COMP10002 Foundations of Algorithms

Semester Two, 2017

Problem Solving, Algorithms, and More

Overview

Chapter 9 – Concepts

A range of generic approaches to solving problems:

- Generate and test
- Divide and conquer
- Simulation
- Approximation
- Adaptation

Generate and test

If solution space can be enumerated and mapped onto a sequence of integers, then systematically try candidate answers until one that meets specified criteria is found.

If solution space is infinite, then include a loop counter and “no answer found” exit point.

If solution space is multiply-infinite, be sure that the dimensions are treated fairly.

- `isprimefunc.c`

Divide and conquer

- Break the problem into smaller instance(s)
- Solve the instance(s), perhaps recursively
- Combine solutions to create solution to original problem.

For example, the towers of Hanoi problem yields to a recursive divide-and-conquer approach:.

- `hanoi.c`
- `quicksort.c`
- plus another sorting method, coming real soon

Divide and conquer – Subset sums

Is there a subset of the following numbers that adds up to exactly 1,000?

```
 34,  38,  39,  43,  55,  66,  67,  84,  85,  91,
101, 117, 128, 138, 165, 168, 169, 182, 184, 186,
234, 238, 241, 276, 279, 288, 386, 387, 388, 389,
413, 444, 487, 513, 534, 535, 616, 722, 786, 787
```

In general, given *n* integer values, and a target value *k*, determine if there is a subset of the integers that adds up to exactly *k*.

Divide and conquer – Subset sums

Evaluate all subsets of the `n` items, and if any one of them adds to `k`, return "yes".

Either the last value `A[n-1]` is part of the sum, or it is not.

If it is, a subset sum on the first `n-1` items in the array must add to `k-A[n-1]`.

- `subsetsum.c`

Hard problems – An aside

The subset sum problem is an example of a very important class of related hard problems.

All (so far!) take exponential time in size of input, which makes them intractable. We can solve for small problems, and we can quickly check any proposed solution for large problems. But we can't find solutions for large problems.

If any one of these problems can be solved in polynomial time, then they all can!

Monte Carlo methods

Use pseudo-random number generation to allow modeling of a physical system.

Function `srand()` is used to initialize the random-number sequence; then each call to `rand()` returns the next seemingly-unrelated `int` in the sequence.

- `gamble1.c`
- `gamble2.c`
- `random.c`
- `montecarlo.c`
- `drunk.c`

Approximation

Solve a simpler problem, with a known degree of fidelity relative to the original one, or with a clear strategy of estimating the extent of the error.

Many numerical examples: integration, curve fitting, root finding, solutions to DEs and PDEs.

Need to be very careful with the floating point arithmetic being performed. *Numerical Analysis* is the branch of mathematics/computing that studies such problems.

- `linelength.c`
- `bisection.c`
- `spring.c`

Adaptation

Modify the solution or approach already used for another similar problem.

Principled “borrowing”.

Always give a full attribution, and recognize the previous work honestly.

Chapter 9 – Summary

All algorithms start somewhere, with someone saying “surely there is a better way than that”.

Complexity theory is the branch of computing which seeks to prove, for a given problem, that “no there can't be”.

Chapter 12 (Part II) – Concepts

Abstract data types: different algorithms require different suites of operations.

The dictionary abstract data type is an important one,

The priority queue is another.

And as a final triumph for the book, and for the subject: $O(n \log n)$ worst-case time sorting.

Dictionary abstract data type

Operations to be supported:

- $D \leftarrow \mathit{make\_empty}()$
- $D' \leftarrow \mathit{insert}(D, \mathit{key}, \mathit{item})$
- $\mathit{item\_ptr} \leftarrow \mathit{search}(D, \mathit{key})$
- $D' \leftarrow \mathit{delete}(D, \mathit{key})$

plus (maybe):

- $\mathit{item\_ptr} \leftarrow \mathit{find\_smallest\_key}(D)$
- $\mathit{item\_ptr} \leftarrow \mathit{find\_next\_key}(D)$

Dictionary options

Array? But search and ordered processing are slow.

Sorted array? But insert is slow.

