COMP20003
Algorithms and Data Structures
Graph: Shortest Paths

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Semester 2

Example weighted graph

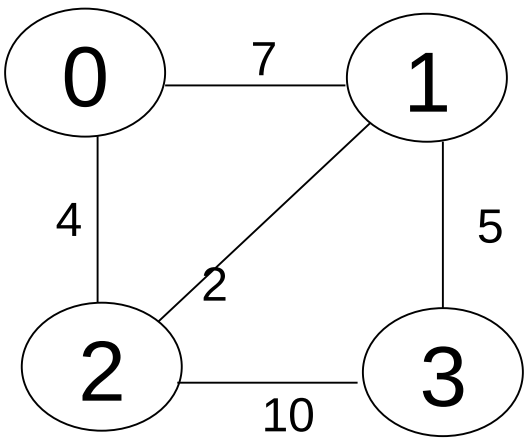
Adjacency List

0→1→2

1→0→2→3

2→0→1→3

3→1→2

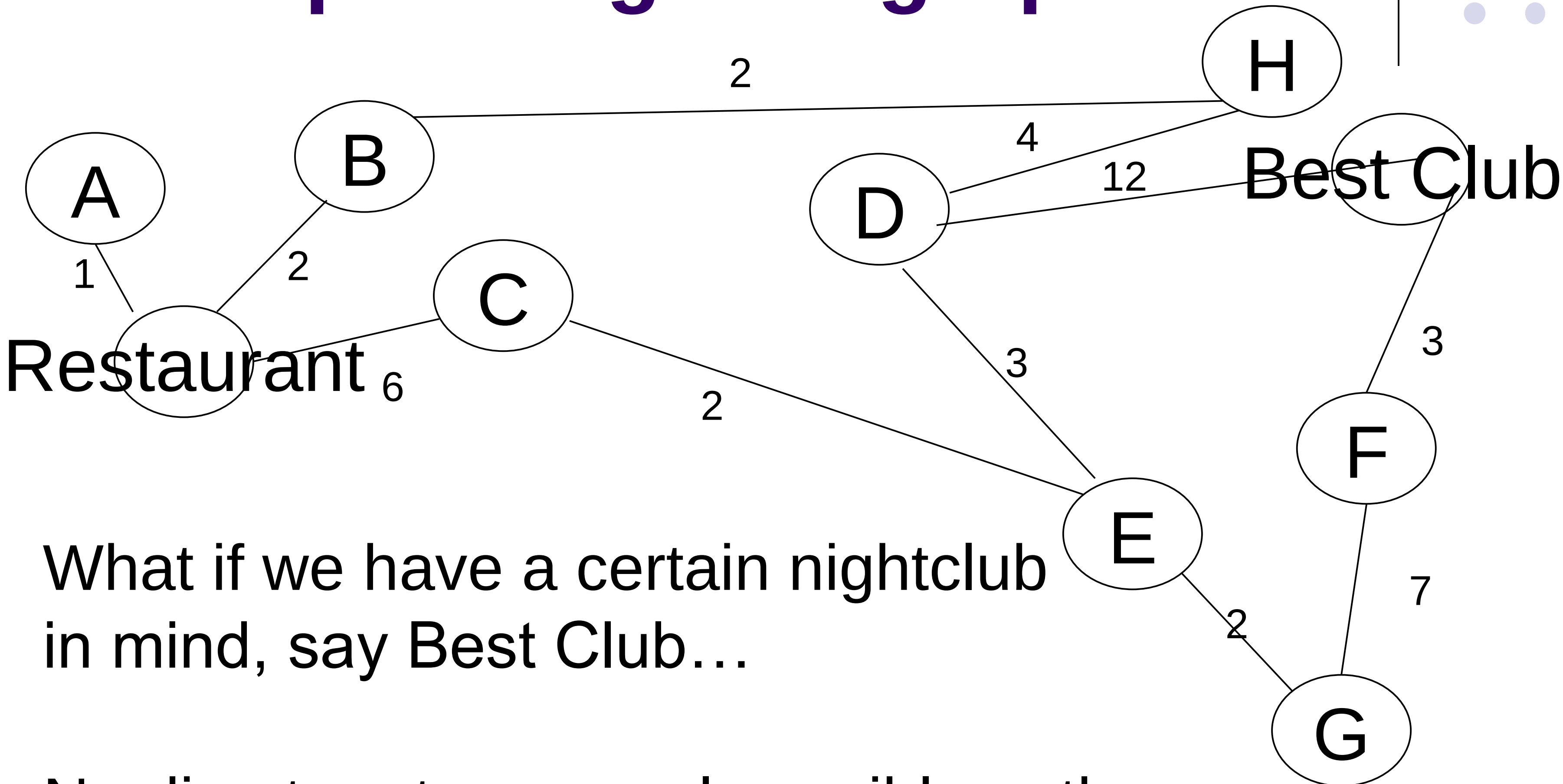


Previous visit order from node 0:

But if these are restaurants and nightclubs, and we want to go to a nearby nightclub from restaurant 0…

…in this case the answer is easy. But if you scale it…

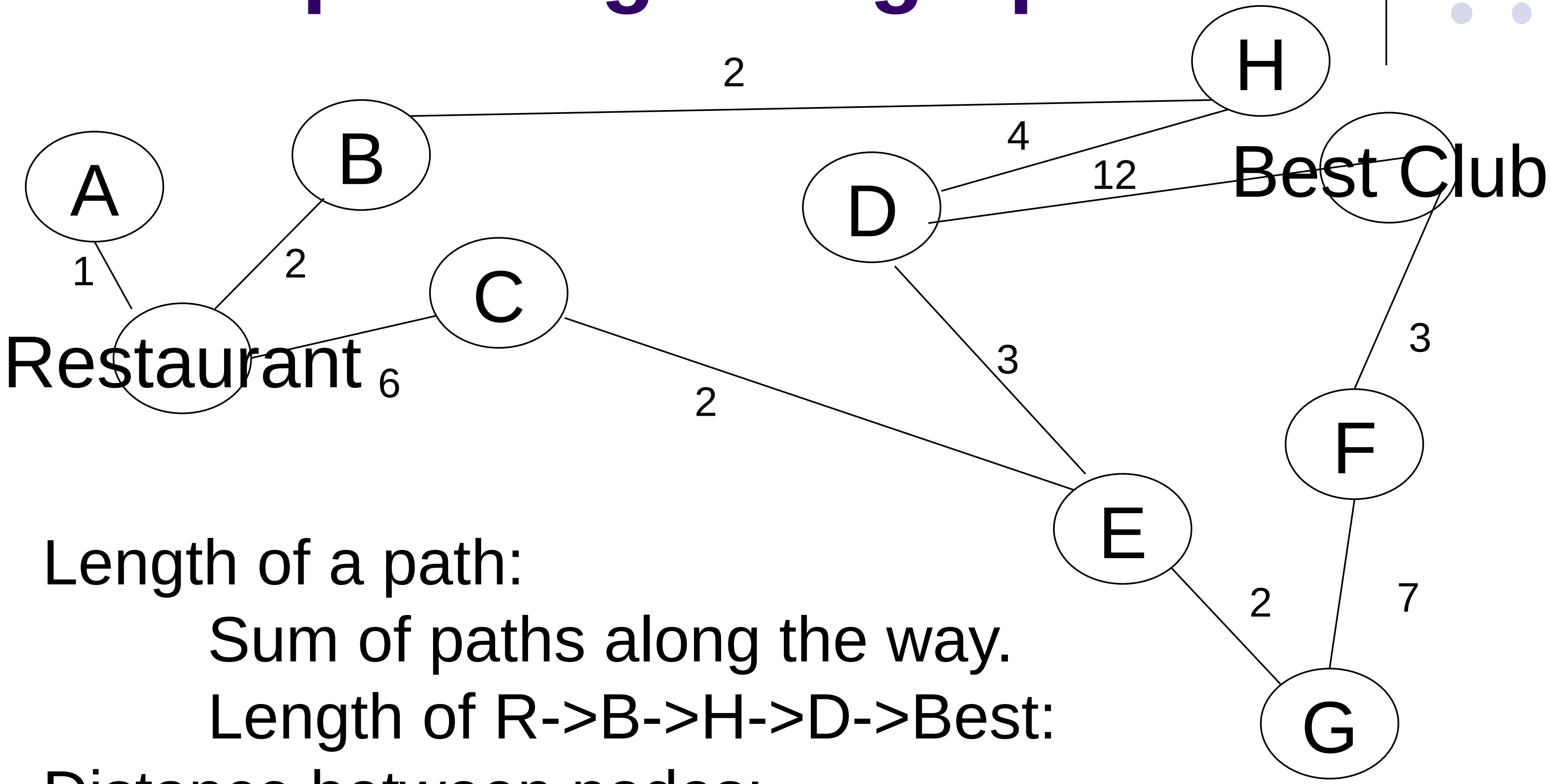
Example weighted graph



What if we have a certain nightclub in mind, say Best Club…

No direct route, several possible paths.

