1. Consider a random experiment with state space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms, prove that for an event A,
$$P(A^c) = 1 - P(A).$$

   (c) Using the axioms, prove that for events $A \subseteq B$,
$$P(A) \leq P(B).$$

[9 marks]

<u>**Solution**</u> (a) [3 marks] The axioms are
A1: For events A, $P(A) \geq 0$,
A2: $P(\Omega) = 1$,
A3: For $A_1, A_2, \ldots$ disjoint events,
$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

(b) [3 marks] $A^c \cap A = \emptyset$ and $A^c \cup A = \Omega$ and so A3 and A2 imply
$$1 = P(\Omega) = P(A^c \cup A) = P(A^c) + P(A).$$

(c) [3 marks] If $A \subseteq B$, then $B \cap A = A$ and so $B = (A \cap B) \cup (B \cap A^c) = A \cup (B \cap A^c)$ and this unions is disjoint. Thus A3 and A1 imply
$$P(B) = P(A) + P(B \cap A^c) \geq P(A).$$

2. A bag has three dice in it, one is fair and the other two are weighted so the chances of a six coming up are 7/36 and 8/36, respectively. You choose a die at random from the bag and roll it.

   (a) What is the chance you roll a six?

   (b) Given you roll a six, what is the chance you rolled the fair die?

   (c) Given you roll a six, what is the chance that if you roll the same die again you get a six?

[8 marks]

<u>**Solution**</u>

Let B_1 be the even that the fair die is chosen, B_2, B_3 the events that the 7/36, 8/36 weighted die is chosen, and A be the event you roll a six.

(a) [3 marks] The law of total probability and the definition of conditional expectation imply
$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \\ &= P(A|B_1)P(B_1) + P(A|B_1)P(B_1) + P(A|B_1)P(B_1) = 7/36 = 0.194. \end{aligned}$$

(b) [2 marks] The definition of conditional probability implies

$$P(B_1|A) = P(A|B_1)P(B_1)/P(A) = 2/7.$$

(c) [3 marks] Let C be the event the second roll is a six. Similar to the workings above,

$$P(C|A) = \left(\sum_{i=1}^{3} P(C \cap A|B_i)P(B_i)\right)/P(A) = \frac{6^2 + 7^2 + 8^2}{21 \cdot 36} = 149/(21 \cdot 36) = 0.197.$$

3. Let X have density

$$f_X(x) = \frac{C}{\sqrt{x}}, \quad 0 < x < 1.$$

(a) What is the value of C?

(b) Compute $P(1/4 < X < 25/16)$.

(c) Find the cumulative distribution $F_Y(y)$ and probability density function $f_Y(y)$ of $Y = X(1 - X)$.

(d) Find probability density function $f_Z(z)$ of $Z = X^{1/4}$.

(e) Find the mean and variance of X.

(f) Calculate the expected value of Z by

(i) evaluating $\int_{-\infty}^{\infty} \psi(x) f_X(x) dx$ for an appropriate function $\psi(x)$,

(ii) evaluating $\int_{-\infty}^{\infty} z f_Z(z) dz$,

(iii) approximation using an appropriate formula based on Taylor series expansion of $x^{1/4}$.

(g) Calculate the variance of Z and compare it to the approximation using an appropriate formula based on Taylor series expansion of $x^{1/4}$.

[19 marks]

Solution

(a) [2 marks] $1 = \int_0^1 Cx^{-1/2} dx = 2C$ implies $C = 1/2$.

(b) [2 marks] $P(1/4 < X < 25/16) = P(1/4 < X < 1) = 1/2$.

(c) [4 marks] For $0 < y < 1/4$,

$$\begin{aligned} P(Y \leq y) &= P(0 < X < (1 - \sqrt{1-4y})/2) + P((1 + \sqrt{1-4y})/2) < X < 1) \\ &= \sqrt{(1 - \sqrt{1-4y})/2} + 1 - \sqrt{(1 + \sqrt{1-4y})/2}. \end{aligned}$$

And differentiating, we find the density on the same range to be

$$f_Y(y) = \frac{1}{\sqrt{2(1-4y)}} \left[\left(1 - \sqrt{1-4y}\right)^{-1/2} + \left(1 + \sqrt{1-4y}\right)^{-1/2} \right].$$

(d) [2 marks] For $0 < z < 1$, $P(Z \leq z) = P(X \leq z^4) = z^2$; $f_Z(z) = 2z$ on $0 < z < 1$.

