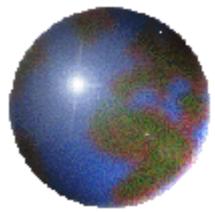


# Linear Algebra



## *Chapter 4*

### VECTOR SPACES



## 4.5 Linear Combinations of Vectors

### Definition

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  be vectors in a vector space  $V$ .

We say that  $\mathbf{v}$ , a vector of  $V$ , is a **linear combination** of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ , if there exist scalars  $c_1, c_2, \dots, c_m$  such that  $\mathbf{v}$  can be written  $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ .



## Example 1

The vector  $(5, 4, 2)$  is a linear combination of the vectors

$(1, 2, 0)$ ,  $(3, 1, 4)$ , and  $(1, 0, 3)$ , since it can be written

$$(5, 4, 2) = (1, 2, 0) + 2(3, 1, 4) - 2(1, 0, 3)$$



## Example 2

Determine whether or not the vector  $(-1, 1, 5)$  is a linear combination of the vectors  $(1, 2, 3)$ ,  $(0, 1, 4)$ , and  $(2, 3, 6)$ .

### Solution



## Example 2

Determine whether or not the vector  $(-1, 1, 5)$  is a linear combination of the vectors  $(1, 2, 3)$ ,  $(0, 1, 4)$ , and  $(2, 3, 6)$ .

### Solution

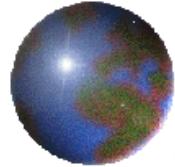
$$\text{Suppose } c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, 3, 6) = (-1, 1, 5)$$

$$(c_1, 2c_1, 3c_1) + (0, c_2, 4c_2) + (2c_3, 3c_3, 6c_3) = (-1, 1, 5)$$

$$(c_1 + 2c_3, 2c_1 + c_2 + 3c_3, 3c_1 + 4c_2 + 6c_3) = (-1, 1, 5)$$

$$\Rightarrow \begin{cases} c_1 + 2c_3 = -1 \\ 2c_1 + c_2 + 3c_3 = 1 \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1 \\ 3c_1 + 4c_2 + 6c_3 = 5 \end{cases}$$

Thus  $(-1, 1, 5)$  is a linear combination of  $(1, 2, 3)$ ,  $(0, 1, 4)$ , and  $(2, 3, 6)$ , where  $(-1, 1, 5) = (1, 2, 3) + 2(0, 1, 4) - 1(2, 3, 6)$ .



## Example 3

Express the vector  $(4, 5, 5)$  as a linear combination of the vectors  $(1, 2, 3)$ ,  $(-1, 1, 4)$ , and  $(3, 3, 2)$ .

### Solution

$$\text{Suppose } c_1(1, 2, 3) + c_2(-1, 1, 4) + c_3(3, 3, 2) = (4, 5, 5)$$

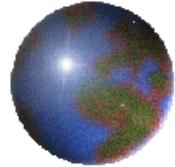
$$(c_1, 2c_1, 3c_1) + (-c_2, c_2, 4c_2) + (3c_3, 3c_3, 2c_3) = (4, 5, 5)$$

$$(c_1 - c_2 + 3c_3, 2c_1 + c_2 + 3c_3, 3c_1 + 4c_2 + 2c_3) = (4, 5, 5)$$

$$\Rightarrow \begin{cases} c_1 - c_2 + 3c_3 = 4 \\ 2c_1 + c_2 + 3c_3 = 5 \Rightarrow c_1 = -2r + 3, c_2 = r - 1, c_3 = r \\ 3c_1 + 4c_2 + 2c_3 = 5 \end{cases}$$

Thus  $(4, 5, 5)$  can be expressed **in many ways** as a linear combination of  $(1, 2, 3)$ ,  $(-1, 1, 4)$ , and  $(3, 3, 2)$ :

$$(4, 5, 5) = (-2r + 3)(1, 2, 3) + (r - 1)(-1, 1, 4) + r(2, 3, 6)$$



## Example 4

Show that the vector  $(3, -4, -6)$  cannot be expressed as a linear combination of the vectors  $(1, 2, 3)$ ,  $(-1, -1, -2)$ , and  $(1, 4, 5)$ .

