MAST10007 Summer 2011 Exam Solutions

1(a) (i)

$$a + b = 10$$
$$a - c = 3$$
$$c + d = 6$$
$$b - d = 1$$

(ii) The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 1 & 0 & -1 & 1 \end{array}\right] R_2 - R_1 \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & -1 & -1 & 0 & -7 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 1 & 0 & -1 & 1 \end{array}\right] R_3 \leftrightarrow R_4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & -1 & -1 & 0 & -7 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] R_3 + R_2 \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & -1 & -1 & 0 & -7 \\ 0 & 0 & -1 & -1 & -6 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] R_4 + R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & -1 & -1 & 0 & -7 \\ 0 & 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the number of unknowns is greater than the number of leading entries, there are an infinite number of solutions.

(b) (i)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2/(-2) \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(ii) There is no leading entry for β. Set $\beta = t$, $t \in \mathbb{R}$. Back substitution gives

$$\gamma = -2, \qquad \alpha = 4$$

Hence the general solution is

$$\{(\alpha, \beta, \gamma) = (4, t, -2), \quad t \in \mathbb{R}\}.$$

2(a) Let A be $p \times q$. Let B be $r \times s$.

For AB to be defined, require $q = r$.

For BA to be defined, require $s = p$

Then AB is $p \times p$, and BA is $q \times q$.

For $AB - BA$ to be defined require $p = q$

Hence $p = q = r = s$ and so A and B must be square matrices of the same size.

(b)
(i) $YZX = Y_{2\times 1} Z_{2\times 3} X_{3\times 1}$

these don't match, so not possible.

(ii) $Y^T Y = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 9 + 4 = 13$

(c) $P^2 = P \Rightarrow (\det P)^2 = \det P$

$\Rightarrow \det P(\det P - 1) = 0 \Rightarrow \det P = 0$ or $\det P = 1$

In the case $\det P = 1$, P is invertible, and so $P^2 = P \Rightarrow P^{-1} P P = I \Rightarrow P = I$.

3(a) We see that $\det M = 1$. Since this is non-zero, M is invertible.

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_3 + R_4$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_2 + R_3 \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_1 + R_2$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

It follows that

$$M^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The given matrix is M^T. Now

$$(M^T)^{-1} = (M^{-1})^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

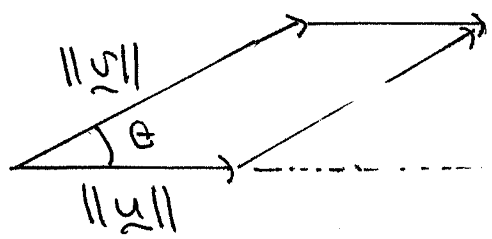
(c) $\det Y = (-1) \times 3 \det X = -15$.

4(a) (i) Let $\underset{\sim}{u} = (u_1, u_2, u_3)$ $\underset{\sim}{v} = (v_1, v_2, v_3)$.

$$\underset{\sim}{u} \times \underset{\sim}{v} = \det \begin{bmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

(ii) $\|\underset{\sim}{u} \times \underset{\sim}{v}\| = \|\underset{\sim}{u}\| \, \|\underset{\sim}{v}\| \sin\theta$

(iii)



We have that

$$\text{area parallelogram} = \text{base} \times \text{height}$$
$$= \|\underset{\sim}{u}\| \left(\|\underset{\sim}{v}\| \sin\theta \right) = \|\underset{\sim}{u} \times \underset{\sim}{v}\|$$
(using (ii))

4 (b) $\text{volume parallelpiped} = \left| \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \right| = 6$

5 (a) (0) dimension 0. The point $\underset{\sim}{0} = (0,0,0)$.

(1) dimension 1. Lines through the origin.

(2) dimension 2. Planes through the origin.

(3) dimension 3. $\mathbb{R}^3$ itself.

(b) Let $\tilde{S}$ be the equivalent set in $\mathbb{R}^4$. Have

$$\tilde{S} = \{ (a, b, b, d), \quad a, b, d \in \mathbb{R} \}$$
$$= \{ a(1,0,0,0) + b(0,1,1,0) + d(0,0,0,1), \quad a, b, d \in \mathbb{R} \}$$
$$= \text{Span}\{ (1,0,0,0), (0,1,1,0), (0,0,0,1) \}$$

All spans of vectors in $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$.

