(1 - \frac{y}{x}\right) = 0 \qquad$ ...(i)

This is a homogeneous differential equation.
Now, put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad$ From (i), $v + x\frac{dv}{dx} + \frac{e^v}{(1+e^v)}(1-v) = 0$

$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(1-v)e^v}{1+e^v} - v \Rightarrow x\frac{dv}{dx} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$

$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(e^v + v)}{1+e^v} \Rightarrow \left(\frac{1+e^v}{e^v + v}\right)dv = \frac{-dx}{x}$

Integrating both sides, we get
$\log(e^v + v) = -\log x + \log c$

$\Rightarrow \quad e^v + v = \frac{c}{x} \Rightarrow e^{\frac{y}{x}} + \frac{y}{x} = \frac{c}{x}$

$\therefore \quad$ Required solution is $e^{\frac{y}{x}} + \frac{y}{x} = \frac{c}{x}$

42. We have, $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$.

This is a homogeneous differential equation.

Now, put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = v - \tan v$

$\Rightarrow \quad x\frac{dv}{dx} = -\tan v \Rightarrow \frac{dv}{\tan v} = -\frac{dx}{x} \Rightarrow \cot v\, dv + \frac{dx}{x} = 0$

Integrating both sides,we get
$\log|\sin v| + \log x = \log C$

$\Rightarrow \quad x \sin v = C \Rightarrow x\sin\left(\frac{y}{x}\right) = C$

When $x = 1$, $y = \frac{\pi}{4}$, we get $1 \cdot \sin\left(\frac{\pi}{4}\right) = C \Rightarrow C = \frac{1}{\sqrt{2}}$

So, $x\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ is the required particular solution.

43. We have, $\cos y\,dx + (1+e^{-x})\sin y\,dy = 0$

$\Rightarrow$ $dx + (1+e^{-x})\tan y\,dy = 0$

$\Rightarrow$ $\dfrac{dx}{1+e^{-x}} + \tan y\,dy = 0$

Integrating both sides, we get

$$\int \frac{e^x}{1+e^x}dx + \int \tan y\,dy = 0$$

$\Rightarrow$ $\log(1+e^x) + \log|\sec y| = \log C$

$\Rightarrow$ $\sec y\,(1+e^x) = C$

When, $x = 0$, $y = \dfrac{\pi}{4}$, we get $C = 2\sqrt{2}$

$\therefore$ Particular solution of the differential equation is, $\sec y(1+e^x) = 2\sqrt{2}$

44. We have, $ye^{x/y}\,dx = (xe^{x/y} + y^2)dy$, $y \neq 0$

$\Rightarrow$ $\dfrac{dx}{dy} = \dfrac{x}{y} + \dfrac{y}{e^{x/y}}$ ...(i)

Putting $\dfrac{x}{y} = t \Rightarrow x = yt \Rightarrow \dfrac{dx}{dy} = t + y\dfrac{dt}{dy}$

$\therefore$ Equation (i) becomes, $t + y\dfrac{dt}{dy} = t + \dfrac{y}{e^t}$

$\Rightarrow$ $y\dfrac{dt}{dy} = ye^{-t} \Rightarrow \dfrac{dt}{dy} = e^{-t} \Rightarrow dy = e^t dt$

Integrating both sides, we get

$y = e^t + C \Rightarrow y = e^{x/y} + C$

Key Points

➲ $\int e^x dx = e^x + C$

45. We have, $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$

$\Rightarrow x\dfrac{dy}{dx} = y + \sqrt{x^2+y^2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{1+\left(\dfrac{y}{x}\right)^2}$ ...(i)

This is a linear homogeneous differential equation.

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ Eq. (i) becomes

$v + x\dfrac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow x\dfrac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow$ $\dfrac{dx}{x} = \dfrac{dv}{\sqrt{1+v^2}} \Rightarrow \int\dfrac{dx}{x} = \int\dfrac{dv}{\sqrt{1+v^2}}$

$\Rightarrow$ $\log x + \log C_1 = \log|v + \sqrt{1+v^2}|$

$\Rightarrow$ $\log x + \log C_1 = \log\left|\dfrac{y}{x} + \sqrt{1+\dfrac{y^2}{x^2}}\right|$

$\Rightarrow$ $\log C_1 x = \log|y + \sqrt{x^2+y^2}| - \log x$

$\Rightarrow$ $\pm C_1 x^2 = y + \sqrt{x^2+y^2} \Rightarrow Cx^2 = y + \sqrt{x^2+y^2}$ [where $C = \pm C_1$]

General solution of the given equation is

$Cx^2 = y + \sqrt{x^2+y^2}$ ...(ii)

Now, putting $y = 0$ and $x = 1$ in (ii), we get $C = 1$

$\therefore$ Required solution is $x^2 = y + \sqrt{x^2+y^2}$.

46. We have $(1+x^2)\dfrac{dy}{dx} + 2xy = 4x^2$

$\Rightarrow$ $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{4x^2}{1+x^2}$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{2x}{1+x^2}$ and $Q = \dfrac{4x^2}{1+x^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx}$

$= e^{\log(1+x^2)} = 1 + x^2$

Hence, the required solution is

$$y(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)\,dx + C$$

$\Rightarrow$ $y(1+x^2) = 4\int x^2 dx + C$

$\Rightarrow$ $y(1+x^2) = \dfrac{4x^3}{3} + C$

Given that $y(0) = 0$

$\therefore$ $0(1+0) = 0 + C \Rightarrow C = 0$

Thus, $y = \dfrac{4x^3}{3(1+x^2)}$ is the required solution.

47. We have, $\dfrac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\therefore \dfrac{dy}{dx} = 1 + x^2 + y^2(1+x^2) = (1+x^2)\,.\,(1+y^2)$

$\Rightarrow$ $\dfrac{dy}{1+y^2} = (1+x^2)dx$

Integrating both sides, we get $\tan^{-1}y = x + \dfrac{x^3}{3} + C$

when $x = 0$, $y = 1$

$\tan^{-1}1 = 0 + 0 + C \Rightarrow C = \dfrac{\pi}{4}$

$\therefore \tan^{-1}y = x + \dfrac{1}{3}x^3 + \dfrac{\pi}{4}$ is the required solution.

