Quantitative Risk Analysis Using Confidence Intervals

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Outline of talk

- Population vs Sample
- Large-Sample Confidence Intervals for Population Mean
- Small-Sample Confidence Intervals for Population Mean
- Confidence Intervals for the difference between Two Population Means

Quantitative Analysis – Quality Risks

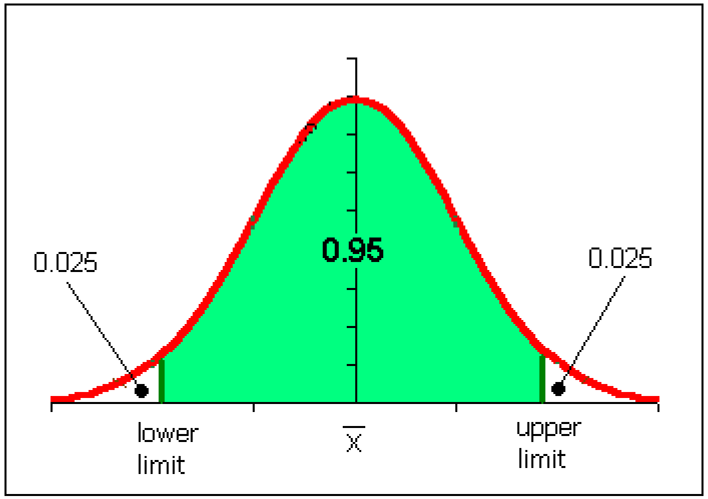
Consider a machine that makes steel bars for use in building construction. The specification for the diameter of the bars is 2.0+/-0.1 cm. During the last hour, the machine has made 1000 rods.

The quality engineer draws a random sample of 50 rods, measures them, and finds that 46 of them (92%) meet the diameter specification.

It is unlikely that the sample of 50 bars represents the population of 1000 perfectly !

Question ?

- Having observed that 92% of the sample bars were good, it indicates that the percentage of acceptable bars in the population as an interval of the form 92% +/ x%. How should x be calculated?
  - Requires the construction of a **Confidence Interval**



Statistical inference

- Most of time it is **NOT** possible to obtain data for the entire population.
  - For example, it is impossible to measure the 28-days compressive strength of every concrete segment manufactured by a factory to determine the mean strength, variance and standard deviation of the concrete segments

- Therefore, population parameters have be estimated using **samples**.
- This process is know as ***statistical inference***.

Introduction – Basic Ideas

The process of making estimates about the truth from a sample.

Observation

Population
Parameters
μ, σ

Sample

Not observable

Sample statistics

sample Mean $\overline{X}$

sample Standard Deviation S

Population vs Sample

Population

quantity (count) = N

mean = μ

variance = σ^2

standard deviation = σ

Sample

quantity (count) = n

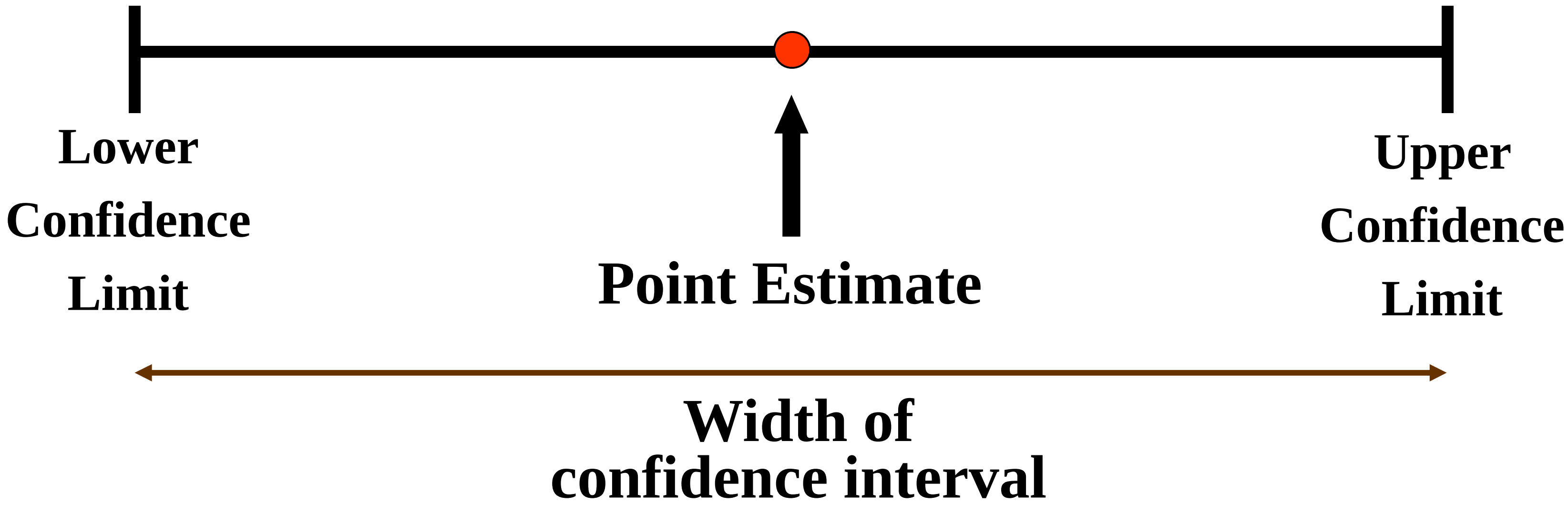
mean = $\overline{X}$

variance = S^2

standard deviation = S

- The sample mean and variance of a random sample can be used as estimators of the population mean and variance respectively.
- The sample mean and variance are referred to as statistics.
- A statistic is any function of observation in a random sample.

- An **interval estimate** provides more information about a population characteristic than does a point estimate. It provides a confidence level for the estimate. Such interval estimates are called **confidence intervals**.



Best Point Estimate of population mean μ is sample mean

The general formula for all confidence intervals is equal to

Population mean

$$\mu = \text{Sample mean } \bar{X} \pm (\text{Critical Value}) \times (\text{Standard Error})$$

Depending on the sampling distribution

Population Mean and Variance

If data for all of the population under investigation is known,

Population Mean

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Population Variance

$$\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$$

N is the size of the population.

The **population standard deviation** σ is the positive square root of the population variance.

Sample Mean and Variance

Sample Mean

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

n is the sample size.

Sample Variance

$$S^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{X})^2}{n-1}$$

The **sample standard deviation** ***S*** is the **positive** square root of the sample variance.

Standard Error

μ = **Sample mean $\overline{X} \pm$ (Critical Value) $\times$ (Standard Error)**

- **Standard Error** of the Mean formula is

$$\text{Standard Error} = \frac{\text{Standard Deviation}}{\sqrt{n}}$$

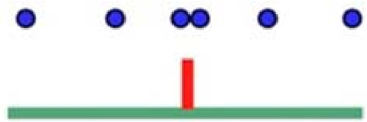
n is the number of measurements

Standard Error = (σ / $\sqrt{n}$) when population σ is known

Standard Error = (S / $\sqrt{n}$) when population σ is not known

Standard Deviation vs Standard Error

- **Standard Deviation** is a measure of spread or variability for a given set of scores.



below mean **mean** **above mean**

- **Standard Error** quantifies how much variability exists between your sample statistic and the population parameter.

