Linear Algebra

Naeem Ul Islam

Contact: naeem@saturn.yzu.edu.tw

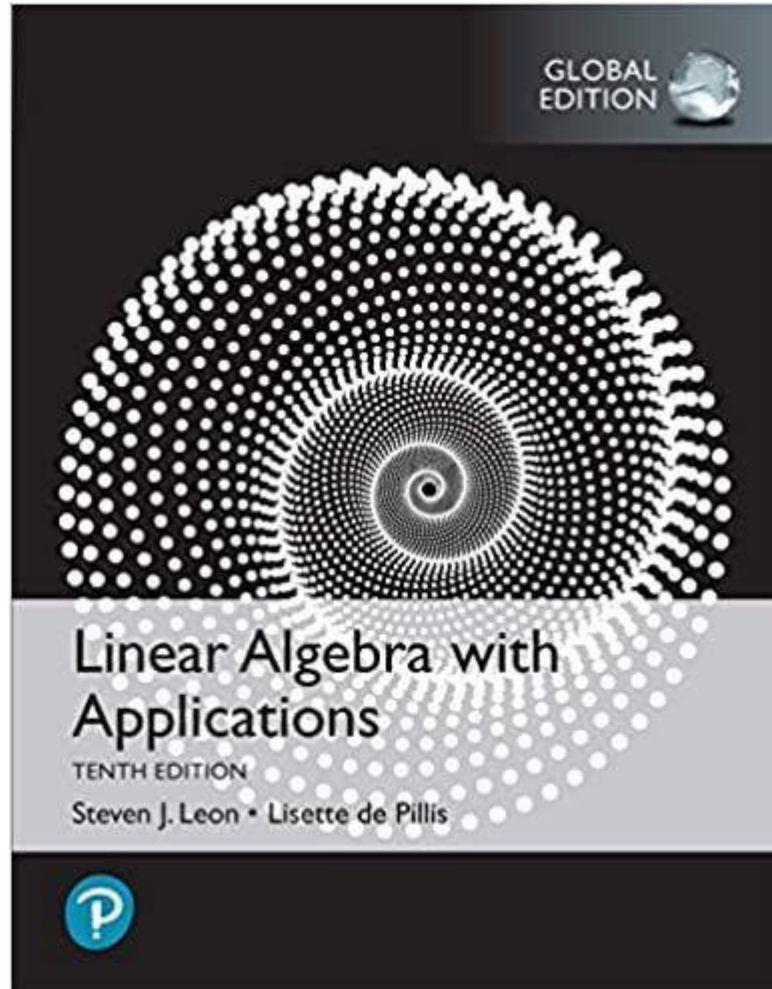
Office: 70928

Linear Algebra

Linear Algebra with applications

By

Steven J. Leon

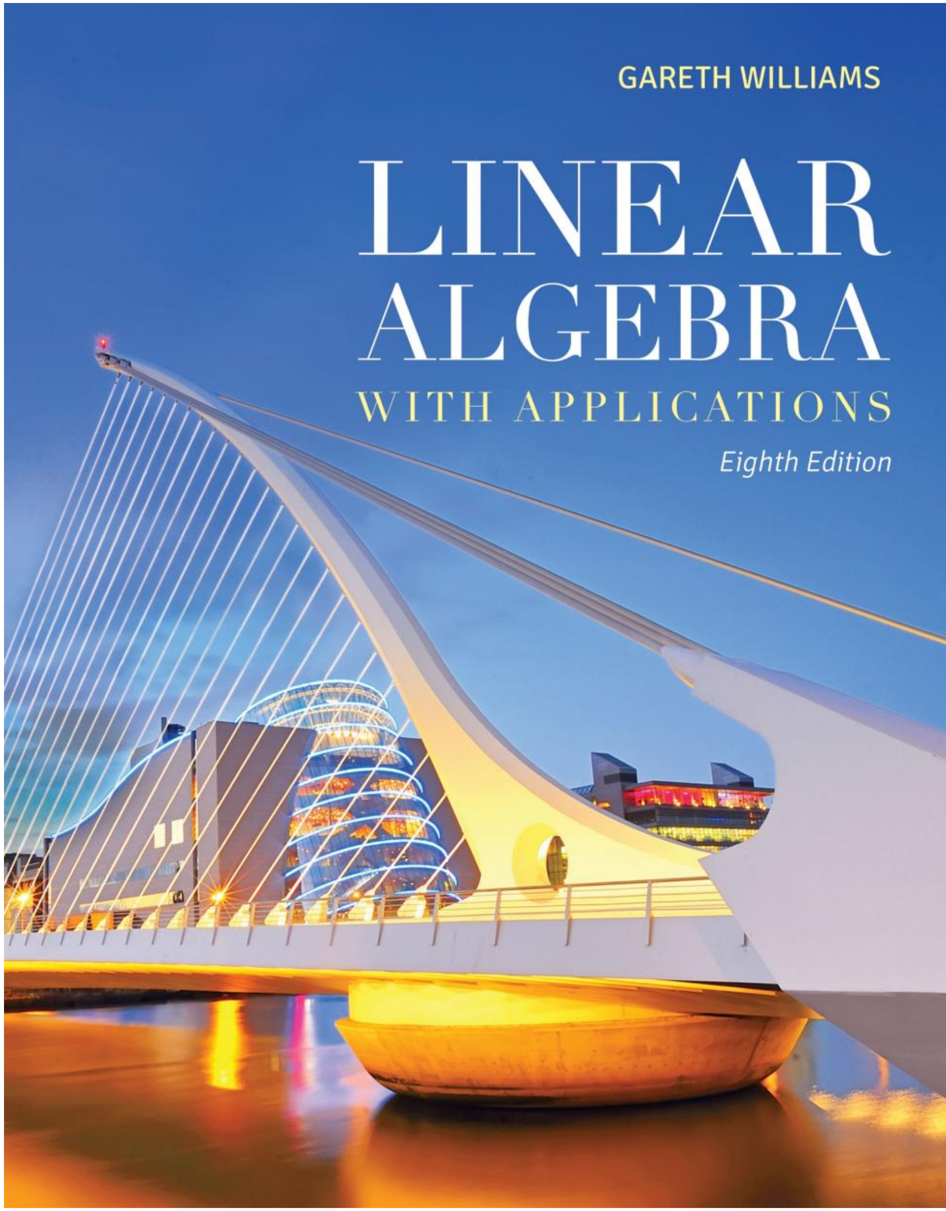


Linear Algebra

Linear Algebra with applications

By

Gareth Williams



Grading

- Quizzes 15%
- Assignments 10%
- Class activities 20%
- Mid Term 25%
- Final Term 30%

Linear Algebra

- Chapter 1 Linear Equations and Vectors:
  - Solving systems of two linear equations to solving general systems.
  - The Gauss-Jordan method of forward elimination is used
  - Concepts of linear independence, basis, and dimension are discussed.

- Chapter 2 Matrices and Linear Transformations:
  - matrix multiplication, transpose, and symmetric matrices
  - Solutions to a homogeneous system of linear equations forms a subspace
  - Applications

Linear Algebra

- Chapter 3 Determinants and Eigenvectors:

  - Determinants and Eigenvectors
  - Applications weather prediction

- Chapter 4 General Vector Spaces:
  - Concepts of subspace, linear dependence, basis, and dimension are defined rigorously and are extended to spaces of matrices and functions
  - Linear transformations, kernel, and range are used to give the reader a geometrical picture of the sets **of solutions to systems of linear equations, both homogeneous and nonhomogeneous**

- Chapter 5 Coordinate Representations:
  - Coordinate Representations of vectors and matrices

- **Chapter 6** Inner Product Spaces：
  - The axioms of inner products are presented and inner products are to define norms of vectors, angles between vectors, and distances in general vector spaces.

Linear Algebra

- Chapter 7 Numerical Methods:
  - Solving linear systems of equations using Gaussian elimination, LU decomposition, and the Jacobi and Gauss-Seidel iterative methods.

- Chapter 8 Linear Programming:

Linear Algebra

Chapter 1

Systems of Linear Equations

1.1 Matrices and Systems of Linear Equations

Definition

- An equation in the variables x and y that can be written in the form **ax + by = c**, where **a, b,** and **c** are real constants (a and b not both zero), is called a linear equation.

