Let $T : V \to V$ be a linear transformation. If $\boldsymbol{v} \neq \mathbf{0}$ satisfies $T\boldsymbol{v} = \lambda\boldsymbol{v}$ then $\boldsymbol{v}$ is called an eigenvector of T and λ is the corresponding eigenvalue. This means that the subspace $\langle \boldsymbol{v} \rangle$ (a line through $\mathbf{0}$) is mapped into itself by T.

Let A be a square matrix. If $\boldsymbol{v} \neq \mathbf{0}$ satisfies $A\boldsymbol{v} = \lambda\boldsymbol{v}$ then $\boldsymbol{v}$ is called an eigenvector of A and λ is the corresponding eigenvalue.

Q1. Let

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

(i) By calculating $A\boldsymbol{v}$, verify that the following vectors $\boldsymbol{v}$ are eigenvectors of A and write down the corresponding eigenvalues:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

(ii) Explain why $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not an eigenvector of A.

The eigenvalues λ of A satisfy the characteristic equation $\det(A - \lambda I) = 0$, where I is the identity matrix. The corresponding eigenvectors are the non-zero vectors $\boldsymbol{v}$ satisfying $(A - \lambda I)\boldsymbol{v} = \mathbf{0}$.

Q2. Use the above theory to calculate the eigenvalues and eigenvectors of the matrix

$$\text{(i)} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \qquad \text{(ii)} \quad A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

An $n \times n$ matrix A with n linearly independent eigenvectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ can be diagonalized according to the formula

$$D = P^{-1}AP$$

where

$$P = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n], \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Diagonalization is possible if and only if A has n linearly independent eigenvectors.

Q3. Consider the matrix A of Q1. Give a reason why A is diagonalizable, and specify P and D in the diagonalization formula (for this you require the information of Q1(i), in particular the eigenvectors listed there and the fact that the corresponding eigenvalues are $2, 2, 1$).

Q4. Consider the matrix

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and corresponding eigenvectors for B. Is the matrix B diagonalizable? Explain geometrically why B only has one eigenvector.

All symmetric matrices $A = A^T$ are orthogonally diagonalizable, meaning that the eigenvectors can be chosen to be an orthonormal set $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\}$. With the matrix of eigenvectors now denoted $Q = [\boldsymbol{u}_1 \ \boldsymbol{u}_2 \cdots \boldsymbol{u}_n]$, then $QQ^T = Q^TQ = I$. This means the formula in the box below Q2 now reads

$$D = Q^T A Q$$

Q5. In Q2.(i) you found eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 3$, and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda = 1$. Show how these can be modified to obtain an orthonormal set $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

An application of the diagonalization formula is the computation of powers of the matrix A. We have $A^k = PD^kP^{-1}$. If λ_1 is the eigenvalue of greatest modulus, this formula implies

$$\lim_{k\to\infty} \lambda_1^{-k} A^k = PE_{1,1}P^{-1}$$

where $E_{1,1}$ has all entries zero except that in the top left corner.

Q6.(i) For the matrix A in Q2 compute A^7, and $\lim_{k\to\infty} 3^{-k}A^k$.

(ii) Again for the matrix A in Q2, explain why $A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be computed without using the formula in the box, and give its value.

The standard form of the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, while the standard form of the equation of a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
By rotating the axes formed by the standard basis in $\mathbb{R}^2$, to the axes formed by the orthogonal unit vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ with coordinates (u, v) so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

the equation $ax^2 + bxy + cy^2 = d$ can be reduced to standard form

$$\frac{u^2}{(d/\lambda_1)} + \frac{v^2}{(d/\lambda_2)} = 1,$$

and so in the coordinate system (u, v) be identified as an ellipse or hyperbola. Introducing the symmetric matrix

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

the new axes — called the principal axes — are in the directions of the normalized eigenvectors of A while λ_1, λ_2 are its eigenvalues.

Q7.(i) Sketch the ellipse $x^2 + \frac{1}{4}y^2 = \frac{1}{4}$, and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ by first writing in standard form.

(ii) Verify that the symmetric matrix $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$ has eigenvalues $\lambda = 3, -2$, with corresponding normalized eigenvectors $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(iii) Use (ii) to identify the conic

$$-x^2 + 4xy + 2y^2 = 3,$$

and specify its principal axes.