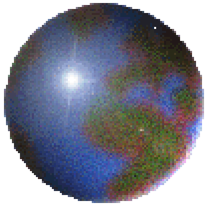
Linear Algebra



Chapter 4

VECTOR SPACES

Spanning Sets

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are said to **span** a vector space if every vector in the space can be expressed as a *linear combination* of these vectors.

In this case $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called a **spanning set**.

Spanning Sets

A set of vectors spans a space if every vector in that space can be written as a linear combination of the given vectors. For example, in R3 , if a set of three vectors spans $R3$, it means any vector (x,y,z) in $R3$ can be expressed as:

$$(x, y, z) = c_1 v_1 + c_2 v_2 + c_3 v_3$$

Spanning Sets

Do the vectors (1,0,1), (0,1,1), and (1,1,2) span $R3$?

Spanning Sets

Do the vectors (1,0,1), (0,1,1), and (1,1,2) span *R*3?

1. Form the matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

2. Perform Gaussian elimination:

- Subtract R_1 from R_3:

$$R_3 \to R_3 - R_1.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

- Subtract R_2 from R_3:

$$R_3 \to R_3 - R_2.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Do not span R3

Spanning Sets

Do the vectors (1,2,3),(4,5,6),(7,8,9) span $R3$?

Do not span R3

Spanning Sets

Do the vectors (1,0,1,2), (0,1,2,3), (1,1,3,5), and (0,0,1,1) span $R4$?

4.6 Linear Dependence and Independence

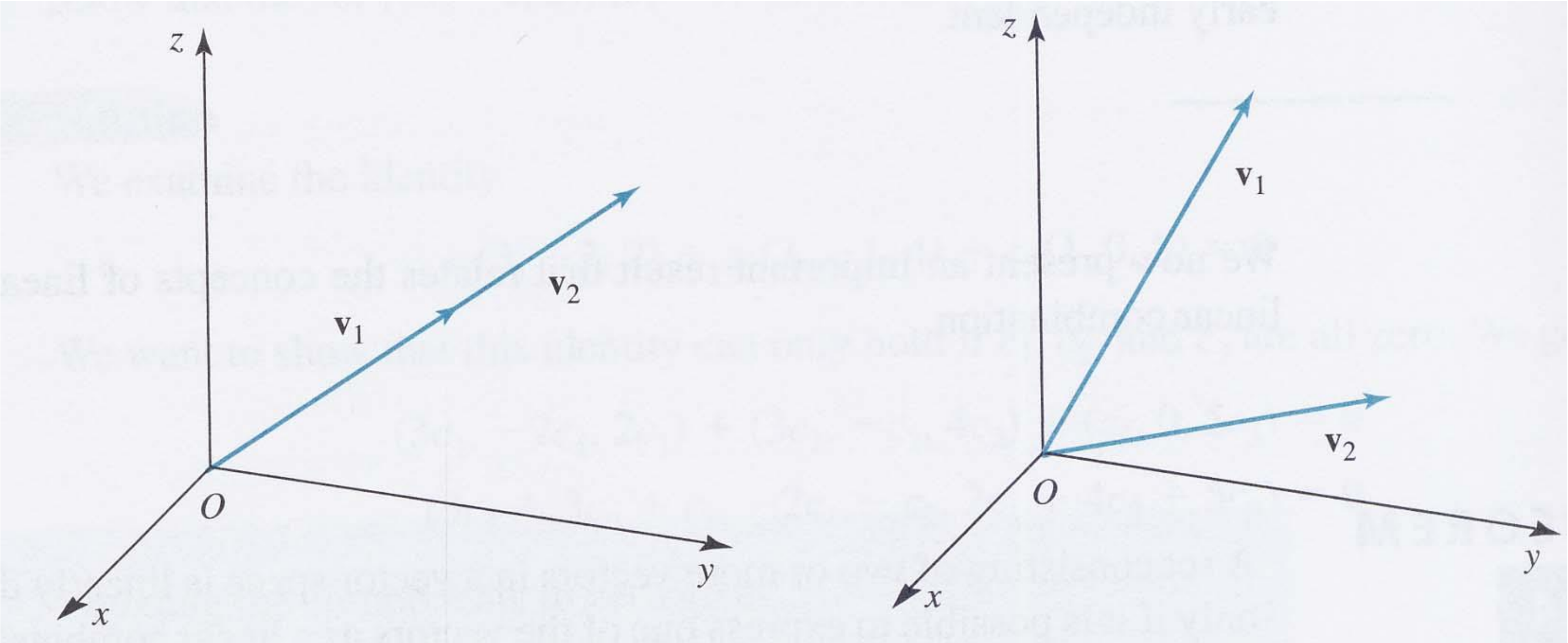
The concepts of dependence and independence of vectors are useful tools in constructing "efficient" spanning sets for vector spaces – sets in which there are no redundant vectors.

