# COMP20003
# Algorithms and Data Structures
# All Pairs Shortest Paths

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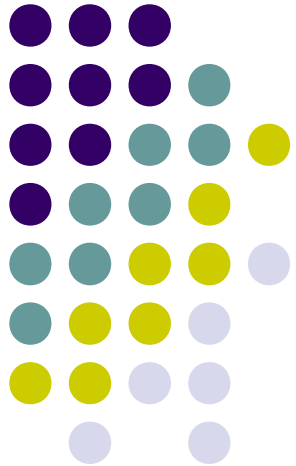
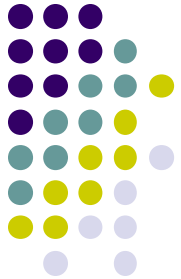
Semester 2

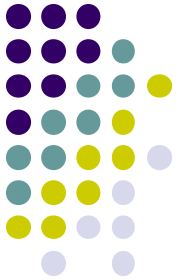


# Shortest Paths: Single Source *vs.* All Pairs



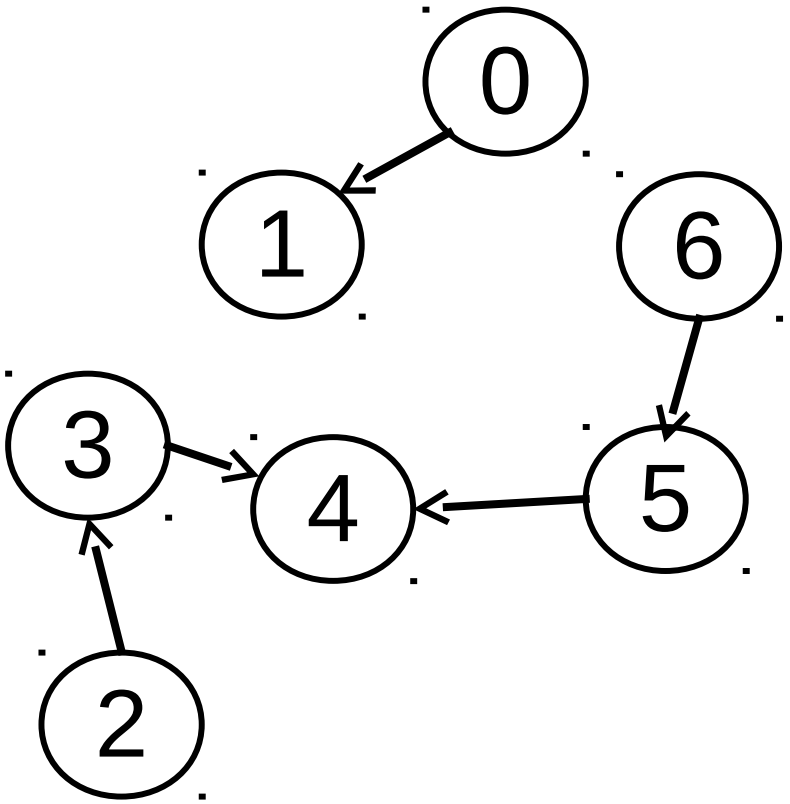
- Single source:
  - Shortest paths from one vertex to all others.
  - Dijkstra's algorithm: O((V+E)log V).
- All pairs:
  - Shortest paths from every vertex to every other vertex.
  - Why not run Dijkstra's algorithm once for every vertex?

