4.8 Rank

Rank enables one to relate matrices to vectors, and vice versa.

Definition

Let A be an $m \times n$ matrix. The rows of A may be viewed as row vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$, and the columns as column vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Each row vector will have n components, and each column vector will have m components, The row vectors will span a subspace of $\mathbf{R}^n$ called the **row space** of A, and the column vectors will span a subspace of $\mathbf{R}^m$ called the **column space** of A.

Example 1

Consider the matrix

$$A=\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 4 & 1 & 6 \\ 5 & 4 & 1 & 0 \end{bmatrix}$$

(1) The row vectors of A are

$$\mathbf{r}_1 = (1, 2, -1, 2), \mathbf{r}_2 = (3, 4, 1, 6), \mathbf{r}_3 = (5, 4, 1, 0)$$

These vectors span a subspace of $\mathbf{R}^4$ called the <u>row space</u> of A.

(2) The column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

These vectors span a subspace of $\mathbf{R}^3$ called the <u>column space</u> of A.

Theorem 4.16

The row space and the column space of a matrix A have the same dimension.

Definition

The dimension of the row space and the column space of a matrix A is called the **rank** of A. The rank of A is denoted **rank(A)**.

Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

Example 2

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution

The third row of A is a linear combination of the first two rows:

$$(2, 5, 8) = 2(1, 2, 3) + (0, 1, 2)$$

Hence the three rows of A are linearly dependent.

The rank of A must be less than 3. Since (1, 2, 3) is not a scalar multiple of (0, 1, 2), these two vectors are linearly independent. These vectors form a basis for the row space of A.

Thus rank(A) = 2.

Theorem 4.17

The nonzero row vectors of a matrix A that is in reduced echelon form are a basis for the row space of A. The rank of A is the number of nonzero row vectors.

Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced echelon form. There are three nonzero row vectors, namely (1, 2, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). According to the previous theorem, these three vectors form a basis for the row space of *A*.

Rank(*A*) = 3.

Theorem 4.18

Let A and B be row equivalent matrices. Then A and B have the same the row space. rank(A) = rank(B).

Theorem 4.19

Let E be a reduced echelon form of a matrix A. The nonzero row vectors of E form a basis for the row space of A. The rank of A is the number of nonzero row vectors in E.

Example 4

Find a basis for the row space of the following matrix A, and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Example 4

Find a basis for the row space of the following matrix A, and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution

Use elementary row operations to find a reduced echelon form of the matrix A. We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The two vectors (1, 0, 7), (0, 1, –2) form a basis for the row space of A. Rank(A) = 2.

Example 5

EXAMPLE 5 Find a basis for the column space of the following matrix A

$$A \quad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

Example 5

EXAMPLE 5 Find a basis for the column space of the following matrix A

$$A \quad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

SOLUTION

The transpose of A is

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A^t. Let us find a basis for the row space of A^t. Compute the reduced echelon form of A^t.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors of this echelon form, namely $(1, 0, 5)$, $(0, 1, -3)$, are a basis for the row space of A^t. Write these vectors in column form to get a basis for the column space of A. The following vectors are a basis for the column space of A.

$$\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Example 6

EXAMPLE 6 Find a basis for the subspace V of $\mathbf{R}^4$ spanned by the vectors

$$(1, 2, 3, 4),\ (-1, -1, -4, -2),\ (3, 4, 11, 8)$$

SOLUTION

Example 6

EXAMPLE 6 Find a basis for the subspace V of $\mathbf{R}^4$ spanned by the vectors

$$(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8)$$

SOLUTION

We construct a matrix A having these vectors as row vectors.

$$A \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Determine the reduced echelon form of A. We get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero vectors of this reduced echelon form, namely $(1, 0, 5, 0)$ and $(0, 1, -1, 2)$, are a basis for the subspace V. The dimension of this subspace is two.

Example 7

Consider a system AX = B of **m** linear equations in **n** variables.

(a) If the augmented matrix and the matrix of coefficients have the same rank "**r**" and **r=n,** then the solution is unique.

(b) If the augmented matrix and the matrix of coefficients have the same rank "**r**" and **r<n,** then there will be many solutions. Where **r** is rank and **n** are the number of variables

(c) If the augmented matrix and the matrix of coefficients do not have the same rank then the solution does not exist.

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$
\begin{array}{rcrcrcr}
x_1 & + & x_2 & + & x_3 & & 2 \\
2x_1 & + & 3x_2 & + & x_3 & & 3 \\
x_1 & - & x_2 & - & 2x_3 & = & -6
\end{array}
$$

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned} x_1 + x_2 + x_3 &\quad 2 \\ 2x_1 + 3x_2 + x_3 &\quad 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

$$\underset{\text{Augmented matrix}}{\underbrace{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}}_{\text{Matrix of coefficients}}} \approx \cdots \approx \underset{\text{Reduced echelon form}}{\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{Reduced echelon form}}}$$

We see that ranks of the augmented matrix and the matrix of coefficients are equal, both being three. The system thus has a unique solution. The reduced echelon form gives that solution to be $x1 = -1, x_1 = 1, x_3 = 2$.