Example weighted graph bfs



Length of a path:

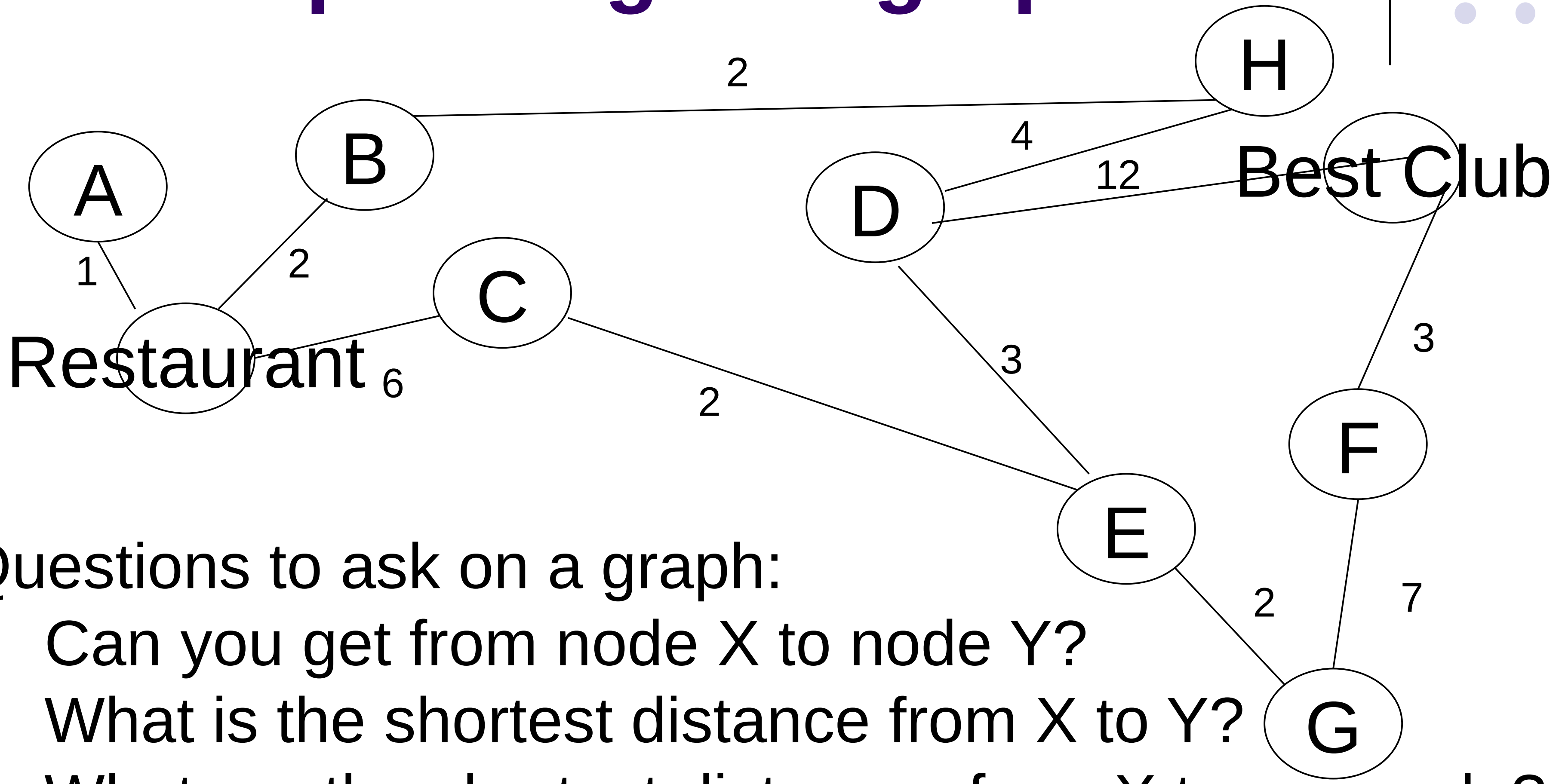
Sum of paths along the way.

Length of R->B->H->D->Best:

Distance between nodes:

Length of shortest path.

