THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 13 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This paper has 9 questions. Attempt as many questions, or parts of questions, as you can.

  The number of marks allocated to each question is shown in the brackets after the question statement. The total number of marks available for this examination is 100.

  There is a table of normal distribution probabilities at the end of the exam.

  Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

1. Consider a random experiment with state space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms, prove that for events A and B,
   $$P(A) = P(A \cap B) + P(A \cap B^c).$$

   (c) Is it possible to have a probability space with events A, B, C such that $P(A) = 1/2$, $P(B) = 1/2$, $P(C) = 3/4$, $P(B \cap C) = 1/4$, $P(A \cap B) = 1/4$, $P(A \cap C) = 3/8$, and $P(A \cap B \cap C) = 1/8$? If your answer is no, then prove it's not possible, and if your answer is yes, then define a probability space and events A, B, C that satisfy the constraints.

   [9 marks]

   Solution (a) [3 marks] The axioms are
   A1: For events A, $P(A) \geq 0$,
   A2: $P(\Omega) = 1$,
   A3: For $A_1, A_2, \ldots$ disjoint events,
   $$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

   (b) [3 marks] $B^c \cap B = \emptyset$ and $A = (A \cap B) \cup (A \cap B^c)$, so A3 implies
   $$P(A) = P(A \cap B) + P(A \cap B^c).$$

   (c) [3 marks] Yes. $\Omega = \{1, \ldots, 8\}$, P is equally likely, and $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 7, 8\}$, and $C = \{2, \ldots, 7\}$.

2. The probabilities with which we choose one of three target shooters are 0.2 for Angela, 0.5 for Brunhilde, and 0.3 for Chelsea. The probabilities of each hitting a target from one shot are 0.3 for Angela, 0.4 for Brunhilde, and 0.1 for Chelsea.

   (a) The chosen shooter fires one shot and misses the target. Find the probability that we have chosen Angela.

   (b) Are the events that Angela was chosen and the target was missed positively related, negatively related, or independent? Justify your answer.

   (c) Suppose Angela and Brunhilde each fired one shot at the target. If one bullet is found in the target, find the probability that it came from Brunhilde's gun. Assume that the two shooters fire at the target independently.

   [8 marks]

   Solution

   (a) (3 marks) Let H be the event the target is hit, and A, B, and C be the events that shooters Angela, Brunhilde, and Chelsea are selected, respectively. By Bayes's theorem and the law of total probability
   $$P(A|H^c) \;=\; \frac{P(H^c|A)P(A)}{P(H^c|A)P(A) + P(H^c|B)P(B) + P(H^c|C)P(C)}$$

$$
\begin{aligned}
&= \frac{0.7 \times 0.2}{0.7 \times 0.2 + 0.6 \times 0.5 + 0.9 \times 0.3} \\
&= \frac{0.14}{0.71} \\
&= 0.1972.
\end{aligned}
$$

(b) (2 marks) Since $P(A|H^c) < P(A)$ the events A and H^c are negatively related.

(c) (3 marks) Let E be the event that one bullet is found in the target, and H_A and H_B be the events that shooters Angela and Brunhilde hit the target, respectively. Then

$$
\begin{aligned}
P(H_B|E) &= \frac{P(H_B \cap E)}{P(H_A \cap E) + P(H_B \cap E)} \\
&= \frac{P(H_A^c \cap H_B)}{P(H_A \cap H_B^c) + P(H_A^c \cap H_B)} \\
&= \frac{P(H_A^c)P(H_B)}{P(H_A)P(H_B^c) + P(H_A^c)P(H_B)} \\
&= \frac{0.7 \times 0.4}{0.3 \times 0.6 + 0.7 \times 0.4} \\
&= \frac{0.28}{0.46} \\
&= 0.6087.
\end{aligned}
$$

3. Let T be the number of failures before the first success in a series of independent Bernoulli trials with probability $p \in (0,1)$, U be the number of failures between the first and the second successes, and V the number of failures between the second and the third successes.

(a) Find the probability mass functions and identify the distribution by name for

(i) $X = T + U + V$;

(ii) $Z = \min\{T, U\}$.

(b) (i) Find the joint probability mass function of (T, Z).

(ii) Compute the probability generating function of $T + Z$.

