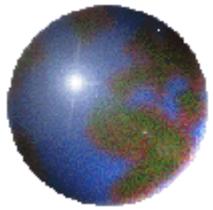
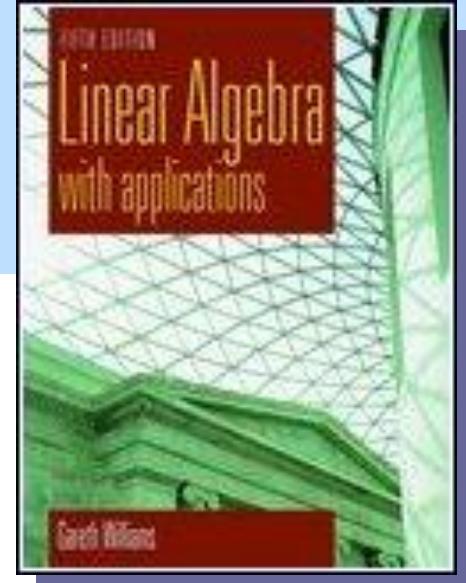


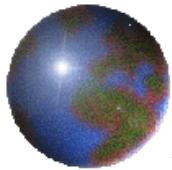
Linear Algebra



Chapter 5



EIGENVALUES AND EIGENVECTORS



5.1 Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a nonzero vector \mathbf{x} in \mathbf{R}^n such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector \mathbf{x} is called an **eigenvector** corresponding to λ .

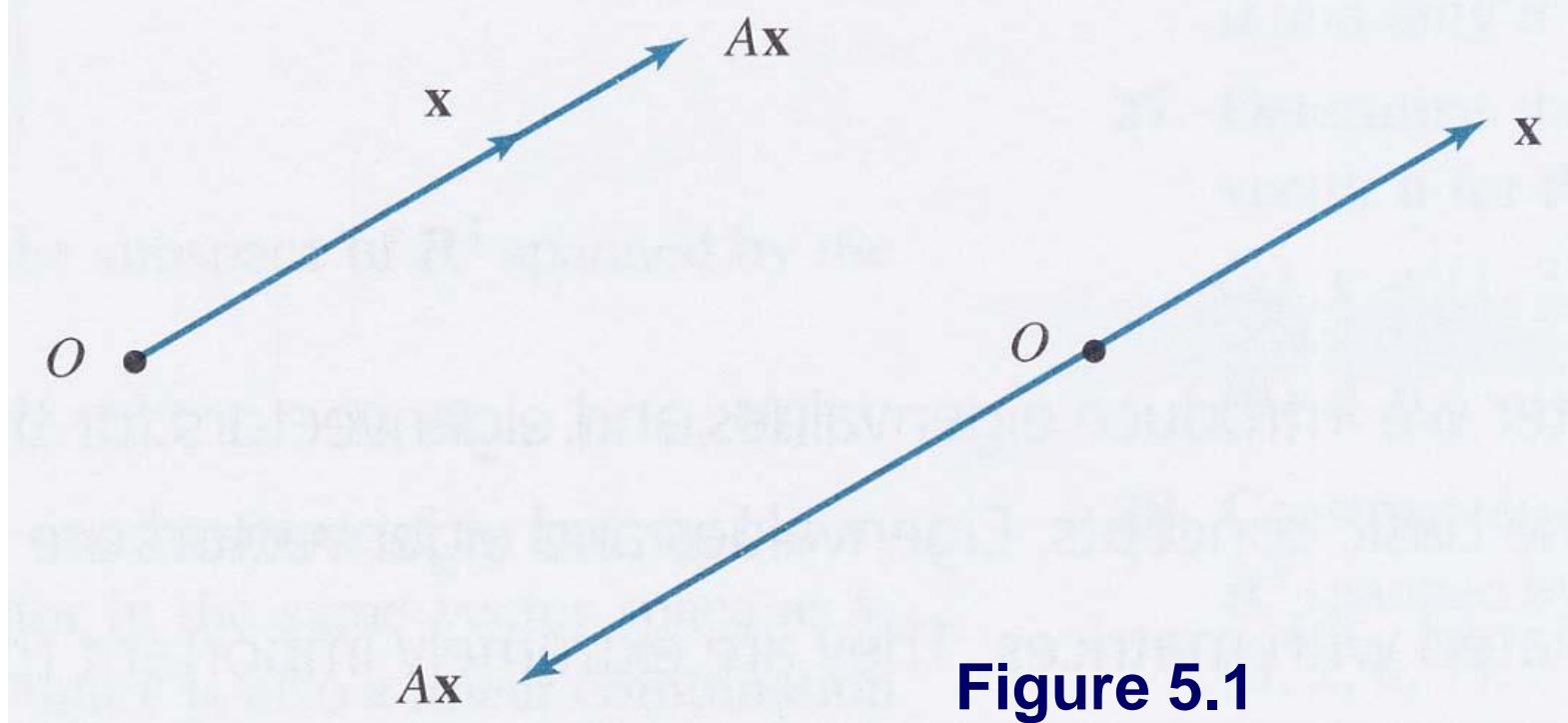
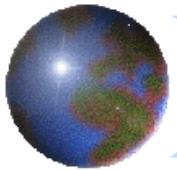


