4.2 Dot Product, Norm, Angle, and Distance

Definition

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbf{R}^n$.
The **dot product** of **u** and **v** is denoted **u.v** and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

The dot product assigns a real number to each pair of vectors.

Example 1

Find the dot product of

$$\mathbf{u} = (1, -2, 4) \text{ and } \mathbf{v} = (3, 0, 2)$$

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Solution

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (1\times 3)+(-2\times 0)+(4\times 2)\\ &= 3+0+8\\ &= 11\end{aligned}$$

Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let c be a scalar. Then

1. **u.v = v.u**
2. **(u + v).w = u.w + v.w**
3. c**u.v** = c**(u.v) = u.**c**v**
4. **u.u** ≥ 0, and **u.u** $= 0$ if and only if **u = 0**

Proof

1. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. We get

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + \cdots + u_n v_n \\ &= v_1 u_1 + \cdots + v_n u_n \quad \text{by the commutative property of real numbers} \\ &= \mathbf{v} \cdot \mathbf{u}\end{aligned}$$

4. $\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + \cdots + u_n u_n = (u_1)^2 + \cdots + (u_n)^2$

$(u_1)^2 + \cdots + (u_n)^2 \geq 0$, thus $\mathbf{u} \cdot \mathbf{u} \geq 0$.

$(u_1)^2 + \cdots + (u_n)^2 = 0$, if and only if $u_1 = 0, \cdots, u_n = 0$.

Thus $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Norm of a Vector in $\mathbf{R}^n$

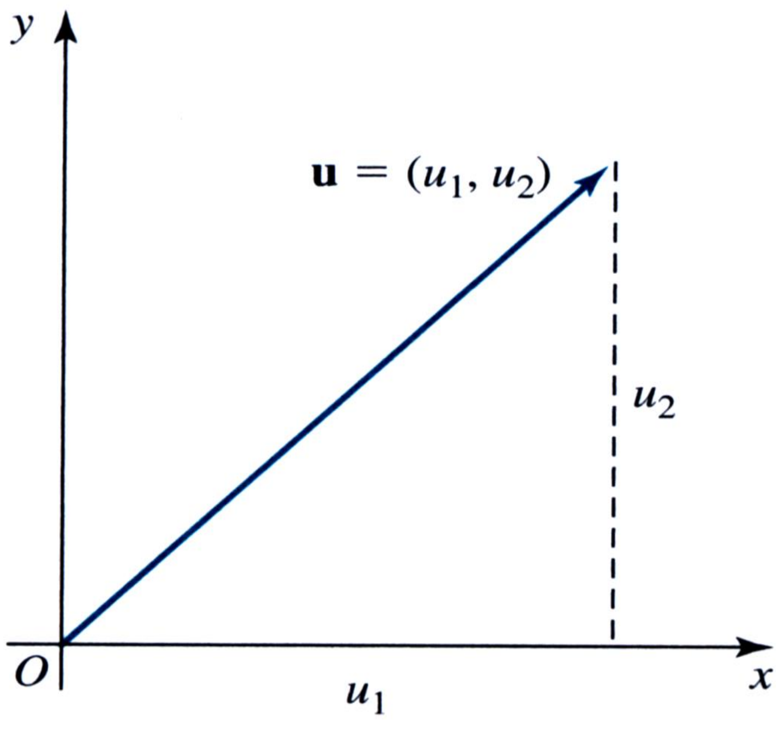

Figure 4.5 length of $\mathbf{u}$

Definition

The **norm** (**length** or **magnitude**) of a vector $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbf{R}^n$ is denoted $\|\mathbf{u}\|$ and defined by

$$\|\mathbf{u}\| = \sqrt{(u_1)^2 + \cdots + (u_n)^2}$$

Note:
The norm of a vector can also be written in terms of the dot product $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Example 2

Find the norm of each of the vectors $\mathbf{u} = (1, 3, 5)$ of $\mathbf{R}^3$ and $\mathbf{v} = (3, 0, 1, 4)$ of $\mathbf{R}^4$.

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Solution

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9 + 0 + 1 + 16} = \sqrt{26}$$

Unit Vector

Definition

A **unit vector** is a vector whose norm is 1.

If **v** is a nonzero vector, then the vector **u**

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is a unit vector in the direction of **v**.

This procedure of constructing a unit vector in the same direction as a given vector is called **normalizing** the vector.

