Tutorial 3

> A system of equations can be written in the form $A\mathbf{x} = \mathbf{b}$, where A is the matrix of coefficients, $\mathbf{x}$ is the column vector of variables, and $\mathbf{b}$ is the column vector of the right hand side. If A is invertible, then the unique solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Q1. It is known that

$$\begin{bmatrix} 1 & -2 & 3 & 3 & 65 \\ 3 & -10 & 8 & 13 & -104 \\ 1 & -1 & 0 & 2 & -78 \\ 2 & 2 & 1 & 0 & 143 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -20 \\ 0 & 1 & 0 & -1 & 58 \\ 0 & 0 & 1 & 0 & 67 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i). Explain why the system

$$\begin{array}{rrrrcr} w & -2x & +3y & +3z & = & 65 \\ 3w & -10x & +8y & +13z & = & -104 \\ w & -x & & +2z & = & -78 \\ 2w & +2x & +y & & = & 143 \end{array}$$

cannot be solved by matrix equation.

(ii). Solve the system in part (i).

> A message made into a square matrix B can be coded by multiplying it by a square matrix A, and so sending the message corresponding to the matrix product $AB = C$. It is possible to decode the message by computing $A^{-1}C$.

Q2. In coding a message, a blank space was represented by 0, an A by 1, a B by 2, a C by 3 and so on. The message was transformed using the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad \text{for which} \quad A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and sent as

$$-19,\ 19,\ 25,\ -21,\ 0,\ 18,\ -18,\ 15,\ 3,\ 10,\ -8,\ 3,\ -2,\ 20,\ -7,\ 12$$

(i). Verify that the stated matrix for A^{-1} is correct.

(ii). What was the message?

> The *determinant* of an $n \times n$ matrix A can be found by:
>
> - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$.
> - Expansion along row i: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij})$.
> - Expansion down column j: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij})$.
>
> *Note:* A_{ij} is the matrix where row i and column j have been deleted from A.

Q3. Let

$$H = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 1 & 3 \\ -2 & -1 & 7 \\ 1 & 0 & -2 \end{bmatrix} \qquad K = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

(i). Find $\det(H)$.

(ii). Find $\det(J)$ by expansion down the second column.

(iii). Find $\det(K)$ by expansion along the third row.

(iv). Is there a simpler expansion for any of (i)–(iii)? If so, what?

If A and B are $n \times n$ matrices, then:

1. $\det(A^T) = \det(A)$
2. $\det(AB) = \det(A)\det(B)$
3. $\det(\alpha A) = \alpha^n \det(A)$
4. If $\det(A) \neq 0$, then A^{-1} exists, and $\det(A^{-1}) = \frac{1}{\det(A)}$.

Q4. Let H, J, K be defined as in Question 4. Without doing further determinant calculations, when defined, find:
(i). $\det(J^2K)$ (ii). $\det(KH)$ (iii). $\det(3J)$ (iv). $\det(K^T(J^{-1})^2)$

Recall that the elementary row operations are:

1. Interchanging two rows, $R_i \to R_j, R_j \to R_i$.
2. Multiplying a row by a (non-zero) constant, $R_i \to \alpha R_i$.
3. Adding a multiple of one row to another, $R_i \to R_i + \alpha R_j$.

Q5. (i). Of the row operations in the box above, which do not affect the determinant? How do the others change it?

(ii). Find the determinant of $A = \begin{bmatrix} 1 & -2 & 7 & 3 \\ 0 & 1 & -2 & 4 \\ -2 & 3 & -3 & 1 \\ -3 & 6 & -21 & 0 \end{bmatrix}$ by first using row operations to reduce it to an upper triangular matrix.

Q6. Find the condition on a, b, c, d such that

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible, and proceed to find the inverse in those cases.