14. Topic 3 — Lecture 13

Slides: pgs. 148–163. Exercises: Topic 3, Q.83–86

Exercise What geometrical property must hold true if the vectors $(1, 1)$ and $(1, -1)$ were to be linearly dependent?

Lines and planes passing through the origin have some special properties which together are said to define a subspace. The defining properties for a subset S of $\mathbb{R}^n$ to be a subspace are that

(0) S is nonempty

(1) closure under vector addition

(2) closure under scalar multiplication.

Example Check that the line $\mathbf{x} = t(1, -1, 1)$, $t \in \mathbb{R}$ defines a subspace

Solution The three defining properties have to be checked.

(0) Since $\mathbf{x} = \mathbf{0}$ is a point on the line, it is nonempty.

(1) Let $\mathbf{x}_1 = t_1(1, -1, 1)$ and $\mathbf{x}_2 = t_2(1, -1, 1)$ be two points on the line, and thus that the set is closed under vector addition. Our task is to show that $\mathbf{x}_1 + \mathbf{x}_2$ is also a point on the line. We have $\mathbf{x}_1 + \mathbf{x}_2 = (t_1 + t_2)(1, -1, 1)$. Setting $t_1 + t_2 = s$, and noting $s \in \mathbb{R}$, we thus see that $\mathbf{x}_1 + \mathbf{x}_2$ is a point on the line.

(2) Let $\mathbf{x}_0 = t_0(1, -1, 1)$ be a point on the line, and let $\alpha \in \mathbb{R}$ be a scalar. Our task is to show that $\alpha\mathbf{x}_0$ is a point on the line, and thus that the set is closed under scalar multiplication. We have $\alpha\mathbf{x}_0 = (\alpha t_0)(1, -1, 1)$. Setting $\alpha t_0 = s$, and noting $s \in \mathbb{R}$, we thus see that $\alpha\mathbf{x}_0$ is a point on the line.

Example Show that the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1$ is not a subspace.

Solution To prove that the set is not a subspace, it suffices to find just one example which contradicts one of the axioms. Thus consider the point $(1/2, 1/2)$ in the set, and suppose we multiply this by the scalar 4. This gives the point $(2, 2)$. But this point is no longer in the set, so the set is not closed under scalar multiplication, and is therefore not a subspace.

15. Topic 3 — Lecture 14

Slides: pgs. 148–163. Exercises: Topic 3, Q.83–86 (Continued)

Exercise Show that the line $y = -2x + 1$ does not define a subspace.

Exercise Show that the plane $2x + y + z = 0$ does define a subpsace.

Cataloguing subspaces in $\mathbb{R}^2$

(i) $\{\mathbf{0}\}$ — the zero vector.

(ii) Lines through the origin.

(iii) $\mathbb{R}^2$ itself.

Cataloguing subspaces in $\mathbb{R}^3$

(i) $\{\mathbf{0}\}$ — the zero vector.

(ii) Lines through the origin.

(iii) Planes through the origin.

(iv) $\mathbb{R}^3$ itself.

16. Topic 3 — Lecture 15

Slides: pgs. 164–174. Exercises: Topic 3, Q.87–90

The span of a set of vectors is the set of linear combinations of those vectors. A little thought shows that all spans are subspaces, and similarly that all subspaces are spans.

Example Give a geometrical description of Span$\{(1,0,-1), (\frac{1}{2}, 0, -\frac{1}{2})\}$.

Solution The two vectors are linearly dependent: $(1,0,-1) = 2(\frac{1}{2}, 0, -\frac{1}{2})$. Hence the span gives vectors of the form $\mathbf{x} = t(1,0,-1)$, $t \in \mathbb{R}$. This is a line through the origin in the direction of $(1,0,-1)$.

The word span can also be used in another sense. We say that set a given set of vectors spans a subspace S if the set of all linear combination of those vectors equals S.

Example Can the vectors $(1,0,-1)$ and $(1,1,1)$ span $\mathbb{R}^3$?

Solution Here we are asking if, for general $(x,y,z) \in \mathbb{R}^3$, we can find scalars α, β such that

$$\alpha(1,0,-1) + \beta(1,1,1) = (x,y,z).$$

Since

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y \\ -1 & 1 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y \\ 0 & 0 & x - 2y + z \end{array}\right]$$

This illustrates the fact that the span of two vectors in $\mathbb{R}^3$ can never equal $mathbbR^3$.

More generally, the span of k vectors in $\mathbb{R}^n$ with $k < n$ can never give $\mathbb{R}^n$

Example Span$\{(1,1,1,1),(1,1,1,0),(1,1,0,0)\} \neq \mathbb{R}^4$.

Question How can we check if 3 or more vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. For example, does Span$\{(1,0,-1),(1,1,1),(1,0,0)\} = \mathbb{R}^3$.

Answer Here we are asking if, for general $(x,y,z) \in \mathbb{R}^3$, we can find scalars α, β such that

$$\alpha(1,0,-1) + \beta(1,1,1) = (x,y,z).$$

For the augmented matrix system we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & y \\ -1 & 1 & 0 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x-2y+z \end{array}\right]$$

Here there are system is consistent, and there are 3 leading entries for three unknowns.

We recognise this as the criterion for the 3 vectors to be linearly independent.