1. Consider a random experiment with state space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms prove that for events A and B,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   (c) For A, B and C events, use part (b) to derive an expression for $P(A \cup B \cup C)$ in terms of $P(A \cap B \cap C), P(A \cap B), P(A \cap C), P(B \cap C), P(A), P(B)$ and $P(C)$. (You may want to check your formula's correct by using a Venn diagram.)

[8 marks]

Solution

(a) [3 marks] The axioms are
A1: For events A, $P(A) \geq 0$,
A2: $P(\emptyset) = 0$ (or $P(\Omega) = 1$),
A3: For $A_1, A_2, \ldots$ disjoint events,
$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

(b) [3 marks] $A \cup B = A \cup (B \cap A^c)$, with the union on the right disjoint. So
$$P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c).$$

On the other hand, $B = (B \cap A) \cup (B \cap A^c)$, with the union disjoint. So
$$P(B) = P(B \cap A) + P(B \cap A^c),$$

and rearranging and substituting this last term into the first expression gives the result.

(c) [2 marks] Using part (b) repeatedly with basic properties of unions and intersections,
$$\begin{aligned} P(A \cup (B \cup C)) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C)) \\ &= P(A) + P(B) + P(C) \\ &\quad - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C). \end{aligned}$$

2. An investment bank rates the performance of stocks in each quarter as Good (G), Satisfactory (S), or Unsatisfactory (U). The historical ratings of stock performance in Q2 of the financial year given the stock's Q1 rating is encoded in the following table:

	Q2 G	Q2 S	Q2 U
Q1 G	50%	40%	10%
S	30%	40%	30%
U	10%	10%	80%

Assume that in 2012, the percentage of stocks rated G, S and U in Q1 were respectively 20%, 30% and 50%.

(a) What is the chance a randomly chosen stock will be rated G in Q2 of 2012?

(b) What is the chance a stock receiving a G rating in Q2 of 2012 was rated U in Q1?

(c) Are the events "rated U in Q1 of 2012" and "rated G in Q2 of 2012" independent?

[5 marks]

Solution

Denote $G1$ the event that a stock gets rated good in Q1, $G2$ the event that a stock gets rated good in Q2, etc.

(a) [2 marks] Using the law of total probability,

$$P(G2) = P(G2|G1)P(G1) + P(G2|S1)P(S1) + P(G2|U1)P(U1) = 0.24.$$

(b) [2 marks] Using Bayes' formula (really the definition of conditional probability),

$$P(U1|G2) = \frac{P(G2|U1)P(U1)}{P(G2)} = 5/24.$$

(c) [1 mark] No. From (a), $P(G2) = 0.24$, but from the table, $P(G2|U1) = 0.1$.

3. Let X have density

$$f_X(x) = \frac{x+1}{2}, \quad -1 < x < 1.$$

(a) Compute $P(-2 < X < 1/2)$.

(b) Find the cumulative distribution $F_Y(y)$ and probability density function $f_Y(y)$ of $Y = X^2$.

(c) Find probability density function $f_Z(z)$ of $Z = X^{1/3}$.

(d) Find the mean and variance of X.

(e) Calculate the expected value of Z by

(i) evaluating $\int_{-\infty}^{\infty} \psi(x) f_X(x) dx$ for an appropriate function $\psi(x)$,

(ii) evaluating $\int_{-\infty}^{\infty} z f_Z(z) dz$,

(iii) approximation using an appropriate formula based on Taylor series expansion of $x^{1/3}$.

(f) Calculate the variance of Z and compare it to the approximation using an appropriate formula based on Taylor series expansion of $x^{1/3}$.

[15 marks]

<u>Solution</u>

(a) [1 mark] $P(-2 < X < 1/2) = \int_{-1}^{1/2} (x+1)/2dx = 9/16$.

