**The University of Melbourne**
**Summer Semester Assessment 2010**

## Department of Mathematics and Statistics
## 620-156 Linear Algebra

**Reading Time:** 15 minutes.
**Writing Time:** 3 hours.

**This paper has:** 7 pages.

**Identical Examination Papers:** None.
**Common Content Papers:** None.

**Authorised Materials**:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

**Instructions to Invigilators:**
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

**Instructions to Students:**
This examination consists of 12 questions. The total number of marks is 80. All questions may be attempted.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Let $k \in \mathbb{R}$ be given, and consider the system of equations

$$\begin{aligned} 2x + 4y + 2kz &= 2 \\ 2x + ky + 8z &= 3 \end{aligned}$$

i. State a reason why this system can never have a unique solution.
ii. Find the value of $k$ such that the system has no solution.
iii. For the explicit value $k = 5$, find all solutions of the system of equations.

(b) Demonstrate graphically that the system of equations

$$\begin{aligned} -2x + y &= 1 \\ x - \frac{1}{2}y &= -2 \end{aligned}$$

has no solution. Explain in words why this is the case.

[7 marks]

2. (a) Let $B$ be any matrix such that $B + B^T$ is defined. Explain why $B$ must be square.

(b) Let

$$X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix}$$

Calculate, if possible

i. $XY$
ii. $YX$

(c) Let $C$ be a matrix of size $p \times q$.

i. What is the size of $CC^T$?
ii. What is the size of $C^TC$?
iii. Suppose $p \geq q$. What is the maximum possible value of the rank of $C$?

[7 marks]

3. Consider the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Use an algorithm based on reduced row echelon form to compute $M^{-1}$.

(b) Use you answer to (a) to find the inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) Compute

$$\det M^3$$

[6 marks]

4. (a) Find the area of the triangle with vertices $(1, -1, 2)$, $(-2, 1, 1)$, $(1, 2, 3)$.

(b) Let

$$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3).$$

i. Write down the formula for $\mathbf{u} \times \mathbf{v}$ in terms of a determinant.

ii. Deduce a formula for

$$\mathbf{a} \cdot (\mathbf{u} \times \mathbf{v})$$

in terms of a determinant. What is the interpretation of the absolute value of this quantity in terms of a volume?

[7 marks]

5. (a) In the vector space $M_{2,2}$ of $2 \times 2$ matrices with real entries, let $S$ denote the set of matrices with determinant equal to 0. Show that $S$ is not a subspace of $M_{2,2}$.

(b) Let the set of anti-symmetric matrices in $M_{2,2}$ be denoted $R$, so that

$$R = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, \ a \in \mathbb{R} \right\}$$

By writing $R$ as a span, conclude that it is a subspace of $M_{2,2}$ (you must quote the appropriate piece of theory). What is the dimension of this subspace?

(c) Show from first principles that the set $R$ is closed under scalar multiplication.

[6 marks]

6. Let $A$ be a $5 \times 5$ matrix and let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be column vectors with five entries. Suppose that

$$[A|\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] \sim \left[\begin{array}{ccccc|cccc} 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Which of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ belong to the column space of $A$?

(b) With the columns of $A$ denoted $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5$ write the vectors found in (a) as linear combinations of the columns of $A$.

(c) Write down a basis for the row space of $A$. What is its dimension?

(d) Compute a basis for the solution space of $A$.

[8 marks]

7. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection in the line $y = x$, let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $\pi/4$ anti-clockwise, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an expansion by $\sqrt{2}$ in both the $x$ and $y$ directions.

(a) Compute the action of the transformation $R \circ S \circ T$ (this means $T$ followed by $S$ followed by $R$) on the unit vectors $(1, 0)$ and $(0, 1)$, and use this to give the standard matrix representation of $R \circ S \circ T$.

(b) What lines in the plane are left unchanged by the action of $R$?

[6 marks]

8. (a) Let $T$ be a linear transformation with standard matrix representation

$$A_T = \begin{bmatrix} -1 & -6 & 3 \\ 1 & 5 & -2 \\ 3 & 6 & 3 \end{bmatrix}$$

i. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$. Calculate $T(x, y, z)$.

ii. Suppose instead that $T : \mathcal{P}_2 \to \mathcal{P}_2$, where $\mathcal{P}_2$ is the vector space of polynomials of degree $\leq 2$. Calculate $T(a_0 + a_1x + a_2x^2)$.

iii. With $T : \mathcal{P}_2 \to \mathcal{P}_2$, calculate a basis for $\mathrm{Im}(T)$, the image of $T$.

(b) You are given that the transition matrix $P_{\mathcal{B},\mathcal{C}}$ from a basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ to a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is

$$P_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

i. Calculate the vector $\mathbf{u} = \mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3$ as a linear combination of vectors in $\mathcal{B}$.

ii. Calculate $P_{\mathcal{C},\mathcal{B}}$.

iii. Suppose

$$\mathbf{c}_1 = (1, 0, 0), \quad \mathbf{c}_2 = (1, 2, 0), \quad \mathbf{c}_3 = (1, 2, 3)$$

Compute $P_{\mathcal{S},\mathcal{B}}$, and use this to give the explicit form of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$.

[6 marks]

9. Suppose that $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis with respect to the dot product for $\mathbb{R}^3$.

(a) Writing $\mathbf{v} \in \mathbb{R}^3$ as the linear combination

$$\mathbf{v} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3$$

derive a formula for $\alpha$ in terms of the dot product of certain vectors.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which orthogonally projects onto the plane spanned by $\mathbf{u}_2$ and $\mathbf{u}_3$. Calculate $[T]_{\mathcal{U}}$ by first computing the action of $T$ on each of the vectors in $\mathcal{U}$. Also, specify in terms of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ Ker $T$ and Im $T$.

(c) Calculate, in terms of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ the transition matrix from the standard basis to the basis $\mathcal{U}$.

[6 marks]

10. (a) Write the formula

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = 5x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$$

in the general form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for certain $a, b, c, d$.

(b) Under what conditions on the matrix in (a) and on $\langle \mathbf{x}, \mathbf{x} \rangle$ does the general form define an inner product? Show that these conditions hold true for the given formula.

(c) Starting with the standard basis for $\mathbb{R}^2$ $\{(1,0), (0,1)\}$, apply the Gram-Schmidt procedure to deduce an orthonormal basis with respect to the inner product in (a).

[7 marks]

11. The enrollment in a certain lab class has decreased by $y$ students after $x$ weeks of the course, as given by the following data:

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |

(a) Find the line of best fit to the data, using the method of least squares.

(b) Draw the line of best fit on a graph, and mark in the data points.

(c) Use your answer to estimate the number of students who have dropped out of the lab class after week $x = 4$.

[7 marks]

12. (a) Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

i. Compute the determinant of $M$. What does this imply about one of the eigenvalues of $M$?

ii. Compute the eigenvalues of $M$.

iii. Is $M$ diagonalizable? Justify your answer by appealing to an appropriate theorem.

(b) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

i. Write down the eigenvalues of $A$.

ii. Compute the corresponding eigenspaces.

[7 marks]

— END OF EXAMINATION QUESTIONS —