PHYC90045 Introduction to Quantum Computing

Lab Session 8

8.1 Introduction

Welcome to Lab-8 of PHYC90045 Introduction to Quantum Computing. The purpose of this lab session is to get some experience with quantum error correction (QEC).

8.2 Quantum Error Correction preliminaries

As you saw in lectures classical concepts of error correction need to be adapted to the quantum framework. The first thing we will explore is the notion of commuting and anti-commuting operators on states, and the products of operators that "stabilise" certain states.

Exercise 8.2.1 Operators and stablisers.

a) In the QUI we will create an "arbitrary" 1-qubit superposition state $|\psi\rangle$ to experiment on – I used $R_x(0.345\pi)$ with global phase set at 0.5π. Show by inspecting the states in the QUI that the following operator commutation properties hold on this state (or additionally any other state by doing some maths for the general case):

$$XZ\,|\psi\rangle = -ZX\,|\psi\rangle \rightarrow XZ = -ZX$$
$$XY\,|\psi\rangle = -YX\,|\psi\rangle \rightarrow XY = -YX$$
$$ZY\,|\psi\rangle = -YZ\,|\psi\rangle \rightarrow ZY = -YZ$$

Beware of the trap: when write $XZ|\psi\rangle$ we mean the Z operator is applied first followed by the X operator, so in the QUI time ordering left to right it's programmed as –Z–X– etc.

b) In the QUI create an "arbitrary" 2-qubit superposition state $|\psi\rangle$ to experiment on (e.g. using some R-gates and CNOT). Show that the following operator commutation property holds:

$$X_1X_2Z_1Z_2\,|\psi\rangle = Z_1Z_2X_1X_2\,|\psi\rangle \rightarrow X_1X_2Z_1Z_2 = Z_1Z_2X_1X_2$$

Again, beware the operator ordering trap.

c) Show that the states $|000\rangle$ and $|111\rangle$ (the 3-bit code) are stabilised as:

$$Z_1Z_2\,|000\rangle = |000\rangle \qquad Z_1Z_2\,|111\rangle = |111\rangle$$
$$Z_2Z_3\,|000\rangle = |000\rangle \qquad Z_2Z_3\,|111\rangle = |111\rangle$$

d) Show using an example in the QUI that an arbitrarily chosen linear combination of states $|000\rangle$ and $|111\rangle$ is stabilised by the operator products Z_1Z_2 and Z_2Z_3 (the "stabilisers"). Hint: to create the states, think back to the GHZ states encountered in Lab-7.

Exercise 8.2.2 In the QUI create a non-trivial linear combination of |000⟩ and |111⟩. Apply an "X-error" on the first qubit and determine the "syndrome" for each of the stabilisers products Z_1Z_2 and Z_2Z_3 as per:

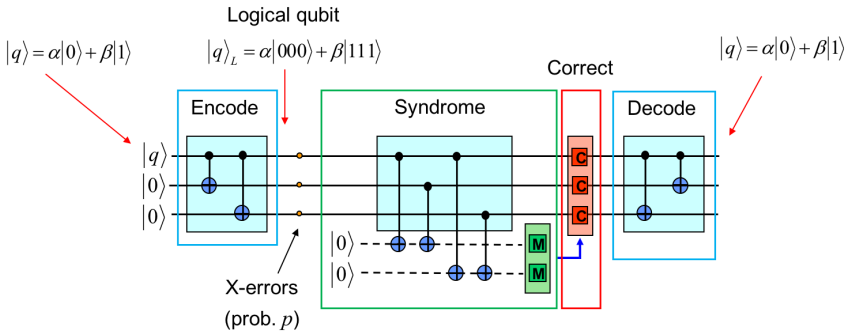
$$Z_1Z_2\,|\psi'\rangle = Z_1Z_2X_1\,|\psi\rangle = -X_1Z_1Z_2\,|\psi\rangle = -X_1\,|\psi\rangle = -\,|\psi'\rangle$$

Now complete the stabiliser table for the 3-bit code:

Error	State	Z_1Z_2	Z_2Z_3
I	$\alpha\|000\rangle + \beta\|111\rangle$	+1	+1
X_1	$\alpha\|100\rangle + \beta\|011\rangle$	-1	
X_2	$\alpha\|010\rangle + \beta\|101\rangle$		
X_3	$\alpha\|001\rangle + \beta\|110\rangle$		

8.3 3-bit Quantum Error Correction code

In the lectures we presented the 3-bit code implemented in the following circuit where ancilla qubits are introduced to measure the stabilisers to obtain the error syndrome for correction:



We have distinguished the various step of the QEC process – encode, error exposure, syndrome, correct, decode. Note that the set of stabilisers chosen in the above is slightly different $\{Z_1Z_2,\ Z_1Z_3\}$ – the syndrome assignments will change, but there will still be 4 distinct ancilla measurement outcomes to instruct us how to correct. Note also that when we measure in the Z-basis (or X-basis) we obtain 0 or 1, not the actual eigenvalues ±1.

