Tutorial 4: Solutions

Q1. (i). $\mathbf{a} + 2\mathbf{c} = (3, 1, -2) + 2(0, 1, -3) = (3, 3, -8)$

(ii). $\mathbf{b} - \mathbf{c} = (2, 0, 1) - (0, 1, -3) = (2, -1, 4)$

(iii). $\sqrt{2}\mathbf{d} = \sqrt{2}(\frac{1}{\sqrt{2}}, 0, -\frac{2}{\sqrt{2}}) = (1, 0, -2)$

(iv). $d(\mathbf{b}, \mathbf{c}) = ||\mathbf{b} - \mathbf{c}|| = ||(2, -1, 4)|| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$

(v). $||\mathbf{a}|| + ||\mathbf{b}|| = ||(3, 1, -2)|| + ||(2, 0, 1)|| = \sqrt{3^2 + 1^2 + (-2)^2} + \sqrt{2^2 + 1^2} = \sqrt{14} + \sqrt{5}$

(vi). $\mathbf{b} \cdot \mathbf{d} = (2, 0, 1) \cdot (\frac{1}{\sqrt{2}}, 0, -\frac{2}{\sqrt{2}}) = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$

Q2. (i). $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - (1, 0) = \left(\frac{1-\sqrt{2}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(ii). $\overrightarrow{OD}$ is the same vector as $\overrightarrow{OB}$ except for sign of x-component $= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(iii). $\overrightarrow{OE}$ is the same vector as $\overrightarrow{OA}$ except for sign of x-component $= (-1, 0)$ so $\overrightarrow{OE} \cdot \overrightarrow{OB} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot (-1, 0) = -\frac{1}{\sqrt{2}}$

(iv). $\overrightarrow{OE} \cdot \overrightarrow{OB} = ||\overrightarrow{OE}||||\overrightarrow{OB}|| \cos\theta$ since $||\overrightarrow{OE}|| = ||\overrightarrow{OE}|| = 1$ then $\cos\theta = \overrightarrow{OE} \cdot \overrightarrow{OB} = -\frac{1}{\sqrt{2}}$. Thus $\cos\theta = \frac{3\pi}{4}$.

(v).

$$\text{proj}_{\overrightarrow{OB}}\overrightarrow{OC} = \frac{\overrightarrow{OB} \cdot \overrightarrow{OC}}{||\overrightarrow{OB}||^2}\overrightarrow{OB} = \frac{\left(\frac{1}{\sqrt{2}} \times 0\right) + \left(\frac{1}{\sqrt{2}} \times 1\right)}{1^2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

(vi). $\overrightarrow{OG} \cdot \overrightarrow{OA} = (0, -1) \cdot (1, 0)$

Q3. (i). $\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} = (4, -6, -6)$

(ii). $\mathbf{c} \times \mathbf{a} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -8 & 4 \\ 3 & 4 & -2 \end{bmatrix} = (0, 0, 0)$

(iii). Area$= ||\mathbf{a} \times \mathbf{d}|| = ||\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}|| = ||(4, -3, 0)|| = 5$

(iv). Area$= \frac{1}{2}||\mathbf{a} \times \mathbf{b}|| = \frac{1}{2}||(4, -6, -6)|| = 2\sqrt{22}$

Q4. (i). $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{bmatrix} 3 & 4 & -2 \\ 0 & -2 & 2 \\ -6 & -8 & 4 \end{bmatrix} = 3\begin{vmatrix} -2 & 2 \\ -8 & 4 \end{vmatrix} - 0 + (-6)\begin{vmatrix} 4 & -2 \\ -2 & 2 \end{vmatrix} = 24 - 24 = 0$

(ii). not possible

(iii). $\mathbf{a}\cdot\mathbf{b}\times\mathbf{d} = \det\begin{bmatrix} 3 & 4 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = 3\times(-2)\times 1 = -6$

(iv). $\mathbf{d}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{d}\times(4,-6,-6) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 4 & -6 & -6 \end{bmatrix} = (6,4,0)$

(v). $|\mathbf{a}\cdot\mathbf{b}\times\mathbf{d}| = |-6| = 6.$

Q5. (i). From the cartesian equation

$$\frac{x+1}{3} = y+2 = \frac{y+2}{1} = \frac{z-1}{4}$$

we can read the direction as $(3,1,4)$.

(ii). We find

$$\frac{x+1}{3} = \frac{-1+1}{3} = 0, \quad y+2 = -2+2 = 0 \text{ and } \frac{z-1}{4} = \frac{1-1}{4} = 0$$

as these are all the same then $(-1,-2,1)$ lies on the line.

(iii). Since $(-1,-2,1)$ lies on the line and using the direction from (i) we have

$$\mathbf{r} = (-1,-2,1) + t(3,1,4),\ t\in\mathbb{R}$$

Q6. (i). The vector form is $\mathbf{r} = (1,0,0) + t(2,-1,-3),\ t\in\mathbb{R}$. By equating the components of the vectors we have the parametric form $x = 1+2t,\ y = -t,\ z = -3t$. The cartesian form is

$$\frac{x-1}{2} = \frac{y-0}{-1} = \frac{z-0}{-3} \quad \text{or} \quad \frac{x-1}{2} = -y = -\frac{z}{3}$$

(ii). The line has direction $(1,0,-2) - (0,0,-1) = (1,0,-1)$.
So

- the vector form is $\mathbf{r} = (0,0,-1) + t(1,0,-1),\ \text{t}\in\mathbb{R}$,

- the parametric form is $x = t,\ y = 0,\ z = -1-t,\ t\in\mathbb{R}$ and
- the cartesian form is

$$\frac{x}{1} = \frac{z+1}{-1},\ y = 0 \quad \text{or} \quad x = -z-1,\ y = 0.$$