Work and the Dot Product

- The **work** done by $\mathbf{F}$ is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

$$|\overrightarrow{PS}| = |\mathbf{F}| \cos \theta$$

- So the work done by $\mathbf{F}$ is defined to be

$$\boxed{1} \qquad W = |\mathbf{D}| \, (|\mathbf{F}| \cos \theta) = |\mathbf{F}||\mathbf{D}| \cos \theta$$

- Notice that work is a scalar quantity; it has no direction. But its value depends on the angle θ between the force and displacement vectors.

Work and the Dot Product

- We use the expression in Equation 1 to define the dot product of two vectors even when they don't represent force or displacement.

Definition The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either **a** or **b** is **0**, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

- This product is called the **dot product** because of the dot in the notation $\mathbf{a} \cdot \mathbf{b}$.

Example 1 – Computing a Dot Product from Lengths and the Contained Angle

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

Example 1 – Computing a Dot Product from Lengths and the Contained Angle

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Solution:

According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

Angle between Vectors (in R^n)

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^n$.

The **cosine of the angle θ** between these vectors is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi$$

Example 4

Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (1, 0, 1)$ in $\mathbf{R}^3$.

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Solution $\mathbf{u}\cdot\mathbf{v} = (1,0,0)\cdot(1,0,1) = 1$

$$\|\mathbf{u}\| = \sqrt{1^2+0^2+0^2} = 1 \qquad \|\mathbf{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

Thus $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°.

Orthogonal Vectors

Definition

Two nonzero vectors are **orthogonal** if the angle between them is a right angle .

Theorem 4.2

Two nonzero vectors **u** and **v** are orthogonal *if and only if* $\mathbf{u}.\mathbf{v} = \mathbf{0}$.

Proof

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal} \Leftrightarrow \cos\theta = 0 \Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0$$

Example 5

Show that the following pairs of vectors are orthogonal.

(a) (1, 0) and (0, 1).

(b) (2, –3, 1) and (1, 2, 4).

Solution

Note

- (1, 0), (0,1) are orthogonal unit vectors in $\mathbf{R}^2$.
- (1, 0, 0), (0, 1, 0), (0, 0, 1) are orthogonal unit vectors in $\mathbf{R}^3$.
- (1, 0, …, 0), (0, 1, 0, …, 0), …, (0, …, 0, 1) are orthogonal unit vectors in $\mathbf{R}^n$.

EXAMPLE 4 (a) Show that the following vectors are orthogonal. (b) Compute the norm of each vector.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLUTION

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

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Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Let the vector (a, b) be orthogonal to (3, –1)

We get $(a,b)\cdot(3,-1)=0$

$$(a\times 3)+(b\times(-1))=0$$

$$3a-b=0$$

$$b=3a$$

Thus any vector of the form $(a, 3a)$ is orthogonal to the vector (3, –1).

Any vector of this form can be written

$$a(1, 3)$$

The set of all such vectors lie on the line defined by the vector (1, 3).

