Linear Algebra

Chapter 2

MATRICES

2.2 Algebraic Properties of Matrix Operations

Theorem 2.2 -1

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Addition and scalar Multiplication

1. $A + B = B + A$ *Commutative property of addition*
2. $A + (B + C) = (A + B) + C$ *Associative property of addition*
3. $A + O = O + A = A$ (*where O is the appropriate zero matrix*)
4. $c(A + B) = cA + cB$ *Distributive property of addition*
5. $(a + b)C = aC + bC$ *Distributive property of addition*
6. $(ab)C = a(bC)$

Theorem 2.2 -2

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Multiplication

1. $A(BC) = (AB)C$ *Associative property of multiplication*
2. $A(B + C) = AB + AC$ *Distributive property of multiplication*
3. $(A + B)C = AC + BC$ *Distributive property of multiplication*
4. $AI_n = I_nA = A$ (*where I_n is the appropriate identity matrix*)
5. $c(AB) = (cA)B = A(cB)$

Note: $AB \neq BA$ in general. *Multiplication of matrices is not commutative.*

Proof of Theorem 2.2 (A+B=B+A)

Consider the (i,j)th elements of matrices $A+B$ and $B+A$:

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

$$\therefore A+B=B+A$$

Example 9

Let $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$.

$$A+B+C = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}.$$

Arithmetic Operations

If A is an $m \times r$ matrix and B is $r \times n$ matrix, the number of scalar multiplications involved in computing the product AB is mrn.

Consider three matrices A, B and C such that the product ABC exists.
Compare the number of multiplications involved in the two ways $(AB)C$ and $A(BC)$ of computing the product ABC

Example 10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. Compute ABC.

Solution.

Which method is better?
Count the number of multiplications.

(1) $(AB)C$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix}.$$

$$(AB)C = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 3 & 11 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

2×6+3×2
=12+6=18

(2) $A(BC)$

$$BC = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}.$$

3×2+2×2
=6+4=10

$\therefore$ $A(BC)$ is better.

Caution

In algebra we know that the following cancellation laws apply.

- If $ab = ac$ and $a \neq 0$ then $b = c$.
- If $pq = 0$ then $p = 0$ or $q = 0$.

However the corresponding results are not true for matrices.

- $AB = AC$ does not imply that $B = C$.
- $PQ = O$ does not imply that $P = O$ or $Q = O$.

Example 11

(1) Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$.

Observe that $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$.

(2) Consider the matrices $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$.

Observe that $PQ = O$, but $P \neq O$ and $Q \neq O$.

Powers of Matrices

Definition

If A is a square matrix, then

$$A^k = \underbrace{AA\cdots A}_{k\text{ times}}$$

Theorem 2.3

If A is an $n \times n$ square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.
3. $A^0 = I_n$ (by definition)

Example 12

If $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4.

Solution

$$A^2 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB$$

Solution

Example 12

If $A=\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4.

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Solution

$$\begin{aligned} &A(A+2B)+3B(2A-B)-A^2+7B^2-5AB \\ &=A^2+2AB+6BA-3B^2-A^2+7B^2-5AB \\ &=\underline{-3AB+6BA}+4B^2 \end{aligned}$$

We can't add the two matrices!

Systems of Linear Equations

A system of *m* linear equations in *n* variables as follows

$$\begin{array}{ccccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots \quad\quad \vdots \quad\quad \vdots & & \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

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A system of m linear equations in n variables as follows

$$\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$\boldsymbol{AX = B}$$

Idempotent and Nilpotent Matrices

(Exercises 24-30 p.83 and 14 p.93)

Definition

(1) A square matrix A is said to be **idempotent** if $A^2=A$.

(2) A square matrix A is said to **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

(1) $A=\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 =$

(2) $B=\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 =$

Idempotent and Nilpotent Matrices (Exercises 24-30 p.83 and 14 p.93)

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Example 14

$$(1)\ A=\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix},\ A^2=\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}=A.$$

$$(2)\ B=\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix},\ B^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\ \text{The degree of nilpotency}:2$$

Class activity

The following matrix is nilpotent. Find the degree of **nilpotency**

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Idempotent and Nilpotent Matrices
(Exercises 24-30 p.83 and 14 p.93)

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is nilpotent, with

$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The index of B is therefore 4.

