MAST10007 ASSIGNMENT 2 SOLUTIONS

Q1 (a) It is required that $\det A \neq 0$ to guarantee that A is invertible.

(b) We know that if the determinant of the matrix is zero, then the matrix is singular. Now

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1-\alpha^2 & 1-\beta^2 & 1-\gamma^2 \end{vmatrix}_{R_3+R_2} = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}_{R_3-R_1} = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0$ (expanding by the final row)

Q2. (a) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Let the rows and columns correspond to nodes 1, 2 & 3 in order. Then the number of walks from node 1 back to itself using exactly 3 edges is equal to the entry in position (11) of A^3. Now

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and thus

$$A^3 = 3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 3 \times 3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 9\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence the number of walks is 9.

(c) The 9 walks are

(1)(1)(1)

(1)(12)(21), (1)(13)(31)

(12)(2)(21), (13)(3)(31)

(12)(21)(1), (13)(31)(1)

(12)(23)(31), (13)(32)(21)

Q3. Let $X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & 1 & -2 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then $[X \mid I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array}$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \\ R_4 - R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 7 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + 2R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 7 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 14 & -7 & 2 & 1 \end{array}\right]$$

which tells us that

$$X^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 14 & -7 & 2 & 1 \end{bmatrix}$$

Similarly

$$[Y|I] = \left[\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_4 \\ R_3 + R_4 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -3 & 7 & 19 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

and thus

$$Y^{-1} = \begin{bmatrix} 1 & -3 & 7 & 19 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Writing the received message down the columns of a 4×4 matrix, denoted Z, gives

$$Z = \begin{bmatrix} 22 & 71 & 72 & 24 \\ 74 & 281 & 269 & 113 \\ 70 & 340 & 300 & 178 \\ 73 & 314 & 293 & 106 \end{bmatrix}$$

The original message in matrix form is

$$(XY)^{-1} Z = Y^{-1} X^{-1} Z$$

$$= \begin{bmatrix} 1 & -3 & 7 & 19 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 7 & -3 & 1 & 0 \\ 14 & -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 22 & 71 & 72 & 24 \\ 74 & 281 & 269 & 113 \\ 70 & 340 & 300 & 178 \\ 73 & 314 & 293 & 106 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 11 & 18 & 11 \\ 8 & 25 & 23 & 9 \\ 5 & 15 & 15 & 14 \\ 3 & 21 & 18 & 7 \end{bmatrix}$$

(here Matlab has been used to carry out the multiplications)

We read off from this that the message is

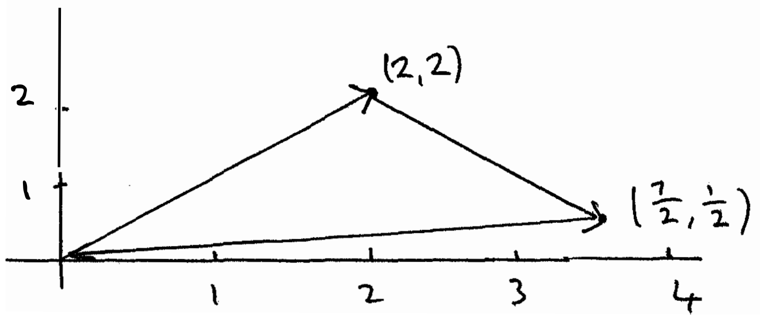
CHECK YOUR WORKING (with the spaces removed).

Q4. (a) The unit vector in the direction of $(1,1)$ is $\frac{1}{\sqrt{2}}(1,1)$, and the unit vector in the direction of $(1,-1)$ is $\frac{1}{\sqrt{2}}(1,-1)$.

Thus travelling $2\sqrt{2}$ kilometers in the direction of $\frac{1}{\sqrt{2}}(1,1)$ takes the hiker to the point $(2,2)$.

And then travelling $\frac{3}{2}\sqrt{2}$ kilometers in the direction of $(1,-1)$ takes the hiker to

$$(2,2) + \tfrac{3}{2}\sqrt{2}\,\tfrac{1}{\sqrt{2}}(1,-1) = (2,2) + \tfrac{3}{2}(1,-1) = \left(\tfrac{7}{2}, \tfrac{1}{2}\right)$$



(b) distance travelled $= 2\sqrt{2} + \frac{3}{2}\sqrt{2} + \frac{1}{2}\|(7,1)\|$ kilometers

$= 2\sqrt{2} + \frac{3}{2}\sqrt{2} + \frac{5}{2}\sqrt{2} = 6\sqrt{2}$ kilometers

(c) area $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2\sqrt{2})(\frac{3}{2}\sqrt{2})$ (kilometers)^2

here a right angle triangle

$= 3$ square kilometers.