# Lecture 6. Perceptron

COMP90051 Statistical Machine Learning

Semester 2, 2019
Lecturer: Ben Rubinstein

# This lecture

- Perceptron
  - Introduction to Artificial Neural Networks
  - The perceptron model
  - Stochastic gradient descent

# The Perceptron Model

A building block for artificial neural networks, yet another linear classifier

# Biological inspiration

- Humans perform well at many tasks that matter
- Originally neural networks were an attempt to mimic the human brain

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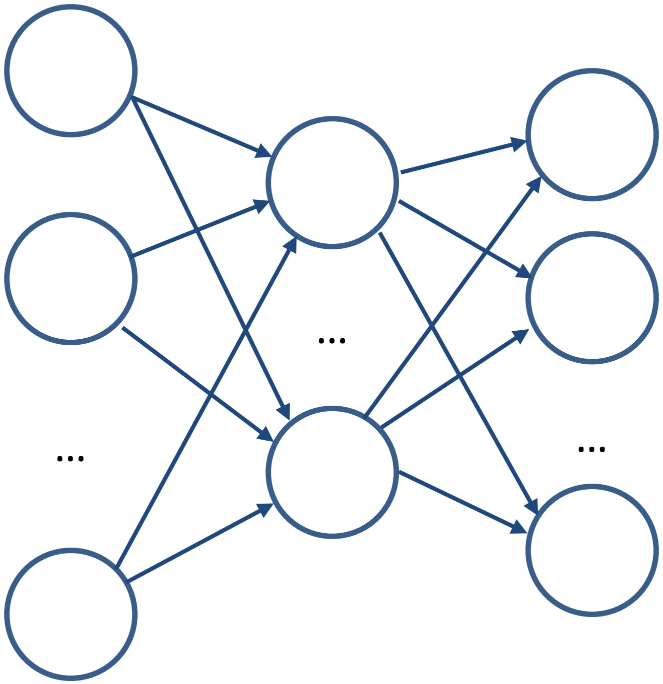
# Artificial neural network

- As a *crude approximation*, the human brain can be thought as a mesh of interconnected processing nodes (neurons) that relay electrical signals

- Artificial neural network is a network of processing elements

- Each element converts inputs to output

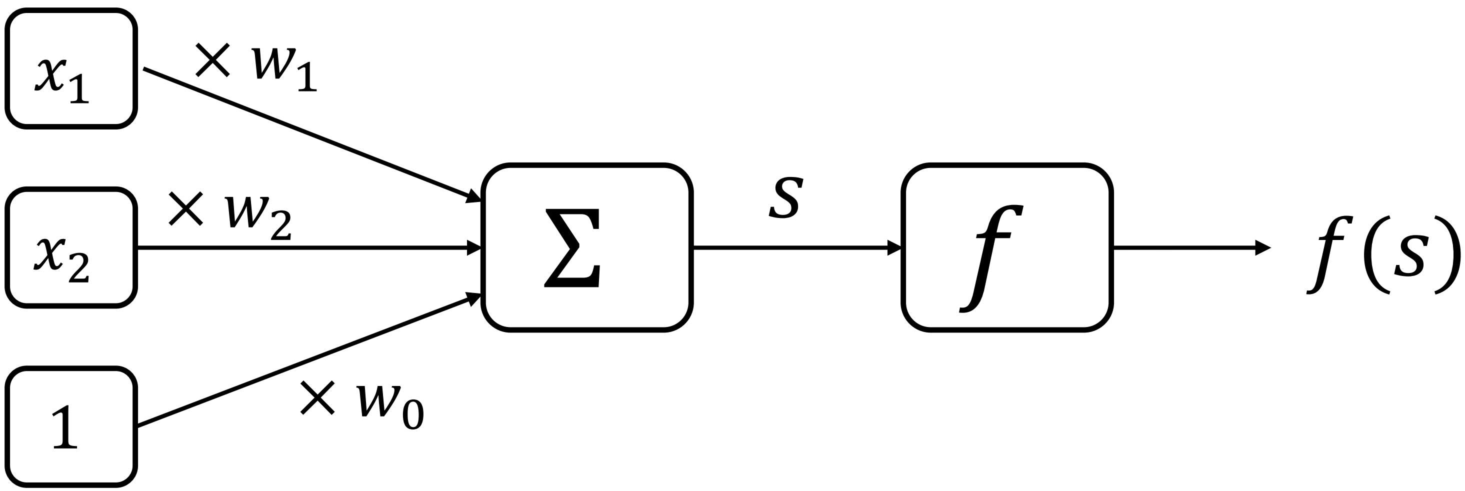
- The output is a function (called *activation function*) of a weighted sum of inputs



# Outline

- In order to use an ANN we need (a) to design network topology and (b) adjust weights to given data
  - In this subject, we will exclusively focus on task (b) for a particular class of networks called feed forward networks
- Training an ANN means adjusting weights for training data given a pre-defined network topology
- We will come back to ANNs and discuss ANN training in the next lecture
- Right now we will turn our attention to an individual network element because it is an interesting model in itself