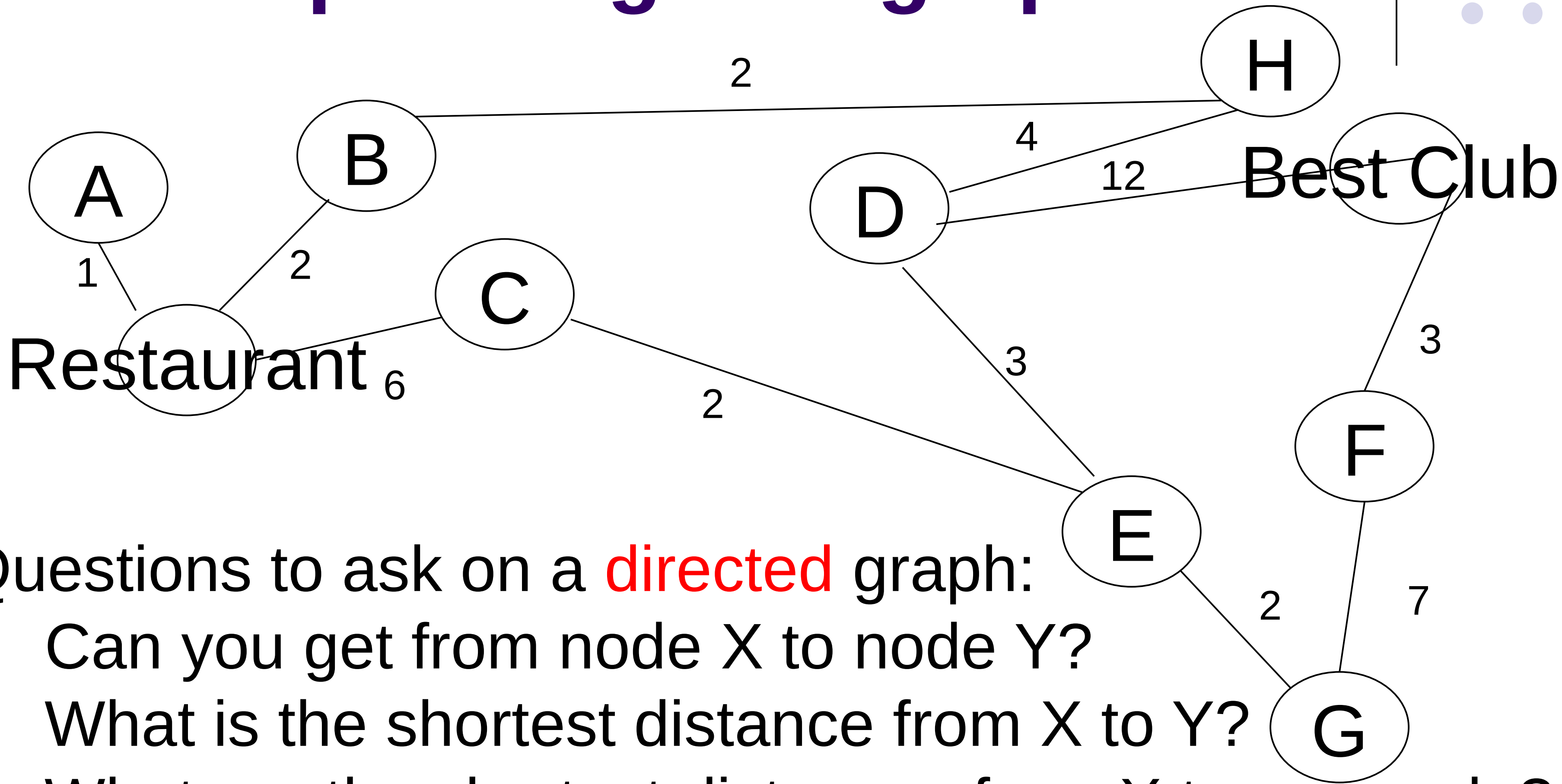
Example weighted graph



Questions to ask on a graph:

- Can you get from node X to node Y?
- What is the shortest distance from X to Y? What are the shortest distances from X to any node?
- What are the shortest distances from any node to any other node?

Example weighted graph



Questions to ask on a directed graph:

- Can you get from node X to node Y?
- What is the shortest distance from X to Y? What are the shortest distances from X to any node?
- What are the shortest distances from any node to any other node?

Single Source Shortest Path Problem

- Given:
  - Directed graph G(V,E)
  - Source vertex s in V
- Determine:
  - Shortest *distance* path

from s to every other vertex in V

Brute force approach

- For each vertex v_i:
  - Enumerate all paths from s to v_i
  - Calculate cost of each path $s \rightarrow\rightarrow v_i$
  - Pick minimum cost.
- How many possible paths from s to v_i?
  - For a dense graph O(V!)
  - V=20: 2432902008176640000 paths
  - Not feasible!

Dijkstra's algorithm for single source shortest path

- Greedy algorithm:
  - Based on idea that any subpath along a shortest path is also a shortest path.
  - NodeA→→→→→NodeX→NodeY
    - If shortest path from A to Y is through X,
    - then this path from A to X is also a shortest path.
- Dijkstra, E. W., *Numerische Mathematik* **1**: 269–271,1959

Dijkstra's algorithm for single source shortest path

- Greedy algorithm:
  - Based on idea that any subpath along a shortest path is also a shortest path.
  - NodeA→→→→→NodeX→NodeY
    - If shortest path from A to Y is through X,
    - then this path from A to X is also a shortest path.
- Assumes no negative edges, so:
    - Distance A→X $\leq$ Distance A→Y

Dijkstra's algorithm for single source shortest path

- Algorithm will give us a shortest path *tree*.
- Root = source node.
  - Every other node is connected to the root through its *shortest* path.

