1 (a) (i) $d+r=318$

$12d+18r=4974$

This linear system is equivalent to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 318 \\ 12 & 18 & 4974 \end{array}\right]$$

(ii) Given $\left[\begin{array}{cc|c} 1 & 1 & 318 \\ 12 & 18 & 4974 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 125 \\ 0 & 1 & 193 \end{array}\right]$

We read off that $d=125$, $r=193$

(b) (i) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & c \end{array}\right] \xrightarrow{R_2-2R_1} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & c-2 \end{array}\right]$

For $c \neq 2$, the last line reads 0 equals a nonzero constant. This is inconsistent and so we conclude the linear system has no solution.

(ii) The linear equations correspond to two parallel planes and thus do not intersect.

2(a) $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$

Hence $(AB)^T = \begin{bmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{bmatrix}$.

On the other hand

$B^T A^T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{bmatrix}$

which is the same expression.

(b) (i) $Y^T A = [2 \ 1 \ 2]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [4 \ 1 \ 2]$

(ii) $Y^TAX = [4 \quad 1 \quad 2]\begin{bmatrix}1\\-1\\1\end{bmatrix} = 5$

Hence $(Y^TAX)^T = 5$

~~(iii) XAY^T Not possible $X_{3\times1}\, A_{3\times3}\, Y^T_{1\times3}$ incompatible~~

(c) $\det\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21}-a_{11} & 2a_{22}-a_{12} & 2a_{23}-a_{13}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

$= 2\det\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} - \underbrace{\det\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{11} & a_{12} & a_{13}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}}_{=0}$ since two rows the same

$= 50$

(a) $\det C = \det\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\det\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$

$= (1\times1\times1)\,(1\times1\times1) = 1$

sing the fact that the determinant of a triangular atrix equals the product of diagonal entries.

(b) $C = \begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}$

Consider

$\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 2 & 2 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{array}\right] R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & -2 & 1 & 0\end{array}\right.$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1+R_3 \\ R_2+R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \; R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] R_3 \times (-1)$$

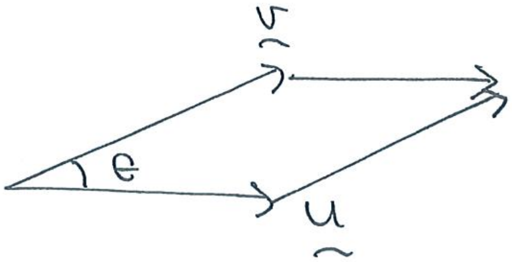
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Hence $C^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

c) $C^{-1}\begin{bmatrix} 27 & 29 \\ 50 & 38 \\ 9 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 27 & 29 \\ 50 & 38 \\ 9 & 21 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 5 & 1 \\ 4 & 20 \end{bmatrix}$

RED HAT

a (i)



(ii) $A =$ base times height

base $= \|\underset{\sim}{u}\|$

height $= \|\underset{\sim}{v}\| \sin\theta$

(iii) $A = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| \sin\theta = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| (1 - \cos^2\theta)^{1/2}$

$$= \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| \left(1 - \left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|}\right)^2\right)^{1/2}$$

$$= \left(\|\underset{\sim}{u}\|^2 \|\underset{\sim}{v}\|^2 - (\underset{\sim}{u}\cdot\underset{\sim}{v})^2\right)^{1/2}$$

4

b) Find the normal

$$\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}$$

what about finding d?

$$= -\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

Hence, using the fact that the plane passes through the origin, the cartesian form of the plane is

$$x + 3y + 2z = 0 = -x - 3y - 2z$$

(a) (i) Consider

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Two linearly independent vectors. Hence dimension is 2.

(ii) It is a subspace of $\mathbb{R}^4$ since each vector in the span is an element of $\mathbb{R}^4$.

(b)
$$x + 2y + z = 0$$
$$x - y + z = 0$$

This linear system corresponds to the matrix equation

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. Thus S is the solution

ce of A.

Since all solutions spaces are subspaces, it must be that S is a subspace.

c) Solution space $= \left\{ (x, y, z) : A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

~~Let's call this~~ set Y. ~~We have to check~~ 3 ~~properties~~ of Y.

~~(a) $(0,0,0) \in Y$.~~ This is true ~~since $A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$~~.

~~Hence Y is non-empty~~.

(1) Closure under vector addition. Let $(x_1, y_1, z_1) \in Y$ and $(x_2, y_2, z_2) \in Y$. This means that

$$A \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Our task is to show that $(x_1, y_1, z_1) + (x_2, y_2, z_2) \in Y$.

Now

$$A\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = A \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + A \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as required.

~~(2) Closure under~~ ~~scalar~~ multipli~~cation~~. Let $\alpha \in \mathbb{R}$, and $(x_0, y_0, z_0) \in Y$ so that

$$\cancel{A \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

~~Our task is~~ ~~to~~ show that ~~$\alpha(x_0, y_0,$~~ $z_0) \in Y$.

