2016 exam (MAST20005), question 2

Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on $[0, \theta]$ with pdf,

$$f(x \mid \theta) = \frac{1}{\theta}, \quad 0 \leqslant x \leqslant \theta,$$

and 0 otherwise.

Recall that the maximum likelihood estimator for θ is $Y = X_{(n)}$ and it can be shown that Y has pdf $g(y) = ny^{n-1}/\theta^n$ if $0 \leqslant y \leqslant \theta$ and 0 otherwise.

(a) Derive an unbiased estimator of θ using the maximum likelihood estimator Y.

(b) Verify that $\Pr(\alpha^{1/n} \leqslant Y/\theta \leqslant 1) = 1 - \alpha$ and use this probability statement to find a $100 \cdot (1 - \alpha)\%$ confidence interval for θ.

(c) Suppose your lecturer's waiting time for the morning tram is uniformly distributed on $[0, \theta]$ and observed weighting times (in minutes) are

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3.1   8.0   8.9   9.4   3.7
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Find a 95% confidence interval for θ.

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(a) $E(Y) = \int_0^\theta y\, g(y)\, dy = \int_0^\theta n \frac{y^n}{\theta^n}\, dy = \left[\frac{n}{n+1} \frac{y^{n+1}}{\theta^n} \right]_0^\theta$

$$= \frac{n}{n+1} \frac{\theta^{n+1}}{\theta^n} - 0$$

$$= \frac{n}{n+1} \theta$$

$\Rightarrow E\left(\frac{n+1}{n} Y \right) = \theta$

$\frac{n+1}{n} Y$ is an unbiased estimator of θ

(b) $\Pr(Y \le c) = \Pr(X_{(n)} \le c) = \left(\Pr(X \le c) \right)^n$

$$= \left(\frac{c}{\theta} \right)^n$$

cdf of uniform

$\Pr\left(\alpha^{\frac{1}{n}} \le \frac{Y}{\theta} \le 1 \right) = \Pr(\theta \alpha^{\frac{1}{n}} \le Y \le \theta)$

$$= \Pr(Y \le \theta) - \Pr(Y < \theta \alpha^{\frac{1}{n}})$$

$$= 1 - \left(\frac{\theta \alpha^{\frac{1}{n}}}{\theta} \right)^n$$

$$= 1 - \alpha$$

$= \Pr(Y \le \theta \le Y \alpha^{-\frac{1}{n}}) \Rightarrow 100 \cdot (1-\alpha)\%$ CI is $(y, y\alpha^{-\frac{1}{n}})$

(c) $y = x_{(5)} = 9.4 \Rightarrow 95\%$ CI is $(y, y \times 0.05^{-\frac{1}{5}}) = (9.4, 17.1)$