

1 (a) (i) Not possible

①

$$(ii) A\tilde{u} = \begin{bmatrix} -3 & 1 & 3 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

①

$$(iii) \tilde{u}\tilde{u}^T = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

①

$$(iv) \tilde{u}^T \tilde{u} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 3$$

①

$$(b) (i) -3x + y + 3z = 0$$

$$4x - 2y = 0 \quad ①$$

$$(ii) 6x + 2y = -1$$

$$2x - y = 2 \quad ①$$

$$2(a) \det H = \det \begin{bmatrix} 1 & 1 & 1 \\ \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \end{bmatrix}$$

$$\stackrel{(12)}{=} \det \begin{bmatrix} 1 & 1 & 1 \\ \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ 1 & 1 & 1 \end{bmatrix}$$

adding row ① to  
row ③ and noting  
 $\sin^2 \theta + \cos^2 \theta = 1$

$$= \det \begin{bmatrix} 1 & 1 & 1 \\ \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ 0 & 0 & 0 \end{bmatrix}$$

subtracting row ①  
from row ③

$$= 0$$

We know that a square matrix is invertible if and only if its determinant is non-zero.

①/2

But the determinant of  $H$  is zero, so it is not invertible.

$$(b) (i) \det AB = \det A \det B = 2 \times \frac{3}{2} = 3 \quad (1)$$

$$(ii) \det A^T A = \det A^T \det A = (\det A)^2 = 4 \quad (1)$$

$$(iii) \det(2B) = 2^5 \det B = 2^5 \times \frac{3}{2} = 48 \quad (1)$$

$$3(a) (i) \text{ Because } \det A = 1 \neq 0 \quad (1)$$

$$(ii) \left[ \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} (1) - (4) \\ (2) - (4) \\ (3) - (4) \end{array}$$

$$\sim \left[ \begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} (1) - (3) \\ (2) - (3) \end{array}$$

$$\sim \left[ \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] (1) - (2)$$

Working (2)

$$\sim \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Hence } A^{-1} = \left[ \begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Answer (1)

(b) As a matrix equation the linear system reads

$$G \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad (1)$$

Applying  $G^{-1}$  to both sides gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = G^{-1} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad (1)$$

4(a) (i) Let  $\underline{a}$  denote such a vector. Then

$$\underline{a} = \det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$
$$= \underline{i} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \quad (1\frac{1}{2})$$
$$= -\underline{i} + 2\underline{j} - \underline{k}$$

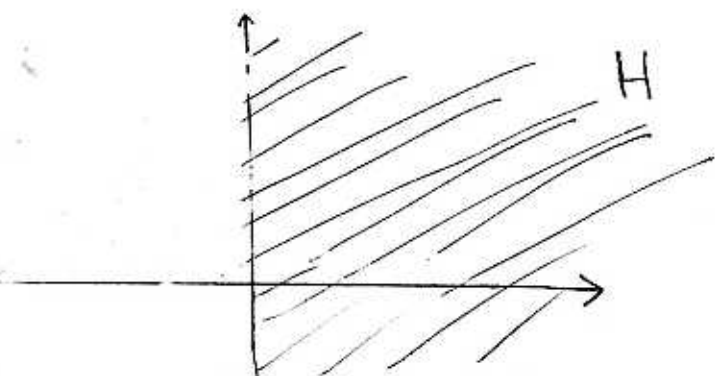
(ii) Since the plane passes through the origin, it is specified by

$$\underline{a} \cdot (x, y, z) = 0$$
$$\Rightarrow (-1, 2, -1) \cdot (x, y, z) = 0 \quad (1\frac{1}{2})$$
$$\Rightarrow x - 2y + z = 0$$