1.1 Matrices and Systems of Linear Equations

Definition

- An equation such as $x+3y=9$ is called a ***linear equation*** (in *two* variables or unknowns).
- The graph of this equation is a straight line in the *xy*-plane.
- A pair of values of x and y that satisfy the equation is called a ***solution***.

Definition

A ***linear equation*** in n variables $x_1, x_2, x_3, \ldots, x_n$ has the form $a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n = b$

where the coefficients $a_1, a_2, a_3, \ldots, a_n$ and b are real numbers.

Solutions for system of linear equations

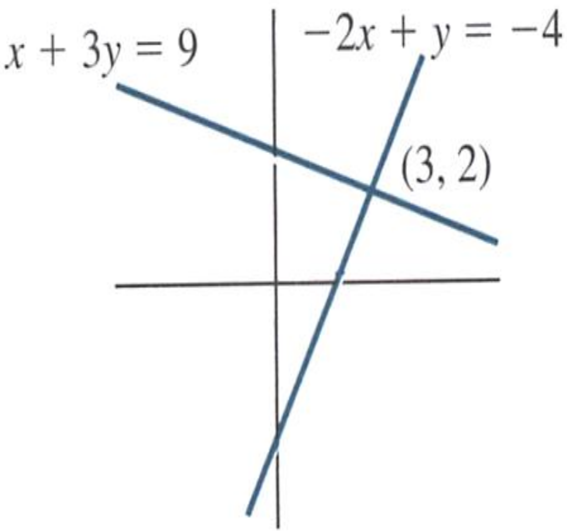


Figure 1.1

Unique solution

$x + 3y = 9$

$-2x + y = -4$

Lines intersect at (3, 2)

Unique solution:

$x = 3, y = 2.$

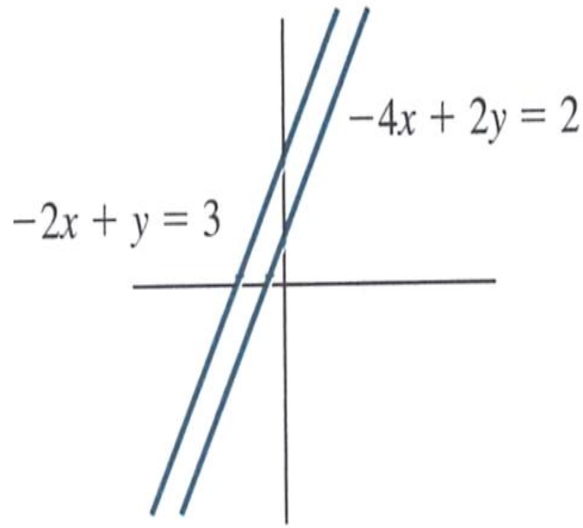


Figure 1.2

No solution

$-2x + y = 3$

$-4x + 2y = 2$

Lines are parallel.

No point of intersection.

No solutions.

