Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST10007 Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 16 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- All answers should be appropriately justified.
- Some notation used in this exam:

  $\mathcal{P}_n$ denotes the (real) vector space of all polynomials of degree at most n.
  $M_{m,n}$ denotes the (real) vector space of all $m \times n$ matrices.
- There are 13 questions. You should attempt all questions.
- The total number of marks available is 100.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination

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Question 1 (10 marks)

For each of the following systems of linear equations, write down the augmented matrix, row reduce until you reach reduced row echelon form, and give the set of all solutions of the system.

(a)
$$\begin{aligned} x-y-2z &= -1\\ 2x-y-4z &= 3\\ x+y+z &= 6 \end{aligned}$$

(b)
$$\begin{aligned} 4x+y+7z &= 8\\ 6x+3y+9z &= 18\\ 5x+2y+8z &= 13 \end{aligned}$$

(c)
$$\begin{aligned} 5x+y-3z &= 1\\ 2x+y &= 1\\ x-y-3z &= 1 \end{aligned}$$

Solution:

(a)

$$\begin{bmatrix} 1 & -1 & -2 & -1\\ 2 & -1 & -4 & 3\\ 1 & 1 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 1 & 0 & 5\\ 0 & 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 1 & 0 & 5\\ 0 & 0 & 3 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -3\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Unique solution:

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 2\\ 5\\ -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 4 & 1 & 7 & 8\\ 6 & 3 & 9 & 18\\ 5 & 2 & 8 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{4} & \frac{7}{4} & 2\\ 6 & 3 & 9 & 18\\ 5 & 2 & 8 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{4} & \frac{7}{4} & 2\\ 0 & \frac{3}{2} & -\frac{3}{2} & 6\\ 0 & \frac{3}{4} & -\frac{3}{4} & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & \frac{1}{4} & \frac{7}{4} & 2\\ 0 & 1 & -1 & 4\\ 0 & \frac{3}{4} & -\frac{3}{4} & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{4} & \frac{7}{4} & 2\\ 0 & 1 & -1 & 4\\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1\\ 0 & 1 & -1 & 4\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution set:

$$\left\{ \begin{bmatrix} 1-2t\\ 4+t\\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

(c)

$$\begin{bmatrix} 5 & 1 & -3 & 1\\ 2 & 1 & 0 & 1\\ 1 & -1 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & 1\\ 5 & 1 & -3 & 1\\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & 1\\ 0 & 6 & 12 & -4\\ 0 & 3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & 1\\ 0 & 1 & 2 & -\frac{2}{3}\\ 0 & 3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 & 1\\ 0 & 1 & 2 & -\frac{2}{3}\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The solution set is the empty set.

Question 2 (6 marks)

In your potions master's cupboard you find 6 litres of frog slime, 13 unicorn hairs and 7 cups of nutritional yeast. You wish to brew some potions, using up all of these ingredients. In your magic potions manual you find:

Vanishing potion:	Hair growing potion:	Very stinky mess:
1 litre frog slime	1 litre frog slime	1 litre frog slime
2 unicorn hairs	10 unicorn hairs	1 unicorn hair
1 cup nutritional yeast	3 cups of nutritional yeast	1 cup of nutritional yeast
Makes 1 cauldron	Makes 1 bottle	Makes 1 batch

(a) Model the problem by a system of linear equations

(b) Solve the system to determine what quantity to make of each recipe.

Solution:

(a) Say we make x cauldrons of the vanishing potion, y bottles of the hair growing potion and z batches of the very stinky mess. Then our problem is described by the following system of linear equations:

$$\begin{aligned} x + y + z &= 6 \\ 2x + 10y + z &= 13 \\ x + 3y + z &= 7. \end{aligned}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 10 & 1 & 13 \\ 1 & 3 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & -1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 1 \\ 0 & 8 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 8 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

You want to make two and a half cauldrons of vanishing potion, half a bottle of hair growing potion and three batches of very stinky mess.

Question 3 (8 marks)

Consider the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

(a) Use cofactor expansion along the *second column* of M to calculate its determinant $\det(M)$.

(b) Determine whether or not M is invertible. If M is invertible, find its inverse M^{-1}.

(c) Suppose that A and B are both 3×3 matrices and that $\det(A) = -2$ and $\det(B) = 4$.

