Quantitative Risk Analysis using Engineering Reliability

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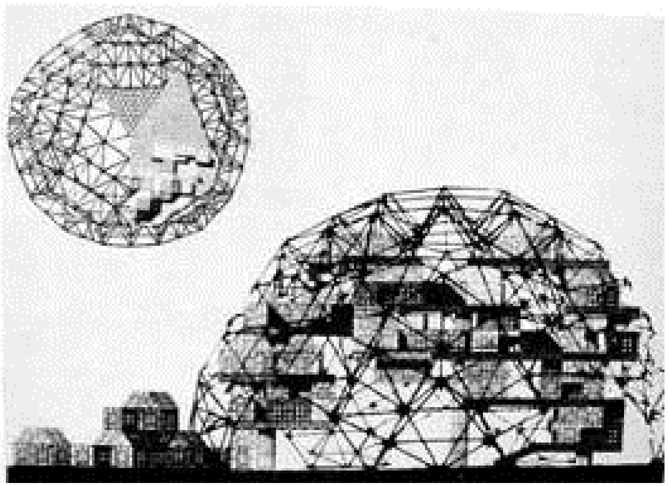
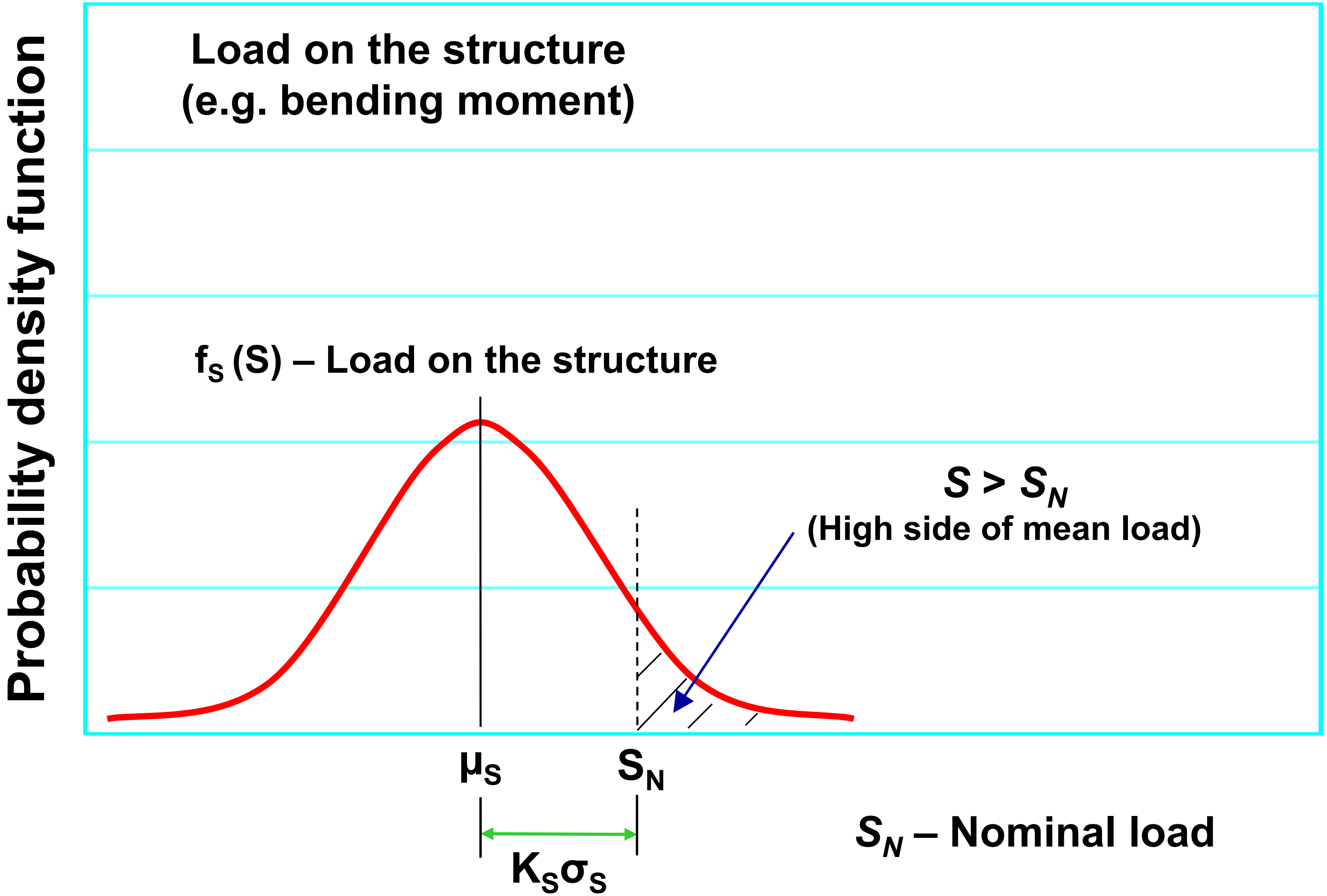


- Why is Reliability important in Engineering?
  - Components in engineering systems are not perfect. Risk can be minimized but cannot be eliminated completely.
  - Practical and economical limitations lead to no-so-prefect designs.
  - Engineers must understand "why" and "how" failures occur.

- In conventional design approaches, the safety factors are used to estimate both the resistance and the loads
  - For example, in concrete design using load and resistance factor design (LRFD) concept.

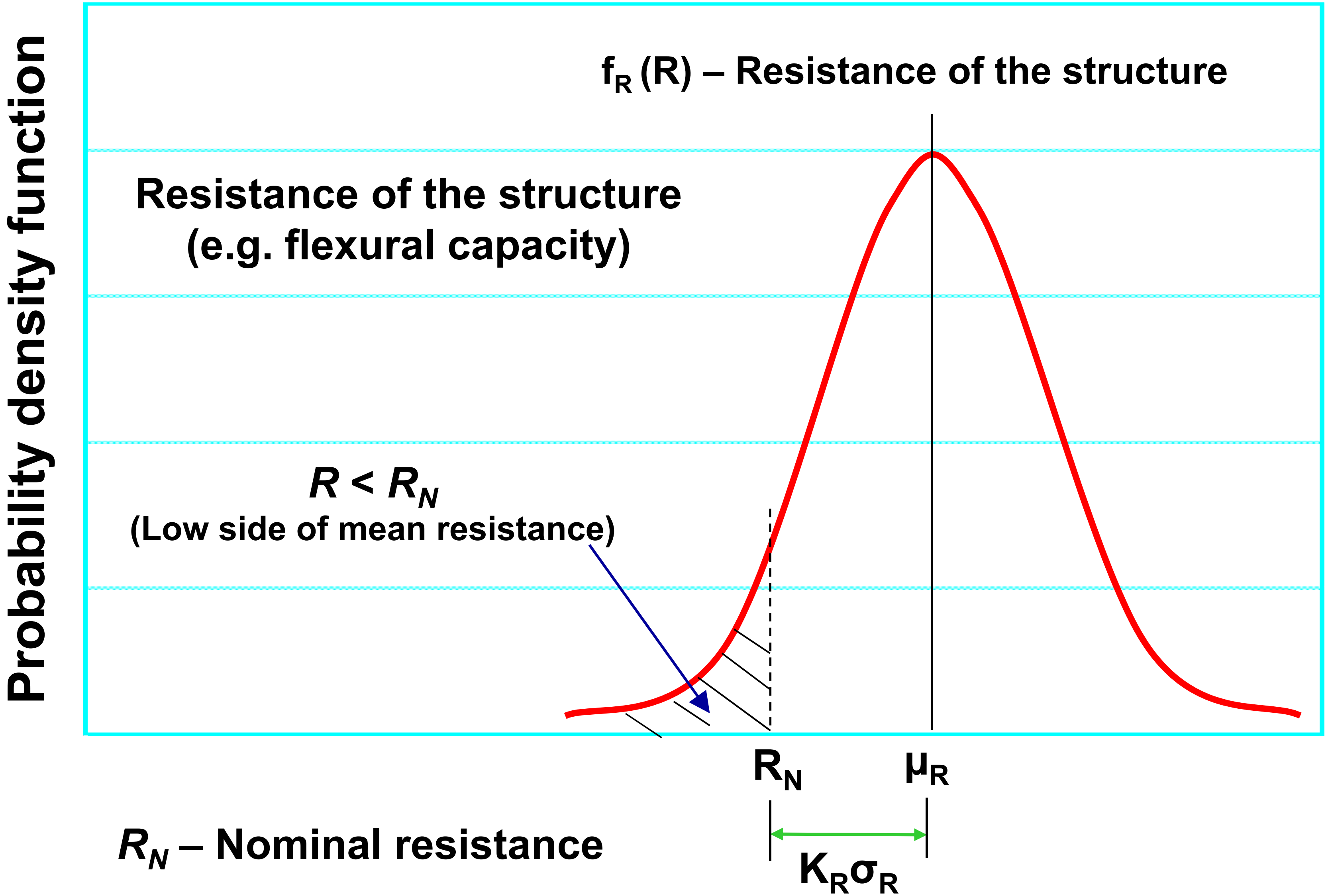
Nominal resistance (R_N) > Nominal load (S_N)

