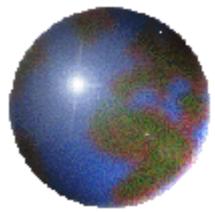
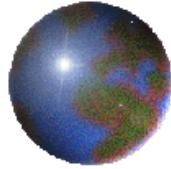


# Linear Algebra



## *Chapter 3*

### DETERMINANTS



## Definition

A square matrix  $A$  is said to be **singular** if  $|A|=0$ .

$A$  is **nonsingular** if  $|A|\neq 0$ .

### Theorem 3.3

Let  $A$  be a square matrix.  $A$  is singular if

- (a) all the elements of a row (column) are zero.
- (b) two rows (columns) are equal.
- (c) two rows (columns) are proportional. (i.e.,  $R_i=cR_j$ )

### Proof

- (a) Let all elements of the  $k$ th row of  $A$  be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

- (c) If  $R_i=cR_j$ , then  $A \underset{R_i=cR_j}{\approx} B$  , row  $i$  of  $B$  is  $[0 \ 0 \ \dots \ 0]$ .  
 $\Rightarrow |A|=|B|=0$



## Example 3

Show that the following matrices are singular.

$$(a) A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b) B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

### Solution



## 3.3 Numerical Evaluation of a Determinant

### Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper – triangular

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

lower – triangular



# Numerical Evaluation of a Determinant

## Theorem 3.5

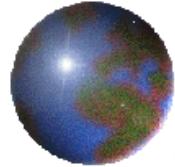
The determinant of a triangular matrix is the product of its diagonal elements.

### Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22} \cdots a_{nn}$$

### Example 1

Let  $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$ , find  $|A|$ . Sol.  $|A| = 2 \times 3 \times (-5) = -30$ .



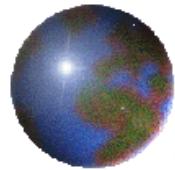
# Numerical Evaluation of a Determinant

## Example 2

Evaluation the determinant.

$$\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$$

**Solution** (elementary row operations )

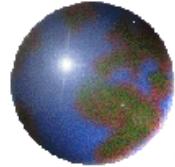


## Example 3

Evaluation the determinant.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$$

### Solution



## Example 4

Evaluation the determinant.

$$\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$$

### Solution

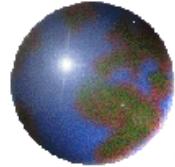


## Example 5

Evaluation the determinant.

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$$

### Solution



### 3.3 Determinants, Matrix Inverse, and Systems of Linear Equations

#### Definition

Let  $A$  be an  $n \times n$  matrix and  $C_{ij}$  be the cofactor of  $a_{ij}$ .

The matrix whose  $(i, j)$ th element is  $C_{ij}$  is called the **matrix of cofactors** of  $A$ .

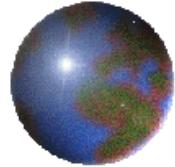
The transpose of this matrix is called the **adjoint** of  $A$  and is denoted  $\text{adj}(A)$ .

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^t$$

adjoint matrix



## Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix  $A$ .

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

**Solution** The cofactors of  $A$  are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14$$

$$C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3$$

$$C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9$$

$$C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$$

$$C_{23} = \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12$$

$$C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1$$

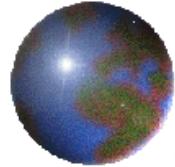
$$C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors of  $A$  is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of  $A$  is

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$



# Theorem 3.6

Let  $A$  be a square matrix with  $|A| \neq 0$ .  $A$  is invertible with

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

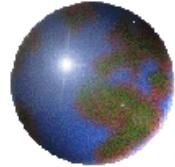
## Proof

Consider the matrix product  $\mathbf{A} \cdot \text{adj}(\mathbf{A})$ . The  $(i, j)$ th element of this product is

$(i, j)$ th element = (row  $i$  of  $A$ )  $\times$  (column  $j$  of  $\text{adj}(A)$ )

$$\begin{aligned} &= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix} \\ &= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} \end{aligned}$$

Proof Excluded



# Proof of Theorem 3.6

Proof Excluded

If  $i = j$ ,  $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |A|$ .

If  $i \neq j$ , let  $A \xrightarrow{R_j \text{ is replaced by } R_i} B$ .

Matrices A and B have the same cofactors  $C_{j1}, C_{j2}, \dots, C_{jn}$ .

So  $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |B| = 0$ .

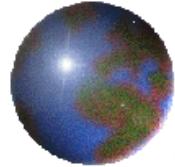
row  $i =$  row  $j$  in  $B$

Therefore  $(i,j)$ th element =  $\begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$   $\therefore A \cdot \text{adj}(A) = |A|I_n$

Since  $|A| \neq 0$ ,  $A \left( \frac{1}{|A|} \text{adj}(A) \right) = I_n$

Similarly,  $\left( \frac{1}{|A|} \text{adj}(A) \right) A = I_n$ .

Thus  $A^{-1} = \frac{1}{|A|} \text{adj}(A)$



# Theorem 3.7

A square matrix  $A$  is invertible if and only if  $|A| \neq 0$ .

