MAST20004 Probability
Outline answers to 2009 exam

1. (a) (i) $\Omega = \{+, -\}$. (ii) $\mathcal{A} = \{\emptyset, \{+\}, \{-\}, \Omega\}$.

   (b) (i) $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{A}$; (ii) $\mathbb{P}(\Omega) = 1$; (iii) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for disjoint A, $B \in \mathcal{A}$ (or the countable additivity, which is equivalent to finite additivity for this case).

   (c) By the finite additivity, $\mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$, which implies $\mathbb{P}(\Omega) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset)$, so $\mathbb{P}(\emptyset) = 0$.

2. (a) 0.4; (b) 0.1; (c) $\mathbb{P}(\text{rated fair}) = \mathbb{P}(\text{rated fair}|\text{actually poor}) = 0.3$, so independent.

3. (a) (i) $f_Y(y) = \begin{cases} \frac{y^{-2/3}}{12} & y \in [-8, 8],\ y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$ (ii) $f_Z(z) = \begin{cases} \frac{z^{-3/4}}{8} & z \in (0, 16], \\ 0 & \text{otherwise.} \end{cases}$

   (b) (i) $1 - e^{-15}$; (ii) N is a geometric rv with parameter $p = 1 - e^{-15}$; (iii) $\mathbb{P}(N = n) = (e^{-15})^n(1 - e^{-15})$; (iv) $\frac{1-e^{-3t}}{1-e^{-15}}$, $t \in [0, 5]$; (v) Let $t = 5n + u$ with $n = 0, 1, 2, \ldots$, $u \in [0, 5)$, then

$$\begin{aligned} \mathbb{P}(T \leq t) &= \sum_{k=0}^{\infty} \mathbb{P}(T \leq t|N = k)\mathbb{P}(N = k) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(N = k) + \mathbb{P}(T \leq t|N = n)\mathbb{P}(N = n) \\ &= 1 - e^{-15n} + (1 - e^{-3u})e^{-15n} = 1 - e^{-3t}; \end{aligned}$$

   (vi) T is exponentially distributed, as expected from the memoryless property of the exponential distribution.

4. (a) $-\frac{10}{37}$; (b) $\frac{136800}{1369}$; (c) $-\frac{100}{37}$; (d) $\frac{1368000}{1369}$; (e) mean is $-\frac{100}{37}$; variance is $\frac{13680000}{1369}$; the same mean but the variance is 10 times larger.

5. (a) The area is the triangle with three vertices (0,0), (0,1), (1,1); (b) $c = 15$; (c) $f_X(x) = \frac{15x^2}{2} - \frac{15x^4}{2}$ for $x \in [0, 1]$; (d) you are expected to integrate it over [0,1] to get 1; (e) $f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & y \in (x, 1), \\ 0 & \text{otherwise;} \end{cases}$ (f) $\frac{5}{336}$.

6. (a) $\lambda\mu$; (b) $\lambda\sigma^2 + \lambda^2\mu^2$; (c) $\lambda\sigma^2$.

7. (a) $t < \frac{1}{2}$; (b) $M_X(t) = (1 - 2t)^{-d/2}$; (c) $\mathbb{E}(X) = d$, $\mathrm{V}(X) = 2d$.

8. (a) You are expected to correctly expand the moment generating function $M_{S_n/n}(t) = [M_{X_1}(t/n)]^n$ to get the limit as specified.

   (b) (i) mean is 0.2 and variance is 0.16; (ii) 0.6826; (iii) Bi(100,0.2); (iv) Y_n can be thought as the sum of n iid rv's, so $Y_n \stackrel{d}{\approx} N(n\mu, n\sigma^2)$.

9. (a) $A(z) = \frac{1}{5}\sum_{i=0}^{\infty}\left(\frac{4}{5}\right)^i z^i$, $\frac{256}{3125}$; (b) 4, 20; (c) $q_{n+1} = A(q_n)$; $q_0 = 0$, $q_1 = \frac{1}{5}$, $q_2 = \frac{5}{21}$; (d) $\frac{1}{4}$.