MAST10007 Linear Algebra **2015 semester 2**

Please note that all answers should be fully justified.

1. (a)

$$\begin{bmatrix} 2 & 1 & 3 & -4 \\ 2 & -2 & 1 & -5 \\ 2 & -8 & -2 & -8 \\ -4 & -2 & -5 & 7 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & -4 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -5 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & -4 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & -4 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the system is consistent and has a unique solution: $(x, y, z) = (-1, 1, -1)$.

(b)

$$\begin{bmatrix} -2 & -1 & -2 & 3 \\ 2 & 7 & 4 & 6 \\ 2 & 4 & 3 & 4 \\ -4 & -2 & -5 & 7 \end{bmatrix} \to \begin{bmatrix} -2 & -1 & -2 & 3 \\ 0 & 6 & 4 & 9 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -2 & -1 & -2 & 3 \\ 0 & 6 & 4 & 9 \\ 0 & 0 & -1 & 5/2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -2 & -1 & -2 & 3 \\ 0 & 6 & 4 & 9 \\ 0 & 0 & -1 & 5/2 \\ 0 & 0 & 0 & -3/2 \end{bmatrix}$$

The system is inconsistent.

(c)

$$\begin{bmatrix} -4 & 2 & -11 & -9 \\ 1 & 1 & 5 & 3 \\ -3 & 3 & -6 & -6 \\ -1 & 1 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 3 \\ -4 & 2 & -11 & -9 \\ -3 & 3 & -6 & -6 \\ -1 & 1 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & 6 & 9 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7/2 & 5/2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent and has infinitely many solutions as given by:

$$\begin{aligned} x &= 5/2 - 7t/2 \\ y &= 1/2 - 3t/2 \\ z &= t \\ t &\in \mathbb{R} \end{aligned}$$

2. (a)

$$\det(A) = \begin{vmatrix} 0 & 0 & -2 & -7 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 5 & 6 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 6 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = (10-12)(7-6) = -2$$

(b) Since $\det(A) \neq 0$, A is invertible.

$$\begin{bmatrix} 0 & 0 & -2 & -7 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1 \times 1/2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & -2 & -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R4-5R1} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & -2 & -7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -5/2 & 0 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R4} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -5/2 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -7 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R4+2R3} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -5/2 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 2 & 0 \end{bmatrix} \xrightarrow[R1-R2]{R3+3R4} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -5/2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R4 \times (-1)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -5/2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -2 & 0 \end{bmatrix}$$

Therefore

$$A^{-1} = \begin{bmatrix} 0 & 3 & 0 & -1 \\ 0 & -5/2 & 0 & 1 \\ 3 & 0 & 7 & 0 \\ -1 & 0 & -2 & 0 \end{bmatrix}$$

(c)

$$\begin{aligned} \det(5(AB^{-2})^T) &= 5^4 \det((AB^{-2})^T) = 5^4 \det(AB^{-2}) = 5^4 \det(A)\det(B^{-2}) = 5^4 \det(A)\det(B^2)^{-1} \\ &= 5^4 \det(A)\det(B)^{-2} = 5^4 \times (-2) \times 3^{-2} \end{aligned}$$

(d) Since $\det(B) \neq 0$ B is invertible and therefore its rref is the 4×4 identity matrix.

3. (a) The normal to the plane is $(1, 0, -2)$. A vector equation for the line is therefore

$$(x, y, z) = (0, 0, 7) + t(1, 0, -2) \qquad t \in \mathbb{R}$$

(b) The second line is given by

$$(x, y, z) = (s + 6, 2s + 2, 3s) \qquad s \in \mathbb{R}$$

At a point of intersection we would have:

$$\begin{aligned} t &= s + 6 \\ 0 &= 2s + 2 \\ -2t + 7 &= 3s \end{aligned}$$

To solve this linear system:

$$\begin{bmatrix} 1 & -1 & 6 \\ 0 & -2 & 2 \\ -2 & -3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 6 \\ 0 & -2 & 2 \\ 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the system is consistent, the lines intersect. The intersection point is the point on L given by $t = 5$:

$$(0, 0, 7) + 5(1, 0, -2) = (5, 0, -3)$$

(c) The angle between L and the normal to the plane is

$$\arccos \frac{(1, 0, -2) \cdot (-1, -1, -1)}{\sqrt{5}\sqrt{3}} = \arccos \frac{1}{\sqrt{15}}$$

