Linear Algebra

Chapter 4

VECTOR SPACES

4.5 Linear Combinations of Vectors

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V.
We say that $\mathbf{v}$, a vector of V, is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, if there exist scalars $c_1, c_2, \ldots, c_m$ such that $\mathbf{v}$ can be written $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$.

Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

Suppose

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = -1 \\ -3c_2 + c_3 = 7 \\ 2c_1 + 2c_3 = 8 \\ c_1 + 2c_2 = -1 \end{cases}$$

This system has the unique solution $c_1 = 3$, $c_2 = -2$, $c_3 = 1$.
Therefore

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4$.

Solution

Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4.$

Solution

Suppose $c_1 g + c_2 h = f.$

Then

$$c_1(x^2 + 3x - 1) + c_2(2x^2 - x + 4) = x^2 + 10x - 7$$

$$(c_1 + 2c_2)x^2 + (3c_1 - c_2)x - c_1 + 4c_2 = x^2 + 10x - 7$$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 1 \\ 3c_1 - c_2 = 10 \\ -c_1 + 4c_2 = -7 \end{cases} \quad \Rightarrow c_1 = 3,\ c_2 = -1 \quad \Rightarrow f = 3g - h.$$

Spanning Sets

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are said to **span** a vector space if every vector in the space can be expressed as a *linear combination* of these vectors.

In this case $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called a **spanning set**.

4.6 Linear Dependence and Independence

The concepts of dependence and independence of vectors are useful tools in constructing "efficient" spanning sets for vector spaces – sets in which there are no redundant vectors.

Definition

(a) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$

(b) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ is **linearly independent** if $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$ can only be satisfied when $c_1 = 0, \ldots, c_m = 0$.

Example 1

Show that the set $\{(1, 2, 3), (-2, 1, 1), (8, 6, 10)\}$ is linearly dependent in $\mathbf{R}^3$.

Example 1

Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

Solution

Suppose $c_1(1, 2, 3) + c_2(-2, 1, 1) + c_3(8, 6, 10) = \mathbf{0}$

$$\Rightarrow \quad (c_1, 2c_1, 3c_1) + (-2c_2, c_2, c_2) + (8c_3, 6c_3, 10c_3) = \mathbf{0}$$

$$(c_1 - 2c_2 + 8c_3, 2c_1 + c_2 + 6c_3, 3c_1 + c_2 + 10c_3) = \mathbf{0}$$

$$\Rightarrow \begin{cases} c_1 - 2c_2 + 8c_3 = 0 \\ 2c_1 + c_2 + 6c_3 = 0 \\ 3c_1 + c_2 + 10c_3 = 0 \end{cases} \Rightarrow \quad \begin{matrix} c_1 = 4 \\ c_2 = -2 \\ c_3 = -1 \end{matrix}$$

Thus $\quad 4(1, 2, 3) - 2(-2, 1, 1) - (8, 6, 10) = \mathbf{0}$

The set of vectors is linearly dependent.

Example 2

Show that the set {(3, –2, 2), (3, –1, 4), (1, 0, 5)} is linearly independent in $\mathbf{R}^3$.

Solution

Suppose $c_1(3,-2,2)+c_2(3,-1,4)+c_3(1,0,5)=\mathbf{0}$

$$\Rightarrow\ (3c_1,-2c_1,2c_1)+(3c_2,-c_2,4c_2)+(c_3,0,5c_3)=\mathbf{0}$$

$$(3c_1+3c_2+c_3,-2c_1-c_2,2c_1+4c_2+5c_3)=\mathbf{0}$$

$$\Rightarrow\ \begin{cases} 3c_1+3c_2+c_3=0 \\ \quad -2c_1-c_2=0 \\ 2c_1+4c_2+5c_3=0 \end{cases}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.
Thus the set is linearly independent.

Example 3

Consider the functions $f(x) = x^2 + 1$, $g(x) = 3x - 1$, $h(x) = -4x + 1$ of the vector space P_2 of polynomials of degree ≤ 2.
Show that the set of functions { f, g, h } is linearly independent.

