1.3 The vector Space R^n

- Solutions to systems of linear equations can be points in a plane if the equations have two variables, points in three-space if they are equations in three variables, points in four-space if they have four variables, and so on.

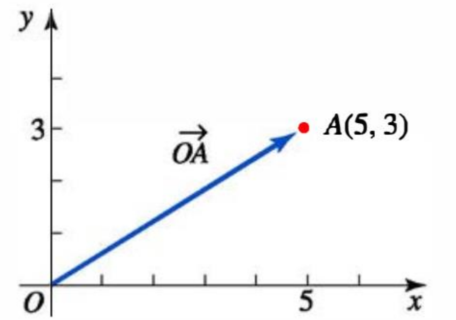

Figure 1.5

- **Point A** is a certain distance in a certain direction from the origin (0, 0). The distance and direction are characterized by the length and direction of the line segment from the origin, 0, to A. We call such a directed line segment a position vector and denote it $\overrightarrow{OA}$
- O is called the initial point of $\overrightarrow{OA}$ and A is called the terminal point.

1.3 The vector Space R^n

EXAMPLE 1 Sketch the position vectors $\overrightarrow{OA} = (4, 1)$, $\overrightarrow{OB} = (-5, -2)$, and $\overrightarrow{OC} = (-3, 4)$. See Figure 1.6.

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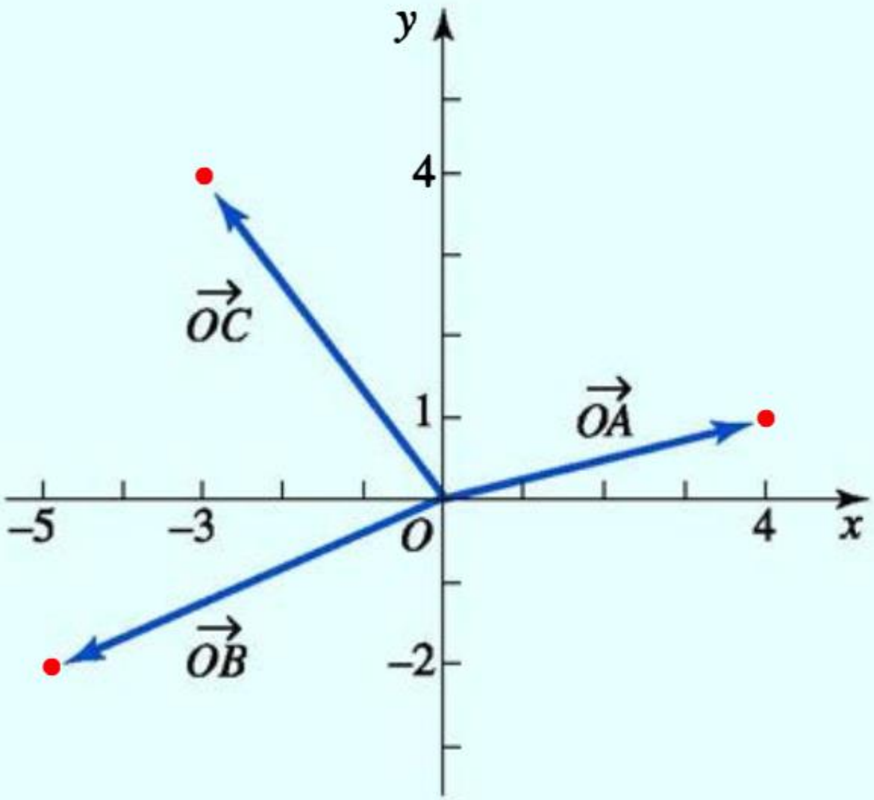

Figure 1.6

Vector addition and Scalar Multiplication

DEFINITION Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be elements of $\mathbf{R}^n$ and let c be a scalar. Addition and scalar multiplication are performed as follows.

Addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$

Scalar multiplication: $c\mathbf{u} = (cu_1, \ldots, cu_n)$

EXAMPLE 2 Let $\mathbf{u} = (-1, 4, 3, 7)$ and $\mathbf{v} = (-2, -3, 1, 0)$ be vectors in $\mathbf{R}^4$. Find $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u}$.

SOLUTION

Vector addition and Scalar Multiplication

EXAMPLE 2 Let $\mathbf{u} = (-1, 4, 3, 7)$ and $\mathbf{v} = (-2, -3, 1, 0)$ be vectors in $\mathbf{R}^4$. Find $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u}$.

