7. Topic 2 — Lecture 7

This lecture will be used to finish off and/ or consolidate the material from Topic 2.

<u>Exercise</u> Let

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1\,.$$

Evaluate the following determinants:

$$\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}, \quad \begin{vmatrix} a & -b & c \\ d & -e & f \\ g & -h & i \end{vmatrix}, \quad \begin{vmatrix} d & e & f \\ 3g & 3h & 3i \\ a & b & c \end{vmatrix}$$

$$\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ d+a & e+b & f+c \\ g-2a & h-2b & i-2c \end{vmatrix}$$

8. Topic 3 — Lecture 8

Slides: pgs. 93–102. Exercises: Topic 3, Q.50–Q.57.

8.1. Vectors in $\mathbb{R}^2$ and $\mathbf{R}^3$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ points can be thought of as being located by a directed straight line from the origin, and this gives us a geometrical representation of a vector.

The basic operations applied to vectors are that of **scalar multiplication** and **vector addition**. Both these can be represented geometrically, by scaling in the case of scalar multiplication, and by the a head to tail rule for addition.

In $\mathbf{R}^3$ it is standard to denote the unit vectors in the directions of the x, y and z axes respectively as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively. We then write

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$$

This array is the standard notation for points in $\mathbb{R}^3$, and is often referred to as a row vector. The same array could be written

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

in which case it is referred to as a column vector.

The (Euclidean) length of a vector $\mathbf{v} = (a, b, c,) \in \mathbb{R}^3$, denoted $||\mathbf{v}||$, is defined as $||\mathbf{v}|| = \sqrt{a^2 + b^2 + c^2}$ which is consistent with Pythagoras' theorem.

8.2. **Vectors in $\mathbb{R}^n$.** Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the unit vectors making up the n perpendicular directions in $\mathbb{R}^n$. We then write

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = (x_1, x_2, \ldots, x_n)$$

which specifies a general vector in $\mathbb{R}^n$.

The length of $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is defined as

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}.$$

Let $c \in \mathbb{R}$ be a scalar. We have that

$$||c\mathbf{x}|| = |c|\,||\mathbf{x}||.$$

Dot product Related to length is the notion of **dot product**. Let $\mathbf{u} = (u_1, \ldots, u_n$ and let $\mathbf{v} = (v_1, \ldots, v_n$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n.$$

Distance between two vectors

The fact that $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ tells us that geometrically the vector $\mathbf{u} - \mathbf{v}$ corresponds (by the head to tail rule) to a vector starting at $\mathbf{v}$ and finishing at $\mathbf{u}$. Hence the distance between vectors $\mathbf{u}$ and $\mathbf{v}$ is simply $||\mathbf{u} - \mathbf{v}||$. Sometimes this distance is denoted $d(\mathbf{u}, \mathbf{v})$.

9. Topic 3 — Lecture 9

Slides: pgs. 103–113. Exercises: Topic 3, Q.58, 59, 63–65.

9.1. **Dot product and angle between two vectors.** In the diagram θ is the angle between the two vectors, and $\theta = \phi - \rho$. Recall $\mathbf{u} \cdots \mathbf{v} = as + bt$. The aim is to eliminate a, s, b, t in favour of $||\mathbf{u}||$, $||\mathbf{v}||$ and θ.

Here

$$a = ||\mathbf{u}|| \cos\phi, \qquad b = ||\mathbf{u}|| \sin\phi$$
$$s = ||\mathbf{v}|| \cos\rho, \qquad t = ||\mathbf{v}|| \sin\rho$$

Hence

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= ||\mathbf{u}||\,||\mathbf{v}|| \Big(\cos\phi\cos\rho + \sin\phi\sin\rho \Big) \\ &= ||\mathbf{u}||\,||\mathbf{v}|| \cos(\phi - \rho) \\ &= ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta\end{aligned}$$

We choose $0 \leq \theta \leq \pi$. Since the vectors are perpendicular when $\theta = \pi/2$, we see that in terms of the dot product, vectors are perpendicular when $\mathbf{u} \cdot \mathbf{v} = 0$.

9.2. **Vector projection.** Here, given a vector $\mathbf{u}$ (which is to be thought of as a direction), the task is to orthogonally project a given vector $\mathbf{v}$ in the direction of $\mathbf{u}$. Denote the projected vector by $\mathbf{p}$.

From trigonometry we have

$$||\mathbf{p}|| = ||\mathbf{v}|| \cos\theta$$

A unit vector in the direction of $\mathbf{u}$ is $\frac{\mathbf{u}}{||\mathbf{u}||}$.

Hence

$$\mathbf{p} = (||\mathbf{v}|| \cos\theta)\frac{\mathbf{u}}{||\mathbf{u}||} = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\mathbf{u}.$$

<u>Exercise</u> Verify that the vector $\mathbf{q} = \mathbf{v} - \mathbf{p}$ is perpendicular to $\mathbf{p}$.

<u>Exercise</u> Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ be the unit vectors corresponding to $\mathbf{u}$ and $\mathbf{v}$ respectively. Show that

$$\mathbf{p} = (\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}}.$$

9.3. **Cross product.** The cross product is a particular rule for multiplying vectors in $\mathbb{R}^3$. The rule can be written in the short hand notation

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Here it is required that the determinant be expanded along the first row, so this means

$$\mathbf{u}\times\mathbf{v} = \mathbf{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$