Week 2

Lecture 3
3.1 Two qubit systems and operations
3.2 Entanglement

Lecture 4
4.1 Dense coding
4.2 Teleportation

Lab 2
Two qubit operations, entanglement, dense coding, teleportation

Lecture overview

In this lecture:

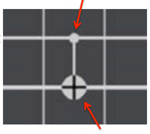
4.1 Dense coding

4.2 Teleportation

- Reiffel: 5.3
- Kaye: Ch 5
- Nielsen and Chuang: 1.3.5, 1.3.7, 2.3

Recap - Two qubit logic gates: CNOT

Two qubit gates can be constructed using an interaction between the two systems. Most important is the Controlled-NOT (CNOT) gate.



How states transform: CNOT truth table

$|00\rangle \rightarrow |00\rangle$
$|01\rangle \rightarrow |01\rangle$
$|10\rangle \rightarrow |11\rangle$
$|11\rangle \rightarrow |10\rangle$

Rule: The target is flipped iff the control qubit is “1”.

As a matrix:

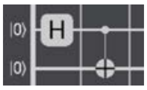
$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$a|00\rangle + b|01\rangle + \boxed{c|10\rangle + d|11\rangle}$$
$$\rightarrow a|00\rangle + b|01\rangle \boxed{+ d|10\rangle + c|11\rangle}$$

Recap: constructing a Bell state

This is one of four states named after the physicist John Bell (who figured out how to experimentally explore reality of entanglement).

Consider the following circuit in the QUI:



Execution: $$|00\rangle \underset{H}{\rightarrow} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \underset{\text{CNOT}}{\rightarrow} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Question: Is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ separable?

Recap: entanglement

Answer: No! We can never find a, b, c, d, i.e.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \neq (a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$$

A state which is not separable is called an **entangled** state.

Entanglement is a uniquely quantum mechanical property, with no direct classical analogue.

Recap: Entanglement Entropy

We would like to have a measure of *how much* entanglement a state has. Some states are more entangled than others:

$|00\rangle$	Not entangled, separable
$\sqrt{0.99}\,|00\rangle + \sqrt{0.01}\,|11\rangle$	Entangled, but close to a separable state
$\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$	Maximally entangled

Entanglement is a type of correlation between two systems, say A and B.

To see how much correlation there is between A and B: We will measure B and ask how many bits of information (as measured by entropy) this can tell us about the state of A?

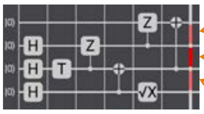
In the QUI we measure the degree of entanglement using an informatic "entropy" measure: *Entanglement Entropy (EE)*

Recap: entanglement in the QUI – time slider

The *time slider* is the vertical bar which moves left and right to show the quantum state at each time step. When there is entanglement it will show it.

The entanglement entropy (EE) is shown in a red colour scale between min and max values possible. Each segment corresponds to the entropy between the system of qubits above and below for that particular bi-partition.

qubit-1
qubit-2
qubit-3
qubit-4



Entanglement entropy between qubit 1 and qubits {2 & 3 & 4} partition

Entanglement entropy between qubits {1 &2} and qubits {3 & 4} partitions

Entanglement entropy between qubit 4 and qubits {1 & 2 & 3} partition

Entanglement and quantum computing

A state which is *not separable* is **entangled**. For example:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

In this lecture we will see how entangled states can be critical in various quantum computing tasks and apply these in the Lab to gain experience in how entangled states work.

In particular we will discuss

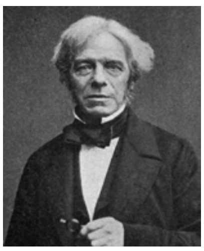
1. Dense Coding
2. No-cloning theorem
3. Quantum teleportation

Entanglement as a resource

When asked what practical use electricity was, Faraday reportedly replied:

"Why sir, there is every probability that you will be able to tax it"

Entanglement is similar, a ***resource*** useful for many quantum information tasks.



Faraday

4.1 Dense coding

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Lecture 4

Dense Coding

Alice would like to send **two classical bits** to Bob.

