COMP30026 Models of Computation

Regular Expressions and Non-Regular Languages

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Lecture 15

Semester 2, 2017

Regular Expressions

Regular expressions is a notation for languages.

You are probably familiar with similar notation in Unix, Awk or Perl.

Example: $0 \cup 1 \cup (0(0 \cup 1)^*0) \cup (1(0 \cup 1)^*1)$ denotes the set of non-empty binary strings that begin and end with the same symbol.

We can avoid excessive parentheses if we agree that the star binds tighter than concatenation, which in turn binds tighter than union.

Regular Expressions

Syntax: The regular expressions over an alphabet $\Sigma = \{a_1, \ldots, a_n\}$ is given by the grammar

$$
\begin{array}{lllllllllll}
regexp & \rightarrow & a_1 & \mid & \cdots & \mid & a_n & \mid & \epsilon & \mid & \emptyset \\
& \mid & regexp \cup regexp & & \mid & & regexp\ regexp & & \mid & & regexp^*
\end{array}
$$

Semantics:

$$
\begin{array}{lcl}
L(a) & = & \{a\} \\
L(\epsilon) & = & \{\epsilon\} \\
L(\emptyset) & = & \emptyset \\
L(R_1 \cup R_2) & = & L(R_1) \cup L(R_2) \\
L(R_1\ R_2) & = & L(R_1) \circ L(R_2) \\
L(R^*) & = & L(R)^*
\end{array}
$$

Regular Expressions – Examples

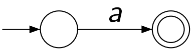
$$
\begin{array}{rcl}
110 & : & \{110\} \\
((0 \cup 1)(0 \cup 1))^* & : & \text{all binary strings of even length} \\
(0 \cup \epsilon)(\epsilon \cup 1) & : & \{\epsilon, 0, 1, 01\} \\
1^* & : & \text{all sequences of 1s} \\
\epsilon \cup 1 \cup (\epsilon \cup 1)^*(\epsilon \cup 1) & : & \text{all sequences of 1s}
\end{array}
$$

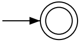
Regular Expressions vs Automata

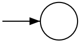
Theorem: L is regular iff L can be described by a regular expression.

Let us first show the 'if' direction, by showing how to convert a regular expression R into an NFA that recognises $L(R)$.

The proof is by structural induction over the form of R.

Case $R = a$: Construct 

Case $R = \epsilon$: Construct 

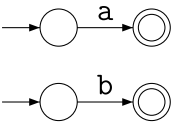
Case $R = \emptyset$: Construct 

Case $R = R_1 \cup R_2$, $R = R_1\ R_2$, or $R = R_1^*$:
We already gave the constructions when we showed that regular languages were closed under the regular operations.

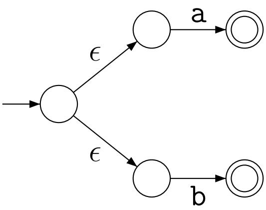
NFAs from Regular Expressions

Let us construct, in the proposed systematic way, an NFA for $(\mathtt{a} \cup \mathtt{b})^*\mathtt{bc}$.

Start from innermost expressions and work out:

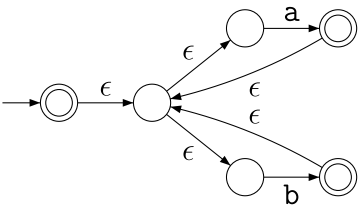


So $a \cup b$ yields:

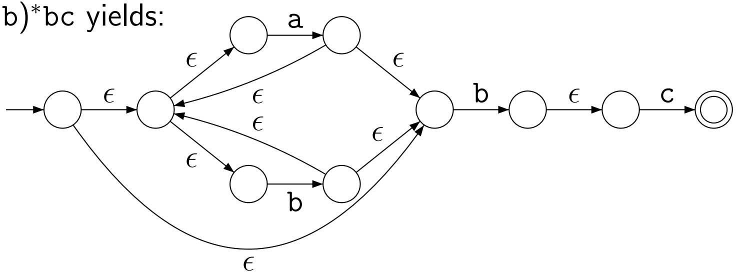


NFAs from Regular Expressions

Then $(\mathtt{a} \cup \mathtt{b})^*$ yields:



Finally $(\mathtt{a} \cup \mathtt{b})^*\mathtt{bc}$ yields:



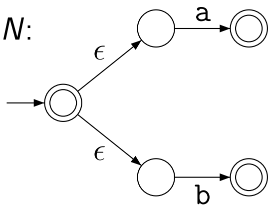
Exercise: Draw a simpler, equivalent NFA.

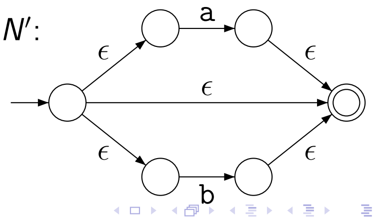
Regular Expressions from NFAs

We now show the 'only if' direction of the theorem.