Definition

(a) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$

(b) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ is **linearly independent** if $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$ can only be satisfied when $c_1 = 0, \ldots, c_m = 0$.

Linear Dependence of $\{v_1, v_2\}$

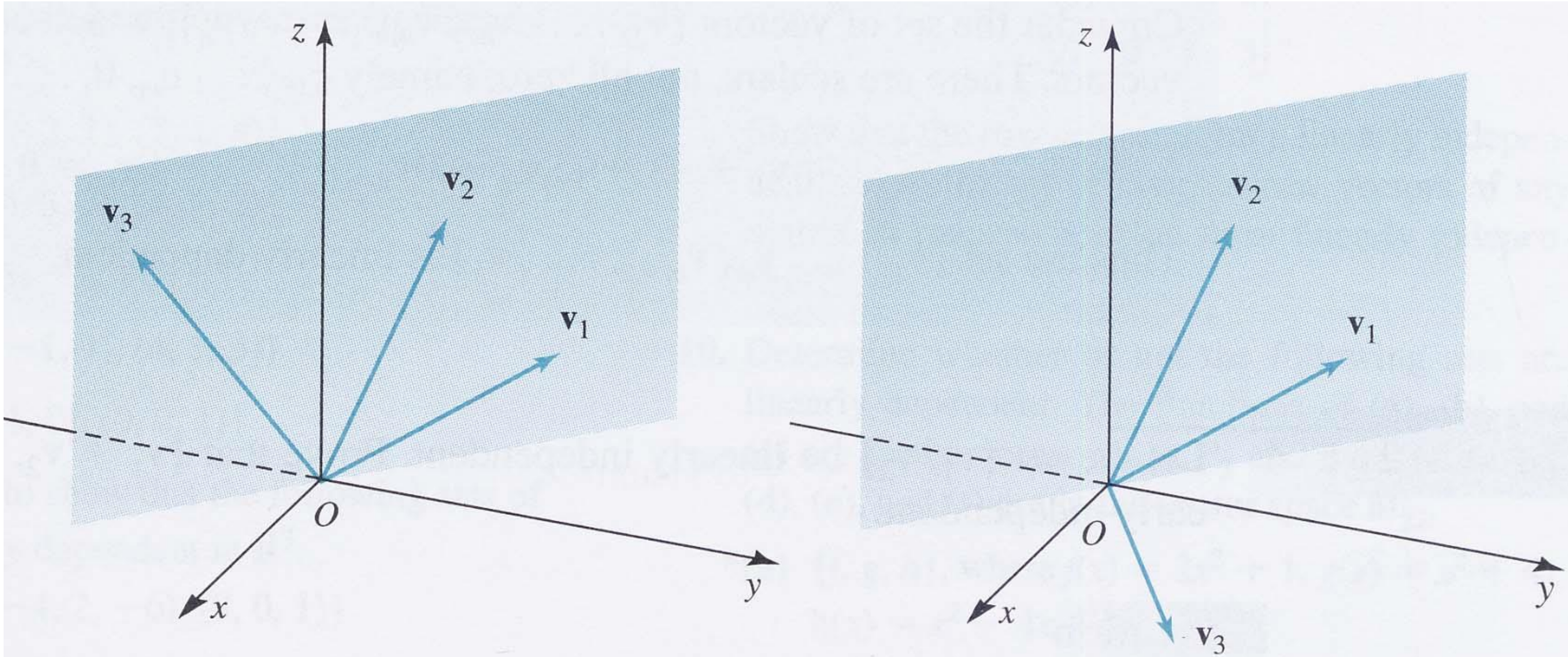


$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent; vectors lie on a line

$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent; vectors do not lie on a line

Figure 4.14 Linear dependence and independence of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbf{R}^3$.

Linear Dependence of $\{v_1, v_2, v_3\}$



$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent; vectors lie in a plane

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent; vectors do not lie in a plane

Figure 4.15 Linear dependence and independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbf{R}^3$.