01 → Wants to send → 01

Alice Bob

Alice and Bob can use a **quantum** NBN, and share some initial entanglement – can they get any advantage?

Dense Coding

Entanglement makes it possible.

(1) Alice and Bob share an entangled state
(2) Alice flips her qubit one of four ways, based on the state she wants to send
(3) Alice sends her qubit to Bob
(4) Bob measures correlations between the qubits, to reveal which of the four (ie. two bits) operations Alice applied

Alice Bob

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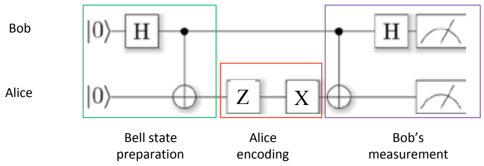
Dense Coding Circuit



Bell state preparation:

$$|00\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle$$

$$\text{CNOT} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

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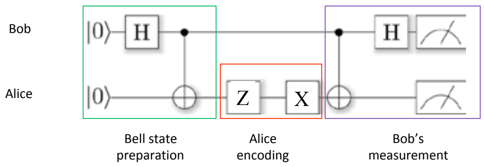
Dense Coding Circuit



$$|00\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$0, 0$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$
$0, 1$	$X_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle + \lvert 10\rangle}{\sqrt{2}}$
$1, 0$	$Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle - \lvert 11\rangle}{\sqrt{2}}$
$1, 1$	$X_2 Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle - \lvert 10\rangle}{\sqrt{2}}$

Alice applies one of four different operations to her qubit, based on the classical information she would like to send.

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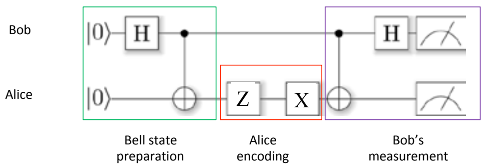
Dense Coding Circuit



		CNOT $\rightarrow$	H $\rightarrow$
$0, 0$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 10\rangle}{\sqrt{2}}$	$\lvert 00\rangle$
$0, 1$	$X_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle + \lvert 10\rangle}{\sqrt{2}}$	$\frac{\lvert 01\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\lvert 01\rangle$
$1, 0$	$Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 00\rangle - \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle - \lvert 10\rangle}{\sqrt{2}}$	$\lvert 10\rangle$
$1, 1$	$X_2 Z_2\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} \rightarrow \frac{\lvert 01\rangle - \lvert 10\rangle}{\sqrt{2}}$	$\frac{\lvert 01\rangle - \lvert 11\rangle}{\sqrt{2}}$	$\lvert 11\rangle$

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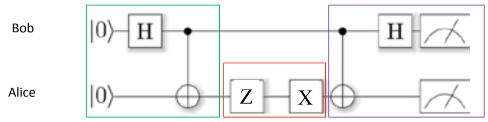
Dense Coding Circuit



Bell state preparation | Alice encoding | Bob's measurement

$H|+\rangle = |0\rangle, H|-\rangle = |1\rangle$

$$0,0 \quad \frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$
$$0,1 \quad X_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|01\rangle+|10\rangle}{\sqrt{2}}$$
$$1,0 \quad Z_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|00\rangle-|11\rangle}{\sqrt{2}}$$
$$1,1 \quad X_2Z_2\frac{|00\rangle+|11\rangle}{\sqrt{2}} \to \frac{|01\rangle-|10\rangle}{\sqrt{2}}$$

CNOT $\rightarrow$

$$\frac{|00\rangle+|10\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}}|0\rangle = |+\rangle|0\rangle$$
$$\frac{|01\rangle+|11\rangle}{\sqrt{2}} = \frac{|0\rangle+|1\rangle}{\sqrt{2}}|1\rangle = |+\rangle|1\rangle$$
$$\frac{|00\rangle-|10\rangle}{\sqrt{2}} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}|0\rangle = |-\rangle|0\rangle$$
$$\frac{|01\rangle-|11\rangle}{\sqrt{2}} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}|1\rangle = |-\rangle|1\rangle$$

H $\rightarrow$

$|00\rangle$
$|01\rangle$
$|10\rangle$
$|11\rangle$

Two bits communicated but only one qubit "sent".

