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The University of Melbourne

Summer Semester Assessment 2014

Department of Mathematics and Statistics
MAST10007 Linear Algebra

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 7 pages (including this page).

Authorised Materials:
No materials are authorised.

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Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 12 questions. The total number of marks is 80.
All questions may be attempted.

Extra Materials required (please tick & supply)

Graph Paper Multiple Choice form Other (please specify)

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) You are given the following table of data:

x	y
-1	3
0	1
1	4

It is desired to fit a curve $y = a + bx + cx^2$ to the data.

i. Write down 3 simultaneous equations for a, b, c deduced from the data.

ii. Show that the equations in (i) have a unique solution, and proceed to compute a, b, c.

(b) Suppose a row echelon form of an augmented matrix for a linear system, in unknowns α, β, γ is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

i. Reduce this to fully reduced row echelon form.

ii. Find the general solution of the linear system in parametric form.

[7 marks]

2. (a) Let A, B, C be matrices with C of size $n \times n$. For $ACB - BCA$ to be well defined, show that A and B must also be of size $n \times n$.

(b) Let

$$X = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Calculate, if possible

i. YZX

ii. ZZ^T

(c) Let A, B be $n \times n$ matrices such that B is singular. Show that AB is also singular.

[7 marks]

3. (a) By applying the algorithm based on fully reduced row echelon form, show that the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is given by

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

(b) You have received the coded form of a mobile number. The code has been constructed by removing the first digit 0, writing the remaining 9 digits as the entries of a 3×3 matrix down successive columns, then multiplying on the left by the matrix A in (a). The coded form of the mobile number you receive, as read off from the columns of the matrix, is

$$15,\ -4,\ 8,\ 1,\ 0,\ 0,\ 19,\ -3,\ 8$$

What is the actual 10 digit mobile number?

[7 marks]

4. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

i. Find a formula for the area of the parallelogram corresponding to the vectors $\mathbf{u}$ and $\mathbf{v}$ in terms of the dot product (**not** cross product) [hint: draw the parallelogram with $||\mathbf{u}||$ as the base and introduce an angle θ].

ii. Use your answer to (i) or otherwise to find the area of the parallelogram corresponding to the vectors

$$(3, -1, 4), \qquad (2, 1, 2).$$

(b) Let $k \in \mathbb{R}$. Show that the volume of the parallelepiped specified by the vectors

$$(k, k+3, k+6), \quad (k+1, k+4, k+7), \quad (k+2, k+5, k+8)$$

is independent of k, and give its value.

[6 marks]

5. (a) i. Calculate the dimension of

$$\operatorname{Span}\{(1,1,0,-1),\ (1,0,1,0)\}.$$

ii. The set $\{a(1,1,0,-1)+b(1,0,1,0) : a,b \in \mathbb{R}\}$ is a subspace of what vector space?

(b) Let $p(x) \in \mathcal{P}_2$ and $p'(x)$ denote the derivative of $p(x)$. Consider

$$S = \{p(x) \in \mathcal{P}_2 : p'(1) = 0\}.$$

With the correspondence $a + bx + cx^2 \leftrightarrow (a,b,c)$, write S as an equivalent set in $\mathbb{R}^3$, then write the set as a span of two vectors. Why does it follow that the equivalent set in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

(c) Show from first principles that the set

$$R = \{(x, y, -2x) : x, y \in \mathbb{R}\}$$

is closed under vector addition.

[7 marks]

6. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 2 & 5 & -1 & 1 & 8 \\ 0 & -3 & 3 & 4 & 1 \\ 6 & 12 & 0 & -14 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

You are given that B is the reduced row echelon form of A.

(a) Write down a basis for the column space of A in terms of the original columns of A.

(b) Do the columns of A span $\mathbb{R}^4$? Explain your answer.

(c) Are the vectors

$$\mathbf{v}_1 = (1,2,0,6), \quad \mathbf{v}_2 = (2,5,-3,12), \quad \mathbf{v}_3 = (0,-1,3,0)$$

linearly independent? If not, express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(d) Deduce the dimension of the solution space of A from knowledge of the dimension of the column space of A.