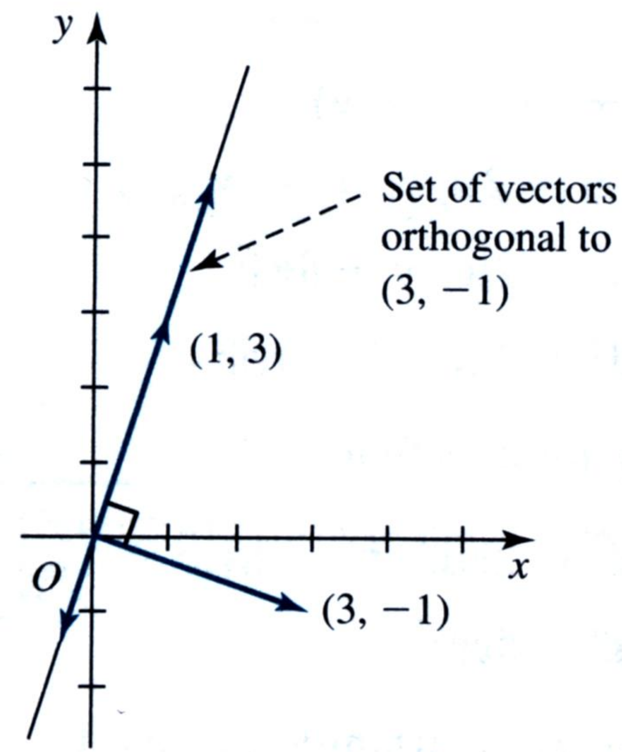


Figure 4.7

Distance between Points

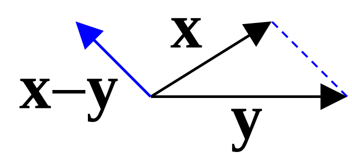
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbf{R}^n$.

The **distance** between **x** and **y** is denoted $d(\mathbf{x}, \mathbf{y})$ and is defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Note: We can also write this distance as follows.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



Note: *It is clear that* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (*the symmetric property*)

Example 7. Determine the distance between the points $\mathbf{x} = (1, -2, 3, 0)$ and $\mathbf{y} = (4, 0, -3, 5)$ in $\mathbf{R}^4$.

Solution

4.9 Orthonormal Vectors and Projections

Definition

A set of vectors in a vector space V is said to be an **orthogonal set** if every pair of vectors in the set is orthogonal. The set is said to be an **orthonormal set** if it is orthogonal and each vector is a unit vector.

Example 1

Show that the set $\left\{(1, 0, 0), \left(0, \frac{3}{5}, \frac{4}{5}\right), \left(0, \frac{4}{5}, -\frac{3}{5}\right)\right\}$ is an orthonormal set.

Solution

Theorem 4.20

An orthogonal set of nonzero vectors in a vector space is linearly independent.

Proof Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be an orthogonal set of nonzero vectors in a vector space V. Let us examine the identity

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

Let $\mathbf{v}_i$ be the ith vector of the orthogonal set. Take the dot product of each side of the equation with $\mathbf{v}_i$ and use the properties of the dot product. We get

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) \cdot \mathbf{v}_i = \mathbf{0} \cdot \mathbf{v}_i$$

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_m\mathbf{v}_m \cdot \mathbf{v}_i = 0$$

Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_2$ are mutually orthogonal, $\mathbf{v}_j \cdot \mathbf{v}_i = 0$ unless $j = i$. Thus

$$c_i\mathbf{v}_i \cdot \mathbf{v}_i = 0$$

Since $\mathbf{v}_i$ is a nonzero, then $\mathbf{v}_i \cdot \mathbf{v}_i \neq 0$. Thus $c_i = 0$.
Letting $i = 1, \ldots, m$, we get $c_1 = 0$, $c_m = 0$, proving that the vectors are linearly independent.

Definition

A basis that is an orthogonal set is said to be an **orthogonal basis**.
A basis that is an orthonormal set is said to be an **orthonormal basis**.

Standard Bases

- $\mathbf{R}^2$: {(1, 0), (0, 1)}
- $\mathbf{R}^3$: {(1, 0, 0), (0, 1, 0), (0, 0, 1)}
- $\mathbf{R}^n$: {(1, …, 0), …, (0, …, 1)}

} orthonormal bases

Theorem 4.21

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for a vector space V. Let $\mathbf{v}$ be a vector in V. $\mathbf{v}$ can be written as a linearly combination of these basis vectors as follows:

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n$$

Example 2

The following vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ form an orthonormal basis for $\mathbf{R}^3$. Express the vector $\mathbf{v} = (7, -5, 10)$ as a linear combination of these vectors.

$$\mathbf{u}_1 = (1, 0, 0), \mathbf{u}_2 = \left(0, \frac{3}{5}, \frac{4}{5}\right), \mathbf{u}_3 = \left(0, \frac{4}{5}, -\frac{3}{5}\right)$$