Example 7

EXAMPLE 7 Consider the following system of linear equations from Section 1.1.

$$\begin{aligned} x_1 + x_2 + x_3 &\quad\ \ 2 \\ 2x_1 + 3x_2 + x_3 &\quad\ \ 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

The augmented matrix of this system of equations, and its reduced echelon form are as follows.

$$\begin{array}{ccc} \text{Augmented matrix} & & \text{Reduced echelon form} \\ \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix} & \approx \cdots \approx & \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ \underbrace{}_{\text{Matrix of coefficients}} & & \underbrace{}_{\text{Reduced echelon form}} \end{array}$$

The system of linear equations can be viewed as the linear combination

$$x_1\begin{bmatrix}1\\2\\1\end{bmatrix} + x_2\begin{bmatrix}1\\3\\-1\end{bmatrix} + x_3\begin{bmatrix}1\\1\\-2\end{bmatrix} \quad \begin{bmatrix}2\\3\\-6\end{bmatrix}$$

The existence of solutions depends upon whether $\begin{bmatrix}2\\3\\-6\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\3\\-1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\-2\end{bmatrix}$. The uniqueness depends upon whether the linear combination is unique.

4.2 Dot Product, Norm, Angle, and Distance

Definition

Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$ be two vectors in $\mathbf{R}^n$.
The **dot product** of **u** and **v** is denoted **u.v** and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

The dot product assigns a real number to each pair of vectors.

Example 1

Find the dot product of

$$\mathbf{u} = (1, -2, 4) \text{ and } \mathbf{v} = (3, 0, 2)$$

Example 1

Find the dot product of

$$\mathbf{u} = (1, -2, 4) \text{ and } \mathbf{v} = (3, 0, 2)$$

Solution

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (1\times 3)+(-2\times 0)+(4\times 2)\\ &= 3+0+8\\ &= 11\end{aligned}$$

Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let c be a scalar. Then

1. **u.v = v.u**
2. **(u + v).w = u.w + v.w**
3. c**u.v** = c**(u.v)** = **u.**c**v**
4. **u.u** ≥ 0, and **u.u** = 0 if and only if **u = 0**

Proof

1. Let $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$. We get

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + \cdots + u_n v_n \\ &= v_1 u_1 + \cdots + v_n u_n \quad \text{by the commutative property of real numbers} \\ &= \mathbf{v} \cdot \mathbf{u}\end{aligned}$$

4. $\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + \cdots + u_n u_n = (u_1)^2 + \cdots + (u_n)^2$

$(u_1)^2 + \cdots + (u_n)^2 \geq 0$, thus $\mathbf{u} \cdot \mathbf{u} \geq 0$.

$(u_1)^2 + \cdots + (u_n)^2 = 0$, if and only if $u_1 = 0, \cdots, u_n = 0$.

Thus $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Norm of a Vector in R^n

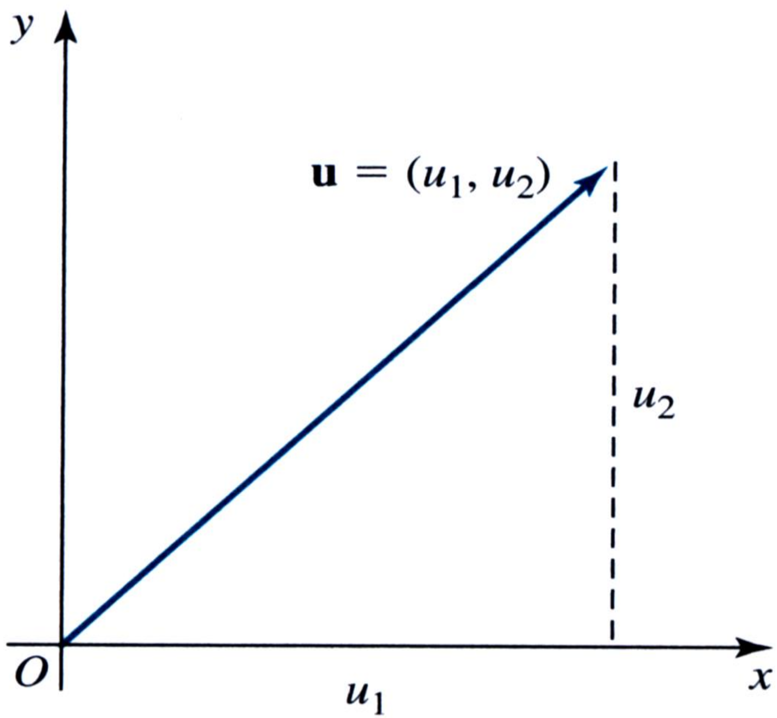


Figure 4.5 length of $\mathbf{u}$

Definition

The **norm** (**length** or **magnitude**) of a vector $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbf{R}^n$ is denoted $\|\mathbf{u}\|$ and defined by

$$\|\mathbf{u}\| = \sqrt{(u_1)^2 + \cdots + (u_n)^2}$$

Note:

