COMP20003
Algorithms and Data Structures
Greedy Algorithms and the MST

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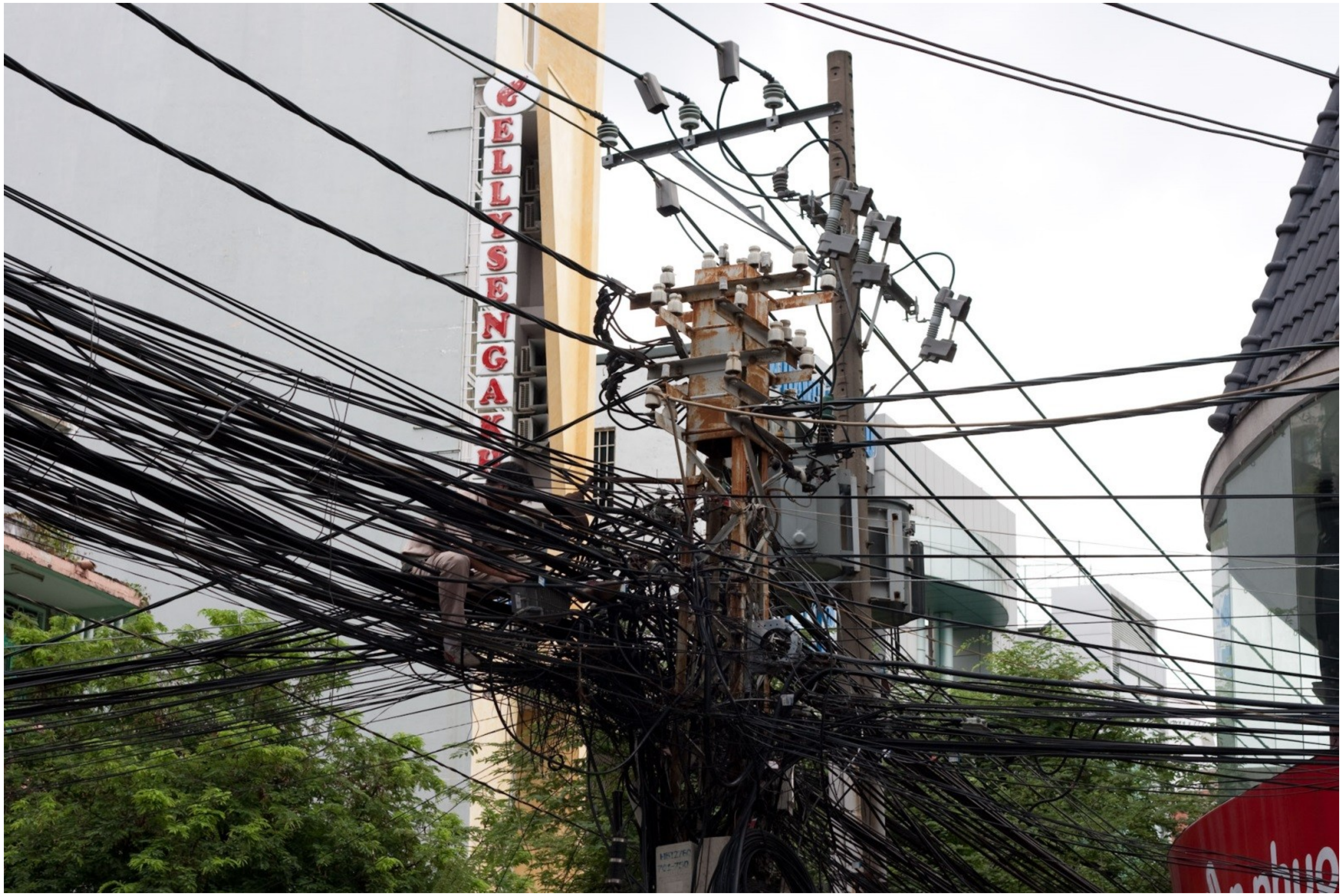
Semester 2

Greedy Algorithms

- Greedy algorithms: used in optimization problems.
- Greedy algorithms keep take the next best step repeatedly, until the best solution is reached.
  - Dijkstra’s algorithm is greedy: takes the next best edge to add to the path tree.

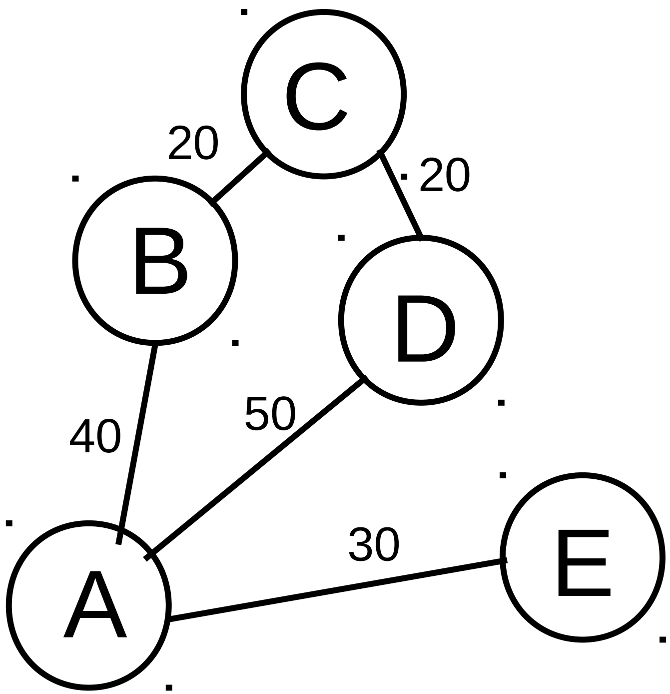
Minimum Spanning Tree

- Undirected weighted graphs.
- Minimum spanning tree: a subgraph that is:
  - A tree (no cycles).
  - Contains every vertex (spans).
  - Minimum sum of edge weights.
- Also called:
  - Minimum weight spanning tree (sum of weights).
  - Minimal spanning tree (might be more than one).

