THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST10007 Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 6 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- All answers should be appropriately justified.
- Some notation used in this exam:

  $\mathcal{P}_n$ denotes the (real) vector space of all polynomials of degree at most n.
  $M_{m,n}$ denotes the (real) vector space of all $m \times n$ matrices.
- There are 13 questions. You should attempt all questions.
- The total number of marks available is 100.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination

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Question 1 (10 marks)

For each of the following systems of linear equations, write down the augmented matrix, row reduce until you reach reduced row echelon form, and give the set of all solutions of the system.

(a)
$$\begin{aligned} x - y - 2z &= -1 \\ 2x - y - 4z &= 3 \\ x + y + z &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 4x + y + 7z &= 8 \\ 6x + 3y + 9z &= 18 \\ 5x + 2y + 8z &= 13 \end{aligned}$$

(c)
$$\begin{aligned} 5x + y - 3z &= 1 \\ 2x + y &= 1 \\ x - y - 3z &= 1 \end{aligned}$$

Question 2 (6 marks)

In your potions master's cupboard you find 6 litres of frog slime, 13 unicorn hairs and 7 cups of nutritional yeast. You wish to brew some potions, using up all of these ingredients. In your magic potions manual you find:

Vanishing potion:
1 litre frog slime
2 unicorn hairs
1 cup nutritional yeast
Makes 1 cauldron

Hair growing potion:
1 litre frog slime
10 unicorn hairs
3 cups of nutritional yeast
Makes 1 bottle

Very stinky mess:
1 litre frog slime
1 unicorn hair
1 cup of nutritional yeast
Makes 1 batch

(a) Model the problem by a system of linear equations

(b) Solve the system to determine what quantity to make of each recipe.

Question 3 (8 marks)

Consider the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

(a) Use cofactor expansion along the *second column* of M to calculate its determinant $\det(M)$.

(b) Determine whether or not M is invertible. If M is invertible, find its inverse M^{-1}.

(c) Suppose that A and B are both 3×3 matrices and that $\det(A) = -2$ and $\det(B) = 4$.

(i) Calculate $\det(2B^{\mathrm{T}}A^{-3})$.

(ii) What is the rank of $2B^{\mathrm{T}}A^{-3}$?

Question 4 (8 marks)

Let L denote the line in $\mathbb{R}^3$ with vector equation

$$(x, y, z) = (-1, 2, -3) + t(2, 0, 1), \quad t \in \mathbb{R}$$

and let Π denote the plane in $\mathbb{R}^3$ with vector equation

$$(x, y, z) = (4, -3, 1) + r(1, 1, 0) + s(2, 1, -1), \quad r, s \in \mathbb{R}$$

(a) Find the Cartesian equation of the plane that is perpendicular to the line L and contains the point $(2, 4, -3)$.

(b) Find the co-ordinates of the point where the line L intersects the plane with Cartesian equation $-x + 2y + 3z = -2$.

(c) Find the distance between the point $(-2, 1, 5)$ and the plane Π.

Question 5 (6 marks)

The following two matrices are row equivalent:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 6 \\ 2 & 4 & 0 & -3 & 2 & 2 \\ 3 & 6 & 1 & -2 & 1 & 6 \\ 1 & 2 & 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write down a basis for the row space of A.

(b) Write down a basis for the column space of A.

(c) Find a basis for the solution space of A.

(d) State the rank-nullity theorem and verify that it is true for the matrix A.

Question 6 (8 marks)

Let V be the subspace of $\mathcal{P}_3$ spanned by the the following set

$$\{1 - x^2 + x^3,\ 2 + x - x^2 + x^3,\ 1 + 2x + x^2 - x^3\}$$

(a) Show that $f(x) = x + x^2 - x^3 \in V$.

(b) Show that $g(x) = 1 + x - x^2 + x^3 \notin V$.

(c) Find a basis for V which contains $f(x)$.

(d) Find a basis for $\mathcal{P}_3$ which contains $g(x)$.

Question 7 (8 marks)

Which of the following are subspaces of $M_{2,2}$? Be sure to justify your answer in each case.

(a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a = b \right\}$

(b) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b = 1 \right\}$

(c) $\{A \in M_{2,2} \mid \det(A) = 0\}$

(d) $\{A \in M_{2,2} \mid A^2 = A\}$

Question 8 (8 marks)

Let $\mathcal{B} = \{\boldsymbol{u}, \boldsymbol{v}\}$ be a basis for a 2-dimensional vector space V and let

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

be the matrix of a linear transformation $T \colon V \to V$ with respect to the basis $\mathcal{B}$.
Let $\mathcal{C}$ be the basis given by $\mathcal{C} = \{\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} - 2\boldsymbol{v}\}$.
(You do not need to prove that $\mathcal{C}$ is a basis of V.)

(a) Compute $\mathcal{P}_{\mathcal{B},\mathcal{C}}$ the change of basis matrix that converts $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

(b) Compute $\mathcal{P}_{\mathcal{C},\mathcal{B}}$ the change of basis matrix that converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

(c) Compute $[T]_{\mathcal{C}}$ the matrix of T with respect to the basis $\mathcal{C}$.

Question 9 (6 marks)

Let V be a vector space with an inner product and let $S \subseteq V$ be a subset of V.

(a) Define what it means to say that S is *orthonormal*.

(b) Prove that if S is orthonormal, then it is linearly independent.

Question 10 (8 marks)

(a) Determine whether or not the following define inner products on $\mathbb{R}^2$:

(i) $\langle (x,y),(a,b)\rangle = 4xa + 2xb + 2ya + yb$

(ii) $\langle (x,y),(a,b)\rangle = 5xa + 2xb + 2ya + yb$

(b) Let W be the subspace $\mathbb{R}^4$ having (ordered) basis

$$\mathcal{B} = \{(1,1,1,1),(-1,1,-1,1),(1,2,3,-4)\}$$

Obtain an orthonormal basis for W (with respect to the dot product) by applying the Gram-Schmidt procedure to $\mathcal{B}$.

Question 11 (6 marks)

Let A be an $n \times n$ matrix and suppose that the characteristic polynomial of A is given by

$$x^3 + x^2 - 5x - 6$$

(a) What is the value of n?

(b) Is A invertible?

(c) Write A^4 as a linear combination of A^2, A and I.

Question 12 (8 marks)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the xy-plane given by a stretch with factor 3 along the line $y = -x$ and a compression by $\frac{1}{2}$ along the line $y = x$ (see Figure 1).

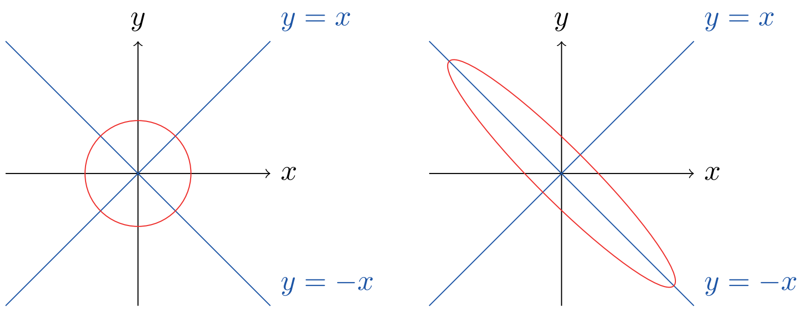

Figure 1: The unit circle in $\mathbb{R}^2$ (left) and its image under T (right).

(a) Find the eigenvalues of T.

(b) Find the corresponding eigenspaces of T.

(c) Choose a basis $\mathcal{B}$ of eigenvectors of T and give the matrix representation of T with respect to this basis.

(d) Finally, find the standard matrix representation $[T]_{\mathcal{S}}$ of T.

Question 13 (10 marks)

(a) Determine whether each of the following matrices is diagonalisable (over $\mathbb{R}$). The characteristic polynomial of each matrix is given.

(i) $\begin{bmatrix} 8 & -7 & 2 \\ 8 & -5 & 4 \\ -2 & 6 & 3 \end{bmatrix}, \quad (x+1)(x-3)(x-4)$

(ii) $\begin{bmatrix} 7 & -4 & -8 \\ 16 & -9 & -24 \\ -4 & 2 & 7 \end{bmatrix}, \quad (x+1)(x-3)(x-3)$

(iii) $\begin{bmatrix} -3 & 7 & -3 \\ 1 & -5 & 2 \\ 4 & -15 & 6 \end{bmatrix}, \quad (x+1)(x^2+x+1)$

(b) (i) Let $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$.

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(ii) Sketch the curve $8x^2 - 4xy + 5y^2 = 1$. You should give the directions of the principal axes and principal axis-intercepts.

End of Exam—Total Available Marks = 100.