(c) Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two general points in the set. Then

$$x_1 + 2y_1 + z_1 = 0 \quad \text{and} \quad x_2 + 2y_2 + z_2 = 0.$$

We have that

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$= (x_3, y_3, z_3)$ with $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$, $z_3 = z_1 + z_2$

and

$$x_3 + 2y_3 + z_3 = (x_1 + 2y_1 + z_1) + (x_2 + 2y_2 + z_2) = 0$$

Hence (x_3, y_3, z_3) is in the set and so it is closed under vector addition.

6. (a) $\underset{\sim}{v}_2$, $\underset{\sim}{v}_3$

(b) $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_4\}$ These correspond to the columns with leading entries.

$\{\underset{\sim}{a}_1, \underset{\sim}{a}_3, \underset{\sim}{a}_4\}$ Since $\underset{\sim}{a}_3$ can be written as a linear combination of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$.

The column space has dimension 3.

(c) $$[\underset{\sim}{a}_1\ \underset{\sim}{a}_3\ \underset{\sim}{a}_5 \mid \underset{\sim}{v}_2] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For $\alpha \underset{\sim}{a}_1 + \beta \underset{\sim}{a}_3 + \gamma \underset{\sim}{a}_5 = \underset{\sim}{v}_2$ we must have, by back substitution

$$\gamma = -2, \quad \beta = -3 - \gamma = -1, \quad \alpha = 1 + \beta = 0.$$

Hence $$-\underset{\sim}{a}_3 - 2\underset{\sim}{a}_5 = \underset{\sim}{v}_2$$

(d) Let the variables be denoted $x_1, \ldots, x_5$. There is no leading entry for x_5 so we set $x_5 = t$, $t \in \mathbb{R}$. There is no leading entry for x_3 so we set $x_3 = s$, $s \in \mathbb{R}$.

Back substitution gives

$$x_4 = x_5 = t$$
$$x_2 = -x_3 - x_5 = -s - t$$
$$x_1 = x_3 = s$$

Hence the solution set is

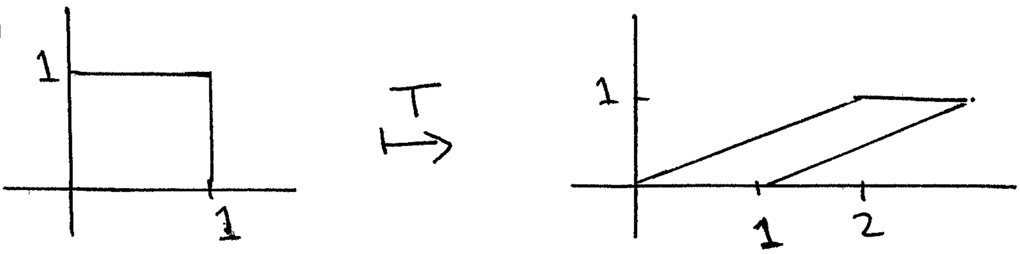
$$\{(x_1, x_2, x_3, x_4, x_5) = s(1,-1,1,0,0) + t(0,-1,0,1,1),\ s,t \in \mathbb{R}\}$$

We read off that a basis is

$$\{(1,-1,1,0,0), (0,-1,0,1,1)\}.$$

7 (a) (i) $A_T = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(ii)



(iii) $\det A_T$ gives the (signed) area of the transformed unit square. From the formula area = base × height, this area equals 1.

7(b) (i) $S(1,0) = (1,0)$ $S(0,1) = (0,-1)$

Hence $A_S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) The x-axis $= \text{span}\{(1,0)\}$ and
y-axis $= \text{span}\{(0,1)\}$
are left unchanged by the action of S.

(iii) $A_S \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A_S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

8(a)
(i) $T(x,y,z) = (x,y,0)$
$= x(1,0,0) + y(0,1,0)$

Hence $\text{Im}T = \text{Span}\{(1,0,0),(0,1,0)\}$
and a basis is $\{(1,0,0),(0,1,0)\}$. Dimension 2.

(ii) $\text{Ker}T = \{(x,y,z) : T(x,y,z) = \underset{\sim}{0}\}$
$= \{(0,0,z),\ z \in \mathbb{R}\}$
$= \text{Span}\{(0,0,1)\}$

A basis is $\{(0,0,1)\}$. Dimension 1.

(iii) T is an orthogonal projection onto the xy-plane.