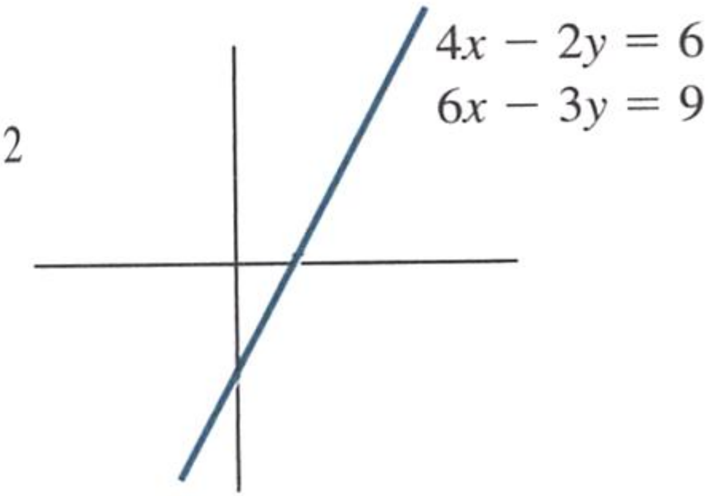


Figure 1.3

Many solution

$4x - 2y = 6$

$6x - 3y = 9$

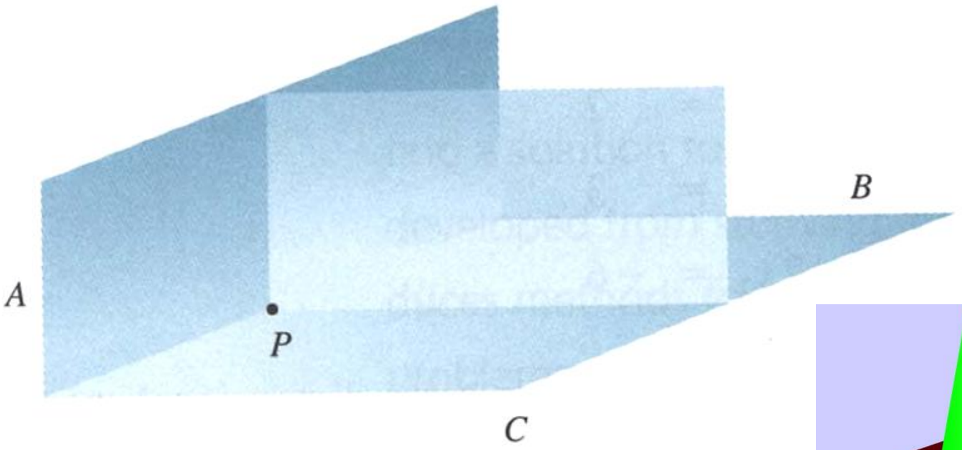
Both equations have the same graph. Any point on the graph is a solution.

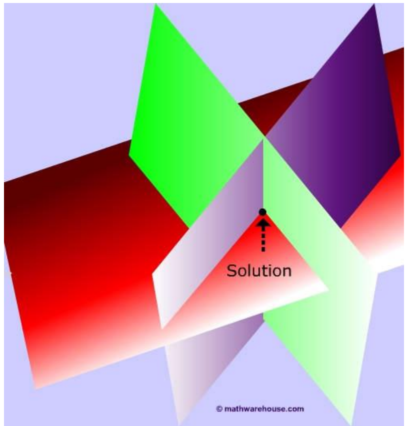
Many solutions.

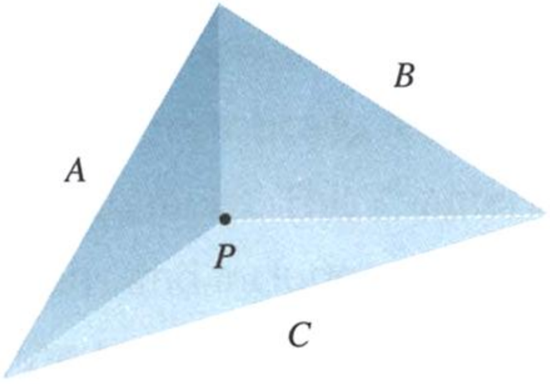
A linear equation in three variables corresponds to a plane in three-dimensional space.

※ Systems of three linear equations in three variables:

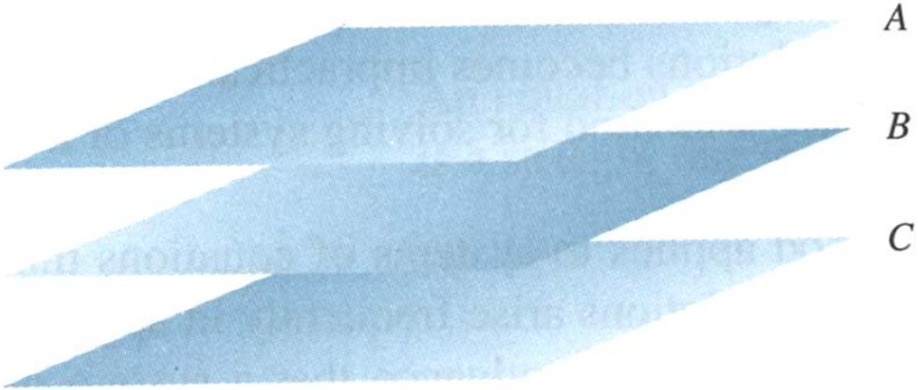
- *Unique solution*

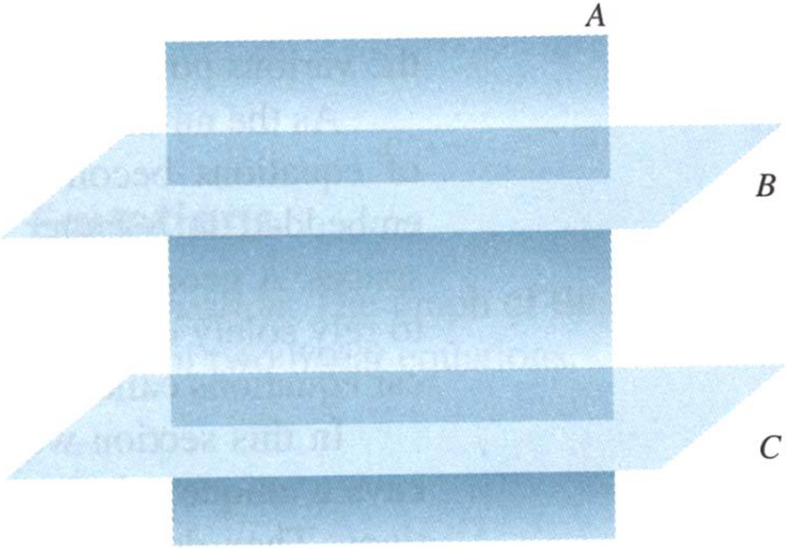




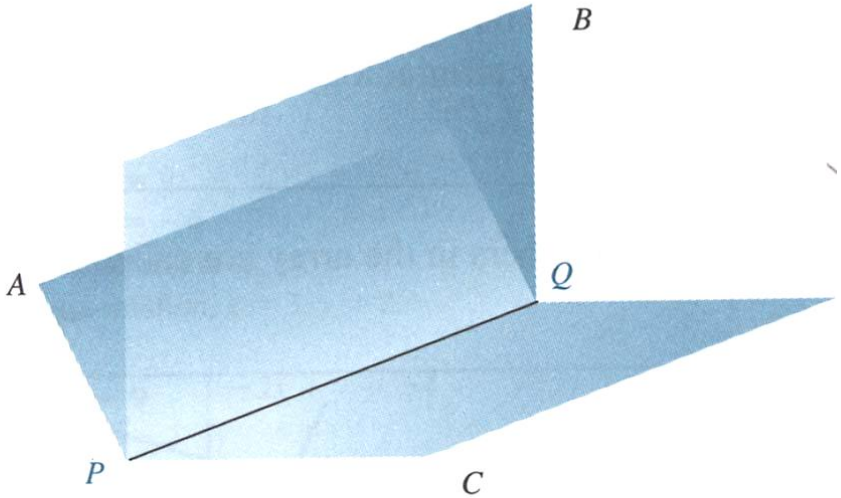


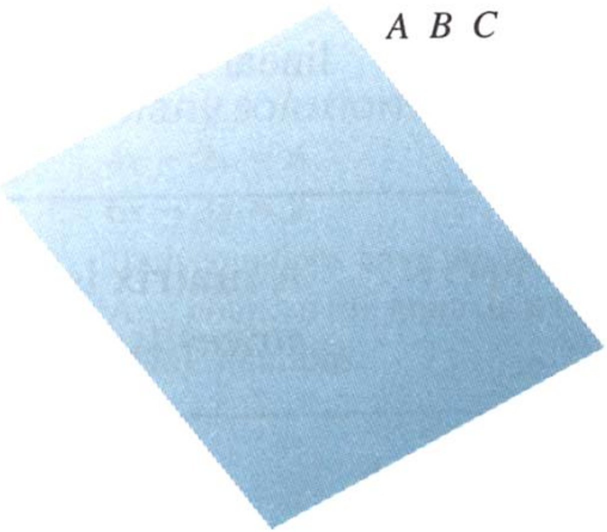
- *No solutions*





- *Many solutions*





A solution to a system of a three linear equations will be points that lie on all three planes.

The following is an example of a system of three linear equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + 3x_2 + x_3 &= 3 \\
x_1 - x_2 - 2x_3 &= -6
\end{aligned}
$$

How to solve a system of linear equations? For this we introduce a method called Gauss-Jordan elimination. (Section1.2)

Definition

- A ***matrix*** is a rectangular array of numbers.
- The numbers in the array are called the ***elements*** of the matrix.