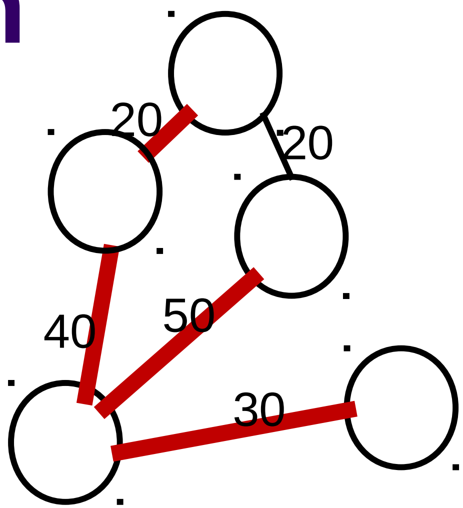


Saigon X, Health and Safety? What's That? Reena Mahtani

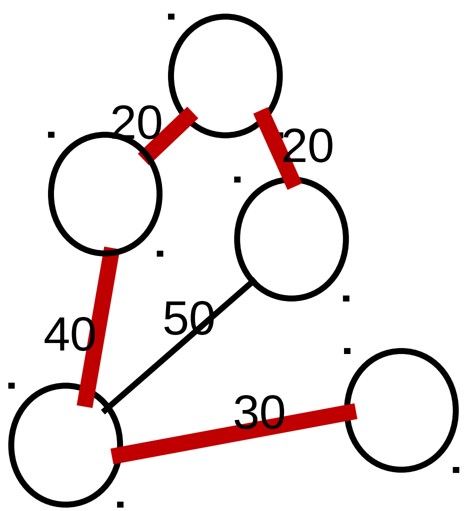
MST *vs.* shortest path



Shortest paths



Min Span Tree



MST and Graph characteristics

- Graph must be connected.
- MST *must* have exactly *V-1* edges.
- No cycles in MST.

Building a MST: General approach

- Start with isolated vertices (all), no edges.
- Begin with any vertex (Prim's) or the least cost edge (Kruskal's).
  - This is a MST subtree.
- Keep adding vertices/edges to extend this MST subtree.
  - Shortest connections.
  - No cycles.

Famous MST algorithms

- Prim's
  - R.C. Prim, *Bell System Technical Journal* ***36****(*6)*,* 1389-1401, 1957.
- Kruskal's
  - J.B. Kruskal, *Proceedings of the American Mathematical Society* **7**, 48-50, 1956.
- Borůvka's (1926, published in Czech)
  - Nešetřil, Jaroslav; Milková, Eva; Nešetřilová, Helena (2001). "Otakar Borůvka on minimum spanning tree problem: translation of both the 1926 papers, comments, history". *Discrete Mathematics* **233** (1–3): 3–36

Prim’s MST algorithm

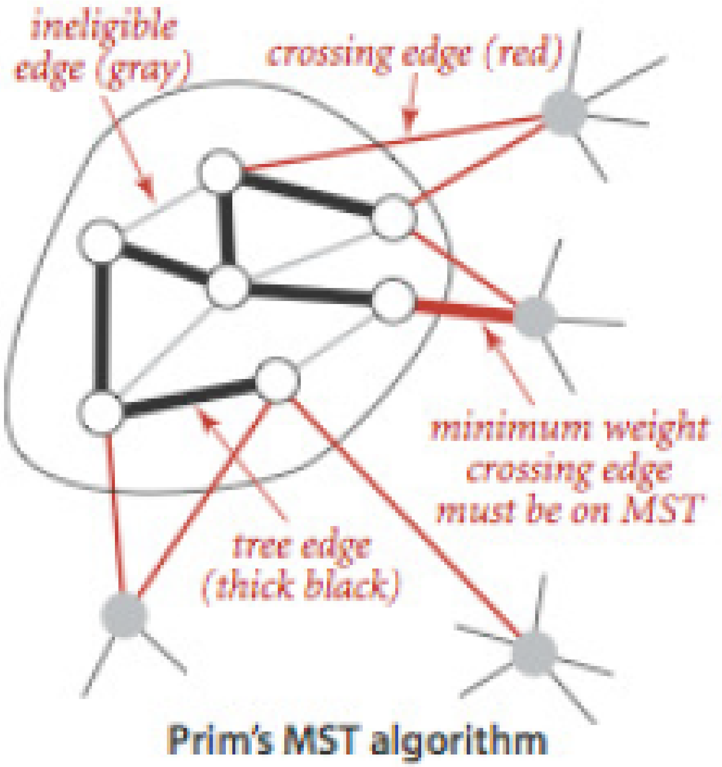
- Preferred method for dense graphs.
- Easiest with matrix representation.
- Prim’s algorithm relies on picking the next best edge that joins two set of vertices:
  - Vertices already in the tree (S).
  - Vertices not yet in the tree (V-S).
- These two sets form a “cut”.

Definitions

- A Cut *(V,V-S)* of *G* is a partition of *V*.
- Cross: an edge *(u,v)* in *E* with one endpoint in *S* and the other in *V-S*.
- Light edge: The minimum weight edge crossing the cut.
- Respect: A cut respects a set *A* of edges if no edge in *A* crosses the cut.

