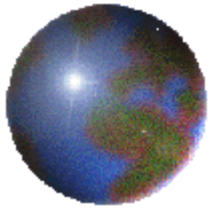


Linear Algebra



Chapter 3

DETERMINANTS



3.1 Introduction to Determinants

Definition

The **determinant** of a 2×2 matrix A is denoted $|A|$ and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2×2 matrix is given by *the difference of the products of the two diagonals* of the matrix.

The notation **det**(A) is also used for the determinant of A .



3.1 Introduction to Determinants

Applications

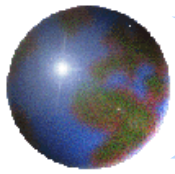
- Solvability of Linear Systems
- Matrix Inversion
- Cramer's Rule
- Eigenvalues and Eigenvectors
- Linear Independence
- Computation of Cross Products



3.1 Introduction to Determinants

Example 1: Find determinant of the following matrix

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$



3.1 Introduction to Determinants

Example 1

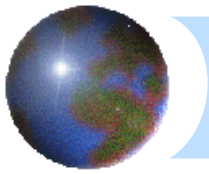
$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$



3.1 *Introduction to Determinants*

- The determinant of a 3×3 matrix is defined in terms of determinants of 2×2 matrices.
- The determinant of a 4×4 matrix is defined in terms of determinants of 3×3 matrices, and so on.
- For these definitions we need the following concepts of minor and cofactor.



Definition

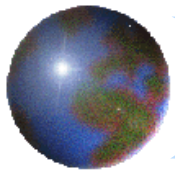
Let A be a square matrix.

The **minor** of the element a_{ij} is denoted by M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of A .

The **cofactor** of a_{ij} is denoted C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that $C_{ij} = M_{ij}$ or $-M_{ij}$.



Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A .

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution



Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A .

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

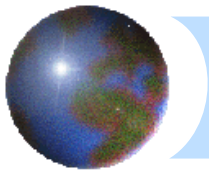
$$\text{Minor of } a_{11} : M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times (-2)) = 3$$

$$\text{Cofactor of } a_{11} : C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$$

$$\text{Minor of } a_{32} : M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1 \times 2) - (3 \times 4) = -10$$

$$\text{Cofactor of } a_{32} : C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$$





Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

$$\text{If } A \text{ is } 3 \times 3, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4 \times 4, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\vdots$$

$$\text{If } A \text{ is } n \times n, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of $|A|$.

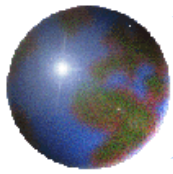


Example 3

Evaluate the determinant of the following matrix A .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution



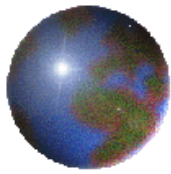
Example 3

Evaluate the determinant of the following matrix A .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$



Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

$$i\text{th row expansion: } |A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$j\text{th column expansion: } |A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= -3 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} \\ &= -3[(2 \times 1) - (-1 \times 2)] + 0[(1 \times 1) - (-1 \times 4)] - 1[(1 \times 2) - (2 \times 4)] \\ &= -12 + 0 + 6 = -6 \end{aligned}$$

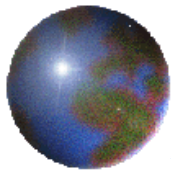


Example 5

Evaluate the determinant of the following 4×4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution



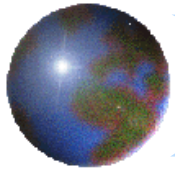
Example 5

Evaluate the determinant of the following 4×4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\ &= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\ &= 3 \begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\ &= 3(2) \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3 - 2) = 6 \end{aligned}$$

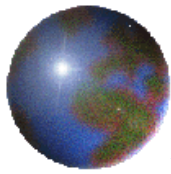


Example 6

Solve the following equation for the variable x .

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution



Example 6

Solve the following equation for the variable x .

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution

Expand the determinant to get the equation

$$x(x-2) - (x+1)(-1) = 7$$

Proceed to simplify this equation and solve for x (*factorization*).

$$x^2 - 2x + x + 1 = 7$$

$$x^2 - x - 6 = 0$$

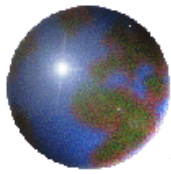
$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } 3$$

Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are two solutions to this equation, $x = -2$ or 3 .



Computing Determinants of 2×2 and 3×3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

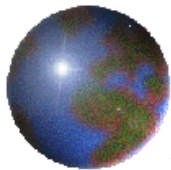
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)



Computing Determinants of 2×2 and 3×3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

The diagram illustrates the expansion of a 3x3 matrix determinant. The first matrix is the original matrix. The second matrix shows the same elements with blue arrows indicating the positive diagonal products (a₁₁a₂₂a₃₃, a₁₂a₂₃a₃₁, a₁₃a₂₁a₃₂) and red arrows indicating the negative diagonal products (a₁₃a₂₂a₃₁, a₁₁a₂₃a₃₂, a₁₂a₂₁a₃₃).

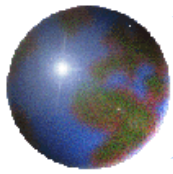
$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

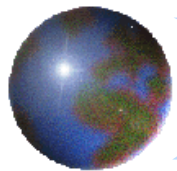


Practice Problem

Example 4

Find the determinant of the following matrix using the diagonal product approach.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$



3.2 Properties of Determinants

Theorem 3.2

Let A be an $n \times n$ matrix and c be a nonzero scalar.

- (a) If a matrix B is obtained from A by multiplying the elements of a row (column) by c then $|B| = c|A|$.
- (b) If a matrix B is obtained from A by interchanging two rows (columns) then $|B| = -|A|$.
- (c) If a matrix B is obtained from A by adding a multiple of one row (column) to another row (column), then $|B| = |A|$.

