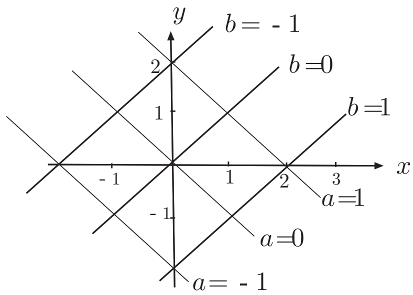
Tutorial 9: Solutions

Q1. (i).



The point $a = 1,\ b = 2$ corresponds to

$$(1, 1) + 2(1, -1) = (3, -1)$$

(ii). $\mathbf{u} = 0\mathbf{b_1} + 1\mathbf{b_2} + -1\mathbf{b_3} = 0(1) + 1(1 + x) + -1(1 + x + x^2) = -x^2$

(iii). As $(3, 1) = 3(1, 1) - 2(0, 1) \quad \Rightarrow \quad [(3, 1)]_{\mathcal{B}} = (3, -2)$
$(1, -1) = 1(1, 1) - 2(0, 1) \quad \Rightarrow \quad [(1, -1)]_{\mathcal{B}} = (1, -2)$

Q2. (i). Since

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad [\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C},\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}.$$

(ii). These are just the corresponding columns of $P_{\mathcal{C},\mathcal{B}}$ so

$$[\mathbf{b_1}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad [\mathbf{b_2}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{b_3}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

(iii). Since

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

As $P_{\mathcal{B},\mathcal{C}} = P^{-1}_{\mathcal{C},\mathcal{B}}$ then we have

$$P_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{c_1} + 7\mathbf{c_3}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ -5 & -2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

so we see $[\mathbf{c_1} + 7\mathbf{c_3}]_{\mathcal{B}} = \mathbf{v}$ from (i) as it should since we found in (i) that $[\mathbf{v}]_{\mathcal{C}} = \mathbf{c_1} + 7\mathbf{c_3}$.

Q3. We read off that Let $\mathcal{C} = \{(1, -2, 2), (0, 3, 4), (0, -2, 0)\}$. It is known that

$$[\mathbf{c_1}\ \mathbf{c_2}\ \mathbf{c_3} \mid \mathbf{u_1}\ \mathbf{u_2}] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{9}{4} \end{bmatrix},$$

For $\mathbf{u_1}$ we have $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = -\frac{1}{2}$ and so $[\mathbf{u_1}]_{\mathcal{C}} = (0, 0, -\frac{1}{2})$ and similarly for $\mathbf{u_2}$ we have $[\mathbf{u_2}]_{\mathcal{C}} = (2, \frac{1}{2}, \frac{9}{4})$.

Q4. (i). We have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Because we have a standard basis we can just put these vectors as the columns in a matrix

$$A_T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(ii).

$$P_{\mathcal{S},\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(iii). Using $[T]_{\mathcal{B}} = P_{\mathcal{C},\mathcal{B}}^{-1}\,[T]_{\mathcal{C}}\,P_{\mathcal{C},\mathcal{B}}$ we have

$$[T]_{\mathcal{B}} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Q5. (i). We have that $T(1,1) = (1,1)$ and $T(1,-1) = -2(1,1) + (1,-1)$. Hence

$$[T(1,1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad [T(1,-1)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

and so

$$[A_T]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

(ii). We have

$$[A_T]_{\mathcal{S}} = P_{\mathcal{S},\mathcal{B}}[A_T]_{\mathcal{B}}P_{\mathcal{B},\mathcal{S}} = P_{\mathcal{S},\mathcal{B}}[A_T]_{\mathcal{B}}P_{\mathcal{S},\mathcal{B}}^{-1}$$

But from the above question

$$P_{\mathcal{S},\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad P_{\mathcal{S},\mathcal{B}}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

and so

$$[A_T]_{\mathcal{S}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\left(\frac{-1}{2}\right)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

Q6. (i). With $\mathcal{S}$ denoting the standard basis, we apply the formula $[T]_{\mathcal{B}} = P_{\mathcal{S},\mathcal{B}}^{-1}[T]_{\mathcal{S}}P_{\mathcal{S},\mathcal{B}}$. Here

$$P_{\mathcal{S},\mathcal{B}} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_{\mathcal{S},\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 0 & -3 & 3 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

and so

$$[T]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 0 & -3 & 3 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii). We see that T is an orthogonal projection onto the plane spanned by $\{(-2,1,1),(1,1,1)\}$, and furthermore stretches by a factor of 4 in the direction of $(1,1,1)$.