MAST 10007 Assignment 4 - Solutions

1(a) The dimension of the row space is equal to the rank and so is three.

(b) A basis for the row space is

$$\{(1,0,0,-1,58),\ (0,1,0,1,-20),\ (0,0,1,0,67)\}$$

The theory being used is that the non-zero rows in RE form are a basis for the row space.

(c) A basis for the column space is

$$\left\{ \begin{bmatrix} -10 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

These are the columns of A corresponding to the leading entries in RE form.

(d) From the given information we have

$$\begin{bmatrix} -10 & 3 & 8 & 13 \\ -2 & 1 & 3 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & 0 & -1 \\ 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(we simply block out the columns not present)

The rank is 3 so the 4 vectors cannot span $\mathbb{R}^4$, which has dimension 4.

(e) From the given information

$$\begin{bmatrix} -10 & 3 & 13 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 0 & -1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has rank 2 and so

$\text{Span}\{(-10,-2,-1,2), (3,1,1,2), (13,3,2,0)\}$

has dimension 2.

(f) Re-interpreting the first equation in (e) as the augmented matrix

$$\left[\begin{array}{cc|c} -10 & 3 & 13 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We see that

$$(13,3,2,0) = (3,1,1,2) - (-10,-2,-1,2)$$

(g) Let the unknowns be denoted $x_1, x_2, \dots, x_5$. Since there are no leading entries for x_4 and x_5 we set

$$x_4 = s, \quad x_5 = t \qquad s,t \in \mathbb{R}$$

Back substitution gives

$$x_3 + 67x_5 = 0 \quad \Rightarrow \quad x_3 = -67t$$

$$x_2 + x_4 - 20x_5 = 0 \quad \Rightarrow \quad x_2 = -s + 20t$$

$$x_1 - x_4 + 58x_5 = 0 \quad \Rightarrow \quad x_1 = s - 58t$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -58 \\ 20 \\ -67 \\ 0 \\ 1 \end{bmatrix} \qquad s,t \in \mathbb{R}.$$

We read off the basis

$\{(1,-1,0,1,0), (-58,20,-67,0,1)\}$

2(a) Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $A = -A^T$ implies $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$

Equating entries gives

$$a = -a \quad \Rightarrow \quad a = 0$$

$$b = -c$$

$$d = -d \quad \Rightarrow \quad d = 0$$

Hence $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

(b) We recall the correspondence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow (a, b, c, d)$$

Hence

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \longleftrightarrow (0, b, -b, 0)$$

But

$$(0, b, -b, 0) = b(0, 1, -1, 0)$$

and so the sought subset of $\mathbb{R}^4$ can be written

$$\text{Span}\{(0, 1, -1, 0)\}$$

This has dimension 1.

(c) Consider a general element of S

$$\underset{\sim}{u} = \begin{bmatrix} 0 & b_1 \\ -b_1 & 0 \end{bmatrix}$$ with $b_1 \in \mathbb{R}$. Consider a

general scalar α. We have

$$\alpha \underset{\sim}{u} = \begin{bmatrix} 0 & \alpha b_1 \\ -\alpha b_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_2 \\ -b_2 & 0 \end{bmatrix}$$ with $b_2 = \alpha b_1 \in \mathbb{R}$

Hence $\alpha \underset{\sim}{u} \in S$ and so S is closed under scalar multiplication.

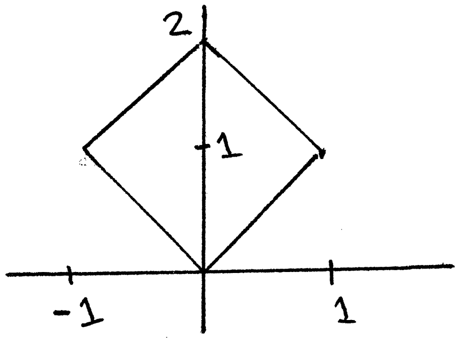
3(a) $$T\underset{\sim}{e}_1 = A_T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T\underset{\sim}{e}_2 = A_T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) The unit square is the span of $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ with scalars between 0 and 1. Since T is linear and thus T applied to a linear combination gives the linear combination of T applied to the individual vectors, we see that the image of the unit square is the set

$$\alpha\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $0 \leq \alpha, \beta \leq 1$



(c) T is a reflection in the line $y = x$ (this effectively interchanges $\underset{\sim}{i}$ and $\underset{\sim}{j}$) and then a rotation by 45° anti-clockwise. There is also an expansion by $\sqrt{2}$ in both the x & y directions (when this is done does not matter).