Cut during MST construction

- Cut:
  - *S*: set of vertices already in the MST.
  - *V-S*: not yet in the MST.
    - Fringe: part of *V-S* one step away from the MST.
    - Vertices in *V-S* have a cost(distance) from the MST subtree so far constructed.
    - Distances between non-MST vertices and MST vertices are updated as vertices are added to MST.



From R. Sedgewick, Algorithms 4th edition

Prim's MST construction

- Start:
  - *S* = {any vertex}
  - *S*-*V* = {all the others}
  - The cut *S*/*V*-*S* respects edges in the MST as it is being constructed .
  - The cut itself changes.

Prim’s MST construction

- Respect:
  - The cut S/V-S respects edges in the MST being constructed.
    - Fringe: vertices in V-S one step away from the MST.
    - Vertices in V-S have a cost(distance) from the MST subtree so far constructed (some may be ∞).

Prim's MST construction

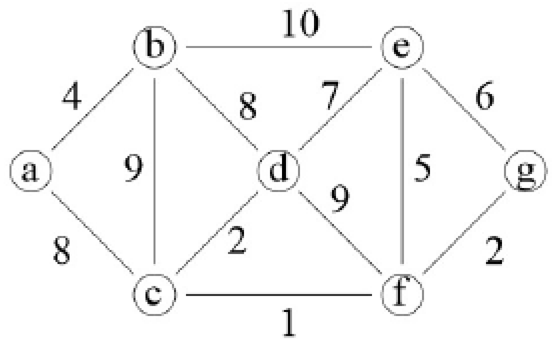
- Pick lightest edge crossing the cut:
  - Crossing edge *(u,v)* has *u* in *S* and *v* in *V-S*.
    - Add v to S.
    - Keep track of path (pred[ ]).
    - Update distances between non-MST vertices and MST vertices (could be closer now) (wt[ ]).
- Repeat until V-S = {0}.
- Reconstruct connections and distances from pred[ ] and wt[ ].

Prim's: Pseudocode

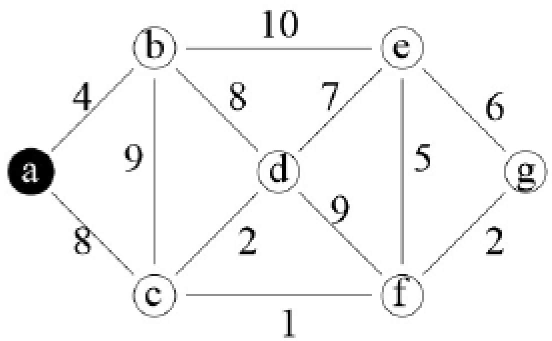
```
prim(G,wt,root)
{
   for every u in V { dist[u] = ∞; inmst[u] = FALSE;}
   dist[root] = 0; pred[root] = NULL;
   PQ = makePQ(V); /* all vertices in PQ */
   while(!empty PQ){
     u = deletemin(PQ);
     for every (v adjacent to u){
         if ((inmst[v]==FALSE)&& (wt[u][v]< dist[v])){
            dist[v] = wt(u,v); /* update distance */
            decreasewt(PQ,v,dist[v]);/* update PQ dist*/
            pred[v]=u; /* update path information */
         }
     }
     inmst[u] = TRUE;
   }
 } /* at end: MST = {{v,pred[v]}: v in V – {root}} */
```

- Fringe vertices are in a priority queue.
- This is a Priority-First Search.

Prim's example



Connected graph



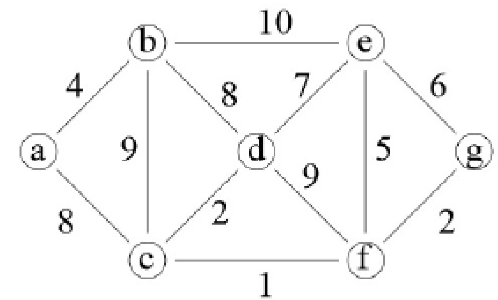
Step 0

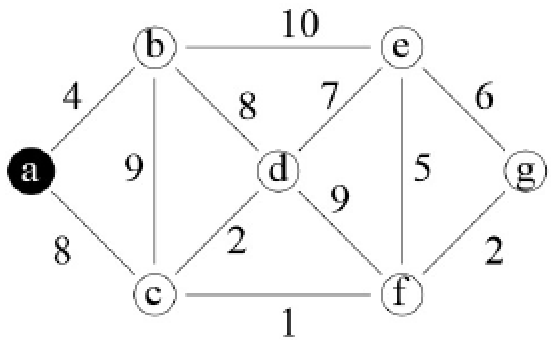
S={a}

V \ S = {b,c,d,e,f,g}

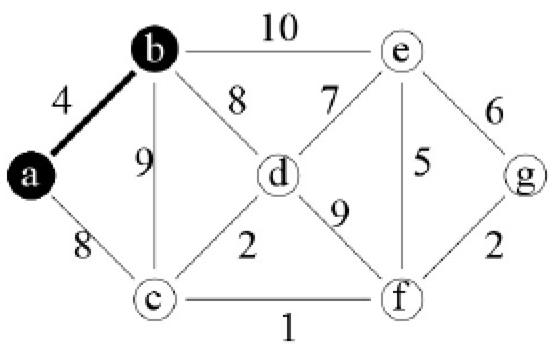
lightest edge = {a,b}

Example from Mordechai Golin, Hong Kong University of Science and Technology
http://www.cse.ust.hk/faculty/golin/COMP271Sp03/Notes/MyL10.pdf