(i) Calculate $\det(2B^{\mathrm{T}}A^{-3})$.

(ii) What is the rank of $2B^{\mathrm{T}}A^{-3}$?

Solution:

(a)

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$
$$= 3 - 3 - 2 = -2$$

(b) As $\det(M) = -2 \neq 0$ the matrix M is invertible. To Calculate its inverse, consider

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+R_2]{R_1-R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{(-1)R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_1+R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 3/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

Hence $M^{-1} = \begin{bmatrix} 5/2 & -1/2 & 1/2 \\ 3/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$.

(c)

$$\det(2B^{\mathrm{T}}A^{-3}) = 2^3\det(B^{\mathrm{T}})\det(A^{-3}) = 2^3\det(B)\det(A)^{-3} = 2^3 \times 4 \times (-2)^{-3} = -4.$$

Since $\det(2B^{\mathrm{T}}A^{-3}) = -4 \neq 0$, the matrix $2B^{\mathrm{T}}A^{-3}$ is invertible and therefore has rank 3.

Question 4 (8 marks)

Let L denote the line in $\mathbb{R}^3$ with vector equation

$$(x, y, z) = (-1, 2, -3) + t(2, 0, 1), \quad t \in \mathbb{R}$$

and let Π denote the plane in $\mathbb{R}^3$ with vector equation

$$(x, y, z) = (4, -3, 1) + r(1, 1, 0) + s(2, 1, -1), \quad r, s \in \mathbb{R}$$

(a) Find the Cartesian equation of the plane that is perpendicular to the line L and contains the point $(2, 4, -3)$.

(b) Find the co-ordinates of the point where the line L intersects the plane with Cartesian equation $-x + 2y + 3z = -2$.

(c) Find the distance between the point $(-2, 1, 5)$ and the plane Π.

Solution:

(a) We can take the direction vector, $(2, 0, 1)$, of the line as a normal vector to the plane. The point-normal form gives the Cartesian equation of the plane: $(x, y, z) \cdot (2, 0, 1) = (2, 0, 1) \cdot (2, 4, -3) \implies 2x + z = 1$ is the required Cartesian equation of the plane.

(b) Substituting the co-ordinates of a point on the line into the equation of the plane gives

$$-(-1 + 2t) + 2(2) + 3(-3 + t) = -2 \implies t = 2.$$

Hence the point of intersection has co-ordinates $(-1, 2, -3) + 2(2, 0, 1) = (3, 2, -1)$.

(c) We can take as a normal vector to the plane the vector $\mathbf{n} = (1, 1, 0) \times (2, 1, -1) = (-1, 1, -1)$. The vector from the point $(4, -3, 1)$ on the plane to the point $(-2, 1, 5)$ is $\boldsymbol{v} = (-2, 1, 5) - (4, -3, 1) = (-6, 4, 4)$. The projection of $\boldsymbol{v}$ onto $\mathbf{n}$ is

$$\text{proj}_{\mathbf{n}}\boldsymbol{v} = \frac{\boldsymbol{v} \cdot \mathbf{n}}{||\mathbf{n}||^2}\mathbf{n} = \frac{6}{3}(-1, 1, -1) = 2(-1, 1, -1).$$

The shortest distance is given by $||\text{proj}_{\mathbf{n}}\boldsymbol{v}|| = 2\sqrt{3}$.

Question 5 (6 marks)

The following two matrices are row equivalent:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 6 \\ 2 & 4 & 0 & -3 & 2 & 2 \\ 3 & 6 & 1 & -2 & 1 & 6 \\ 1 & 2 & 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write down a basis for the row space of A.

(b) Write down a basis for the column space of A.

(c) Find a basis for the solution space of A.

(d) State the rank-nullity theorem and verify that it is true for the matrix A.

