School of Computing and Information Systems
COMP30026 Models of Computation Tutorial Week 6

28 August – 1 September 2017

Plan

There is quite a bit to do this week. Try to get through it all, even though we won't cover all in the tute. If you look to skip something, Questions 41 and 42 are just more of the same when you have done 38. Question 44 is large, but it is a good test of your logical fitness; have you got the stamina for it? Question 45 is not actually a question; it is large and non-trivial example of using resolution. Check it and make sure you understand the details.

Some reading material on resolution theorem proving is available (see "Readings Online" on the LMS). Note, however, that Dowsing, Rayward-Smith and Walter use a different unification method. In Lecture 9 we introduced unification as a process of solving term equations—a view that is both simpler and more abstract than the view taken by Dowsing *et al.*

The exercises

37. Determine whether $\neg\forall x\exists y\,(\neg P(x) \wedge P(y))$ is valid and/or satisfiable. Convert the formula to clausal form.

38. For each of the following pairs of terms, determine whether the pair is unifiable. If it is, give the most general unifier.

    (a) $(h(f(x), g(y, f(x)), y), h(f(u), g(v, v), u))$
    (b) $(h(f(g(x, y)), y, g(y, y)), h(f(u), g(a, v), u))$
    (c) $(h(g(x, x), g(y, z), g(y, f(z))), h(g(u, v), g(v, u), v))$
    (d) $(h(v, g(v), f(u, a)), h(g(x), y, x))$
    (e) $(h(f(x, x), y, y, x), h(v, v, f(a, b), a))$

    (Never forget our usual convention: for constants we use letters from the beginning of the alphabet, here a and b, whereas for variables we use letters from the end of the alphabet.)

39. Consider the two statements

    S_1: "No politician is honest."
    S_2: "Some politicians are not honest."

    (a) Using the predicate symbols P and H for being a politician and being honest, respectively, express the two statements as first order predicate formulas F_1 and F_2.
    (b) Is $F_1 \Rightarrow F_2$ satisfiable?
    (c) Is $F_1 \Rightarrow F_2$ valid?
    (d) Consider the two statements

    S_3: "No Australian politician is honest."
    S_4: "All honest politicians are Australian."

    Using the predicate symbol A for "is Australian", express S_3 and S_4 in clausal form.
    (e) Using resolution, show that S_1 is a logical consequence of S_3 and S_4.
    (f) Prove or disprove the statement "S_2 is a logical consequence of S_3 and S_4."

40. Consider the following unsatisfiable set of clauses:

$$\{\{P(x)\}, \{\neg P(x), \neg Q(y)\}, \{Q(x), \neg R(y)\}, \{R(x), S(a)\}, \{R(b), \neg S(x)\}\}$$

What is the simplest refutation proof, if "simplest" means "the refutation tree has minimal depth"? What is the simplest refutation proof, if "simplest" means "the refutation tree has fewest nodes"?

41. Using the unification algorithm, determine whether $Q(f(g(x), y, f(y, z, z)), g(f(a, y, z)))$ and $Q(f(u, g(a), v), u)$ are unifiable. If they are, give a most general unifier. (As usual, we use letters from the end of the alphabet for variables, and letters from the beginning of the alphabet for constants.)

42. Determine whether $P(f(g(x), f(g(x), g(a))), x)$ and $P(f(u, f(v, v)), u)$ are unifiable. If they are, give a most general unifier.

43. Consider the following predicates:

   - $E(x, y)$, which stands for "x envies y"
   - $F(x, y)$, which stands for "x is more fortunate than y"

   (a) Using 'a' for Adam, express, in first-order predicate logic, the sentence "Adam envies everyone more fortunate than him."

   (b) Using 'e' for Eve, express, in first-order predicate logic, the sentence "Eve is no more fortunate than any who envy her."

   (c) Formalise an argument for the conclusion that "Eve is no more fortunate than Adam." That is, express this statement in first-order predicate logic and show that it is a logical consequence of the other two.

44. For this question use the following predicates:

   - $G(x)$ for "x is a green dragon"
   - $R(x)$ for "x is a red dragon"
   - $H(x)$ for "x is a happy dragon"
   - $S(x)$ for "x is a dragon capable of spitting fire"
   - $P(x, y)$ for "x is a parent of y"
   - $C(x, y)$ for "x is a child of y"

   (a) Express the following statements as formulas in first-order predicate logic:

      i. x is a parent of y if and only if y is a child of x.
      ii. A dragon is either green or red; not both.
      iii. A dragon is green if and only if at least one of its parents is green.
      iv. Green dragons can spit fire.
      v. A dragon is happy if all of its children can spit fire.

   (b) Translate each of the five formulas to clausal form.