The angle between the line and the plane is therefore

$$\frac{\pi}{2} - \arccos \frac{1}{\sqrt{15}}$$

(The supplementary angle is also fine.)

(d)

$$\mathrm{proj}_{(1,0,-2)}(3, 1, -10) = \frac{(3, 1, -10) \cdot (1, 0, -2)}{(1, 0, -2) \cdot (1, 0, -2)}(1, 0, -2) = \frac{23}{5}(1, 0, -2)$$

The nearest point is therefore

$$(0, 0, 7) + \frac{23}{5}(1, 0, -2) = \frac{1}{5}(23, 0, -11)$$

4. (a) i. $\{(1,2,0,1,-1,0),(0,0,1,-2,-1,0),(0,0,0,0,0,1)\}$

   ii. $\{(3,1,4,4,1),(-1,-7,-8,-1,-8),(-3,-7,-10,-4,-8)\}$

(b) i. The subset of $\mathbb{R}^n$ given by $\{(x_1,\dots,x_n) \mid C \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0\}$.

   ii. Let $(x_1,\dots,x_n),(y_1,\dots,y_n)$ be elements of the solution space and $\alpha \in\ R$. Then

$$C \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} = C(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}) = C\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + C\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = 0+0=0$$

$$C \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} = \alpha C \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \alpha 0 = 0$$

   Therefore the solution space is a subspace of $\mathbb{R}^n$.

5. In each case we verify the conditions of the Subspace Theorem or give a counter-example.

(a) i. S is non-empty since (for example) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2,2}$ satisfies $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

ii. Let $A, B \in S$. Then

$$\begin{aligned} (A+B) \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + B \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(Since } A, B \in S\text{)} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore $A + B \in S$.

iii. Let $A \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (\alpha A) \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \alpha (A \begin{bmatrix} 1 \\ 2 \end{bmatrix}) \\ &= \alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(Since } A \in S\text{)} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore $\alpha A \in S$

Since S satisfies the conditions of the Subspace Theorem, it is a subspace of $M_{2,2}$.

(b) i. S is non-empty since (for example) $p(x) = 0 \in M_{2,2}$ satisfies $p(1) + p(2) + p(3) = 0$.

ii. Let $p, q \in S$. Then

$$\begin{aligned} (p+q)(1) + (p+q)(2) + (p+q)(3) &= p(1) + q(1) + p(2) + q(2) + p(3) + q(3) \\ &= (p(1) + p(2) + p(3)) + (q(1) + q(2) + q(3)) \\ &= 0 + 0 \qquad \text{(Since } p, q \in S\text{)} \\ &= 0 \end{aligned}$$

Therefore $p + q \in S$.

iii. Let $p \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (\alpha p)(1) + (\alpha p)(2) + (\alpha p)(3) &= \alpha p(1) + \alpha p(2) + \alpha p(3) \\ &= \alpha (p(1) + p(2) + p(3)) \\ &= \alpha \times 0 \qquad \text{(Since } p \in S\text{)} \end{aligned}$$

Therefore $\alpha p \in S$

Since S satisfies the conditions of the Subspace Theorem, it is a subspace of $\mathcal{P}_2$.

(c) Let $\boldsymbol{w} = (3, 0, 0)$. Then $\boldsymbol{w} \in S$ since $\boldsymbol{w} \cdot (1, 1, 1) = 3$, but $2\boldsymbol{w} \notin S$ since $(2\boldsymbol{w}) \cdot (1, 1, 1) = (6, 0, 0) \cdot (1, 1, 1) = 6$. Therefore S is not a subspace of $\mathbb{R}^3$.