Example 1

Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

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Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

Solution

Suppose $c_1(1,2,3)+c_2(-2,1,1)+c_3(8,6,10)=\mathbf{0}$

$$\Rightarrow \quad (c_1, 2c_1, 3c_1)+(-2c_2, c_2, c_2)+(8c_3, 6c_3, 10c_3)=\mathbf{0}$$

$$(c_1-2c_2+8c_3,\ 2c_1+c_2+6c_3,\ 3c_1+c_2+10c_3)=\mathbf{0}$$

$$\Rightarrow \begin{cases} c_1-2c_2+8c_3=0 \\ 2c_1+c_2+6c_3=0 \\ 3c_1+c_2+10c_3=0 \end{cases} \Rightarrow \quad \begin{aligned} c_1 &= 4 \\ c_2 &= -2 \\ c_3 &= -1 \end{aligned}$$

Thus $4(1,2,3)-2(-2,1,1)-(8,6,10)=\mathbf{0}$

The set of vectors is linearly dependent.

Example 2

Show that the set $\{(3, -2, 2), (3, -1, 4), (1, 0, 5)\}$ is linearly independent in $\mathbf{R}^3$.

Solution

Example 3

Consider the functions $f(x) = x^2 + 1$, $g(x) = 3x - 1$, $h(x) = -4x + 1$ of the vector space P_2 of polynomials of degree ≤ 2.

Show that the set of functions $\{ f, g, h \}$ is linearly independent.

Solution

Using reduce echelon form **approach**

Step 1: Represent the functions as vectors

As before, we represent the functions in the vector space P_2 (with the basis $\{x^2, x, 1\}$):

$$f(x) = x^2 + 1 \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad g(x) = 3x - 1 \quad \Rightarrow \quad \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad h(x) = -4x + 1 \quad \Rightarrow \quad \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}.$$

Thus, the corresponding matrix is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 1 & -1 & 1 \end{bmatrix}.$$

Step 2: Perform row reduction

We row reduce A to check if it has a pivot in each column.

1. **Write the initial matrix:**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 1 & -1 & 1 \end{bmatrix}.$$

Using reduce echelon form **approach**

2. **Eliminate the first element of the third row** (make the (3,1) entry 0) by subtracting the first row from the third row:

$$R_3 \to R_3 - R_1.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & -1 & 1 \end{bmatrix}.$$

3. **Normalize the second row** (make the (2,2) entry 1) by dividing the second row by 3:

$$R_2 \to \frac{R_2}{3}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix}.$$

4. **Eliminate the second element of the third row** (make the (3,2) entry 0) by adding the second row to the third row:

$$R_3 \to R_3 + R_2.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Using reduce echelon form **approach**

5. **Normalize the third row** (make the (3,3) entry 1) by multiplying the third row by 3:

$$R_3 \to 3R_3.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3: Interpret the row-reduced form

The row-reduced form of the matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix has a pivot in each column, indicating that the columns of A (and hence the functions f, g, h) are linearly independent.

Theorem 4.7

A set consisting of two or more vectors in a vector space is linearly dependent *if and only if* it is possible to express one of the vectors as a linearly combination of the other vectors.

Example 4

The set of vectors {$\mathbf{v}_1$=(1, 2, 1) , $\mathbf{v}_2$ =(-1, -1, 0) , $\mathbf{v}_3$ = (0, 1,1)}

is linearly dependent, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Thus, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

m

Theorem 4.8

Let V be a vector space. Any set of vectors in V that contains the zero is linearly dependent.

Proof

Consider the set { $\mathbf{0}$, $\mathbf{v}_2$, …, $\mathbf{v}_m$ }, which contains the zero vectors. Let us examine the identity

$$c_1\mathbf{0}+c_2\mathbf{v}_2+\cdots+c_m\mathbf{v}_m=\mathbf{0}$$

We see that the identity is true for $c_1 = 1$, $c_2 = 0$, …, $c_m = 0$ (not all zero).

Thus the set of vectors is linearly dependent, proving the theorem.

Theorem 4.9

Let the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be linearly dependent in a vector space V. Any set of vectors in V that contains these vectors will also be linearly dependent.

Example 5

The set of vectors

$\{\mathbf{v}_1=(1, 2, 1)$, $\mathbf{v}_2 =(-1, -1, 0)$, $\mathbf{v}_3 = (0, 1,1)$, $\mathbf{v}_4 =(1, 1, 1)\}$
is linearly dependent, since it contains the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ which are linearly dependent.

Definition

A finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for a vector space V if the set spans V and is linearly independent.

Standard Basis

The set of n vectors

$$\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 1)\}$$

is a basis for $\mathbf{R}^n$. This basis is called the **standard basis** for $\mathbf{R}^n$.

