MAST10007 Linear Algebra
MATLAB Test

Test duration: 45 minutes

This paper has 6 pages

Please complete ***all*** the following details.

Name: ....

Student Number: ....

Tutor's Name: ....

Lab Time: ....

Instructions to Students:

This test is designed to evaluate your comprehension of concepts in linear algebra, and your ability to calculate efficiently with the aid of MATLAB. Some questions test your understanding of the material covered in lectures, and do not necessarily require MATLAB. No partial credit is given, so please carefully check anything typed into MATLAB, and check the output of programs used.

Any rough working must be done on this paper, but only the final answer is marked.

Answer all multiple choice questions by circling the correct answer(s).

The number of marks for each question is indicated and the total number of marks is 25.

Some MATLAB commands:

- rref(A) gives the fully reduced row echelon form of A
- A' is the transpose of the matrix A
- det(A) gives the determinant of the matrix A
- eye(n) gives the identity matrix of size $n \times n$
- inv(A) gives the inverse of A
- ones(p, q) gives the $p \times q$ matrix of all 1's
- zeros(p, q) gives the $p \times q$ matrix of all 0's
- diag(v) gives the diagonal matrix with diagonal v
- The command v = B(:,3) selects the third column of B, for example.
- dot(u, v) gives the dot product of the vectors u and v.

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

1. (a) Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Suppose A is the adjacency matrix for a graph, with vertices A, B, C and D, and with rows and columns in this order. Draw the graph corresponding to A.

(b) In the setting of (a), calculate the number of walks from vertex A to vertex D using exactly 10 edges.

(c) Let

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Compute the solution space of B, writing down your answer only.

(d) Consider the matrix B of (c). Circle any vector below belonging to the column space of B,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 1 \\ -3 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ -1 \\ -3 \\ 4 \end{bmatrix}$$

(e) Your friend has given you a coded message of where to meet. The original message had letters replaced by numbers according to $A \leftrightarrow 1$, $B \leftrightarrow 2$ etc., and the letters were placed down the columns of a 3×3 matrix in order, then coded by multiplication on the left by

$$C = \begin{bmatrix} 1 & 17 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

You receive the message

1145, 141, 42, 1380, 206, 71, 1449, 210, 71

Where are you to meet?

[5 marks]

2. For each of the following statements, write true if the statement is true, and false if the statement is false.

   (a) Let A be a $p \times q$ matrix. The dimension of the row space plus the dimension of the solution of A equals p.

   (b) Let A be an $N \times N$ matrix, and suppose the dimension of the solution space of A is zero. Then A is singular.

   (c) The plane with vector equation

   $$(x, y, z) = t(1, -1, 1) + s(1, 0, 1) + (0, 1, 0), \qquad t, s \in \mathbb{R}$$

   passes through the origin.

   (d) Let $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$ be linearly independent. Then

   $$\text{Span}\{\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2\}$$

   has dimension 2.

   (e) The intersection of planes

   $$\{(x, y, z) : z = 2x - y + 1\} \cap \{(x, y, z) : z = x + y\}$$

   is a subspace of $\mathbb{R}^3$.

   (f) Let $T : \mathcal{P}_2 \to \mathcal{P}_1$ be specified by $T(a + bx + cx^2) = b + 2cx$. We have $\text{Ker}T = \{0\}$.

[6 marks]

3. Consider the matrix

$$X = \begin{bmatrix} 11 & 11 & 11 & 11 & 11 & 11 & -2 & 0 & 0 & 0 & 0 & 0 \\ 11 & 11 & 11 & 11 & 11 & 11 & 0 & -2 & 0 & 0 & 0 & 0 \\ 11 & 11 & 11 & 11 & 11 & 11 & 0 & 0 & -2 & 0 & 0 & 0 \\ 11 & 11 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & -2 & 0 & 0 \\ 11 & 11 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 & -2 & 0 \\ 11 & 11 & 11 & 11 & 11 & 11 & 0 & 0 & 0 & 0 & 0 & -2 \\ 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 7 & 9 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 7 & 7 & 10 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 11 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 & 12 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 & 7 & 13 & 7 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$$

(a) Let $\mathbf{a}$ denote the vector corresponding to the second column of X, let $\mathbf{b}$ denote the vector corresponding to the seventh row of X, and let $\mathbf{c}$ denote the vector corresponding to the third row of X. Calculate $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$.

(b) Calculate $\det X^{-1}$, giving your answer as an exact fraction.

(c) Calculate the projection of the vector corresponding to second row onto the direction of the vector corresponding to the seventh row.

(d) Add to X the matrix corresponding to the MatLab code

$$\texttt{Y} = [\texttt{zeros}(6,6) \quad 2 * \texttt{eye}(6,6); -\texttt{diag}([1,2,3,4,5,6]) \quad \texttt{zeros}(6,6)]$$

What is the dimension of the solution space of $X + Y$?

[4 marks]

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation. Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4 \in \mathbb{R}^4$ be the vectors in the set

$$\mathcal{B} = \{(1,2,3,4), (0,2,3,4), (0,0,3,4), (0,0,0,4)\}$$

in order. It is known that $\mathcal{B}$ is a basis for $\mathbb{R}^4$.

(a) Let $U = [\mathbf{b}_1\, \mathbf{b}_2\, \mathbf{b}_3\, \mathbf{b}_4]$. What is the rank of U?

(b) Let $\mathcal{S}$ denote the standard basis. Of the following matrices, identify which are the change of basis matrices $P_{\mathcal{S},\mathcal{B}}$ and $P_{\mathcal{B},\mathcal{S}}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 0 & -1/2 & 1/3 & 0 \\ 0 & 0 & -1/3 & 1/4 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

(c) Suppose

$$T\mathbf{b}_1 = \mathbf{b}_1, \quad T\mathbf{b}_2 = \mathbf{0}, \quad T\mathbf{b}_3 = \mathbf{b}_2 - \mathbf{b}_1, \quad T\mathbf{b}_4 = -2\mathbf{b}_4$$

Give the explicit form of $[T]_{\mathcal{B},\mathcal{B}}$.

(d) Give the explicit form of $[T]_{\mathcal{S},\mathcal{S}}$ (also denoted A_T), where $\mathcal{S}$ is the standard basis.

(e) Calculate $T\mathbf{x}$, where $\mathbf{x} = (1,1,1,1)$.

[6 marks]

5. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ has a standard matrix representation

$$A_T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Write down the image of the three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ under T.

(b) Draw on a graph the image under the linear transformation T of the unit cube formed by the three unit vectors. Your graph should consist of the image of the corners, and the edges connecting the corners.

(c) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ have standard matrix $A_S = A_T^T A_T$, where the superscript T denotes transpose. What is the dimension of the image of S?

(d) Given the explicit form of the standard matrix for the linear transformation corresponding to first applying A_S, then applying A_T.

[4 marks]