Solution:

(a) The nonzero rows of B are the first 4 rows, so a basis for the row space of A consists of the first 4 rows of B, i.e.

$$\{(1,2,0,0,0,3),(0,0,1,0,0,0),(0,0,0,1,0,2),(0,0,0,0,1,1)\}.$$

(b) The leading entries in B are in columns 1, 3, 4 and 5 so a basis for the column space of A consists of columns 1, 3, 4 and 5 of A, i.e.

$$\{(1,2,3,1,1),(2,0,1,1,1),(1,-3,-2,1,0),(1,2,1,-1,1)\}.$$

(c) We have free parameters $x_2 = s$ and $x_6 = t$ corresponding to columns 2 and 6 in B that do not have a leading entry. We see that

$$(x_1,x_2,x_3,x_4,x_5,x_6) = (-2s-3t,s,0,-2t,-t,t) = s(-2,1,0,0,0,0)+t(-3,0,0,-2,-1,1)$$

So a basis for the solution space of A is

$$\{(-2,1,0,0,0,0),(-3,0,0,-2,-1,1)\}$$

(d) The rank-nullity states that if M is an $m \times n$ matrix then $\text{nullity}(M)+\text{rank}(M)=n$. Since $\text{rank}(A)=4$, $\text{nullity}(A)=2$ and $n=6$, and $2+4=6$, the theorem is verified for A.

Question 6 (8 marks)

Let V be the subspace of $\mathcal{P}_3$ spanned by the the following set

$$\{1 - x^2 + x^3,\ 2 + x - x^2 + x^3,\ 1 + 2x + x^2 - x^3\}$$

(a) Show that $f(x) = x + x^2 - x^3 \in V$.

(b) Show that $g(x) = 1 + x - x^2 + x^3 \notin V$.

(c) Find a basis for V which contains $f(x)$.

(d) Find a basis for $\mathcal{P}_3$ which contains $g(x)$.

Solution:
Let $v_1 = 1 - x^2 + x^3$, $v_2 = 2 + x - x^2 + x^3$, and $v_3 = 1 + 2x + x^2 - x^3$.
(a) Since $f(x) = x + x^2 - x^3 = v_2 - 2v_1$ then $f(x) \in V$.
(b) If $g(x) = 1 + x - x^2 + x^3 = c_1v_1 + c_2v_2 + c_3c_3$ then

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 1, \\ c_2 + 2c_3 &= 1, \\ -c_1 - c_2 + c_3 &= -1, \\ c_1 + c_2 - c_3 &= 1. \end{aligned}$$

The third equation is the negative of the first equation and subtracting the first equation from the fourth gives

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 1, \\ c_2 + 2c_3 &= 1, \\ -c_2 - 2c_3 &= 0, \end{aligned}$$

which has no solution since $1 \neq 0$.
(c) If $c_1v_1 + c_2v_2 + c_3v_3 = 0$ then

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0, \\ c_2 + 2c_3 &= 0, \\ -c_1 - c_2 + c_3 &= 0, \\ c_1 + c_2 - c_3 &= 0. \end{aligned}$$

The third equation is the negative of the first equation and subtracting the first equation from the fourth gives

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0, \\ c_2 + 2c_3 &= 0, \\ -c_2 - 2c_3 &= 0, \end{aligned} \quad \text{giving} \quad \begin{aligned} c_1 &= 4c_3 - c_3 = 3c_3, \\ c_2 &= -2c_3, \end{aligned}$$

so that $c_1 = -5$, $c_2 = -2$, $c_3 = 1$ is a solution. Indeed

$$3(1 - x^2 + x^3) - 2(2 + x - x^2 + x^3) + (1 + 2x + x^2 - x^3) = 0 + 0x + 0x^2 + 0x^3.$$

So $\{v_1, v_2\}$ is a basis for V.
Thus, since $f(x) = v_2 - 2v_1$, then $\{v_1, f(x)\} = \{1 - x^2 + x^3, x + x^2 - x^3\}$ is a basis for V.
(d) $\{1, x, x^2, 1 + x - x^2 + x^3\}$ is a basis of $\mathcal{P}_3$ which contains $g(x)$.

Question 7 (8 marks)

Which of the following are subspaces of $M_{2,2}$? Be sure to justify your answer in each case.

(a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a = b \right\}$

(b) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b = 1 \right\}$

(c) $\{A \in M_{2,2} \mid \det(A) = 0\}$

(d) $\{A \in M_{2,2} \mid A^2 = A\}$

Solution:

Let E_{ij} be the matrix with 1 in the (i, j) entry and 0 elsewhere.

(a) Since $S_1 = \text{span}\{E_{11} + E_{12}, E_{21}, E_{22}\}$ then S_1 is a subspace of $M_{2,2}$.

