# *Homogeneous System of linear Equations*

**Definition**

A system of linear equations is said to be ***homogeneous*** if all the constant terms are zeros.

Example:
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ is a solution.

**Theorem 1.1**

A system of homogeneous linear equations in $n$ variables always has the solution $x_1 = 0$, $x_2 = 0$. …, $x_n = 0$. This solution is called the **trivial solution**.

# Homogeneous System of linear Equations

**Note.** Non trivial solution

Example:
$$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

The system has other nontrivial solutions.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\therefore x_1 = -3r,\ \ x_2 = 4r,\ \ x_3 = r$$

**Theorem 1.2**

A system of homogeneous linear equations that has more variables than equations has many solutions.

# Homework

- Exercise set 1.2: (page 21)
  2, 5, 6, 7, 8, 14

# 1.3 Gaussian Elimination

**Definition**

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)

# Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\
-x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\
2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4
\end{aligned}
$$

**Solution**

Starting with the augmented matrix, create zeros below the pivot in the first column.

$$
\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix}
\underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx}
\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}
$$

At this stage, we create a zero only below the pivot.

$$
\underset{R3+(-2)R2}{\approx}
\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}
\underset{\frac{1}{2}R3}{\approx}
\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}
$$

Echelon form

We have arrived at the echelon form.

The corresponding system of equation is

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\
x_3 + 3x_4 &= 1 \\
x_4 &= 2
\end{aligned}
$$

We get

$$
\begin{aligned}
x_3 + 3(2) &= 1 \\
x_3 &= -5
\end{aligned}
$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

$$
\begin{aligned}
x_1 + 2x_2 + 3(-5) + 2(2) &= -1 \\
x_1 + 2x_2 &= 10 \\
x_1 &= -2x_2 + 10
\end{aligned}
$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10,\ \ x_2 = r,\ \ x_3 = -5,\ \ x_4 = 2$$

# Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

**Solution**

We arrive at the echelon form as in the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \cdots \approx \underset{\text{Echelon form}}{\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}$$

This marks the end of the forward elimination of variables from equations. We now commence the back substitution using matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+(-3)R3}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R1+(-3)R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ x_3 &= -5 \\ x_4 &= 2 \end{aligned}$$

Let $x_2 = r$. We get same solution as previously,

$$x_1 = -2r + 10,\ \ x_2 = r,\ \ x_3 = -5,\ \ x_4 = 2$$