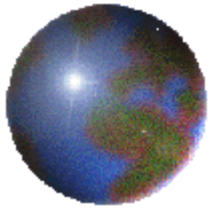


Linear Algebra



Chapter 2

MATRICES



2.1 Addition, Scalar Multiplication, and Multiplication of Matrices

- a_{ij} : the element of matrix A in row i and column j .
- For a square $n \times n$ matrix A , the **main diagonal** is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

Two matrices are **equal** if they are of the same size and if their corresponding elements are equal.

Thus $A = B$ if $a_{ij} = b_{ij} \quad \forall i, j$.

(\forall for every, for all)



Addition of Matrices

Definition

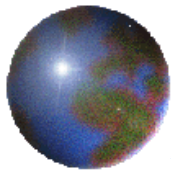
Let A and B be matrices of the same size.

Their **sum** $A + B$ is the matrix obtained by adding together the corresponding elements of A and B .

The matrix $A + B$ will be of the same size as A and B .

If A and B are not of the same size, they cannot be added, and we say that **the sum does not exist**.

Thus if $C = A + B$, then $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$.

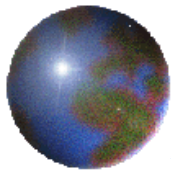


Example 1

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}$.

Determine $A + B$ and $A + C$, if the sum exist.

Solution



Example 1

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}$.

Determine $A + B$ and $A + C$, if the sum exist.

Solution

$$\begin{aligned} (1) \ A + B &= \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 4+5 & 7-6 \\ 0-3 & -2+1 & 3+8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 9 & 1 \\ -3 & -1 & 11 \end{bmatrix}. \end{aligned}$$

(2) Because A is 2×3 matrix and C is a 2×2 matrix, they are not of the same size, $A + C$ does not exist.



Scalar Multiplication of matrices

Definition

Let A be a matrix and c be a scalar. The **scalar multiple** of A by c , denoted cA , is the matrix obtained by multiplying every element of A by c . The matrix cA will be the same size as A .

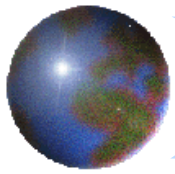
Thus if $B = cA$, then $b_{ij} = ca_{ij} \forall i, j$.

Example 2

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 4 \\ 7 & -3 & 0 \end{bmatrix}.$$

$$3A = \begin{bmatrix} 3 \times 1 & 3 \times (-2) & 3 \times 4 \\ 3 \times 7 & 3 \times (-3) & 3 \times 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 21 & -9 & 0 \end{bmatrix}.$$

Observe that A and $3A$ are both 2×3 matrices.



Negation and Subtraction

Definition

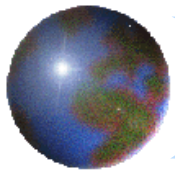
We now define subtraction of matrices in such a way that makes it compatible with addition, scalar multiplication, and negative. Let

$$A - B = A + (-1)B$$

Example 3

Suppose $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -1 \\ 0 & 4 & 6 \end{bmatrix}$.

$$A - B = \begin{bmatrix} 5-2 & 0-8 & -2-(-1) \\ 3-0 & 6-4 & -5-6 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1 \\ 3 & 2 & -11 \end{bmatrix}.$$



Multiplication of Matrices

Definition

Let the number of columns in a matrix A be the same as the number of rows in a matrix B . The product AB then exists.

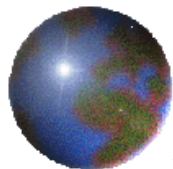
Let $A: m \times n$ matrix, $B: n \times k$ matrix,

The **product matrix** $C=AB$ has elements

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

C is a $m \times k$ matrix.

If the number of columns in A does not equal the number of row B , we say that **the product does not exist**.



Example 4

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 6 & -2 & 5 \end{bmatrix}.$$

Determine AB , BA , and AC , if the products exist.

Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (1 \times 5) + (3 \times 3) & (1 \times 0) + (3 \times (-2)) & (1 \times 1) + (3 \times 6) \\ (2 \times 5) + (0 \times 3) & (2 \times 0) + (0 \times (-2)) & (2 \times 1) + (0 \times 6) \end{bmatrix} \\ &= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}. \end{aligned}$$

BA and AC do not exist. **Note.** In general, $AB \neq BA$.



