

## Example 3

Consider the set  $\{(1, 2, 3), (-2, 4, 1)\}$  of vectors in  $\mathbf{R}^3$ . These vectors generate a subspace  $V$  of  $\mathbf{R}^3$  consisting of all vectors of the form

$$\mathbf{v} = c_1(1, 2, 3) + c_2(-2, 4, 1)$$

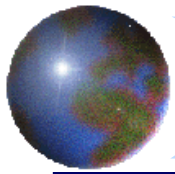
The vectors  $(1, 2, 3)$  and  $(-2, 4, 1)$  **span** this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector, the vectors are **linearly independent**.

Therefore  $\{(1, 2, 3), (-2, 4, 1)\}$  is a **basis** for  $V$ .

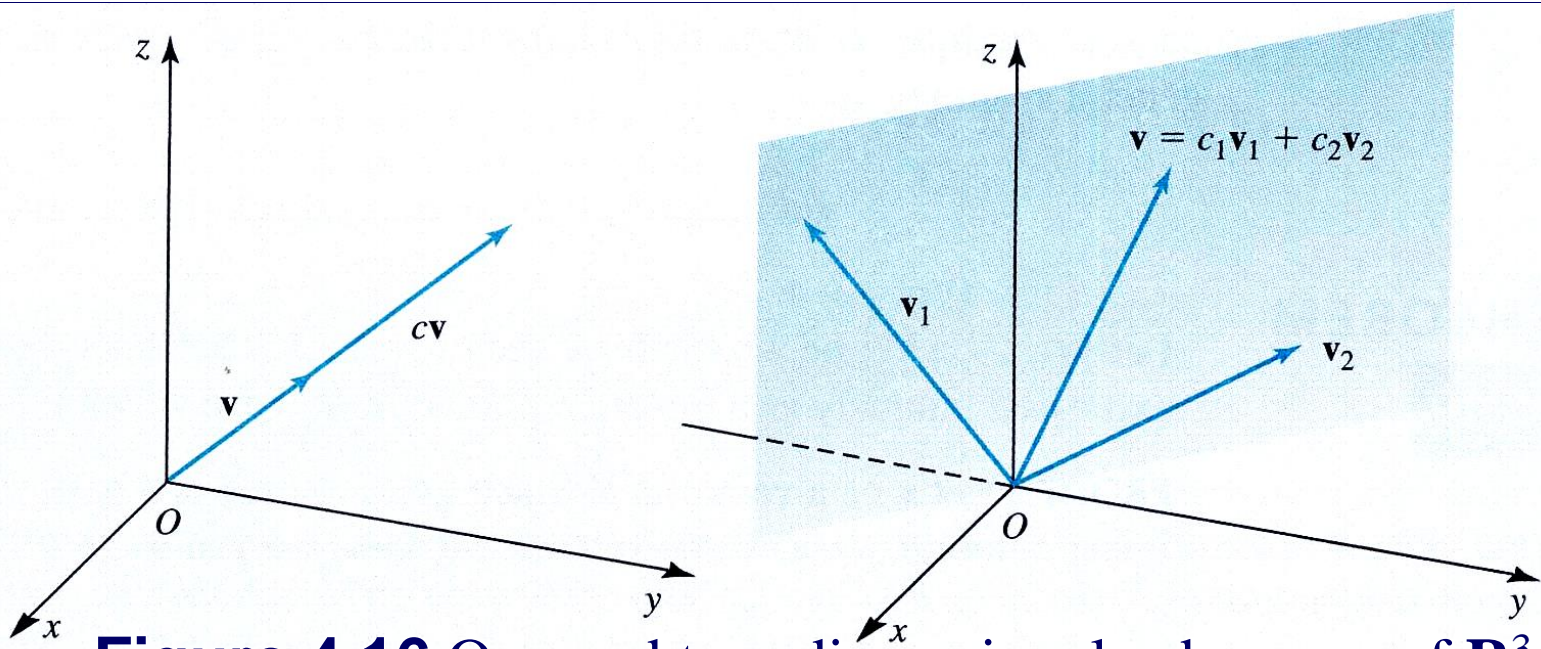
Thus  $\dim(V) = 2$ .

We know that  $V$  is, in fact, a plane through the origin.

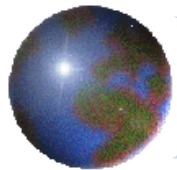


## Theorem 4.12

- (a) The origin is a subspace of  $\mathbf{R}^3$ . The dimension of this subspace is defined to be zero.
- (b) The one-dimensional subspaces of  $\mathbf{R}^3$  are lines through the origin.
- (c) The two-dimensional subspaces of  $\mathbf{R}^3$  are planes through the origin.



**Figure 4.16** One and two-dimensional subspaces of  $\mathbf{R}^3$



## Theorem 4.13

Let  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis for a vector space  $V$ . Then each vector in  $V$  can be expressed **uniquely** as a linear combination of these vectors.

### Proof

Let  $\mathbf{v}$  be a vector in  $V$ . Since  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis, we can express  $\mathbf{v}$  as a linear combination of these vectors. Suppose we can write

$$\mathbf{v} = a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n \text{ and } \mathbf{v} = b_1 \mathbf{v}_1 + \dots + b_n \mathbf{v}_n$$

Then

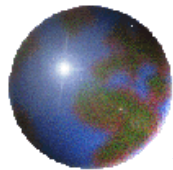
$$a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n = b_1 \mathbf{v}_1 + \dots + b_n \mathbf{v}_n$$

giving

$$(a_1 - b_1) \mathbf{v}_1 + \dots + (a_n - b_n) \mathbf{v}_n = \mathbf{0}$$

Since  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis, the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly independent. Thus  $(a_1 - b_1) = 0, \dots, (a_n - b_n) = 0$ , implying that  $a_1 = b_1, \dots, a_n = b_n$ . There is thus only one way of expressing  $\mathbf{v}$  as a linear combination of the basis.