Proof (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn}$$

$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \dots + ca_{kn}C_{kn}$$

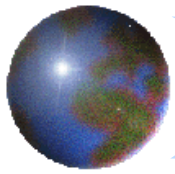
$$\therefore |B| = c|A|.$$



Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

Solution

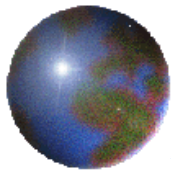


Example 1

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \stackrel{C2+2C3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3) \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$



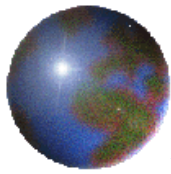
Example 2

If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

$$(a) B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix} \quad (b) B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix} \quad (c) B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$$

Solution



Definition

A square matrix A is said to be **singular** if $|A|=0$.

A is **nonsingular** if $|A| \neq 0$.

Theorem 3.3

Let A be a square matrix. A is singular if

- (a) all the elements of a row (column) are zero.
- (b) two rows (columns) are equal.
- (c) two rows (columns) are proportional. (i.e., $R_i = cR_j$)

Proof

- (a) Let all elements of the k th row of A be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

- (c) If $R_i = cR_j$, then $A \xrightarrow{R_i - cR_j} B$, row i of B is $[0 \ 0 \ \dots \ 0]$.

$$\Rightarrow |A| = |B| = 0$$



Example 3

Show that the following matrices are singular.

$$(a) A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b) B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution



Theorem 3.4

Let A and B be $n \times n$ matrices and c be a nonzero scalar.

(a) $|cA| = c^n |A|$.

(b) $|AB| = |A||B|$.

(c) $|A^t| = |A|$.

(d) $|A^{-1}| = \frac{1}{|A|}$ (assuming A^{-1} exists)

Proof

(a) Each row of cA is a row of A multiplied by c . Apply Theorem 3.2(a) to each of the n rows of cA to get $|cA| = c^n |A|$.

(d) $AA^{-1} = I_n$. Thus $|AA^{-1}| = |I_n|$. $|A||A^{-1}| = 1$, by Part (b). Therefore $|A| \neq 0$, and $|A^{-1}| = \frac{1}{|A|}$.



Example 4

If A is a 2×2 matrix with $|A| = 4$, use Theorem 3.4 to compute the following determinants.

- (a) $|3A|$ (b) $|A^2|$ (c) $|5A^t A^{-1}|$, assuming A^{-1} exists

Solution

(a) $|3A| = (3^2)|A| = 9 \times 4 = 36.$

(b) $|A^2| = |AA| = |A| |A| = 4 \times 4 = 16.$

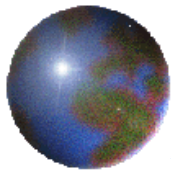
(c) $|5A^t A^{-1}| = (5^2)|A^t A^{-1}| = 25|A^t A^{-1}| = 25|A| \frac{1}{|A|} = 25.$

Example 5

Prove that $|A^{-1} A^t A| = |A|$

Solution

$$|A^{-1} A^t A| = |(A^{-1} A^t) A| = |A^{-1} A^t| |A| = |A^{-1}| |A^t| |A| = \frac{1}{|A|} |A| |A| = |A|$$



Example 6

Prove that if A and B are square matrices of the same size, with A being singular, then AB is also singular. Is the converse true?

Solution

(\Rightarrow)

$$|A| = 0 \quad \Rightarrow \quad |AB| = |A||B| = 0$$

Thus the matrix AB is singular.

(\Leftarrow)

$$|AB| = 0 \Rightarrow |A||B| = 0 \Rightarrow |A| = 0 \text{ or } |B| = 0$$

Thus AB being singular implies that either A or B is singular.

The inverse is not true.



3.3 Numerical Evaluation of a Determinant

Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

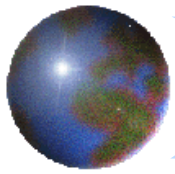
It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

upper – triangular

$$\begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

lower – triangular



Numerical Evaluation of a Determinant

Theorem 3.5

The determinant of a triangular matrix is the product of its diagonal elements.

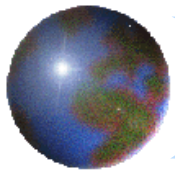
Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22} \cdots a_{nn}$$

Example 1

Let $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$, find $|A|$.

Sol. $|A| = 2 \times 3 \times (-5) = -30$.



Numerical Evaluation of a Determinant

Example 2

Evaluation the determinant.
$$\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$$

Solution (elementary row operations)

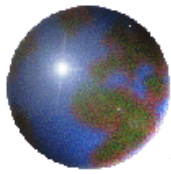


Example 3

Evaluation the determinant.

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$$

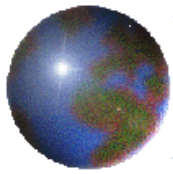
Solution



Example 4

Evaluation the determinant. $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

Solution

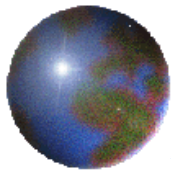


Example 5

Evaluation the determinant.

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$$

Solution



3.3 Determinants, Matrix Inverse, and Systems of Linear Equations

Definition

Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij} .

The matrix whose (i, j) th element is C_{ij} is called the **matrix of cofactors** of A .

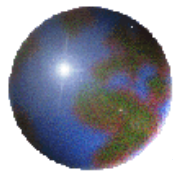
The transpose of this matrix is called the **adjoint** of A and is denoted $\text{adj}(A)$.

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^t$$

adjoint matrix



Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A .

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution The cofactors of A are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14 \quad C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3 \quad C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9 \quad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7 \quad C_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1 \quad C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors

of A is
$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$