Matrices

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 \\ 0 & 5 \\ -8 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 5 & 6 \\ 0 & -2 & 5 \\ 8 & 9 & 12 \end{bmatrix}$$

Row and Column

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}$$

$\begin{bmatrix} 2 & 3 & -4 \end{bmatrix}$	$\begin{bmatrix} 7 & 5 & -1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$	$\begin{bmatrix} -4 \\ -1 \end{bmatrix}$
row 1	row 2	column 1	column 2	column 3

Submatrix

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 3 & 0 \\ 5 & 1 & -2 \end{bmatrix}$$

matrix A

$$P = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$$

submatrice s of A

Size and Type

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 & 7 \\ -9 & 0 & 1 \\ -3 & 5 & 8 \end{bmatrix} \qquad \begin{bmatrix} 4 & -3 & 8 & 5 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}$$

$\underline{\text{Size : } 2\times 3}$ $\quad$ 3×3 matrix $\quad$ 1×4 matrix $\quad$ 3×1 matrix

a square matrix $\quad$ a row matrix $\quad$ a column matrix

Location

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix} \qquad a_{13} = -4, \; a_{21} = 7$$

The element a_{ij} *is in* row i , column j

The element in location (1,3) is –4

Identity Matrices

diagonal size

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Relations between system of linear equations and matrices

- **matrix of coefficients and augmented matrix**

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

matrix of coefficients   augmented matrix

Observe that the matrix of coefficients is a submatrix of the augmented matrix. The augmented matrix completely describes the system.

- Transformations called elementary transformations can be used to change a system of linear equations into another system of linear equations that has the same solution.
- These transformations are used to solve systems of linear equations by eliminating variables.
- In practice it is simpler to work in terms of matrices using analogous transformations called elementary row operations.

Elementary Row Operations of Matrices

- These transformations are as follows:

- Elementary Transformation

1. Interchange two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

- Elementary Row Operation

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Example 1

Solving the following system of linear equation.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + 3x_2 + x_3 &= 3 \\
x_1 - x_2 - 2x_3 &= -6
\end{aligned}
$$

Example 1

Solving the following system of linear equation.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$\approx$ row equivalent

Solution

Equation Method

Initial system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Eq2+(–2)Eq1

Eq3+(–1)Eq1

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

Analogous Matrix Method

Augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

$$\underset{\substack{R2+(-2)R1 \\ R3+(-1)R1}}{\approx} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

Eq1+(–1)Eq2

Eq3+(2)Eq2

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -2x_2 - 3x_3 &= -8 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ -5x_3 &= -10 \end{aligned}$$

$$\underset{\substack{R1+(-1)R2 \\ R3+(2)R2}}{\approx} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

(–1/5)Eq3

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= -1 \\ x_3 &= 2 \end{aligned}$$

$$\underset{(-1/5)R3}{\approx} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Eq1+(–2)Eq3

Eq2+Eq3

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

$$\underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution is

$x_1 = -1, x_2 = 1, x_3 = 2.$

The solution is

$x_1 = -1, x_2 = 1, x_3 = 2.$

Example 2

Solving the following system of linear equation.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

Example 2

Solving the following system of linear equation.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{1 \quad -2 \quad 4 \quad 12} \\ 2 \quad -1 \quad 5 \quad 18 \\ -1 \quad 3 \quad -3 \quad -8 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{3} & \boxed{-3} & \boxed{-6} \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & \boxed{1} & \boxed{-1} & \boxed{-2} \\ 0 & 1 & 1 & 4 \end{bmatrix} \underset{\substack{R1+(2)R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{2} & \boxed{6} \end{bmatrix}$$

$$\underset{\left(\frac{1}{2}\right)R3}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \boxed{1} & \boxed{3} \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

solution $\begin{cases} x_1 = 2 \\ x_2 = 1. \\ x_3 = 3 \end{cases}$

Example 3

Solve the system

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - \ \ 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\quad &= -7 \end{aligned}$$

