Linear Combinations of Vectors

EXAMPLE 6 Determine whether the vector $(8, 0, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, -1, 1)$.

SOLUTION

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SOLUTION

We examine the following identity for values of c_1, c_2, and c_3.

$$c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, -1, 1) = (8, 0, 5)$$

Multiplying the scalars and then adding the vectors, we get

$$(c_1 + 2c_3, 2c_1 + c_2 - c_3, 3c_1 + 4c_2 + c_3) = (8, 0, 5)$$

Equating the components of the vectors, we get the following system of linear equations:

$$\begin{aligned} c_1 \quad\quad\;\; + 2c_3 &= 8 \\ 2c_1 + \;\; c_2 - \;\; c_3 &= 0 \\ 3c_1 + 4c_2 + \;\; c_3 &= 5 \end{aligned}$$

It can be shown that this system of equations has the unique solution,

$$c_1 = 2, c_2 = -1, c_3 = 3$$

Thus, the vector (8, 0, 5) can be written in one way as a linear combination,

$$(8, 0, 5) = 2(1, 2, 3) - 1(0, 1, 4) + 3(2, -1, 1)$$

Vector addition and Scalar Multiplication of Column Vectors

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \text{and} \quad c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

For example, in $\mathbf{R}^2$,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \quad \text{and} \quad 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

The Subspaces of R^n

There are two properties that have to be checked to see if a subset is a subspace or not.

1. The sum of two arbitrary vectors of the subset must lie in the subset,
2. The scalar multiple of an arbitrary vector must also lie in the subset.

In other words, the subset must be closed under addition and under scalar multiplication

The Subspaces of R^n

EXAMPLE 2 Consider the subset V of $\mathbf{R}^3$ of vectors of the form $(a, 2a, 3a)$, where the second component is twice the first, and the third is three times the first. Let us show that V is a subspace of $\mathbf{R}^3$. Let $(a, 2a, 3a)$ and $(b, 2b, 3b)$ be two vectors in V, and let k be a scalar. Then

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$$\begin{aligned}(a, 2a, 3a) + (b, 2b, 3b) &= (a + b, 2a + 2b, 3a + 3b) \\ &= (a + b, 2(a + b), 3(a + b))\end{aligned}$$

This is a vector in V since the second component is twice the first, and the third is three times the first. V is closed under addition. Further,

$$k(a, 2a, 3a) = (ka, 2ka, 3ka)$$

This vector is in V. V is closed under scalar multiplication.

The Subspaces of R^n

EXAMPLE 3 Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

SOLUTION

The Subspaces of R^n

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SOLUTION

W consists of all elements of $\mathbf{R}^3$ for which the second component is the square of the first. For example, the vector (2, 4, 3) is in W, whereas the vector (2, 5, 3) is not.

Let us check for closure under addition. Let (a, a^2, b) and (c, c^2, d) be elements of W. We get

$$\begin{aligned}(a, a^2, b) + (c, c^2, d) &= (a + c, a^2 + c^2, b + d)\\ &\neq (a + c, (a + c)^2, b + d)\end{aligned}$$

Thus $(a, a^2, b) + (c, c^2, d)$ is not an element of W. The set W is not closed under addition. Thus, W is not a subspace.

The Subspaces of R^n

EXAMPLE 3 Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

SOLUTION

This completes the proof that W is not a subspace. Let us, however, illustrate the check for closure under scalar multiplication. Let k be a scalar. We get

$$\begin{aligned} k(a, a^2, b) &= (ka, ka^2, kb) \\ &\neq (ka, (ka)^2, kb) \end{aligned}$$

Thus $k(a, a^2, b)$ is not an element of W. W is not closed under scalar multiplication either.

