THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2017

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches, and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Approved hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the exam.
- This paper has 9 questions. Attempt as many questions, or parts of questions, as you can.

  The number of marks allocated to each question is shown in the brackets after the question statement.

  The total number of marks available for this exam is 100.

  There is a table of normal distribution probabilities at the end of this question paper.

  Working and/or reasoning must be given to obtain full credit. Clarity, neatness, and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the exam.

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1. Consider a random experiment with sample space Ω.
   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.
   (b) Using the axioms, prove that for events A and B where $A \subseteq B$,
   $$P(A) \leq P(B).$$
   (c) Using the axioms and part (b), prove that for events A and B,
   $$P(A \cup B) \leq P(A) + P(B).$$

[9 marks]

2. A certain disease affects 3 out of every 1,000 people in Australia. A test for the disease has a false positive rate of 2%, and a false negative rate of 1% (that is, 2% of those that do not have the disease test positive, and 1% of those that have the disease test negative).
   (a) What proportion of those tested, test positive?
   (b) What is the chance that someone who has had a positive test result will have the disease?
   (c) What is the chance that someone who has had a negative test result will have the disease?

[9 marks]

3. Let X and Y have joint probability density function given by
   $$f_{(X,Y)}(x,y) = Cxy, \qquad 0 < x < y < 1,$$
   where C is a constant.
   (a) Find the constant C.
   (b) Find $f_X(x)$, the marginal density function of X.
   (c) Evaluate $P\left(Y > 3/4 | X = 1/2\right)$.
   (d) Evaluate $P\left(Y > 3/4 | X > 1/2\right)$.
   (e) Are X and Y independent? Justify your answer.
   (f) Compute $E[(XY)^{-1/2}]$.

[13 marks]

4. For parameter $a > 1$, let X have density function $f_X(x) = ax^{-(a+1)}$ on the interval $[1, \infty)$. Let $Y = \sqrt{X}$ and $Z = \log(Y)$.
   (a) Find $F_Y(y)$, the distribution function of Y.
   (b) Find $f_Y(y)$, the density function of Y, state the values of y for which it is defined, and show that it is a density function.
   (c) Compute the expected value of Y, or show that it does not exist.
   (d) Compute the variance of Y, or show that it does not exist.
   (e) Approximate $E(Z)$ and $Var(Z)$ using suitable Taylor series expansions.
   (f) Compute the exact value of $E(Z)$.

[15 marks]

5. Let U, V, and W be independent random variables, all distributed uniformly on the interval $(0, 1)$.

   (a) Find the probability density functions for

      (i) $X = \max\{U, V\}$;

      (ii) $Y = \min\{U, V\}$;

      (iii) $Z = \max\{U, V, W\}$.

   (b) Find the joint probability density function of (X, Y).

   (c) Compute the moment generating function of $X + Y$.

[10 marks]

6. The price of a stock at the beginning of the trading day is 50 dollars. The price of the same stock one hour into the trading day and four hours into the trading day is $(S_1, S_4) = (50e^X, 50e^Y)$, where (X, Y) is a bivariate normal random variable with mean parameters $(\mu_X, \mu_Y) = (1, 4)$, variance parameters $(\sigma_X^2, \sigma_Y^2) = (2, 8)$, and $Cov(X, Y) = 2$.

   (a) What is the probability the stock has a higher price four hours into the trading day than at the beginning?

   (b) Given the price of the stock four hours into the trading day is $50e^8$ dollars, what is the probability that the price one hour into the trading day was greater than $50e^3$ dollars?

   (c) What is the mean of S_1?

   (d) What is the mean of S_4?

   (e) What is the covariance of S_1 and S_4?

[15 marks]

7. Consider a roulette wheel with 37 slots, each having a number $0 - 36$. There are 18 red slots, 18 black slots, and 1 green slot. Each roulette game consists of choosing a slot uniformly at random. For a given game, we consider two kinds of wagers.

   - If you make a wager of $\$w$ on a specific number and the ball lands on that number, then you keep your wager and win an additional $\$35w$. Otherwise you lose your wager (so your winnings are either $-\$w$ or $\$35w$).
   - If you make a wager of $\$w$ on "black" and the ball lands on a black slot, then you keep your wager and win an additional $\$w$. Otherwise you lose your wager (so your winnings are either $-\$w$ or $\$w$).

   Let W_1 be the winnings from wagering \$5 on a single number in 25 consecutive games, and let W_2 be the winnings from wagering \$125 on "black" in a single game.

   (a) Find $E[W_1]$ and $Var(W_1)$.

   (b) Find $E[W_2]$ and $Var(W_2)$.

   (c) Compute $P(W_1 > 0)$ using an appropriate approximation.

   (d) Compute $P(W_2 > 0)$.

   (e) If you had to choose between wagering \$5 on a single number in 25 consecutive games and wagering \$125 on "black" in a single game, using the information from the previous parts of the question, which would you choose and why?

[11 marks]

8. (a) Let U be uniformly distributed on the interval $(0, 1)$ and let X have density function $3(1 - x)^2$ on $0 < x < 1$. Find a function ψ so that $\psi(U)$ has the same distribution as X.

   (b) Let X be as in part (a) and let $U_1, U_2, \ldots, U_{1000}$ be independent and each uniformly distributed on the interval $(0, 1)$. Write down a function of $U_1, \ldots, U_{1000}$ which would make a good estimate of $E[e^{X^2}]$.

   (c) Let Y be a Bernoulli random variable with parameter p with $0 < p < 1$. If V is uniformly distributed on the interval $(0, 1)$, find a function ϕ such that $\phi(V)$ has the same distribution as Y.

   (d) Let X be as in part (a) and let the distribution of W given $X = x$ be Bernoulli with parameter x. Let U, V be independent and uniformly distributed on the interval $(0, 1)$. Find a function χ such that $\chi(U, V)$ has the same distribution as W.

[8 marks]

9. Consider the branching process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n is the population size of the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of the offspring distribution is $A(z) = C(1 + 2z + 3z^2)$ for some constant C.

   (a) What is the probability mass function of the offspring distribution?

   (b) Find the expected value of the offspring distribution.

   (c) Find a simple expression for $E[X_n]$.

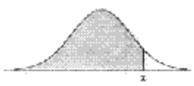
   (d) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_n and q_{n+1}.

   (e) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

   (f) Find a simple expression for $P(X_2 > 0)$.

[10 marks]

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817