Makes use of pre-existing entanglement.

4.2 Teleportation

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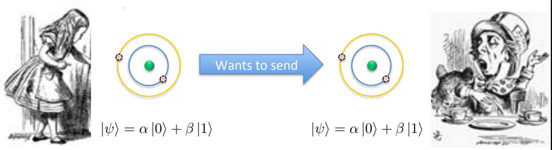
Lecture 4

A Quantum Computing Bus?

To understand the role entanglement can play in quantum information processing, we will consider how it can be to transmit quantum information around our quantum computer (and potentially between quantum computers)



Alice Bob

Communication around the quantum computer is an important primitive. We could physically move quantum systems, but there is a (potentially) better way: **teleportation**

Sending classical information

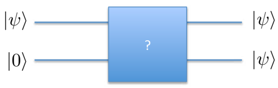
How would we do this classically? Measure everything about the state, then send that information down (classical) bus and recreate a perfect copy elsewhere.

Image by ChtiTux, Used here under CC-by-SA 1.0 license.

Problem: we can't do this in quantum mechanics because classical measurement (1) *collapses the system*, and (2) *this clones the system* which we can't do in quantum mechanics.

No-cloning theorem

Can we make a circuit which clones the input state?



That is, we ask if it is possible to make a ***unitary*** transformation s.t.

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

No-cloning theorem: the answer is **no**.

Proof of no-cloning theorem

If we had a cloning circuit, we could use it on two arbitrary states, $|\psi\rangle$ and $|\phi\rangle$

$$U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle \qquad U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

Inner product on LHS: $\langle 0|\langle\phi|U^\dagger U|\psi\rangle|0\rangle = \langle\phi|\psi\rangle$

Inner product on RHS: $\langle\psi|\langle\psi|\phi\rangle|\phi\rangle = \langle\psi|\phi\rangle^2$

But the only solutions to $x^2=x$ are x=0 or x=1. We can only have a circuit clone states *which are orthogonal (x=0 case), not arbitrary states*.

There can be no unitary transformation which clones two arbitrary states.

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Teleportation

Entanglement makes it possible.

Alice

(1) Alice has a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
(2) Alice and Bob share an entangled state
(3) Alice measures correlations between her qubit and half of the entangled state
(4) Alice sends the results of the measurements to Bob
(5) Bob uses them to reconstruct the original state in his qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Bob

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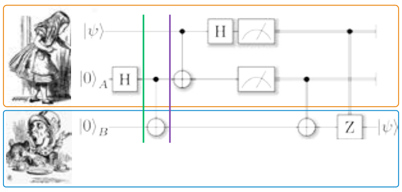
Teleportation



Double lines indicate classical operations

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle$$

Hadamard (A) $\longrightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$

CNOT(A-B) $\longrightarrow (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Alice's state Bell state preparation (shared)

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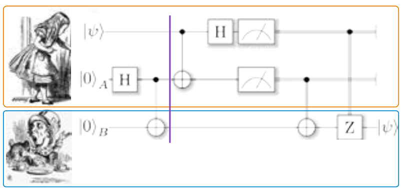
Teleportation



Total system state: $(\alpha|0\rangle + \beta|1\rangle)\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Alice's state $|\psi\rangle$ Shared entangled state A & B

Expand: $\longrightarrow \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

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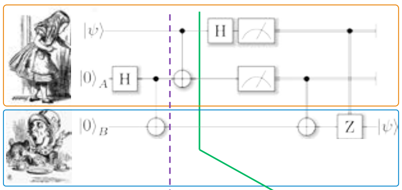
Teleportation



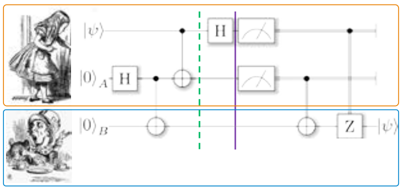
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right)$$

$$\xrightarrow{\text{CNOT[1,2]}} \alpha\frac{|000\rangle + |011\rangle}{\sqrt{2}} + \beta\frac{|110\rangle + |101\rangle}{\sqrt{2}}$$