The norm of a vector can also be written in terms of the dot product $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Example 2

Find the norm of each of the vectors **u** = (1, 3, 5) of $\mathbf{R}^3$ and **v** = (3, 0, 1, 4) of $\mathbf{R}^4$.

Example 2

Find the norm of each of the vectors $\mathbf{u} = (1, 3, 5)$ of $\mathbf{R}^3$ and $\mathbf{v} = (3, 0, 1, 4)$ of $\mathbf{R}^4$.

Solution

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9 + 0 + 1 + 16} = \sqrt{26}$$

Unit Vector

Definition

A **unit vector** is a vector whose norm is 1.

If **v** is a nonzero vector, then the vector **u**

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is a unit vector in the direction of **v**.

This procedure of constructing a unit vector in the same direction as a given vector is called **normalizing** the vector.

Example

EXAMPLE 1 Find the norm of the vector $(2, -1, 3)$. Normalize this vector.

Example

EXAMPLE 1 Find the norm of the vector $(2, -1, 3)$. Normalize this vector.

SOLUTION

$\|(2, -1, 3)\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$. The norm of $(2, -1, 3)$ is $\sqrt{14}$. The normalized vector is

$$\frac{1}{\sqrt{14}}(2, -1, 3)$$

This vector may also be written $\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$. This vector is a unit vector in the direction of $(2, -1, 3)$.

Example 3

(a) Show that the vector (1, 0) is a unit vector.

(b) Find the norm of the vector (2, –1, 3). Normalize this vector.

Solution

Example 3

(a) Show that the vector (1, 0) is a unit vector.

(b) Find the norm of the vector (2, –1, 3). Normalize this vector.

Solution

(a) $\|(1,0)\| = \sqrt{1^2+0^2} = 1$. Thus (1, 0) is a unit vector. It can be similarly shown that (0, 1) is a unit vector in $\mathbf{R}^2$.

(b) $\|(2,-1,3)\| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}$. The norm of (2, –1, 3) is $\sqrt{14}$.

The normalized vector is

$$\frac{1}{\sqrt{14}}(2,-1,3)$$

The vector may also be written $\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

This vector is a unit vector in the direction of (2, –1, 3).

Work and the Dot Product

- We defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object.

Work and the Dot Product

- Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 1.

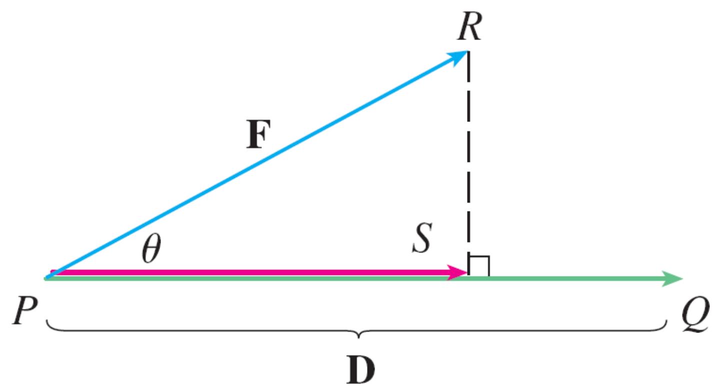


Figure 1

- If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. So here we have two vectors: the force **F** and the displacement **D**.

Work and the Dot Product

- The **work** done by **F** is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion, which, from Figure 1, is

$$|\overrightarrow{PS}| = |\mathbf{F}| \cos \theta$$

- So the work done by **F** is defined to be

$$\boxed{1} \qquad W = |\mathbf{D}| \, (|\mathbf{F}| \cos \theta) = |\mathbf{F}||\mathbf{D}| \cos \theta$$

- Notice that work is a scalar quantity; it has no direction. But its value depends on the angle θ between the force and displacement vectors.

Work and the Dot Product

- We use the expression in Equation 1 to define the dot product of two vectors even when they don't represent force or displacement.

Definition The **dot product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leqslant \theta \leqslant \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

- This product is called the **dot product** because of the dot in the notation $\mathbf{a} \cdot \mathbf{b}$.

