Tutorial 10

An *inner product* over a vector space V is defined by the four axioms (for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V, c \in \mathbb{R}$):

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$
3. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Rightarrow \mathbf{u} = \mathbf{0}$

All inner products in the case that $V = \mathbb{R}^n$ can be written in the form $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$, where we regard $\mathbf{u}$, $\mathbf{v}$ as column vectors, and A is an $n \times n$ matrix. Conversely, this form defines an inner product if and only if A is symmetric and axiom 4. holds.

Q1. With $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, let $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 + x_3y_3$.

(i). Find the 3×3 matrix A such that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}$.

(ii). By completing the square, show that axiom 4 holds for $\langle \mathbf{x}, \mathbf{y} \rangle$, and hence it defines an inner product.

Vectors $\mathbf{u}$, $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is said to be an orthogonal set if each pair of distinct vectors in the set is orthogonal.
The set is said to be an orthonormal set if it is an orthogonal set and if each vector has norm (length) one. Writing $X = [\mathbf{v_1}\, \mathbf{v_2} \cdots \mathbf{v_n}]$, we have that $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is an orthonormal set if and only if $X^T X = I_n$.

Q2. Verify that in $\mathbb{R}^3$, with the dot product

$$\left\{ \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{20}}, \frac{3}{\sqrt{20}} \right), \left(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{20}}, -\frac{1}{\sqrt{20}} \right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$$

is an orthonormal set.

Let $\{\mathbf{u_1}, \mathbf{u_2} \dots, \mathbf{u_k}\}$ be an orthonormal basis for a vector space W. Then any $\mathbf{w} \in W$ an be written as a linear combination of $\mathbf{u_1}, \mathbf{u_2} \dots, \mathbf{u_k}$ according to the formula

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{u_1} \rangle \mathbf{u_1} + \langle \mathbf{w}, \mathbf{u_2} \rangle \mathbf{u_2} + \cdots + \langle \mathbf{w}, \mathbf{u_k} \rangle \mathbf{u_k}$$

Q3. Let $\{\mathbf{u_1}, \mathbf{u_2}\}$ be an orthonormal basis for a vector space W.

(i). What is the dimension of W?

(ii). Write down a formula for the norm of $\mathbf{v}$ where $\mathbf{v}$ is any vector in W.

(iii). Write down the norm of $\mathbf{u_1}$, the norm of $\mathbf{u_2}$, and the inner product between $\mathbf{u_1}$ and $\mathbf{u_2}$ which follows from $\{\mathbf{u_1}, \mathbf{u_2}\}$ being an orthonormal basis.

(iv). For $\mathbf{w} \in W$, write $\mathbf{w} = \alpha_1 \mathbf{u_1} + \alpha_2 \mathbf{u_2}$. Take the inner product of both sides with respect to $\mathbf{u_1}$ to determine α_1, and the inner product of both sides with respect to $\mathbf{u_2}$ to determine α_2. Check that this is consistent with the formula in the box above.

(v). Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{u_1} = \frac{1}{2}(1, -1, 1, 1), \quad \mathbf{u_2} = \frac{1}{2}(-1, 1, 1, 1)$$

Check that $\mathbf{u_1}$ and $\mathbf{u_2}$ are orthonormal using the dot product on $\mathbb{R}^4$ and use your answer to (iv) to write the vector $(-1, 1, 5, 5) \in W$ as a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$.

(vi). Use $\text{Proj}_W(\mathbf{w}) = \alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2}$ where $\alpha_i = \langle \mathbf{w}, \mathbf{u_i}\rangle = \mathbf{w}\cdot\mathbf{u_i}$ to find the *orthogonal projection* of $\mathbf{w} = (-1, 1, 3, 5)$ onto W.

The *Gram-Schmidt* procedure converts any basis $\{\mathbf{v_1}, \dots, \mathbf{v_k}\}$ for an inner product space into an *orthonormal* basis. If there are three vectors in the basis the algorithm reads
Step 1. Normalize $\mathbf{v_1}$: $\mathbf{u_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1}$.
Step 2. Form $\mathbf{w_2} = \mathbf{v_2} - \langle \mathbf{v_2}, \mathbf{u_1}\rangle\mathbf{u_1}$ and normalize $\mathbf{w_2}$ to get $\mathbf{u_2} = \frac{1}{||\mathbf{w_2}||}\mathbf{w_2}$
Step 3. Form $\mathbf{w_3} = \mathbf{v_3} - \langle \mathbf{v_3}, \mathbf{u_1}\rangle\mathbf{u_1} - \langle \mathbf{v_3}, \mathbf{u_2}\rangle\mathbf{u_2}$ and normalize $\mathbf{w_3}$ to get $\mathbf{u_3} = \frac{1}{||\mathbf{w_3}||}\mathbf{w_3}$
The set of vectors $\{\mathbf{u_1} \dots, \mathbf{u_k}\}$ is an orthonormal basis.

Q4. Use the Gram-Schmidt procedure to make $\{(1,0),(0,1)\}$ into an orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $\langle \mathbf{x}, \mathbf{y}\rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$.

The least squares line of best fit to data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, is given by

$$y = a + bx$$

where $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ satisfies the matrix equation

$$A^TA\mathbf{u} = A^T\mathbf{y}, \quad \text{where} \quad A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

If we want to fit the quadratic curve $y = a + bx + cx^2$, the solution is the same if

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Q5. A grocery store takes delivery of stock and records sales from this day. At the end of business on day x, the number of packets y of a certain brand of cereal remaining on the shelf were recorded to give the following data

x	1	2	3
y	14	10	10

(i). Compute the least squares line of best fit to this data.

(ii). Use your answer to (i) to estimate the number of days before the product runs out.