Standard Error of Sample Mean

Examples

- Population σ known
  Population standard deviation $\sigma = 12$
  Sample size $n = 36$
  Standard Error $= (\sigma / \sqrt{n}) = (12 / \sqrt{36}) = 12 / 6 = 2$

- Population σ **not** known
  Sample standard deviation $S = 20$
  Sample size $n = 16$
  Standard Error $= (S / \sqrt{n}) = (20 / \sqrt{16}) = 20 / 4 = 5$

Determine Critical Value

$$\mu = \text{Sample mean } \bar{X} \pm \text{(Critical Value)} \times \text{(Standard Error)}$$

Large – Sample : Z-distribution

Small – Sample : t-distribution

Central Limit Theorem (CLT)

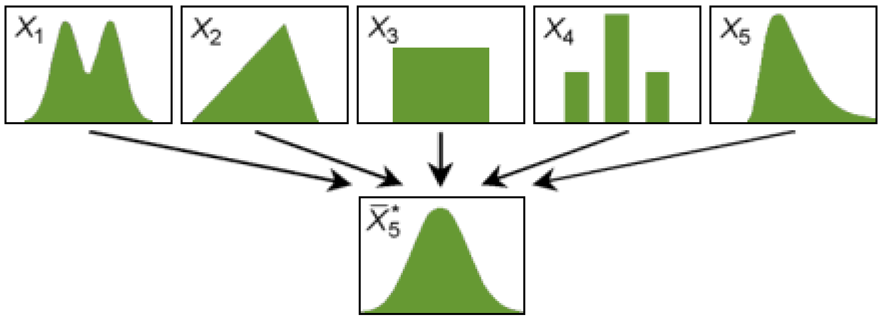
<u>The ultimate benefit of the Central Limit Theorem is use of the Z formula for sample means.</u>

Central Limit Theorem (CLT)

- Irrespective of the shape of the underlying distribution of the population,

By increasing the sample size,

Sample means & proportions will approximate normal distributions if the sample sizes are sufficiently large. It all comes back to Z statistic!



Central Limit Theorem (CLT)

Z statistics– score / standard score

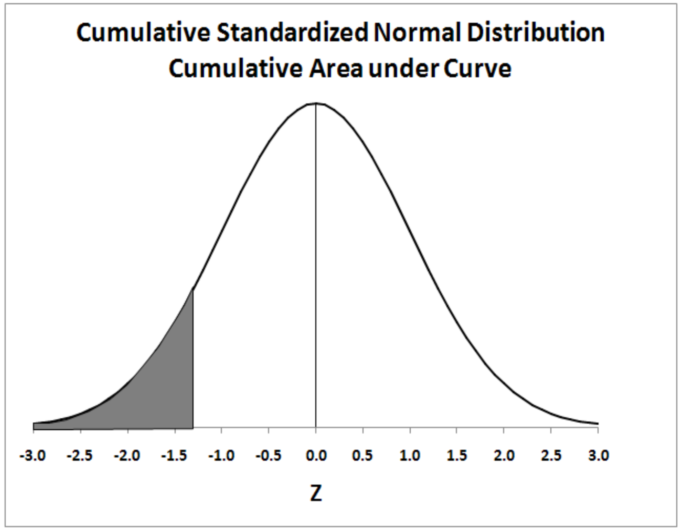
- Used to compare means from different normally distributed sets of data
- The Z score indicates how many standard deviations a sample mean $\overline{X}$ is away from the population mean μ
- The formula

$$Z = \frac{\overline{X} - \mu}{\text{Standard deviation}}$$

$\overline{X}$ Raw score or observation to be standardized

μ population mean

Central Limit Theorem (CLT)



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-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
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-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
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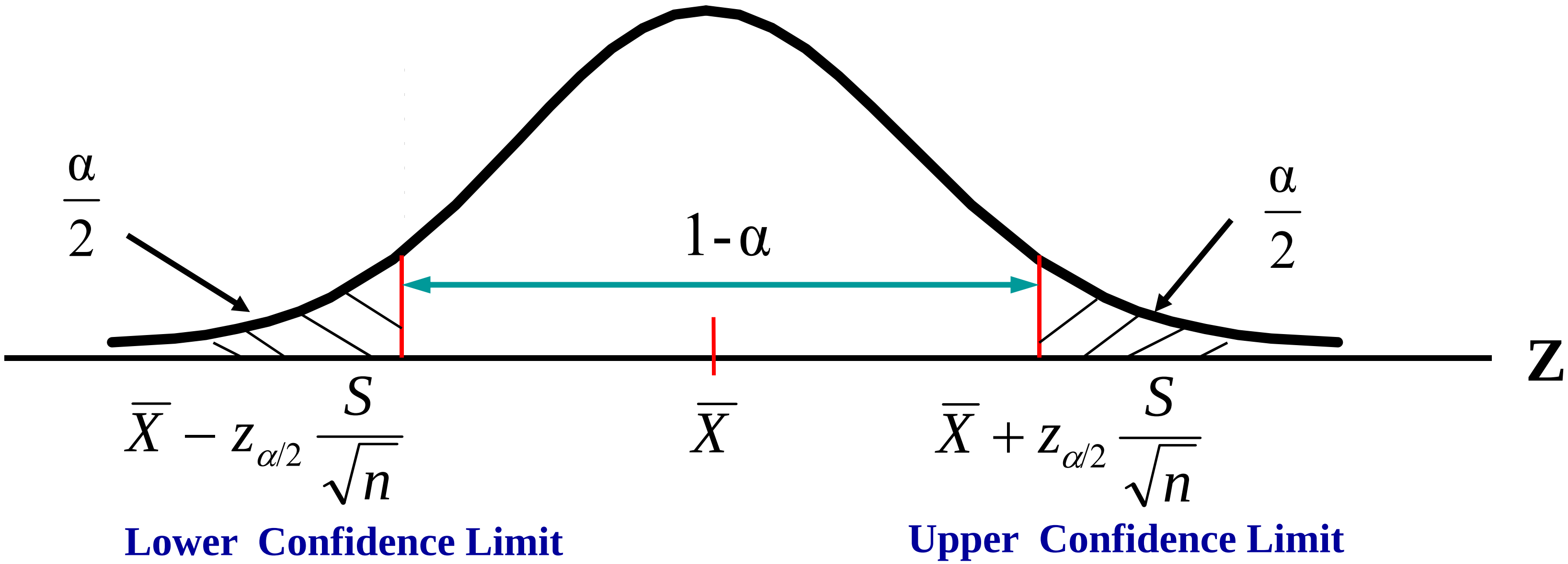
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Large-Sample Confidence Intervals for Population μ