SOLUTION

We get

$$\mathbf{u} + \mathbf{v} = (-1, 4, 3, 7) + (-2, -3, 1, 0) = (-3, 1, 4, 7)$$
$$3\mathbf{u} = 3(-1, 4, 3, 7) = (-3, 12, 9, 21)$$

Note that the resulting vector under each operation is in the original vector space $\mathbf{R}^4$.

Vector addition and Scalar Multiplication

EXAMPLE 3 This example gives us a geometrical interpretation of vector addition. Consider the sum of the vectors (4, 1) and (2, 3). We get

$$(4, 1) + (2, 3) = (6, 4)$$

Algebraically we can add these vectors as

$$(4, 1) + (2, 3) = (6, 4)$$

1.3 The vector Space R^n

Vector addition and Scalar Multiplication

EXAMPLE 3 This example gives us a geometrical interpretation of vector addition. Consider the sum of the vectors (4, 1) and (2, 3). We get

$$(4, 1) + (2, 3) = (6, 4)$$

In Figure 1.8 we interpret these vectors as position vectors. Construct the parallelogram having the vectors (4, 1) and (2, 3) as adjacent sides. The vector (6, 4), the sum, will be the diagonal of the parallelogram.

Vector addition and Scalar Multiplication

Such vectors are used in the physical sciences to describe forces. In this example (4, 1) and (2, 3) might be forces, acting on a body at the origin, O. The vectors would give the directions of the forces and their lengths (using the Pythagorean Theorem) would be the magnitudes of the forces, $\sqrt{4^2 + 1^2} = 4.12$ and $\sqrt{2^2 + 3^2} = 3.61$. The vector sum (6, 4) is the resultant force, a single force that would be equivalent to the two forces. The magnitude of the resultant force would be $\sqrt{6^2 + 4^2} = 7.21$.

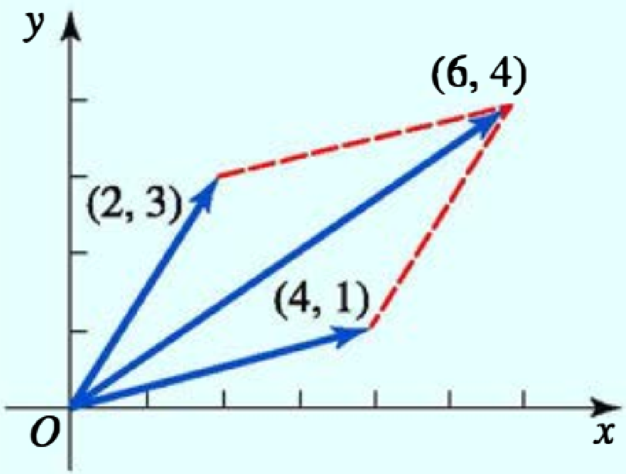

Figure 1.8

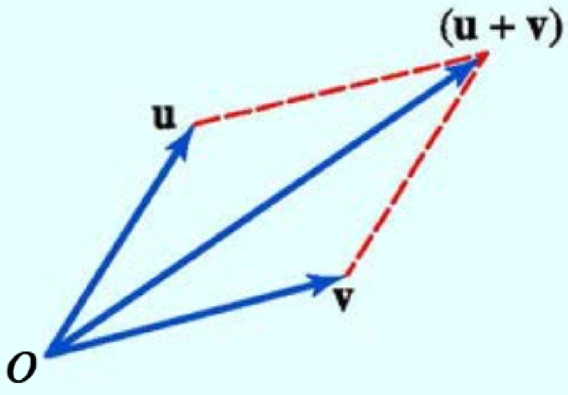

Figure 1.9

Vector addition and Scalar Multiplication

EXAMPLE 4 This example gives us a geometrical interpretation of scalar multiplication. Consider the scalar multiple of the vector (3, 2) by 2. We get

$$2(3, 2) = (6, 4)$$

Vector addition and Scalar Multiplication

Observe in Figure 1.10 that (6, 4) is a vector in the same direction as (3, 2), and 2 times it in length.

