

Final exam solutions

MAST20005 Statistics

Semester 2, 2017

1. (a) $\mathbb{E}(X) = 3\theta$ and $\text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 5\theta - 9\theta^2 = \theta(5 - 9\theta)$
(b) i. Let the sample frequencies of the three values be n_0, n_1, n_2 . The likelihood is $L(\theta) = (1 - 2\theta)^{n_0} \theta^{n_1} \theta^{n_2} = (1 - 2\theta)^{n_0} \theta^{n_1+n_2}$, and therefore the log-likelihood is:

$$l(\theta) = n_0 \log(1 - 2\theta) + (n_1 + n_2) \log \theta.$$

Note that $n_0 + n_1 + n_2 = n$, so an alternative expression for the log-likelihood is:

$$l(\theta) = n_0 \log(1 - 2\theta) + (n - n_0) \log \theta.$$

- ii. $\frac{\partial l}{\partial \theta} = -\frac{2n_0}{1-2\theta} + \frac{n-n_0}{\theta} = 0 \Rightarrow \hat{\theta} = \frac{n-N_0}{2n} = \frac{N_1+N_2}{2n}$.
iii. A sufficient statistic is n_0 , the number of 0's. An equivalent alternative is $n_1 + n_2 = n - n_0$, which is the sum of the number of 1's and 2's.
iv. Note that $N_0 \sim \text{Bi}(n, 1 - 2\theta)$, which means that $\mathbb{E}(\hat{\theta}) = \frac{n-n(1-2\theta)}{2n} = \theta$, and therefore the MLE is unbiased.
v. $-\frac{\partial^2 l}{\partial \theta^2} = \frac{4n_0}{(1-2\theta)^2} + \frac{n-n_0}{\theta^2} \Rightarrow I(\theta) = \mathbb{E}\left(-\frac{\partial^2 l}{\partial \theta^2}\right) = \frac{4n(1-2\theta)}{(1-2\theta)^2} + \frac{n-n(1-2\theta)}{\theta^2} = \frac{4n}{1-2\theta} + \frac{2n}{\theta} = \frac{4n\theta+2n(1-2\theta)}{\theta(1-2\theta)} = \frac{2n}{\theta(1-2\theta)}$. Therefore, the lower bound is $\frac{1}{I(\theta)} = \frac{\theta(1-2\theta)}{2n}$.
vi. $\text{var}(\hat{\theta}) = \frac{1}{4n^2} \text{var}(N_0) = \frac{1}{4n^2} n(1-2\theta)(2\theta) = \frac{\theta(1-2\theta)}{2n}$. Therefore, the MLE achieves the lower bound.
- (c) i. We have $n_0 = 8, n_1 = 4, n_2 = 8$. This gives $\hat{\theta} = \frac{12}{2 \times 20} = 0.30$. To get a standard error we substitute this for θ into the expression for the variance of $\hat{\theta}$, which gives, $\text{se}(\hat{\theta}) = \sqrt{0.3 \times 0.4/40} = 0.055$.
ii. Based on the above results, we know that $\hat{\theta} \approx N(\theta, 0.055^2)$. Therefore, an approximate 95% confidence interval is given by $\hat{\theta} \pm 1.96 \times 0.055 = (0.19, 0.41)$.
Note: Using either the observed information function or the Fisher information function will lead to the same result. In particular, $J(\hat{\theta}) = I(\hat{\theta}) = \frac{4n^3}{n_0(n-n_0)} = 1000/3$. An exact alternative is an interval based on F_0 , which follows a binomial distribution. This gives a very similar result:

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> round((1 - rev(binom.test(8, 20)$conf.int)) / 2, 2)
[1] 0.18 0.40
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- iii. The expected counts (based on $\hat{\theta}$) are: 8, 6, 6. The test statistic is:

$$\chi^2 = \frac{(8-8)^2}{8} + \frac{(4-6)^2}{6} + \frac{(8-6)^2}{6} = \frac{4}{3} = 1.33.$$

We have estimated one parameter so have a test with 1 degree of freedom. The 0.95 quantile of a χ_1^2 distribution is 3.84. Since $1.33 < 3.84$, we cannot reject H_0 .

iv. It will not be possible, there will be no degrees of freedom remaining to carry out the test. The resulting model will be simply a binomial distribution. (Once we estimate the probability parameter for the binomial, we get a ‘perfect’ (saturated) fit to the data and cannot test for deviations.)