Solution

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)\mathbf{u}_n$$

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Solution

$$c_1 = \mathbf{v} \cdot \mathbf{u}_1 = (7, -5, 10) \cdot (1,\ 0, 0) = 7$$

$$c_2 = \mathbf{v} \cdot \mathbf{u}_2 = (7, -5, 10) \cdot \left(0, \frac{3}{5}, \frac{4}{5}\right) = 5$$

$$c_3 = \mathbf{v} \cdot \mathbf{u}_3 = (7, -5, 10) \cdot \left(0, \frac{4}{5}, -\frac{3}{5}\right) = -10$$

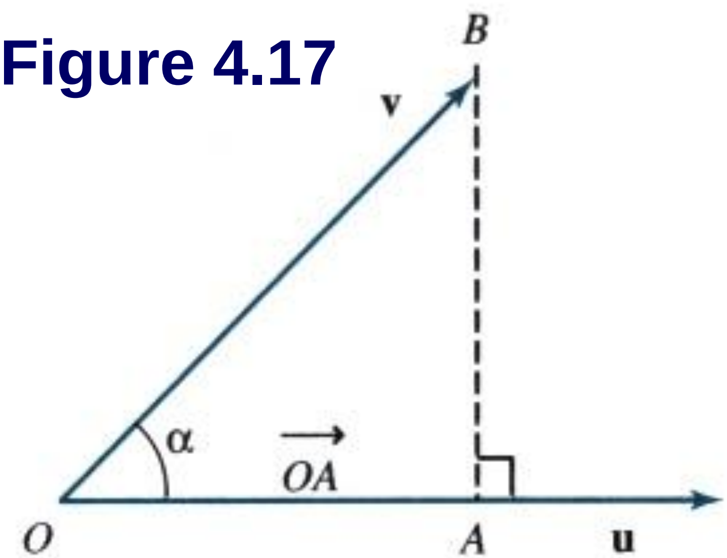
Thus

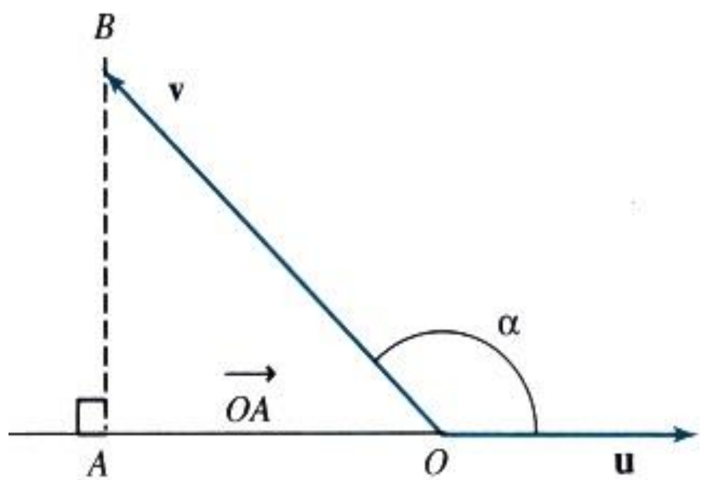
$$\mathrm{v} = 7\mathrm{u}_1 + 5\mathrm{u}_2 - 10\mathrm{u}_3$$

Projection of One vector onto Another Vector

Let **v** and **u** be vectors in $\mathbf{R}^n$ with angle α ($0 \le \alpha \le \pi$) between them.

Figure 4.17





$\overrightarrow{OA}$: the projection of **v** onto **u**

$$\overline{OA} = \overline{OB}\cos\alpha = \|\mathrm{v}\|\cos\alpha$$

$$= \|\mathrm{v}\| \frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{v}\|\,\|\mathrm{u}\|} = \frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{u}\|}$$

$$\therefore \overrightarrow{OA} = \left(\frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{u}\|}\right)\left(\frac{\mathrm{u}}{\|\mathrm{u}\|}\right) = \frac{\mathrm{v}\cdot\mathrm{u}}{\mathrm{u}\cdot\mathrm{u}}\mathrm{u}$$

$$\text{Note : If } \alpha > \pi/2 \text{ then } \frac{\mathrm{v}\cdot\mathrm{u}}{\mathrm{u}\cdot\mathrm{u}} < 0.$$