Note that, given an NFA N, we can easily build an equivalent NFA with at most one accept state. We transform $N = (Q, \Sigma, \delta, q_0, F)$ to $N' = (Q \cup \{q_f\}, \Sigma, \delta', q_0, \{q_f\})$ by adding a new q_f, with ϵ transitions to q_f from each state in F. q_f becomes the only accept state:

$$\delta'(q, v) = \begin{cases} \delta(q, v) \cup \{q_f\} & \text{if } q \in F \text{ and } v = \epsilon \\ \delta(q, v) & \text{otherwise} \end{cases}$$





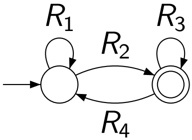
Regular Expressions from NFAs

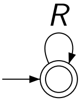
We sketch how an NFA can be turned into a regular expression in a systematic process of "state elimination".

In the process, arcs get labelled with regular expressions.

Start by making sure the NFA has a single accept state.

Repeatedly eliminate states that are neither start nor accept states.

The process produces either 

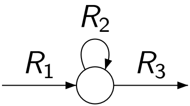
or 

We get $(R_1 \cup R_2 R_3^* R_4)^* R_2 R_3^*$ in the first case; R^* in the second.

Note that some Rs may be ϵ or $\emptyset$.

The State Elimination Process

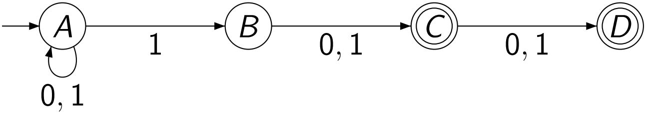
Consider a node



Any such pair of incoming/outgoing arcs get replaced by a single arc that bypasses the node. The new arc gets the label $R_1 R_2^* R_3$.

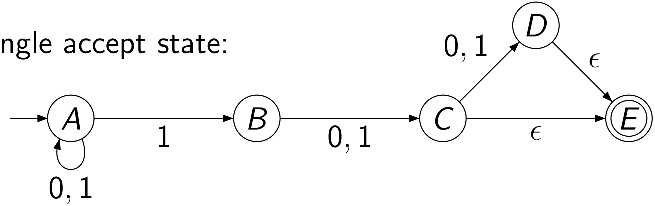
If there are m incoming and n outgoing arcs, these arcs are replaced by $m \times n$ bypassing arcs when the node is removed.

Let us illustrate the process on this example:

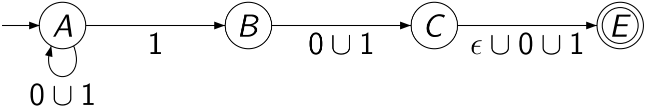


State Elimination Example

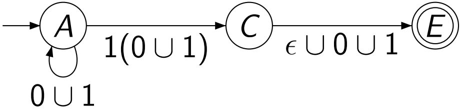
Create a single accept state:



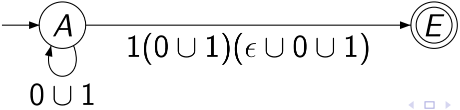
Eliminate D (and use regular expressions with all arcs):



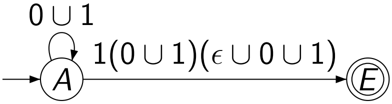
Now eliminate B:

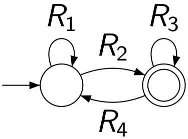


and then C:



State Elimination Example

Note that 

is of form 

with

- $R_1 = 0 \cup 1$
- $R_2 = 1(0 \cup 1)(\epsilon \cup 0 \cup 1)$
- $R_3 = R_4 = \emptyset$

Hence the instance of the general "recipe" $(R_1 \cup R_2 R_3^* R_4)^* R_2 R_3^*$ is

$$(0 \cup 1)^* 1(0 \cup 1)(\epsilon \cup 0 \cup 1)$$

Sipser (see "Readings Online" on the LMS) provides more details of this kind of translation.

Some Useful Laws for Regular Expressions

$$A \cup A = A$$

$$A \cup B = B \cup A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \circ B) \circ C = A \circ (B \circ C)$$

$$\emptyset \cup A = A \cup \emptyset = A$$

$$\epsilon \circ A = A \circ \epsilon = A$$

$$\emptyset \circ A = A \circ \emptyset = \emptyset$$

More Useful Laws for Regular Expressions

$$(A \cup B) \circ C = A \circ C \cup B \circ C$$

$$A \circ (B \cup C) = A \circ B \cup A \circ C$$

$$(A^*)^* = A^*$$

$$\emptyset^* = \epsilon^* = \epsilon$$

$$(\epsilon \cup A)^* = A^*$$

$$(A \cup B)^* = (A^* B^*)^*$$

Limitations of Finite-State Automata

Consider the language

$$\{0^n1^n \mid n \geq 0\} = \{\epsilon, 01, 0011, 000111, \ldots\}$$

Intuitively we cannot build a DFA to recognise this language, because a DFA has no memory of its actions so far.

