# COMP20003
# Algorithms and Data Structures
# Priority Queues


Nir Lipovetzky

Department of Computing and Information Systems

University of Melbourne


Semester 2

# Queues

- A queue Q has the following operations:
  - **`makeQ();`**
  - **`enQ(Q,item);`**
  - **`deQ(Q,first);`**
  - **`emptyQ(Q);`**

# Priority Queues

- A priority queue PQ has the following operations:
  - `makePQ();`
  - `enQ(PQ,item);`
  - `deletemax(PQ);  /* or deletemin() */`
  - `emptyPQ(PQ);`
  - `changeWeight(PQ,item);`

- `Also delete(PQ,item), replace(PQ,item).`

# Simple implementations of priority queue

- Unsorted array:
  - Construct:
  - Get highest priority:
- Sorted array:
  - Construct:
  - Get highest priority:

# Simple implementations of priority queue

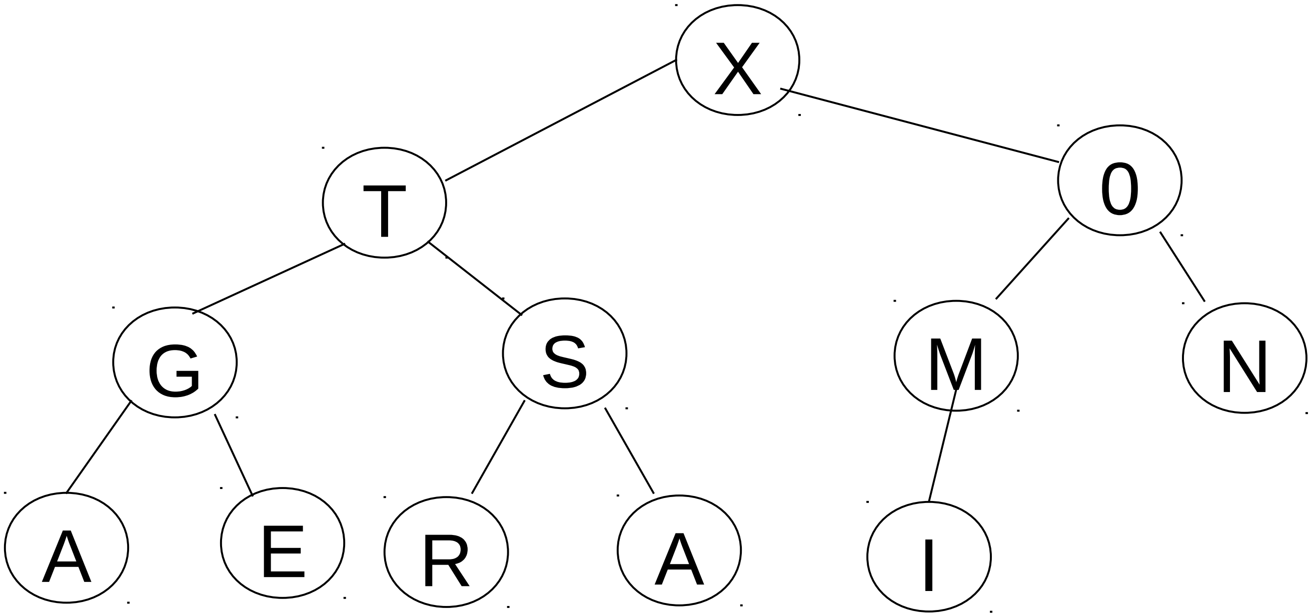
- Unsorted list:
  - Construct:
  - Get highest priority:

- Sorted list:
  - Construct:
  - Get highest priority:

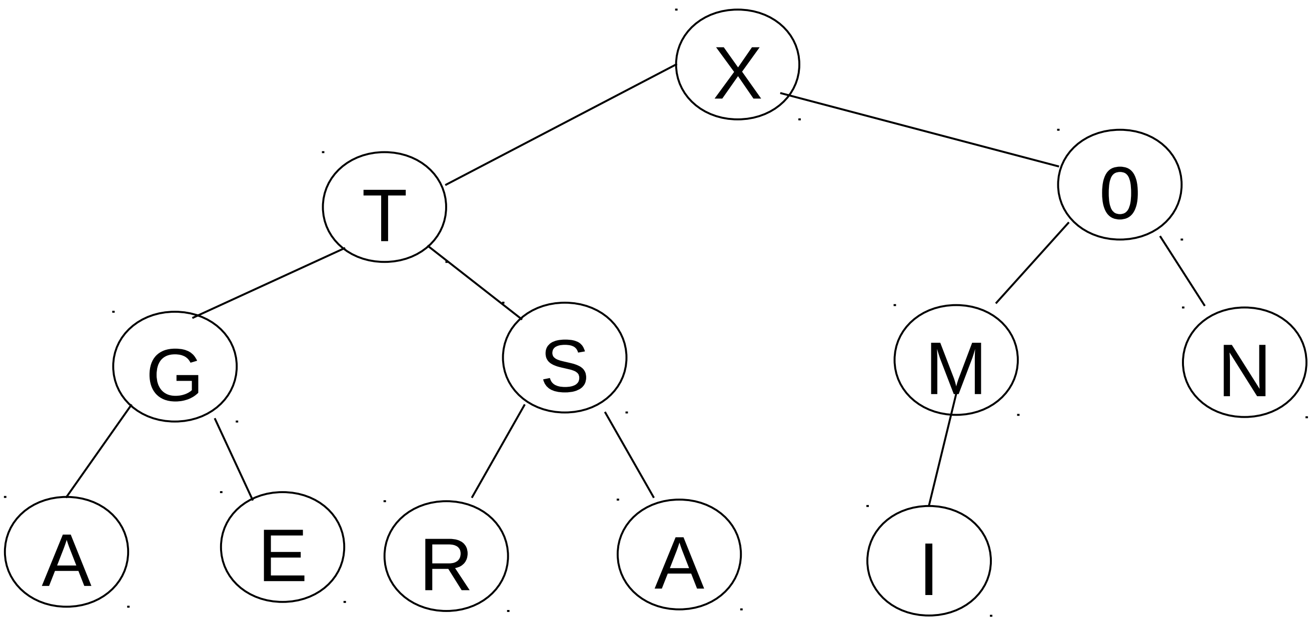
# A better implementation of priority queue: The Heap

- Heap data structure:
  - A complete tree.
  - *n.b.* a complete tree is…
  - Every node satisfies the “heap condition”:
    - **`parent->key >= child->key`**, for all children
    - Root is therefore …
  - Complete tree represented as an array.
  - *n.b.* we first look at binary heaps, but
  - A heap need not be binary

# Example heap



# Example heap

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# Example heap

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|---|---|---|---|---|---|---|---|---|---|---|
| X | T | O | G | S | M | N | A | E | R | A |

# Example heap

Parent:
A[i]
i=1

The arithmetic is easier
if we use A[1] as the root.

Children:
A[i]
i=2,3
i= parent*2, parent*2 + 1

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# Example heap

Parent:
A[i]
i=2

Children:
A[i]
i=4,5
i= parent*2, parent*2 + 1

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# Example heap

Parent:
A[i]
i=3

Children:
A[i]
i=6,7
i= parent*2, parent*2 + 1

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| X | T | O | G | S | M | N | A | E | R | A |

# Example heap

- Recall the heap condition:
- **parent->key >= child->key**, for all children

- For array representation, this means that:
  - A[i] >= A[2*i] && A[i] >= A[2*i+1]

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| X | T | O | G | S | M | N | A | E | R | A |

# `deletemax()`

- Return highest priority item:
  - Return root.
- Fix heap:
  - Put last item into root position.
  - Reduce size of PQ by one.
  - Fix heap condition for root: **`downheap()`**.

# downheap()

```
downheap(int k)
{
   int j,v;
   v = A[k];      /* value, or priority */
   while(k<=n/2)  /*A[k] has children*/
   { /* compare with largest child */
      j= k+k;
      if(j<n && a[j]<a[j+1]) j++;
        if (v>= a[j]) break;/* heap OK */
      a[k] = a[j]; k = j;
   }
   a[k] = v;
}
```

# deletemax()

For a maxheap of integers:

```
int deletemax()
{
        int v = a[1];
        a[1] = a[n--];
        downheap(1);
        return(v);
}
```

Exercise: construct a maxheap of pointers to struct; return a pointer to the maximum priority item.

# Fixing heap with upheap( )

- Inserting a new item into an *already-formed heap*, **upheap()** makes more sense:

```
upheap(int k)
{
   int v;
   v = A[k];
   A[0] = INT_MAX; /* sentinel, limits.h */
   while(A[k/2] <= v) /* note integer arith */
        {A[k] = A[k/2]; k = k/2;}
   A[k] = v;
}
```

# **`uphead()`** *vs.* **`downheap()`**

- Add new item in last place in heap:
  - **`upheap()`**
  - *O()*
- Replace root in heap:
  - **`downheap()`**
  - *O()*

# Heapsort

- Heap suggests a method for sorting:
  - Construct heap.
  - Swap root (max) with last element.
  - Remove last element from further consideration, *i.e.* decrease size of heap by 1.
  - Fix heap using….

    … **`downheap()`**
  - Repeat until finished.