- **For a large (n > 30) random sample from a population**

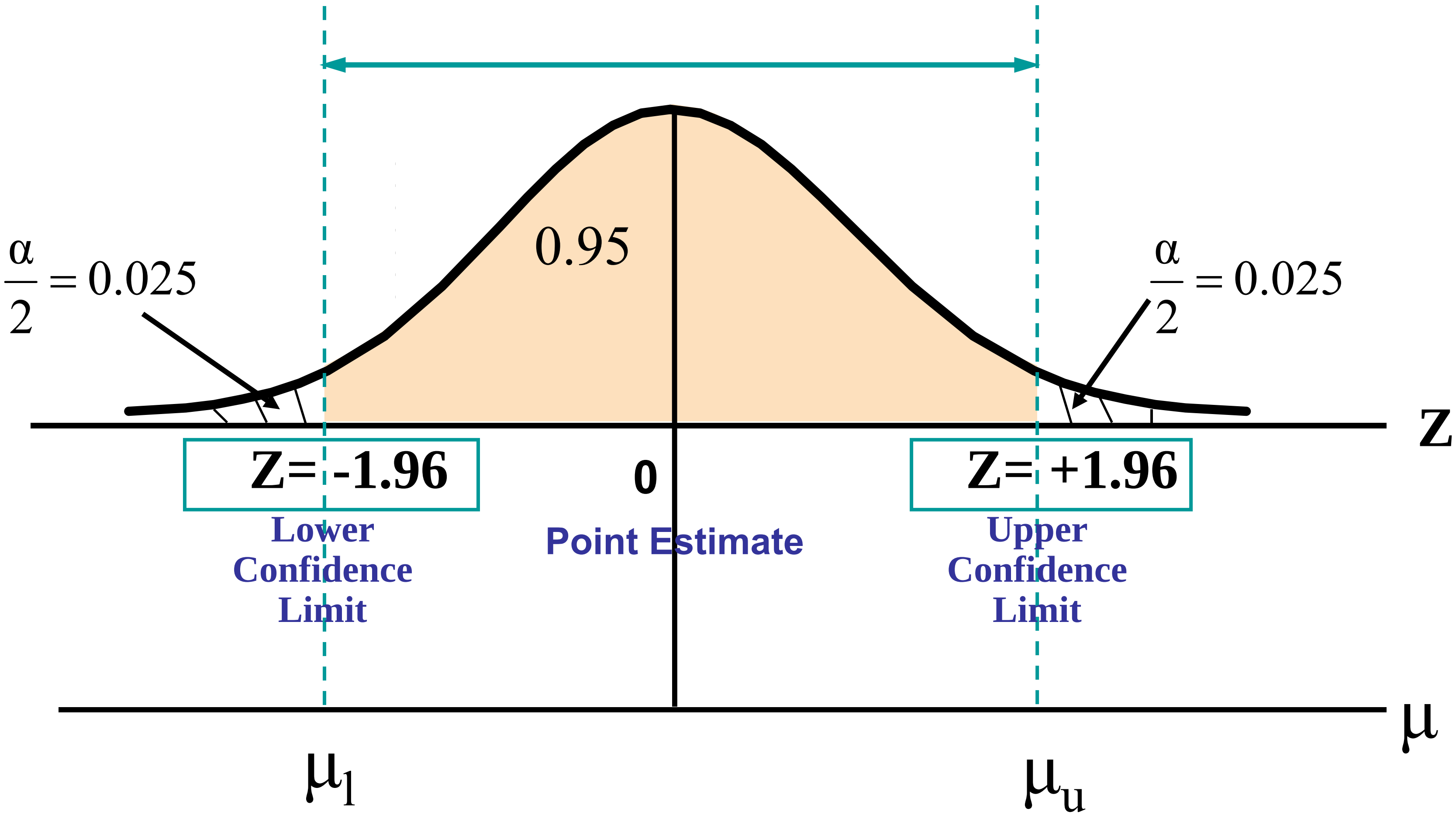
$$\mu = \overline{X} \pm z_{\alpha/2} \frac{S}{\sqrt{n}}$$

α is the proportion of the distribution in the two tails areas outside the confidence interval

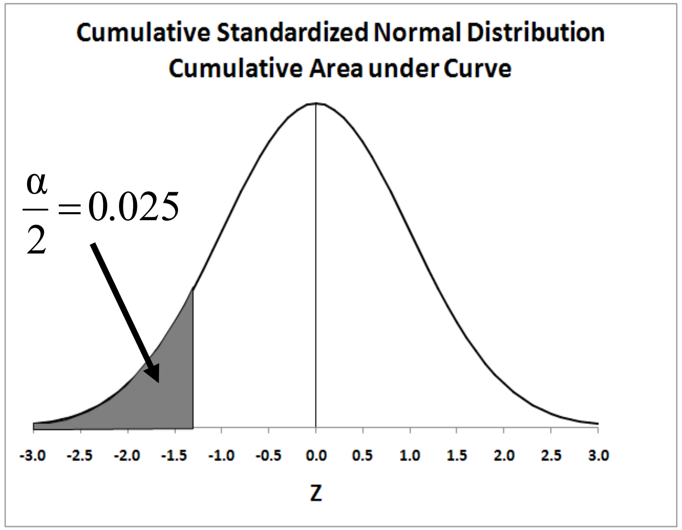


Confidence Interval Estimates for Population μ

- Suppose confidence level = 95% (α = 0.05)
- Also written (1 - α) = 0.95



Confidence Interval Estimates for Population μ



$$z_{\alpha/2=0.025} = -1.96$$

$$\overline{X} \pm 1.96 \frac{S}{\sqrt{n}}$$

is a 95% confidence interval for μ

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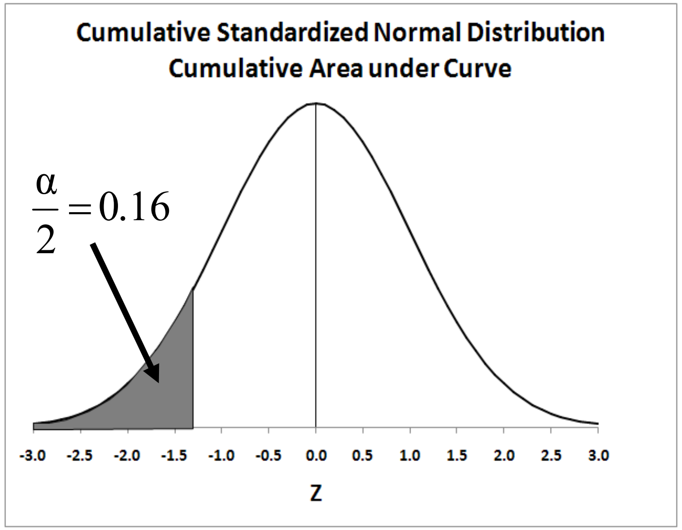
Confidence Interval Estimates for Population μ



$$z_{\alpha/2=0.16} = -1 \qquad \overline{X} \pm \frac{S}{\sqrt{n}}$$

is a 68% confidence interval for μ

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0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Large-Sample Confidence Intervals for Population μ

$\overline{X} \pm \frac{S}{\sqrt{n}}$ **is a 68% confidence interval for μ**

$\overline{X} \pm 1.645 \frac{S}{\sqrt{n}}$ **is a 90% confidence interval for μ**

$\overline{X} \pm 1.96 \frac{S}{\sqrt{n}}$ **is a 95% confidence interval for μ**

$\overline{X} \pm 2.58 \frac{S}{\sqrt{n}}$ **is a 99% confidence interval for μ**

Example 1 (Quality Risks)

In a sample of 50 microdrills drilling a low-carbon alloy steel, the average lifetime (expressed as the number of holes drilled before failure) was 12.68 with a standard deviation of 6.83.