Solution

Suppose

$$c_1 f + c_2 g + c_3 h = \mathbf{0}$$

Since for any real number x,

$$c_1(x^2+1) + c_2(3x-1) + c_3(-4x+1) = \mathbf{0}$$

$$x^2 c_1 + x(3c_2 - 4c_3) + (c_1 - c_2 + c_3) = 0$$

$$c_1 = 0$$

$$3c_2 - 4c_3 = 0$$

$$c_1 - c_2 + c_3 = 0$$

OR

Consider three convenient values of x. We get

$$x = 0: c_1 - c_2 + c_3 = 0$$

$$x = 1: 2c_1 + 2c_2 - 3c_3 = 0$$

$$x = -1: 2c_1 - 4c_2 + 5c_3 = 0$$

It can be shown that this system of three equations has the unique solution

$$c_1 = 0, c_2 = 0, c_3 = 0$$

Thus $c_1 f + c_2 g + c_3 h = \mathbf{0}$ implies that $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
The set $\{ f, g, h \}$ is linearly independent.

Theorem 4.7

A set consisting of two or more vectors in a vector space is linearly dependent *if and only if* it is possible to express one of the vectors as a linearly combination of the other vectors.

Example 4

The set of vectors {$\mathbf{v}_1$=(1, 2, 1) , $\mathbf{v}_2$ =(-1, -1, 0) , $\mathbf{v}_3$ = (0, 1,1)}

is linearly dependent, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Thus, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

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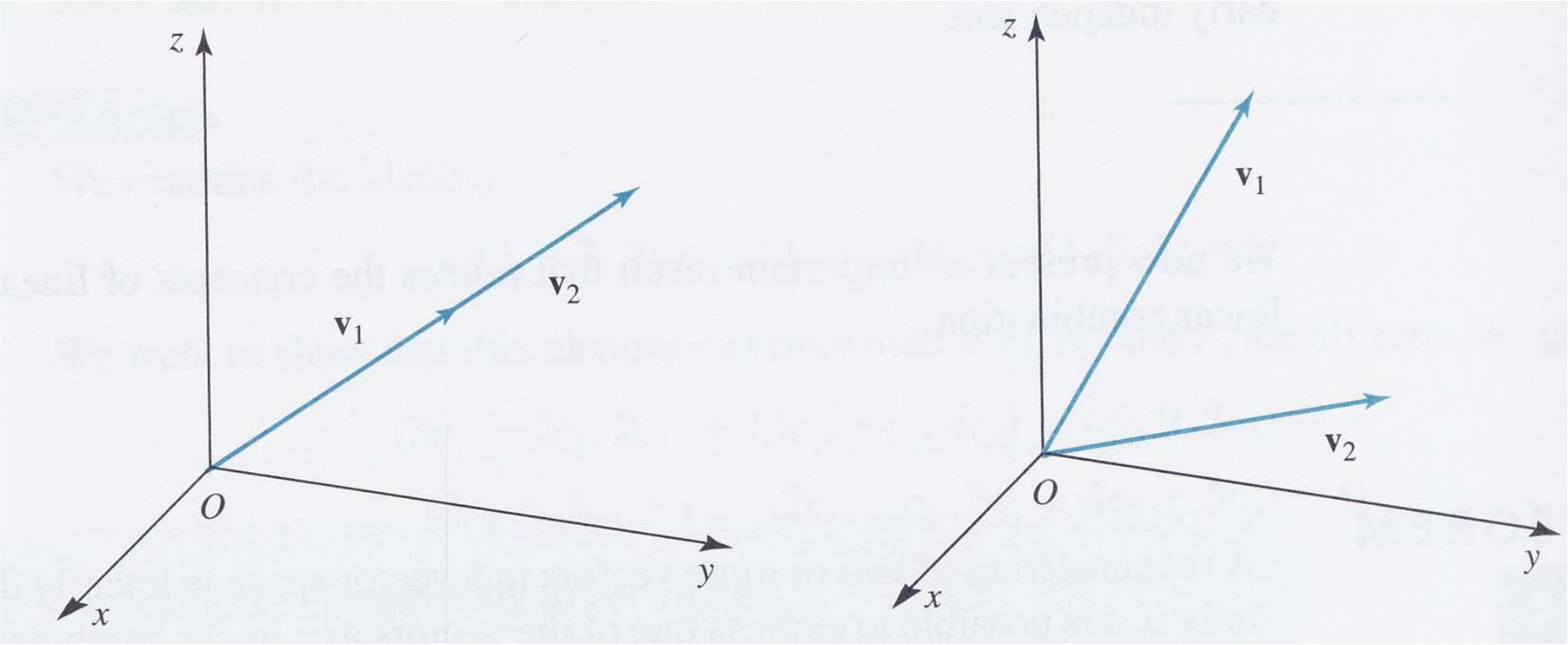
Linear Dependence of $\{v_1, v_2\}$

