Relations between system of linear equations and matrices

matrix of coefficients and augmented matrix

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &= 2 \\ 2x_1 + 3x_2 + \quad x_3 &= 3 \\ x_1 - \quad x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

matrix of coefficients   augmented matrix

Observe that the matrix of coefficients is a submatrix of the augmented matrix. The augmented matrix completely describes the system.

Relations between system of linear equations and matrices

- Transformations called elementary transformations can be used to change a system of linear equations into another system of linear equations that has the same solution.
- These transformations are used to solve systems of linear equations by eliminating variables.
- In practice it is simpler to work in terms of matrices using analogous transformations called elementary row operations.

Elementary Row Operations of Matrices

- These transformations are as follows:

- Elementary Transformation

1. Interchange two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

- Elementary Row Operation

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Example 1

Solving the following system of linear equation.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + 3x_2 + x_3 &= 3 \\
x_1 - x_2 - 2x_3 &= -6
\end{aligned}
$$

Example 1

Solving the following system of linear equation.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$\approx$ row equivalent

Solution

Equation Method

Initial system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Eq2+(–2)Eq1

Eq3+(–1)Eq1

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

Analogous Matrix Method

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

$\approx$ R2+(–2)R1 R3+(–1)R1

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

Eq1+(–1)Eq2

Eq3+(2)Eq2

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

(–1/5)Eq3

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ -5x_3 &= -10 \end{aligned}$$

R1+(–1)R2

R3+(2)R2

$$\approx \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Eq1+(–2)Eq3

Eq2+Eq3

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ x_3 &= 2 \end{aligned}$$

(–1/5)R3

$$\approx \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

R1+(–2)R3

R2+R3

$$\approx \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is

$x_1 = -1, x_2 = 1, x_3 = 2.$

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$x_1 = -1, x_2 = 1, x_3 = 2.$

Example 2

Solving the following system of linear equation.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

Example 2

Solving the following system of linear equation.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{1 \quad -2 \quad 4 \quad 12} \\ 2 \quad -1 \quad 5 \quad 18 \\ -1 \quad 3 \quad -3 \quad -8 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{3} & \boxed{-3} & \boxed{-6} \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{1} & \boxed{-1} & \boxed{-2} \\ 0 & 1 & 1 & 4 \end{bmatrix} \underset{\substack{R1+(2)R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{2} & \boxed{6} \end{bmatrix}$$

$$\underset{\left(\frac{1}{2}\right)R3}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{1} & \boxed{3} \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{solution} \begin{cases} x_1 = 2 \\ x_2 = 1. \\ x_3 = 3 \end{cases}$$

Example 3

Solve the system

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - \ 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\qquad &= -7 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{4 \quad 8 \quad -12 \quad 44} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\left(\frac{1}{4}\right)\text{R1}}{\approx} \begin{bmatrix} \boxed{1 \quad 2 \quad -3 \quad 11} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\substack{\text{R2}+(-3)\text{R1} \\ \text{R3}+2\text{R1}}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & \boxed{3} & \boxed{-6} & \boxed{15} \end{bmatrix}$$

$$\underset{\text{R2} \leftrightarrow \text{R3}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{3} & \boxed{-6} & \boxed{15} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\left(\frac{1}{3}\right)\text{R2}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{1} & \boxed{-2} & \boxed{5} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\text{R1}+(-2)\text{R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & \boxed{1} & \boxed{-1} \end{bmatrix}$$

$$\underset{\substack{\text{R1}+(-1)\text{R3} \\ \text{R2}+2\text{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The solution is $x_1 = 2,\ x_2 = 3,\ x_3 = -1.$

Summary

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - \ \ 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\qquad &= -7 \end{aligned}$$

$$[A:B] = \left[\begin{array}{ccc|c} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array}\right]$$

A B

Use row operations to $[A: B]$:

$$\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

i.e., $[A:B] \approx \cdots \approx [I_n : X]$

Def. $[I_n : X]$ is called the ***reduced echelon form*** of $[A:B]$.

Note. 1. If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution, then A is row equivalent to I_n $(A \approx I_n)$.

