PHYC90045 Introduction to Quantum Computing

This Week

Lecture 7

Introduction to Grover's algorithm for amplitude amplification, geometric interpretation

Lecture 8

Amplitude Amplification, Succeeding with Certainty, Quantum Counting

Lab

Grover's algorithm

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Amplitude Amplification

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Lecture 8

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Amplitude Amplification

- This lecture: Amplitude amplificaiton
  - Amplitude Amplification
  - Succeeding with certainty
  - Quantum Counting

References:

Rieffel, Chapter 9.1-9.2
Kaye, Chapter 8.1-8.2
Nielsen and Chuang, Chapter 6.1-6.2

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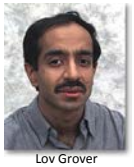
Grover's Algorithm (1996)

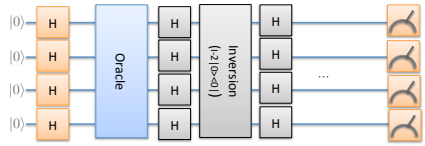
- Unordered search, find one marked item among many
- Classically, this requires N/2 queries to the oracle
- Quantum mechanically, requires only O(√N) queries.



Lov Grover

Simple problem = search for one integer marked by the oracle.

High level structure:



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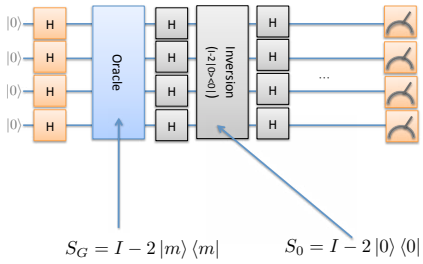
Some notation



$S_G = I - 2\,|m\rangle\,\langle m|$ $S_0 = I - 2\,|0\rangle\,\langle 0|$

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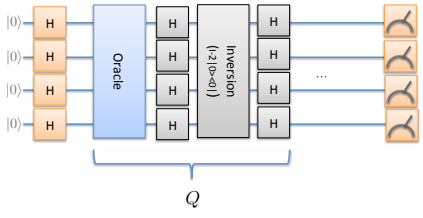
Some notation



n is the number of input qubits.
N is the total dimension (N=2ⁿ).
M is the number of solutions.

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Oracles for NP-problems

The phone book isn't a great example: Adding in all the names would take O(N) time.

In general though, many problems (specifically those in the class NP) can have easily **checkable** solutions even if it is hard to solve the problem originally. Examples:

- Factoring
- Travelling Salesman with route less than distance d
- Hamiltonian cycle

Straightforward application of Grover's algorithm provides a **polynomial** improvement over random guessing… and potentially a better (but still polynomial speedup) known as **amplitude amplication**.

Part of Norfolk Island's telephone book, with people listed by nickname (Photo: Wikicommons)

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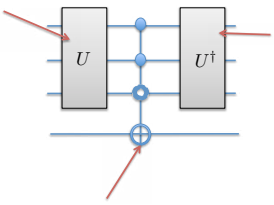
Oracle for a hash function

A hash function whose output is hard to predict based on the input.



"Uncompute" the hash function – ensures the input register remains unchanged.

The oracle recognises the 'correct' solution, but does not know in advance which input leads to the correct solution

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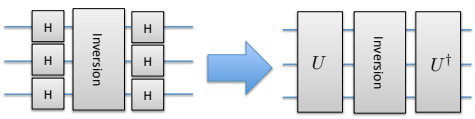
Amplitude amplification

What happens if we replace the Hadamard gates with some other U? Perhaps, for example, we can create a U which gives the correct outcome with probability greater than 1/N. Can we get any advantage?