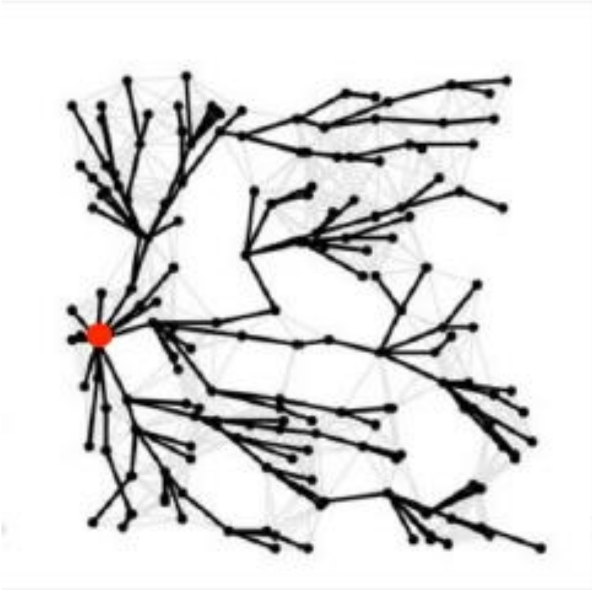


Image from R. Sedgewick, Lecture Notes
http://www.cs.princeton.edu/courses/archive/fall05/cos226/lectures/shortest-path.pdf

Dijkstra's Algorithm: Overview

- *F*or every vertex *v*, maintain *estimate* **`dist[v]`**, of minimum distance $\delta(s,v)$.
- **`dist[v]`**: length of *a known path s→v*, but not necessarily the shortest path.
- **`dist[v]`**$\geq \delta(s,v)$. Always.
- Where **`dist[v]=`**∞, there is no estimate (yet).
- Initially **`dist[s]`**=0, all other **`dist[v]=`**∞.

Dijkstra's Algorithm: Overview

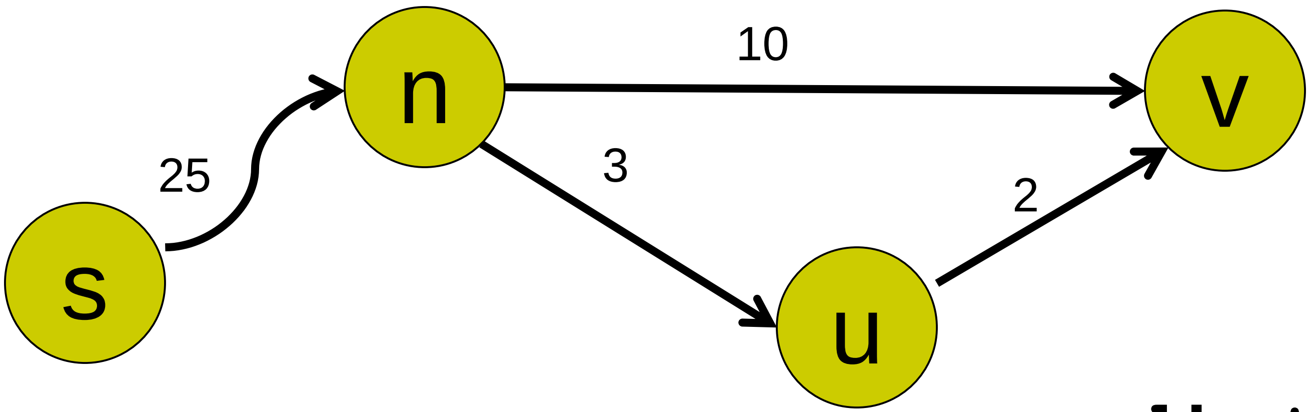
- Process vertices one-by-one, updating **`dist[v]`** until **`dist[v]`** = $\delta(s,v)$ for every vertex *v*.
- (Along the way, keep track of path information in **`pred[]`**.)
- When algorithm finishes:
  - Have shortest distances in **`dist[]`**.
  - Can reconstruct shortest path from **`pred[]`**.

Relaxation: Updating estimated distances

- Relaxation:
  - Estimate the solution by answering an easier problem (relax the conditions).
  - Keep updating the relaxed estimate until it is the solution to the original problem.
- For shortest paths:
  - Estimate: known distance of *some* path.
  - Solution: shortest possible distance.