Example 1

Show that the set {(1, 0, –1), (1, 1, 1), (1, 2, 4)} is a basis for $\mathbf{R}^3$.

Solution

Example 1

Show that the set {(1, 0, –1), (1, 1, 1), (1, 2, 4)} is a basis for $\mathbf{R}^3$.

Solution

(span)

Let (x_1, x_2, x_3) be an arbitrary element of $\mathbf{R}^3$.

Suppose

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(1, 2, 4)$$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = x_1 \\ a_2 + 2a_3 = x_2 \\ -a_1 + a_2 + 4a_3 = x_3 \end{cases} \Rightarrow \begin{cases} a_1 = 2x_1 - 3x_2 + x_3 \\ a_2 = -2x_1 + 5x_2 - 2x_3 \\ a_3 = x_1 - 2x_2 + x_3 \end{cases}$$

Thus the set spans the space.

(linearly independent)

Consider the identity

$$b_1(1, 0, -1) + b_2(1, 1, 1) + b_3(1, 2, 4) = (0, 0, 0)$$

The identity leads to the system of equations

$$\begin{aligned} b_1 + b_2 + b_3 &= 0 \\ b_2 + 2b_3 &= 0 \\ -b_1 + b_2 + 4b_3 &= 0 \end{aligned}$$

$\Rightarrow$ $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$ is the unique solution.
Thus the set is linearly independent.

$\Rightarrow$ {(1, 0, –1), (1, 1, 1), (1, 2, 4)} spans $\mathbf{R}^3$ and is linearly independent.

$\Rightarrow$ It forms a basis for $\mathbf{R}^3$.

Example 2

Show that { f, g, h }, where $f(x) = x^2 + 1$, $g(x) = 3x - 1$, and $h(x) = -4x + 1$ is a basis for P_2.

Solution

(linearly independent) see Example 3 of the previous section.

(span). Let p be an arbitrary function in P_2.

p is thus a polynomial of the form

$$p(x) = bx^2 + cx + d$$

Suppose $p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$

for some scalars a_1, a_2, a_3.

This gives

$$bx^2 + cx + d = a_1(x^2 + 1) + a_2(3x - 1) + a_3(-4x + 1)$$

$$= a_1 x^2 + (3a_2 - 4a_3)x + (a_1 - a_2 + a_3)$$

Example 2

Show that { f, g, h }, where $f(x) = x^2 + 1$, $g(x) = 3x - 1$, and $h(x) = -4x + 1$ is a basis for P_2.

Solution

(linearly independent) see Example 3 of the previous section.

(span). Let p be an arbitrary function in P_2.

p is thus a polynomial of the form

Step 1: Represent the polynomials as vectors

In P_2, a general polynomial $p(x) = ax^2 + bx + c$ can be written as the vector (a, b, c). Using this representation:

- $f(x) = x^2 + 1 \quad \rightarrow \quad (1, 0, 1)$,
- $g(x) = 3x - 1 \quad \rightarrow \quad (0, 3, -1)$,
- $h(x) = -4x + 1 \quad \rightarrow \quad (0, -4, 1)$.

Thus, the set $\{f, g, h\}$ corresponds to the vectors:

$$\{(1, 0, 1), (0, 3, -1), (0, -4, 1)\}.$$

Step 2: Form a matrix with these vectors as rows

Construct the matrix A whose rows are the vectors:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -4 & 1 \end{bmatrix}.$$

Example 2

Show that { f, g, h }, where $f(x) = x^2 + 1$, $g(x) = 3x - 1$, and $h(x) = -4x + 1$ is a basis for P_2.

Solution

Step 3: Check for linear independence using row reduction

Perform Gaussian elimination to row reduce A:

1. Subtract 0 times R_1 from R_2 and R_3 (no changes):

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -4 & 1 \end{bmatrix}.$$

2. Divide R_2 by 3 to make the pivot 1:

$$R_2 \to \frac{R_2}{3}.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -4 & 1 \end{bmatrix}.$$

3. Add $4R_2$ to R_3:

$$R_3 \to R_3 + 4R_2.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

4. Multiply R_3 by 3 to make the pivot 1:

$$R_3 \to 3R_3.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Subtract R_3 from R_1 and add $\frac{1}{3}R_3$ to R_2:

$$R_1 \to R_1 - R_3, \quad R_2 \to R_2 + \frac{1}{3}R_3.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It spans p2, hence basis

$$\Rightarrow \begin{cases} a_1 = b \\ 3a_2 - 4a_3 = c \\ a_1 - a_2 + a_3 = d \end{cases} \Rightarrow \begin{cases} a_1 = b \\ a_2 = 4b - 4d - c \\ a_3 = 3b - 3d - c \end{cases}$$

Thus the polynomial *p* can be expressed

$$p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$$

The functions *f*, *g*, and *h* span P_2.

