The University of Melbourne
Department of Mathematics and Statistics
Summer Semester Exam 2012

MAST10007 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 7 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Hand-held electronic calculators may be used.. Candidates are reminded that no written, printed or electronically stored material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may **not** remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 12 questions. The total number of marks is 80. All questions may be attempted.

This paper may be held by the Baillieu Library.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Manny and Louise were born on the same day, but in different years. In 7 years time Manny will be twice as old as he was when Louise was 9 years younger than she is now, and Louise will be one year older than Manny is now.

   (a) Let L and M denote Louise and Manny's present ages, and let l and m denote their ages in a past era as relevant to the wording. One equation relating these unknowns is
   $$M - L = m - l$$
   Give three more equations for the four unknowns.

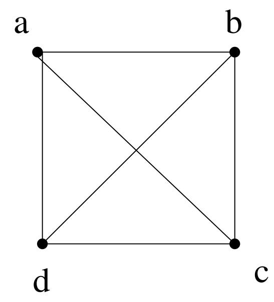
   (b) Show that the equations in (a) are equivalent to the augmented matrix
   $$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -7 \\ 0 & 1 & 0 & -1 & 9 \\ 1 & -1 & 0 & 0 & 6 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right]$$

   (c) Find the values of the four unknowns.

[7 marks]

2. (a) Let A be a $p \times q$ matrix. Determine the size of B for the matrix product ABA to be well defined.

   (b) Consider the graph



   i. Write down the adjacency matrix A of the graph.
   ii. Use A to calculate the number of walks from vertex a back to itself using exactly 3 edges.

   (c) Find the rank of the matrix $Z = XY$, where
   $$X = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}.$$

[7 marks]

3. Suppose on Valentine's day earlier this week you received a coded message from an Italian friend. You know that the message, consisting of 4 letters, was written as numbers by the correspondence $A \leftrightarrow 1$, $B \leftrightarrow 2$ etc., then put into a 2×2 matrix down successive columns, and finally coded by multiplication on the left by the matrix product

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

The message received was

$$24, \ 11, \ 51, \ 24$$

(a) Calculate

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \right)^{-1}$$

(b) Use your answer to (a) to decode the message.

(c) Your response, is the English word HUGS. Using the same coding procedure, what message would your Italian friend receive?

[7 marks]

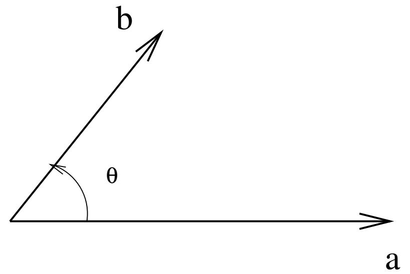
4. (a) Consider the plane through the origin in $\mathbb{R}^3$ defined by

$$\text{Span}\,\{(1, 0, -1), (1, 1, 0)\}.$$

i. Give the vector equation of the plane.

ii. Calculate the Cartesian equation of the plane.

(b) Consider the parallelogram corresponding to the following two vectors



i. Derive a formula for the area of the parallelogram in terms of the length of **a**, the length of **b** and the angle θ.

ii. Calculate the area of the parallelogram with corners at $(1, 1, 1)$, $(2, 2, 2)$, $(1, 2, 2)$, $(2, 3, 3)$.

[7 marks]

5. (a) Subspaces of $\mathbb{R}^2$ can have what dimensions? For each possibility, use geometrical terms to describe the subspace.

   (b) i. A polynomial is said to be palindromic if its coefficients are the same when read left to right or right to left. Show that a general palindromic polynomial in $\mathcal{P}_2$ is of the form
$$a + bx + ax^2.$$

   ii. By making a correspondence between members of $\mathcal{P}_2$ and vectors $(a, b, c) \in \mathbb{R}^3$, write the subspace of $\mathbb{R}^3$ corresponding to palindromic polynomials in $\mathcal{P}_2$ as a span. What is the dimension of this span?

   iii. Show that $S = \{(a, b, a) \in \mathbb{R}^3\}$ is closed under vector addition.

[6 marks]

6. Let A be a 4×5 matrix with columns given by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_5$, and let $\mathbf{b}$ be a column vector with 4 rows. Suppose that

$$[A|\mathbf{b}] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   (a) What is the dimension of the column space of A? Give a reason.