Relaxation: Updating estimated distances

- Example:

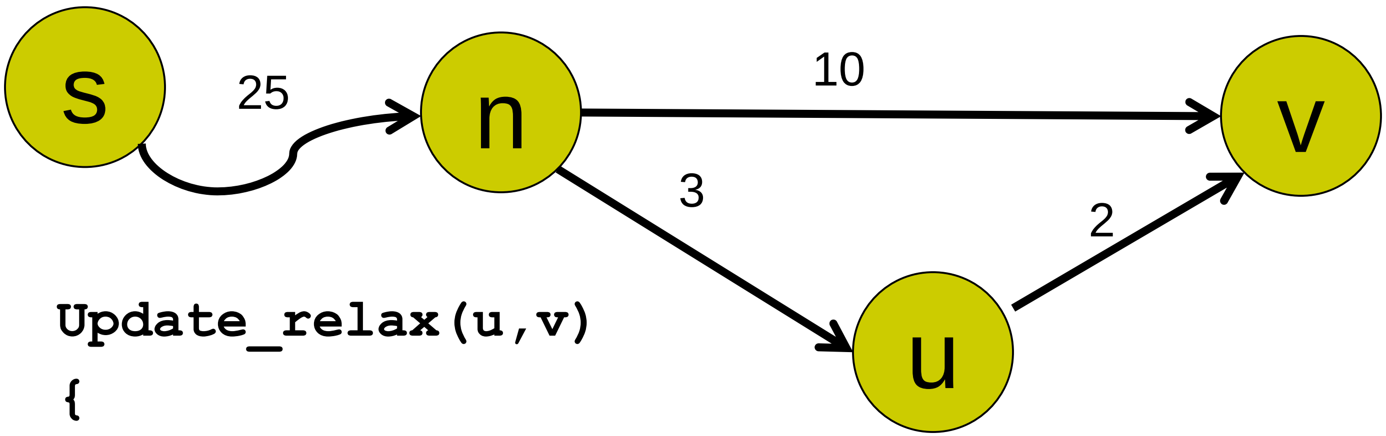



```
dist[v]: 35
pred[v]: n
```

```
dist[u]: 28
dist[v]: 30
pred[v]: u
```

Relaxation: Updating estimated distances

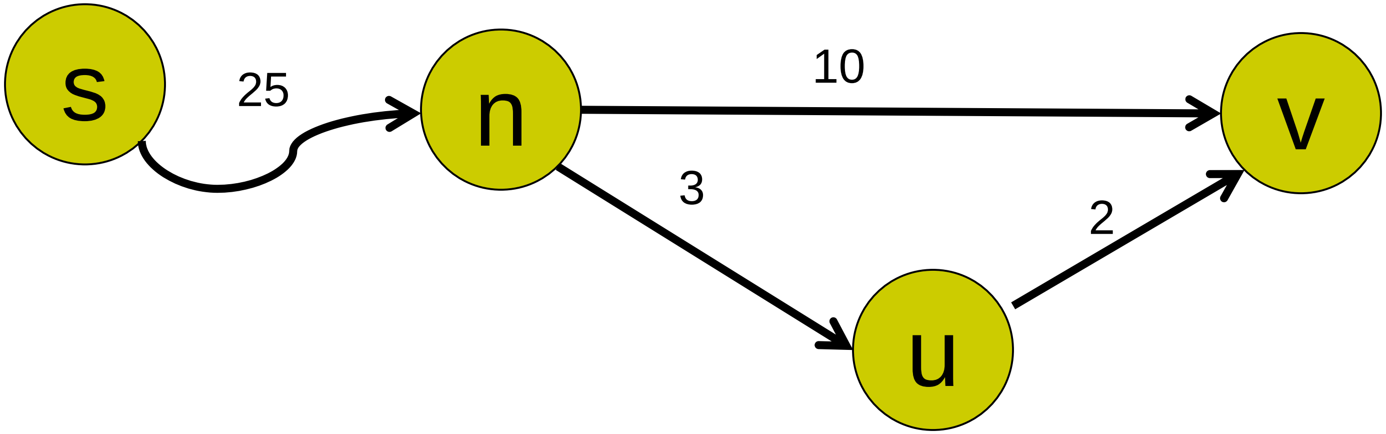
- Example:




```
Update_relax(u,v)
{
   if dist[u] + edgeweight(u,v) < dist[v]
   {
     dist[v] = dist[u] + edgeweight(u,v);
     pred[v] = u;
   }
 }
```

Relaxation: Updating estimated distances

- Example:



Note: **`pred[v] = u;`**

`pred[u] = n;`

...

`pred[j] = s;`

Reconstruct path $s \rightsquigarrow v$ going backwards through **`pred[]`**.

Dijkstra's algorithm: successive relaxations

- How do we pick the next node to look at?
- Process vertices in order of estimated closeness to source, value of **`dist[v]`**.
- Priority queue to store vertex *v* and **`dist[v]`** value.

