COMP30026 Models of Computation

Busy Beavers

Harald Søndergaard

Lecture Appendix

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Preliminaries

For the purpose of this (non-examinable) lecture, let us assume that we are dealing with two-way Turing machines.

That is, the tape is infinite in both directions.

Busy Beavers

Consider Turing machines with tape alphabet $\{\sqcup, 1\}$.

Starting with an empty tape, a machine may write a string of 1s and halt.

If the machine halts, its productivity is the number of 1s written.

If it does not halt, its productivity is 0.

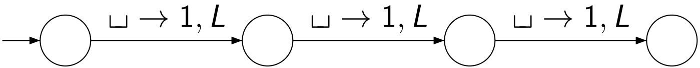
Define

$$\Sigma(n) = \left\{ \begin{array}{l} \text{the productivity of the most} \\ \text{productive } n\text{-state machines} \end{array} \right.$$

An optimally productive machine is a busy beaver.

Busy Beavers

Here is a trivial machine T_3 which has productivity 3:



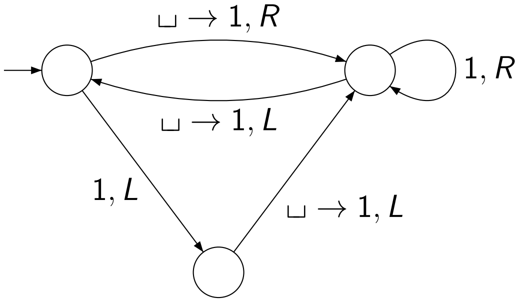
(The Turing machine model used here is the two-way tape machine with anonymous halting state.)

Of course there are more productive 4-state machines.

But this shows that there are $(n+1)$-state machines T_n with productivity n.

A Busy Beaver with 3 States

Here is a proof that $\Sigma(3) \geq 6$:



About Busy Beavers

It is known that

$$\begin{array}{rcrl} \Sigma(1) & = & 1 & \text{(trivial)} \\ \Sigma(2) & = & 4 & \text{(easy)} \\ \Sigma(3) & = & 6 & \text{(fairly hard)} \\ \Sigma(4) & = & 13 & \text{(hard)} \end{array}$$

In October 2016, what is known about $\Sigma(5)$ is that it is at least 4098.

Out of about 88 million 5-state machines, some 40 remain to be classified.

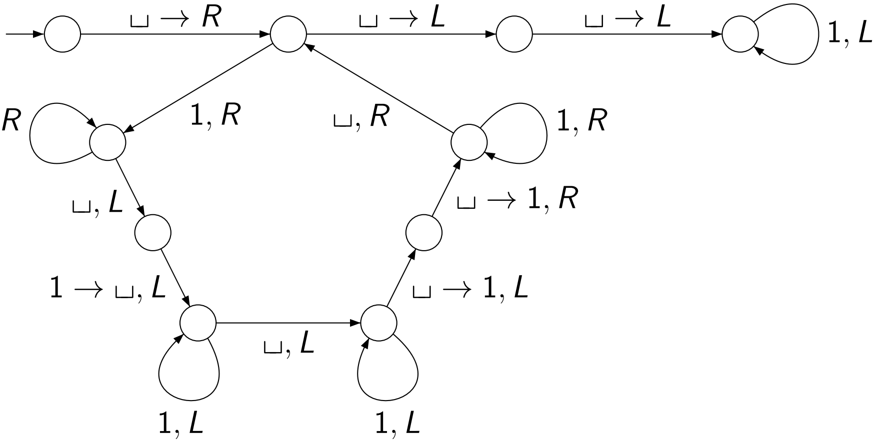
$\Sigma(6) > 10^{18267}$.

Less than one third of the 6-state machines have been classified.

Clearly Σ is a very fast-growing function.

Σ Is Not Computable

Here is a 10-state Turing machine *D* to double a string of 1s:



Σ Is Not Computable

If we run D after T_n, we have a machine of productivity $2n$.

$$\longrightarrow \boxed{T_n} \longrightarrow \boxed{D} \longrightarrow$$

We have

$$\Sigma(n+10) \geq 2n \tag{1}$$

To show that Σ is not computable, assume that we have a machine BB which computes Σ:

$$n \mapsto \Sigma(n)$$

Let k be the number of states in BB. The following machine:

$$\longrightarrow \boxed{T_n} \longrightarrow \boxed{BB} \longrightarrow \boxed{BB} \longrightarrow$$

shows that

$$\Sigma(n+2k-1) \geq \Sigma(\Sigma(n)) \tag{2}$$

Σ Is Not Computable

In summary:

$$\Sigma(n+10) \geq 2n \quad (1)$$
$$\Sigma(n+2k-1) \geq \Sigma(\Sigma(n)) \quad (2)$$

Note that Σ is total. Σ is also isotone and strictly increasing, so

$$\Sigma(j) \geq \Sigma(i) \Rightarrow j \geq i.$$

Hence for all n,

$$n + 2k - 1 \geq \Sigma(n).$$

In particular,

$$n + 2k + 9 \geq \Sigma(n+10) \geq 2n$$

by (1). So $2k + 9 \geq n$ for all n.

But this is clearly false—take $n = 2k + 10$.

We conclude that BB does not exist.

A Link to Undecidability

We can think of the busy beaver problem as a decidability problem.

If we can decide $\Sigma(n) \leq k$ for all n and k, then we can compute Σ, and vice versa.

So in showing Σ uncomputable we have produced an undecidable problem, without relying on any other results about undecidability.

Turing Machine Halting Is Undecidable

Consider the question of whether a given Turing machine halts when started on a blank tape.

We can reduce the problem of computing Σ to this "Turing machine halting-on-blank-tape" problem.

Namely, if the halting problem was decidable then we could compute $\Sigma(n)$ as follows:

1. Generate all n-state Turing machines.
2. Filter out all the non-terminating machines.
3. Run the rest until all have halted.
4. Pick the most productive machine.
5. Print its productivity.