COMP30026 Assignment 1

Challenge 1

Let A be "The person A is a knight".
Let B be "The person B is a knight".
Let C be "The person C is a knight".
Then what the person A says can be translated as: $A \Rightarrow (\neg B \wedge \neg C)$.
And now we can have the truth table:

A	B	C	$\neg B \wedge \neg C$	$A \Rightarrow (\neg B \wedge \neg C)$	
0	0	0	1	1	
0	0	1	0	1	
0	1	0	0	1	
0	1	1	0	1	
1	0	0	1	1	(⁎)
1	0	1	0	0	
1	1	0	0	0	
1	1	1	0	0	

If the person A is a knight, what he says must be true. So in this case, both A and $A \Rightarrow (\neg B \wedge \neg C)$ must be true, which corresponds to the line (⁎) in the truth table.
If the person A is a knave, what he says must be false. So in this case, both A and $A \Rightarrow (\neg B \wedge \neg C)$ must be false, which does not occur in the truth table.
Therefore, there is only one case that can be true, in which A is a knight, and B and C are knaves.

Challenge 2

1. $\neg\varphi \equiv \neg(\ ((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \Rightarrow S\ )$
$\equiv \neg(\ (\neg P \vee S) \wedge (\neg Q \vee R) \wedge (\neg R \vee P) \Rightarrow S\ )$
$\equiv \neg(\ \neg((\neg P \vee S) \wedge (\neg Q \vee R) \wedge (\neg R \vee P)) \vee S\ )$
$\equiv \neg(\ (P \wedge \neg S) \vee (Q \wedge \neg R) \vee (R \wedge \neg P) \vee S\ )$
$\equiv (\neg P \vee S) \wedge (\neg Q \vee R) \wedge (\neg R \vee P) \wedge \neg S$

2. Let P, Q, R and S all be false, then we have:

P	Q	R	S	$P \Rightarrow S$	$Q \Rightarrow R$	$R \Rightarrow P$	$(P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)$	$((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \Rightarrow S$
0	0	0	0	1	1	1	1	0

This set of assignments shows that the value of φ can be false, which means φ is non-valid.

3. $\neg\psi \equiv \neg(\ (((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \Rightarrow S\ )$
$\equiv \neg(\ ((\neg(P \vee Q) \vee S) \wedge (\neg\neg P \vee \neg R \vee Q) \wedge (R \vee S)) \Rightarrow S\ )$
$\equiv \neg(\ \neg\ ((\neg(P \vee Q) \vee S) \wedge (P \vee \neg R \vee Q) \wedge (R \vee S)) \vee S\ )$
$\equiv \neg(\ ((P \vee Q) \wedge \neg S) \vee (\neg P \wedge R \wedge \neg Q) \vee (\neg R \wedge \neg S) \vee S\ )$
$\equiv ((\neg P \wedge \neg Q) \vee S) \wedge (P \vee \neg R \vee Q) \wedge (R \vee S) \wedge \neg S$
$\equiv (\neg P \vee S) \wedge (\neg Q \vee S) \wedge (P \vee \neg R \vee Q) \wedge (R \vee S) \wedge \neg S$

4. Here we prove that ψ is valid by the resolution on $\neg\psi$.

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¬P, S    ¬Q, S    P, ¬R, Q    R, S    ¬S
    \        \      /           /       /
     \       P, ¬R, S         /       /
      \          \           /      /
       \           P, S           /
        \          /            /
         \       /            /
            S              /
              \         /
                  ⊥
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Challenge 3

Let φ : $\forall x \forall y(P(x, y) \Rightarrow P(h(x), h(h(y))))$

Let ψ: $\forall x(P(x, h(x)) \wedge P(h(h(x)), x))$

To show the original formula is non-valid, we need an interpretation that makes φ true and ψ false.

Let P(x, y) be "x is less than y".

Let h(x) = x + 1.

Let the domain D = $\mathbb{Z}$, representing the set of all the integers.

Then for φ, if x < y, then P(x, y) is true, and x+1 is less than y+1+1, which means P(h(x), h(h(y))) is also true. If x ≥ y, then P(x, y) is false, and φ is true no matter true of false P(h(x), h(h(y))) is. Thus, φ is true.

For ψ, we can pick any integer k, where h(h(k)) = k+2 > k that makes P(h(h(k)), k) false, which means ψ is false.

Therefore $\varphi \Rightarrow \psi$ is false, the original formula is non-valid.

To show the original formula is satisfiable, we need an interpretation that makes $\varphi \Rightarrow \psi$ true.

Let P(x, y) be "x is equal to y".

Let h(x) = x^2.

Let the domain D = {0}.

For φ, x and y only take the value of 0, and so do h(x) and h(h(y)), then $\forall x \forall y(P(x, y) \Rightarrow P(h(x), h(h(y))))$ is true.

For ψ, x only can be 0, and so do h(x) and h(h(x)), then $\forall x(P(x, h(x)) \wedge P(h(h(x)), x))$ is true.

Therefore, $\varphi \Rightarrow \psi$ is true, the original formula is satisfiable.

Challenge 4

1. $\forall x(S(x) \Rightarrow (\forall y \neg P(y, x) \Rightarrow H(x)))$

2. $\forall x(S(x) \Rightarrow (\forall y(P(y, x) \Rightarrow R(y)) \Rightarrow H(x)))$

3. Eliminate ‘⇒’:

   $\forall x(\neg S(x) \vee (\exists y(P(y, x) \wedge \neg R(y))) \vee H(x))$

   Skolemized by mapping y to f(x) :

   $\forall x(\neg S(x) \vee (P(f(x), x) \wedge \neg R(f(x))) \vee H(x))$

   Drop Universal Quantifiers:

   $\neg S(x) \vee (P(f(x), x) \wedge \neg R(f(x))) \vee H(x)$

Convert to CNF:

$$(\neg S(x) \vee P(f(x), x) \vee H(x)) \wedge (\neg S(x) \vee \neg R(f(x)) \vee H(x))$$

Then we have a set of sets of literals:

$$\{ \{\neg S(x), P(f(x), x), H(x)\}, \{\neg S(x), \neg R(f(x)), H(x)\} \}.$$

4. Negate S1:

$$\neg(\forall x(S(x) \Rightarrow (\forall y\ \neg P(y, x) \Rightarrow H(x))))$$

Eliminate '⇒':

$$\neg(\forall x(\neg S(x) \vee \exists y\ p(y, x) \vee H(x)))$$

Push Negation:

$$\exists x(S(x) \wedge \forall y\ \neg P(y, x) \wedge \neg H(x))$$

Skolemized by mapping x to a :

$$S(a) \wedge \forall y\ \neg P(y, a) \wedge \neg H(a)$$

Drop Universal Quantifiers:

$$S(a) \wedge \neg P(y, a) \wedge \neg H(a)$$

It is already in CNF.

Then we have a set of sets of literals:

$$\{ \{S(a)\}, \{\neg P(y, a)\}, \{\neg H(a)\} \}$$

5. Apply the resolution to $S2 \wedge \neg S1$:

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¬S(x), P(f(x), x), H(x)     ¬S(x), ¬R(f(x)), H(x)     S(a)     ¬P(y, a)     ¬H(a)
                 \                                    /          /            /
                  \                              / x → a   /            /
                   \                            /          /            /
                    \                          /          /            /
                      P(f(a), a), H(a)                / y → f(a)   /
                                 \                /            /
                                  \             /            /
                                     H(a)                  /
                                        \             /
                                         \          /
                                              ⊥
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Therefore, S1 follows from S2.