Tutorial 10: Solutions

Q1. (i). The coefficient of $x_i y_j$ in $\langle \mathbf{x}, \mathbf{y} \rangle$ is the entry (ij) in the matrix A. Hence

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii). We have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{x} \rangle &= x_1^2 - 2x_1x_2 + 4x_2^2 + x_3^2 \\ &= (x_1 - x_2)^2 + 3x_2^2 + x_3^2 \end{aligned}$$

Hence $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$, and thus $\mathbf{x} = \mathbf{0}$.

Q2. Let $\mathbf{u_1} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{20}}, \frac{3}{\sqrt{20}}\right)$, $\quad \mathbf{u_2} = \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{20}}, -\frac{1}{\sqrt{20}}\right)$, $\quad \mathbf{u_3} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then

$$\begin{aligned} \mathbf{u_1} \cdot \mathbf{u_2} &= \left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(-\frac{1}{\sqrt{20}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(-\frac{1}{\sqrt{20}}\right) = \frac{3}{10} - \frac{3}{20} - \frac{3}{20} = 0 \\ \mathbf{u_1} \cdot \mathbf{u_3} &= \left(\frac{1}{\sqrt{10}}\right) \times 0 + \left(\frac{3}{\sqrt{20}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{3}{\sqrt{20}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0 \\ \mathbf{u_2} \cdot \mathbf{u_3} &= \left(\frac{3}{\sqrt{10}}\right) \times 0 + \left(-\frac{1}{\sqrt{20}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{20}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0 \end{aligned}$$

Since each distinct pair of vectors is orthogonal, the vectors form an orthogonal set. Furthermore

$$\begin{aligned} ||\mathbf{u_1}||^2 &= \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{3}{\sqrt{20}}\right)^2 + \left(\frac{3}{\sqrt{20}}\right)^2 = 1 \\ ||\mathbf{u_2}||^2 &= \left(\frac{3}{\sqrt{10}}\right)^2 + \left(-\frac{1}{\sqrt{20}}\right)^2 + \left(-\frac{1}{\sqrt{20}}\right)^2 = 1 \\ ||\mathbf{u_3}||^2 &= 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \end{aligned}$$

as the length of each vector is 1 the basis is orthonormal.

Q3. Let $\{\mathbf{u_1}, \mathbf{u_2}\}$ be an orthonormal basis for a vector space W.

(i). As the basis for W has two vectors the dimension is 2.

(ii). The length of $\mathbf{v}$ written $||\mathbf{v}||$ is $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

(iii). As $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthonormal basis $||\mathbf{u_1}|| = ||\mathbf{u_2}|| = 1$, and $\langle \mathbf{u_1}, \mathbf{u_2} \rangle = 0$.

(iv). Starting with $\mathbf{w} = \alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}$ and taking the inner product with respect to $\mathbf{u_1}$ gives

$$\langle \mathbf{w}, \mathbf{u_1} \rangle = \langle \alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}, \mathbf{u_1} \rangle = \langle \alpha_1 \mathbf{u_1}, \mathbf{u_1} \rangle + \langle \alpha_2 \mathbf{u_2}, \mathbf{u_1} \rangle = \alpha_1 \langle \mathbf{u_1}, \mathbf{u_1} \rangle + \alpha_2 \langle \mathbf{u_2}, \mathbf{u_1} \rangle = \alpha_1$$

Similarly, taking the inner product with respect to $\mathbf{u_2}$ gives

$$\langle \mathbf{w}, \mathbf{u_2} \rangle = \langle \alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}, \mathbf{u_2} \rangle = \langle \alpha_1 \mathbf{u_1}, \mathbf{u_2} \rangle + \langle \alpha_2 \mathbf{u_2}, \mathbf{u_2} \rangle = \alpha_1 \langle \mathbf{u_1}, \mathbf{u_2} \rangle + \alpha_2 \langle \mathbf{u_2}, \mathbf{u_2} \rangle = \alpha_2$$

Thus we have

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{u_1} \rangle \mathbf{u_1} + \langle \mathbf{w}, \mathbf{u_2} \rangle \mathbf{u_2}$$