Definition of Nominal Load



S_N – **Nominal load**

Definition of Nominal Resistance



R_N – Nominal resistance

Deterministic Approach

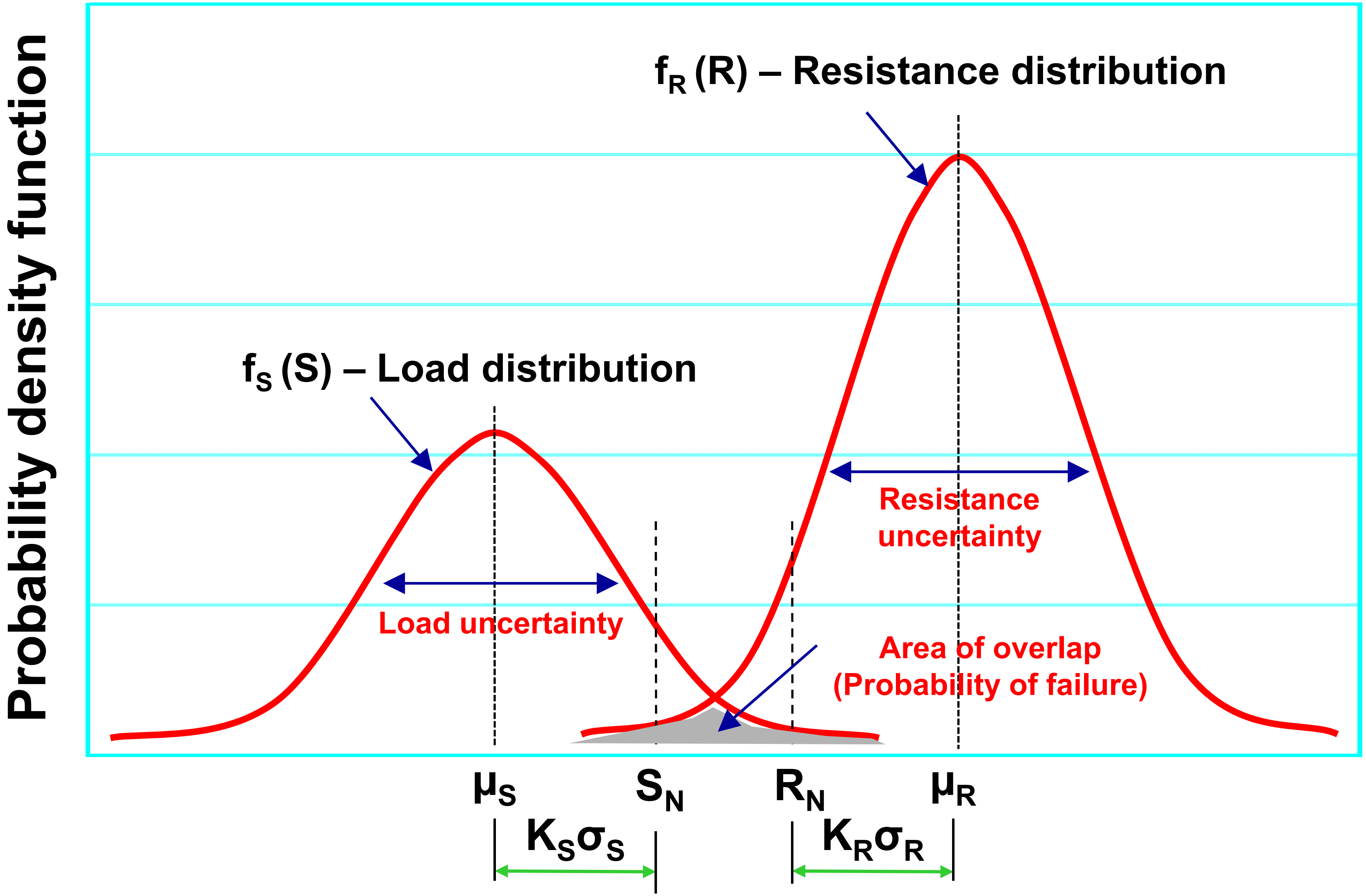
- The deterministic approach

$$SF = \frac{R_N}{S_N} \geq 1$$

SF - the safety factor

- However, the actual f_R and f_S are difficult to obtain. Engineers normally use only means and standard deviations to formulate the acceptable design methodology.

Probabilistic Approaches



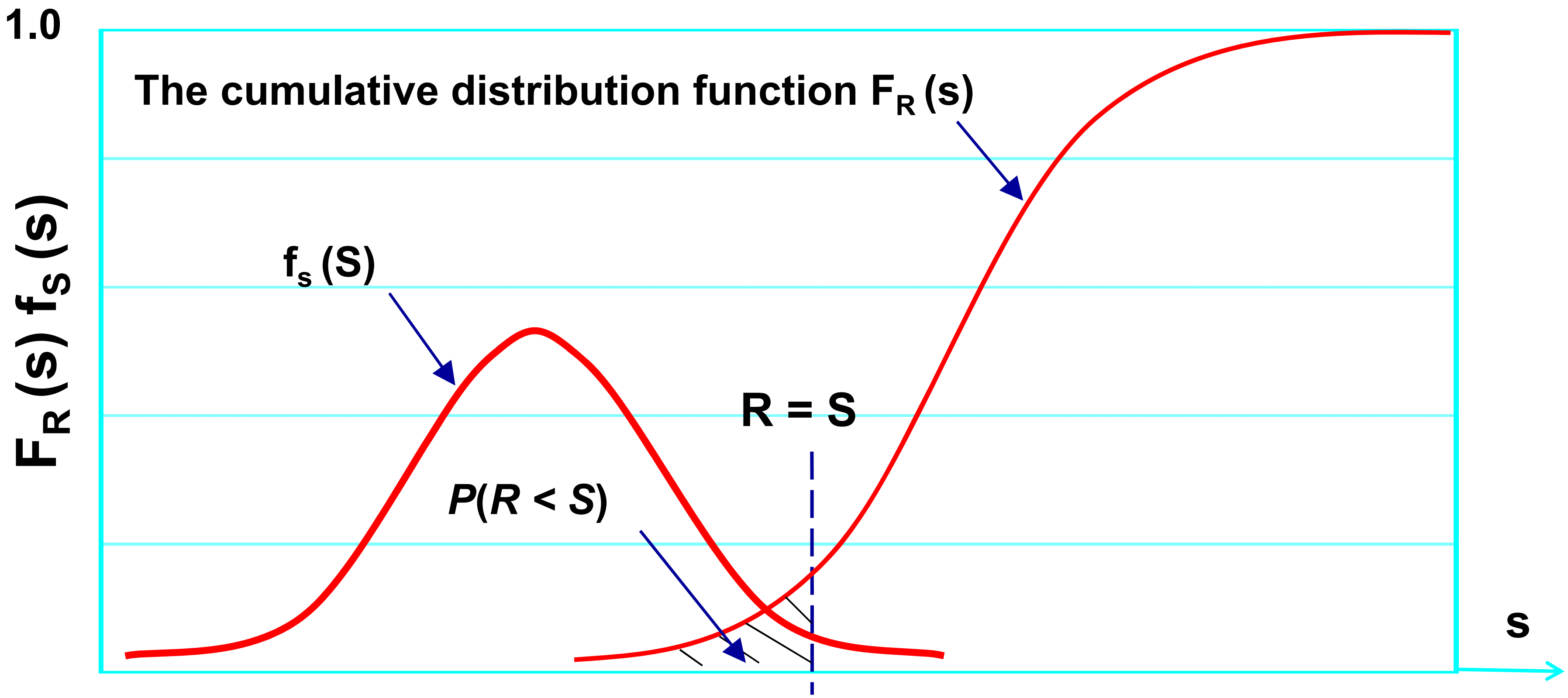
Probabilistic approaches are required to quantify the probability of failure !

- The area of overlap depends on three factors
  - The relative position of the two curves represented by μ_R and μ_S.
  - The dispersion of the two curves characterized by the σ_R and σ_S.
  - The shapes of the two curves (*e.g.* skewness of two distribution curves)

- The objective of safe design - Select the design variables in such a way that the area of overlap as small as possible.

- Risk-based design concept
  - Measure the risk in terms of the probability of the failure event

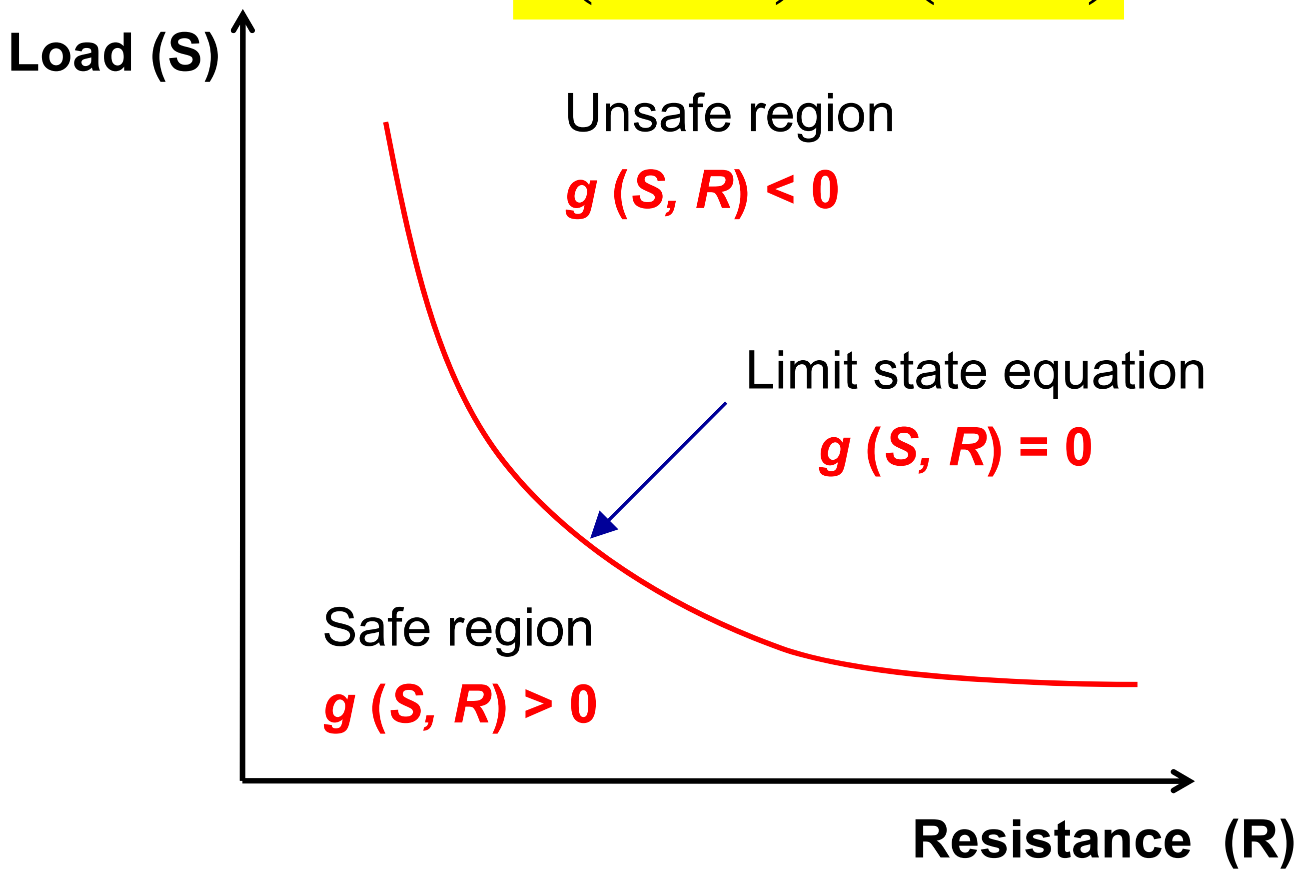
$$p_f = P(\text{failure}) = P(R < S) = \int_0^{\infty} F_R(s) f_S(s) ds$$



- Performance function:

$$Z = R - S = g(S, R) = g(X_1, X_2, \ldots, X_n)$$

$$P(\text{failure}) = P(Z < 0)$$



$$P(\text{failure}) = P(Z < 0)$$

Probability of failure:

$$p_f = \int \bullet\bullet\bullet \int_{g(\ )<0} f_X\left(x_1, x_2, \ldots, x_n\right) dx_1 dx_2 \bullet\bullet\bullet dx_n$$

