COMP30026 Models of Computation

Propositional Logic

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Our Aim

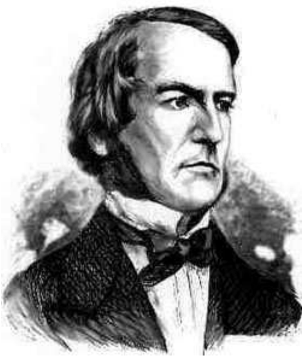
- Introduce/recapitulate propositional logic
- Use it as a vehicle for launching more generally applicable logic concepts.
- Use it for simple, mechanised reasoning.

The coverage of propositional logic serves as a blueprint for similar (but more complex and powerful) methods using predicate logic.

Propositional = Boolean Logic

Philosophers have been interested in the “rules of reasoning” for thousands of years. Aristotle’s syllogisms had particular importance for European scholars.

George Boole is usually considered the father of modern logic. Boole took an algebraic view of logic, pointing out that there are important abstract analogies between certain arithmetic operations and the logical connectives.



(Classical) Propositional Logic: Syntax

We shall build propositional formulas from this set of symbols:

$$\underbrace{A, B, C, \ldots Z,}_{\text{prop. letters}} \underbrace{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus,}_{\text{connectives}} \mathbf{f}, \mathbf{t}, (,).$$

Well-formed formulas (wffs) are generated by the grammar

$$\begin{array}{lcl} \mathit{wff} & \rightarrow & A \mid B \mid C \mid \ldots \mid Z \mid \mathbf{f} \mid \mathbf{t} \\ & \mid & (\neg \mathit{wff}) \\ & \mid & (\mathit{wff} \wedge \mathit{wff}) \\ & \mid & (\mathit{wff} \vee \mathit{wff}) \\ & \mid & (\mathit{wff} \Rightarrow \mathit{wff}) \\ & \mid & (\mathit{wff} \Leftrightarrow \mathit{wff}) \\ & \mid & (\mathit{wff} \oplus \mathit{wff}) \end{array}$$

Propositional Logic: Notational Conveniences

We shall drop outermost parentheses.

We shall assume that $\neg$ binds tighter than $\wedge$ and $\vee$.

These bind tighter than $\oplus$, which binds tighter than $\Rightarrow$ and $\Leftrightarrow$.

This allows us to write, without ambiguity

$$((P \wedge (\neg Q)) \Rightarrow (P \vee (P \Leftrightarrow Q)))$$

as

$$P \wedge \neg Q \Rightarrow P \vee (P \Leftrightarrow Q)$$

Note: O'Donnell et al. (and Makinson) use $\rightarrow$ instead of $\Rightarrow$, and $\leftrightarrow$ instead of $\Leftrightarrow$. Makinson also uses 0 for **f** and 1 for **t**. On the whiteboard I often use 0 and 1 too, as they are faster to write.

Propositional Logic: Semantics

A proposition is false (**f**) or true (**t**).

A truth assignment maps each propositional letter to **t** or **f**.

We can give the semantics of the connectives via truth tables:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$	$A \oplus B$
f	**f**	**t**	**f**	**f**	**t**	**t**	**f**
f	**t**	**t**	**f**	**t**	**t**	**f**	**t**
t	**f**	**f**	**f**	**t**	**f**	**f**	**t**
t	**t**	**f**	**t**	**t**	**t**	**t**	**f**

This gives meaning to all propositional formulas, as we let A and B stand for the values of arbitrary (compound) propositions.

Sidebar: Connectives Defined in Haskell

Haskell has a type `Bool`, and some connectives are pre-defined:

```
data Bool = False | True

not :: Bool -> Bool
not True  = False
not False = True

(&&) :: Bool -> Bool -> Bool
False && _ = False
True  && x = x

(||) : Bool -> Bool -> Bool
False || x = x
True  || _ = True
```

Conjunction and Disjunction

$P \wedge Q$ is the conjunction of P and Q.

$P \vee Q$ is their disjunction.

An "or" in English sometimes translates to disjunction:

I'll eat if there is peanut butter or jam in the fridge.

Other times it translates to exclusive or:

Would you like the fish or the chicken?

Implication

The proposition $P \Rightarrow Q$ is best read "if P then Q" (or sometimes "P only if Q" or "Q whenever P"). Usually, "implies" is misleading.

A	B	$A \Rightarrow B$
f	**f**	**t**
f	**t**	**t**
t	**f**	**f**
t	**t**	**t**

1. If the volume is increased, the pressure falls.
2. If Melbourne is in Queensland then Brisbane is in Victoria.
3. Melbourne and Brisbane are in different states and if Melbourne is in Queensland then so is Brisbane.

We talk about material implication.

Note that $A \Rightarrow B$ has the same truth table as $\neg A \vee B$.

Sidebar: More Connectives in Haskell

```haskell
infix 1 ==>
infix 1 <=>
infix 2 <+>

(==>) :: Bool -> Bool -> Bool
False ==> _ = True
True  ==> x = x

(<=>) :: Bool -> Bool -> Bool
x <=> y  =  x == y

(<+>) :: Bool -> Bool -> Bool
x <+> y  =  x /= y
```

Quiz

Which of these claims hold?

1. $P \Rightarrow Q$ has the same truth table as $\neg Q \Rightarrow \neg P$
2. $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ has the same truth table as $P \Rightarrow (Q \wedge R)$
3. $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ has the same truth table as $(P \wedge Q) \Rightarrow R$

Other Binary Connectives

We can also define $\downarrow$, or "nor', as well as $\uparrow$, or "nand".

A	B	$A \downarrow B$	$A \uparrow B$
f	**f**	**t**	**t**
f	**t**	**f**	**t**
t	**f**	**f**	**t**
t	**t**	**f**	**f**

"Nand" is sometimes called Sheffer's stroke.

Some Ternary Connectives

A	B	C	if A then B else C	$median(A, B, C)$
f	**f**	**f**	**f**	**f**
f	**f**	**t**	**t**	**f**
f	**t**	**f**	**f**	**f**
f	**t**	**t**	**t**	**t**
t	**f**	**f**	**f**	**f**
t	**f**	**t**	**f**	**t**
t	**t**	**f**	**t**	**t**
t	**t**	**t**	**t**	**t**

On Boolean Short-Circuit Definitions

Most programming languages offer the Boolean connectives ‘and’ and ‘or’, but usually these are not commutative!

In C, Haskell, and many other languages, `0 == 1 && 1/0 == 42` has a behaviour that is different from `1/0 == 42 && 0 == 1`. One evaluates to ‘false’, the other causes a run-time error. The first version avoids the runtime error, because conjunction is not a strict function in typical programming languages: If the first argument is false, the second won’t be evaluated.

To model the behaviour properly, we really need three-valued propositional logic.

Exit Puzzle

On the island of Knights and Knaves, everyone is a knight or knave. Knights always tell the truth. Knaves always lie.

On the 1st of August there is a census on the island!

You are a census taker, going from house to house. Fill in what you know about each of these three houses.

- **In house 1:** Husband: We are both knaves.
- **In house 2:** Wife: At least one of us is a knave.
- **In house 3:** Husband: If I am a knight then so is my wife.

Next Up

Next up: We introduce some important logical concepts and think about how to automate simple deduction.