MAST10007 Assignment 5 Solutions

1 (a) We are given $P_{C,B} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

For a general vector $\underset{\sim}{u}$ we have

$$P_{C,B}[\underset{\sim}{u}]_B = [\underset{\sim}{u}]_C$$

Here $\underset{\sim}{u} = \underset{\sim}{b}_1 + \underset{\sim}{b}_2 + 2\underset{\sim}{b}_3$ and so $[\underset{\sim}{u}]_B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Hence

$$[\underset{\sim}{u}]_C = P_{C,B}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

and so $\underset{\sim}{u} = \frac{1}{2}\underset{\sim}{c}_1 + \underset{\sim}{c}_2 + \frac{1}{2}\underset{\sim}{c}_3$

(b) $P_{B,C} = P_{C,B}^{-1} = 2\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

Now

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{R_3 \leftrightarrow R_1} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right] R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array}\right]_{R_3 + R_2} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right]_{R_3/2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] R_2 - R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right]$$

Hence $P_{B,C} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ \end{bmatrix}$

[redacted]

(c) We are given $\underset{\sim}{c}_1 = (1,2,3)$, $\underset{\sim}{c}_2 = (1,2,0)$, $\underset{\sim}{c}_3 = (1,0,0)$.

It follows that

$$P_{S,C} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

To make use of this, we note

$$P_{S,B} = P_{S,C} P_{C,B} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 \\ -2 & 0 & 4 \\ 0 & -3 & 3 \end{bmatrix}$$

We read off that $\underset{\sim}{b}_1 = \frac{1}{2}(0,-2,0)$ $\underset{\sim}{b}_2 = \frac{1}{2}(0,0,-3)$ $\underset{\sim}{b}_3 = \frac{1}{2}(2,4,3)$.

2. (a) We have

$$T\underset{\sim}{i} = (1,0,-1) + (1,1,1) = (2,1,0)$$

$$T\underset{\sim}{j} = (1,1,1)$$

$$T\underset{\sim}{k} = -1 \times (1,0,-1) + (1,1,1) = (0,1,2)$$

(b) $$A_T = [T\underset{\sim}{i} \;\; T\underset{\sim}{j} \;\; T\underset{\sim}{k}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(c) We know that $\operatorname{Im} T$ is the same thing as the column space of A_T. To compute the column space:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} R_2 - \tfrac{1}{2}R_1 \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} R_3 - 2R_2 \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The leading entries are in the first and second columns, and so

$$\operatorname{Im} T = \operatorname{Span}\{(2,1,0),(1,1,1)\}$$

It follows from this that $\dim \operatorname{Im} T = 2$.

(d) We know that $\operatorname{Ker} T$ is the same thing as the solution space of A_T. From the RE form of A_T we read off that there is no leading entry for z, so we set $z = t$, $t \in \mathbb{R}$. Back substitution then gives

$$y = -2t, \qquad x = -\tfrac{1}{2}y = t.$$

Hence the solution space and thus $\operatorname{Ker} T$ is

$$\{(x,y,z) = t(1,-2,1), \quad t \in \mathbb{R}\}$$

We read off from this that $\dim \operatorname{Ker} T = 1$.

(e) We have

volume of the image of the unit cube $= |\det[T\underline{i}\ T\underline{j}\ T\underline{k}]|$
$= |\det A_T| = 0,$

using the working from (c).