(e) Find a basis for the solution space of A.

(f) Write down the fully reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

and justify your answer.

[8 marks]

7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that

$$T(1,1) = (1,1) + 2(1,-1) \qquad T(1,-1) = (1,-1)$$

   i. With $\mathcal{B} = \{(1,1),(1,-1)\}$ write down $[T]_{\mathcal{B},\mathcal{B}}$.
   ii. Illustrate on a diagram how T maps the parallelogram defined by the vectors of $\mathcal{B}$.
   iii. Use your diagram to explain why $\det[T]_{\mathcal{B},\mathcal{B}} = 1$.

   (b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which reflects in the line $y = x$.

   i. Calculate the standard matrix A_S of S.
   ii. State the one-dimensional subspaces of $\mathbb{R}^2$ that are left unchanged by the action of S.
   iii. Verify that the vectors corresponding to the direction of the lines you found in (ii) are eigenvectors of A_S, and state the corresponding eigenvalues.

[6 marks]

8. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x,y,z) = \frac{1}{3}(x+y+z, x+y+z, x+y+z)$$

   i. Compute A_T, the standard matrix form of T.
   ii. Write the image of T as a span of the smallest number of vectors possible, and state its dimension.
   iii. Write the kernel of T as a span.

   (b) i. You are given that the change of basis matrix $P_{\mathcal{B},\mathcal{S}}$ from the standard basis to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is

$$P_{\mathcal{B},\mathcal{S}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

   Express the vector $\mathbf{x} = (1,2,3)$ as a linear combination of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$.

   ii. Determine the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ in (i) by first computing $P_{\mathcal{S},\mathcal{B}}$.
   iii. Give a reason why the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

   cannot be a change of basis matrix.

[7 marks]

9. (a) Consider the plane through the origin in $\mathbb{R}^3$ defined by

$$W = \text{Span}\left\{\frac{1}{\sqrt{2}}(1,1,0),\ \frac{1}{\sqrt{6}}(1,-1,2)\right\}.$$

i. Verify that the vectors in the span are an orthonormal set.
ii. By using your answer to (i), or otherwise, specify the point in W closest to the vector $(1,1,1)$.

(b) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + \frac{1}{2}(x_1 y_2 + x_2 y_1) + \frac{1}{3} x_2 y_2.$$

Show that this satisfies the axiom required for an inner product in $\mathbb{R}^2$ relating to $\langle \mathbf{x}, \mathbf{x} \rangle$ for $\mathbf{x} \in \mathbb{R}^2$. Make sure you clearly state this axiom.

[6 marks]

10. Measurements of the height y metres at distances x kilometres along a straight road from a marker are given by

x	y
-1	20
0	30
1	30

(a) Use the method of least squares to find an equation $y = a + bx$ which best fits this data.

(b) Indicate on a diagram what is being minimized by the least square solution.

[6 marks]

11. Consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

(a) Calculate the eigenvalues of A.

(b) Find the corresponding eigenvectors of A

(c) Give a reason why the matrix A is diagonalisable in terms of a property of its eigenvectors.

[6 marks]

12. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(a) State the formula relating the sum of the diagonal entries of A to the sum of the eigenvalues of A.

(b) You are given that $\lambda = 3$ is an eigenvalue of A repeated twice. Use your answer to (a) or otherwise to compute the third eigenvalue.

(c) Find the normalized eigenvector corresponding to the third eigenvalue.

(d) You are given that normalized eigenvectors corresponding to $\lambda = 3$ are

$$\frac{1}{\sqrt{2}}(-1, 0, 1), \qquad (0, 1, 0)$$

Use this information and the results of your above calculations to identify the quantities on the right hand side of the decomposition

$$A = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^T + \lambda_3 \mathbf{v}_3 \mathbf{v}_3^T$$

Verify that the right hand side equals A.

(e) Let $\mathbf{x} = 2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$, where each $\mathbf{v}_i$ is as in (d). Compute $[A\mathbf{x}]_V$, where $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

[7 marks]

— END OF EXAMINATION QUESTIONS —