Exercise: Is the language $\{0^n1^n \mid 0 \leq n \leq 999999999\}$ regular?

What about $\left\{ w \;\middle|\; \begin{array}{l} w \text{ has an equal number of occurrences} \\ \text{of the substrings 01 and 10} \end{array} \right\}$?

The Pumping Lemma for Regular Languages

This is our main tool for proving languages non-regular.

Loosely, it says that if we have a regular language A and consider a sufficiently long string $s \in A$, then a recogniser for A must traverse some loop to accept s. So A must contain infinitely many strings exhibiting repetition of some substring in s.

Pumping Lemma: If A is regular then there is a number p such that for any string $s \in A$ with $|s| \geq p$, s can be written as $s = xyz$, satisfying

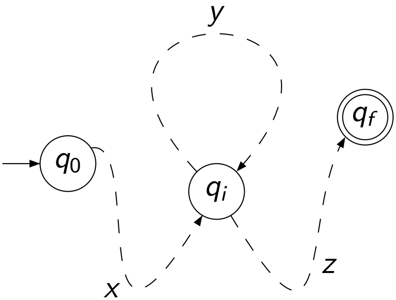
1. $xy^iz \in A$ for all $i \geq 0$
2. $y \neq \epsilon$
3. $|xy| \leq p$

We call p the pumping length.

Proving the Pumping Lemma

Let DFA $M = \{Q, \Sigma, \delta, q_0, F)$ recognise A. Let the number of states of M be p, let $|s| \geq p$.

In an accepting run for s, some state must be re-visited. Let q_i be the first such state. At the first visit, x has been consumed, at the second, xy, (strictly longer than x). This suggests a way of splitting s into x, y and z such that $xz, xyz, xyyz, \ldots$ are all in A.



Notice that $y \neq \epsilon$. Also, if input consumed has length k then the number of states visited is $k + 1$. As the first $p + 1$ states visited must contain a repetition, $|xy| \leq p$.

Using the Pumping Lemma

The pumping lemma says:

$$A \text{ regular} \Rightarrow \exists p \forall s \in A : \left\{ \begin{array}{l} s \text{ can be written} \\ xyz \text{ such that } \ldots \end{array} \right.$$

We can use its contrapositive to show that a language is non-regular:

$$\forall p \exists s \in A : \left\{ \begin{array}{l} s \text{ can't be written} \\ xyz \text{ such that } \ldots \end{array} \right\} \Rightarrow A \text{ not regular}$$

Coming up with such an s is sometimes easy, sometimes difficult.

Pumping Example 1

We show that $B = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Assume it is, and let p be the pumping length.

Consider $0^p 1^p \in B$ with length greater than p.

By the pumping lemma, $0^p 1^p = xyz$, with $xy^i z$ in B for all $i \geq 0$.

But y cannot consist of all 0s, since $xyyz$ then has more 0s than 1s.

Similarly y cannot consist of all 1s. And if y has at least one 0 and one 1, then some 1 comes before some 0 in $xyyz$.

So we inevitably arrive at a contradiction if we assume that B is regular.

Pumping Example 2

$C = \{w \mid w$ has an equal number of 0s and 1s$\}$ is not regular.

Assume it is, and let p be the pumping length.

Consider $0^p 1^p \in C$ with length greater than p.

By the pumping lemma, $0^p 1^p = xyz$, with xy^iz in C for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$. Since $|xy| \leq p$, y consists entirely of 0s.

But then $xyyz \notin C$, a contradiction.

A simpler alternative proof: If C were regular then also B from before would be regular, since $B = C \cap 0^*1^*$ and regular languages are closed under intersection.

Pumping Example 3

We show that $D = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Assume it is, and let p be the pumping length.

Consider $0^p10^p1 \in D$ with length greater than p.

By the pumping lemma, $0^p10^p1 = xyz$, with xy^iz in D for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$.

Since $|xy| \leq p$, y consists entirely of 0s.

But then $xyyz \notin D$, a contradiction.

Example 4 – Pumping Down

We show that $E = \{0^i 1^j \mid i > j\}$ is not regular.

Assume it is, and let p be the pumping length.

Consider $0^{p+1}1^p \in E$.

By the pumping lemma, $0^{p+1}1^p = xyz$, with xy^iz in E for all $i \geq 0$, $y \neq \epsilon$, and $|xy| \leq p$.

Since $|xy| \leq p$, y consists entirely of 0s.

But then $xz \notin E$, a contradiction.

Next Up

Context-free grammars and languages.