(e) [2 marks] $E[X] = \int_0^1 \sqrt{x}/2 dx = 1/3$; $E[X^2] = \int_0^1 x^{3/2}/2 dx = 1/5$: $Var(X) = 4/45$.

(f) [4 marks] (i) $E[Z] = \int_0^1 x^{1/4}/(2\sqrt{x}) dx = 2/3$.

(ii) $E[Z] = \int_0^1 2z^2 dz = 2/3$.

(iii) For μ, σ^2 the mean and variance of X, $E[\psi(X)] \approx \psi(\mu) + \psi''(\mu)\sigma^2/2$ and setting $\psi(x) = x^{1/4}$, we find $\psi(\mu) = 3^{-1/4}$ and $\psi''(\mu) = -(3/16)3^{7/4}$ and so our estimate is

$$E[Z] \approx 3^{-1/4} - (3/32)3^{7/4}(4/45) = 0.7028.$$

(g) [3 marks] $E[Z^2] = 1/2$ and so $Var(Z) = 1/2 - 4/9 = 1/18 = 0.555$. For μ, σ^2 the mean and variance of X, $Var(\psi(X)) \approx \psi'(\mu)^2 Var(X)$ and so

$$Var(Z) \approx (3^{3/2}/16)(4/45) = 0.02887.$$

4. The amount of time T (in hours) that a certain electrical component takes to fail has an exponential distribution with parameter $\lambda > 0$. The component is found to be working at midnight on a certain day. Let N be the number of *full* days after this time before the component fails (so if the component fails before midnight the next day, $N = 0$).

   (a) What is the probability that the component lasts at least 24 hours?

   (b) Find the probability mass function of N.

   (c) Identify the distribution of N by name.

   (d) Given that the component is found to have failed on a certain day, what is the distribution function of the number of hours H past midnight that the component failed?

[7 marks]

Solution

(a) [2 marks] $P(T > 24) = e^{-24\lambda}$.

(b) [2 marks] For $k = 0, 1, \ldots$,

$$P(N = k) = P(24k \le T < 24(k+1)) = e^{-24k\lambda} - e^{-24(k+1)\lambda}.$$

(c) [1 mark] Rewriting $P(N = k) = (e^{-24\lambda})^k(1 - e^{-24\lambda})$, we see that N is geometric with parameter $1 - e^{-24\lambda}$.

(d) [2 marks] For any non-negative integer k, and $0 < t < 24$,

$$P(T < 24k + t | 24k \le T < 24(k+1)) = \frac{P(24k \le T < 24k + t)}{P(24k \le T < 24(k+1))} = \frac{1 - e^{-\lambda t}}{1 - e^{-24\lambda}}.$$

Since this term does not depend on k, this is the distribution function of H.

5. The density of (X, Y) is given by

$$f(x, y) = Ce^{-x},\ 0 < y < x.$$

   (a) What is the constant C?

   (b) What is $P(X < 2 - Y)$?

   (c) What is the density of Y?

   (d) What is the density of X given $Y = y$?

   (e) What is $P(X^2 \ge 1/4 | Y \le 3/4)$?

(f) Find $E[X]$ and $E[Y]$.

(g) What is the covariance of X and Y?

(h) Are X and Y independent (and why)?

(i) What is $E\left[\frac{Y}{X}\right]$?

(j) Find the distribution function of Y/X and name the distribution.

[18 marks]

Solution

(a) [1 mark] $1 = \int_0^\infty \int_0^x Ce^{-x}dydx = C$. So $C = 1$.

(b) [2 marks] $P(X < 2 - Y) = \int_0^1 \int_0^x e^{-x}dydx + \int_1^2 \int_0^{2-x} e^{-x}dydx = (1 - e^{-1})^2 = 0.399$

(c) [1.5 mark] $f_Y(y) = \int_y^\infty e^{-x}dx = e^{-y}$, $y > 0$.