## Proof

( $\Rightarrow$ ) Assume that  $A$  is invertible.

$$\Rightarrow AA^{-1} = I_n.$$

$$\Rightarrow |AA^{-1}| = |I_n|.$$

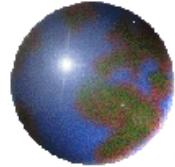
$$\Rightarrow |A|/|A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

Proof Excluded

( $\Leftarrow$ ) Theorem 3.6 tells us that if  $|A| \neq 0$ , then  $A$  is invertible.

$A^{-1}$  exists if and only if  $|A| \neq 0$ .

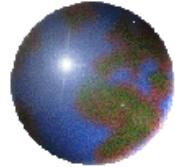


## Example 2

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

### Solution



## Example 3

Use the formula for the inverse of a matrix to compute the inverse of the matrix

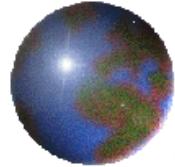
$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

### Solution

$|A| = 25$ , so the inverse of  $A$  exists. We found  $\text{adj}(A)$  in Example 1

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{25} \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$



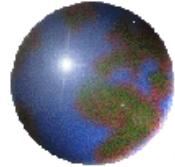
# Homework

- ➊ Exercise 3.3 page 178-179: 1, 3, 5, 7.

- ➋ *Exercise*

Show that if  $A = A^{-1}$ , then  $|A| = \pm 1$ .

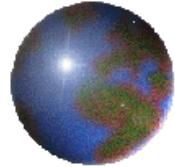
Show that if  $A^t = A^{-1}$ , then  $|A| = \pm 1$ .



## Theorem 3.8

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables.

- (1) If  $|A| \neq 0$ , there is a unique solution.
- (2) If  $|A| = 0$ , there may be many or no solutions.



# Theorem 3.8

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables.

- (1) If  $|A| \neq 0$ , there is a unique solution.
- (2) If  $|A| = 0$ , there may be many or no solutions.

## Proof

- (1) If  $|A| \neq 0$

$\Rightarrow A^{-1}$  exists (Thm 3.7)

Proof Excluded

$\Rightarrow$  there is then a unique solution given by  $X = A^{-1}B$  (Thm 2.9).

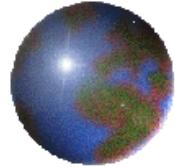
- (2) If  $|A| = 0$

$\Rightarrow$  since  $A \approx \dots \approx C$  implies that if  $|A| \neq 0$  then  $|C| \neq 0$  (Thm 3.2).

$\Rightarrow$  the reduced echelon form of  $A$  is not  $I_n$ .

$\Rightarrow$  The solution to the system  $AX = B$  is not unique.

$\Rightarrow$  many or no solutions.



## Example 4

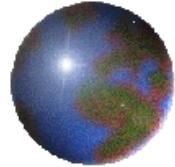
Determine whether or not the following system of equations has an unique solution.

$$3x_1 + 3x_2 - 2x_3 = 2$$

$$4x_1 + x_2 + 3x_3 = -5$$

$$7x_1 + 4x_2 + x_3 = 9$$

### Solution

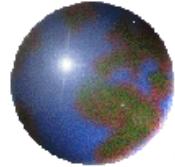


## Theorem 3.9 Cramer's Rule

Let  $AX = B$  be a system of  $n$  linear equations in  $n$  variables such that  $|A| \neq 0$ . The system has a unique solution given by

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n|}{|A|}$$

Where  $A_i$  is the matrix obtained by replacing column  $i$  of  $A$  with  $B$ .



## Example 5

Solving the following system of equations using Cramer's rule.

$$x_1 + 3x_2 + x_3 = -2$$

$$2x_1 + 5x_2 + x_3 = -5$$

$$x_1 + 2x_2 + 3x_3 = 6$$

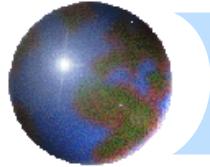
### Solution

The matrix of coefficients  $A$  and column matrix of constants  $B$  are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

It is found that  $|A| = -3 \neq 0$ . Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$

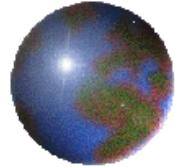


Giving  $|A_1| = -3, |A_2| = 6, |A_3| = -9$

Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2, \quad x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is  $x_1 = 1, x_2 = -2, x_3 = 3$ .



## Example 6

Determine values of  $\lambda$  for which the following system of equations has nontrivial solutions. Find the solutions for each value of  $\lambda$ .

$$(\lambda + 2)x_1 + (\lambda + 4)x_2 = 0$$

$$2x_1 + (\lambda + 1)x_2 = 0$$

### Solution

homogeneous system

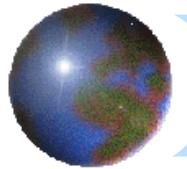
$\Rightarrow x_1 = 0, x_2 = 0$  is the trivial solution.

$\Rightarrow$  nontrivial solutions exist  $\Rightarrow$  many solutions

$$\Rightarrow \begin{vmatrix} \lambda+2 & \lambda+4 \\ 2 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+2)(\lambda+1) - 2(\lambda+4) = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda-2)(\lambda+3) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = 2.$$



$\lambda = -3$  results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

This system has many solutions,  $x_1 = r$ ,  $x_2 = r$ .

$\lambda = 2$  results in the system

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions,  $x_1 = -3r/2$ ,  $x_2 = r$ .