They form a basis for P_2.

Theorem 4.10

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.
If $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a set of more than n vectors in V, then this set is linearly dependent.

Proof

Suppose

$$c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m = 0 \qquad (1)$$

We will show that values of $c_1, \ldots, c_m$ are not all zero.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Thus each of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n \\ &\vdots \\ \mathbf{w}_m &= a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n \end{aligned}$$

Proof Excluded

Substituting for $\mathbf{w}_1, \ldots, \mathbf{w}_m$ into Equation (1) we get

$$c_1(a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n) + \cdots + c_m(a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n) = \mathbf{0}$$

Rearranging, we get

$$(c_1a_{11} + c_2a_{21} + \cdots + c_ma_{m1})\mathbf{v}_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ma_{mn})\mathbf{v}_n = \mathbf{0}$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linear independent,

$$\begin{gathered} a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m = 0 \\ \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m = 0 \end{gathered}$$

Since $m > n$, there are many solutions in this system.

Thus the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is linearly dependent.

Proof Excluded

Theorem 4.11

Any two basis for a vector space V consist of the same number of vectors.

Proof

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be two bases for V.

By Theorem 4.10,

$$m \leq n \text{ and } n \leq m$$

Thus $n = m$.

Proof Excluded

Definition

If a vector space V has a basis consisting of n vectors, then the **dimension** of V is said to be n. We write **dim(V)** for the dimension of V.

- V is **finite dimensional** if such a finite basis exists.
- V is **infinite dimensional** otherwise.

Example 3

Consider the set {{1, 2, 3), (-2, 4, 1)} of vectors in $\mathbf{R}^3$.
These vectors generate a subspace V of $\mathbf{R}^3$ consisting of all vectors of the form

$$\mathbf{v} = c_1(1, 2, 3) + c_2(-2, 4, 1)$$

The vectors (1, 2, 3) and (-2, 4, 1) span this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector, the vectors are linearly independent.
Therefore {{1, 2, 3), (-2, 4, 1)} is a basis for V.
Thus $\dim(V) = 2$.
We know that V is, in fact, a plane through the origin.

Theorem 4.12

(a) The origin is a subspace of $\mathbf{R}^3$. The dimension of this subspace is defined to be zero.

(b) The one-dimensional subspaces of $\mathbf{R}^3$ are lines through the origin.

(c) The two-dimensional subspaces of $\mathbf{R}^3$ are planes through the origin.