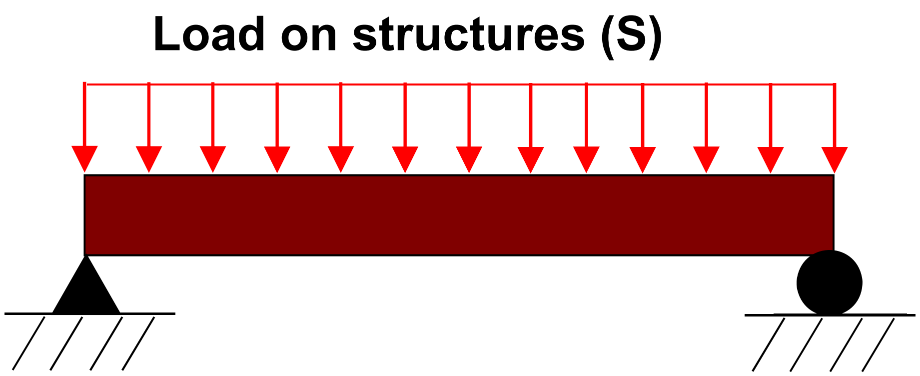
- Two types of analytical approximate approaches
  - First-order reliability methods (FORM)
  - Secord-order reliability methods (SORM)

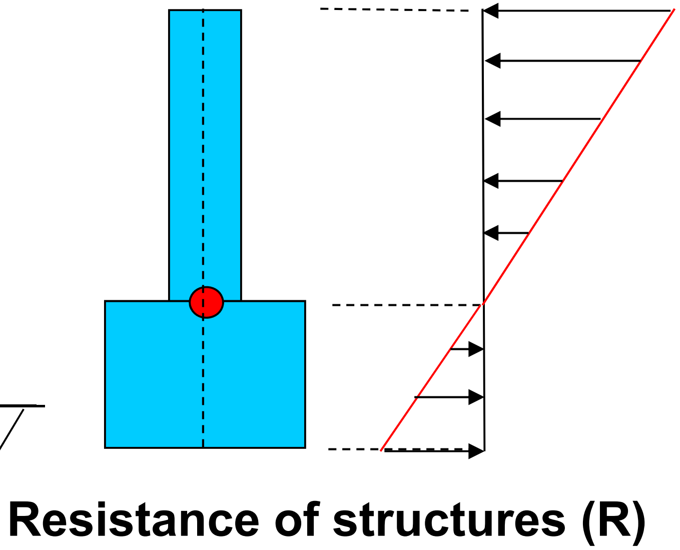
First-order reliability methods (FORM)

- Second moment concept

Performance function: $Z = R - S$

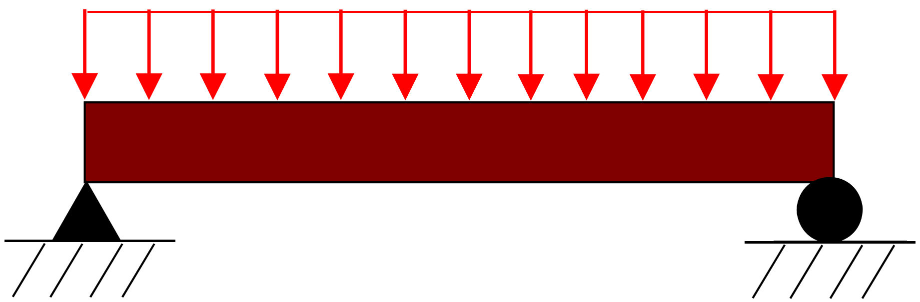
Probability of failure: $p_f = P(Z < 0)$

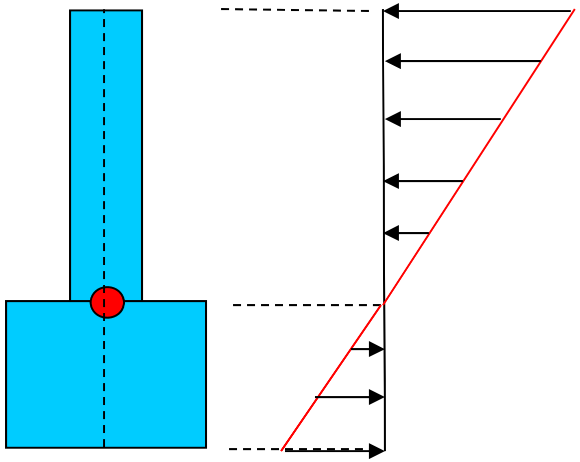




- Special case: *R* and *S* are independent normally distributed random variables

Load on structures (S) - N (μ_S, σ_S)





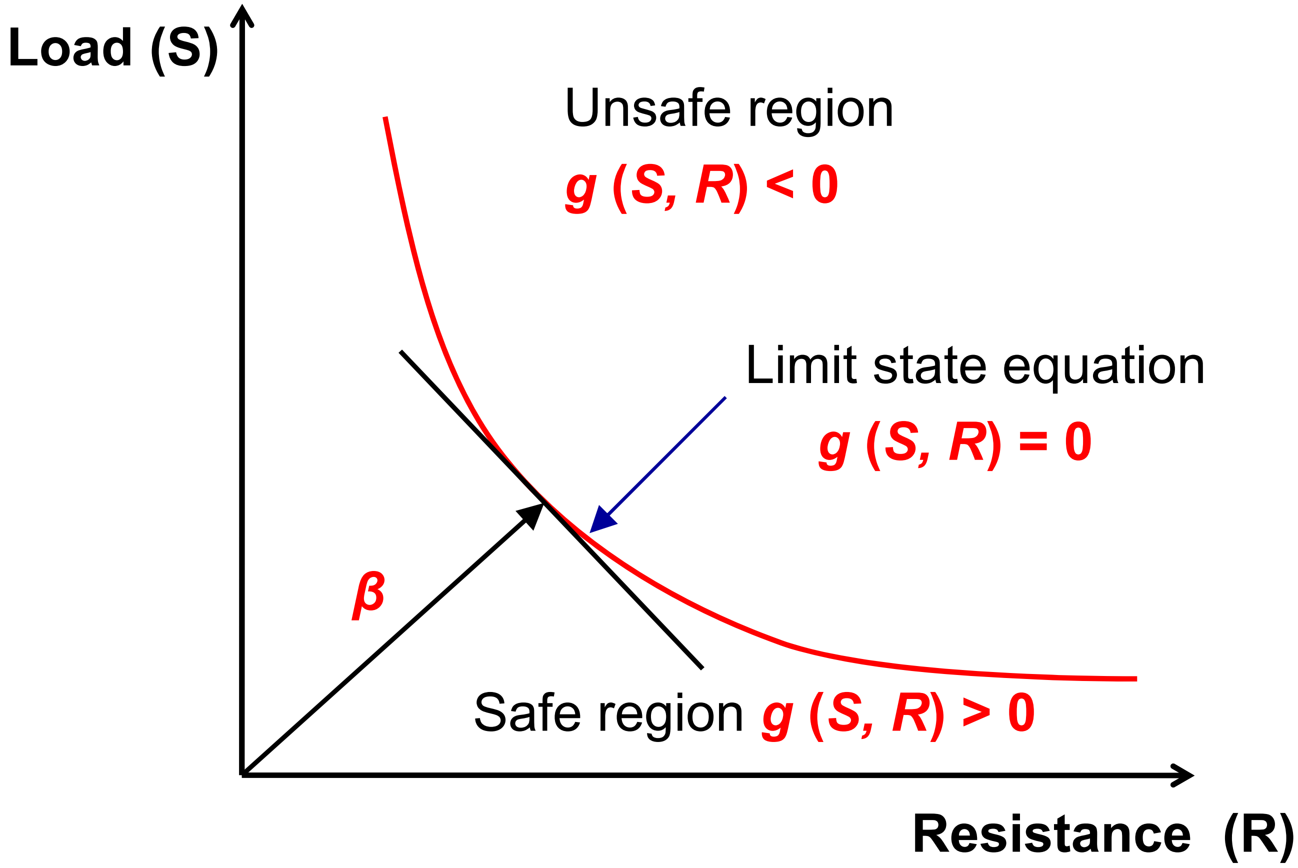
Resistance of structures (R) - N (μ_R, σ_R)

$$p_f = P(Z < 0) = 1 - \Phi(\beta)$$

$\Phi(\bullet)$ - Standard normal distribution function (zero mean and unit variance)

Safety Index (Reliability index): $\beta = \dfrac{\mu_Z}{\sigma_Z} = \dfrac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$

Probabilistic Approaches



First-order reliability methods (FORM)

A tension member in a truss has an ultimate tensile strength R with μ_R = 120 kN and σ_R = 10 kN. The tension load P in this member has a mean value of μ_P = 80 kNm and standard deviation σ_P = 20 kN. Assuming that the normal distribution of R and P, evaluate the probability of failure.

First-order reliability methods (FORM)

Solution:

Safety Index (Reliability index):

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} = \frac{120 - 80}{\sqrt{10^2 + 20^2}} = 1.79$$

$$p_f = 1 - \Phi(1.79) = 0.03673$$

- Sums and differences of independent normal variables

$$Y = a_1 X_1 + a_2 X_2 + \ldots. + a_n X_n$$

$$\mu_Y = \sum_{i=1}^{n} a_i \mu_{X_i}$$

$$\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2$$

- Sums and differences of independent normal variables
  - Example: Assume a random variable Y can be represented by the following relationship and X_1, X_2 and X_3 are statistically independent normal variables

$$Y = X_1 + 2X_2 - 4X_3$$

	Mean	**STD**
$\boldsymbol{X_1}$	1.0	0.1
$\boldsymbol{X_2}$	1.5	0.2
$\boldsymbol{X_3}$	0.8	0.15

Solution:

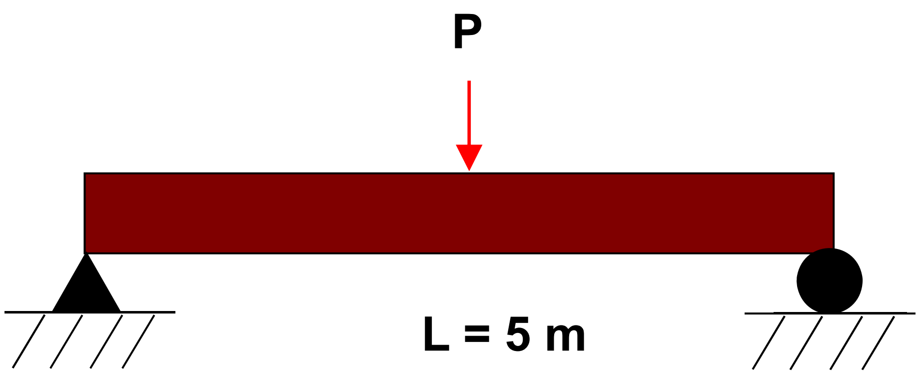
Solution:

$$\mu_Y = 1.0 + 2\times 1.5 - 4\times 0.8 = 0.8$$

$$\sigma_Y = \sqrt{0.1^2 + 2^2\times 0.2^2 + (-4)^2\times 0.15^2} = 0.728$$

Example - Single load case with normal variables

A simply supported timber beam of length 5 m is loaded with a central load P with μ_p = 3 kN and σ_p = 1 kN. The applied moment $S = P \times L/4$. The bending strength of similar beams has been found to have a mean strength μ_R = 10 kNm with a coefficient of variation (COV) of 0.15. Assuming that the beam self-weight and any variation in the length of beam can be ignore, evaluate the probability of failure.