Example 5

Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$. Determine AB .

$$AB = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} [2 \ 1] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [2 \ 1] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [7 \ 0] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [7 \ 0] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ [-3 \ -2] \begin{bmatrix} -1 \\ 3 \end{bmatrix} & [-3 \ -2] \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 0+5 \\ -7+0 & 0+0 \\ 3-6 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -7 & 0 \\ -3 & -10 \end{bmatrix}$$

Example 6

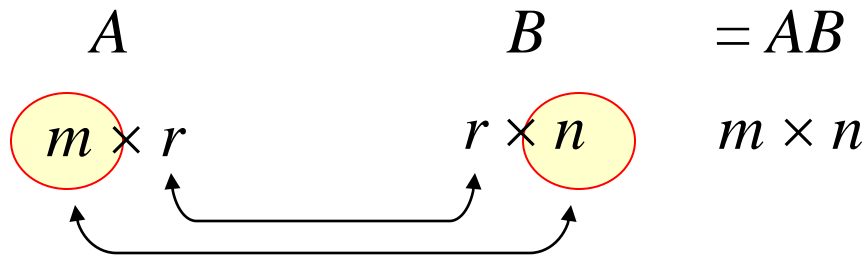
Let $C = AB$, $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$. Determine c_{23} .

$$c_{23} = [-3 \ 4] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$



Size of a Product Matrix

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.



Example 7

If A is a 5×6 matrix and B is an 6×7 matrix.

Because A has six columns and B has six rows. Thus AB exists.

And AB will be a 5×7 matrix.



Special Matrices

Definition

A **zero matrix** is a matrix in which all the elements are zeros.

A **diagonal matrix** is a square matrix in which all the elements not on the main diagonal are zeros.

An **identity matrix** is a diagonal matrix in which every diagonal element is 1.

$$O_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

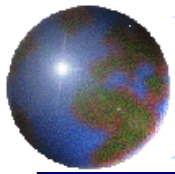
zero matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

diagonal matrix A

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

identity matrix



Theorem 2.1

Let A be $m \times n$ matrix and O_{mn} be the zero $m \times n$ matrix. Let B be an $n \times n$ square matrix. O_n and I_n be the zero and identity $n \times n$ matrices. Then

$$A + O_{mn} = O_{mn} + A = A$$

$$BO_n = O_n B = O_n$$

$$BI_n = I_n B = B$$

Example 8

Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$.

$$A + O_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} = A$$

$$BO_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$



Homework

✚ Exercises: 1, 3, 5, 9, 10, 11 from page 71 to 73.

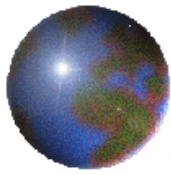
Exercise 17 p.72

Let A be a matrix whose third row is all zeros. Let B be any matrix such that the product AB exists.

Prove that the third row of AB is all zeros.

Solution

$$(AB)_{3i} = [a_{31} \ a_{32} \ \cdots \ a_{3n}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = [0 \ 0 \ \cdots \ 0] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = 0, \forall i.$$



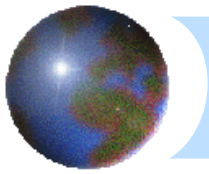
2.2 Algebraic Properties of Matrix Operations

Theorem 2.2 -1

Let A , B , and C be matrices and a , b , and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Addition and scalar Multiplication

1. $A + B = B + A$ *Commutative property of addition*
2. $A + (B + C) = (A + B) + C$ *Associative property of addition*
3. $A + O = O + A = A$ *(where O is the appropriate zero matrix)*
4. $c(A + B) = cA + cB$ *Distributive property of addition*
5. $(a + b)C = aC + bC$ *Distributive property of addition*
6. $(ab)C = a(bC)$



Theorem 2.2 -2

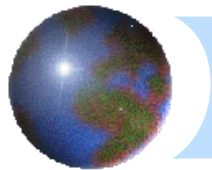
Let A , B , and C be matrices and a , b , and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Multiplication

- | | |
|----------------------------|--|
| 1. $A(BC) = (AB)C$ | <i>Associative property of multiplication</i> |
| 2. $A(B + C) = AB + AC$ | <i>Distributive property of multiplication</i> |
| 3. $(A + B)C = AC + BC$ | <i>Distributive property of multiplication</i> |
| 4. $AI_n = I_nA = A$ | <i>(where I_n is the appropriate identity matrix)</i> |
| 5. $c(AB) = (cA)B = A(cB)$ | |

Note: $AB \neq BA$ in general.

Multiplication of matrices is not commutative.



Proof of Theorem 2.2 ($A+B=B+A$)

Consider the (i,j) th elements of matrices $A+B$ and $B+A$:

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

$$\therefore A+B=B+A$$

Example 9

$$\text{Let } A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}.$$

$$\begin{aligned} A+B+C &= \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}. \end{aligned}$$

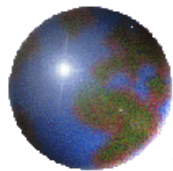


Arithmetic Operations

If A is an $m \times r$ matrix and B is $r \times n$ matrix, the number of scalar multiplications involved in computing the product AB is mnr .

Consider three matrices A , B and C such that the product ABC exists.

Compare the number of multiplications involved in the two ways $(AB)C$ and $A(BC)$ of computing the product ABC



Example 10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. Compute ABC .