Inversion about the mean

Apply general U instead of H

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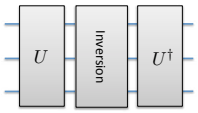
New inversion step



Apply general U instead of H

$$|\phi\rangle = U|0\rangle$$

Then we can break this up as:

$$|\phi\rangle = g_0|\phi_g\rangle + b_0|\phi_b\rangle$$

Good: In the subspace spanned by all solutions

Bad: Not in the subspace spanned by all solutions

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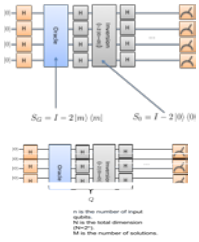
Maths of the Geometric Interpretation

$$\begin{aligned} US_0U^\dagger|\psi\rangle &= U(I - 2|0\rangle\langle 0|)U^\dagger|\psi\rangle \\ &= |\psi\rangle - 2\langle 0|U^\dagger|\psi\rangle U|0\rangle \\ &= |\psi\rangle - 2\langle\psi|U|0\rangle^* U|0\rangle \end{aligned}$$

where $|\phi\rangle = U|0\rangle$

$|\phi\rangle = g_0|\phi_g\rangle + b_0|\phi_b\rangle$



$$\begin{aligned} Q|\phi\rangle_g &= -US_0U^\dagger S_G|\phi_g\rangle \\ &= US_0U^\dagger|\phi_g\rangle \\ &= |\phi_g\rangle - 2g_0^*U|0\rangle \\ &= |\phi_g\rangle - 2g_0^*g_0|\phi_g\rangle - 2g_0^*b_0|\phi_b\rangle \\ &= (1-2t)|\phi_g\rangle - 2\sqrt{t(1-t)}|\phi_b\rangle \end{aligned}$$

$t = |g_0|^2$

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Maths of Amplitude Amplification

Similarly, $Q|\phi_b\rangle = (1-2t)|\phi_b\rangle + 2\sqrt{t(1-t)}|\phi_g\rangle$

And from previous slide: $Q|\phi_g\rangle = (1-2t)|\phi_g\rangle - 2\sqrt{t(1-t)}|\phi_b\rangle$

Q recursive step:

$$Q = \begin{bmatrix} (1-2t) & -2\sqrt{t(1-t)} \\ 2\sqrt{t(1-t)} & (1-2t) \end{bmatrix}$$

Compare to a rotation matrix:

$$R(2\theta) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\sin\theta = \sqrt{t} = g_0$

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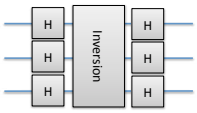
Grover vs Amplitude Amplification

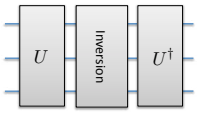
Grover



Angle of rotation:

$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Amplitude Amplification



Angle of rotation:

$$\sin\theta = \sqrt{t} = g_0$$

If you can construct a U with a higher probability of success than random guessing 1/N, then amplitude amplificaton can help.

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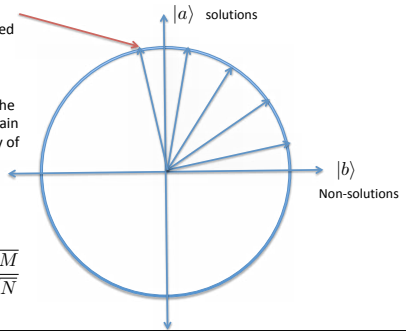
How to achieve 100% Success

The optimal, 100% probability of measuring marked can be missed.

Can we modify the algorithm to obtain 100% probability of success?

$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$



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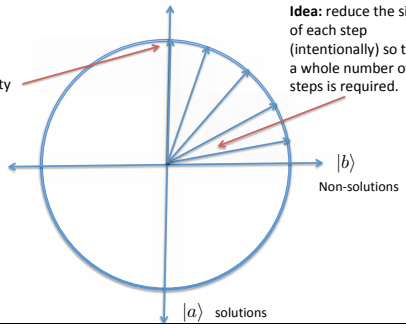
Grover with 100% success

Using amplitude amplification

This step gives 100% probability of finding the marked state

Idea: reduce the size of each step (intentionally) so that a whole number of steps is required.