# Shortest Paths: Single Source *vs.* All Pairs

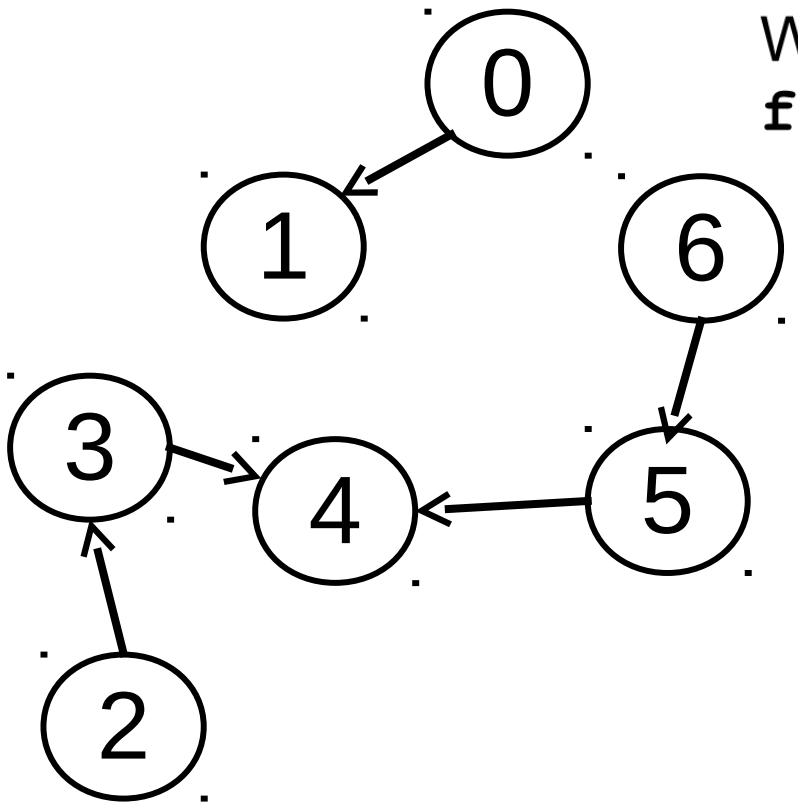


- Using Dijkstra's multiple times:
  - Dijkstra's algorithm: $O((V+E) \log V)$
  - Once for every vertex: $O((V^2+VE) \log V)$.
  - $O(V^3 \log V)$ for dense graphs.
- Can we do better?

# Transitive closure: Unconnected directed graph



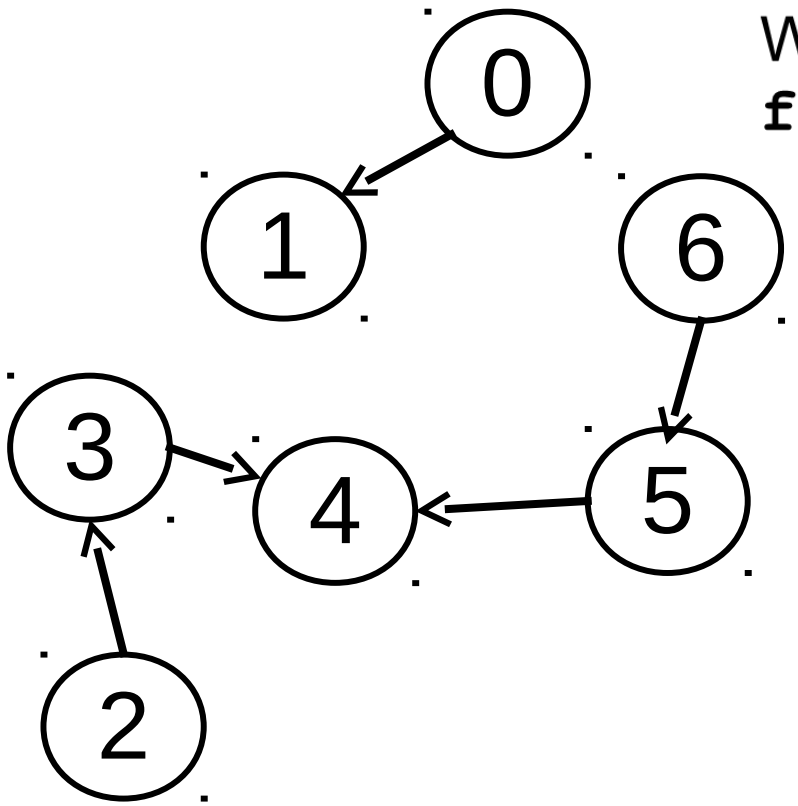
# Transitive closure: Unconnected directed graph