Find a 95% confidence interval for the mean lifetime of microdrills under these conditions.

Solution:

Solution:

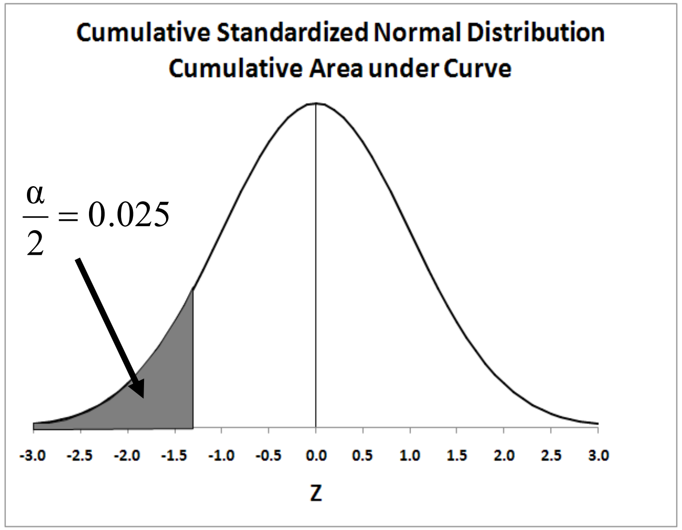
$$\text{Mean}: \overline{X} = 12.68$$

$$\text{Standard Deviation}: S = 6.83$$

$$\text{Sample Size}: n = 50$$

$$\text{Use confidence interval equation}: \overline{X} \pm Z \frac{S}{\sqrt{n}}$$

Solution (continued):



$$z_{\alpha/2=0.025} = -1.96$$

$$\overline{X} \pm 1.96 \frac{S}{\sqrt{n}}$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

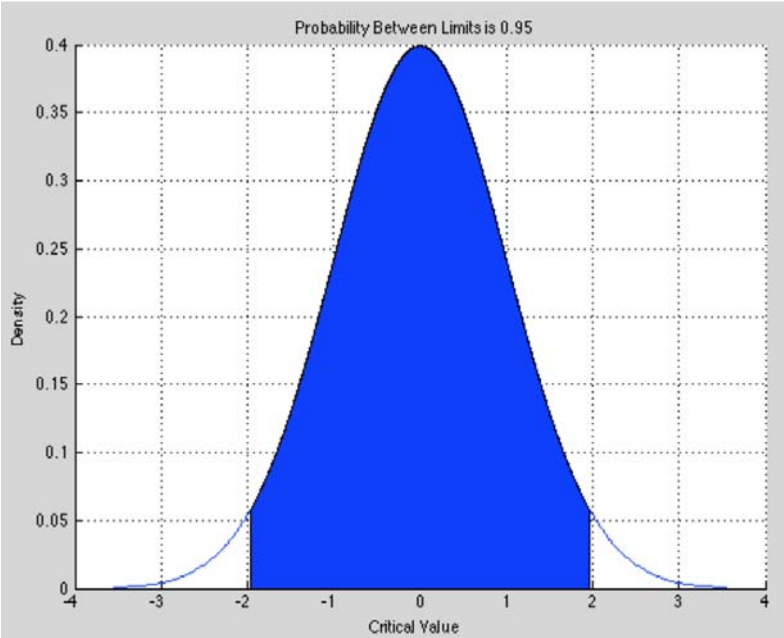
Solution (continued):

$$\overline{X} \pm 1.96\frac{S}{\sqrt{n}} = 12.68 \pm 1.96 \text{ x } \frac{6.83}{\sqrt{50}} = 12.68 \pm 1.89$$

95% Confidence Inverval for the mean lifetime of microdrills : (10.79, 14.57)

Large-Sample Confidence Intervals for Population μ

```
%calculate interval
N = 50;                  % sample size
mu = 12.68;              % sample mean (normally distributed)
stdev = 6.83;            % sample standard deviation
stderror = stdev/sqrt(N);     % sample standard error
alpha = 0.05;
alphaon2 = alpha/2;        % alpha over 2
Z_alphaon2 = norminv(alphaon2);
display(Z_alphaon2);
normspec([norminv(alphaon2) -norminv(alphaon2)],0,1);
grid on;
CI = [mu+Z_alphaon2*stderror, mu-Z_alphaon2*stderror];
display(CI);
```




```
Z_alphaon2 =

   -1.9600

CI =

   10.7869   14.5731
```

Example 2 (Quality Risks)

In a sample of 100 batteries produced by a certain method, the average lifetime was 150 hours and the standard deviation was 25 hours.

(a) Find a 95% confidence interval for the mean lifetime of batteries produced by this method.

(b) An engineer claims that the mean lifetime is between 147 and 153 hours. With what level of confidence can this statement be made?

(c) Approximately how many batteries must be sampled so that a 99% confidence interval will specify the mean to within ± 2 hours?

Solution:

Solution:

(a) Find a 95% confidence interval for the mean lifetime of batteries produced by this method.

$$\bar{X} = 150 \text{ hours}$$

$$S = 25 \text{ hours}$$

$$n = 100$$

$$\bar{X} \pm 1.96 \frac{S}{\sqrt{n}} = 150 \pm 1.96 \times \frac{25}{\sqrt{100}} = 150 \pm 4.9$$

95% Confidence interval for the mean lifetime of batteries: (145.1, 154.9)

Solution (continued):

(b) An engineer claims that the mean lifetime is between 147 and 153 hours. With what level of confidence can this statement be made?

Determine the Z value which gives an interval of the mean to within the range (147, 153)

Hence:

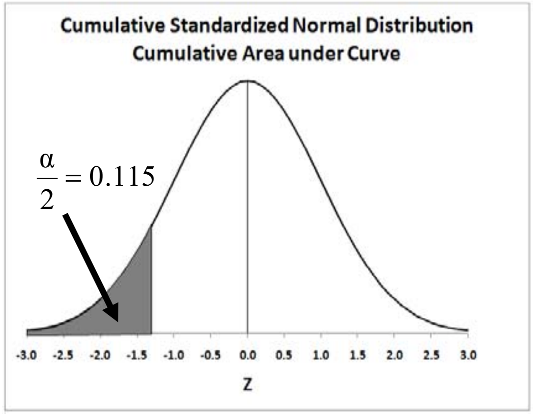
$$\bar{X} \pm Z\frac{S}{\sqrt{n}} = 150 \pm Z\frac{25}{\sqrt{100}} = (147, 153)$$

$$\Rightarrow 150 - Z\frac{25}{\sqrt{100}} = 147$$

$$\Rightarrow Z = -1.2$$

Solution (continued):

For Z = -1.2, the $\alpha/2$ value is approximately 0.115



The level of confidence:

$1 - \alpha = 1 - 2 \times 0.115 = 77\%$

Hence, the level of confidence is 77%

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Solution (continued):

(c) Approximately how many batteries must be sampled so that a 99% confidence interval will specify the mean to within ±2 hours??