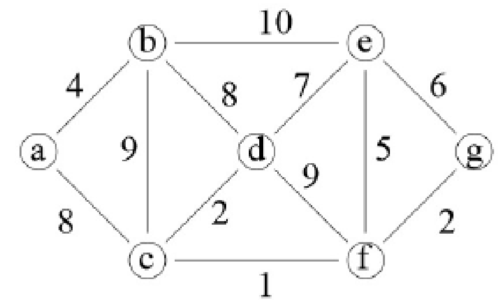


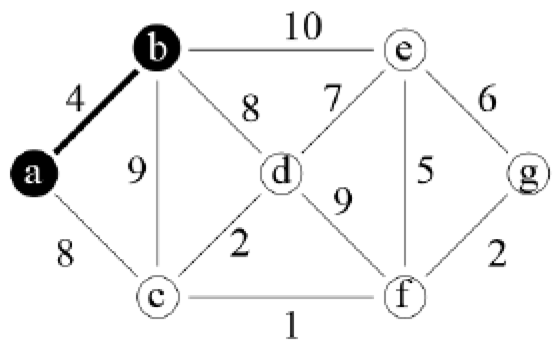


Step 1.1 before
S={a}
V \ S = {b,c,d,e,f,g}
A={}
lightest edge = {a,b}



Step 1.1 after
S={a,b}
V \ S = {c,d,e,f,g}
A={{a,b}}
lightest edge = {b,d}, {a,c}





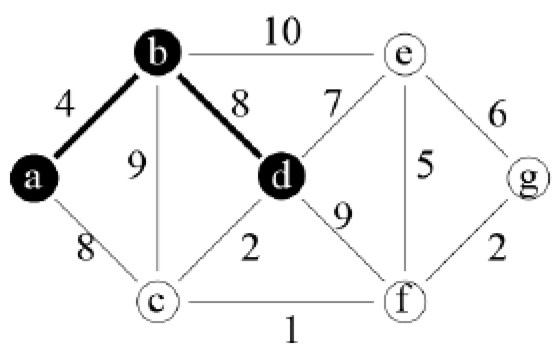
Step 1.2 before

S={a,b}

V \ S = {c,d,e,f,g}

A={{a,b}}

lightest edge = {b,d}, {a,c}



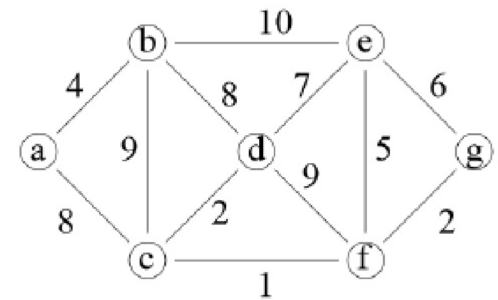
Step 1.2 after

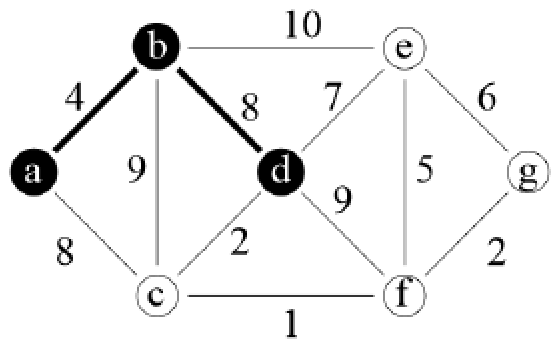
S={a,b,d}

V \ S = {c,e,f,g}

A={{a,b},{b,d}}

lightest edge = {d,c}





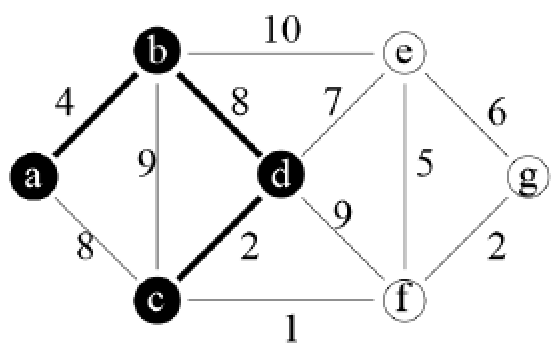
Step 1.3 before

S={a,b,d}

V \ S = {c,e,f,g}

A={{a,b},{b,d}}

lightest edge = {d,c}



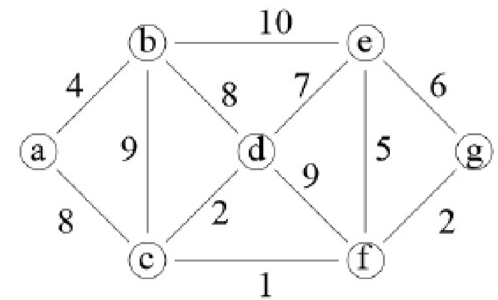
Step 1.3 after

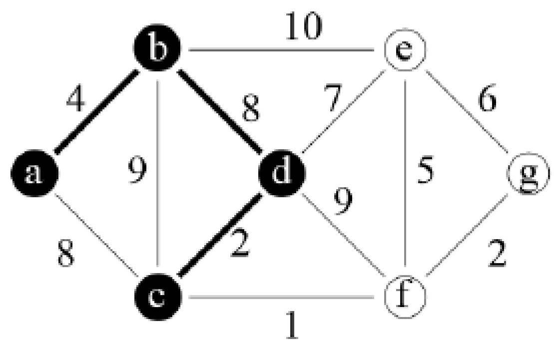
S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}





