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The University of Melbourne

Department of Mathematics and Statistics

MAST20004 Probability

Semester 1 Exam — June 24, 2011

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has **nine** (9) questions.
Attempt as many questions, or parts of questions, as you can.
The number of marks allocated to each question is shown in the brackets after the question statement.
The total number of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. (a) Consider a random experiment with sample space Ω.
   (i) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.
   (ii) Using the axioms, prove that if C and D are events with $C \subseteq D$, then $P(C) \leq P(D)$.

   (b) For two events A and B it is known that the probabilities $P(A) = P(B) = 1/5$ and $P(A \cap B) = 1/10$.
   (i) Find $P(B^c)$.
   (ii) Find $P(A \cup B)$.
   (iii) Find $P(A \cap B^c)$.
   (iv) Find $P(A \cup B^c)$.
   (v) Find $P(A^c \cup B^c)$.
   (vi) Are A and B positively or negatively related? Justify your answer.

[12 marks]

2. A bank uses a credit test to decide if loan applicants are likely to default. Each applicant is rated either $A1$, $A2$, $A3$ or B, in proportions 10%, 40%, 40% and 10% respectively. Applicants rated $A1$, $A2$ or $A3$ receive a loan, while applicants rated B do not receive a loan. Past experience indicates that $1/40$ of applicants rated $A1$, $1/25$ of applicants rated $A2$ and $1/20$ of applicants rated $A3$ eventually default.
   (a) What are the probabilities that an arbitarily-chosen applicant is rated $A1$, $A2$ and $A3$, given that they receive a loan?
   (b) What is the probability that an applicant who receives a loan eventually defaults?
   (c) What is the probability that an applicant who defaults was originally rated $A1$?

[6 marks]

3. (a) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 1/5 & \text{if } x \in (-2, 3) \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Find the probability distribution function $F_Y(y)$ and probability density function $f_Y(y)$ of $Y = X^2$.
   (ii) List any points where $F_Y(y)$ is not differentiable.
   (iii) Verify that $\int_{S_Y} f_Y(y)dy = 1$.

   You should be careful to state the intervals over which your expressions apply.

(b) You enter a casino and decide to bet on "red" (which has a probability of 18/37 of coming up) in roulette. Your overall strategy is to keep betting on "red" until you win three times. Let N be the number of bets that you lose before this happens.

   (i) Giving your reasons, name the distribution of the random variable N and give the value of any parameter(s).

   (ii) What the probability mass function of N?

   (iii) What is the expected value of N?

   (iv) Assume that you bet \$5 on each spin of the roulette wheel, so that your net winnings will be $\$W = \$5(3 - N)$. Calculate your expected net winnings using this strategy.

   (v) The variance of N is approximately 6.50. What is the variance in your net winnings?

[14 marks]

4. $X \stackrel{d}{=} \mathrm{R}(1, 2)$ and $Y = \log X$.

   (a) Write down $E(X)$ and $V(X)$.

   (b) Describe the set S_Y of possible values of the random variable Y.

   (c) Write down the density function $f_Y(y)$ of Y.

   (d) Calculate $E(Y)$,

      (i) by evaluating $\int_{S_X} \log x f_X(x)dx$, and

      (ii) by evaluating $\int_{S_Y} y f_Y(y)dy$.

   Hint: You may use the fact that

$$\frac{d}{dx}(x \log x - x) = \log x.$$

   (e) Using an appropriate second-order Taylor series expansion of $\log x$, give an approximation for $E(Y)$. Compare this with the exact value that you calculated in part (d).

[12 marks]

5. Let X and Y have joint probability density function of the form

$$f_{(X,Y)}(x, y) = \begin{cases} kxy & \text{if } 0 \leq x + y \leq 1,\ x \geq 0 \text{ and } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the value of k?

   (b) Giving your reasons, state whether X and Y are dependent or independent.

   (c) Find the marginal probability density functions of X and Y.

   (d) Calculate $E(X)$ and $E(Y)$.

   (e) Calculate $\mathrm{Cov}(X, Y)$.

   (f) Find the conditional probability density function of Y given $X = \frac{1}{4}$.