For 99% Confidence interval, Z = 2.58 (Slide 24)

Hence the 99% Confidence interval is $150 \pm 2.58 \times \frac{25}{\sqrt{n}}$

For the mean to within ± 2 hours

The interval is 150 ± 2

$$150 \pm 2 = 150 \pm 2.58 \times \frac{25}{\sqrt{n}} \qquad \Rightarrow 2 = 2.58 \times \frac{25}{\sqrt{n}}$$

$$n = 1041 \text{ (Rounded up to integer)}$$

1041 batteries are required

- What can we do if $\overline{X}$ is the mean of a small sample?
  - S may not close to σ
  - X may not be approximately normal (CLT)

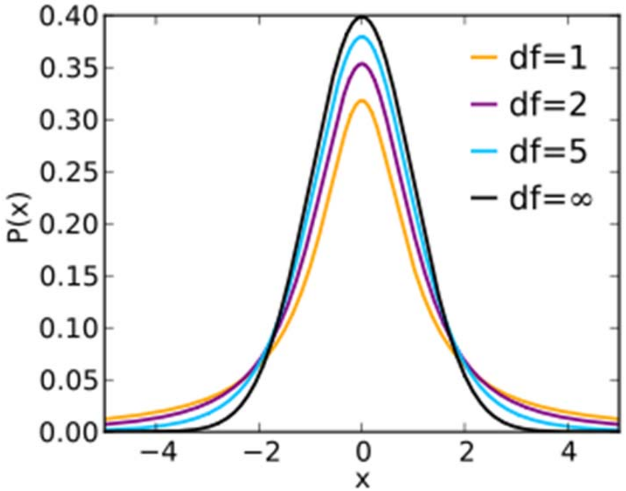
Student's t distribution

Discovered in 1908 by William Sealy Gossett

- **For a small ($n \leq 30$) random sample from a population**

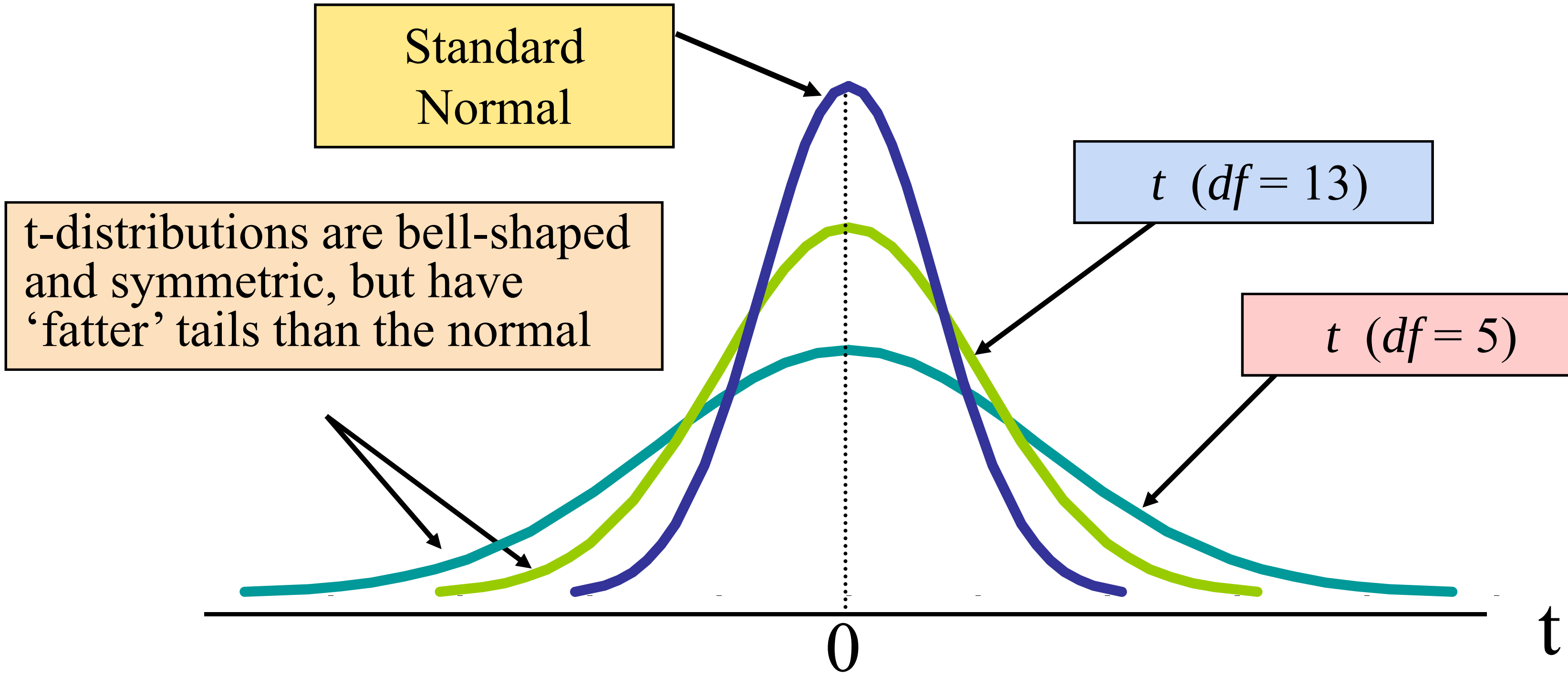
$$\mu = \overline{X} \pm t_{n-1,\, \alpha/2} \frac{S}{\sqrt{n}}$$

where t is the critical value of the t distribution with n-1 degrees of freedom and an area of α/2 in each tail.