# Perceptron model




Compare this model to logistic regression

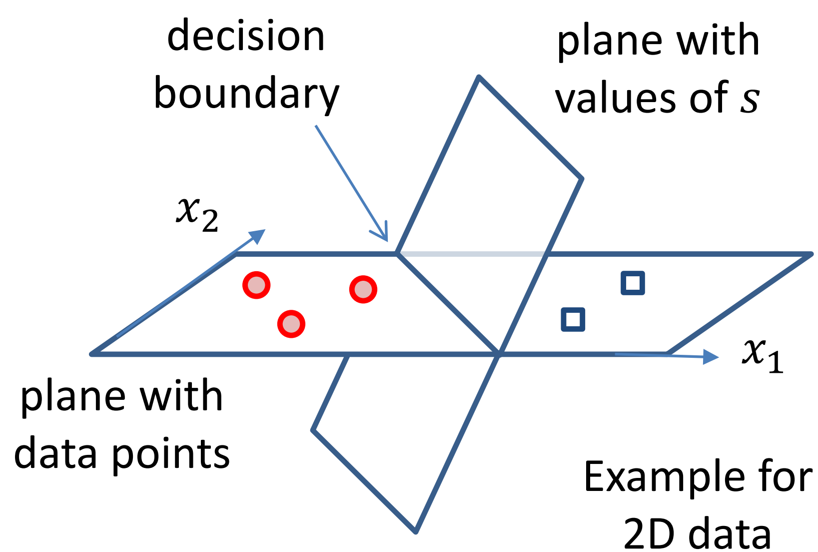
- $x_1, x_2$ – inputs
- $w_1, w_2$ – synaptic weights
- $w_0$ – bias weight
- $f$ – activation function

# Perceptron is a linear binary classifier

Perceptron is a binary classifier:

Predict class A if $s \geq 0$
Predict class B if $s < 0$
where $s = \sum_{i=0}^{m} x_i w_i$

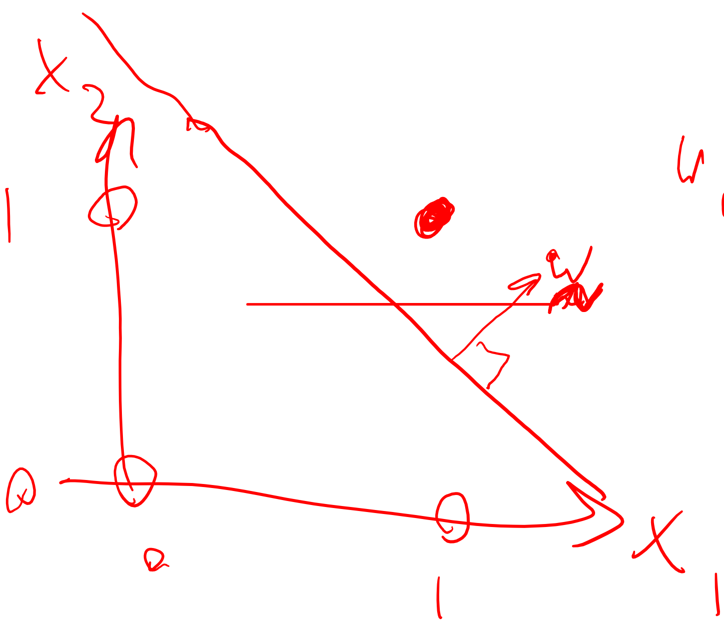
Perceptron is a <u>linear classifier</u>: $s$ is a linear function of inputs, and the decision boundary is linear




Example for 2D data

Exercise: find weights of a perceptron capable of perfect classification of the following dataset

| $x_1$ | $x_2$ | $y$ | |
|---|---|---|---|
| 0 | 0 | Class B | 0 |
| 0 | 1 | Class B | 1 |
| 1 | 0 | Class B | 1 |
| 1 | 1 | Class A | 2 |



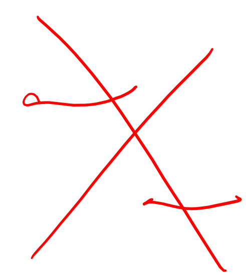

$w_1 = w_2 = 1$

$w_0 = -1.5$

art: OpenClipartVectors at pixabay.com (CC0)

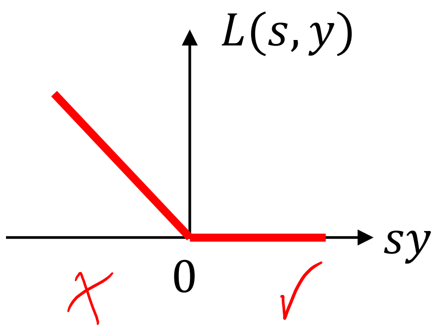
# Loss function for perceptron

- "Training": finds weights to minimise some loss. Which?
- Our task is binary classification. Let's arbitrarily encode one class as $+1$ and the other as $-1$. So each training example is now $\{\boldsymbol{x}, y\}$, where $y$ is either $+1$ or $-1$
- Recall that, in a perceptron, $s = \sum_{i=0}^{m} x_i w_i$, and the sign of $s$ determines the predicted class: $+1$ if $s > 0$, and $-1$ if $s < 0$
- Consider a single training example. If $y$ and $s$ have same sign then the example is classified correctly. If $y$ and $s$ have different signs, the example is misclassified