Solution

$$\begin{bmatrix} \boxed{4 \quad 8 \quad -12 \quad 44} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\left(\frac{1}{4}\right)\text{R1}}{\approx} \begin{bmatrix} \boxed{1 \quad 2 \quad -3 \quad 11} \\ 3 \quad 6 \quad -8 \quad 32 \\ -2 \quad -1 \quad 0 \quad -7 \end{bmatrix} \underset{\substack{\text{R2}+(-3)\text{R1} \\ \text{R3}+2\text{R1}}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & \boxed{3} & \boxed{-6} & \boxed{15} \end{bmatrix}$$

$$\underset{\text{R2} \leftrightarrow \text{R3}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{3} & \boxed{-6} & \boxed{15} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\left(\frac{1}{3}\right)\text{R2}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & \boxed{1} & \boxed{-2} & \boxed{5} \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\text{R1}+(-2)\text{R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & \boxed{1} & \boxed{-1} \end{bmatrix}$$

$$\underset{\substack{\text{R1}+(-1)\text{R3} \\ \text{R2}+2\text{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The solution is $x_1 = 2,\ x_2 = 3,\ x_3 = -1.$

Summary

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - \ \ 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\qquad &= -7 \end{aligned}$$

$$[A:B] = \left[\begin{array}{ccc|c} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{array}\right]$$

A B

Use row operations to $[A: B]$:

$$\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

i.e., $[A:B] \approx \cdots \approx [I_n : X]$

Def. $[I_n : X]$ is called the ***reduced echelon form*** of $[A:B]$.

Note. 1. If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution, then A is row equivalent to I_n ($A \approx I_n$).

2. If $A \approx I_n$, then the system has unique solution.

Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_1 - x_2 + 4x_3 &= b_2 \\ -x_1 + 2x_2 - 4x_3 &= b_3 \end{aligned} \quad \text{for} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \text{ in turn}$$

Solution

Example 4 Many Systems

Solving the following three systems of linear equation, all of which have the same matrix of coefficients.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_1 - x_2 + 4x_3 &= b_2 \\ -x_1 + 2x_2 - 4x_3 &= b_3 \end{aligned} \quad \text{for} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \text{ in turn}$$

Solution

$$\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{\approx} \begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+(-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+2R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$$

The solutions to the three systems are

$$\begin{cases} x_1 = 1 \\ x_2 = -1, \\ x_3 = 2 \end{cases} \begin{cases} x_1 = 0 \\ x_2 = 3, \\ x_3 = 1 \end{cases} \begin{cases} x_1 = -2 \\ x_2 = 1 \\ x_3 = 2 \end{cases}.$$

1.2 Gauss-Jordan Elimination

More generalization of Gauss-Jordan Elimination

Definition

A matrix is in ***reduced echelon form*** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a ***leading* 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

The following matrices are all in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are not in reduced echelon form for the reasons stated.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros not at bottom of matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in row 2

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(?) (?) (?) (?)

- Elementary row operations reduced echelon form
- The reduced echelon form of a matrix is unique.

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\checkmark) \qquad (\times) \qquad (\checkmark) \qquad (\times)$$

- There are usually many sequences of row operations that can be used to transform a given matrix to reduced echelon form-they all, however, lead to the same reduced echelon form.
- We say that the reduced echelon form of a matrix is unique.

Gauss-Jordan Elimination

- System of linear equations
  $\Rightarrow$ augmented matrix
  $\Rightarrow$ reduced echelon form
  $\Rightarrow$ solution

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot (leading 1)

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\underset{R3+(-4)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.

Example 2

Solve, if possible, the system of equations

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+2R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -3x_3 + 4 \\ x_2 &= -2x_3 + 1 \end{aligned}$$

The general solution to the system is

$$x_1 = -3r + 4$$

$$x_2 = -2r + 1$$

$x_3 = r$, where r is real number (called a parameter).

Example 3

Solve the system of equations

$$\begin{aligned} 2x_1 - 4x_2 + 12x_3 - 10x_4 &= 58 \\ -x_1 + 2x_2 - 3x_3 + 2x_4 &= -14 \\ 2x_1 - 4x_2 + 9x_3 - 6x_4 &= 44 \end{aligned}$$

$\Rightarrow$ many sol.