(b) [3 marks] For $0 < y < 1$,

$$F_Y(y) = P(X^2 \leq x) = \int_{-\sqrt{y}}^{\sqrt{y}} (1+x)/2dx = \sqrt{y}.$$

$$f_Y(y) = F'_Y(y) = \frac{1}{2} y^{-1/2}.$$

(c) [2 marks] Note that $\phi(x) = x^{1/3}$ is a one-to-one function. So for $-1 < z < 1$,

$$f_Z(z) = \left(\frac{z^3+1}{2}\right) 3z^2.$$

(d) [2 marks]

$$E[X] = (1/2) \int_{-1}^{1} x^2 + xdx = 1/3.$$

$$V(X) = (1/2) \int_{-1}^{1} x^3 + x^2 dx - (1/3)^2 = 2/9.$$

(e) [4 marks]

$$(i)\; E[X^{1/3}] = \int_{-1}^{1} x^{1/3} \frac{x+1}{2} dx = 3/7.$$

$$(ii)\; E[Z] = \int_{-1}^{1} 3z^3 \left(\frac{z^3+1}{2}\right) dz = 3/7 \approx 0.42857.$$

$$(iii)\; E[X^{1/3}] \approx E[X]^{1/3} - \frac{2}{9} E[X]^{-5/3} V(X) \approx 0.3852.$$

(f) [3 marks] $V(Z) = E[X^{2/3}] - (3/7)^2 = 102/245 \approx 0.4163$

$V(X^{1/3}) \approx (1/3E[X]^{-2/3})^2 V(X) = 2/3^{8/3} \approx 0.1068.$

4. The standard "pass" bet in craps has some of the best odds found in a casino on a simple bet. If a player wagers D dollars, then they win D dollars with probability 0.493 and lose D dollars otherwise.

   (a) If a player wagers D dollars in a pass bet, what is the mean of their winnings?

   (b) If a player wagers D dollars in a pass bet, what is the variance of their winnings?

   You go into a casino with \$100 dollars in your pocket and consider two betting strategies. You either make one craps pass bet for all \$100 or you make 100 one dollar pass bets in a row. Let Y be your winnings using the first strategy and W be your winnings using the second strategy.

   (c) What is the mean and variance of Y?

   (d) What is the mean and variance W?

   (e) Compute $P(Y > 0)$ and approximate $P(W > 0)$.

   (f) Find a number x with $-100 < x < 100$ such that $P(Y > x) \approx P(W > x)$.

[13 marks]

<u>Solution</u>

(a) [1 mark] Expected winnings are $D(.493 - .507) = -0.014D$.

(b) [1 mark] Variance of winnings is $D^2(1 - (.014)^2) = D^2(0.999804)$.

(c) [1 mark] The expected value and variance of Y are respectively (a) and (b) with $D = 100$; $E[Y] = -1.4$ and $V(Y) = 9998$.

[1 mark] for correctness of calculations in (a-c).

(d) [3 marks] $W = \sum_{i=1}^{100} X_i$, where the X_i are independent and each is ± 1 depending on the outcome of the ith bet and using parts (a) and (b) with $D = 1$, $E[X_i] = -0.14$ and $V(X_i) = 0.9804$. So

$$E[W] = -1.4, \quad V(W) = 99.98.$$

(e) [3 marks] Since $Y > 0 \iff Y = D$, $P(Y > 0) = 0.493$. On the other hand, using the CLT, W is roughly normal with mean and variance as in (d) so if Z is standard normal,

$$P(W > 0) \approx P(Z > 1.4/\sqrt{99.98}) = P(Z > .1400) \approx 0.4443.$$

(f) [3 marks] For $-100 < x < 100$, $Y > x \iff Y = D$, so $P(Y > x) = 0.493$. Similar to (e),

$$P(W > x) \approx P(Z > (x + 1.4)/\sqrt{99.98}),$$

and since $P(Z > .02) \approx 0.493$, we need

$$x = .02\sqrt{99.98} - 1.4 \approx -1.2.$$

5. The density of (X, Y) is given by

$$f(x,y) = Cx^2y^2,\ 0 < y < 1,\ -y < x < y.$$

(a) What is the constant C?

(b) What is the density of Y?

(c) What is the mean of Y?

(d) What is the variance of Y?

(e) What is $P(X^2 \geq 1/4|Y \leq 3/4)$?

(f) What is the density of X given $Y = y$?

(g) Assuming that $E[X] = 0$, what is the covariance of X and Y?

(h) Are X and Y independent (and why)?

(i) What is $P(X > Y^2)$?