Binary search tree? But not robust, and hence vulnerable.

Balanced search tree? Yes, can do $O(\log n)$ time for all operations, but complex to implement. (And you have to take another subject.)

Hashing: insert, search, and delete in $O(1)$ average time. No ordered processing, but still useful for some applications.

Hashing

Main idea: take each key, and use a hash function $h()$ to deterministically construct a seemingly-random integer in a constrained range $[0 \ldots t-1]$, where $t > n$, the number of items to be stored.

Use those integers to index an array A of size t, putting x in location $A[h(x)]$.

If lucky, every thing fits just fine, and no collisions.

Don't expect to be lucky – the birthday "paradox".

Hashing

If the location indicated by the hash function is used, put the item "somewhere else" that makes sense.

When searching for x, look first in the location $A[h(x)]$ indicated by the hash function.

If not there, look in all possible "somewhere else" locations.

If not there, then not in the dictionary.

Try to keep the set of "somewhere else" options small!

Hashing – Collisions

When the location indicated by $h(x)$ is already occupied:

- linear advance: move cyclicly forward to the next vacant cell in the array (simple, but not that good, chains get longer, and deletion is problematic)
- cuckoo hashing: displace the previous object to allow the new one to be in its right spot, then re-insert it using a different hash function (effective, but needs a suite of hash functions, still needs complex deletion)
- separate chaining: use a secondary data structure to maintain the set of colliding objects: linked list, tree, or dynamic array (may require additional overhead space).

Hashing – Collisions

Separate chaining is easy to implement, and robust against overflow – doesn't require that $h > n$ the way the other two methods do.

When records are large, overhead of extra pointers per item is small.

Whenever table loading gets too high, double the size of the array of lists, rehash all current objects, and resume.

Hashing – Good and bad

Use all characters of the key, and seek to involve all bit positions in the intermediate value, before a final `%t` operation to bring into the right range.

Never just add character values, or shift-n-add. It isn't enough "randomness", and performance will degrade.

Test the hash function on the data it will be applied to, by checking the distribution of bucket loadings.

- `hashing.c`
- `hashscaffold.c`
- `hashingbad.c`

Hashing – Analysis

If hash function is good (and if data is not pathological), after n objects are inserted, average list length is n/t items long. (And can do stats on standard deviation and etc.)

If $n/t < K$ for some K, then insert and search are $O(1)$ – provided that the hash function itself can be evaluated in $O(1)$ time per key. Might not be possible for strings.

Further enhancement – move any item that is accessed to the front of its list. Accelerates search, unless access pattern is uniform.

Divide and conquer sorting

Already saw Quicksort. There is another beautiful divide-and-conquer sorting algorithm:

```
$mergesort(A[0 \ldots n-1])$
    **if** $n \le 1$
        **return**
    $mid \leftarrow (n/2)$
    $mergesort(A[0 \ldots mid-1])$
    $mergesort(A[mid \ldots n-1])$
    $merge(A[0 \ldots mid-1], A[mid \ldots n-1])$
```

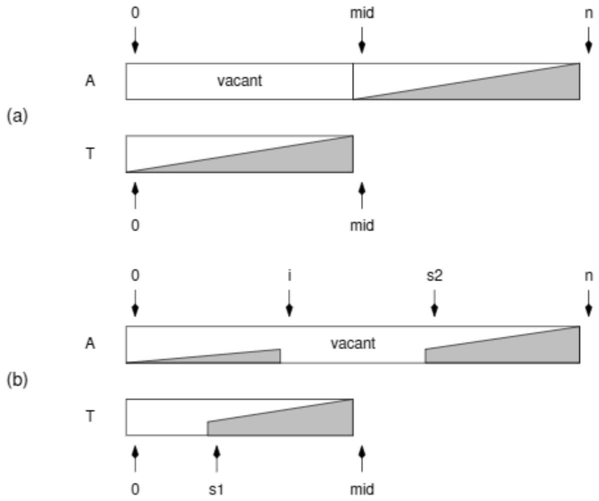
Where Quicksort is “hard split, easy join”, this one is “easy split, hard join”.

