17. Topic 3 — Lecture 16

Slides: pgs. 175–185. Exercises: Topic 3, Q.91–95

Exercise (a) It is true that $\text{Span}\{(1,1),(1,-1),(0,1)\} = \mathbb{R}^2$. What does this mean?

(b) What has to be checked to verify this statement?

Solution sets as subspaces

The linear system $A\mathbf{v} = \mathbf{0}$ is referred to as being homogeneous, the reason being that the RHS is the zero vector.

We can check that the solution space of a homogeneous linear system satisfies the axioms defining a subspace:

(0) $\mathbf{0}$ is in the solution space.

(1) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the solution space, then $\mathbf{v}_1 + \mathbf{v}_2$ is in the solution space.

(2) If $\mathbf{v}_0$ is in the solution space, and $\alpha \in \mathbb{R}$ is a general scalar, then $\alpha\mathbf{v}_0$ is in the solution space.

The above result is not too surprising when we recall that the method of parameters allows us to write the solution space as a span.

Question Suppose A is 3×3. Relate the number of leading entries in the RE form of A to the allowed subspaces of $\mathbb{R}^3$.

Basis

For a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ to be a basis for the subspace V, it is required that

(1) $\text{Span}\,\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = V$.

(2) The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are all linearly independent.

The number of vectors in the basis is called the dimension.

18. Topic 3 — Lecture 17

Slides pgs. 187–202 Exercises: Topic 3, Q.96–102

True or false:

(1) $\text{Span}\,\{(1,0,-1),(\frac{1}{2},0,-\frac{1}{2})\}$ is a plane through the origin.

(2) $\text{Span}\,\{(1,0,-1),(\frac{1}{2},0,\frac{1}{2})\}$ is a plane through the origin.

(3) All spans are subspaces.

(4) All planes through the origin in $\mathbb{R}^3$ can be written as the span of just two vectors in $\mathbb{R}^3$.

Question Given a span of some vectors, how do we find a basis?

Answer Form a matrix with the vectors as columns, and choose the vectors corresponding to the columns with leading entries in RE form. It is also possible to use the information in the RE form to express the vectors from the span which are not in the basis as a linear combination of the basis vectors.

Row and column spaces

Given a matrix A, the span of the vectors form by the columns is called the column space.

The span of the vectors formed by the rows is called the row space.

The method of determining a basis for the column space is to choose the vectors corresponding to the columns with leading entries in RE form, just as we did in choosing a basis for a span.

In the case of a row space, a basis is given by the rows of the RE matrix corresponding to the leading entries.

Solution space

The solution space is the set of solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We have

$$\dim \text{column space} + \dim \text{solution space} = \text{number of columns}$$

The dimension of the column space is just the rank. The dimension of the solution space is sometimes called the nullity. Thus

$$\text{rank} + \text{nullity} = \text{number of columns}.$$

19. Topic 3 — Lecture 18

Slides pgs. 203–210, Exercises: Topic 3, Q.103.

Let's first discuss a further property of a basis. We know that one property of a basis is that it spans the corresponding set. For example, the fact that $\{(1,1,1),(1,1,0),(1,0,0)\}$ is a basis for $\mathbb{R}^3$ tells us that for any $(x,y,z) \in \mathbb{R}^3$ we can find scalars α, β, γ such that

$$\alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0) = (x,y,z). \tag{(A)}$$

Question Is the choice of α, β, γ unique?

Answer Suppose there were different scalars α', β', γ' say such that

$$\alpha'(1,1,1) + \beta'(1,1,0) + \gamma'(1,0,0) = (x,y,z).$$

Subtracting (A) gives

$$(\alpha - \alpha')(1,1,1) + (\beta - \beta')(1,1,0) + (\gamma - \gamma')(1,0,0) = (0,0,0).$$

By linear independence of the three vectors, the only solution of this vector equation is $\alpha - \alpha' = 0$, $\beta - \beta' = 0$ and $\gamma - \gamma' = 0$, so we conclude that there are no different scalars.

Coordinate vector

By definition

$$(2,3) = 2\mathbf{e}_1 + 3\mathbf{e}_2,$$

where $\mathbf{e}_1$ is the unit vector in the x-direction, and $\mathbf{e}_2$ is the unit vector in the y-direction.

Let's consider a different basis of $\mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ say, with $\mathbf{b}_1 = (1,-1)$ and $\mathbf{b}_2 = (1,1)$. Now we have

$$(2,3) = -\frac{1}{2}(1,-1) + \frac{5}{2}(1,1).$$

This is sometimes written

$$[(2,3)]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}.$$

In words this says the coordinates of $(2,3)$ with respect to the basis $\mathcal{B}$ are $\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$.

Matrices and polynomials regarded as vector spaces

The essential properties of vectors in $\mathbf{R}^n$ are that they are closed under the operations of vector addition and scalar multiplication.

There are other mathematical sets that share these closure properties, with the usual rule for addition and scalar multiplication:

1. The set of all $p \times q$ matrices, denoted $\mathcal{M}_{p,q}$.
2. The set of all polynomials of degree less that or equal to n, denoted $\mathcal{P}_n$.

Actually, these new vector spaces are equivalent to

1. The Euclidean vector space $\mathbb{R}^{p+q}$.
2. The Euclidean vector space $\mathbb{R}^{n+1}$.

For example

1. $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \leftrightarrow (a,b,c,d,e,f)$
2. $a + bx + cx^2 + dx^3 \leftrightarrow (a,b,c,d)$.

By first making this correspondence, questions relating to linear dependence, basis, subspaces etc. can be answered in the usual way.