Linear Algebra

Chapter 3

DETERMINANTS

Definition

A square matrix A is said to be **singular** if $|A|=0$.

A is **nonsingular** if $|A|\neq 0$.

Theorem 3.3

Let A be a square matrix. A is singular if

(a) all the elements of a row (column) are zero.

(b) two rows (columns) are equal.

(c) two rows (columns) are proportional. (i.e., $Ri=cRj$)

Proof

(a) Let all elements of the kth row of A be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

(c) If $Ri=cRj$, then $A \underset{Ri-cRj}{\approx} B$, row i of B is [0 0 … 0].

$\Rightarrow |A|=|B|=0$

Example 3

Show that the following matrices are singular.

$$(a)\ A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b)\ B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution

3.3 Numerical Evaluation of a Determinant

Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\underbrace{\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{upper}-\text{triangular}} \qquad \underbrace{\begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}}_{\text{lower}-\text{triangular}}$$

Numerical Evaluation of a Determinant

Theorem 3.5

The determinant of a triangular matrix is the product of its diagonal elements.

Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}$$

Example 1

Let $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$, find $|A|$.

Sol. $|A| = 2 \times 3 \times (-5) = -30.$

Numerical Evaluation of a Determinant

Example 2

Evaluation the determinant. $\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$

Solution (elementary row operations)

Example 3

Evaluation the determinant. $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$

Solution

Example 4

Evaluation the determinant. $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

Solution

Example 5

Evaluation the determinant. $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$

Solution

3.3 Determinants, Matrix Inverse, and Systems of Linear Equations

Definition

Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}.

The matrix whose (i, j)th element is C_{ij} is called the **matrix of cofactors** of A.

The transpose of this matrix is called the **adjoint** of A and is denoted adj(A).

$$\underset{\text{matrix of cofactors}}{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}} \qquad \underset{\text{adjoint matrix}}{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{t}}$$

Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A.

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution The cofactors of A are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14 \qquad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3 \qquad C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9 \qquad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \qquad C_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 \qquad C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1 \qquad C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors of A is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

Theorem 3.6

Let A be a square matrix with $|A| \neq 0$. A is invertible with

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$$

Proof

Consider the matrix product $A \cdot \operatorname{adj}(A)$. The (i, j)th element of this product is

$$(i, j)\text{th element} = (\text{row } i \text{ of } A) \times (\text{column } j \text{ of } \operatorname{adj}(A))$$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix}$$

$$= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

Proof Excluded

Proof of Theorem 3.6

If $i = j,\ a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |A|.$

Proof Excluded

If $i \neq j$, let $A \underset{Rj \text{ is replaced by } Ri}{\Longrightarrow} B.$

Matrices A and B have the same cofactors $C_{j1}, C_{j2}, \ldots, C_{jn}$.

So $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |B| = 0.$

row i = row j in B

Therefore $(i.j)\text{th element} = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\quad \therefore\ A \cdot \text{adj}(A) = |A| I_n$

Since $|A| \neq 0$, $\quad A\left(\dfrac{1}{|A|}\text{adj}(A)\right) = I_n$

Similarly, $\left(\dfrac{1}{|A|}\text{adj}(A)\right)A = I_n$.

Thus $A^{-1} = \dfrac{1}{|A|}\text{adj}(A)$

Theorem 3.7

A square matrix A is invertible if and only if $|A| \neq 0$.

Proof

($\Rightarrow$) Assume that A is invertible.

$$\Rightarrow \ AA^{-1} = I_n.$$

$$\Rightarrow \ |AA^{-1}| = |I_n|.$$

$$\Rightarrow \ |A||A^{-1}| = 1$$

$$\Rightarrow \ |A| \neq 0.$$

Proof Excluded

($\Leftarrow$) Theorem 3.6 tells us that if $|A| \neq 0$, then A is invertible.

$$A^{-1} \text{ exists if and only if } |A| \neq 0.$$

Example 2

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

Solution

Example 3

Use the formula for the inverse of a matrix to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution

$|A| = 25$, so the inverse of A exists. We found adj(A) in Example 1

$$\operatorname{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{25}\begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$

Homework

- Exercise 3.3 page 178-179: 1, 3, 5, 7.

- ***Exercise***

  Show that if $A = A^{-1}$, then $|A| = \pm 1$.

  Show that if $A^t = A^{-1}$, then $|A| = \pm 1$.

Theorem 3.8

Let $AX = B$ be a system of n linear equations in n variables.

(1) If $|A| \neq 0$, there is a unique solution.

(2) If $|A| = 0$, there may be many or no solutions.

Theorem 3.8

Let $AX = B$ be a system of n linear equations in n variables.

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Proof

Proof Excluded

(1) If $|A| \neq 0$

$\Rightarrow A^{-1}$ exists (Thm 3.7)

$\Rightarrow$ there is then a unique solution given by $X = A^{-1}B$ (Thm 2.9).

(2) If $|A| = 0$

$\Rightarrow$ since $A \approx \ldots \approx C$ implies that if $|A| \neq 0$ then $|C| \neq 0$ (Thm 3.2).

$\Rightarrow$ the reduced echelon form of A is not I_n.

$\Rightarrow$ The solution to the system $AX = B$ is not unique.

$\Rightarrow$ many or no solutions.

Example 4

Determine whether or not the following system of equations has an unique solution.

$$\begin{aligned} 3x_1 + 3x_2 - 2x_3 &= 2 \\ 4x_1 + x_2 + 3x_3 &= -5 \\ 7x_1 + 4x_2 + x_3 &= 9 \end{aligned}$$

Solution

Theorem 3.9 Cramer's Rule

Let $AX = B$ be a system of n linear equations in n variables such that $|A| \neq 0$. The system has a unique solution given by

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ \ldots,\ x_n = \frac{|A_n|}{|A|}$$

Where A_i is the matrix obtained by replacing column i of A with B.

Example 5

Solving the following system of equations using Cramer's rule.

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= -2 \\
2x_1 + 5x_2 + x_3 &= -5 \\
x_1 + 2x_2 + 3x_3 &= 6
\end{aligned}
$$

Solution

The matrix of coefficients A and column matrix of constants B are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

It is found that $|A| = -3 \neq 0$. Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$

Giving $|A_1| = -3, |A_2| = 6, |A_3| = -9$

Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1,\ x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2,\ x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is $x_1 = 1, x_2 = -2, x_3 = 3.$

Example 6

Determine values of λ for which the following system of equations has nontrivial solutions.Find the solutions for each value of λ.

$$(\lambda+2)x_1+(\lambda+4)x_2=0$$
$$2x_1+(\lambda+1)x_2=0$$

Solution

homogeneous system

$\Rightarrow x_1 = 0,\ x_2 = 0$ is the trivial solution.

$\Rightarrow$ nontrivial solutions exist $\Rightarrow$ many solutions

$$\Rightarrow \quad \begin{vmatrix} \lambda+2 & \lambda+4 \\ 2 & \lambda+1 \end{vmatrix} = 0$$

$\Rightarrow (\lambda+2)(\lambda+1)-2(\lambda+4)=0 \Rightarrow \lambda^2+\lambda-6=0 \Rightarrow (\lambda-2)(\lambda+3)=0$

$\Rightarrow \lambda = -3$ or $\lambda = 2$.

λ = – 3 results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

This system has many solutions, $x_1 = r$, $x_2 = r$.

λ = 2 results in the system

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions, $x_1 = -3r/2$, $x_2 = r$.