11. Topic 3 — Lecture 10

Slides pgs. 113–124. Problem sheet exercises, Topic 3 Q.60-62, Q.66-67.

11.1. **Cross product.** Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. The vector $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. This can be verified by showing that

$$\mathbf{w} \cdot \mathbf{u} = 0, \quad \mathbf{w} \cdot \mathbf{v} = 0.$$

The direction of $\mathbf{w}$ relative to the plane containing $\mathbf{u}$ and $\mathbf{v}$ can be predicted by the <u>right hand rule</u>: using your right hand, curl your fingers from vector $\mathbf{u}$ to $\mathbf{v}$. If you now straighten your thumb: it will point in the direction of $\mathbf{w}$.

11.2. **Area of a parallelogram.** Associated with the addition of two vectors is a parallelogram. Its area A can be written in terms of the dot product (general dimensions) according to

$$A = \Big((||\mathbf{u}|| \, ||\mathbf{v}||)^2 - (\mathbf{u} \cdot \mathbf{v})^2 \Big)^{1/2}.$$

A messy calculation can verify that in $\mathbb{R}^3$ this is also equal to

$$A = ||\mathbf{u} \times \mathbf{v}||$$

giving a geometric interpretation to the length of the cross product.
<u>Exercise</u> Compute the area of the parallelogram formed by $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, -1, 1)$ using both of the above formulas.

The cross (or dot) product rule can be used to compute the area of a triangle formed by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. The triangle has half the area of the parallelogram.

11.3. **Lines and planes.** One method of specifying lines and planes involves the use of a parameter (for lines) and multiple parameters (for planes).

A line through the origin is uniquely determined by a <u>single vector</u> $\mathbf{v}$, scaled by a parameter t.
<u>Exercise</u> Describe in words the geometrical object $t\mathbf{v} + \mathbf{v}_0$, where $t \in \mathbb{R}$. When does this geometrical object pass through the origin?

12. Topic 3 — Lecture 11

Slides pgs. 124–131. Problem sheet exercises, Topic 3 Q.69-73.

12.1. **Parametric form.** Specifying a line (or plane) by the use of a parameter(s) is said to specify the parametric form when written in terms of its components. Otherwise it is referred to as the vector form.

Example The equation $\mathbf{x} = (1, 0, -1)t + (0, 1, 0)$, $t \in \mathbb{R}$ is the vector form of a line in $\mathbb{R}^3$. This can equivalently be written in the parametric form $x = t$, $y = 1$ and $z = -t$.

12.2. **Cartesian form.** In $\mathbb{R}^2$ an example of the Cartesian form of a line is $y = 2x + 3$. It could be obtained from the parametric form $y = t$, $x = \frac{1}{2}(t - 3)$, $t \in \mathbb{R}$. Conversely, the parametric form can be obtained from the Cartesian form by considering it as linear system, and solving using the method of parameters.

In $\mathbb{R}^3$ starting with the parametric form, e.g. $x = -3t + 1$, $y = -t + 2$, $z = t$, the Cartesian form is obtained by obtaining an expression for t from each, and equating. Thus

$$t = \frac{1}{3}(1 - x), \quad t = 2 - y, \quad t = z.$$

Equating gives

$$\frac{1}{3}(1 - x) = 2 - y = z,$$

which is shorthand notation for the simultaneous equations

$$\frac{1}{3}(1 - x) = 2 - y, \quad \text{and} \quad 2 - y = z.$$

12.3. **Planes.** Two linearly independent vectors can be used to specify a plane.

As for lines, we have a vector form, e.g.

$$(x, y, z) = s(1, -1, 0) + t(2, 0, 1) + (-1, 1, 1), \qquad s, , t \in \mathbb{R}.$$

In words, this plane is in the directions of $(1, -1, 0)$ and $(2, 0, 1)$, and it passes through $(-1, 1, 1)$.

For the parametric form we simply equate components:

$$x = s + 2t - 1, \quad y = -s + 1, \quad z = t + 1.$$

To get the Cartesian form of the plane, the parameters must be eliminated. One method is to solve for s and t using two of the parametric equations, and substitute in the remaining. Thus, $t = z - 1$, $s = 1 - y$. Substituting in the first equation gives $x + y - 2z = -2$.

An alternative approach is to characterise a plane in terms of a point it passes through $\mathbf{v}_0$ and a normal $\mathbf{n}$ perpendicular to the plane. Both these vectors can be can be deduced from the vector form. For example, consider the vector form

$$\mathbf{x} = s(1, -1, 0) + t(2, 0, 1) + (-1, 1, 1), \qquad s, t \in \mathbb{R}.$$

Setting $s = t = 0$ shows us that we can take $\mathbf{v}_0 = (-1, 1, 1)$. A normal can be computed according to $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (-1, -1, 2)$. Substituting in $(\mathbf{x} - \mathbf{v}_0) \cdot \mathbf{n} = 0$ again gives $x + y - 2z = -2$.

12.4. **Linear combinations.** We have seen that $\mathbf{x} = \alpha_1 \mathbf{u} + \alpha_2 \mathbf{v}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ specifies a plane through the origin. We say that $\mathbf{x}$ is obtained by forming a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Examples

1. Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The vector $(1, 2)$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$.

2. The vector $(1, 2)$ is a linear combination of $(1, 1)$ and $(1, -1)$.

13. Topic 3 — Lecture 12

Slides pgs. 135–147. Problem sheet exercises, Topic 3 Q.76–82.

Practice question Identify a vector normal to the plane $x - 2y + 2z = 0$.

Question How can we show that a particular vector $(1, 2, 3)$ say, is not a linear combination of some given vectors $(1, -2, 2)$ and $(-1, 1, 2)$ say.

Answer We have to show that there are no scalars α, β such that

$$(1, 2, 3) = \alpha(1, -2, 2) + \beta(-1, 1, 2).$$

This is equivalent to showing that the linear system

$$\begin{aligned} \alpha - \beta &= 1 \\ -2\alpha + \beta &= 2 \\ 2\alpha + 2\beta &= 3 \end{aligned}$$

has no solution.

Task Formulate a linear systems criterion, for three vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ to be related by linear combination.

Answer There is a nonzero solution to the linear system implied by the vector equation

$$\alpha_1 \mathbf{a} + \alpha_2 \mathbf{b} + \alpha_3 \mathbf{c} = \mathbf{0}.$$

If the vectors are not related by linear combination, and thus the only solution of the linear system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we say the vectors are linearly independent.

Exercise You are told that

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 2 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Explain why the vectors $(1,1,2)$, $(1,-1,2)$ and $(3,1,6)$ are linearly dependent, and express the third as a linear combination of the first two.

Question Form a matrix A out to the column vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Suppose each vector is in $\mathbb{R}^4$, and that the rank of A is equal to 3. Are the vectors linearly independent? Give a reason.