The University of Melbourne
Department of Mathematics and Statistics
Summer Semester Exam 2011

MAST10007 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 6 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

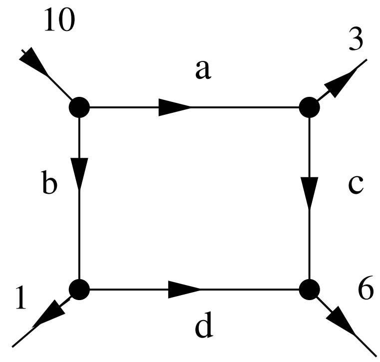
Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 12 questions. The total number of marks is 80. All questions may be attempted.

This paper may be held by the Baillieu Library.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) You are given the following flow diagram.



The total flow into each vertex must equal the total flow out.

i. Write down the four simultaneous equations corresponding to the flows.

ii. Show that the equations in (i) have an infinite number of solutions (it is **not** required that you specify the solution set).

(b) Suppose the row echelon form of an augmented matrix for a linear system, in unknowns α, β and γ, is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i. Reduce this to fully reduced row echelon form.

ii. Find the general solution of the linear system.

[7 marks]

2. (a) Let A and B be matrices. For $AB - BA$ to be well defined, show that A and B must be square matrices of the same size.

(b) Let

$$X = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad Y = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Calculate, if possible

i. YZX

ii. Y^TY

(c) Suppose P is a square matrix such that $P^2 = P$. Find the possible values of $\det P$, and in the case $\det P = 1$, specify P.

[7 marks]

3. (a) Consider the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Give a reason why M is invertible, then use an algorithm based on fully reduced row echelon form to compute M^{-1}.

(b) Use your answer to (a) to find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(c) Let X be a 3×3 matrix and suppose $\det X = 5$. Let Y be obtained from X by multiplication of the first row by 2, and interchange of the second and third rows. Calculate $\det Y$.

[7 marks]

4. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

i. Write down a formula for $\mathbf{u} \times \mathbf{v}$ in terms of a determinant.

ii. Write down a formula for $||\mathbf{u} \times \mathbf{v}||$ in terms of $||\mathbf{u}||$, $||\mathbf{v}||$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

iii. From your formula in (ii), show that $||\mathbf{u} \times \mathbf{v}||$ gives the area of the parallelogram corresponding to the vectors $\mathbf{u}$ and $\mathbf{v}$ [hint: draw the parallelogram with $||\mathbf{u}||$ as the base].

(b) Find the volume of the parallelpiped defined by the three vectors $\mathbf{i}$, $\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

[6 marks]

5. (a) Subspaces of $\mathbb{R}^3$ can have what dimensions? For each possibility, describe the subspace geometrically (i.e. in terms of lines etc).

(b) Let the set of symmetric matrices in $M_{2,2}$ be denoted S so that

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix}, \quad a, b, d \in \mathbb{R} \right\}.$$

Write S as an equivalent set in $\mathbb{R}^4$. Then write the set as a span. Why can we then conclude that this equivalent set in $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$?

(c) Show from first principles that $\{(x, y, z) : x + 2y + z = 0\}$ is closed under vector addition.

[7 marks]

6. Let A be a 4×5 matrix with columns given by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_5$, and let V be a 4×4 matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_4$. Suppose that

$$[A|V] \sim \left[\begin{array}{ccccc|cccc} 1 & 0 & -1 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(a) Which of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ belong to the column space of A?

(b) Give two sets of vectors from $\mathbf{a}_1, \ldots, \mathbf{a}_5$ that form a basis for the column space of A, and give a reason in each case. What is the dimension of the column space?

(c) Express $\mathbf{v}_2$ as a linear combination of $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$.

(d) Compute a basis for the solution space of A.

[8 marks]

7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that $T(1, 0) = (1, 0)$ and $T(0, 1) = (2, 1)$.

i. Write down the standard matrix A_T of T.

ii. Illustrate on a diagram how T maps the unit square.

iii. Use your diagram to explain why $\det A_T = 1$.

(b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which reflects in the x-axis.

i. Calculate the standard matrix A_S of S.

ii. State the one-dimensional subspaces of $\mathbb{R}^2$ that are left unchanged by the action of S.

iii. Verify that the vectors corresponding to the direction of the lines you found in (ii) are eigenvectors of A_S.

[6 marks]

8. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x, y, z) = (x, y, 0)$.
   i. Give a basis for the image of T and state its dimension.
   ii. Give a basis for the kernel of T and state its dimension.
   iii. Describe T geometrically.

   (b) i. You are given that the change of basis matrix $P_{\mathcal{B},\mathcal{S}}$ from the standard basis to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is

$$P_{\mathcal{B},\mathcal{S}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

   Express the vector $\mathbf{x} = (3, 2, 1)$ as a linear combination of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$.
   ii. Determine the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ in (i) by first computing $P_{\mathcal{S},\mathcal{B}}$.
   iii. Give a reason why the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   cannot be a change of basis matrix.

[7 marks]

9. (a) i. Use the Gram-Schmidt procedure with the dot product to deduce the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\} = \{\frac{1}{\sqrt{2}}(1, 0, 1, 0), \frac{1}{2}(1, 1, -1, 1)\}$ for the subspace W of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 1, 0)$ and $(3, 1, 1, 1)$.
   ii. Find the orthogonal projection of the vector $\mathbf{u} = (0, 0, 1, 1)$ onto the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ as found in (i).

   (b) It is known that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = 4x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

   is an inner product for vectors in $\mathbf{R}^2$.
   i. Decide whether the vectors $(1, 0)$ and $(0, 1)$ are orthogonal using this inner product.
   ii. Calculate the norm of $(3, 4)$ with respect to this inner product.
   iii. State which of the four axioms defining an inner product implies that the eigenvalues of

$$\begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix}$$

   must be positive.

[6 marks]

10. A small square paddock of area 20×20 square metres is given xy-coordinates so that its centre is at $(0,0)$, its bottom left corner at $(-10,-10)$ and its top right corner at $(10,10)$. It is desired to dig a narrow straight trench across the paddock so that it passes as close as possible to the points specified by the data

x	y
-4	-2
1	1
3	2

(a) Use the method of least squares to find the equation of the trench.

(b) What is the y-coordinate, to the nearest metre, of the trench at the right hand boundary $x = 10$ of the paddock?

[6 marks]

11. Let

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding normalized eigenvectors of A.

(b) Verify that the eigenvectors are orthogonal.

(c) Give a geometrical description of the linear transformation corresponding to A.

(d) Write the vector $(2,4)$ as a linear combination of the vectors $(3,1)$ and $(-1,3)$. From this calculate

$$A^{10} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(there is no need to simplify the arithmetic).

[7 marks]

12. For each of the following, write in your exam book true if the statement is true, and false if the statement is false.

(a) The solution space of a 4×6 matrix is a subspace of $\mathbb{R}^4$.

(b) The orthogonal projection $\mathbf{p}$ of a vector $\mathbf{v}$ onto a subspace W is the closest point in W to $\mathbf{v}$.

(c) The plane with vector equation $(x,y,z) = t(2,1,0) + s(-1,1,2) + (1,2,2)$ does not pass through the origin.

(d) A matrix A with distinct real eigenvalues can always be diagonalized.

(e) Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for $\mathbb{R}^3$, and let $\mathcal{S}$ be the standard basis. With U denoting the matrix having $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ as its columns, $P_{\mathcal{U},\mathcal{S}} = U^T$.

(f) Let A be an $N \times N$ matrix, and suppose the dimension of the solution space is 0. Then A is singular.

[6 marks]