19. Topic 3 — Lecture 19

Slides: pgs. 211–217. Exercises: Topic 3, Q.113–129

Exercise What is the dimension of $\mathcal{P}_3$, the set of polynomials of degree less than or equal to 3?

Exercise Explain why $\{x+2, -x^2+1, x^3\}$ cannot be a basis for $\mathcal{P}_3$.

Exercise A matrix is said to be lower triangular if all element above the diagonal are zero. Show that the set of lower triangular 2×2 matrices forms a subspace of $\mathcal{M}_{2,2}$.

Solution: method 1 The set of lower triangular 2×2 matrices is specified by

$$L = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \, a, b, c \in \mathbb{R} \right\}.$$

We need to check the 3 axioms specifying a vector space.

(0) We observe that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in L$, and so L is nonempty.

(1) Let

$$\mathbf{v}_1 = \begin{bmatrix} a_1 & 0 \\ b_1 & c_1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, so that $\mathbf{v}_1, \mathbf{v}_2 \in L$. Then

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} a_1 + a_2 & 0 \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ b_0 & c_0 \end{bmatrix}$$

where

$$a_1 + a_2 = a_0 \in \mathbb{R}, \quad b_1 + b_2 = b_0 \in \mathbb{R}, \quad c_1 + c_2 = c_0 \in \mathbb{R}.$$

Thus we have demonstrated that $\mathbf{v}_1 + \mathbf{v}_2 \in L$, and thus L is closed under vector addition.

(2) Let $\mathbf{v} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$, so that $\mathbf{v} \in L$. Let $\alpha \in \mathbb{R}$ be a general scalar. We have

$$\alpha\mathbf{v} = \begin{bmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{bmatrix} = \begin{bmatrix} \tilde{a} & 0 \\ \tilde{b} & \tilde{c} \end{bmatrix}$$

where

$$\tilde{a} = \alpha a \in \mathbb{R}, \, \tilde{b} = \alpha b \in \mathbb{R}, \, \tilde{c} = \alpha c \in \mathbb{R}.$$

Thus we have demonstrated that $\alpha\mathbf{v} \in \mathbb{L}$, and thus L is closed under scalar multiplication.

Solution: method 2 The set L is equivalent to the subset of $\mathbb{R}^4$ specified by

$$L' = \{(a, 0, b, c) : \, a, b, c \in \mathbb{R}\}.$$

Our task now is to verify that L' is a subspace of $\mathbb{R}^4$.

(0) We observe that $(0,0,0,0)$ is an element of L', and so L' is nonempty.

(1) Let

$$\mathbf{v}_1 = (a_1, 0, b_1, c_1), \quad \mathbf{v}_2 = (a_2, 0, b_2, c_2)$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$ be two general elements in L'. We have

$$\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2, 0, b_1 + b_2, c_1 + c_2) = (a_0, 0, b_0, c_0)$$

where

$$a_1 + a_2 = a_0 \in \mathbb{R}, \quad b_1 + b_2 = b_0 \in \mathbb{R}, \quad c_1 + c_2 = c_0 \in \mathbb{R}.$$

Thus we have demonstrated that $\mathbf{v}_1 + \mathbf{v}_2 \in L'$, and thus L' is closed under vector addition.

(2) Let $\mathbf{v} = (a, 0, b, c)$ where $a, b, c \in \mathbb{R}$, so that $\mathbf{v} \in L'$. Let $\alpha \in \mathbb{R}$ be a general scalar. We have

$$\alpha\mathbf{v} = (\alpha a, 0, \alpha b, \alpha c) = (\tilde{a}, 0, \tilde{b}, \tilde{c})$$

where

$$\tilde{a} = \alpha a \in \mathbb{R}, \; \tilde{b} = \alpha b \in \mathbb{R}, \; \tilde{c} = \alpha c \in \mathbb{R}.$$

Thus we have demonstrated that $\alpha\mathbf{v} \in \mathbb{L}'$, and thus L' is closed under scalar multiplication.

<u>Exercise</u> Determine if the matrices

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -4 \\ 0 & -2 \end{bmatrix},$$

are linearly independent.

20. Topic 4 — Lecture 20

Slides: pgs. 220–234. Exercises: Topic 4, Q.130-39 (continued)

<u>Revision exercises</u>

(1) Draw a set of typical basis vectors in $\mathbb{R}^3$.

(2) Find the dimension of

$$\operatorname{Span}\{(1,-1,0,3), (0,1,1,1), (-1,3,2,-1)\}$$

(3) Let $\mathbf{v}_j$ $(j = 1, \ldots, 5)$ be vectors in $\mathbb{R}^3$, and define

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix}$$

You are told that

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give a basis for the column space of A.

(b) Give a basis for the row space of A.

(c) Let $B = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_4\end{bmatrix}$. Does $\mathbf{v}_3$ belong to the column space of B?

(d) Write the solution space of A in vector form.

<u>Linear transformation</u>

Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a mapping from $\mathbb{R}^p$ to $\mathbb{R}^q$. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^p$. A linear transformation has the defining property that

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}).$$

Linear transformations have the property that they map the zero vector to the zero vector. Thus $T(\mathbf{0}) = \mathbf{0}$.

Linear transformations also have the property that they map lines through the origin to other lines through the origin,

$$T(t\mathbf{u}) = tT(\mathbf{u}).$$

More generally they map all lines to other lines

$$T(t\mathbf{u} + \mathbf{u}_0) = tT(\mathbf{u}) + T(\mathbf{u}_0).$$

It follows that the image of the mapping of a parallelogram region by a linear transformation is fully determined by the image of the two vectors corresponding to the parallelogram.

<u>Exercise</u> Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. You are told that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}, \qquad T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

Sketch the image of the unit square.

21. Topic 4 — Lecture 21

Slides: pgs. 220–234. Exercises: Topic 4, Q.130-39 (continued)

<u>Matrices and linear transformations</u> Linear transformations are completely determined by the mapping rule for the basis vectors. Consider for example a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that

$$T\begin{bmatrix}1\\0\end{bmatrix} = \mathbf{v}_1, \qquad T\begin{bmatrix}0\\1\end{bmatrix} = \mathbf{v}_2$$

Then

$$T\begin{bmatrix}x\\y\end{bmatrix} = T\Big(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\Big) = xT\begin{bmatrix}1\\0\end{bmatrix} + yT\begin{bmatrix}0\\1\end{bmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2.$$

This can be written in the matrix form

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A_T = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$$

<u>Examples of linear transformations in the plane</u>

1. Reflection in the x-axis.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence

$$A_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2. Reflection in the line $y = x$.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence

$$A_T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3. Rotation by angle θ

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Hence

$$A_T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

4. Shear by c units in the x direction.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta \\ 1 \end{bmatrix}.$$

Hence

$$A_T = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}.$$

5. Scale by c units in the x-direction.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0\theta \\ 1 \end{bmatrix}.$$

Hence

$$A_T = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}.$$