Tutorial 5: Solutions

Q1.
(i) (ii)

Q2. (i). Rearranging gives $2(-1,1)+2(1,1)-(0,4)=(0,0)$. Hence from the definition, the set of vectors $\{(-1,1),(1,1),(0,4)\}$ is linearly dependent.

(ii). We calculate the rank of the matrix with the vectors as columns;

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} \\ R_2+R_1 \\ R_3-R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

This has rank 3 and so the vectors are linearly independent.

(iii). The reduced row echelon form has rank 3 and so the set of vectors formed from the columns

$$\{(1,3,1,2),(-2,-10,-1,2),(3,8,0,1),(3,13,2,0)\}$$

are linearly dependent.

Q3. (i). With $\mathbf{v_1}=(x_1,-2x_1)$ and $\mathbf{v_2}=(x_2,-2x_2)$, a general linear combination is

$$\begin{aligned} \alpha\mathbf{v_1}+\beta\mathbf{v_2} &= \alpha(x_1,-2x_1)+\beta(x_2,-2x_2) \\ &= (\alpha x_1+\beta x_2,-2(\alpha x_1+\beta x_2)) \\ &= (t,-2t)\in S \quad \text{with } t=\alpha x_1+\beta x_2 \end{aligned}$$

Hence S is a subspace, Geometrically, it is a line through the origin.

(ii).
Consider the points $(1,2)$ and$(1,-1)$ in S. Then $(1,2)+(1,-1)=(2,1)$. This is not in S so it is not closed under vector addition and so is not a subspace.

(iii). Here V is a parabola through the origin. Consider the points $(1,1)$ and $(-1,1)\in V$. Then $(1,1)+(-1,1)=(0,2)$ this is not in V as it is not of the form (t,t^2) for any t.

(iv). We know that all vector spaces must contain the zero vector $\mathbf{x}=\mathbf{0}$. Substituting this in the linear equation tells us that $\mathbf{b}=\mathbf{0}$. W is the solution set of the linear system

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 4 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Here $\mathbf{b}\neq\mathbf{0}$ and so W is not a subspace.

(v). Let $S=\{(x,y,z):\in\mathbb{R}^3 : x-4y+3z=0\}$. If you think S is a subspace show by testing the following conditions :

(o) $\mathbf{0}=(0,0,0)\in S$ as $0-4(0)+3(0)=0$.

(i) Using $\mathbf{v}$ as above we have $\alpha\mathbf{v}=(\alpha x_1,\alpha y_1,\alpha z_1)$ and

$$x-4y+3z=\alpha x_1-4\alpha y_1+3\alpha z_1=\alpha(x_1-4y_1+3z_1)=\alpha(0)=0$$

so we have $\alpha\mathbf{v}\in S$

(ii Let $v = (x_1, y_1, z_1) \in S$ and $w = (x_2, y_2, z_2) \in S$ then

$$v + w = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

and we have

$$\begin{aligned} x - 4y + 3z &= x_1 + x_2 - 4(y_1 + y_2 + 3(z_1 + z_2) \\ &= x_1 - 4y_1 + 3z_1 + x_2 - 4y_2 + 3z_2 \\ &= 0 + 0 = 0 \end{aligned}$$

so we have $v + w \in S$

So as all conditions are satisfied then S is a subspace.

Q4. (i).

(ii). Regarded as vectors in $\mathbb{R}^2$, $(1, -1)$ and $(2, 3)$ are linearly independent. Hence Span$\{(1, -1, 0), (2, 3, 0)\}$ is equal to the xy–plane in $\mathbb{R}^3$.

(iii). Span$\{(1, 1, 1), (3, 2, 1)\}$ is the plane through the origin containing the vectors $(1, 1, 1)$ and $(3, 2, 1)$. A normal vector to the plane is

$$\mathbf{n} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \mathbf{i}(-1) - \mathbf{j}(-2) + \mathbf{k}(-1) = -\mathbf{i} + 2\mathbf{j} - \mathbf{k} = (-1, 2, -1)$$

The cartesian form for the plane is thus

$$\mathbf{r} \cdot \mathbf{n} = 0 \quad \Rightarrow \quad (x, y, z) \cdot (-1, 2, -1) = 0 \quad \Rightarrow \quad x - 2y + z = 0$$

(iv). The criteria is that the matrix formed with the vectors as columns has rank 3. This matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_3 \\ \\ R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_2 + R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The matrix has a rank of 3 so the span is $\mathbb{R}^3$, as claimed.