Linear Algebra

Chapter 4

VECTOR SPACES

4.5 Linear Combinations of Vectors

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

We say that $\mathbf{v}$, a vector of V, is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, if there exist scalars $c_1, c_2, \ldots, c_m$ such that $\mathbf{v}$ can be written $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$.

Example 1

The vector (5, 4, 2) is a linear combination of the vectors (1, 2, 0), (3, 1, 4), and (1, 0, 3), since it can be written

$(5, 4, 2) = (1, 2, 0) + 2(3, 1, 4) – 2(1, 0, 3)$

Example 2

Determine whether or not the vector $(–1, 1, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, 3, 6)$.

Solution

Example 2

Determine whether or not the vector (–1, 1, 5) is a linear combination of the vectors (1, 2, 3), (0, 1, 4), and (2, 3, 6).

Solution

Suppose $c_1(1,2,3)+c_2(0,1,4)+c_3(2,3,6)=(-1,1,5)$

$$(c_1, 2c_1, 3c_1)+(0, c_2, 4c_2)+(2c_3, 3c_3, 6c_3)=(-1,1,5)$$

$$(c_1+2c_3, 2c_1+c_2+3c_3, 3c_1+4c_2+6c_3)=(-1,1,5)$$

$$\Rightarrow \begin{cases} c_1 \qquad\quad + 2c_3 = -1 \\ 2c_1 + \ \ c_2 + 3c_3 = \ \ 1 \\ 3c_1 + 4c_2 + 6c_3 = \ \ 5 \end{cases} \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1$$

Thus (–1, 1, 5) is a linear combination of (1, 2, 3), (0, 1, 4), and (2, 3, 6), where $(-1,1,5)=(1,2,3)+2(0,1,4)-1(2,3,6)$.

Example 3

Express the vector (4, 5, 5) as a linear combination of the vectors (1, 2, 3), (–1, 1, 4), and (3, 3, 2).

Solution

Suppose $c_1(1,2,3)+c_2(-1,1,4)+c_3(3,3,2)=(4,5,5)$

$$(c_1, 2c_1, 3c_1)+(-c_2, c_2, 4c_2)+(3c_3, 3c_3, 2c_3)=(4,5,5)$$
$$(c_1-c_2+3c_3, 2c_1+c_2+3c_3, 3c_1+4c_2+2c_3)=(4,5,5)$$
$$\Rightarrow \begin{cases} c_1-\ \ c_2+3c_3=4 \\ 2c_1+\ \ c_2+3c_3=5 \\ 3c_1+4c_2+2c_3=5 \end{cases} \Rightarrow c_1=-2r+3, c_2=r-1, c_3=r$$

Thus (4, 5, 5) can be expressed in many ways as a linear combination of (1, 2, 3), (–1, 1, 4), and (3, 3, 2):

$$(4,5,5)=(-2r+3)(1,2,3)+(r-1)(-1,1,4)+r(2,3,6)$$

Example 4

Show that the vector (3, –4, –6) cannot be expressed as a linear combination of the vectors (1, 2, 3), (–1, –1, –2), and (1, 4, 5).

Solution

Suppose

$$c_1(1,2,3)+c_2(-1,\ -1,-2)+c_3(1,4,5)=(3,-4,-6)$$

$\Rightarrow$

$$\begin{cases} c_1 - c_2 + c_3 = 3 \\ 2c_1 - c_2 + 4c_3 = -4 \\ 3c_1 - 2c_2 + 5c_3 = -6 \end{cases}$$

This system has no solution.

Thus (3, –4, –6) is not a linear combination of the vectors (1, 2, 3), (–1, –1, –2), and (1, 4, 5).

Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

Suppose

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = -1 \\ -3c_2 + c_3 = 7 \\ 2c_1 + 2c_3 = 8 \\ c_1 + 2c_2 = -1 \end{cases}$$

This system has the unique solution $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Therefore

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4$.

Solution

Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4.$

Solution

Suppose $c_1 g + c_2 h = f.$

Then

$$c_1(x^2 + 3x - 1) + c_2(2x^2 - x + 4) = x^2 + 10x - 7$$

$$(c_1 + 2c_2)x^2 + (3c_1 - c_2)x - c_1 + 4c_2 = x^2 + 10x - 7$$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 1 \\ 3c_1 - c_2 = 10 \\ -c_1 + 4c_2 = -7 \end{cases} \quad \Rightarrow c_1 = 3,\ c_2 = -1 \quad \Rightarrow f = 3g - h.$$

Spanning Sets

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are said to **span** a vector space if every vector in the space can be expressed as a *linear combination* of these vectors.

