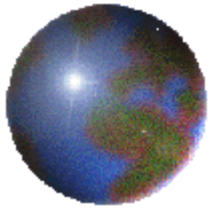


Linear Algebra



Chapter 4

VECTOR SPACES



4.5 Linear Combinations of Vectors

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be vectors in a vector space V .

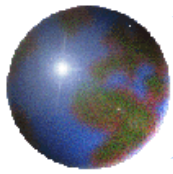
We say that \mathbf{v} , a vector of V , is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, if there exist scalars c_1, c_2, \dots, c_m such that \mathbf{v} can be written $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$.



Example 1

The vector $(5, 4, 2)$ is a linear combination of the vectors $(1, 2, 0)$, $(3, 1, 4)$, and $(1, 0, 3)$, since it can be written

$$(5, 4, 2) = (1, 2, 0) + 2(3, 1, 4) - 2(1, 0, 3)$$



Example 2

Determine whether or not the vector $(-1, 1, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, 3, 6)$.

Solution



Example 2

Determine whether or not the vector $(-1, 1, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, 3, 6)$.

Solution

Suppose $c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, 3, 6) = (-1, 1, 5)$

$$(c_1, 2c_1, 3c_1) + (0, c_2, 4c_2) + (2c_3, 3c_3, 6c_3) = (-1, 1, 5)$$

$$(c_1 + 2c_3, 2c_1 + c_2 + 3c_3, 3c_1 + 4c_2 + 6c_3) = (-1, 1, 5)$$

$$\Rightarrow \begin{cases} c_1 + 2c_3 = -1 \\ 2c_1 + c_2 + 3c_3 = 1 \\ 3c_1 + 4c_2 + 6c_3 = 5 \end{cases} \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1$$

Thus $(-1, 1, 5)$ is a linear combination of $(1, 2, 3)$, $(0, 1, 4)$, and $(2, 3, 6)$, where $(-1, 1, 5) = (1, 2, 3) + 2(0, 1, 4) - 1(2, 3, 6)$.



Example 3

Express the vector $(4, 5, 5)$ as a linear combination of the vectors $(1, 2, 3)$, $(-1, 1, 4)$, and $(3, 3, 2)$.

Solution

Suppose $c_1(1, 2, 3) + c_2(-1, 1, 4) + c_3(3, 3, 2) = (4, 5, 5)$

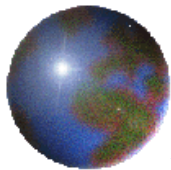
$$(c_1, 2c_1, 3c_1) + (-c_2, c_2, 4c_2) + (3c_3, 3c_3, 2c_3) = (4, 5, 5)$$

$$(c_1 - c_2 + 3c_3, 2c_1 + c_2 + 3c_3, 3c_1 + 4c_2 + 2c_3) = (4, 5, 5)$$

$$\Rightarrow \begin{cases} c_1 - c_2 + 3c_3 = 4 \\ 2c_1 + c_2 + 3c_3 = 5 \\ 3c_1 + 4c_2 + 2c_3 = 5 \end{cases} \Rightarrow c_1 = -2r + 3, c_2 = r - 1, c_3 = r$$

Thus $(4, 5, 5)$ can be expressed **in many ways** as a linear combination of $(1, 2, 3)$, $(-1, 1, 4)$, and $(3, 3, 2)$:

$$(4, 5, 5) = (-2r + 3)(1, 2, 3) + (r - 1)(-1, 1, 4) + r(2, 3, 6)$$



Example 4

Show that the vector $(3, -4, -6)$ cannot be expressed as a linear combination of the vectors $(1, 2, 3)$, $(-1, -1, -2)$, and $(1, 4, 5)$.

Solution

Suppose

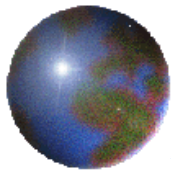
$$c_1(1, 2, 3) + c_2(-1, -1, -2) + c_3(1, 4, 5) = (3, -4, -6)$$

\Rightarrow

$$\begin{cases} c_1 - c_2 + c_3 = 3 \\ 2c_1 - c_2 + 4c_3 = -4 \\ 3c_1 - 2c_2 + 5c_3 = -6 \end{cases}$$

This system has no solution.

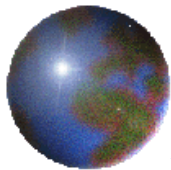
Thus $(3, -4, -6)$ is not a linear combination of the vectors $(1, 2, 3)$, $(-1, -1, -2)$, and $(1, 4, 5)$.



Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of 2×2 matrices.

Solution



Example 5

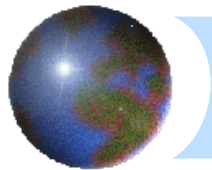
Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of 2×2 matrices.

Solution

Suppose
$$c_1 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

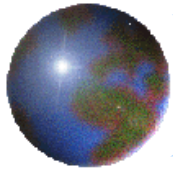


$$\begin{cases} c_1 + 2c_2 = -1 \\ -3c_2 + c_3 = 7 \\ 2c_1 + 2c_3 = 8 \\ c_1 + 2c_2 = -1 \end{cases}$$

This system has the unique solution $c_1 = 3$, $c_2 = -2$, $c_3 = 1$.

Therefore

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$



Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and

$$h(x) = 2x^2 - x + 4.$$

Solution



Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4$.

Solution

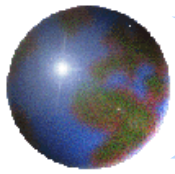
Suppose $c_1g + c_2h = f$.

