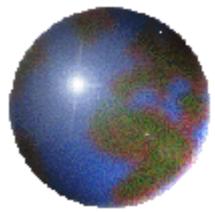


Linear Algebra



Chapter 3

DETERMINANTS



Example 3

Evaluate the determinant of the following matrix A .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned}|A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\&= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\&= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)] \\&= -2 + 2 - 6 \\&= -6\end{aligned}$$



Example 4

Evaluation the determinant.

$$\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$$

Solution

$$\begin{array}{l} \left| \begin{array}{ccc} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{array} \right| = \left| \begin{array}{ccc} 1 & -2 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{array} \right| \\ \text{R2 + R1} \quad \text{R3 + (-2)R1} \\ \\ \left| \begin{array}{ccc} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{array} \right| \\ \text{R2} \leftrightarrow \text{R3} \quad (-1) \left| \begin{array}{ccc} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{array} \right| \\ \\ = (-1) \times 1 \times 2 \times (-1) = 2 \end{array}$$



3.3 Determinants, Matrix Inverse, and Systems of Linear Equations

Definition

Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij} .

The matrix whose (i, j) th element is C_{ij} is called the **matrix of cofactors** of A .

The transpose of this matrix is called the **adjoint** of A and is denoted $\text{adj}(A)$.

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

matrix of cofactors

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^t$$

adjoint matrix

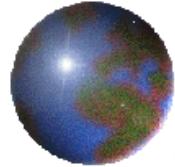


Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A .

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution



Example 1

Give the matrix of cofactors and the adjoint matrix of the following matrix A .

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution The cofactors of A are as follows.

$$C_{11} = \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = 14$$

$$C_{12} = -\begin{vmatrix} -1 & -2 \\ 1 & 5 \end{vmatrix} = 3$$

$$C_{13} = \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ -3 & 5 \end{vmatrix} = -9$$

$$C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$$

$$C_{23} = \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12$$

$$C_{32} = -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 1$$

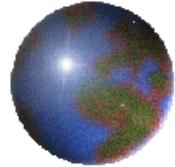
$$C_{33} = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = 8$$

The matrix of cofactors of A is

$$\begin{bmatrix} 14 & 3 & -1 \\ -9 & 7 & 6 \\ -12 & 1 & 8 \end{bmatrix}$$

The adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$



Theorem 3.6

Let A be a square matrix with $|A| \neq 0$. A is invertible with

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

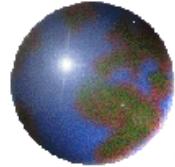
Proof

Consider the matrix product $\mathbf{A} \cdot \text{adj}(\mathbf{A})$. The (i, j) th element of this product is

(i, j) th element = (row i of A) \times (column j of $\text{adj}(A)$)

$$\begin{aligned} &= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \begin{bmatrix} C_{j1} \\ C_{j2} \\ \vdots \\ C_{jn} \end{bmatrix} \\ &= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} \end{aligned}$$

Proof Excluded



Proof of Theorem 3.6

Proof Excluded

If $i = j$, $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |A|$.

If $i \neq j$, let $A \xrightarrow{R_j \text{ is replaced by } R_i} B$.

Matrices A and B have the same cofactors $C_{j1}, C_{j2}, \dots, C_{jn}$.

So $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = |B| = 0$.

row $i =$ row j in B

Therefore (i,j) th element = $\begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\therefore A \cdot \text{adj}(A) = |A|I_n$

Since $|A| \neq 0$, $A \left(\frac{1}{|A|} \text{adj}(A) \right) = I_n$

Similarly, $\left(\frac{1}{|A|} \text{adj}(A) \right) A = I_n$.

Thus $A^{-1} = \frac{1}{|A|} \text{adj}(A)$



Theorem 3.7

A square matrix A is invertible if and only if $|A| \neq 0$.

Proof

(\Rightarrow) Assume that A is invertible.

$$\Rightarrow AA^{-1} = I_n.$$

$$\Rightarrow |AA^{-1}| = |I_n|.$$

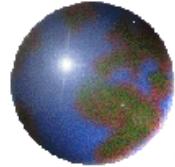
$$\Rightarrow |A|/|A^{-1}| = 1$$

$$\Rightarrow |A| \neq 0.$$

Proof Excluded

(\Leftarrow) Theorem 3.6 tells us that if $|A| \neq 0$, then A is invertible.

A^{-1} exists if and only if $|A| \neq 0$.



Example 2

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

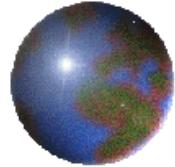
Solution

$|A| = 5 \neq 0$. A is invertible.

$|B| = 0$. B is singular. The inverse does not exist.

$|C| = 0$. C is singular. The inverse does not exist.

$|D| = 2 \neq 0$. D is invertible.



