1. Topic 1 — Lecture 1

Slides: pgs. 3–11. Problem Sheet Exercises: Topic 1, Q.1, Q.13, Q.14

1.1. **Simultaneous linear equations.**

Question: What distinguishes **linear equations**?

Some word problems lead to simultaneous linear equations.

A systematic method to approach such problems.

- Introduce notation for the unknowns.
- Convert the given information into equations.
- Solve the equations.
- Check your answer.

Example: Problem sheet exercises, Q.14.

Let F, D and P denote the ages of Frank, Dave and Phil.

The sum of Dave's and Phil's ages is 13 more than Frank's. Hence $D+P = F+13$.

Frank's age plus Phil's age is 19 more than Dave's. Hence $F + P = D + 19$.

The sum of their ages is 71. Hence $F + D + P = 71$.

Question: How can these equations be solved systematically? (See lecture 2: augmented matrices.) If there are only two unknowns, substitution can be used.

Example: Solve the simultaneous equations $3x + y = -1$ and $2x + y/4 = 1$.

Solution. From the first equation, $y = -1 - 3x$. Substituting in the second equation gives $5x/4 = 5/4$, and hence $x = 1$. Now that x has been determined, we substitute back in the first equation to get $y = -4$.

Remark: We should substitute these back into the original to check our answer.

Network flow problems lead to linear equations due to the rule: sum of flows in equals sum of flows out.

A systematic method to approach such problems.

- Label each node (also referred to as a vertex) of the graph, and covert the given information into a diagram.
- For each node, write down a linear equation from the flow rule.

It turns out that the 4 equations obtained in Example 2 **do not** uniquely determine the 4 unknowns. We will see in lecture 3 that in this circumstance the general solution will involve a parameter.

Curve fitting. Intuitively, specifying two points uniquely determines the line, specifying three points uniquely specifies a parabola (a quadratic equation), specifying four points uniquely specifies a cubic equation. How does this tie in with simultaneous equations?

2. Lecture 2

Slides: pgs. 12–20. Problem Sheet Exercises: Topic 1, Q.2, Q.3, Q.4.

2.1. **Augmented matrix form.** The idea here is to rewrite simultaneous equations in a matrix form, which is augmented by drawing a vertical straight line, and writing a further column.

A systematic way to form an augmented matrix is

- Write each linear equation so that the unknowns are on the left, and the constant on the right.
- For each equation require that the variables be written in the same order.
- Reading the coefficients on the LHS, and the constants on the RHS gives the augmented matrix.

Our aim is to use elementary row operations to reduce the augmented matrix to row echelon form.

For the meaning of row echelon form, the idea of a leading entry (also called a leading coefficient) is needed. This is the first non-zero element in each row of the coefficient matrix. An augmented matrix is said to be in row echelon form if

- All rows of zeros in the coefficient matrix are below all rows with non-zero entries.
- All leading entries have zeros in the same column below.
- The pattern formed by the leading entries is a staircase (read right to left).

Exercise Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Mark in the leading entries.

The elementary row operations are

- Interchange two rows (corresponds to changing the order of the equations).
- Multiply a row by a non-zero constant (corresponds to multiplying the LHS and RHS of the equation by this constant).

- Adding a multiple of one row to another (corresponds to adding a multiple of one equation to another equation).

The idea is to use elementary row operations to convert the augmented matrix to row echelon form.

Exercise Convert the augmented matrix corresponding to the linear system

$$\begin{aligned} x + 2y - z &= -1 \\ 2x + 7y - z &= 3 \\ -3x - 12y + z &= 0 \end{aligned}$$

to row echelon form.

Once an augmented matrix is converted to row echelon form, the linear system can be solved by back substitution.

Exercise Solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

where the columns of the coefficient matrix correspond to the unkowns x, y and z.

2.2. **Fully reduced row echelon form (also known as reduced row echelon form.** In (fully) reduced row echelon form, the leading entries must all be 1, and the entries above the leading entries must all be 0.

Exercise Reduce the augmented matrix in row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

to (fully) reduced row echelon form.

3. Lecture 3

Slides: pgs. 21–35. Problem Sheet Exercises: all remaining questions from Topic 1.

3.1. Different types of solutions.

Linear systems permit one of the following 3 types of solutions.

- A unique solution.
- Infinitely many solutions.
- No solution.

There is a unique solution when the number of leading entries in RE form equals the number of unknowns.

There is no solution when the coefficient matrix has a row of zeros, but the corresponding entry on the RHS is nonzeros.

Exercise Solve the linear system (if possible) corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

where the columns of the coefficient matrix correspond to the unkowns x, y and z.

There are an infinite number of solutions when the number of leading entries in RE form is less than the number of unknowns. To find these solutions, for each variable without a leading entry, a real parameter is introduced.

Definition The number of leading entries in a coefficient matrix is called the **rank** of the matrix.

Exercise Solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

where the columns of the coefficient matrix correspond to the unkowns x, y and z.

Exercise Solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where the columns of the coefficient matrix corresponds to the unkowns x_1, x_2, x_3, x_4.

Exercise Find the value of k for which the linear system

$$u + 2v = 1$$

$$2u + kv = 2$$

has (a) a unique solution; (b) has an infinite number of solutions. Make sure you quote appropriate theory in the setting out of your solution. Also, interpret your solution geometrically.