Some answers to MAST10007 (620-156) exam
Semester 2 2010

1. (a) not defined

   (b) $B^TA = \begin{bmatrix} 11 & 15 & 28 \\ 3 & 3 & 8 \end{bmatrix}$

   (c) $BA = \begin{bmatrix} 9 & 15 & 22 \\ -1 & -3 & -2 \end{bmatrix}$

   (d) $2C + BC = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

2. (a) (i) $\det(G) = -20$, (ii) $\det(G^{-1}) = -\frac{1}{20}$, (iii) $\det(G^T) = -20$.

   (b) (i) $\det(NM) = -\frac{3}{2}$, (ii) $\det(3NM^2) = \frac{81}{4}$

   (c) Since A is invertible, $\det(A) \neq 0$ and hence A^{-1} exists.

   We have $AA^{-1} = I$ and hence $\det(AA^{-1}) = \det(I) = 1$. But $\det(AA^{-1}) = \det(A)\det(A^{-1})$ hence $\det(A)\det(A^{-1}) = 1$ and so $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{x}$ as required.

3. (a)

$$\frac{1}{4}\begin{bmatrix} -6 & -6 & 4 \\ -3 & -1 & 2 \\ 11 & 9 & -6 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 \\ -3 \\ 18 \end{bmatrix}$$

   (c) (i) $k \neq -1$, (ii) no value of k exists, (iii) $k = -1$

   (d) solution set for (iii) is $\left\{\left(2 - 2w, w, \frac{1}{3} - \frac{2}{3}s, s, t\right) : w, s, t \in \mathbb{R}\right\}$

4. (a) $(x, y, z) = (1, -5, 3) + t(2, 3, \frac{1}{2}) \qquad t \in \mathbb{R}$

   (b) $x - 2y + 8z = 35$

   (c) $\frac{4}{53}\sqrt{61^2 + 41^2 + 2^2} = \frac{4}{53}\sqrt{5406}$

5. (a) First, S is non-empty. Second, if $p_1(x) = a_1+2b_1x+3c_1x^2 \in S, p_2(x) = a_2+2b_2x+3c_2x^2 \in S$, then $p_1(x)+p_2(x) = (a_1+a_2)+2(b_1+b_2)x+3(c_1+c_2)x^2 \in S$. Thirdly, for any $p(x) = a+2bx+3cx^2 \in S$ and any scalar α, $\alpha p(x) = (\alpha a)+2(\alpha b)x+3(\alpha c)x^2 \in S$.

   So by the Subspace Theorem, S is a subspace of $\mathcal{P}_2$.

   (b) S is not a subspace of $M_{2,3}$. For example,

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \in S, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in S$$

   but

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \notin S.$$

   (c) Not a subspace since it is not closed under scalar multiplication, for example:

$$(2f)(0) = 2(f(0)) = 2 \times 1 = 2 \neq 1$$

6. (a)(i) S is linearly dependent because $2 + 2x^2 = 2(2 + x + x^2) - 2(1 + x)$.

   (a)(ii) S is linearly independent.

   (b) Any polynomial in $\mathrm{Span}(S)$ is of the form

$$a(1 + x) + b(2 + x + x^2) + c(2 + 2x^2) = (a + 2b + 2c) + (a + b)x + (b + 2c)x^2$$

   and so satisfies $a_0 = a_1 + a_2$.

   Conversely, if $a_0 = a_1 + a_2$, then

$$a_0+a_1x+a_2x^2 = (a_1+a_2)+a_1x+a_2x^2 = a_1(1+x)+0(2+x+x^2)+\frac{a_2}{2}(2+2x^2) \in \mathrm{Span}(S).$$

   Therefore,

$$\mathrm{Span}(S) = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}, a_0 = a_1 + a_2\}.$$

7. (a) (i) 3.

   (ii) No.

