Linear Algebra

Chapter 2

MATRICES

2.1 Addition, Scalar Multiplication, and Multiplication of Matrices

- a_{ij}: the element of matrix A in row i and column j.
- For a square $n \times n$ matrix A, the **main diagonal** is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Definition

Two matrices are **equal** if they are of the same size and if their corresponding elements are equal.

Thus $A = B$ if $a_{ij} = b_{ij} \quad \forall\ i, j$.

($\forall$ for every, for all)

Addition of Matrices

Definition

Let A and B be matrices of the same size.

Their **sum** $A + B$ is the matrix obtained by adding together the corresponding elements of A and B.

The matrix $A + B$ will be of the same size as A and B.

If A and B are not of the same size, they cannot be added, and we say that **the sum does not exist**.

Thus if $C = A + B$, then $c_{ij} = a_{ij} + b_{ij} \ \forall i, j$.

Example 1

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}$.

Determine $A + B$ and $A + C$, if the sum exist.

Solution

Example 1

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 4 \\ 2 & 7 \end{bmatrix}$.

Determine $A + B$ and $A + C$, if the sum exist.

Solution

(1) $$A + B = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -6 \\ -3 & 1 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} 1+2 & 4+5 & 7-6 \\ 0-3 & -2+1 & 3+8 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 9 & 1 \\ -3 & -1 & 11 \end{bmatrix}.$$

(2) Because A is 2×3 matrix and C is a 2×2 matrix, they are not of the same size, $A + C$ does not exist.

Scalar Multiplication of matrices

Definition

Let A be a matrix and c be a scalar. The **scalar multiple** of A by c, denoted cA, is the matrix obtained by multiplying every element of A by c. The matrix cA will be the same size as A.

Thus if $B = cA$, then $b_{ij} = ca_{ij}\ \forall i, j$.

Example 2

Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 7 & -3 & 0 \end{bmatrix}$.

$$3A = \begin{bmatrix} 3\times 1 & 3\times(-2) & 3\times 4 \\ 3\times 7 & 3\times(-3) & 3\times 0 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 12 \\ 21 & -9 & 0 \end{bmatrix}.$$

Observe that A and $3A$ are both 2×3 matrices.

Negation and Subtraction

Definition

We now define subtraction of matrices in such a way that makes it compatible with addition, scalar multiplication, and negative. Let

$$A - B = A + (-1)B$$

Example 3

Suppose $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 8 & -1 \\ 0 & 4 & 6 \end{bmatrix}$.

$$A - B = \begin{bmatrix} 5-2 & 0-8 & -2-(-1) \\ 3-0 & 6-4 & -5-6 \end{bmatrix} = \begin{bmatrix} 3 & -8 & -1 \\ 3 & 2 & -11 \end{bmatrix}.$$

Multiplication of Matrices

Definition

Let the number of columns in a matrix A be the same as the number of rows in a matrix B. The product AB then exists.

Let A: $m\times n$ matrix, B: $n\times k$ matrix,
The **product matrix $C=AB$** has elements

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

C is a $m\times k$ matrix.

If the number of columns in A does not equal the number of row B, we say that **the product does not exist**.

Example 4

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} 6 & -2 & 5 \end{bmatrix}$.

Determine AB, BA, and AC, if the products exist.

Solution.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1\times 5)+(3\times 3) & (1\times 0)+(3\times(-2)) & (1\times 1)+(3\times 6) \\ (2\times 5)+(0\times 3) & (2\times 0)+(0\times(-2)) & (2\times 1)+(0\times 6) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}.$$

BA and AC do not exist. **Note.** In general, $AB \neq BA$.

Example 5

Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix}$. Determine AB.

$$AB = \begin{bmatrix} 2 & 1 \\ 7 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} & \begin{bmatrix} -3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 0+5 \\ -7+0 & 0+0 \\ 3-6 & 0-10 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -7 & 0 \\ -3 & -10 \end{bmatrix}$$

Example 6

Let $C = AB$, $A = \begin{bmatrix} 2 & 1 \\ \boxed{-3 \quad 4} \end{bmatrix}$ and $B = \begin{bmatrix} -7 & 3 & \boxed{2} \\ 5 & 0 & \boxed{1} \end{bmatrix}$ Determine c_{23}.

$$c_{23} = \begin{bmatrix} -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-3 \times 2) + (4 \times 1) = -2$$

Size of a Product Matrix

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

A	B	$= AB$
$m \times r$	$r \times n$	$m \times n$

Example 7

If A is a 5×6 matrix and B is an 6×7 matrix.
Because A has six columns and B has six rows. Thus AB exits.
And AB will be a 5×7 matrix.

