The University of Melbourne
Department of Mathematics and Statistics
Summer Semester Exam 2016

MAST10007 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 7 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 12 questions. The total number of marks is 80. All questions may be attempted.

This paper may be held by the Baillieu Library.

1. (a) A concert venue sold r tickets at the regular price of \$18, and also d tickets at the discount price of \$12. A total of 318 tickets were sold, and total revenue of \$4,974 was taken.

   i. Identify two linear equations from this situation, and from them read off the augmented matrix
$$\left[\begin{array}{cc|c} 1 & 1 & 318 \\ 12 & 18 & 4974 \end{array}\right]$$

   ii. You are told that
$$\left[\begin{array}{cc|c} 1 & 1 & 318 \\ 12 & 18 & 4974 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 125 \\ 0 & 1 & 193 \end{array}\right]$$
   Using this fact, or otherwise, specify r and d.

   (b) Consider the linear system for unknowns x, y, z with augmented matrix form
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & c \end{array}\right]$$

   i. Show, using RE form, that for $c \neq 2$ the system has no solution.

   ii. Give a geometrical reason for the fact in (i).

[7 marks]

2. (a) Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$
   Show by explicit computation that
$$(AB)^T = B^T A^T.$$

   (b) Let
$$X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
   Calculate

   i. $Y^T A$

   ii. $(Y^T A X)^T$

   (c) You are told that $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 25$. Calculate
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} - a_{11} & 2a_{22} - a_{12} & 2a_{23} - a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

[7 marks]

3. (a) Give a reason why the matrix defined by

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

must have determinant equal to 1. Here you are **not** to first calculate the matrix product explicitly.

(b) By first calculating the matrix product, or otherwise, and using working based on reduced row echelon form, show that

$$C^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

(c) A message consisting of 2 three letter words has been coded by replacing the letters by numbers according to $A \leftrightarrow 1$, $B \leftrightarrow 2$, etc., placing these numbers down the columns of a 3×2 matrix, then multiplying on the left by C as specified in (a). The coded message is

$$\begin{bmatrix} 27 & 29 \\ 50 & 38 \\ 9 & 21 \end{bmatrix}$$

What is the original message?

[7 marks]

4. (a) Consider two vectors **u**, and **v** pictured as in the following diagram

Exam1-16d.pdf

i. Copy this diagram into your exam booklet, and illustrate the parallelogram corresponding to these two vectors.

ii. Let A denote the area of the parallelogram. Explain the reasoning behind the formula

$$A = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta$$

iii. Making use of the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$, deduce from (ii) that

$$A = \sqrt{||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u}\cdot\mathbf{v})^2}$$

(b) Find the cartesian form of the plane specified by

$$\text{span}\,\{(2,0,-1),(1,-1,1)\}$$

[6 marks]

5. (a) i. Calculate the dimension of

$$\text{Span}\,\{(1,1,0,-1),\ (1,0,1,0),(0,1,-1,-1)\}.$$

ii. The set $\{a(1,1,0,-1)+b(1,0,1,0)+c(0,1,-1,-1)\ :\ a,b,c\in\mathbb{R}\}$ is a subspace of what vector space?

(b) Consider

$$S=\{(x,y,z):x+2y+z=0 \text{ and } x-y+z=0\}.$$

Specify S as the solution space of a particular matrix A. From this fact does it then follow that S is a subspace? Give a reason.

(c) Let B be a 2×3 matrix. Show from first principles that the solution space of B is closed under vector addition.

[7 marks]

6. Let

$$A=\begin{bmatrix}1 & 1 & 5 & 0\\ 1 & -1 & 1 & -2\\ 1 & 1 & 5 & 1\\ 0 & 2 & 4 & 3\\ 1 & 1 & 5 & 2\end{bmatrix},\qquad B=\begin{bmatrix}1 & 0 & 3 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

You are given that B is the reduced row echelon form of A.

(a) What is the rank of A?

(b) Write the column space as the span of three vectors, explaining in your answer how you deduced the three vectors.

(c) Write down a basis for the row space of A, stating too the theory you have used.

(d) Do the vectors $(1,1,5,0),(1,1,5,1),(0,2,4,3),(1,1,5,2)$ span $\mathbb{R}^4$? Give a reason.

(e) Write $(5,1,5,4,5)$ as a linear combination of $(1,1,1,0,1)$ and $(1,-1,1,2,1)$.

(f) Find a basis for the solution space of A, and specify the nullity.

[8 marks]

7. (a) i. Illustrate on a diagram the image of the unit square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ when transformed by a shear transformation S of -1 units parallel to the x-axis.

   ii. Use your diagram to deduce that the standard matrix of S is

$$A_S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

   (b) Let K denote the linear transformation corresponding to applying first a linear transformation S, then a linear transformation T. Let the standard matrix of K be denoted A_K. With S as in (a), find the standard matrix A_T of T such that

$$A_K = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

[6 marks]

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined in terms of the cross product according to

$$T(x, y, z) = (x, y, z) \times (1, 1, 1)$$

   (a) i. Compute $T(2, -2, 2)$.

   ii. Explain why $\operatorname{Ker} T = \{t(1, 1, 1) : t \in \mathbb{R}\}$

   (b) i. Show that the standard matrix for T is

$$A_T = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

   ii. Use (i) to calculate $\operatorname{Im} T$, and explain how the resulting subspace is related to the line specified by $\operatorname{Span}(1, 1, 1)$.

[6 marks]

9. (a) You are given that the change of basis matrix $P_{\mathcal{B},\mathcal{S}}$ from the standard basis to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is

$$P_{\mathcal{B},\mathcal{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Express the vector $\mathbf{x} = (1, -1, 1)$ as a linear combination of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$.

(b) Determine the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ in (a) by first computing $P_{\mathcal{S},\mathcal{B}}$.

(c) Would it be possible for

$$P_{\mathcal{S},\mathcal{C}} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

to be a change of basis matrix for some basis $\mathcal{C}$? Give a reason.

[6 marks]

10. (a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Show that this satisfies the axiom required for an inner product in $\mathbb{R}^2$ relating to $\langle \mathbf{x}, \mathbf{x} \rangle$ for $\mathbf{x} \in \mathbb{R}^2$. Make sure you clearly state this axiom, and any theorem you may use to verify the axiom.

(b) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

i. Calculate the eigenvalues of A.

ii. Give a reason why the matrix A is diagonalisable by quoting a theorem involving a feature of its eigenvalues.

[6 marks]

11. The total number of students who had dropped out of a particular tutorial was recorded at the end of week x to be equal to y students on three different occasions to give the following data

x	y
1	2
2	3
3	3

(a) Find the least squares line of best fit $y = a + bx$ to this data.

(b) Use your answer to (a) to estimate, using an integer, the number of students who had dropped out by the end of week 5.

[6 marks]

12. (a) Show that

$$\mathcal{B} = \{\frac{1}{\sqrt{2}}(-1, 0, 1), (0, 1, 0), \frac{1}{\sqrt{2}}(1, 0, 1)\}$$

is an orthonormal set with respect to the dot product.

(b) Let $\mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and define

$$A = 3\mathbf{v}\mathbf{v}^T$$

By making use of (a), or otherwise, **verify** that the eigenvalues of A are

$$3 \quad \text{and} \quad 0 \text{ repeated twice}$$

with corresponding normalised eigenvectors

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Let A in (b) be the standard matrix for a linear transformation T.

i. Give a geometrical description of T.

ii. With $\mathcal{B}$ as in (a), specify $[T]_{\mathcal{B},\mathcal{B}}$.

[7 marks]

— END OF EXAMINATION QUESTIONS —