2. (a) i. Let the sample frequencies of the three values be N_0, N_1, N_2 . The MM estimate is obtained by solving $3\theta = \bar{X}$, which gives $\tilde{\theta} = \frac{1}{3}\bar{X} = \frac{N_1+2N_2}{3n}$.
 ii. $\mathbb{E}(\tilde{\theta}) = \frac{1}{3}\mathbb{E}(\bar{X}) = \frac{1}{3}\mathbb{E}(X) = \theta$, and therefore the MM estimator is unbiased.
 iii. $\text{var}(\tilde{\theta}) = \frac{1}{9}\text{var}(\bar{X}) = \frac{1}{9}\frac{\text{var}(X)}{n} = \frac{\theta(5-9\theta)}{9n}$. Note that,

$$\text{var}(\tilde{\theta}) = \frac{\theta(\frac{5}{9} - \theta)}{n} > \frac{\theta(\frac{1}{2} - \theta)}{n} = \text{var}(\hat{\theta}) = \frac{1}{I(\theta)},$$

So we can see that the MM estimator does **not** achieve the lower bound.

iv. From above, we see that both estimators are unbiased but the MLE has lower variance, so the MLE is a better estimator in this scenario.

- (b) i. We have $n_0 = 8, n_1 = 4, n_2 = 8$. This gives $\tilde{\theta} = \frac{20}{3 \times 20} = \frac{1}{3} = 0.333$. To get a standard error we substitute this for θ into the expression for the variance of $\tilde{\theta}$, which gives,

$$\text{se}(\tilde{\theta}) = \sqrt{\frac{1}{3} \times \frac{2}{180}} = \frac{1}{\sqrt{270}} = 0.061.$$

ii. The MM estimator is based on the sample mean and so due to the Central Limit Theorem will be approximately normally distributed. Based on the above results, we know that $\tilde{\theta} \approx N(\theta, 0.061^2)$. Therefore, an approximate 95% confidence interval is given by $\tilde{\theta} \pm 1.96 \times 0.061 = (0.21, 0.45)$.

3. (a) These are paired samples so we first calculate the differences and treat those as coming from a single normal distribution. We get the following summary statistics for the differences: $\bar{d} = 3.4$ and $s_d = 3.75$. The sample size is $n = 10$ and we need the 0.975 quantile of t_9 which is 2.26. A 95% confidence interval is given by: $3.4 \pm 2.26 \times 3.75/\sqrt{10} = (0.72, 6.1)$.

- (b) i. $H_0: \mu_1 = \mu_2$ and $H_1: \mu_1 \neq \mu_2$.
 ii. Use the t-test statistic on the differences, $T = \frac{\bar{D}}{S_D/\sqrt{10}}$.
 iii. $T \sim t_9$ under the null and its 0.975 quantile is 2.26. Therefore, reject H_0 if $|t| > 2.26$.
 iv. $t = 3.4/(3.75/\sqrt{10}) = 2.87 > 2.26$ so therefore reject H_0 .