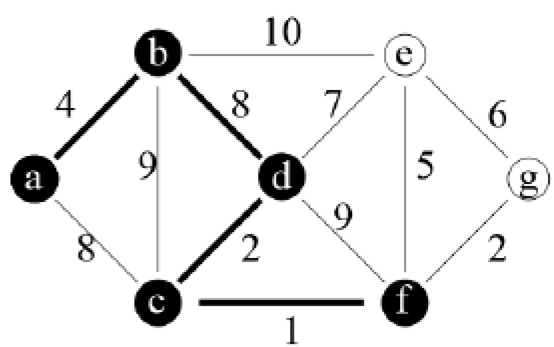
Step 1.4 before

S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}

lightest edge = {c,f}



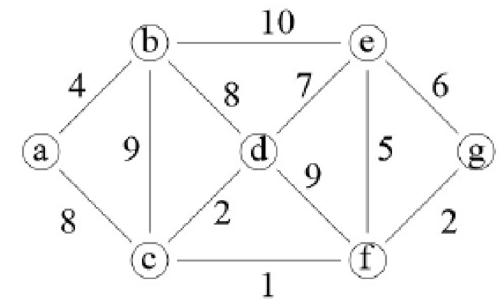
Step 1.4 after

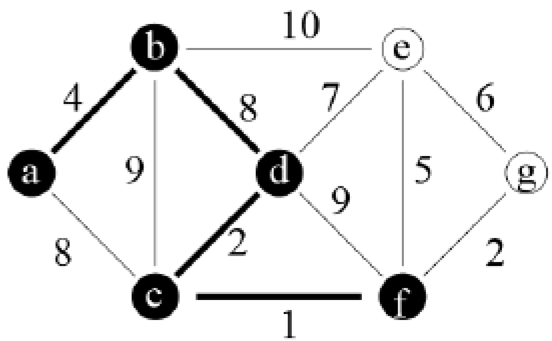
S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}





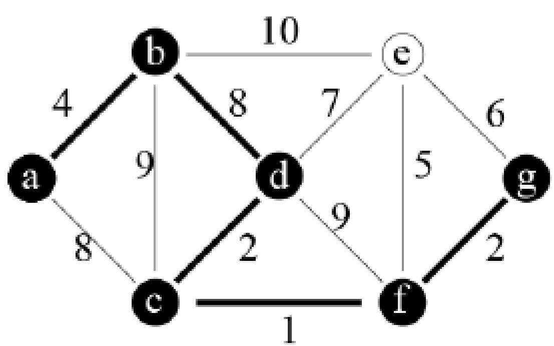
Step 1.5 before

S={a,b,c,d,f}

V \ S = {e,g}

A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}



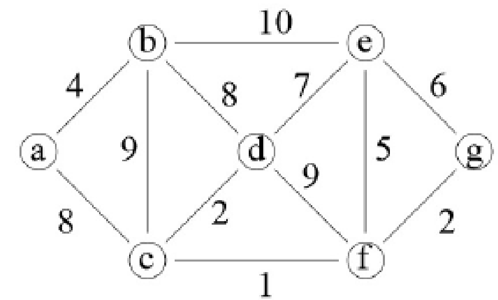
Step 1.5 after

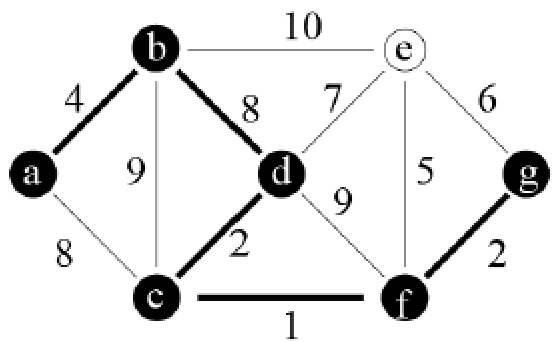
S={a,b,c,d,f,g}

V \ S = {e}

A={{a,b},{b,d},{c,d},{c,f},
{f,g}}

lightest edge = {f,e}





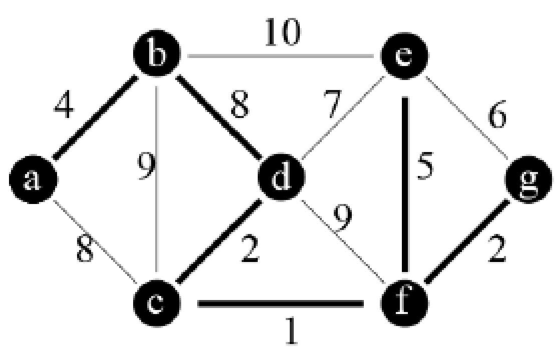
Step 1.6 before

S={a,b,c,d,f,g}

V \ S = {e}

A={{a,b},{b,d},{c,d},{c,f},
{f,g}}

lightest edge = {f,e}



Step 1.6 after

S={a,b,c,d,e,f,g}

V \ S = {}

A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}

MST completed

Prim's: Analysis

Initialize arrays pred[] dist[]: O(v)

Make PQ of vertices: if heap O(v)

Loop while PQ not empty V*

Deletemin (if heap) O(log v)

Update adjacent weights,

adjust wt in PQ O(degree of u * log v)

= O(V* (log V + deg(u) * logV))

= O(V log V + V deg(u) * log V)