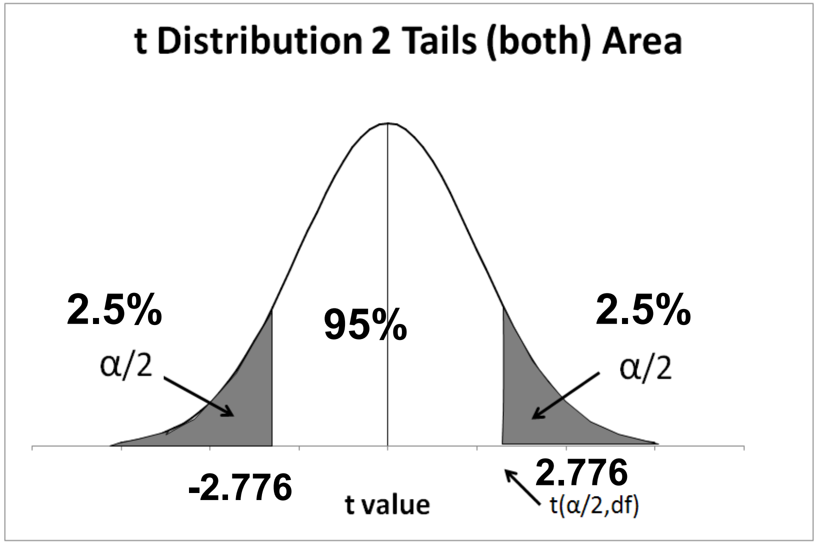


Small-Sample Confidence Intervals for Population μ

- t distribution is symmetrical around its mean of zero, like Z distribution.
- Compared to Z distribution, a larger portion of the probability areas are in the tails.
- As n increases, the t distribution approached the Z distribution.
- t values depends on the degree of freedom.



Small-Sample Confidence Intervals for Population μ



$$t_{n-1=4,\ \alpha/2=0.025} = 2.776$$

$$\overline{X} \pm 2.776 \frac{S}{\sqrt{n}}$$

is a 95% confidence interval for μ

Degrees of Freedom	Combined Area α in Two Tails					
	0.250	0.100	0.050	0.025	0.010	0.005
1	2.4142	6.3138	12.7062	25.4517	63.6567	127.3213
2	1.6036	2.9200	4.3027	6.2053	9.9248	14.0890
3	1.4226	2.3534	3.1824	4.1765	5.8409	7.4533
4	1.3444	2.1318	2.7764	3.4954	4.6041	5.5976
5	1.3009	2.0150	2.5706	3.1634	4.0321	4.7733
6	1.2733	1.9432	2.4469	2.9687	3.7074	4.3168
7	1.2543	1.8946	2.3646	2.8412	3.4995	4.0293
8	1.2403	1.8595	2.3060	2.7515	3.3554	3.8325
9	1.2297	1.8331	2.2622	2.6850	3.2498	3.6897
10	1.2213	1.8125	2.2281	2.6338	3.1693	3.5814
11	1.2145	1.7959	2.2010	2.5931	3.1058	3.4966
12	1.2089	1.7823	2.1788	2.5600	3.0545	3.4284
13	1.2041	1.7709	2.1604	2.5326	3.0123	3.3725
14	1.2001	1.7613	2.1448	2.5096	2.9768	3.3257
15	1.1967	1.7531	2.1314	2.4899	2.9467	3.2860
16	1.1937	1.7459	2.1199	2.4729	2.9208	3.2520
17	1.1910	1.7396	2.1098	2.4581	2.8982	3.2224
18	1.1887	1.7341	2.1009	2.4450	2.8784	3.1966
19	1.1866	1.7291	2.0930	2.4334	2.8609	3.1737
20	1.1848	1.7247	2.0860	2.4231	2.8453	3.1534

$$\overline{X} \pm 1.96 \frac{S}{\sqrt{n}}$$ **is a 95% confidence interval for μ (large-Sample)**

Example 3 (Quality Risks)

Measurements of the nominal shear strength (in kN) for a sample of 15 prestressed concrete beams. The results are

580	400	428	825	850	875	920	550
575	750	636	360	590	735	950	

Find a 99% confidence interval for the mean shear strength.

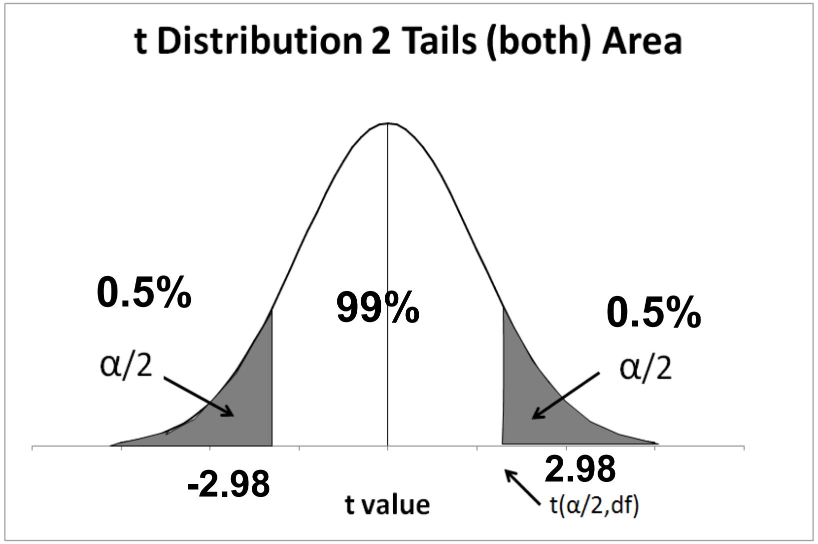
Solution:

Solution:

$$n = 15$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = 668.27kN$$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2} = 192.09kN$$



Degrees of Freedom	Combined Area α in Two Tails					
	0.250	0.100	0.050	0.025	0.010	0.005
1	2.4142	6.3138	12.7062	25.4517	63.6567	127.3213
2	1.6036	2.9200	4.3027	6.2053	9.9248	14.0890
3	1.4226	2.3534	3.1824	4.1765	5.8409	7.4533
4	1.3444	2.1318	2.7764	3.4954	4.6041	5.5976
5	1.3009	2.0150	2.5706	3.1634	4.0321	4.7733
6	1.2733	1.9432	2.4469	2.9687	3.7074	4.3168
7	1.2543	1.8946	2.3646	2.8412	3.4995	4.0293
8	1.2403	1.8595	2.3060	2.7515	3.3554	3.8325
9	1.2297	1.8331	2.2622	2.6850	3.2498	3.6897
10	1.2213	1.8125	2.2281	2.6338	3.1693	3.5814
11	1.2145	1.7959	2.2010	2.5931	3.1058	3.4966
12	1.2089	1.7823	2.1788	2.5600	3.0545	3.4284
13	1.2041	1.7709	2.1604	2.5326	3.0123	3.3725
14	1.2001	1.7613	2.1448	2.5096	2.9768	3.3257
15	1.1967	1.7531	2.1314	2.4899	2.9467	3.2860
16	1.1937	1.7459	2.1199	2.4729	2.9208	3.2520
17	1.1910	1.7396	2.1098	2.4581	2.8982	3.2224
18	1.1887	1.7341	2.1009	2.4450	2.8784	3.1966
19	1.1866	1.7291	2.0930	2.4334	2.8609	3.1737
20	1.1848	1.7247	2.0860	2.4231	2.8453	3.1534