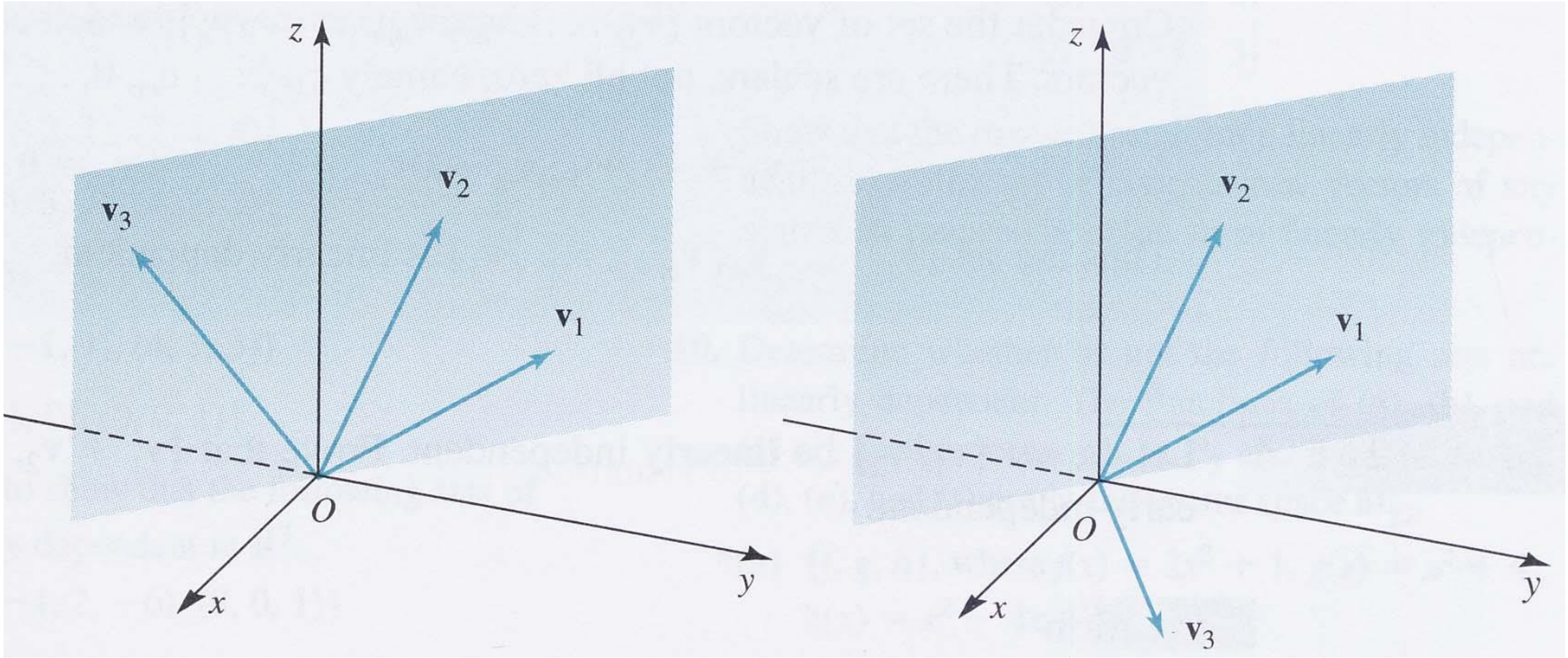


$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent; vectors lie on a line

$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent; vectors do not lie on a line

Figure 4.14 Linear dependence and independence of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbf{R}^3$.

Linear Dependence of {v_1, v_2, v_3}



{$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} linearly dependent; vectors lie in a plane

{$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} linearly independent; vectors do not lie in a plane

Figure 4.15 Linear dependence and independence of {$\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$} in $\mathbf{R}^3$.

Theorem 4.8

Let V be a vector space. Any set of vectors in V that contains the zero is linearly dependent.

Proof

Consider the set $\{ \mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m \}$, which contains the zero vectors. Let us examine the identity

$$c_1\mathbf{0} + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$$

We see that the identity is true for $c_1 = 1$, $c_2 = 0$, …, $c_m = 0$ (not all zero).