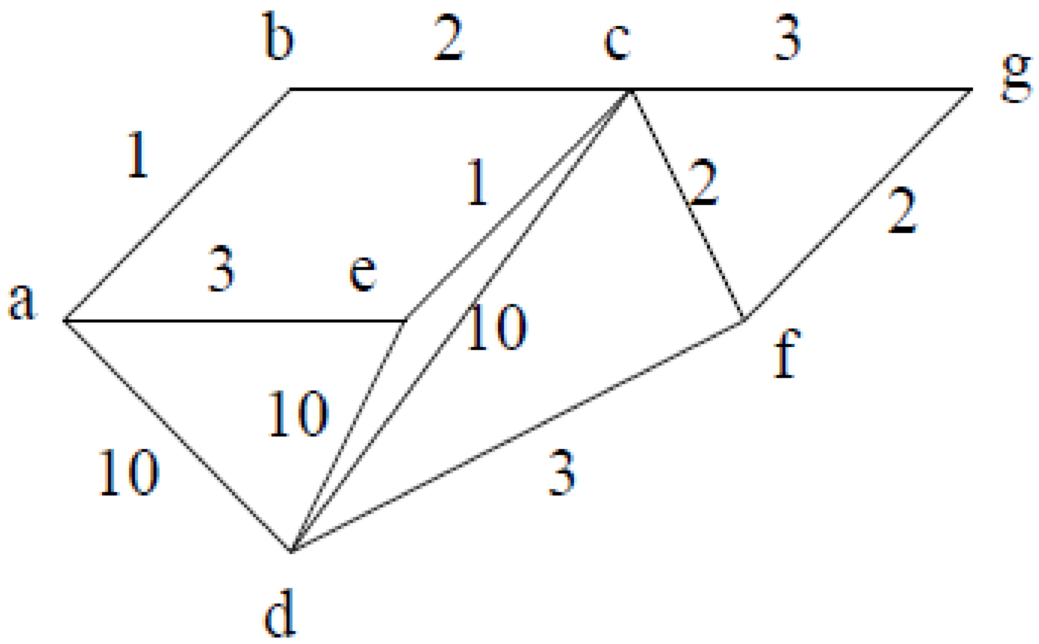
Noting that V deg(u) = E,

= O(V log V + E log V)

= O((V+E)logV)

= O(E log V) for dense graphs with heap PQ

Prim’s *vs.* Dijkstra’s



Kruskal’s MST algorithm

- Prim’s algorithm adds the next closest vertex.
- Kruskal’s algorithm adds the next lowest weight edge *that doesn’t form a cycle*.

Kruskal's Algorithm for MST

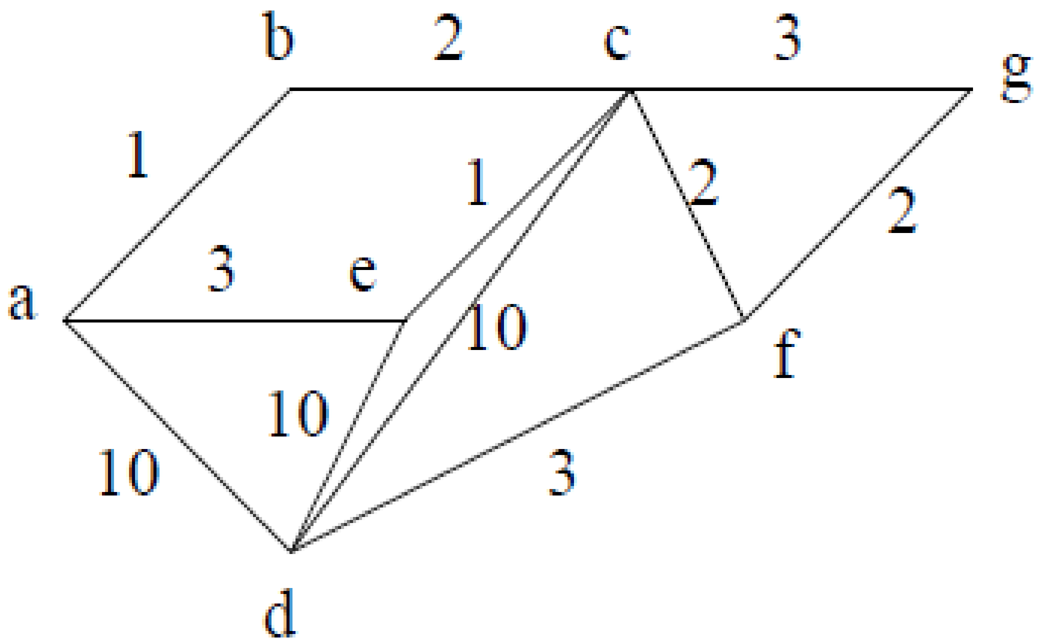
```
E1: edges in MST so far
E2: remaining edges

E1=EMPTY,E2=E
Sort edges in E2 by weight
while E1 contains fewer then |V|-1 edges and E2 not
EMPTY
      Pick min cost edge e(i,j) from E2
      E2=E2 - e(ij)
      if V(i),V(j) are not in same MST-so far, then
            e(i,j) goes into E1
            unite MSTs with V(i) and V(j)
```

```
if V(i),V(j) are not in same MST-so far, then
            unite MSTs with V(i) and V(j)
```

- Prevents cycles (not in same MST-so far)
- Unites MSTs (new edge in MST-so far)

Kruskal's



Kruskal's algorithm

- Sort edges:
  - E log E
- E* get next edge and check for cycle
- E*merge subsets