Solution (continued):

$$t_{n-1=14,\alpha/2=0.005} = 2.98$$

$$\overline{X} \pm 2.98\frac{S}{\sqrt{n}} = 668.27 \pm 2.98\frac{192.09}{\sqrt{15}}$$

$$\Rightarrow (520.47, 816.07)$$

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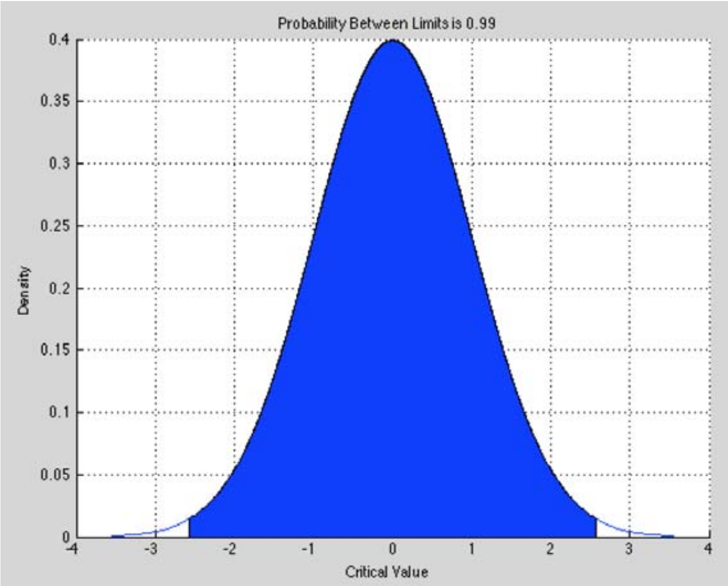
Small-Sample Confidence Intervals for Population μ

example_03.dat

	A VarName1
	NUMBER
1	580
2	400
3	428
4	825
5	850
6	875
7	920
8	550
9	575
10	750
11	636
12	360
13	590
14	735
15	950

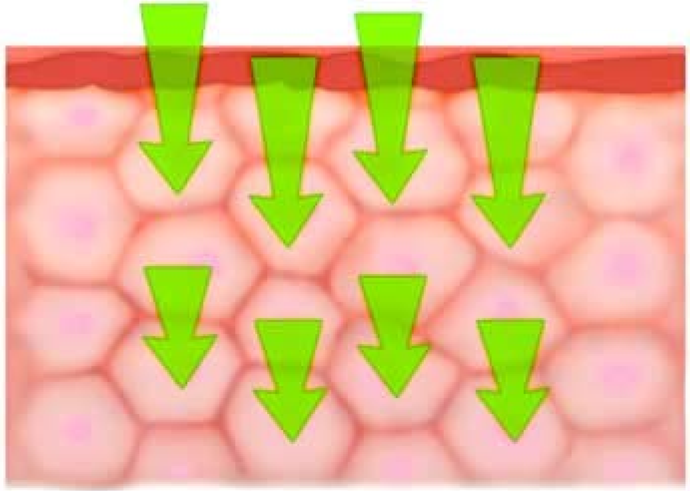
```
data = load('example_03.dat');

N = length(data);    %sample size
mu = mean(data);     %sample mean
stdev = std(data);   %sample mean

confidence = 0.99;
alpha = 1-confidence;
alphaon2 = alpha/2;
Z_alphaon2 = norminv(alphaon2);
display(Z_alphaon2);
normspec([norminv(alphaon2) -norminv(alphaon2)],0,1);
grid on;
stderror = stdev/sqrt(N);
CI=[mu+tinv(alphaon2,N-1)*stderror, mu-tinv(alphaon2,N-1)*stderror];
display(CI);
```




```
Z_alphaon2 =

   -2.5758

CI =

  520.6237   815.9097
```

Example 4 (Environmental Risks)

In an experiment to measure the rate of absorption of pesticides through skin, 500 μg of uniconazole was applied to the skin of four rats. After 10 hours, the amounts absorbed (in μg) were 0.5,2.0,1.4 and 1.1. Find a 90% confidence interval for the mean amount absorbed.



Solution:

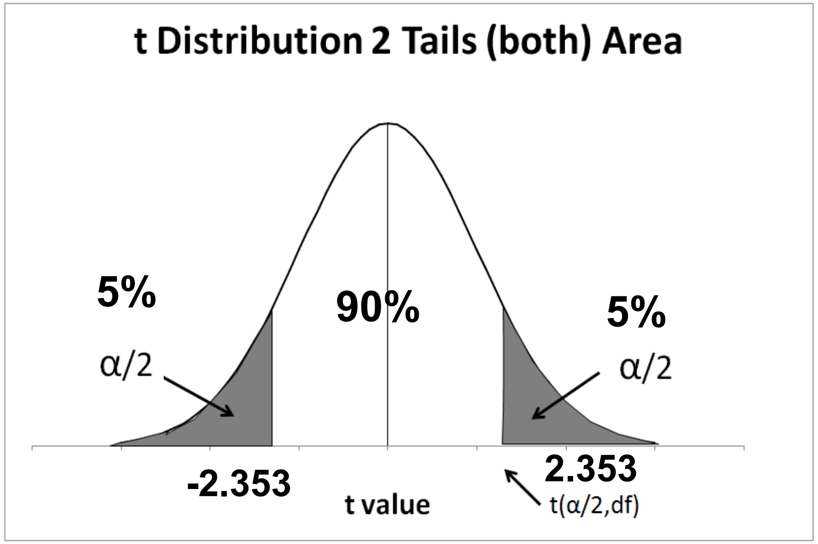
Small-Sample Confidence Intervals for Population μ

Solution:

$$n = 4$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = 1.25\mu g$$

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2} = 0.6245\mu g$$



Degrees of Freedom	Combined Area α in Two Tails					
	0.250	0.100	0.050	0.025	0.010	0.005
1	2.4142	6.3138	12.7062	25.4517	63.6567	127.3213
2	1.6036	2.9200	4.3027	6.2053	9.9248	14.0890
3	1.4226	2.3534	3.1824	4.1765	5.8409	7.4533
4	1.3444	2.1318	2.7764	3.4954	4.6041	5.5976
5	1.3009	2.0150	2.5706	3.1634	4.0321	4.7733
6	1.2733	1.9432	2.4469	2.9687	3.7074	4.3168
7	1.2543	1.8946	2.3646	2.8412	3.4995	4.0293
8	1.2403	1.8595	2.3060	2.7515	3.3554	3.8325
9	1.2297	1.8331	2.2622	2.6850	3.2498	3.6897
10	1.2213	1.8125	2.2281	2.6338	3.1693	3.5814
11	1.2145	1.7959	2.2010	2.5931	3.1058	3.4966
12	1.2089	1.7823	2.1788	2.5600	3.0545	3.4284
13	1.2041	1.7709	2.1604	2.5326	3.0123	3.3725
14	1.2001	1.7613	2.1448	2.5096	2.9768	3.3257
15	1.1967	1.7531	2.1314	2.4899	2.9467	3.2860
16	1.1937	1.7459	2.1199	2.4729	2.9208	3.2520
17	1.1910	1.7396	2.1098	2.4581	2.8982	3.2224
18	1.1887	1.7341	2.1009	2.4450	2.8784	3.1966
19	1.1866	1.7291	2.0930	2.4334	2.8609	3.1737
20	1.1848	1.7247	2.0860	2.4231	2.8453	3.1534

Solution (continued):

$$t_{n-1=3,\alpha/2=0.05} = 2.3534$$

$$\overline{X} \pm 2.3534\frac{S}{\sqrt{n}} = 1.25 \pm 2.3534\frac{0.6245}{\sqrt{4}}$$

$$\Rightarrow (0.515, 1.985)$$

Confidence Intervals for the difference between two means

From a farm in Western Australia, 50 soil samples were each taken at the depths 50 cm and 250 cm respectively.

