# Data Structures and Algorithms

## 8. Search Algorithms

Basic Stuff You're Gonna Need to Search over a Graph
Where To Search Next?

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# Path-finding in graphs

**Search algorithms** over **directed graphs**:

- The **search nodes** (vertex) of the graph represent some information
- The edges $(v, v')$ capture transitions

One way to understand search algorithms is to think about **path-finding** over **graphs**.

# Classification of Search Algorithms

**Blind search vs. heuristic (or informed) search:**

- Blind search algorithms: Only use the basic ingredients for general search algorithms.
  - *e.g., Depth First Search (DFS), Breadth-first search (BrFS), Uniform Cost (Dijkstra), Iterative Deepening (ID)*
- Heuristic search algorithms: Additionally use heuristic functions which estimate the distance (or remaining cost) to reach an target vertex.
  - *e.g., A*, IDA*, Hill Climbing, Best First, WA*, DFS B&B, LRTA*, . . .*

**Systematic search vs. local search:**

- Systematic search algorithms: Consider a large number of search nodes simultaneously.
- Local search algorithms: Work with one (or a few) candidate solutions (search nodes) at a time.

$\rightarrow$ This is not a black-and-white distinction; there are *crossbreeds* (e.g., enforced hill-climbing).

# Blind Search

**Blind search:**

→ Here, we cover the subset of search algorithms. Only some Blind search algorithms are covered. Some Algorithms do not appear in Skiena's book.

# Agenda

1. Basics
2. Blind Systematic Search Algorithms

## Search Terminology

**Search node $n$:** Contains a *vertex* reached by the search, plus information about how it was reached.

**Path cost $g(n)$:** The cost of the path reaching $n$.

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- **Search nodes $\sigma$**: Search vertexes, plus information on where/when/how they are encountered during search.

**What is in a search node?**

Different search algorithms store different information in a search node $\sigma$, but typical information includes:

- *Vertex*$(\sigma)$: Associated search vertex.
- *parent*$(\sigma)$: Pointer to search node from which $\sigma$ is reached.
- $g(\sigma)$: Cost of $\sigma$ (cost of path from the root node to $\sigma$).

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# Criteria for Evaluating Search Strategies

**Guarantees:**

Completeness: Is the strategy guaranteed to find a solution when there is one?

Optimality: Are the returned solutions guaranteed to be optimal?

**Complexity:**

Time Complexity: How long does it take to find a solution? (Measured in generated nodes.)

Space Complexity: How much memory does the search require? (Measured in nodes.)

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Branching factor $b$: How many successors does each node have?

Goal depth $d$: The number of actions required to reach the shallowest goal vertex.

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- Blind search does not require any input beyond the graph.

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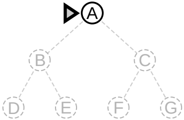
**Blind search strategies we'll discuss:**

- Breadth-first search. Advantage: time complexity.
  Variant: Uniform cost search.
- Depth-first search. Advantage: space complexity.
- Iterative deepening search. Combines advantages of breadth-first search and depth-first search. Uses depth-limited search as a sub-procedure.

# Breadth-First Search: Illustration and Guarantees

**Strategy:** Expand nodes in the order they were produced (FIFO frontier).

**Illustration:**



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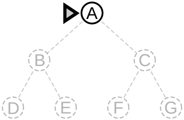


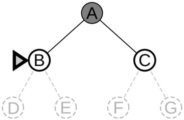

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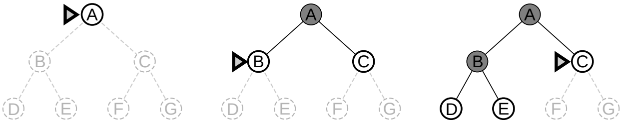

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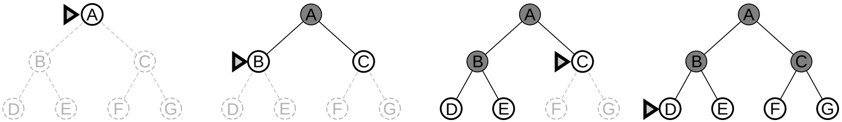

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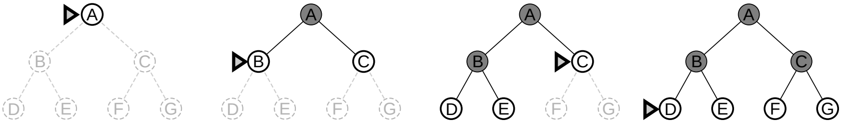