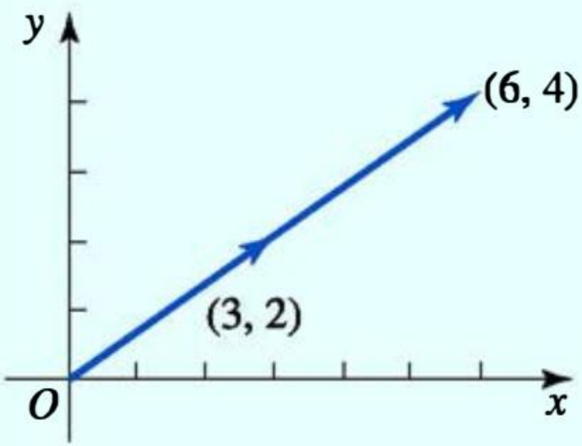

Figure 1.10

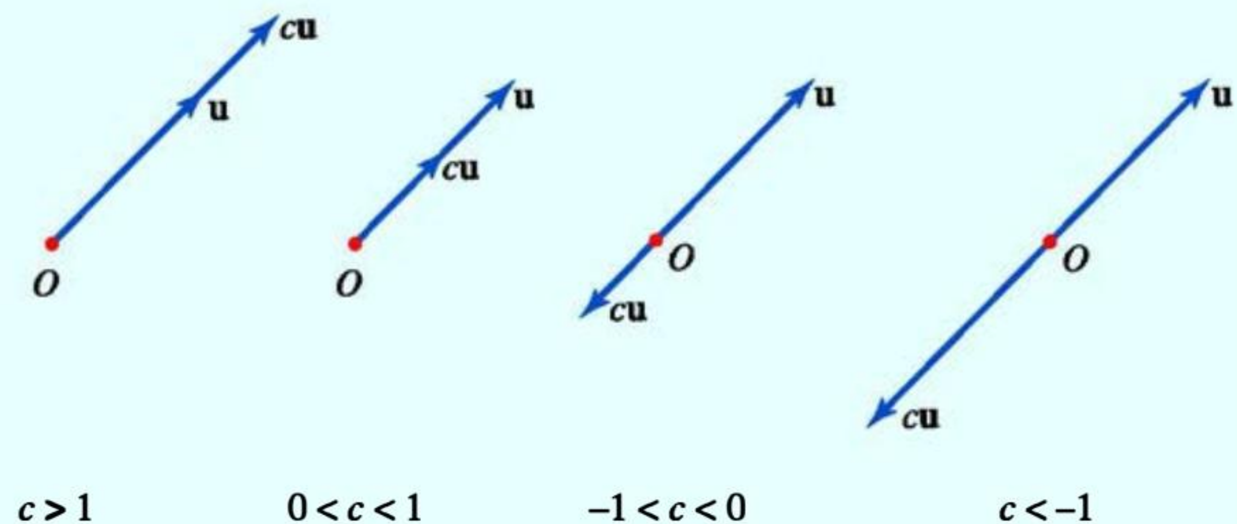

Figure 1.11

Special Vectors

The vector (0, 0, ..., 0), having **n** zero components, is called the zero vector of $\boldsymbol{R^n}$ and is denoted **0**. For example, (0, 0, 0) is the **zero vector** of $\boldsymbol{R^3}$

Special Vectors

Subtraction Subtraction is performed on elements of $\mathbf{R}^n$ by subtracting corresponding components. For example, in $\mathbf{R}^3$,

$$(5, 3, -6) - (2, 1, 3) = (3, 2, -9)$$

1.3 The vector Space R^n

THEOREM 1.3

Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbf{R}^n$ and let c and d be scalars.

(a)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
(b)	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
(c)	$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	Property of the zero vector
(d)	$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Property of the negative vector
(e)	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive properties
(f)	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	
(g)	$c(d\mathbf{u}) = (cd)\mathbf{u}$	
(h)	$1\mathbf{u} = \mathbf{u}$	Scalar multiplication by 1

Linear Combinations of Vectors

EXAMPLE 5 Let $\mathbf{u} = (2, 5, -3)$, $\mathbf{v} = (-4, 1, 9)$, $\mathbf{w} = (4, 0, 2)$. Determine the linear combination $2\mathbf{u} - 3\mathbf{v} + \mathbf{w}$.

SOLUTION

Linear Combinations of Vectors

EXAMPLE 5 Let $\mathbf{u} = (2, 5, -3)$, $\mathbf{v} = (-4, 1, 9)$, $\mathbf{w} = (4, 0, 2)$. Determine the linear combination $2\mathbf{u} - 3\mathbf{v} + \mathbf{w}$.

