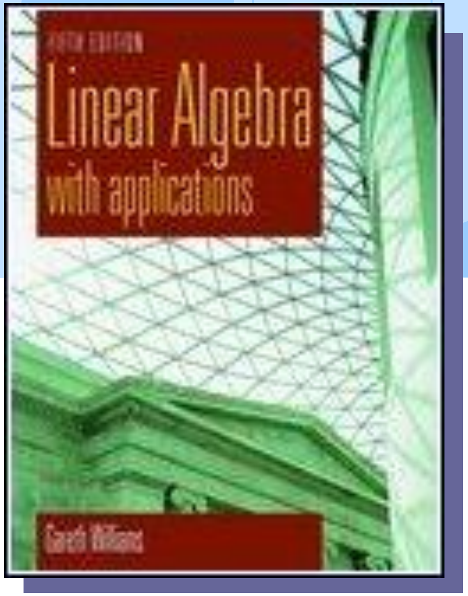
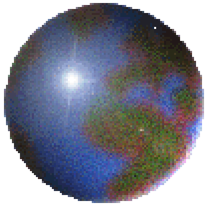
Linear Algebra





Chapter 5

EIGENVALUES AND EIGENVECTORS

5.1 Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A <u>*scalar*</u> λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ in $\mathbf{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an **eigenvector** corresponding to λ.