(b) Since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin S_2$ then S_2 is not a subspace of $M_{2,2}$.

(c) Since $E_{11} \in S_4$ and $E_{22} \in S_4$ and $E_{11} + E_{22} \notin S_4$, then S_4 is not a subspace of $M_{2,2}$.

(d) Since $E_{11} + E_{22} \in S_6$ and $2(E_{11} + E_{22}) \notin S_6$, then S_6 is not a subspace of $M_{2,2}$.

Question 8 (8 marks)

Let $\mathcal{B} = \{\boldsymbol{u}, \boldsymbol{v}\}$ be a basis for a 2-dimensional vector space V and let

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

be the matrix of a linear transformation $T\colon V \to V$ with respect to the basis $\mathcal{B}$.
Let $\mathcal{C}$ be the basis given by $\mathcal{C} = \{\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} - 2\boldsymbol{v}\}$.
(You do not need to prove that $\mathcal{C}$ is a basis of V.)

(a) Compute $\mathcal{P}_{\mathcal{B},\mathcal{C}}$ the change of basis matrix that converts $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates.

(b) Compute $\mathcal{P}_{\mathcal{C},\mathcal{B}}$ the change of basis matrix that converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

(c) Compute $[T]_{\mathcal{C}}$ the matrix of T with respect to the basis $\mathcal{C}$.

Solution:

(a) Let $c_1 = u + v$, $c_2 = u - 2v$. The change of basis matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$ is

$$P_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

(b) The change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ is

$$P_{\mathcal{C},\mathcal{B}} = P_{\mathcal{B},\mathcal{C}}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) The matrix of T with respect to the basis $\mathcal{C}$ is

$$[T]_{\mathcal{C}} = P_{\mathcal{C},\mathcal{B}}[T]_{\mathcal{C}}P_{\mathcal{B},\mathcal{C}} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 8 \\ -2 & 1 \end{bmatrix}$$

Question 9 (6 marks)

Let V be a vector space with an inner product and let $S \subseteq V$ be a subset of V.

(a) Define what it means to say that S is *orthonormal*.

(b) Prove that if S is orthonormal, then it is linearly independent.

Solution:

(a)

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \begin{cases} 1 & \text{if } \boldsymbol{u} = \boldsymbol{v} \\ 0 & \text{if } \boldsymbol{u} \neq \boldsymbol{v} \end{cases}$$

(b) Suppose that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in S$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ are such that $\sum_1^k \alpha_i \boldsymbol{u}_i = \boldsymbol{0}$. We need to show that $\alpha_i = 0$ for all i. Let $j \in \{1, \ldots, k\}$. We have

$$\begin{aligned} \sum_1^k \alpha_i \boldsymbol{u}_i = \boldsymbol{0} &\implies \langle \sum_1^k \alpha_i \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \langle \boldsymbol{0}, \boldsymbol{u}_j \rangle \\ &\implies \sum_1^k \alpha_i \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = 0 \\ &\implies \alpha_j \langle \boldsymbol{u}_j, \boldsymbol{u}_j \rangle = 0 \\ &\implies \alpha_j = 0 \end{aligned}$$

Question 10 (8 marks)

(a) Determine whether or not the following define inner products on $\mathbb{R}^2$:

(i) $\langle (x,y),(a,b)\rangle = 4xa + 2xb + 2ya + yb$

(ii) $\langle (x,y),(a,b)\rangle = 5xa + 2xb + 2ya + yb$

(b) Let W be the subspace $\mathbb{R}^4$ having (ordered) basis

$$\mathcal{B} = \{(1,1,1,1), (-1,1,-1,1), (1,2,3,-4)\}$$

Obtain an orthonormal basis for W (with respect to the dot product) by applying the Gram-Schmidt procedure to $\mathcal{B}$.

Solution:

(a) (i) NOT an inner product. To see this note that

$$\langle (1,-2),(1,-2)\rangle = 4 - 4 - 4 + 4 = 0$$

but $(1,-2) \neq \mathbf{0}$.

(ii) IS an inner product. To see this note that it can be written as

$$\langle (x,y),(a,b)\rangle = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

The matrix $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ is symmetric, has (strictly) positive entries in the top left and bottom right and has $\det(A) = 1 > 0$.