Example - Single Load Case

Solution:

Solution:

The applied moment $S = \dfrac{P \times L}{4}$

$$\mu_S = \frac{\mu_P \times L}{4} = \frac{3 \times 5}{4} = 3.75 \text{ kNm} \qquad \sigma_S^2 = \left(\frac{\sigma_P \times L}{4} \right)^2 = 1.56 \,(\text{kNm})^2$$

$$\mu_R = 10 \text{ kNm}$$

$$\sigma_R^2 = \left[(\text{COV}) \mu_R \right]^2 = \left[0.15 \times 10 \right]^2 = 2.25 (\text{kNm})^2$$

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} = 3.2$$

$$p_f = 1 - \Phi(\beta) = 7 \times 10^{-4} = 0.07\%$$

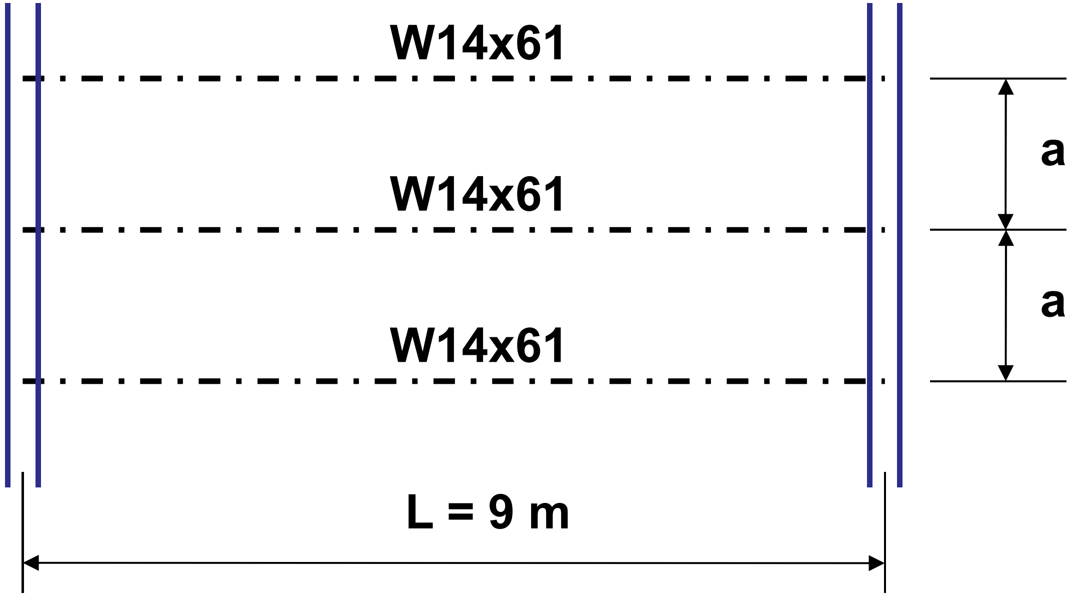
The probability of failure is extremely low

Example - Multiple Load Case

A simply supported steel beam W14x61 (capacity μ_R = 560.7 kNm, σ_R = 72.9 kNm) with a 9 m span has been designed to carry a dead load (μ_D = 2.6 kN/m^2 , σ_D = 0.35 kN/m^2) and a live load (μ_L = 2.75 kN/m^2 , σ_L = 1 kN/m^2). Assuming dead load (*D*), live load (*L*) and beam capacity (*R*) are statistically independent normal variables, and the applied moment

$$M_a = \frac{S \times a \times L^2}{8}, S = D + L$$

Evaluate the probability of failure.



Solution:

Solution:

Total load $S = D + L$

$$\mu_S = \mu_D + \mu_L = 2.6 + 2.75 = 5.35\text{kN/m}^2$$

$$\sigma_S = \sqrt{(\sigma_D)^2 + (\sigma_L)^2} = \sqrt{0.35^2 + 1^2} = 1.06\text{kN/m}^2$$

The applied moment $M_a = \dfrac{S \times a \times L^2}{8}$

$$\mu_{M_a} = \frac{\mu_S \times a \times L^2}{8} = \frac{5.35 \times 3 \times 9^2}{8} = 162.5\text{kNm}$$

$$\sigma_{M_a} = \sqrt{\left(\frac{a \times L^2}{8}\right)^2 \sigma_S^2} = \sqrt{\left(\frac{3 \times 9^2}{8}\right)^2 1.06^2} = 32.2\text{kNm}$$

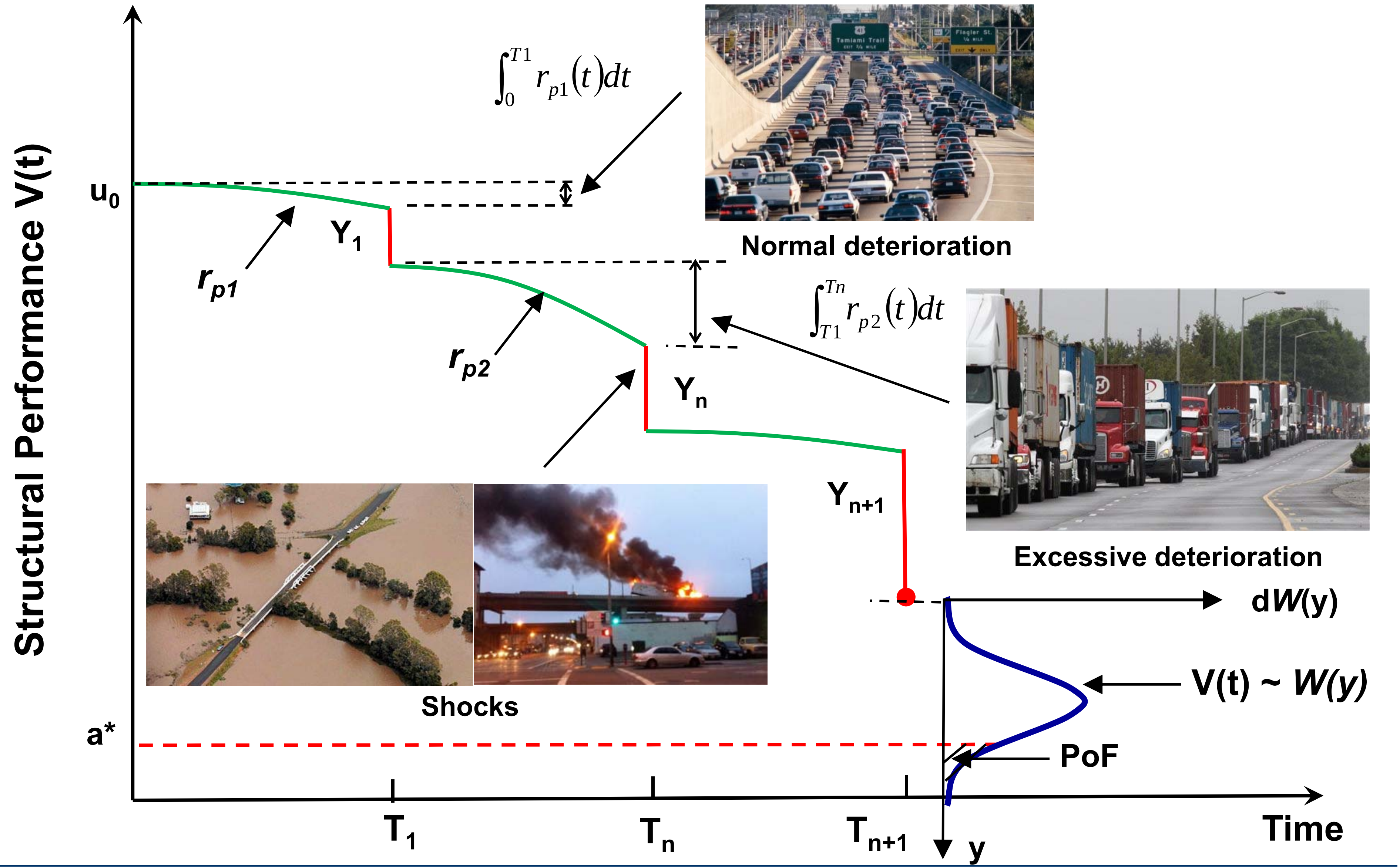
Solution (continued):

$$\beta = \frac{\mu_R - \mu_{M_a}}{\sqrt{\sigma_R^2 + \sigma_{M_a}^2}} = \frac{560.7 - 162.5}{\sqrt{72.9^2 + 32.2^2}} = 5$$

