MAST20004 – SEMESTER 1 – SOLUTIONS

1 (a) Axioms

(i) For all events $A \in \mathcal{A}$, $P(A) \geq 0$
(ii) $P(\Omega) = 1$
(iii) For any sequence of disjoint events A_i, $i = 1, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

I will accept

(iiia) For any two disjoint events A & B.

$$P(A \cup B) = P(A) + P(B)$$

or (iiib) For any set $A_1, \ldots, A_n$ of disjoint events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

[3 marks]

(b) (i) $\mathcal{A}$ has $2^4 = 16$ elements [1 mark]

(ii) By Axiom 3, we know that

$$P(\{A\}) + P(\{B\}) = P(\{A, B\}) \Rightarrow \underline{P(\{B\}) = 1/}$$
$$P(\{A\}) + P(\{C\}) = P(\{A, C\}) \Rightarrow \underline{P(\{C\}) = 5/}$$

Then

$$P(\{A, B, C\}) = P(\{A\}) + P(\{B\}) + P(\{C\}) = \tfrac{11}{12}.$$
$$P(\{A, B, C\}) + P(\{D\}) = P(\Omega) = 1 \Rightarrow \underline{P(\{D\}) = 1/1}$$

We then derive the probabilities of the other elements of $\mathcal{A}$.

$P(\emptyset) = 0$, $P(\{A, D\}) = 1/3$, $P(\{B, C\}) = 2/3$, $P(\{B, D\}) =$
$P(\{C, D\}) = 1/2$, $P(\{A, B, D\}) = 7/12$, $P(\{A, C, D\}) = 3/$
$P(\{B, C, D\}) = 3/4$, $P(\Omega) = 1$

[5 marks]

2(a)

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}.$$

[2 marks]

(b). (i) Let D be the event that the person has the disease and C be the event that they test positive. Then the information that we are given can be written as.

$P(C|D) = 0.95$ $\quad P(C|D^c) = 0.05$

$P(D) = 0.002$

Using Bayes' Formula.

$$P(D|C) = \frac{P(C|D)\,P(D)}{P(C|D)P(D) + P(C|D^c)P(D^c)}$$

$$= \frac{0.95 \times 0.002}{0.95 \times 0.002 + 0.05 \times 0.998}$$

$$= .0367.$$

[3 marks]

(ii). The test is not likely to be useful. because the probability that a person has the disease when they test positive is still very low. - There is a high rate of false positives.

[1 mark]

3 (a). X has a hypergeometric distribution. with parameters $N = 45$, $D = 7$ and $n = 7$.

[2 marks]

(b). For $x = 0, \dots, 7$

$$p_X(x) = \frac{\binom{7}{x}\binom{38}{\cancel{7}-x}}{\binom{45}{7}}$$

[2 marks]

(c). M is a binomial random variable with parameters $n = 80{,}000{,}000$ and $p = \dfrac{\binom{7}{7}\binom{38}{0}}{\binom{45}{7}}$.

$$= \frac{1}{45{,}379{,}620}$$ [2 marks]

(d) $P(M>0) = 1 - P(M=0)$

$$= 1 - (1-p)^n$$

$$= 1 - \left(\frac{45{,}379{,}619}{45{,}379{,}620}\right)^{80{,}000{,}000}.$$ [2 marks].

(e) M is approximately distributed as a Poisson random variable with parameter $np =$

$$\frac{80{,}000{,}000}{45{,}379{,}620} = 1.7629.$$

So $P(M>0) = 1 - P(M=0)$

$$= 1 - e^{-1.7629}.$$

$$= 0.8285.$$ [2 marks].

(f) N is the number of 'failures' before the first success in a sequence of independent Bernoulli Trials, with success probability $p = 0.8285$. So N has a geometric distribution with parameter 0.8285.

[2 marks]

(g) $E[N] = \dfrac{(1-p)}{p} = .2070$

$V[N] = \dfrac{(1-p)}{p^2} = .2498$.

(2 marks).