Exercise 8.3.1 The encode circuit is relatively easy to follow (see 8.2.2 above). Refer to your lecture notes to understand how the syndrome circuit works in terms of measuring the stabilisers using ancilla qubits.

Exercise 8.3.2 In the QUI, program the encode and syndrome steps of the circuit based on the stabilisers $\{Z_1Z_2,\ Z_2Z_3\}$. Starting from an arbitrarily chosen state $|\psi\rangle = [\alpha|0\rangle +$

$\beta|1\rangle]\otimes|0\rangle\otimes|0\rangle$ (e.g. using the state in 8.2.1a) run the circuit with the four X-error scenarios and record the ancilla result for each. Fill in the table:

Scenario	Error	State	$Z_1 Z_2$	$Z_2 Z_3$	Ancillas
No error	I	$\alpha\|000\rangle + \beta\|111\rangle$	+1	+1	00
Error on 1st	X_1	$\alpha\|100\rangle + \beta\|011\rangle$			
Error on 2nd	X_2	$\alpha\|010\rangle + \beta\|101\rangle$			
Error on 3rd	X_3	$\alpha\|001\rangle + \beta\|110\rangle$			

Exercise 8.3.3 Now add the decode circuit, leaving some room for a correction step. Run the circuit with various error scenarios and manually apply the correction required to show the original state $|\psi\rangle = [\alpha|0\rangle + \beta|1\rangle]\otimes|0\rangle\otimes|0\rangle$ is recovered. When comparing with the system state before the error was applied don't forget the final state has the final two qubits (ancillas) changed – to make the direct comparison insert the appropriate X operator(s) to flip the ancillas back to their initial values.

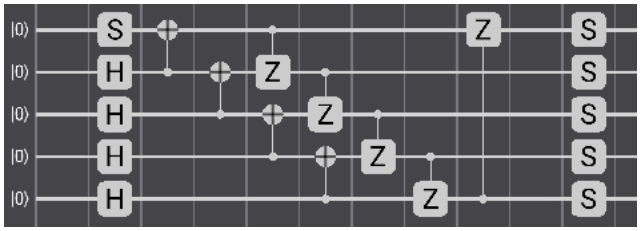
Exercise 8.3.4 Now we can automate the process. Using your knowledge of Toffoli based classical logic build a circuit to perform the correction based on the measured ancilla values. Yes, this is possible in the QUI. Test your circuit on the four error scenarios. Again, to achieve an easy direct comparison manually flip the ancillas where required (OK, this you have to do manually in the QUI).

8.4 5-bit Quantum Error Correction code

The 5-bit code, which can correct X, Z and Y errors, can be represented in terms of the following set of stabilisers (slightly different to lectures, but effect is the same):

$$ZXXZI$$
$$IZXXZ$$
$$ZIZXX$$
$$XZIZX$$

The encoding circuit for the 5-bit code is:



Exercise 8.4.1 Program the above 5-bit encoding circuit into the QUI and run it (with appropriate input states) to obtain the logical states $|0\rangle_L$ and $|1\rangle_L$ in the expressions below (cross out the states not present and add signs).

$$|0_L\rangle = \frac{1}{4}(|00000\rangle \quad |00001\rangle \quad |00010\rangle \quad |00011\rangle \quad |00100\rangle \quad |00101\rangle \quad |00110\rangle \quad |00111\rangle \quad |01000\rangle \quad |01001\rangle \quad |01010\rangle \quad |01011\rangle \quad |01100\rangle \quad |01101\rangle \quad |01110\rangle \quad |01111\rangle \quad |10000\rangle \quad |10001\rangle \quad |10010\rangle \quad |10011\rangle \quad |10100\rangle \quad |10101\rangle \quad |10110\rangle \quad |10111\rangle \quad |11000\rangle \quad |11001\rangle \quad |11010\rangle \quad |11011\rangle \quad |11100\rangle \quad |11101\rangle \quad |11110\rangle \quad |11111\rangle)$$

