

Homogeneous System of linear Equations

Definition

A system of linear equations is said to be **homogeneous** if all the constant terms are zeros.

Example:

$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that $x_1 = 0, x_2 = 0, x_3 = 0$ is a solution.

Theorem 1.1

A system of homogeneous linear equations in n variables always has the solution $x_1 = 0, x_2 = 0, \dots, x_n = 0$. This solution is called the **trivial solution**.



Homogeneous System of linear Equations

Note. Non trivial solution

Example:
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

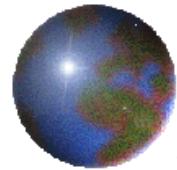
The system has other nontrivial solutions.

$$\left[\begin{array}{cccc} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{array} \right] \approx \dots \approx \left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{array} \right]$$

$$\therefore x_1 = -3r, \quad x_2 = 4r, \quad x_3 = r$$

Theorem 1.2

A system of homogeneous linear equations that has more variables than equations has many solutions.



Homework

- ➊ Exercise set 1.2: (page 21)
2, 5, 6, 7, 8, 14



1.3 Gaussian Elimination

Definition

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)



Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

Solution

Starting with the augmented matrix, create zeros below the pivot in the first column.

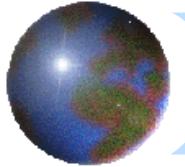
$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{array} \right] \approx R2 + R1 \left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{array} \right] R3 + (-2)R1$$

At this stage, we create a zero only below the pivot.

$$\approx R3 + (-2)R2 \left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{array} \right] \approx \frac{1}{2}R3 \left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

We have arrived at the echelon form.

Echelon form



The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$x_3 + 3x_4 = 1$$

$$x_4 = 2$$

We get

$$x_3 + 3(2) = 1$$

$$x_3 = -5$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

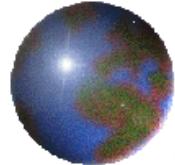
$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$

$$x_1 + 2x_2 = 10$$

$$x_1 = -2x_2 + 10$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$



Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

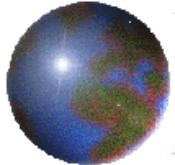
Solution

We arrive at the echelon form as in the previous example.

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{array} \right] \approx \dots \approx \left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

Echelon form

This marks the end of the forward elimination of variables from equations. We now commence the **back substitution** using matrices.



$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \approx R1 + (-2)R3 \quad \left[\begin{array}{ccccc} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$
$$R2 + (-3)R3 \approx R1 + (-3)R2 \quad \left[\begin{array}{ccccc} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$x_1 + 2x_2 = 10$$

$$x_3 = -5$$

$$x_4 = 2$$

Let $x_2 = r$. We get same solution as previously,

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$