# Loss function for perceptron

- Consider a single training example. If $y$ and $s$ have the same sign then the example is classified correctly. If $y$ and $s$ have different signs, the example is misclassified
- The perceptron uses a loss function in which there is no penalty for correctly classified examples, while the penalty (loss) is equal to $s$ for misclassified examples*
- Formally:
  - $L(s, y) = 0$ if both $s, y$ have the same sign
  - $L(s, y) = |s|$ if both $s, y$ have different signs




- This can be re-written as $L(s, y) = \max(0, -sy)$

* This is similar, but not identical to another widely used loss function called ***hinge loss***

# Stochastic gradient descent

- Randomly shuffle/split all training examples in $B$ batches
- Choose initial $\boldsymbol{\theta}^{(1)}$
- For $i$ from 1 to $T$
- For $j$ from 1 to $B$
- Do gradient descent update using data from batch $j$

Iterations over the entire dataset are called *epochs*

- Advantage of such an approach: computational feasibility for large datasets

# Perceptron training algorithm

Choose initial guess $\boldsymbol{w}^{(0)}$, $k = 0$

For $i$ from 1 to $T$ (epochs)

For $j$ from 1 to $N$ (training examples)

Consider example $\{\boldsymbol{x}_j, y_j\}$

Update*: $\boldsymbol{w}^{(k++)} = \boldsymbol{w}^{(k)} - \eta \boldsymbol{\nabla} L(\boldsymbol{w}^{(k)})$

$$L(\boldsymbol{w}) = \max(0, -sy)$$

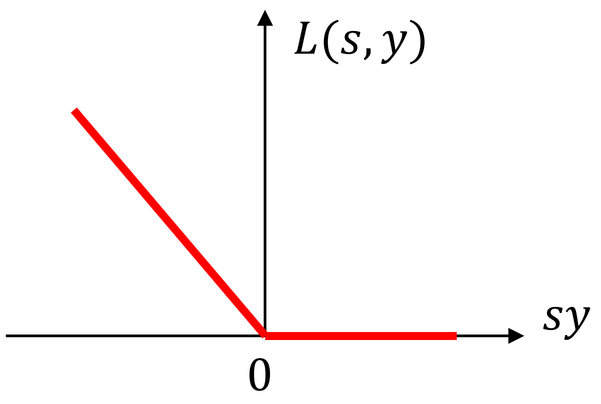
$$s = \sum_{i=0}^{m} x_i w_i$$

$\eta$ is learning rate

*There is no derivative when $s = 0$, but this case is handled explicitly in the algorithm, see next slides

# Perceptron training rule

- We have $\frac{\partial L}{\partial w_i} = 0$ when $sy > 0$
  - We don't need to do update when an example is correctly classified
- We have $\frac{\partial L}{\partial w_i} = -x_i$ when $y = 1$ and $s < 0$
- We have $\frac{\partial L}{\partial w_i} = x_i$ when $y = -1$ and $s > 0$
- $s = \sum_{i=0}^{m} x_i w_i$



# Perceptron training algorithm

When classified correctly, weights are unchanged

When misclassified: $\boldsymbol{w}^{(k+1)} = -\eta(\pm\boldsymbol{x})$
($\eta > 0$ is called *learning rate)*

<u>If $y = 1$, but $s < 0$</u>
$w_i \leftarrow w_i + \eta x_i$
$w_0 \leftarrow w_0 + \eta$

<u>If $y = -1$, but $s \geq 0$</u>
$w_i \leftarrow w_i - \eta x_i$
$w_0 \leftarrow w_0 - \eta$

<u>*Convergence Theorem*</u>: if the training data is linearly separable, the algorithm is guaranteed to converge to a solution. That is, there exist a finite $K$ such that $L(\boldsymbol{w}^K) = 0$

# Perceptron convergence theorem

- Assumptions
  - Linear separability: There exists $\boldsymbol{w}^*$ so that $y_i(\boldsymbol{w}^*)'\boldsymbol{x}_i \geq \gamma$ for all training data $i = 1, \dots, N$ and some positive $\gamma$.
  - Bounded data: $\|\boldsymbol{x}_i\| \leq R$ for $i = 1, \dots, N$ and some finite $R$.