[15 marks]

6. The number N of claims received by an insurance company in one day is distributed according to a Poisson distribution with parameter $\lambda = 8$. We assume that claim sizes are independent random variables $X_1, \ldots, X_N$, that are also independent of N. To a good approximation, the size of each claim, in units of \$1,000, follows a normal distribution with mean 10 and variance 25. Let C be the total value of claims received by the insurance company in a given day. That is, $C = \sum_{i=1}^{N} X_i$.
   (a) Calculate $E(C)$.
   (b) Calculate $V(C)$.
   (c) Calculate the moment generating function of C, $M_C(t) = E(e^{tC})$.

[9 marks]

7. In this question, you may use the fact that, for $s > 0$, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du. \qquad (*)$$

A random variable X that has a probability density function

$$f(x) = \begin{cases} \dfrac{x^{-1/2} e^{-x/2}}{\Gamma(1/2) 2^{1/2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is known as a χ^2-random variable with 1 degree of freedom.
   (a) For what values of t is the moment generating function $M_X(t)$ of such a random variable X defined?
   (b) Derive $M_X(t)$. (Hint: You will need to use an appropriate integral substitution to reduce the integral to something that looks like the right hand side of (*)).
   (c) Hence calculate the mean and variance of X.

[10 marks]

8. (a) Let $X_1, X_2, \ldots$ be independent and identically-distributed random variables with $E(X_i) = \mu$ and let $S_n = \sum_{i=1}^{n} X_i$. Assume that the moment generating function $M_{X_i}(t)$ of X_i exists for some $t \neq 0$. The Law of Large Numbers states that

$$Z_n = \frac{S_n}{n}$$

converges in distribution to a random variable which takes the constant value μ with probability one.

Prove the Law of Large Numbers by showing that the moment generating function $M_{Z_n}(t)$ converges, as $n \to \infty$, to the moment generating function $e^{\mu t}$ of a random variable that takes the value μ with probability one.

(b) Let $X_1, \ldots, X_{600}$ be independent Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = 0.4$ and let $S_{600} = \sum_{i=1}^{600} X_i$.

(i) Write down $E(X_i)$ and $V(X_i)$.

(ii) Use the Central Limit Theorem to derive an approximation for the probability that $210 \leq S_{600} \leq 270$.
(You may use the following information about a standard normal random variable Z: $P(Z \leq 0) = .5000$, $P(Z \leq 0.5) = .6915$, $P(Z \leq 1) = .8413$, $P(Z \leq 1.5) = .9332$, $P(Z \leq 2) = .9772$, $P(Z \leq 2.5) = .9938$ and $P(Z \leq 3) = .9987$).

(iii) Physically, what does the random variable S_{600} represent? Name its distribution, giving the values of any parameters.

(iv) Let Y_n be a $Bi(n,p)$ random variable. Motivated by your answer to part (iii), explain how the Central Limit Theorem can be used to derive the fact that Y_n converges in distribution to a normally distributed random variable with mean np and variance $np(1-p)$ as $n \to \infty$.

[14 marks]

9. A short-stay bed and breakfast apartment is rented on a weekly basis. During any given week, it can be either 'occupied' or 'unoccupied'. There is a probability 0.2 that client who is occupying the apartment in week n will want to occupy it during week $n+1$. Such a client is given priority over any possible new arrival in week $n+1$. In any week, there is a 0.5 probability that a new client will want to rent the apartment. If an existing client wants the apartment, such a client will be turned away. Otherwise, the apartment will be rented to the new client. Let $X_n = 0$ if the apartment is unoccupied in week n and 1 if it is occupied in week n. Then $\{X_n : n = 0, 1, 2, \ldots\}$ is a discrete time Markov Chain.

(a) Write down the transition probability matrix P for this Markov chain.

(b) What is the probability that the apartment will be occupied in week 4 if it was occupied in week 1.

(c) Calculate the equilibrium probability that the apartment is occupied.

(d) If the owner of the apartment makes \$1500 for every week that she rents it out, how much money can she expect to make in a year? [8 marks]

End of the exam