Example 1 – Computing a Dot Product from Lengths and the Contained Angle

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

- If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

According to the definition,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

Angle between Vectors (in R^n)

Definition

Let **u** and **v** be two nonzero vectors in $\mathbf{R}^n$.

The **cosine of the angle θ** between these vectors is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi$$

Example 4

Determine the angle between the vectors **u** = (1, 0, 0) and **v** = (1, 0, 1) in $\mathbf{R}^3$.

Angle between Vectors (in R^n)

Definition

Let **u** and **v** be two nonzero vectors in $\mathbf{R}^n$.

The **cosine of the angle θ** between these vectors is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi$$

Example 4

Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (1, 0, 1)$ in $\mathbf{R}^3$.

Solution $\mathbf{u}\cdot\mathbf{v} = (1,0,0)\cdot(1,0,1) = 1$

$$\|\mathbf{u}\| = \sqrt{1^2+0^2+0^2} = 1 \qquad \|\mathbf{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

Thus $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}$, the angle between **u** and **v** is 45°.

Orthogonal Vectors

Definition

Two nonzero vectors are **orthogonal** if the angle between them is a right angle .

Theorem 4.2

Two nonzero vectors **u** and **v** are orthogonal *if and only if* **u**.**v** = **0**.

Proof

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal} \iff \cos\theta = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0$$

Example 5

Show that the following pairs of vectors are orthogonal.

(a) (1, 0) and (0, 1).

(b) (2, –3, 1) and (1, 2, 4).

Solution

Example 5

Show that the following pairs of vectors are orthogonal.

(a) (1, 0) and (0, 1).

(b) (2, –3, 1) and (1, 2, 4).

Solution

(a) $(1, 0).(0, 1) = (1 \times 0) + (0 \times 1) = 0.$

The vectors are orthogonal.

(b) $(2, -3, 1).(1, 2, 4) = (2 \times 1) + (-3 \times 2) + (1 \times 4) = 2 - 6 + 4 = 0.$

The vectors are orthogonal.

Note

- (1, 0), (0,1) are orthogonal unit vectors in $\mathbf{R}^2$.
- (1, 0, 0), (0, 1, 0), (0, 0, 1) are orthogonal unit vectors in $\mathbf{R}^3$.
- (1, 0, …, 0), (0, 1, 0, …, 0), …, (0, …, 0, 1) are orthogonal unit vectors in $\mathbf{R}^n$.

EXAMPLE 4 (a) Show that the following vectors are orthogonal. (b) Compute the norm of each vector.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLUTION

EXAMPLE 4 (a) Show that the following vectors are orthogonal. (b) Compute the norm of each vector.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

SOLUTION

(a) The dot product of **u** and **v** is obtained by multiplying corresponding components and adding.

$$\mathbf{u} \cdot \mathbf{v} = (1 \times -1) + (2 \times 3) + (-5 \times 1) = 0$$

The dot product is zero, thus the vectors are orthogonal.

(b) The norm of each vector is obtained by taking the square root of the sum of the squares of the components.

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + (-5)^2} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{11}$$

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Let the vector (a, b) be orthogonal to (3, –1)

We get $$(a,b)\cdot(3,-1)=0$$
$$(a\times 3)+(b\times(-1))=0$$
$$3a-b=0$$
$$b=3a$$

Thus any vector of the form $(a, 3a)$ is orthogonal to the vector (3, –1).

Any vector of this form can be written
$$a(1, 3)$$

The set of all such vectors lie on the line defined by the vector (1, 3).

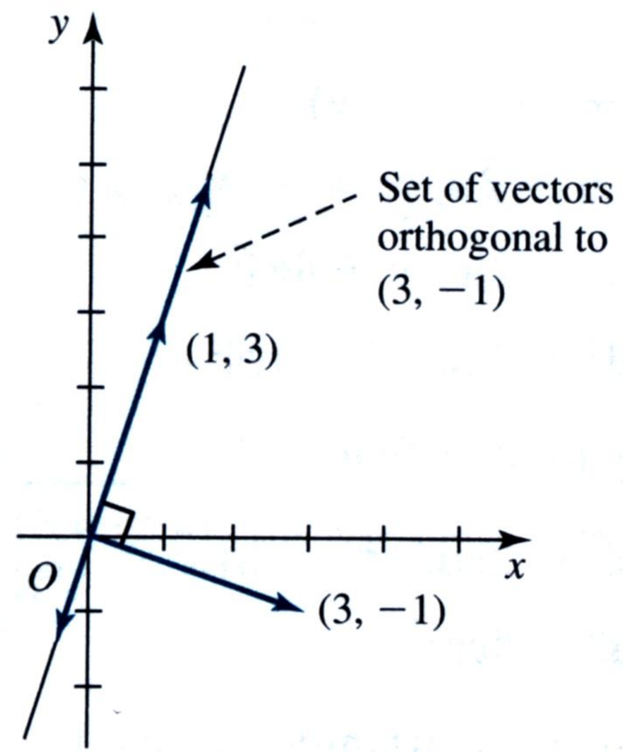


Figure 4.7

Distance between Points

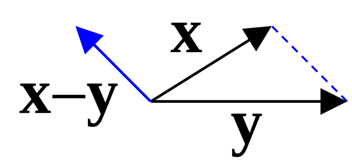
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbf{R}^n$.