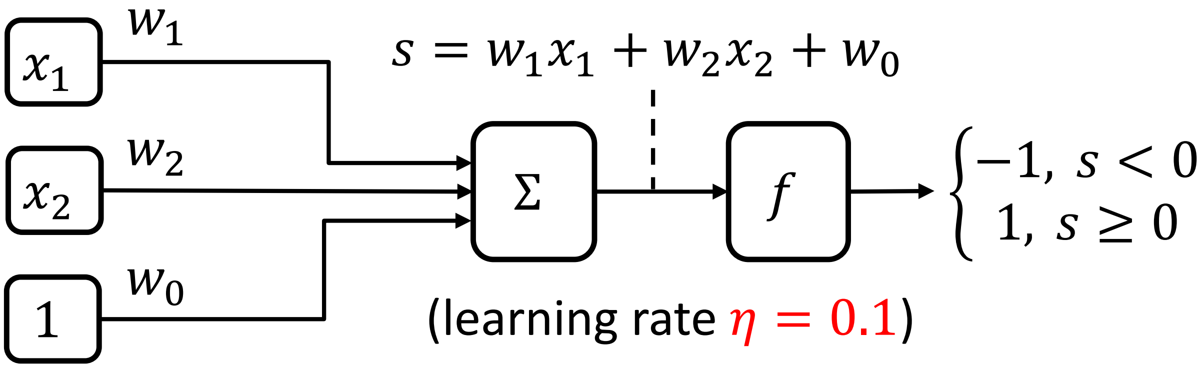
- Proof sketch
  - Establish that $(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)} \geq (\boldsymbol{w}^*)'\boldsymbol{w}^{(k-1)} + \gamma$
  - It then follows that $(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)} \geq k\gamma$
  - Establish that $\left\|\boldsymbol{w}^{(k)}\right\|^2 \leq kR^2$
  - Note that $\cos\left(\boldsymbol{w}^*, \boldsymbol{w}^{(k)}\right) = \frac{(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)}}{\|\boldsymbol{w}^*\|\|\boldsymbol{w}^{(k)}\|}$
  - Use the fact that $\cos(\dots) \leq 1$
  - Arrive at $k \leq \frac{R^2\|\boldsymbol{w}^*\|^2}{\gamma}$

# Pros and cons of perceptron learning

- If the data is linearly separable, the perceptron training algorithm will converge to a correct solution
  - There is a formal proof ← good!
  - It will converge to some solution (separating boundary), one of infinitely many possible ← bad!
- However, if the data is not linearly separable, the training will fail completely rather than give some approximate solution
  - Ugly ☹

