THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST10007 Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 6 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- All answers should be appropriately justified.
- Some notation used in this exam:

  $\mathcal{P}_n$ denotes the (real) vector space of all polynomials of degree at most n.
  $M_{m,n}$ denotes the (real) vector space of all $m \times n$ matrices.
- There are 13 questions. You should attempt all questions.
- The total number of marks available is 100.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination

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Question 1 (8 marks)

For each of the following linear systems determine whether the system is consistent or inconsistent. If the system is consistent, give the full set of solutions.

(a)

$$\begin{array}{rcrcrcr} 2x & + & y & + & 3z & = & -4 \\ 2x & - & 2y & + & z & = & -5 \\ 2x & - & 8y & - & 2z & = & -8 \\ -4x & - & 2y & - & 5z & = & 7 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} -2x & - & y & - & 2z & = & 3 \\ 2x & + & 7y & + & 4z & = & 6 \\ 2x & + & 4y & + & 3z & = & 4 \\ -4x & - & 2y & - & 5z & = & 7 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} -4x & + & 2y & - & 11z & = & -9 \\ x & + & y & + & 5z & = & 3 \\ -3x & + & 3y & - & 6z & = & -6 \\ -x & + & y & - & 2z & = & -2 \end{array}$$

Question 2 (8 marks)

Let

$$A = \begin{bmatrix} 0 & 0 & -2 & -7 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 5 & 6 & 0 & 0 \end{bmatrix}$$

(a) Calculate the determinant of A.

(b) Determine whether or not A is invertible. If A is invertible, find its inverse A^{-1}.

Suppose that B is a matrix of size 4×4 with $\det(B) = 3$.

(c) Calculate the determinant of the matrix $5(AB^{-2})^T$.

(d) What is the reduced row-echelon form of B?

Question 3 (8 marks)

Let L be the line in $\mathbb{R}^3$ that goes through the point $(0,0,7)$ and is perpendicular to the plane given by the equation $x - 2z = 0$.

(a) Find a vector equation for the line L.

(b) Find the point of intersection the line L and the line given by the equation

$$x - 6 = \frac{y-2}{2} = \frac{z}{3}$$

(Or show that the lines do not intersect.)

(c) Find the angle between L and the plane with equation $x + y + z = 5$.

(d) Find the point on L that is closest to the point $(3, 1, -3)$.

Question 4 (6 marks)

(a) The following two matrices are row equivalent.

$$A = \begin{bmatrix} 3 & 6 & -1 & 5 & -2 & -3 \\ 1 & 2 & -7 & 15 & 6 & -7 \\ 4 & 8 & -8 & 20 & 4 & -10 \\ 4 & 8 & -1 & 6 & -3 & -4 \\ 1 & 2 & -8 & 17 & 7 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Write down a basis for the row space of A.

(ii) Write down a basis for the column space of A.

(b) Let C be an $m \times n$ matrix.

(i) Give the definition of the *solution space* of the matrix C.

(ii) Show that the solution space is a vector space.

Question 5 (6 marks)

For each of the following, decide if the set S is a subspace of the given vector space V. Justify your answers by using appropriate theorems or providing a counter-example.

(a) $V = M_{2,2}$ and $S = \left\{ A \in M_{2,2} \mid A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(b) $V = \mathcal{P}_2$ and $S = \{p(x) \in \mathcal{P}_2 \mid p(1) + p(2) + p(3) = 0\}$.

(c) $V = \mathbb{R}^3$ and $S = \{\boldsymbol{w} \in \mathbb{R}^3 \mid \boldsymbol{w} \cdot (1,1,1) = 3\}$.

Question 6 (6 marks)

Consider the subspace V of $\mathcal{P}_2$ given by

$$V = \{p \in \mathcal{P}_2 \mid p(1) = 0\}$$

(You do not need to show that it is a subspace.)