Special Matrices

Definition

A **zero matrix** is a matrix in which all the elements are zeros.

A **diagonal matrix** is a square matrix in which all the elements not on the main diagonal are zeros.

An **identity matrix** is a diagonal matrix in which every diagonal element is 1.

$$0_{mn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

zero matrix

diaginal matrix A

identity matrix

Theorem 2.1

Let A be $m \times n$ matrix and O_{mn} be the zero $m \times n$ matrix. Let B be an $n \times n$ square matrix. O_n and I_n be the zero and identity $n \times n$ matrices. Then

$$A + O_{mn} = O_{mn} + A = A$$
$$BO_n = O_nB = O_n$$
$$BI_n = I_nB = B$$

Example 8

Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$.

$$A + O_{23} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 5 & 8 \end{bmatrix} = A$$

$$BO_2 = \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_2$$

$$BI_2 = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} = B$$

Homework

- Exercises: 1, 3, 5, 9, 10, 11 from page 71 to 73.

Exercise 17 p.72

Let A be a matrix whose third row is all zeros. Let B be any matrix such that the product AB exists.
Prove that the third row of AB is all zeros.

Solution

$$(AB)_{3i} = [a_{31}\ a_{32} \cdots a_{3n}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = [0\ 0 \cdots 0] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = 0, \forall i.$$

2.2 Algebraic Properties of Matrix Operations

Theorem 2.2 -1

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Addition and scalar Multiplication

1. $A + B = B + A$ *Commutative property of addition*
2. $A + (B + C) = (A + B) + C$ *Associative property of addition*
3. $A + O = O + A = A$ (*where O is the appropriate zero matrix*)
4. $c(A + B) = cA + cB$ *Distributive property of addition*
5. $(a + b)C = aC + bC$ *Distributive property of addition*
6. $(ab)C = a(bC)$

Theorem 2.2 -2

Let A, B, and C be matrices and a, b, and c be scalars. Assume that the size of the matrices are such that the operations can be performed.

Properties of Matrix Multiplication

1. $A(BC) = (AB)C$ *Associative property of multiplication*
2. $A(B + C) = AB + AC$ *Distributive property of multiplication*
3. $(A + B)C = AC + BC$ *Distributive property of multiplication*
4. $AI_n = I_nA = A$ (*where* I_n *is the appropriate identity matrix*)
5. $c(AB) = (cA)B = A(cB)$

Note: $AB \neq BA$ in general. *Multiplication of matrices is not commutative.*

Proof of Theorem 2.2 (A+B=B+A)

Consider the (i,j)th elements of matrices $A+B$ and $B+A$:

$$(A+B)_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (B+A)_{ij}.$$

$$\therefore\ A+B=B+A$$

Example 9

Let $A=\begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix}$, and $C=\begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$.

$$A+B+C = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -7 \\ 8 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+0 & 3-7-2 \\ -4+8+5 & 5+1-1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 9 & 5 \end{bmatrix}.$$

Arithmetic Operations

If A is an $m \times r$ matrix and B is $r \times n$ matrix, the number of scalar multiplications involved in computing the product AB is mrn.

Consider three matrices A, B and C such that the product ABC exists.
Compare the number of multiplications involved in the two ways $(AB)C$ and $A(BC)$ of computing the product ABC

Example 10

Let $A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$, $B=\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & -2\end{bmatrix}$, and $C=\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}$. Compute ABC.

Solution.

Which method is better?

Count the number of multiplications.

(1) $(AB)C$

$$AB=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & -2\end{bmatrix}=\begin{bmatrix}-2 & 1 & -1\\ 1 & 3 & 11\end{bmatrix}.$$

$$(AB)C=\begin{bmatrix}-2 & 1 & -1\\ 1 & 3 & 11\end{bmatrix}\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}-9\\ 1\end{bmatrix}.$$

$2\times6+3\times2$

$=12+6=18$

(2) $A(BC)$

$$BC=\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & 2\end{bmatrix}\begin{bmatrix}4\\ -1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ -4\end{bmatrix}$$

$3\times2+2\times2$

$=6+4=10$

$$A(BC)=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}-1\\ -4\end{bmatrix}=\begin{bmatrix}-9\\ 1\end{bmatrix}.$$

$\therefore$ $A(BC)$ is better.

Caution

In algebra we know that the following cancellation laws apply.

