620-156 Summer 2010 Exam Solutions

1 (a) (i) In the corresponding augmented matrix, there cannot be a leading entry for all the variables

or

The number of equations is less than the number of unknowns.

(ii) In augmented matrix form

$$\left[\begin{array}{ccc|c} 2 & 4 & 2k & 2 \\ 2 & k & 8 & 3 \end{array}\right] \underset{R_2-R_1}{\sim} \left[\begin{array}{ccc|c} 2 & 4 & 2k & 2 \\ 0 & k-4 & 8-2k & 1 \end{array}\right]$$

We observe that with $k=4$ the second line reads $0 \; 0 \; 0 \mid 1$ which is inconsistent, and so there is no solution.

(iii) With $k=5$

$$\left[\begin{array}{ccc|c} 2 & 4 & 10 & 2 \\ 2 & 5 & 8 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 10 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

No leading entry for x_3. Set $x_3 = t$, $t \in \mathbb{R}$.

Back substitution gives $x_2 = 1 + 2t$

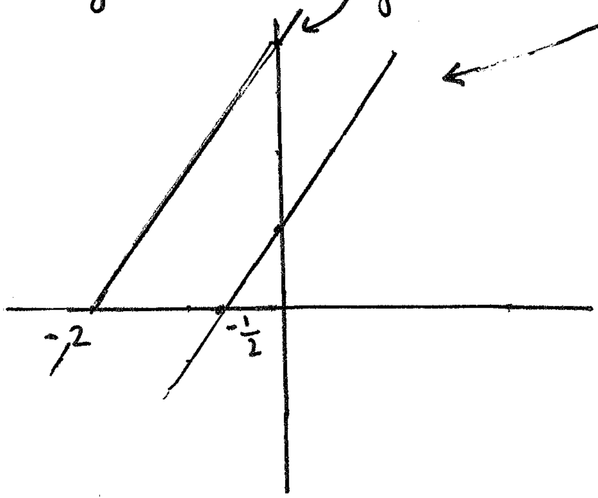
$x_1 = 1 - 2x_2 - 5x_3 = -1 - 9t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -9 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad t \in \mathbb{R}$$

(b)

$-2x + y = 1 \Rightarrow y = 2x + 1$

$x - \frac{1}{2}y = -2 \Rightarrow y = 2x + 4$



The lines are parallel.

2 (a) Suppose B has size $p \times q$

Then B^T has size $q \times p$.

To add matrices, they must be of the same size, and so we require

$$p = q$$

Hence B must be square.

(b) (i) $XY = X_{3\times1} Y_{2\times3}$ not possible

(ii) $YX = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

(c) (i) $CC^T = C_{p\times q} (C^T)_{q\times p} \Rightarrow p \times p$ in size

(ii) $C^T C = (C^T)_{q\times p} C_{p\times q} \Rightarrow q \times q$ in size

(iii) $\text{rank } C \leq \min\{p, q\} = q$.

3 (a)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_2+R_1 \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] R_2-R_3 \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] R_1-R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] R_2 \mapsto -R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Hence $M^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

(b)

We observe

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = M^T$$

In general

$$(M^T)^{-1} = (M^{-1})^T$$

So we have

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

(c) $\det M^3 = (\det M)^3 = -1$ (after reading off that $\det M = -1$ from (a))

4 (a) (i) Write $\vec{A} = (1,-1,2)$, $\vec{B} = (-2,1,1)$, $\vec{C} = (1,2,3)$.

Then $\vec{BA} = (-2,1,1) - (1,-1,2) = (-3,2,-1)$

$\vec{CA} = (1,2,3) - (1,-1,2) = (0,3,1)$.