The Subspaces of R^n

EXAMPLE 4 Consider the following homogeneous system of linear equations:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

It can be shown that there are many solutions $x_1 = 2r, x_2 = 5r, x_3 = r$. We can write these solutions as vectors in $\mathbf{R}^3$ as follows

$$(2r, 5r, r)$$

The Subspaces of R^n

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$$(2r, 5r, r)$$

The set of solutions W thus consists of vectors for which the first component is twice the third, and the second component is five times the third component. The vector $(4, 10, 2)$ for example is in W while $(4, 9, 2)$ is not. Let us show that W is closed under addition and under scalar multiplication. Let $\mathbf{u} = (2r, 5r, r)$ and $\mathbf{v} = (2s, 5s, s)$ be arbitrary vectors in W. We get

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (2r + 2s, 5r + 5s, r + s) \\ &= (2(r + s), 5(r + s), r + s) \end{aligned}$$

and

$$\begin{aligned} k\mathbf{u} &= k(2r, 5r, r) \\ &= (2kr, 5kr, kr) \end{aligned}$$

In $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ the first component is twice the third, and the second component is five times the third. They are both in W. W is closed under addition and under scalar multiplication. It is a subspace of $\mathbf{R}^3$.

The Subspaces of R^n

EXAMPLE 5 Consider the following system of homogeneous linear equations.

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 0 \\ x_1 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 + 4x_4 &= 0 \end{aligned}$$

It can be shown that there are many solutions $x_1 = 3r - 2s, x_2 = 4r, x_3 = r, x_4 = s$. Write these solutions as vectors in $\mathbf{R}^4$,

$$(3r - 2s, 4r, r, s)$$

It can be shown that this set W of vectors is closed under addition and scalar multiplication, and is thus a subspace of $\mathbf{R}^4$. Separate the variables in the general solution.

$$(3r - 2s, 4r, r, s) = r(3, 4, 1, 0) + s(-2, 0, 0, 1)$$

This implies that every vector in W can be expressed as a linear combination of $(3, 4, 1, 0)$ and $(-2, 0, 0, 1)$. The vectors $(3, 4, 1, 0)$ and $(-2, 0, 0, 1)$ span W. W will be a plane through the origin in a four-space.

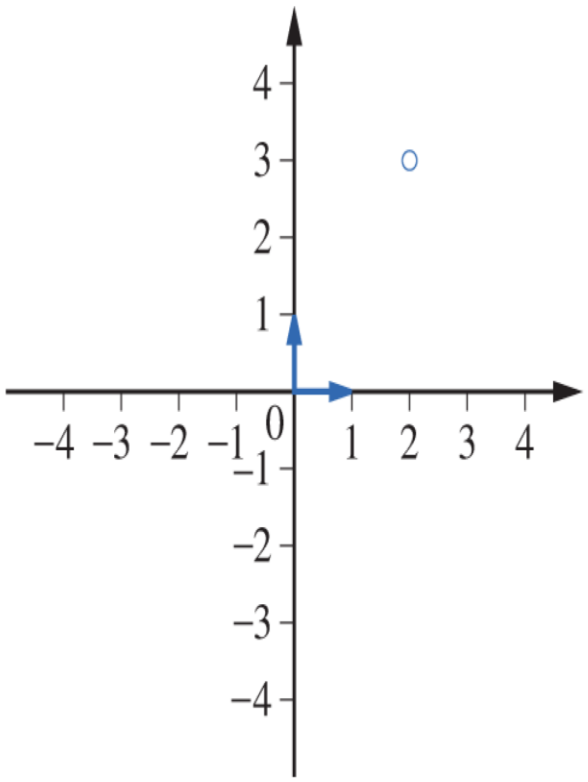
We shall see in Section 2.2 that the set of solutions to every homogeneous system of linear equations is a subspace. An implication of this result is that the sum of any two solutions is a solution. The scalar multiple of a solution is also a solution. However, we shall find that the set of solutions to a nonhomogeneous system is not a subspace. We shall discuss ways of relating solutions of nonhomogeneous systems to solutions of "corresponding" systems of homogeneous linear equations.

