Linear Algebra

Chapter 3

DETERMINANTS

3.1 Introduction to Determinants

Definition

The **determinant** of a 2 × 2 matrix A is denoted **|*A*|** and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2 × 2 matrix is given by *the difference of the products of the two diagonals* of the matrix.

The notation **det(*A*)** is also used for the determinant of A.

3.1 Introduction to Determinants

Applications

- Solvability of Linear Systems
- Matrix Inversion
- Cramer's Rule
- Eigenvalues and Eigenvectors
- Linear Independence
- Computation of Cross Products

3.1 Introduction to Determinants

Example 1: Find determinant of the following matrix

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

Example 1

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$

3.1 Introduction to Determinants

- The determinant of a 3 X 3 matrix is defined in terms of determinants of 2 X 2 matrices.
- The determinant of a 4 X 4 matrix is defined in terms of determinants of 3 X 3 matrices, and so on.
- For these definitions we need the following concepts of minor and cofactor.

Definition

Let A be a square matrix.

The **minor** of the element a_{ij} is denoted by M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of A.

The **cofactor** of a_{ij} is denoted C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that $C_{ij} = M_{ij}$ *or* $-M_{ij}$.

Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

Minor of a_{11} : $M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times (-2)) = 3$

Cofactor of a_{11} : $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

Minor of a_{32} : $M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1 \times 2) - (3 \times 4) = -10$

Cofactor of a_{32} : $C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$

Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

$$\text{If } A \text{ is } 3\times 3, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4\times 4, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\vdots$$

$$\text{If } A \text{ is } n\times n, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of $|A|$.

Example 3

Evaluate the determinant of the following matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

Example 3

Evaluate the determinant of the following matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0\times 1) - (1\times 2)] - 2[(3\times 1) - (1\times 4)] - [(3\times 2) - (0\times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$

Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of <u>any row or column</u> and their cofactors.

ith row expansion: $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

jth column expansion: $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned}
|A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\
&= -3\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} \\
&= -3[(2\times 1) - (-1\times 2)] + 0[(1\times 1) - (-1\times 4)] - 1[(1\times 2) - (2\times 4)] \\
&= -12 + 0 + 6 = -6
\end{aligned}$$

Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution

Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\ &= 3\begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\ &= 3(2)\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3-2) = 6 \end{aligned}$$

Example 6

Solve the following equation for the variable *x*.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution

Example 6

Solve the following equation for the variable x.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution

Expand the determinant to get the equation

$$x(x-2)-(x+1)(-1)=7$$

Proceed to simplify this equation and solve for x *(factorization)*.

$$x^2-2x+x+1=7$$

$$x^2-x-6=0$$

$$(x+2)(x-3)=0$$

$$x=-2 \text{ or } 3$$

Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

There are two solutions to this equation, $x = -2$ or 3.

Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |\mathrm{A}| = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Practice Problem

Example 4

Find the determinant of the following matrix using the diagonal product approach.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

3.2 Properties of Determinants

Theorem 3.2

Let A be an $n \times n$ matrix and c be a nonzero scalar.

(a) If a matrix B is obtained from A by multiplying the elements of a row (column) by c then $|B| = c|A|$.

(b) If a matrix B is obtained from A by interchanging two rows (columns) then $|B| = -|A|$.

(c) If a matrix B is obtained from A by adding a multiple of one row (column) to another row (column), then $|B| = |A|$.

Proof (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$

$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \ldots + ca_{kn}C_{kn}$$

$$\therefore |B| = c|A|.$$

Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

Solution

Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}.$

Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \underset{C2+2C3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$

Example 2

If $A=\begin{bmatrix}1 & 4 & 3\\ 0 & 2 & 5\\ -2 & -4 & 10\end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

$$\text{(a) } B_1=\begin{bmatrix}1 & 12 & 3\\ 0 & 6 & 5\\ -2 & -12 & 10\end{bmatrix} \text{(b) } B_2=\begin{bmatrix}1 & 4 & 3\\ -2 & -4 & 10\\ 0 & 2 & 5\end{bmatrix} \text{(c) } B_3=\begin{bmatrix}1 & 4 & 3\\ 0 & 2 & 5\\ 0 & 4 & 16\end{bmatrix}$$

Solution

Definition

A square matrix A is said to be **singular** if $|A|=0$.

A is **nonsingular** if $|A|\neq 0$.

Theorem 3.3

Let A be a square matrix. A is singular if

(a) all the elements of a row (column) are zero.