We have

$$\begin{array}{c}\text{area of the}\\ \text{triangle}\end{array} = \frac{1}{2}\left(\begin{array}{c}\text{area of the}\\ \text{parallelogram}\\ \text{defined by } \vec{BA}, \vec{CA}\end{array}\right)$$

$$= \frac{1}{2}\, \|\vec{BA} \times \vec{CA}\|$$

Now
$$\vec{BA} \times \vec{CA} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -3 & 2 & -1 \\ 0 & 3 & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -3 & -1 \\ 0 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -3 & 2 \\ 0 & 3 \end{vmatrix}$$

$$= 5\underset{\sim}{i} + 3\underset{\sim}{j} - 9\underset{\sim}{k}$$

$$\Rightarrow \|\vec{BA} \times \vec{CA}\| = \sqrt{25+9+81} = \sqrt{115}$$

$\therefore$ Area of the triangle $= \frac{1}{2}\sqrt{115}$

(b) (i)
$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(ii) Expanding (i),

$$\underset{\sim}{u} \times \underset{\sim}{v} = \underset{\sim}{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$\Rightarrow \underset{\sim}{a} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = a_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - a_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + a_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

We know $|\underset{\sim}{a} \cdot (\underset{\sim}{u} \times \underset{\sim}{v})| =$ volume of the parallelpiped specified by the vectors $\underset{\sim}{a}, \underset{\sim}{u}, \underset{\sim}{v}$.

5 (a) Let $s_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $\det s_1 = \det s_2 = 0$ and so $s_1, s_2 \in S$.

On the other hand

$$s_1 + s_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{so } \det(s_1+s_2) = 1 \neq 0$$

which says $s_1 + s_2 \notin S$ and so S is not closed under vector addition. It therefore is not a subspace.

(b) Since $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} = a\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

we see that

$$R = \text{Span}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

All spans are subspaces so R is a subspace of $M_{2,2}$.

(c) Let $r = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$ be an element of R.

Let $\alpha \in \mathbb{R}$ be a scalar.

Then $\alpha r = \begin{bmatrix} 0 & -a\alpha \\ a\alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \in R$

with $a\alpha = b$. Hence R is closed under scalar multiplication.

6 (a) $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_3$ belong to the column space

(b) $\underset{\sim}{v}_1 = -\underset{\sim}{a}_1 + 2\underset{\sim}{a}_3 + 7\underset{\sim}{a}_5$

$\underset{\sim}{v}_3 = -\underset{\sim}{a}_1 + \underset{\sim}{a}_3 + 2\underset{\sim}{a}_5$

(c) $\{(1,1,0,0,0),\ (0,0,1,-1,0),\ (0,0,0,0,1)\}$

The row space has dimension 3.

(d) Columns 2 & 4 have no leading entry, so we set

$x_4 = t, \quad x_2 = s \quad t, s \in \mathbb{R}$

Back substitution gives

$x_5 = 0$

$x_3 = x_4 = t$

$x_1 = -x_2 = -s$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s \\ s \\ t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

so the sought basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

7 (a) $R \circ S \circ T\,(1,0)$

$= R \circ S\,(\sqrt{2}, 0) = R\,(1,1) = (1,1)$

$R \circ S \circ T\,(0,1)$

$= R \circ S\,(0, \sqrt{2}) = R\,(-1,1) = (1,-1)$

Standard matrix rep. of $R \circ S \circ T$

$$= \left[[R \circ S \circ T(1,0)]_S \quad [R \circ S \circ T(0,1)]_S \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) The lines $y = x$ and $y = -x$ are left unchanged by the action of R.

8 (a) (i) $[(x,y,z)]_S = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

so $[T(x,y,z)]_S = A_T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x-6y+3z \\ x+5y-2z \\ 3x+6y+3z \end{bmatrix}$

Hence

$T(x,y,z) = (-x-6y+3z,\ x+5y-2z,\ 3x+6y+3z)$

(ii) Making use of the above working, we see

$$T(a_0+a_1x+a_2x^2) = (-a_0-6a_1+3a_2) + (a_0+5a_1-2a_2)x + (3a_0+6a_1+3a_2)x^2$$

(iii) In the standard basis, Im T equals the column space of A_T. Now

$$\begin{bmatrix} -1 & -6 & 3 \\ 1 & 5 & -2 \\ 3 & 6 & 3 \end{bmatrix} \begin{matrix} \\ R_2+R_1 \\ R_3+3R_2 \end{matrix} \sim \begin{bmatrix} -1 & -6 & 3 \\ 0 & -1 & 1 \\ 0 & -12 & 12 \end{bmatrix} R_3-12R_2 \sim \begin{bmatrix} -1 & -6 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Leading entries are in rows 1 and 2 so a basis for the column space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 6 \end{bmatrix} \right\}$