The **distance** between **x** and **y** is denoted $d(\mathbf{x}, \mathbf{y})$ and is defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Note: We can also write this distance as follows.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



Note: *It is clear that* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (*the symmetric property*)

Example 7. Determine the distance between the points $\mathbf{x} = (1, -2, 3, 0)$ and $\mathbf{y} = (4, 0, -3, 5)$ in $\mathbf{R}^4$.

Solution

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(1-4)^2 + (-2-0)^2 + (3+3)^2 + (0-5)^2}$$

$$= \sqrt{9 + 4 + 36 + 25}$$

$$= \sqrt{74}$$

4.9 Orthonormal Vectors and Projections

Definition

A set of vectors in a vector space V is said to be an **orthogonal set** if every pair of vectors in the set is orthogonal. The set is said to be an **orthonormal set** if it is orthogonal and each vector is a unit vector.

Example 1

Show that the set $\left\{(1, 0, 0), \left(0, \frac{3}{5}, \frac{4}{5}\right), \left(0, \frac{4}{5}, -\frac{3}{5}\right)\right\}$ is an orthonormal set.

Solution

Example 1

Show that the set $\left\{(1,0,0),\left(0,\frac{3}{5},\frac{4}{5}\right),\left(0,\frac{4}{5},-\frac{3}{5}\right)\right\}$ is an orthonormal set.

Solution

(1) orthogonal:

$$(1,0,0)\cdot\left(0,\tfrac{3}{5},\tfrac{4}{5}\right)=0;$$

$$(1,0,0)\cdot\left(0,\tfrac{4}{5},-\tfrac{3}{5}\right)=0;$$

$$\left(0,\tfrac{3}{5},\tfrac{4}{5}\right)\cdot\left(0,\tfrac{4}{5},-\tfrac{3}{5}\right)=0;$$

(2) unit vector:

$$\|(1,0,0)\|=\sqrt{1^2+0^2+0^2}=1$$

$$\left\|\left(0,\tfrac{3}{5},\tfrac{4}{5}\right)\right\|=\sqrt{0^2+\left(\tfrac{3}{5}\right)^2+\left(\tfrac{4}{5}\right)^2}=1$$

$$\left\|\left(0,\tfrac{4}{5},-\tfrac{3}{5}\right)\right\|=\sqrt{0^2+\left(\tfrac{4}{5}\right)^2+\left(-\tfrac{3}{5}\right)^2}=1$$

Thus the set is thus an orthonormal set.

Theorem 4.20

An orthogonal set of nonzero vectors in a vector space is linearly independent.

Proof Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be an orthogonal set of nonzero vectors in a vector space V. Let us examine the identity

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

Let $\mathbf{v}_i$ be the ith vector of the orthogonal set. Take the dot product of each side of the equation with $\mathbf{v}_i$ and use the properties of the dot product. We get

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) \cdot \mathbf{v}_i = \mathbf{0} \cdot \mathbf{v}_i$$

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_i + c_2\mathbf{v}_2 \cdot \mathbf{v}_i + \cdots + c_m\mathbf{v}_m \cdot \mathbf{v}_i = 0$$

Since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_2$ are mutually orthogonal, $\mathbf{v}_j \cdot \mathbf{v}_i = 0$ unless $j = i$. Thus

$$c_i\mathbf{v}_i \cdot \mathbf{v}_i = 0$$

Since $\mathbf{v}_i$ is a nonzero, then $\mathbf{v}_i \cdot \mathbf{v}_i \neq 0$. Thus $c_i = 0$.

Letting $i = 1, \ldots, m$, we get $c_1 = 0$, $c_m = 0$, proving that the vectors are linearly independent.

Definition

A basis that is an orthogonal set is said to be an **orthogonal basis**.
A basis that is an orthonormal set is said to be an **orthonormal basis**.