### Solution

Suppose

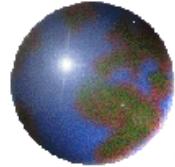
$$c_1(1, 2, 3) + c_2(-1, -1, -2) + c_3(1, 4, 5) = (3, -4, -6)$$

$\Rightarrow$

$$\begin{cases} c_1 - c_2 + c_3 = 3 \\ 2c_1 - c_2 + 4c_3 = -4 \\ 3c_1 - 2c_2 + 5c_3 = -6 \end{cases}$$

This system has no solution.

Thus  $(3, -4, -6)$  is not a linear combination of the vectors  $(1, 2, 3)$ ,  $(-1, -1, -2)$ , and  $(1, 4, 5)$ .



## Example 5

Determine whether the matrix  $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$  is a linear combination of the matrices  $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$  in the vector space  $M_{22}$  of  $2 \times 2$  matrices.

### Solution



## Example 5

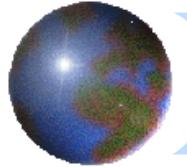
Determine whether the matrix  $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$  is a linear combination of the matrices  $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$  in the vector space  $M_{22}$  of  $2 \times 2$  matrices.

### Solution

Suppose  $c_1 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$

Then

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

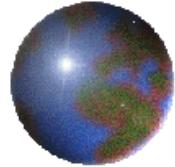


$$\begin{cases} c_1 + 2c_2 = -1 \\ -3c_2 + c_3 = 7 \\ 2c_1 + 2c_3 = 8 \\ c_1 + 2c_2 = -1 \end{cases}$$

This system has the unique solution  $c_1 = 3$ ,  $c_2 = -2$ ,  $c_3 = 1$ .

Therefore

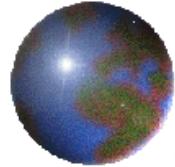
$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$



## Example 6

Determine whether the function  $f(x) = x^2 + 10x - 7$  is a linear combination of the functions  $g(x) = x^2 + 3x - 1$  and  $h(x) = 2x^2 - x + 4$ .

### Solution



## Example 6

Determine whether the function  $f(x) = x^2 + 10x - 7$  is a linear combination of the functions  $g(x) = x^2 + 3x - 1$  and  $h(x) = 2x^2 - x + 4$ .

### Solution

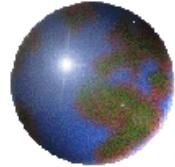
Suppose  $c_1g + c_2h = f$ .

Then

$$c_1(x^2 + 3x - 1) + c_2(2x^2 - x + 4) = x^2 + 10x - 7$$

$$(c_1 + 2c_2)x^2 + (3c_1 - c_2)x - c_1 + 4c_2 = x^2 + 10x - 7$$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 1 \\ 3c_1 - c_2 = 10 \\ -c_1 + 4c_2 = -7 \end{cases} \Rightarrow c_1 = 3, \quad c_2 = -1 \Rightarrow f = 3g - h.$$

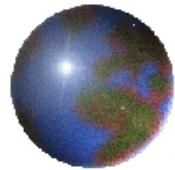


# Spanning Sets

## Definition

The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  are said to **span** a vector space if every vector in the space can be expressed as a *linear combination* of these vectors.

In this case  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is called a **spanning set**.

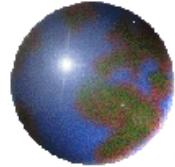


## 4.6 Linear Dependence and Independence

The concepts of dependence and independence of vectors are useful tools in constructing “efficient” spanning sets for vector spaces – sets in which there are no redundant vectors.

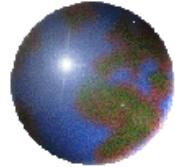
### Definition

- (a) The set of vectors  $\{ \mathbf{v}_1, \dots, \mathbf{v}_m \}$  in a vector space  $V$  is said to be **linearly dependent** if there exist scalars  $c_1, \dots, c_m$ , not all zero, such that  $c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m = 0$
- (b) The set of vectors  $\{ \mathbf{v}_1, \dots, \mathbf{v}_m \}$  is **linearly independent** if  $c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m = 0$  can only be satisfied when  $c_1 = 0, \dots, c_m = 0$ .



## Example 1

Show that the set  $\{(1, 2, 3), (-2, 1, 1), (8, 6, 10)\}$  is linearly dependent in  $\mathbf{R}^3$ .



## Example 1

Show that the set  $\{(1, 2, 3), (-2, 1, 1), (8, 6, 10)\}$  is linearly dependent in  $\mathbf{R}^3$ .