Example

EXAMPLE 1 Find the norm of the vector $(2, -1, 3)$. Normalize this vector.

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SOLUTION

$\|(2, -1, 3)\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$. The norm of $(2, -1, 3)$ is $\sqrt{14}$. The normalized vector is

$$\frac{1}{\sqrt{14}}(2, -1, 3)$$

This vector may also be written $\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$. This vector is a unit vector in the direction of $(2, -1, 3)$.

Example 3

(a) Show that the vector (1, 0) is a unit vector.

(b) Find the norm of the vector (2, –1, 3). Normalize this vector.

Solution

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Solution

(a) $\|(1,0)\| = \sqrt{1^2+0^2} = 1$. Thus (1, 0) is a unit vector. It can be similarly shown that (0, 1) is a unit vector in $\mathbf{R}^2$.

(b) $\|(2,-1,3)\| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}$. The norm of (2, –1, 3) is $\sqrt{14}$.

The normalized vector is

$$\frac{1}{\sqrt{14}}(2,-1,3)$$

The vector may also be written $\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

This vector is a unit vector in the direction of (2, –1, 3).

Work and the Dot Product

- We defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object.

Work and the Dot Product

- Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 1.

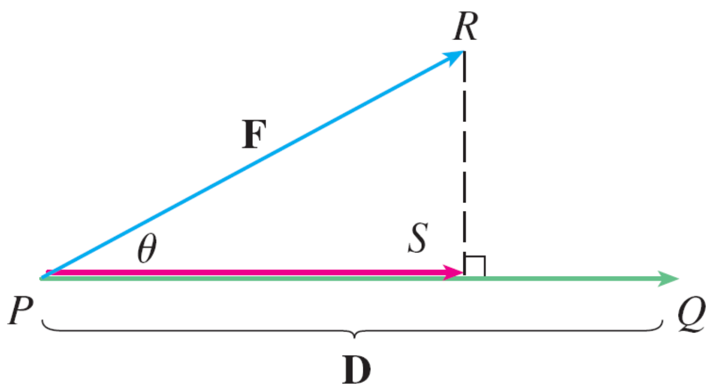

Figure 1

- If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. So here we have two vectors: the force $\mathbf{F}$ and the displacement $\mathbf{D}$.

Work and the Dot Product

- The **work** done by **F** is defined as the magnitude of the displacement, |**D**|, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

$$|\overrightarrow{PS}| = |\mathbf{F}| \cos\theta$$

- So the work done by **F** is defined to be

$$\boxed{1} \qquad W = |\mathbf{D}|\,(|\mathbf{F}|\cos\theta) = |\mathbf{F}||\mathbf{D}|\cos\theta$$

- Notice that work is a scalar quantity; it has no direction. But its value depends on the angle θ between the force and displacement vectors.

Work and the Dot Product

- We use the expression in Equation 1 to define the dot product of two vectors even when they don't represent force or displacement.

Definition The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either **a** or **b** is **0**, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

- This product is called the **dot product** because of the dot in the notation $\mathbf{a} \cdot \mathbf{b}$.

Example 1 – Computing a Dot Product from Lengths and the Contained Angle

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

Example 1 – Computing a Dot Product from Lengths and the Contained Angle

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

Angle between Vectors (in R^n)

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^n$.

The **cosine of the angle θ** between these vectors is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi$$

Example 4

Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (1, 0, 1)$ in $\mathbf{R}^3$.

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Example 4

Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (1, 0, 1)$ in $\mathbf{R}^3$.

Solution $\mathbf{u}\cdot\mathbf{v} = (1,0,0)\cdot(1,0,1) = 1$

$$\|\mathbf{u}\| = \sqrt{1^2+0^2+0^2} = 1 \qquad \|\mathbf{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

Thus $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}$, the angle between **u** and **v** is 45°.

Orthogonal Vectors

Definition

Two nonzero vectors are **orthogonal** if the angle between them is a right angle .

Theorem 4.2

Two nonzero vectors **u** and **v** are orthogonal *if and only if* **u**.**v** = **0**.

Proof

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal} \iff \cos\theta = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0$$

Example 5

Show that the following pairs of vectors are orthogonal.

(a) $(1, 0)$ and $(0, 1)$.

(b) $(2, -3, 1)$ and $(1, 2, 4)$.

Solution

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(a) (1, 0) and (0, 1).