[10 marks]

Solution

(a) (i) (2 marks) T, U, and V are independent and geometric random variables with probability p of success. Therefore $X \stackrel{d}{=} Nb(3, p)$ and, for $x = 0, 1, \ldots,$

$$
p_X(x) = \binom{x+2}{2} p^3 (1-p)^x
$$

(ii) (3 marks) For $z = 0, 1, 2 \ldots,$

$$
\begin{aligned}
P(Z \leq z) &= P(\min(T, U) \leq z) \\
&= 1 - P(\min(T, U) > z)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - P(T > z, U > z) \\
&= 1 - P(T > z)P(U > z) \\
&= 1 - \left((1-p)^{z+1}\right)^2 \\
&= 1 - (1-p)^{2z+2}.
\end{aligned}
$$

Therefore, for $z = 0, 1, 2, \ldots$,

$$
\begin{aligned}
p_Z(z) &= P(Z \le z) - P(Z \le z-1) \\
&= 1 - (1-p)^{2z+2} - \left(1 - (1-p)^{2z}\right) \\
&= (1-p)^{2z}(1 - (1-p)^2),
\end{aligned}
$$

and we recognize Z as geometric with parameter $1 - (1-p)^2$.

(b) (i) (3 marks) Noting that $Z \le T$, we have for $k \ge 0$,

$$P(T = k, Z = k) = P(T = k, U \ge k) = (1-p)^{2k}p,$$

and for $0 \le j < k$,

$$P(T = k, Z = j) = P(T = k, U = j) = (1-p)^{k+j}p^2.$$

(ii) (2 marks) For $0 < z \le 1$,

$$
\begin{aligned}
E[z^{T+Z}] &= \sum_{k=0}^{\infty} z^{2k} P(T = k, Z = k) + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} z^{j+k}(1-p)^{j+k}p^2 \\
&= p\sum_{k=0}^{\infty}\left(z^2(1-p)^2\right)^k + p^2\sum_{j=0}^{\infty}(z(1-p))^j \sum_{k=j+1}^{\infty}(z(1-p))^k \\
&= \frac{p}{1 - z^2(1-p)^2} + p^2\sum_{j=0}^{\infty}(z(1-p))^{2j+1}\sum_{k=j+1}^{\infty}(z(1-p))^{k-j-1} \\
&= \frac{p}{1 - z^2(1-p)^2} + \frac{p^2 z(1-p)}{(1 - z^2(1-p)^2)(1 - z(1-p))} \\
&= \frac{p(1 - (1-p)^2 z)}{(1 - z^2(1-p)^2)(1 - z(1-p))}.
\end{aligned}
$$

[Note that in fact the probability generating function is defined for $|z| < 1/(1-p)$.]

4. Let X and Y have joint probability density function given by $ce^{-(x+y)}$, $0 < x < y$.

   (a) Evaluate the constant c.

   (b) Find $f_X(x)$, the marginal density function of X, and identify this distribution by name.

   (c) Evaluate $P(Y > 2 | X = 1)$.

   (d) Evaluate $P(Y > 2 | X > 1)$.

   (e) Are X and Y independent? Justify your answer.

   (f) Compute $E[e^{(X+Y)/2}]$. [14 marks]

Solution

(a) (3 marks)

$$\begin{aligned}
\int_0^\infty \int_0^y c e^{-(x+y)}\, dx\, dy &= c\int_0^\infty \left[-e^{-(x+y)}\right]_0^y dy \\
&= c\int_0^\infty \left(e^{-y} - e^{-2y}\right) dy \\
&= c\left[-e^{-y} + \frac{e^{-2y}}{2}\right]_0^\infty \\
&= \frac{c}{2}.
\end{aligned}$$

Therefore $c = 2$.

(b) (3 marks) For $0 < x < y$,

$$\begin{aligned}
f_X(x) &= \int_x^\infty 2e^{-(x+y)} dy \\
&= \left[-2e^{-(x+y)}\right]_x^\infty \\
&= 2e^{-2x},
\end{aligned}$$

and X is exponential with rate 2.