Warshall algorithm:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

# Transitive closure: Unconnected directed graph




Warshall algorithm:

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  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Transitive closure: Unconnected directed graph

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Warshall algorithm:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

# Transitive closure: Unconnected directed graph

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Warshall algorithm:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

**i=0** Column 0 all 0, so no A[s][i]

**i=1** Row 1 all 0 so no A[i][t]

**i=2** Column 2 all 0, so no A[s][i]

**i=3** A[2][3]&& A[3][4], so A[2][4]

**i=4** Row 4 all 0, so no A[i][t]

**i=5** A[6][5]&& A[5][4], so A[6][4]

**i=6** Column 6 all 0, so no A[s][i]

# Transitive closure: Unconnected directed graph

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Warshall algorithm:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

**i=0** Column 0 all 0, so no A[s][i]

**i=1** Row 1 all 0 so no A[i][t]

**i=2** Column 2 all 0, so no A[s][i]

**i=3** A[2][3]&& A[3][4], so A[2][4]

**i=4** Row 4 all 0, so no A[i][t]

**i=5** A[6][5]&& A[5][4], so A[6][4]

**i=6** Column 6 all 0, so no A[s][i]

# Transitive closure: Unconnected directed graph

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Warshall algorithm:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i] && A[i][t])
        A[s][t] = TRUE;
```

**i=0** Column 0 all 0, so no A[s][i]

**i=1** Row 1 all 0 so no A[i][t]

**i=2**

**i=3** Column 2 all 0, so no A[s][i]
A[2][3]&& A[3][4], so A[2][4]

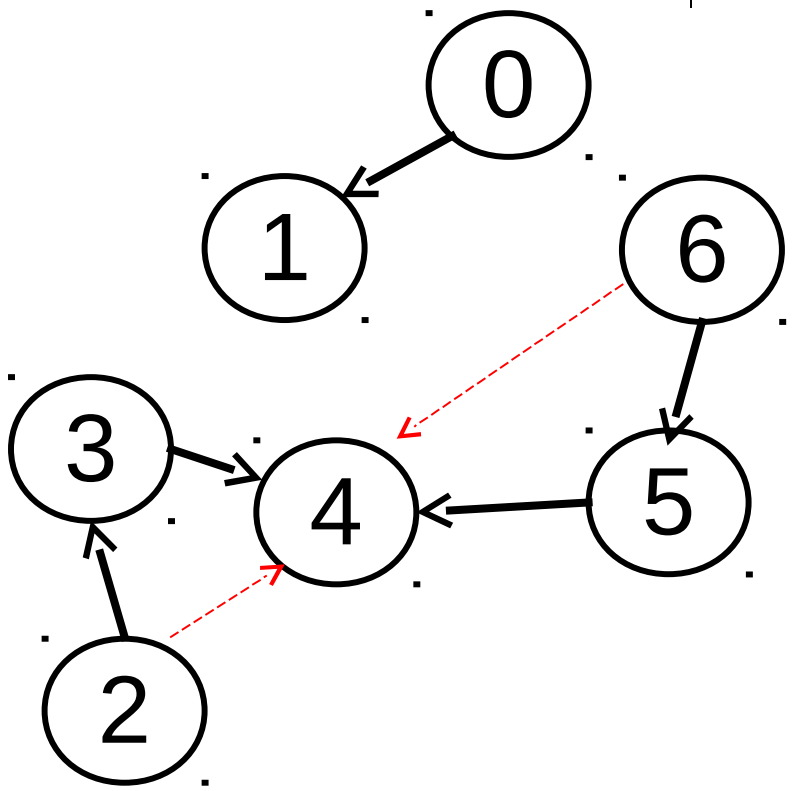
**i=4** Row 4 all 0, so no A[i][t]

**i=5**

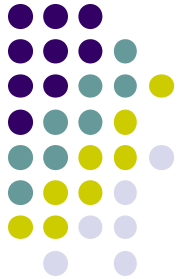
**i=6** A[6][5]&& A[5][4], so A[6][4]
Column 6 all 0, so no A[s][i]

# Transitive closure: Unconnected directed graph

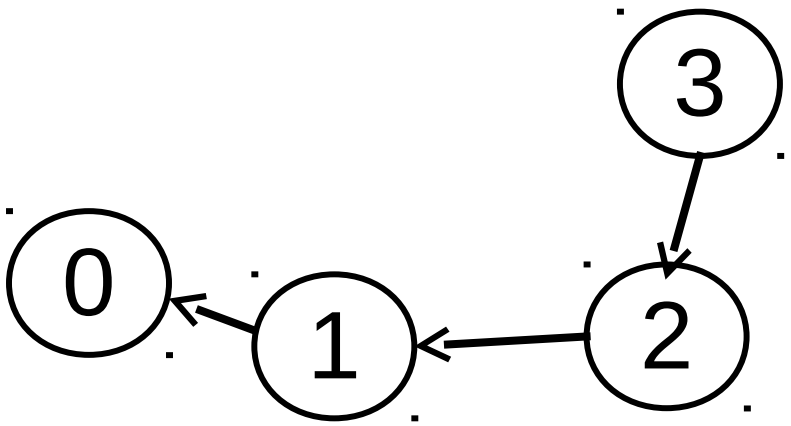
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |



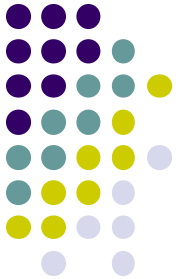
# Transitive Closure with multi-segment paths



| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 |



# Warshall algorithm: Analysis



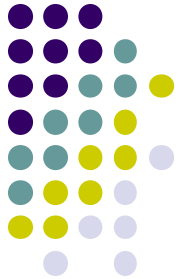
Warshall algorithm:

```
for(i=0;i<V;i++)
   for(s=0;s<V;s++)
      for(t=0;t<V;t++)
        if(A[s][i] && A[i][t])
           A[s][t] = TRUE;
```

$\Theta(?)$ for graph of $V$ vertices and $E$ edges.

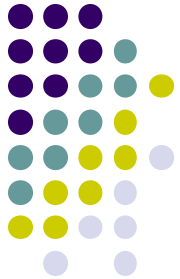
How does this compare with running Dijkstra's algorithm $V$ times?

# Floyd-Warshall algorithm




- Warshall, Stephen (January 1962). "A theorem on Boolean matrices". *Journal of the ACM* **9** (1): 11–12.
- Floyd, Robert W. (June 1962). "Algorithm 97: Shortest Path". *Communications of the ACM* **5** (6): 345.

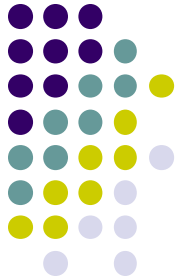
# Use Warshall *framework* to get *shortest path lengths*



- Warshall algorithm, boolean matrix, no self-loops:

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i]&&A[i][t]) A[s][t]=TRUE;
```

# Use Warshall framework to get *shortest path lengths*



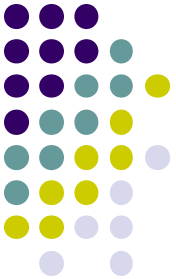
- Warshall algorithm (boolean matrix, no self-loops):

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i]&&A[i][t]) A[s][t]=TRUE;
```

- Floyd-Warshall algorithm (weights, **A[i][i]**= 0, no path=∞)

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(                    )    A[s][t]=  ;
```