Standard Bases

$$\left.\begin{array}{l} \bullet\ \mathbf{R}^2\text{: } \{(1, 0), (0, 1)\} \\ \bullet\ \mathbf{R}^3\text{: } \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \\ \bullet\ \mathbf{R}^n\text{: } \{(1, \ldots, 0), \ldots, (0, \ldots, 1)\} \end{array}\right\} \text{ orthonormal bases}$$

Theorem 4.21

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for a vector space V. Let $\mathbf{v}$ be a vector in V. $\mathbf{v}$ can be written as a linearly combination of these basis vectors as follows:

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n$$

Example 2

The following vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ form an orthonormal basis for $\mathbf{R}^3$. Express the vector $\mathbf{v} = (7, -5, 10)$ as a linear combination of these vectors.

$$\mathbf{u}_1 = (1, 0, 0), \mathbf{u}_2 = \left(0, \frac{3}{5}, \frac{4}{5}\right), \mathbf{u}_3 = \left(0, \frac{4}{5}, -\frac{3}{5}\right)$$

Solution

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n$$

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Solution

$$\mathbf{v} \cdot \mathbf{u}_1 = (7, -5, 10) \cdot (1, 0, 0) = 7$$

$$\mathbf{v} \cdot \mathbf{u}_2 = (7, -5, 10) \cdot \left(0, \frac{3}{5}, \frac{4}{5}\right) = 5$$

$$\mathbf{v} \cdot \mathbf{u}_3 = (7, -5, 10) \cdot \left(0, \frac{4}{5}, -\frac{3}{5}\right) = -10$$

Thus

$$\mathbf{v} = 7\mathbf{u}_1 + 5\mathbf{u}_2 - 10\mathbf{u}_3$$

Example 3

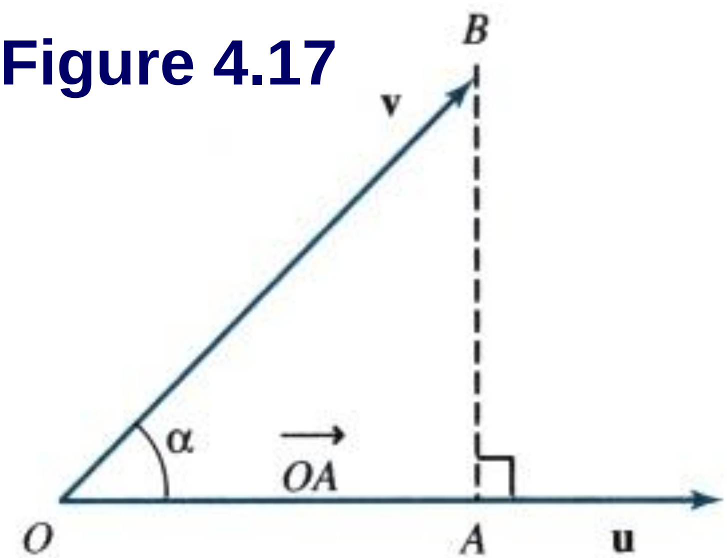
The following vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ form an orthonormal basis for $\mathbf{R}^3$. Express the vector $\mathbf{v} = (3, -1, 4)$ as a linear combination of these vectors.

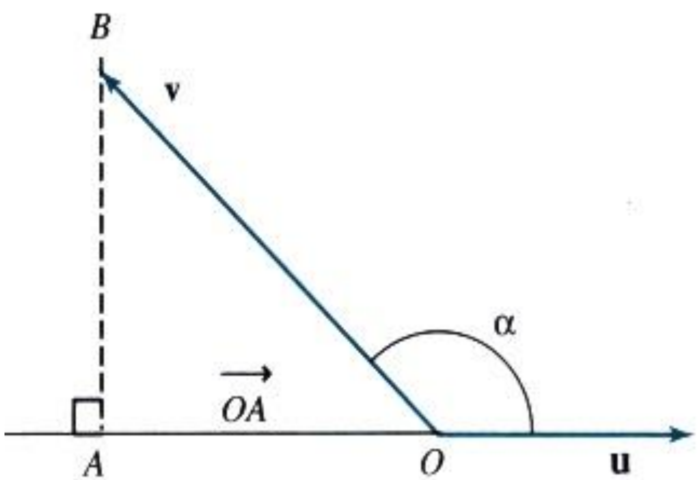
$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Projection of One vector onto Another Vector

Let **v** and **u** be vectors in $\mathbf{R}^n$ with angle α $(0 \le \alpha \le \pi)$ between them.