Figure 5.1



Computation of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector \mathbf{x} . Thus $A\mathbf{x} = \lambda\mathbf{x}$. This equation may be written

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

given

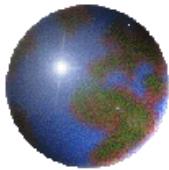
$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

Solving the equation $|A - \lambda I_n| = 0$ for λ leads to all the eigenvalues of A .

On expanding the determinant $|A - \lambda I_n|$, we get a polynomial in λ .

This polynomial is called the **characteristic polynomial** of A .

The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation** of A .

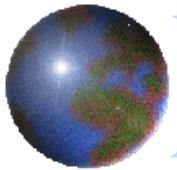


Example 1

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution



Example 1

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution Let us first derive the characteristic polynomial of A .

We get

$$A - \lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 - \lambda & -6 \\ 3 & 5 - \lambda \end{bmatrix}$$

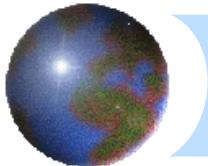
$$|A - \lambda I_2| = (-4 - \lambda)(5 - \lambda) + 18 = \lambda^2 - \lambda - 2$$

We now solve the characteristic equation of A .

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 2 \text{ or } -1$$

The eigenvalues of A are 2 and -1 .

The corresponding eigenvectors are found by using these values of λ in the equation $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$. There are many eigenvectors corresponding to each eigenvalue.



- For $\lambda = 2$

We solve the equation $(A - 2I_2)\mathbf{x} = \mathbf{0}$ for \mathbf{x} .

The matrix $(A - 2I_2)$ is obtained by subtracting 2 from the diagonal elements of A . We get

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

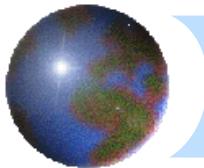
This leads to the system of equations

$$-6x_1 - 6x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

giving $x_1 = -x_2$. The solutions to this system of equations are $x_1 = -r$, $x_2 = r$, where r is a scalar. Thus the eigenvectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



- For $\lambda = -1$

We solve the equation $(A + 1I_2)x = 0$ for x .

The matrix $(A + 1I_2)$ is obtained by adding 1 to the diagonal elements of A . We get

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

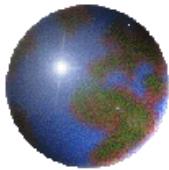
This leads to the system of equations

$$-3x_1 - 6x_2 = 0$$

$$3x_1 + 6x_2 = 0$$

Thus $x_1 = -2x_2$. The solutions to this system of equations are $x_1 = -2s$ and $x_2 = s$, where s is a scalar. Thus the **eigenvectors** of A corresponding to $\lambda = -1$ are nonzero vectors of the form

$$\mathbf{v}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Example 2

Find the eigenvalues and eigenvectors of the matrix

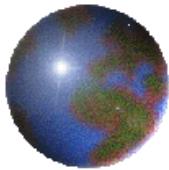
$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Solution The matrix $A - \lambda I_3$ is obtained by subtracting λ from the diagonal elements of A . Thus

$$A - \lambda I_3 = \begin{bmatrix} 5 - \lambda & 4 & 2 \\ 4 & 5 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I_3|$. Using row and column operations to simplify determinants, we get

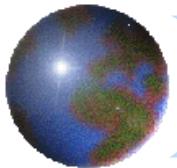
$$|A - \lambda I_3| = \begin{vmatrix} 5 - \lambda & 4 & 2 \\ 4 & 5 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda & -1 + \lambda & 0 \\ 4 & 5 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{vmatrix}$$



5.3 Diagonalization of Matrices

Definition

Let A and B be square matrices of the same size. B is said to be **similar** to A if there exists an invertible matrix C such that $B = C^{-1}AC$. The transformation of the matrix A into the matrix B in this manner is called a **similarity transformation**.



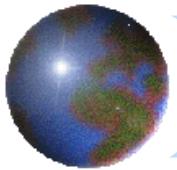
Example 3

Consider the following matrices A and C with C is invertible.
Use the similarity transformation $C^{-1}AC$ to transform A into a matrix B .

$$A = \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} B &= C^{-1}AC = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -10 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$



Theorem 5.3

Similar matrices have the same eigenvalues.

Proof

Let A and B be similar matrices. Hence there exists a matrix C such that $B = C^{-1}AC$.

