Gauss-Jordan Elimination for finding the Inverse of a Matrix

Let A be an $n \times n$ matrix.

1. Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.

2. Compute the reduced echelon form of $[A : I_n]$.

    If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.

    If the reduced echelon form is not of the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n, then A has no inverse.
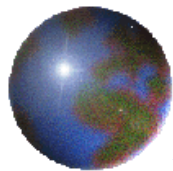
An $n \times n$ matrix A is invertible if and only if its reduced echelon form is I_n.
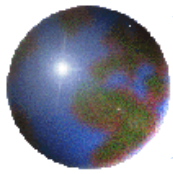
Example 20

Determine the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

Solution

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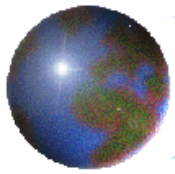
Solution

$$[A:I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R2+(-2)R1 \\ R3+R1 \end{matrix} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{(-1)R2}{\approx} \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \approx \\ R1+R2 \\ R3+(-2)R2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix} \approx \\ R1+R3 \\ R2+(-1)R3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Thus, $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$.
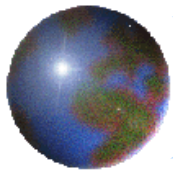
Example 21

Determine the inverse of the following matrix, if it exist.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$$

Solution

Properties of Matrix Inverse

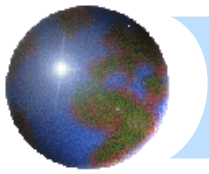Let A and B be invertible matrices and c a nonzero scalar, Then

1. $(A^{-1})^{-1} = A$

2. $(cA)^{-1} = \dfrac{1}{c} A^{-1}$

3. $(AB)^{-1} = B^{-1} A^{-1}$

4. $(A^n)^{-1} = (A^{-1})^n$

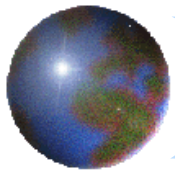5. $(A^t)^{-1} = (A^{-1})^t$

Example 22

If $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, then it can be shown that $A^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$. Use this information to compute $(A^t)^{-1}$.

Solution

$$(A^t)^{-1} = (A^{-1})^t = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$

Theorem 2.6

Let $AX = B$ be a system of n linear equations in n variables.
If A^{-1} exists, the solution is unique and is given by $X = A^{-1}B$.

Proof

($X = A^{-1}B$ is a solution.)
Substitute $X = A^{-1}B$ into the matrix equation.

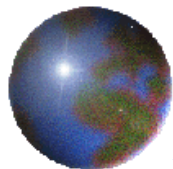$$AX = A(A^{-1}B) = (AA^{-1})B = I_n B = B.$$

(The solution is unique.)
Let Y be any solution, thus $AY = B$. Multiplying both sides of this
equation by A^{-1} gives

$$A^{-1}A\ Y = A^{-1}B$$
$$I_n Y = A^{-1}B$$
$$Y = A^{-1}B. \quad Then \ \ Y = X.$$

Example 22

Solve the system of equations
$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 - 5x_3 &= 3 \\ -x_1 + 3x_2 + 5x_3 &= -2 \end{aligned}$$

Solution

This system can be written in the following matrix form:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$
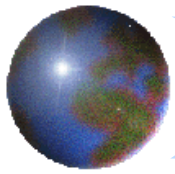
If the matrix of coefficients is invertible, the unique solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

This inverse has already been found in Example 20. We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The unique solution is $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

Elementary Matrices

Definition

An **elementary matrix** is one that can be obtained from the identity matrix I_n through a single elementary row operation.

Example 23

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
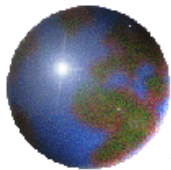
$R2 \leftrightarrow R3$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$5R2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R2 + 2R1$

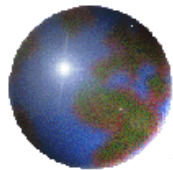$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot A = E_1 A$$

$$\xrightarrow{5R2} \begin{bmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_2 A$$

$$\xrightarrow{R2 + 2R1} \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = E_3 A$$
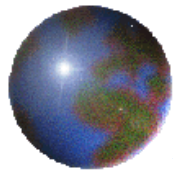
Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$
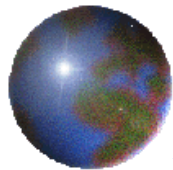
Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
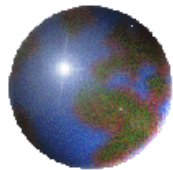
Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
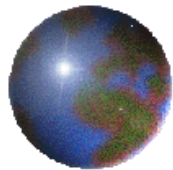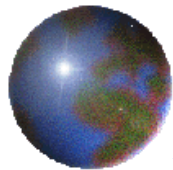
Elementary Matrices

Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
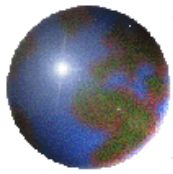
Determine if the following matrix is an elementary matrix.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Notes for elementary matrices

⊕ Each elementary matrix is invertible.

Example 24

$$I \underset{R1+2R2}{\approx} E_1 \implies E_1 \underset{R1-2R2}{\approx} I, \quad \text{i.e.,} \quad E_2 E_1 = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⊕ If *A* and *B* are row equivalent matrices and *A* is invertible, then *B* is invertible.