Kruskal's Algorithm for MST

```
E1: edges in MST so far
E2: remaining edges

E1=EMPTY,E2=E
Sort edges in E2 by weight
while E1 contains fewer then n-1 edges and E2 not
EMPTY
      Pick min cost edge e(i,j) from E2
      E(2)=E(2)-{e(ij)
      if V(i),V(j) are not in same MST-so far, then
            unite MSTs with V(i) and V(j)
```

??

```
if V(i),V(j) are not in same MST-so far, then
            unite MSTs with V(i) and V(j)
```

- Prevents cycles (not in same MST-so far)
- Unites MSTs (new edge in MST-so far)
- Sounds easy, but…
- ….requires data structure and algorithm
  - Disjoint-set data structure
  - Union-find algorithm

Union-find

- Have disjoint (non-overlapping) subsets.
  - Find: Which subset is an element in?
  - Union: Join two subsets into a single subset.
- For Kruskal’s algorithm:
  - Find: Is the new edge in an existing subset?
    - If yes, this is a cycle! – don’t use!
  - Union: Does the new edge join two subjects?
    - If yes, join the two subsets.

Union-find

- Have disjoint (non-overlapping) subsets.
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  - Union: Join two subsets into a single subset.

Union-find

- Have disjoint (non-overlapping) subsets.
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- Naïve union-find: array

Union-find: array

- Have disjoint (non-overlapping) subsets.
  - Find: Which subset is an element in?
  - Union: Join two subsets into a single subset.
- Naïve union-find: array

	0	1	2	3	4	5	6	7
id[]								

Union-find: array

- Start: Singleton Sets

- id[]

0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7

- Put 2 and 3 in same set, and 4 and 6: change entry in id[]: choose representatives

0	1	2	3	4	5	6	7
0	1	2	2	4	5	4	7

Union-find: array

0	1	2	3	4	5	6	7
0	1	2	2	4	5	4	7
4	2	2	2	4	4	4	2
2	2	2	2	2	2	2	2

Union-find: array

- Naïve algorithm, using array:
- Find:
  - id[p] == id[q]?
  - O(?)
- Union:
  - id[p and all in same subset] = id[q]
  - O(?)

Speeding up the Union in Union-Find

- Speed up union: tree-based approach.
  - id[] is a parent array
  - Root is the representative of the subset.
  - To union two subsets – make the root of one the parent of the root of the other.
  - O(?)

Find in Tree-based Union-Find

- Find:
  - Trace back through parent array to root.
  - Nodes are in the same subset if they have the same root.
  - O(?)
  - Time for trace depends on depth of tree.

Improvements in Union-Find

- Find:
  - Time for trace depends on depth of tree.
  - Weighted: merge smaller tree into larger – keeps tree broader
  - Path compression
- Analysis: E union-finds on V vertices
  - Naïve: O(EV)
  - Weighted or path compress: O(V + E log V)
  - Weighted AND path compress: O(E+V) α(V)

    ≈ O(E+V)
  -

Union-Find Analysis

- Analysis: E union-finds on V vertices
  - Naïve: O(EV)
    - Array: O(1) find; O(n) union
    - Tree: O(1) union; O(n) find
  - Weighted OR path compress: O(V + E log V)
  - Weighted AND path compression:
    - $O(E^*\alpha(E,V) + V)$
    - $\alpha(n)$: inverse Ackermann function, small constant
    - $\approx O(E+V)$

Kruskal's: Analysis with best union-find

- Sort edges:
  - E log E
- E finds and E unions:
  - E+V
- O(E log E + E + V) = O(E log E)

- Time is dominated by sorting the edges!

Kruskal's: Analysis with best union-find

- Time is dominated by sorting the edges!

- Any ideas for what we might do?

Improvement to Kruskals: Partial sort

- Where sorting dominates performance, partial sorting can help…
- … only need the smallest V-1 edges
- *e.g.* quicksort-like partition, but
  - Doesn't work if graph is not connected.
  - Doesn't work if longest edge needs to be in MST, *e.g.* tight clusters connected by one or more long edges.

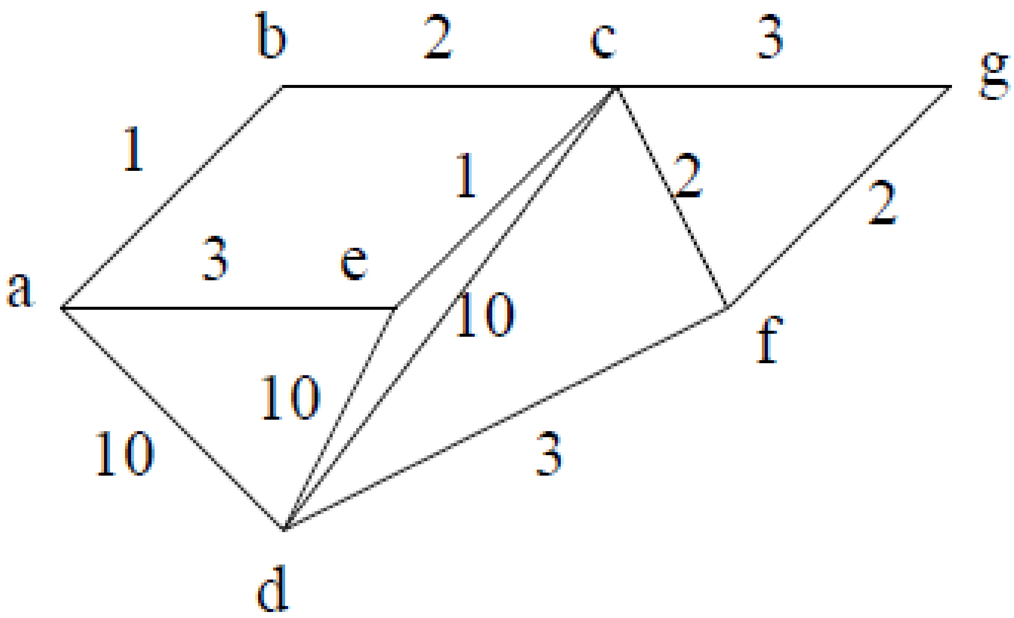
	Prim	Kruskal
General	(E+V) log V	E log E
Dense Graph	E log V	E log E
V<<E, Prim’s is faster		
Sparse Graph	V log V	V log V
Kruskal’s is faster because of the data structures		

