# MAST10007 Linear Algebra
# MATLAB Test

**Test duration: 45 minutes**

**This paper has 6 pages (only the 4 containing the questions are numbered)**

Please complete ***all*** the following details.

**Name:** ................................................

**Student Number:** ................................................

**Tutor's Name:** ................................................

**Lab Time:** ................................................

**Instructions to Students:**

This test is designed to evaluate your comprehension of concepts in linear algebra, and your ability to calculate efficiently with the aid of MATLAB. Some questions test your understanding of the material covered in lectures, and do not necessarily require MATLAB. No partial credit is given, so please carefully check anything typed into MATLAB, and check the output of code used.

Any rough working must be done on this paper, but only the final answer is marked.

Answer all multiple choice questions by circling the correct answer(s).

The number of marks for each question is indicated and the total number of marks is 20.

Some MATLAB commands:

- rref($A$) gives the fully reduced row echelon form of $A$
- $A'$ is the transpose of the matrix $A$
- det($A$) gives the determinant of the matrix $A$
- eye($n$) gives the identity matrix of size $n \times n$
- inv($A$) gives the inverse of $A$
- ones($p, q$) gives the $p \times q$ matrix of all 1's
- zeros($p, q$) gives the $p \times q$ matrix of all 0's
- diag(v) gives the diagonal matrix with diagonal v
- The command v = B(:,3) selects the third column of B, for example.
- dot($u, v$) gives the dot product of the vectors $u$ and $v$.

Do not turn over until instructed to do so.

Rough working — will not be graded

1. Consider the matrix

$$X = \begin{bmatrix} -2 & 0 & 0 & 0 & 11 & 11 & 11 & 11 \\ 0 & -2 & 0 & 0 & 11 & 11 & 11 & 11 \\ 0 & 0 & -2 & 0 & 11 & 11 & 11 & 11 \\ 0 & 0 & 0 & -2 & 11 & 11 & 11 & 11 \\ 6 & 5 & 5 & 5 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 1 & 1 & 1 & 1 \\ 5 & 5 & 4 & 5 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Let $\mathbf{a}$ denote the vector corresponding to the third column of $X$, let $\mathbf{b}$ denote the vector corresponding to the fourth row of $X$, and let $\mathbf{c}$ denote the vector corresponding to the third row of $X$. Calculate $2\mathbf{a} - \mathbf{b} + 4\mathbf{c}$.

(b) Let $I$ denote the $8 \times 8$ identity matrix. Calculate $\det(3I + X)^{-1}$, giving your answer as an exact fraction.

(c) Calculate the orthogonal projection of the vector corresponding to the second column onto the direction of the vector corresponding to the seventh column, giving your answer in terms of exact fractions.

(d) Add to $X$ the matrix corresponding to the MatLab code

$$\texttt{Y} = [\texttt{2} * \texttt{eye(4)} \quad \texttt{zeros(4,4);diag([-1,0,1,2])} \quad \texttt{zeros(4,4)}]$$

What is the dimension of the column space of $X + Y$?

[4 marks]

2. Let

$$A = \begin{bmatrix} 1 & 3 & 2 & 6 & 3 & 1 \\ -2 & -6 & -2 & -8 & 3 & 1 \\ 3 & 9 & 0 & 6 & 6 & 2 \\ -1 & -3 & 1 & 0 & 9 & 3 \end{bmatrix}$$

(a) What is the dimension of the row space.

(b) Write the vector corresponding to the final row of $A$ as a linear combination of the earlier rows. For this purpose, denote the rows by $\mathbf{r}_1, \ldots, \mathbf{r}_4$ in order.

(c) Let $t \in \mathbb{R}$. Circle the vectors that are in the solution space of $A$.

$$\begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 9 \\ 3 \end{bmatrix} \qquad t\begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} -2 \\ -6 \\ -2 \\ -8 \\ 3 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -17 \\ 7 \\ 4 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

(d) Let $t \in \mathbb{R}$. Circle the vectors that are in the column space of $A$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -5, \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 4, \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 13 \\ -8 \\ 5 \\ 10, \end{bmatrix} \qquad \mathbf{v}_4 = t\begin{bmatrix} 0 \\ -9 \\ 3 \\ -12 \end{bmatrix}$$

(e) Let $\mathbf{a}_j$ ($j = 1, \ldots, 6$) denote the columns of $A$. Express $\mathbf{v}_2$ as defined in (d) as a linear combination of some of the $\mathbf{a}_j$.

[6 marks]

3. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has a standard matrix representation

$$A_T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Write down the image of the three unit vectors **i**, **j**, **k** under $T$.

(b) The image of the unit cube formed by the three unit vectors is given by

(A) A shear by 1 unit parallel to the $y$-axis.

(B) A reflection of the original cube in the plane $y = z$, and an expansion by a factor of 3 in the $x$-direction.

(C) None of the above.

(c) Give the standard matrix representation of $T^{-1}$.

(d) Acting on the unit cube, the transformation $T^{-1}$ corresponds to

(A) A shear by -1 units parallel to the $y$-axis, and a compression by a factor of 1/3 in the $x$-direction.

(B) A reflection of the original cube in the plane $y = z$, and an compression by a factor of 1/3 in the $x$-direction.

(C) None of the above.

[5 marks]

4. Consider the basis for $\mathbb{R}^5$:

$$\mathcal{B} = \left\{(-1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, 0, 0, 0), (\frac{1}{2}, 0, -\frac{1}{2}, 0, 0), (\frac{1}{2}, 0, 0, -\frac{1}{2}, 0), (\frac{1}{2}, 0, 0, 0, -\frac{1}{2})\right\}$$

(a) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ and compute $P_{\mathcal{B},\mathcal{S}}$, writing down the final answer only (here $\mathcal{S}$ denotes the standard basis for $\mathbb{R}^5$).

(b) Give the co-ordinates in the basis $\mathcal{B}$ of each of the vectors $\mathbf{v}_1 = (-1, 2, 3, 4, 9)$, $\mathbf{v}_2 = (2, -2, 2, 2, 2)$, $\mathbf{v}_3 = (0, 1, 0, 1, 0)$

(c) Write down the standard matrix representation of the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ where

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \\ (x_1 + x_2 + x_3 + x_4 + x_5)/2 \\ (x_1 + x_2 + x_3 + x_4 + x_5)/2 \end{bmatrix}$$

(d) Compute $[T]_{\mathcal{B}}$, and from this compute $T^{99}\mathbf{b}_2$ where $\mathbf{b}_2$ is the second basis vector in $\mathcal{B}$, writing down the final answer only.

[5 marks]