48. We have, $\dfrac{dy}{dx} = \dfrac{xy}{x^2+y^2}$

This is a homogeneous linear differential equation

$\therefore$ Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ $v + x\dfrac{dv}{dx} = \dfrac{x \cdot vx}{x^2 + v^2x^2}$

$\Rightarrow$ $v + x\dfrac{dv}{dx} = \dfrac{v}{1+v^2} \Rightarrow x\dfrac{dv}{dx} = \dfrac{v}{1+v^2} - v$

$\Rightarrow$ $x\dfrac{dv}{dx} = \dfrac{-v^3}{1+v^2} \Rightarrow \dfrac{dx}{x} = -\left(\dfrac{1+v^2}{v^3}\right)dv$

Integrating both sides, we get

$$\int \frac{dx}{x} = -\int v^{-3} dv - \int \frac{1}{v} dv$$

$$\Rightarrow \log x = \frac{1}{2v^2} - \log v + C$$

$$\Rightarrow \log x = \frac{x^2}{2y^2} - \log y + \log x + C$$

$$\Rightarrow \log y = \frac{x^2}{2y^2} + C$$

When $y = 1, x = 0 \Rightarrow \log 1 = 0 + C \Rightarrow C = 0$

$\therefore$ Particular solution is $y = e^{\frac{x^2}{2y^2}}$

49. We have, $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x;\ x \neq 0$

$$\Rightarrow \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = \left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) + 1 \quad \text{...(i)}$$

This is a linear homogeneous differential equation

Put $y = vx \Rightarrow \frac{dy}{dx} = v \cdot 1 + x \cdot \frac{dv}{dx}$

Now (i) becomes

$$\cos v.\left[v + x\frac{dv}{dx}\right] = v\cos v + 1$$

$$\Rightarrow x\cos v\frac{dv}{dx} = 1 \Rightarrow \cos v\, dv = \frac{dx}{x}$$

Integrating both sides, we get

$\sin v = \log x + C \Rightarrow \sin\left(\frac{y}{x}\right) = \log x + C$

50. The given differential equation is,

$e^x \tan y\, dx + (2 - e^x)\sec^2 y\, dy = 0$

$\Rightarrow (2 - e^x)\sec^2 y\, dy = -e^x \tan y\, dx$

$$\Rightarrow \frac{\sec^2 y}{\tan y} dy = \frac{-e^x}{2 - e^x} dx$$

Integrating both sides, we get

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{-e^x}{2 - e^x} dx$$

$\Rightarrow \log \tan y = \log(2 - e^x) + C$

When $y = \frac{\pi}{4}, x = 0$

$$\therefore \log \tan\frac{\pi}{4} = \log(2 - e^0) + C$$

$\Rightarrow 0 = \log 1 + C \Rightarrow C = 0$

$\therefore$ Particular solution is $\log \tan y = \log(2 - e^x)$

$\Rightarrow e^x + \tan y - 2 = 0$

51. We have, $\frac{dy}{dx} + 2y\tan x = \sin x$

It is linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

where $P = 2\tan x$, and $Q = \sin x$

Now, I.F. $= e^{\int 2\tan x\, dx} = e^{2\log|\sec x|} = \sec^2 x$

$\therefore y(\sec^2 x) = \int (\sec^2 x)(\sin x) dx$

$\Rightarrow y(\sec^2 x) = \int \sec x \tan x\, dx$

$\Rightarrow y(\sec^2 x) = \sec x + C$

When $x = \frac{\pi}{3}, y = 0$

$(0)[\sec^2(\pi/3)] = \sec(\pi/3) + C \Rightarrow C = -2$

$\therefore y(\sec^2 x) = \sec x - 2$ *i.e.*, $y = \cos x - 2\cos^2 x$ is the required solution.

52. We have, $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad \text{...(i)}$$

Put, $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (i) becomes

$$v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v} = \frac{1 - v^4}{v(v^2 - 3)}$$

$$\Rightarrow \frac{v(v^2 - 3)dv}{1 - v^4} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{(v^3 - 3v)dv}{(1 - v^2)(1 + v^2)} = \int \frac{dx}{x} \quad \text{...(ii)}$$

$$\text{Now, let } \frac{v^3 - 3v}{(1 - v^2)(1 + v^2)} = \frac{Av + B}{1 - v^2} + \frac{Cv + D}{1 + v^2} \quad \text{...(iii)}$$

Comparing coeff. of like powers, we get

$A - C = 1, A + C = -3, B - D = 0$ and $B + D = 0$

Solving these equations, we get $A = -1, B = 0, C = -2, D = 0$

From (ii) and (iii), we have

$$\int \frac{-v}{1 - v^2} dv - \int \frac{2v}{1 + v^2} dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log(1 - v^2) - \log(1 + v^2) = \log x + \log C_1$$

$$\Rightarrow \frac{\sqrt{1 - v^2}}{1 + v^2} = C_1 x \Rightarrow x\frac{\left(\sqrt{x^2 - y^2}\right)}{x^2 + y^2} = C_1 x$$

$\Rightarrow x^2 - y^2 = C^2_1(x^2 + y^2)^2$

i.e., $x^2 - y^2 = C(x^2 + y^2)^2$ (where $C^2_1 = C$)

which is the required solution.

Commonly Made Mistake

- Remember the difference between $y = vx$ and $x = vy$ while solving differential equation.

53. We have, $\frac{dy}{dx} = \frac{(\tan^{-1} x - y)}{1 + x^2}$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1 + x^2} = \frac{\tan^{-1} x}{1 + x^2}$$

which is a linear differential equation

where $P=\frac{1}{1+x^2}, Q=\frac{\tan^{-1}x}{1+x^2}$

$\therefore$ $\text{I.F.}=e^{\int Pdx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

$\therefore$ Solution is $y.(\text{I.F})=\int Q(\text{I.F})dx$

$\Rightarrow$ $ye^{\tan^{-1}x}=\int\frac{\tan^{-1}x}{1+x^2}\cdot e^{\tan^{-1}x}dx$ ...(i)

Let $I=\int\frac{\tan^{-1}x}{1+x^2}e^{\tan^{-1}x}dx$

Put $\tan^{-1}x=t \Rightarrow \frac{dx}{1+x^2}=dt$

$\therefore I=\int t\cdot e^t dt=t\int e^t dt-\int\left(\frac{d}{dt}(t)\int e^t dt\right)dt$

$\Rightarrow$ $I=te^t-\int e^t dt=te^t-e^t+C$

$\Rightarrow$ $I=e^t(t-1)+C$

$\Rightarrow$ $I=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$ ...(ii)

Putting (ii) in (i), we get

$ye^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$

$\Rightarrow$ $y=\tan^{-1}x-1+Ce^{-\tan^{-1}x}$

54. We have, $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$\Rightarrow$ $(x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$