### Solution

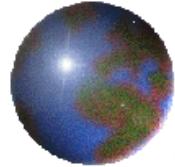
$$\text{Suppose } c_1(1, 2, 3) + c_2(-2, 1, 1) + c_3(8, 6, 10) = \mathbf{0}$$

$$\begin{aligned}\Rightarrow & (c_1, 2c_1, 3c_1) + (-2c_2, c_2, c_2) + (8c_3, 6c_3, 10c_3) = \mathbf{0} \\ & (c_1 - 2c_2 + 8c_3, 2c_1 + c_2 + 6c_3, 3c_1 + c_2 + 10c_3) = \mathbf{0}\end{aligned}$$

$$\Rightarrow \begin{cases} c_1 - 2c_2 + 8c_3 = 0 \\ 2c_1 + c_2 + 6c_3 = 0 \\ 3c_1 + c_2 + 10c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -2 \\ c_3 = -1 \end{cases}$$

$$\text{Thus } 4(1, 2, 3) - 2(-2, 1, 1) - (8, 6, 10) = \mathbf{0}$$

The set of vectors is linearly dependent.



## Example 2

Show that the set  $\{(3, -2, 2), (3, -1, 4), (1, 0, 5)\}$  is linearly independent in  $\mathbf{R}^3$ .

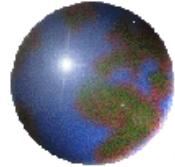
### Solution

$$\text{Suppose } c_1(3, -2, 2) + c_2(3, -1, 4) + c_3(1, 0, 5) = \mathbf{0}$$

$$\begin{aligned}\Rightarrow (3c_1, -2c_1, 2c_1) + (3c_2, -c_2, 4c_2) + (c_3, 0, 5c_3) &= \mathbf{0} \\ (3c_1 + 3c_2 + c_3, -2c_1 - c_2, 2c_1 + 4c_2 + 5c_3) &= \mathbf{0}\end{aligned}$$

$$\Rightarrow \begin{cases} 3c_1 + 3c_2 + c_3 = 0 \\ -2c_1 - c_2 = 0 \\ 2c_1 + 4c_2 + 5c_3 = 0 \end{cases}$$

This system has the unique solution  $c_1 = 0$ ,  $c_2 = 0$ , and  $c_3 = 0$ . Thus the set is linearly independent.



## Example 3

Consider the functions  $f(x) = x^2 + 1$ ,  $g(x) = 3x - 1$ ,  $h(x) = -4x + 1$  of the vector space  $P_2$  of polynomials of degree  $\leq 2$ .

Show that the set of functions  $\{ f, g, h \}$  is linearly independent.

### Solution

Suppose

$$c_1f + c_2g + c_3h = \mathbf{0}$$

Since for any real number  $x$ ,

$$c_1(x^2 + 1) + c_2(3x - 1) + c_3(-4x + 1) = \mathbf{0}$$

$$x^2c_1 + x(3c_2 - 4c_3) + (c_1 - c_2 + c_3) = 0$$

$$c_1 = 0$$

$$3c_2 - 4c_3 = 0$$

$$c_1 - c_2 + c_3 = 0$$

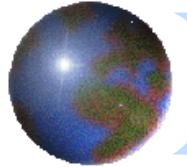
OR

Consider three convenient values of  $x$ . We get

$$x = 0: c_1 - c_2 + c_3 = 0$$

$$x = 1: 2c_1 + 2c_2 - 3c_3 = 0$$

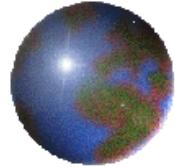
$$x = -1: 2c_1 - 4c_2 + 5c_3 = 0$$



It can be shown that this system of three equations has the unique solution

$$c_1 = 0, c_2 = 0, c_3 = 0$$

Thus  $c_1f + c_2g + c_3h = \mathbf{0}$  implies that  $c_1 = 0, c_2 = 0, c_3 = 0$ .  
The set  $\{ f, g, h \}$  is linearly independent.



## Theorem 4.7

A set consisting of two or more vectors in a vector space is linearly dependent *if and only if* it is possible to express one of the vectors as a linear combination of the other vectors.

### Example 4

The set of vectors  $\{\mathbf{v}_1=(1, 2, 1), \mathbf{v}_2=(-1, -1, 0), \mathbf{v}_3 = (0, 1, 1)\}$  is linearly dependent, since  $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ . Thus,  $\mathbf{v}_3$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .