The University of Melbourne
Department of Mathematics and Statistics
Summer Semester Exam 2017

MAST10007 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 7 pages.

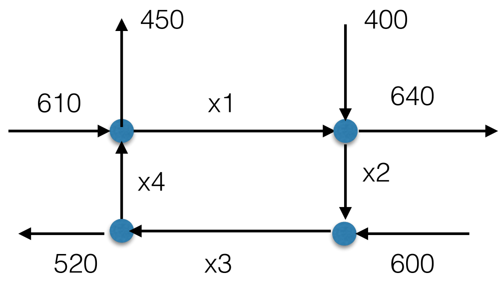
Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 12 questions. The total number of marks is 80. All questions may be attempted.

1. (a) Consider the following flow diagram:



At each vertex, the flow in must equal the flow out.

i. Specify 4 linear equations, with all the variables on the left hand side, and constants on the right hand side, which can be read off from the diagram.

ii. You are told

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 160 \\ 1 & -1 & 0 & 0 & 240 \\ 0 & -1 & 1 & 0 & 600 \\ 0 & 0 & 1 & -1 & 520 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 160 \\ 0 & 1 & 0 & -1 & -80 \\ 0 & 0 & 1 & -1 & 520 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Use this information or otherwise to specify the general solution to the linear system.

iii. If all the variables are required to be non-negative, what is the smallest allowed value of x_4?

(b) Specify values of the parameter $k \in \mathbb{R}$ for which the linear system

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 2 \\ 2x_2 + kx_3 &= k \end{aligned}$$

has a unique solution for the variables x_1, x_2 and x_3, and in this circumstance, specify the value of x_3 in terms of k.

[7 marks]

2. (a) Let

$$A = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

i. Evaluate BC.

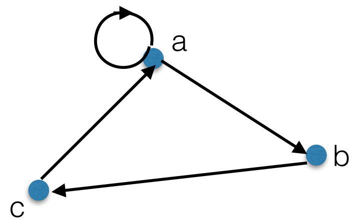
ii. Evaluate $C^T B^T BC$

iii. Evaluate $(ABC)^{-1}$.

(b) Let A and B be matrices of size $m \times n$ and $p \times q$ respectively. Show that for $AB - BA$ to be defined, both A and B must be square matrices of the same size.

[7 marks]

3. (a) Consider the following graph:



i. Specify the adjacency matrix, A say, for this graph.

ii. Without doing any actual matrix multiplication, predict the value of the entry in row 1, column 1 of A^4. Explain your reasoning.

(b) i. Use a method based on an augmented matrix to compute the inverse of the matrix

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

ii. Let C be the 2×2 matrix

$$C = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

Let

$$\tilde{C} = \begin{bmatrix} \gamma & \delta \\ \alpha & \beta \end{bmatrix},$$

i.e. $\tilde{C}$ is obtained from C by interchanging the first and second row. Specify the rule for obtaining $\tilde{C}^{-1}$ from C^{-1}.

[7 marks]

4. (a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

   i. Write down a formula for $\mathbf{u} \times \mathbf{v}$ in terms of a determinant.

   ii. Show that $\mathbf{u}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$.

   iii. You may assume that $||\mathbf{u} \times \mathbf{v}||$ is equal to the area of the parallelogram corresponding to the vectors $\mathbf{u}, \mathbf{v}$. Deduce from this that $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$ is equal to volume of the parallelepiped (i.e. three dimensional prism) formed by the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

   (b) Find the area of the triangle with vertices $(0, 0, 0)$, $(1, 2, 0)$ and $(1, 2, 3)$.

[6 marks]

5. (a) Calculate a basis for

$$\{a(1, 1, 0, -1) + b(2, 0, 2, 0) + c(0, 2, -2, -2) : a, b, c \in \mathbb{R}\}$$

   and specify the dimension.

   (b) Show that the set

$$S = \{(x, y, z) : x + 2y + z = 0 \text{ and } x - y + z = 0\}$$

   is closed under vector addition.

   (c) Show that the subset of $\mathbb{R}^2$ defined as the union of the quarter-planes $x, y \geq 0$ and $x, y \leq 0$ is not closed under vector addition.

[7 marks]

6. Let A be a 4×5 matrix with columns given by the vectors $\mathbf{a}_1, \cdots, \mathbf{a}_5$ and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be column vectors with 4 rows. Suppose that

$$\left[\ A \mid \mathbf{b}_1 \quad \mathbf{b}_2\ \right] = \left[\begin{array}{ccccc|cc} 1 & 0 & -2 & 0 & 0 & 7 & 4 \\ 0 & 2 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

   (a) What is the dimension of the row space of A? Give a reason.