$\Rightarrow$ $\frac{dx}{dy}=\frac{x-e^{\tan^{-1}y}}{-(1+y^2)} \Rightarrow \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$

$\therefore$ $\text{I.F.}=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

$\therefore$ Solution is $x\cdot e^{\tan^{-1}y}=\int\frac{(e^{\tan^{-1}y})^2}{1+y^2}dy+C$

$=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+C$

$\Rightarrow$ $x\cdot e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C_1 \Rightarrow x=\frac{e^{\tan^{-1}y}}{2}+C_1e^{-\tan^{-1}y}$

55. We have, $(1-y^2)(1+\log x)\,dx+2xy\,dy=0$

$\Rightarrow$ $(1-y^2)(1+\log x)\,dx=-2xy\,dy$

$\Rightarrow$ $\frac{(1+\log x)}{x}dx=-\frac{2y}{1-y^2}dy$

On integrating both sides, we get

$\frac{(1+\log x)^2}{2}=\log|1-y^2|+C$

When $x=1, y=0$

$\therefore$ $\frac{(1+\log 1)^2}{2}=\log(1)+C \Rightarrow C=\frac{1}{2}$

$\Rightarrow$ $\frac{(1+\log x)^2}{2}=\log|1-y^2|+\frac{1}{2}$

$\Rightarrow$ $(1+\log x)^2=2\log|1-y^2|+1$ is the required solution.

Answer Tips

- $\int\mu^n d\mu=\frac{\mu^{n+1}}{n+1}$ with $n\neq-1$

56. We have, $y+x\frac{dy}{dx}=x-y\frac{dy}{dx}$

$\Rightarrow$ $x\frac{dy}{dx}+y\frac{dy}{dx}=x-y \Rightarrow \frac{dy}{dx}=\frac{x-y}{x+y}$ ...(i)

This is a linear homogeneous D.E.

$\therefore$ Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{x-vx}{x+vx}=\frac{1-v}{1+v}$

$\Rightarrow$ $x\frac{dv}{dx}=\frac{1-v}{1+v}-v=\frac{1-v-v^2-v}{1+v}=\frac{1-2v-v^2}{1+v}$

$\Rightarrow$ $\frac{(1+v)}{v^2+2v-1}dv=-\frac{dx}{x}$

Integrating both sides, we get

$\frac{1}{2}\log|v^2+2v-1|=-\log|x|+\log C$

$\Rightarrow$ $\frac{1}{2}\log|v^2+2v-1|+\log|x|=\log C$

$\Rightarrow$ $\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{2y}{x}-1\right|+\log|x|=\log C$

$\Rightarrow$ $\log\left|\frac{y^2}{x^2}+\frac{2y}{x}-1\right|+2\log|x|=2\log C$

$\Rightarrow$ $\log\left|\frac{y^2+2xy-x^2}{x^2}\times x^2\right|=\log C^2$

$\Rightarrow$ $y^2+2xy-x^2=\pm C^2$

$\Rightarrow$ $y^2+2xy-x^2=C_1$ (where $C_1=\pm C^2$)

57. We have, $y^2\,dx+(x^2-xy+y^2)\,dy=0$

$\Rightarrow$ $\frac{dy}{dx}=\frac{-y^2}{x^2-xy+y^2}$

This is homogeneous differential equation.

$\therefore$ Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$, we get

$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x^2-vx^2+v^2x^2}$

$\Rightarrow$ $v+x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2} \Rightarrow x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2}-v$

$\Rightarrow$ $x\frac{dv}{dx}=\frac{-v-v^3}{1-v+v^2} \Rightarrow \frac{1-v+v^2}{v(1+v^2)}dv=-\frac{1}{x}dx$

Integrating both sides, we get

$$\int\frac{1+v^2}{v(1+v^2)}dv-\int\frac{v}{v(1+v^2)}dv=-\int\frac{1}{x}dx$$

$\Rightarrow \quad \int\frac{1}{v}dv-\int\frac{1}{1+v^2}dv=-\int\frac{1}{x}dx$

$\Rightarrow \quad \log|v| - \tan^{-1}v = -\log|x| + \log C$

$\Rightarrow \quad \log\left|\frac{vx}{C}\right| = \tan^{-1}v \quad \Rightarrow \quad \left|\frac{vx}{C}\right| = e^{\tan^{-1}v}$

$\Rightarrow \quad |y| = Ce^{\tan^{-1}(y/x)}$ is the required solution.

Concept Applied

- A differential equation of the form $f(x, y)dy = g(x, y)dx$ is said to be homogeneous differential equation if the degree of $f(x, y)$ and $g(x, y)$ is same.

58. We have, $(\cot^{-1}y + x)\, dy = (1 + y^2)dx$

$\Rightarrow \quad \frac{dx}{dy}=\frac{\cot^{-1}y+x}{1+y^2}$

$\Rightarrow \quad \frac{dx}{dy}+\left(-\frac{1}{1+y^2}\right)x=\frac{\cot^{-1}y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy}+Px=Q$, where, $P=-\frac{1}{1+y^2}$ and $Q=\frac{\cot^{-1}y}{1+y^2}$

$\therefore \quad$ I.F. $= e^{-\int\frac{1}{1+y^2}dy}=e^{\cot^{-1}y}$

$\therefore \quad$ Solution is,

$$xe^{\cot^{-1}y}=\int\frac{\cot^{-1}y}{(1+y^2)}e^{\cot^{-1}y}dy$$

[Put $t = \cot^{-1}y \Rightarrow dt = -\frac{1}{1+y^2}dy$]

$xe^{\cot^{-1}y}=-\int te^{t}dt$

$\Rightarrow \quad xe^{\cot^{-1}y}=-e^{t}(t-1)+C$

$\Rightarrow \quad xe^{\cot^{-1}y}=e^{\cot^{-1}y}(1-\cot^{-1}y)+C$

59. We have, $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$

$\Rightarrow \quad (\sin y + y\cos y)dy = x(2\log x + 1)dx$

On integrating both sides, we get

$$-\cos y + y\sin y - (-\cos y) = 2\left[\log x\times\frac{x^2}{2}-\int\frac{1}{x}\times\frac{x^2}{2}dx\right]+\frac{x^2}{2}+C$$

$\Rightarrow \quad y\sin y = x^2\log x-\frac{x^2}{2}+\frac{x^2}{2}+C$

$\Rightarrow \quad y\sin y = x^2\log x + C$

when $x = 1, y = \frac{\pi}{2}$

$\therefore \quad \frac{\pi}{2}\sin\frac{\pi}{2}=1\cdot\log(1)+C \quad \Rightarrow \quad \frac{\pi}{2}=C$

$\therefore \quad y\sin y = x^2\log x + \pi/2$ is the required solution.

