# Linear Algebra

## *Chapter 3*

## DETERMINANTS

# 3.1 Introduction to Determinants

**Definition**

The **determinant** of a 2 × 2 matrix $A$ is denoted **|*A*|** and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2 × 2 matrix is given by *the difference of the products of the two diagonals* of the matrix.

The notation **det(*A*)** is also used for the determinant of $A$.

# 3.1 Introduction to Determinants

## Applications

- Solvability of Linear Systems
- Matrix Inversion
- Cramer's Rule
- Eigenvalues and Eigenvectors
- Linear Independence
- Computation of Cross Products

**Example 1: Find determinant of the following matrix**

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

# 3.1 Introduction to Determinants

**Example 1**

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$

# 3.1 Introduction to Determinants

- The determinant of a 3 X 3 matrix is defined in terms of determinants of 2 X 2 matrices.
- The determinant of a 4 X 4 matrix is defined in terms of determinants of 3 X 3 matrices, and so on.
- For these definitions we need the following concepts of minor and cofactor.

## Definition

Let A be a square matrix.

The **minor** of the element $a_{ij}$ is denoted by $M_{ij}$ and is the determinant of the matrix that remains after deleting row $i$ and column $j$ of $A$.

The **cofactor** of $a_{ij}$ is denoted $C_{ij}$ and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that $C_{ij} = M_{ij}$ *or* $-M_{ij}$ .

# Example 2

Determine the minors and cofactors of the elements $a_{11}$ and $a_{32}$ of the following matrix $A$.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

**Solution**

# Example 2

Determine the minors and cofactors of the elements $a_{11}$ and $a_{32}$ of the following matrix $A$.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

**Solution**

$$\text{Minor of } a_{11}: M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1\times 1) - (2\times(-2)) = 3$$

$$\text{Cofactor of } a_{11}: C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$\text{Minor of } a_{32}: M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1\times 2) - (3\times 4) = -10$$

$$\text{Cofactor of } a_{32}: C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$

## Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

$$\text{If } A \text{ is } 3\times 3, \ |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4\times 4, \ |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\vdots$$

$$\text{If } A \text{ is } n\times n, \ |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of $|A|$.

# *Example 3*

Evaluate the determinant of the following matrix $A$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

**Solution**

# Example 3

Evaluate the determinant of the following matrix *A*.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

**Solution**

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0\times 1) - (1\times 2)] - 2[(3\times 1) - (1\times 4)] - [(3\times 2) - (0\times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$

# Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

*i*th row expansion: $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

*j*th column expansion: $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

## Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

## Solution

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= -3\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} \\ &= -3[(2\times 1) - (-1\times 2)] + 0[(1\times 1) - (-1\times 4)] - 1[(1\times 2) - (2\times 4)] \\ &= -12 + 0 + 6 = -6 \end{aligned}$$

# Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

**Solution**

# Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

**Solution**

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\ &= 3\begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\ &= 3(2)\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3-2) = 6 \end{aligned}$$

# Example 6

Solve the following equation for the variable *x*.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

**Solution**

# Example 6

Solve the following equation for the variable *x*.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

**Solution**

Expand the determinant to get the equation

$$x(x-2)-(x+1)(-1)=7$$

Proceed to simplify this equation and solve for *x (factorization)*.

$$x^2-2x+x+1=7$$

$$x^2-x-6=0$$

$$(x+2)(x-3)=0$$

$$x=-2 \text{ or } 3$$

Use the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

There are two solutions to this equation, $x = -2$ or 3.

# Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |\mathrm{A}| = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

# Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

# Homework Practice

- Exercise 3.1 pages 161-162:
  1, 3, 5, 7, 9, 11, 13

# 3.2 Properties of Determinants

## Theorem 3.2

Let $A$ be an $n \times n$ matrix and $c$ be a nonzero scalar.

**(a)** If a matrix $B$ is obtained from $A$ by multiplying the elements of a row (column) by $c$ then $|B| = c|A|$.

**(b)** If a matrix $B$ is obtained from $A$ by interchanging two rows (columns) then $|B| = -|A|$.

**(c)** If a matrix $B$ is obtained from $A$ by adding a multiple of one row (column) to another row (column), then $|B| = |A|$.

**Proof** (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn}$$
$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \dots + ca_{kn}C_{kn}$$
$$\therefore |B| = c|A|.$$

# *Example 1*

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

**Solution**

# Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

**Solution**

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \underset{C2+2C3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$

# Example 2

If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

(a) $B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix}$ (b) $B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix}$ (c) $B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$

**Solution**

# Example 2

If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

(a) $B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix}$ (b) $B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix}$ (c) $B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$

**Solution**

(a) $A \underset{3C2}{\approx} B_1$ Thus $|B_1| = 3|A| = 36$.

(b) $A \underset{R2 \leftrightarrow R3}{\approx} B_2$ Thus $|B_2| = -|A| = -12$.

(c) $A \underset{R3+2R1}{\approx} B_3$ Thus $|B_3| = |A| = 12$.

## Definition

A square matrix $A$ is said to be **singular** if $|A|=0$.

$A$ is **nonsingular** if $|A|\neq 0$.

### *Theorem 3.3*

Let $A$ be a square matrix. A is singular if

(a) all the elements of a row (column) are zero.

(b) two rows (columns) are equal.

(c) two rows (columns) are proportional. (i.e., $Ri=cRj$)

### Proof

(a) Let all elements of the $k$th row of $A$ be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

(c) If $Ri=cRj$, then $A \underset{Ri-cRj}{\approx} B$ , row $i$ of $B$ is [0 0 … 0].

$\Rightarrow |A|=|B|=0$

# Example 3

Show that the following matrices are singular.

$$\text{(a) } A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \qquad \text{(b) } B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

**Solution**

# Example 3

Show that the following matrices are singular.

$$(a)\ A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \qquad (b)\ B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

**Solution**

# Theorem 3.4

Let $A$ and $B$ be $n \times n$ matrices and $c$ be a nonzero scalar.

(a) $|cA| = c^n|A|$.

(b) $|AB| = |A||B|$.

(c) $|A^t| = |A|$.

(d) $\left|A^{-1}\right| = \dfrac{1}{|A|}$ (assuming $A^{-1}$ exists)

## Proof

**(a)** Each row of $cA$ is a row of $A$ multiplied by $c$. Apply Theorem 3.2(a) to each of the $n$ rows of $cA$ to get $|cA| = c^n |A|$.

**(d)** $AA^{-1} = I_n$. Thus $|AA^{-1}| = |I_n|$. $|A||A^{-1}| = 1$, by Part (b). Therefore $|A| \neq 0$, and $|A^{-1}| = \dfrac{1}{|A|}$.