Tutorial 4

The *norm* of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is

$$||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

and the *distance* between two vectors $\mathbf{u}$ and $\mathbf{v}$, $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

The *dot product* of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The *angle* θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by the formula:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta, \qquad 0 \leq \theta \leq \pi.$$

The *projection* of $\mathbf{v}$ onto $\mathbf{u}$ is

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||^2}\mathbf{u}.$$

The vector $\overrightarrow{AB}$ is the vector $\mathbf{B} - \mathbf{A}$ thought of as starting at $\mathbf{A}$ and finishing at $\mathbf{B}$.

Q1. Let $\mathbf{a} = (3, 1, -2)$, $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{c} = \mathbf{j} - 3\mathbf{k}$ and $\mathbf{d} = (\frac{1}{\sqrt{2}}, 0, -\frac{2}{\sqrt{2}})$. Find

(i). $\mathbf{a} + 2\mathbf{c}$ (ii). $\mathbf{b} - \mathbf{c}$ (iii). $\sqrt{2}\mathbf{d}$

(iv). $d(\mathbf{b}, \mathbf{c})$ (v). $||\mathbf{a}|| + ||\mathbf{b}||$ (vi). $\mathbf{b} \cdot \mathbf{d}$

Q2. In the (unit) octagon, with vertices $A, B, \ldots, H$, and centre O, we know that $\overrightarrow{OA} = (1, 0)$ and $\overrightarrow{OB} = \frac{1}{\sqrt{2}}(1, 1)$. Using vector methods, find the following quantities:

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{OD}$ (iii) $\overrightarrow{OE} \cdot \overrightarrow{OB}$

(iv) The angle between $\overrightarrow{OE}$ and $\overrightarrow{OB}$ (v) The projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$

(vi) $\overrightarrow{OG} \cdot \overrightarrow{OA}$

The *cross product* is equal to

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = ||\mathbf{u}||\,||\mathbf{v}|| \sin\theta\, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the (right-handed) unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Geometrically, $||\mathbf{u} \times \mathbf{v}||$ is the *area* of the parallelogram with edges $\mathbf{u}$ and $\mathbf{v}$.

Q3. Let $\mathbf{a} = (3, 4, -2)$, $\mathbf{b} = (0, -2, 2)$, $\mathbf{c} = (-6, -8, 4)$ and $\mathbf{d} = (0, 0, 1)$. Then find

(i). $\mathbf{a} \times \mathbf{b}$ (ii). $\mathbf{c} \times \mathbf{a}$

(iii). The area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{d}$ (iv). The area of the triangle with sides $\mathbf{a}$ and $\mathbf{b}$

The *scalar triple product*

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

and geometrically $|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|$ is the *volume* of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Q4. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be defined as in Question 3. Where possible, calculate

(i). $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

(ii). $(\mathbf{a}\cdot\mathbf{c})\times\mathbf{b}$

(iii). $\mathbf{a}\cdot\mathbf{b}\times\mathbf{d}$

(iv). $\mathbf{d}\times(\mathbf{a}\times\mathbf{b})$

(v). The volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$.

Let $a, b, c, x_0, y_0, z_0 \in \mathbb{R}$. A *line* in $\mathbb{R}^3$ passing through (x_0, y_0, z_0) and in the direction of (a, b, c) is defined in either vector form

$$\mathbf{r} = (x, y, z) = (x_0, y_0, z_0) + t(a, b, c), \qquad t \in \mathbb{R}$$

parametric form

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

or cartesian form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \qquad a, b, c \neq 0.$$

The equivalence of the forms is seen by setting each fraction in the cartesian form to equal t.

Q5. Consider the straight line with cartesian equation

$$\frac{x+1}{3} = y + 2 = \frac{z-1}{4}\,.$$

(i). Write down a vector in the direction of the line.

(ii). Does the point $P(-1, -2, 1)$ lie on the line?

(iii). Write down a vector equation for the line.

Q6. Write the following lines in vector, parametric and cartesian form.

(i). The line through the point $(1, 0, 0)$ and parallel to the vector $(2, -1, -3)$.

(ii). The line which passes through $(0, 0, -1)$ and $(1, 0, -2)$.