Rewrite (ex):

$$\frac{|+\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) + \frac{|+\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) + \frac{|-\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) + \frac{|-\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle)$$

Teleportation

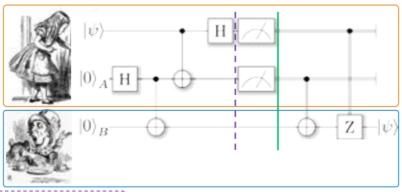


$$\frac{|+\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) + \frac{|+\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) + \frac{|-\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) + \frac{|-\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle)$$

Hadamard

$$\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) + \frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) + \frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) + \frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle)$$

Teleportation



$$\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle + \beta|1\rangle) + \frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle + \beta|0\rangle) + \frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle - \beta|1\rangle) + \frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle - \beta|0\rangle)$$

Alice measures her two qubits.

Bob's qubit collapses to one of the four possibilities.

Alice now tells Bob her outcomes (double lines indicate classical communication).

Bob will perform simple corrections shown.

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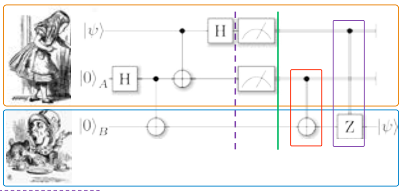
Teleportation



$$\frac{|0\rangle|0\rangle}{2}(\alpha|0\rangle+\beta|1\rangle) + \frac{|0\rangle|1\rangle}{2}(\alpha|1\rangle+\beta|0\rangle) + \frac{|1\rangle|0\rangle}{2}(\alpha|0\rangle-\beta|1\rangle) + \frac{|1\rangle|1\rangle}{2}(\alpha|1\rangle-\beta|0\rangle)$$

Alice measures	Bob's qubit
$0,0$	$\alpha|0\rangle+\beta|1\rangle$
$0,1$	$\alpha|1\rangle+\beta|0\rangle \rightarrow \alpha|0\rangle+\beta|1\rangle$
$1,0$	$\alpha|0\rangle-\beta|1\rangle \rightarrow \alpha|0\rangle+\beta|1\rangle$
$1,1$	$\alpha|1\rangle-\beta|0\rangle \rightarrow \alpha|0\rangle-\beta|1\rangle \rightarrow \alpha|0\rangle+\beta|1\rangle$

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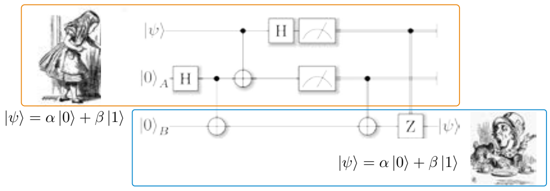
Teleportation



i.e. after correction Bob has successfully reconstructed Alice's original state.

Alice measures	Bob's qubit	
$0,0$	$\alpha|0\rangle+\beta|1\rangle$	$\alpha|0\rangle+\beta|1\rangle \rightarrow \alpha|0\rangle+\beta|1\rangle$
$0,1$	$\alpha|1\rangle+\beta|0\rangle$	$X(\alpha|1\rangle+\beta|0\rangle) \rightarrow \alpha|0\rangle+\beta|1\rangle$
$1,0$	$\alpha|0\rangle-\beta|1\rangle$	$Z(\alpha|0\rangle-\beta|1\rangle) \rightarrow \alpha|0\rangle+\beta|1\rangle$
$1,1$	$\alpha|1\rangle-\beta|0\rangle$	$ZX(\alpha|1\rangle-\beta|0\rangle) \rightarrow \alpha|0\rangle+\beta|1\rangle$

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Teleportation

Entanglement makes it possible.

(1) Alice has a qubit $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$
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(3) Alice measures correlations between her qubit and half of the entangled state
(4) Alice sends the results of the measurements to Bob
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Alice

Bob

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