The characteristic polynomial of B is $|B - \lambda I_n|$. Substituting for B and using the multiplicative properties of determinants, we get

$$\begin{aligned}|B - \lambda I| &= |C^{-1}AC - \lambda I| = |C^{-1}(A - \lambda I)C| \\&= |C^{-1}| |A - \lambda I| |C| = |A - \lambda I| |C^{-1}| |C| \\&= |A - \lambda I| |C^{-1}C| = |A - \lambda I| |I| \\&= |A - \lambda I|\end{aligned}$$

The characteristic polynomials of A and B are identical. This means that their eigenvalues are the same.



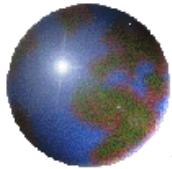
Definition

A *square matrix A* is said to be **diagonalizable** if there exists a matrix C such that $D = C^{-1}AC$ is a diagonal matrix.

Theorem 5.4

Let A be an $n \times n$ matrix.

- (a) If A has n linearly independent eigenvectors, it is diagonalizable. *The matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $C^{-1}AC$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.*
- (b) If A is diagonalizable, then it has n linearly independent eigenvectors



Example 4

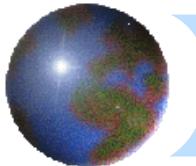
- (a) Show that the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable.
- (b) Find a diagonal matrix D that is similar to A .
- (c) Determine the similarity transformation that diagonalizes A .

Solution

- (a) The eigenvalues and corresponding eigenvector of this matrix were found in Example 1 of Section 5.1. They are

$$\lambda_1 = 2, \mathbf{v}_1 = r \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda_2 = -1, \text{ and } \mathbf{v}_2 = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since A , a 2×2 matrix, has two linearly independent eigenvectors, it is diagonalizable.



- (b) A is similar to the diagonal matrix D , which has diagonal elements $\lambda_1 = 2$ and $\lambda_2 = -1$. Thus

$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \text{ is similar to } D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

- (c) Select two convenient linearly independent eigenvectors, say

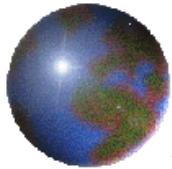
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let these vectors be the column vectors of the diagonalizing matrix C .

$$C = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

We get

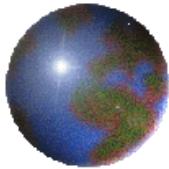
$$\begin{aligned} C^{-1}AC &= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D \end{aligned}$$



Example 4

- (a) Show that the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ is diagonalizable.
- (b) Find a diagonal matrix D that is similar to A .
- (c) Determine the similarity transformation that diagonalizes A .

Solution



Note

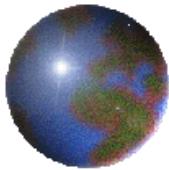
If A is similar to a diagonal matrix D under the transformation $C^{-1}AC$, then it can be shown that $A^k = CD^kC^{-1}$.

This result can be used to compute A^k . Let us derive this result and then apply it.

$$D^k = (C^{-1}AC)^k = \underbrace{(C^{-1}AC) \quad \cdots \quad (C^{-1}AC)}_{k \text{ times}} = C^{-1}A^kC$$

This leads to

$$A^k = CD^kC^{-1}$$



Example 5

Compute A^9 for the following matrix A .

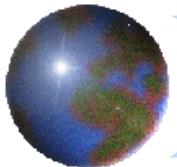
$$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

Solution

A is the matrix of the previous example. Use the values of C and D from that example. We get

$$D^9 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^9 = \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix} = \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A^9 &= CD^9C^{-1} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -514 & -1026 \\ 513 & 1025 \end{bmatrix} \end{aligned}$$



Example 6

Show that the following matrix A is not diagonalizable.

$$A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$$

Solution

$$A - \lambda I_2 = \begin{bmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I_2| = 0 \Rightarrow (5 - \lambda)(-1 - \lambda) + 9 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)(\lambda - 2) = 0$$

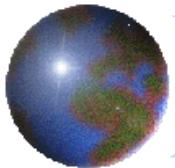
There is a single eigenvalue, $\lambda = 2$. We find the corresponding eigenvectors. $(A - 2I_2)\mathbf{x} = \mathbf{0}$ gives $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0} \Rightarrow 3x_1 - 3x_2 = 0$.

Thus $x_1 = r$, $x_2 = r$. The eigenvectors are nonzero vectors of the form

$$r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace is a one-dimensional space. A is a 2×2 matrix, but it does not have two linearly independent eigenvectors.

Thus A is not diagonalizable.



Theorem 5.5

Let A be an $n \times n$ symmetric matrix.

- (a) All the eigenvalues of A are real numbers.
- (b) The dimension of an eigenspace of A is the multiplicity of the eigenvalues as a root of the characteristic equation.
- (c) A has n linearly independent eigenvectors.

Exercise 5.3 p 301: 1, 3, 5.