Spanning set for a vector space

Definition

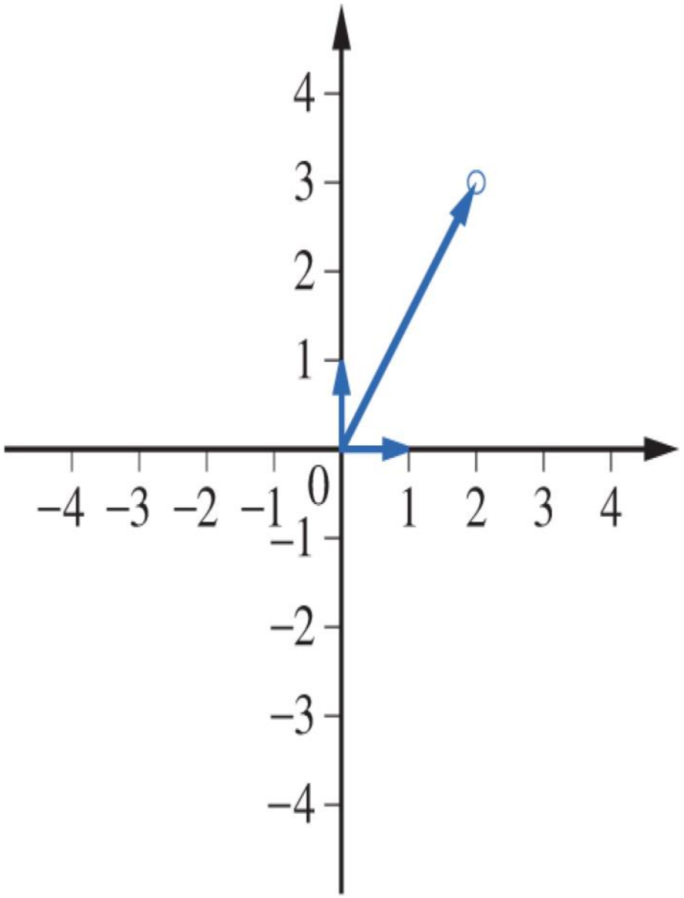
The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **spanning set** for V iff every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Spanning set for a vector space



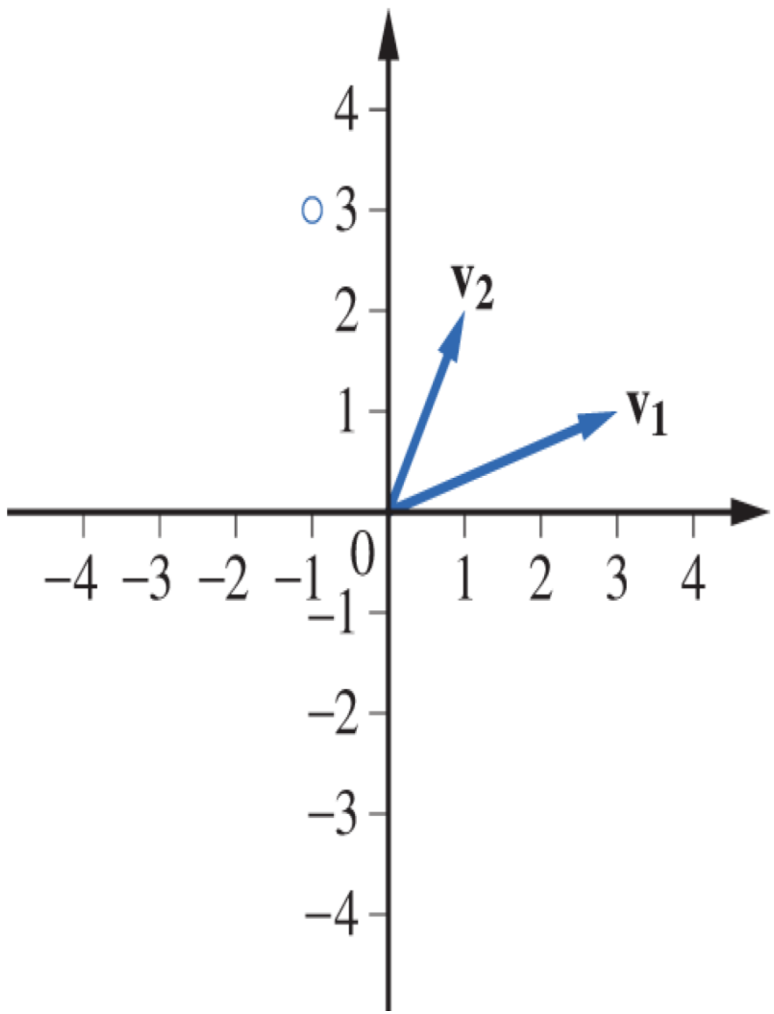
Terminal point of first vector (1, 0)
Terminal point of second vector (0, 1)
Target point (2, 3)