60. We have, $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}, |x|\neq 1$

$\Rightarrow \quad \frac{dy}{dx}+\frac{2xy}{x^2-1}=\frac{2}{(x^2-1)^2}$

This is a linear differential equation of the form,

$\frac{dy}{dx}+Py=Q$, where $P=\frac{2x}{x^2-1}$ and $Q=\frac{2}{(x^2-1)^2}$

$\therefore \quad$ I.F. $=e^{\int Pdx}=e^{\int\frac{2x}{x^2-1}dx}=e^{\log(x^2-1)}=x^2-1$

Hence, solution of differential equation is given by

$$y(x^2-1)=\int\frac{2(x^2-1)}{(x^2-1)^2}dx$$

$\Rightarrow \quad y(x^2-1)=2\int\frac{dx}{x^2-1}$

$\Rightarrow \quad y(x^2-1)=2\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$

$\Rightarrow \quad y(x^2-1)=\log\left|\frac{x-1}{x+1}\right|+C$

Answer Tips

- $\int e^{\log}x=x+C$ where C is an arbitary constant.

61. We have, $e^x\sqrt{1-y^2}\,dx+\frac{y}{x}dy=0$

$\Rightarrow \quad x\,e^{x}dx+\frac{y}{\sqrt{1-y^2}}dy=0$

Integrating both sides, we get

$$x\cdot e^x-\int 1.e^x dx-\frac{1}{2}\int(1-y^2)^{-\frac{1}{2}}(-2y)dy=C$$

$\Rightarrow xe^x-e^x-\frac{1}{2}\frac{(1-y^2)^{\frac{1}{2}}}{1/2}=C$

$\Rightarrow e^x(x-1)-\sqrt{1-y^2}=C$

When $x=0, y=1, e^0(0-1)-\sqrt{1-1}=C$

$\Rightarrow \quad C = -1$

$\therefore \quad e^x(x-1)-\sqrt{1-y^2}=-1$ is the required solution.

62. We have, $\text{cosec}\,x\log y\frac{dy}{dx}+x^2y^2=0$

$\Rightarrow \quad \frac{\log y}{y^2}dy+\frac{x^2}{\text{cosec}\,x}dx=0$

Integrating both sides, we get

$$\int\frac{\log y}{y^2}dy+\int x^2\sin x\,dx=0$$

[Put $\log y=t\Rightarrow\frac{1}{y}dy=dt$ and $y=e^t$]

$\Rightarrow \int t.e^{-t}dt+\int x^2\sin x\,dx=0$

$$\Rightarrow\ t.\frac{e^{-t}}{-1}-\int 1.\frac{e^{-t}}{-1}dt+x^2(-\cos x)-\int 2x(-\cos x)dx=C$$

$$\Rightarrow\ -t\,e^{-t}-e^{-t}-x^2\cos x+2x\sin x-2\int 1.\sin x\,dx=C$$

$$\Rightarrow -\frac{1+\log y}{y}-x^2\cos x+2x\sin x+2\cos x=C$$

This is the required solution.

63. We have, $\frac{dy}{dx}=1+x+y+xy$

$$\Rightarrow \frac{dy}{dx}=(1+x)+(1+x)y=(1+x)(1+y)$$

$$\Rightarrow \frac{dy}{1+y}=(1+x)dx$$

Integrating both sides, we get

$$\int\frac{dy}{1+y}=\int(1+x)dx+C$$

$$\Rightarrow \log(1+y)=x+\frac{x^2}{2}+C \quad \text{...(i)}$$

When $x = 1$, $y = 0$

$$\therefore\ \log 1=1+\frac{1}{2}+C\ \Rightarrow C=-\frac{3}{2}$$

$\therefore$ The particular solution of (i) is $\log(1+y)=x+\frac{x^2}{2}-\frac{3}{2}$.

64. We have,

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$

$$\Rightarrow\ \frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{e^{\tan^{-1}x}}{1+x^2}$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{1+x^2}$ and $Q=\frac{e^{\tan^{-1}x}}{1+x^2}$

$$\therefore\ \text{I.F.}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

So, the required solution is,

$$y\cdot e^{\tan^{-1}x}=\int\frac{e^{2\tan^{-1}x}}{1+x^2}dx+C \quad \text{...(i)}$$

Put $\tan^{-1}x = t$

$$\Rightarrow\ \frac{1}{1+x^2}dx=dt$$

$\therefore$ (i) becomes,

$$y\cdot e^{\tan^{-1}x}=\int e^{2t}dt+C$$

$\Rightarrow\ y\cdot e^{\tan^{-1}x}=\frac{e^{2\tan^{-1}x}}{2}+C$ is the required solution.

65. We have, $x(1+y^2)\,dx-y(1+x^2)\,dy=0$

$$\Rightarrow\ \frac{x}{1+x^2}dx-\frac{y}{1+y^2}dy=0\ \Rightarrow\ \frac{2x}{1+x^2}dx=\frac{2y}{1+y^2}dy$$

Integrating both sides, we get $\log(1+y^2)=\log(1+x^2)+\log C$

$\Rightarrow\ 1+y^2=C(1+x^2)$

When $x = 0$, $y = 1$

$\therefore\ 1+1=C(1+0)\Rightarrow C=2$

$\therefore\ 1+y^2=2(1+x^2)$ is the required particular solution.

66. We have, $\log\left(\frac{dy}{dx}\right)=3x+4y$

$$\frac{dy}{dx}=e^{3x+4y}=e^{3x}\cdot e^{4y}\ \Rightarrow e^{-4y}\,dy=e^{3x}\,dx$$

Integrating both sides, we get

$$\int e^{3x}dx-\int e^{-4y}dy=0\ \Rightarrow\ \frac{e^{3x}}{3}-\frac{e^{-4y}}{-4}=C$$

When $x = 0$, $y = 0$

$$\therefore\ \frac{1}{3}+\frac{1}{4}=C\ \Rightarrow C=\frac{7}{12}$$

$$\Rightarrow\ \frac{e^{3x}}{3}+\frac{e^{-4y}}{4}=\frac{7}{12}$$

$\Rightarrow\ 4e^{3x}+3e^{-4y}=7$ is the required particular solution.