$$|1_L\rangle = \frac{1}{4}(|00000\rangle \quad |00001\rangle \quad |00010\rangle \quad |00011\rangle \quad |00100\rangle \quad |00101\rangle \quad |00110\rangle \quad |00111\rangle \quad |01000\rangle \quad |01001\rangle \quad |01010\rangle \quad |01011\rangle \quad |01100\rangle \quad |01101\rangle \quad |01110\rangle \quad |01111\rangle \quad |10000\rangle \quad |10001\rangle \quad |10010\rangle \quad |10011\rangle \quad |10100\rangle \quad |10101\rangle \quad |10110\rangle \quad |10111\rangle \quad |11000\rangle \quad |11001\rangle \quad |11010\rangle \quad |11011\rangle \quad |11100\rangle \quad |11101\rangle \quad |11110\rangle \quad |11111\rangle)$$

Show that the stabilisers are stabilised by these states.

Exercise 8.4.2 From the example in the lecture notes build the syndrome extraction circuit. There are a total of 16 stabiliser combinations, so you will need 4 ancilla qubits (testing the very limits of the QUI configuration).

Scenario	Error	$Z_1 X_2 X_3 Z_4 I_5$	$I_1 Z_2 X_3 X_4 Z_5$	$Z_1 I_2 Z_3 X_4 X_5$	$X_1 Z_2 I_3 Z_4 X_5$	Ancillas
No error	I	+1	+1	+1	+1	0000
X error on 1st	X_1					
X error on 2nd	X_2					
X error on 3rd	X_3					
Y error on 1st	Y_1					
Y error on 2nd	Y_2					
Y error on 3rd	Y_3					
Z error on 1st	Z_1					
Z error on 2nd	Z_2					
Z error on 3rd	Z_3					

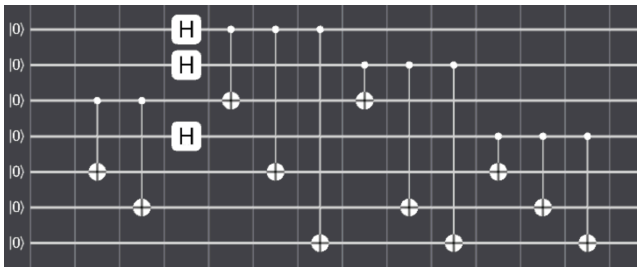
Exercise 8.4.3 Now try to build the correction circuit to implement the appropriate correction for each ancilla measurement scenario for the X-error scenarios. Save your circuit as "Lab-8 5 bit QEC template". See if you can extend for Y and Z errors.

8.5 The Steane 7-bit Quantum Error Correction code

In the lectures we presented the 7-bit code, which can correct X, Z and Y errors, in terms of the following stabilisers:

$$\begin{Bmatrix} I & I & I & X & X & X & X \\ I & X & X & I & I & X & X \\ X & I & X & I & X & I & X \\ I & I & I & Z & Z & Z & Z \\ I & Z & Z & I & I & Z & Z \\ Z & I & Z & I & Z & I & Z \end{Bmatrix}$$

The encoding circuit for the Steane code is:



Exercise 8.5.1 Program the encoding circuit into the QUI and verify that the logical states have the following structure, and stabilise the set listed above.

$$|0_L\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}(|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle + |111000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle)$$

8.6 Logical operations on the 5-bit Quantum Error Correction code

Now we can start doing some logical operations within a QEC code – i.e. perform logical gates on the logical encoding itself. To make things easier and tractable in the QUI we will focus on the 5 qubit code.

Exercise 8.6.1 Load the circuit "Lab-8 5 bit QEC template". Modify the circuit to perform one round of syndrome and correction, followed by a transverse logical operation (e.g. try each of X_L = XXXXX, Y_L = YYYYY, Z_L = ZZZZZ) followed by another round of syndrome and correction as per the following sequence:

Note: as we don't have a mechanism in QUI (yet) to re-set the ancillas you will have to do that by hand after the first round of error correction.

a) In the no-error scenario, does your circuit work, i.e. is the logical operation performed correctly on the state?

b) Add single X, Y or Z errors in the locations specified and check how your circuit performs. Can you correctly perform the logical operation in the presence of errors? What happens to the output when there is more than one error in any given (indicated) location?