(b)

$$\begin{aligned}
u_1 &= b_1/\|b_1\| = \frac{1}{2}(1,1,1,1) \\
w_2 &= b_2 - (b_2 \cdot u_1)u_1 = (-1,1,-1,1) - 0(1,1,1,1) = (-1,1,-1,1) \\
u_2 &= w_2/\|w_2\| = \frac{1}{2}(-1,1,-1,1) \\
w_3 &= b_3 - (b_3 \cdot u_1)u_1 - (b_3 \cdot u_2)u_2 \\
&= (1,2,3,-4) - \frac{1}{4}(1,1,1,1)\cdot(1,2,3,-4)(1,1,1,1) - \frac{1}{4}(-1,1,-1,1)\cdot(1,2,3,-4)(-1,1,-1,1) \\
&= (1,2,3,-4) - \frac{1}{2}(1,1,1,1) + \frac{3}{2}(-1,1,-1,1) = (-1,3,1,-3) \\
u_3 &= w_3/\|w_3\| = \frac{1}{\sqrt{20}}(-1,3,1,-3)
\end{aligned}$$

The orthonormal basis is:

$$\{\frac{1}{2}(1,1,1,1),\ \frac{1}{2}(-1,1,-1,1), \frac{1}{2\sqrt{5}}(-1,3,1,-3)\}$$

Question 11 (6 marks)

Let A be an $n \times n$ matrix and suppose that the characteristic polynomial of A is given by

$$x^3 + x^2 - 5x - 6$$

(a) What is the value of n?

(b) Is A invertible?

(c) Write A^4 as a linear combination of A^2, A and I.

Solution:

(a) $n =$ degree of char polynomial $= 3$

(b) Yes, A is invertible since 0 is not a root of the char poly ($\det(A) = -6$)

(c) By Cayley-Hamilton, we have $A^3 + A^2 - 5A - 6I = 0$. Therefore

$$A^3 = -A^2 + 5A + 6I$$
$$A^4 = -A^3 + 5A^2 + 6A = (A^2 - 5A - 6I) + 5A^2 + 6A = 6A^2 + A - 6I$$

Question 12 (8 marks)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the xy-plane given by a stretch with factor 3 along the line $y = -x$ and a compression by $\frac{1}{2}$ along the line $y = x$ (see Figure 1).

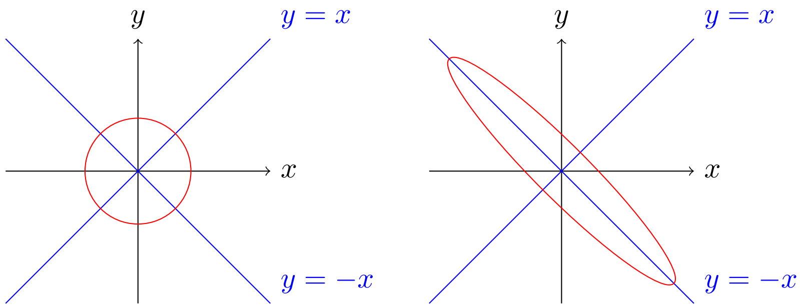


Figure 1: The unit circle in $\mathbb{R}^2$ (left) and its image under T (right).

(a) Find the eigenvalues of T.

(b) Find the corresponding eigenspaces of T.

(c) Choose a basis $\mathcal{B}$ of eigenvectors of T and give the matrix representation of T with respect to this basis.

(d) Finally, find the standard matrix representation $[T]_{\mathcal{S}}$ of T.

Solution:

(a) The eigenvalues are 3 and $\frac{1}{2}$.

(b) The eigenspaces are the two blue lines shown in the picture, i.e., $x = y$ and $x = -y$.

(c) Several solutions are possible. Say $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$. Then

$$[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(d) To find the two columns of $[T]_{\mathcal{S}\mathcal{S}}$, we need to calculate, in this order, the effect of T on

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \text{ and on } \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\left(-\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

These are given by

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) \text{ and on } \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\left(-\begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right).$$

So,

$$[T]_{\mathcal{S}\mathcal{S}} = \frac{1}{4}\begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix}.$$

Question 13 (10 marks)

(a) Determine whether each of the following matrices is diagonalisable (over $\mathbb{R}$). The characteristic polynomial of each matrix is given.

