Tutorial 3: Solutions

Q1. (i). From the information given reduced row echelon form of the coefficient matrix has a row of zeros and so the coefficient matrix is singular (i.e. does not have an inverse).

(ii). From the given information the reduced row echelon form of the linear system is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -20 \\ 0 & 1 & 0 & -1 & 58 \\ 0 & 0 & 1 & 0 & 67 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There is no leading entry corresponding to the variable z so we set $z = t, \quad t \in \Re$ and so

$$\begin{aligned} w &= -20 - t \\ x &= 58 + t \\ y &= 67 \\ z &= t \end{aligned}$$

Hence the solution set is $\{(w, x, y, z) : w = -20 - t,\ x = 58 + t, y = 67,\ z = t, t \in \Re\}$.

Q2. (i). We verify that $AA^{-1} = I$.

(ii). Using the sent integers to successively form the columns of the 4×4 matrix C gives

$$C = \begin{bmatrix} -19 & 0 & 3 & -2 \\ 19 & 18 & 10 & 20 \\ 25 & -18 & -8 & -7 \\ -21 & 15 & 3 & 12 \end{bmatrix}$$

We then compute

$$A^{-1}C = \begin{bmatrix} 4 & 15 & 8 & 23 \\ 15 & 21 & 15 & 15 \\ 0 & 18 & 13 & 18 \\ 25 & 0 & 5 & 11 \end{bmatrix}$$

Now making the correspondence blank = 0, A = 1 etc,. and reading down the columns we read off the message DO YOUR HOMEWORK.

Q3. (i). $\det \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \frac{1}{2} \times \left(-\frac{1}{3}\right) - \left(-\frac{3}{4}\right) \times \frac{2}{3} = -\frac{1}{6} - \frac{1}{2} = -\frac{2}{3}$

(ii). $\det J = -1 \times \begin{vmatrix} -2 & 7 \\ 1 & -2 \end{vmatrix} + (-1) \times \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + 0 = -1 \times (-3) + (-1) \times (-7) = 10$

(iii). $\det K = 2 \times \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} - (-1) \times \begin{vmatrix} -1 & 0 \\ 3 & 0 \end{vmatrix} + -1 \times \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} = -9$

(iv). In the case of K, expanding by the final column gives

$\det K = 1 \times \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} = 1 \times (-9) = -9$. This is simpler than (iii)

Q4. (i). $\det(J^2K) = \det(J)^2 \det(K) = 100 \times (-9) = -900$

(ii). KH is not defined since K is 3×3 and H is 2×2. Hence $\det(KH)$ is not defined.

(iii). $\det(3J) = 3^3 \det(J) = 27 \times 10 = 270$.

(iv). $\det(K^T(J^{-1})^2) = \frac{\det(K)}{(\det J)^2} = \frac{-9}{100}$.

Q5. (i).

Operation 3:	does not affect the determinant.
Operation 1:	changes the sign.
Operation 2:	multiplies the determinant by α.

(ii).

$$\begin{bmatrix} 1 & -2 & 7 & 3 \\ 0 & 1 & -2 & 4 \\ -2 & 3 & -3 & 1 \\ -3 & 6 & -21 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_1 \\ R_4 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 7 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & -1 & 11 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix} \; R_3 + R_2 \sim \begin{bmatrix} 1 & -2 & 7 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 9 & 11 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

It follows from this that $\det A = 1 \times 1 \times 9 \times 9 = 81$.

Q6. We know the matrix in invertible if and only if the determinant is non-zero. Expanding by the final row shows that the determinant is equal to

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Hence the matrix is invertible provided $ad - bc \neq 0$. In general, we see from the rules of matrix multiplication that if a matrix is of the form

$$\begin{bmatrix} A_{n\times n} & 0_{n\times(N-n)} \\ 0_{(N-n)\times n} & \mathbb{I}_{N-n} \end{bmatrix}$$

then its inverse is given by

$$\begin{bmatrix} A^{-1} & 0_{n\times(N-n)} \\ 0_{(N-n)\times n} & \mathbb{I}_{N-n} \end{bmatrix}$$

Hence the inverse is

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$