Tutorial 11: Solutions

Q1. (i). Calculating $A\mathbf{v}_i$ for each vector we have

$$A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3&0&-1\\0&2&0\\2&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}=2\begin{bmatrix}0\\1\\0\end{bmatrix}\quad\Rightarrow\quad\lambda=2$$

$$A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}3&0&-1\\0&2&0\\2&0&0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}=2\begin{bmatrix}1\\0\\1\end{bmatrix}\quad\Rightarrow\quad\lambda=2$$

$$A\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}3&0&-1\\0&2&0\\2&0&0\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}1\\0\\2\end{bmatrix}\quad\Rightarrow\quad\lambda=1$$

(ii).

$$A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}3&0&-1\\0&2&0\\2&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}3\\0\\2\end{bmatrix}\neq\lambda\begin{bmatrix}1\\0\\0\end{bmatrix}$$

for any λ. Hence $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not an eigenvector of A.

Q2. (i). Eigenvalues are given by

$$\begin{bmatrix}2-\lambda&1\\1&2-\lambda\end{bmatrix}=(2-\lambda)^2-1=(1-\lambda)(3-\lambda)=0$$

so $\lambda=1$ or $\lambda=3$.
For $\lambda=1$: $(A-\lambda I)\mathbf{v}=\mathbf{0}$ is equivalent to the augmented matrix

$$\left[\begin{array}{cc|c}1&1&0\\1&1&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right].$$

Since there is no leading entry for v_2 we set $v_2=t$ and then $v_1=-t$. The solution space is therefore $t\begin{bmatrix}-1\\1\end{bmatrix}$. Taking $t=1$ gives $\begin{bmatrix}-1\\1\end{bmatrix}$ for the corresponding eigenvector.
For $\lambda=3$: $(A-\lambda I)\mathbf{v}=\mathbf{0}$ is equivalent to the augmented matrix

$$\left[\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right]\sim\left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right].$$

Since there is no leading entry for v_2 we set $v_2=t$ and then $v_1=t$. The solution space is therefore $t\begin{bmatrix}1\\1\end{bmatrix}$. Taking $t=1$ gives $\begin{bmatrix}1\\1\end{bmatrix}$ for the corresponding eigenvector.

(ii). Eigenvalues are given by

$$\begin{bmatrix}2-\lambda&1\\4&2-\lambda\end{bmatrix}=(2-\lambda)^2-4=-\lambda(4-\lambda)=0$$

so $\lambda=0$ or $\lambda=4$.
For $\lambda=0$: $(A-\lambda I)\mathbf{v}=\mathbf{0}$ is equivalent to the augmented matrix

$$\left[\begin{array}{cc|c}2&1&0\\4&2&0\end{array}\right]\sim\left[\begin{array}{cc|c}2&1&0\\0&0&0\end{array}\right].$$

Since there is no leading entry for v_2 we set $v_2 = t$ and then $v_1 = -\frac{1}{2}t$. The solution space is therefore $t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$. Taking $t = 2$ gives $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ for the corresponding eigenvector.
For $\lambda = 4$: $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is no leading entry for v_2 we set $v_2 = t$ and then $v_1 = \frac{1}{2}t$. The solution space is therefore $t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$. Taking $t = 2$ gives $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for the corresponding eigenvector.

Q3. The matrix of eigenvectors for A in Q1 is

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

the rank of P is 3 therefore the eigenvectors are linearly independent. Hence A is diagonalizable and

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q4. The eigenvalues are given by

$$\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0$$

so $\lambda = 1$.
For $\lambda = 1$: $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Since there is no leading entry for v_1 we set $v_1 = t$ while $v_2 = 0$. Setting $t = 1$ gives the eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
There only being one eigenvector, the matrix is not diagonalizable. Geometrically, B corresponds to a shear by 1 unit in the x-direction. The only direction left unchanged is the x-axis, explaining why there is only one eigenvector.

Q5. First we note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ so these vectors are orthogonal and we just need to normalize each vector.

$$||\mathbf{v_1}|| = \sqrt{2} \quad \Rightarrow \quad \mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,1), \quad ||\mathbf{v_2}|| = \sqrt{2} \quad \Rightarrow \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1,-1)$$

Q6. (i). Using Q2, we have $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Hence

$$A^7 = PD^7P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^7 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3^7 & -1 \\ 3^7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3^7+1 & 3^7-1 \\ 3^7-1 & 3^7+1 \end{bmatrix}.$$

Also

$$\lim_{k\to\infty} 3^{-k}A^k = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(ii). Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 3$, we have $A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Q7. (i). The second equation is already in standard form. The standard form of the first equation is obtained by dividing both sides by 1/4 to get

$$\frac{x^2}{(1/4)} + y^2 = 1 \quad \text{or equivalently} \quad \frac{x^2}{(1/2)^2} + y^2 = 1.$$

In relation to the second equation, asymptotes occur when $x^2/9 - y^2/4 = 0$ and thus $y = \pm(2/3)x$. Intersection with the x-axis occurs when $y = 0$ and thus $x = \pm 3$.

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(ii). We see that $||\frac{1}{\sqrt{5}}(1,2)|| = \frac{1}{\sqrt{5}}||(1,2)|| = 1$ and $||\frac{1}{\sqrt{5}}(-2,1)|| = \frac{1}{\sqrt{5}}||(-2,1)|| = 1$ so the vectors are normalized. These are eigenvectors since

$$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

(iii). According to the theory, introducing coordinates (u, v) by the formula

$$\begin{bmatrix} x \\ y \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

we obtain

$$3u^2 - 2v^2 = 3 \quad \text{or equivalently} \quad u^2 - \frac{v^2}{(3/2)} = 1.$$

This is a hyperbola, with principal axes in the directions of $\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.