Student Number: ____________________

The University of Melbourne

Semester 1 Assessment 2014

Department of Mathematics and Statistics

MAST10007 Linear Algebra

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 6 pages (including this page).

Authorised Materials:
☐ The following items are authorised:
OR
☐ Students may have unrestricted access to all materials.
OR
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Subjects with common content: MAST10008 Accelerated Mathematics 1

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination. No calculators, computers or mobile phones may be used.

Instructions to Students:
This examination consists of 14 questions. The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.
The number of marks for each question is indicated on the examination paper.
Use of calculators is not allowed.

Extra materials required:

☐ Graph Paper ☐ Multiple Choice Form ☐ Other

1. Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

Evaluate, if possible:

(a) AB,

(b) BA,

(c) $A + B$,

(d) $A^{\mathrm{T}}A$,

(e) A^{-1},

(f) B^{-1}.

[3 marks]

2. Determine the values (if any) of k in $\mathbb{R}$ for which the following linear system has:

(a) no real solution,

(b) a unique real solution,

(c) infinitely many real solutions.

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 3 \\ x & + & (k-1)y & + & z & = & 2k-1 \\ 2x & + & ky & + & (k+3)z & = & 3k+1 \end{array}$$

[6 marks]

3. (a) Use elementary row operations to calculate the inverse of the 3×3 matrix:

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -5 & 4 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Using the result from (a), solve the following equation for the 3×3 matrix X:

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

[6 marks]

4. (a) Find a vector equation for the line which is the intersection of the planes $x + y + z = 1$ and $2x + z = 2$.

   (b) Find equations for the plane which passes through the origin and the points $P(-1, -1, -1)$ and $Q(2, 1, 2)$ in:

      (i) vector form,

      (ii) Cartesian form.

   (c) Find the area of the triangle with vertices at P, Q and the origin.

[9 marks]

5. (a) Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

   (b) Suppose that a 5×5 matrix B has determinant $\det(B) = 12$. If we obtain C from B by applying the following sequence of row operations:

$$\begin{aligned} B &\xrightarrow{R_1 \leftrightarrow R_3} B_1 \\ &\xrightarrow{R_2 \leftrightarrow R_3} B_2 \\ &\xrightarrow{R_3 - 4R_4} B_3 \\ &\xrightarrow{R_3/3} B_4 \\ &\xrightarrow{R_4 \leftrightarrow R_5} C, \end{aligned}$$

   what is the value of $\det(C)$? Give a brief explanation.

[6 marks]

6. Determine which of the following sets are subspaces of the vector space $M_{2,2}$ of all real 2×2 matrices. Justify your answers by either using appropriate theorems, or providing a specific counterexample.

   (a)

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} : b + c = 0 \right\}.$$

   (b)

$$T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} : ad + bc = 0 \right\}.$$

[6 marks]

7. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & 12 \\ 2 & 5 & 7 & -1 & 0 \\ -1 & -2 & -8 & 1 & 12 \\ 3 & -2 & 96 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 26 & 0 & -3 \\ 0 & 1 & -9 & 0 & 5 \\ 0 & 0 & 0 & 1 & 19 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix B is the reduced row echelon form of the matrix A. Using this information, or otherwise, answer the following, giving reasons for your answers.

(a) Write down a basis for the column space of A.

(b) Write down (or calculate) the dimension of the row space of A.

(c) Are the rows of A linearly independent? Explain your answer.

(d) Write down a basis for the row space of A.

(e) Do the vectors $(1,2,-1,3), (3,5,-2,-2), (-1,7,-8,96), (0,-1,1,1), (12,0,12,0)$ span $\mathbb{R}^4$? Give a reason.

(f) Write $(-1,7,-8,96)$ as a linear combination of $(1,2,-1,3)$ and $(3,5,-2,-2)$.

(g) Find a basis for the solution space of A.

[10 marks]

8. Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations where $S(x,y) = (x+2y, y)$ and T is reflection in the line $y = -x$.

(a) Find the standard matrices for (i) S, (ii) T and (iii) $T \circ S$.