Kruskal’s algorithm: an overview (Skiena)

A large-scale view of Kruskal’s algorithm:

http://www.cs.sunysb.edu/~skiena/combinatorica/animations/mst.html

Kruskal's *vs.* Prim's



More advanced MSTs

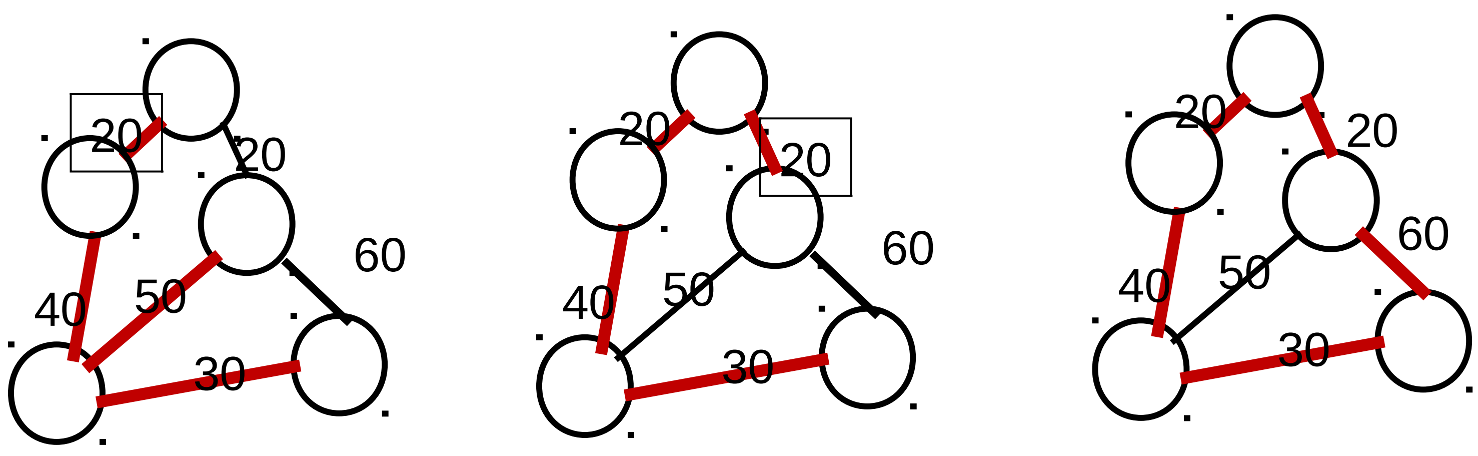
- Euclidean MSTs:
  - Given points on a plane, build MST.
  - Could construct complete graph, then use Prim’s. – Slow!
  - Other more clever algorithms exist.

More advanced MSTs

- Randomized MST algorithm
  - Random partition of the graph
  - Expected time linear, but bad worst case.
  - Karger, David R.; Klein, Philip N.; Tarjan, Robert E. (1995). "A randomized linear-time algorithm to find minimum spanning trees". *JACM* **42** (2): 321–328.
  - Linear MST algorithms exist for restricted types of graphs.
- The general solution for linear time MST creation is an open research problem.

MST and the Travelling Salesperson Problem

- Travelling salesperson problem (TSP):
  - Given a list of cities and the distances between each pair of cities, find:
    - shortest possible route that
    - visits each city exactly once
    - and returns to the origin city.



MST and the Travelling Salesperson Problem

- Travelling salesperson problem (TSP):
  - Given a list of cities and the distances between each pair of cities, find:
    - shortest possible route that
    - visits each city exactly once
    - and returns to the origin city.
- Much harder than MST!
- Greedy (nearest neighbor) doesn't work!

Graph algorithms

- Graph search
- Algorithms on undirected graphs
- Algorithms on directed graphs

Graph algorithms

- Graph search
  - Depth-first search
  - Breadth-first search
  - Priority-first search
  - (Connected components)
- Algorithms on undirected graphs
- Algorithms on directed graphs

Graph algorithms

- Graph search
- Algorithms on undirected graphs
- Algorithms on directed graphs
  - Single source shortest path (Dijkstra's)
  - Transitive closure (Warshall)
  - All pairs shortest path (Floyd-Warshall)

Graph algorithms

- Graph search
- Algorithms on undirected graphs
    - Minimum spanning tree
        - Prim’s
        - Kruskal’s
    - Travelling salesperson
- Algorithms on directed graphs

Graphs in the real world

- Many real-world problems can be modelled as graphs.
- Many specialized types of graphs allow modelling of complex problems.
- People have been working on graph algorithms for a long time, so
- Huge library of algorithms available.

Take away lesson

- If you can model a problem as a graph, there is a very good chance that there is already an algorithm to solve the problem…
- … or evidence that the problem is intractible.