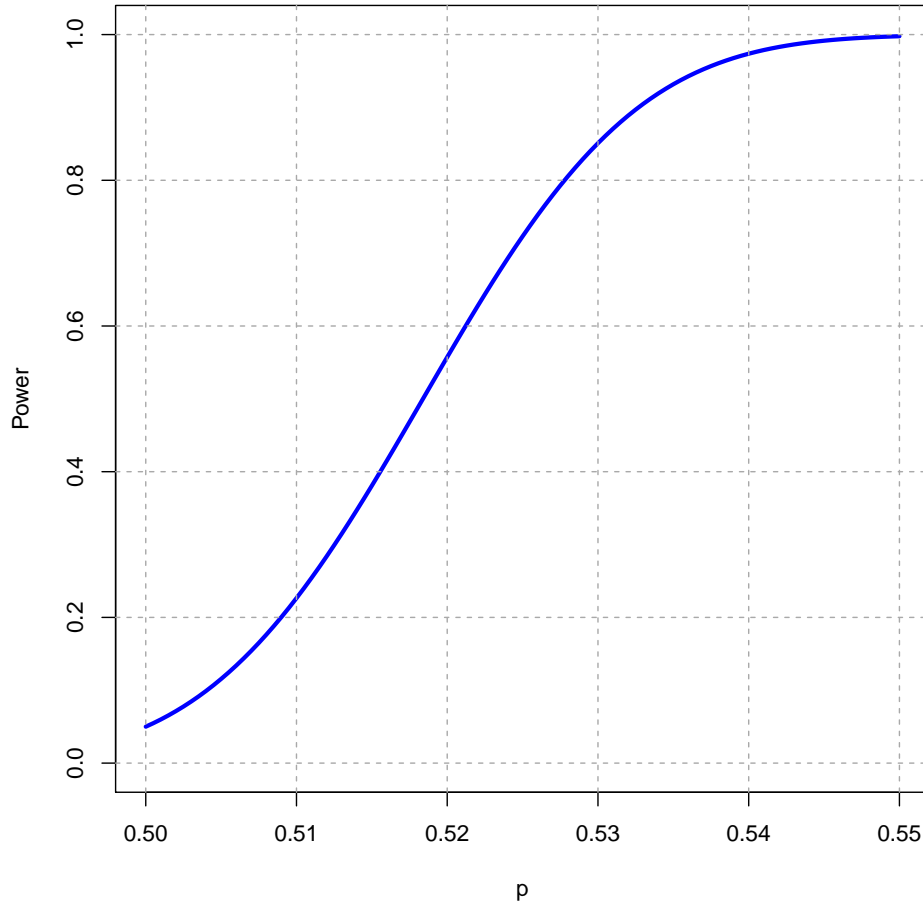
4. (a) Using the standard approximation for a single proportion, $\hat{p}_1 = 260/500 = 0.520$, $\text{se}(\hat{p}_1) = \sqrt{\frac{0.52 \times 0.48}{500}} = 0.0223$, so a 95% confidence interval is $\hat{p}_1 \pm 1.96 \text{se}(\hat{p}_1) = (0.48, 0.56)$.

- (b) We need to account for the sampling variation in both polls in any comparison. Using the standard comparison of two proportions, $\hat{p}_2 = 255/520 = 0.490$, $\hat{p}_2 - \hat{p}_1 = -0.0296$, $\text{se}(\hat{p}_2 - \hat{p}_1) = \sqrt{\frac{0.52 \times 0.48}{500} + \frac{0.49 \times 0.51}{520}} = 0.0313$, so a 95% confidence interval is $\hat{p}_2 - \hat{p}_1 \pm 1.96 \text{se}(\hat{p}_2 - \hat{p}_1) = (-0.091, 0.032)$. Differences in both a positive and negative direction are plausible, which means we do not have strong evidence for a change between the two polls. We certainly cannot draw as strong a conclusion as what the newspaper reported.

- (c) It needs to be $n \geq 1.96^2 / (4 \times (0.01)^2) = 9604$.

- (d) Let p be the proportion of the population who support the Purple Party. The test is $H_0: p = 0.5$ versus $H_1: p > 0.5$. Use the standard test for a single proportion: reject H_0 if $\frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} > c$. We have $p_0 = 0.5$, $n = 2000$ and $c = \Phi^{-1}(0.95) = 1.645$, so the rejection rule simplifies to: $2\sqrt{n}(\hat{p} - 0.5) > 1.645$. Under H_1 , we have $\hat{p} \approx N(p, p(1-p)/n)$. With a bit of algebraic manipulation, we can show the power function is:

$$K(p) = \Pr(2\sqrt{n}(\hat{p} - 0.5) > 1.645 \mid p) = \Phi\left(\frac{2\sqrt{n}(p - 0.5) - 1.645}{2\sqrt{p(1-p)}}\right)$$



- (e) Use $n = 2000$ and $p = 0.53$ to give $\beta = 1 - K(0.53) = 1 - \Phi(1.04) = 1 - 0.851 = 0.149$
5. (a) This is a sample from a binomial distribution so the MLE is $\hat{p} = \frac{1}{40} = 0.025$.
- (b) The conjugate prior for a binomial likelihood is a **beta distribution**.
- (c) We want $p \sim \text{Beta}(\alpha, \beta)$ where the α and β can be interpreted as pseudocounts. We require that $\alpha + \beta = 5$ and also that $\mathbb{E}(p) = \frac{\alpha}{\alpha + \beta} = 0.01$. Solving these gives $\alpha = 0.05$ and $\beta = 4.95$.
- (d) The posterior is $p \mid \text{data} \sim \text{Beta}(1 + \alpha, 39 + \beta) = \text{Beta}(1.05, 43.95)$.
- (e) Posterior mean: $\mathbb{E}(p \mid \text{data}) = \frac{1.05}{45} = 0.023$. Central 95% credible interval: (0.0007, 0.0826).