(c) (3 marks) First find $f_{Y|X}(y|x)$, the conditional density function of Y given X. For $0 < x < y$,

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{(X,Y)}(x,y)}{f_X(x)} \\
&= \frac{2e^{-(x+y)}}{2e^{-2x}} \\
&= e^{x-y}.
\end{aligned}$$

Now

$$\begin{aligned}
P(Y > 2|X = 1) &= \int_2^\infty e^{1-y} dy \\
&= \left[-e^{1-y}\right]_2^\infty \\
&= e^{-1}.
\end{aligned}$$

(d) (2 marks)

$$\begin{aligned}
P(Y > 2|X > 1) &= \frac{P(Y > 2, X > 1)}{P(X > 1)} \\
&= \frac{\int_2^\infty \int_1^y 2e^{-(x+y)}\, dx\, dy}{\int_1^\infty 2e^{-2x}\, dx}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_2^\infty \left[-2e^{-(x+y)}\right]_1^y dy}{\left[-e^{-2x}\right]_1^\infty} \\
&= \frac{\int_2^\infty \left(2e^{-1-y} - 2e^{-2y}\right) dy}{e^{-2}} \\
&= \frac{\left[-2e^{-1-y} + e^{-2y}\right]_2^\infty}{e^{-2}} \\
&= \frac{2e^{-3} - e^{-4}}{e^{-2}} \\
&= 2e^{-1} - e^{-2}.
\end{aligned}
$$

(e) (1 mark) They are not independent since the support is not rectangular. (Alternatively note that parts (c) and (d) have different answers.)

(f) (2 marks) We compute

$$E[e^{(X+Y)/2}] = \int_0^\infty \int_x^\infty 2\exp\{-(x+y)/2\}dydx = \int_0^\infty 4e^{-x}dx = 4.$$

5. Let X be uniform on the interval $(0, \pi)$. Let $Y = \cos X$ and $Z = Y^2$.

(a) Find $F_Y(y)$, the distribution function of Y.

(b) Find $f_Y(y)$, the density function of Y, state the values of y for which it is defined, and show that it is a density function.

Recall that $\arccos : [-1, 1] \to [0, \pi]$ and, for $-1 < x < 1$,

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}.$$

(c) Evaluate $E(Y)$.

(d) Find $F_Z(z)$, the distribution function of Z.

(e) Find $f_Z(z)$, the density function of Z, state the values of z for which it is defined.

(f) Approximate $E(Z)$ and $Var(Z)$ using suitable Taylor series expansions.

[17 marks]

Solution

(a) (2 marks) For $y \in (-1, 1)$,

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\cos X \le y) \\
&= P(X \ge \arccos y) \\
&= 1 - P(X < \arccos y) \\
&= 1 - \frac{\arccos y}{\pi}.
\end{aligned}
$$

(b) (3 marks) Differentiating the expression for $F_Y(y)$ gives, for $y \in (-1,1)$,

$$f_Y(y) \;=\; \frac{1}{\pi\sqrt{1-y^2}}.$$

Clearly, for $y \in (-1,1)$, $f_Y(y) \geq 0$. Also,

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\pi\sqrt{1-t^2}}\,dt &= \left[\frac{-\arccos(t)}{\pi}\right]_{-1}^{1} \\
&= \frac{\arccos(-1)-\arccos(1)}{\pi} \\
&= \frac{\pi-0}{\pi} \\
&= 1.
\end{aligned}$$

Thus f_Y is a density function.

(c) (2 marks)

$$\begin{aligned}
E(Y) &= \int_{-1}^{1} \frac{y}{\pi\sqrt{1-y^2}}\,dy \\
&= \left[-\frac{\sqrt{1-y^2}}{\pi}\right]_{-1}^{1} \\
&= 0.
\end{aligned}$$

(d) (3 marks) For $z \in [0,1)$,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(Y^2 \leq z) \\
&= P(-\sqrt{z} \leq Y \leq \sqrt{z}) \\
&= F_Y(\sqrt{z}) - F_Y(-\sqrt{z}) \\
&= 1 - \frac{\arccos(\sqrt{z})}{\pi} - \left(1 - \frac{\arccos(-\sqrt{z})}{\pi}\right) \\
&= -\frac{\arccos(\sqrt{z})}{\pi} + \frac{\pi - \arccos(\sqrt{z})}{\pi} \\
&= 1 - \frac{2\arccos(\sqrt{z})}{\pi}.
\end{aligned}$$