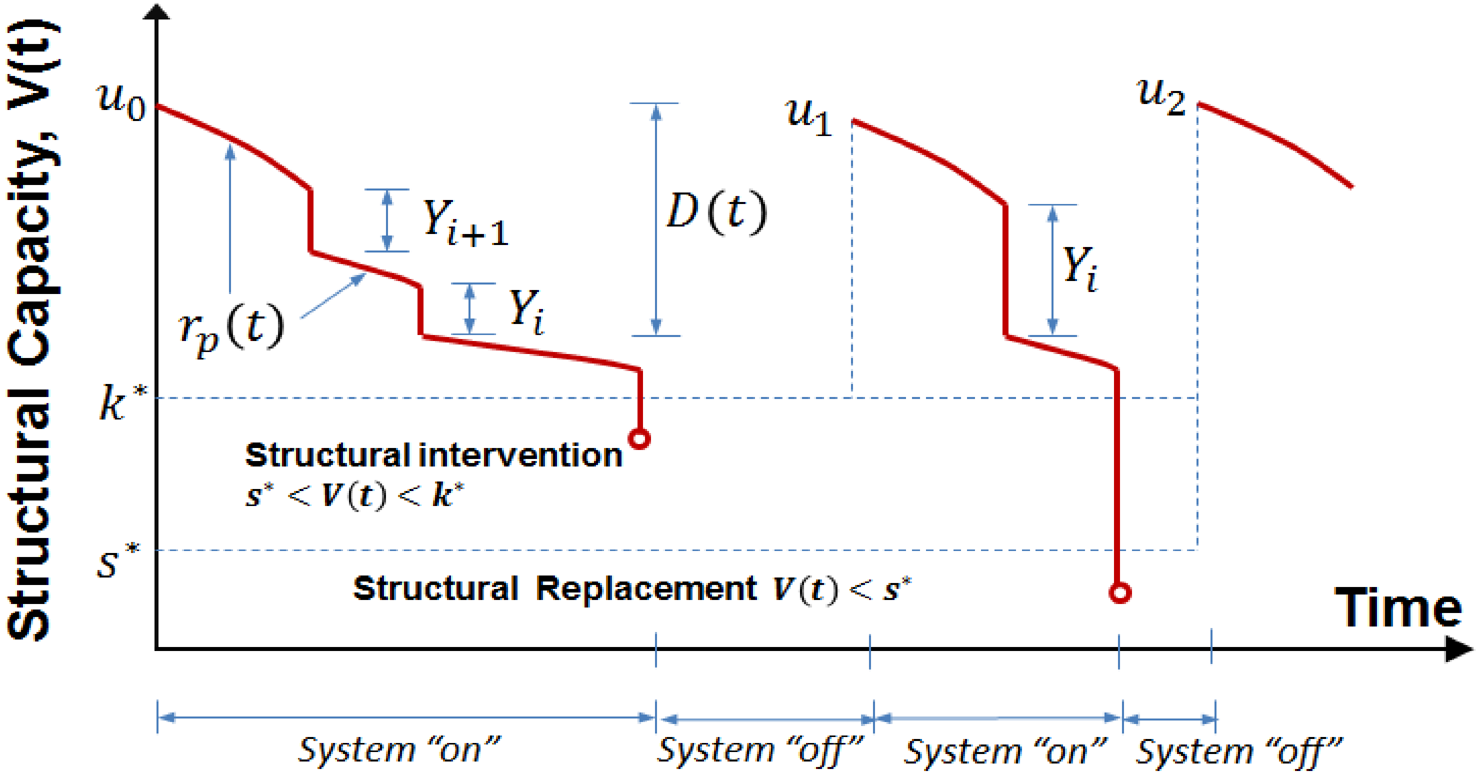
$$p_f = 1 - \Phi(\beta) \approx 0.3 \times 10^{-6}$$

The probability of failure is extremely low

Life-cycle deterioration of infrastructures

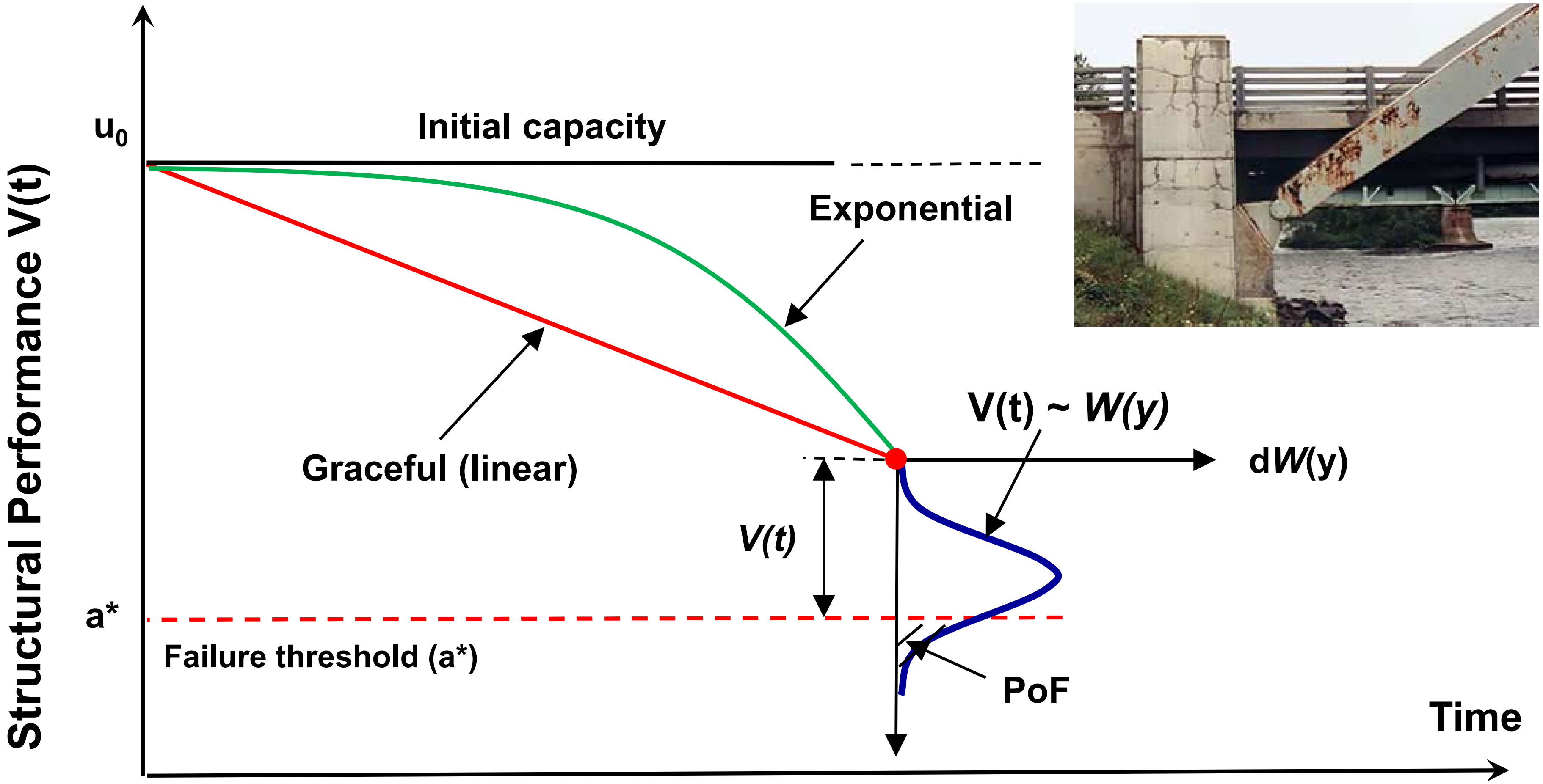


Life-cycle deterioration of infrastructures

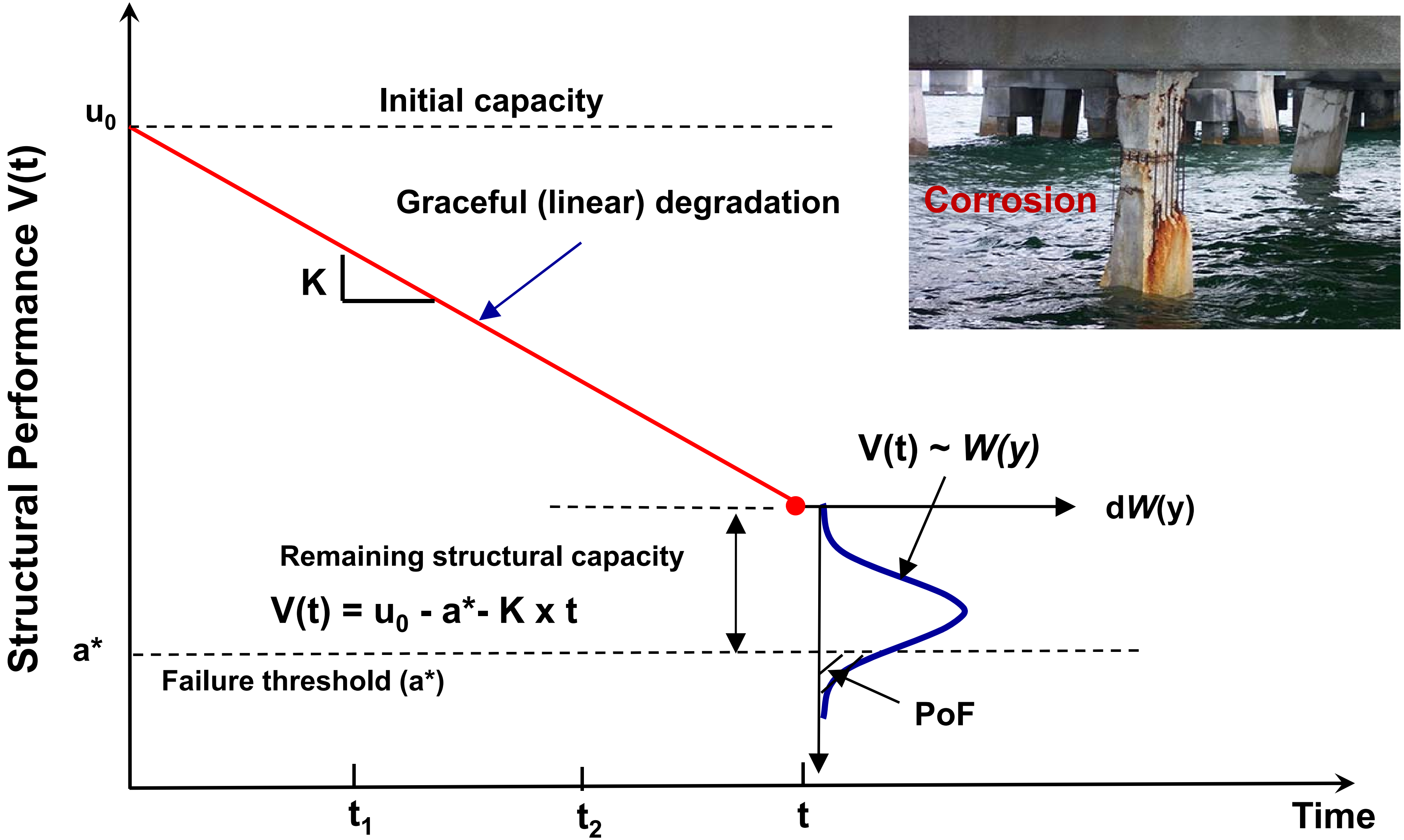


- Maintain the system operating in acceptable conditions during the maximum length of time.
- Maximize the system availability at minimum cost.