In P_2 this corresponds to $\{(-1+x+3x^2), (-6+5x+6x^2)\}$

8(b) (i) $[\underset{\sim}{u}]_B = P_{B,C}[\underset{\sim}{u}]_C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(ii) $P_{C,B} = (P_{B,C})^{-1}$

Now

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence $P_{C,B} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) From knowledge of $\{\underset{\sim}{c}_1, \underset{\sim}{c}_2, \underset{\sim}{c}_3\}$ we read off

$P_{S,C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Now $P_{S,B} = P_{S,C}P_{C,B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Thus $\underset{\sim}{b}_1 = (1,0,0)$ $\underset{\sim}{b}_2 = (0,2,0)$ $\underset{\sim}{b}_3 = (0,0,3)$

9 (a) $\underset{\sim}{v} = \alpha \underset{\sim}{u}_1 + \beta \underset{\sim}{u}_2 + \gamma \underset{\sim}{u}_3$

Take the dot product of both sides with respect to $\underset{\sim}{u}_1$

$$\underset{\sim}{u}_1 \cdot \underset{\sim}{v} = \alpha\, \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 + \beta\, \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 + \gamma\, \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_3$$

But $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_1 = 1$, $\underset{\sim}{u}_1 \cdot \underset{\sim}{u}_2 = \underset{\sim}{u}_1 \cdot \underset{\sim}{u}_3 = 0$ since $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3\}$ is an orthonormal set. So

$$\alpha = \underset{\sim}{u}_1 \cdot \underset{\sim}{v}.$$

(b) $T\underset{\sim}{u}_1 = \underset{\sim}{0}$, $T\underset{\sim}{u}_2 = \underset{\sim}{u}_2$ $T\underset{\sim}{u}_3 = \underset{\sim}{u}_3$

$$[T]_{\mathcal{U}} = \left[[T\underset{\sim}{u}_1]_{\mathcal{U}} \; [T\underset{\sim}{u}_2]_{\mathcal{U}} \; [T\underset{\sim}{u}_3]_{\mathcal{U}} \right]$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{Ker } T = \text{span}\{\underset{\sim}{u}_1\}$

$\text{Im } T = \text{span}\{u_2, \underset{\sim}{u}_3\}$

(c)

$$P_{\mathcal{U},S} = P_{S,\mathcal{U}}^{-1} = [\underset{\sim}{u}_1 \; \underset{\sim}{u}_2 \; \underset{\sim}{u}_3]^{-1}$$
$$= [\underset{\sim}{u}_1 \; \underset{\sim}{u}_2 \; \underset{\sim}{u}_3]^T \quad \text{since } \{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3\} \text{ is orthonormal}$$
$$= \begin{bmatrix} \underset{\sim}{u}_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

10 (a) $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 5x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$

$$= [x_1 \quad x_2] \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(b) Conditions

- the matrix must be symmetric
- $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \geq 0$
- If $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 0$, then $\underset{\sim}{x} = \underset{\sim}{0}$

By inspection, in the specific case the matrix is symmetric.

Also

$$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 5x_1^2 - 2x_1x_2 + 5x_2^2$$

$$= 5\left(x_1 - \tfrac{2}{5}x_1x_2\right) + 5x_2^2$$

$$= 5\left(x_1 - \tfrac{1}{5}x_2\right)^2 - \tfrac{1}{5}x_2^2 + 5x_2^2$$

$$= 5\left(x_1 - \tfrac{1}{5}x_2\right)^2 + \tfrac{24}{5}x_2^2$$

≥ 0 since each term is non-negative

Suppose $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 0$

The final formula tells us that then

$$5\left(x_1 - \tfrac{1}{5}x_2\right)^2 + \tfrac{24}{5}x_2^2 = 0$$

$\Rightarrow$ (since each term is non-negative) ① $\left(x_1 - \tfrac{1}{5}x_2\right)^2 = 0$

② $x_2^2 = 0 \quad \Rightarrow \quad x_2 = 0$

Subst. in ① $\Rightarrow x_1 = 0$.

