THE UNIVERSITY OF MELBOURNE
SCHOOL OF COMPUTING AND INFORMATION SYSTEMS
COMP30026 MODELS OF COMPUTATION

Sample Answers to Tutorial Exercises, Week 5

27. These are the clauses generated:

    (a) For each node i generate the clause $B_i \vee G_i \vee R_i$. That's $n+1$ clauses of size 3 each.

    (b) For each node i generate three clauses: $(\neg B_i \vee \neg G_i) \wedge (\neg B_i \vee \neg R_i) \wedge (\neg G_i \vee \neg R_i)$. That comes to $3n+3$ clauses of size 2 each.

    (c) For each pair (i, j) of nodes with $i < j$ we want to express $E_{ij} \Rightarrow (\neg(B_i \wedge B_j) \wedge \neg(G_i \wedge G_j) \wedge \neg(R_i \wedge R_j)$. This means for each pair (i, j) we generate three clauses: $(\neg E_{ij} \vee \neg B_i \vee \neg B_j) \wedge (\neg E_{ij} \vee \neg G_i \vee \neg G_j) \wedge (\neg E_{ij} \vee \neg R_i \vee \neg R_j)$. There are $n(n+1)/2$ pairs, so we generate $3n(n+1)/2$ clauses, each of size 3.

    Altogether we generate $3n + 3 + 6n + 6 + 9n(n + 1)/2$ literals, that is, $9(n + 1)(n/2 + 1)$.

28. Let us give names to the propositions:

    - C: Ann clears 2 meters
    - F: Ann gets the flu
    - K: The selectors are sympathetic
    - S: Ann is selected
    - T: Ann trains hard

    The four assumptions then become:

    (a) $C \Rightarrow S$
    (b) $T \Rightarrow (F \Rightarrow K)$
    (c) $(T \wedge \neg F) \Rightarrow C$
    (d) $K \Rightarrow S$

    It is easy to see that S is not a logical consequence of these, as we can give all five variables the value *false*, and all the assumptions will thereby be true.

    To see that $T \Rightarrow S$ is a logical consequence of the assumptions, we can negate it, obtaining $T \wedge \neg S$. Then, translating everything to clausal form, we can use resolution to derive an empty clause.

    Alternatively, note that $T \Rightarrow (F \Rightarrow K)$ is equivalent to $(T \wedge F) \Rightarrow K$. Since also $K \Rightarrow S$, we have $(T \wedge F) \Rightarrow S$. Similarly, $(T \wedge \neg F) \Rightarrow C$ together with $C \Rightarrow S$ gives us $(T \wedge \neg F) \Rightarrow S$.

    But from $(T \wedge F) \Rightarrow S$ and $(T \wedge \neg F) \Rightarrow S$ we get $T \Rightarrow S$. (You may want to check that by massaging the conjunction of the two formulas.)

29. Let us give names to the propositions:

    - A: The commissioner apologises
    - F: The commissioner can attend the function
    - R: The commissioner resigns

The four statements then become

(a) $F \Rightarrow (A \wedge R)$
(b) $(R \wedge A) \Rightarrow F$
(c) $R \Rightarrow F$
(d) $F \Rightarrow A$

The first translation may not be obvious. But to say "X does not happen unless Y happens" is the same as saying "it is not possible to have X happen and at the same time Y does not happen." That is, $\neg(X \wedge \neg Y)$, which is equivalent to $X \Rightarrow Y$. Note that (a) entails (d) and (c) entails (b).

30. (a) $\neg P$ becomes $\boxed{P \oplus \mathbf{t}}$. With XNF it is perhaps more natural to use 0 for $\mathbf{f}$ and 1 for $\mathbf{t}$. We really are dealing with arithmetic modulo 2, $\oplus$ playing the role of addition, and $\wedge$ playing the role of multiplication.
(b) $P \wedge Q$ is unchanged, or we can write simply $\boxed{PQ}$.
(c) $P \wedge \neg Q$ can be written $P(Q \oplus \mathbf{t})$, so by "multiplying out" we get $\boxed{PQ \oplus P}$.
(d) $P \Leftrightarrow Q$ becomes $\boxed{P \oplus Q \oplus \mathbf{t}}$ (since biimplication is the negation of exclusive or).
(e) $P \vee Q$ becomes $\boxed{P \oplus Q \oplus PQ}$, as truth tables will confirm. But how could we *discover* that solution? Well, we now know how to deal with negation and disjunction, and so we can make use of the fact that $P \vee Q \equiv \neg(\neg P \wedge \neg Q)$. This way we arrive at $\mathbf{t} \oplus ((\mathbf{t} \oplus P)(\mathbf{t} \oplus Q))$. Now all we need to do is to simplify that formula (come on, do it).
(f) Using the insight from part (e), we transform $P \vee (Q \wedge R)$ to $\boxed{P \oplus QR \oplus PQR}$.