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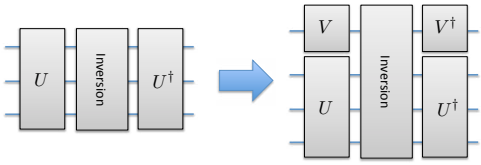
Reducing the angle

We want to reduce the angle of rotation in Grover's algorithm/ amplitude amplification so that we require a whole number of steps to achieve 100% probability of success.

Trick: Introduce a new qubit.



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How much reduction?

Previously:

$$U|0\rangle = g_0|\phi_g\rangle + b_0|\phi_b\rangle$$

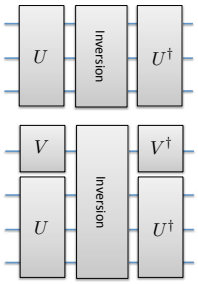
With new qubit:

$$V \otimes U|0\rangle = V|0\rangle \otimes (g_0|\phi_g\rangle + b_0|\phi_b\rangle)$$

If we arrange so that:

$$V|0\rangle = \sqrt{1-\left(\frac{g_0'}{g_0}\right)^2}|0\rangle + \frac{g_0'}{g_0}|1\rangle$$

e.g. Y-rotation by an angle: $\cos\frac{\alpha}{2} = \frac{g_0'}{g}$



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New rotation angle

$$V \otimes U|0\rangle = V|0\rangle \otimes (g_0|\phi_g\rangle + b_0|\phi_b\rangle)$$

$$V|0\rangle = \sqrt{1-\left(\frac{g_0'}{g_0}\right)^2}|0\rangle + \frac{g_0'}{g_0}|1\rangle$$

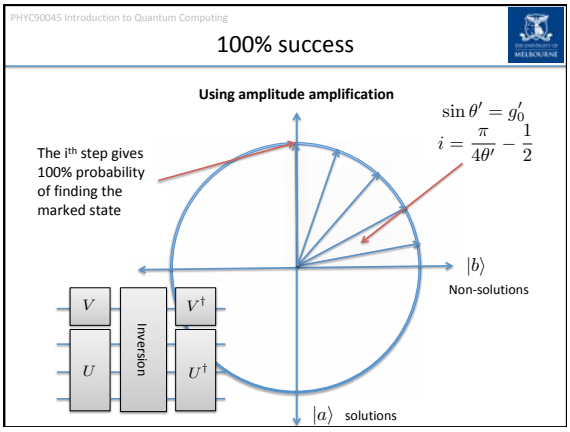
Gives:

$$V \otimes U|0\rangle = g_0'|1\rangle|\phi_g\rangle + \ldots$$

We can choose the initial amplitude to be anything value less than the original

Our new "good" states, but now have a preceding "1" on the extra qubit we added

Choose g_0' s.t. $i = \frac{\pi}{4\theta'} - \frac{1}{2}$ is a whole number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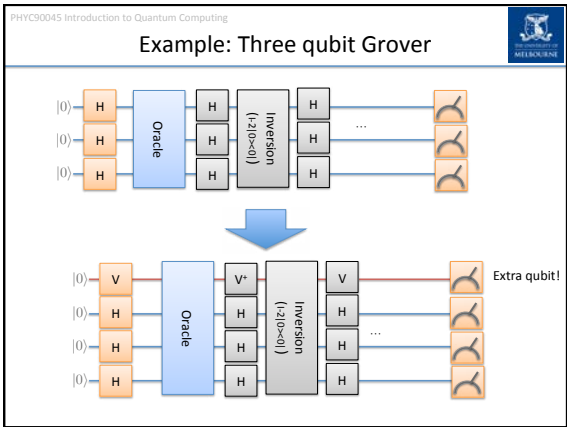
100% success
Using amplitude amplification
The i^th step gives 100% probability of finding the marked state
$\sin\theta' = g_0'$
$i = \frac{\pi}{4\theta'} - \frac{1}{2}$
$|b\rangle$
Non-solutions
$|a\rangle$ solutions
V
U
Inversion
$V^\dagger$
$U^\dagger$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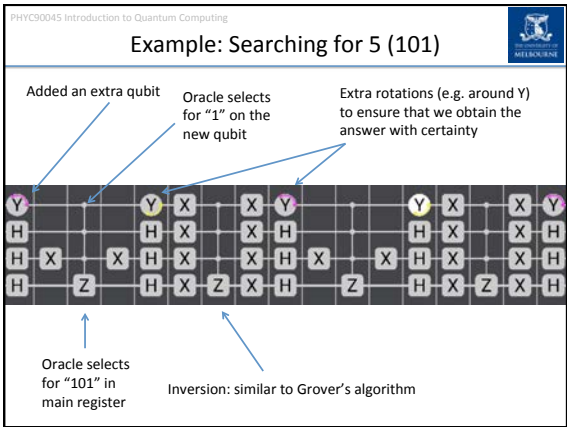
Example: Three qubit Grover
|0⟩
H
Oracle
Inversion (I-2|0><0|)
V
V+
Extra qubit!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Example: Searching for 5 (101)
Added an extra qubit
Oracle selects for "1" on the new qubit
Extra rotations (e.g. around Y) to ensure that we obtain the answer with certainty
Oracle selects for "101" in main register
Inversion: similar to Grover's algorithm

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Example: Searching for 5 (101)

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Example: Searching for 5 (101)

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Example: Searching for 5 (101)

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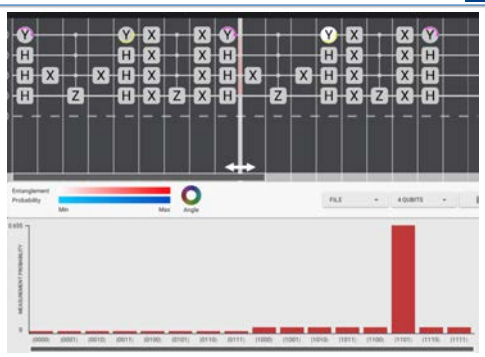
Example: Searching for 5 (101)



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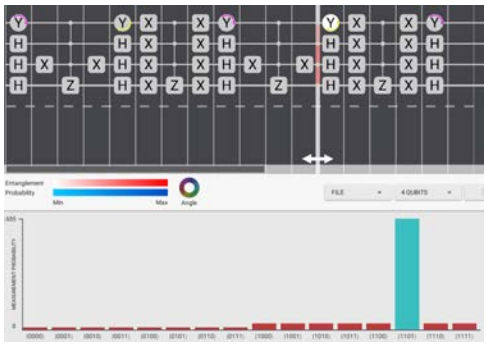
Example: Searching for 5 (101)



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Example: Searching for 5 (101)

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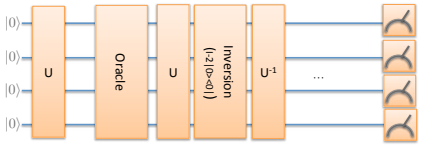
Amplitude Amplification

Given an black box (oracle), U_f, which computes the function $f:\{0,1\}^n \rightarrow \{0,1\}$

Find an x s.t. $f(x) = 1$

- Unordered search, generalisation of Grover's algorithm
- Classically, this requires N/2 uses of the oracle
- Quantum mechanically, requires only O(√N).



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Amplitude Amplification is optimal

Proof in your textbooks.

Grover's algorithm is optimal in terms of the number of applications of the oracle.

For many oracle problems the required number of uses of the oracle scales like:

$$O(\sqrt{N})$$

This means that for a broad range of problems the best speedup we can achieve using a quantum computer is **not** exponential, but polynomial (which can be quite significant).

For problems with identifiable structure, we might hope for more speedup. *More on this next week.*

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Quantum Counting

Will show you this algorithm now, but will leave some of the details until after next week's lectures/lab.