67. We have, $(x^2-yx^2)dy+(y^2+x^2y^2)dx=0$

$\Rightarrow\ x^2(1-y)dy+y^2(1+x^2)dx=0$

$$\Rightarrow\int\frac{(1-y)}{y^2}dy+\int\frac{(1+x^2)}{x^2}dx=0$$

$$\Rightarrow\int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy+\int\left(\frac{1}{x^2}+1\right)dx=0$$

$$\Rightarrow-\frac{1}{y}-\log|y|-\frac{1}{x}+x=C$$

$\Rightarrow\ -x-xy\log|y|-y+x^2y=C(xy) \quad \text{...(i)}$

when $x = 1$, $y = 1$

$\therefore\ -(1)-(1)(1)\log|1|-(1)+(1)^2(1)=C(1)$

$\Rightarrow\ C=-1$

$\therefore$ Equation (i) becomes

$x^2y=x+xy\log|y|+y-xy$

68. We have, $\frac{dy}{dx}+y\cot x=2\cos x$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\cot x$, $Q=2\cos x$

$$\therefore\ \text{I.F.}=e^{\int\cot x\,dx}=e^{\log|\sin x|}=|\sin x|$$

$$\therefore\ y|\sin x|=\int|\sin x|(2\cos x)dx$$

$$\Rightarrow y|\sin x|=\int\sin 2x\,dx$$

$$\Rightarrow y(\sin x)=-\frac{1}{2}\cos 2x+C$$

when $x=\frac{\pi}{2}, y=0$

$$\therefore\ 0(\sin\frac{\pi}{2})=-\frac{1}{2}\cos 2\left(\frac{\pi}{2}\right)+C\Rightarrow C=-\frac{1}{2}$$

$$\therefore\ y(\sin x)=-\frac{1}{2}\cos 2x-\frac{1}{2}$$

i.e., $2y\sin x+\cos 2x+1=0$ is the required solution.

Concept Applied

➲ $\int\sin 2x\,dx=\frac{-1}{2}\cos 2x+C$

69. We have, $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

$\Rightarrow \frac{dy}{dx}+\frac{1}{x\log x}y=\frac{2}{x^2}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{x\log x}, Q=\frac{2}{x^2}$

$\therefore$ I.F. $=e^{\int\frac{1}{x\log x}dx}=e^{\log\log x}=\log x$

$\therefore$ $y(\log x)=\int(\log x)\frac{2}{x^2}dx$

$\Rightarrow y(\log x)=\log x\int\frac{2}{x^2}dx-\int\left(\frac{d}{dx}(\log x)\int\frac{2}{x^2}dx\right)dx$

$\Rightarrow y(\log x)=\log x\left(-\frac{2}{x}\right)+\int\frac{2}{x^2}dx$

$\Rightarrow y(\log x)=\log x\left(-\frac{2}{x}\right)-\frac{2}{x}+C$

70. We have, $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$

$\Rightarrow \frac{dy}{1+y}=-\frac{\cos x}{2+\sin x}dx$

Integrating both sides, we get

$\log(y+1)=-\log(2+\sin x)+\log C$

$\Rightarrow \log(y+1)=\log\frac{C}{2+\sin x}$

$\Rightarrow$ $y+1=C/(2+\sin x)$ $\Rightarrow$ $(y+1)(2+\sin x)=C$

Given : $y(0)=1 \Leftrightarrow x=0, y=1$

$\therefore$ $(1+1).(2+\sin 0)=C$ $\Rightarrow$ $C=4$

$\therefore$ $(y+1)(2+\sin x)=4$

$\Rightarrow y=\frac{4}{2+\sin x}-1$ ...(i)

Put $x=\frac{\pi}{2}$ in (i), $y\left(\frac{\pi}{2}\right)=\frac{4}{2+1}-1=\frac{1}{3}$.

71. We have, $(x-y)\frac{dy}{dx}=x+2y$

$\Rightarrow \frac{dy}{dx}=\frac{x+2y}{x-y}$ ...(i)

This is a linear homogeneous differential equation.

$\therefore$ Put $y=vx \Rightarrow \frac{dy}{dx}=v.1+x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}=\frac{1+2v}{1-v}$

$\Rightarrow x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+v+v^2}{1-v}$

$\Rightarrow \frac{1-v}{1+v+v^2}dv=\frac{dx}{x}$

Integrating both sides, we get

$$\int\frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}dv=\log x+C$$

$$\Rightarrow -\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv+\frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$=\log x+C$

$\Rightarrow -\frac{1}{2}\log(v^2+v+1)+\frac{3}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left[\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right]=\log x+C$

$\Rightarrow -\frac{1}{2}\log\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\cdot x}\right)=\log x+C$

$\Rightarrow -\frac{1}{2}\log(y^2+xy+x^2)+\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\cdot x}\right)=C.$

72. We have, $x\frac{dy}{dx}-y+x\operatorname{cosec}\left(\frac{y}{x}\right)=0$

$\Rightarrow$ $\frac{dy}{dx}-\frac{y}{x}=-\operatorname{cosec}\left(\frac{y}{x}\right)$ ...(i)

Put $y=vx \Rightarrow \frac{dy}{dx}=v.1+x.\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}-v=-\operatorname{cosec}v$ $\Rightarrow$ $x\frac{dv}{dx}=-\operatorname{cosec}v$

$\Rightarrow -\sin v\,dv=\frac{dx}{x}$

Integrating both sides, we get $\cos v=\log x+C$

$\Rightarrow \cos\left(\frac{y}{x}\right)=\log x+C$

When $x=1, y=0$

$\Rightarrow \cos\left(\frac{0}{1}\right)=\log 1+C \Rightarrow C=1$

$\therefore \cos\left(\frac{y}{x}\right)=\log x+1$

This is the required particular solution.

73. We have, $x\frac{dy}{dx}+y=x\cos x+\sin x$

$\Rightarrow$ $\frac{dy}{dx}+\frac{1}{x}.y=\frac{x\cos x+\sin x}{x}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{x}$, $Q=\frac{x\cos x+\sin x}{x}$

$\therefore$ I.F. $=e^{\int Pdx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x.$

$\therefore$ $y\cdot x=\int\frac{x\cos x+\sin x}{x}\cdot x\,dx+C$

$\Rightarrow xy=\int x\cos x\,dx+\int\sin x\,dx+C$

$=x\cdot\sin x-\int 1\cdot\sin x\,dx+\int\sin x\,dx+C$

$=x\sin x+C$

Given $y\left(\frac{\pi}{2}\right)=1$

$\therefore$ $\frac{\pi}{2}.1=\frac{\pi}{2}\sin\frac{\pi}{2}+C \Rightarrow C=0$

$\therefore \quad xy = x\sin x$

$\Rightarrow \quad y = \sin x$ is the required solution.