6. This is a standard contingency table scenario. The observed and expected frequencies are:

		Symptoms		
		Worse	Same	Better
Placebo	O	15	10	18
	E	9.46	6.45	27.09
Drug	O	7	5	45
	E	12.54	8.55	35.91

The test statistic is:

$$\chi^2 = \frac{(15 - 9.46)^2}{9.46} + \dots + \frac{(45 - 35.91)^2}{35.91} = 14.47.$$

The 0.99 quantile of a χ^2_2 distribution is 9.21. Since $14.47 > 9.21$, we reject H_0 .

7. (a) i. According to the exponential model, $\mathbb{E}(X) = \text{sd}(X) = \theta$. The Central Limit Theorem then implies $\hat{\theta} = \bar{X} \approx N(\theta, \theta^2/n)$.
- ii. $\hat{\theta} = \bar{x} = 3.9$.
- iii. $\text{se}(\hat{\theta}) = \hat{\theta}/\sqrt{n} = 3.9/\sqrt{9} = 1.3$. An alternative solution is to use the sample standard deviation directly, $\text{se}(\hat{\theta}) = s/\sqrt{n} = 0.82$. This avoids making the exponential distribution assumption so is more robust, but is less accurate if the exponential distribution is indeed true. Both are valid solutions to the question.
- (b) i. The median of an exponential distribution with mean θ is $\theta \log 2$. You can derive this easily from the cdf: $0.5 = F(m) = 1 - e^{-m/\theta} \Rightarrow m = \theta \log 2$. The sample median is asymptotically unbiased, which means $\mathbb{E}(\hat{M}) \approx \theta \log 2$. Therefore, it is biased for θ .
- ii. According to the above, letting $c = 1/\log 2$ makes T asymptotically unbiased.
- iii. Asymptotically, $\hat{M} \approx N(m, \frac{1}{4nf(m)^2})$. The pdf is $f(x) = \frac{1}{\theta}e^{-x/\theta}$, so we have $f(m) = f(\theta \log 2) = \frac{1}{\theta}e^{-\log 2} = \frac{1}{2\theta}$. This gives, $\hat{M} \approx N(\theta \log 2, \frac{\theta^2}{n})$. Therefore, $T \approx N(\theta, \frac{\theta^2}{n(\log 2)^2})$.
- iv. Both \bar{X} and T are asymptotically unbiased but \bar{X} has smaller variance: $\text{var}(T) \approx \frac{\theta^2}{n(\log 2)^2} = 2.08 \frac{\theta^2}{n} \approx 2.08 \text{var}(\bar{X}) > \text{var}(\bar{X})$. Therefore, \bar{X} is the better estimator.
- v. $t = \hat{m}/\log 2 = 1.443x_{(5)} = 1.443 \times 3.5 = 5.05$
- vi. $\text{se}(t) = \frac{t}{\sqrt{n \log 2}} = 2.43$
8. (a) $L(\theta) = \prod_{i=1}^n \frac{2x_i}{\theta^2} \propto \theta^{-2n}$ with $\theta \geq x_{(n)}$. This is maximised on the boundary, $\hat{\theta} = X_{(n)}$.
- (b) $F(x) = \int_0^x f(y) dy = \left(\frac{x}{\theta}\right)^2$. The pdf of the MLE is $G(x) = \Pr(X_{(n)} \leq x) = \Pr(X \leq x)^n = F(x)^n = \left(\frac{x}{\theta}\right)^{2n}$. The p -quantile of this distribution satisfies $p = G(\pi_p) = \left(\frac{\pi_p}{\theta}\right)^{2n}$. Rearranging gives $\pi_p = \theta p^{1/(2n)}$. For a central 95% confidence interval, we need bounds based on $p = 0.025$ and $p = 0.975$,

$$\begin{aligned} 0.95 &= \Pr(\theta \times 0.025^{1/(2n)} \leq X_{(n)} \leq \theta \times 0.975^{1/(2n)}) \\ &= \Pr(X_{(n)} \times 0.975^{-1/(2n)} \leq \theta \leq X_{(n)} \times 0.025^{-1/(2n)}). \end{aligned}$$

Therefore, a 95% confidence interval is given by $(x_{(n)} \times 0.975^{-1/(2n)}, x_{(n)} \times 0.025^{-1/(2n)})$.

- (c) $n = 6$, $x_{(6)} = 4.8$, $0.975^{-1/12} = 1.002112$, $0.025^{-1/12} = 1.359894$, and a central 95% confidence interval for θ is $(4.81, 6.53)$.