   (iii) If v_i is the i^{th} vector, then either $\{v_1, v_2, v_4\}$ or $\{v_1, v_3, v_4\}$ work.

   (b) (i) 3.

   (ii) If v_i is the i^{th} column vector, then

$$v_3 = 2v_1 + v_2, \quad \text{or} \quad v_5 = -2v_1 + 4v_2 - 3v_4,$$

   or any linear combination of those two equalities.

   (iii) Any subset of 3 columns of the original matrix, except the the first three!

   (iv) $\{(-2, -1, 1, 0, 0),\ (2, -4, 0, 3, 1)\}$.

8. (a)

$$[T]_{\mathcal{S}} = \begin{pmatrix} 5 & 3 & -10 \\ -6 & -4 & 12 \\ 0 & 0 & 0 \end{pmatrix}$$

(b)

$$[P]_{\mathcal{S},B} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$[P]_{\mathcal{B},\mathcal{S}} = [P]_{\mathcal{S},B}^{-1} = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(d)

$$[T]_{\mathcal{B}} = [P]_{\mathcal{B},\mathcal{S}}[T]_{\mathcal{S}}[P]_{\mathcal{S},B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

9. (a)

$$A = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 5 \end{pmatrix}$$

(b) $\mathbf{x}^T A\mathbf{x} = 3x_1^2 + x_2^2 + 5x_3^2 - 6x_1x_3 = 3(x_1 - x_3)^2 + x_2^2 + 2x_3^2$ is a sum of positive multiples of squares, so is only zero when $x_1 - x_3 = 0$, $x_2 = 0$, and $x_3 = 0$. That is precisely when $\mathbf{x} = 0$.

(c)

$$\left\{ \frac{1}{\sqrt{3}}(1,0,0),\ (0,1,0),\ \frac{1}{\sqrt{2}}(1,0,1) \right\}$$

10. (a) (i) Diagonalizable since real symmetric (or has distinct eigenvalues)

(ii) Characteristic eqn is $(x^2 - 25) + 36 = 0$. Not diagonalizable, because it has no real eigenvalues.

(iii) Char eqn is: $0 = (x+5)(x-7) + 36 = x^2 - 2x + 1 = (x-1)^2$.

For eigenvalue 1, the eigenspace has dimension 1. Therefore the matrix is not diagonalizable, because there is no basis consisting of eigenvectors.

(b) (i) For eigenvalue -1: a basis for the eigenspace is $\{(1,0,1),(0,1,0)\}$

For eigenvalue 3: a basis for the eigenspace is $\{(1,0,-1)\}$

(ii) We can take

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

11. (a) $\boldsymbol{v} \neq \mathbf{0}$ and there exists $\lambda \in \mathbb{R}$ such that $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$.

(b)

$$\begin{aligned} \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = \mathbf{0} &\implies T(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2) = \mathbf{0} \\ &\implies \alpha_1 \lambda_1 \boldsymbol{v}_1 + \alpha_2 \lambda_2 \boldsymbol{v}_2 = \mathbf{0} \\ &\implies \lambda_1 \alpha_2 \boldsymbol{v}_2 - \alpha_2 \lambda_2 \boldsymbol{v}_2 = \mathbf{0} \\ &\implies (\lambda_1 \alpha_2 - \alpha_2 \lambda_2) \boldsymbol{v}_2 = \mathbf{0} \\ &\implies \alpha_2(\lambda_1 - \lambda_2) = 0 \text{ since } \boldsymbol{v} \neq \mathbf{0} \\ &\implies \alpha_2 = 0 \text{ since } \lambda_1 \neq \lambda_2 \end{aligned}$$

Similarly, we obtain $\alpha_1 = 0$.

12. (a) A simplified equation for the conic is

$$-3(x')^2 + 5(y')^2 = 2$$

so the curve is a hyperbola.

(b) The directions of the principal axes are: $(1, -\sqrt{3})$ and $(\sqrt{3}, 1)$

(c)