BC Free Response HW 11 (No Calculator)

Q1 This is 4 separate problems.

a) Let f be a function having derivatives for all orders of real numbers. The 3rd degree Taylor polynomial for f about $x=-2$ is given by $\sum_{k=0}^{3}(2k+1)(x+2)^k$. Find $f(-2)+f'(-2)+f''(-2)+f'''(-2)$.

$$\sum_{k=0}^{3}(2k+1)(x+2)^k = 1+3(x+2)+5(x+2)^2+7(x+2)^3$$

$$f(-2)=1, \quad f'(-2)=3, \quad \frac{f''(-2)}{2!}=5 \Rightarrow f''(-2)=10, \quad \frac{f'''(-2)}{3!}=7 \Rightarrow f'''(-2)=42$$

$$\therefore\ f(-2)+f'(-2)+f''(-2)+f'''(-2)=1+3+10+42=\boxed{56}$$

b) If $\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{2n-1}}$ converges, find its value. If not, explain why not.

$$\sum_{n=0}^{\infty}\frac{2^{n+1}}{3^{2n-1}} = \sum_{n=0}^{\infty}\frac{2^n\cdot 2}{3^{2n}\cdot 3^{-1}} = 2(3)\sum_{n=0}^{\infty}\frac{2^n}{3^{2n}} = 6\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n = 6\left(\frac{1}{1-\frac{2}{9}}\right) = \boxed{\frac{54}{7}}$$

geometric $r=2/9$ & $a_1=1$

c) Suppose k is an integer and $\sum_{n=1}^{\infty}\left(\frac{k}{5}\right)^n$ and $\sum_{n=1}^{\infty}\left(\frac{1}{n^{|2k-1|}}\right)$ both converge. What are all possible values of k? Explain how you arrive at your answer.

$\sum_{n=1}^{\infty}\left(\frac{k}{5}\right)^n$ is a geometric seires so $\left|\frac{k}{5}\right|<1 \Rightarrow \underline{-5<k<5}$

$\sum_{n=1}^{\infty}\left(\frac{1}{n^{|2k-1|}}\right)$ is a p-series $\therefore\ |2k-1|>1$

$$-1>2k-1>1$$

$$\underline{0>k\ \&\ k>1}$$

So the possible values of $\boxed{k \text{ are } -4,-3,-2,-1,\ 2,\ 3,\ 4}$

d) Determine if the series $\sum_{n=1}^{\infty}\frac{e^{n/2}}{\cos(\pi n)\cdot(n-1)!}$ is alternating. Show the ratio test can be applied to this series as well as giving its conclusion.

$\cos\pi n$, n is an integer $\therefore$ $\boxed{\text{always } \pm 1 \text{ so it is an alternating series}}$

$$\lim_{n\to\infty}\left|\frac{e^{n+1/2}}{n!}\cdot\frac{(n-1)!}{e^{n/2}}\right| = \lim_{n\to\infty}\frac{e^{n/2}e^{1/2}}{n(n-1)!}\cdot\frac{(n-1)!}{e^{n/2}} = \lim_{n\to\infty}\frac{e^{1/2}}{n} = 0<1$$

$$\therefore\ \boxed{\text{series is convergent}}$$

Q2 A particle moves along the x-axis so that at any time t its position is given by

$$x(t) = \begin{cases} t\cos 2\pi t, & 0 \le t < 1 \\ \ln(kt), & t \ge 1 \end{cases}$$

a) Find the velocity at time t.

$$v(t) = -2\pi t \sin 2\pi t + \cos 2\pi t$$

$$v(t) = \frac{k}{kt} = \frac{1}{t}$$

$$\boxed{v(t) = \begin{cases} -2\pi t \sin 2\pi t + \cos 2\pi t, & 0 \le t < 1 \\ \frac{k}{kt} = \frac{1}{t}, & t \ge 1 \end{cases}}$$

b) Find the acceleration at time t.

$$a(t) = -4\pi^2 t \cos 2\pi t - 2\pi \sin 2\pi t - 2\pi \sin 2\pi t$$

$$a(t) = \frac{-1}{t^2}$$

$$\boxed{a(t) = \begin{cases} -4\pi^2 t \cos 2\pi t - 4\pi \sin 2\pi t, & 0 \le t < 1 \\ \frac{-1}{t^2}, & t \ge 1 \end{cases}}$$

c) Determine if the particle is speeding up or slowing down at time $t = \frac{1}{2}$. Justify your answer.

$$v\left(\frac{1}{2}\right) = -2\pi\left(\frac{1}{2}\right)\sin 2\pi\left(\frac{1}{2}\right) + \cos 2\pi\left(\frac{1}{2}\right) = 0 + (-1) = -1 < 0$$

$$a\left(\frac{1}{2}\right) = -4\pi^2\left(\frac{1}{2}\right)\cos 2\pi\left(\frac{1}{2}\right) - 4\pi \sin 2\pi\left(\frac{1}{2}\right) = 2\pi^2 - 0 = 2\pi^2 > 0$$

$\boxed{\text{Since velocity and acceleration have opposite signs, the particle is slowing down}}$

d) Write, but do not solve an equation using a single trig function to determine the values of t for which the particle is at rest.

$$v(t) = -2\pi t \sin 2\pi t + \cos 2\pi t = 0$$

$$\cos 2\pi t = 2\pi t \sin 2\pi t$$

$$2\pi t = \frac{\cos 2\pi t}{\sin 2\pi t}$$

$$\boxed{\cot 2\pi t = 2\pi t} \text{ or } \boxed{\tan 2\pi t = \frac{1}{2\pi t}} \quad 0 \le t \le 1$$

e) What is the value of k such that the velocity of the particle is continuous? Explain your reasoning.

If v is continuous, then $x(t)$ must be differentiable which means that $x(t)$ is also continuous

$$\lim_{t\to 1^-} x(t) = \lim_{t\to 1^+} x(t)$$

$$(1)\cos 2\pi(1) = \ln(1)k$$

$$\cos 2\pi = \ln k$$

$$1 = \ln k$$

$$\boxed{k = e}$$

$$\lim_{t\to 1^-} v(t) = \lim_{t\to 1^+} v(t)$$

$$-2\pi(1)\sin 2\pi(1) + \cos 2\pi(1) = \frac{1}{1}$$

$$\cos 2\pi = 1$$

$$1 = 1 \therefore \boxed{\text{continuous}}$$