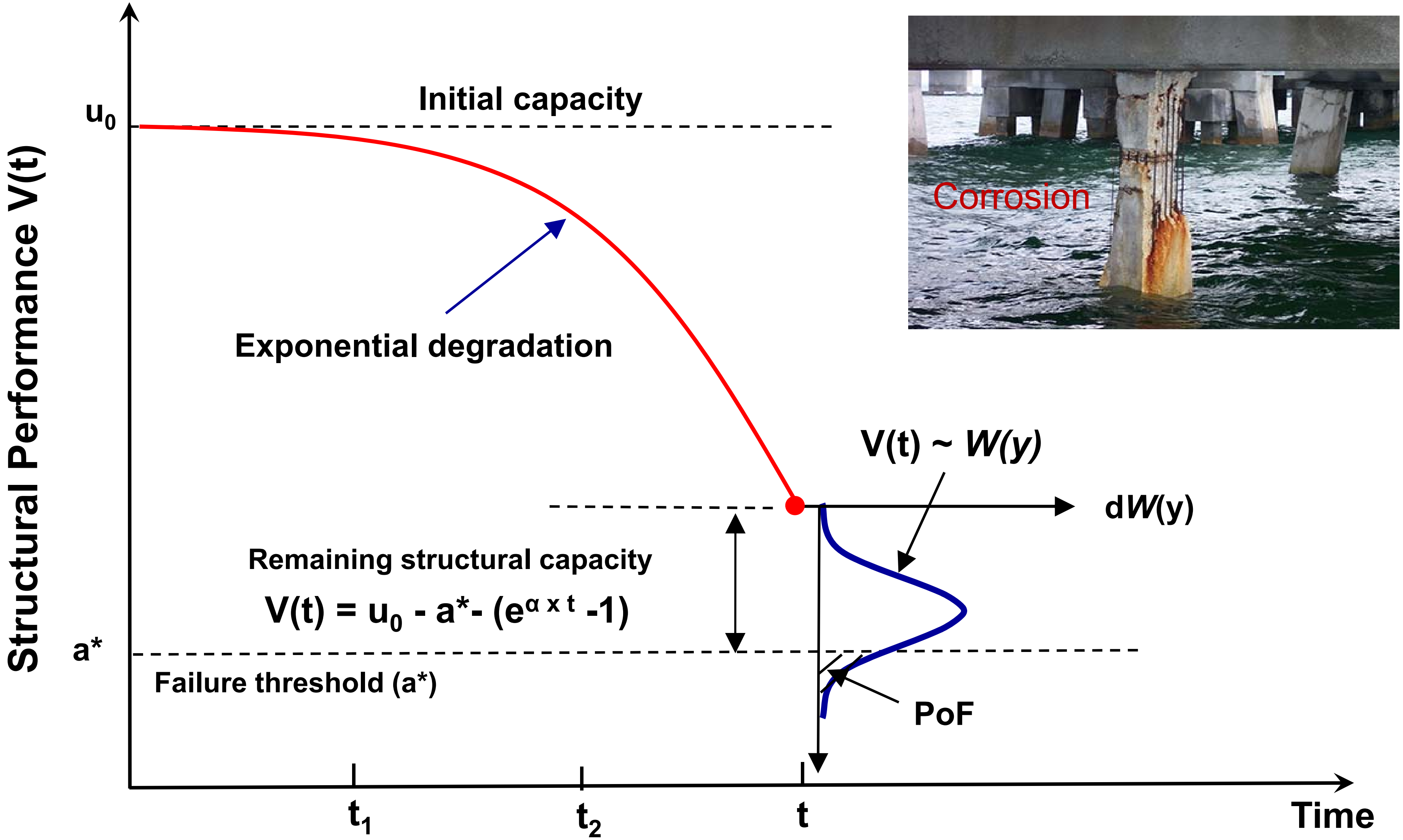
Deterministic Progressive Deterioration



Graceful (Linear) Deterioration



Graceful (Linear) Deterioration



Deterministic Progressive Deterioration

- The remaining structural capacity at a given time ***V(t)*** :

$$\boldsymbol{V(t) = u_0 - a^* - D(t)}$$

The accumulated damage ***D(t)*** is given by:

- Graceful (linear) degradation

$$V(t) = u_0 - a^* - K \times t$$

- Exponential degradation

$$V(t) = u_0 - a^* - (e^{\alpha \times t} - 1)$$

Deterministic Progressive Deterioration

- Probability of failure at time ***t*** : $p(t)=\left[\int_{V(t,a^*)}^{\infty} dW(y)\right]=\left[1-\int_{0}^{V(t,a^*)} dW(y)dy\right]$

Distribution of ***W*** describes the probability of having a certain damage level as a result of progressive deterioration (*e.g.* loss of structural capacity due to corrosion).

If ***W*** is exponentially distributed,

Probability density function (PDF):

$$\mathrm{PDF}(y) = \theta\ e^{-\theta\, y}$$

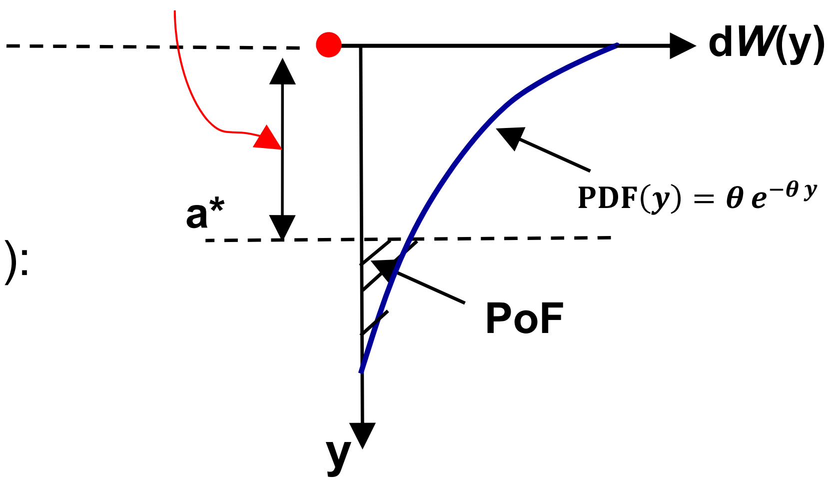
Cumulative distribution function (CDF):

$$\mathrm{CDF}(y) = 1 - e^{-\theta\, y}$$

Probability of failure (PoF):

$$\mathrm{PoF}(y) = 1 - \mathrm{CDF}(y)$$

Remaining structural capacity V(t)



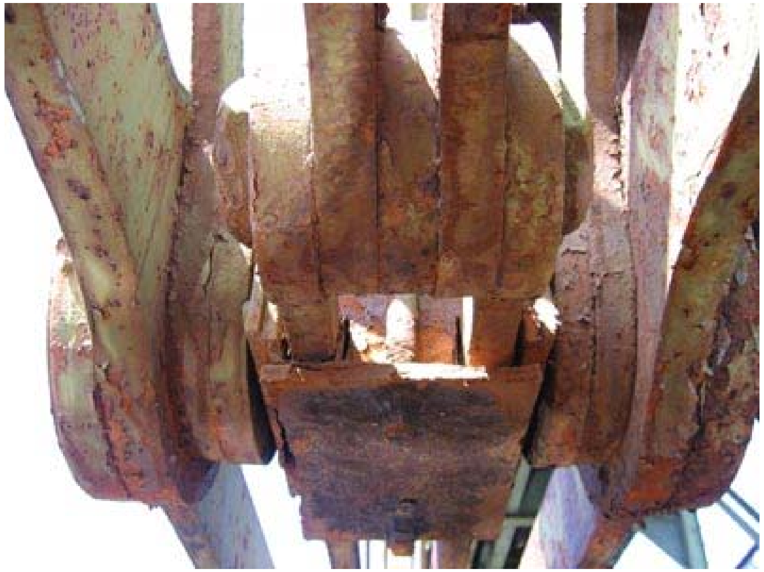
Example 1 – Deterministic Progressive deterioration

Consider a case of a steel bridge that deteriorates continuously with time (*e.g.* corrosion). The initial structure performance is u_0 = 100% with a threshold limit a^* = 25%. Estimate the probability an intervention is required when t = 30 years if the progressive deterioration of the bridge can be modelled as

(a) Graceful (linear) deterioration with a rate K = 0.75% per year.

(b) Exponential deterioration with a rate α = 0.046/year.

Assume the remaining structural capacity is governed by an exponential distribution $W(y, \theta)$ with an average rate θ = 0.05.



Solution:

(a) Graceful (linear) deterioration with a rate K = 0.75/year.

$$V(t=30) = u_0 - a^* - K \times t = 100-25-0.75\times 30 = 52.5$$

Cumulative distribution function (CDF): $\mathrm{CDF}(V) = 1 - e^{-\theta V}$

Probability of failure (PoF):

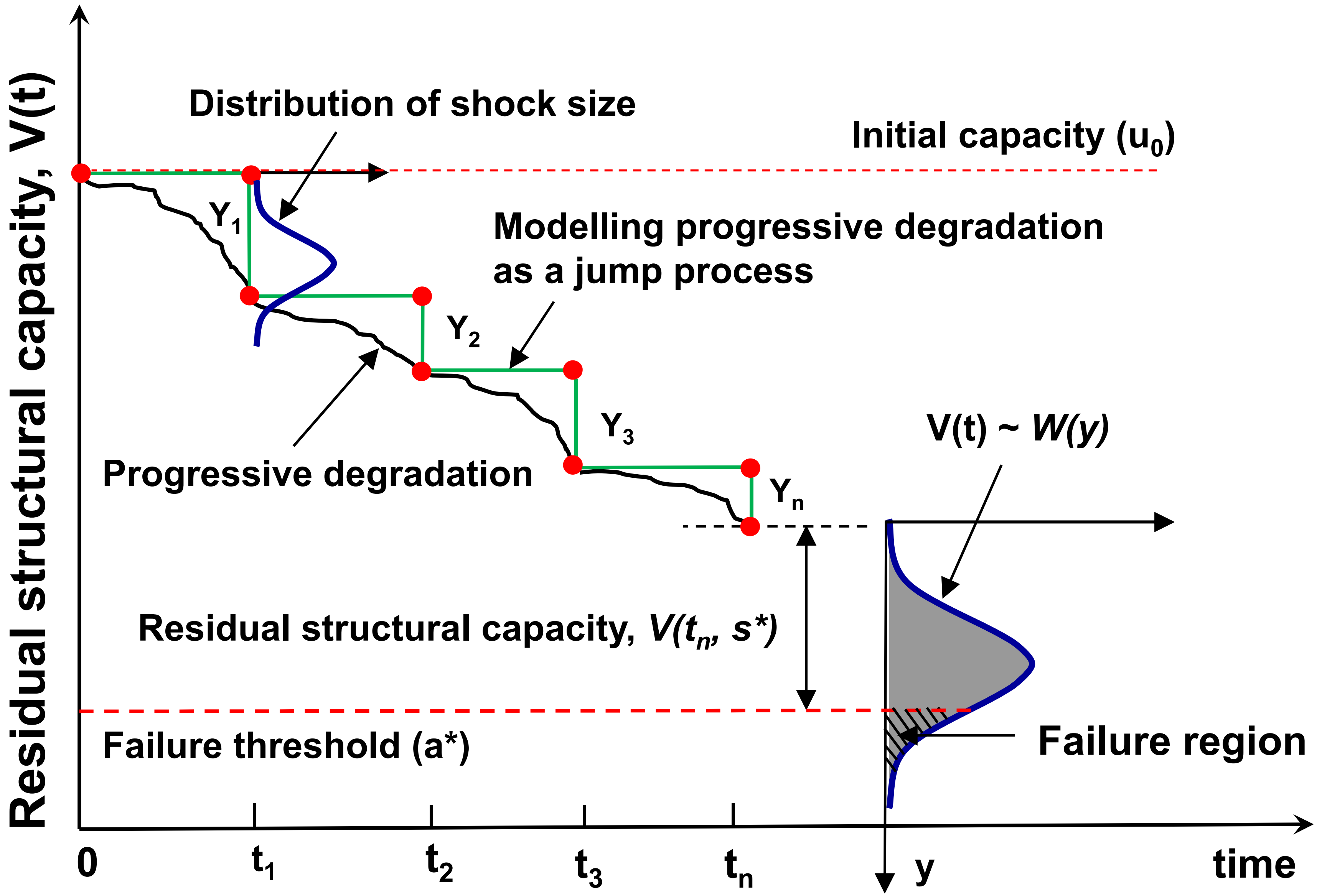
$$\mathrm{PoF(V)} = 1 - \mathrm{CDF}(V) = 1 - (1 - e^{-\theta V}) = 1 - (1 - e^{-0.05 X\, 52.5}) = 7.24\%$$

