4.8 Rank

Rank enables one to relate matrices to vectors, and vice versa.

Definition

Let A be an $m \times n$ matrix. The rows of A may be viewed as row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$, and the columns as column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Each row vector will have n components, and each column vector will have m components, The row vectors will span a subspace of $\mathbf{R}^n$ called the **row space** of A, and the column vectors will span a subspace of $\mathbf{R}^m$ called the **column space** of A.

Example 1

Consider the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 4 & 1 & 6 \\ 5 & 4 & 1 & 0 \end{bmatrix}$$

(1) The row vectors of A are

$$\mathbf{r}_1 = (1, 2, -1, 2), \mathbf{r}_2 = (3, 4, 1, 6), \mathbf{r}_3 = (5, 4, 1, 0)$$

These vectors span a subspace of $\mathbf{R}^4$ called the <u>row space</u> of A.

(2) The column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

These vectors span a subspace of $\mathbf{R}^3$ called the <u>column space</u> of A.

Theorem 4.16

The row space and the column space of a matrix A have the same dimension.

Definition

The dimension of the row space and the column space of a matrix A is called the **rank** of A. The rank of A is denoted **rank(A)**.

Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

The third row of A is a linear combination of the first two rows:

$$(2, 5, 8) = 2(1, 2, 3) + (0, 1, 2)$$

Hence the three rows of A are linearly dependent.

The rank of A must be less than 3. Since (1, 2, 3) is not a scalar multiple of (0, 1, 2), these two vectors are linearly independent. These vectors form a basis for the row space of A.

Thus rank(A) = 2.

Theorem 4.17

The nonzero row vectors of a matrix A that is in reduced echelon form are a basis for the row space of A. The rank of A is the number of nonzero row vectors.

Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced echelon form. There are three nonzero row vectors, namely (1, 2, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). According to the previous theorem, these three vectors form a basis for the row space of *A*.

Rank(*A*) = 3.

Theorem 4.18

Let A and B be row equivalent matrices. Then A and B have the same the row space. rank(A) = rank(B).

Theorem 4.19

Let E be a reduced echelon form of a matrix A. The nonzero row vectors of E form a basis for the row space of A. The rank of A is the number of nonzero row vectors in E.

Example 4

Find a basis for the row space of the following matrix A, and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Example 4

Find a basis for the row space of the following matrix A, and determine its rank.

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Use elementary row operations to find a reduced echelon form of the matrix A. We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The two vectors (1, 0, 7), (0, 1, –2) form a basis for the row space of A. Rank(A) = 2.

Example 5

EXAMPLE 5 Find a basis for the column space of the following matrix A

$$A \quad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

Example 5

EXAMPLE 5 Find a basis for the column space of the following matrix A

$$A \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

SOLUTION

The transpose of A is

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A^t. Let us find a basis for the row space of A^t. Compute the reduced echelon form of A^t.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors of this echelon form, namely $(1, 0, 5)$, $(0, 1, -3)$, are a basis for the row space of A^t. Write these vectors in column form to get a basis for the column space of A. The following vectors are a basis for the column space of A.

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Example 6

EXAMPLE 6 Find a basis for the subspace V of $\mathbf{R}^4$ spanned by the vectors

$$(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8)$$

SOLUTION

Example 6

EXAMPLE 6 Find a basis for the subspace V of $\mathbf{R}^4$ spanned by the vectors

$$(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8)$$

SOLUTION

We construct a matrix A having these vectors as row vectors.

$$A \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Determine the reduced echelon form of A. We get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero vectors of this reduced echelon form, namely $(1, 0, 5, 0)$ and $(0, 1, -1, 2)$, are a basis for the subspace V. The dimension of this subspace is two.

Example 7

Consider a system AX = B of **m** linear equations in **n** variables.

(a) If the augmented matrix and the matrix of coefficients have the same rank “**r**” and **r=n,** then the solution is unique.

(b) If the augmented matrix and the matrix of coefficients have the same rank “**r**” and **r<n,** then the solution is unique. Where **r** is rank and **n** are the number of variables

(c) If the augmented matrix and the matrix of coefficients do not have the same rank then the solution does not exist.

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned}
x_1 + x_2 + x_3 &\quad 2 \\
2x_1 + 3x_2 + x_3 &\quad 3 \\
x_1 - x_2 - 2x_3 &= -6
\end{aligned}$$

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & & 2 \\ 2x_1 & + & 3x_2 & + & x_3 & & 3 \\ x_1 & - & x_2 & - & 2x_3 & = & -6 \end{array}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

$$\overset{\text{Augmented matrix}}{\underbrace{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}}_{\text{Matrix of coefficients}}} \approx \cdots \approx \overset{\text{Reduced echelon form}}{\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{Reduced echelon form}}}$$

We see that ranks of the augmented matrix and the matrix of coefficients are equal, both being three. The system thus has a unique solution. The reduced echelon form gives that solution to be $x1 = -1, x_1 = 1, x_3 = 2$.

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned} x_1 + x_2 + x_3 & 2 \\ 2x_1 + 3x_2 + x_3 & 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

$$\overset{\text{Augmented matrix}}{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}} \approx \cdots \approx \overset{\text{Reduced echelon form}}{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$$

$$\underbrace{\phantom{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}}_{\text{Matrix of coefficients}} \qquad \underbrace{\phantom{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}}_{\text{Reduced echelon form}}$$

The system of linear equations can be viewed as the linear combination

$$x_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$$

The existence of solutions depends upon whether $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. The uniqueness depends upon whether the linear combination