The University of Melbourne
School of Computing and Information Systems
COMP30026 Models of Computation

Selected Tutorial Solutions, Week 7

47. The way we used Haskell was brute force. For example, Fermat's conjecture was a universal statement claiming something to be true for all natural numbers n (namely $2^{2^n}+1$ is prime). Since it is easy to use Haskell to generate the natural numbers systematically, we could refute it by proving its negation. The negation is this existential statement: there is some n for which $2^{2^n}+1$ is not prime. (This turned out to be the case for $n = 5$, since $2^{32}+1 = 6700417 \cdot 641$.)

    Here is why there cannot be any prime triples other than $(3, 5, 7)$. Assume that $p > 3$. Out of p, $p+2$, and $p+4$, one must be divisible by 3, and so it is not a prime. The following table shows all the possible remainders of the three numbers, after division by 3:

p	0	1	2
$p+2$	2	0	1
$p+4$	1	2	0

    In all cases, one of the three is divisible by 3.

    Notice that this proof is not by brute force. Its critical step is to identify an essential property of prime triples and use that, rather than simply enumerate-and-test.

48. The formula $\exists x\ (P(x) \Rightarrow \neg H(x))$ can be written $\exists x\ (\neg P(x) \vee \neg H(x))$, which in turn can be rephrased as $\neg\forall x\ (P(x) \wedge H(x))$. In other words, it says that it is *not* the case that everybody is an honest politician.

    Now let us look at the interpretation I that has domain $D = \{\mathit{turnbull}, \mathit{shorten}, \mathit{skippy}\}$, with $H(\mathit{turnbull}) = H(\mathit{shorten}) = H(\mathit{skippy}) = \mathit{true}$, $P(\mathit{turnbull}) = P(\mathit{shorten}) = \mathit{true}$, and $P(\mathit{skippy}) = \mathit{false}$. The interpretation I makes $\exists x\ (P(x) \wedge \neg H(x))$ false, while making $\exists x\ (P(x) \Rightarrow \neg H(x))$ true. In the "honest politician" reading, I makes the latter formula true, because *skippy* is not a politician, and therefore not an honest politician (*skippy* is an honest kangaroo).

    Now you may disagree with the interpretation I, but that's irrelevant. Our task was simply to find *some* interpretation that makes one statement false and the other true.

49. (a) We can negate both sides of the biimplication, so we just need to show:

    $$A \subseteq B \Leftrightarrow A \setminus B = \emptyset$$

    The left-hand side is, by definition: $\forall x(x \in A \Rightarrow x \in B)$. The right-hand side can be written: $\neg\exists y(y \in A \wedge y \notin B)$. Pushing the negation in, we get $\forall y(y \notin A \vee y \in B)$, or equivalently, $y \in A \Rightarrow y \in B$.

    (b) It is easier to look at the logical expressions. The left-hand side is $\{x \mid x \in A \wedge x \in B\}$. The right-hand side is

    $$\begin{array}{ll} & \{x \mid x \in A \wedge x \notin A \setminus B\} \\ = & \{x \mid x \in A \wedge \neg(x \in A \wedge x \notin B)\} \\ = & \{x \mid x \in A \wedge (x \notin A \vee x \in B)\} \\ = & \{x \mid x \in A \wedge x \in B\} \end{array}$$

50. These are simpler expressions:

    (a) $A \oplus B = A$ is equivalent to $B = \emptyset$.

    (b) $A \oplus B = A \setminus B$ is equivalent to $B \subseteq A$.

    (c) $A \oplus B = A \cup B$ is equivalent to $A \cap B = \emptyset$.

    (d) $A \oplus B = A \cap B$ is equivalent to $A \cup B = \emptyset$.

    (e) $A \oplus B = A^c$ is equivalent to $B = X$, assuming a universal set X.

51. We certainly do not have $A \times A = A$. In fact, no member of A is a member of $A \times A$, and no member of $A \times A$ is a member if A. So $\times$ is not absorptive.

    Neither is it commutative. Let $A = \{0\}$ and $B = \{1\}$. Then $A \times B = \{(0,1)\}$ while $B \times A = \{(1,0)\}$, and those singleton sets are different, because the members are.

    If we also define $C = \{2\}$ then $A \times (B \times C) = \{(0,(1,2))\}$ while $(A \times B) \times C = \{((0,1),2)\}$. Again, these are different. However, it is not uncommon to identify both of $(0,(1,2))$ and $((0,1),2)$ with the triple $(0,1,2)$ ("flattening" the nested pairings). If we agree to do that then $\times$ is associative, and we can simply write $A \times B \times C$ for the set of triples.

52. The statement is false, as we have, for example, $\{42\} \times \emptyset = \emptyset \times \{42\} = \emptyset$, but $\emptyset \neq \{42\}$.