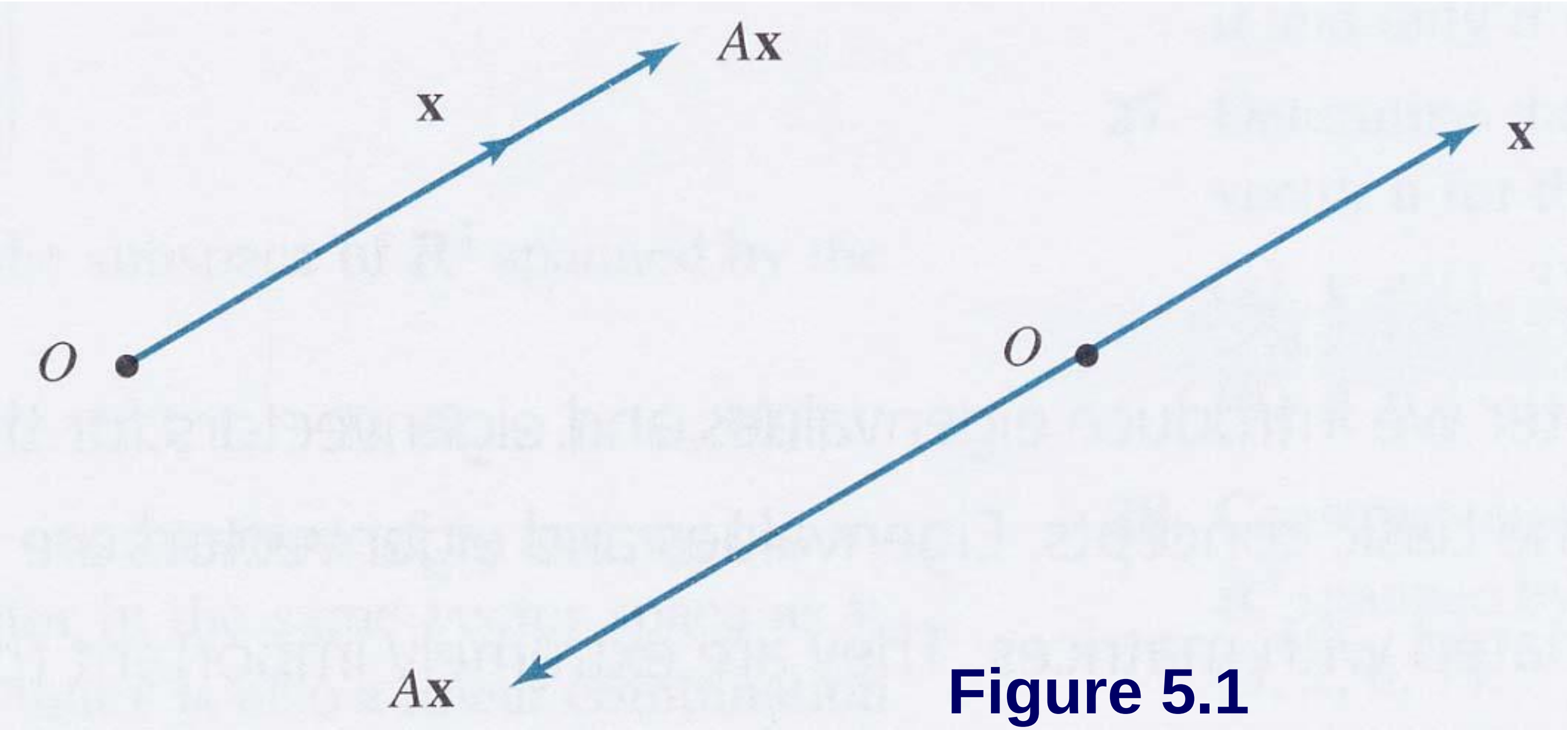


Figure 5.1

Computation of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Thus $A\mathbf{x} = \lambda\mathbf{x}$. This equation may be written

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

given

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

Solving the equation $|A - \lambda I_n| = 0$ for λ leads to all the eigenvalues of A.

On expending the determinant $|A - \lambda I_n|$, we get a polynomial in λ.

This polynomial is called the **characteristic polynomial** of A.

The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation** of A.

Example 1

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution

Example 1

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution Let us first derive the characteristic polynomial of A. We get

$$A - \lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

$$|A - \lambda I_2| = (-4-\lambda)(5-\lambda) + 18 = \lambda^2 - \lambda - 2$$

We now solve the characteristic equation of A.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 2 \text{ or } -1$$

The eigenvalues of A are 2 and –1.

The corresponding eigenvectors are found by using these values of λ in the equation$(A - \lambda I_2)\mathbf{x} = \mathbf{0}$. There are many eigenvectors corresponding to each eigenvalue.

- For $\underline{\lambda = 2}$

  We solve the equation $(A - 2I_2)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

The matrix $(A - 2I_2)$ is obtained by subtracting 2 from the diagonal elements of A. We get

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

This leads to the system of equations

$$-6x_1 - 6x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

giving $x_1 = -x_2$. The solutions to this system of equations are $x_1 = -r$, $x_2 = r$, where r is a scalar. Thus the eigenvectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$\mathrm{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- For $\lambda = -1$

  We solve the equation $(A + 1I_2)x = 0$ for x.

The matrix $(A + 1I_2)$ is obtained by adding 1 to the diagonal elements of A. We get

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

This leads to the system of equations

$$-3x_1 - 6x_2 = 0$$

$$3x_1 + 6x_2 = 0$$

Thus $x_1 = -2x_2$. The solutions to this system of equations are $x_1 = -2s$ and $x_2 = s$, where s is a scalar. Thus the **eigenvectors** of A corresponding to $\lambda = -1$ are nonzero vectors of the form

$$\mathrm{v}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example 2

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Solution The matrix $A - \lambda I_3$ is obtained by subtracting λ from the diagonal elements of A. Thus

$$A - \lambda I_3 = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I_3|$. Using row and column operations to simplify determinants, we get

$$|A - \lambda I_3| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(9-\lambda)(2-\lambda)-8] = (1-\lambda)[\lambda^2 - 11\lambda + 10]$$

$$= (1-\lambda)(\lambda-10)(\lambda-1) = -(\lambda-10)(\lambda-1)^2$$

We now solving the characteristic equation of A:

$$-(\lambda-10)(\lambda-1)^2 = 0$$

$$\lambda = 10 \text{ or } 1$$

The eigenvalues of A are 10 and 1.

The corresponding eigenvectors are found by using three values of λ in the equation $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$.

- $\lambda_1 = 10$

We get

$$(A - 10I_3)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

The solution to this system of equations are $x_1 = 2r$, $x_2 = 2r$, and $x_3 = r$, where r is a scalar.
Thus the eigenspace of $\lambda_1 = 10$ is the one-dimensional space of vectors of the form.

$$r\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

- $\lambda_2 = 1$

  Let $\lambda = 1$ in $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$. We get

$$(A - 1I_3)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

The solution to this system of equations can be shown to be $x_1 = -s - t$, $x_2 = s$, and $x_3 = 2t$, where s and t are scalars. Thus the eigenspace of $\lambda_2 = 1$ is the space of vectors of the form.

$$\begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix}$$

Separating the parameters *s* and *t*, we can write

$$\begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Thus the eigenspace of $\lambda = 1$ is a two-dimensional subspace of $\mathbf{R}^3$ with basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

If an eigenvalue occurs as a *k* times repeated root of the characteristic equation, we say that it is of **multiplicity** *k*. Thus λ=10 has multiplicity 1, while λ=1 has multiplicity 2 in this example.