(b) Let Δ be the triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$. Draw a large diagram showing Δ and its images under S and $T \circ S$ on the same set of coordinate axes. You should label Δ, $S(\Delta)$ and $T \circ S(\Delta)$ clearly.

[6 marks]

9. Let $\mathcal{P}_2$ be the vector space of polynomials of degree ≤ 2. A function $T : \mathcal{P}_2 \to \mathbb{R}^2$ is defined by

$$T(p(x)) = (p(1), p(2)).$$

(a) Prove that T is a linear transformation.

(b) Find the matrix of T with respect to the standard bases $\mathcal{B} = \{1, x, x^2\}$ for $\mathcal{P}_2$ and $\mathcal{S} = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$.

(c) Find bases for the kernel and the image of T.

[8 marks]

10. (a) Explain why the following formulas **do not** define inner products on $\mathbb{R}^2$, by giving an explicit counterexample to one of the inner product space axioms:

   (i) $\langle (u_1, u_2), (v_1, v_2) \rangle = 3u_1v_2 + u_2v_1$,

   (ii) $\langle (u_1, u_2), (v_1, v_2) \rangle = u_1v_1 - 3u_1v_2 - 3u_2v_1 + u_2v_2$,

   where $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$.

   (b) We define an inner product on the vector space $V = \mathcal{P}_2$ of polynomials of degree at most 2 by the formula

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx,$$

   for all $p(x), q(x) \in \mathcal{P}_2$. Using this inner product:

   (i) Find $\|x\|$.

   (ii) Decide whether the polynomials 1 and x^2 are orthogonal.

   (iii) Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace W of V spanned by $\{1, x^2\}$.

   (iv) Find the polynomial $p(x)$ in W closest to x, i.e. minimizing $\|p(x) - x\|$.

[11 marks]

11. Consider the ordered bases $\mathcal{S} = \{(1, 0), (0, 1)\}$ and $\mathcal{B} = \{(1, 1), (-1, 3)\}$ for $\mathbb{R}^2$.

   (a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{S}$.

   (ii) Find the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{B}$.

   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x-y, 3x+5y)$.

   (i) Find the matrix $[T]_{\mathcal{S}}$ of the transformation T with respect to the ordered basis $\mathcal{S}$.

   (ii) Find the matrix $[T]_{\mathcal{B}}$ of the transformation T with respect to the ordered basis $\mathcal{B}$.

   (iii) If $\boldsymbol{v} = (7, -1)$ find $[\boldsymbol{v}]_{\mathcal{B}}$ and $[T(\boldsymbol{v})]_{\mathcal{B}}$.

[10 marks]

12. Consider the matrix

$$C = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}.$$

   (a) Find all eigenvalues and corresponding eigenvectors for the matrix C.

   (b) Find an invertible matrix P and a diagonal matrix D such that $C = PDP^{-1}$.

   (c) Let $\boldsymbol{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use your results from (b) to find a formula for $C^n\boldsymbol{v}_0$ valid for each integer $n \geq 1$.

   (d) Describe the limiting behaviour of $C^n\boldsymbol{v}_0$ as $n \to \infty$.

[9 marks]

13. (a) Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 4 \\ 5 & 4 & 3 \end{bmatrix}.$$

Determine which of these matrices are diagonalizable, giving reasons for your answers.

(b) Consider the linear transformation $P : \mathbb{R}^3 \to \mathbb{R}^3$ given by orthogonal projection onto the plane $3x - y - 2z = 0$, using the dot product on $\mathbb{R}^3$ as inner product.

Describe the eigenspaces and eigenvalues of P, giving brief reasons for your answers. (Hint: you do not need to find a matrix representing the transformation.)

[6 marks]

14. Let V, W be subspaces of a vector space U such that $V \cap W = \{\mathbf{0}\}$. Prove that if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent vectors in V and $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ are linearly independent vectors in W, then $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ are linearly independent vectors in U.

[4 marks]

END OF EXAMINATION PAPER