# Cost of heapsort

- Construct heap.
- Sucessively move max to end of array and fix array.
  - *n* * **`deletemax()`**:
  - *n * O(logn) –> O(n log n)*

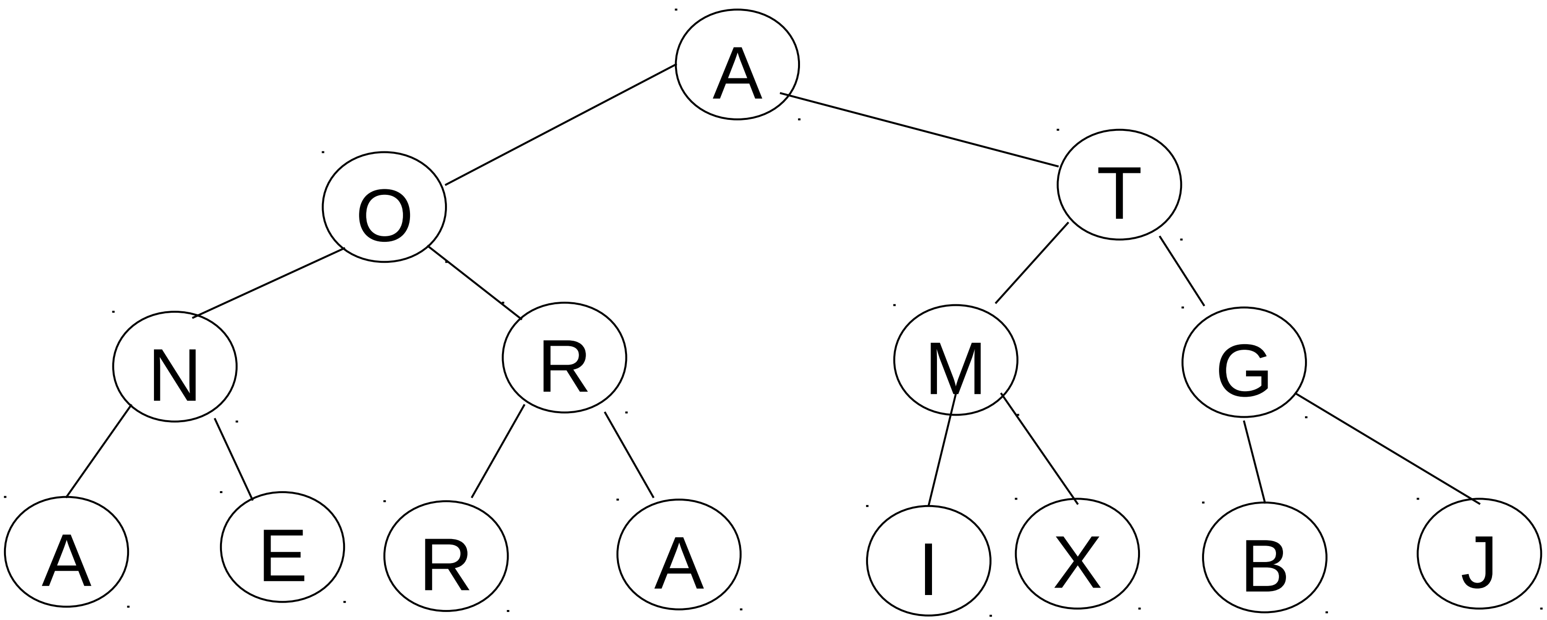
# Making a heap: two strategies

- Strategy 1:
  - Insert items one-by-one into the array.
  - **upheap()** as each new item is inserted.
  - Insert *n* items into heap of size *n*:
    - Each insertion: *O( )*
    - How many insertions?
    - Overall: *O( )*

# Making a heap: two strategies

- Strategy 2:
  - Insert items into unordered array.
  - When all items are in, **`downheap()`** for each subheap with roots from **`A[n/2]`** to **`A[1]`**.

# Strategy 2: How does it work?

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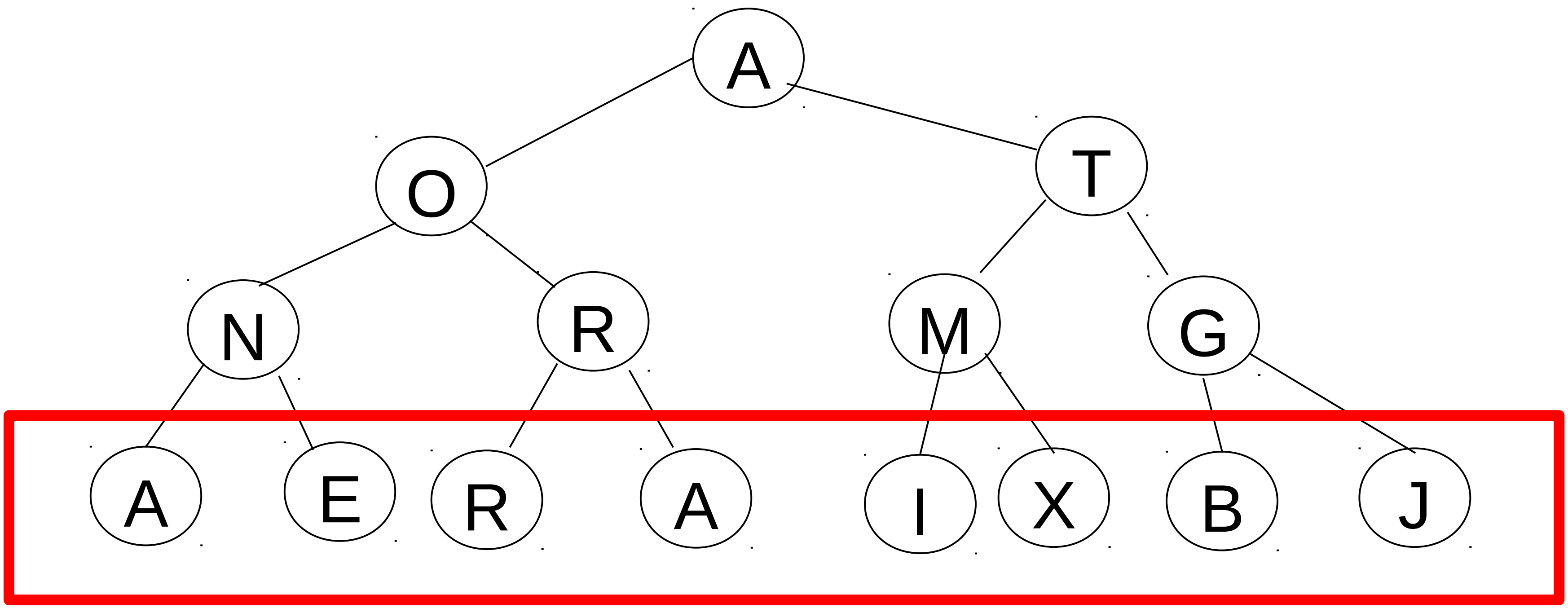
# Strategy 2: How does it work? downheap( ) bottom row

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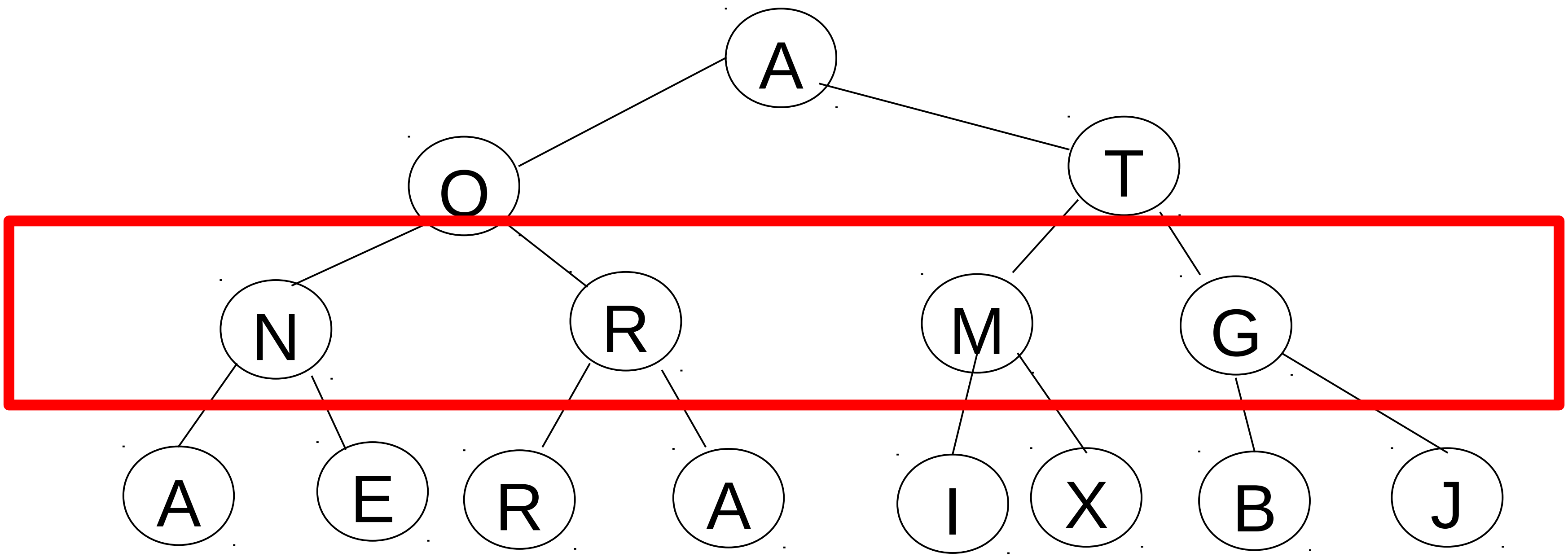
# Strategy 2: How does it work? downheap( ) next row up

