Tutorial 8: Solutions

Q1. (i). Not a linear transformation. For example, what once were straight lines are now curved, which is not permitted under a llinear transformation.

(ii). Linear transformation. Consists of a shear with $c = \frac{1}{2}$ in the $x-$direction.

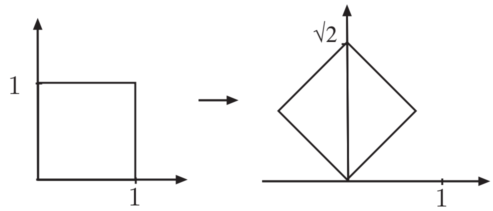
(iii). Linear transformation. Consists of a dilation in x direction, then a rotation.

(iv). Not a linear transformation. The squares transform to different sizes.

(v). Not a linear transformation. The squares transform to different sizes.

(vi). Linear transformation. This is a rotation.

Q2. (i).

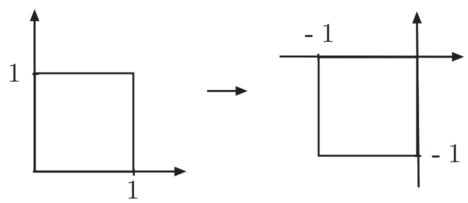


$$R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence we have

$$A_R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(ii).

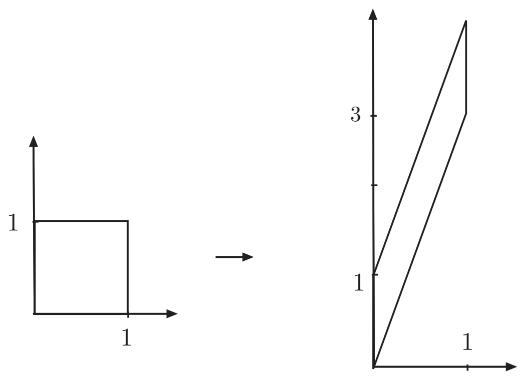


$$M\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad M\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence we have

$$A_M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

(iii).



$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence we have

$$A_S = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Q3. The matrix required is

$$\begin{aligned} A_M A_S A_R &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 2\sqrt{2} & -\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -2\sqrt{2} & \sqrt{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Q4. (i). We know that the standard matrix for an anti-clockwise rotation by angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

But a clockwise rotation by θ is just an anti-clockwise rotation by $-\theta$. Hence

$$R_\theta = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(ii).

$$\begin{aligned} R_\theta R_\phi &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -(\cos\theta\sin\phi + \sin\theta\cos\phi) & \cos\theta\cos\phi - \sin\sin\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \\ &= R_{\theta+\phi} \end{aligned}$$

Q5. (i). We can do this by inspection

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 - x_4 \\ x_1 + 3x_3 + 4x_4 \\ x_2 + x_3 + 3x_4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

or consider the effect of T on the basis vectors

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

In either case we have that

$$A_T = \begin{bmatrix} 2 & -3 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

(ii). We know that the image of T is equal to the column space of A_T. To compute the latter, note

$$\begin{aligned} \begin{bmatrix} 2 & -3 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_2 \\ R_3 \\ R_1 \end{matrix} &\sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 2 & -3 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix} \\ &\sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -6 & -9 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} \\ &\sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Leading entries are in columns 1 and 2, so a basis for the image is $\{(2,1,0), (-3,0,1)\}$.

(iii). The kernel of T is the solution space of A_T. From the row echelon form, there is no leading entry corresponding to x_3 and x_4.
Set $x_3 = s,\ x_4 = t$. We then have $x_1 = -3s - 4t,\ x_2 = -2s - 3t$. Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s - 4t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

and so a basis for $\ker(T)$ is $\{(-3,-2,1,0), (-4,-3,0,1)\}$

(iv). $\operatorname{rank}(T) + \operatorname{nullity}(T) = 2 + 2 = 4 =$ number of columns in A_T.