2.3 Symmetric Matrices

Definition

The **transpose** of a matrix A, denoted A^t, is the matrix whose columns are the rows of the given matrix A.

i.e., $A: m\times n \Rightarrow A^t: n\times m,\ (A^t)_{ij} = A_{ji}\ \ \forall i, j.$

Example 15

$$A=\begin{bmatrix} 2 & 7 \\ -8 & 0 \end{bmatrix}, B=\begin{bmatrix} 1 & 2 & -7 \\ 4 & 5 & 6 \end{bmatrix}, \text{and } C=\begin{bmatrix} -1 & 3 & 4 \end{bmatrix}.$$

The **transpose** of these matrices

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$$A=\begin{bmatrix} 2 & 7 \\ -8 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 & 2 & -7 \\ 4 & 5 & 6 \end{bmatrix},\ \text{and } C=\begin{bmatrix} -1 & 3 & 4 \end{bmatrix}.$$

$$A^t=\begin{bmatrix} 2 & -8 \\ 7 & 0 \end{bmatrix}\qquad B^t=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -7 & 6 \end{bmatrix}\qquad C^t=\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

Theorem 2.4: Properties of Transpose

Let A and B be matrices and c be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.

1. $(A + B)^t = A^t + B^t$ *Transpose of a sum*
2. $(cA)^t = cA^t$ *Transpose of a scalar multiple*
3. $(AB)^t = B^tA^t$ *Transpose of a product*
4. $(A^t)^t = A$

Symmetric Matrix

Definition

A **symmetric matrix** is a matrix that is equal to its transpose.

$$A = A^t, \quad \text{i.e.,} \quad a_{ij} = a_{ji} \ \forall i, j$$

Example 16

match

$$\begin{bmatrix} 2 & 5 \\ 5 & -4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -4 \\ 1 & 7 & 8 \\ -4 & 8 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 7 & 3 & 9 \\ -2 & 3 & 2 & -3 \\ 4 & 9 & -3 & 6 \end{bmatrix}$$

match

Remark: If and only if

- Let p and q be statements.
  Suppose that p implies q (if p then q), written $p \Rightarrow q$, and that also $q \Rightarrow p$, we say that

  "p if and only if q" (*in short* iff)

Example 17

Let A and B be symmetric matrices of the same size. Prove that the product AB is symmetric if and only if $AB = BA$.

Proof

*We have to show (a) AB is symmetric $\Rightarrow$ $AB = BA$, and the converse, (b) AB is symmetric $\Leftarrow$ $AB = BA$.

Example 17

Let A and B be symmetric matrices of the same size. Prove that the product AB is symmetric if and only if $AB = BA$.

Proof

*We have to show (a) if AB is symmetric $\Rightarrow$ $AB = BA$, and the converse, (b) AB is symmetric $\Leftarrow$ if $AB = BA$.

($\Rightarrow$) Let AB be symmetric, then

$AB = (AB)^t$ by definition of symmetric matrix

$= B^t A^t$ by Thm 2.4 (3)

$= BA$ since A and B are symmetric

($\Leftarrow$) Let $AB = BA$, then

$(AB)^t = (BA)^t$

$= A^t B^t$ by Thm 2.4 (3)

$= AB$ since A and B are symmetric

Example 18

Let A be a symmetric matrix. Prove that A^2 is symmetric.

Proof

$$(A^2)^t = (AA)^t = (A^t A^t) = AA = A^2$$

2.4 The Inverse of a Matrix

Definition

Let A be an $n \times n$ matrix. If a matrix B can be found such that $AB = BA = I_n$, then A is said to be **invertible** and B is called the **inverse** of A. If such a matrix B does not exist, then A has no inverse. (denote $B = \boldsymbol{A^{-1}}$, and $A^{-k}=(A^{-1})^k$)

Example 19

Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has inverse $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Proof

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Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has inverse $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Proof

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus $AB = BA = I_2$, proving that the matrix A has inverse B.

Theorem 2.5

The inverse of an invertible matrix is unique.

Proof

Theorem 2.5

The inverse of an invertible matrix is unique.

Proof

Let B and C be inverses of A.

Thus $AB = BA = I_n$, and $AC = CA = I_n$.

Multiply both sides of the equation $AB = I_n$ by C.

$$C(AB) = CI_n$$

$$(CA)B = C \longleftarrow \boxed{\text{Thm2.2}}$$

$$I_nB = C$$

$$B = C$$

Thus an invertible matrix has only one inverse.

Gauss-Jordan Elimination for finding the Inverse of a Matrix

Let A be an $n \times n$ matrix.

1. Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
2. Compute the reduced echelon form of $[A : I_n]$.

   If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.

   If the reduced echelon form is not of the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n, then A has no inverse.

An $n \times n$ matrix A is invertible if and only if its reduced echelon form is I_n.

Example 20

Determine the inverse of the matrix $A=\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Solution

Example 20

Determine the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Solution

$$[A : I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{(-1)R2}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx} \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+R3 \\ R2+(-1)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Thus, $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$.

Example 21

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

Solution