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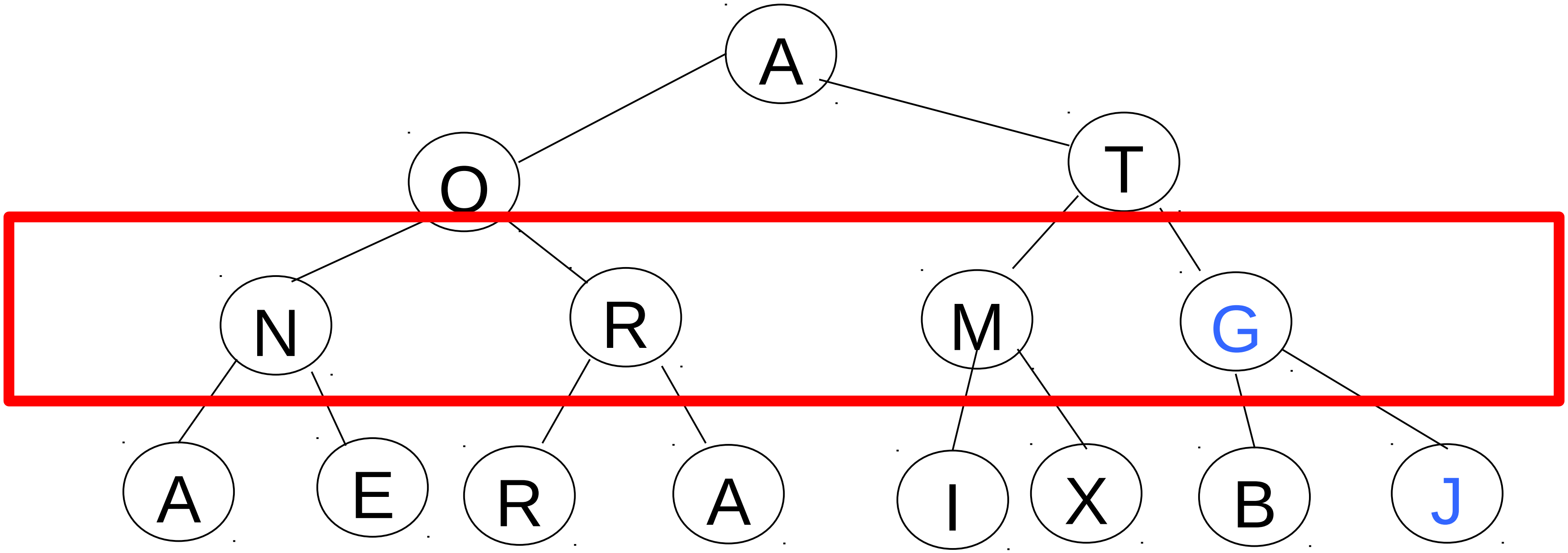


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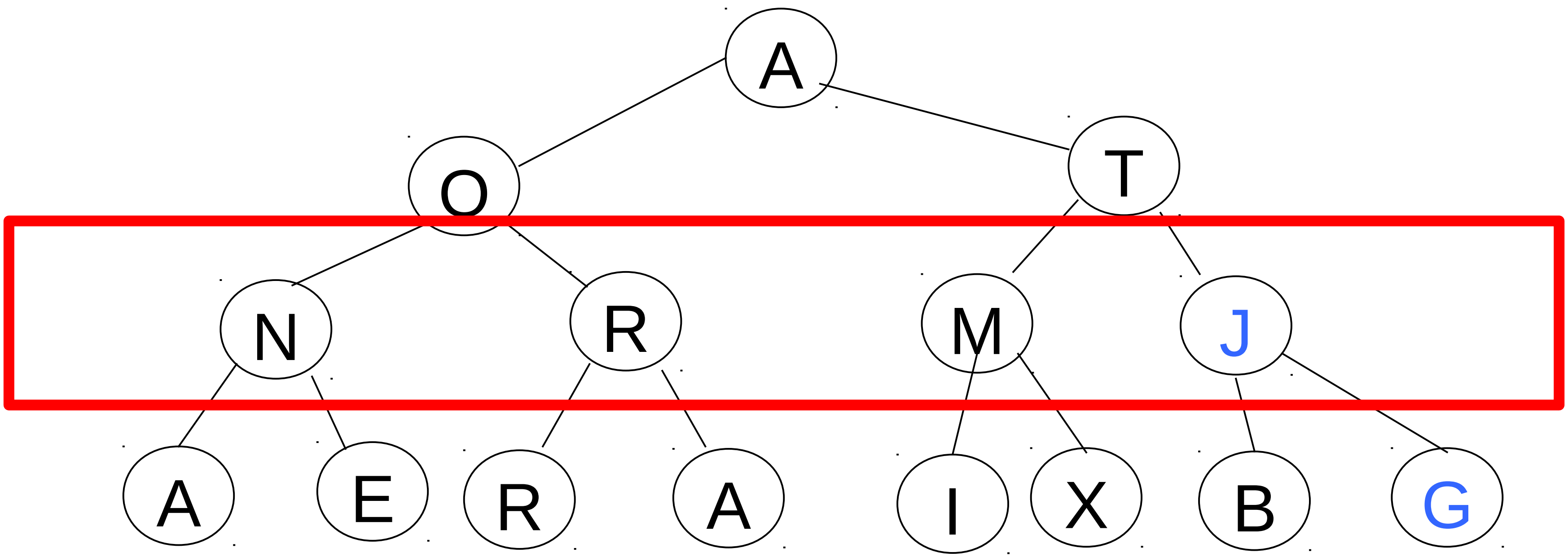


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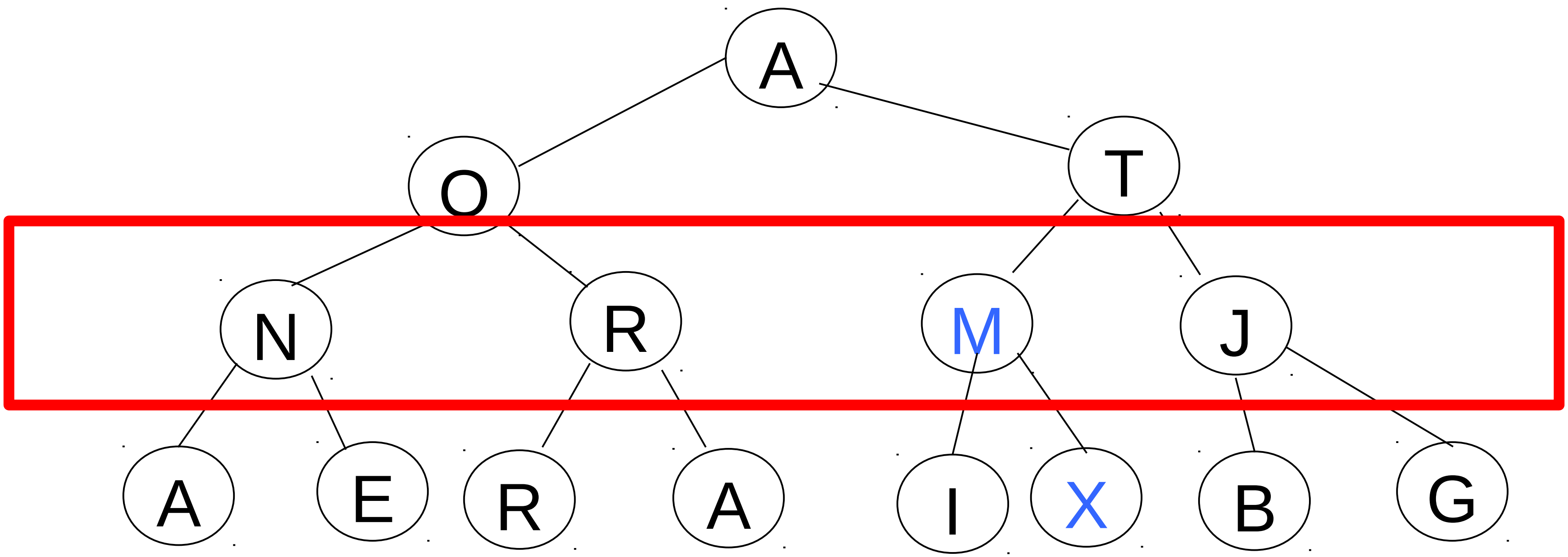


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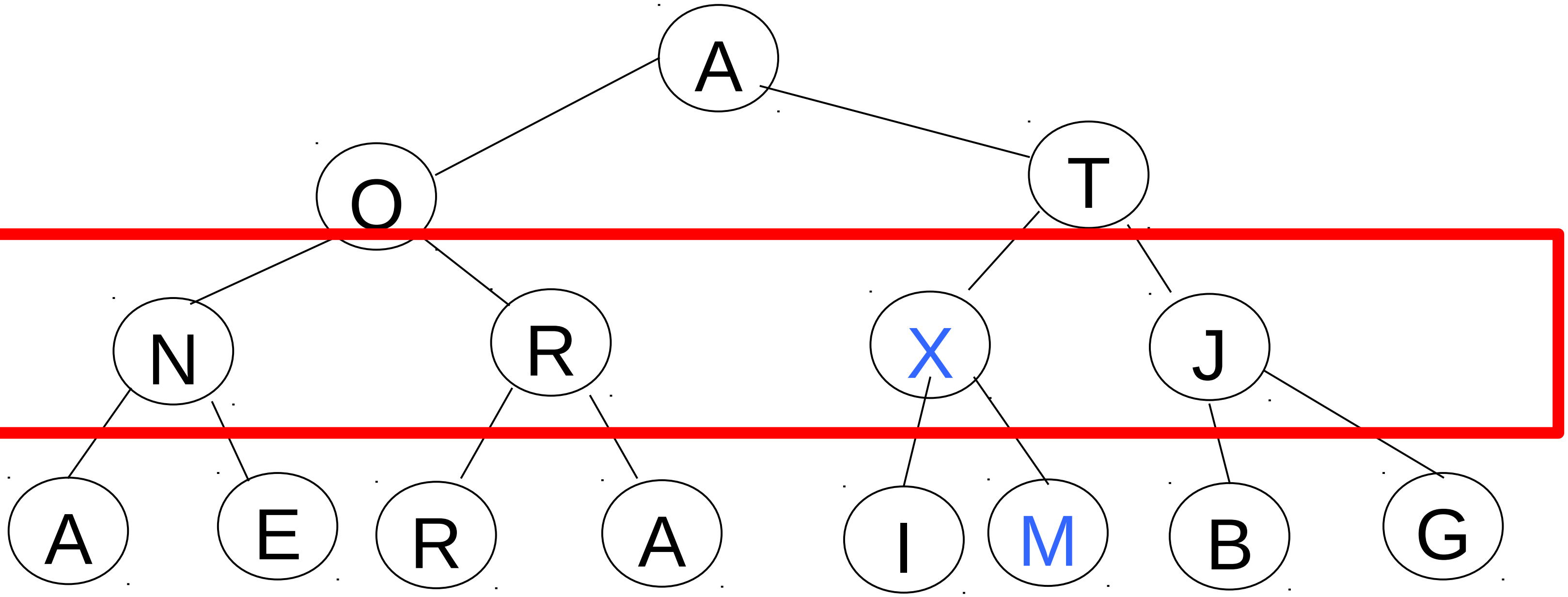