Figure 4.17





$\overrightarrow{OA}$: the projection of **v** onto **u**

$$\overline{OA} = \overline{OB}\cos\alpha = \|\mathrm{v}\|\cos\alpha$$

$$= \|\mathrm{v}\| \frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{v}\|\,\|\mathrm{u}\|} = \frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{u}\|}$$

$$\therefore \overrightarrow{OA} = \left(\frac{\mathrm{v}\cdot\mathrm{u}}{\|\mathrm{u}\|}\right)\left(\frac{\mathrm{u}}{\|\mathrm{u}\|}\right) = \frac{\mathrm{v}\cdot\mathrm{u}}{\mathrm{u}\cdot\mathrm{u}}\mathrm{u}$$

$$\text{Note : If } \alpha > \pi / 2 \text{ then } \frac{\mathrm{v}\cdot\mathrm{u}}{\mathrm{u}\cdot\mathrm{u}} < 0.$$

So we define $\boxed{\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}.}$

Definition

The **projection** of a vector **v** onto a nonzero vector **u** in $\mathbf{R}^n$ is denoted $\text{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\overline{OA} = ||\mathrm{v}|| \cos\alpha = \mathrm{v}.\frac{\mathrm{u}}{||\mathrm{u}||} = \mathrm{v}.\frac{\mathrm{u}}{||\mathrm{u}||}\left(\frac{\mathrm{u}}{||u||}\right)$$

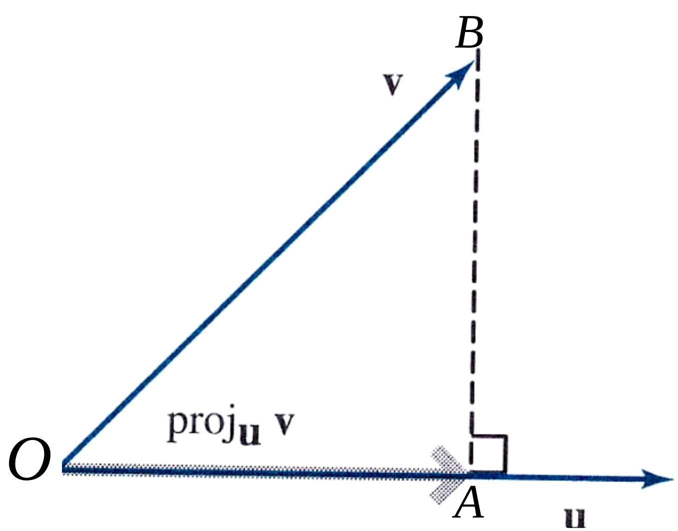


Figure 4.18

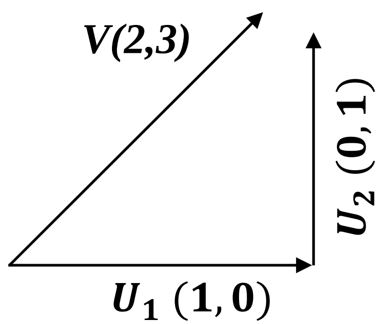
$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n$$

$$\mathbf{v} = \left(\mathbf{v}\cdot\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}\right)\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} + \left(\mathbf{v}\cdot\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}\right)\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} + \cdots + \left(\mathbf{v}\cdot\frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}\right)\frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}$$

Definition

The **projection** of a vector **v** onto a nonzero vector **u** in $\mathbf{R}^n$ is denoted $\mathrm{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$



$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\mathrm{proj}_{\mathbf{u}_1}\mathbf{v} = \frac{(2,3)\cdot(1,0)}{1\cdot 1}(1,0) = (2,0)$$

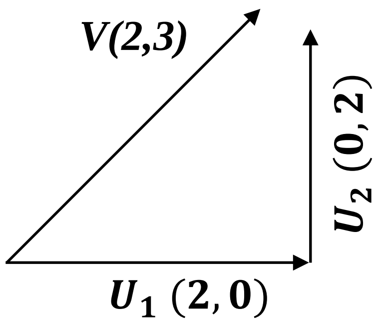
$$\mathrm{proj}_{\mathbf{u}_2}\mathbf{v} = \frac{(2,3)\cdot(0,1)}{1\cdot 1}(0,1) = (0,3)$$

$$\mathrm{proj}_{\mathbf{u}_1}\mathbf{v} + \mathrm{proj}_{\mathbf{u}_2}\mathbf{v} = (2,0) + (0,3) = (2,3)$$

Definition

The **projection** of a vector **v** onto a nonzero vector **u** in $\mathbf{R}^n$ is denoted $\text{proj}_{\mathbf{u}}\mathbf{v}$ and is defined by

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$



$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} = \frac{(2,3)\cdot(2,0)}{2\cdot 2}(2,0) = (2,0)$$

$$\text{proj}_{\mathbf{u}_2}\mathbf{v} = \frac{(2,3)\cdot(0,2)}{2\cdot 2}(0,2) = (0,3)$$

$$\text{proj}_{\mathbf{u}_1}\mathbf{v} + \text{proj}_{\mathbf{u}_2}\mathbf{v} = (2,0) + (0,3) = (2,3)$$

Example 3

Determine the projection of the vector **v** = (6, 7) onto the vector **u** = (1, 4).

Solution

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

Example 3

Determine the projection of the vector **v** = (6, 7) onto the vector **u** = (1, 4).