Let S be the subset of V given by

$$S = \left\{1 - x, 1 - x^2, 1 + x - 2x^2\right\}$$

(a) Determine whether or not the set S is linearly dependent. If S is linearly dependent, express one of its elements as a linear combination of the other elements.

(b) Determine whether or not the set S is a spanning set for the vector space V. If S is a spanning set, find a subset of S that is a basis for V.

Question 7 (10 marks)

(a) Show that the map $T : M_{2,2} \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + d, b + c)$$

is a linear transformation.

(b) Write down the matrix representation $[T]_{\mathcal{C},\mathcal{B}}$ of the linear transformation T, where the ordered bases are

$$\mathcal{C} = \{(1, 1), (1, -1)\} \quad \text{and} \quad \mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

for $\mathbb{R}^2$ and $M_{2,2}$ respectively.

(c) Determine the rank of T.

(d) Find a basis for the kernel of T and determine the nullity of T.

(e) State the Rank-Nullity Theorem for linear transformations. Verify that the theorem holds for the linear transformation T.

Question 8 (10 marks)
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z, x + 2y, x)$$

Consider the following two bases of $\mathbb{R}^3$

$$\mathcal{B} = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\} \quad \text{and} \quad \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

(a) Find the matrix $[T]_\mathcal{C}$ of T with respect to the basis $\mathcal{C}$.

(b) Calculate the transition matrix $P_{\mathcal{C},\mathcal{B}}$ (which converts coordinates with respect to $\mathcal{B}$ to coordinates with respect to $\mathcal{C}$).

(c) Calculate the transition matrix $P_{\mathcal{B},\mathcal{C}}$.

(d) Using the transition matrices, calculated above, find the matrix $[T]_\mathcal{B}$ of T with respect to the basis $\mathcal{B}$.

(e) Calculate the matrix $[T]_\mathcal{B}$ directly, to verify your answer from part (d).

Question 9 (6 marks)
Let $T : V \to W$ be a linear transformation and let $v_1, v_2, ..., v_k \in V$. Assume that $\{T(v_1), T(v_2), ..., T(v_k)\}$ is a basis for W and that T is injective (i.e., one-to-one). Prove that $\{v_1, v_2, \ldots, v_k\}$ is a basis for V.

Question 10 (7 marks)

(a) State the definition of an *inner product* on a real vector space V.

(b) Show that the following defines an inner product on $\mathbb{R}^2$:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 8x_2y_2$$

(c) Calculate the angle (with respect to the inner product from part (b)) between the vectors $(1, 0)$ and $(0, 1)$.

Question 11 (8 marks)
Consider the matrix

$$A = \begin{bmatrix} 6 & 0 & 16 \\ 0 & 4 & 0 \\ -1 & 0 & -4 \end{bmatrix}$$

(a) Calculate the characteristic polynomial of the matrix A.

(b) Find all the eigenvalues of the matrix A.

(c) For each eigenvalue find a basis for the corresponding eigenspace.

(d) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Question 12 (8 marks)
Let V be the subspace of $\mathbb{R}^4$ with basis

$$\{(1,7,1,7),(0,7,2,7),(1,8,1,6)\}$$

(a) Apply the Gram-Schmidt procedure to obtain an orthonormal basis for V (with respect to the dot product on $\mathbb{R}^4$).

(b) Find the projection of the vector $(7,7,9,5)$ onto V.

(c) Let $\mathcal{B}$ be the orthonormal basis found in part (a). Find a vector $\boldsymbol{u} \in \mathbb{R}^4$ such that $\mathcal{B} \cup \{\boldsymbol{u}\}$ is an orthonormal basis for $\mathbb{R}^4$.

Question 13 (9 marks)

(a) Decide whether the following matrices are diagonalisable. You should justify your answer in each case.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) Consider the conic section given by the equation

$$3x^2 + 4xy = 1$$

(i) Determine what kind of conic section is given by the above equation.
(ii) Determine the directions of the principal axes.
(iii) Sketch the curve in the x-y plane, showing the principal axes and any axis intercepts.

End of Exam—Total Available Marks = 100.