Example 3

Use the formula for the inverse of a matrix to compute the inverse of the matrix

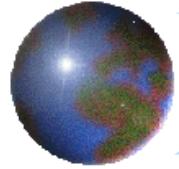
$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

Solution

$|A| = 25$, so the inverse of A exists. We found $\text{adj}(A)$ in Example 1

$$\text{adj}(A) = \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{25} \begin{bmatrix} 14 & -9 & -12 \\ 3 & 7 & 1 \\ -1 & 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{14}{25} & -\frac{9}{25} & -\frac{12}{25} \\ \frac{3}{25} & \frac{7}{25} & \frac{1}{25} \\ -\frac{1}{25} & \frac{6}{25} & \frac{8}{25} \end{bmatrix}$$



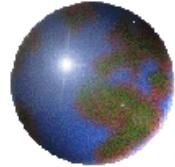
Homework

- ➊ Exercise 3.3 page 178-179: 1, 3, 5, 7.

- ➋ *Exercise*

Show that if $A = A^{-1}$, then $|A| = \pm 1$.

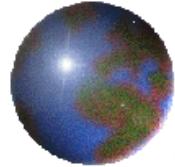
Show that if $A^t = A^{-1}$, then $|A| = \pm 1$.



Theorem 3.8

Let $AX = B$ be a system of n linear equations in n variables.

- (1) If $|A| \neq 0$, there is a unique solution.
- (2) If $|A| = 0$, there may be many or no solutions.



Example 4

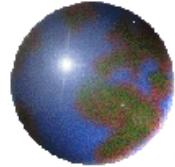
Determine whether or not the following system of equations has an unique solution.

$$3x_1 + 3x_2 - 2x_3 = 2$$

$$4x_1 + x_2 + 3x_3 = -5$$

$$7x_1 + 4x_2 + x_3 = 9$$

Solution



Example 4

Determine whether or not the following system of equations has an unique solution.

$$3x_1 + 3x_2 - 2x_3 = 2$$

$$4x_1 + x_2 + 3x_3 = -5$$

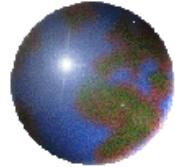
$$7x_1 + 4x_2 + x_3 = 9$$

Solution

Since

$$\begin{vmatrix} 3 & 3 & -2 \\ 4 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix} = 0$$

Thus the system does not have an unique solution.

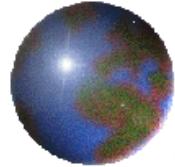


Theorem 3.9 Cramer's Rule

Let $AX = B$ be a system of n linear equations in n variables such that $|A| \neq 0$. The system has a unique solution given by

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n|}{|A|}$$

Where A_i is the matrix obtained by replacing column i of A with B .



Example 5

Solving the following system of equations using Cramer's rule.

$$x_1 + 3x_2 + x_3 = -2$$

$$2x_1 + 5x_2 + x_3 = -5$$

$$x_1 + 2x_2 + 3x_3 = 6$$

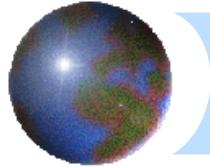
Solution

The matrix of coefficients A and column matrix of constants B are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

It is found that $|A| = -3 \neq 0$. Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$

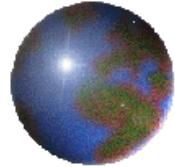


Giving $|A_1| = -3, |A_2| = 6, |A_3| = -9$

Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2, \quad x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is $x_1 = 1, x_2 = -2, x_3 = 3$.



Example 6

Determine values of λ for which the following system of equations has nontrivial solutions. Find the solutions for each value of λ .

$$(\lambda + 2)x_1 + (\lambda + 4)x_2 = 0$$

$$2x_1 + (\lambda + 1)x_2 = 0$$

Solution

homogeneous system

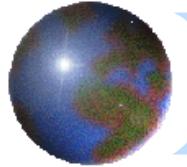
$\Rightarrow x_1 = 0, x_2 = 0$ is the trivial solution.

\Rightarrow nontrivial solutions exist \Rightarrow many solutions

$$\Rightarrow \begin{vmatrix} \lambda + 2 & \lambda + 4 \\ 2 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda + 2)(\lambda + 1) - 2(\lambda + 4) = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda - 2)(\lambda + 3) = 0$$

$$\Rightarrow \lambda = -3 \text{ or } \lambda = 2.$$



$\lambda = -3$ results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

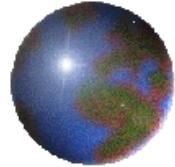
This system has many solutions, $x_1 = r$, $x_2 = r$.

$\lambda = 2$ results in the system

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions, $x_1 = -3r/2$, $x_2 = r$.



Homework

- ➊ Exercise 3.3 pages 179-180:
9, 11, 13, 15.