Tutorial 5

For $\mathbf{v} \in \mathbb{R}^n$ and α a scalar (i.e. $\alpha \in \mathbb{R}$), $\alpha\mathbf{v}$ is said to be a *scalar multiple* of $\mathbf{v}$.

For $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^n$ and α, β scalars,

$$\alpha\mathbf{v_1} + \beta\mathbf{v_2}$$

is said to be a *linear combination* of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Q1. (i). For the vector $(2,1) \in \mathbb{R}^2$, sketch

$$2(2,1) \qquad \text{and} \qquad (-1)(2,1).$$

(ii). For the vectors $(-1,1)$ and $(1,1) \in \mathbb{R}^2$ sketch

$$2(-1,1) + 2(1,1) \qquad \text{and} \qquad (-1)(-1,1) + 2(1,1).$$

A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is *linearly dependent* if there is a linear combination equalling the zero vector and with not all scalars zero, i.e. if

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k} = \mathbf{0}, \qquad a_i \neq 0 \text{ for some } i.$$

A set of vectors is *linearly independent* if the only solution of the equation

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k} = \mathbf{0},$$

is $a_1 = \cdots = a_k = 0$. Forming the matrix

$$\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_k}\end{bmatrix}$$

where the vector $\mathbf{v_i}$ is the i-th column, the set of vectors are linearly independent if the rank is equal to k, and linearly dependent if the rank is less than k.

Q2. (i). We have

$$2(-1,1) + 2(1,1) = (0,4).$$

Therefore, are the vectors $\{(-1,1),(1,1),(0,4)\}$ linearly independent?

(ii). Determine whether the set of vectors

$$\{(1,-1,1),(1,0,1),(1,1,0)\}$$

is linearly dependent or independent in $\mathbb{R}^3$.

(iii). It is known that

$$\begin{bmatrix} 1 & -2 & 3 & 3 \\ 3 & -10 & 8 & 13 \\ 1 & -1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Use this to make a statement about the linear dependence of the vectors in $\mathbb{R}^4$ corresponding to the columns.

> A non-empty subset S of $\mathbb{R}^n$ is said to be a subspace of $\mathbb{R}^n$ if it has the properties:
> (i) For α a scalar and $\mathbf{v} \in S$, we have $\alpha\mathbf{v} \in S$.
> (ii) For $\mathbf{v}_1, \mathbf{v}_2 \in S$ we have $\mathbf{v_1} + \mathbf{v_2} \in S$ (closure under scalar multiplication and vector addition).
>
> To prove that S is a subspace from first principles, these properties must be checked for general vectors and scalars.
> To prove that S is not a subspace, only one counter-example need be given.
> Geometrically, a subspace in $\mathbb{R}^3$ is a point (the origin), a line through the origin, or a plane through the origin, or $\mathbb{R}^3$ itself.
> The solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a subspace.

Q3. (i). Let $S = \{(x, y) \in \mathbb{R}^2 : y = -2x\}$. Take two elements of this set, $\mathbf{v_1} = (x_1, -2x_1)$ and $\mathbf{v_2} = (x_2, -2x_2)$, form a general linear combination, and show that the result is also in S. Hence conclude that S is a subspace. Describe S geometrically.

(ii). Let $S = \{(x, y) \in \mathbb{R}^2 : y = 2x\} \cup \{(x, y) \in \mathbb{R}^2 : y = -x\}$, so S consists of two different straight lines through the origin. Demonstrate that S is not a subspace by providing a counter example to the closure property. Illustrate your example graphically.

(iii). Do you suspect $V = \{(t, t^2), t \in \mathbb{R}^2\}$ is a subspace of $\mathbb{R}^2$? [Hint: since $x = t$ and $y = t^2$, what is y in terms of x?] Formally prove your answer by providing a counter-example to the closure property.

(iv). Give a reason why the solution set of the linear system $A\mathbf{x} = \mathbf{b}$ is not a subspace for $\mathbf{b} \neq \mathbf{0}$. Use this fact to show that $W = \{(a, b, c) \colon 2a + c = b, a + b = c + 2, a + 4b = 4c\}$ is not a subspace.

(v). Let $S = \{(x, y, z) :\in \mathbb{R}^3 : x - 4y + 3z = 0\}$. If you think S is a subspace show by testing the following conditions :

(o) non-empty (that is, is $\mathbf{0} \in S$?),

(i) closed under scalar multiplication (if $\mathbf{v} \in S$ and $\alpha \in \mathbb{R}$ is $\alpha\mathbf{v} \in S$?). and

(ii) closed under vector addition (if $\mathbf{v} \in S$ and $\mathbf{w} \in S$ is $\mathbf{v} + \mathbf{w} \in S$?)

If you think S is not a subspace show by an appropriate counter-example.

> The *span* of a set of vectors, $\text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$, is the set of all linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_k}$. For V a subspace of V, we say $\mathbf{v_1}, \ldots, \mathbf{v_k}$ *span* V if $\text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_k}\} = V$.

Q4. (i). Sketch the region specified by

$$\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for α and β *non-negative* real numbers.

(ii). Describe geometrically $\text{Span}\{(1, -1, 0), (2, 3, 0)\}$.

(iii). Write $\text{Span}\{(1, 1, 1), (3, 2, 1)\}$ as a single linear equation.

(iv). State a criteria that can be used to verify if $\text{Span}\{(0, 0, 1), (1, -1, 1), (1, 1, 0)\} = \mathbb{R}^3$, and use it to proceed to show that this statement is true.