(b) (2, –3, 1) and (1, 2, 4).

Solution

(a) $(1, 0).(0, 1) = (1 \times 0) + (0 \times 1) = 0$.

The vectors are orthogonal.

(b) $(2, -3, 1).(1, 2, 4) = (2 \times 1) + (-3 \times 2) + (1 \times 4) = 2 - 6 + 4 = 0$.

The vectors are orthogonal.

Note

- (1, 0), (0,1) are <u>orthogonal</u> <u>unit</u> vectors in $\mathbf{R}^2$.
- (1, 0, 0), (0, 1, 0), (0, 0, 1) are orthogonal unit vectors in $\mathbf{R}^3$.
- (1, 0, …, 0), (0, 1, 0, …, 0), …, (0, …, 0, 1) are orthogonal unit vectors in $\mathbf{R}^n$.

EXAMPLE 4 (a) Show that the following vectors are orthogonal. (b) Compute the norm of each vector.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLUTION

EXAMPLE 4 (a) Show that the following vectors are orthogonal. (b) Compute the norm of each vector.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLUTION

(a) The dot product of **u** and **v** is obtained by multiplying corresponding components and adding.

$$\mathbf{u} \cdot \mathbf{v} = (1 \times -1) + (2 \times 3) + (-5 \times 1) = 0$$

The dot product is zero, thus the vectors are orthogonal.

(b) The norm of each vector is obtained by taking the square root of the sum of the squares of the components.

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + (-5)^2} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{11}$$

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Let the vector (a, b) be orthogonal to (3, –1)

We get

$$(a, b)\cdot(3, -1) = 0$$
$$(a\times 3) + (b\times(-1)) = 0$$
$$3a - b = 0$$
$$b = 3a$$

Thus any vector of the form $(a, 3a)$ is orthogonal to the vector (3, –1).

Any vector of this form can be written

$$a(1, 3)$$

The set of all such vectors lie on the line defined by the vector (1, 3).

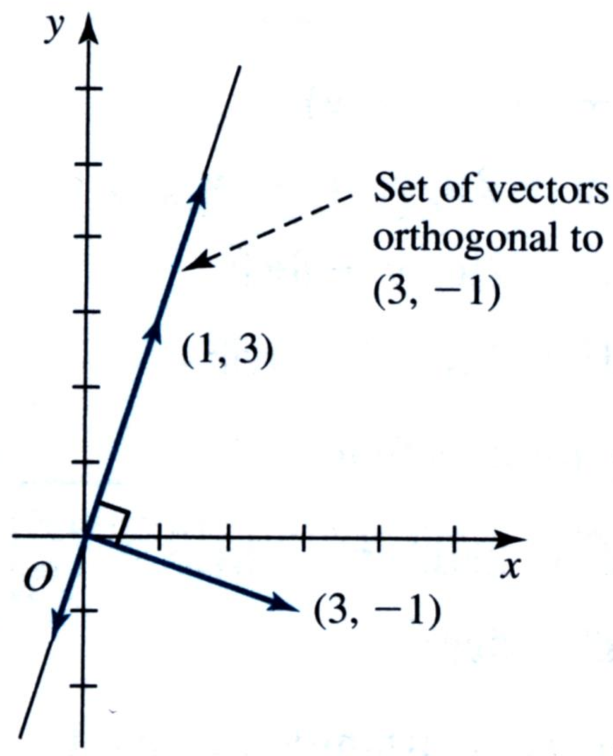


Figure 4.7

Distance between Points

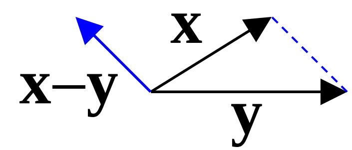
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbf{R}^n$.

The **distance** between **x** and **y** is denoted $d(\mathbf{x}, \mathbf{y})$ and is defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Note: We can also write this distance as follows.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



Note: *It is clear that* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (*the symmetric property*)

Example 7. Determine the distance between the points $\mathbf{x} = (1, -2, 3, 0)$ and $\mathbf{y} = (4, 0, -3, 5)$ in $\mathbf{R}^4$.

Solution

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(1-4)^2 + (-2-0)^2 + (3+3)^2 + (0-5)^2}$$

$$= \sqrt{9 + 4 + 36 + 25}$$

$$= \sqrt{74}$$