# Use Warshall framework to get *shortest path lengths*



- Warshall algorithm (boolean matrix, no self-loops):

```
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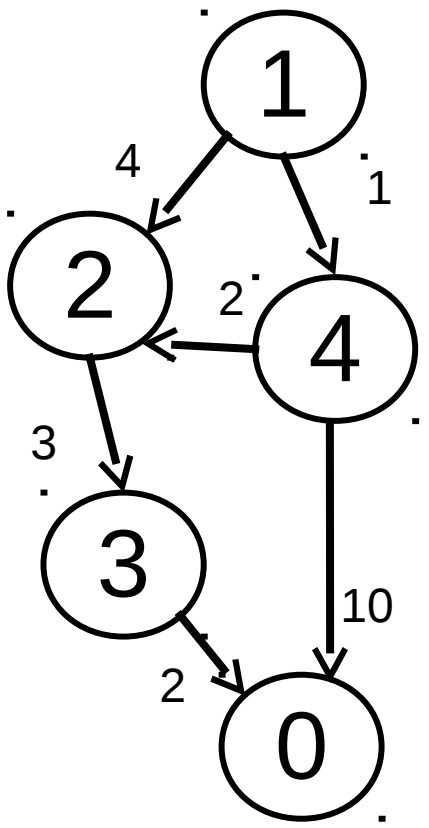
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i]&&A[i][t]) A[s][t]=TRUE;
```

- Floyd-Warshall algorithm (weights, **A[i][i]**= 0, no path=∞)

```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      if(A[s][i]+A[i][t] < A[s][t])
        A[s][t] = (A[s][i]+A[i][t]);
```

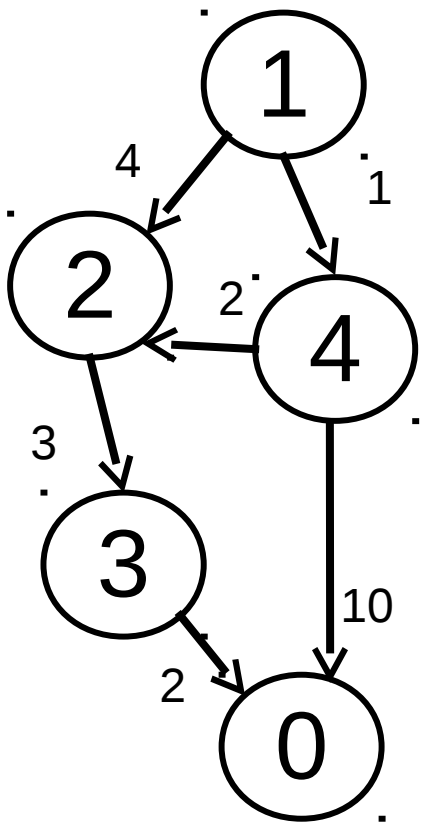
# All pairs shortest paths: Unconnected directed graph




```
Floyd-Warshall algorithm
    for(i=0;i<V;i++)
        for(s=0;s<V;s++)
            for(t=0;t<V;t++)
                if(A[s][i]+A[i][t] < A[s][t])
                    A[s][t] = (A[s][i]+A[i][t]);
```

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | ∞ | ∞ | ∞ | ∞ |
| 1 | ∞ | 0 | 4 | ∞ | 1 |
| 2 | ∞ | ∞ | 0 | 3 | ∞ |
| 3 | 2 | ∞ | ∞ | 0 | ∞ |
| 4 | 10 | ∞ | 2 | ∞ | 0 |

# All pairs shortest paths: Unconnected directed graph

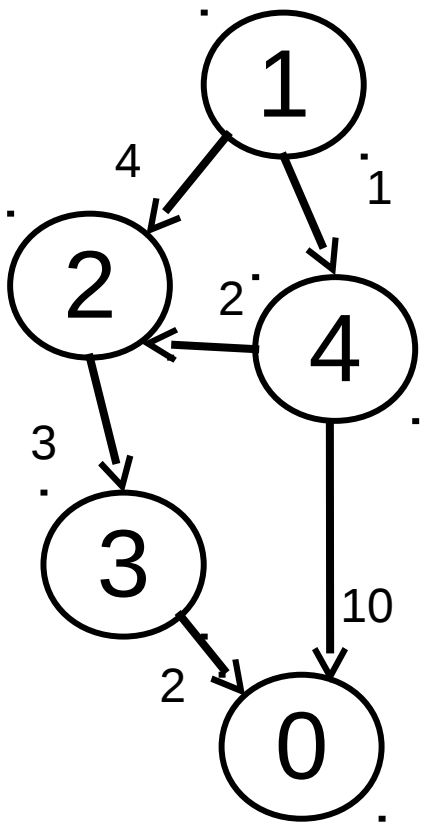



```
Floyd-Warshall algorithm
    for(i=0;i<V;i++)
        for(s=0;s<V;s++)
            for(t=0;t<V;t++)
                if(A[s][i]+A[i][t] < A[s][t])
                    A[s][t] = (A[s][i]+A[i][t]);
```

