# 2014 exam (MAST20005), question 1

Let $X_1, \ldots, X_n$ be a random sample from the probability density function:

$$f(x \mid \theta) = \frac{x}{\theta^2} e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty.$$

(a) Determine a sufficient statistic for $\theta$.

(b) Write the log-likelihood function and the score function.

(c) Determine the maximum likelihood estimator of $\theta$.

(d) Give the Cramér–Rao lower bound of unbiased estimators of $\theta$.
*Hint:* If $X$ follows a Gamma$(\alpha, \beta)$ distribution with pdf $(x^{\alpha-1}e^{-x/\beta})/(\beta^\alpha \Gamma(\alpha))$, with $x, \alpha, \beta > 0$, then $\mathbb{E}(X) = \alpha\beta$.

(e) A random sample of size $n = 35$ gave $\bar{x} = 10.5$. Determine the maximum likelihood estimate of $\theta$ and an approximate 95% confidence interval for $\theta$.

The following R output may help.

```
> z <- c(0.95, 0.975, 0.99, 0.995)
> qnorm(z)
[1] 1.644854 1.959964 2.326348 2.575829
```

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(a) $$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-\frac{x_i}{\theta}}$$

$$= \frac{1}{\theta^{2n}} \left(\prod x_i\right) e^{-\frac{1}{\theta}\sum x_i}$$

$$= \left[\prod x_i\right] \left[\frac{1}{\theta^{2n}} e^{-\frac{1}{\theta}\sum x_i}\right]$$

$\Rightarrow$ $\sum x_i$ is sufficient for $\theta$, by the factorisation theorem.

(b) $$\ell(\theta) = -2n \log\theta - \frac{1}{\theta}\sum x_i + \text{const.}$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum x_i$$

(c) $$\frac{\partial \ell}{\partial \theta} = 0 \Rightarrow \frac{2n}{\theta} = \frac{1}{\theta^2}\sum x_i \Rightarrow \hat{\theta} = \frac{1}{2n}\sum x_i = \frac{1}{2}\bar{x}$$

$$\hat{\theta} = \frac{1}{2}\bar{X} \quad \text{(estimator)}$$

(d) $$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum x_i$$

$$I(\theta) = E\left(-\frac{\partial^2 \ell}{\partial \theta^2}\right) = E\left(-\frac{2n}{\theta^2} + \frac{2}{\theta^3}\sum X_i\right)$$

$$= -\frac{2n}{\theta^2} + \frac{2}{\theta^3}\sum E(X_i)$$

n.b. $X_i \sim \text{Gamma}(2, \theta) \Rightarrow E(X_i) = 2\theta$

$$= -\frac{2n}{\theta^2} + \frac{2}{\theta^3} n \times 2\theta$$

$$= \frac{2n}{\theta^2}(-1+2) = \frac{2n}{\theta^2}$$

CR-LB is $\frac{1}{I(\theta)} = \frac{\theta^2}{2n}$

(e) $$\hat{\theta} = \frac{1}{2}\bar{x} = \frac{1}{2} \times 10.5 = 5.25$$

$$se(\hat{\theta}) = \sqrt{\frac{\hat{\theta}^2}{2n}} = \sqrt{\frac{5.25^2}{2 \times 35}} = 0.628$$

95% CI (approx.): $\hat{\theta} \pm 1.96\, se(\hat{\theta}) = (4.02, 6.48)$