(j) What is $E\left[\frac{X^2}{Y}\right]$?

[18 marks]

Solution

(a) [2 marks] Since densities integrate to one,

$$1 = \int_0^1 \int_{-y}^{y} Cx^2y^2 dxdy = \frac{C}{9},$$

so $C = 9$.

(b) [2 marks]

$$f_Y(y) = 9\int_{-y}^{y} x^2y^2 dx = 6y^5, \quad 0 < y < 1.$$

(c) [1 mark] $E[Y] = 6/7$.

(d) [1 mark] $V(Y) = 6/8 - (6/7)^2 = 3/196 = 0.0153$

(e) [3 marks]

$$\begin{aligned} P(X^2 \geq 1/4|Y \leq 3/4) &= \frac{\int_{1/2}^{3/4}\int_{1/2}^{y} 9x^2y^2dxdy + \int_{1/2}^{3/4}\int_{-y}^{-1/2} 9x^2y^2dxdy}{\int_0^{3/4} 6y^5dy} \\ &= \frac{2\int_{1/2}^{3/4}\int_{1/2}^{y} 9x^2y^2dxdy}{\int_0^{3/4} 6y^5dy} \\ &= \frac{361}{729} \approx 0.4952. \end{aligned}$$

(f) [2 marks] $f_{X|Y}(x|y) = 9x^2y^2/(6y^5) = 3x^2/(2y^3)$, for $0 < y < 1, -y < x < y$.

(g) [2 marks] $Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY] = 0$ since by symmetry or calculation

$$E[XY] = \int_0^1 \int_{-y}^{y} 9x^3y^3 dxdy = 0.$$

(h) [1 mark] X and Y are not independent since the support of (X, Y) is not rectangular; i.e., information about Y affects the distribution of X (also fine to show directly the joint density does not factorize).

(i) [2 marks] $P(X > Y^2) = \int_0^1 \int_{y^2}^{y} 9x^2y^2 dxdy = 1/6$.

(j) [2 marks] $E\left[\frac{X^2}{Y}\right] = \int_0^1 \int_{-y}^{y} 9x^4 y dxdy = 18/35 \approx 0.514$.

6. For fixed $n \geq 1$, let X be the number of heads in n independent tosses of a fair coin and let Y be the number of tails in the same n tosses.

   (a) What is the covariance and correlation of X and Y?

   Assume that X is the number of heads and Y is the number of tails in N independent tosses of a fair coin, where now $N \geq 1$ is an integer valued random variable having mean μ and variance σ^2. Find expressions in terms of μ and σ^2 for

   (b) the expected value of both X and Y.

   (c) the variance of both X and Y.

   (d) the covariance of X and Y.

   (e) the correlation of X and Y.

[12 marks]

__Solution__

(a) [3 marks] $Y = n - X$, so $Cov(X, Y) = -V(X) = -n/4$. $Corr(X, Y) = -1$.

(b) [2 marks] $E[X] = E\{E[X|N]\} = EN/2 = \mu/2$.

(c) [3 marks]

$$V(X) = V(E[X|N]) + E[V(X|N)] = \frac{\sigma^2 + \mu}{4}.$$

And the same formulas hold with Y replacing X since $X \stackrel{d}{=} Y$.

(d) [3 marks] Since $Y = N - X$:

$$Cov(X, Y) = Cov(X, N) - Cov(X, X) = Cov(X, N) - (\sigma^2 + \mu)/4.$$

To compute $Cov(X, N) = E[XN] - \mu^2/2$, we condition:

$$E[XN] = E\{NE[X|N]\} = E[N^2]/2 = (\sigma^2 + \mu^2)/2,$$

and so $Cov(X, N) = \sigma^2/2$, and putting everything together we get

$$Cov(X, Y) = \frac{\sigma^2 - \mu}{4}.$$

(e) [1 mark]

$$Corr(X, Y) = \frac{(d)}{(c)} = \frac{\sigma^2 - \mu}{\sigma^2 + \mu^2}.$$

7. Let X have density xe^{-x}, $x > 0$ and U be uniform on the interval $(0, 1)$, independent of X. Let $Y = UX$.

   (a) What is the density of Y given $X = x$? (No calculation should be necessary.)