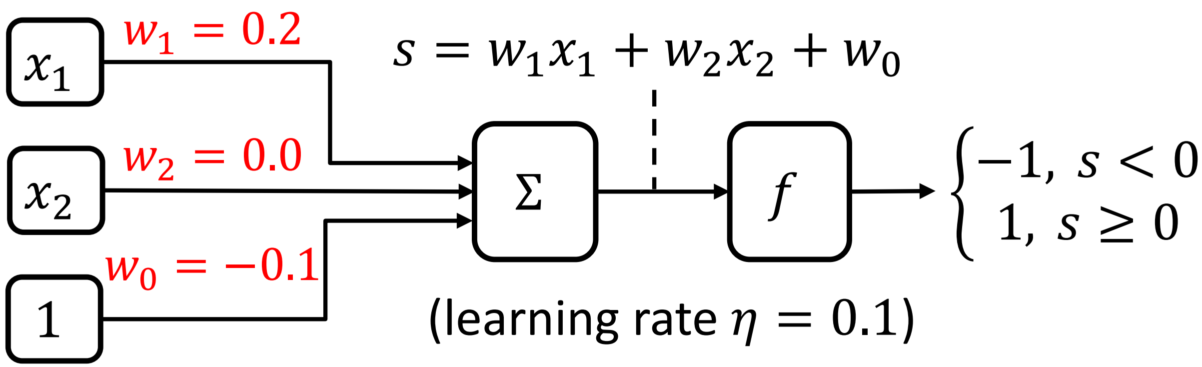
# Perceptron learning example

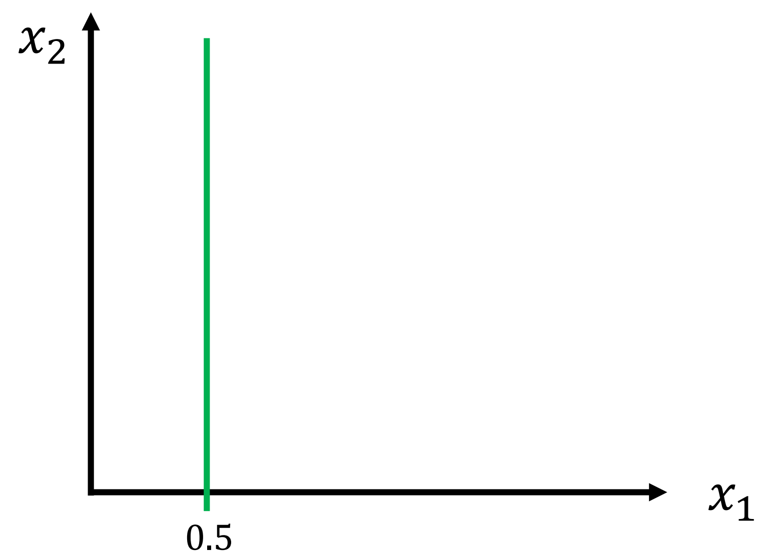
Basic setup



# Perceptron learning example

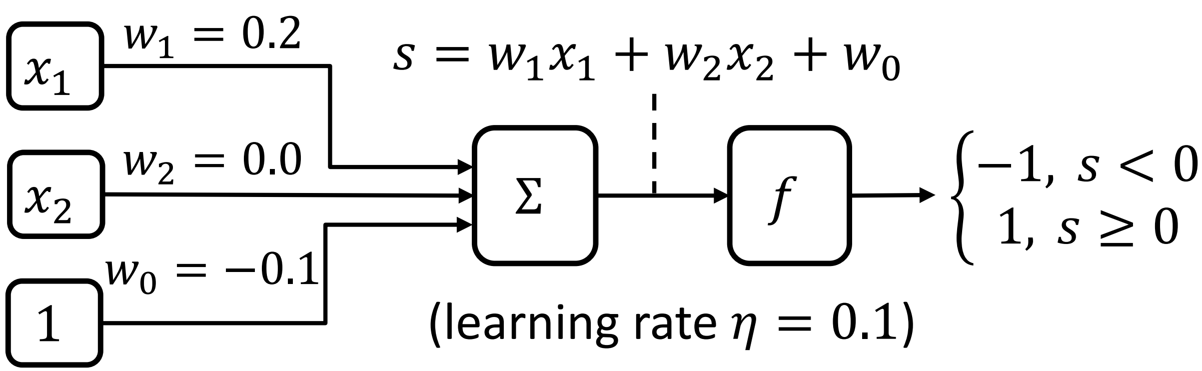
Start with random weights



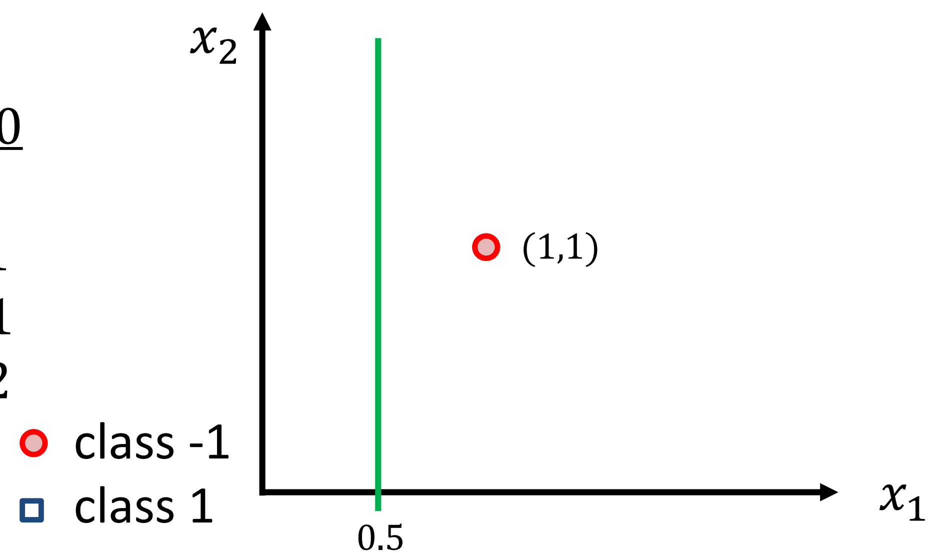
# Perceptron learning example

Consider training example 1



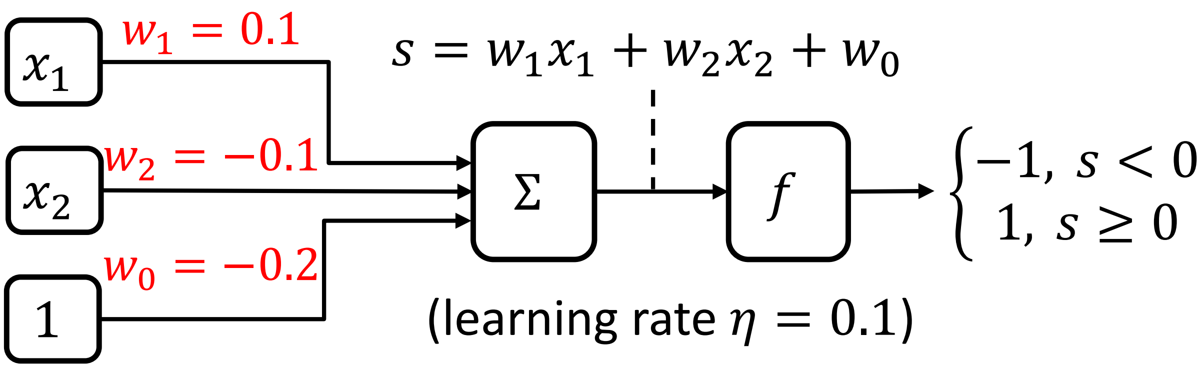

$\underline{0.2x_1 + 0.0x_2 - 0.1 > 0}$

