

4.8 Rank

Rank enables one to relate matrices to vectors, and vice versa.

Definition

Let A be an $m \times n$ matrix. The rows of A may be viewed as row vectors $\mathbf{r}_1, \dots, \mathbf{r}_m$, and the columns as column vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$. Each row vector will have n components, and each column vector will have m components. The row vectors will span a subspace of \mathbf{R}^n called the **row space** of A , and the column vectors will span a subspace of \mathbf{R}^m called the **column space** of A .



Example 1

Consider the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 4 & 1 & 6 \\ 5 & 4 & 1 & 0 \end{bmatrix}$$

(1) The row vectors of A are

$$\mathbf{r}_1 = (1, 2, -1, 2), \mathbf{r}_2 = (3, 4, 1, 6), \mathbf{r}_3 = (5, 4, 1, 0)$$

These vectors span a subspace of \mathbf{R}^4 called the row space of A .

(2) The column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

These vectors span a subspace of \mathbf{R}^3 called the column space of A .



Theorem 4.16

The row space and the column space of a matrix A have the same dimension.

Definition

The dimension of the row space and the column space of a matrix A is called the **rank** of A . The rank of A is denoted **rank**(A).



Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution



Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

The third row of A is a linear combination of the first two rows:

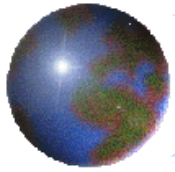
$$(2, 5, 8) = 2(1, 2, 3) + (0, 1, 2)$$

Hence the three rows of A are linearly dependent.

The rank of A must be less than 3. Since $(1, 2, 3)$ is not a scalar multiple of $(0, 1, 2)$, these two vectors are linearly independent.

These vectors form a basis for the row space of A .

Thus $\text{rank}(A) = 2$.



Theorem 4.17

The nonzero row vectors of a matrix A that is in reduced echelon form are a basis for the row space of A . The rank of A is the number of nonzero row vectors.



Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced echelon form. There are three nonzero row vectors, namely $(1, 2, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

According to the previous theorem, these three vectors form a basis for the row space of A .

$\text{Rank}(A) = 3$.



Theorem 4.18

Let A and B be row equivalent matrices. Then A and B have the same row space. $\text{rank}(A) = \text{rank}(B)$.

Theorem 4.19

Let E be a reduced echelon form of a matrix A . The nonzero row vectors of E form a basis for the row space of A . The rank of A is the number of nonzero row vectors in E .

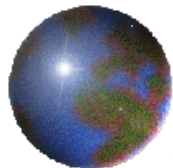


Example 4

Find a basis for the row space of the following matrix A, and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution



Example 4

Find a basis for the row space of the following matrix A , and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Use elementary row operations to find a reduced echelon form of the matrix A . We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

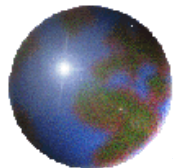
The two vectors $(1, 0, 7)$, $(0, 1, -2)$ form a basis for the row space of A . $\text{Rank}(A) = 2$.



Example 5

EXAMPLE 5 Find a basis for the column space of the following matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$



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$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

SOLUTION

The transpose of A is

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A^t . Let us find a basis for the row space of A^t . Compute the reduced echelon form of A^t .

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors of this echelon form, namely $(1, 0, 5)$, $(0, 1, -3)$, are a basis for the row space of A^t . Write these vectors in column form to get a basis for the column space of A . The following vectors are a basis for the column space of A .

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

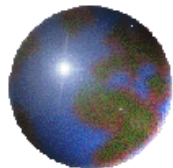


Example 6

EXAMPLE 6 Find a basis for the subspace V of \mathbf{R}^4 spanned by the vectors

$$(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8)$$

SOLUTION



Example 6

EXAMPLE 6 Find a basis for the subspace V of \mathbf{R}^4 spanned by the vectors

$$(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8)$$

SOLUTION

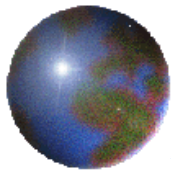
We construct a matrix A having these vectors as row vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Determine the reduced echelon form of A . We get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

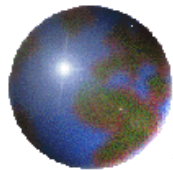
The nonzero vectors of this reduced echelon form, namely $(1, 0, 5, 0)$ and $(0, 1, -1, 2)$, are a basis for the subspace V . The dimension of this subspace is two.



Example 7

Consider a system $AX = B$ of \mathbf{m} linear equations in \mathbf{n} variables.

- (a) If the augmented matrix and the matrix of coefficients have the same rank “ \mathbf{r} ” and $\mathbf{r}=\mathbf{n}$, then the solution is unique.
- (b) If the augmented matrix and the matrix of coefficients have the same rank “ \mathbf{r} ” and $\mathbf{r}<\mathbf{n}$, then the solution is unique. Where \mathbf{r} is rank and \mathbf{n} are the number of variables
- (c) If the augmented matrix and the matrix of coefficients do not have the same rank then the solution does not exist.



Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{array}{rclcl} x_1 & + & x_2 & + & x_3 & & 2 \\ 2x_1 & + & 3x_2 & + & x_3 & & 3 \\ x_1 & - & x_2 & - & 2x_3 & = & -6 \end{array}$$



Example 7

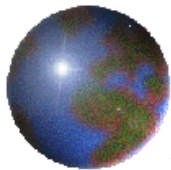
EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 + 3x_2 + x_3 &= 3 \\x_1 - x_2 - 2x_3 &= -6\end{aligned}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

Augmented matrix		Reduced echelon form
$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$	$\approx \dots \approx$	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
$\underbrace{\hspace{10em}}$		$\underbrace{\hspace{10em}}$
Matrix of coefficients		Reduced echelon form

We see that ranks of the augmented matrix and the matrix of coefficients are equal, both being three. The system thus has a unique solution. The reduced echelon form gives that solution to be $x_1 = -1, x_2 = 1, x_3 = 2$.



Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned}x_1 + x_2 + x_3 &= 2 \\2x_1 + 3x_2 + x_3 &= 3 \\x_1 - x_2 - 2x_3 &= -6\end{aligned}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

<p>Augmented matrix</p> $\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array} \right]$ <p>Matrix of coefficients</p>	$\approx \dots \approx$	<p>Reduced echelon form</p> $\left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$ <p>Reduced echelon form</p>
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The system of linear equations can be viewed as the linear combination

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$$

The existence of solutions depends upon whether $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a linear combination of

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The uniqueness depends upon whether the linear combination