Given an black box (oracle), U_f, which computes the function $f:\{0,1\}^n \rightarrow \{0,1\}$

How many x s.t. $f(x) = 1$?

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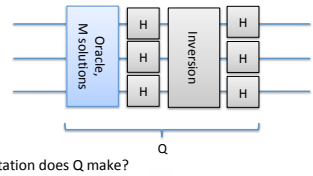
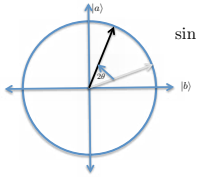
Equivalent question



What angle rotation does Q make?



$$\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

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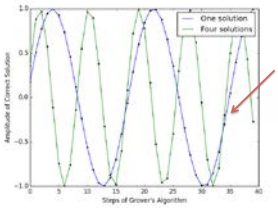
Plotting amplitude as function of step number

After k steps: $\theta_k = (2k+1)\theta$ $\sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$



Number of solutions is reflected in the period/frequency

Incorrect solutions would follow cosine, rather than sin

Amplitude at step 'k' is:

$$g_k = \sin(2k+1)\theta$$

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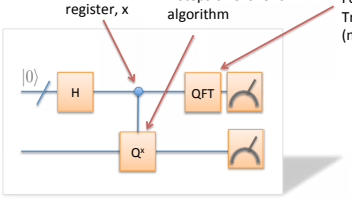
Finding the period of a periodic function

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_x |x\rangle \otimes Q^x |0\rangle$$

Control register, x

x steps of Grover's algorithm

Quantum Fourier Transform (next week)



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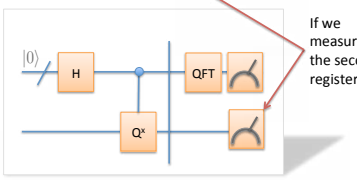
Finding the period of a periodic function

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle \otimes Q^x |0\rangle$$
$$= \frac{1}{\sqrt{N}} \sum_x |x\rangle \otimes (\sin(2x+1)\theta\,|\psi_g\rangle + \cos(2x+1)\theta\,|\psi_b\rangle)$$



If we measure the second register

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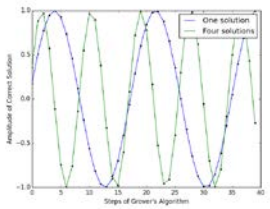
After measurement of the second register

$$|\psi\rangle = \sum \sin(2x+1)\theta\,|x\rangle \otimes |\psi_g\rangle \quad \text{(not normalized)}$$



Next step: Use Quantum Fourier Transformation to find the period

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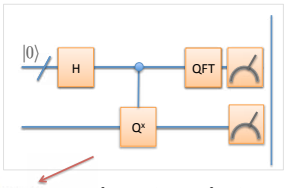
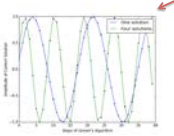
After the Fourier transformation



Dimension: N'

Dimension: N



After Fourier transforming a periodic function, we get a good approximation to theta. If we measure value "j":

$$\theta = \frac{j\pi}{N'} \qquad \sin\theta = \frac{\sqrt{M}}{\sqrt{N}}$$

Which we can solve to obtain the number of solutions, M

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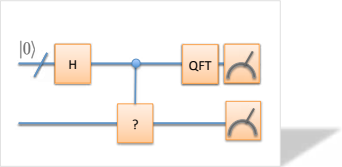
Phase Estimation and HSP Problems



The Hadamard and Fourier transform part is a known as **phase estimation**, and extremely useful for period functions (and eigenvalues which are periodic).

As we will see in the next lecture, this pattern is often repeated.

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This Week

Lecture 7

Introduction to Grover's algorithm for amplitude amplification, geometric interpretation

Lecture 8

Amplitude Amplification, Succeeding with Certainty, Quantum Counting

Lab

Grover's algorithm