   (b) What is the dimension of the solution space of A? Give a reason.

   (c) Write the row space of A as a span.

   (d) Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_4$.

   (e) Find the general solution of the matrix equation $A\mathbf{x} = \mathbf{b}$.

   (f) For $\mathbf{c}$ a column vector with 4 rows, suppose that $\mathbf{c}$ does not belong to the column space of A. What can be said about the solution of the matrix equation $A\mathbf{x} = \mathbf{c}$? Give a reason.

[7 marks]

7. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that $T(1,0) = (0,-1)$ and $T(0,1) = (1,0)$.

    i. Write down the standard matrix A_T of T.

    ii. Illustrate on a diagram how T maps the unit square formed by $\mathbf{i}$ and $\mathbf{j}$.

    iii. From your diagram describe T geometrically. Hence explain why T does not have any eigenspaces.

   (b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is a shear by 1 unit in the x direction.

    i. Draw a diagram of the action of S on the unit square formed by $\mathbf{i}$ and $\mathbf{j}$.

    ii. A shear does not change the area of the unit square. What does this tell us about the determinant of the standard matrix corresponding to S, A_S.

    iii. Write down the standard matrix for the transformation in the plane which first shears by one unit in the x direction, then shears by -1 unit in the x-direction. Give a reason.

[7 marks]

8. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ specified by

$$T\mathbf{x} = \frac{1}{2}(\mathbf{x} \cdot (1,0,1))(1,0,1) + \frac{1}{3}(\mathbf{x} \cdot (-1,1,1))(-1,1,1)$$

   (a) Describe T geometrically.

   (b) Calculate $T\mathbf{i}$, $T\mathbf{j}$, $T\mathbf{k}$.

   (c) Show that the standard matrix A_T is given by

$$A_T = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

   (d) Write $\operatorname{Im} T$ as a span, and give a basis.

[6 marks]

9. You are given that the change of basis matrix $P_{\mathcal{C},\mathcal{B}}$ from the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ to the basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is

$$P_{\mathcal{C},\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Express the vector $\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ as a linear combination of the vectors in $\mathcal{C}$.

(b) Calculate the change of basis matrix $P_{\mathcal{B},\mathcal{C}}$.

(c) Suppose

$$\mathbf{c}_1 = (1, 0, 0), \qquad \mathbf{c}_2 = (1, 2, 0), \qquad \mathbf{c}_3 = (1, 2, 3)$$

Write down the explicit form of the change of basis matrix $P_{\mathcal{S},\mathcal{C}}$, where $\mathcal{S}$ denotes the standard basis in $\mathbb{R}^3$.

(d) Let $\mathbf{y} = \mathbf{b}_1 + \mathbf{b}_3$. With $\mathcal{C}$ specified as in (c), compute $\mathbf{y}$ in the standard basis.

[6 marks]

10. (a) Consider the usual dot product in $\mathbb{R}^3$,

$$(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3$$

Let $\langle (u_1, u_2), (v_1, v_2) \rangle$ be defined by eliminating the third components in this formula, by requiring that all points (x, y, z) lie on the plane $x + y + z = 0$. Show that

$$\langle (u_1, u_2), (v_1, v_2) \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2.$$

(b) Write the above formula in the form

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for certain a, b, c, d.

(c) What properties of the above 2×2 matrix tell us that $\langle (u_1, u_2), (v_1, v_2) \rangle$ defines an inner product? (There is no need to check these properties.)

(d) Using the inner product of (b), find the orthogonal projection of the vector $(1, 1)$ in the direction of the vector $(1, 2)$.

[6 marks]

11. A biologist has measured that a particular species of crickets chirp at a rate R per minute, dependent on the temperature T degrees Celcius, according to the following table

T	R
2	5
4	9
5	10

(a) Use the method of least squares to find the line of best fit for this data. Show all your working.

(b) Illustrate the data, together with the line found in (a) on a graph.

[7 marks]

12. Let

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find the corresponding eigenspaces.

(c) Give a reason why A is diagonalizable.

(d) Calculate A^5.

[7 marks]