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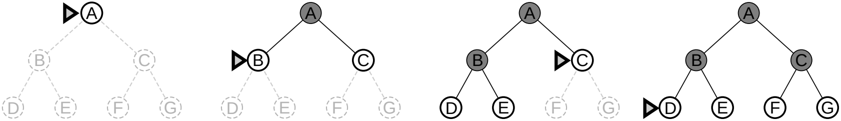


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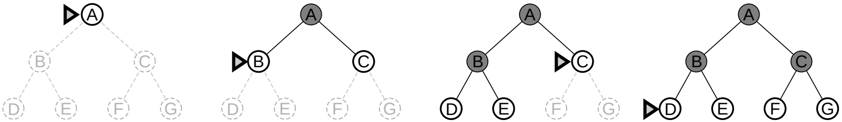


**Guarantees:**

- Completeness? Yes.
- Optimality? Yes, for uniform action costs. Breadth-first search always finds a shallowest goal vertex. If costs are not uniform, this is not necessarily optimal.

# Breadth-First Search: Complexity

**Time Complexity:** Say that $b$ is the maximal branching factor, and $d$ is the depth of goal vertex.

- Upper bound on the number of generated nodes? $b + b^2 + b^3 + \cdots + b^d$: In the worst case, the algorithm generates all nodes in the first $d$ layers.
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# Breadth-First Search: Example Data

**Setting:** $b = 10$; $10000$ nodes/second; $1000$ bytes/node.

**Yields data:** (inserting values into previous equations)

| Depth | Nodes | Time | | Memory | |
|---|---|---|---|---|---|
| 2 | 110 | .11 | milliseconds | 107 | kilobytes |
| 4 | 11110 | 11 | milliseconds | 10.6 | megabytes |
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**Strategy:** Expand the most recent nodes in (LIFO frontier).

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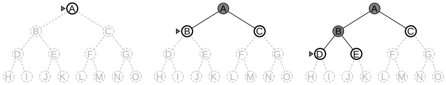

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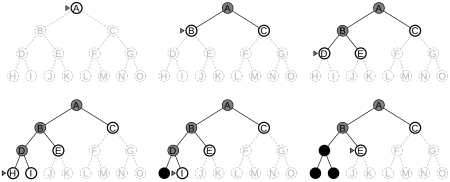

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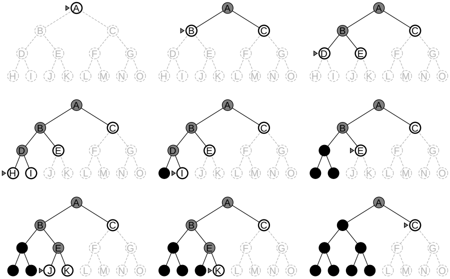

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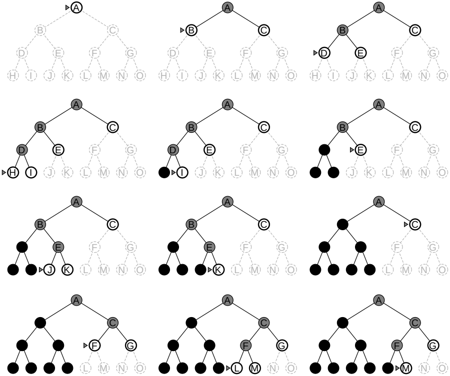

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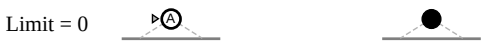

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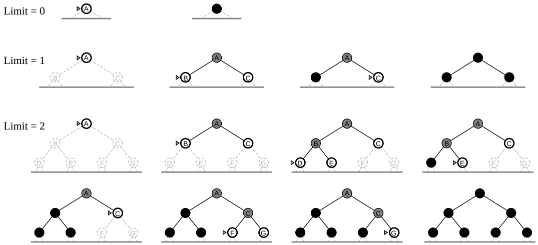

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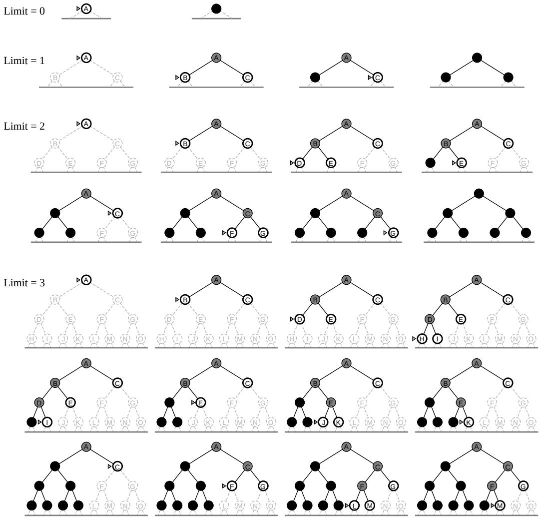

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