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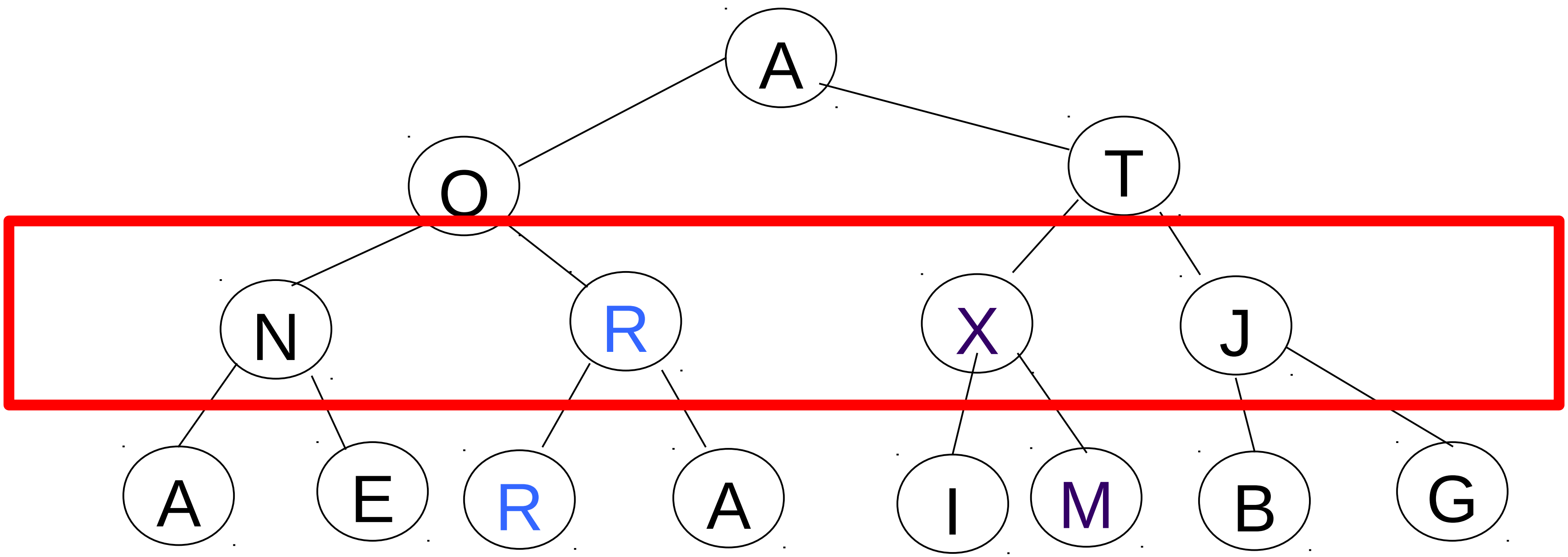


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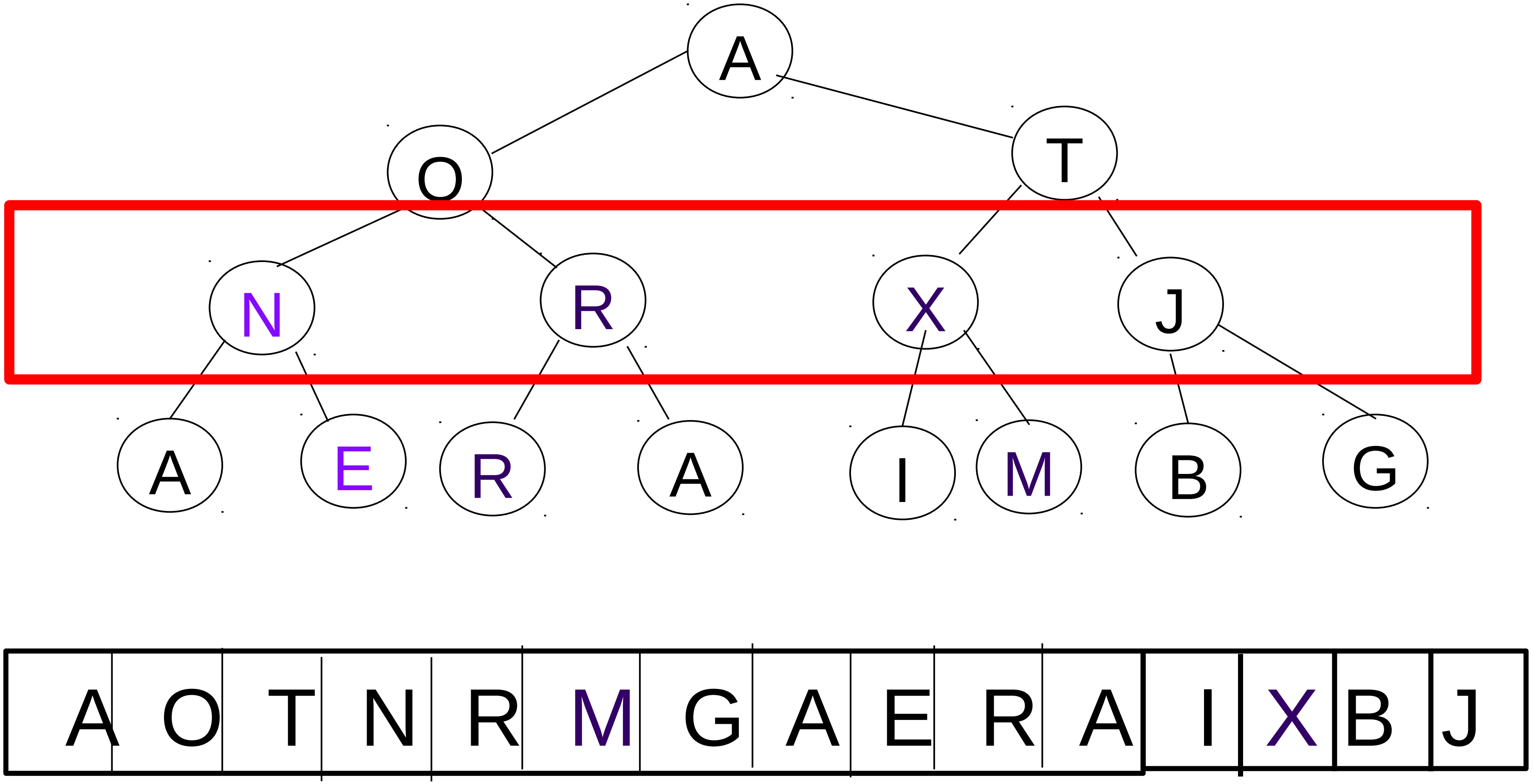

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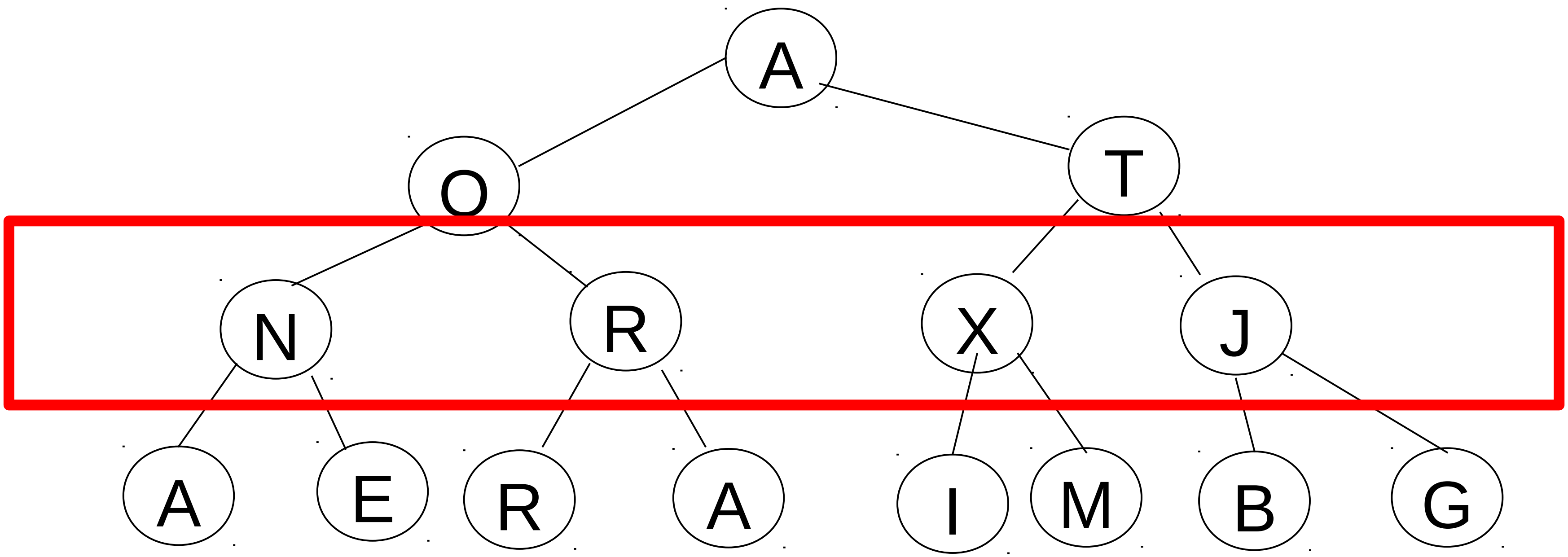