**Proof Excluded**

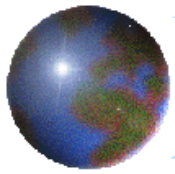


## Theorem 4.14

If a vector space is known to be of dimension  $n$ . The following theorem tells us that we do not have to check both the linear dependence and spanning conditions to see if a given set is a basis.

Let  $V$  be a vector space of dimension  $n$ .

- (a) If  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a set of  $n$  linearly independent vectors in  $V$ , then  $S$  is a basis for  $V$ .
- (b) If  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a set of  $n$  vectors  $V$  that spans  $V$ , then  $S$  is a basis for  $V$ .

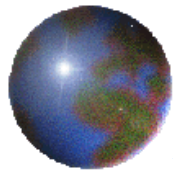


## Example 4

Prove that the set  $B = \{(1, 3, -1), (2, 1, 0), (4, 2, 1)\}$  is a basis for  $\mathbf{R}^3$ .

### Solution

- The dimension of  $\mathbf{R}^3$  is three.
- The number of vector given above are 3, so, we have the correct number of vectors for a basis.
- We can check either span or linear independence to find whether this set is basis for  $\mathbf{R}^3$
- Let us check for linear independence. Suppose



## Example 4

Prove that the set  $B = \{(1, 3, -1), (2, 1, 0), (4, 2, 1)\}$  is a basis for  $\mathbf{R}^3$ .

### Solution

Let us check for **linear independence**. Suppose

$$c_1(1, 3, -1) + c_2(2, 1, 0) + c_3(4, 2, 1) = (0, 0, 0)$$

This identity leads to the system of equations

$$c_1 + 2c_2 + 4c_3 = 0$$

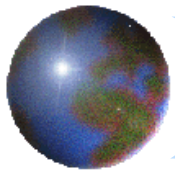
$$3c_1 + c_2 + 2c_3 = 0$$

$$-c_1 + c_3 = 0$$

This system has the unique solution  $c_1 = 0, c_2 = 0, c_3 = 0$ .

Thus the vectors are linearly independent.

The set  $\{(1, 3, -1), (2, 1, 0), (4, 2, 1)\}$  is therefore a basis for  $\mathbf{R}^3$ .



## Theorem 4.14

The following theorem tells us that any linearly independent set of vectors can be extended to give a basis for a vector space.

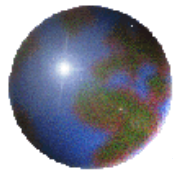
Let  $V$  be a vector space,  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a set of vectors in  $V$ .

(a)  $\dim(V) = |S|$ .

(b)  $S$  is a linearly independent set.

(c)  $S$  spans  $V$ .

}  $S$  is a basis of  $V$ .

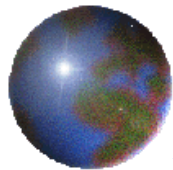


## Example 5

State (with a brief explanation) whether the following statements are true or false.

- (a) The vectors  $(1, 2)$ ,  $(-1, 3)$ ,  $(5, 2)$  are linearly dependent in  $\mathbf{R}^2$ .
- (b) The vectors  $(1, 0, 0)$ ,  $(0, 2, 0)$ ,  $(1, 2, 0)$  span  $\mathbf{R}^3$ .
- (c)  $\{(1, 0, 2), (0, 1, -3)\}$  is a basis for the subspace of  $\mathbf{R}^3$  consisting of vectors of the form  $(a, b, 2a - 3b)$ .
- (d) Any set of two vectors can be used to generate a two-dimensional subspace of  $\mathbf{R}^3$ .

**Solution**



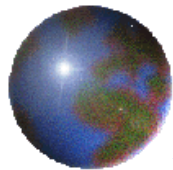
## Example 5

State (with a brief explanation) whether the following statements are true or false.

(a) The vectors  $(1, 2)$ ,  $(-1, 3)$ ,  $(5, 2)$  are linearly dependent in  $\mathbf{R}^2$ .

### Solution

(a) True: The dimension of  $\mathbf{R}^2$  is two. Thus any three vectors are linearly dependent.



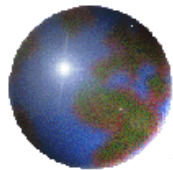
## Example 5

State (with a brief explanation) whether the following statements are true or false.

(b) The vectors  $(1, 0, 0)$ ,  $(0, 2, 0)$ ,  $(1, 2, 0)$  span  $\mathbf{R}^3$ .

### Solution

b. False: The three vectors are linearly dependent. Thus they cannot span a three-dimensional space.



## Example 5

State (with a brief explanation) whether the following statements are true or false.

(c)  $\{(1, 0, 2), (0, 1, -3)\}$  is a basis for the subspace of  $\mathbf{R}^3$  consisting of vectors of the form  $(a, b, 2a - 3b)$ .

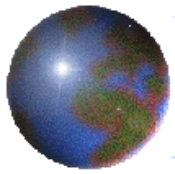
Check subspace of  $\mathbf{R}^3$  (close under addition and multiplication)

### Solution

(c) True: The vectors span the subspace since

$$(a, b, 2a - 3b) = a(1, 0, 2) + b(0, 1, -3)$$

The vectors are also linearly independent since they are not colinear.



## *Example 5*

State (with a brief explanation) whether the following statements are true or false.

(d) Any set of two vectors can be used to generate a two-dimensional subspace of  $\mathbf{R}^3$ .

### **Solution**

(d) False: The two vectors must be linearly independent.