MAST10007 Linear Algebra
Semester 1, 2014
Brief answers
(Note: you are expected to show all working and give complete explanations.)

1.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 4 \\ 1 & -2 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} \\ BA &= \text{not defined} \\ A+B &= \text{not defined} \\ A^{\mathrm{T}}A &= \begin{bmatrix} 11 & 3 \\ 3 & 6 \end{bmatrix} \\ A^{-1} &= \text{not defined} \\ B^{-1} &= \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

2. Writing down the augmented matrix and row-reducing gives:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & k-2 & 0 & 2k-4 \\ 0 & 0 & k+1 & k-1 \end{array}\right]$$

This leads to the cases:
(a) $k=-1$ (b) $k \in \mathbb{R} \setminus \{-1, 2\}$ (c) $k=2$

3. (a)

$$A^{-1} = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix}.$$

(b)

$$X = A^{-1} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 7 & 6 \\ 0 & 2 & 3 \\ -2 & 6 & 7 \end{bmatrix}.$$

4. (a) One possible vector equation for the line is

$$(x, y, z) = (1, 0, 0) + t(-\frac{1}{2}, -\frac{1}{2}, 1), \text{ where } t \in \mathbb{R}.$$

(b) (i) A vector equation is

$$\mathbf{r} = (x, y, z) = s(-1, -1, -1) + t(2, 1, 2) \text{ for } s, t \in \mathbb{R}.$$

(ii) The Cartesian equation is

$$-x + z = 0$$

(c) The area of the triangle is

$$\frac{1}{2}\|\overrightarrow{OP} \times \overrightarrow{OQ}\| = \frac{1}{2}\|(-1, 0, 1)\| = \frac{\sqrt{2}}{2}.$$

5. (a) We may calculate $\det(A)$ either via co-factor expansion, or via row reductions, or by a combination of these. Answer: $\det(A) = -8$.
   (b) Answer: $\det(C) = -4$.

6. (a) S is a subspace.
   (0) $0 \in S$ so S is non-empty.
   (1) Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in S,$$
   so $b + c = 0$ and $b' + c' = 0$. Then
$$A + B = \begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix} \in S$$
   since
$$(b + b') + (c + c') = (b + c) + (b' + c') = 0 + 0 = 0.$$
   (2) Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S \text{ and } \alpha \in \mathbb{R}.$$
   Then
$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \in S$$
   since $\alpha b + \alpha c = \alpha(b + c) = 0$.
   Therefore, by the subspace theorem, S is a subspace of V.

   (b) T is not a subspace.
   We have $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in T$,
   but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $ad + bc = 1$, so $A + B \notin T$.

7. (a)
$$\{(1, 2, -1, 3), (3, 5, -2, -2), (0, -1, 1, 1)\}.$$
   (b) The dimension of the row space is 3.
   (c) The rows of A are not linearly independent, otherwise they would form a basis for the row space of A, so the dimension of the row space would be 4.
   (d)
$$\{(1, 0, 26, 0, -3), (0, 1, -9, 0, 5), (0, 0, 0, 1, 19)\}.$$
   (e) The given vectors are the columns of A so they span the column space of A, which is a 3-dimensional subspace of $\mathbb{R}^4$. Hence they do not span $\mathbb{R}^4$.
   (f)
$$(-1, 7, -8, 96) = 26(1, 2, -1, 3) - 9(3, 5, -2, -2).$$
   (g)
$$\{(-26, 9, 1, 0, 0), (3, -5, 0, -19, 1)\}.$$

8. (a) i.

$$[S] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

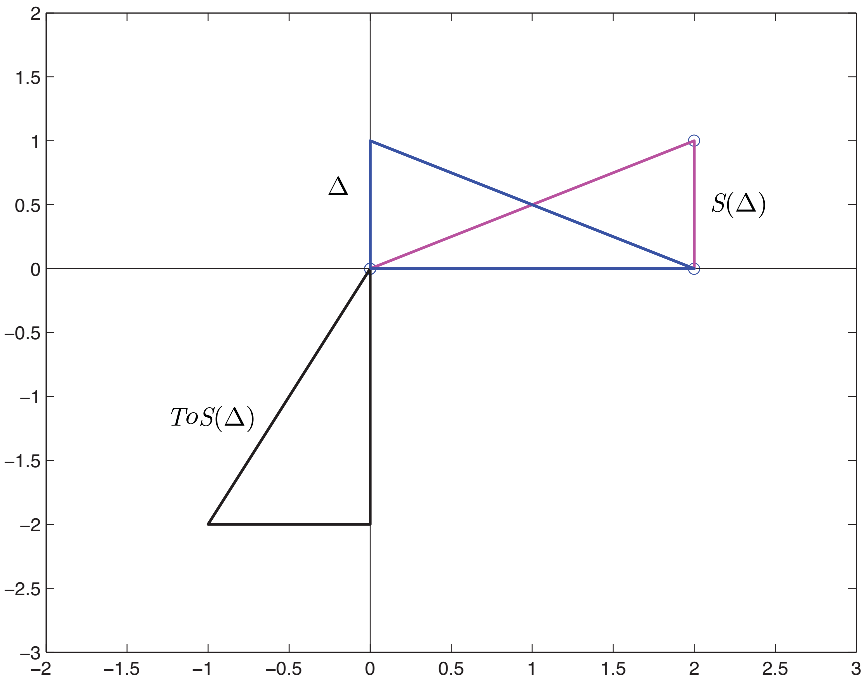
ii.