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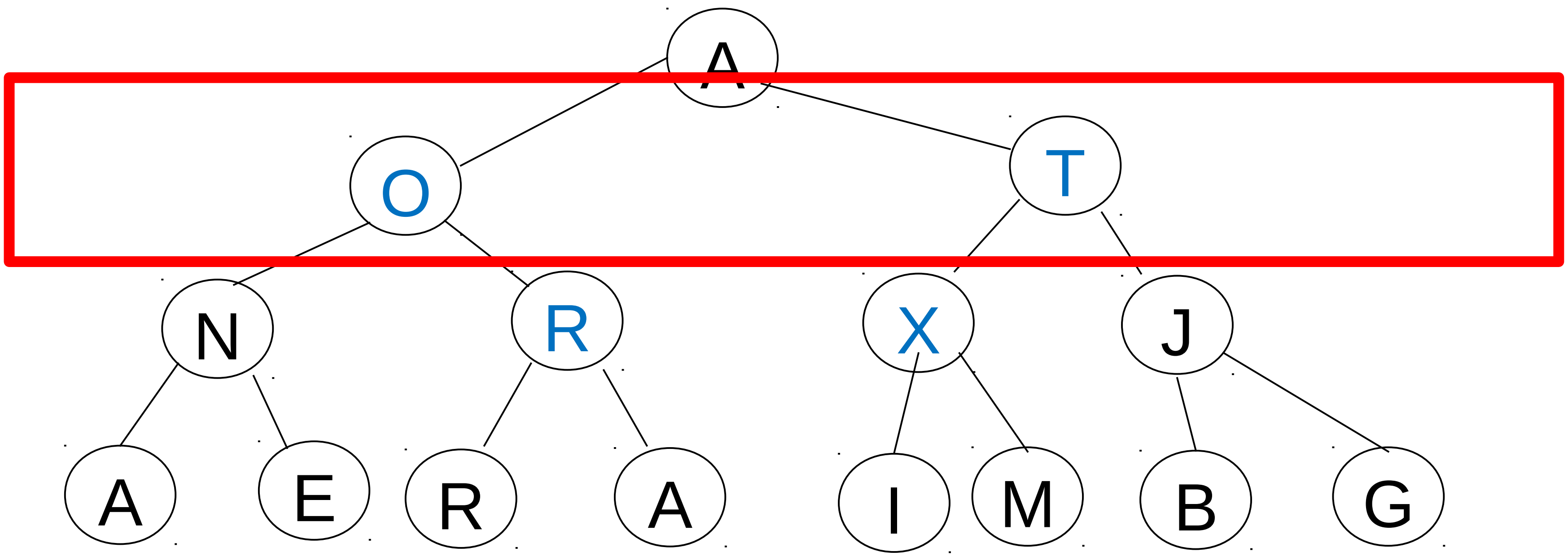


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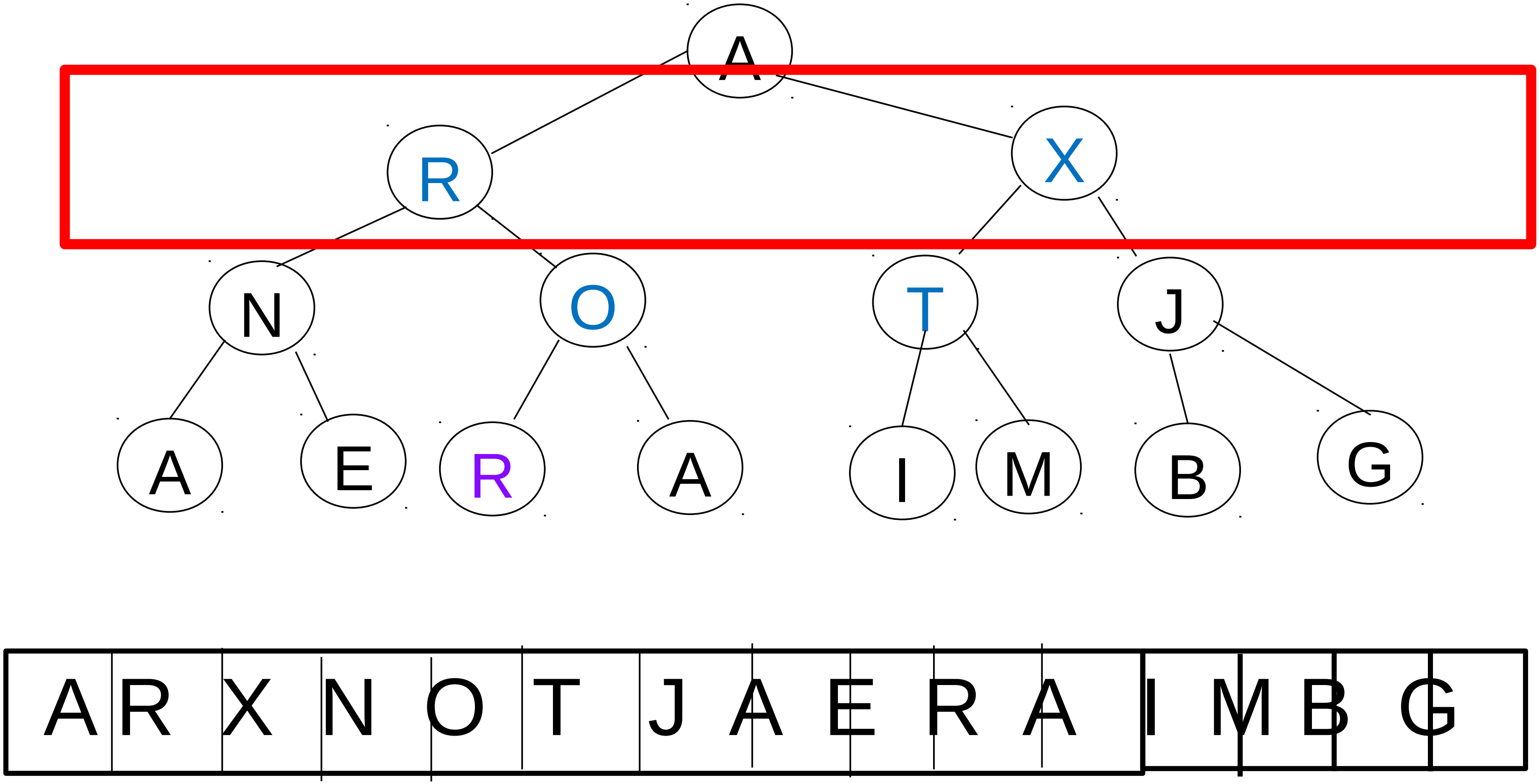

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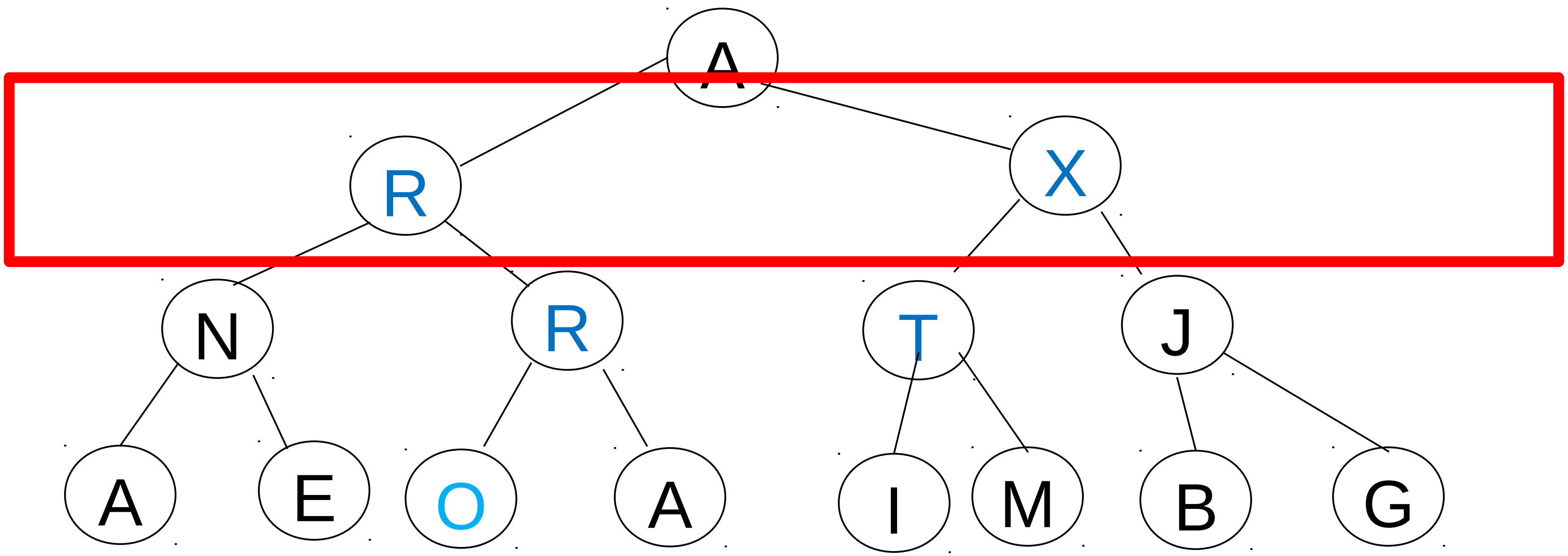


