MAST20004 Probability
Outline answers to 2011 exam

1. (a) (i) (C1) for any event A, $\mathbb{P}(A) \geq 0$; (C2) $\mathbb{P}(\Omega) = 1$; (C3) For any sequence of disjoint events $\{A_n\}$, $\mathbb{P}\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.
   (ii) $D = C \cup (C^c \cap D) \cup \emptyset \cup \emptyset \cup \ldots$, by (C3),
   $$\mathbb{P}(D) = \mathbb{P}\left(C \cup (C^c \cap D) \cup \emptyset \cup \emptyset \cup \ldots\right) = \mathbb{P}(C) + \mathbb{P}(C^c \cap D) + \mathbb{P}(\emptyset) + \ldots$$
   and it follows from (C1) that $\mathbb{P}(D) \geq \mathbb{P}(C)$.
   (b) (i) 4/5; (ii) 3/10; (iii) 1/10; (iv) 9/10; (v) 9/10; (vi) positively related.

2. (a) $\mathbb{P}(A1|B^c) = 1/9$, $\mathbb{P}(A2|B^c) = 4/9$, $\mathbb{P}(A3|B^c) = 4/9$; (b) 77/1800; (c) 5/77.

3. (a) (i) $F_Y(y) = \begin{cases} 0 & y < 0, \\ \frac{2\sqrt{y}}{5} & 0 \leq y \leq 4, \\ \frac{\sqrt{y}+2}{5} & 4 < y \leq 9, \\ 1 & y > 9; \end{cases}$ $f_Y(y) = \begin{cases} \frac{1}{5\sqrt{y}} & 0 \leq y \leq 4, \\ \frac{1}{10\sqrt{y}} & 4 < y \leq 9, \\ 0 & \text{otherwise;} \end{cases}$ (ii) 0, 4, 9; (iii) You are expected to correctly integrate f_Y to get 1.
   (b) (i) Negative binomial with $r = 3$ and $p = 18/37$; (ii) $p_N(n) = \binom{-3}{n} p^3 (p-1)^n$, $n = 0, 1, \ldots$; (iii) 3.1666; (iv) -0.8333; (v) 162.5.

4. (a) 3/2, 1/12; (b) $(0, \log 2)$; (c) $f_Y(y) = \begin{cases} e^y & y \in (0, \log 2), \\ 0 & \text{otherwise;} \end{cases}$ (d) (i), (ii) you are expected to demonstrate how to get the same answer $2\log 2 - 1 = 0.3863$ using the two different methods; (e) 0.3869, a good approximation to the exact value.

5. (a) $k = 24$; (b) $f_{X,Y}(x, y) = 0$ while $f_X(x)f_Y(y) \neq 0$ for $0 \leq x, y \leq 1$ and $x + y > 1$, so not independent; (c) $f_Y(y) = \begin{cases} 12y(1-y)^2 & 0 \leq y \leq 1, \\ 0 & \text{otherwise;} \end{cases}$ (d) $\mathbb{E}(X) = \mathbb{E}(Y) = 2/5$; (e) 1/15; (f) $f_{Y|X}(y|x = 1/4) = \begin{cases} \frac{32y}{9} & 0 \leq y \leq 3/4, \\ 0 & \text{otherwise.} \end{cases}$

6. (a) 80; (b) 1000; (c) $\exp\left\{-8(1 - e^{10t + 25t^2/2})\right\}$.

7. (a) $M_X(t) = \mathbb{E}[e^{tX}]$ is defined for $t < 1/2$; (b) you are expected to make a suitable substitution and work out the integral to get $M_X(t) = \frac{1}{\sqrt{1-2t}}$; (c) $\mathbb{E}(X) = 1$, $\mathrm{V}(X) = 2$.

8. (a) $M_{Z_n}(t) = \cdots = [1 + \mu t/n + o(1/n)]^n \to e^{\mu t}$ as $n \to \infty$, so Z_n converges in distribution to a rv that takes the constant value μ with probability 1.
   (b) (i) $\mathbb{E}(X_i) = 0.4$, $\mathrm{V}(X_i) = 0.24$; (ii) 0.9876; (iii) $\mathrm{Bi}(600, 0.4)$; (iv) Y_n can be regarded as the sum of n iid rv's and the CLT says that Y_n can be approximated by a normal rv with the same mean and the same variance, so $Y_n \stackrel{d}{\approx} N(np, np(1-p))$ for n large.

9. (a) $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$; (b) $P^3 = \begin{bmatrix} 0.445 & 0.555 \\ 0.444 & 0.556 \end{bmatrix}$, so the probability asked is $p_{11} = 0.556$; (c) $(\pi_0, \pi_1) = (4/9, 5/9)$, so the equilibrium probability that the apartment is occupied is 5/9; (d) \$43,333.