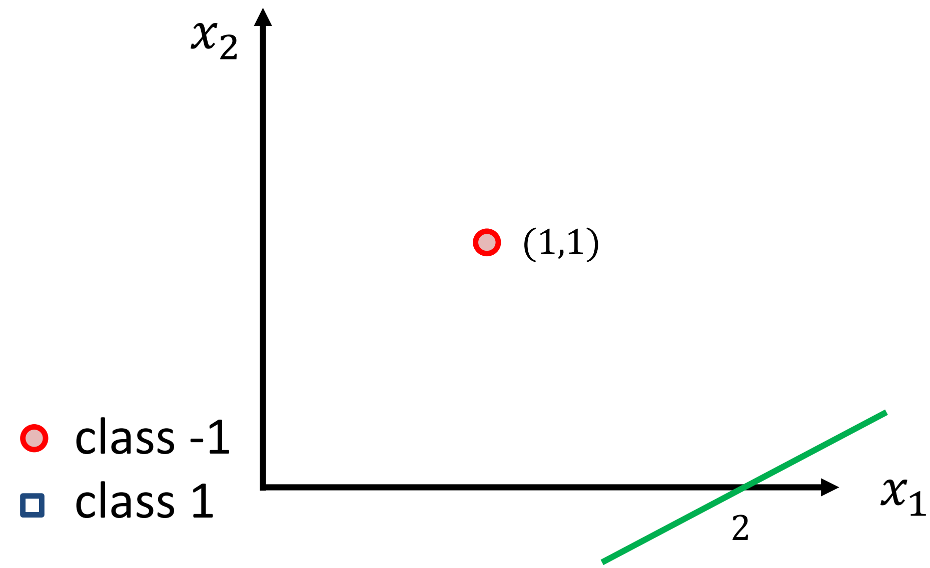
$$
\begin{aligned}
w_1 &\leftarrow w_1 - \eta x_1 = 0.1 \\
w_2 &\leftarrow w_2 - \eta x_2 = -0.1 \\
w_0 &\leftarrow w_0 - \eta \quad = -0.2
\end{aligned}
$$



# Perceptron learning example

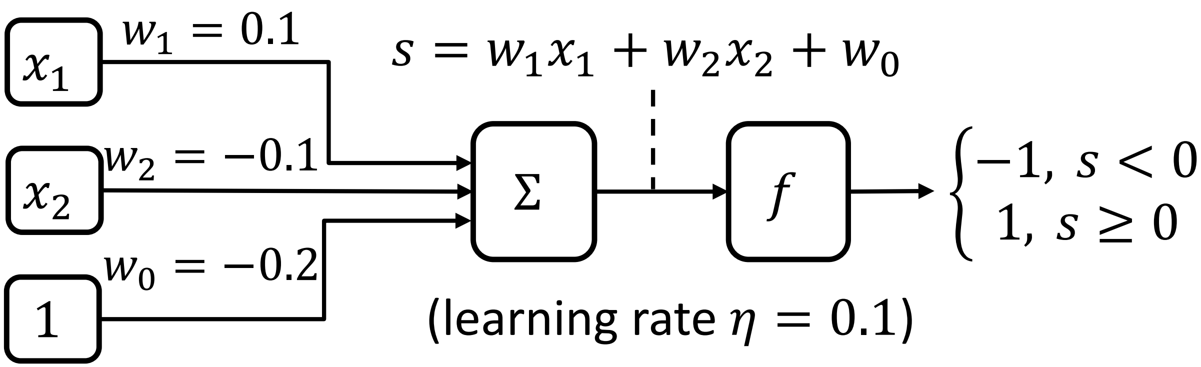
## Update weights



# Perceptron learning example

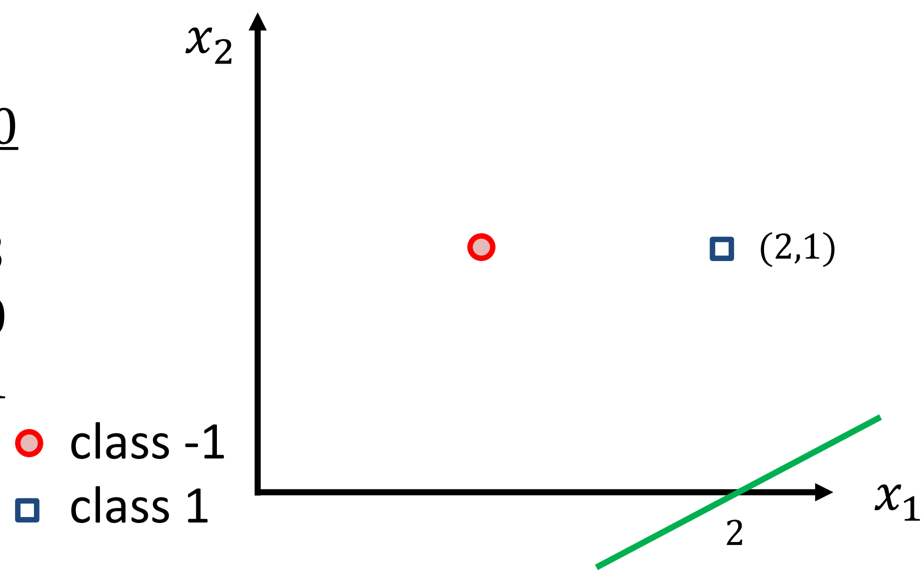
Consider training example 2




(learning rate $\eta = 0.1$)

