4. Topic 2 — Lecture 4

Slides: pgs. 36–46. Problem Sheet Exercises: Topic 2, Q.17–Q.21.

4.1. **Basic definitions.** Definition Saying that the size of matrix in $n \times m$ means that there are n rows and m columns.

Definition Writing that $A = [A_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ means the entry in row i and column j of the matrix A is A_{ij}, and that A has size $n \times m$.

Exercise What is the largest value of the rank of a 4×2 matrix?

4.2. **Entering matrices in Matlab.** Suppose we type

```
A = [12  -18;3*pi  1026;0.12  3*theta]
```

This corresponds to the 3×2 matrix

$$A = \begin{bmatrix} 12 & -18 \\ 3\pi & 1026 \\ 0.12 & 3 * \text{theta} \end{bmatrix}$$

Matlab knows the value of π, but it treats `theta` as a variable.

There are many shortcuts for entering matrices. For example, `B = ones(3,2)` corresponds to

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

4.3. **Scalar multiplication and addition of matrices.** Definition A scalar refers to an element of a field. In this course this will mostly mean that a scalar is a real number (other possibilities include complex numbers, or the field consisting of just the two elements $\{0, 1\}$ in binary arithmetic).

Definition Let α be a scalar, and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We define αA by

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

The same rule holds for a matrix A of general size.

Definition Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. We define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

The same rule hold for matrices A and B of general size *provided* both are of the same size.

4.4. **Matrix multiplication.** We first define a rule for a matrix of size $n \times m$ multiplied by a matrix of size $m \times 1$. The rule is that if the matrix of $n \times m$ was a coefficient matrix, and the matrix of size $n \times m$ the symbols for the corresponding unknowns, then a matrix of size $n \times 1$ results containing the corresponding linear equations.

Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - z \\ 4y + 3z \end{bmatrix}$$

Dot product rule Let A be of size $n_1 \times n$ and let B be of size $n \times n_2$. We have that AB is of size $n_1 \times n_2$ and the entry in row j and column k of AB is equal to the dot product of row j of A with column k of B.

Example Consider

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 6 & -1 \\ 3 & -2 & 8 \\ 0 & 1 & -1 \end{bmatrix}$$

This is the product of a 2×3 matrix times a 3×3 matrix, so the result is a 2×3 matrix. The entry in row 2, column 2 for example is equal to $(0, 4, 3) \cdot (6, -2, 1) = -5$.

4.5. **Extracting matrix entries in Matlab.** Typing `A(2,3)` returns $A_{2,3}$ — the entry in position row 2 and column 3 of A.

Typing `A*B(2,3)` returns $(AB)_{2,3}$ — the entry in position row 2 and column 3 of AB.

Exercise Let $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. (a) What is the size of AB? Compute AB. (b) What is the size of BA? Compute BA.

5. Lecture 5

Slides: pgs. 47–65. Problem Sheet Exercises: Topic 2, Q.22, Q.24-Q38.

5.1. **Adjacency matrices.** Adjacency matrices are symmetric square matrices with entries taking on the values 0 or 1 only. Each row and column of the matrix is thought of as corresponding to a vertex (also known as a node) of the graph. The entry is 1 in position (ij) is there is an edge between vertices i and j, and it is zero otherwise.

<u>Definition</u> A walk (also referred to as a path), is a sequence of edges linking one vertex to another, and the length of a walk is the number of edges it contains (some may be repeated).

<u>Theorem</u> Let A be an adjacency matrix. The entry (ij) in the matrix power A^k is equal to the number of walks from vertex i to vertex j of length k.

The proof of this theorem requires **induction**, and is not given.

<u>Exercise</u> (a) Draw the graph corresponding to the adjacency matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
(b) Use the above theorem to explain by the entry in position (11) of A^7 must equal 0.

5.2. **Matrix inverses.** This only applies to a square matrices. A square matrix A is said to have inverse B if $AB = BA = I$, where I is the identity. We write $B = A^{-1}$.