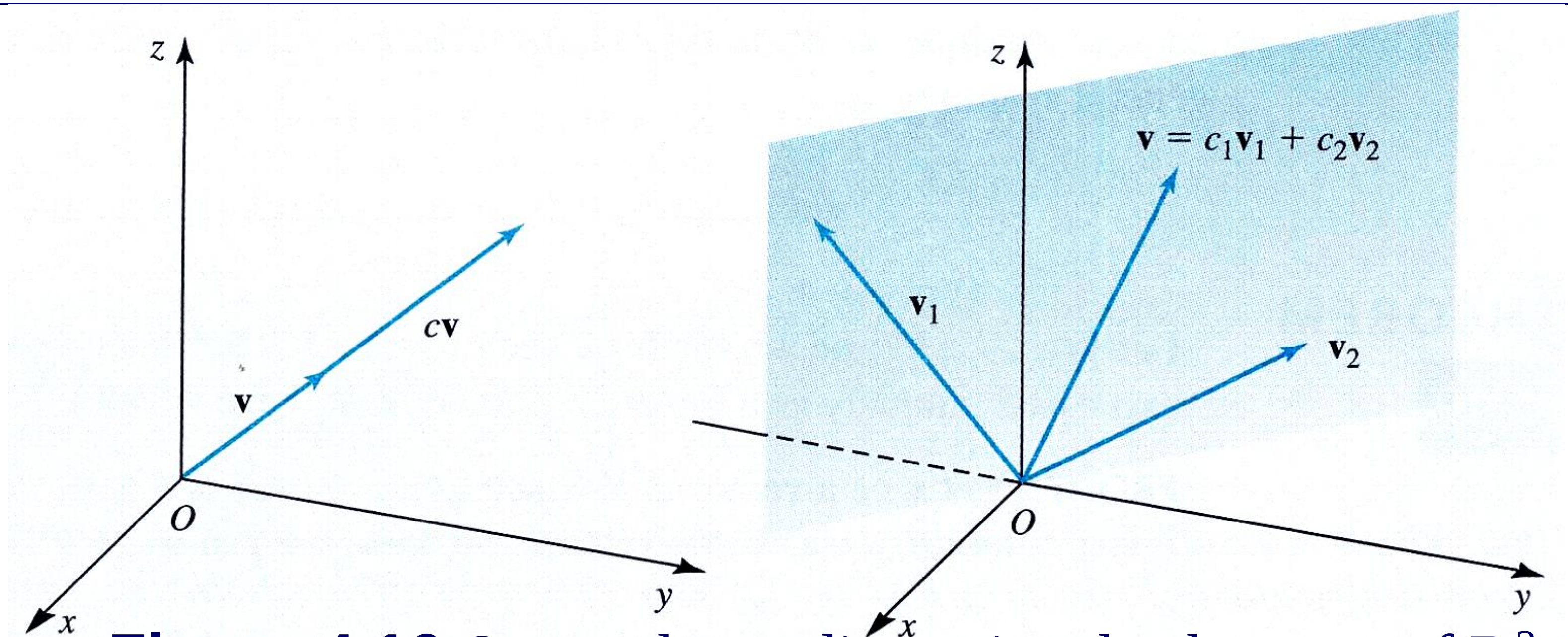

Figure 4.16 One and two-dimensional subspaces of $\mathbf{R}^3$

Theorem 4.13

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then each vector in V can be expressed uniquely as a linear combination of these vectors.

Proof

Let $\mathbf{v}$ be a vector in V. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, we can express $\mathbf{v}$ as a linear combination of these vectors. Suppose we can write

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \text{ and } \mathbf{v} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

Then

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

giving

$$(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{0}$$

Proof Excluded

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Thus $(a_1 - b_1) = 0, \ldots, (a_n - b_n) = 0$, implying that $a_1 = b_1, \ldots, a_n = b_n$. There is thus only one way of expressing $\mathbf{v}$ as a linear combination of the basis.

Theorem 4.14

If a vector space is known to be of dimension n. The following theorem tells us that we do not have to check both the linear dependence and spanning conditions to see if a given set is a basis.

Let V be a vector space of dimension n.

(a) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of n linearly independent vectors in V, then S is a basis for V.

(b) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of n vectors V that spans V, then S is a basis for V.

Example 4

Prove that the set B={(1, 3, -1), (2, 1, 0), (4, 2, 1)} is a basis for $\mathbf{R}^3$.

Solution

- The dimension of R3 is three.
- The number of vector given above are 3, so, we have the correct number of vectors for a basis.
- We can check either span or linear independence to find whether this set is basis for R3
- Let us check for linear independence. Suppose

Example 4

Prove that the set B={(1, 3, -1), (2, 1, 0), (4, 2, 1)} is a basis for $\mathbf{R}^3$.

Solution

Let us check for linear independence. Suppose

$$c_1(1,3,-1)+c_2(2,1,0)+c_3(4,2,1)=(0,0,0)$$

This identity leads to the system of equations

$$\begin{aligned} c_1+2c_2+4c_3&=0\\ 3c_1+c_2+2c_3&=0\\ -c_1+c_3&=0 \end{aligned}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Thus the vectors are linearly independent.
The set {(1, 3, -1), (2, 1, 0), (4, 2, 1)} is therefore a basis for $\mathbf{R}^3$.

Theorem 4.14

The following theorem tells us that any linearly independent set of vectors can be extended to give a basis for a vector space.

Let V be a vector space, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of vectors in V.

(a) $\dim(V) = |S|$.

(b) S is a linearly independent set.

(c) S spans V.

$\left.\right\}$ S is a basis of V.

Example 5

State (with a brief explanation) whether the following statements are true or false.

(a) The vectors (1, 2), (–1, 3), (5, 2) are linearly dependent in $\mathbf{R}^2$.

(b) The vectors (1, 0, 0), (0, 2, 0), (1, 2, 0) span $\mathbf{R}^3$.

(c) {(1, 0, 2), (0, 1, -3)} is a basis for the subspace of $\mathbf{R}^3$ consisting of vectors of the form $(a, b, 2a - 3b)$.

(d) Any set of two vectors can be used to generate a two-dimensional subspace of $\mathbf{R}^3$.

Solution