Depth (cm)	Average NO_3 (mg/L)	STD
50	88.5	49.4
250	110.6	51.5

What is the difference between the NO_3 concentrations at the two depths?

Let $X_1, \ldots., X_{nx}$ be a large random sample of size n_x and standard deviation s_x.

Let $Y_1, \ldots., Y_{nY}$ be a large random sample of size n_Y and standard deviation s_Y.

$$\mu_X - \mu_Y = \overline{X} - \overline{Y} \pm z_{\alpha/2} \sqrt{\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}}$$

Example 5 – Soil Pollution Risks

From a farm in Western Australia, 50 soil samples were each taken at the depths 50 cm and 250 cm respectively.

Depth (cm)	Average NO_3 (mg/L)	STD
50	88.5	49.4
250	110.6	51.5

Find a 95% confidence interval for the difference between the NO_3 concentrations at the two depths.

Solution:

Solution:

Let $X_1, \ldots . X_{50}$ represent the concentrations of the 50 specimens taken at 50cm

Let $Y_1, \ldots . Y_{50}$ represent the concentrations of the 50 specimens taken at 250cm

$$\bar{X} = 88.5, \bar{Y} = 110.6, s_X = 49.4, s_Y = 51.5$$

The sample size: $n_X = n_Y = 50$

95% confidence interval $Z = 1.96$

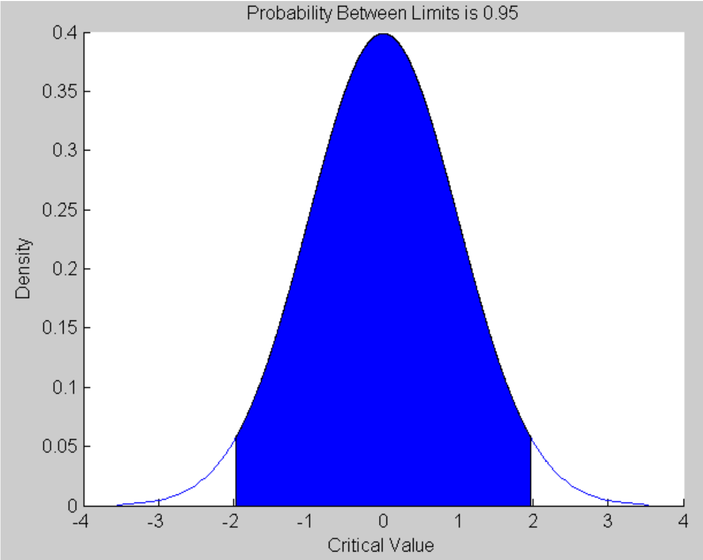
The 95% confidence interval for the difference $\mu_Y - \mu_X$

$$\bar{Y} - \bar{X} \pm Z\sqrt{\frac{s_Y{}^2}{n_Y} + \frac{s_X{}^2}{n_X}} = 110.6 - 88.5 \pm 1.96\sqrt{49.4^2/50 + 51.5^2/50}$$

$$\Rightarrow (2.32, 41.88)$$

Confidence Intervals for the difference between two means

```
mu_x = 88.5;
mu_y = 110.6;

stdev_x = 49.4;
stdev_y = 51.5;

Nx = 50;
Ny = 50;

confidence = 0.95;
alpha = 1-confidence;
alphaon2 = alpha/2;
Z_alphaon2 = norminv(alphaon2);
display(Z_alphaon2);
normspec([norminv(alphaon2) -norminv(alphaon2)],0,1);
CI = [mu_y-mu_x+norminv(alphaon2)*sqrt(stdev_x^2/Nx+stdev_y^2/Ny),...
    mu_y-mu_x-norminv(alphaon2)*sqrt(stdev_x^2/Nx+stdev_y^2/Ny)];
display(CI);
```




```
Z_alphaon2 =

   -1.9600


CI =

    2.3197   41.8803
```

Example 6 – Environmental Risks

In a study to measure the effect of an herbicide on phosphate content bean plants, 75 plants treated by the herbicide has an average phosphate concentration of 3.72% with a standard deviation of 0.51%, and 100 untreated plants had an average phosphate concentration of 4.82% with a standard deviation of 0.42%.

Find a 95% confidence interval for the difference in mean phosphate concentration between treated and untreated plants.

Solution:

Solution:

Let $X_1,\ldots.X_{75}$ represent the concentrations of the 75 treated plants

Let $Y_1,\ldots.Y_{100}$ represent the concentrations of the 100 untreated plants

Then $\bar{X} = 3.72\%, \bar{Y} = 4.82\%, s_X = 0.51\%, s_Y = 0.42\%$

The sample size: $n_X = 75, n_Y = 100$

95% confidence interval $Z = 1.96$

The 95% confidence interval for the difference $\mu_Y - \mu_X$

$$\bar{Y} - \bar{X} \pm Z\sqrt{\frac{s_Y^{\ 2}}{n_Y} + \frac{s_X^{\ 2}}{n_X}} = 4.82\% - 3.72\% \pm 1.96\sqrt{0.42\%^2/100 + 0.51\%^{\ 2}/75}$$

$$\Rightarrow (0.96\%, 1.24\%)$$

Example 7 – Material Protection

To compare two different corrosion inhibitors, 47 specimens of stainless steel in the presence of inhibitor A had a mean weight loss of 242 mg and a standard deviation of 20 mg, and 42 specimens in the presence of inhibitor B had a mean weight loss of 220 mg and a standard deviation of 31 mg.

Find a 95% confidence interval for the difference in mean weight loss between the two inhibitors.

Solution:

Solution:

Let $X_1, \ldots. X_{47}$ represent the concentrations of the 47 specimens in inhibitor A

Let $Y_1, \ldots. Y_{42}$ represent the concentrations of the 42 specimens in inhibitor B

Then $\bar{X} = 242, \bar{Y} = 220, s_X = 20, s_Y = 31$

The sample size: $n_X = 47, n_Y = 42$

95% confidence interval $Z = 1.96$

The 95% confidence interval for the difference $\mu_Y - \mu_X$

$$\bar{X} - \bar{Y} \pm Z\sqrt{\frac{s_Y{}^2}{n_Y} + \frac{s_X{}^2}{n_X}} = 242 - 220 \pm 1.96\sqrt{20^2/47 + 31^2/42}$$

$$\Rightarrow (11.02, 32.98)$$