The University of Melbourne
Summer Semester Assessment 2009

Department of Mathematics and Statistics
620-156 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 7 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 13 questions. The total number of marks is 80. All questions may be attempted.

This paper may be held by the Baillieu Library.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} -3 & 1 & 3 \\ 4 & -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Calculate, if possible:

(i) AB

(ii) $A\mathbf{v}$

(iii) $\mathbf{v}\mathbf{v}^T$

(iv) $\mathbf{v}^T\mathbf{v}$.

(b) Write down the linear system in the variables x, y, z which is equivalent to the augmented matrix

(i)

$$[A|\mathbf{0}]$$

where $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(ii)

$$[B|\mathbf{c}]$$

where $\mathbf{c} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

[6 marks]

2. (a) Let

$$H = \begin{bmatrix} 1 & 1 & 1 \\ \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \end{bmatrix}$$

Use row operations to calculate $\det H$. Is the matrix H invertible? Clearly state the theorem used in your answer.

(b) Let A and B be two 5×5 matrices such that $\det A = 2$ and $\det B = \frac{3}{2}$. Calculate:

(i) $\det(AB)$

(ii) $\det(A^T A)$

(iii) $\det(2B)$

[6 marks]

3. Consider the matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (i) Give a reason why G^{-1} must exist.

(ii) Compute G^{-1}.

(b) Write the linear system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_2 + x_3 + x_4 &= 2 \\ x_3 + x_4 &= 2 \\ x_4 &= 2 \end{aligned}$$

as a matrix equation involving the matrix G, and proceed to solve the matrix equation by using your answer to (a)(ii).

[6 marks]

4. (a) We are told that a plane passing through the origin contains the vectors $(1, 1, 1)$ and $(3, 2, 1)$.

(i) Calculate a vector which is perpendicular to the plane.

(ii) Find the cartesian equation of the plane.

(b) Find the area of the triangle formed by the three points $(1, -1, 2)$, $(-2, 1, 1)$, $(1, 2, 3)$.

[6 marks]

5. (a) Let $H = \{(x, y) : x \geq 0\}$. Sketch H and determine whether or not it is a subspace of the vector space $\mathbb{R}^2$.

(b) The trace of a matrix is defined as the sum of its diagonal entries. For matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ from $M_{2,2}$, write down the condition for the trace to equal zero as a single linear equation. Let the set of such matrices be denoted S. By making a correspondence between elements of $M_{2,2}$ and four-tuples (a, b, c, d), do you suspect that S is a subspace of $M_{2,2}$? You must clearly say what piece of knowledge regarding subspaces in $\mathbb{R}^4$ you are using.

(c) Show from first principles that the set S in (b) is closed under scalar multiplication.

[6 marks]

6. It is known that

$$\begin{bmatrix} 1 & -2 & 3 & 3 & 65 \\ 3 & -10 & 8 & 13 & -104 \\ 1 & -1 & 0 & 2 & -78 \\ 2 & 2 & 1 & 0 & 143 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -20 \\ 0 & 1 & 0 & -1 & 58 \\ 0 & 0 & 1 & 0 & 67 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let the matrix on the left be denoted A.

(a) What is the rank of A?

(b) Write down a basis for the row space of A, and state the piece of theory being used.

(c) Write down a basis for the column space of A.

(d) Do the vectors $(1, 3, 1, 2), (-2, -10, -1, 2), (3, 8, 0, 1), (3, 13, 2, 0)$ span $\mathbb{R}^4$? Give a reason.

(e) What is the dimension of $\text{Span}\{(1, 3, 1, 2), (-2, -10, -1, 2), (3, 8, 0, 1), (3, 13, 2, 0)\}$?

(f) Write $(3, 13, 2, 0)$ as a linear combination of $(1, 3, 1, 2)$ and $(-2, -10, -1, 2)$.

(g) Find a basis for the solution space of A.

[8 marks]

7. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$, and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by π anticlockwise about the origin.

   (a) For the vector $(x, y) = (1, 1)$, plot its image under the action of S, and also plot its image under the action of R.

   (b) Calculate the standard matrix representations of S and R.

[4 marks]

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation specified by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 - 6x_3 \\ -2x_1 + x_2 + 5x_3 \\ 3x_1 + 3x_2 + 6x_3 \end{bmatrix}$$

   (a) Write down the standard matrix representation of T.

   (b) Calculate a basis $\mathcal{B}$ for $\mathrm{Ker}(T)$, the kernel of T. Describe $\mathrm{Ker}(T)$ geometrically.

   (c) Is T invertible? Give a reason.

   (d) Calculate a basis $\mathcal{C}$ for $\mathrm{Im}(T)$, the image of T. Describe $\mathrm{Im}(T)$ geometrically.

   (e) Show that $(4, -3, 0)$ is in $\mathrm{Im}(T)$ and find its co-ordinates with respect to $\mathcal{C}$.

[8 marks]

9. (a) Verify that

$$\mathcal{B} = \{(1, 1, 1), (1, 0, 1), (-1, 1, 0)\}$$

   is a basis for $\mathbb{R}^3$.

   (b) Let $\mathcal{S} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ denote the standard basis for $\mathbb{R}^3$. Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from the basis $\mathcal{B}$ to the standard basis, and use this to compute the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from the standard basis to the basis $\mathcal{B}$.

   (c) Use your answer to (b) to find the coordinates of the vector $\mathbf{v} = (3, -1, 1)$ with respect to $\mathcal{B}$.

[6 marks]

10. Consider the matrix $A = \begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$.

(a) Calculate the factorized form of the characteristic polynomial of A.

(b) Find the eigenvalues and eigenvectors of A.

(c) Give a reason why A is diagonalizable. Use your answer to (b) to write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

[6 marks]

11. (a) Write the formula

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = 5x_1y_1 - x_2y_2$$

in the form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for certain a, b, c, d.

(b) Under what conditions does the above matrix formula define an inner product? Show that one of these conditions does not hold for $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ as defined in (a).

(c) Let

$$\mathbf{u}_1 = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right), \quad \mathbf{u}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right), \quad \mathbf{u}_3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

The set $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis (with respect to the dot product) for $\mathbb{R}^3$.

(i) Use this fact to express $\mathbf{x} = (1, 2, 3)$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(ii) Write down, in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ the transition matrix from the standard basis to the basis $\mathcal{U}$.

[6 marks]

12. Four weather stations are placed along a straight road x kilometres from a marker, and the minimum temperature y °C is measured on a particular day, giving the following data

x	-3	0	1	2
y	-2	1	0	3

(a) Find the line of best fit to the data, using the method of least squares.

(b) Use your answer to estimate the minimum temperature -2 kilometres along the road from the marker, giving your answer to the nearest °C.

[6 marks]

13. Let

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding normalized eigenvectors for A.

(b) Describe geometrically the transformation represented by A.

(c) Compute $A^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (there is no need to simplify the arithmetic).

[6 marks]

— END OF EXAMINATION QUESTIONS —