Solution

$$\begin{bmatrix} 2 & -4 & 12 & -10 & 58 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R1}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ -1 & 2 & -3 & 2 & -14 \\ 2 & -4 & 9 & -6 & 44 \end{bmatrix}$$

$$\underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 3 & -3 & 15 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -2 & 6 & -5 & 29 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -3 & 4 & -14 \end{bmatrix}$$

$$\underset{\substack{R1+(-6)R2 \\ R3+3R2}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & -2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 - 2x_2 &= -2 \\ x_3 &= 6 \\ x_4 &= 1 \end{aligned} \Rightarrow \begin{cases} x_1 = 2r - 2 \\ x_2 = r \\ x_3 = 6 \\ x_4 = 1 \end{cases}, \text{ for some } r.$$

Example 4

Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 + x_5 &= 2 \\ 2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 &= 6 \\ -x_1 - 2x_2 + x_3 - x_4 + 3x_5 &= 4 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 2 & 4 & -2 & 6 & 3 & 6 \\ -1 & -2 & 1 & -1 & 3 & 4 \end{bmatrix} \underset{\substack{\text{R2+(−2)R1} \\ \text{R3+R1}}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\underset{\text{R2}\leftrightarrow\text{R3}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\left(\frac{1}{2}\right)\text{R}2}{\approx} \begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{\text{R1+(−3)R2}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & -5 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{\text{R1+5R3} \\ \text{R2+(−2)R3}}}{\approx} \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 &= -2x_2 + x_3 + 3 \\ x_4 &= -1 \\ x_5 &= 2 \end{aligned} \qquad \Rightarrow \begin{aligned} x_1 &= -2r + s + 3 \\ x_2 &= r,\ x_3 = s,\ x_4 = -1,\ , \text{ for some } r \text{ and } s. \\ x_5 &= 2 \end{aligned}$$

Example 5

This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-1)R1}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{R2\leftrightarrow R3}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{R1+R2 \\ R4+(-2)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix} \underset{\substack{R1+(-1)R3 \\ R2+R3 \\ R4+(-4)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\underset{\left(\frac{1}{13}\right)R4}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{0x_1+0x_2+0x_3=1}$$

The system has no solution.

Homogeneous System of linear Equations

Definition

A system of linear equations is said to be ***homogeneous*** if all the constant terms are zeros.

Example: $$\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$$

Observe that $x_1 = 0,\ x_2 = 0,\ x_3 = 0$ is a solution.

Theorem 1.1

A system of homogeneous linear equations in n variables always has the solution $x_1 = 0$, $x_2 = 0$. …, $x_n = 0$. This solution is called the **trivial solution**.

Homogeneous System of linear Equations

Note. Non trivial solution

Example: $\begin{cases} x_1 + 2x_2 - 5x_3 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases}$

The system has other nontrivial solutions.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

$$\therefore x_1 = -3r,\ x_2 = 4r,\ x_3 = r$$

Theorem 1.2

A system of homogeneous linear equations that has more variables than equations has many solutions.

Homework

- Exercise set 1.2: (page 21)
  2, 5, 6, 7, 8, 14

1.3 Gaussian Elimination

Definition

A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.

(This implies that all the elements below a leading 1 are zero.)

Example 6

Solving the following system of linear equations using the method of Gaussian elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

Solution

Starting with the augmented matrix, create zeros below the pivot in the first column.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

At this stage, we create a zero only below the pivot.

$$\underset{R3+(-2)R2}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \underset{\frac{1}{2}R3}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Echelon form

We have arrived at the echelon form.

The corresponding system of equation is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ x_4 &= 2 \end{aligned}$$

We get

$$\begin{aligned} x_3 + 3(2) &= 1 \\ x_3 &= -5 \end{aligned}$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

$$\begin{aligned} x_1 + 2x_2 + 3(-5) + 2(2) &= -1 \\ x_1 + 2x_2 &= 10 \\ x_1 &= -2x_2 + 10 \end{aligned}$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10,\ \ x_2 = r,\ \ x_3 = -5,\ \ x_4 = 2$$

Example 7

Solving the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

Solution

We arrive at the echelon form as in the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \cdots \approx \underset{\text{Echelon form}}{\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}$$

This marks the end of the forward elimination of variables from equations. We now commence the back substitution using matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+(-3)R3}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\underset{R1+(-3)R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ x_3 &= -5 \\ x_4 &= 2 \end{aligned}$$

Let $x_2 = r$. We get same solution as previously,

$$x_1 = -2r + 10,\ \ x_2 = r,\ \ x_3 = -5,\ \ x_4 = 2$$