(v). We have $\mathbf{u_1} \cdot \mathbf{u_2} = 0,\ \mathbf{u_1} \cdot \mathbf{u_1} = \mathbf{u_2} \cdot \mathbf{u_2} = 1$. Using the above gives

$$\mathbf{w} \cdot \mathbf{u_1} = (-1,1,5,5) \cdot \frac{1}{2}(1,-1,1,1) = \frac{1}{2}(-1-1+5+5) = 4$$

$$\mathbf{w} \cdot \mathbf{u_2} = (-1,1,5,5) \cdot \frac{1}{2}(-1,1,1,1) = \frac{1}{2}(1+1+5+5) = 6$$

So we have $\mathbf{w} = 4\mathbf{u_1} + 6\mathbf{u_2}$.

(vi). We have

$$\alpha_1 = \mathbf{w} \cdot \mathbf{u_1} = (-1,1,3,5) \cdot \frac{1}{2}(1,-1,1,1) = \frac{1}{2}(-1-1+3+5) = 3$$

and

$$\alpha_2 = \mathbf{w} \cdot \mathbf{u_2} = (-1,1,3,5) \cdot \frac{1}{2}(-1,1,1,1) = \frac{1}{2}(1+1+3+5) = 5$$

so the projection is

$$3\mathbf{u_1} + 5\mathbf{u_2} = (-1,1,4,4)$$

Q4. First we normalize $\mathbf{v_1} = (1,0)$

$$||\mathbf{v_1}|| = \sqrt{2-0-0+0} = \sqrt{2} \quad \Rightarrow \quad \mathbf{u_1} = \frac{\mathbf{v_1}}{||\mathbf{v_1}||} = \frac{1}{\sqrt{2}}(1,0)$$

Next using $\langle \mathbf{v_2}, \mathbf{u_1} \rangle = \langle (0,1)1_{\overline{\sqrt{2}(1,0)\rangle = \frac{1}{\sqrt{2}}(0-0-1+0) = -\frac{1}{\sqrt{2}}}}$ we find

$$\mathbf{w_2} = \mathbf{v_2} - \langle \mathbf{v_2}, \mathbf{u_1} \rangle \mathbf{u_1} = (0,1) - (-\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}(1,0) = (0,1) + (\frac{1}{2},0) = \frac{1}{2}(1,2)$$

Now we normalize $\mathbf{w_2}$

$$||\mathbf{w_2}|| = \sqrt{\frac{1}{4}(2-2-2+4)} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \mathbf{u_2} = \frac{\mathbf{w_2}}{||\mathbf{w_2}||} = \frac{\frac{1}{2}(1,2)}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}(1,2)$$

Hence an orthonormal basis is $\{\frac{1}{\sqrt{2}}(1,0), \frac{1}{\sqrt{2}}(1,2)\}$.

Q5. (i). Here we have $(x_1,y_1) = (1,14)$, $(x_2,y_2) = (2,10)$, $(x_3,y_3) = (3,10)$. Thus in the notation of the least squares formulation

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 14 \\ 10 \\ 10 \end{bmatrix} \quad \Rightarrow \quad A^T A = \begin{bmatrix} 3 & 6 \\ 1 & 14 \end{bmatrix} \quad A^T \mathbf{y} = \begin{bmatrix} 34 \\ 64 \end{bmatrix}$$

So we have to solve the equations

$$\begin{bmatrix} 3 & 6 \\ 1 & 14 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 34 \\ 64 \end{bmatrix}$$

Writing this as an augmented matrix system;

$$\left[\begin{array}{cc|c} 3 & 6 & 34 \\ 1 & 14 & 64 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 0 & 46 \\ 0 & 2 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{46}{3} \\ 0 & 1 & -2 \end{array}\right]$$

So we have $a = \frac{46}{3},\ b = -2$ and the line of best fit is $y = \frac{46}{3} - 2x$

(ii). Let $y = 0$ in the answer to (i) gives $x = \frac{23}{3} \approx 7.7$ so we would predict the product would run out in about 8 days.