~~Now~~

$$\cancel{A\left(\alpha \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) = \alpha A \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

~~as required~~.

6 (a) rank $A = 3$

(b) column space $= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$

These vectors are the column in A corresponding to the leading entries in B

(c) Basis for row space
$= \{(1,0,3,0), (0,1,2,0), (0,0,0,1)\}$

These are the non-zero rows in RE form.

(d) Being rows of A, these four vectors are in the row space of A. But

dim(row space) = rank = 3

and so the four vectors cannot span $\mathbb{R}^4$.

(e) We have

$$(5,1,5,4,5) = 3(1,1,1,0,1) + 2(1,-1,1,2,1)$$

(f) Let the unknowns be denoted x_1, x_2, x_3, x_4.
No leading entry for x_3. Set $x_3 = t$, $t \in \mathbb{R}$
Back substitution gives

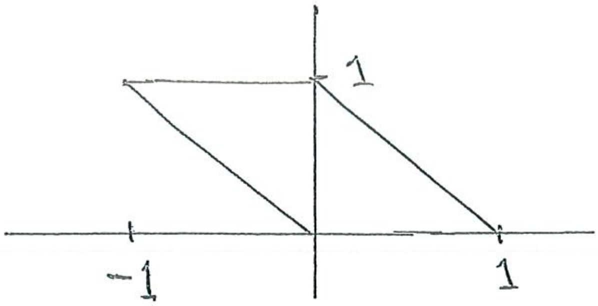
$x_4 = 0$

$x_2 = -2t$

$x_1 = -3t$

Hence the sol^n space is $\{t(-3,-2,1,0) \,;\, t \in \mathbb{R}\}$
A basis is $\{(-3,-2,1,0)\}$ and thus the dimension (nullity) is 1.

1 (a) (i)



(ii) We read off that

$$S\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}, \quad S\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

and the stated formula follows from fact that

$$A_S=\begin{bmatrix}S\begin{bmatrix}1\\0\end{bmatrix} & S\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}$$

(b) (i) $A_{TS} = A_T A_S$

(ii) Seek A_T such that $A_T\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$

Thus $$A_T=\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

. (a) (i) $$T(2,-2,2)=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\2 & -2 & 2\\1 & 1 & 1\end{vmatrix}$$
$$=\underset{\sim}{i}\begin{vmatrix}-2 & 2\\1 & 1\end{vmatrix}-\underset{\sim}{j}\begin{vmatrix}2 & 2\\1 & 1\end{vmatrix}+\underset{\sim}{k}\begin{vmatrix}2 & -2\\1 & 1\end{vmatrix}$$
$$=-4\underset{\sim}{i}+4\underset{\sim}{k}$$

singularity (ii) First, $T(t(1,1,1)) = t(1,1,1)\times(1,1,1)$
$= (0,0,0)$

) matrix.

Hence all stated vectors are in the kernel.

Also, we know that
$$\|(x,y,z)\times(1,1,1)\| = \|(x,y,z)\|\,\|(1,1,1)\|\sin\theta$$
ere θ is the angle between (x,y,z) and $(1,1,1)$.

For this to equal 0, we must have that (x,y,z) and $(1,1,1)$ are in the same direction so the stated set is all the vectors in the kernel.

(b)(i) $A_T = [T\underset{\sim}{i} \quad T\underset{\sim}{j} \quad T\underset{\sim}{k}]$

$$T\underset{\sim}{i} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -\underset{\sim}{j} + \underset{\sim}{k} = (0,-1,1)$$

$$T\underset{\sim}{j} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \underset{\sim}{j}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i} - \underset{\sim}{k} = (1,0,-1)$$

$$T\underset{\sim}{k} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -\underset{\sim}{i} + \underset{\sim}{j} = (-1,1,0)$$

The stated result follows.

(ii) We know $\mathrm{Im}\, T = $ column space A_T

Now
$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} R_3 + R_1$$

$$\sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} R_3 + R_2 \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\mathrm{Im}\, T = \mathrm{span}\{(0,-1,1), (1,0,-1)\}$

The plane corresponding to $\mathrm{Im}\, T$ is perpendicular to $(1,1,1)$, by properties of the cross product

Why? More reasoning needed?

(a) We know $[\underset{\sim}{x}]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Hence $\underset{\sim}{x} = \underset{\sim}{b}_1 + \underset{\sim}{b}_3$

(b) $P_{S,B} = P_{B,S}^{-1}$. Now

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_3 \\ R_2 - R_3 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence $P_{S,B} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

We read off that

$\underset{\sim}{b}_1 = (1,0,0) \quad \underset{\sim}{b}_2 = (-1,1,0) \quad \underset{\sim}{b}_3 = (0,-1,1)$

(c) The columns are not linearly independent.

(a) The axiom relating to $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle$ requires

(i) $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \geq 0$

(ii) $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 0 \iff \underset{\sim}{x} = \underset{\sim}{0}$

Writing

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = [x_1 \ x_2] \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

we know this is equivalent to showing that both eigenvalues of $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$ are > 0.