Commonly Made Mistake

- Remember the difference of differential equations of the form $\frac{dy}{dx}+Py=Q$ and $\frac{dx}{dy}+Px=Q$

74. We have, $x\frac{dy}{dx}+y=x\cos x+\sin x$

$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\cos x+\frac{\sin x}{x}$

It is a linear differential equation.

$\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$

$\therefore\ y\cdot x=\int x\left(\cos x+\frac{\sin x}{x}\right)dx+c\ =\int(x\cos x+\sin x)dx+c$

$=x\sin x-\int\sin x dx+\int\sin x dx+c\ = x\sin x+c$

$\Rightarrow\ y=\sin x+\frac{c}{x}$

Given that, $y=1$ when $x=\frac{\pi}{2}\ \therefore\ 1=1+\frac{c}{\pi/2}\ \Rightarrow\ c=0$

$\therefore\quad y = \sin x$ is the required solution.

75. We have, $(x-y)\frac{dy}{dx}=x+2y$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{x+2y}{x-y} \qquad \text{...(i)}$$

Put $y = vx \quad\Rightarrow\ \frac{dy}{dx}=v+x\frac{dv}{dx}$

Putting $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}=\frac{1+2v}{1-v}$

$\Rightarrow\ x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+2v-v+v^2}{1-v}\ \Rightarrow\ \int\frac{1-v}{v^2+v+1}dv=\int\frac{1}{x}dx$

$\Rightarrow\ \int\frac{2-2v}{v^2+v+1}dv=2\log|x|+c\ \Rightarrow\ \int\frac{3-(2v+1)}{v^2+v+1}dv=2\log|x|+c$

$\Rightarrow\ \int\frac{3}{v^2+v+1}dv-\int\frac{2v+1}{v^2+v+1}dv=\log|x|^2+c$

$\Rightarrow\ 3\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt3}{2}\right)^2}dv-\log|v^2+v+1| = \log|x^2|+c$

$\Rightarrow\ \frac{3}{\left(\frac{\sqrt3}{2}\right)}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt3}{2}}\right)=\log|x^2(v^2+v+1)|+c$

$$\Rightarrow\ 2\sqrt3\tan^{-1}\left(\frac{2v+1}{\sqrt3}\right)=\log|x^2(v^2+v+1)|+c \qquad \text{...(ii)}$$

Substituting $v=\frac{y}{x}$ in (ii), we get

$$2\sqrt3\tan^{-1}\left(\frac{2y+x}{\sqrt3x}\right)=\log\left|x^2\frac{(y^2+yx+x^2)}{x^2}\right|+c \qquad \text{...(iii)}$$

Now, at $y = 0$ and $x = 1$, we have

$2\sqrt3\tan^{-1}\left(\frac{1}{\sqrt3}\right)=\log|1|+c$

$\Rightarrow\ c=2\sqrt3\cdot\frac{\pi}{6}=\frac{\pi}{\sqrt3}$

Substituting $c=\frac{\pi}{\sqrt3}$ in (iii), we get

$2\sqrt3\tan^{-1}\left(\frac{2y+x}{\sqrt3x}\right)=\log|y^2+xy+x^2|+\frac{\pi}{\sqrt3}$

$\Rightarrow\ 6\tan^{-1}\left(\frac{2y+x}{\sqrt3x}\right)=\sqrt3\log(x^2+xy+y^2)+\pi$

76. We have, $(\tan^{-1}y-x)dy=(1+y^2)dx$

$\Rightarrow\ \frac{dx}{dy}=\frac{\tan^{-1}y-x}{1+y^2}\ \Rightarrow\ \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{\tan^{-1}y}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1}y}{1+y^2}$

$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

$\therefore\quad$ Required solution is,

$$x\cdot e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}\cdot\tan^{-1}y}{1+y^2}dy+C \qquad \text{...(i)}$$

Put $\tan^{-1}y=t\Rightarrow\left(\frac{1}{1+y^2}\right)dy=dt$

$\therefore\quad$ (i) becomes, $x\cdot e^{\tan^{-1}y}=\int e^t\cdot t dt+C$

$\Rightarrow\quad x\cdot e^{\tan^{-1}y}=t\cdot e^t-\int1\cdot e^t dt+C\ \Rightarrow\ x\cdot e^{\tan^{-1}y}=t\cdot e^t-e^t+C$

$\Rightarrow\quad x\cdot e^{\tan^{-1}y}=\tan^{-1}y\cdot e^{\tan^{-1}y}-e^{\tan^{-1}y}+C$

$\Rightarrow\quad x=\tan^{-1}y-1+Ce^{-\tan^{-1}y}$

Answer Tips

- $\int\frac{1}{1+x^2}dx=\tan^{-1}x+C$, where C is arbitrary constant.

77. We have, $$\frac{dy}{dx}=\frac{y^2}{xy-x^2}=\frac{y^2/x^2}{(xy-x^2)/x^2} \qquad \text{... (i)}$$

This is a homogeneous differential equation

$\therefore$ Put $y=vx\Rightarrow\frac{dy}{dx}=v\cdot1+x\frac{dv}{dx}$

$\therefore\quad$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{v^2}{v-1}\ \Rightarrow x\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v}{v-1}\ \Rightarrow\frac{v-1}{v}dv=\frac{dx}{x}$

$\Rightarrow\left(1-\frac{1}{v}\right)dv=\frac{dx}{x}$

Integrating, we get

$v-\log v=\log x+C\ \Rightarrow\ v=\log vx+C$

$\Rightarrow\quad \frac{y}{x}=\log y+C$

$\Rightarrow\quad y=x(\log y+C)$ is the required solution.

78. We have, $(\tan^{-1}y - x)dy = (1+y^2)dx$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1}y}{1+y^2}$$

This is a linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1}y}{1+y^2}$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ Required solution is,

$$x\cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}\cdot \tan^{-1}y}{1+y^2}dy + C \quad \text{...(i)}$$

Put $\tan^{-1}y = t \Rightarrow \left(\frac{1}{1+y^2}\right)dy = dt$

$\therefore$ (i) becomes, $x\cdot e^{\tan^{-1}y} = \int e^t\cdot t\,dt + C$

$$\Rightarrow x\cdot e^{\tan^{-1}y} = t\cdot e^t - \int 1\cdot e^t dt + C \Rightarrow x\cdot e^{\tan^{-1}y} = t\cdot e^t - e^t + C$$

$$\Rightarrow x\cdot e^{\tan^{-1}y} = \tan^{-1}y\cdot e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

$$\Rightarrow x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$$

We get the solution as

$$x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y} \quad \text{...(ii)}$$

Now, putting $x = 1, y = 0$ in (ii), we get

$1 = \tan^{-1}0 - 1 + Ce^{-\tan^{-1}0} \Rightarrow C = 2$

So, required particular solution is $x = \tan^{-1}y - 1 + 2e^{-\tan^{-1}y}$.

