Tutorial Sheet 1

A system of equations is ***linear*** if the variables only appear multiplied by constants, never by each other, themselves or other functions. So

- $\log_2(30)x - (e^3 - 1)y = \sin(\frac{\pi}{7})$ is linear in x and y, but
- $x - xy = 1$ is not.

Q1. Decide whether or not the following systems are linear:

(a) A Singaporean refinery processes oil from different sources, containing different amounts of natural gas (N), aromatics (A), petroleum (P), and heavy hydrocarbons (H); this is represented by the system:

$$\begin{array}{lccccl} \text{Bass Strait:} & 0.9\,G & +\,0.80\,A & +\,0.19\,P & +\,0.01\,H & = \ 25,000 \text{ barrels /day} \\ \text{East Timor:} & 0.5\,G & +\,0.05\,A & +\,0.40\,P & +\,0.05\,H & = \ \ 4,000 \text{ barrels /day} \\ & & & \vdots & & \end{array}$$

(b) Ocean currents in the mid-depths of the antarctic circle locally undergo *circulation*, where the new position (y_1, y_2, y_3) of a particle, which was at (x_1, x_2, x_3), a fraction of a second later is approximately given by:

$$\begin{aligned} y_1 &= x_1 + \sigma(x_2 - x_1) \\ y_2 &= x_2 + rx_1 - x_2 - x_1x_3 \\ y_3 &= x_1x_2 - bx_3 \end{aligned}$$

where σ, r and b are constants.

(c) A (fictional) study on the effect of dosage of a particular medication on blood pressure gathered D (mg), the dosage applied, and ΔP (mmHg) the average change in systolic blood pressure for those patients with very high (> 160) blood pressure (< 140 is considered satisfactory, 100–120 normal). It gave the results:

D	10	20	30	40	50	60	$\cdots$
ΔP	3	11	20	32	38	40	$\cdots$

which are being used to fit the full cubic model: $\Delta P = \alpha + \beta D + \gamma D^2 + \delta D^3$.

The ***elementary row operations*** are:

1. Interchanging two rows.
2. Multiplying a row by a (non-zero) constant.
3. Adding a multiple of one row to another.

We can use these operations to reduce a matrix to ***row echelon form.***

A matrix is in row echelon form if and only if:

1. For any row with a leading entry, all elements below that entry and in the same column as it, are zero.
2. For any two rows, the leading entry of the lower row is further to the right than the leading entry in the higher row.
3. Any row that consists solely of zeros is lower than any row with a non-zero entry.

Q2. Consider the system of equations:

$$\begin{array}{rcrcr} -2x & - & y & = & 44 \\ 5x & + & 8y & = & -22 \end{array}$$

(a) Write these in the form of an augmented matrix.

(b) Reduce the matrix to row-echelon form.

(c) Hence solve the system of equations.

Q3. I was making chocolates over Christmas, but already had the raw ingredients (dark, plain and white chocolate). I had 650g, 560g, and 420g of Dark, Plain and White Chocolate, respectively, and wanted to use all of it. Truffles use 75g of each kind; Torrone uses 50g Dark, 170g Plain and 240g White; and Fudge uses 150g Dark, 80g Plain and 10g White. Use matrices to find the number of each kind of chocolate I made.

A system of equations is ***consistent*** if in its row echelon form (r.e.f.) it does not have any rows of the form $[0\ 0\ \cdots\ 0|k]$ for some $k \neq 0$. Let r be the number of non-zero rows in the r.e.f., and n be the number of variables. A consistent system has

- a ***unique*** solution if $n = r$
- ***infinitely many*** solutions if $n > r$ (and in this case will involve $(n - r)$ parameters)

Q4. Systems of linear equations have been written as augmented matrices and already row-reduced, as shown below. For each matrix:

(a) Is the system consistent?
(b) How many solutions does it have?
(c) Solve the system where possible.

(i). $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ (ii). $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right]$

(iii). $\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 7 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (iv). $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Q5. If k is a real constant, then we can define a system by

$$\begin{array}{rcrcrcr} 2x & - & 4y & + & 4z & = & 12 \\ 3x & + & y & - & 8z & = & 4 \\ -5x & + & 11y & + & kz & = & -32 \end{array}$$

(i) Use matrices to reduce the system to row-echelon form.

(ii) For what value(s) of k does the system have a unique solution? Infinitely many solutions?

(iii) Solve the system when there is an infinite number of solutions.

If you finish the above problems before the end of class go on with the Topic 1 questions from the Exercise booklet. You should aim to finish a selection of the exercises in Topic 1 before your next tutorial practice class.