   (b) What is the dimension of the solution space of A? Give a reason.

   (c) Write down the column space of A as a span.

   (d) Express $\mathbf{b}_1$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_4$.

   (e) Find the general solution of the matrix equation $A\mathbf{x} = \mathbf{b}_1$.

   (f) Explain why $\mathbf{b}_2$ does not belong to the column space of A.

[7 marks]

7. (a) i. Illustrate on a diagram the image of the unit square $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ when transformed by $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T is a compression by a factor of $1/2$ in the y direction, and an expansion by a factor of 2 in the x direction.

ii. From your diagram or otherwise specify

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

iii. Let A_T denote the standard matrix of T. Give geometrical reasons why $\det A_T = 1$.

(b) i. Give a geometrical description of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$A_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

ii. Specify the standard matrix for the linear transformation $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y \end{bmatrix},$$

and give a geometrical interpretation of S_1.

iii. Let T_1 be the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates by $\pi/5$ anti-clockwise. Specify the standard matrix for T_1 applied five times.

[7 marks]

8. Let $\mathbf{w}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$ and $\mathbf{w}_2 = \frac{1}{\sqrt{2}}(0, 1, -1)$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined in terms of the dot product according to

$$T\mathbf{x} = (\mathbf{w}_1 \cdot \mathbf{x})\mathbf{w}_1 + (\mathbf{w}_2 \cdot \mathbf{x})\mathbf{w}_2$$

(a) i. Compute $T(1, 1, 1)$.

ii. Compute the cross product $\mathbf{n} = (1, 1, 1) \times (0, 1, -1)$. Explain why $\operatorname{Ker} T = \operatorname{Span} \mathbf{n}$.

(b) i. Show that the standard matrix for T is

$$A_T = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

ii. Verify that $(-2, 1, 1)$ belongs to the solution space of A_T.

[6 marks]

9. Suppose

$$\mathbf{c}_1 = (1, 0, -1), \quad \mathbf{c}_2 = (1, 0, 1), \quad \mathbf{c}_3 = (0, 1, 0)$$

and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$.

(a) Explain why $\mathcal{C}$ is a basis for $\mathbb{R}^3$.

(b) You are told that $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Calculate $\mathbf{x}$ in terms of the standard basis.

(c) Calculate the transition matrix $P_{\mathcal{C},\mathcal{S}}$, where $\mathcal{S}$ denotes the standard basis of $\mathbb{R}^3$.

(d) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for $\mathbb{R}^3$. You are given that the change of basis matrix $P_{\mathcal{C},\mathcal{B}}$ from the basis $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C},\mathcal{B}} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Express $\mathbf{x} = \mathbf{b}_1 + 2\mathbf{b}_2$ as a linear combination of the vectors in $\mathcal{C}$.

(e) With $\mathcal{B}$ as in (d), calculate $\mathbf{b}_1$ in terms of the standard basis.

[7 marks]

10. (a) Let

$$\langle (u_1, u_2), (v_1, v_2) \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

where $u_3 = u_1 - u_2$ and $v_3 = v_1 - v_2$. Show that this can be rewritten

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(b) Demonstrate that

$$\langle (u_1, u_2), (u_1, u_2) \rangle \geq 0$$

and show furthermore that the only circumstance that $\langle (u_1, u_2), (u_1, u_2) \rangle = 0$ is when $u_1 = u_2 = 0$.

(c) Using the inner product in (a), find a unit vector perpendicular to the vector $(1, 1)$.

[6 marks]

11. A person on a diet has lost a total of Y kg in weight after X days, as recorded in the following table

X	Y
2	1
3	2
4	2

(a) Find the least squares line of best fit $Y = a + bX$ to this data.

(b) Estimate, using an integer, the amount of weight lost after 6 days.

[6 marks]

12. (a) i. Show that

$$\mathcal{B} = \{\frac{1}{\sqrt{2}}(-1, 1), \frac{1}{\sqrt{2}}(1, 1)\}$$

is an orthonormal set with respect to the dot product.

ii. Let Q denote the matrix with columns given by the vectors in $\mathcal{B}$. Calculate the matrix A defined by

$$A = QDQ^T,$$

where D is the diagonal matrix with entries on the diagonal $1, 2$.

iii. Determine the eigenvalues and corresponding normalised eigenvectors of A.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to reflection in the line $y = -x$.

i. Specify two one-dimensional subspaces that are left unchanged by T.

ii. Specify the eigenvalues of A_T, the standard matrix of T. Explain your reasoning.

[7 marks]

— END OF EXAMINATION QUESTIONS —