(b) Exponential deterioration with a rate α = 0.046/year.

$$V(t=30) = u_0 - a^* - (e^{\alpha \times t} - 1) = 100 - 25 - (e^{0.0461 \times 30} - 1) = 72$$

$$\mathrm{PoF(V)} = 1 - \mathrm{CDF}(V) = 1 - (1 - e^{-\theta V}) = 1 - (1 - e^{-0.05 X\, 72}) = 2.7\%$$

Random Progressive deterioration



Random Progressive Deterioration

- Assume progressive deterioration is a jump process using small jumps in which the size of every jump is random, and jumps occur at fix time intervals (e.g. annual inspections).
- By assuming the damage caused by each shock is exponentially distributed, simulation process involves the following steps:

(1) Set accumulated deterioration $D = 0$; residual capacity $V = u_0 - a^*$;

(2) $t_i = t_{i-1} + \Delta t$; obtain the damage size (Y_i) from exponential distribution;

(3) Compute damage accumulation $D = D + Y_i$;

(4) Compute residual capacity $V = V - D$;

(5) Goto Step (2) until reaching a particular time point (t_n);

(6) Probability of failure at time t_n :

$$p(t) = \left[\int_{V(t,a^*)}^{\infty} dW(y) \right] = \left[1 - \int_{0}^{V(t,a^*)} dW(y) dy \right]$$

Example 2 – Random Progressive deterioration

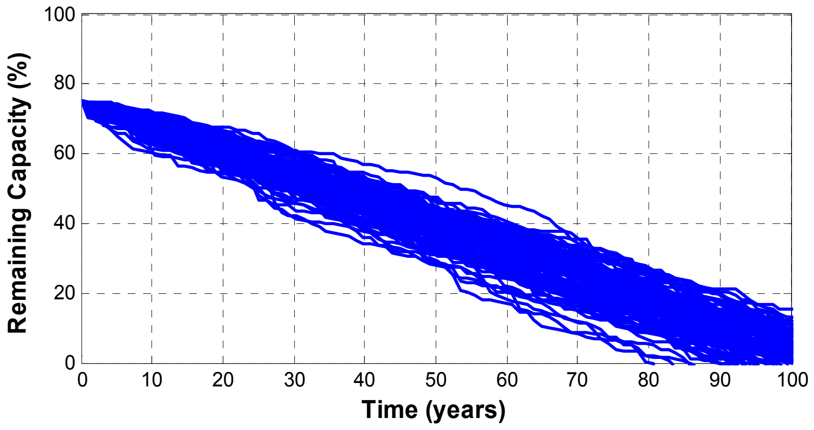
Consider a case of a steel bridge that deteriorates continuously with time (*e.g.* corrosion). The initial structure performance is u_0 = 100% with a threshold limit a^* = 25%. Estimate the probability an intervention is required if the progressive deterioration of the bridge can be modelled as a jump process in which the size of every jump is exponentially distributed with an average rate λ = 0.75%. Assume every jump is randomly distributed with fixed-time interval Δt = 1 year.

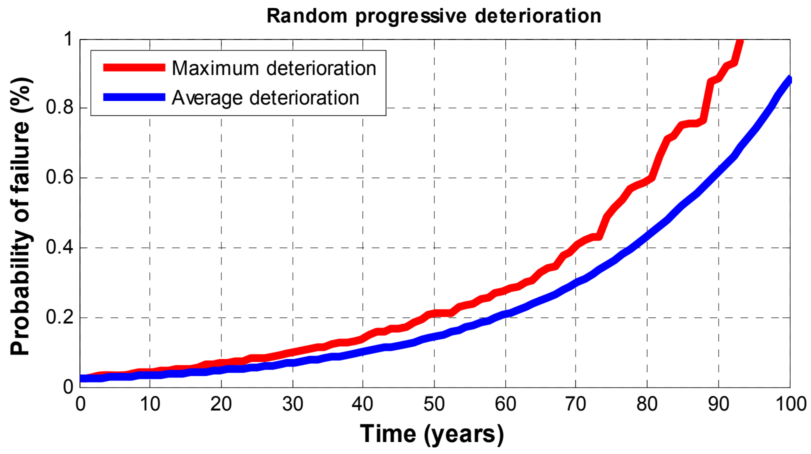
Assume the remaining structural capacity is governed by an exponential distribution $W(y, \theta)$ with an average rate θ = 0.05,

Example 2 – Random Progressive deterioration

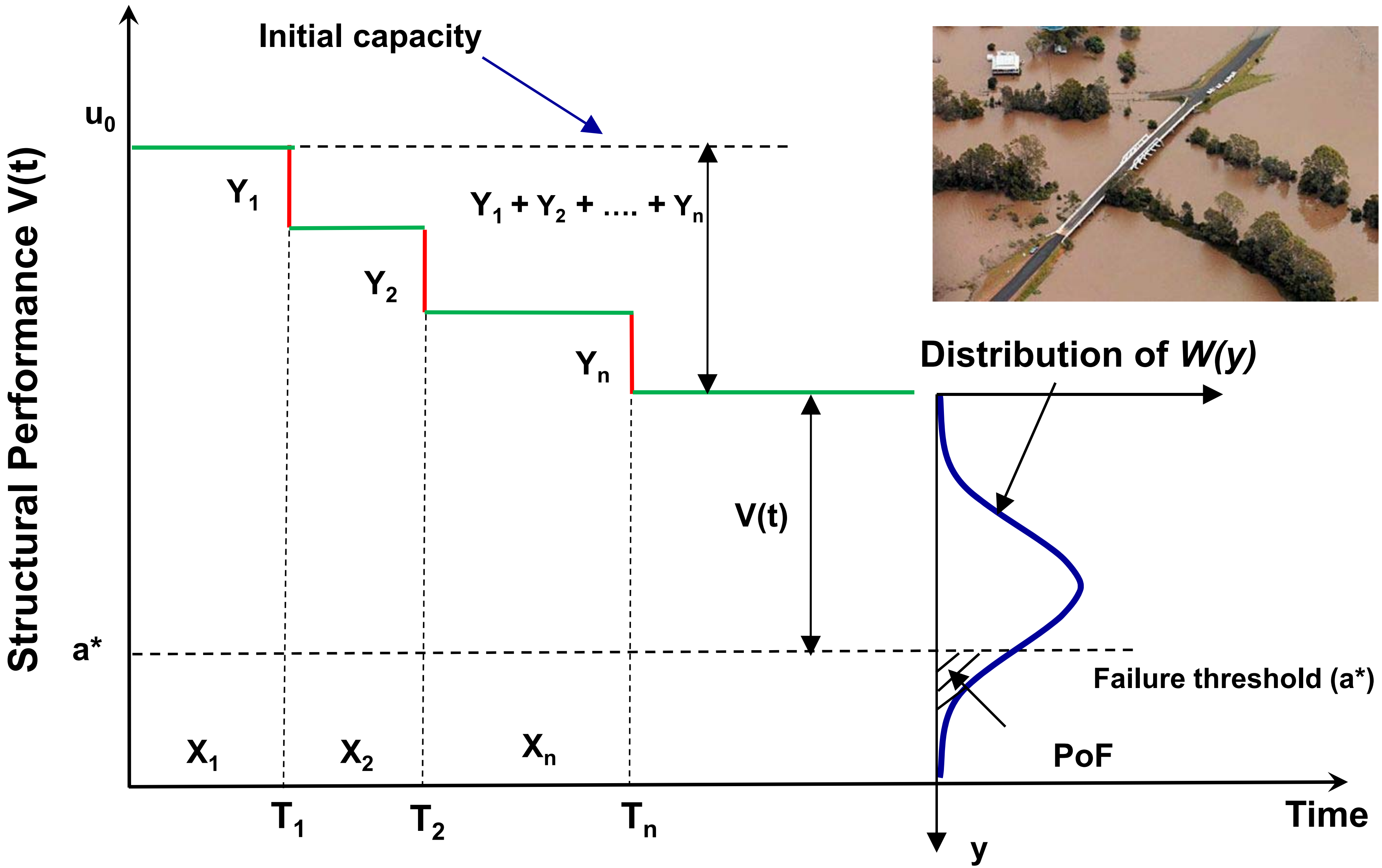
Data used in the example:

- Initial capacity, $u_0 = 100\%$;
- Threshold limit, $a^* = 25\%$;
- Damage distributed exponentially with $\theta = 0.05$;
- Jump size, $\lambda = 0.75\%$;
- Number of random sample, $n = 100$ *-sample*





Failure after Shocks



Failure after Shocks

- The cumulated deterioration (D_n)

$$D_n = \sum_{i=1}^{n} Y_i$$

- The total time (S_n)

$$S_n = \sum_{i=1}^{n} X_i$$

- Two main challenges in modelling shock-based degradation
  - Not enough data to understand the distributions of the time between shocks X_i, and shock sizes Y_i .
  - The reliability estimation is numerically intractable.