4 (a) $E[X] = 1/2$
$V(X) = 1/12$. [2 marks]

(b) X can range between 0 and 1 and $Y = e^x$.
So Y ranges between 1 and e.
That is $S_Y = [1, e]$. [1 mark]

(c) For $y \in [1,e]$, $F_Y(y) = P(Y \le y)$

$= P(e^X \le y)$

$= P(X \le \log y)$

$= \log y$.

So $F_Y(y) = \begin{cases} 0 & , y \in [-\infty, 1) \\ \log y & , y \in [1, e] \\ 1 & y \in [e, \infty). \end{cases}$

$f_Y(y) = \begin{cases} \frac{1}{y} & , y \in [1, e] \\ 0 & \text{otherwise}. \end{cases}$

[3 marks]

(d) (i) $E[Y] = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1$.

(ii) $E[Y] = \int_1^e y \frac{1}{y} \, dy = [y]_1^e = e - 1$.

[2 marks]

(e) $E[Y^2] = \int_1^e y^2 \frac{1}{y} \, dy = \left[\frac{y^2}{2}\right]_1^e = 1/2(e^2 - 1)$

$\therefore V(Y) = 1/2(e^2 - 1) - (e-1)^2 = -1/2 e^2 + 2e - 3/2$.

[2 marks]

(f) $Y \simeq e^{1/2} + (X-1/2)e^{1/2} + (X-1/2)^2 e^{1/2}/2$

So $E[Y] \simeq e^{1/2} + 1/2\, V(X)\, e^{1/2}$

$= e^{1/2}\,\frac{25}{24}. \quad = \quad 1.7174$

cf $e-1 = 1.7183$

The variance is small, so we expect the approximation to the mean to be good.

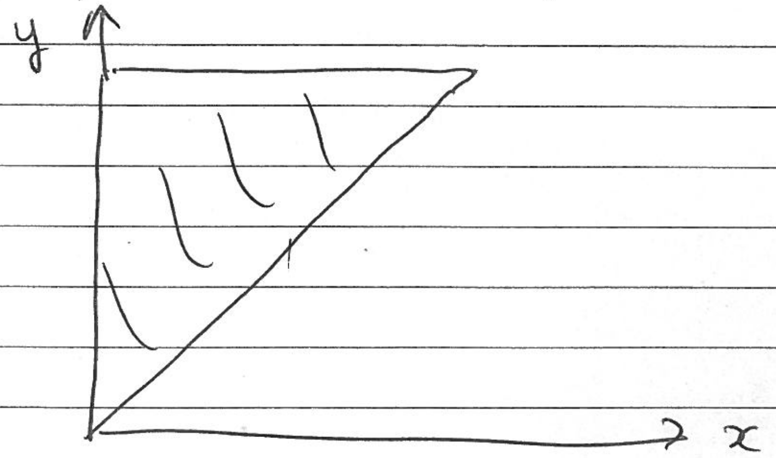
$V(Y) \simeq V(X)\,(e^{1/2})^2$

$= \frac{1}{12}\cdot e \quad = \quad .2265$

cf $-\frac{e^2}{2} + 2e - 3/2 = .2420.$

The approximation is reasonably close. [4 marks]

5 (a) (i) X and Y are clearly dependent since $f_{(X,Y)}(3/4, 1/4) = 0$ but $f_X(3/4) > 0$ and $f_Y(1/4) > 0$



[2 marks]

(ii). $1 = \int_0^1 \int_0^y kxy \; dx \; dy$

$= \int_0^1 \left[\frac{kx^2y}{2}\right]_0^y dy$

$= \int_0^1 \frac{ky^3}{2} \; dy$

$= \left[\frac{ky^4}{8}\right]_0^1 = \frac{k}{8}.$

$\therefore \; k = 8.$ [2 marks]

(iii) For $x \in (0,1)$

$$f_X(x) = \int_x^1 8xy \, dy$$

$$= \left[4xy^2 \right]_x^1$$

$$= 4x - 4x^3.$$

For $y \in (0,1)$

$$f_Y(y) = \int_0^y 8xy \, dx$$

$$= \left[4x^2 y \right]_0^y$$

$$= 4y^3$$

[2 marks]

(iv) $$f_{Y|X}(y \mid x = 1/2) = \left. \frac{8xy}{4x - 4x^3} \right|_{x=1/2}.$$

$$= \frac{4y}{3/2}$$

$$= \frac{8y}{3}.$$

[2 marks].