(e) (2 marks) Differentiating the expression for $F_Z(z)$ gives, for $z \in (0,1)$,

$$f_Z(z) \;=\; \frac{1}{\pi\sqrt{z-z^2}}.$$

(f) (5 marks) $E(X) = \frac{\pi}{2}$ and $Var(X) = \frac{\pi^2}{12}$. Let $\psi(x) = \cos^2 x$. Then $\psi'(x) = -2\cos x \sin x = -\sin 2x$ and $\psi''(x) = -2\cos 2x$. Therefore

$$\begin{aligned} E(Z) &= E(\cos^2 X) \\ &\approx \cos^2\left(\frac{\pi}{2}\right) - \cos\pi \times \frac{\pi^2}{12} \\ &= \frac{\pi^2}{12}, \end{aligned}$$

and

$$\begin{aligned} Var(Z) &= Var(\cos^2 X) \\ &\approx \sin^2\pi \times \frac{\pi^2}{12} \\ &= 0. \end{aligned}$$

6. Two casino games are as follows. In Game 1 the chance of winning \$1 is 12/25 and the chance of losing \$1 is 13/25. In Game 2, the chance of winning \$23 is 1/25 and the chance of losing \$1 is 24/25. Let W_1 be the total winnings after playing Game 1 100 times. Let W_2 be the total winnings after playing Game 2 100 times. Losses count as negative in "winnings".

(a) Find $E[W_1]$ and $Var(W_1)$.
(b) Find $E[W_2]$ and $Var(W_2)$.
(c) Compute $P(W_1 > 0)$ using a normal approximation.
(d) Compute $P(W_2 > 0)$ using a Poisson approximation.
(e) Which game would you play and why?

[13 marks]

Solution

(a) (2.5 marks) Write $W_1 = \sum_{i=1}^{100} X_i$ where X_i is the winnings on the ith game. By the description, $E[X_i] = -1/25 = 0.04$ and $Var(X_i) = 1 - (1/25)^2 = 624/625 = 0.9984$. Therefore $E[W_1] = -4$ and $Var(W_1) = 100(624/625) = 99.84$.

(b) (2.5 marks) Similarly we can write $W_2 = \sum_{i=1}^{100} Y_i$ with $E[Y_i] = -1/25 = -.04$ and $Var(Y_i) = (23^2+24)/25 - 1/625 = 22.1184$ so that $E[W_2] = -4$ and $Var(W_2) = 2211.84$.

(c) (3 marks) We can use the CLT for W_1. For Z standard normal,

$$P(W_1 > 0) \approx P(Z > 4/\sqrt{99.84}) \approx 0.34446.$$

Or using the continuity correction,

$$P(W_1 > 0) \approx P(Z > 4.5/\sqrt{99.84}) \approx 0.326225.$$

(d) (3 marks) For W_2, if N denotes the number of times we win, then $P(W_2 > 0) = P(N \geq 5) = 1 - P(N \leq 4)$. But N is Binomial with 100 trials and success probability 0.04. Therefore N is approximately Poisson with mean 4. Therefore

$$P(W_2 > 0) = 1 - P(N \leq 4) \approx 1 - \sum_{i=0}^{4} e^{-4}\frac{4^i}{i!} \approx 0.371163.$$

[Not required for solution, but note that the exact solution is $1 - P(N \le 4) = 0.371136$ *and that the normal approximation (with continuity correction) for* W_2 *gives*

$$P(W_2 > 0) \approx P(Z > 4.5/\sqrt{2119.84}) \approx 0.46107,$$

so the exact or Poisson approximation are the correct approaches.]

(e) (2 marks) The means are the same, so best to look at the chance of being up after 100 rounds. Game 2 is (slightly) better.

7. Let Z be a standard normal random variable and let N have distribution given by $P(N = 1) = P(N = 2) = 1/2$ with N and Z independent. Set $X = Z^N$.

   (a) Compute $P(X > 1)$.
   (b) Compute $P(N = 1|X = -1)$.
   (c) Compute $P(N = 1|X > 1)$.
   (d) Compute $P(N = 1|X = 1)$.
   (e) Describe the distribution of $E[X|N]$.
   (f) Compute $E[X]$.
   (g) Describe the distribution of $Var[X|N]$.
   (h) Compute $Var(X)$.
   (i) Find the covariance and correlation of X and N.