So we define $\boxed{\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}.}$

Definition

The **projection** of a vector **v** onto a nonzero vector **u** in $\mathbf{R}^n$ is denoted $\text{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\overline{OA} = ||\text{v}|| \cos\alpha = \text{v}.\frac{\text{u}}{||\text{u}||} = \text{v}.\frac{\text{u}}{||\text{u}||}\left(\frac{\text{u}}{||u||}\right)$$

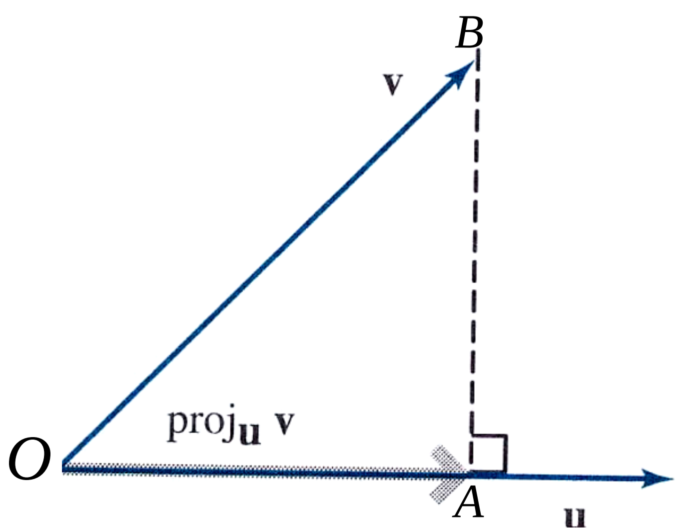


Figure 4.18

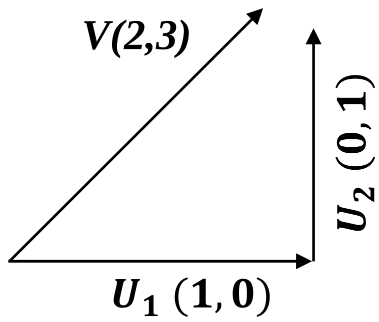
$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n$$

$$\mathbf{v} = \left(\mathbf{v}\cdot\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}\right)\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} + \left(\mathbf{v}\cdot\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}\right)\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} + \cdots + \left(\mathbf{v}\cdot\frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}\right)\frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}$$

Definition

The **projection** of a vector **v** onto a nonzero vector **u** in $\mathbf{R}^n$ is denoted $\text{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$



$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} = \frac{(2,3)\cdot(1,0)}{1\cdot 1}(1,0) = (2,0)$$

$$\text{proj}_{\mathbf{u}_2}\mathbf{v} = \frac{(2,3)\cdot(0,1)}{1\cdot 1}(0,1) = (0,3)$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} + \text{proj}_{\mathbf{u}_2}\mathbf{v} = (2,0) + (0,3) = (2,3)$$

Definition

The **projection** of a vector $\mathbf{v}$ onto a nonzero vector $\mathbf{u}$ in $\mathbf{R}^n$ is denoted $\text{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

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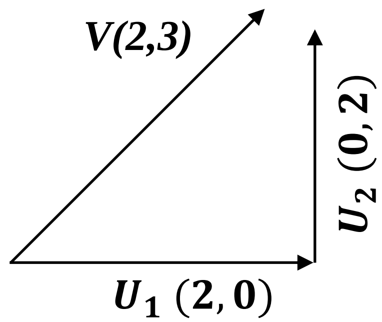

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} = \frac{(2,3)\cdot(2,0)}{2\cdot 2}(2,0) = (2,0)$$

$$\text{proj}_{\mathbf{u}_2}\mathbf{v} = \frac{(2,3)\cdot(0,2)}{2\cdot 2}(0,2) = (0,3)$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} + \text{proj}_{\mathbf{u}_2}\mathbf{v} = (2,0) + (0,3) = (2,3)$$

Example 3

Determine the projection of the vector $\mathbf{v} = (6, 7)$ onto the vector $\mathbf{u} = (1, 4)$.

