Tutorial 2: Solutions

Q1. (i). $A+B = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 4 & -5 \end{bmatrix}$

(ii). $C+D$ not possible

(iii). $4+D$ not possible

(iv). $3D = 3\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \\ 9 \end{bmatrix}$

(v). $AD = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix}.$

(vi). DA not possible

(vii). $BC = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -13 & -26 \end{bmatrix}$

(viii). $CB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -11 \\ 2 & 0 & -1 \\ 6 & 15 & -28 \end{bmatrix}$

MATLAB calculates $4+D$ as adding 4 to each entry of D so would give $\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$

MATLAB also reads $3D$ and AD, for example, as names for functions or variables, so to show multiplication you would need to type in $3 * D$ and $A * D$ etc.

Q2. (i). $\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}^{-1} = -\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

(ii). Here $ad - bc = (-3)(-10) - 5 \times 6 = 0$ and so the inverse does not exist.

(iii).
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2+2R_1 \\ R_3+R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 2 & 1 & 0 \\ 0 & 4 & 2 & 1 & 0 & 1 \end{array}\right] R_3 - \tfrac{4}{7}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 2 & 1 & 0 \\ 0 & 0 & \frac{10}{7} & -\frac{1}{7} & -\frac{4}{7} & 1 \end{array}\right] \begin{array}{c} \\ \frac{1}{7}R_2 \\ \frac{7}{10}R_3 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{4}{10} & \frac{7}{10} \end{array}\right] R_2 - \tfrac{1}{7}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{21}{70} & \frac{14}{70} & -\frac{7}{70} \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{4}{10} & \frac{7}{10} \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{10} & \frac{2}{10} & -\frac{1}{10} \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{4}{10} & \frac{7}{10} \end{array}\right] R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{10} & -\frac{4}{10} & \frac{2}{10} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{2}{10} & -\frac{1}{10} \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{4}{10} & \frac{7}{10} \end{array}\right]$$

Hence $\begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -4 & 2 \\ 3 & 2 & -1 \\ -1 & -4 & 7 \end{bmatrix}$

Q3. For $B + B^T$ to be defined, both matrices must be of the same size. As B^T has rows and columns interchanged from matrix B, this means the number of rows and columns must be the same and therefore the matrix must be square.

Q4. Now a matrix $B = (A+I)^{-1}$ if $(A+I)B = I$.
Hence to show that $(A+I)^{-1} = \frac{1}{2}(A^2 - A + I)$ it suffices to show

$$(A+I) \times \frac{1}{2}(A^2 - A + I) = I.$$

Now

$$\begin{aligned} (A+I) \times \frac{1}{2}(A^2 - A + I) &= \frac{1}{2}(A^3 - A^2 + A + A^2 - A + I) \\ &= \frac{1}{2}(A^3 + I) \\ &= \frac{1}{2}(I + I) \qquad \text{since } A^3 = I \\ &= I \qquad \text{as required.} \end{aligned}$$

Q5. For the system defined by:

(i). $\begin{bmatrix} 2 & -3 \\ 9 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 10 \end{bmatrix}$

(ii). It follows from the result of (i) that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 9 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 10 \end{bmatrix}$ provided the inverse exists. From the formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} 2 & -3 \\ 9 & -4 \end{bmatrix}^{-1} = \frac{1}{2 \times (-4) - (-3) \times 9} \begin{bmatrix} -4 & 3 \\ -9 & 2 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} -4 & 3 \\ -9 & 2 \end{bmatrix}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{19} \begin{bmatrix} -4 & 3 \\ -9 & 2 \end{bmatrix} \begin{bmatrix} 17 \\ 10 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} -38 \\ -133 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$

Q6. (i). Let

$$A = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.01 & 0 & 0.99 \\ 0.01 & 0.9 & 0.09 \end{bmatrix}$$

Then the linear system can be written

$$A \begin{bmatrix} M \\ C \\ S \end{bmatrix} = \begin{bmatrix} 400 \\ 10 \\ 120 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} M \\ C \\ S \end{bmatrix} = A^{-1} \begin{bmatrix} 400 \\ 10 \\ 120 \end{bmatrix}$$

Using the given value of A^{-1} this tells us that

$$\begin{bmatrix} M \\ C \\ S \end{bmatrix} \approx \begin{bmatrix} 1.25 & -0.1 & -0.14 \\ -0.013 & -0.1 & 1.11 \\ -0.01 & 1.0 & 0 \end{bmatrix} \begin{bmatrix} 400 \\ 10 \\ 120 \end{bmatrix} \approx \begin{bmatrix} 482 \\ 127 \\ 6 \end{bmatrix}$$

Converting to percentages gives:

$$\begin{aligned} \frac{482}{482 + 127 + 6} \times 100 &\approx 78.5\% \text{ for M} \\ \frac{127}{482 + 127 + 6} \times 100 &\approx 20.5\% \text{ for C} \\ \frac{6}{482 + 127 + 6} \times 100 &\approx 1\% \text{ for S} \end{aligned}$$

(ii). If the number of murder arrests doubles, we see from the middle column of A^{-1} that it would have the greatest effect on the workload of the Supreme court, which increases from 6 to 16 cases. (The new values are approximately $M = 481$, $C = 126$ and $S = 16$.)