Dijkstra's algorithm (C-ish pseudocode)

```
/* Find shortest paths in graph G from
source s*/
/* vertices identified by number for
convenience */
dijkstra(G,s)
{
      int  dist[V], pred[V];
      initialize(G,V,s,pred,dist);
      run(G,V,s,pred,dist);

      reconstruct(s,pred,dist);
}
```

Dijkstra’s algorithm (C-ish pseudocode)

```
initialize(G,V,s,pred,dist)
{
      int i;
      for(i=0;i<V;i++)
            dist[i] = MAX_INT;
      dist[s] = 0;
      for(i=0;i<V;i++) pred[i]) = NULL;
}
```

Dijkstra's algorithm (C-ish pseudocode)

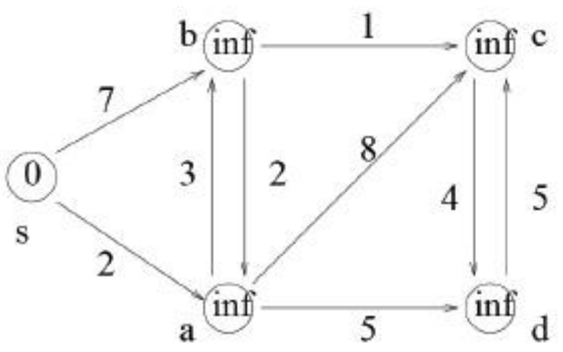
```
run(G,V,s,pred,dist)
{
  pq_node_t  *pq;
  pq = makePQ(G); /* vertices into min PQ,
             dist  from s as in Graph, as key */
  while(!emptyPQ(pq))
  {
    u = deletemin(pq);
    for(/*each v reached from u */)
      if(dist[u] + edgeweight(u,v) < dist[v])
          update(v,pred,dist,pq);
   }
   /* vertex u has been processed,
    * i.e. dist[u] =\delta(s,u) */
}
```

Dijkstra's algorithm (C-ish pseudocode)

```
update(v,pred,dist,pq)
{
     dist[v] = dist[u] + edgeweight(u,v);
     pred[v] = u;
     decreaseweight(pq,v,dist[v]);
}
```

Dijkstra's algorithm: example

Example:

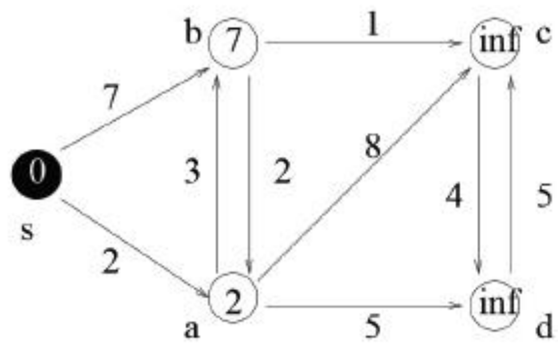


Step 0: Initialization.

v	s	a	b	c	d
$d[v]$	0	∞	∞	∞	∞
$pred[v]$	nil	nil	nil	nil	nil
$color[v]$	W	W	W	W	W

Priority Queue:

v	s	a	b	c	d
$d[v]$	0	∞	∞	∞	∞



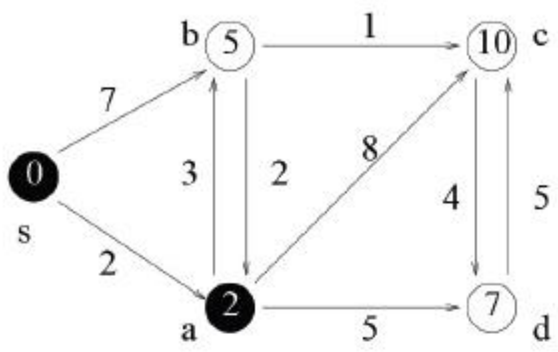
Step 1: As $Adj[s] = \{a, b\}$, work on a and b and update information.

v	s	a	b	c	d
$d[v]$	0	2	7	∞	∞
$pred[v]$	nil	s	s	nil	nil
$color[v]$	B	W	W	W	W

Priority Queue:

v	a	b	c	d
$d[v]$	2	7	∞	∞

From lecture notes by Mordechai Golin, Univ Science and Technology, Hong Kong: http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL09.pdf

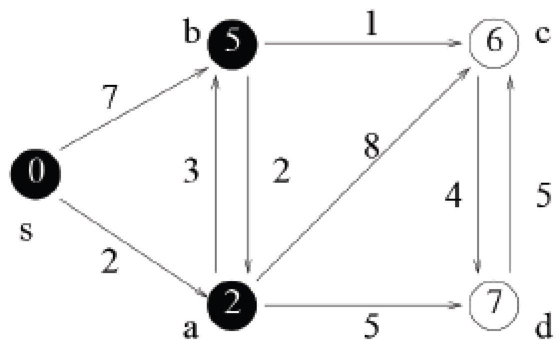


Step 2: After Step 1, a has the minimum key in th priority queue. As $Adj[a] = \{b, c, d\}$, work on b, c, and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	10	7
$pred[v]$	nil	s	a	a	a
$color[v]$	B	B	W	W	W