[13 marks]

Solution

(a) (2 marks) By the law of total probability and definition of conditional probability,

$$\begin{aligned} P(X > 1) &= \frac{1}{2}(P(X > 1|N = 1) + P(X > 1|N = 2)) \\ &= \frac{1}{2}(P(Z > 1) + P(Z^2 > 1)) \\ &= \frac{3}{2}P(Z > 1) = \frac{3}{2}(0.1587) = 0.23805. \end{aligned}$$

(b) (1 mark) If $X = -1$, then $N = 1$ since if $N = 2$ then $Z^N > 0$. Thus $P(N = 1|X = -1) = 1$.

(c) (2 marks) By the definition of conditional probability and Part (a),

$$\begin{aligned} P(N = 1|X > 1) &= \frac{P(N = 1, X > 1)}{P(X > 1)} \\ &= \frac{P(X > 1|N = 1)(1/2)}{\frac{3}{2}P(Z > 1)} \\ &= \frac{P(Z > 1)(1/2)}{\frac{3}{2}P(Z > 1)} = 1/3. \end{aligned}$$

(d) (2 marks) The solution is given by

$$\frac{f_{N,X}(1,1)}{f_X(1)},$$

where $f_{N,X}$ is the joint pmf/pdf of (N, X) and f_X is the marginal pdf of X. The joint pmf/pdf satisfies

$$f_{N,X}(n,1) = f_{X|N}(1|n)P(N = n).$$

Given $N = 1$, $X = Z$ is standard normal and given $N = 2$, $X = Z^2$ has density, for $x > 0$,

$$\frac{d}{dx}P(Z^2 \le x) = \frac{d}{dx}(1 - 2P(Z > \sqrt{x})) = \frac{1}{\sqrt{2\pi}}x^{-1/2}e^{-x/2}.$$

Therefore, $f_{N,X}(1,1) = \frac{e^{-1/2}}{\sqrt{2\pi}}\left(\frac{1}{2}\right)$, and

$$f_X(1) = (f_{N,X}(1,1) + f_{N,X}(2,1)) = \frac{e^{-1/2}}{\sqrt{2\pi}},$$

and so $P(N = 1|X = 1) = 1/2$.

(e) (1 mark) Since $X = Z^N$ with Z and N independent, $E[X|N = 1] = E[Z] = 0$ and $E[X|N = 2] = E[Z^2] = 1$, and so

$$P(E[X|N] = 0) = P(E[X|N] = 1) = 1/2.$$

(f) (1 mark) $E[X] = E\big[E[X|N]\big] = 1/2$.

(g) (2 mark) Since $X = Z^N$ with Z and N independent, $Var(X|N = 1) = Var(Z) = 1$ and $Var(X|N = 2) = Var(Z^2) = E[Z^4] - E[Z^2]^2 = 3 - 1 = 2$, and so

$$P(Var(X|N) = 1) = P(Var(X|N) = 2) = 1/2.$$

(h) (1 mark) $Var(X) = Var(E[X|N]) + E[Var(X|N)] = 1/4 + 3/2 = 7/4$.

(i) (1 mark) $Cov(X, N) = E[XN] - E[X]E[N] = E[NE[X|N]] - 3/4 = 1 - 3/4 = 1/4$.

8. A computer can simulate independent standard normal random variables $Z_1, Z_2, \ldots$.

   (a) How can you transform the independent standard normal random variables Z_1, Z_2 to generate a vector (X_1, X_2) having the standard bivariate normal distribution with correlation ρ?

   (b) Given you have a standard bivariate normal vector (X_1, X_2) with correlation ρ, how can you transform (X_1, X_2) to obtain a vector (Y_1, Y_2) having the bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation ρ?

   (c) If (X_1, X_2) has the standard bivariate normal distribution with correlation ρ, describe a method to use the computer-generated independent normal random variables $Z_1, Z_2, \ldots, Z_{200}$ to estimate $E[\max\{X_1, X_2\}]$.

   (d) How can you transform the random variables Z_1, Z_2 to obtain a number U that is uniformly distributed between 0 and 1?

[6 marks]

Solution

(a) (1 mark) From labs, we know that setting

$$\begin{aligned} X_1 &= Z_1, \\ X_2 &= \rho Z_1 + \sqrt{1-\rho^2}\, Z_2, \end{aligned}$$

produces a bivariate standard normal vector with correlation ρ.

(b) (1 mark) From the definition of the general parameter standard bivariate normal, we have

$$\begin{aligned} Y_1 &= \sigma_1 X_1 + \mu_1, \\ Y_2 &= \sigma_2 X_2 + \mu_2, \end{aligned}$$

are distributed bivariate normal with the required parameters.