SOLUTION

We get

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} + \mathbf{w} &= 2(2, 5, -3) - 3(-4, 1, 9) + (4, 0, 2) \\ &= (4, 10, -6) - (-12, 3, 27) + (4, 0, 2) \\ &= (4 + 12 + 4, 10 - 3 + 0, -6 - 27 + 2) \\ &= (20, 7, -31) \end{aligned}$$

We say that the vector $(20, 7, -31)$ is a linear combination

$$(20, 7, -31) = 2(2, 5, -3) - 3(-4, 1, 9) + (4, 0, 2)$$

of the three vectors $(2, 5, -3)$, $(-4, 1, 9)$, and $(4, 0, 2)$.

Linear Combinations of Vectors

EXAMPLE 6 Determine whether the vector $(8, 0, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, -1, 1)$.

SOLUTION

Linear Combinations of Vectors

EXAMPLE 6 Determine whether the vector $(8, 0, 5)$ is a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 4)$, and $(2, -1, 1)$.

SOLUTION

We examine the following identity for values of c_1, c_2, and c_3.

$$c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, -1, 1) = (8, 0, 5)$$

Multiplying the scalars and then adding the vectors, we get

$$(c_1 + 2c_3, 2c_1 + c_2 - c_3, 3c_1 + 4c_2 + c_3) = (8, 0, 5)$$

Equating the components of the vectors, we get the following system of linear equations:

$$\begin{aligned} c_1 \qquad\quad + 2c_3 &= 8 \\ 2c_1 + \ \ c_2 - \ \ c_3 &= 0 \\ 3c_1 + 4c_2 + \ \ c_3 &= 5 \end{aligned}$$

It can be shown that this system of equations has the unique solution,

$$c_1 = 2, c_2 = -1, c_3 = 3$$

Thus, the vector $(8, 0, 5)$ can be written in one way as a linear combination,

$$(8, 0, 5) = 2(1, 2, 3) - 1(0, 1, 4) + 3(2, -1, 1)$$

If the system had no solutions then it would not have been possible to express it as a linear combination of the other vectors. In the exercises that follow you will also see that it is possible that some vectors can be written in many ways as combinations of other vectors.

Vector addition and Scalar Multiplication of Column Vectors

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \text{and} \quad c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

For example, in $\mathbf{R}^2$,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \quad \text{and} \quad 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

The Subspaces of R^n

There are two properties that have to be checked to see if a subset is a subspace or not.

1. The sum of two arbitrary vectors of the subset must lie in the subset,
2. The scalar multiple of an arbitrary vector must also lie in the subset.

In other words, the subset must be closed under addition and under scalar multiplication

The Subspaces of R^n

EXAMPLE 2 Consider the subset V of $\mathbf{R}^3$ of vectors of the form $(a, 2a, 3a)$, where the second component is twice the first, and the third is three times the first. Let us show that V is a subspace of $\mathbf{R}^3$. Let $(a, 2a, 3a)$ and $(b, 2b, 3b)$ be two vectors in V, and let k be a scalar. Then

The Subspaces of R^n

EXAMPLE 2 Consider the subset V of $\mathbf{R}^3$ of vectors of the form $(a, 2a, 3a)$, where the second component is twice the first, and the third is three times the first. Let us show that V is a subspace of $\mathbf{R}^3$. Let $(a, 2a, 3a)$ and $(b, 2b, 3b)$ be two vectors in V, and let k be a scalar. Then

$$\begin{aligned}(a, 2a, 3a) + (b, 2b, 3b) &= (a + b, 2a + 2b, 3a + 3b) \\ &= (a + b, 2(a + b), 3(a + b))\end{aligned}$$

This is a vector in V since the second component is twice the first, and the third is three times the first. V is closed under addition. Further,

$$k(a, 2a, 3a) = (ka, 2ka, 3ka)$$

This vector is in V. V is closed under scalar multiplication.

The Subspaces of R^n

EXAMPLE 3 Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

SOLUTION

The Subspaces of R^n

EXAMPLE 3 Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

SOLUTION

W consists of all elements of $\mathbf{R}^3$ for which the second component is the square of the first. For example, the vector $(2, 4, 3)$ is in W, whereas the vector $(2, 5, 3)$ is not.