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

iii.

$$[T \circ S] = [T] \circ [S] = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

(b) Here is a sketch showing Δ, $S(\Delta)$ and $T \circ S(\Delta)$:



9. (a) i. For all $p, q \in \mathcal{P}_2$ and all $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} T(p+q) &= ((p+q)(1), (p+q)(2)) \\ &= (p(1)+q(1), p(2)+q(2)) \\ &= (p(1), p(2)) + (q(1), q(2)) = T(p) + T(q) \end{aligned}$$

and

$$T(\lambda p) = (\lambda p(1), \lambda p(2)) = \lambda(p(1), p(2)) = \lambda T(p).$$

Hence T is a linear transformation.

ii. The standard matrix for T is

$$[T] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

iii. $\operatorname{Im} T = \mathbb{R}^2$ has a basis

$$\{(1,0), (0,1)\} \text{ or } \{(1,1), (1,2)\}.$$

$\ker T$ has basis

$$\{2 - 3x + x^2\}.$$

10. (a) i. This is not symmetric. (Give a counterexample.)
    ii. This fails the positivity condition. (Give a counterexample.)

(b) i. $\langle x, x\rangle = \int_{-1}^{1} x^2\,dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{2}{3}$ so $\|x\| = \sqrt{\frac{2}{3}}$

ii. $\langle 1, x^2\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \neq 0$ so $1, x^2$ are not orthogonal.

iii. The orthonormal basis for W given by Gram-Schmidt is

$$\{\boldsymbol{u}_1 = 1/\sqrt{2}, \boldsymbol{u}_2 = \sqrt{45/8}(x^2 - 1/3)\}.$$

iv. To find $p(x)$ we must project x onto the subspace W, so

$$p(x) \quad = \quad \langle x, \boldsymbol{u}_1\rangle\boldsymbol{u}_1 + \langle x, \boldsymbol{u}_2\rangle\boldsymbol{u}_2 = 0.$$

11. (a) (i)

$$P_{S,B} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

(ii)

$$P_{B,S} = P_{S,B}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

(b) (i)

$$[T]_S = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

(ii)

$$\begin{aligned} [T]_B &= P_{B,S}[T]_S P_{S,B} \\ &= \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

(iii)

$$[\boldsymbol{v}]_B = P_{B,S}[\boldsymbol{v}]_S = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and

$$[T(\boldsymbol{v})]_B = [T]_B[\boldsymbol{v}]_B = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

12. (a) The eigenvalues are 1 and 0.1 with corresponding eigenvectors $(1, 2), (-1, 1)$.

(b)

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

(c)

$$C^n\boldsymbol{v}_0 = PD^nP^{-1}\boldsymbol{v}_0 = \frac{1}{3}\begin{bmatrix} 2 + (0.1)^n \\ 4 - (0.1)^n \end{bmatrix}.$$

(d) As $n \to \infty$,

$$C^n\boldsymbol{v}_0 \to \frac{1}{3}\begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

13. (a) A has distinct eigenvalues $1, 2, 3$ so is diagonalizable.

    B has only one eigenvalue: $\lambda = 1$, and there is only one linearly independent eigenvector. Hence B is not diagonalizable.

    C is a real symmetric matrix so is diagonalizable.

    (b) Any vector in the plane $3x - y - 2z = 0$ is left invariant by the transformation P, so this plane is an eigenspace corresponding to eigenvalue 1.

    Any vector which is a multiple of the normal to the plane, $(3, -1, -2)$, is mapped to the origin, and thus the normal line $\text{Span}\{(3, -1, -2)\}$ is an eigenspace with eigenvalue 0.

14. Assume that

$$\alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_n \boldsymbol{v}_n + \beta_1 \boldsymbol{w}_1 + \ldots + \beta_m \boldsymbol{w}_m = \mathbf{0}$$

where the α_i and β_j are scalars. Then

$$\alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_n \boldsymbol{v}_n = -\beta_1 \boldsymbol{w}_1 - \ldots - \beta_m \boldsymbol{w}_m \in V \cap W = \{\mathbf{0}\}$$

since the left hand side is in V and the right hand side is in W. Hence

$$\alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_n \boldsymbol{v}_n = \mathbf{0},$$

so each $\alpha_i = 0$ since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent. Similarly,

$$\beta_1 \boldsymbol{w}_1 + \ldots + \beta_m \boldsymbol{w}_m = \mathbf{0},$$

so each $\beta_j = 0$ since $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$ are linearly independent. Hence

$$\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$$

are linearly independent.