Solution

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Theorem 4.22

The Gram-Schmidt Orthogonalization Process

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. The set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ defined as follows is orthogonal. To obtain an orthonormal basis for V, normalize each of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1} \mathbf{v}_2$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1} \mathbf{v}_3 - \text{proj}_{\mathbf{u}_2} \mathbf{v}_3$$

$$\cdots$$

$$\mathbf{u}_n = \mathbf{v}_n - \text{proj}_{\mathbf{u}_1} \mathbf{v}_n - \cdots - \text{proj}_{\mathbf{u}_{n-1}} \mathbf{v}_n$$

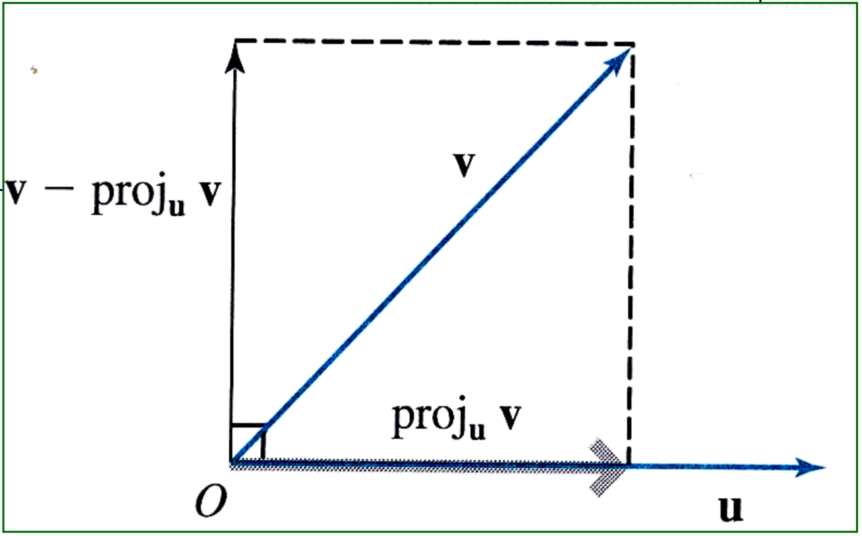

Figure 4.19

Example 4

The set {(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)} is linearly independent in $\mathbf{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbf{R}^4$. Construct an orthonormal basis for V.

Solution

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Solution

Let $\mathbf{v}_1 = (1, 2, 0, 3)$, $\mathbf{v}_2 = (4, 0, 5, 8)$, $\mathbf{v}_3 = (8, 1, 5, 6)\}$.
Use the Gram-Schmidt process to construct an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ from these vectors.

$$\text{Let } \mathbf{u}_1 = \mathbf{v}_1 = (1, 2, 0, 3)$$

$$\text{Let } \mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1$$

$$= (2, -4, 5, 2)$$

$$\text{Let } \mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{u}_2}\mathbf{v}_3$$

$$= \mathbf{v}_3 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_2)}{(\mathbf{u}_2 \cdot \mathbf{u}_2)}\mathbf{u}_2 = (4, 1, 0,$$

The set {(1, 2, 0, 3), (2, –4, 5, 2), (4, 1, 0, –2)} is an orthogonal basis for *V*.

Normalize them to get an orthonormal basis:

$$\|(1,2,0,3)\| = \sqrt{1^2+2^2+0^2+3^2} = \sqrt{14}$$

$$\|(2,-4,5,2)\| = \sqrt{2^2+(-4)^2+5^2+2^2} = 7$$

$$\|(4,1,0,-2)\| = \sqrt{4^2+1^2+0^2+(-2)^2} = \sqrt{21}$$

$\Rightarrow$ orthonormal basis for *V*:

$$\left\{\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},0,\frac{3}{\sqrt{14}}\right),\left(\frac{2}{7},-\frac{4}{7},\frac{5}{7},\frac{2}{7}\right),\left(\frac{4}{\sqrt{21}},\frac{1}{\sqrt{21}},0,-\frac{2}{\sqrt{21}}\right)\right\}$$