- If $ab = ac$ and $a \neq 0$ then $b = c$.
- If $pq = 0$ then $p = 0$ or $q = 0$.

However the corresponding results are not true for matrices.

- $AB = AC$ does not imply that $B = C$.
- $PQ = O$ does not imply that $P = O$ or $Q = O$.

Example 11

(1) Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 8 \\ 3 & -2 \end{bmatrix}$.

Observe that $AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, but $B \neq C$.

(2) Consider the matrices $P = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, and $Q = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}$.

Observe that $PQ = O$, but $P \neq O$ and $Q \neq O$.

Powers of Matrices

Definition

If A is a square matrix, then

$$A^k = \underbrace{AA\cdots A}_{k \text{ times}}$$

Theorem 2.3

If A is an $n \times n$ square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$.
2. $(A^r)^s = A^{rs}$.
3. $A^0 = I_n$ (by definition)

Example 12

If $A=\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4.

Solution

$$A^2=\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB$$

Solution

Example 12

If $A=\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$, compute A^4.

Solution

$$A^2=\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$A^4=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 11 & -10 \\ -5 & 6 \end{bmatrix}.$$

Example 13 Simplify the following matrix expression.

$$A(A+2B)+3B(2A-B)-A^2+7B^2-5AB$$

Solution

$$\begin{aligned} &A(A+2B)+3B(2A-B)-A^2+7B^2-5AB \\ &=A^2+2AB+6BA-3B^2-A^2+7B^2-5AB \\ &=\underline{-3AB+6BA}+4B^2 \end{aligned}$$

We can't add the two matrices！

Systems of Linear Equations

A system of *m* linear equations in *n* variables as follows

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Systems of Linear Equations

A system of m linear equations in n variables as follows

$$\begin{array}{c}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n = b_m
\end{array}$$

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ \text{and } B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can write the system of equations in the matrix form

$$\boldsymbol{AX = B}$$

Idempotent and Nilpotent Matrices
(Exercises 24-30 p.83 and 14 p.93)

Definition

(1) A square matrix A is said to be **idempotent** if $A^2=A$.

(2) A square matrix A is said to **nilpotent** if there is a positive integer p such that $A^p=0$. The least integer p such that $A^p=0$ is called the **degree of nilpotency** of the matrix.

Example 14

$$(1)\ A=\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}, A^2 =$$

$$(2)\ B=\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}, B^2 =$$

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Example 14

$$(1)\ A=\begin{bmatrix}3 & -6\\ 1 & -2\end{bmatrix},\ A^2=\begin{bmatrix}3 & -6\\ 1 & -2\end{bmatrix}=A.$$

$$(2)\ B=\begin{bmatrix}3 & -9\\ 1 & -3\end{bmatrix},\ B^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.\ \text{The degree of nilpotency}: 2$$

Class activity

The following matrix is nilpotent. Find the degree of **nilpotency**

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quiz 1

Solve the following systems of linear equations by Gaussian elimination method:

$$\begin{aligned} 2x - 2y + 3z &= 2 \\ x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \end{aligned}$$

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$$\begin{aligned} 2x - 2y + 3z &= 2 \\ x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \end{aligned}$$

SOLUTION

(i) $2x - 2y + 3z = 2$, $x + 2y - z = 3$, $3x - y + 2z = 1$

The given system equations is

$$2x - 2y + 3z = 2 \quad \text{-----(1)}$$
$$x + 2y - z = 3 \quad \text{-----(2)}$$
$$3x - y + 2z = 1 \quad \text{-----(3)}$$

The augmented matrix of the above system is $[A \mid B] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right]$

Transforming augmented matrix to echelon form we get

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array}\right]$$

The given system of equations using this row-echelon form reduces to

$$x + 2y - z = 3 \quad \text{-----(4)}$$
$$-6y + 5z = -4 \quad \text{-----(5)}$$
$$-y = -4 \quad \text{-----(6)}$$
$$\Rightarrow \quad y = 4$$

Substituting $y = 4$ in equation (5) we get

$$-6 \times 4 + 5z = -4 \quad \Rightarrow \quad 5z = -4 + 24 = 20$$
$$z = \frac{20}{5} = 4$$

Substituting $y = 4$ and $z = 4$ in equation (1) we get

$$x + 2 \times 4 - 4 = 3 \quad \Rightarrow \quad x + 8 - 4 = 3 \quad \Rightarrow \quad x = 3 - 4 = -1$$

$\therefore$ The required solutions of the system is $x = -1$, $y = 4$, $z = 4$