

# Homogeneous System of linear Equations

## Definition

A system of linear equations is said to be *homogeneous* if all the constant terms are zeros.

Example: 
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that  $x_1 = 0, x_2 = 0, x_3 = 0$  is a solution.

## Theorem 1.1

A system of homogeneous linear equations in  $n$  variables always has the solution  $x_1 = 0, x_2 = 0, \dots, x_n = 0$ . This solution is called the **trivial solution**.



# Homogeneous System of linear Equations

Note. Non trivial solution

Example: 
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

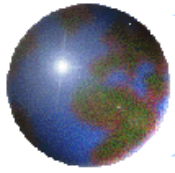
The system has other nontrivial solutions.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \dots \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\therefore x_1 = -3r, \quad x_2 = 4r, \quad x_3 = r$$

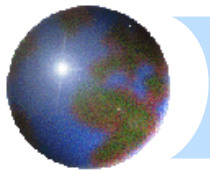
## Theorem 1.2

A system of homogeneous linear equations that has more variables than equations has many solutions.



# Homework

✚ Exercise set 1.2: (page 21)  
2, 5, 6, 7, 8, 14



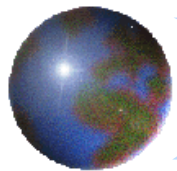
## 1.3 *Gaussian Elimination*

### Definition

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)



# Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

## Solution

Starting with the augmented matrix, create zeros below the pivot in the first column.

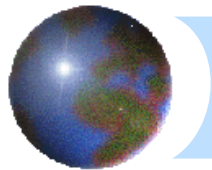
$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \xrightarrow[R3 + (-2)R1]{R2 + R1} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

At this stage, we create a zero only below the pivot.

$$\xrightarrow{R3 + (-2)R2} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R3} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We have arrived at the echelon form.

Echelon form



The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$x_3 + 3x_4 = 1$$

$$x_4 = 2$$

We get

$$x_3 + 3(2) = 1$$

$$x_3 = -5$$

**Substituting**  $x_4 = 2$  and  $x_3 = -5$  into the first equation,

$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$

$$x_1 + 2x_2 = 10$$

$$x_1 = -2x_2 + 10$$

Let  $x_2 = r$ . The system has many solutions. The solutions are

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$



## Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

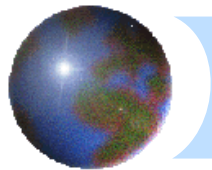
### Solution

We arrive at the echelon form as in the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \dots \approx \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Echelon form

This marks the end of the forward elimination of variables from equations. We now commence the **back substitution** using matrices.



$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{array}{l} R1 + (-2)R3 \\ R2 + (-3)R3 \end{array}} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\xrightarrow{\begin{array}{l} R1 + (-3)R2 \end{array}} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$x_1 + 2x_2 = 10$$

$$x_3 = -5$$

$$x_4 = 2$$

Let  $x_2 = r$ . We get same solution as previously,

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$