Mergesort – Merging

```
$merge(A[0 \ldots mid-1], A[mid \ldots n-1])$
    $T[0 \ldots mid-1] \leftarrow A[0 \ldots mid-1]$
    $i, s1, s2 \leftarrow 0, 0, mid$
    while $s1 < mid$ and $s2 < n$
        if $T[s1] < A[s2]$
            $A[i] \leftarrow T[s1]$
            $i, s1 \leftarrow i+1, s1+1$
        else
            $A[i] \leftarrow A[s2]$
            $i, s2 \leftarrow i+1, s2+1$
    $A[i \ldots s2-1] \leftarrow T[s1 \ldots mid-1]$
```

Needs a temporary array T half the size of original input.
Same array can be used in all recursive calls.

Mergesort – Merging



Mergesort – Analysis

Counting the comparisons required:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ M(n) + T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) & \text{if } n > 1 \end{cases}$$

where $M(n)$ is the cost of merging.

With $M(n) = n - 1$, $T(n) = n \log_2 n - n + 1$ for n a power of two.

That is, Mergesort takes $O(n \log n)$ time in the worst case.

Brilliant! ... except for the extra space. Can we have it all?

Heaps

In an array $A[0 \ldots n-1]$, define the two *children* of item $A[i]$ to be in $A[2i+1]$ and $A[2i+2]$.

With the *parent* of $A[i]$ in $A[(i-1)/2]$.

This balanced tree is implicit, based on array positions. The root is in $A[0]$, and the leaves in $A[n/2 \ldots n-1]$. No extra space is required for pointers.

Add one more requirement to make it a heap – no child may be larger than its parent.

Heaps – Example

Initially (with $n = 9$):

13 16 14 10 15 17 18 30 25

Converted into a heap:

30 25 18 16 15 17 14 10 13

How? And how long does it take?

Heaps – Construction

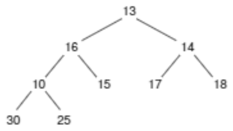
```
$build\_heap(A[0 \ldots n-1])$
    for $i \leftarrow n/2 - 1$ downto $0$
        $sift\_down(A, i, n)$

$sift\_down(A, p, n)$
    $child \leftarrow 2p + 1$
    if $child < n$
        if $child + 1 < n$ and $A[child] < A[child + 1]$
            $child \leftarrow child + 1$
        if $A[p] < A[child]$
            $swap(A[p], A[child])$
            $sift\_down(A, child, n)$
```

Heaps – Heapsort!

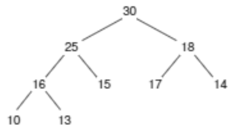
And now, for the moment of truth ...

```
$heapsort(A[0 \ldots n-1])$
    $build\_heap(A[0 \ldots n-1])$
    **for** $a \leftarrow n-1$ **downto** $1$
        $swap(A[0], A[a])$
        $sift\_down(A, 0, a)$
```

Heapsort – Example



(a)



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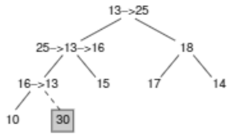

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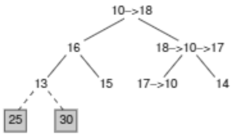

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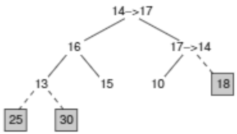

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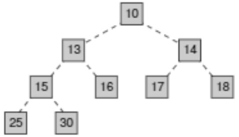

(f)

Heapsort – Analysis

Phase 1 (*build_heap*):

$$T_1(n) = 0\frac{n}{2} + 2\frac{n}{4} + 4\frac{n}{8} + 6\frac{n}{16} + \cdots + (2\log n)\frac{n}{n} = 2n$$

Phase 2 (swapping maximums):

$$T_2(n) = n \cdot (2\log n)$$

In-place sorting in $O(n\log n)$ time in the worst case.

Triumph? It doesn't get any better!

Chapter 12 (Part II) – Program examples

- `heapsort.c`
- `mergesort.c`

Chapter 12 (Part II) – Summary

Algorithms are fun!

And there are plenty more where these ones came from . . .