Tutorial 7: Solutions

Q1. (i). $3p(x) = 3(1 - x + 3x^2) = 3 - 3x + 9x^2$
$p(x) + 2q(x) = 1 - x + 3x^2 + 2(2 - x^2) = 5 - x + x^2$

(ii). $3\mathbf{u} = 3(1, -1, 3) = (3, -3, 9)$
$\mathbf{u} + 2\mathbf{v} = (1, -1, 3) + 2(2, 0, -1) = (5, -1, 1)$
We see that under the operations of scalar multiplication and vector addition the polynomials $a + bx + cx^2$ behave as the 3-tuples (a, b, c).

(iii). $(-1)A = -1 \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$
$-A + B = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

(iv). $(-1)\mathbf{u} = -(2, 1, 0, -1) = (-2, -1, 0, 1)$
$-\mathbf{u} + \mathbf{v} = (-2, -1, 0, 1) + (-1, 0, 2, 0) = (-3, -1, 2, 1)$
We see that under the operations of scalar multiplication and vector addition the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the 4-tuples (a, b, c, d) are in one-to-one correspondence.

Q2. (i). These equations are equivalent to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ -R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

There is no leading entry for c, so we set $c = t$ and then $a = -t,\ b = 2t$. Hence the analogous subspace of $\mathcal{P}_2$ is $\{-t + 2tx + tx^2,\ t \in \mathbb{R}\}$

(ii). These equations are equivalent to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

There is no leading entry for b or d. Set $b = s,\ d = t$ and then $a = -s - 3,\ d = -2t$. Hence the analogous subspace of $M_{2,2}$ consists of matrices

$$\begin{bmatrix} -s - 3t & s \\ -2t & t \end{bmatrix} = s \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} -3 & 0 \\ -2 & 1 \end{bmatrix}$$

with $s, t \in \mathbb{R}$.

(iii). From the working of Q2(i), a basis for the subspace of $\mathbb{R}^3$ is $\{(-1, 2, 1)\}$. The corresponding basis for the subspace of $\mathcal{P}_2$ is $\{-1 + 2x + x^2\}$.

(iv). From the working of Q2(ii), a basis for the subspace of $\mathbb{R}^4$ is $\{(-1, 1, 0, 0), (-3, 0, -2, 1)\}$. The corresponding basis for the subspace of $\mathcal{M}_{2,2}$ is

$$\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ -2 & 1 \end{bmatrix} \right\}$$

Q3. (i). The condition is $a + d = 1$ this does not form a subspace as $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has a trace of 0 and so is not in this set.

(ii). The condition is $a + d = 0$. Since $\text{trace}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$ then the set is non empty.

let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ with $\text{trace}(A) = a_1 + d_1 = 0$ and $\text{trace}(B) = a_2 + d_2 = 0$ then

$$\text{trace}(A+B) = \text{trace} \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} = a_1 + a_2 + d_1 + d_2 = a_1 + d_1 + a_2 + d_2 = 0 + 0 = 0$$

so this is closed under scalar addition.

$$\text{trace}(\alpha A) = \text{trace}\begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix} = \alpha a_1 + \alpha d_1 = \alpha(a_1 + d_1) = \alpha 0 = 0$$

so this is closed under scalar multiplication. This set is therefore a subspace.

(iii). Let $p(x) = 1$ and $q(x) = x^2$ the discriminants of $p(x)$ and $q(x)$ are 0 but $p(x)+q(x) = 1+x^2$ has a discriminant of -4 and so this set is not closed under scalar addition and so not a subspace.

(iv). They are symmetric if $c = b$. As $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has $b = c = 0$ then the set is non empty.

let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ with $b_1 = c_1$ and $b_2 = c_2$ then

$$(A + B) = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \quad \text{and} \quad b_1 + b_2 = c_1 + c_2$$

so this is closed under scalar addition.

$$\alpha A = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix} \quad \text{and} \quad \alpha b_1 = \alpha c_1$$

so this is closed under scalar multiplication. This set is therefore a subspace.

Q4. (i). As $\mathcal{B}$ has 4 polynomials and the dimension of $\mathcal{P}_4$ is 5 then $\mathcal{B}$ is not a basis for $\mathcal{P}_4$.

(ii). We need the equation

$$x\begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

to have a unique solution for all a, b and d. This gives the system of equations represented by the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ -2 & 1 & 3 & b \\ -2 & 1 & 3 & b \\ 0 & 0 & 5 & d \end{array}\right] \begin{array}{c} \\ R_2 + 2R_1 \\ R_2 + 2R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 7 & b+2a \\ 0 & 1 & 7 & b+2a \\ 0 & 0 & 5 & d \end{array}\right] \begin{array}{c} \\ \\ R_4 \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 7 & b+2a \\ 0 & 0 & 5 & d \\ 0 & 0 & 0 & 0 \end{array}\right]$$

this has a unique solution for all a, b and d and so $\mathcal{B}'$ is a basis for all symmetric 2×2 matrices.

Q5. (i). As this is the standard basis for $\mathcal{P}_3$ we can read off $[\mathbf{a}]_{\mathcal{B}} = (2, -1, 0, 1)$

(ii). Let

$$a\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix}$$

this gives the system represented by the augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 3 \\ -1 & 1 & -2 & 0 & -2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -1 & 4 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 & -5 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -1 & 4 \end{array}\right] \begin{array}{c} -R_1 \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & -2 & 0 & -5 \\ 0 & 0 & 4 & 0 & 5 \\ 0 & 0 & 4 & -1 & 4 \end{array}\right] \begin{array}{c} \\ R_2 + \frac{1}{2}R_3 \\ R_3/4 \\ R_3 - R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Reading off the solution for a, b, c and d we have $[\mathbf{a}]_{\mathcal{B}} = \left(-3, -\frac{5}{2}, \frac{5}{4}, 1\right)$.