(v). $P(Y \leq 3/4 \mid X \leq 1/2)$

$$= \frac{P(Y \leq 3/4 \text{ and } X \leq 1/2)}{P(X \leq 1/2)}$$

$$= \frac{\int_0^{1/2} \int_x^{3/4} 8xy \, dy \cdot dx}{\int_0^{1/2} \int_x^1 8xy \, dy \, dx}.$$

$$= \frac{\int_0^{1/2} \left[4xy^2 \right]_x^{3/4} dx}{\int_0^{1/2} \left[4xy^2 \right]_x^1 dx}$$

$$= \frac{\int_0^{1/2} \frac{9x}{4} - 4x^3 \, dx}{\int_0^{1/2} 4x - 4x^3 \, dx.}$$

$$= \frac{\left[\frac{9x^2}{8} - x^4\right]_0^{1/2}}{\left[2x^2 - x^4\right]_0^{1/2}}$$

$$= \frac{9/32 - 1/16}{1/2 - 1/16}$$

$$= \frac{7/32}{7/16}$$

$$= 1/2.$$

[3 marks].

(b). (i) $F_Z(z) = P(Z \le z)$

$$= \int_{S_X} P(Z \le z \mid X = x) \; f_X(x) \, dx.$$

$$= \int_{S_X} P(Y \le z - x \mid X = x) \; f_X(x) \, dx.$$

$$= \int_{S_X} P(Y \le z - x) \; f_X(x) \, dx$$ (by independence

$$= \int_{S_X} F_Y(z-x) \; f_X(x) \, dx.$$

[3 marks].

(ii).

$$F_Z(z) = \int_0^{\infty} \left(1 - e^{-\alpha(z-x)}\right) I(z - x > 0) \, \alpha e^{-\alpha x} dx$$

$$= \int_0^{z} \alpha e^{-\alpha x} - \alpha e^{-\alpha z} \, dx$$

$$= -e^{-\alpha x} - \alpha e^{-\alpha z} x \Big]_0^{z}$$

$$= 1 - e^{-\alpha z} - \alpha z e^{-\alpha z}.$$

[2 marks

6/ (a) $E[Z \mid Y=1] = E[X] = 1/2$

$E[Z \mid Y=2] = E[X^2] = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3$

[2 marks]

(b) $V[Z \mid Y=1] = V[X] = 1/12$

$V[Z \mid Y=2] = V[X^2] = E[X^4] - E[X^2]^2$

$= \int_0^1 x^4\,dx - 1/9$

$= \left[\frac{x^5}{5}\right]_0^1 - 1/9$

$= 4/45$

[2 marks]

(c) $E[Z \mid Y] = \begin{cases} 1/2 & \text{with probability } 1/2 \\ 1/3 & \text{"} \quad 1/2 \end{cases}$

$V[Z \mid Y] = \begin{cases} 1/12 & \text{"} \quad 1/2 \\ 4/45 & \text{"} \quad 1/2 \end{cases}$

[2 marks]

(d) $E[Z] = E[E[Z \mid Y]] = 1/2(1/2 + 1/3) = 5/12$

[2 marks]

(e). $V[Z] = E[V[Z \mid Y]] + V[E[Z \mid Y]] =$

$= 1/2\left[\frac{1}{12} + \frac{4}{45}\right] + 1/2\left[\frac{1}{2} - \frac{5}{12}\right]^2 + 1/2\left[\frac{1}{3} - \frac{5}{12}\right]^2$

$= \frac{1}{2}\left[\frac{15+16}{180}\right] + 1/2\left[\frac{1}{144}\right] + 1/2\left[\frac{1}{144}\right]$

$= \frac{31}{360} + \frac{1}{144}$

$= \frac{248+20}{2880}$

$= \frac{268}{2880}$

$= \frac{67}{720}.$

[2 marks].