(i) $\begin{bmatrix} 8 & -7 & 2 \\ 8 & -5 & 4 \\ -2 & 6 & 3 \end{bmatrix}, \quad (x+1)(x-3)(x-4)$

(ii) $\begin{bmatrix} 7 & -4 & -8 \\ 16 & -9 & -24 \\ -4 & 2 & 7 \end{bmatrix}, \quad (x+1)(x-3)(x-3)$

(iii) $\begin{bmatrix} -3 & 7 & -3 \\ 1 & -5 & 2 \\ 4 & -15 & 6 \end{bmatrix}, \quad (x+1)(x^2+x+1)$

(b) (i) Let $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$.

Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(ii) Sketch the curve $8x^2 - 4xy + 5y^2 = 1$. You should give the directions of the principal axes and principal axis-intercepts.

Solution:

(a) (i) DIAGONALISABLE since it has three distinct eigenvalues: -1,3,4

(ii) The eigenvalues are -1,3,3. For the $\lambda = 3$ eigenspace:

$$\begin{bmatrix} 4 & -4 & -8 \\ 16 & -12 & -24 \\ -4 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & -8 \\ 0 & 4 & 8 \\ 0 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & -8 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace has dimension 1. Therefore the matrix is NOT DIAGONALISABLE

(iii) The only (real) eigenvalue is 1. Therefore NOT DIAGONALISABLE

(b) (i) The characteristic polynomial of A is

$$\chi_A(x) = \det(xI - A) = \begin{vmatrix} x-8 & 2 \\ 2 & x-5 \end{vmatrix} = (x-8)(x-5)-4 = x^2-13x+36 = (x-4)(x-9)$$

The eigenvalues are 4,9. For each we find a unit eigenvector.
For eigenvalue 4:

$$4I - A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

A unit eigenvector is: $\frac{1}{\sqrt{5}}(1,2)$
For eigenvalue 9:

$$9I - A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

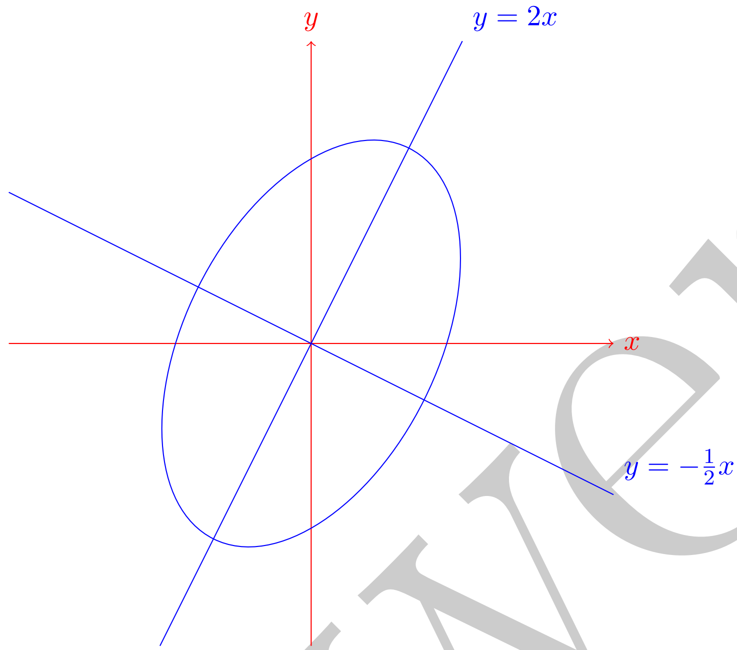
A unit eigenvector is: $\frac{1}{\sqrt{5}}(-2,1)$
Therefore we can take

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \qquad P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

(ii) The equation of the curve is

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x & y \end{bmatrix} = 1 \quad \text{where} \quad A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

From the previous part we know that the curve is an ellipse with principal in the directions of $(1, 2)$ and $(-2, 1)$.



The principal axis intercepts are at: $\pm\frac{1}{\sqrt{15}}(1, 2)$ and $\pm\frac{1}{3\sqrt{5}}(-2, 1)$

End of Exam—Total Available Marks = 100.