Spanning set for a vector space



$c_1 = 2 \qquad c_2 = 3$

Spanning set for a vector space



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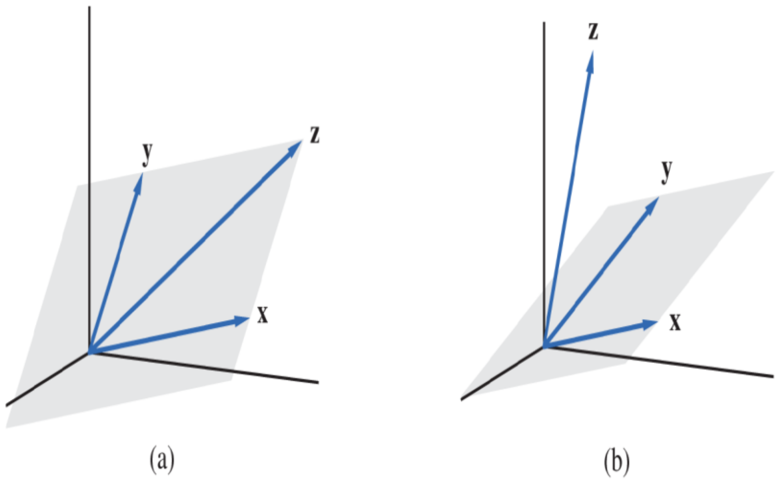

Spanning set for a vector space

Example 11

Which of the following are spanning sets for R^3?

(a) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 3)^T\}$

(b) $\{(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T\}$

(c) $\{(1, 0, 1)^T, (0, 1, 0)^T\}$

(d) $\{(1, 2, 4)^T, (2, 1, 3)^T, (4, -1, 1)^T\}$

Note

- The **“standard” spanning set** for R^3:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set for a vector space

Example 11(a)

- let $(a, b, c)^T \in R^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ (a) is a spanning set for R^3.

Spanning set for a vector space

Example 11(b)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 &= b \\ \alpha_1 &= c \end{aligned}$$

Put the values of $\alpha_1, \alpha_2, \alpha_3$ in the above equation

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

Example 11(b)

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b-c)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (a-b)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c \\ c \\ c \end{bmatrix} + \begin{bmatrix} b-c \\ b-c \\ 0 \end{bmatrix} + \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ (b) is a spanning set for R^3.

Spanning set for a vector space

Example 11(c)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

if $a \neq c$, then $\notin$ Span$\{(1, 0, 1)^T, (0, 1, 0)^T\}$

$\therefore$ (c) is **not** a spanning set for R^3

Spanning set for a vector space

Example 11(d)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned}$$

Example 11(d)

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 2 & 1 & -1 & b \\ 4 & 3 & 1 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & -3 & -9 & -2a+b \\ 0 & -5 & -15 & -4a+c \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & (2a-b)/3 \\ 0 & 0 & 0 & -(2a-5b+3c)/3 \end{array}\right]$$

if $2a - 3c + 5b \neq 0$, then the system is <u>inconsistent</u>

$\therefore$ (d) is **not** a spanning set for R^3