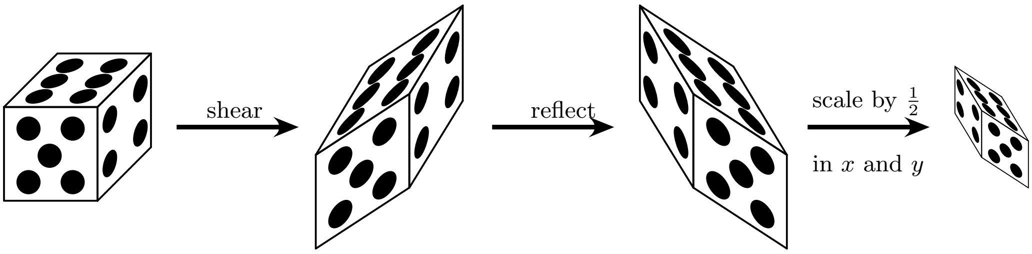
Tutorial 8

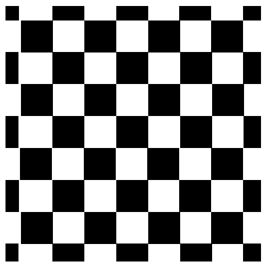
T is a *linear transformation* of a vector space if for all vectors $\mathbf{u}, \mathbf{v}$ and scalars α, β:

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}).$$

In $\mathbb{R}^n$, linear transformations are rotations, reflections, compressions/expansions, shears or some combination of these. For example, the dice below shows a shear followed by a reflection, followed by two compressions.

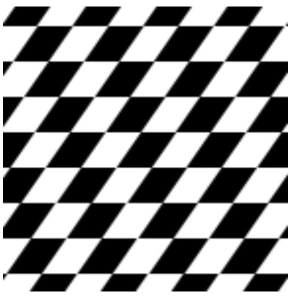


Q1. Which of the following transformations of the standard checkerboard (to the right) are linear? Identify which type(s) of linear transformation where possible.

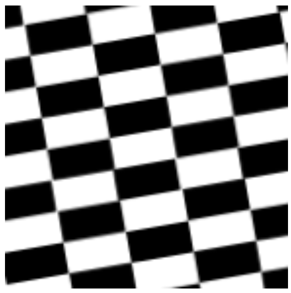


(i).

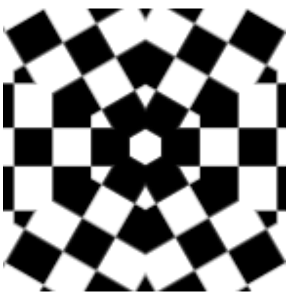
(ii).



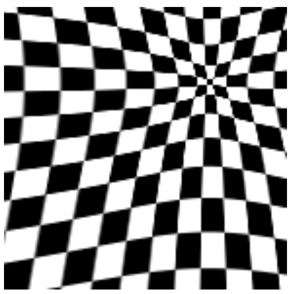
(iii).



(iv).



(v).



(vi).

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the matrix form, A_T is an $m \times n$ matrix so that $A_T\mathbf{v} = T(\mathbf{v})$. This is found by applying T to the standard basis vectors $(\mathbf{e_1}, \ldots, \mathbf{e_n})$, and then writing these as columns:

$$A_T = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}.$$

Q2. First draw the effect of the linear transformation on the square with corners $(0,0), (0,1), (1,0), (1,1)$, and then find its matrix representation:

(i). R, rotation around the origin by $\pi/4$.

(ii). M, reflection in the line $y = -x$

(iii). S, shear along the y-axis (factor of 3).

> Let S and T be linear transformations with matrix forms A_S and A_T, respectively. Then the linear transformation where S is applied first, and then T, has the matrix form $A_T A_S$.

Q3. Using the linear transformations from Question 2, give the matrix form for T, where T is composed of R, then S, then M. What is the effect on the vector $(1, -2)$?

Q4. Consider the linear transformation $R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ where it is *clockwise* rotation around the origin by θ.

(i). Find a matrix form for R_θ.

(ii). Show that $R_\theta R_\phi = R_{\theta+\phi}$

> Let $T \colon U \to V$ be a linear transformation. Then
>
> - The *kernel* (also *nullspace*) of T is
>
> $$\ker(T) = \{\mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0}\}$$
>
> - The *image* (or *range*) of T is
>
> $$\mathrm{Im}(T) = \{\mathbf{v} \in V | \mathbf{v} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in U\}$$
>
> The kernel is a subspace of U with $\mathrm{nullity}(T) = \dim(\ker(T))$; and the image is a subspace of V with $\mathrm{rank}(T) = \dim(\mathrm{Im}(T))$.

Q5. Consider the linear transformation T with

$$T(x_1, x_2, x_3, x_4) = (2x_1 - 3x_2 - x_4, x_1 + 3x_3 + 4x_4, x_2 + 2x_3 + 3x_4).$$

(i). Find a matrix representation for T.

(ii). Find a basis for the image of T.

(iii). Find a basis for the kernel of T.

(iv). What is $\mathrm{rank}(T) + \mathrm{nullity}(T)$?