

Relations between system of linear equations and matrices

⊕ matrix of coefficients and augmented matrix

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + x_3 = 3$$

$$x_1 - x_2 - 2x_3 = -6$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

matrix of coefficients

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

augmented matrix

Observe that the matrix of coefficients is a submatrix of the augmented matrix. The augmented matrix completely describes the system.



Relations between system of linear equations and matrices

- ⊕ Transformations called elementary transformations can be used to **change a system of linear equations into another system of linear equations that has the same solution.**
- ⊕ These transformations are used to solve systems of linear equations by **eliminating variables.**
- ⊕ In practice it is simpler to work in terms of matrices using analogous transformations called elementary row operations.



Elementary Row Operations of Matrices

⊕ These transformations are as follows:

⊕ Elementary Transformation

1. Interchange two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

⊕ Elementary Row Operation

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.



Example 1

Solving the following system of linear equation.

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + x_3 = 3$$

$$x_1 - x_2 - 2x_3 = -6$$



Example 1

Solving the following system of linear equation.

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 + 3x_2 + x_3 &= 3 \\x_1 - x_2 - 2x_3 &= -6\end{aligned}$$

\approx row equivalent

Solution

Equation Method

Initial system:

$$\begin{array}{l} \text{Eq2}+(-2)\text{Eq1} \quad \boxed{x_1 + x_2 + x_3 = 2} \\ \text{Eq3}+(-1)\text{Eq1} \quad \boxed{2x_1 + 3x_2 + x_3 = 3} \\ \qquad \qquad \qquad \boxed{x_1 - x_2 - 2x_3 = -6} \end{array}$$

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\x_2 - x_3 &= -1 \\-2x_2 - 3x_3 &= -8\end{aligned}$$

Analogous Matrix Method

Augmented matrix:

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array} \right]$$

$$\begin{array}{l} \approx \\ \text{R2}+(-2)\text{R1} \\ \text{R3}+(-1)\text{R1} \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array} \right]$$



$$\begin{array}{l} \text{Eq1} + (-1)\text{Eq2} \rightarrow x_1 + x_2 + x_3 = 2 \\ \text{Eq3} + (2)\text{Eq2} \rightarrow x_2 - x_3 = -1 \\ \hline -2x_2 - 3x_3 = -8 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array} \right]$$

$$\begin{array}{l} x_1 + 2x_3 = 3 \\ x_2 - x_3 = -1 \\ (-1/5)\text{Eq3} \rightarrow -5x_3 = -10 \end{array}$$

$$\approx \left[\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array} \right]$$

$$\begin{array}{l} \text{Eq1} + (-2)\text{Eq3} \rightarrow x_1 + 2x_3 = 3 \\ \text{Eq2} + \text{Eq3} \rightarrow x_2 - x_3 = -1 \\ \hline x_3 = 2 \end{array}$$

$$\approx \left[\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\begin{array}{l} x_1 = -1 \\ x_2 = 1 \\ x_3 = 2 \end{array}$$

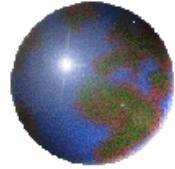
$$\approx \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

The solution is

$$x_1 = -1, x_2 = 1, x_3 = 2.$$

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$$x_1 = -1, x_2 = 1, x_3 = 2.$$



Example 2

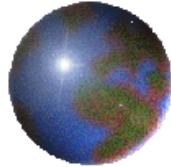
Solving the following system of linear equation.

$$x_1 - 2x_2 + 4x_3 = 12$$

$$2x_1 - x_2 + 5x_3 = 18$$

$$-x_1 + 3x_2 - 3x_3 = -8$$

Solution



Example 2

Solving the following system of linear equation.