79. We have,

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin\left(\frac{y}{x}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)} \quad \text{...(i)}$$

This is a homogeneous differential equation.

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} \Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v^2}{v - \cos v} \Rightarrow \frac{v - \cos v}{2\sin v - v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{-1}{2}\frac{(2\cos v - 2v)}{2\sin v - v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\frac{-1}{2}\log(2\sin v - v^2) = \log x + C_1$$

$$\Rightarrow \log x^2 + 2C_1 + \log\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right) = 0$$

$$\Rightarrow \log\left[x^2\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right)\right] = -2C_1$$

$$\Rightarrow 2x^2\sin\frac{y}{x} - y^2 = e^{-2C_1} = C \text{ (say)}$$

which is the required solution.

80. We have, $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{(1+x^2)(1+y^2)}}{xy} \Rightarrow \int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{\sqrt{1+x^2}}{x^2}x\,dx$$

Putting $1 + x^2 = v^2 \Rightarrow 2x\,dx = 2v\,dv$

$$\Rightarrow \frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}dy = -\int \frac{v^2}{v^2-1}dv \Rightarrow \sqrt{1+y^2} = -\int\left(1 + \frac{1}{v^2-1}\right)dv$$

$$\Rightarrow \sqrt{1+y^2} = -v - \frac{1}{2}\log\left|\frac{v-1}{v+1}\right| + C$$

$$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = C$$

81. We have, $x\frac{dy}{dx} + y - x + xy\cot x = 0, (x \neq 0)$

$$\Rightarrow x\frac{dy}{dx} + (1 + x\cot x).y = x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1 + x\cot x}{x}.y = 1 \quad \text{...(i)}$$

This is linear D.E. of the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{1 + x\cot x}{x} = \frac{1}{x} + \cot x$ and $Q = 1$

$\therefore$ Now I.F. $= e^{\int P dx} = e^{\log(x\sin x)} = x\sin x$

$\therefore$ The solution. of (i) is $y\cdot x\sin x = \int 1\cdot x\sin x\,dx + C$

$= x(-\cos x) + \int 1\cdot\cos x\,dx + C$. $\Rightarrow xy\sin x = -x\cos x + \sin x + C$

The required solution is

$$y\cdot x\sin x = x(-\cos x) + \sin x + C \quad \text{...(i)}$$

Putting $x = \frac{\pi}{2}, y = 0$ in (i), we get

$$0 = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C \Rightarrow C = -1$$

$xy\sin x = \sin x - x\cos x - 1$ is the required particular solution.

82. We have, $x^2\,dy + (xy + y^2)\,dx = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{xy + y^2}{x^2} \quad \text{...(i)}$$

This is a homogeneous linear differential equation

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v\cdot 1 + x\cdot\frac{dv}{dx}$

$\therefore$ (i) becomes $v + x\frac{dv}{dx} = -\frac{x\cdot vx + v^2x^2}{x^2} \Rightarrow x\frac{dv}{dx} = -(2v + v^2)$

Separating the variables, we get

$$\frac{dv}{2v + v^2} + \frac{dx}{x} = 0 \Rightarrow \frac{dv}{v(v+2)} + \frac{dx}{x} = 0$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{v}-\frac{1}{v+2}\right]dv+\frac{dx}{x}=0$$

Integrating, we get

$$\frac{1}{2}[\log v-\log(v+2)]+\log x=\log C \Rightarrow \log\left(\frac{v}{v+2}\right)+2\log x=\log C$$

$$\Rightarrow \log\left(\frac{v}{v+2}\right)+\log x^2=\log C \Rightarrow \log\left(\frac{vx^2}{v+2}\right)=\log C$$

$$\Rightarrow \frac{vx^2}{v+2}=C$$

$$\Rightarrow \frac{\frac{y}{x}.x^2}{\frac{y}{x}+2}=C \Rightarrow x^2y = C(2x+y) \quad ...(ii)$$

Putting $x = 1, y = 1$ in (ii), we get

$$1^2.1=C(2\cdot1+1)\Rightarrow C=\frac{1}{3}$$

$\therefore$ The required particular solution is

$$3x^2y=2x+y \Leftrightarrow y=\frac{2x}{3x^2-1}$$

83. We have, $\left(x\sin^2\left(\frac{y}{x}\right)-y\right)dx+x\,dy=0$

$$\Rightarrow \frac{dy}{dx}+\sin^2\left(\frac{y}{x}\right)-\frac{y}{x}=0 \quad ...(i)$$

Put $y = vx \Rightarrow \frac{dy}{dx}=v\cdot1+x\frac{dv}{dx}$ $\therefore$ (i) becomes

$$v+x\frac{dv}{dx}+\sin^2 v-v=0$$

$$\Rightarrow x\frac{dv}{dx}+\sin^2 v=0 \Rightarrow \operatorname{cosec}^2 v\,dv+\frac{dx}{x}=0$$

Integrating both sides, we get

$$\int\operatorname{cosec}^2 v\,dv+\int\frac{dx}{x}=C \Rightarrow -\cot v+\log x=C$$

$$\Rightarrow -\cot\left(\frac{y}{x}\right)+\log x=C \quad ...(ii)$$

Put $x = 1, y = \pi/4$ in (ii), we get

$$-\cot\frac{\pi}{4}+\log 1=C \Rightarrow C=-1$$

$\therefore -\cot\left(\frac{y}{x}\right)+\log x+1=0$ is the required particular solution.

