MAST10007 Assignment 3 - Solutions

1 (a) The vector equation is equivalent to the matrix equation

$$\begin{bmatrix} \underset{\sim}{x}_1 & \underset{\sim}{x}_2 & \underset{\sim}{x}_3 & \underset{\sim}{x}_4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1^{st} column

or more explicitly

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 2 & -1 & 1 & -1 \\ -1 & -1 & 1 & -4 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As an augmented matrix, this reads

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 2 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & -4 & 0 \\ 1 & -1 & 0 & -1 & 0 \end{array}\right]$$

(b)

$$\begin{bmatrix} 1 & -1 & 1 & -2 \\ 2 & -1 & 1 & -1 \\ -1 & -1 & 1 & -4 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & 2 & -6 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 2R_2 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} R_3 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) From the answer to (b) we have that

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 2 & -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & -4 & 0 \\ 1 & -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There is no leading entry for α_4. Hence we set

$$\alpha_4 = t, \qquad t \in \mathbb{R}$$

Proceeding now by back substitution gives

$$\alpha_3 = t$$

$$\alpha_2 = \alpha_3 - 3\alpha_4 = -2t$$

$$\alpha_1 = \alpha_2 - \alpha_3 + 2\alpha_4 = -t$$

Hence the general solution is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-t, -2t, t, t) = t(-1, -2, 1, 1) \qquad t \in \mathbb{R}$$

(d) The row echelon form of $[\underset{\sim}{x}_1\ \underset{\sim}{x}_3\ \underset{\sim}{x}_4]$ is the same as the row echelon form of $[\underset{\sim}{x}_1\ \underset{\sim}{x}_2\ \underset{\sim}{x}_3\ \underset{\sim}{x}_4]$ with the second column deleted. Hence

$$[\underset{\sim}{x}_1\ \underset{\sim}{x}_3\ \underset{\sim}{x}_4] \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of this matrix is 3 so the vectors are linearly independent.

2. (a) $R = \{(x, y, z) : 2x + z = 0\}$

This can be rewritten

$$R = \{(x, y, -2x) : x, y \in \mathbb{R}\}$$

We suspect that this is a subspace since it is equal to $\mathrm{Span}\{(1,0,-2), (0,1,0)\}$.

We are asked to check this using the definition of a subspace:

(0) Since $(0,0,0) \in R$, we see that R is nonempty.

(1) Let $\underset{\sim}{u} = (x_1, y_1, -2x_1)$ and $\underset{\sim}{v} = (x_2, y_2, -2x_2)$ be two general vectors in R. Then

$$\begin{aligned} \underset{\sim}{u} + \underset{\sim}{v} &= (x_1, y_1, -2x_1) + (x_2, y_2, -2x_2) \\ &= ((x_1 + x_2), y_1 + y_2, -2(x_1 + x_2)) \\ &= (x_3, y_3, -2x_3) \quad \text{with } x_1 + x_2 = x_3 \in \mathbb{R} \\ &\in R \qquad\qquad\qquad\quad\; y_1 + y_2 = y_3 \in \mathbb{R} \end{aligned}$$

Hence R is closed under vector addition

(2) Let $\underset{\sim}{u} = (x_1, y_1, -2x_1)$ be a general vector in R, and let α be a general scalar. We have that

$$\begin{aligned} \alpha \underset{\sim}{u} &= \alpha (x_1, y_1, -2x_1) \\ &= (\alpha x_1, \alpha y_1, -2\alpha x_1) \\ &= (x', y', -2x') \quad \text{with } \alpha x_1 = x' \in \mathbb{R} \\ &\in R \qquad\qquad\qquad\; \alpha y_1 = y' \in \mathbb{R} \end{aligned}$$

Hence R is closed under scalar multiplication

All 3 defining properties of a subspace hold true, so R is a subspace.

2 (b) $S = \{(x, y, z) : \frac{x}{2} = y + 1 = 3z\}$

We recognise this as the equation of a line not passing through the origin. We know that only lines passing through the origin are subspaces, so we suspect S is not a subspace. To prove this from the definition of a subspace, we see that

$$(0, -1, 0) \in S.$$

Also $\alpha = 0$ is a scalar. We have that

$$\alpha(0, -1, 0) = 0(0, -1, 0) = (0, 0, 0) \notin S$$

Hence S is not closed under scalar multiplication

3 Vector form: $(x, y, z) = s(0, 1, 1) + t(1, 0, -2)$ with $s, t \in \mathbb{R}$

Parametric form: Equating components

$$x = t$$
$$y = s$$
$$z = s - 2t$$

Cartesian form: Substituting for s and t in the equation for z in the parametric form gives

$$z = y - 2x \Rightarrow 2x - y + z = 0$$

Alternatively: $\underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$$= -2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$$

Equation of the plane: $(x, y, z) \cdot \underset{\sim}{n} = \underset{\sim}{0}$

$\Rightarrow (x, y, z) \cdot (-2, 1, -1) = 0 \quad \Rightarrow \quad -2x + y - z = 0$

$\Rightarrow \quad 2x - y + z = 0$ as before.