Failure after Shocks

- By assuming X_i and Y_i are independent and identically distributed (iid), as well as exponentially distributed, simulation process involves the following steps:

  (1) Set accumulated deterioration $D = 0$; total time $S_n = 0$; residual capacity $V = u_0 - a^*$;

  (2) Obtain the time between shocks (X_i) from exponential distribution; total time $S_i = S_{i-1} + X_i$;

  (2) obtain the damage size (Y_i) at total time S_i from exponential distribution;

  (3) Compute damage accumulation $D = D + Y_i$;

  (4) Compute residual capacity $V = V - D$;

  (5) Goto Step (2) until reaching a particular time point (t_n);

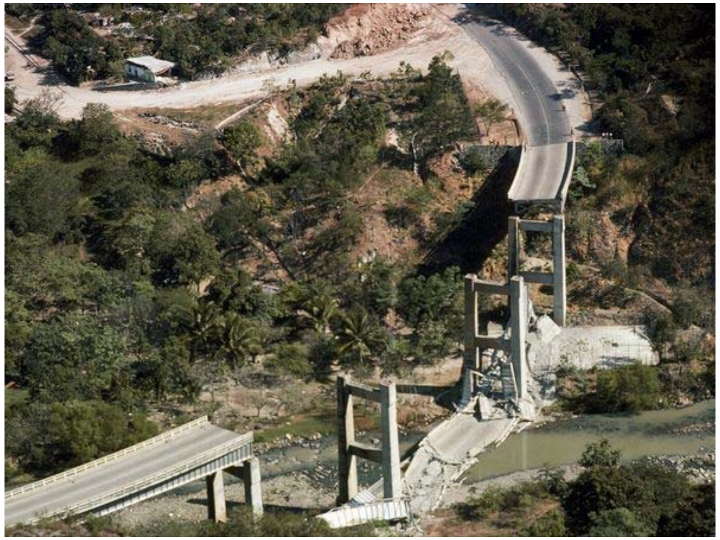
  (6) Probability of failure at time t_n :

$$p(t) = \left[\int_{V(t,a^*)}^{\infty} dW(y) \right] = \left[1 - \int_{0}^{V(t,a^*)} dW(y) dy \right]$$

Example 3 - Failure after Multiple Shocks

Consider a case of a bridge that is subject to shock-based degradation (*e.g.* earthquake) that occur randomly in time. The inter-arrival times of disturbances are exponentially distributed with mean $\mu = 2$ *years*. Suppose the initial capacity of a bridge is $u_0 = 100\%$ with a threshold limit $a^* = 25\%$. Estimate the probability an intervention is required if the shock size is exponentially distributed with an average rate $\lambda = 2\%$.

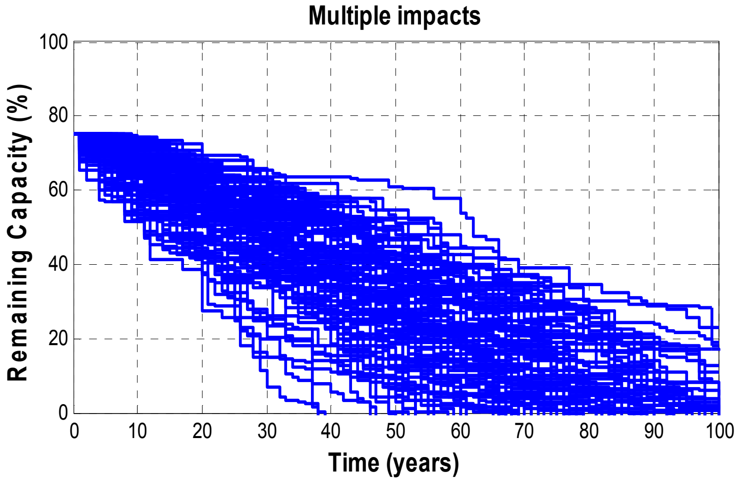
Assume the remaining structural capacity is governed by an exponential distribution $W(y, \theta)$ with an average rate $\theta = 0.05$

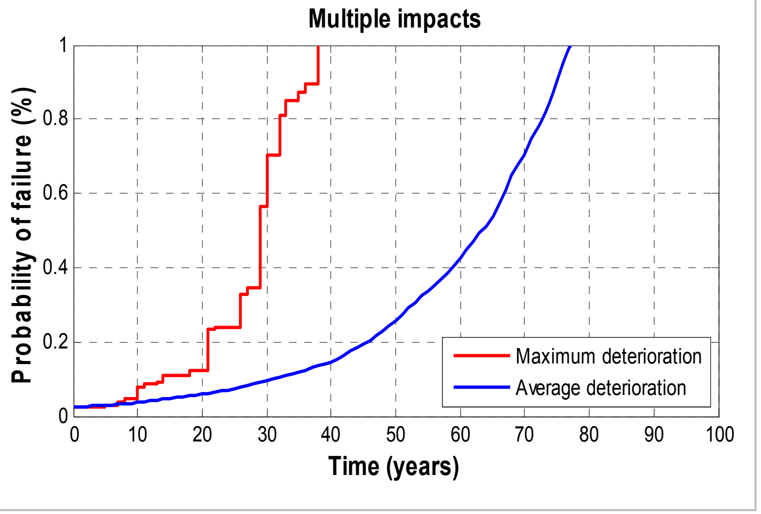


Example 3 – Failure after Multiple Shocks

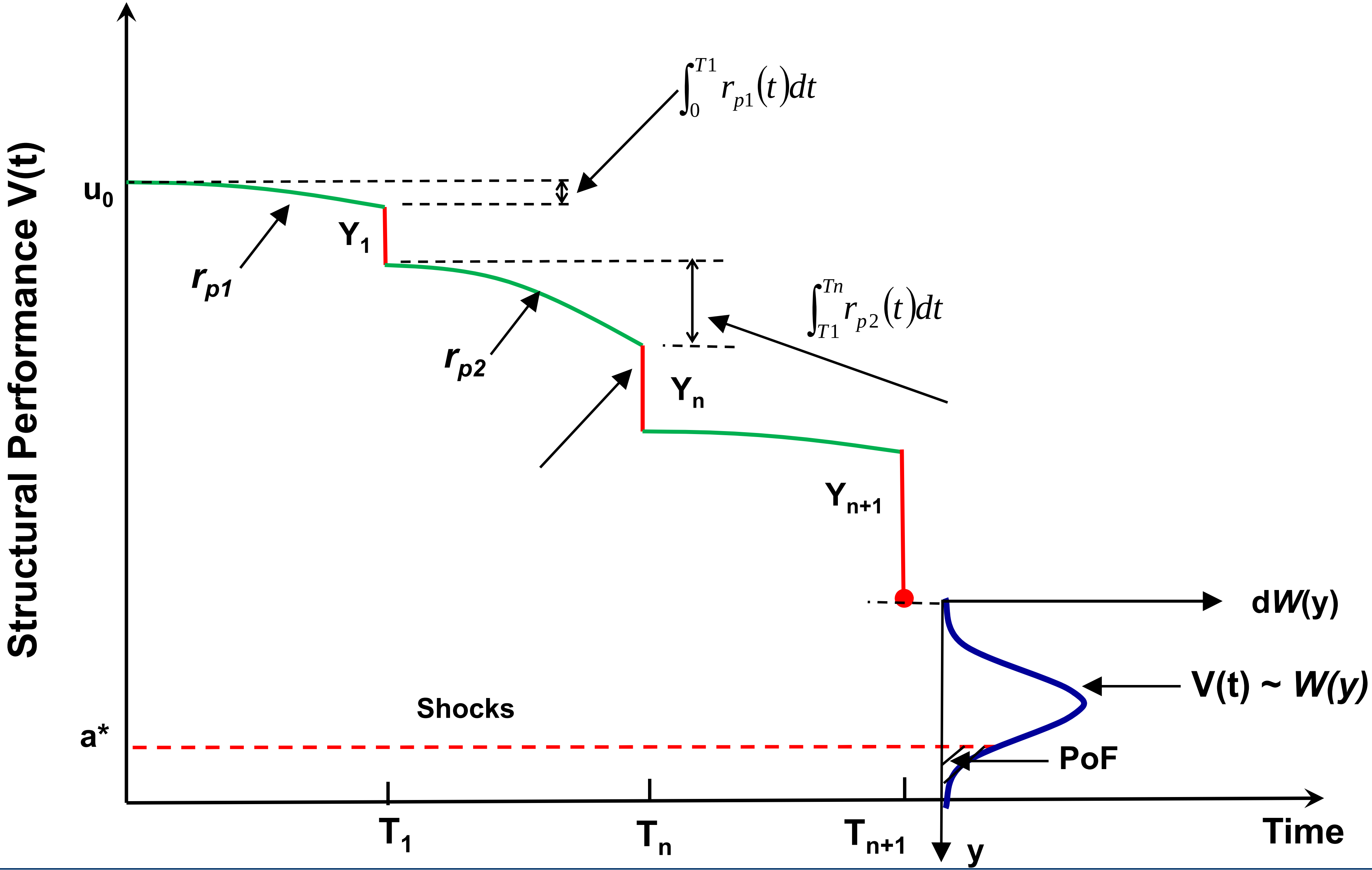
Data used in the example:

- Initial capacity, $u_0 = 100\%$;
- Threshold limit, $a^* = 25\%$;
- Damage distributed exponentially with $\theta = 0.05$.
- Shock size, $\lambda = 2\%$;
- The inter-arrival times with mean $\mu = 2$ *years*
- Number of random sample, $n = 100$ *-sample*





Combined Deterioration



Example 4 – Combined Deterioration

Estimate the probability an intervention is required for a steel bridge that is subject to both progressive and shock-based degradation. Suppose the initial capacity of bridge is $u_0 = 100\%$ with a threshold limit $a^* = 25\%$. The progressive deterioration of the bridge can be modelled as a jump process in which the size of every jump is exponentially distributed with an average rate $\lambda = 0.75\%$. Assume every jump is randomly distributed with fixed-time interval.

The shocks that occur randomly in time can be modelled as a Poisson process in which the inter-arrival times are exponentially distributed with mean $\mu = 2$ *years* and the average shock size $2 = 4\%$.

Finally, assume the remaining structural capacity is governed by an exponential distribution $W(y, \theta)$ with an average rate $\theta = 0.05$

Example 4 – Combined deterioration

Data used in the example:

- Jump size, $\lambda = 0.75\%$;
- Initial capacity, $u_0 = 100\%$;
- Threshold limit, $a^* = 25\%$;
- Damage distributed exponentially with $\theta = 0.05$.
- Shock size, $\lambda = 2\%$;
- The inter-arrival times with mean $\mu = 2$ years
- Number of random sample, $n = 100$-sample