Thus the set of vectors is linearly dependent, proving the theorem.

4.7 Bases and Dimension

Definition

A finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for a vector space V if the set spans V and is linearly independent.

Standard Basis

The set of n vectors

$$\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 1)\}$$

is a basis for $\mathbf{R}^n$. This basis is called the **standard basis** for $\mathbf{R}^n$.

Example 1

Show that the set {(1, 0, –1), (1, 1, 1), (1, 2, 4)} is a basis for $\mathbf{R}^3$.

Solution

(span)

Let (x_1, x_2, x_3) be an arbitrary element of $\mathbf{R}^3$.

Suppose

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(1, 2, 4)$$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = x_1 \\ a_2 + 2a_3 = x_2 \\ -a_1 + a_2 + 4a_3 = x_3 \end{cases} \Rightarrow \begin{cases} a_1 = 2x_1 - 3x_2 + x_3 \\ a_2 = -2x_1 + 5x_2 - 2x_3 \\ a_3 = x_1 - 2x_2 + x_3 \end{cases}$$

Thus the set spans the space.

(linearly independent)

Consider the identity

$$b_1(1, 0, -1) + b_2(1, 1, 1) + b_3(1, 2, 4) = (0, 0, 0)$$

The identity leads to the system of equations

$$\begin{aligned} b_1 + b_2 + b_3 &= 0 \\ b_2 + 2b_3 &= 0 \\ -b_1 + b_2 + 4b_3 &= 0 \end{aligned}$$

$\Rightarrow$ $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$ is the unique solution.

Thus the set is linearly independent.

$\Rightarrow$ {(1, 0, –1), (1, 1, 1), (1, 2, 4)} spans $\mathbf{R}^3$ and is linearly independent.

$\Rightarrow$ It forms a basis for $\mathbf{R}^3$.

Example 2

Show that $\{ f, g, h \}$, where $f(x) = x^2 + 1$, $g(x) = 3x - 1$, and $h(x) = -4x + 1$ is a basis for P_2.

Solution

(linearly independent) see Example 3 of the previous section.

(span). Let p be an arbitrary function in P_2.

p is thus a polynomial of the form

$$p(x) = bx^2 + cx + d$$

Suppose $\quad p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$

for some scalars a_1, a_2, a_3.

This gives

$$\begin{aligned} bx^2 + cx + d &= a_1(x^2 + 1) + a_2(3x - 1) + a_3(-4x + 1) \\ &= a_1 x^2 + (3a_2 - 4a_3)x + (a_1 - a_2 + a_3) \end{aligned}$$

$$\Rightarrow \begin{cases} a_1 = b \\ 3a_2 - 4a_3 = c \\ a_1 - a_2 + a_3 = d \end{cases} \Rightarrow \begin{cases} a_1 = b \\ a_2 = 4b - 4d - c \\ a_3 = 3b - 3d - c \end{cases}$$

Thus the polynomial *p* can be expressed

$$p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$$

The functions *f*, *g*, and *h* span P_2.

They form a basis for P_2.

Theorem 4.10

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.
If $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a set of more than n vectors in V, then this set is linearly dependent.

Proof

Suppose

$$c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m = 0 \qquad (1)$$

We will show that values of $c_1, \ldots, c_m$ are not all zero.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Thus each of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n \\ &\vdots \\ \mathbf{w}_m &= a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n \end{aligned}$$

Proof Excluded

Substituting for $\mathbf{w}_1, \ldots, \mathbf{w}_m$ into Equation (1) we get

$$c_1(a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n) + \cdots + c_m(a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n) = \mathbf{0}$$

Rearranging, we get

$$(c_1a_{11} + c_2a_{21} + \cdots + c_m a_{m1})\mathbf{v}_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_m a_{mn})\mathbf{v}_n = \mathbf{0}$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linear independent,

$$\begin{gathered} a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m = 0 \\ \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m = 0 \end{gathered}$$

Since $m > n$, there are many solutions in this system.

Thus the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is linearly dependent.

Proof Excluded

Theorem 4.11

Any two bases for a vector space V consist of the same number of vectors.

Proof

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be two bases for V.

By Theorem 4.10,

$$m \leq n \text{ and } n \leq m$$

Thus $n = m$.

Proof Excluded