COMP30026 Models of Computation

Predicate Logic: Semantics

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The Meaning of a Predicate Logic Formula

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4. It is true if $D = \{0\}$.

The Meaning of a Formula

In some cases, the meaning of a formula is independent of what its predicate (and function) names denote, and of what sort of things the variables range over.

For example, $\forall x\ P(x) \lor \exists y\ (\neg P(y))$ is inherently true, no matter what (it is valid).

Similarly, $\forall x\ P(x) \land (\neg P(a))$ is false no matter what a and P stand for (the formula is unsatisfiable).

Interpretations (or Structures)

An interpretation (or structure) consists of

1. A non-empty set D (the domain, or universe);
2. An assignment, to each n-ary predicate symbol P, of an n-place function $\mathbf{p} : D^n \to \{\mathbf{f}, \mathbf{t}\}$;
3. An assignment, to each n-ary function symbol g, of an n-place function $\mathbf{g} : D^n \to D$;
4. An assignment to each constant a of some fixed element of D.

Free Variables and Valuations

To give meaning to formulas that may have free variables, such as

$$\exists x \; P(f(y), x)$$

we need two things:

- A valuation $\sigma : var \rightarrow D$ for free variables;
- An interpretation as just discussed.

Connectives are always given their usual meaning.

Terms and Valuations

We just said that a valuation is a function $\sigma : var \to D$.

But, given an interpretation $\mathcal{I}$, we get a valuation function from terms automatically, by natural extension:

$$\begin{array}{rcl} \sigma(a) & = & \mathrm{d} \\ \sigma(g(t_1, \ldots, t_n)) & = & \mathbf{g}(\sigma(t_1), \ldots, \sigma(t_n)) \end{array}$$

where d is the element of D that $\mathcal{I}$ assigns to a, and $\mathbf{g} : D^n \to D$ is the function that $\mathcal{I}$ assigns to g.

Example: Consider the term $t = f(y, g(x, a))$. Let our interpretation assign to a the value 3, to f the multiplication function, and to g addition. If $\sigma(x) = 9$ and $\sigma(y) = 5$ then $\sigma(t) = 60$.

Truth of a Formula

The truth of a closed formula should depend only on the given interpretation.

Our only interest in formulas with free variables (and hence in valuations) is that we want to define the truth of a formula compositionally, as done on the next slide.

Notation:

$$\sigma_{x \mapsto d}(y) = \begin{cases} d & \text{if } y = x \\ \sigma(y) & \text{otherwise} \end{cases}$$

Read this as "the map σ, updated to map x to d."

Making a Formula True

Given an interpretation $\mathcal{I}$ (with domain D), and a valuation σ,

- σ makes $P(t_1, \ldots, t_n)$ true iff $\mathbf{p}(\sigma(t_1), \ldots, \sigma(t_n)) = \mathbf{t}$, where $\mathbf{p}$ is the meaning that $\mathcal{I}$ gives P.
- σ makes $\neg F$ true iff σ does not make F true.
- σ makes $F_1 \wedge F_2$ true iff σ makes both of F_1 and F_2 true.
- σ makes $\forall x\ F$ true iff $\sigma_{x \mapsto d}$ makes F true for every $d \in D$.

If we now define

$$\exists x\ F \equiv \neg \forall x\ \neg F$$

then the meaning of every other formula follows from this.

Models and Validity of Formulas

A wff F is true in interpretation $\mathcal{I}$ iff every valuation makes F true (for $\mathcal{I}$). If not true then it is false in interpretation $\mathcal{I}$.

A model for F is an interpretation $\mathcal{I}$ such that F is true in $\mathcal{I}$. We write $\mathcal{I} \models F$.

A wff F is logically valid iff every interpretation is a model for F. In that case we write $\models F$.

F_2 is a logical consequence of F_1 iff $\mathcal{I} \models F_2$ whenever $\mathcal{I} \models F_1$. We write $F_1 \models F_2$.

F_1 and F_2 are logically equivalent iff $F_1 \models F_2$ and $F_2 \models F_1$. We write $F_1 \equiv F_2$.

Summarising: Satisfiability and Validity

A closed, well-formed formula F is

- satisfiable iff $\mathcal{I} \models F$ for some interpretation $\mathcal{I}$;
- valid iff $\mathcal{I} \models F$ for every interpretation $\mathcal{I}$;
- unsatisfiable iff $\mathcal{I} \not\models F$ for every interpretation $\mathcal{I}$;
- non-valid iff $\mathcal{I} \not\models F$ for some interpretation $\mathcal{I}$.

Example of Non-Validity

Consider the formula

$$(\forall y \exists x\ P(x, y)) \Rightarrow (\exists x \forall y\ P(x, y))$$

It is not valid.

For example, consider the interpretation with $D = \mathbb{Z}$, and the predicate P meaning “less than”.

Or, let $D = \{0, 1\}$ and let P mean “equals”.

The formula is satisfiable, as it is true, for example, in the interpretation where $D = \{0, 1\}$ and P means “less than or equal”.

Example of Validity

$F = (\exists y \forall x \; P(x, y)) \Rightarrow (\forall x \exists y \; P(x, y))$ is valid.

If we negate F (and rewrite it) we get

$$(\exists y \forall x \; P(x, y)) \wedge (\exists x \forall y \; \neg P(x, y))$$

The right conjunct is made true only if there is some $d_0 \in D$ for which $\mathbf{p}(d_0, d)$ is false for all $d \in D$.

But the left conjunct requires that $\mathbf{p}(d_0, d)$ be made true for at least some d.

Since F's negation is unsatisfiable, F is valid.

Another Example of Validity

Consider

$$F = (\forall x \; P(x)) \Rightarrow P(t)$$

F is valid no matter what the term t is.

To see this, again it is easiest to consider

$$\neg F = (\forall x \; P(x)) \land \neg P(t)$$

The term t denotes some element of the domain D, so $\neg F$ cannot be satisfied.

Rules of Passage for the Quantifiers

We cannot in general "push quantifiers in".

For example, there is no immediate simplification of a formula of the form $\exists x\ (P(x) \land Q(x))$.

However, we do get, for formulas F_1 and F_2:

$$
\begin{array}{rcl}
\exists x\ (\neg F_1) & \equiv & \neg \forall x\ F_1 \\
\forall x\ (\neg F_1) & \equiv & \neg \exists x\ F_1 \\
\exists x\ (F_1 \lor F_2) & \equiv & (\exists x\ F_1) \lor (\exists x\ F_2) \\
\forall x\ (F_1 \land F_2) & \equiv & (\forall x\ F_1) \land (\forall x\ F_2)
\end{array}
$$

It follows that

$$\exists x\ (F_1 \Rightarrow F_2) \equiv (\forall x\ F_1) \Rightarrow (\exists x\ F_2)$$

More Rules of Passage for Quantifiers

If G is a formula with no free occurrences of x, then we also get

$$\begin{array}{rcl}
\exists x\ G & \equiv & G \\
\forall x\ G & \equiv & G \\
\exists x\ (F \wedge G) & \equiv & (\exists x\ F) \wedge G \\
\forall x\ (F \vee G) & \equiv & (\forall x\ F) \vee G \\
\forall x\ (F \Rightarrow G) & \equiv & (\exists x\ F) \Rightarrow G \\
\forall x\ (G \Rightarrow F) & \equiv & G \Rightarrow (\forall x\ F)
\end{array}$$

no matter what F is. In particular F may have free occurrences of x.

Next Up

Clausal form and how resolution is extended to first-order predicate logic.