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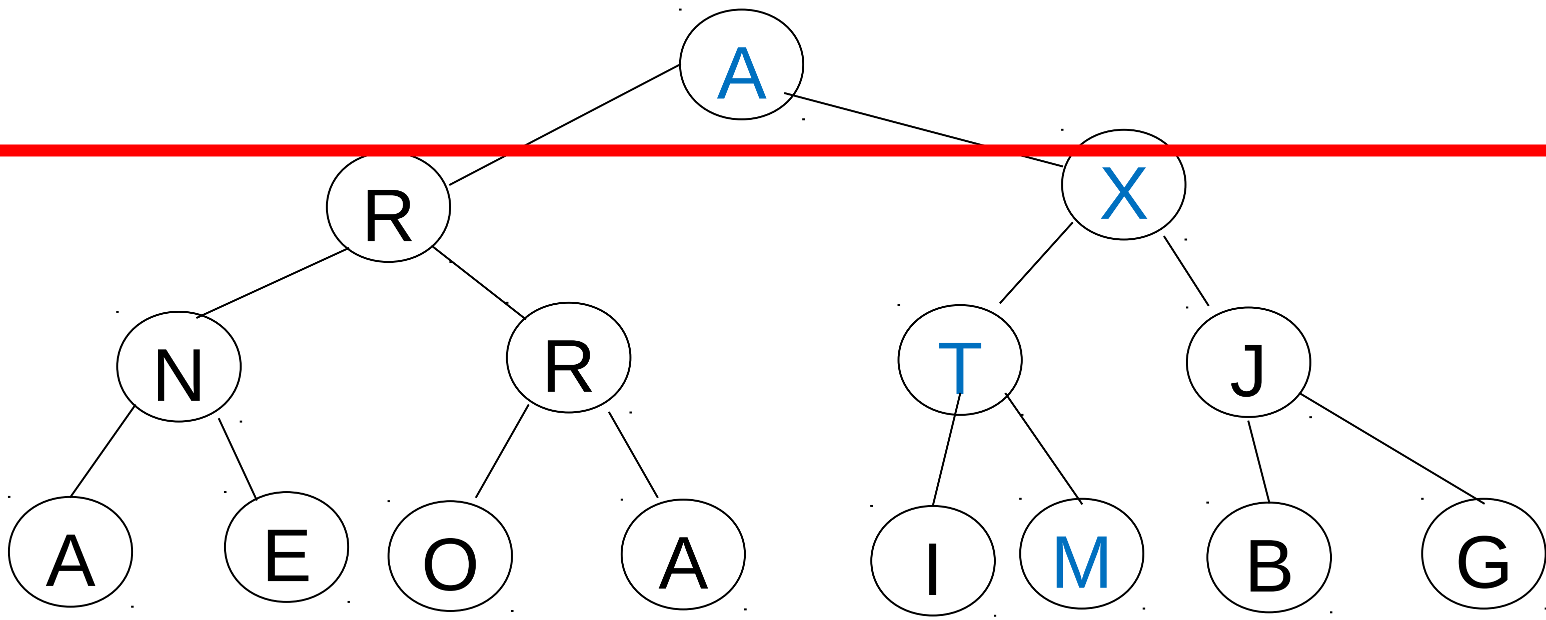


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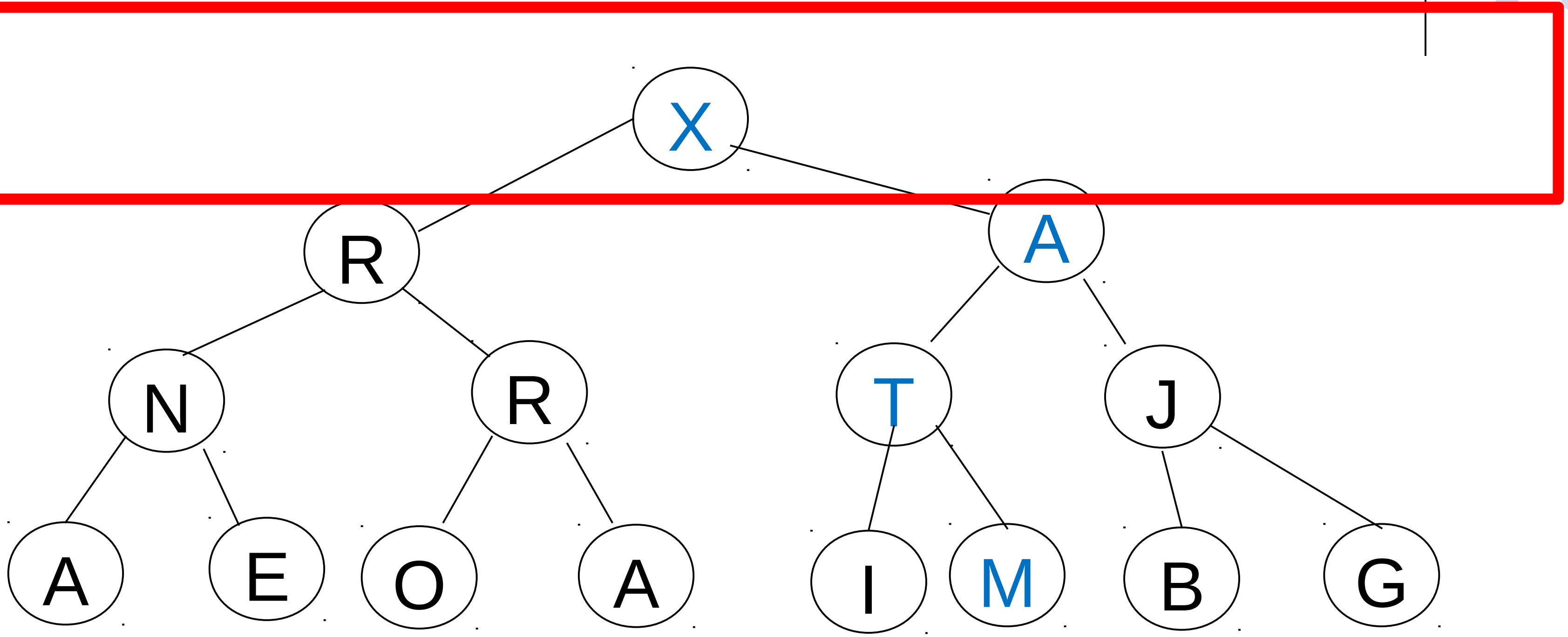


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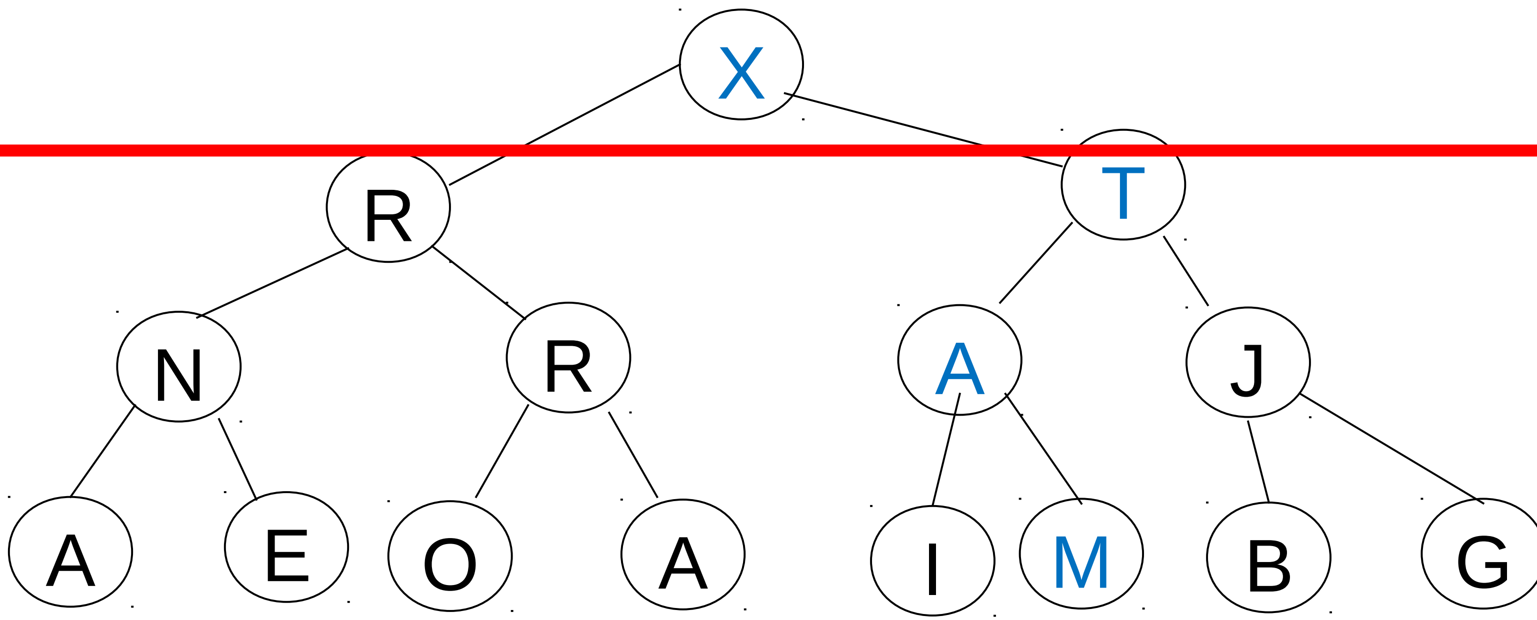