Solution.

(1) $(AB)C$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

(2) $A(BC)$

$$BC = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

Which method is better?

Count the number of multiplications.

$$\begin{aligned} &2 \times 6 + 3 \times 2 \\ &= 12 + 6 = 18 \end{aligned}$$

$$\begin{aligned} &3 \times 2 + 2 \times 2 \\ &= 6 + 4 = 10 \end{aligned}$$

$\therefore A(BC)$ is better.



Caution

In algebra we know that the following cancellation laws apply.

⊕ If $ab = ac$ and $a \neq 0$ then $b = c$.

⊕ If $pq = 0$ then $p = 0$ or $q = 0$.

However the corresponding results are not true for matrices.

⊕ $AB = AC$ **does not imply** that $B = C$.

⊕ $PQ = O$ **does not imply** that $P = O$ or $Q = O$.

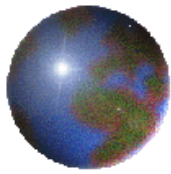
Example 11

(1) Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$.

Observe that $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$.

(2) Consider the matrices $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$.

Observe that $PQ = O$, but $P \neq O$ and $Q \neq O$.



Powers of Matrices

Definition

If A is a square matrix, then

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Theorem 2.3

If A is an $n \times n$ square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.
3. $A^0 = I_n$ (by definition)



Example 12

If $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4 .

Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB$$

Solution



Example 12

If $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4 .

Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB$$

Solution

$$\begin{aligned} & A(A + 2B) + 3B(2A - B) - A^2 + 7B^2 - 5AB \\ &= A^2 + 2AB + 6BA - 3B^2 - A^2 + 7B^2 - 5AB \\ &= \underline{-3AB + 6BA} + 4B^2 \end{aligned}$$

We can't add the two matrices !



Systems of Linear Equations

A system of m linear equations in n variables as follows

$$\begin{array}{cccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$



Systems of Linear Equations

A system of m linear equations in n variables as follows

$$\begin{array}{cccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$***AX = B***$$



Idempotent and Nilpotent Matrices

(Exercises 24-30 p.83 and 14 p.93)

Definition

- (1) A square matrix A is said to be **idempotent** if $A^2=A$.
- (2) A square matrix A is said to **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

$$(1) A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 =$$

$$(2) B = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 =$$



Idempotent and Nilpotent Matrices

(Exercises 24-30 p.83 and 14 p.93)

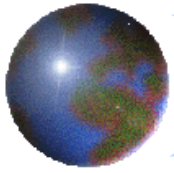
Definition

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- (2) A square matrix A is said to **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

$$(1) A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = A.$$

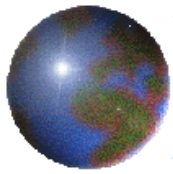
$$(2) B = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The degree of nilpotency : 2}$$



Class activity

The following matrix is nilpotent. Find the degree of **nilpotency**

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



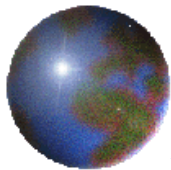
Quiz 1

Solve the following systems of linear equations by Gaussian elimination method:

$$2x - 2y + 3z = 2$$

$$x + 2y - z = 3$$

$$3x - y + 2z = 1$$



Quiz 1

Solve the following systems of linear equations by Gaussian elimination method:

$$2x - 2y + 3z = 2$$

$$x + 2y - z = 3$$

$$3x - y + 2z = 1$$

SOLUTION

(i) $2x - 2y + 3z = 2$, $x + 2y - z = 3$, $3x - y + 2z = 1$

The given system equations is $2x - 2y + 3z = 2$ -----(1)

$$x + 2y - z = 3 \quad \text{-----(2)}$$

$$3x - y + 2z = 1 \quad \text{-----(3)}$$

The augmented matrix of the above system is $[A | B] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array} \right]$

Transforming augmented matrix to echelon form we get

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array} \right]$$

The given system of equations using this row - echelon form reduces to

$$x + 2y - z = 3 \quad \text{-----(4)}$$

$$-6y + 5z = -4 \quad \text{-----(5)}$$

$$-y = -4 \quad \text{-----(6)}$$

$$\Rightarrow y = 4$$

Substituting $y = 4$ in equation (5) we get

$$-6 \times 4 + 5z = -4 \quad \Rightarrow \quad 5z = -4 + 24 = 20$$

$$z = \frac{20}{5} = 4$$

Substituting $y = 4$ and $z = 4$ in equation (1) we get

$$x + 2 \times 4 - 4 = 3 \quad \Rightarrow \quad x + 8 - 4 = 3 \quad \Rightarrow \quad x = 3 - 4 = -1$$

∴ The required solutions of the system is $x = -1$, $y = 4$, $z = 4$