In this case $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called a **spanning set**.

4.6 Linear Dependence and Independence

The concepts of dependence and independence of vectors are useful tools in constructing "efficient" spanning sets for vector spaces – sets in which there are no redundant vectors.

Definition

(a) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$

(b) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ is **linearly independent** if $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$ can only be satisfied when $c_1 = 0, \ldots, c_m = 0$.

Example 1

Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

Example 1

Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

Solution

Suppose $c_1(1,2,3)+c_2(-2,1,1)+c_3(8,6,10)=\mathbf{0}$

$$\Rightarrow \quad (c_1, 2c_1, 3c_1)+(-2c_2, c_2, c_2)+(8c_3, 6c_3, 10c_3)=\mathbf{0}$$

$$(c_1-2c_2+8c_3,\ 2c_1+c_2+6c_3,\ 3c_1+c_2+10c_3)=\mathbf{0}$$

$$\Rightarrow \begin{cases} c_1-2c_2+8c_3=0 \\ 2c_1+c_2+6c_3=0 \\ 3c_1+c_2+10c_3=0 \end{cases} \Rightarrow \quad \begin{matrix} c_1=4 \\ c_2=-2 \\ c_3=-1 \end{matrix}$$

Thus $\quad 4(1,2,3)-2(-2,1,1)-(8,6,10)=\mathbf{0}$

The set of vectors is linearly dependent.

Example 2

Show that the set {(3, –2, 2), (3, –1, 4), (1, 0, 5)} is linearly independent in $\mathbf{R}^3$.

Solution

Suppose $c_1(3,-2,2)+c_2(3,-1,4)+c_3(1,0,5)=\mathbf{0}$

$$\Rightarrow (3c_1,-2c_1,2c_1)+(3c_2,-c_2,4c_2)+(c_3,0,5c_3)=\mathbf{0}$$

$$(3c_1+3c_2+c_3,-2c_1-c_2,2c_1+4c_2+5c_3)=\mathbf{0}$$

$$\Rightarrow \begin{cases} 3c_1+3c_2+c_3=0 \\ -2c_1-c_2=0 \\ 2c_1+4c_2+5c_3=0 \end{cases}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.
Thus the set is linearly independent.

Example 3

Consider the functions $f(x) = x^2 + 1$, $g(x) = 3x - 1$, $h(x) = -4x + 1$ of the vector space P_2 of polynomials of degree ≤ 2.

Show that the set of functions $\{ f, g, h \}$ is linearly independent.

Solution

Suppose

$$c_1 f + c_2 g + c_3 h = \mathbf{0}$$

Since for any real number x,

$$c_1(x^2+1) + c_2(3x-1) + c_3(-4x+1) = \mathbf{0}$$

$$x^2 c_1 + x(3c_2 - 4c_3) + (c_1 - c_2 + c_3) = 0$$

$$c_1 = 0$$

$$3c_2 - 4c_3 = 0$$

$$c_1 - c_2 + c_3 = 0$$

OR

Consider three convenient values of x. We get

$$x = 0: c_1 - c_2 + c_3 = 0$$

$$x = 1: 2c_1 + 2c_2 - 3c_3 = 0$$

$$x = -1: 2c_1 - 4c_2 + 5c_3 = 0$$

It can be shown that this system of three equations has the unique solution

$$c_1 = 0, c_2 = 0, c_3 = 0$$

Thus $c_1 f + c_2 g + c_3 h = \mathbf{0}$ implies that $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. The set $\{ f, g, h \}$ is linearly independent.

Theorem 4.7

A set consisting of two or more vectors in a vector space is linearly dependent *if and only if* it is possible to express one of the vectors as a linearly combination of the other vectors.

Example 4

The set of vectors $\{\mathbf{v}_1=(1, 2, 1)$, $\mathbf{v}_2 =(-1, -1, 0)$, $\mathbf{v}_3 = (0, 1,1)\}$

is linearly dependent, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Thus, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

m