(c) (2 marks) We generate 100 independent bivariate vectors, each distributed as (X_1, X_2), then compute their respective maximums, and finally average over these 100 observations, appealing to the law of large numbers. In detail, according to the scheme above, for $k = 1, 2, \ldots, 100$, $(Z_k, \rho Z_k + \sqrt{1-\rho^2}\, Z_{100+k})$ are independent bivariate vectors, each distributed as (X_1, X_2). We can then estimate

$$E[\max\{X_1, X_2\}] \approx \frac{\sum_{k=1}^{100} \max\{Z_k, \rho Z_k + \sqrt{1-\rho^2}\, Z_{100+k}\}}{100}.$$

[Other pairings of the 200 *independent standard normals into* 100 *bivariate vectors are possible, another natural one is* $(Z_{2k}, \rho Z_{2k} + \sqrt{1-\rho^2}\, Z_{2k+1})$*.]*

(d) (2 marks) From lectures, we know that the angle $\Theta \in (0, 2\pi)$ that the ray from $(0,0)$ to (Z_1, Z_2) makes with the positive x-axis is uniform on $(0, 2\pi)$. Thus $\Theta/(2\pi)$ is uniform on $(0,1)$ and so we set

$$\Theta = \begin{cases} \arctan(Z_2/Z_1) & Z_1 > 0, Z_2 > 0, \\ \pi + \arctan(Z_2/Z_1) & Z_1 < 0, \\ 2\pi + \arctan(Z_2/Z_1) & Z_1 > 0, Z_2 < 0, \end{cases}$$

and $U = \Theta/(2\pi)$.

Alternatively, we can set $U = F_{Z_1}(Z_1) = \int_{-\infty}^{Z_1} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$.

9. Consider the branching process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n is the population size of the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of the offspring distribution is $A(z) = (1-p) + pz^2$ for some $0 < p < 1$.

   (a) What is the probability mass function of the offspring distribution?

   (b) Find a simple expression for $E[X_n]$ in terms of p.

   (c) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_n and q_{n+1}. Hence or otherwise, find an expression for q_n, $n = 0, 1, \ldots$.

   (d) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

   (e) Find a simple expression for $P(X_3 = 0)$ in terms of p.

   (f) What is $P(X_{100} = 3)$?

[10 marks]

Solution

(a) (1 mark) Denote by N a variable having the offspring distribution. By the definition of the probability generating function, the coefficient of z^n is $P(N = n)$. Thus $P(N = 2) = 1 - P(N = 0) = p$.

(b) (2 marks) Since the expectation of N is $2p$ and the sequence $(E[X_n])_{n\geq 0}$ satisfies $E[X_n] = E[N]^n$, we have $E[X_n] = (2p)^n$.

(c) (2 marks) If A_n denotes the generating function of X_n, then $A_n(0) = q_n$. Moreover, lecture facts imply $A_{n+1} = A(A_n)$ and so

$$q_{n+1} = A(q_n).$$

(d) (2 marks) We know from lectures that $\lim_{n\to\infty} q_n = q$ is the minimum non-negative solution of $A(z) = z$. The equation $z = (1-p) + pz^2$, has solutions $z = 1, (1-p)/p$, and so $q = 1$ if $p \leq 1/2$ and $q = (1-p)/p$ if $p > 1/2$. Note that $q = 1$ for $p \leq 1/2$ also follows by noting that in this case, the offspring distribution has mean no greater than one and that this implies extinction is assured.

(e) (2 marks) We compute $A_3(0) = A(A(A(0)))$. Note first

$$A_3(z) = (1-p) + p[(1-p) + pA(z)^2]^2,$$

and since $A(0) = (1-p)$, we have

$$A_3(0) = (1-p) + p[(1-p) + p(1-p)^2]^2.$$

Note that expanding this expression, we have

$$A_3(0) = (1-p) + p(1-p)^2 + p(2p(1-p))(1-p)^2 + p^3(1-p)^4,$$

which corresponds to summing over the probability of all possible outcomes in the event $\{X_3 = 0\}$.

(f) (1 mark) It's easy to see that X_n is even for $n \geq 1$, so $P(X_{100} = 3) = 0$.

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817