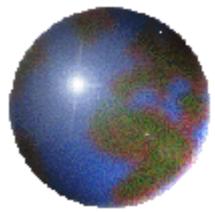


# Linear Algebra



## *Chapter 3*

### DETERMINANTS



## 3.1 Introduction to Determinants

### Definition

The **determinant** of a  $2 \times 2$  matrix  $A$  is denoted  $|A|$  and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a  $2 \times 2$  matrix is given by *the difference of the products of the two diagonals* of the matrix.

The notation **det(A)** is also used for the determinant of  $A$ .



## 3.1 *Introduction to Determinants*

### Applications

- Solvability of Linear Systems
- Matrix Inversion
- Cramer's Rule
- Eigenvalues and Eigenvectors
- Linear Independence
- Computation of Cross Products



## 3.1 Introduction to Determinants

**Example 1: Find determinant of the following matrix**

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

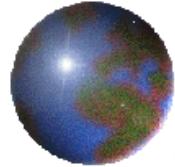


## 3.1 Introduction to Determinants

### Example 1

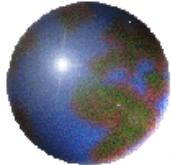
$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$



## 3.1 Introduction to Determinants

- The determinant of a  $3 \times 3$  matrix is defined in terms of determinants of  $2 \times 2$  matrices.
- The determinant of a  $4 \times 4$  matrix is defined in terms of determinants of  $3 \times 3$  matrices, and so on.
- For these definitions we need the following concepts of minor and cofactor.



## Definition

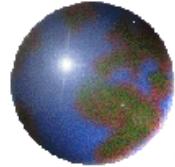
Let  $A$  be a square matrix.

The **minor** of the element  $a_{ij}$  is denoted by  $M_{ij}$  and is the determinant of the matrix that remains after deleting row  $i$  and column  $j$  of  $A$ .

The **cofactor** of  $a_{ij}$  is denoted  $C_{ij}$  and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that  $C_{ij} = M_{ij}$  or  $-M_{ij}$ .



## Example 2

Determine the minors and cofactors of the elements  $a_{11}$  and  $a_{32}$  of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

### Solution



## Example 2

Determine the minors and cofactors of the elements  $a_{11}$  and  $a_{32}$  of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

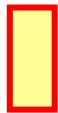
### Solution

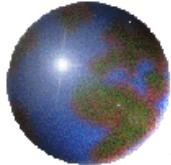
Minor of  $a_{11}$  :  $M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times (-2)) = 3$

Cofactor of  $a_{11}$  :  $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

Minor of  $a_{32}$  :  $M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1 \times 2) - (3 \times 4) = -10$

Cofactor of  $a_{32}$  :  $C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-10) = 10$





## Definition

The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

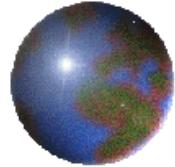
$$\text{If } A \text{ is } 3 \times 3, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A \text{ is } 4 \times 4, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

⋮

$$\text{If } A \text{ is } n \times n, |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

These equations are called **cofactor expansions** of  $|A|$ .

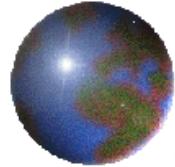


## Example 3

Evaluate the determinant of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

**Solution**



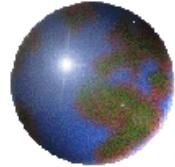
## Example 3

Evaluate the determinant of the following matrix  $A$ .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

### Solution

$$\begin{aligned}|A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\&= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\&= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)] \\&= -2 + 2 - 6 \\&= -6\end{aligned}$$



# Theorem 3.1

The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

*i*th row expansion:  $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

*j*th column expansion:  $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

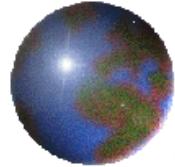
## Example 4

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

### Solution

$$\begin{aligned}|A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\&= -3 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} \\&= -3[(2 \times 1) - (-1 \times 2)] + 0[(1 \times 1) - (-1 \times 4)] - 1[(1 \times 2) - (2 \times 4)] \\&= -12 + 0 + 6 = -6\end{aligned}$$

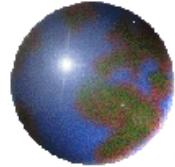


## Example 5

Evaluate the determinant of the following  $4 \times 4$  matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

**Solution**



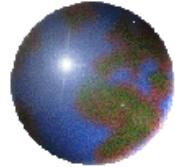
## Example 5

Evaluate the determinant of the following  $4 \times 4$  matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

### Solution

$$\begin{aligned}|A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \\&= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43}) \\&= 3 \begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\&= 3(2) \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3 - 2) = 6\end{aligned}$$

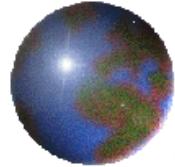


## Example 6

Solve the following equation for the variable  $x$ .

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

### Solution



## Example 6

Solve the following equation for the variable  $x$ .

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

### Solution

Expand the determinant to get the equation

$$x(x-2) - (x+1)(-1) = 7$$

Proceed to simplify this equation and solve for  $x$  (*factorization*).

$$x^2 - 2x + x + 1 = 7$$

$$x^2 - x - 6 = 0$$

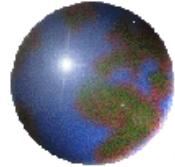
$$(x+2)(x-3) = 0$$

Use the quadratic formula

$$x = -2 \text{ or } 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are two solutions to this equation,  $x = -2$  or  $3$ .