   (c) Prove, using resolution, that all green dragons are happy.

45. Work through the following more substantial resolution example in your own time and make sure that you understand each step. You only need to discuss this in a tutorial or the LMS Discussion Forum if there are steps you don't understand.

Dowsing, Rayward-Smith and Walter (see **Readings Online**) give the following example of a non-trivial proof using resolution. It is concerned with group theory. A *group* is a set endowed with a binary operation $\circ$. If we use $P(x, y, z)$ to mean $x \circ y = z$ then we can write the so-called *group axioms* as follows:

$$\begin{array}{lr}
\forall x \forall y \exists z (P(x, y, z)) & \text{(closure)} \\
\forall x, y, z, u, v, w\, ([P(x, y, u) \wedge P(y, z, v)] \Rightarrow [P(x, v, w) \Leftrightarrow P(u, z, w)]) & \text{(associativity)} \\
\exists x \forall y (P(x, y, y) \wedge \exists z (P(z, y, x))) & \text{(left identity} \\
& \text{and left inverse)}
\end{array}$$

Notice that the associativity axiom says that if $x \circ y = u$ and $y \circ z = v$ then $x \circ v = u \circ z$. In other words, $x \circ (y \circ z) = (x \circ y) \circ z$. The last axiom says that there is some special element x in the set, with the property that $x \circ y = y$ for all y (that is, this element is a *neutral element* for $\circ$). Moreover, for each y there is a z such that $z \circ y$ yields that special element (in other words, each element has a *left inverse*). For an example of a group, consider the set $\mathbb{Z}$ of integers, endowed with the operation $+$.

We can translate the group axioms to clausal form. The first axiom (closure) becomes

$$\{P(x, y, f(x, y))\}$$

The second axiom (associativity) produces two clauses:

$$\begin{array}{c}
\{\neg P(x, y, u), \neg P(y, z, v), \neg P(x, v, w), P(u, z, w)\} \\
\{\neg P(x, y, u), \neg P(y, z, v), \neg P(u, z, w), P(x, v, w)\}
\end{array}$$

The last axiom (left identity and left inverse) also produces two clauses:

$$\begin{array}{l}
\{P(a, y, y)\} \\
\{P(g(y), y, a)\}
\end{array}$$

Suppose we want to prove that every element of a group also has a right inverse. That is, we want to prove

$$\exists x \forall y \exists z (P(y, z, x))$$

from the axioms. To do this we first negate our formula, obtaining:

$$\forall x \exists y \forall z (\neg P(y, z, x))$$

In clausal form this becomes $\{\neg P(h(x), z, x)\}$. On the next page is a proof by resolution. It is a mechanical proof of a non-trivial theorem. When there is ambiguity, I have used underlining to show which atom takes part in the resolution step.

Make sure you understand each resolution step. Did the refutation make use of all the axioms? If you try to do the proof on your own without looking at the proof above, you will find that there are many blind alleys (most of them will end in failure due to the occur check). So you will most likely take a long time, and do lots of back-tracking. With a computer of course we find the refutation in a flash.

Notice how clauses have had their variables renamed to avoid name clashes. Try to track how the variables x' and z' from the original query get bound during this proof.

$$\neg P(x,y,u), \neg P(y,z,v), \neg P(x,v,w), P(u,z,w) \qquad \neg P(h(x'),z',x')$$

$$u = h(x') \quad w = x' \quad z = z'$$

$$\neg P(x,y,h(x')), \neg P(y,z',v), \underline{\neg P(x,v,x')} \qquad P(g(u),u,a)$$

$$x = g(u) \quad v = u \quad x' = a$$

$$\neg P(g(u),y,h(a)), \neg P(y,z',u) \qquad P(a,u',u')$$

$$y = a \quad z' = u' \quad u = u'$$

$$\neg P(x_2,y_2,u_2), \neg P(y_2,z_2,v_2), \neg P(u_2,z_2,w_2), P(x_2,v_2,w_2) \qquad \neg P(g(u'),a,h(a))$$

$$x_2 = g(u') \quad v_2 = a \quad w_2 = h(a)$$

$$\neg P(g(u'),y_2,u_2), \neg P(y_2,z_2,a), \underline{\neg P(u_2,z_2,h(a))} \qquad P(a,v',v')$$

$$u_2 = a \quad v' = h(a) \quad z_2 = h(a)$$

$$\underline{\neg P(g(u'),y_2,a)}, \neg P(y_2,h(a),a) \qquad P(g(u_3),u_3,a)$$

$$u' = u_3 \quad y_2 = u_3$$

$$\neg P(u_3,h(a),a) \qquad P(g(u_4),u_4,a)$$

$$u_3 = g(h(a)) \quad u_4 = h(a)$$

$$\perp$$