   (b) What is the joint density of (X, Y)? Verify that your answer integrates to one.

   (c) Hence or otherwise, find the marginal density of Y and identify the distribution by name.

   (d) What is the density of $X - UX$? (No further calculations are required.)

[11 marks]

Solution

(a) [2 marks] Since U and X are independent, given $X = x$, the distribution of U is unchanged and so $UX|X = x$ is uniform on $(0, x)$. [Specifying the pdf or cdf here is also a fine answer.]

(b) [4 marks] The joint density g is given by

$$g(t, x) = f_{UX|X}(t|x) f_X(x) = \frac{1}{x} x e^{-x} = e^{-x}, \quad 0 < t < x.$$

We check

$$\int_0^\infty \int_t^\infty e^{-x} dx dt = 1.$$

(c) [3 marks] Integrating the joint density in (b), we find the marginal density of Y is for $t > 0$

$$\int_t^\infty e^{-x} dx = e^{-t},$$

which is the density of the exponential distribution with rate one.

(d) [2 marks] $X - UX = (1 - U)X$, and since $1 - U$ is uniform on $(0, 1)$ and independent of X, the density is the same as (a).

8. Let N be geometric with parameter $0 < p < 1$; that is

$$P(N = n) = (1 - p)^n p, \quad n = 0, 1, 2, \ldots$$

and has probability generating function

$$P_N(s) = \frac{p}{1 - s(1 - p)}.$$

Let X be such that the density of $X|N = n$ is given by

$$f_{X|N}(x|n) = \frac{x^n e^{-x}}{n!}, \quad x > 0.$$

(a) Derive the conditional moment generating function of $X|N$.

(b) Use the answer to part (a) to derive the moment generating function of X and identify the distribution by name.

(c) Use the moment generating function in part (b) to derive a formula for the moments of X.

(d) Find the conditional probability mass function of N given $X = x$ and identify this distribution by name.

[11 marks]

<u>Solution</u>

(a) [3 marks] $X|N$ is gamma $N + 1$ and so has mgf

$$M_{X|N}(t) = (1 - t)^{-(N+1)},$$

defined for $x < 1$.

(b) [2 marks] $M_X(t) = E[M_{X|N}(t)] = P_{N+1}((1-t)^{-1})$, where P_{N+1} is the probability generating function of $N + 1$, a geometric variable started from one. And so

$$M_X(t) = \frac{p}{p - t},$$

for $t < p$, which means X is exponential rate p.

(c) [3 marks] $M_x^{(k)}(t) = pk!/(p - t)^{k+1}$ and so

$$E[X^k] = M_X^{(k)}(0) = \frac{k!}{p^k}.$$

(d) [3 marks] Using the information above and formulas derived therein, for $x > 0$ and $n = 0, 1, 2, \ldots,$

$$P(N = n|X = x) = \frac{f_{X|N}(x|n)P(N = n)}{f_X(x)} = \frac{x^n e^{-x}(1 - p)^n p}{pe^{-px}n!} = \frac{(x(1 - p))^n e^{-x(1-p)}}{n!},$$

and so $N|X = x$ is Poisson with mean $(1 - p)x$.

9. The chance that it rains in Melbourne given that it rained yesterday is 3/4. If it didn't rain yesterday, then the chance it rains today is 3/5. If we set $X_n = 0$ if it doesn't rain n days from now and $X_n = 1$ if it does rain n days from now, then $(X_n)_{n\geq 0}$ is a Markov chain.

   (a) Write down the transition matrix for the Markov chain.

   (b) If it didn't rain today, what is the chance it doesn't rain 3 days from now?

   (c) What is the equilibrium chance that it rains in Melbourne?

[7 marks]

Solution

(a) [2 marks]

$$P = \begin{bmatrix} 2/5 & 3/5 \\ 1/4 & 3/4 \end{bmatrix}.$$

(b) [2 mark] $(P^3)_{1,1} = 593/2000 = 0.2965$.

(c) [3 marks] The equilibrium distribution π solves $\pi P = \pi$; so

$$\pi = [5/17, 12/17],$$

and the equilibrium chance it rains in Melbourne is 12/17.