6. (a) Taking coordinates with respect to the standard basis of $\mathcal{P}_2$ and row-reducing the corresponding matrix we obtain

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows that the set is linearly dependent and that

$$1 + x - 2x^2 = -(1 - x) + 2(1 - x^2)$$

(b) Note that

$$\begin{aligned} V &= \{p \in \mathcal{P}_2 \mid p(1) = 0\} \\ &= \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}, a + b + c = 0\} \\ &= \{(-b - c) + bx + cx^2 \mid b, c \in \mathbb{R}\} \\ &= \{b(-1 + x) + c(-1 + x^2) \mid b, c \in \mathbb{R}\} \\ &= \operatorname{Span}\{-1 + x, -1 + x^2\} \\ &= \operatorname{Span}(S) \end{aligned}$$

Therefore S is a spanning set for V.
From the row-echelon matrix in part (a), we have that a basis for $\operatorname{Span}(V)$ is

$$\{1 - x, 1 - x^2\}$$

7. (a) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{2,2}$ and $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &= T\left(\begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}\right) = (a+x+d+w, b+y+c+z) \\ &= (a+d, b+c) + (x+w, y+z) \\ &= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} T\left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right) = (\alpha a + \alpha d, \alpha b + \alpha c) = \alpha(a+d, b+c) \\ &= \alpha T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \end{aligned}$$

Therefore T is a linear transformation.

(b) We have that

$$\begin{aligned} [T]_{\mathcal{C},\mathcal{B}} &= \left[\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} \ \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} \ \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} \ \left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{C}} \right] \\ &= [\,[(1,0)]_{\mathcal{C}} \ [(0,1)]_{\mathcal{C}} \ [(0,1)]_{\mathcal{C}} \ [(2,0)]_{\mathcal{C}}\,] \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

(c) Reducing to row echelon form gives:

$$[T]_{\mathcal{C},\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

Therefore the rank of T is 2.

(d) Continuing the row reduction from above to reduced row echelon form:

$$[T]_{\mathcal{C},\mathcal{B}} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

A basis for the solution space of $[T]_{\mathcal{C},\mathcal{B}}$ is therefore given by:

$$\{(0,-1,1,0), (-2,0,0,1)\}$$

The corresponding basis for the kernel of T is: $\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\}$

The nullity of T is the dimension of the kernel, which is 2.

(e) For a linear transformation $T : U \to V$,

$$\text{rank}(T) + \text{nullity}(T) = \dim(U)$$

For the above linear transformation we have $\text{rank}(T) = 2$, $\text{nullity}(t) = 2$ and $\dim(U) = \dim(M_{2,2}) = 4$.

8. (a)

$$[T]_{\mathcal{C}} = [\ [T(1,0,0)]_{\mathcal{C}}\ [T(0,1,0)]_{\mathcal{C}}\ [T(0,0,1)]_{\mathcal{C}}\ ] = [\ [(1,1,1)]_{\mathcal{C}}\ [(2,2,0)]_{\mathcal{C}}\ [(3,0,0)]_{\mathcal{C}}\ ] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b)

$$P_{\mathcal{C},\mathcal{B}} = [\ [(1,1,1)]_{\mathcal{C}}\ [(1,1,0)]_{\mathcal{C}}\ [(1,0,0)]_{\mathcal{C}}\ ] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$P_{\mathcal{B},\mathcal{C}} = P_{\mathcal{C},\mathcal{B}}^{-1}$$

To calculate the invese

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Therefore

$$P_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(d)

$$[T]_{\mathcal{B}} = P_{\mathcal{B},\mathcal{C}}\,[T]_{\mathcal{C}}\,P_{\mathcal{C}.\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(e)

$$[T]_{\mathcal{B}} = [\ [T(1,1,1)]_{\mathcal{B}}\ [T(1,1,0)]_{\mathcal{B}}\ [T(1,0,0)]_{\mathcal{B}}\ ] = [\ [(6,3,1)]_{\mathcal{B}}\ [(3,3,1)]_{\mathcal{B}}\ [(1,1,1)]_{\mathcal{B}}\ ] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

9. To show that $\{v_1, ..., v_k\}$ is a basis for V we have to show that $\{v_1, ..., v_k\}$ is linearly independent and spans V.