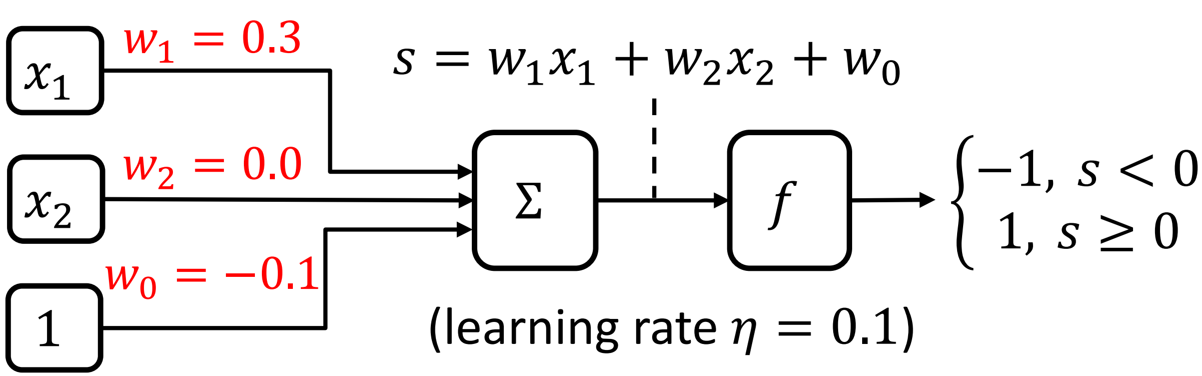
$\underline{0.1x_1 - 0.1x_2 - 0.2 < 0}$

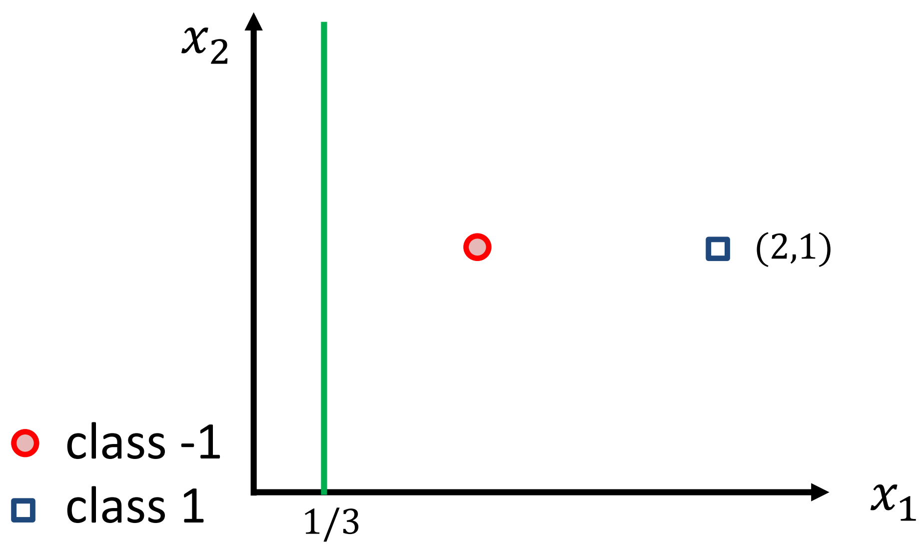
$$\begin{aligned} w_1 &\leftarrow w_1 + \eta x_1 = 0.3 \\ w_2 &\leftarrow w_2 + \eta x_2 = 0.0 \\ w_0 &\leftarrow w_0 + \eta = -0.1 \end{aligned}$$