$$x_1 - 2x_2 + 4x_3 = 12$$

$$2x_1 - x_2 + 5x_3 = 18$$

$$-x_1 + 3x_2 - 3x_3 = -8$$

Solution

$$\left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array} \right]$$

$$\text{R2} \approx (-2)\text{R1}$$

$$\text{R3} + \text{R1}$$

$$\left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{array} \right]$$

$$\left(\frac{1}{3} \right) \text{R2} \approx \left[\begin{array}{cccc} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{array} \right]$$

$$\text{R1} \approx (2)\text{R2}$$

$$\text{R3} + (-1)\text{R2}$$

$$\left[\begin{array}{cccc} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{array} \right]$$

$$\left(\frac{1}{2} \right) \text{R3} \approx \left[\begin{array}{cccc} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\text{R1} \approx (-2)\text{R3}$$

$$\text{R2} + \text{R3}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

solution $\begin{cases} x_1 = 2 \\ x_2 = 1. \\ x_3 = 3 \end{cases}$



Example 3

Solve the system

$$\begin{aligned}4x_1 + 8x_2 - 12x_3 &= 44 \\3x_1 + 6x_2 - 8x_3 &= 32 \\-2x_1 - x_2 &= -7\end{aligned}$$

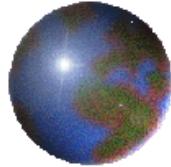
Solution

$$\approx \left[\begin{array}{cccc} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right] \left(\frac{1}{4} \right) R1 \left[\begin{array}{cccc} 1 & 2 & -3 & 11 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right] \begin{matrix} \approx \\ R2 + (-3)R1 \end{matrix} \left[\begin{array}{cccc} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ -2 & -1 & 0 & -7 \end{array} \right] \begin{matrix} \approx \\ R3 + 2R1 \end{matrix} \left[\begin{array}{cccc} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & -6 & 15 \end{array} \right]$$

$$\approx \left[\begin{array}{cccc} 1 & 2 & -3 & 11 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 1 & -1 \end{array} \right] \left(\frac{1}{3} \right) R2 \left[\begin{array}{cccc} 1 & 2 & -3 & 11 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array} \right] \begin{matrix} \approx \\ R1 + (-2)R2 \end{matrix} \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{matrix} \approx \\ R1 + (-1)R3 \end{matrix} \left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right].$$

The solution is $x_1 = 2, x_2 = 3, x_3 = -1$.



Summary

$$\begin{aligned}4x_1 + 8x_2 - 12x_3 &= 44 \\3x_1 + 6x_2 - 8x_3 &= 32 \\-2x_1 - x_2 &= -7\end{aligned}$$

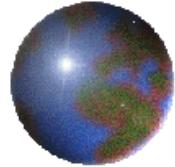
$$[A : B] = \left[\begin{array}{ccc|c} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right] \quad \begin{matrix} A \\ B \end{matrix}$$

Use row operations to $[A : B]$:

$$\left[\begin{array}{cccc} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array} \right] \approx \dots \approx \left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]. \text{ i.e., } [A : B] \approx \dots \approx [I_n : X]$$

Def. $[I_n : X]$ is called the *reduced echelon form* of $[A : B]$.

- Note.**
1. If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution,
then A is row equivalent to I_n ($A \approx I_n$).
 2. If $A \approx I_n$, then the system has unique solution.

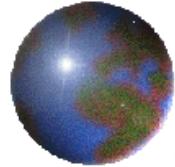


Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned}x_1 - x_2 + 3x_3 &= b_1 \\2x_1 - x_2 + 4x_3 &= b_2 \quad \text{for} \\-x_1 + 2x_2 - 4x_3 &= b_3\end{aligned}\quad \left[\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \right] = \left[\begin{matrix} 8 \\ 11 \\ -11 \end{matrix} \right], \left[\begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \right], \left[\begin{matrix} 3 \\ 3 \\ -4 \end{matrix} \right] \text{ in turn}$$

Solution



Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned}x_1 - x_2 + 3x_3 &= b_1 \\2x_1 - x_2 + 4x_3 &= b_2 \quad \text{for} \\-x_1 + 2x_2 - 4x_3 &= b_3\end{aligned}\quad \left[\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \right] = \left[\begin{matrix} 8 \\ 11 \\ -11 \end{matrix} \right], \left[\begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \right], \left[\begin{matrix} 3 \\ 3 \\ -4 \end{matrix} \right] \text{ in turn}$$

Solution

$$\left[\begin{matrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{matrix} \right] \approx \left[\begin{matrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{matrix} \right]$$