31. (a) Negation is just '$\oplus\, \mathbf{t}$', so we can write $\neg(P \oplus Q)$ as $\boxed{\mathbf{t} \oplus P \oplus Q}$.
(b) For $(P \oplus Q) \wedge R$, we just "multiply out", to get $\boxed{PR \oplus QR}$.
(c) Given $(PQ \oplus PQR \oplus R) \wedge (P \oplus Q)$ we again multiply out. There will be six products, but some will cancel out:
$$\begin{aligned} & PQP \oplus PQRP \oplus PR \oplus PQQ \oplus PQRQ \oplus QR \\ = \; & PQ \oplus PQR \oplus PR \oplus PQ \oplus PQR \oplus QR \\ = \; & \boxed{PR \oplus QR} \end{aligned}$$
(d) Given $Q \wedge (P \oplus PQ \oplus \mathbf{t})$ we multiply out, to get $PQ \oplus PQ \oplus Q$, that is, $\boxed{Q}$.
(e) From the previous question we know that $A \vee B \equiv A \oplus B \oplus AB$. Applying that to $Q \vee (P \oplus PQ)$ we obtain the formula $Q \oplus P \oplus PQ \oplus (Q(P \oplus PQ))$. Multiplying out, we get
$$Q \oplus P \oplus PQ \oplus PQ \oplus PQ = \boxed{P \oplus Q \oplus PQ}$$

32. An empty LHS is true because that's the neutral element for $\wedge$. An empty RHS is false because that's the neutral element for $\vee$.

(a) We get, step by step:

$$\begin{array}{rcll} & \vdash & \neg(\neg P \wedge Q) \vee (\neg P \vee R) & \text{Split } \vee \text{ on right} \\ & \vdash & \neg(\neg P \wedge Q), \neg P \vee R & \text{Move } \neg(\neg P \wedge Q) \text{ to left, dropping negation} \\ \neg P \wedge Q & \vdash & \neg P \vee R & \text{Split } \vee \text{ on right} \\ \neg P \wedge Q & \vdash & \neg P, R & \text{Split } \wedge \text{ on left} \\ \neg P, Q & \vdash & \neg P, R & \end{array}$$

The sequent is now fully reduced. Since we find some literal ($\neg P$) that appears on both sides, the original formula was valid.

(b) We do this for $\Rightarrow$ only. Suppose we have $S \vdash (F \Rightarrow F') : S'$, that is, we find $(F \Rightarrow F') : S'$ on the RHS. This is the same as $(\neg F \vee F') : S'$, so the correct manipulation is to split the $\vee$ and then move F to the other side, dropping the negation. That is, we change the sequent $S \vdash (F \Rightarrow F') : S'$ to $F : S \vdash F' : S'$.

For the case where we find the arrow on the left-hand side, we have: $(F \Rightarrow F') : S \vdash S'$, which is the same as $(\neg F \vee F') : S \vdash S'$. Here we need to generate two sequents, splitting the $\vee$ on the left: $\neg F : S \vdash S'$ and $F' : S \vdash S'$. In the former, F is then moved to the right-hand side, dropping the negation, so we end up with $S \vdash F : S'$ and $F' : S \vdash S'$.

34. In each case below we use two interpretations over the same domain. That is just a coincidence—we could have chosen differently.

(a) Let the domain be very simple, say $\{0\}$. Interpreting P as '$=$' makes $\forall x \forall y (P(x, y))$ true. Interpreting P as '$\neq$' makes $\forall x \forall y (P(x, y))$ false.

(b) Let the domain be the set of natural numbers, $\mathbb{N}$. Interpreting P as '$=$' makes this true. Interpreting P as '$<$' makes it false.

(c) The same interpretation will work here.

35. Let I be an interpretation that satisfies $\forall x (P(x))$. If the domain of I is D then the relation that P is assigned to is the *full* relation, that is, the one that holds for every $d \in D$. Since D is assumed to be non-empty, there is some $d \in D$ for which P holds, so I satisfies $\exists y (P(y))$.

The converse does not hold. Let I be the interpretation with domain $\mathbb{Z}$ (the set of integers), and let P stand for the relation "is zero". Then I satisfies $\exists y (P(y))$ but not $\forall x (P(x))$.

36. We can use the rules of passage for the quantifiers. First note that the sub-formula

$$\Big(P(x) \Rightarrow \forall y \forall z (Q(y, z)) \Big)$$

has no free occurrence of y or z. Hence we can simply drop the quantifiers $\forall y \exists z$ that we find in front of that sub-formula, which leaves us with

$$\forall x \Big(P(x) \Rightarrow \forall y \forall z (Q(y, z)) \Big).$$

Since the right-hand side of the arrow has no free occurrence of x, we can push the remaining universal quantifier in, which yields $\exists x (P(x)) \Rightarrow \forall y \forall z (Q(y, z))$. This is of the required form. If you wonder about the universal quantifier turning into an existential quantifier, complete this exercise by filling in the details (start by rewriting the implication to a disjunction).