<u>Matlab notation</u> Typing `inv(A)` gives the inverse of A. Typing `eye(3)` gives the 3×3 identity.

Consider the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is simple to check that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

An important point is that the inverse exists only if $ad - bc$ (which is the determinant) is non-zero.

5.3. **Linear systems and matrix inverses.** <u>Exercise</u> (a) Write the linear system

$$2x + 5y = 4, \;\; x + 3y = -1$$

in the form $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. (b) Compute A^{-1} and use this to solve for x and y.

Let A be 3×3. To solve the matrix equation $AB = I$, the task is equivalent to solving the three linear systems

$$A \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

This in turn is equivalent to solving the augmented matrix system $[A|I]$. This system can be solved provided the rank of A is equal to the number of rows. Going to fully reduced form gives $[I|B]$, and we conclude $A = B^{-1}$.

Exercise (Booklet, Q.34(c). Find the inverse of $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

6. Lecture 6

Slides: pgs. 66–89. Problem Sheet Exercises: Topic 2, all remaining questions

6.1. **Coding messages.**

Theorem In the case of 2×2 matrices A with integer coefficients, if $\det A = 1$, then B^{-1} has integer coefficients. This same property holds true in the $n \times n$ case.

In coding, letters can be written as numbers, the numbers put into a matrix form, and the message 'scrambled' by multiplication on the left by a non-singular matrix A. The coded message is then decoded by multiplying by on the left by A^{-1}.

Example Let's first write the word FAME as letters in a 2×2 matrix: $\begin{bmatrix} 6 & 13 \\ 1 & 5 \end{bmatrix}$. Choose $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. What is the matrix form of the sent message, and how is it decoded?

6.2. **Matrix transpose.**

Definition By writing A^T, we are referring to the matrix A with its rows and columns interchanged (i.e. what was the first row becomes the first column etc.).

Exercise Let A be of size $p \times q$ and let B be of size $q \times r$. (a) Show that $(AB)^T$ and $B^T A^T$ have the same size. (b) Show that in fact $(AB)^T = B^T A^T$.

Exercise Let A be a square invertible matrix. Show that $(A^T)^{-1} = (A^{-1})^T$.

6.3. **Determinants.**

The determinant of a square matrix has a scalar value. It's defined to have the following properties:

(a) $\det I = 1$ (the determinant of the identity is unity.

(b) Let A' be related to A by multiplying row j by the scalar α. Then $\det A' = \alpha \det A$.

(c) Let $A_{\mathbf{a}+\mathbf{b};j}$ be the matrix obtained by A by replacing row j by $(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$. Then $\det A_{\mathbf{a}+\mathbf{b};j} = \det A_{\mathbf{a};j} + \det A_{\mathbf{b};j}$.

(c) Let A' be obtained from A by interchanging two rows. Then $\det A' = -\det A$.

In the 2×2 case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, choosing $\det A = ad - bc$ fulfils these properties (exercise: check property (c)).

For these properties to hold in the $N \times N$ case, a recursive definition of the determinant can be used which builds on knowledge of the evaluation in the 2×2 case.

This requires the following definitions: Let $A(i, j)$ denote the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A.

Let $C_{i,j} = (-1^{i+j} \det A(i, j)$. The $C_{i,j}$ are called cofactors.

Definition The determinant of an $n \times n$ matrix A is defined recursively, by expanding along any row or down any column according to the rule that the determinant is equal to the sum of the entries multiplied by their cofactors.

Exercise Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Compute $\det A$.

6.4. **Computing determinants using elementary row operations.** .

It follows from properties (b) and (c) of a determinant, that the elementary row operation of multiplying a row by α multiplies the determinant by α, the elementary row operation of adding a multiple of one row to another leaves the determinant unchanged, and the elementary row operation of interchanging two rows changes the sign of the determinant. Also, the recursive formula for a determinant tells us that the determinant of a matrix with all zero below the diagonal is equal to the product of the diagonal entries. Hence, reduction to RE form provides another way to compute determinants.