10 (c) Step 1: Normalize $(1,0)$ with respect to $\langle \underset{\sim}{x}, \underline{y} \rangle$.

$$\underset{\sim}{v}_1 = \frac{(1,0)}{\sqrt{\langle (1,0),(1,0) \rangle}} = \frac{(1,0)}{\sqrt{5}}$$

Step 2: Compute the orthogonal complement to $(0,1)$ in the direction of $\underset{\sim}{v}_1$.

$$\underset{\sim}{u} = (0,1) - \langle (0,1), \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1$$

$$= (0,1) - \tfrac{1}{5} \langle (0,1),(1,0) \rangle (1,0)$$

$$= (0,1) - \tfrac{1}{5}(-1)(1,0) = \left(\tfrac{1}{5}, 1\right)$$

Now normalize $\underset{\sim}{u}$ to get $\underset{\sim}{v}_2$

$$\underset{\sim}{v}_2 = \frac{1}{\sqrt{\langle (\frac{1}{5},1),(\frac{1}{5},1) \rangle}} \left(\tfrac{1}{5},1\right) = \frac{1}{\sqrt{24/5}} \left(\tfrac{1}{5}, 1\right)$$

Hence, the sought orthonormal basis is

$$\frac{1}{\sqrt{5}}(1,0) \qquad \sqrt{\frac{5}{24}}\left(\frac{1}{5}, 1\right)$$

11 (a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \underset{\sim}{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T \underset{\sim}{y} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

According the the method of least squares the line of best fit is

$$y = a + bx$$

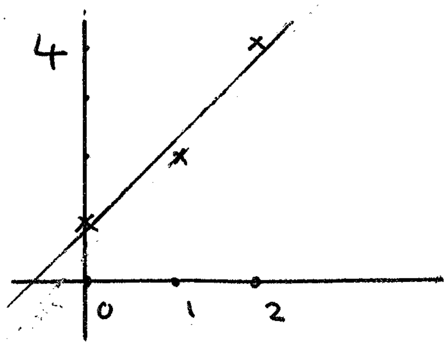
with

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T \underset{\sim}{y}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 3/2 \end{bmatrix}$$

$$\therefore y = \frac{5}{6} + \frac{3}{2} x$$

(b)



(c) When $x = 4$, $y = \frac{5}{6} + \frac{3}{2} \times 4 = 6\frac{5}{6}$

$\therefore$ It predicts 7 students would have dropped out.

12 (a)
(i)
$$\begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 1 & -4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence

$$\det M = 0.$$

This implies $\lambda = 0$ is an eigenvalue.

(ii)
$$\det(M - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 5 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -4-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & -4-\lambda \end{vmatrix} + 5 \begin{vmatrix} 1 & 1-\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left((\lambda-1)(\lambda+4)-1\right) + 5 = (1-\lambda)(\lambda^2+3\lambda-5)+5$$

$$= \lambda^2 + 3\lambda \cancel{-5} \cancel{+5} - \lambda^3 - 3\lambda^2 + 5\lambda$$

$$= -\lambda^3 - 2\lambda^2 + 8\lambda = -\lambda(\lambda^2 + 2\lambda^2 - 8) = -\lambda(\lambda+4)(\lambda-2)$$

Hence the eigenvalues are

$$\lambda = 0, \ \lambda = -4, \ \lambda = 2$$

(iii) We know that a matrix with distinct eigenvalues can always be diagonalized. Since M has distinct eigenvalues, it can be diagonalized.

(b) (i) Since A is triangular, we read off that the eigenvalues are $\lambda = 1, 2, 2$ (i.e. $\lambda = 2$ is repeated).

(ii) Eigenspaces:

$\underline{\lambda = 1}$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \sim \end{matrix} \begin{bmatrix} -3 & 5 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 + \tfrac{1}{3}R_1$$

$$\sim \begin{bmatrix} -3 & 5 & 1 \\ 0 & 8/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

No leading entry for x_3. Set $x_3 = t$, $t \in \mathbb{R}$.
Back substitution gives

$$x_2 = -\frac{1}{8}t$$

$$3x_1 = -\frac{5}{8}t + t = \frac{3t}{8} \Rightarrow x_1 = \frac{1}{8}t$$

$$\Rightarrow \text{eigenspace} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$\lambda = 2$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No leading entry for x_3. Set $x_3 = t$, $t \in \mathbb{R}$.
Back substitution gives

$$x_2 = 0$$
$$x_1 = 0$$

$$\Rightarrow \text{eigenspace} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$