   First we will check linear independence. Since we are given that $\{T(v_1), ..., T(v_k)\}$ forms a basis for W we know that $c_1T(v_1) + ... + c_kT(v_k) = 0$ has only the trivial solution, meaning that $c_1 = .. = c_k = 0$. Since T is injective we know that if $c_1T(v_1) + ... + c_kT(v_k) = T(c_1v_1 + ... + c_kv_k) = \mathbf{0}$ then $c_1v_1 + ...c_kv_k = \mathbf{0}$. We can therefore conclude that $v_1, ..., v_k$ are linearly independent since the equation $c_1v_1 + ...c_kv_k = 0$ has only the trivial solution.

   Now we want to check that $v_1, ..., v_k$ spans V. First, let v be in V and consider $T(v) = w$. Since $\{T(v_1), ..., T(v_k)\}$ forms a basis for W there exist $c_1, ..., c_k$ so that $w = c_1T(v_1)+...+c_kT(v_k) = T(c_1v_1+...+c_kv_k)$. This is the same as saying $T(v) = T(c_1v_1+...c_kv_k)$ and since T is injective, that $v = c_1v_1 + ...c_kv_k$. Therefore $\{v_1, ..., v_k\}$ is a spanning set for V.

10. (a) standard defn... look in the lecture notes!

(b) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$, $X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$

Checking the axioms, we have:

$$\begin{aligned}
\langle (x_1, y_1), (x_2, y_2) \rangle &= X_1^T A X_2 = (X_1^T A X_2)^T = X_2^T A^T X_1 && (1) \\
&= X_2^T A^T X_1 = \langle (x_2, y_2), (x_1, y_1) \rangle \\
\langle k(x_1, y_1), (x_2, y_2) \rangle &= (kX_1^T) A X_2 = kX_1^T A X_2 = k \langle (x_1, y_1), (x_2, y_2) \rangle && (2) \\
\langle (x_1, y_1), (x_2, y_2) + (x_3, y_3) \rangle &= X_1^2 A (X_2 + X_3) = X_1^T A X_2 + X_1^T A X_3 && (3) \\
&= \langle (x_1, y_1), (x_2, y_2) \rangle + \langle (x_1, y_1), (x_3, y_3) \rangle \\
\langle (x, y), (x, y) \rangle &= x^2 + 4xy + 8y^2 = (x + 2y)^2 + 4y^2 \geq 0 && (4) \\
\langle (x, y), (x, y) \rangle = 0 &\iff (x + 2y)^2 + 4y^2 \\
&\iff x = y = 0
\end{aligned}$$

(c)

$$\theta = \arccos \frac{\langle (1, 0), (0, 1) \rangle}{\sqrt{\langle (1, 0), (1, 0) \rangle \langle (0, 1), (0, 1) \rangle}} = \arccos \frac{2}{\sqrt{8}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

11. (a) The char poly is given by $\det(\lambda I - A)$:

$$\begin{aligned}
\det(\lambda I - A) &= \begin{vmatrix} \lambda-6 & 0 & -16 \\ 0 & \lambda-4 & 0 \\ 1 & 0 & \lambda+4 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda-6 & 0 & -16 \\ 0 & 1 & 0 \\ 1 & 0 & \lambda+4 \end{vmatrix} \\
&= (\lambda-4)\begin{vmatrix} \lambda-6 & -16 \\ 1 & \lambda+4 \end{vmatrix} \\
&= (\lambda-4)\left\{\lambda^2-2\lambda-8\right\} \\
&= (\lambda-4)(\lambda-4)(\lambda+2)
\end{aligned}$$

(b) The eigenvalues are the roots of the char poly: $4, 4, -2$.