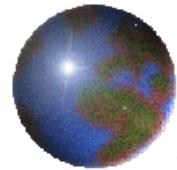
R2+(-2)R1
R3+R1

$$\approx \left[\begin{matrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{matrix} \right] \approx \left[\begin{matrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{matrix} \right]$$

R1+R2
R3+(-1)R2
R1+(-1)R3
R2+2R3

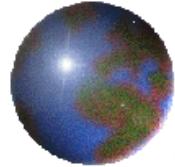
The solutions to the three systems are

$$\begin{cases} x_1 = 1 \\ x_2 = -1, \\ x_3 = 2 \end{cases}, \begin{cases} x_1 = 0 \\ x_2 = 3, \\ x_3 = 1 \end{cases}, \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 2 \end{cases}.$$



Homework

- ➊ Exercise 1.1: (page 12)
1, 2, 4, 5, 6, 10, 11, 12, 13



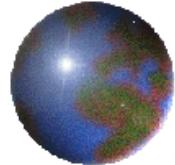
1.2 Gauss-Jordan Elimination

More generalization of Gauss-Jordan Elimination

Definition

A matrix is in *reduced echelon form* if

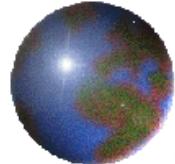
1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a *leading 1*.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.



The following matrices are all in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



The following matrices are not in reduced echelon form for the reasons stated.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros
not at bottom
of matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

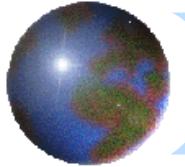
First nonzero
element in row
2 is not 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in
row 3 not to the
right of leading
1 in row 2

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

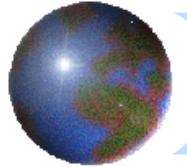
Nonzero
element above
leading 1 in
row 2



- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (?) \qquad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad (?) \qquad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad (?) \qquad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (?)$$

- Elementary row operations reduced echelon form
- The reduced echelon form of a matrix is **unique**.



● Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(✓)

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(✗)

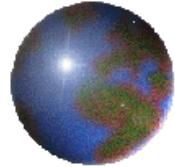
$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

(✓)

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

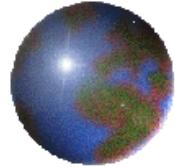
(✗)

- There are usually many sequences of row operations that can be used to transform a given matrix to reduced echelon form-they all, however, lead to the same reduced echelon form.
- We say that the reduced echelon form of a matrix is **unique**.



Gauss-Jordan Elimination

- ◆ System of linear equations
 - ⇒ augmented matrix
 - ⇒ reduced echelon form
 - ⇒ solution

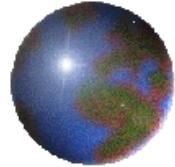


Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution



Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

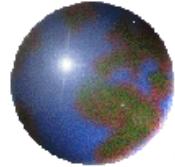
Solution

$$\begin{array}{l} \approx \\ R1 \leftrightarrow R2 \end{array} \left[\begin{array}{ccccc} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{array} \right] \left(\begin{array}{c} 1 \\ 0 \\ 3 \end{array} \right) \approx \left[\begin{array}{ccccc} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{array} \right]$$

$$\begin{array}{l} \approx \\ R3 + (-4)R1 \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{array} \right] \left(\begin{array}{c} 1 \\ 0 \\ \frac{1}{2} \end{array} \right) \approx \left[\begin{array}{ccccc} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{array} \right]$$

$$\begin{array}{l} \approx \\ R1 + R2 \\ R3 + (-2)R2 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right] \left(\begin{array}{c} 1 \\ 0 \\ -2 \end{array} \right) \approx \left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$$

The matrix is the reduced echelon form of the given matrix.