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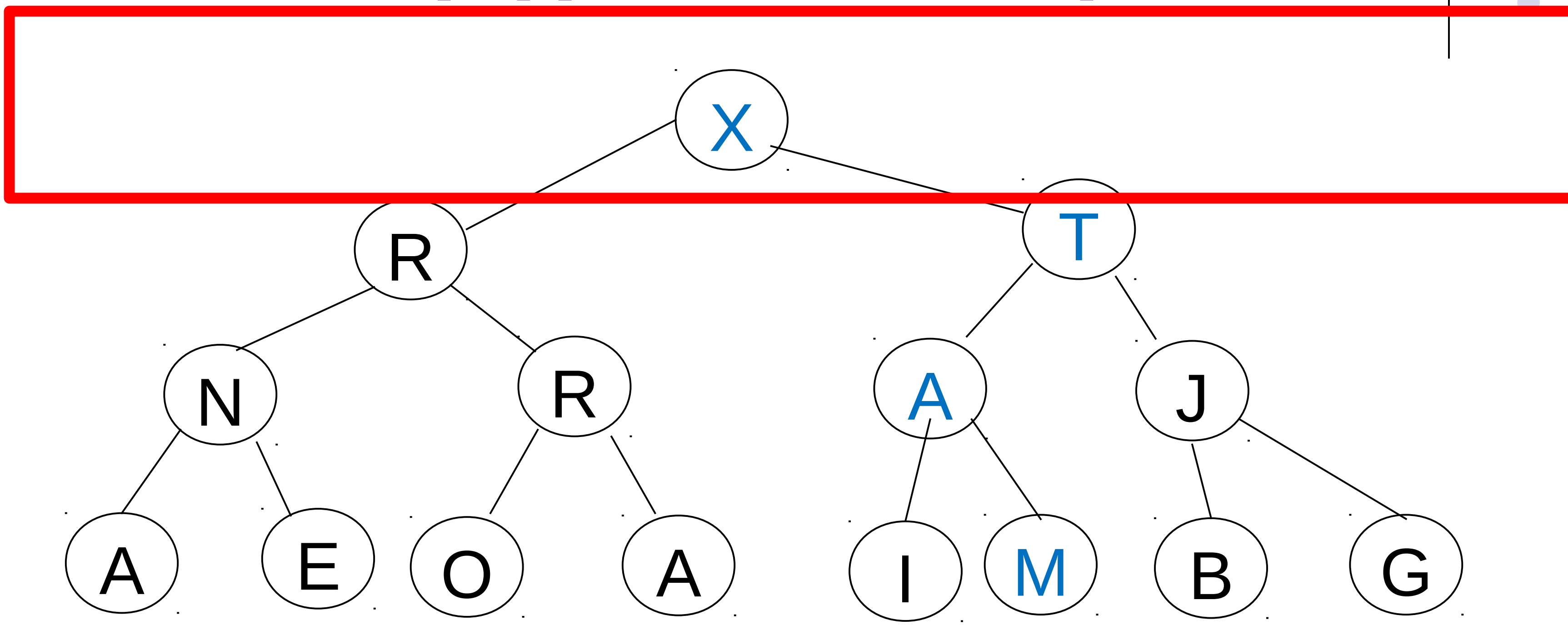


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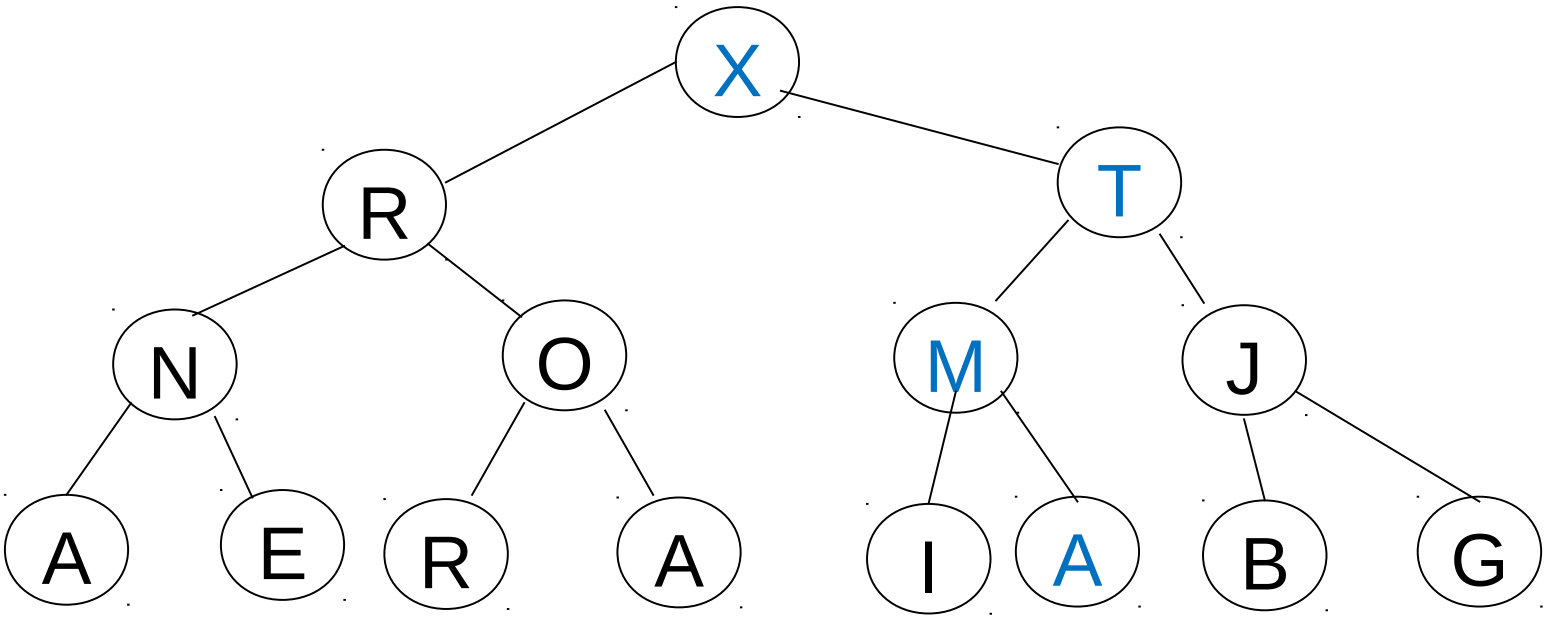
# Strategy 2: How does it work? downheap( ) next row up

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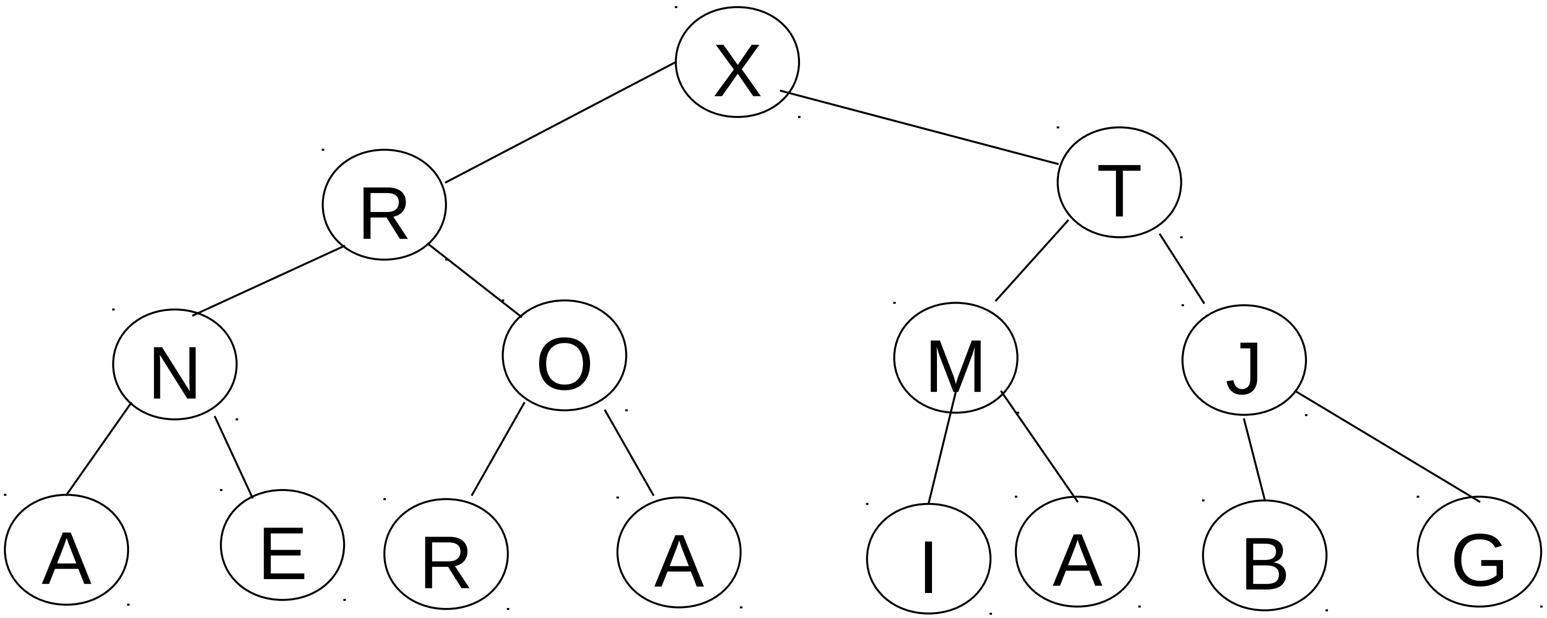
| X | R | T | N | O | M | J | A | E | R | A | I | A | B | G |