Solution

$$\mathbf{v}\cdot\mathbf{u} = (6,7)\cdot(1,4) = 6 + 28 = 34$$

$$\mathbf{u}\cdot\mathbf{u} = (1,4)\cdot(1,4) = 1 + 16 = 17$$

Thus

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{34}{17}(1,4) = (2,8)$$

The projection of **v** onto **u** is (2, 8).

Theorem 4.22

The Gram-Schmidt Orthogonalization Process

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. The set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ defined as follows is orthogonal. To obtain an orthonormal basis for V, normalize each of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{u}_2}\mathbf{v}_3$$

$$\cdots$$

$$\mathbf{u}_n = \mathbf{v}_n - \text{proj}_{\mathbf{u}_1}\mathbf{v}_n - \cdots - \text{proj}_{\mathbf{u}_{n-1}}\mathbf{v}_n$$

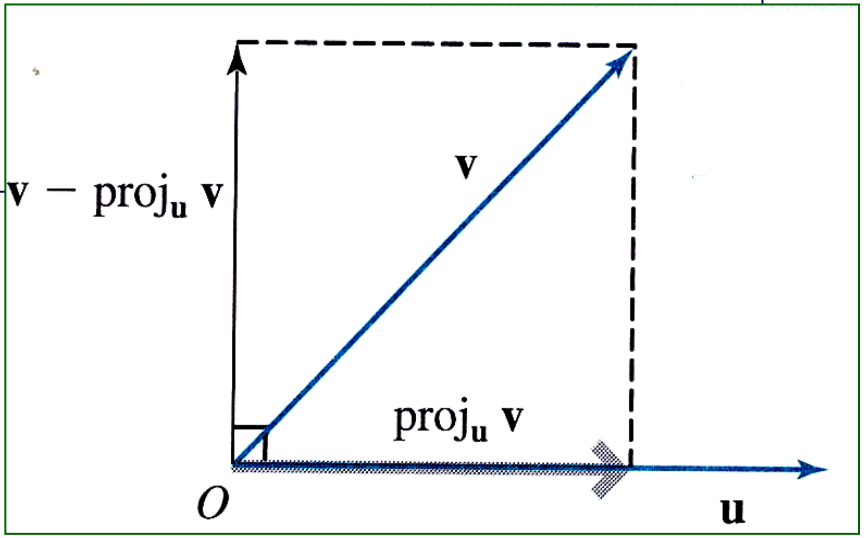


Figure 4.19

Example 4

The set {(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)} is linearly independent in $\mathbf{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbf{R}^4$. Construct an orthonormal basis for V.

Solution

Example 4

The set {(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)} is linearly independent in $\mathbf{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbf{R}^4$. Construct an orthonormal basis for V.

Solution

Let $\mathbf{v}_1 = (1, 2, 0, 3)$, $\mathbf{v}_2 = (4, 0, 5, 8)$, $\mathbf{v}_3 = (8, 1, 5, 6)$}.
Use the Gram-Schmidt process to construct an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ from these vectors.

$$\text{Let } \mathbf{u}_1 = \mathbf{v}_1 = (1, 2, 0, 3)$$

$$\text{Let } \mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2 \cdot \mathbf{u}_2)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 = (2, -4, 5, 2)$$

$$\text{Let } \mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{u}_2}\mathbf{v}_3$$

$$= \mathbf{v}_3 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_1)}{(\mathbf{u}_1 \cdot \mathbf{u}_1)}\mathbf{u}_1 - \frac{(\mathbf{v}_3 \cdot \mathbf{u}_2)}{(\mathbf{u}_2 \cdot \mathbf{u}_2)}\mathbf{u}_2 = (4, 1, 0, -2)$$

The set {(1, 2, 0, 3), (2, –4, 5, 2), (4, 1, 0, –2)} is an orthogonal basis for *V*.

Normalize them to get an orthonormal basis:

$$\|(1,2,0,3)\| = \sqrt{1^2+2^2+0^2+3^2} = \sqrt{14}$$

$$\|(2,-4,5,2)\| = \sqrt{2^2+(-4)^2+5^2+2^2} = 7$$

$$\|(4,1,0,-2)\| = \sqrt{4^2+1^2+0^2+(-2)^2} = \sqrt{21}$$

$\Rightarrow$ orthonormal basis for *V*:

$$\left\{\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0, \frac{3}{\sqrt{14}}\right), \left(\frac{2}{7}, -\frac{4}{7}, \frac{5}{7}, \frac{2}{7}\right), \left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, 0, -\frac{2}{\sqrt{21}}\right)\right\}$$

Example 5 Use Gram-Schmidt orthogonalization process for the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Example 6 Use Gram-Schmidt orthogonalization process for the following vectors

Given the following linearly independent vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We will use Gram-Schmidt to find the orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.