The University of Melbourne
Semester 2 Assessment 2010

Department of Mathematics and Statistics
MAST10007 (620-156) Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 7 pages

Identical Examination Papers: None
Common Content Papers: None

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 12 questions.

The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Let
$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$
Evaluate, if possible:
   (a) AB
   (b) $B^T A$
   (c) BA
   (d) $2C + BC$

[6 marks]

2. (a) Let
$$G = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 0 & 6 \\ 4 & 1 & -5 \end{bmatrix}$$
   (i) Calculate the determinant of G using either row operations or cofactor expansion.
   (ii) Write down $\det(G^{-1})$.
   (iii) Write down $\det(G^T)$.

   (b) Let M and N be two 3×3 matrices such that $\det(M) = -\frac{1}{2}$ and $\det(N) = 3$. Calculate:
   (i) $\det(NM)$
   (ii) $\det(3NM^2)$

   (c) If A is an invertible matrix such that $\det(A) = x$, prove that $\det(A^{-1}) = \dfrac{1}{x}$. You should justify each step carefully.

[8 marks]

3. (a) Use row operations to find the inverse, if it exists, of the square matrix

$$\begin{bmatrix} 3 & 0 & 2 \\ -1 & 2 & 0 \\ 4 & 3 & 3 \end{bmatrix}$$

At each step you should clearly indicate the row operations that you have performed.

(b) Using your answer to part (a), solve the following linear system:

$$\begin{array}{rcrcrcr} 6x & & & + & 4z & = & 12 \\ -2x & + & 4y & & & = & 8 \\ 4x & + & 3y & + & 3z & = & 5 \end{array}$$

(c) A linear system has coefficient matrix A which has row reduced echelon form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1+k \end{array}\right]$$

For which values of k (if any) does the system have

(i) no solutions,

(ii) exactly one solution,

(iii) infinitely many solutions?

(d) For each of cases in part (c) above where a solution exists, write down the solution.

[10 marks]

4. (a) Let $\mathcal{L}$ be the line in $\mathbb{R}^3$ given by the equations:

$$\frac{x-1}{2} = \frac{y+5}{3} = 2z - 6 \qquad (\text{for all } (x, y, z) \in \mathcal{L})$$

Find a vector equation of $\mathcal{L}$.

(b) Find the Cartesian equation of the plane Π in $\mathbb{R}^3$ that contains both the point $(1, 3, 5)$ and the line $\mathcal{L}$.

(c) Find the distance from the point $(-3, -1, 3)$ to the line $\mathcal{L}$.

[6 marks]

5. For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either appealing to appropriate theorems, or providing a counter-example.

(a) $V = \mathcal{P}_2$ (the vector space of all polynomials of degree at most two)
$S = \{a + 2bx + 3cx^2 \mid a, b, c \in \mathbb{R}\}$

(b) $V = M_{2,3}$ (vector space of all 2×3 matrices with real entries)
$$S = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \;\middle|\; (a+b+c)(d+e+f) = 0 \right\}$$

(c) $V = C(\mathbb{R}, \mathbb{R})$ (vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$)
$S = \{f \in C(\mathbb{R}, \mathbb{R}) \mid f(0) = 1\}$

[6 marks]

6. (a) Determine whether or not the given set S is linearly independent in the given vector space V. If the set is linearly dependent, express one of its vectors as a linear combination of the other vectors in the set.

(i) $S = \{1 + x, 2 + x + x^2, 2 + 2x^2\}$
$V = \mathcal{P}_2$ (the vector space of all polynomials of degree at most two)

(ii)
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$V = M_{2,2}$ (the vector space of all 2×2 matrices of real entries)

(b) Let S and V as in (a)(i) above. Show that

$$\mathrm{Span}(S) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}, a_0 = a_1 + a_2\}$$

[8 marks]

7. (a) Let $W \subseteq \mathbb{R}^4$ the span of the vectors in the set

$$S = \{(1,-2,3,1),\ (0,1,0,1),\ (3,-5,9,4),\ (0,-2,1,-1)\}$$

(i) What is the dimension of W ?

(ii) Are the vectors in S linearly independent ?

(iii) Find a basis B for W satisfying $B \subseteq S$.

(b) The following two matrices are row-equivalent:

$$A = \begin{bmatrix} -3 & 1 & -5 & 2 & 4 \\ 2 & 2 & 6 & 0 & 4 \\ -2 & 0 & -4 & 1 & 1 \\ 3 & 2 & 8 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) What is the rank of A ?

(ii) Write one of the columns of A as a linear combination of the others.

(iii) Write down a basis for the column space of A.

(iv) Find a basis for the solution space of A.

[10 marks]

8. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by

$$T(a_0 + a_1 x + a_2 x^2) = (5a_0 + 3a_1 - 10a_2) + (-6a_0 - 4a_1 + 12a_2)x$$

(a) Find the matrix $[T]_{\mathcal{S}}$ of T with respect to the standard basis $\mathcal{S} = \{1, x, x^2\}$ for $\mathcal{P}_2$.

(b) The set $\mathcal{B} = \{1 - x, -1 + 2x, 2 + x^2\}$ is also a basis for $\mathcal{P}_2$. Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{S}$.

(c) Give an expression for $P_{\mathcal{B},\mathcal{S}}$ in terms of $P_{\mathcal{S},\mathcal{B}}$ and use it to calculate the transition matrix $P_{\mathcal{B},\mathcal{S}}$.

(d) Calculate the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

[10 marks]

9. Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ for vectors in $\mathbb{R}^3$. Define:

$$\langle \mathbf{x}, \mathbf{y} \rangle = 3x_1y_1 + x_2y_2 + 5x_3y_3 - 3x_1y_3 - 3x_3y_1$$

(a) Find a 3×3 symmetric matrix A so that

$$\mathbf{x}^T A \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

(b) Show that $\mathbf{x}^T A \mathbf{x} > 0$ when $\mathbf{x} \neq 0$.

(c) Starting with the basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3$, apply the Gram-Schmidt procedure to obtain a basis of $\mathbb{R}^3$ that is orthonormal with respect to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ defined above.

[12 marks]

10. (a) For each of the following matrices, decide whether or not the matrix is diagonalizable over $\mathbb{R}$. (You should justify your answer.)

(i) $\begin{bmatrix} 7 & 5 \\ 5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -6 \\ 6 & -5 \end{bmatrix}$ (iii) $\begin{bmatrix} -5 & -8 \\ \frac{9}{2} & 7 \end{bmatrix}$

(b) Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

The characteristic equation of A is $(x + 1)^2(x - 3) = 0$

(i) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(ii) Write down matrices D and Q, such that D is diagonal, Q is orthogonal and

$$QDQ^T = A$$

[12 marks]

11. Let V be a real vector space and $T : V \to V$ a linear transformation.

(a) What does it mean to say that $\boldsymbol{v} \in V$ is an eigenvector of T ?

(b) Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ are eigenvectors of T having eigenvalues λ_1 and λ_2 respectively. Prove that if $\lambda_1 \neq \lambda_2$ then the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly independent.

[6 marks]

12. (a) Identify the conic defined by the following equation:

$$3x^2 + 4\sqrt{3}\,xy - y^2 = 2$$

(b) Find the direction of the principle axes of the conic.

(c) Sketch the conic in the xy-plane. Include the principle axes and the distances between the origin and the points where the conic intersects the principle axes.

[6 marks]

— END OF EXAMINATION QUESTIONS —