School of Computing and Information Systems
COMP30026 Models of Computation Tutorial Week 4

14–18 August 2017

Plan

Make sure to go over these questions before you get to the tute. Then try to get through at least Questions 16–22 in the tute, and get started on Question 23. Questions 24–26 are just more of the same; use them for practice if needed. If we don't get through 24–27 in the tute, that's fine. We won't skip Question 27, but we may have to carry it over to next week.

The exercises

15. Let Φ and Ψ be propositional formulas. What is the difference between '$\Phi \equiv \Psi$' and '$\Phi \Leftrightarrow \Psi$' — do we really need both? Show that $\Phi \equiv \Psi$ iff $\Phi \Leftrightarrow \Psi$ is valid.

16. For each of the following propositional formulas, determine whether it is satisfiable, and if it is, whether it is a tautology:

    (a) $P \Leftrightarrow ((P \Rightarrow Q) \Rightarrow P)$

    (b) $(P \Rightarrow \neg Q) \wedge ((P \vee Q) \Rightarrow P)$

    (c) $((P \Rightarrow Q) \Rightarrow Q) \wedge (Q \oplus (P \Rightarrow Q))$

17. By negating a satisfiable proposition, can you get a tautology? A satisfiable proposition? A contradiction? Illustrate your affirmative answers.

18. Put the following formulas in reduced CNF:

    (a) $\neg(A \wedge \neg(B \wedge C))$

    (b) $A \vee (\neg B \wedge (C \vee (\neg D \wedge \neg A)))$

    (c) $((A \vee B) \Rightarrow (C \wedge D))$

    (d) $A \wedge (B \Rightarrow (A \Rightarrow B))$

19. Find the reduced CNF of $\neg((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$ and express the result as a set of sets of literals. Then determine whether a refutation of the set is possible.

20. Using resolution, show that the set $\{\{A, B, \neg C\}, \{\neg A\}, \{A, B, C\}, \{A, \neg B\}\}$ of clauses is unsatisfiable.

21. Use resolution to show that each of these formulas is a tautology:

    (a) $(P \vee Q) \Rightarrow (Q \vee P)$

    (b) $(\neg P \Rightarrow P) \Rightarrow P$

    (c) $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

    (d) $P \Leftrightarrow ((P \Rightarrow Q) \Rightarrow P)$

22. For each of the following clause sets, write down a propositional formula in CNF to which it corresponds. Which of the resulting formulas are satisfiable? Give models of those that are.

    (a) $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$
    (b) $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$
    (c) $\{\{A\}, \emptyset\}$
    (d) $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$

23. The Grok worksheet for Week 4 asks you to improve on a Haskell program that performs resolution for propositional logic. Study the program and improve it as suggested.

24. Letting Φ and Ψ be two different formulas from the set

$$\{(P \land Q) \lor R, \quad (P \lor Q) \land R, \quad P \land (Q \lor R), \quad P \lor (Q \land R)\}$$

    list all combinations that satisfy $\Phi \models \Psi$.

25. In Lecture 6 it was claimed that the formula

$$(P \land Q \land R) \lor (\neg P \land \neg Q \land \neg R) \lor (\neg P \land R) \lor (Q \land \neg R)$$

    is logically equivalent to the simpler

$$\neg P \lor Q$$

    with both being in reduced disjunctive normal form (RDNF). Show that the claim is correct.

26. In the previous question we saw that this formula Φ:

$$(P \land Q \land R) \lor (\neg P \land \neg Q \land \neg R) \lor (\neg P \land R) \lor (Q \land \neg R)$$

    is *independent* of R. We may say that the formula depends on R syntactically (because R occurs in it), but not semantically. Find a smallest possible CNF formula, equivalent to Φ, that depends syntactically on P, Q and R.

27. Recall the graph colouring problem from Week 3's Grok worksheet. How can we encode the three-colouring problem in propositional logic, in CNF to be precise? (One reason we might want to do so is that we can then make use of a so-called SAT solver to determine colourability.) Using propositional variables

    - B_i to mean node i is blue,
    - G_i to mean node i is green,
    - R_i to mean node i is red;
    - E_{ij} to mean i and j are different but connected by an edge,

    write formulas in CNF for these statements:

    (a) Every node (0 to n inclusive) is coloured.
    (b) Every node has at most one colour.
    (c) No two connected nodes have the same colour.

    For a graph with $n + 1$ nodes, what is the size of the CNF formula?