(b) This is the same as the area of the triangle formed by  $(0,0,0)$ ,  $(-3,2,-1)$ ,  $(0,3,1)$ , which in turn is half the area of the parallelogram formed by  $(-3,2,-1)$  and  $(0,3,1)$ :

$$\begin{aligned} \frac{1}{2} \|(-3,2,-1) \times (0,3,1)\| &= \frac{1}{2} \sqrt{\|(-3,2,-1)\|^2 \|(0,3,1)\|^2 - ((-3,2,-1) \cdot (0,3,1))^2} \\ &= \frac{1}{2} \sqrt{14 \times 10 - 25} \\ &= \frac{1}{2} \sqrt{115} \end{aligned} \quad (3)$$

5(a)



Consider the point  $(1,0) \in H$ . Scalar multiplication by  $(-1)$  gives the point  $(-1,0) \notin H$ . The set is not closed under scalar multiplication, and so is not a subspace. (1)

(b)  $a+d=0$

With the correspondence  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow (a,b,c,d) \in \mathbb{R}^4$

we recognise  $a+d=0$  as a homogeneous linear eq<sup>n</sup>. We know that <sup>the solution space of</sup> homogeneous linear equations forms a subspace of  $\mathbb{R}^4$ , so we suspect  $S$  is a subspace of  $M_{2,2}$ . (2)

(c) Consider  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$  so that  $a+d=0$ .

Then, with  $\alpha$  a scalar

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

and

$$\alpha a + \alpha d = \alpha(a+d) = 0 \quad \text{since } a+d=0$$

Hence

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \quad \text{and so } S \text{ is closed}$$

under scalar multiplication. (2)

6 (a) rank  $A = 3$  (1)

(b)  $\{(1, 0, 0, 1, -20), (0, 1, 0, -1, 58), (0, 0, 1, 0, 67)\}$ , (1)

using the fact that the non-zero rows in RE form are a basis for the row space

(c)  $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -10 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 0 \\ 1 \end{bmatrix} \right\}$  (1)

(d) No. The rank of the matrix formed by these vectors is 3. (1)

(e) Three (1)

(f)  $(3, 13, 2, 0) = (1, 3, 1, 2) + (-1)(-2, -10, -1, 2)$  (1)

(g) Let the variables be denoted  $x_1, x_2, \dots, x_5$ .

There is no leading entry for  $x_4$  &  $x_5$  so we set  $x_5 = t$ ,  $x_4 = s$ . Back substitution gives

$$x_3 = -67x_5 = -67t$$

$$x_2 = x_4 - 58x_5 = s - 58t$$

$$x_1 = -x_4 + 20x_5 = -s + 20t$$

Hence

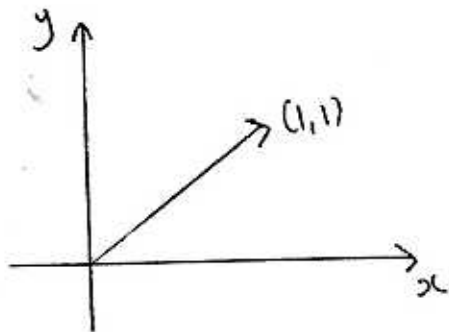
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 20 \\ -58 \\ -67 \\ 0 \\ 1 \end{bmatrix}$$

(2)

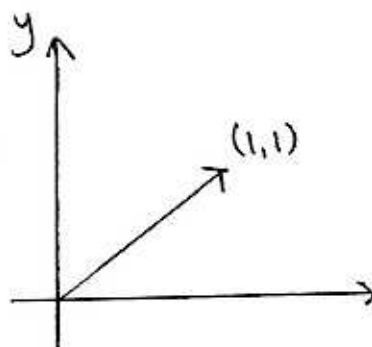
so a basis for the sol<sup>n</sup> space is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ -58 \\ -67 \\ 0 \\ 1 \end{bmatrix} \right\}$$

7. (a)

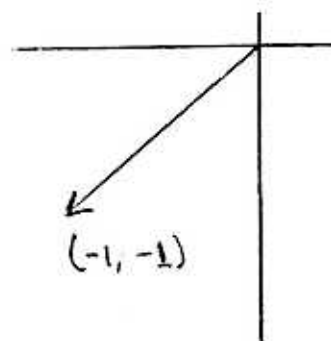


$\xrightarrow{S}$



(1)