Let us check for closure under addition. Let (a, a^2, b) and (c, c^2, d) be elements of W. We get

$$\begin{aligned}(a, a^2, b) + (c, c^2, d) &= (a + c, a^2 + c^2, b + d) \\ &\neq (a + c, (a + c)^2, b + d)\end{aligned}$$

Thus $(a, a^2, b) + (c, c^2, d)$ is not an element of W. The set W is not closed under addition. Thus, W is not a subspace.

The Subspaces of R^n

EXAMPLE 3 Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

SOLUTION

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Let us check for closure under addition. Let (a, a^2, b) and (c, c^2, d) be elements of W. We get

$$\begin{aligned}(a, a^2, b) + (c, c^2, d) &= (a + c, a^2 + c^2, b + d) \\ &\neq (a + c, (a + c)^2, b + d)\end{aligned}$$

Thus $(a, a^2, b) + (c, c^2, d)$ is not an element of W. The set W is not closed under addition. Thus, W is not a subspace.

This completes the proof that W is not a subspace. Let us, however, illustrate the check for closure under scalar multiplication. Let k be a scalar. We get

$$\begin{aligned}k(a, a^2, b) &= (ka, ka^2, kb) \\ &\neq (ka, (ka)^2, kb)\end{aligned}$$

Thus $k(a, a^2, b)$ is not an element of W. W is not closed under scalar multiplication either.

The Subspaces of R^n

EXAMPLE 4 Consider the following homogeneous system of linear equations:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

It can be shown that there are many solutions $x_1 = 2r, x_2 = 5r, x_3 = r$. We can write these solutions as vectors in $\mathbf{R}^3$ as follows

$$(2r, 5r, r)$$

The Subspaces of R^n

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$$(2r, 5r, r)$$

The set of solutions W thus consists of vectors for which the first component is twice the third, and the second component is five times the third component. The vector (4, 10, 2) for example is in W while (4, 9, 2) is not. Let us show that W is closed under addition and under scalar multiplication. Let $\mathbf{u} = (2r, 5r, r)$ and $\mathbf{v} = (2s, 5s, s)$ be arbitrary vectors in W. We get

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (2r + 2s, 5r + 5s, r + s) \\ &= (2(r + s), 5(r + s), r + s) \end{aligned}$$

and

$$\begin{aligned} k\mathbf{u} &= k(2r, 5r, r) \\ &= (2kr, 5kr, kr) \end{aligned}$$

In $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ the first component is twice the third, and the second component is five times the third. They are both in W. W is closed under addition and under scalar multiplication. It is a subspace of $\mathbf{R}^3$.

The Subspaces of R^n

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and

$$\begin{aligned} k\mathbf{u} &= k(2r, 5r, r) \\ &= (2kr, 5kr, kr) \end{aligned}$$

In $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ the first component is twice the third, and the second component is five times the third. They are both in W. W is closed under addition and under scalar multiplication. It is a subspace of $\mathbf{R}^3$.

The Subspaces of R^n

EXAMPLE 5 Consider the following system of homogeneous linear equations.

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 0 \\ x_1 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 + 4x_4 &= 0 \end{aligned}$$

It can be shown that there are many solutions $x_1 = 3r - 2s, x_2 = 4r, x_3 = r, x_4 = s$. Write these solutions as vectors in $\mathbf{R}^4$,

$$(3r - 2s, 4r, r, s)$$

It can be shown that this set W of vectors is closed under addition and scalar multiplication, and is thus a subspace of $\mathbf{R}^4$. Separate the variables in the general solution.

$$(3r - 2s, 4r, r, s) = r(3, 4, 1, 0) + s(-2, 0, 0, 1)$$

This implies that every vector in W can be expressed as a linear combination of $(3, 4, 1, 0)$ and $(-2, 0, 0, 1)$. The vectors $(3, 4, 1, 0)$ and $(-2, 0, 0, 1)$ span W. W will be a plane through the origin in a four-space.

We shall see in Section 2.2 that the set of solutions to every homogeneous system of linear equations is a subspace. An implication of this result is that the sum of any two solutions is a solution. The scalar multiple of a solution is also a solution. However, we shall find that the set of solutions to a nonhomogeneous system is not a subspace. We shall discuss ways of relating solutions of nonhomogeneous systems to solutions of "corresponding" systems of homogeneous linear equations.