Eigenvalues are given by

$$\begin{vmatrix} 1-\lambda & 1/2 \\ 1/2 & \frac{1}{3}-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left(\frac{1}{3}-\lambda\right) - \frac{1}{4} = 0$$

$$\Rightarrow \lambda^2 - \frac{4}{3}\lambda + \frac{1}{12} = 0 \Rightarrow \lambda = \frac{4/3 \pm \left(\frac{16}{9} - \frac{3}{9}\right)^{1/2}}{2}$$

$$= \frac{4 \pm \sqrt{13}}{} \ \cancel{\neq} \ \cancel{4/3 \pm \frac{2}{3}\sqrt{2}}.$$ Hence both eigenvalues are

b) (i) Eigenvalues given by

$$\det\begin{bmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)(4-\lambda) - 3 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda-5)(\lambda-1) = 0.$

Hence $\lambda = 5$, $\lambda = 1$.

(ii) We know that if ~~the eigenvalues of A are distinct, then~~ A has linearly indep. eigenvectors, so A... is diagonalisable. It follows that A is diagonalisable.

1. (a) Define $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\underset{\sim}{y} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ $\underset{\sim}{a} = \begin{bmatrix} a \\ b \end{bmatrix}$

We know that the least squares solution to the linear system

$$A\underset{\sim}{a} = \underset{\sim}{y}$$

is

$$A^T A \underset{\sim}{a} = A^T \underset{\sim}{y}$$

Now $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^T \underset{\sim}{y} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$

Thus $\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 17 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 17 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 10 \\ 3 \end{bmatrix} \quad \Rightarrow \quad a = \frac{5}{3}, \; b = \frac{1}{2}$

Hence least squares line of best fit is

$$y = \frac{5}{3} + \frac{1}{2}x$$

(b) Substituting $x = 5$ gives $y = \frac{5}{3} + \frac{5}{2} \simeq 4$ students dropped out

2. (a) Let $\underset{\sim}{v}_1 = \frac{1}{\sqrt{2}}(-1,0,1)$

$\underset{\sim}{v}_2 = (0,1,0)$

$\underset{\sim}{v}_3 = \frac{1}{\sqrt{2}}(1,0,-1)$

We have $\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_1 = \frac{1}{2}((-1)^2+0^2+1^2) = 1$

$\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_2 = 1^2 = 1$

$\underset{\sim}{v}_3 \cdot \underset{\sim}{v}_3 = \frac{1}{2}(1^2+0^2+1^2) = 1$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 = \frac{1}{\sqrt{2}}(-1,0,1)\cdot(0,1,0) = \frac{1}{\sqrt{2}}(0+0+0) = 0$

$\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_3 = \frac{1}{2}(-1,0,1)\cdot(1,0,1) = \frac{1}{2}(-1+1) = 0$

$\underset{\sim}{v}_2 \cdot \underset{\sim}{v}_3 = (0,1,0)\cdot\frac{1}{\sqrt{2}}(1,0,1) = \frac{1}{\sqrt{2}}(0+0+0) = 0$

Hence $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3\}$ is an orthonormal set.

(b) In the notation of (a),

$A = 3\underset{\sim}{v}\,\underset{\sim}{v}^T = 3\underset{\sim}{v}_1\underset{\sim}{v}_1^T$

Hence $A\underset{\sim}{v}_1 = 3\underset{\sim}{v}_1\underset{\sim}{v}_1^T\underset{\sim}{v}_1 = 3\underset{\sim}{v}_1$ since $\underset{\sim}{v}_1^T\underset{\sim}{v}_1 = \underset{\sim}{v}_1\cdot\underset{\sim}{v}_1 = 1$

$A\underset{\sim}{v}_2 = 3\underset{\sim}{v}_1\underset{\sim}{v}_1^T\underset{\sim}{v}_2 = \underset{\sim}{0} = 0\underset{\sim}{v}_2$ " $\underset{\sim}{v}_1^T\underset{\sim}{v}_2 = \underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = 0$

$A\underset{\sim}{v}_3 = 3\underset{\sim}{v}_1\underset{\sim}{v}_1^T\underset{\sim}{v}_3 = \underset{\sim}{0} = 0\underset{\sim}{v}_3$ " $\underset{\sim}{v}_1^T\underset{\sim}{v}_3 = \underset{\sim}{v}_1\cdot\underset{\sim}{v}_3 = 0$

(c)(i) T corresponds to an orthogonal projection onto the direction of $\underset{\sim}{v}_1$ scaled by a factor of 3.

(ii) $[T]_{B,B} = \left[[T\underset{\sim}{v}_1]_B, [T\underset{\sim}{v}_2]_B, [T\underset{\sim}{v}_3]_B\right]$

$= \left[[3\underset{\sim}{v}_1]_B\ [\underset{\sim}{0}]_B\ [\underset{\sim}{0}]_B\right]$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$