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | ∞ | ∞ | ∞ | ∞ |
| 1 | 11 | 0 | 3 | 7 | 1 |
| 2 | 5 | ∞ | 0 | 3 | ∞ |
| 3 | 2 | ∞ | ∞ | 0 | ∞ |
| 4 | 10 | ∞ | 2 | 5 | 0 |

Shortest paths len ≤ 2

# All pairs shortest paths: Unconnected directed graph

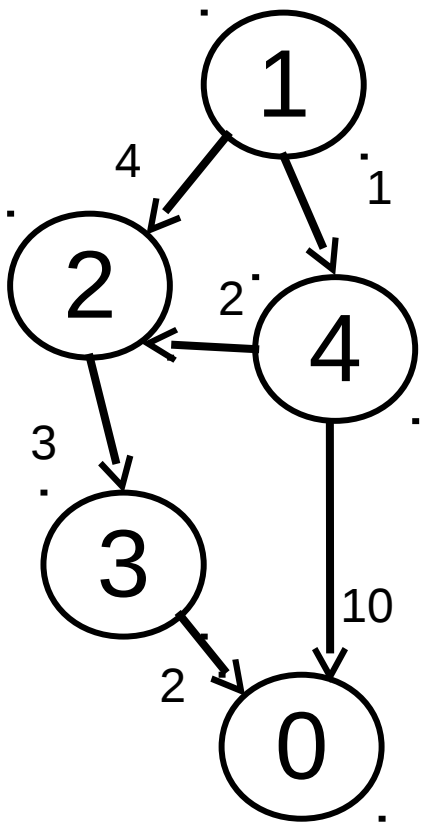



```
Floyd-Warshall algorithm
    for(i=0;i<V;i++)
        for(s=0;s<V;s++)
            for(t=0;t<V;t++)
                if(A[s][i]+A[i][t] < A[s][t])
                    A[s][t] = (A[s][i]+A[i][t]);
```

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | ∞ | ∞ | ∞ | ∞ |
| 1 | 9 | 0 | 3 | 6 | 1 |
| 2 | 5 | ∞ | 0 | 3 | ∞ |
| 3 | 2 | ∞ | ∞ | 0 | ∞ |
| 4 | 7 | ∞ | 2 | 5 | 0 |

Shortest paths len ≤ 3

# All pairs shortest paths: Unconnected directed graph

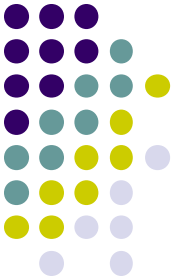



```
Floyd-Warshall algorithm
    for(i=0;i<V;i++)
        for(s=0;s<V;s++)
            for(t=0;t<V;t++)
                if(A[s][i]+A[i][t] < A[s][t])
                    A[s][t] = (A[s][i]+A[i][t]);
```

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | ∞ | ∞ | ∞ | ∞ |
| 1 | 8 | 0 | 3 | 6 | 1 |
| 2 | 5 | ∞ | 0 | 3 | ∞ |
| 3 | 2 | ∞ | ∞ | 0 | ∞ |
| 4 | 7 | ∞ | 2 | 5 | 0 |

Shortest paths len ≤ 4

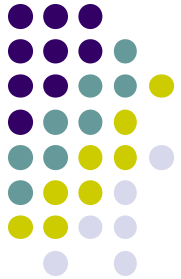
# Floyd-Warshall Algorithm: Analysis



- Floyd-Warshall algorithm
- 
```
for(i=0;i<V;i++)
  for(s=0;s<V;s++)
    for(t=0;t<V;t++)
      A[s][t]=(min(A[s][i] +A[i][t], A[s][t]);
```

$\Theta$ (?)

