Tutorial 6

Recall that the *span* of a set of vectors, $\text{Span}(\mathbf{v_1}, \dots, \mathbf{v_k})$, is the set of all linear combinations of $\mathbf{v_1}, \dots, \mathbf{v_k}$, and that a set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_k}\}$ is *linearly dependent* if there is a linear combination equalling the zero vector, with scalars not all zero, i.e.

$$\text{If } a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k} = \mathbf{0}, \qquad a_i \neq 0 \text{ for some } i.$$

A set of vectors is *linearly independent* if the only solution of this equation is $a_1 = \cdots = a_k = 0$.

A *basis* for a vector space (or subspace) V is a linearly independent spanning set. So, $\mathbf{v_1}, \dots, \mathbf{v_k}$ are a basis for V if and only if:

- $\text{Span}(\mathbf{v_1}, \dots, \mathbf{v_k}) = V$; and
- $\mathbf{v_1}, \dots, \mathbf{v_k}$ are linearly independent.

In $\mathbb{R}^n$ the set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis if and only if $k = n$ and

$$\det[\mathbf{v}_1 \cdots \mathbf{v}_n] \neq 0.$$

The dimension of a space, $\dim(V)$, is the number of vectors in a basis, so in particular the dimension of $\mathbb{R}^n$ is n.

Q1. For each of the following, determine whether the vectors are linearly independent.

(i). $(-2, 1, 0, -\pi, 1)$ and $(1, -\frac{1}{2}, 0, \frac{\pi}{2}, -\frac{1}{2})$.

(ii). $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

(iii). $(1, 2, 4)$, $(0, 1, 1)$, $(-1, 0, 1)$ and $(1, 2, -4)$.

(iv). $(1, 2, 0, -1)$, $(0, 1, 0, 1)$, $(0, 0, -1, 2)$ and $(3, 5, -1, -2)$.

Q2. Which of the vectors in Q1. are a basis of $\mathbb{R}^n$ for some n.

Consider the set of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}$. Let A be the matrix with columns $\mathbf{v_1}, \dots, \mathbf{v_k}$, and R be the row echelon form of A. Then a basis for $\text{Span}(\mathbf{v_1}, \dots, \mathbf{v_k})$ is the *columns* of A corresponding to the *leading entries* of R.

Q3. For the following vectors (a) find a basis for the span of the vectors; and (b) describe the span geometrically.

(i). $(2, 0, -1)$, $(-1, 0, 3)$ and $(0, 0, -3)$.

(ii). $(2, -1, 0)$, $(1, 1, 2)$ and $(0, 3, 4)$.

(iii). $(1, 1, 1)$, $(2, 4, 4)$ and $(3, 4, 5)$

Let A be a matrix. Then

- the *row space* of A is the span of the rows of A;
- the *column space* is the span of the columns of A; and
- the *solution space* (or *nullspace*) is the set of all solutions $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$.

All of these can be easily found by examining R, the row reduced form of A:

- The non-zero rows of R are a basis for the row space of A.
- The columns in A corresponding to the leading entries in R are a basis for the column space of A.
- The solution space of A is all solutions to $[R|\mathbf{0}]$.

The *rank* of A is dim(row space of A), and the *nullity* is dim(nullspace of A). These must add up to n, the number of columns of A.

Q4. It is known that

$$A = \begin{bmatrix} 2 & 0 & -2 & 3 & 0 & 4 \\ -11 & 8 & 43 & 9 & 12 & 17 \\ -3 & -1 & -1 & 0 & 0 & 3 \\ 2 & -1 & -6 & 1 & -1 & 1 \\ 1 & 2 & 7 & 0 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i). Write down a basis for the row space of A.

(ii). Write down a basis for the column space of A.

(iii). Do the vectors $(2, -11, -3, 2, 1)$, $(0, 8, -1, -1, 2)$, $(3, 9, 0, 1, 0)$, $(0, 12, 0, -1, 2)$ and $(4, 17, 3, 1, -3)$ span $\mathbb{R}^5$? Explain your answer.

(iv). What is the dimension of the solution space of A?

(v). Find a basis for the solution space of A.

(vi). Write the vectors $(-2, 43, -1, -6, 7)$ and $(4, 17, 3, 1, -3)$ as linear combinations of the other columns of A.

Q5. (i). In Question 4 above, verify that dim(row space of A) = dim(column space of A).

(ii). Explain why for a general matrix B, dim(row space of B) = dim(column space of B).

For a basis $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ and a vector $\mathbf{a}$, if

$$\mathbf{a} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_k\mathbf{v_k}$$

then the coordinates of $\mathbf{a}$ relative to basis $\mathcal{B}$, denoted $[\mathbf{a}]_\mathcal{B}$ are the column matrix formed by $\alpha_1, \dots, \alpha_k$:

$$[\mathbf{a}]_\mathcal{B} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix}.$$

Q6. Write the vector $(-5, -15, -46)$ in terms of the basis $\mathcal{B} = \{(1, 2, 4), (0, 1, 1), (-1, 0, 1)\}$.