Answer Tips

- $\int\operatorname{cosec}^2 x\,dx=-\cot(x)+C$

84. We have, $\frac{dy}{dx}-3y\cot x=\sin 2x$ ...(i)

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$ where $P = -3\cot x$, $Q = \sin 2x$

$$\therefore \text{I.F.}=e^{\int Pdx}=e^{-3\int\cot x\,dx}=e^{-3\log|\sin x|}=|\sin^{-3}x|$$

$$\therefore y\cdot\sin^{-3}x=\int\sin 2x\cdot\sin^{-3}x\,dx+C$$

$$\Rightarrow \frac{y}{\sin^3 x}=\int\frac{2\sin x\cos x}{\sin^3 x}dx+C$$

$$=\int\frac{2\cos x}{\sin^2 x}dx+C \quad (\text{Put } \sin x = t \Rightarrow \cos x\,dx = dt)$$

$$=2\int\frac{dt}{t^2}+C=-\frac{2}{t}+C=-\frac{2}{\sin x}+C$$

$$\Rightarrow y=-2\sin^2 x+C\sin^3 x \quad ...(ii)$$

Put $x=\frac{\pi}{2}, y=2$ in (ii), we get $2 = -2\cdot1 + C\cdot1 \Rightarrow C = 4$

$\therefore$ $y = 4\sin^3 x - 2\sin^2 x$ is the required particular solution.

CBSE Sample Questions

1. (c) : The given differential equation is $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4$

$$\Rightarrow 4\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2}=0.$$

Here, $m = 2$ and $n = 1$

Hence, $m + n = 3$ **(1)**

2. For $n = 3$, the given differential equation becomes homogeneous. **(1)**

3. We have, $\frac{d}{dx}\left(\frac{dy}{dx}\right)=5\Rightarrow\frac{d^2y}{dx^2}=5$

$\therefore$ Order = 2 **(1/2)**

Degree = 1 **(1/2)**

$\therefore$ Required sum = 3 **(1)**

4. There is no arbitrary constant in a particular solution of differential equation. **(1)**

5. The given differential equation is

$\frac{dy}{dx}=x^3\operatorname{cosec}y \Rightarrow \int\frac{dy}{\operatorname{cosec}y}=\int x^3dx$ **(1/2)**

$\Rightarrow \int\sin y\,dy=\int x^3dx \Rightarrow -\cos y=\frac{x^4}{4}+C$ **(1)**

Now, $y(0) = 0 \Rightarrow -1 = C$

So, the required solution is, $\cos y=1-\frac{x^4}{4}$ **(1/2)**

6. Given, $ydx + (x - y^2)dy = 0$

Reducing the given differential equation to the form

$\frac{dx}{dy}+Px=Q$ we get, $\frac{dx}{dy}+\frac{x}{y}=y$ **(1/2)**

Integrating factor (I.F.) = $e^{\int Pdy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$ **(1/2)**

Thus, solution is given by, $xy=\int y^2dy+C$ **(1)**

$\Rightarrow xy=\frac{y^3}{3}+C$, which is the required general solution. **(1)**

7. We have, $xdy - ydx = \sqrt{x^2+y^2}\,dx$

$$\Rightarrow x\frac{dy}{dx}-y=\sqrt{x^2+y^2} \Rightarrow x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x} \qquad \text{...(i)}$$

Put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Putting in (i), we get

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2+v^2x^2}}{x} \qquad (1)$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\frac{x\left(v+\sqrt{1+v^2}\right)}{x}$$

$$\Rightarrow \quad x\frac{dv}{dx}=v+\sqrt{1+v^2}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\sqrt{1+v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x} \qquad (1)$$

Integrating both sides, we get

$$\int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}$$

$$\Rightarrow \quad \log\left(v+\sqrt{1+v^2}\right)=\log x+\log c$$

$$\Rightarrow \quad \log\left(v+\sqrt{1+v^2}\right)=\log cx \quad \Rightarrow \quad \left(v+\sqrt{1+v^2}\right)=cx$$

$$\Rightarrow \quad \left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right)=cx \quad \Rightarrow \quad y+\sqrt{x^2+y^2}=cx^2 \qquad (1)$$

8. The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y}{x}-\sin\left(\frac{y}{x}\right) \qquad \text{...(i)}$$

Since, the equation is a homogeneous differential equation. Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ (1)

From (i), we get

$$v+x\frac{dv}{dx}=v-\sin v$$

$$\Rightarrow \quad \frac{dv}{\sin v}=-\frac{dx}{x} \Rightarrow \operatorname{cosec} v\,dv=-\frac{dx}{x} \qquad (1/2)$$

Integrating both sides, we get

$\log|\operatorname{cosec} v-\cot v| = -\log|x| + \log K$, $K > 0$ (Here, $\log K$ is a constant) (1/2)

$\Rightarrow \quad \log|(\operatorname{cosec} v-\cot v)x| = \log K \Rightarrow |(\operatorname{cosec} v-\cot v)x| = K$

$\Rightarrow \quad (\operatorname{cosec} v-\cot v)x = \pm K \Rightarrow \left(\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right)x=C,$

which is the required general solution. (1)

9. The differential equation is a linear differential equation.

$\therefore \quad \text{I.F.} = e^{\int\cot x\,dx}=e^{\log\sin x}=\sin x$ (1)

The general solution is given by

$$y\sin x=\int 2\frac{\sin x}{1+\sin x}dx+c$$

$$\Rightarrow \quad y\sin x=2\int\frac{\sin x+1-1}{1+\sin x}dx+c$$

$$\Rightarrow \quad y\sin x=2\int[1-\frac{1}{1+\sin x}]dx+c \qquad (1/2)$$

$$\Rightarrow \quad y\sin x=2\int\left[1-\frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]dx+c$$

$$\Rightarrow \quad y\sin x=2\int[1-\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}]dx+c$$

$$\left[\because 1+\cos\theta=2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \quad y\sin x=2\int[1-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)]dx+c$$

$$\Rightarrow \quad y\sin x=2\left[x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]+c \qquad \text{... (i)} \quad (1)$$

Now, $y=0$, when $x=\frac{\pi}{4}$ $\therefore$ From (i), we get

$$0=2\left[\frac{\pi}{4}+\tan\frac{\pi}{8}\right]+c \Rightarrow c=-\frac{\pi}{2}-2\tan\frac{\pi}{8}$$

Hence, the particular solution is given by

$$y=\operatorname{cosec}x\left[2\left\{x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}-\left(\frac{\pi}{2}+2\tan\frac{\pi}{8}\right)\right] \qquad (1/2)$$

10. The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y+2x^2}{x} \Rightarrow \frac{dy}{dx}-\frac{1}{x}y=2x \qquad (1/2)$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Px=Q, \text{ where } P=-\frac{1}{x} \text{ and } Q=2x$$

$$\therefore \text{ I.F.} = e^{\int Pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x} \qquad (1)$$

$\therefore$ Required solution is

$$y\times\frac{1}{x}=\int\left(2x\times\frac{1}{x}\right)dx \Rightarrow \frac{y}{x}=2x+C \qquad (1)$$

$$\Rightarrow \quad y = 2x^2 + Cx \qquad (1/2)$$