# Floyd-Warshall agorithm: Maximum length of path

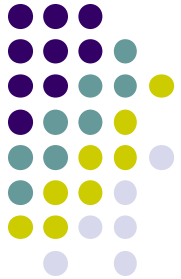


- Note:

  No shortest path has *length* (number of segments, *not* distance) greater than V-1.

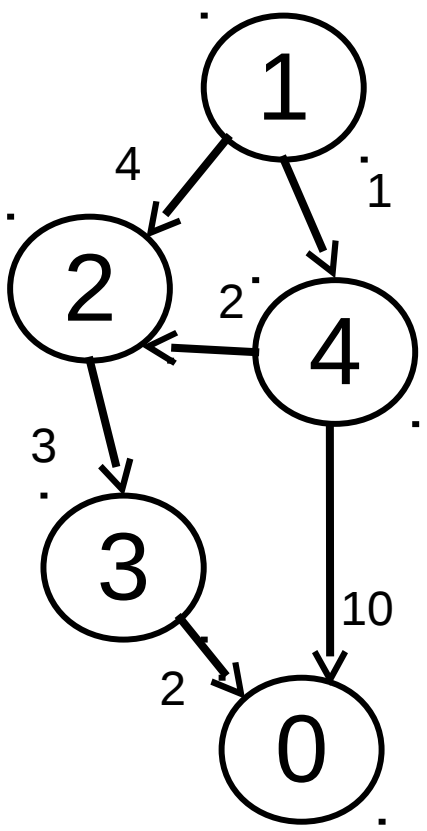
Why not?

# Floyd-Warshall agorithm: What is the path?



- Floyd-Warshall gives
  - *Distance* of shortest path, for all a→x,
  - But does not established the actual paths!
- Path information can be obtained through a small addition to the code:
  - Keep another 2-dimensional array.
  - For each update to distance array, update path array to show node that made the path shorter

# All pairs shortest paths: Unconnected directed graph



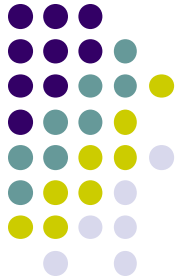

next along shortest path

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | \ | \ | \ | \ |
| 1 | 4 | 0 | 4 | 4 | 1 |
| 2 | 3 | \ | 0 | 3 | \ |
| 3 | 0 | \ | \ | 0 | \ |
| 4 | 2 | \ | 2 | \ | 0 |

shortest path lengths

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | ∞ | ∞ | ∞ | ∞ |
| 1 | 8 | 0 | 3 | 6 | 1 |
| 2 | 5 | ∞ | 0 | 3 | ∞ |
| 3 | 2 | ∞ | ∞ | 0 | ∞ |
| 4 | 10 | ∞ | 2 | ∞ | 0 |

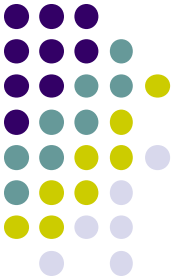
# Floyd-Warshall agorithm: What is the path?



- Path information can be obtained through a small addition to the code:
  - For details and Java code, see:

    Sedgewick, R., Algorithms in Java, 3rd edition, Part 5: Graph Algorithms, Addison-Wesley, 308.

# Floyd-Warshall agorithm: A big assumption



- Assumed graph representation is matrix.
- For sparse graphs, adjacency list representation, use Johnson's algorithm.
  - Run Dijkstra's single source algorithm for each vertex.
  - Use Fibonacci heap for priority queue.
  - D.S. Johnson, "Efficient algorithms for shortest paths in sparse networks", *Journal of the ACM* **24**(1), 1-13, 1977