# Perceptron learning example

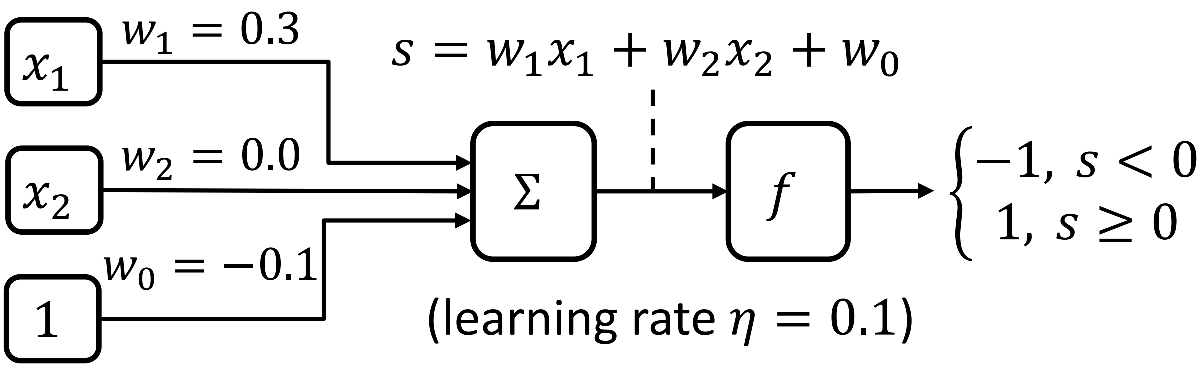
Update weights



# Perceptron learning example

## Further examples



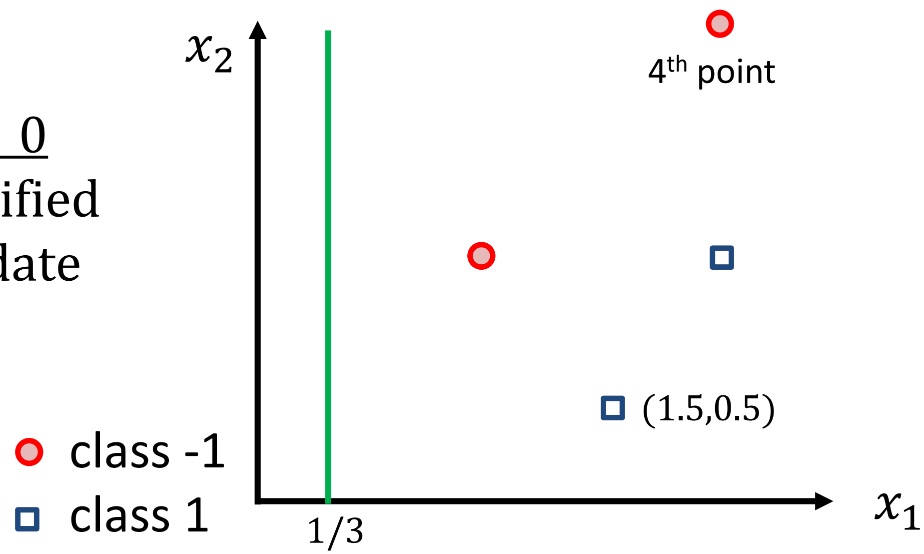

(learning rate $\eta = 0.1$)

$\underline{0.3x_1 - 0.0x_2 - 0.1 > 0}$

3rd point: correctly classified

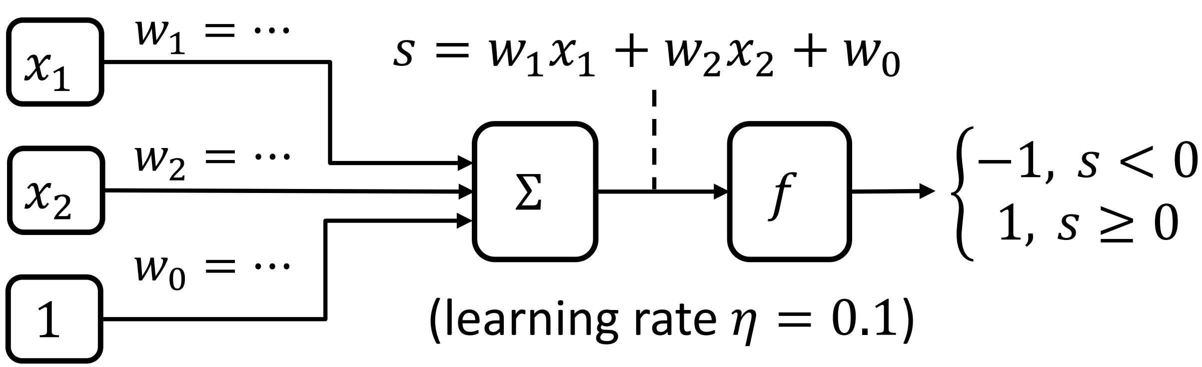
4th point: incorrect, update

etc.

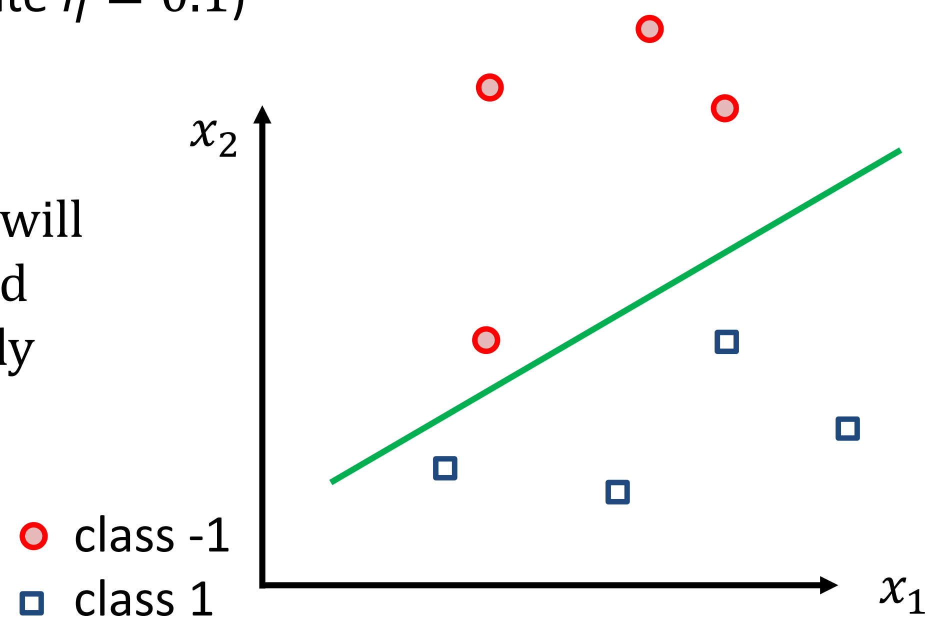


# Perceptron learning example

Further examples




Eventually, all the data will be correctly classified (provided it is linearly separable)



# Summary

- Perceptron
  - Introduction to Artificial Neural Networks
  - The perceptron model
  - Training algorithm

- Next lecture: Multiple layers, Backprop training