MAST20004 Probability
Outline answers to 2010 exam

1. (a) (i) (C1) For all $A \in \mathcal{A}$, $\mathbb{P}(A) \geq 0$; (C2) $\mathbb{P}(\Omega) = 1$; (C3) For a sequence of disjoint events $\{A_i\}$, $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.
   (ii) Let $A_1 = \Omega$, $A_i = \emptyset$ for all $i \geq 2$, we have from (C3) that
   $$\mathbb{P}(\Omega) = \mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \mathbb{P}(\Omega) + \sum_{i=2}^{\infty} \mathbb{P}(A_i) = \mathbb{P}(\Omega) + \sum_{i=2}^{\infty} \mathbb{P}(\emptyset).$$
   Hence, the only solutions to the above equation is $\mathbb{P}(\emptyset) = 0$.
   (b) (i) $\mathbb{P}(\Omega) = \frac{11}{12} \neq 1$ or $\mathbb{P}(\{B, C\}) \neq \mathbb{P}(\{B\}) + \mathbb{P}(\{C\})$.
   (ii) Put $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\{B, C\}) = \frac{2}{3}$.

2. (a) 0.224; (b) 1/7; (c) $\mathbb{P}(RP) = 0.196 \neq \mathbb{P}(RP|S)$, so not independent; (d) the test is not very useful since when an applicant is rated RS, there is only 1/7 probability that he/she is actually suitable for the job.

3. (a) $f_Y(y) = \begin{cases} 0 & y < 0, \\ \frac{1}{6y^{3/4}} & y \in [0, 1), \\ \frac{1}{12y^{3/4}} & y \in [1, 16), \\ 0 & y \geq 16. \end{cases}$
   (b) (i) Hypergeometric distribution with $N = 45$, $D = 7$ and $n = 7$; (ii) $\frac{\binom{7}{x}\binom{38}{7-x}}{\binom{45}{7}}$ for $x = 0, \ldots, 7$; (iii) $\frac{1}{45379620}$; (iv) $1 - \left(\frac{45379619}{45379620}\right)^{100000000}$; (v) 0.8896.

4. (a) $\mathbb{P}(R = 0.4) = 0.5$, $\mathbb{P}(R = 0.6) = 0.5$, $\mathbb{E}R = 0.5$, $\mathrm{V}(R) = 0.01$; (b) $\mathbb{P}(X = x) = \binom{n}{x} 0.4^x 0.6^{n-x} \frac{1}{2} + \binom{n}{x} 0.6^x 0.4^{n-x} \frac{1}{2}$; (c) $\mathbb{E}X = 0.5n$, $\mathrm{V}(X) = 0.24n + 0.01n^2$; (d) $\int_0^1 \binom{n}{x} u^x (1-u)^{n-x} du$.

5. (a) $f_{X,Y}(x, y) = \begin{cases} \frac{1}{8} & x \geq 0, y \geq 0, x + y < 4, \\ 0 & \text{otherwise}, \end{cases}$, the area is the triangle with vertices (0,0), (4,0), (0,4); (b) $f_{X,Y}(x, y) = 0$ and $f_X(x) f_Y(y) \neq 0$ for $0 \leq x, y \leq 4$ and $x + y > 4$, not indept; (c) $f_Y(y) = \frac{4-y}{8}$ for $0 \leq y \leq 4$, you are expected to check the conditions of a pdf for f_Y; (d) $x \in [0, 4 - y]$; (e) $f_{X|Y}(x, y) = \begin{cases} \frac{1}{4-y} & x \in [0, 4-y], \\ 0 & \text{otherwise}; \end{cases}$ (f) $-4/9$.

6. (a) Two values, 0 and 1; (b) $\eta(0) = 1$, $\eta(1) = 1/2$; (c) $\zeta(0) = 0$, $\zeta(1) = \frac{1}{12}$; (d) $\mathbb{E}(Z|Y) = \begin{cases} 1 & \text{with prob } 0.5, \\ 1/2 & \text{with prob } 0.5, \end{cases}$ $\mathrm{V}(Z|Y) = \begin{cases} 0 & \text{with prob } 0.5, \\ 1/12 & \text{with prob } 0.5; \end{cases}$ (e) 3/4; (f) 5/48.

7. (a) you are expected to correctly show how to get the mgf $M_Z(t) = \mathbb{E}[e^{tZ}] = e^{t^2/2}$; (b) $M_X(t) = e^{\frac{\sigma^2 t^2}{2} + \mu t}$; (c) $M_L(t) = \exp\left\{\frac{t^2}{2} \sum_{i=1}^n \sigma_i^2 a_i^2 + t \sum_{i=1}^n a_i \mu_i\right\}$, the mgf of $N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n \sigma_i^2 a_i^2\right)$.

8. (a) $M_{Z_n}(t) = \left[1 + \frac{t^2}{2n} + o(1/n)\right]^n \to e^{t^2/2}$ as $n \to \infty$, the mgf of $N(0,1)$, hence Z_n converges in distribution to $N(0,1)$ as $n \to \infty$;

   (b) 0.9544.

9. (a) $q_{n+1} = \frac{1}{12}(3 + 5q_n + 3q_n^2 + q_n^3)$, $q_0 = 0$, $q_1 = \frac{1}{4}$, $q_2 = \frac{285}{768}$;

   (b) $\sqrt{7} - 2$.