# Strategy 2: Finished heap after bottom-up heap construction

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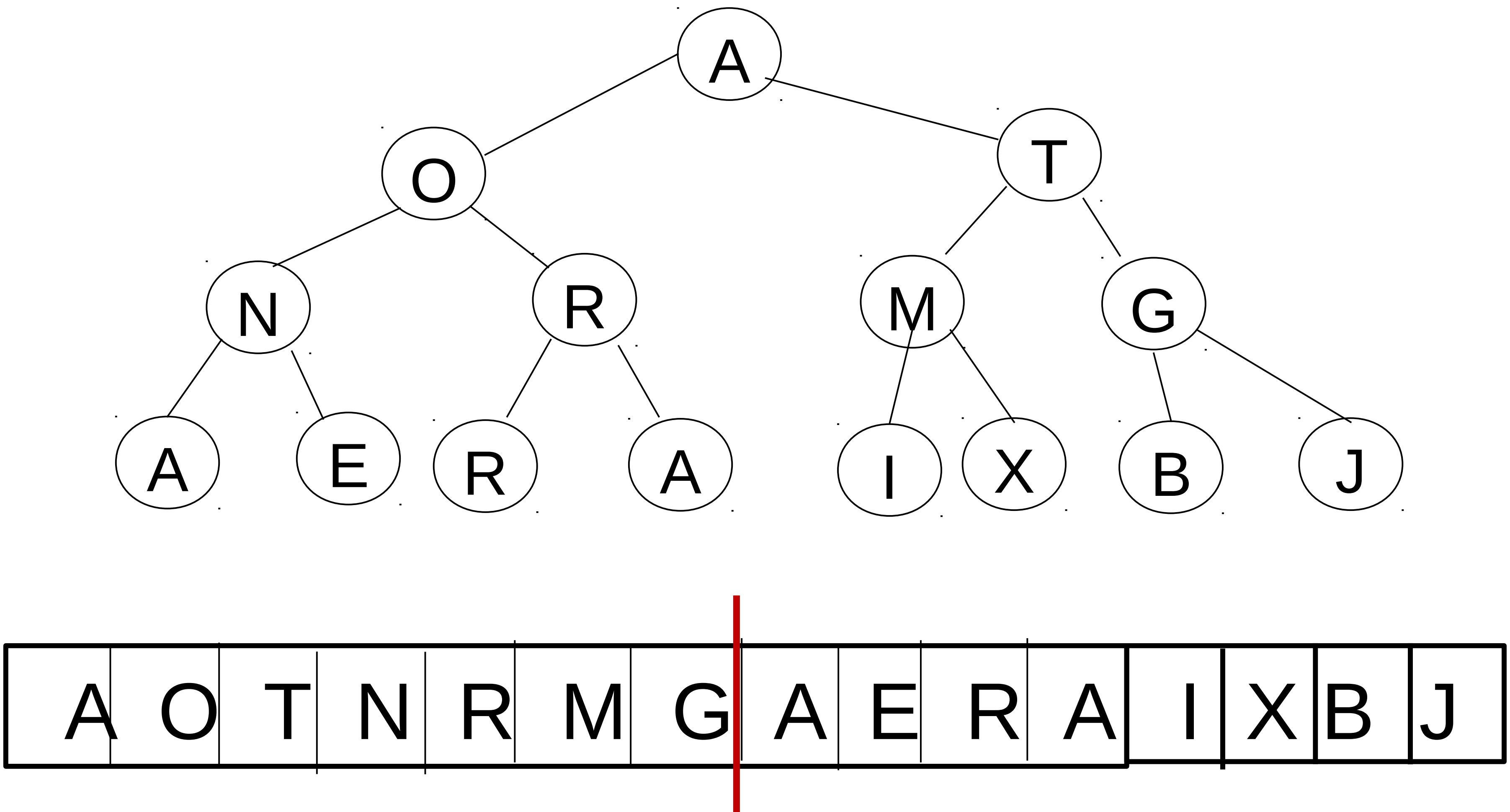
| X | R | T | N | O | M | J | A | E | R | A | I | A | B | G |

# Strategy 2: Analysis

- Strategy 2:
  - Insert items into unordered array.
  - When all items are in, **`downheap()`** for each subheap with roots from **`A[n/2]`** to **`A[1]`.**
  - Insert *n* items into heap of size *n*:
    - Start with Insert into unordered array: *O( )*
    - Then **`downheap( )`** subheaps from **`A[n/2]`** to **`A[1]`**

- **`downheap( )`** subheaps from **`A[n/2]`** to **`A[1]`**
- Bottom n/2 nodes are already heaps.
  - Cost to fix:
- Next level up nodes:
  - n/4 nodes, max cost each = 2 (cmp both children)
  - n/8 nodes, max cost each = 2 levels * 2 cmps
  - n/16, 3 levels*2 cmps
  - At the root, may need up to log n cmps to fix up – but there is only one node at root level.

# Strategy 2: Analysis



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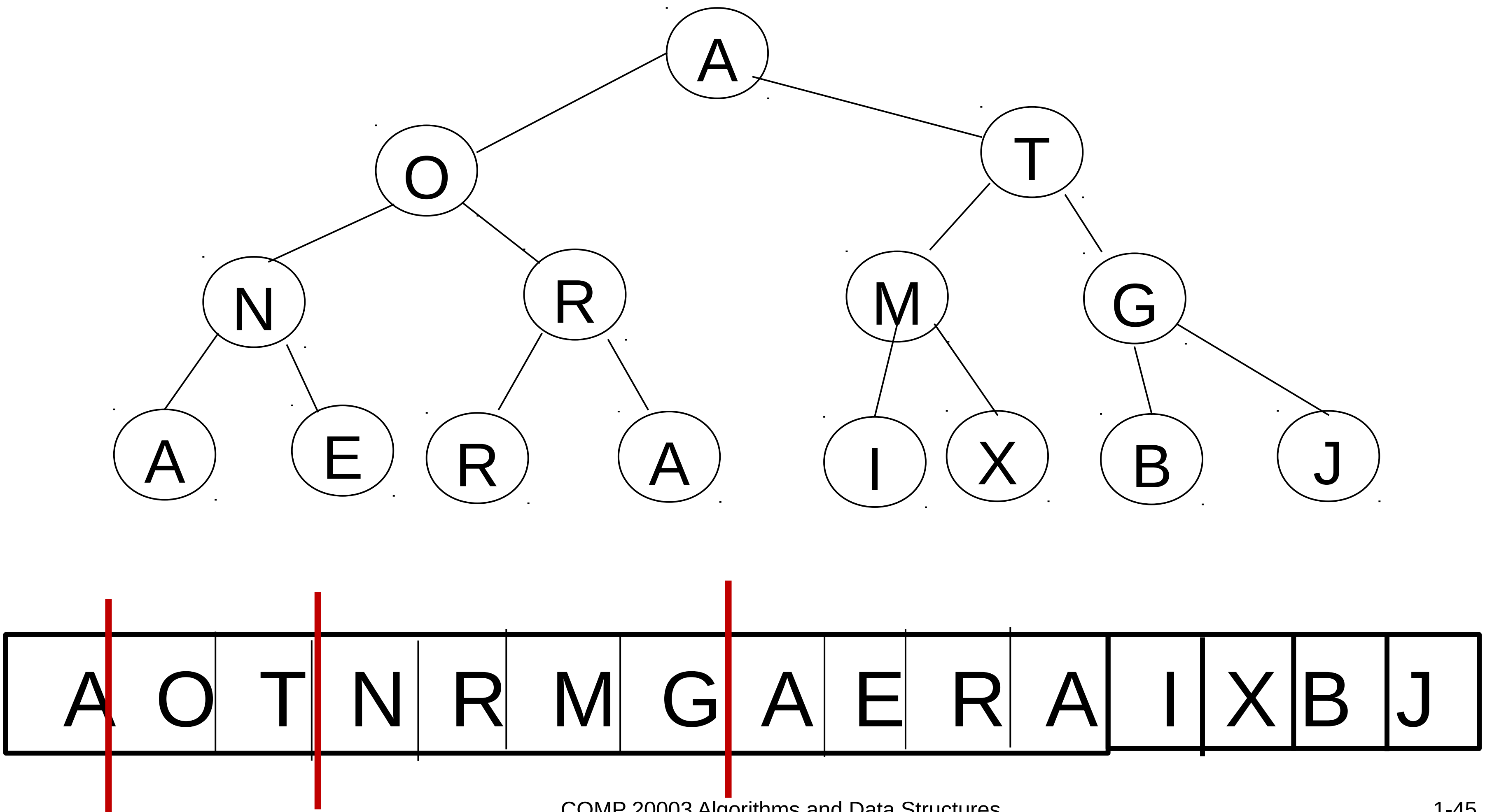

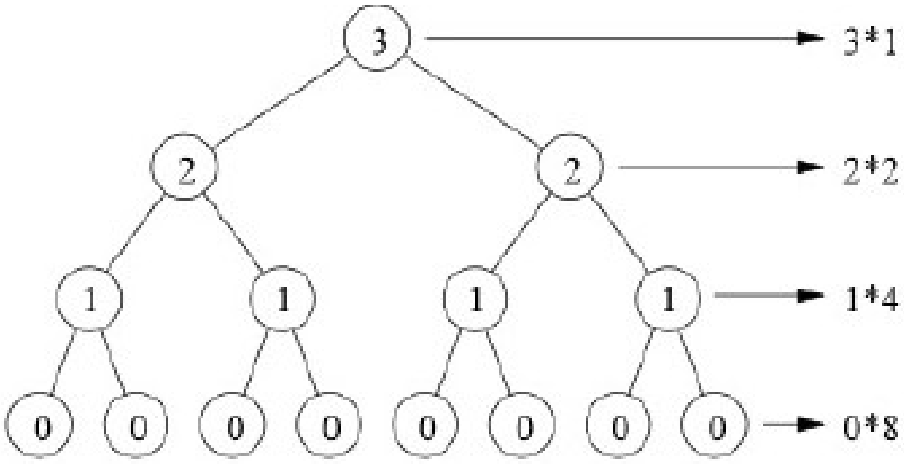
3
2
2
1
1
1
1
0
0
0
0
0
0
0
0
3*1
2*2
1*4
0*8

# Analysis of buildheap()

- Loose bound:
  - **`downheap()`** *O(logn)*
  - *n* operations
  - On first glance: *O(n log n)*
- BUT: observe
  - only the root ever goes has a *log n* **`downheap()`.**
  - The n/2 leaves have 0 work for **`downheap()`**
  - n/4 leaves at level <u>h-1</u> have max 1 **`downheap()`**

# Analysis of buildheap()

- Overall:
  - at most *ceil*(n/2$^{(h+1)}$) nodes exist at height h
  - When $h = 0$, n/2 nodes
  - When $h = 1$, n/4 nodes
  - When $h = floor(log\ n)$, 1 node
- Total cost =
  - $\sum_{(h=0 \rightarrow floor(\log n))} \underbrace{ceil(n/2^{(h+1)})}_{\text{number of nodes at this level}} * \underbrace{O(h)}_{\text{work for each node at this level to restore heap}}$

# Analysis of `buildheap()`

$\sum_{(h=0 \rightarrow \text{floor}(\log n))} \text{ceil}(n/2^{(h+1)})*O(h)$

$= O(n \sum h/2^h)$

(converging geometric series)

$= O(n)$

See Cormen, Leiserson, and Rivest for more detail

# Heapsort

- We will be using Priority Queues in the context of graph algorithms.
- But note that the Priority Queue suggests an efficient sorting algorithm. Heapsort.

# Applications

- Bandwidth Management: VoIP, IPTV
- Shortest Path Algorithms: Pathfinding, navigation, games
- Job Scheduling: OS, Clusters
- Minimum Spanning Tree algorithm: network design
- Huffman Code: Entropy encoding, compression jpeg, mp3