5.3 Diagonalization of Matrices

Definition

Let A and B be square matrices of the same size. B is said to be **similar** to A if there exists an invertible matrix C such that $B = C^{-1}AC$. The transformation of the matrix A into the matrix B in this manner is called a **similarity transformation**.

Example 3

Consider the following matrices A and C with C is invertible. Use the similarity transformation $C^{-1}AC$ to transform A into a matrix B.

$$A=\begin{bmatrix}7 & -10\\ 3 & -4\end{bmatrix} \quad C=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

Solution

$$B=C^{-1}AC=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}7 & -10\\ 3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}\begin{bmatrix}7 & -10\\ 3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -10\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$$

Theorem 5.3

Similar matrices have the same eigenvalues.

Proof

Let A and B be similar matrices. Hence there exists a matrix C such that $B = C^{-1}AC$.

The characteristic polynomial of B is $|B - \lambda I_n|$. Substituting for B and using the multiplicative properties of determinants, we get

$$
\begin{aligned}
|B-\lambda I| &= |C^{-1}AC-\lambda I| = |C^{-1}(A-\lambda I)C| \\
&= |C^{-1}||A-\lambda I||C| = |A-\lambda I||C^{-1}||C| \\
&= |A-\lambda I||C^{-1}C| = |A-\lambda I||I| \\
&= |A-\lambda I|
\end{aligned}
$$

The characteristic polynomials of A and B are identical. This means that their eigenvalues are the same.

Definition

A *square matrix* A is said to be **diagonalizable** if there exists a matrix C such that $D = C^{-1}AC$ is a diagonal matrix.

Theorem 5.4

Let A be an $n \times n$ matrix.

(a) If A has n linearly independent eigenvectors, it is diagonalizable. *The matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $C^{-1}AC$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.*

(b) If A is diagonalizable, then it has n linearly independent eigenvectors

Example 4

(a) Show that the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable.

(b) Find a diagonal matrix *D* that is similar to *A*.

(c) Determine the similarity transformation that diagonalizes *A*.

Solution

(a) The eigenvalues and corresponding eigenvector of this matrix were found in Example 1 of Section 5.1. They are

$$\lambda_1 = 2, \mathbf{v}_1 = r\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \text{ and } \mathbf{v}_2 = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since *A*, a 2×2 matrix, has two linearly independent eigenvectors, it is diagonalizable.

(b) A is similar to the diagonal matrix D, which has diagonal elements $\lambda_1 = 2$ and $\lambda_2 = -1$. Thus

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \text{is similar to } D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(c) Select two convenient linearly independent eigenvectors, say

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{and } \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let these vectors be the column vectors of the diagonalizing matrix C.

$$C = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

We get

$$C^{-1}AC = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Note

If A is similar to a diagonal matrix D under the transformation $C^{-1}AC$, then it can be shown that $A^k = CD^kC^{-1}$.
This result can be used to compute A^k. Let us derive this result and then apply it.

$$D^k = (C^{-1}AC)^k = \underbrace{(C^{-1}AC) \quad \cdots \quad (C^{-1}AC)}_{k \text{ times}} = C^{-1}A^kC$$

This leads to

$$\boxed{A^k = CD^kC^{-1}}$$

Example 5

Compute A^9 for the following matrix A.

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution

A is the matrix of the previous example. Use the values of C and D from that example. We get

$$D^9 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^9 = \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix} = \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^9 = CD^9C^{-1}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -514 & -1026 \\ 513 & 1025 \end{bmatrix}$$

Example 6

Show that the following matrix A is not diagonalizable.

$$A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$$

Solution

$$A - \lambda I_2 = \begin{bmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I_2| = 0 \Rightarrow (5-\lambda)(-1-\lambda) + 9 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)(\lambda - 2) = 0$$

There is a single eigenvalue, $\lambda = 2$. We find he corresponding eigenvectors. $(A - 2I)\,\mathbf{x} = \mathbf{0}$ gives $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad 3x_1 - 3x_2 = 0.$

Thus $x_1 = r$, $x_2 = r$. The eigenvectors are nonzero vectors of the form

$$r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace is a one-dimensional space. A is a 2×2 matrix, but it does not have two linearly independent eigenvectors. Thus A is not diagonalizable.

Theorem 5.5

Let A be an $n \times n$ symmetric matrix.

(a) All the eigenvalues of A are real numbers.

(b) The dimension of an eigenspace of A is the multiplicity of the eigenvalues as a root of the characteristic equation.

(c) A has n linearly independent eigenvectors.

Exercise 5.3 p 301: 1, 3, 5.