2. If $A \approx I_n$, then the system has unique solution.

Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_1 - x_2 + 4x_3 &= b_2 \\ -x_1 + 2x_2 - 4x_3 &= b_3 \end{aligned} \quad \text{for} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \text{ in turn}$$

Solution

Example 4 Many Systems

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Solution

$$\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+2R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$$

The solutions to the three systems are $\begin{cases} x_1 = 1 \\ x_2 = -1, \\ x_3 = 2 \end{cases} \begin{cases} x_1 = 0 \\ x_2 = 3, \\ x_3 = 1 \end{cases} \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 2 \end{cases}.$

Homework

- Exercise 1.1: (page 12)
  1, 2, 4, 5, 6, 10, 11, 12, 13

More generalization of Gauss-Jordan Elimination

Definition

A matrix is in ***reduced echelon form*** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a ***leading 1***.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

The following matrices are all in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are not in reduced echelon form for the reasons stated.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros not at bottom of matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in row 2

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(?) (?) (?) (?)

- Elementary row operations reduced echelon form
- The reduced echelon form of a matrix is unique.

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\checkmark) \qquad\qquad (\times) \qquad\qquad (\checkmark) \qquad\qquad (\times)$$

- There are usually many sequences of row operations that can be used to transform a given matrix to reduced echelon form-they all, however, lead to the same reduced echelon form.
- We say that the reduced echelon form of a matrix is unique.

Gauss-Jordan Elimination

- System of linear equations
  $\Rightarrow$ augmented matrix
  $\Rightarrow$ reduced echelon form
  $\Rightarrow$ solution

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot (leading 1)

$$\underset{R1\leftrightarrow R2}{\approx}\begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\underset{R3+(-4)R1}{\approx}\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx}\begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.

Example 2

Solve, if possible, the system of equations

$$
\begin{aligned}
3x_1 - 3x_2 + 3x_3 &= 9 \\
2x_1 - x_2 + 4x_3 &= 7 \\
3x_1 - 5x_2 - x_3 &= 7
\end{aligned}
$$

Solution

$$
\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix}
\underset{\left(\frac{1}{3}\right)R1}{\approx}
\begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix}
\underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx}
\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}
$$

$$
\underset{\substack{R1+R2 \\ R3+2R2}}{\approx}
\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\Rightarrow
\begin{aligned} x_1 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 1 \end{aligned}
\Rightarrow
\begin{aligned} x_1 &= -3x_3 + 4 \\ x_2 &= -2x_3 + 1 \end{aligned}
$$

The general solution to the system is

$$
\begin{aligned}
x_1 &= -3r + 4 \\
x_2 &= -2r + 1 \\
x_3 &= r
\end{aligned}
\quad \text{, where } r \text{ is real number (called a parameter).}
$$

Example 3

Solve the system of equations

$$\begin{aligned} 2x_1 - 4x_2 + 12x_3 - 10x_4 &= 58 \\ -x_1 + 2x_2 - 3x_3 + 2x_4 &= -14 \\ 2x_1 - 4x_2 + 9x_3 - 6x_4 &= 44 \end{aligned}$$

$\Rightarrow$ many sol.

Solution

$$\begin{bmatrix} 2 & -4 & 12 & -10 & 58 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R1}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix}$$

$$\underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix}$$

$$\underset{\substack{R1+(-6)R2 \\ R3+3R2}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 - 2x_2 &= -2 \\ x_3 &= 6 \\ x_4 &= 1 \end{aligned} \Rightarrow \begin{cases} x_1 = 2r - 2 \\ x_2 = r \\ x_3 = 6 \\ x_4 = 1 \end{cases}, \text{ for some } r.$$

Example 4

Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 + x_5 &= 2 \\ 2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 &= 6 \\ -x_1 - 2x_2 + x_3 - x_4 + 3x_5 &= 4 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 2 & 4 & -2 & 6 & 3 & 6 \\ -1 & -2 & 1 & -1 & 3 & 4 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\underset{R2\leftrightarrow R3}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R1+(-3)R2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & -5 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1+5R3 \\ R2+(-2)R3}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 &= -2x_2 + x_3 + 3 \\ x_4 &= -1 \\ x_5 &= 2 \end{aligned} \qquad \Rightarrow \begin{aligned} x_1 &= -2r + s + 3 \\ x_2 &= r,\ x_3 = s,\ x_4 = -1, \text{ , for some } r \text{ and } s. \\ x_5 &= 2 \end{aligned}$$

Example 5

This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-1)R1}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{R2 \leftrightarrow R3}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{R1+R2 \\ R4+(-2)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+R3 \\ R4+(-4)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\underset{\left(\frac{1}{13}\right)R4}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{0x_1+0x_2+0x_3=1}$$

The system has no solution.