(b) two rows (columns) are equal.

(c) two rows (columns) are proportional. (i.e., $Ri=cRj$)

Proof

(a) Let all elements of the kth row of A be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

(c) If $Ri=cRj$, then $A \underset{Ri-cRj}{\approx} B$, row i of B is [0 0 … 0].

$\Rightarrow |A|=|B|=0$

Example 3

Show that the following matrices are singular.

$$(a)\ A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b)\ B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution

Theorem 3.4

Let A and B be $n \times n$ matrices and c be a nonzero scalar.

(a) $|cA| = c^n|A|$.

(b) $|AB| = |A||B|$.

(c) $|A^t| = |A|$.

(d) $\left|A^{-1}\right| = \dfrac{1}{|A|}$ (assuming A^{-1} exists)

Proof

(a) Each row of cA is a row of A multiplied by c. Apply Theorem 3.2(a) to each of the n rows of cA to get $|cA| = c^n |A|$.

(d) $AA^{-1} = I_n$. Thus $|AA^{-1}| = |I_n|$. $|A||A^{-1}| = 1$, by Part (b). Therefore $|A| \neq 0$, and $|A^{-1}| = \dfrac{1}{|A|}$.

Example 4

If A is a 2×2 matrix with $|A| = 4$, use Theorem 3.4 to compute the following determinants.

(a) $|3A|$ (b) $|A^2|$ (c) $|5A^tA^{-1}|$, assuming A^{-1} exists

Solution

(a) $|3A| = (3^2)|A| = 9 \times 4 = 36.$

(b) $|A^2| = |AA| = |A|\,|A| = 4 \times 4 = 16.$

(c) $|5A^tA^{-1}| = (5^2)|A^tA^{-1}| = 25|A^t||A^{-1}| = 25|A|\dfrac{1}{|A|} = 25.$

Example 5

Prove that $|A^{-1}A^tA| = |A|$

Solution

$$\left|A^{-1}A^tA\right| = \left|(A^{-1}A^t)A\right| = \left|A^{-1}A^t\right|\left|A\right| = \left|A^{-1}\right|\left|A^t\right|\left|A\right| = \frac{1}{|A|}\left|A\right|\left|A\right| = \left|A\right|$$

Example 6

Prove that if A and B are square matrices of the same size, with A being singular, then AB is also singular. Is the converse true?

Solution

$(\Rightarrow)$

$$|A| = 0 \quad \Rightarrow \quad |AB| = |A||B| = 0$$

Thus the matrix AB is singular.

$(\Leftarrow)$

$$|AB| = 0 \Rightarrow |A||B| = 0 \Rightarrow |A| = 0 \text{ or } |B| = 0$$

Thus AB being singular implies that either A or B is singular.

The inverse is not true.

3.3 Numerical Evaluation of a Determinant

Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

$$\text{upper}-\text{triangular} \qquad \text{lower}-\text{triangular}$$

Numerical Evaluation of a Determinant

Theorem 3.5

The determinant of a triangular matrix is the product of its diagonal elements.

Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}$$

Example 1

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}, \text{ find } |A|.$$

$$\text{Sol. } \quad |A| = 2 \times 3 \times (-5) = -30.$$

Numerical Evaluation of a Determinant

Example 2

Evaluation the determinant. $\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$

Solution (elementary row operations)

Example 3

Evaluation the determinant. $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$

Solution

Example 4

Evaluation the determinant. $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

Solution

Example 5

Evaluation the determinant. $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$

Solution

3.3 Determinants, Matrix Inverse, and Systems of Linear Equations

Definition

Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}.

The matrix whose (i, j)th element is C_{ij} is called the **matrix of cofactors** of A.

The transpose of this matrix is called the **adjoint** of A and is denoted $\operatorname{adj}(A)$.

$$\underset{\text{matrix of cofactors}}{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}} \qquad \underset{\text{adjoint matrix}}{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{t}}$$

Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A.

$$A=\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution The cofactors of A are as follows.

$$C_{11}=\begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix}=14 \qquad C_{12}=-\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix}=3 \qquad C_{13}=\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix}=-1$$

$$C_{21}=-\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix}=-9 \qquad C_{22}=\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}=7 \qquad C_{23}=-\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix}=6$$

$$C_{31}=\begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix}=-12 \qquad C_{32}=-\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix}=1 \qquad C_{33}=\begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix}=8$$

The matrix of cofactors of A is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is

$$\mathrm{adj}(A)=\begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$