(d) [1.5 mark] For $0 < y < x$,

$$f(x|y) = \frac{f(x, y)}{f_Y(y)} = e^{-(x-y)}.$$

(e) [3 marks]

$$\begin{aligned} P(X^2 \geq 1/4|Y \leq 3/4) &= \frac{P(X^2 \geq 1/4, Y \leq 3/4)}{P(Y \leq 3/4)} \\ &= \frac{\int_{1/2}^{3/4} \int_0^x e^{-x}dydx + \int_{3/4}^\infty \int_0^{3/4} e^{-x}dydx}{\int_0^{3/4} e^{-y}dy} \\ &= \frac{(3/2)e^{-1/2} - e^{-3/4}}{1 - e^{-3/4}} = 0.829. \end{aligned}$$

(f) [2 marks] Y is exponential rate one and X is gamma with rate one and parameter 2 and so $E[X] = 2$ and $E[Y] = 1$.

(g) [2 marks] $E[XY] = \int_0^\infty \int_0^x xye^{-x}dydx = 3$ so $Cov(X, Y) = 3 - 2 = 1$

(h) [1 mark] Not independent since covariance is non-zero.

(i) [1 mark] $E[Y/X] = \int_0^\infty \int_0^x y/xe^{-x}dydx = 1/2$.

(j) [3 marks] Y/X is between 0 and 1 so for $0 < t < 1$ we compute

$$P(Y/X \leq t) = \int_0^\infty \int_{y/t}^\infty e^{-x}dxdy = t,$$

so Y/X is uniform on $(0, 1)$.

6. The price of a stock at the start of a trading day is 25 dollars. The price of the stock two hours into the trading day is $25e^X$ and the price of the stock four hours into the trading day is $25e^Y$, where (X, Y) is bivariate normal with $E[X] = -2$, $Var(X) = 2$, $E[Y] = -4$, $Var(Y) = 4$ and $Cov(X, Y) = 2$.

   (a) Find the chance that the price of the stock two hours after the start of the day is at least $25e^{-2\sqrt{2}}$.

(b) If you buy one share of the stock and sell it at after two hours, how much money would you expect to make (losses count negative)?

(c) What is the correlation of X and Y?

(d) Given the price of the stock four hours after the start of the day is $25e^2$ dollars, what is the chance the price of the stock two hours after the start of the day was more than 25 dollars?

[10 marks]

Solution

(a) [2 marks] We want to compute

$$P(25e^X > 25e^{-2\sqrt{2}}) = P(X > -2\sqrt{2}) = P(X_s > \sqrt{2} - 2) = P(X_s > -0.59) \approx 0.72.$$

where X_s is the standardized version of X which is standard normal and we get the value from the z-table.

(b) [3 marks] Again setting X_s the standardized version of X which is standard normal,

$$E[25e^X] - 25 = 25(e^{-2}E[e^{\sqrt{2}X_s}] - 1) = 25(e^{-1} - 1) = -15.8.$$

using the formula for the MGF of a standard normal.

(c) [2 marks] The correlation of X and Y is $Cov(X,Y)/\sqrt{Var(X)Var(Y)} = 1/(\sqrt{2})$.

(d) [3 marks] We want

$$P(25e^X > 25|25e^Y = 25e^2) = P(X_s > \sqrt{2}|Y_s = 3).$$

Since (X_s, Y_s) is standard bivariate normal with correlation $1/\sqrt{2}$, we can write

$$X_s = Y_s/\sqrt{2} + Z/\sqrt{2},$$

where Z is standard normal independent of Y_s and so

$$P(X_s > \sqrt{2}|Y_s = 3) = P(Z > -1) = 0.8413.$$

according to the normal table.

7. Roll a fair die and let N be the number of rolls *before* the first six appears. Now toss a fair coin N times and let X be the number of heads in the N tosses.

   (a) Compute $E[X]$.

   (b) Compute $Var(X)$.

   (c) Find the probability generating function of X and identify the distribution of X by name.

   (d) Find the conditional probability mass function of N given $X = x$, for $x = 0, 1, 2, \ldots$.

[11 marks]

Solution

N is geometric with parameter $1/6$ and $X|N = n$ is binomial$(n, 1/2)$.

(a) [2 marks] Using the information above, $E[X|N] = N/2$ and

$$E[X] = E[E[X|N]] = 5/2,$$

since N is geometric$(1/6)$ and so $E[N] = 5$.