Priority Queue:

v	b	c	d
$d[v]$	5	10	7

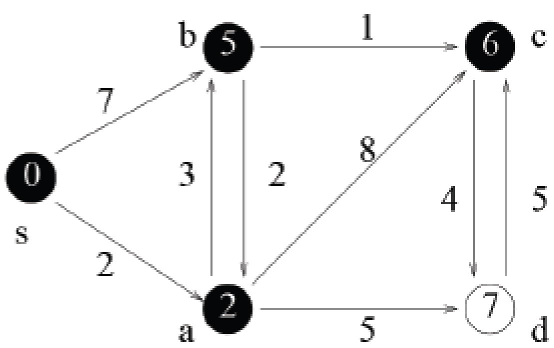


Step 3: After Step 2, b has the minimum key in the priority queue. As $Adj[b] = \{a, c\}$, work on a, c and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	W	W

Priority Queue:

v	c	d
$d[v]$	6	7

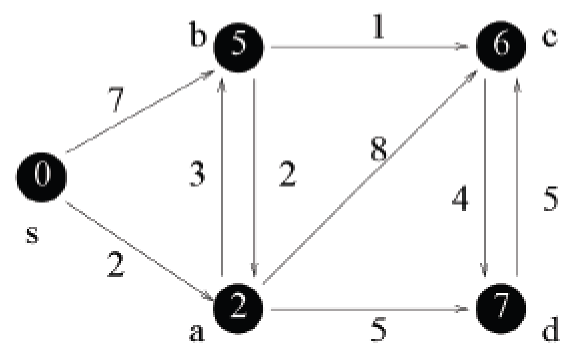


Step 4: After Step 3, c has the minimum key in th
ority queue. As $Adj[c] = \{d\}$, work on d and up
information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	B	W

Priority Queue:

v	d
$d[v]$	7



Step 5: After Step 4, d has the minimum key in the priority queue. As $Adj[d] = \{c\}$, work on c and update information.

v	s	a	b	c	d
$d[v]$	0	2	5	6	7
$pred[v]$	nil	s	a	b	a
$color[v]$	B	B	B	B	B

Priority Queue: $Q = \emptyset$.

Dijkstra's Algorithm: Analysis

- Cost depends on implementation of PQ.
- Using a heap:
  - **`makePQ()`** $O(V)$
  - V **`deletemin()operations`** **`@`**$O(\log V)$
  - $O(E)$ **`decreaseweight()`**ops **`@`**$O(\log V)$
  - Total: $O((V+E) \log V)$
- More or less using other PQ implementations.

Dijkstra's Algorithm: Limitations

- Assumes no negative edges:
  - Good for physical distances.
  - Distances are static
- Negative edges:
  - Use Bellman-Ford algorithm.
  - Cannot deal with negative cycles.
  - O(V*E)

Dijkstra’s Algorithm: Limitations

- Negative *cycles*:
  - What is the shortest path?
  - Problem is not well-formed, intractible.
  - Bellman-Ford detects negative cycles (algorithm does not terminate, keeps shortening paths).

Applications

- More applications.
  - Robot navigation.
  - Texture mapping.
  - Typesetting in TeX.
  - Urban traffic planning.
  - Optimal pipelining of VLSI chip.
  - Telemarketer operator scheduling.
  - Routing of telecommunications messages.
  - Network routing protocols (OSPF, BGP, RIP).
  - Exploiting arbitrage opportunities in currency exchange.
  - Optimal truck routing through given traffic congestion pattern.

Negative Cycle Detection: Arbitrage

- Common example in CS materials is arbitrage:
  - currency 1 → currency 2 → currency 3 → currency 1'
  - If currency 1' > currency 1, you have made money.
- Model problem as a graph:
  - Vertices = currency
  - Edges = - $\log_2$(exchange rate)
  - Detect negative cycle and change money → get rich!
- Not realistic!
  - D.J.Fenn *et al*., "The Mirage of Triangular Arbitrage in the Foreign Currency Exchange Market", *Int. J. Theoretical and Applied Finance* **12**(8), 1105-1123, 2009.

Edsger W. Dijkstra

- The question of whether computers can think is like the question of whether submarines can swim.
- *Computer science is no more about computers than astronomy is about telescopes.*
- How do we convince people that in programming simplicity and clarity —in short: what mathematicians call "elegance"— are not a dispensable luxury, but a crucial matter that decides between success and failure?
- Elegance is not a dispensable luxury but a quality that decides between success and failure.

Turing award 1972

- A very nice explanation of Dijkstra's algorithm by Mordechai Golin can be found at http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL09.pdf