(c) To find the eigenvector for $\lambda = -2$. The eigenspace is the solution space of

$$A + 2I = \begin{bmatrix} 8 & 0 & 16 \\ 0 & 6 & 0 \\ -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -2 \\ 0 & 6 & 0 \\ 8 & 0 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is therefore: $\{(-2, 0, 1)\}$
For $\lambda = 2$ the eigenspace is the solution space of

$$A + 2I = \begin{bmatrix} 2 & 0 & 16 \\ 0 & 0 & 0 \\ -1 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, a basis for the eigenspace is: $\{(0, 1, 0), (-8, 0, 1)\}$

(d) We can take

$$P = \begin{bmatrix} -2 & 0 & -8 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

12. (a)

$$
\begin{aligned}
u_1 &= \frac{1}{\|(1,7,1,7)\|}(1,7,1,7) = \frac{1}{10}(1,7,1,7) \\
w_2 &= (0,7,2,7) - \mathrm{proj}_{u_1}(0,7,2,7) = (0,7,2,7) - ((0,7,2,7)\cdot u_1)u_1 \\
&= (0,7,2,7) - 10u_1 = (-1,0,1,0) \\
u_2 &= \frac{1}{\|w_2\|}w_2 = \frac{1}{\sqrt{2}}(-1,0,1,0) \\
w_3 &= (1,8,1,6) - \mathrm{proj}_{u_1}(1,8,1,6) - \mathrm{proj}_{u_2}(1,8,1,6) \\
&= (1,8,1,6) - ((1,8,1,6)\cdot u_1)u_1 - ((1,8,1,6)\cdot u_2)u_2 \\
&= (1,8,1,6) - (1,7,1,7) - (0,0,0,0) = (0,1,0,-1) \\
u_3 &= \frac{1}{\|w_3\|}w_3 = \frac{1}{\sqrt{2}}(0,1,0,-1)
\end{aligned}
$$

The orthonormal basis obtained is

$$
\{\frac{1}{10}(1,7,1,7), \frac{1}{\sqrt{2}}(-1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,-1)\}
$$

(b)

$$
\begin{aligned}
\mathrm{proj}_V(7,7,9,5) &= ((7,7,9,5)\cdot u_1)u_1 + ((7,7,9,5)\cdot u_2)u_2 + ((7,7,9,5)\cdot u_3)u_3 \\
&= (1,7,1,7) + (-1,0,1,0) + (0,1,0,-1) \\
&= (0,8,2,6)
\end{aligned}
$$

(c) From the previous part, we know that $(7,7,9,5)-(0,8,2,6) = (7,-1,7,-1)$ is orthogonal to V. So we can take

$$
\boldsymbol{u} = \frac{1}{\|(7,-1,7,-1)\|}(7,-1,7,-1) = \frac{1}{10}(7,-1,7,-1)
$$

13. (a) The eigenvalues of A are $1, 1, -1$. The eigenspace with eigenvalue 1 is given by the solution space of

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has dimension 1. Therefore A is NOT diagonalisable.

The matrix B is diagonalisable since its eigenvalues $\{1, 2, -1\}$ are distinct.

The matrix C is diagonalisable since it is real symmetric.

(b) i. The curve is given by

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

We calculate the eigenvalues:

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

The eigenvalues are: $4, -1$

It follows that the curve is a hyperbola.

ii. We calculate the eigenspaces.

For $\lambda = 4$:

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

The eigenspace is spanned by $\{(2, 1)\}$

For $\lambda = -1$:

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

The eigenspace is spanned by $\{(1, -2)\}$

The principal axes are in the directions given by $(2, 1)$ and $(1, -2)$ respectively.

iii.