(b) (i) $[\underset{\sim}{x}]_B = P_{B,S}[\underset{\sim}{x}]_S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ y-z \\ z \end{bmatrix}$

Hence $\underset{\sim}{x} = (x-y)\underset{\sim}{b}_1 + (y-z)\underset{\sim}{b}_2 + z\underset{\sim}{b}_3$

(ii) $P_{S,B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (from working in Q3)

Hence $\underset{\sim}{b}_1 = (1,0,0)$ $\underset{\sim}{b}_2 = (1,1,0)$ $\underset{\sim}{b}_3 = (1,1,1)$

(iii) This matrix has the first two columns linearly dependent, and so the corresponding vectors cannot define a basis.

9(a) (i) Let $\underset{\sim}{b}_1 = (1,0,1,0)$ $\underset{\sim}{b}_2 = (3,1,1,1)$.

We have

$$\underset{\sim}{v}_1 = \frac{1}{\|\underset{\sim}{b}_1\|}(1,0,1,0) = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2 = \underset{\sim}{b}_2 - (\underset{\sim}{v}_1 \cdot \underset{\sim}{b}_2)\underset{\sim}{v}_1$$

$$= (3,1,1,1) - \frac{1}{2} \times 4 (1,0,1,0)$$

$$= (1, 1, -1, 1)$$

$$\Rightarrow \underset{\sim}{v}_2 = \frac{1}{2}(1, 1, -1, 1)$$

(ii) Let $\underset{\sim}{p}$ denote the projected vector. We have

$$\underset{\sim}{p} = (\underset{\sim}{v}_1 \cdot \underset{\sim}{u})\underset{\sim}{v}_1 + (\underset{\sim}{v}_2 \cdot \underset{\sim}{u})\underset{\sim}{v}_2$$

$$= \frac{1}{2}(1,0,1,0) + \frac{1}{4} \times 0 = \frac{1}{2}(1,0,1,0).$$

(b)(i) $(x_1, x_2) = (1,0)$ $(y_1, y_2) = (0,1)$

$\therefore \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 4 \times 1 \times 0 - 2 \times 1 \times 1 - 2 \times 0 \times 0 + 5 \times 0 \times 1 = -2 \neq 0$.

Hence $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are not orthogonal in this inner product space

(ii) $\|(3,4)\| = \sqrt{\langle (3,4),(3,4)\rangle} = \sqrt{4\times 9 - 4\times 12 + 5\times 16}$

$= \sqrt{68}$

(iii) Axiom 4: $\langle \underset{\sim}{x}, \underset{\sim}{x}\rangle \geq 0$

and if $\langle \underset{\sim}{x}, \underset{\sim}{x}\rangle = 0$ then $\underset{\sim}{x} = \underset{\sim}{0}$.

10 (a) $A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \underset{\sim}{y} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

We want to solve

$$A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T \underset{\sim}{y}$$

Now $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 26 \end{bmatrix}$

$A^T \underset{\sim}{y} = \begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 15 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/26 \end{bmatrix} \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 15/26 \end{bmatrix}$

Hence the line of best fit is

$$y = \frac{1}{3} + \frac{15}{26} x$$

(b) Setting $x = 10$ gives $y = \frac{1}{3} + \frac{150}{26}$

$\simeq 6\text{m}$

11 (a) $\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{bmatrix}$

$= -(4-\lambda)(4+\lambda) - 9 = \lambda^2 - 25$

Eigenvalues when $\det(A-\lambda I) = 0$. Hence $\lambda = \pm 5$.

Eigenvector for $\lambda = 5$:

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}_{R_2+3R_1} \sim \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$$

No leading entry for y. Set $y = t$. Then $x = 3t$.

Hence eigenvector $t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\Rightarrow$ normalized eigenvector $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Eigenvector for $\lambda = -5$:

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}_{R_2-3R_1} \sim \begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}$$

No leading entry for y. Set $y = t$. Then $x = -\frac{1}{3}t$

Hence eigenvector $\frac{t}{3}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\Rightarrow$ normalized eigenvector $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) We have $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{10}(-3+3) = 0$.

Hence the eigenvectors are orthogonal.

(c) A stretches by a factor of 5 in the direction of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, & by a factor of -5 in the direction of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(d) $(2,4) = (3,1) + (-1,3)$

Hence $A^{10}\begin{bmatrix}2\\4\end{bmatrix} = A^{10}\begin{bmatrix}3\\1\end{bmatrix} + A^{10}\begin{bmatrix}-1\\3\end{bmatrix}$

$= 5^{10}\begin{bmatrix}3\\1\end{bmatrix} + 5^{-10}\begin{bmatrix}-1\\3\end{bmatrix}$

12 (a) False

(b) True

(c) False

(d) True

(e) True

(f) False