Then

$$c_1(x^2 + 3x - 1) + c_2(2x^2 - x + 4) = x^2 + 10x - 7$$

$$(c_1 + 2c_2)x^2 + (3c_1 - c_2)x - c_1 + 4c_2 = x^2 + 10x - 7$$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 1 \\ 3c_1 - c_2 = 10 \\ -c_1 + 4c_2 = -7 \end{cases} \quad \Rightarrow c_1 = 3, c_2 = -1 \quad \Rightarrow f = 3g - h.$$



Spanning Sets

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are said to **span** a vector space if every vector in the space can be expressed as a linear combination of these vectors.

In this case $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is called a **spanning set**.



4.6 Linear Dependence and Independence

The concepts of dependence and independence of vectors are useful tools in constructing “efficient” spanning sets for vector spaces – sets in which there are no redundant vectors.

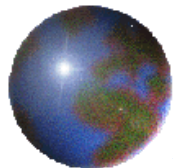
Definition

- (a) The set of vectors $\{ \mathbf{v}_1, \dots, \mathbf{v}_m \}$ in a vector space V is said to be **linearly dependent** if there exist scalars c_1, \dots, c_m , not all zero, such that $c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m = \mathbf{0}$
- (b) The set of vectors $\{ \mathbf{v}_1, \dots, \mathbf{v}_m \}$ is **linearly independent** if $c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m = \mathbf{0}$ can only be satisfied when $c_1 = 0, \dots, c_m = 0$.



Example 1

Show that the set $\{(1, 2, 3), (-2, 1, 1), (8, 6, 10)\}$ is linearly dependent in \mathbf{R}^3 .



Example 1

Show that the set $\{(1, 2, 3), (-2, 1, 1), (8, 6, 10)\}$ is linearly dependent in \mathbf{R}^3 .

Solution

$$\text{Suppose } c_1(1, 2, 3) + c_2(-2, 1, 1) + c_3(8, 6, 10) = \mathbf{0}$$

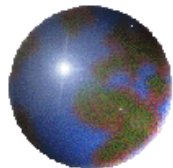
$$\Rightarrow (c_1, 2c_1, 3c_1) + (-2c_2, c_2, c_2) + (8c_3, 6c_3, 10c_3) = \mathbf{0}$$

$$(c_1 - 2c_2 + 8c_3, 2c_1 + c_2 + 6c_3, 3c_1 + c_2 + 10c_3) = \mathbf{0}$$

$$\Rightarrow \begin{cases} c_1 - 2c_2 + 8c_3 = 0 \\ 2c_1 + c_2 + 6c_3 = 0 \\ 3c_1 + c_2 + 10c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -2 \\ c_3 = -1 \end{cases}$$

$$\text{Thus } 4(1, 2, 3) - 2(-2, 1, 1) - (8, 6, 10) = \mathbf{0}$$

The set of vectors is linearly dependent.



Example 2

Show that the set $\{(3, -2, 2), (3, -1, 4), (1, 0, 5)\}$ is linearly independent in \mathbf{R}^3 .

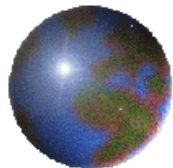
Solution

$$\text{Suppose } c_1(3, -2, 2) + c_2(3, -1, 4) + c_3(1, 0, 5) = \mathbf{0}$$

$$\begin{aligned} \Rightarrow (3c_1, -2c_1, 2c_1) + (3c_2, -c_2, 4c_2) + (c_3, 0, 5c_3) &= \mathbf{0} \\ (3c_1 + 3c_2 + c_3, -2c_1 - c_2, 2c_1 + 4c_2 + 5c_3) &= \mathbf{0} \end{aligned}$$

$$\Rightarrow \begin{cases} 3c_1 + 3c_2 + c_3 = 0 \\ -2c_1 - c_2 = 0 \\ 2c_1 + 4c_2 + 5c_3 = 0 \end{cases}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.
Thus the set is linearly independent.



Example 3

Consider the functions $f(x) = x^2 + 1$, $g(x) = 3x - 1$, $h(x) = -4x + 1$ of the vector space P_2 of polynomials of degree ≤ 2 .

Show that the set of functions $\{f, g, h\}$ is linearly independent.

Solution

Suppose

$$c_1f + c_2g + c_3h = \mathbf{0}$$

Since for any real number x ,

$$c_1(x^2 + 1) + c_2(3x - 1) + c_3(-4x + 1) = \mathbf{0}$$

$$x^2c_1 + x(3c_2 - 4c_3) + (c_1 - c_2 + c_3) = 0$$

$$c_1 = 0$$

$$3c_2 - 4c_3 = 0$$

$$c_1 - c_2 + c_3 = 0$$

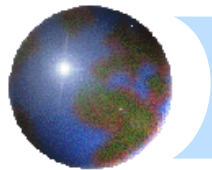
OR

Consider three convenient values of x . We get

$$x = 0 : c_1 - c_2 + c_3 = 0$$

$$x = 1 : 2c_1 + 2c_2 - 3c_3 = 0$$

$$x = -1 : 2c_1 - 4c_2 + 5c_3 = 0$$



It can be shown that this system of three equations has the unique solution

$$c_1 = 0, c_2 = 0, c_3 = 0$$

Thus $c_1f + c_2g + c_3h = \mathbf{0}$ implies that $c_1 = 0, c_2 = 0, c_3 = 0$.

The set $\{ f, g, h \}$ is linearly independent.



Theorem 4.7

A set consisting of two or more vectors in a vector space is linearly dependent if and only if it is possible to express one of the vectors as a linearly combination of the other vectors.

Example 4

The set of vectors $\{\mathbf{v}_1=(1, 2, 1) , \mathbf{v}_2=(-1, -1, 0) , \mathbf{v}_3 = (0, 1,1)\}$

is linearly dependent, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Thus, \mathbf{v}_3 is a linear combination of \mathbf{v}_1 and \mathbf{v}_2 .

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