$\xrightarrow{T}$



(1)

$$(b) \quad S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence

$$A_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hence

$$A_R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

$$8. (a) A_T = \begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \quad (1)$$

$$(b) \begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \begin{matrix} \\ \textcircled{2} + \frac{2}{3}\textcircled{1} \\ \textcircled{3} - \textcircled{1} \end{matrix} \sim \begin{bmatrix} 3 & -1 & -6 \\ 0 & \frac{1}{3} & 1 \\ 0 & 4 & 12 \end{bmatrix} \begin{matrix} \\ \\ \textcircled{3} - 12 \times \textcircled{2} \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & -6 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We know that  $\text{Ker}(T)$  is the same as the solution space of  $A_T$ . For this,  $x_3$  has no leading entry, so set  $x_3 = t$ .

Back substitution gives  $\frac{1}{3}x_2 = -x_3 = -t \Rightarrow x_2 = -3t$ ,

$$3x_1 = x_2 + 6x_3 \Rightarrow 3x_1 = -3t + 6t = 3t \Rightarrow x_1 = t$$

Hence the sol<sup>n</sup> space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \quad (3)$$

so a basis is  $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$

This is a line through the origin in  $\mathbb{R}^3$ .

(c)  $T$  is not invertible since  $\text{Rank } T = 2 < 3$  (dimension of  $\mathbb{R}^3$ ).  $\textcircled{1}$

(d) We know that  $\text{Im}(T)$  is the same as the column space of  $A_T$ . Since the leading entries are in columns ① & ②, a basis is

$$\textcircled{\frac{1}{2}} \left\{ \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

This corresponds to a plane through the origin in  $\mathbb{R}^3$ .

(e) We observe that

$$(4, -3, 0) = (3, -2, 3) + (-1)(-1, 1, 3)$$

and so is in the span of the basis vectors for  $\text{Im}(T)$ , and is thus an element of  $\text{Im}(T)$ . We read off from this that

$$[(4, -3, 0)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\textcircled{\frac{1}{2}}$$

9. (a) We know that  $\mathcal{B}$  is a basis if and only if

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

has rank 3. Now

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} \textcircled{2} - \textcircled{1} \\ \textcircled{3} - \textcircled{1} \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

which indeed has rank 3.

$$\textcircled{1}$$

(b) We can write down the transition matrix  $P_{\mathcal{S}, \mathcal{B}}$  from  $\mathcal{B}$  to the standard basis:

$$P_{\mathcal{S}, \mathcal{B}} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\textcircled{1}$$



We know that

$$P_{B,S} = P_{S,B}^{-1}$$

$$\begin{aligned} & \left[ \begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \textcircled{2} - \textcircled{1} \\ \textcircled{3} - \textcircled{1} \end{array} \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right] \textcircled{(-1)\textcircled{2}} \\ & \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} \textcircled{1} + \textcircled{3} \\ \textcircled{2} + 2\textcircled{3} \end{array} \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right] \textcircled{1} - \textcircled{2} \\ & \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right] \end{aligned}$$

Hence  $P_{S,B}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = P_{B,S}$  ②

(c)  $[\tilde{v}]_B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$  ②

$$10. (a) \det(A - \lambda I) = \begin{vmatrix} 7-\lambda & -2 \\ 15 & -4-\lambda \end{vmatrix}$$

$$= (7-\lambda)(-1)(4+\lambda) - (-2) \times 15$$

$$= (\lambda-7)(\lambda+4) + 30 = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) \quad (2)$$

(b) Eigenvalues  $\lambda = 2, \lambda = 1$ .

$$\lambda = 2 \text{ eigenvector: } \begin{bmatrix} 5 & -2 & | & 0 \\ 15 & -6 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Set  $y = t$ . Back substitution gives  $5x = 2y = 2t$

$$\Rightarrow x = \frac{2}{5}t$$

Taking for example  $t = 5$  gives the eigenvector  $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$  ①

$$\lambda = 1 \text{ eigenvector: } \begin{bmatrix} 6 & -2 & | & 0 \\ 15 & -5 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Set  $y = t$ . Back substitution gives  $x = \frac{1}{3}t$