# Computing Determinants of $2 \times 2$ and $3 \times 3$ Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

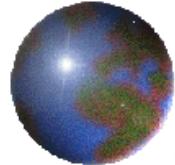
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)



# Computing Determinants of $2 \times 2$ and $3 \times 3$ Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} \text{red} \\ \text{blue} \end{array} \begin{array}{l} \text{red} \\ \text{blue} \end{array} \begin{array}{l} a_{11} \\ a_{21} \\ a_{31} \end{array} \begin{array}{l} a_{12} \\ a_{22} \\ a_{32} \end{array} \begin{array}{l} a_{13} \\ a_{23} \\ a_{33} \end{array} \begin{array}{l} a_{11} \\ a_{21} \\ a_{31} \end{array} \begin{array}{l} a_{12} \\ a_{22} \\ a_{32} \end{array}$$

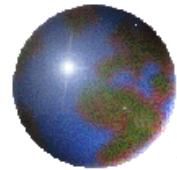
$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

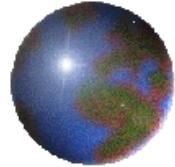
(diagonal products from right to left)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$



# Homework Practice

- ➊ Exercise 3.1 pages 161-162:  
1, 3, 5, 7, 9, 11, 13



## 3.2 Properties of Determinants

### Theorem 3.2

Let  $A$  be an  $n \times n$  matrix and  $c$  be a nonzero scalar.

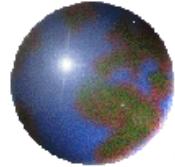
- (a) If a matrix  $B$  is obtained from  $A$  by multiplying the elements of a row (column) by  $c$  then  $|B| = c|A|$ .
- (b) If a matrix  $B$  is obtained from  $A$  by interchanging two rows (columns) then  $|B| = -|A|$ .
- (c) If a matrix  $B$  is obtained from  $A$  by adding a multiple of one row (column) to another row (column), then  $|B| = |A|$ .

#### Proof (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn}$$

$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \dots + ca_{kn}C_{kn}$$

$$\therefore |B| = c|A|.$$

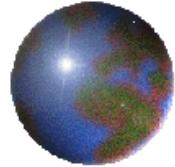


## Example 1

Evaluate the determinant

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$$

### Solution

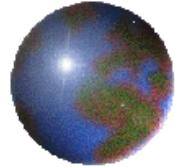


## Example 1

Evaluate the determinant  $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$ .

### Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \stackrel{C_2 + 2C_3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3) \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$



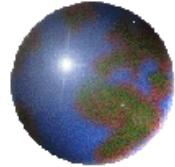
## Example 2

If  $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$ ,  $|A| = 12$  is known.

Evaluate the determinants of the following matrices.

$$(a) B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix} (b) B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix} (c) B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$$

## Solution



## Example 2

If  $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$ ,  $|A| = 12$  is known.

Evaluate the determinants of the following matrices.

$$(a) B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix} (b) B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix} (c) B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$$

### Solution

(a)  $A \underset{3C2}{\approx} B_1$  Thus  $|B_1| = 3|A| = 36$ .

(b)  $A \underset{R2 \leftrightarrow R3}{\approx} B_2$  Thus  $|B_2| = -|A| = -12$ .

(c)  $A \underset{R3+2R1}{\approx} B_3$  Thus  $|B_3| = |A| = 12$ .



## Definition

A square matrix  $A$  is said to be **singular** if  $|A|=0$ .

$A$  is **nonsingular** if  $|A|\neq 0$ .

### Theorem 3.3

Let  $A$  be a square matrix.  $A$  is singular if

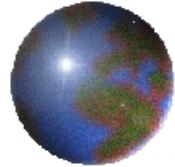
- (a) all the elements of a row (column) are zero.
- (b) two rows (columns) are equal.
- (c) two rows (columns) are proportional. (i.e.,  $Ri=cRj$ )

### Proof

- (a) Let all elements of the  $k$ th row of  $A$  be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

- (c) If  $Ri=cRj$ , then  $A \underset{Ri=cRj}{\approx} B$  , row  $i$  of  $B$  is  $[0 \ 0 \ \dots \ 0]$ .  
 $\Rightarrow |A|=|B|=0$

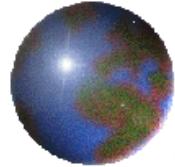


## Example 3

Show that the following matrices are singular.

$$(a) A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b) B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

### Solution

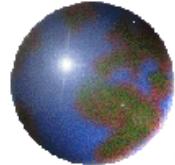


## Example 3

Show that the following matrices are singular.

$$(a) A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad (b) B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

### Solution



# Theorem 3.4

Let  $A$  and  $B$  be  $n \times n$  matrices and  $c$  be a nonzero scalar.

- (a)  $|cA| = c^n|A|$ .
- (b)  $|AB| = |A||B|$ .
- (c)  $|A^t| = |A|$ .
- (d)  $|A^{-1}| = \frac{1}{|A|}$  (assuming  $A^{-1}$  exists)

## Proof

- (a)** Each row of  $cA$  is a row of  $A$  multiplied by  $c$ . Apply Theorem 3.2(a) to each of the  $n$  rows of  $cA$  to get  $|cA| = c^n |A|$ .
- (d)**  $AA^{-1} = I_n$ . Thus  $|AA^{-1}| = |I_n|$ .  $|A||A^{-1}| = 1$ , by Part (b). Therefore  $|A| \neq 0$ , and  $|A^{-1}| = \frac{1}{|A|}$ .