(b) [3 marks] Using the conditional variance formula and formulas for means and variances of binomial and geometric distributions,

$$Var(X) = E[Var(X|N)] + Var(E[X|N]) = 5/4 + 30/4 = 35/4.$$

(c) [4 marks] The PGF of X is

$$E[z^X] = E[E[z^X|N]] = E[(1/2 + z/2)^N] = \frac{1/6}{1 - (5/12)(z+1)} = \frac{2/7}{1 - (5/7)z},$$

which we recognize as the PGF of a geometric distribution with parameter $2/7$.

(d) [2 marks] For $x = 0, 1, \ldots$ and $n = x, x+1, \ldots$,

$$\begin{aligned} P(N = n|X = x) &= P(X = x|N = n)P(N = n)/P(X = x) \\ &= \binom{n}{x} 2^{-n}(5/6)^n(1/6)(5/7)^{-x}(7/2). \end{aligned}$$

8. A gambler is going to make \$100 worth of bets at a roulette table. The gambler is considering two strategies:

Strategy 1: make a single \$100 bet on black where the chance of winning \$100 is $18/37$ and the chance of losing \$100 is $19/37$.

Strategy 2: make 100 sequential \$1 dollar bets on black where for each bet, the chance of winning \$1 dollar is $18/37$ and the chance of losing \$1 is $19/37$.

Let W_1 be the winnings (counting losses as negative) of the gambler if Strategy 1 is used and W_2 the winnings if Strategy 2 is used.

(a) Compute the mean and variance of W_1.

(b) Compute the mean and variance of W_2.

(c) Approximate the chance that $W_2 > 0$.

(d) Which strategy should the gambler use to maximize the chance of ending with more than they started?

[9 marks]

Solution

(a) [2 marks] $E[W_1] = 100(18/37) - 100(19/37) = -100/37$. $Var(W_1) = 100^2 - 100^2/37^2 = 9993$.

(b) [3 marks] $(W_2 + 100)/2$ is binomial$(100, 18/37)$ so $E[W_2] = -100/37$ and $Var(W_2) = 100(36 \times 38/37^2) = 99.93$.

(c) [3 marks] W_2 can be represented as a sum of independent random variables (the winnings at each play), so by the CLT, W_2 is approximately normal with the mean and variance given in part (b). If Z is standard normal, then

$$P(W_2 > 0) \approx P(Z > 5/(3\sqrt{38}) = P(Z > 0.267) = 1 - 0.60 = 0.40.$$

(d) [1 mark] $P(W_1 > 0) = 18/37 > 0.40$, so the first strategy is better.

9. Consider the branching process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n is the population size of the nth generation. Assume $P(X_0 = 1) = 1$ and that the common offspring distribution is geometric with parameter $1/2$.

   (a) Show that the generating function of the offspring distribution is

   $$A_1(z) = \frac{1}{2-z}, \quad -2 < z < 2.$$

   [Note that part of the question is to justify the range of z where the formula is valid.]

   (b) Find $E[X_n]$.

   (c) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_n and q_{n+1}. Hence or otherwise, find an expression for q_n, $n = 0, 1, \ldots$.

   (d) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

   (e) Find an expression for the generating function $A_n(z)$ of X_n.

[9 marks]

Solution

(a) [2 marks] $A_1(z) = E[z^{X_1}] = \sum_{k=0}^{\infty}(1/2)^{k+1}z^k = (1/2)/(1-(1/2)z) = 1/(2-z)$; the sum converges if $|z/2| < 1$.

(b) [2 marks] Since the expectation of a geometric $1/2$ is one, the sequence $(E[X_n])_{n\geq 0}$ satisfies $E[X_n] = E[E[X_n|X_{n-1}]] = 1 \times E[X_{n-1}]$ and $E[X_0] = 1$, so $E[X_n] = 1$.

(c) [2 marks] If A_n denotes the generating function of X_n, then $A_n(0) = q_n$. Moreover, lecture facts imply $A_n = A_1(A_{n-1})$ and so $q_n = A_1(q_{n-1})$ which implies, using (a), that $q_n = 1/(2 - q_{n-1})$. So $q_n = n/(n+1)$.

(d) [1 mark] We know from lectures that $\lim_{n\to\infty} q_n = q$ is the minimum non-negative root of $A_1(z) = z$, thus $q = 1$ and extinction is assured. Alternatively, just appeal to the fact that the offspring distribution has mean one and that this implies extinction is assured.

(d) [2 marks] Considering small cases leads to the guess

$$A_n(z) = \frac{n-(n-1)z}{n+1-nz},$$

which is easily verified by induction.

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817