7/ (a) $f_R(r) = re^{-r^2/2} \qquad r > 0$

$f_\Theta(\theta) = \frac{1}{2\pi} \qquad 0 < \theta < 2\pi$

[2 marks]

(b) $F_R(r) = \int_0^r ue^{-u^2/2}\,du$

$= -e^{-u^2/2}\Big]_0^r = 1 - e^{-r^2/2} \qquad r > 0$

$F_\Theta(\theta) = \frac{\theta}{2\pi} \qquad 0 < \theta < 2\pi$

[2 marks].

(c) Since $f_{(R,\theta)}(r,\theta) = f_R(r)\, f_\theta(\theta)$, we conclude that R and θ are independent.
[1 mark].

(d) (i) To get a random variable with distribution $F_R(r)$, we transform U_1 according to F_R^{-1}.
So we put $R = \sqrt{-2\log(1-U)}$ [2 marks]

(ii) We transform U_2 according to F_θ^{-1}.
So we put $\theta = 2\pi U$. [1 mark].

(e) The random variable (X,Y) is bivariate standard normal with $\rho = 0$. [1 mark]

8/ (a) (i)

$$M_Z(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-zt} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(z-t)^2} e^{\frac{t^2}{2}}\, dz.$$

$$= e^{t^2/2}\, \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(z-t)^2} dz}_{=1.}$$

$$= e^{t^2/2}$$ [3 marks]

(ii) If $X = \mu + \sigma Z$

$$M_X(t) = e^{\mu t} M_Z(\sigma t)$$

$$= e^{\frac{\sigma^2 t^2}{2} + \mu t}.$$ [2 marks]

(iii) If $S_n = \sum_{i=1}^{n} X_i$, then

$$M_{S_n}(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

$$= \prod_{i=1}^{n} e^{\frac{\sigma_i^2 t^2}{2} + \mu_i t}$$

$$= \exp\left(\sum_{i=1}^{n} \frac{\sigma_i^2 t^2}{2} + \sum_{i=1}^{n} \mu_i t \right).$$

which we recognise as the moment generating function of a normal distribution with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

So $S_n \stackrel{d}{=} N\left(\sum_{i=1}^{n} \mu_i ; \sum_{i=1}^{n} \sigma_i^2\right)$. [4 marks].

(d) $M_{A_n}(t) = M_{\frac{S_n}{n}}(t) = M_{S_n}\left(\frac{t}{n}\right)$

$$= \exp\left(\sum_{i=1}^{n} \frac{\sigma_i^2 t^2}{2n^2} + \sum_{i=1}^{n} \mu_i \frac{t}{n} \right).$$

which we recognise as the moment generating function of a normal distribution with mean $\frac{1}{n} \sum_{i=1}^{n} \mu_i$ and variance $\frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2$ [2 marks].

(e). $M_{A_n} = \exp\left(\frac{\sigma^2 t^2}{n} + \mu t \right)$

which is the moment generating function of a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$. [2 marks)

9/ (a) $q_0 = 0$

$q_1 = A(q_0)$

$= \frac{2}{5}$

$q_2 = A(q_1) = \frac{1}{10}\left(\frac{8}{125} + \frac{12}{25} + \frac{4}{5} + 4 \right)$

$= \frac{334}{625}$

[3 marks]

(b) To get the extinction probability we solve

$$q = A(q)$$

$$\Rightarrow \quad q = \frac{1}{10}(q^3 + 3q^2 + 2q + 4).$$

$$\Rightarrow \quad 0 = \frac{1}{10}(q^3 + 3q^2 - 8q + 4)$$

$$\Rightarrow \quad 0 = (q-1)(q^2 + 4q - 4)$$

$$\Rightarrow \quad q = 1, \quad \frac{-4 \pm \sqrt{16+16}}{2}$$

$$\Rightarrow \quad q = 1, \quad -2 \pm 2\sqrt{2}.$$

We observe that $-2+2\sqrt{2} \in (0,1)$. This is the minimal nonnegative solution, so the extinction probability is $-2+2\sqrt{2}$.