Example 2

Solve, if possible, the system of equations

$$3x_1 - 3x_2 + 3x_3 = 9$$

$$2x_1 - x_2 + 4x_3 = 7$$

$$3x_1 - 5x_2 - x_3 = 7$$

Solution

$$\left[\begin{array}{cccc} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{array} \right] \xrightarrow{\left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right) R1} \left[\begin{array}{cccc} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{array} \right] \approx \left[\begin{array}{cccc} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{array} \right]$$

\approx

$$\xrightarrow{R1+R2} \left[\begin{array}{cccc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow x_1 + 3x_3 = 4 \Rightarrow x_1 = -3x_3 + 4$$
$$\Rightarrow x_2 + 2x_3 = 1 \Rightarrow x_2 = -2x_3 + 1$$

The $\xrightarrow{R3+2R2}$ general solution to the system is

$$x_1 = -3r + 4$$

$$x_2 = -2r + 1$$

$$x_3 = r \quad , \text{ where } r \text{ is real number (called a parameter)}$$



Example 3

Solve the system of equations

$$2x_1 - 4x_2 + 12x_3 - 10x_4 = 58$$

$$-x_1 + 2x_2 - 3x_3 + 2x_4 = -14$$

$$2x_1 - 4x_2 + 9x_3 - 6x_4 = 44$$

⇒ many sol.

Solution

$$\left[\begin{array}{cccc|c} 2 & -4 & 12 & -10 & 58 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{array} \right] \xrightarrow{\left(\frac{1}{2} \right) R_1} \left[\begin{array}{cccc|c} 1 & -2 & 6 & -5 & 29 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{array} \right]$$

$$\approx \left[\begin{array}{cccc|c} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -3 & 4 & -14 \end{array} \right] \xrightarrow{\left(\frac{1}{3} \right) R_2} \left[\begin{array}{cccc|c} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -3 & 4 & -14 \end{array} \right]$$

$$\xrightarrow{R_3 + (-2)R_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{R_1 + (-6)R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 + (-1)R_3} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$x_1 - 2x_2 = -2$$

$$x_3 = 6$$

$$x_4 = 1$$

$$\Rightarrow \quad \quad \quad \Rightarrow \begin{cases} x_1 = 2r - 2 \\ x_2 = r \\ x_3 = 6 \\ x_4 = 1 \end{cases}, \text{ for some } r.$$



Example 4

Solve the system of equations

$$x_1 + 2x_2 - x_3 + 3x_4 + x_5 = 2$$

$$2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 = 6$$

$$-x_1 - 2x_2 + x_3 - x_4 + 3x_5 = 4$$

Solution

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 3 & 1 & 2 \\ 2 & 4 & -2 & 6 & 3 & 6 \\ -1 & -2 & 1 & -1 & 3 & 4 \end{array} \right] \approx \left[\begin{array}{cccccc} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array} \right]$$

R2+(-2)R1 R3+R1

$$\approx \left[\begin{array}{cccccc} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right] \underset{\left(\frac{1}{2}\right)R2}{\approx} \left[\begin{array}{cccccc} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

R2↔R3

$$\approx \left[\begin{array}{cccccc} 1 & 2 & -1 & 0 & -5 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right] \approx \left[\begin{array}{cccccc} 1 & 2 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

R1+(-3)R2 R1+5R3 R2+(-2)R3

$$x_1 = -2x_2 + x_3 + 3$$

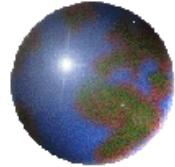
$$x_1 = -2r + s + 3$$

$$\Rightarrow x_4 = -1$$

$$\Rightarrow x_2 = r, x_3 = s, x_4 = -1, , \text{for some } r \text{ and } s.$$

$$x_5 = 2$$

$$x_5 = 2$$



Example 5

This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 3 \\2x_1 - 2x_2 + 5x_3 &= 4 \\x_1 + 2x_2 - x_3 &= -3 \\2x_2 + 2x_3 &= 1\end{aligned}$$

Solution

$$\left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{array} \right] \approx \left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{array} \right] \approx \left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{array} \right]$$

$\text{R2}+(-2)\text{R1}$ $\text{R2} \leftrightarrow \text{R3}$

$$\left(\frac{1}{3} \right) \text{R2} \approx \left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{array} \right] \approx \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{array} \right] \approx \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{array} \right]$$

$\text{R1}+(-1)\text{R3}$ $\text{R2}+\text{R3}$ $\text{R4}+(-4)\text{R3}$

$$\left(\frac{1}{13} \right) \text{R4} \approx \left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

0x₁+0x₂+0x₃=1

The system has no solution.