Taking for example  $t = 3$  gives the eigenvector  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  ①

(c)  $A$  is diagonalizable because the eigenvectors ① are linearly independent (this is always the case when the eigenvalues are distinct)

$$A = PDP^{-1} \quad \text{with} \quad P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

①

$$11 \text{ (a)} \quad 5x_1 y_1 - x_2 y_2 = [x_1 \ x_2] \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (1)$$

(b) We require that

(i)  $b = c$  (i.e. the matrix is symmetric)

$$(ii) \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0 \quad (2)$$

with equality if and only if  $u = v = 0$ .

Here

$$\begin{aligned} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 5u & -v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\ &= 5u^2 - v^2 \end{aligned} \quad (1)$$

But with  $u = 0$ ,  $v = 1$  (for example) this is negative, which violates (ii).

$$\begin{aligned} \text{(c) (i)} \quad \underline{x} &= (\underline{u}_1 \cdot \underline{x}) \underline{u}_1 + (\underline{u}_2 \cdot \underline{x}) \underline{u}_2 + (\underline{u}_3 \cdot \underline{x}) \underline{u}_3 \\ &= \frac{1}{3} (3) \underline{u}_1 + \frac{1}{3} (6) \underline{u}_2 + \frac{1}{3} (9) \underline{u}_3 \\ &= \underline{u}_1 + 2\underline{u}_2 + 3\underline{u}_3 \end{aligned} \quad (1)$$

$$\text{(ii)} \quad P_{u,s} = \begin{bmatrix} \underline{u}_1 \cdot \underline{x} \\ \underline{u}_2 \cdot \underline{x} \\ \underline{u}_3 \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad (1)$$

12. (a)  $y = a + bx$  where, with

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \underline{\tilde{y}} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \underline{\tilde{y}}$$

$$\text{Now } A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}$$

$$\Rightarrow (A^T A)^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/14 \end{bmatrix}$$

$$\text{and so } \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/14 \end{bmatrix} A^T \underline{\tilde{y}}$$

$$\text{But } A^T \underline{\tilde{y}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/14 \end{bmatrix} \begin{bmatrix} 2 \\ 12 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6/7 \end{bmatrix}$$

Hence  $y = \frac{1}{2} + \frac{6}{7}x$  is the line of best fit

$$(b) \quad y = \frac{1}{2} + \frac{6}{7}x$$

With  $x = -2$  this gives  $y = \frac{1}{2} - \frac{12}{7} \approx -1.0^\circ\text{C}$ .

(2)

13. (a) Eigenvalues:  $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0$$

$$\Rightarrow (\lambda-3)(\lambda+3) - 16 = 0$$

$$\Rightarrow \lambda^2 - 9 - 16 = 0 \Rightarrow \lambda = \pm 5.$$

Eigenvectors:  $\lambda = 5$

$$\begin{bmatrix} -2 & 4 & | & 0 \\ 4 & -8 & | & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \begin{array}{l} \text{Set } y = t \\ \Rightarrow x = 2t \end{array}$$

$$\Rightarrow \text{eigenvector } t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \text{normalized eigenvector } \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = -5$

$$\begin{bmatrix} 8 & 4 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \begin{array}{l} \text{Set } y = t \\ \Rightarrow x = -\frac{t}{2} \end{array}$$

$$\Rightarrow \text{eigenvector } t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \text{normalized eigenvector } \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

We see that  $\left( \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \cdot \left( \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) = \frac{1}{5} (-2+2) = 0$

thus demonstrating that the vectors are orthogonal.

(b) The transformation scales by a factor of  $-5$  in the direction of

②  $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , and scales by a factor of  $5$  in the direction of  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c)  $A^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5^{10} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  since  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is an  
② eigenvector.