Tutorial 9

(Revision, tutorial 6) For a basis $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_k}\}$ of a vector space U and a vector $\mathbf{a} \in U$, if

$$\mathbf{a} = \alpha_1 \mathbf{b_1} + \alpha_2 \mathbf{b_2} + \cdots + \alpha_k \mathbf{b_k}$$

then the coordinate of $\mathbf{a}$ relative to the basis $\mathcal{B}$, denoted $[\mathbf{a}]_{\mathcal{B}}$, is the column matrix formed by $\alpha_1, \ldots, \alpha_k$:

$$[\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}.$$

The scalars $\alpha_1, \ldots, \alpha_k$ are the solution of the augmented matrix system

$$[\mathbf{b_1} \ \ \mathbf{b_2} \ \ \cdots \ \ \mathbf{b_k} \,|\, \mathbf{a}].$$

Q1. (i). In the xy-plane, sketch the coordinate system $\begin{bmatrix} a \\ b \end{bmatrix}$ corresponding to the basis

$$\{(1,1),\ (1,-1)\}$$

by drawing the lines $a = 0, \pm 1$ and $b = 0, \pm 1$. What point in the xy-plane corresponds to $a = 1$, $b = 2$?

(ii). Consider the basis $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$ for the vector space $\mathcal{P}_2$. Compute $\mathbf{u}$ given by

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(iii). Let $\mathcal{B} = \{(1,1),\ (0,1)\}$ for $\mathbb{R}^2$. Compute $\mathbf{u}$ given by

$$[(3,1)]_{\mathcal{B}}, \qquad [(1,-1)]_{\mathcal{B}}.$$

Suppose $[\mathbf{v}]_{\mathcal{B}}$ is known. Then the coordinates of $\mathbf{v}$ in terms of the basis $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_k}\}$ can be computed from knowledge of the change of basis matrix $P_{\mathcal{C},\mathcal{B}}$ which has the defining property

$$[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C},\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$$

for all $\mathbf{v} \in U$. Note that $P_{\mathcal{B},\mathcal{C}} = P_{\mathcal{C},\mathcal{B}}^{-1}$.

Q2. Let $P_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ be a change of basis matrix from $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ to $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}\}$.

(i). Find the coordinates in $\mathcal{C}$ of $\mathbf{v} = \mathbf{b_1} + 2\mathbf{b_2} + \mathbf{b_3}$.

(ii). Find the coordinates in $\mathcal{C}$ of the vectors $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$.

(iii). Check that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Use this to write down $P_{\mathcal{B},\mathcal{C}}$, and from this compute the coordinates in $\mathcal{B}$ of $\mathbf{c_1} + 7\mathbf{c_3}$. Check that your answer is consistent with (i).

Q3. Let $\mathcal{C} = \{(1,-2,2),(0,3,4),(0,-2,0)\}$. You are given that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 1 \\ -2 & 3 & -2 & 1 & -7 & 0 & 0 \\ 2 & 4 & 0 & 0 & 6 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{9}{4} & \frac{3}{8} & -\frac{7}{4} \end{bmatrix},$$

What are the coordinates in $\mathcal{C}$ of $\mathbf{u_1} = (0,1,0)$ and $\mathbf{u_2} = (2,-7,6)$?

> Let $T\colon V \to V$ be a linear transformation, and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V. Then T has matrix representations in terms $\mathcal{B}$ and $\mathcal{C}$, which are related by the change of basis matrix:
>
> $$\begin{aligned} [T]_{\mathcal{B}} &= P_{\mathcal{B},\mathcal{C}}\,[T]_{\mathcal{C}}\,P_{\mathcal{C},\mathcal{B}} \\ &= P_{\mathcal{C},\mathcal{B}}^{-1}\,[T]_{\mathcal{C}}\,P_{\mathcal{C},\mathcal{B}} \end{aligned}$$
>
> Note: $[T]_{\mathcal{B}}$ is also denoted $[T]_{\mathcal{B},\mathcal{B}}$, and similarly $[T]_{\mathcal{C}}$.

Q4. (i). Write down the standard matrix representation of $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T(x,y) = (2x+y, x+2y).$$

(ii). Let $\mathcal{S}$ be the standard basis of $\mathbb{R}^2$, and let $\mathcal{B}$ be the basis

$$\mathcal{B} = \{(1,1),\,(1,-1)\}.$$

Write down $P_{\mathcal{S},\mathcal{B}}$.

(iii). From your answer to (i) and (ii), compute $[T]_{\mathcal{B}}$ for T defined in (i).

Q5. (i). Let $\mathcal{B} = \{(1,1),(1,-1)\}$. Find the matrix $[A_T]_{\mathcal{B}}$ for the linear transformation T which is shear parallel to $y = x$ by a factor of -2 (*Hint:* first draw the effect on a square with sides which are the basis vectors).

(ii). Find $[A_T]_{\mathcal{S}}$.

Q6. (i). A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has standard matrix

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Find the matrix representation of T with respect to the basis

$$\mathcal{B} = \{(1,-1,0),(-2,1,1),(1,1,1)\}$$

(ii). Give a geometrical description of the action of T with respect to the basis $\mathcal{B}$.